Abstract and Applied Analysis

Nonlinear Analysis of Dynamical Complex Networks

Guest Editors: Zidong Wang, Bo Shen, Hongli Dong, and Jun Hu

Nonlinear Analysis of Dynamical Complex Networks

Abstract and Applied Analysis

Nonlinear Analysis of Dynamical Complex Networks

Guest Editors: Zidong Wang, Bo Shen, Hongli Dong, and Jun Hu

Copyright © 2013 Hindawi Publishing Corporation. All rights reserved.

This is a special issue published in "Abstract and Applied Analysis." All articles are open access articles distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Editorial Board

Ravi P. Agarwal, USA
Bashir Ahmad, Saudi Arabia
M. O. Ahmedou, Germany
Nicholas D. Alikakos, Greece
Debora Amadori, Italy
Pablo Amster, Argentina
Douglas R. Anderson, USA
Jan Andres, Czech Republic
Giovanni Anello, Italy
Stanislav Antontsev, Portugal
Mohamed Kamal Aouf, Egypt
Narcisa C. Apreutesei, Romania
Natig M. Atakishiyev, Mexico
Ferhan M. Atici, USA
Ivan G. Avramidi, USA
Soohyun Bae, Korea
Zhanbing Bai, China
Chuanzhi Bai, China
Dumitru Baleanu, Turkey
Józef Banaś, Poland
Gerassimos Barbatis, Greece
Martino Bardi, Italy
Roberto Barrio, Spain
Feyzi Başar, Turkey
Abdelghani Bellouquid, Morocco
Daniele Bertaccini, Italy
Michiel Bertsch, Italy
Lucio Boccardo, Italy
Igor Boglaev, New Zealand
Martin J. Bohner, USA
Julian F. Bonder, Argentina
Geraldo Botelho, Brazil
Elena Braverman, Canada
Romeo Brunetti, Italy
Janusz Brzdek, Poland
Detlev Buchholz, Germany
Sun-Sig Byun, Korea
Fabio M. Camilli, Italy
Jinde Cao, China
Anna Capietto, Italy
Jianqing Chen, China
Wing-Sum Cheung, Hong Kong
Michel Chipot, Switzerland
Changbum Chun, Korea
Soon Y. Chung, Korea
Jaeyoung Chung, Korea
Silvia Cingolani, Italy
Jean M. Combes, France
Monica Conti, Italy
Diego Córdoba, Spain
Juan C. Cortés, Spain
Graziano Crasta, Italy
Bernard Dacorogna, Switzerland
Vladimir Danilov, Russia
Mohammad T. Darvishi, Iran
Luis F. Pinheiro de Castro, Portugal
T. Diagana, USA
Jesús I. Díaz, Spain
Josef Diblík, Czech Republic
Fasma Diele, Italy
Tomas Dominguez, Spain
Alexander I. Domoshnitsky, Israel
Marco Donatelli, Italy
Bo-Qing Dong, China
Ondřej Došlý, Czech Republic
Wei-Shih Du, Taiwan
Luiz Duarte, Brazil
Roman Dwilewicz, USA
Paul W. Eloe, USA
Ahmed El-Sayed, Egypt
Luca Esposito, Italy
Jose A. Ezquerro, Spain
Khalil Ezzinbi, Morocco
Dashan Fan, USA
Angelo Favini, Italy
Marcia Federson, Brazil
Stathis Filippas, Equatorial Guinea
Alberto Fiorenza, Italy
Tore Flåtten, Norway
Ilaria Fragala, Italy
Bruno Franchi, Italy
Xianlong Fu, China
Massimo Furi, Italy
Giovanni P. Galdi, USA
Isaac Garcia, Spain
Jesús García Falset, Spain
José A. García-Rodríguez, Spain
Leszek Gasinski, Poland
György Gát, Hungary
Vladimir Georgiev, Italy
Lorenzo Giacomelli, Italy
Jaume Giné, Spain
Valery Y. Glizer, Israel
Laurent Gosse, Italy
Jean P. Gossez, Belgium
Jose L. Gracia, Spain
Maurizio Grasselli, Italy
Qian Guo, China
Yuxia Guo, China
Chaitan P. Gupta, USA
Uno Hämarik, Estonia
Ferenc Hartung, Hungary
Behnam Hashemi, Iran
Norimichi Hirano, Japan
Jiaxin Hu, China
Chengming Huang, China
Zhongyi Huang, China
Gennaro Infante, Italy
Ivan Ivanov, Bulgaria
Hossein Jafari, Iran
Jaan Janno, Estonia
Aref Jeribi, Tunisia
Un C. Ji, Korea
Zhongxiao Jia, China
L. Jdar, Spain
Jong Soo Jung, Republic of Korea
Henrik Kalisch, Norway
Hamid Reza Karimi, Norway
Satyanad Kichenassamy, France
Tero Kilpelinen, Finland
Sung Guen Kim, Republic of Korea
Ljubisa Kocinac, Serbia
Andrei Korobeinikov, Spain
Pekka Koskela, Finland
Victor Kovtunenko, Austria
Ren-Jieh Kuo, Taiwan
Pavel Kurasov, Sweden
Miroslaw Lachowicz, Poland
Kunquan Lan, Canada
Ruediger Landes, USA
Irena Lasiecka, USA
Matti Lassas, Finland
Chun-Kong Law, Taiwan
Ming-Yi Lee, Taiwan
Gongbao Li, China

Elena Litsyn, Israel
Yansheng Liu, China
Shengqiang Liu, China
Carlos Lizama, Chile
Milton C. Lopes Filho, Brazil
Julian López-Gómez, Spain
Guozhen Lu, USA
Jinhu Lü, China
Grzegorz Lukaszewicz, Poland
Shiwang Ma, China
Wanbiao Ma, China
Eberhard Malkowsky, Turkey
Salvatore A. Marano, Italy
Cristina Marcelli, Italy
Paolo Marcellini, Italy
Jesús Marín-Solano, Spain
Jose M. Martell, Spain
MieczysThlaw MastyThlo, Poland
Ming Mei, Canada
Taras Mel'nyk, Ukraine
Anna Mercaldo, Italy
Changxing Miao, China
Stanislaw Migorski, Poland
Mihai Mihăilescu, Romania
Feliz Minhós, Portugal
Dumitru Motreanu, France
Roberta Musina, Italy
G. M. N'Guérékata, USA
Maria Grazia Naso, Italy
Sylvia Novo, Spain
Micah Osilike, Nigeria
Mitsuharu Ôtani, Japan
Turgut Öziş, Turkey
Filomena Pacella, Italy
Nikolaos S. Papageorgiou, Greece
Sehie Park, Korea
Alberto Parmeggiani, Italy
Kailash C. Patidar, South Africa
Kevin R. Payne, Italy
Ademir Fernando Pazoto, Brazil
Josip E. Pečarić, Croatia
Shuangjie Peng, China
Sergei V. Pereverzyev, Austria
Maria Eugenia Perez, Spain
Josefina Perles, Spain
Allan Peterson, USA
Andrew Pickering, Spain
Cristina Pignotti, Italy
Somyot Plubtieng, Thailand
Milan Pokorny, Czech Republic
Sergio Polidoro, Italy
Ziemowit Popowicz, Poland
Maria M. Porzio, Italy
Enrico Priola, Italy
Vladimir S. Rabinovich, Mexico
Irena Rachůnková, Czech Republic
Maria Alessandra Ragusa, Italy
Simeon Reich, Israel
Abdelaziz Rhandi, Italy
Hassan Riahi, Malaysia
Juan P. Rincón-Zapatero, Spain
Luigi Rodino, Italy
Yuriy V. Rogovchenko, Norway
Julio D. Rossi, Argentina
Wolfgang Ruess, Germany
Bernhard Ruf, Italy
Marco Sabatini, Italy
Satit Saejung, Thailand
Stefan G. Samko, Portugal
Martin Schechter, USA
Javier Segura, Spain
Sigmund Selberg, Norway
Valery Serov, Finland
Naseer Shahzad, Saudi Arabia
Andrey Shishkov, Ukraine
Stefan Siegmund, Germany
Abdel-Maksoud A. Soliman, Egypt
Pierpaolo Soravia, Italy
Marco Squassina, Italy
Svatoslav Staněk, Czech Republic
Stevo Stevic, Serbia
Antonio Suárez, Spain
Wenchang Sun, China
Robert Szalai, UK
Sanyi Tang, China
Chun-Lei Tang, China
Youshan Tao, China
Gabriella Tarantello, Italy
Nasser-eddine Tatar, Saudi Arabia
Susanna Terracini, Italy
Gerd Teschke, Germany
Alberto Tesei, Italy
Bevan Thompson, Australia
Sergey Tikhonov, Spain
Claudia Timofte, Romania
Thanh Tran, Australia
Juan J. Trujillo, Spain
Gabriel Turinici, France
Milan Tvrdy, Czech Republic
Mehmet Unal, Turkey
Csaba Varga, Romania
Carlos Vazquez, Spain
Gianmaria Verzini, Italy
Jesus Vigo-Aguiar, Spain
Qing-Wen Wang, China
Yushun Wang, China
Shawn X. Wang, Canada
Youyu Wang, China
Jing Ping Wang, UK
Peixuan Weng, China
Noemi Wolanski, Argentina
Ngai-Ching Wong, Taiwan
Patricia J. Y. Wong, Singapore
Zili Wu, China
Yong Hong Wu, Australia
Shanhe Wu, China
Tie-cheng Xia, China
Xu Xian, China
Yanni Xiao, China
Fuding Xie, China
Gonang Xie, China
Naihua Xiu, China
Daoyi Xu, China
Zhenya Yan, China
Xiaodong Yan, USA
Norio Yoshida, Japan
Beong In Yun, Korea
Vjacheslav Yurko, Russia
Agacik Zafer, Turkey
Sergey V. Zelik, UK
Jianming Zhan, China
Meirong Zhang, China
Weinian Zhang, China
Chengjian Zhang, China
Zengqin Zhao, China
Sining Zheng, China
Tianshou Zhou, China
Yong Zhou, China
Qiji J. Zhu, USA
Chun-Gang Zhu, China
Malisa R. Zizovic, Serbia
Wenming Zou, China

Contents

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 530124, 3 pages
http://dx.doi.org/10.1155/2013/530124

Editorial

Nonlinear Analysis of Dynamical Complex Networks

Zidong Wang,[1,2] Bo Shen,[1] Hongli Dong,[3,4] and Jun Hu[4,5]

[1] *School of Information Science and Technology, Donghua University, Shanghai 200051, China*
[2] *Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, UK*
[3] *College of Electrical and Information Engineering, Northeast Petroleum University, Daqing 163318, China*
[4] *Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China*
[5] *Department of Applied Mathematics, Harbin University of Science and Technology, Harbin 150080, China*

Correspondence should be addressed to Zidong Wang; zidong.wang@brunel.ac.uk

Received 10 October 2013; Accepted 10 October 2013

Copyright © 2013 Zidong Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Complex networks are composed of a large number of highly interconnected dynamical units and therefore exhibit very complicated dynamics. Examples of such complex networks include the Internet, that is, a network of routers or domains, the World Wide Web (WWW), that is, a network of websites, the brain, that is, a network of neurons, and an organization, that is, a network of people. Since the introduction of the small-world network principle, a great deal of research has been focused on the dependence of the asymptotic behavior of interconnected oscillatory agents on the structural properties of complex networks. It has been found out that the general structure of the interaction network may play a crucial role in the emergence of synchronization phenomena in various fields such as physics, technology, and the life sciences.

Complex networks have already become an ideal research area for control engineers, mathematicians, computer scientists, and biologists to manage, analyze, and interpret functional information from real-world networks. Sophisticated computer system theories and computing algorithms have been exploited or emerged in the general area of computer mathematics, such as analysis of algorithms, artificial intelligence, automata, computational complexity, computer security, concurrency and parallelism, data structures, knowledge discovery, DNA and quantum computing, randomization, semantics, symbol manipulation, numerical analysis and mathematical software. This special issue aims to bring together the latest approaches to understanding complex networks from a dynamic system perspective. Topics include, but are not limited to the following aspects of dynamics analysis for complex networks: (a) synchronization and control, (b) topology structure and dynamics, (c) stability analysis, (d) robustness and fragility, and (e) Applications in real-world complex networks.

We have solicited submissions to this special issue from electrical engineers, control engineers, mathematicians, and computer scientists. After a rigorous peer review process, 17 papers have been selected that provide overviews, solutions, or early promises to manage, analyze, and interpret dynamical behaviors of complex systems. These papers have covered both the theoretical and practical aspects of complex systems in the broad areas of dynamical systems, mathematics, statistics, operational research, and engineering.

This special issue starts with a survey paper on the recent advances of multiobjective control and filtering problems for nonlinear stochastic systems with variance constraints. Specifically, in the paper entitled "*Variance-constrained multiobjective control and filtering for nonlinear stochastic systems: a survey*" by L. Ma et al., the focus is to provide a timely review on the recent advances of the multiobjective control and filtering issues for nonlinear stochastic systems with variance constraints. Firstly, the concepts of nonlinear stochastic systems are recalled along with the introduction of some recent advances. Then, the covariance control theory, which serves as a practical method for multiobjective control design as well as a foundation for linear system theory, is reviewed comprehensively. The multiple design requirements frequently applied in engineering practice for the use of evaluating

system performances are introduced, including robustness, reliability and dissipativity. Several design techniques suitable for the multiobjective variance-constrained control and filtering problems for nonlinear stochastic systems are discussed. The design objects (nonlinear stochastic system), design requirements (multiple performance specifications including variance constraints), several design techniques and a special case of the addressed problem, mixed H_2/H_∞ design problem, have been discussed in great detail with some recent advances. Subsequently, some latest results on the variance-constrained multiobjective control and filtering problems for nonlinear stochastic systems are summarized. Finally, concluding remarks are drawn and several possible future research directions are pointed out.

Complex networks have been extensively used in theoretical analysis of dynamical complex systems, such as the Internet, World Wide Web, communication networks, and social networks. Accordingly, the synchronization of dynamics complex networks has attracted a great deal of attention. In the paper entitled "*Achieving synchronization in arrays of coupled differential systems with time-varying couplings*" by X. Yi et al., the synchronization problem is studied for complex dynamical networks described by linearly coupled ordinary differential equation systems. The time-varying coupling is used to represent the interaction between individuals. A general sufficient condition is derived such that the directed time-varying graph reaches consensus. Finally, a numerical simulation is provided to show the effectiveness of the theoretical results. The synchronization and reconstruction problem is investigated in "*Chaos synchronization based on unknown input proportional multiple-integral fuzzy observer*" by T. Youssef et al. for chaotic systems. An unknown input proportional multiple-integral observer is designed for synchronization of chaotic systems with immeasurable decision variables and unknown input. By using the Lyapunov stability theory and the linear matrix inequality (LMI) technique, sufficient conditions are given to ensure the synchronization. In the work entitled "*Synchronization of switched complex bipartite neural networks with infinite distributed delays and derivative coupling*" by Q. Bian et al., the synchronization problem is investigated for two coupled switched complex bipartite neural networks (SCBNNs) with distributed delays and derivative coupling. By constructing effective controllers, some synchronization criteria are proposed to ensure the synchronization of these two SCBNNs. The distributed consensus problem is studied in "*Distributed impulsive consensus of the multiagent system without velocity measurement*" by Z. Liu et al. for continuous-time multiagent system under intermittent communication. An impulsive consensus algorithm is developed, where the local algorithm of each agent is only based on the position information. Also, some necessary and sufficient conditions for consensus are given. Finally, a numerical example is given to illustrate the effectiveness of the theoretical analysis. Subsequently, in the paper entitled "*Distributed consensus for discrete-time directed networks of multiagents with time-delays and random communication links*" by Y. Liu et al., the leader-following consensus problem is addressed for discrete-time directed multiagent systems with time-delay and random communication links. By constructing new Lyapunov functionals and employing some analytical techniques, sufficient conditions for the leader-following consensus in mean-square sense are established for multiagent system.

Over the past decades, the design of the controllers has long been the mainstream of research topics and much effort has been made for dynamical complex networks. In the paper entitled "*Sliding intermittent control for BAM neural networks with delays*" by J. Hu et al., the exponential stability problem is firstly studied for a class of delayed bidirectional associative memory (BAM) neural networks with delays. By taking the advantages of the periodically intermittent control idea and the impulsive control scheme, a sliding intermittent controller is designed for the addressed BAM system with time-delays. It is shown that such a sliding intermittent control method can comprise several kinds of control schemes as special cases. Also, some sufficient conditions are proposed such that the closed-loop delayed BAM neural networks are globally exponentially stable. The design of nonfragile gain-scheduled controller is discussed in "*Nonfragile gain-scheduled control for discrete-time stochastic systems with randomly occurring sensor saturations*" by W. Li et al. for a class of discrete stochastic systems with randomly occurring sensor saturations (ROSSs). The sensor saturations occur in a random way. By constructing the probability-dependent Lyapunov functional, a nonfragile gain-scheduled controller with the gain including both constant and time-varying parameters is designed such that the closed-loop system is exponentially stable in the mean-square sense. In the work entitled "*Optimal guaranteed cost control of a class of discrete-time nonlinear systems with Markovian switching and mode-dependent mixed time-delays*" by Y. Liu, the guaranteed cost control problem is addressed for a class of nonlinear discrete-time systems with Markovian jumping parameters and mixed time-delays. The mixed time-delays include both the mode-dependent discrete delay and the infinite distributed delay with mode-dependent lower bound. By constructing novel Lyapunov-Krasovskii functionals, some sufficient conditions for the existence of guaranteed cost controllers are derived. Also, a convex optimization approach is developed to minimize cost function so as to obtain the optimal guaranteed cost controller. By using the dwell time approach, in the paper entitled "*Stabilization and controller design of 2D discrete switched systems with state delays under asynchronous switching*" by S. Huang et al., the stability analysis and controller design are conducted for 2D discrete switched delayed systems represented by Roesser's model. Accordingly, sufficient conditions are given such that the resulting closed-loop systems are exponentially stable.

In the past years, the stability analysis of the dynamical complex networks has attracted much attention. In the work entitled "*An analysis of stability of a class of neutral-type neural networks with discrete time delays*" by Z. Orman and S. Arik, the problems of existence, uniqueness, and global asymptotic stability are discussed for a class of neutral-type neural network with discrete time-delays. By employing a Lyapunov functional and using the homeomorphism mapping theorem, some new delay-independent sufficient conditions are derived to guarantee the existence, uniqueness, and global

asymptotic stability of the equilibrium point. It is shown that the advantage of the proposed results is that the developed results can be expressed in terms of network parameters only. Finally, some examples are provided to compare the proposed results with the existing results and to illustrate the effectiveness of the main results. In the work entitled "*Robust almost periodic dynamics for interval neural networks with mixed time-varying delays and discontinuous activation functions*" by H. Wu et al., the delay-dependent robust exponential stability problem is studied for almost periodic solution of interval neural networks with mixed time-varying delays and discontinuous activation functions. According to the nonsmooth Lyapunov stability theory and employing the LMI technique, a new delay-dependent criterion is given to guarantee the existence and globally exponential stability of almost periodic solution. Also, the proposed results are extended to prove the existence and robust stability of periodic solution for neural networks with mixed time-varying delays and discontinuous activations. Finally, a numerical example is provided to show the feasibility of the developed results. The problem of bounded-input bounded-output (BIBO) stability is investigated in "*BIBO stability analysis for delay switched systems with nonlinear perturbation*" by J. Wei et al. for a class of switched systems with mixed neutral delays and nonlinear perturbation. By constructing the Lyapunov-Krasovskii functional, new BIBO stabilization criteria are derived. Finally, a numerical simulation is given to demonstrate the usefulness of the proposed results.

As is well known, the estimation theory has important applications in a variety of areas. In the work entitled "*Deconvolution filtering for nonlinear stochastic systems with randomly occurring sensor delays via probability-dependent method*" by Y. Luo et al., the deconvolution filtering problem is studied for a class of discrete-time stochastic systems with randomly occurring sensor delays and external stochastic noises. By constructing the probability-dependent Lyapunov functional and employing convex optimization approach, sufficient condition is given to ensure the stability of the addressed stochastic systems. The proposed gain-scheduled filters include both constant parameters and time-varying gains which can be updated online according to the measurable missing probabilities in real time. It is shown that the desired filters can be easily obtained by solving a set of linear matrix inequalities (LMIs). Finally, a simulation example is provided to illustrate the feasibility and effectiveness of the proposed filtering scheme. Subsequently, in the paper entitled "*Estimate of number of periodic solutions of second-order asymptotically linear difference system*" by H. Bin and Z. Huang, the number of periodic solutions is discussed for second-order linear difference system. By using the Morse theory and twist number, three cases are discussed. As the system is resonant at infinity, the perturbation method is used to study the compactness condition of functional. Some new results are derived concerning the lower bounds of the nonconstant periodic solutions for discrete system. In the paper entitled "*Convergence rate of numerical solutions for nonlinear stochastic pantograph equations with Markovian switching and jumps*" by Z. W. Lu et al., the study of convergence rate is addressed for nonlinear stochastic pantograph equations with Markovian switching and Poisson jump. Sufficient conditions of existence and uniqueness of the solutions are given for nonlinear stochastic pantograph equations with Markovian switching and jumps. It is shown that Euler-Maruyama scheme for nonlinear stochastic pantograph equations with Markovian switching and Brownian motion is of convergence with strong order 1/2. The mean-square convergence is preferable to be used for nonlinear stochastic pantograph equations with Markovian switching and pure jumps. Accordingly, the order of mean-square convergence is close to 1/2. Moreover, the impact of human dynamics on the information propagation in online social networks is discussed in "*Information propagation in online social network based on human dynamics*" by Q. Yan et al. Also, an extended susceptible-infected (SI) propagation model is proposed to incorporate bursty human activity patterns and limited attention. The proposed result can be used to optimize/control the information propagation in online social networks.

Acknowledgments

This special issue is a timely reflection of the research progress in the area of nonlinear analysis of dynamical complex networks. We would like to acknowledge all authors for their efforts in submitting high-quality papers. We are also very grateful to the reviewers for their thorough and on-time reviews of the papers.

Zidong Wang
Bo Shen
Hongli Dong
Jun Hu

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 724018, 13 pages
http://dx.doi.org/10.1155/2013/724018

Review Article

Variance-Constrained Multiobjective Control and Filtering for Nonlinear Stochastic Systems: A Survey

Lifeng Ma,[1] Zidong Wang,[2,3] Hongli Dong,[4,5] and Guoliang Wei[6]

[1] *School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China*
[2] *School of Information Science and Technology, Donghua University, Shanghai 200051, China*
[3] *Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, UK*
[4] *Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China*
[5] *College of Electrical and Information Engineering, Northeast Petroleum University, Daqing 163318, China*
[6] *Shanghai Key Lab of Modern Optical System, Department of Control Science and Engineering, University of Shanghai for Science and Technology, Shanghai 200093, China*

Correspondence should be addressed to Zidong Wang; zidong.wang@brunel.ac.uk

Received 2 October 2013; Accepted 6 October 2013

Academic Editor: Bo Shen

Copyright © 2013 Lifeng Ma et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The multiobjective control and filtering problems for nonlinear stochastic systems with variance constraints are surveyed. First, the concepts of nonlinear stochastic systems are recalled along with the introduction of some recent advances. Then, the covariance control theory, which serves as a practical method for multi-objective control design as well as a foundation for linear system theory, is reviewed comprehensively. The multiple design requirements frequently applied in engineering practice for the use of evaluating system performances are introduced, including robustness, reliability, and dissipativity. Several design techniques suitable for the multi-objective variance-constrained control and filtering problems for nonlinear stochastic systems are discussed. In particular, as a special case for the multi-objective design problems, the mixed H_2/H_∞ control and filtering problems are reviewed in great detail. Subsequently, some latest results on the variance-constrained multi-objective control and filtering problems for the nonlinear stochastic systems are summarized. Finally, conclusions are drawn, and several possible future research directions are pointed out.

1. Introduction

It is widely recognized that, in almost all engineering applications, nonlinearities are inevitable and could not be eliminated thoroughly. Hence, the nonlinear systems have gained more and more research attention, and lots of results have been published. On the other hand, due to the wide appearance of the stochastic phenomena in almost every aspect of our daily life, stochastic systems which have found successful applications in many branches of science and engineering practice have stirred quite a lot of research interests during the past few decades. Therefore, the control and filtering problems for nonlinear stochastic systems have been studied extensively so as to meet ever-increasing demand toward systems with both nonlinearities and stochasticity.

In many engineering control/filtering problems, the performance requirements are naturally expressed by the upper bounds on the steady state covariance which is usually applied to scale the control/estimation precision, one of the most important performance indices of stochastic design problems. As a result, a large number of control and filtering methodologies have been developed to seek a convenient way to solve the variance-constrained design problems, among which the LQG control and Kalman filtering are two representative minimum variance design algorithms.

On the other hand, in addition to the variance constraints, real-world engineering practice also desires the simultaneous satisfaction of many other frequently seen performance requirements including stability, robustness, reliability, and energy constraints, to name but a few key ones. It gives the

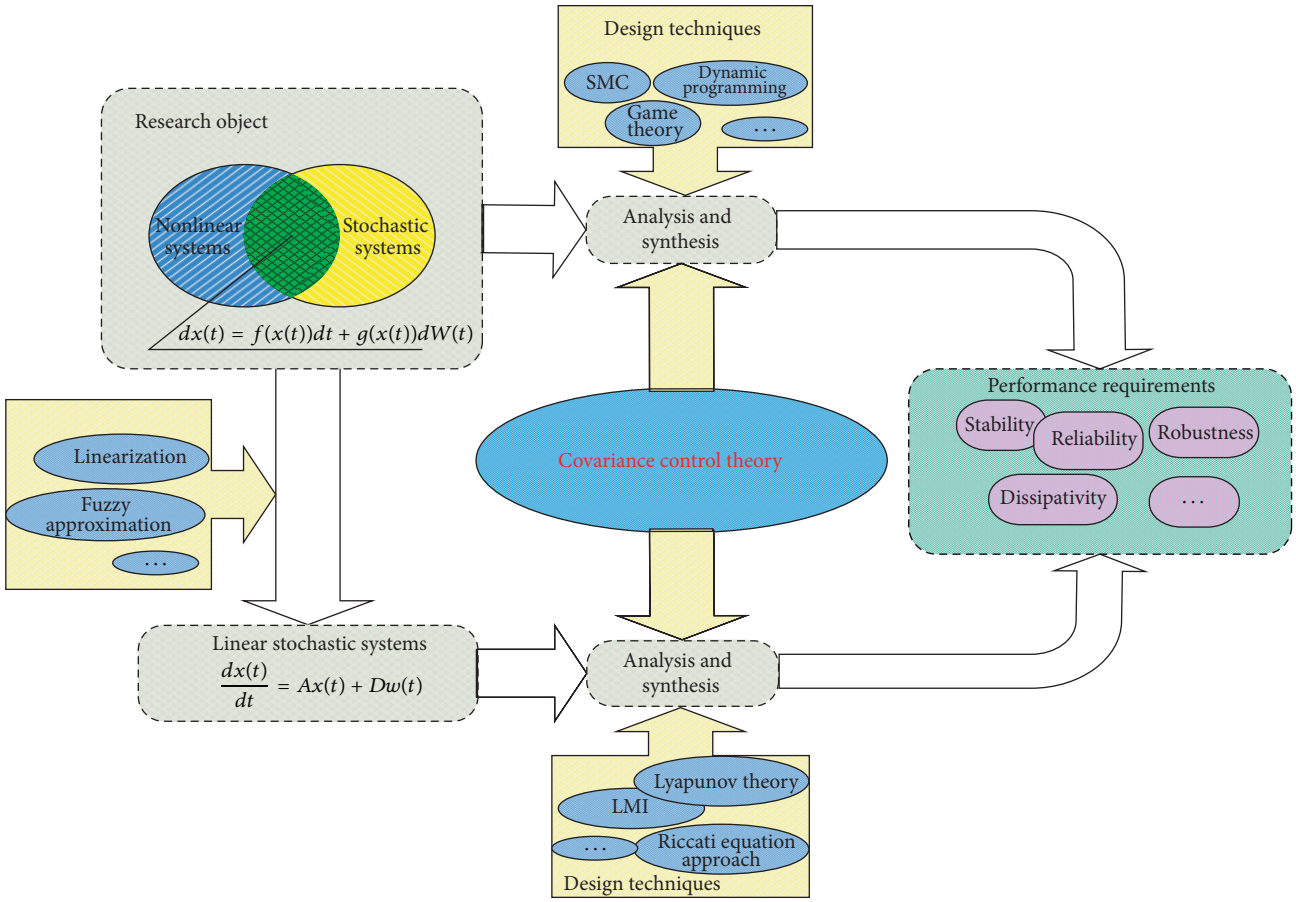


Figure 1: Architecture of surveyed contents.

rise to the so-called multiobjective design problems, in which multiple cost functions or performance requirements are simultaneously considered with constraints being imposed on the system. An example of multiobjective control design would be to minimize the system steady-state variance indicating the performance of control precision, subject to a prespecified external disturbance attenuation level evaluating system robustness. Obviously, multiobjective design methods have the ability to provide more flexibility in dealing with the tradeoffs and constraints in a much more explicit manner on the prespecified performance requirements than those conventional optimization methodologies like LQG control scheme or H_∞ design technique, which does not seem to have the ability of handling multiple performance specifications.

When coping with the multiobjective design problem with variance constraints for stochastic systems, the well-known covariance control theory provides us with a useful tool for the system analysis and synthesis. For linear stochastic systems, it has been shown that multiobjective control/filtering problems can be formulated using linear matrix inequalities (LMIs), due to their ability to include desirable performance objectives such as variance constraints, H_2 performance, H_∞ performance, and pole placement as convex constraints. However, as the nonlinear stochastic systems are concerned, the relevant progress so far has been very slow due primarily to the difficulties in dealing with the variance related problems resulting from the complexity of the nonlinear dynamics. A key issue for the nonlinear covariance control study is the existence of the covariance of nonlinear stochastic systems and its mathematical expression, which is extremely difficult to investigate because of the complex coupling of nonlinearities and stochasticity. Therefore, it is not surprising that the multiobjective control and filtering problems for nonlinear stochastic systems with variance constraints have not been adequately investigated despite their clear engineering insights and good application prospect.

In this paper, we focus mainly on the multiobjective control and filtering problems for nonlinear systems with variance constraints and aim to give a comprehensive survey on some recent advances in this area. The design objects (nonlinear stochastic system), design requirements (multiple performance specifications including variance constraints), several design techniques, and a special case of the addressed problem, mixed H_2/H_∞ design problem, have been discussed in great detail with some recent advances. The contents that are reviewed in this paper and the architecture are shown in Figure 1.

The rest of the paper is organized as follows. In Section 2, the nonlinear stochastic systems are reviewed with some

recent advances. Section 3 reviews the covariance control theory. Several widely applied performance requirements in engineering practice and commonly seen design techniques in the addressed multiobjective problems are then discussed. Moreover, a special case of multiobjective control and filtering problems, namely, mixed H_2/H_∞ design problem, is surveyed in great detail. Section 4 gives latest results on multiobjective control and filtering problems of nonlinear stochastic systems with variance constraints. The conclusions and future work are given in Section 5.

2. Analysis and Synthesis of Nonlinear Stochastic Systems

For several decades, nonlinear stochastic systems have been attracting increasing attention in the system and control community due to their extensive applications in a variety of areas ranging from communication and transportation to manufacturing, building automation, computing, automotive, and chemical industry, to mention just a few key areas. In this section, the analysis and synthesis problems for nonlinear systems and stochastic systems are recalled, respectively, and some recent advances in these areas are also given.

2.1. Nonlinear Systems. It is well recognized that in almost all engineering applications, nonlinearities are inevitable and could not be eliminated thoroughly. Hence, the nonlinear systems have gained more and more research attention, and lots of results have been reported; see, for example, [1–4]. When analyzing and designing nonlinear dynamical systems, there are a wide range of nonlinear analysis tools, among which the most common and wildly used is linearization because of the powerful tools we know for linear systems. It should be pointed out that, however, there are two basic limitations of linearization [5]. (i) As is well known, linearization is an approximation in the neighborhood of certain operating points. Thus, the resulting linearized system can only show the local behavior of the nonlinear system in the vicinity of those points. Neither nonlocal behavior of the original nonlinear system far away from those operating points nor global behavior throughout the entire state space can be correctly revealed after linearization. (ii) The dynamics of a nonlinear system are much richer than those of a linear system. There are essentially nonlinear phenomena, like finite escape time, multiple isolated equilibria, subharmonic, harmonic, or almost periodic oscillations, to name just a few key ones which can take place only in the presence of nonlinearity; hence, they cannot be described by linear models [6–9]. Therefore, as a compromise, during the past few decades, there has been tremendous interest in studying nonlinear systems with nonlinearities being taken as the exogenous disturbance input to a linear system, since it could better illustrate the dynamics of the original nonlinear system than the linearized one with less sacrifice of the convenience on the application of existing mathematical tools. The nonlinearities emerging in such systems may arise from the linearization process of an originally highly nonlinear plant or may be an external nonlinear input, which would drastically degrade the system performance or even cause instability; see, for example, [10–13].

On the other hand, in real-world applications, one of the most inevitable and physically important features of some sensors and actuators is that they are always corrupted by different kinds of nonlinearities, either from within the devices themselves or from the external disturbances. Such nonlinearities are generally resulting from equipment limitations as well as the harsh environments such as uncontrollable elements (e.g., variations in flow rates, temperature) and aggressive conditions (e.g., corrosion, erosion, and fouling) [14]. Since the sensor/actuator nonlinearity cannot be simply ignored and often leads to poor performance of the controlled system, a great deal of effort in investigating the analysis and synthesis problems has been devoted by many researchers to the study of various systems with sensor/actuator nonlinearities; see [15–20].

Recently, the system with randomly occurring nonlinearities (RONs) has started to stir quite a lot of research interests as it reveals an appealing fact that, instead of occurring in a deterministic way, a large quantity of nonlinearities in real-world systems would probably take place in a random way. Some of the representative publications can be discussed as follows. The problem of randomly occurring nonlinearities was raised in [21], where an iterative filtering algorithm has been proposed for the stochastic nonlinear system in presence of both RONs and output quantization effects. The filter parameters can be obtained by resorting to solving certain recursive linear matrix inequalities. The obtained results have been soon extended to the case of multiple randomly occurring nonlinearities in [22]. Such a breakthrough on how to deal with nonlinear systems with RONs has been well recognized and quickly followed by other researchers in the area. Using similar techniques, the filtering as well as control problems have been solved for a wide range of systems containing nonlinearities that are occurring randomly, like Markovian jump systems in [23, 24], sliding mode control systems in [25], discrete-time complex networks in [26], sensor networks in [27], time-delay systems in [28], and other types of nonlinear systems [29–31], which therefore has proven that the method developed in [21] is quite general and is applicable to the analysis and synthesis of many different kinds of nonlinear systems.

It should be emphasized that, for nonlinearities, there are many different constraints conditions for certain aim, such as Lipschitz conditions, among which the kind of stochastic nonlinearities described by statistical means has drawn particular research focus since it covers several well-studied nonlinearities in stochastic systems; see [29, 32–35] and the references therein. Several techniques for analysis and synthesis of this type of nonlinear systems have been exploited, including linear matrix inequality approach [32], Riccati equation method [33], recursive matrix inequality approach [34], gradient method [35], sliding mode control scheme [36], and the game theory approach [29].

2.2. Stochastic Systems. As is well known, in the past few decades, there have been extensive study and application

of stochastic systems because the stochastic phenomenon is inevitable and cannot be avoided in the real-world systems. When modeling such kinds of systems, the way neglecting the stochastic disturbances, which is a conventional technique in traditional control theory for deterministic systems, is not suitable anymore. Having realized the necessity of introducing more realistic models, nowadays, a great number of real-world systems such as physical systems, financial systems, and ecological systems, as well as social systems, are more suitable to be modeled by stochastic systems, and therefore the stochastic control problem which deals with dynamical systems, described by difference or differential equations, and subject to disturbances characterized as stochastic processes has drawn much research attention; see [37] and the references therein. It is worth mentioning that a kind of stochastic systems represented as deterministic system adding a stochastic disturbance characterized as white noise has gained special research interests and found extensively applications in engineering based on the fact that it is possible to generate stochastic processes with covariance functions belonging to a large class simply by sending white noise through a linear system; hence, a large class of problems can be reduced to the analysis of linear systems with white noise inputs; see [38–42] for examples.

Parallel to the control problems, the filtering and prediction theory for stochastic systems which aims to extract a signal from observations of signals and disturbances has been well studied and found widely applied in many engineering fields. It also plays a very important role in the solution of the stochastic optimal control problem. The research on filtering problem was originated in [43], where the well-known Wiener-Kolmogorov filter has been proposed. However, the Wiener-Kolmogorov filtering theory has not been widely applied mainly because it requires the solution of an integral equation (the Wiener-Hopf equation) which is not easy to solve either analytically or numerically. In [44, 45], Kalman and Bucy gave a significant contribution to the filtering problem, by giving the celebrated Kalman-Bucy filter which could solve the filtering problem recursively. Kalman-Bucy filter (also known as H_2 filter) has been extensively adopted and widely used in many branches of stochastic control theory, since the fast development of digital computers recently; see [46–49] and the references therein.

3. Multiobjective Control and Filtering with Variance Constraints

In this section, we first review the covariance control theory which provides us with a powerful tool in variance-constrained design problems with multiple requirements specified by the engineering practice. Then, we discuss several important performance specifications including robustness, reliability, and dissipativity. Two common techniques for solving the addressed problems for nonlinear stochastic systems are introduced. The mixed H_2/H_∞ design problem is reviewed in great detail as a special case of multiobjective control/filtering problem with variance constraints.

3.1. Covariance Control Theory. As we have stated in the previous section, engineering control problems always require upper bounds on the steady state covariances [41, 50, 51]. However, many control design techniques used in both theoretical analysis and engineering practice, such as LQG and H_∞ design, do not seem to give a direct solution to this kind of design problem since they lack a convenient avenue for imposing design objectives stated in terms of upper bounds on the variance values. For example, the LQG controllers minimize a linear quadratic performance index without guaranteeing the variance constraints with respect to individual system states. The covariance control theory [52] developed in the late 80s has provided a more direct methodology for achieving the individual variance constraints than the LQG control theory. The covariance control theory aims to solve the variance-constrained control problems while satisfying other performance indices [40, 47, 52, 53]. It has been shown that the covariance control approach is capable of solving multiobjective design problems, which has found applications in dealing with transient responses, round off errors in digital control, residence time/probability in aiming control problems, and stability, robustness in the presence of parameter perturbations [53]. Such an advantage is based on the fact that several control design objectives, such as stability, time-domain and frequency-domain performance specifications, robustness, and pole location, can be directly related to steady-state covariances of the closed-loop systems. Therefore, covariance control theory serves as a practical method for multiobjective control design as well as a foundation for linear system theory.

On the other hand, it is always the case in real-world applications such as the tracking of a maneuvering target, that the filtering precision is characterized by the error variance of estimation [53, 54]. Considering its clear engineering insights, in the past few years, the filtering problem with error variance constraints has received much interests and a large amount of research fruit has been reported in the literature [44, 45, 55, 56]. The celebrated Kalman filtering approach is a typical method which aims to obtain the state information based on the minimization of the variance of the estimation error [44, 45]. Nevertheless, the strict request of a highly accurate model seriously impedes the application of Kalman filtering as in many cases only an approximate model of the system is available. It therefore has brought about remarkable research interests to the robust filtering method which aims to minimize the error variance of estimation against the system uncertainties or external unknown disturbances [57, 58]. Despite certain merits and successful applications, as in the case of LQG control problem, the traditional minimum variance filtering techniques cannot directly impose the designing objectives stated in terms of upper bounds on the error variance values, by which we mean that those techniques try to minimize the filtering error variance in mean square sense rather than to constrain it within a prespecified bound, which is obviously better to meet the requirements of practical engineering. Motivated by the covariance control theory, in [59], the authors have proposed a more direct designing procedure for achieving the individual variance constraint in filtering problems. Due to

its design flexibility, the covariance control theory is capable of solving the error variance-constrained filtering problem while guaranteeing other multiple designing objectives [60]. Therefore, it always serves as one of the most powerful tools in dealing with the multiobjective filtering as well as control problems [61].

It should be pointed out that most of the available literature regarding covariance control theory has been concerned with *linear time invariant* stochastic systems with the linear matrix inequality (LMI) approach. Moreover, when it comes to the variance-constrained controller/filter design problems for much more complicated systems such as *time-varying systems, nonlinear systems, and Markovian Jump systems* unfortunately, the relevant results have been very few due primarily to the difficulties in dealing with the existence problem of the steady-state covariances and their mathematical expressions for the abovementioned complex systems. With the hope to resolve such difficulties, in recent years, special efforts have been devoted in study of the variance-constrained multiobjective design problems for systems of complex dynamics, and several methodologies for analysis and synthesis have been developed. For example, in [47], a Riccati equation method has been proposed to solve the filtering problem for linear time-varying stochastic systems with prespecified error variance bounds. In [62–64], by means of the technique of sliding mode control (SMC), robust controller design problem has been solved for linear parameter perturbed systems, since SMC has certain robustness to matched disturbances or parameter perturbations. We shall return to this SMC problem later, and more details will be discussed in the following section.

When it comes to nonlinear stochastic systems, limited work has been done in the covariance-constrained analysis and design problems, just as what we have anticipated. A multiobjective filter has been designed in [65] for systems with Lipschitz-type nonlinearity, but the variance bounds cannot be prespecified. Strictly speaking, such an algorithm cannot be referred to as variance-constrained filtering in view of lack of capability for directly imposing specified constraints on variance. An LMI approach has been proposed in [32] to cope with robust filtering problems for a class of stochastic systems with nonlinearities characterized by statistical means, attaining an assignable H_2 performance index. In [61], for a special class of nonlinear stochastic systems, namely, systems with multiplicative noises (also called bilinear systems or systems with state/control dependent noises), a state feedback controller has been put forward in a unified LMI framework in order to ensure that the multiple objectives including stability, H_∞ specification, and variance constraints are simultaneously satisfied. This paper is always regarded as the origination of covariance control theory for nonlinear systems, as within the established theoretical framework, quite a lot of performance requirements can be taken into consideration simultaneously. Furthermore, with the developed techniques, the obtained elegant results could be easily extended to a wide range of nonlinear stochastic systems; see, for example, [29, 35, 66–68]. We shall return to such a type of nonlinear stochastic systems later to present more details of recent progresses in Section 4.

3.2. Multiple Performance Requirements. In the following, several performance indices originated from the engineering practice and frequently applied in multiobjective design problems are introduced.

3.2.1. Robustness. In real-world engineering practice, various reasons such as variations of the operating point, aging of devices, and identification errors, would lead to the parameter uncertainties which result in the perturbations of the elements of a system matrix when modeling the system in a state-space form. Such a perturbation in system parameters cannot be avoided and would cause degradation (sometimes even instability) to the system performance. Therefore, in the past decade, considerable attention has been devoted to different issues for linear or nonlinear uncertain systems, and a great number of papers have been published; see [2, 48, 69–74] for some recent results.

On another research frontier of robust control, the H_∞ design method which is used to design controller/filter with guaranteed performances with respect to the external disturbances as well as internal perturbations has received an appealing research interest during the past decades; see [75–78], for instance. Since Zames' original work [75], significant advances have been made in the research area of H_∞ control and filtering. The standard H_∞ control problem has been completely solved by Doyle et al. for linear systems by deriving simple state-space formulas for all controllers [76]. For nonlinear systems, the H_∞ performance evaluation can be conducted through analyzing the L_2 gain of the relationship between the external disturbance and the system output, which is a necessary step to decide whether further controller design is needed. In the past years, the nonlinear H_∞ control problem has also received considerable research attention, and many results have been available in the literature [77–81]. On the other hand, the H_∞ filtering problem has also gained considerable research interests along with the development of H_∞ control theory; see [26, 79, 82–85]. It is well known that the existence of a solution to the H_∞ filtering problem is in fact associated with the solvability of an appropriate algebraic Riccati equality (for the linear cases) or a so-called Hamilton-Jacobi equation (for the nonlinear ones). So far, there have been several approaches for providing solutions to nonlinear H_∞ filtering problems, few of which, however, are capable of handling multiple performance requirements in an H_∞ optimization framework.

It is worth mentioning that, in contrast to the H_∞ design framework within which multiple requirements can hardly be under simultaneous consideration, the covariance control theory has provided a convenient avenue for the robustness specifications to be perfectly integrated into the multiobjective design procedure; see [61, 80], for example. For nonlinear stochastic systems, control and filtering problems have been solved with the occurrence of parameter uncertainties and stochastic nonlinearities while guaranteeing the H_∞ and variance specifications; see [35, 66, 67, 80] for some recent publications.

3.2.2. Reliability. In practical control systems especially networked control systems (NCSs), due to a variety of reasons including the erosion caused by severe circumstance, abrupt changes of working conditions, the intense external disturbance, and the internal physical equipment constraints and aging, the process of signal sampling and transmission has always confronted with different kinds of failures such as measurements missing, signal quantization, and sensor and actuator saturations. Such a phenomenon is always referred to as incomplete information, which would drastically degrade the system performance. In recent years, as requirements increase toward the reliability of engineering systems, the reliable control problem which aims to stabilize the systems accurately and precisely in spite of incomplete information caused by possible failures has therefore attracted considerable attention. In [86, 87], binary switching sequences and Markovian jumping parameters have been introduced to model the measurements missing phenomena. A more general model called the multiple measurements missing model has been proposed in [88] by employing a diagonal matrix to characterize the different missing probabilities for individual sensors. The incomplete information caused by sensor and actuator saturations is also receiving considerable research attention, and some results have been reported in the literature [20, 89, 90], where the saturation has been modeled as so-called sector bound nonlinearities. As far as signal quantization is mentioned, in [19], a sector bound scheme has been proposed to handle the logarithmic quantization effects in feedback control systems, and such an elegant scheme has then been extensively employed later on; see, for example, [91, 92] and the references therein.

It should be pointed out that, for nonlinear stochastic systems, the relevant results of reliable control/filtering with variance constraints are relatively fewer, and some representative results can be summarized as follows. By means of linear matrix inequality approach, a reliable controller has been designed for nonlinear stochastic system in [66] against actuator faults with variance constraints. In the case of sensor failures, the gradient method and LMI method have been applied, respectively, in [35] and [67] to design multiobjective filters, respectively, satisfying multiple requirements including variance specifications simultaneously. However, despite its clear physical insight and importance in engineering application, the control problem for nonlinear stochastic systems with incomplete information has not yet been studied sufficiently.

3.2.3. Dissipativity. In recent years, the theory of dissipative systems, which plays an important role in system and control areas, has been attracting a great deal of research interests, and many results have been reported so far; see [93–99]. Originated in [97], the dissipative theory serves as a powerful tool in characterizing important system behaviors such as stability and passivity and has close connections with bounded real lemma, passivity lemma, and circle criterion. It is worth mentioning that, due to its simplicity in analysis and convenience in simulation, the LMI method has gained particular attention in dissipative control problems. For example, in [96, 98], an LMI method was used to design the state feedback controller ensuring both the asymptotic stability and strictly quadratic dissipativity. For singular systems, [93] has established a unified LMI framework to satisfy admissibility and dissipativity of the system simultaneously. In [95], the dissipative control problem has been solved for time-delay systems.

Although the dissipativity theory provides us a useful tool for the analysis of systems with multiple performance criteria, unfortunately, when it comes to nonlinear stochastic systems, few of the literature has been concerned with the multiobjective design problem for nonlinear stochastic systems, except [100], where a multiobjective control law has been proposed to simultaneously meet the stability, variance constraints, and dissipativity of closed-loop system. So far, the variance-constrained design problem with dissipativity being taken into consideration has not yet been studied adequately and is still remaining challenging.

3.3. Design Techniques for Nonlinear Stochastic Systems with Variance Constraints. The complexity of nonlinear system dynamics challenges us to come up with systematic design procedures to meet control objectives and design specifications. It is clear that we cannot expect one particular procedure to apply to all nonlinear systems; therefore, quite a lot of tools have been developed to deal with control and filtering problems for nonlinear stochastic systems, including T-S fuzzy model approximation approach, linearization, gain scheduling, sliding mode control, and backstepping, to name but a few key ones. In the sequel, we will investigate two nonlinear design tools that can be well combined with the covariance control theory for the purpose of providing a theoretical framework within which the variance-constrained control and filtering problems can be solved systematically for nonlinear stochastic systems.

3.3.1. T-S Fuzzy Model. The T-S fuzzy model approach occupies an important place in the study of nonlinear systems for its excellent capability in nonlinear system descriptions. Such a model allows one to perfectly approximate a nonlinear system by a set of local linear subsystems with certain fuzzy rules, thereby carrying out the analysis and synthesis work within the linear system framework. Therefore, T-S fuzzy model is extensively applied in both theoretical research and engineering practice of nonlinear systems; see [101–104] for some latest publications. However, despite its engineering significance, few of the literature has taken the system state variance into consideration mainly due to the technical difficulties in dealing with the variance related problems. Some tentative work can be summarized as follows. In [105], a minimum variance control algorithm as well as direct adaptive control scheme has been applied in a stochastic T-S fuzzy ARMAX model to track the desired reference signal. However, as we mentioned above, such a minimum variance control algorithm lacks the ability of directly imposing design requirements on the variances of individual state component. Therefore, in order to cope with this problem, in [106],

a fuzzy controller has been designed to stabilize a nonlinear continuous-time system, while simultaneously minimizing the control input energy and satisfying variance constraints placed on the system state. The result has then been extended in [107] to the output variance constraints case. Recently, such a T-S fuzzy model based variance-constrained algorithm has found successful application in nonlinear synchronous generator systems; see [108] for more details.

3.3.2. Sliding Mode Control. In the past few decades, the sliding mode control (also known as variable structure control) problem originated in [109] has been extensively studied and widely applied, because of its advantage of strong robustness against model uncertainties, parameter variations, and external disturbances. In the sliding mode control, trajectories are forced to reach a sliding manifold in finite time and to stay on the manifold for all future time. It is worth mentioning that in the existing literature about sliding mode control problem for nonlinear systems, the nonlinearities and uncertainties taken into consideration are mainly under the matching conditions, that is, when nonlinear and uncertain terms enter the state equation at the same point as the control input and motion on the sliding manifold is independent of those matched terms; see [110, 111] for examples. Under such an assumption, the covariance-constrained control problems have been solved in [62–64] for a type of continuous stochastic systems with matching condition nonlinearities.

Along with the development of continuous-time sliding mode control theory, in recent years, as most control strategies are implemented in a discrete-time setting (e.g., networked control systems), the sliding mode control problem for discrete-time systems has gained considerable research interests, and a large amount of literature has appeared on this topic. For example, in [112, 113], the integral type SMC schemes have been proposed for sample-data systems and a class of nonlinear discrete-time systems, respectively. Adaptive laws were applied in [114, 115] to synthesize sliding mode controllers for discrete-time systems with stochastic as well as deterministic disturbances. In [116], a simple methodology for designing sliding mode controllers was proposed for a class of linear multi-input discrete-time systems with matching perturbations. Using dead-beat control technique, [117] presented a discrete variable structure control method with a finite-time step to reach the switching surface. In cases when the system states were not available, the discrete-time SMC problems were solved in [118, 119] via output feedback. It is worth mentioning that in [120], the discrete-time sliding mode reaching condition was first revised, and then a reaching law approach was developed which has proven to be a convenient way to handle robust control problems; see [121, 122] for some latest publications. Recently, for discrete-time systems that are not only confronted with nonlinearities but also corrupted by more complicated situations like propagation time delays, randomly occurring parameter uncertainties, and multiple data packet dropouts, the SMC strategies have been designed in [25, 81, 83] to solve the robust control problems and have shown good performances against all the mentioned negative factors. Currently, the sliding mode control problems for discrete-time systems still remain a hotspot in systems and control science; however, when it comes to the variance-constrained problems, the related work is much fewer. As preliminary work, [36] has proposed an SMC algorithm guaranteeing the required H_2 specification for discrete-time stochastic systems in presence of both matched and unmatched nonlinearities. In this paper, although only the H_2 performance is handled, it is worth mentioning that, with the proposed method, other performance indices can be considered simultaneously within the established unified framework by employing similar design techniques.

3.4. A Special Case of Multiobjective Design: Mixed H_2/H_∞ Control/Filtering. As a special case of multiobjective control problem, the mixed H_2/H_∞ control/filtering has gained a great deal of research interests for several decades. So far, there have been several approaches to tackling the mixed H_2/H_∞ control/filtering problem. For linear deterministic systems, the mixed H_2/H_∞ control problems have been extensively studied. For example, an algebraic approach has been presented in [123] and a time domain Nash game approach has been proposed in [39, 124] to solve the addressed mixed H_2/H_∞ control/filtering problems, respectively. Moreover, some efficient numerical methods for mixed H_2/H_∞ control problems have been developed based on a convex optimization approach in [42, 125–127], among which the linear matrix inequality approach has been employed widely to design both linear state feedback and output feedback controllers subject to H_2/H_∞ criterion due to its effectiveness in numerical optimization. It is noted that the mixed H_2/H_∞ control theories have already been applied to various engineering fields [49, 128, 129].

Parallel to the mixed H_2/H_∞ control problem, the mixed H_2/H_∞ filtering problem has also been well studied, and several approaches have been proposed to tackling the problem. For example, Bernstein and Haddad [123] transformed the mixed H_2/H_∞ filtering problem into an auxiliary minimization problem. Then, by using the Lagrange multiplier technique, they gave the solutions in terms of an upper bound on the H_2 filtering error. In [130, 131], a time domain game approach was proposed to solve the mixed H_2/H_∞ filtering problem through a set of coupled Riccati equations. Recently, LMI method has been widely employed to solve the multiobjective mixed H_2 and H_∞ filtering problems; see [60, 132] for examples.

As far as nonlinear systems are concerned, the mixed H_2/H_∞ control problem as well as filtering problem has gained some research interests; see, for examples, [133–135]. For nonlinear deterministic systems, the mixed H_2/H_∞ control problem has been solved with the solutions characterized in terms of the cross-coupled Hamilton-Jacobi-Isaacs (HJI) partial differential equations. Since it is difficult to solve the cross-coupled HJI partial differential equations either analytically or numerically, in [134], the authors have used the Takagi and Sugeno (T-S) fuzzy linear model to

approximate the nonlinear system, and solutions to the mixed H_2/H_∞ fuzzy output feedback control problem have been obtained via an LMI approach. For nonlinear stochastic systems, unfortunately, the mixed H_2/H_∞ control and filtering problem has not received full investigation, and few results have been reported. In [38], for a special type of nonlinear stochastic system, which is known as bilinear systems (also called systems with state-dependent noise or systems with multiplicative noise), a stochastic mixed H_2/H_∞ control problem has been solved and sufficient conditions have been provided in terms of the existence of the solutions of cross-coupled Riccati equations. Very recently, an LMI approach has been proposed in [135] to solve the mixed H_2/H_∞ control problem for a class of nonlinear stochastic systems which includes several well-studied types of nonlinear systems. For the stochastic systems with much more complicated nonlinearities, by means of game theory approach, the mixed H_2/H_∞ control problem has been solved for systems with RONs in [29] and Markovian jump parameters in [68], respectively. Nevertheless, to the best of authors' knowledge, the mixed H_2/H_∞ control and filtering problems for general nonlinear systems have not yet received enough investigation and still remain as challenging topics.

4. Latest Progress

Very recently, the variance-constrained multiobjective control as well as filtering problem for nonlinear stochastic systems has been intensively studied, and some elegant results have been reported. In this section, we highlight some of the newest work with respect to this topic.

(i) In [67], a robust variance-constrained filter has been designed for a class of nonlinear stochastic systems with both parameter uncertainties and probabilistic missing measurements. In this paper, we have simultaneously considered the exponentially mean-square stability, variance constraints, robustness against the parameter uncertainties, and reliability in case of possible measurements missing. A general framework for solving this problem has been established using an LMI approach.

(ii) For the stochastic system with nonlinearities of both the matched and unmatched forms, in [81], a sliding mode control algorithm has been proposed to solve the robust H_2 control problem. A new discrete-time switching function has been proposed, and then a sufficient condition has been derived to ensure both the exponentially mean-square stability and the H_2 performance in the sliding surface. It is worth mentioning that, using the proposed method in this paper, several typical classes of stochastic nonlinearities can be dealt with via SMC method.

(iii) In [100], a dissipative control problem has been solved for a class of nonlinear stochastic systems while guaranteeing tumultuously exponentially mean-square stability, variance constraints, system dissipativity, and reliability. An algorithm has been proposed to convert the original nonconvex feasibility problem into an optimal minimization problem which is much more easy to solve by standard numerical software.

(iv) For the same type of nonlinear stochastic systems as mentioned above, in [66], a robust variance-constrained controller has been designed with the guaranteed reliability against the possible actuator failures.

(v) When the nonlinear stochastic system is time-varying, [80] has designed a multiobjective controller that meets the H_∞ performance and variance constraint over a finite horizon. By using the recursive linear matrix inequalities method, a sufficient condition for the solvability of the addressed controller design problem has been given. Such an algorithm is so elegant that it is soon followed by many researchers in related fields.

(vi) When it comes to the finite-horizon multiobjective filtering for time-varying nonlinear stochastic systems, [35] has proposed a technique that could handle H_∞ performance and variance constraint at the same time. It is worth mentioning that the design algorithm developed in this paper is forward in time, which is different from those in most of the existing literature where the H_∞ problem can be solved only backward in time and thus can be combined with the variance design and is suitable for online design.

(vii) In [29], the mixed H_2/H_∞ controller design problem has been dealt with for a class of nonlinear stochastic systems with randomly occurring nonlinearities that are characterized by two Bernoulli distributed white sequences with known probabilities. For the multiobjective controller design problem, the sufficient condition of the solvability of the mixed H_2/H_∞ control problem has been established by means of the solvability of four coupled matrix-valued equations. A recursive algorithm has been developed to obtain the value of feedback controller step by step at every sampling instant. Such a design algorithm has been extended to the Markovian Jump systems with probabilistic sensor failures in [68].

5. Conclusions and Future Work

In this paper, the variance-constrained multiobjective control and filtering problems have been reviewed with some recent advances for nonlinear stochastic systems. Latest results on analysis and synthesis problems for nonlinear stochastic systems with multiple performance constraints have been surveyed. Based on the literature review, some related topics for the future research work are listed as follows.

In practical engineering, there are still some more complicated yet important kinds of nonlinearities that have not been studied. Therefore, the variance-constrained multiobjective control and filtering problems for more general nonlinear systems still remain open and challenging.

Another future research direction is to further investigate new performance indices (e.g., system energy constraints) that can be simultaneously considered with other existing ones. Also, variance-constrained multiobjective modeling, estimation, filtering, and control problems could be considered for more complex systems [4, 13, 73, 74, 99].

It would be interesting to study the problems of variance-constrained multiobjective analysis and design for large scale nonlinear interconnected systems that are frequently seen in modern industries.

A practical engineering application of the existing theories and methodologies would be the target tracking problem.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grants 61304010, 61134009, 61329301, 61374127, and 61374039, the Royal Society of the UK, and the Alexander von Humboldt Foundation of Germany.

References

[1] Z. Wang, Y. Liu, and X. Liu, "H_∞ filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities," *Automatica*, vol. 44, no. 5, pp. 1268–1277, 2008.

[2] Y. Liu, Z. Wang, and X. Liu, "Robust H_∞ control for a class of nonlinear stochastic systems with mixed time delay," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 16, pp. 1525–1551, 2007.

[3] D. Yue and Q.-L. Han, "Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 217–222, 2005.

[4] Y. Chen and K. A. Hoo, "Stability analysis for closed-loop management of a reservoir based on identification of reduced-order nonlinear model," *Systems Science and Control Engineering: An Open Access Journal*, vol. 1, no. 1, pp. 12–19, 2013.

[5] H. K. Khalil, *Nonlinear Systems*, Publishing House of Electronics Industry, Beijing, China, 3rd edition, 2007.

[6] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, vol. 42, Springer, New York, NY, USA, 1983.

[7] J. K. Hale and H. Koçak, *Dynamics and Bifurcations*, vol. 3, Springer, New York, NY, USA, 1991.

[8] S. Strogatz, *Nonlinear Dynamics and Chaos*, Addison Wesley, Reading, Mass, USA, 1994.

[9] J. Wait, L. Huelsman, and G. Korn, *Introduction to Operational Amplifiers*, McGraw-Hill, New York, NY, USA, 1975.

[10] L. Guo and H. Wang, "Fault detection and diagnosis for general stochastic systems using B-spline expansions and nonlinear filters," *IEEE Transactions on Circuits and Systems I*, vol. 52, no. 8, pp. 1644–1652, 2005.

[11] Z. Wang, Y. Liu, M. Li, and X. Liu, "Stability analysis for stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Transactions on Neural Networks*, vol. 17, no. 3, pp. 814–820, 2006.

[12] Y. Wang, L. Xie, and C. E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Systems & Control Letters*, vol. 19, no. 2, pp. 139–149, 1992.

[13] S. R. Desai and R. Prasad, "A new approach to order reduction using stability equation and big bang big crunch optimization," *Systems Science and Control Engineering: An Open Access Journal*, vol. 1, no. 1, pp. 20–27, 2013.

[14] R. K. Pearson, "Gray-box modeling of nonideal sensors," in *Proceedings of the American Control Conference*, pp. 4404–4409, Arlington, Va, USA, June 2001.

[15] Y.-Y. Cao, Z. Lin, and B. M. Chen, "An output feedback H_∞ controller design for linear systems subject to sensor nonlinearities," *IEEE Transactions on Circuits and Systems I*, vol. 50, no. 7, pp. 914–921, 2003.

[16] Q.-L. Han, "Absolute stability of time-delay systems with sector-bounded nonlinearity," *Automatica*, vol. 41, no. 12, pp. 2171–2176, 2005.

[17] G. Kreisselmeier, "Stabilization of linear systems in the presence of output measurement saturation," *Systems & Control Letters*, vol. 29, no. 1, pp. 27–30, 1996.

[18] Z. Lin and T. Hu, "Semi-global stabilization of linear systems subject to output saturation," *Systems & Control Letters*, vol. 43, no. 3, pp. 211–217, 2001.

[19] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1698–1711, 2005.

[20] Z. Wang, D. W. C. Ho, H. Dong, and H. Gao, "Robust H_∞ finite-horizon control for a class of stochastic nonlinear time-varying systems subject to sensor and actuator saturations," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1716–1722, 2010.

[21] B. Shen, Z. Wang, H. Shu, and G. Wei, "Robust H_∞ finite-horizon filtering with randomly occurred nonlinearities and quantization effects," *Automatica*, vol. 46, no. 11, pp. 1743–1751, 2010.

[22] B. Shen, Z. Wang, H. Shu, and G. Wei, "H_∞ filtering for uncertain time-varying systems with multiple randomly occurred nonlinearities and successive packet dropouts," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 14, pp. 1693–1709, 2011.

[23] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao, "Robust H_∞ filtering for Markovian jump systems with randomly occurring nonlinearities and sensor saturation: the finite-horizon case," *IEEE Transactions on Signal Processing*, vol. 59, no. 7, pp. 3048–3057, 2011.

[24] H. Dong, Z. Wang, and H. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Transactions on Circuits and Systems I*, vol. 59, no. 10, pp. 2354–2362, 2012.

[25] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3008–3015, 2012.

[26] D. Ding, Z. Wang, J. Hu, and H. Shu, "H_∞ state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 725–736, 2012.

[27] J. Liang, Z. Wang, B. Shen, and X. Liu, "Distributed state estimation in sensor networks with randomly occurring nonlinearities subject to time-delays," *ACM Transactions on Sensor Networks*, vol. 9, no. 1, article 4, 2012.

[28] G. Wei, Z. Wang, and B. Shen, "Probability-dependent gain-scheduled control for discrete stochastic delayed systems with

randomly occurring nonlinearities," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 7, pp. 815–826, 2013.

[29] L. Ma, Z. Wang, Y. Bo, and Z. Guo, "A game theory approach to mixed H_2/H_∞ control for a class of stochastic time-varying systems with randomly occurring nonlinearities," *Systems & Control Letters*, vol. 60, no. 12, pp. 1009–1015, 2011.

[30] Z. Wang, B. Shen, and X. Liu, "H_∞ filtering with randomly occurring sensor saturations and missing measurements," *Automatica*, vol. 48, no. 3, pp. 556–562, 2012.

[31] D. Ding, Z. Wang, J. Hu, and H. Shu, "Dissipative control for state-saturated discrete time-varying systems with randomly occurring nonlinearities and missing measurements," *International Journal of Control*, vol. 86, no. 4, pp. 674–688, 2013.

[32] F. Yang, Z. Wang, D. W. C. Ho, and X. Liu, "Robust H_2 filtering for a class of systems with stochastic nonlinearities," *IEEE Transactions on Circuits and Systems II*, vol. 53, no. 3, pp. 235–239, 2006.

[33] E. E. Yaz and Y. I. Yaz, "State estimation of uncertain nonlinear stochastic systems with general criteria," *Applied Mathematics Letters*, vol. 14, no. 5, pp. 605–610, 2001.

[34] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao, "Variance-constrained H_∞ filtering for a class of nonlinear time-varying systems with multiple missing measurements: the finite-horizon case," *IEEE Transactions on Signal Processing*, vol. 58, no. 5, pp. 2534–2543, 2010.

[35] L. Ma, Y. Bo, Y. Zhou, and Z. Guo, "Error variance-constrained H_∞ filtering for a class of nonlinear stochastic systems with degraded measurements: the finite horizon case," *International Journal of Systems Science*, vol. 43, no. 12, pp. 2361–2372, 2012.

[36] L. F. Ma, Z. D. Wang, and Z. Guo, "Robust H_2 sliding mode control for non-linear discrete-time stochastic systems," *IET Control Theory & Applications*, vol. 3, no. 11, pp. 1537–1546, 2009.

[37] K. J. Åström, *Introduction to Stochastic Control Theory*, Academic Press, New York, NY, USA, 1970.

[38] B.-S. Chen and W. Zhang, "Stochastic H_2/H_∞ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[39] X. Chen and K. Zhou, "Multi-objective H_2 and H_∞ control design," *SIAM Journal on Control and Optimization*, vol. 40, no. 2, pp. 628–660, 2002.

[40] E. G. Collins, Jr. and R. E. Skelton, "A theory of state covariance assignment for discrete systems," *IEEE Transactions on Automatic Control*, vol. 32, no. 1, pp. 35–41, 1987.

[41] B. Huang, "Minimum variance control and performance assessment of time-variant processes," *Journal of Process Control*, vol. 12, no. 6, pp. 707–719, 2002.

[42] C. W. Scherer, "Multi-objective H_2/H_∞ control," *IEEE Transactions on Automatic Control*, vol. 40, no. 6, pp. 1054–1062, 1995.

[43] N. Wiener, *The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications*, John Wiley & Sons, New York, NY, USA, 1949.

[44] R. Kalman and R. Bucy, "A new approach to linear filtering and prediction problems," *ASME Journal of Basic Engineering*, vol. 82, pp. 34–45, 1960.

[45] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *ASME Journal of Basic Engineering*, vol. 83, pp. 95–108, 1961.

[46] D. S. Bernstein and W. M. Haddad, "Steady-state Kalman filtering with an H_∞ error bound," *Systems & Control Letters*, vol. 12, no. 1, pp. 9–16, 1989.

[47] Y. S. Hung and F. Yang, "Robust H_∞ filtering with error variance constraints for discrete time-varying systems with uncertainty," *Automatica*, vol. 39, no. 7, pp. 1185–1194, 2003.

[48] X. Lu, L. Xie, H. Zhang, and W. Wang, "Robust Kalman filtering for discrete-time systems with measurement delay," *IEEE Transactions on Circuits and Systems II*, vol. 54, no. 6, pp. 522–526, 2007.

[49] H. Rotstein, M. Sznaier, and M. Idan, "H_2/H_∞ filtering theory and an aerospace application," *International Journal of Robust and Nonlinear Control*, vol. 6, no. 4, pp. 347–366, 1996.

[50] K. Boukas, "Stabilization of stochastic nonlinear hybrid systems," *International Journal of Innovative Computing, Information and Control*, vol. 1, no. 1, pp. 131–141, 2005.

[51] B. Huang, S. L. Shah, and E. K. Kwok, "Good, bad or optimal? Performance assessment of multivariable processes," *Automatica*, vol. 33, no. 6, pp. 1175–1183, 1997.

[52] A. Hotz and R. E. Skelton, "Covariance control theory," *International Journal of Control*, vol. 46, no. 1, pp. 13–32, 1987.

[53] R. Skelton, T. Iwasaki, and K. Grigoriadis, *A Unified Approach to Linear Control Design*, Taylor & Francis, London, UK, 1997.

[54] H. Ruan, E. Yaz, and T. Zhai, "Current output filter for state estimation of nonlinear stochastic systems with linear measurement," *International Journal of Innovative Computing, Information and Control*, vol. 1, no. 2, pp. 277–287, 2005.

[55] U. Shaked and C. E. de Souza, "Robust minimum variance filtering," *IEEE Transactions on Signal Processing*, vol. 43, no. 11, pp. 2474–2483, 1995.

[56] Z. Wang, D. W. C. Ho, and X. Liu, "Variance-constrained filtering for uncertain stochastic systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1254–1258, 2003.

[57] C.-S. Hsieh, "Extension of unbiased minimum-variance input and state estimation for systems with unknown inputs," *Automatica*, vol. 45, no. 9, pp. 2149–2153, 2009.

[58] U. Shaked, L. Xie, and Y. C. Soh, "New approaches to robust minimum variance filter design," *IEEE Transactions on Signal Processing*, vol. 49, no. 11, pp. 2620–2629, 2001.

[59] Z. Wang, D. W. C. Ho, and X. Liu, "Robust filtering under randomly varying sensor delay with variance constraints," *IEEE Transactions on Circuits and Systems II*, vol. 51, no. 6, pp. 320–326, 2004.

[60] F. Yang, Z. Wang, Y. S. Hung, and H. Shu, "Mixed H_2/H_∞ filtering for uncertain systems with regional pole assignment," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 41, no. 2, pp. 438–448, 2005.

[61] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust variance-constrained H_∞ control for stochastic systems with multiplicative noises," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 1, pp. 487–502, 2007.

[62] K.-Y. Chang and W.-J. Wang, "Robust covariance control for perturbed stochastic multivariable system via variable structure control," *Systems & Control Letters*, vol. 37, no. 5, pp. 323–328, 1999.

[63] K.-Y. Chang and W.-J. Wang, "H_∞ norm constraint and variance control for stochastic uncertain large-scale systems via the sliding mode concept," *IEEE Transactions on Circuits and Systems I*, vol. 46, no. 10, pp. 1275–1280, 1999.

[64] W.-J. Wang and K.-Y. Chang, "Variable structure-based covariance assignment for stochastic multivariable model reference systems," *Automatica*, vol. 36, no. 1, pp. 141–146, 2000.

[65] A. Subramanian and A. H. Sayed, "Multi-objective filter design for uncertain stochastic time-delay systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 149–154, 2004.

[66] L. Ma, Z. Wang, Y. Bo, and Z. Guo, "Robust fault-tolerant control for a class of nonlinear stochastic systems with variance constraints," *Journal of Dynamic Systems, Measurement and Control*, vol. 132, no. 4, Article ID 044501, 6 pages, 2010.

[67] L. Ma, Z. Wang, J. Hu, Y. Bo, and Z. Guo, "Robust variance-constrained filtering for a class of nonlinear stochastic systems with missing measurements," *Signal Processing*, vol. 90, no. 6, pp. 2060–2071, 2010.

[68] L. Ma, Z. Wang, Y. Bo, and Z. Guo, "Finite-horizon H_2/H_∞ control for a class of nonlinear Markovian jump systems with probabilistic sensor failures," *International Journal of Control*, vol. 84, no. 11, pp. 1847–1857, 2011.

[69] L. Guo, F. Yang, and J. Fang, "Multi-objective filtering for nonlinear time-delay systems with nonzero initial conditions based on convex optimization," *Circuits, Systems, & Signal Processing*, vol. 25, no. 5, pp. 591–607, 2006.

[70] P. Shi, M. S. Mahmoud, J. Yi, and A. Ismail, "Worst case control of uncertain jumping systems with multi-state and input delay information," *Information Sciences*, vol. 176, no. 2, pp. 186–200, 2006.

[71] E. Tian and C. Peng, "Delay-dependent stability analysis and synthesis of uncertain T-S fuzzy systems with time-varying delay," *Fuzzy Sets & Systems*, vol. 157, no. 4, pp. 544–559, 2006.

[72] D. Yue and J. Lam, "Non-fragile guaranteed cost control for uncertain descriptor systems with time-varying state and input delays," *Optimal Control Applications & Methods*, vol. 26, no. 2, pp. 85–105, 2005.

[73] S. Elmadssia, K. Saadaoui, and M. Benrejeb, "New delay-dependent stability conditions for linear systems with delay," *Systems Science and Control Engineering: An Open Access Journal*, vol. 1, no. 1, pp. 2–11, 2013.

[74] M. Darouach and M. Chadli, "Admissibility and control of switched discrete-time singular systems," *Systems Science and Control Engineering: An Open Access Journal*, vol. 1, no. 1, pp. 43–51, 2013.

[75] G. Zames, "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 301–320, 1981.

[76] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, 1989.

[77] A. Isidori and W. Kang, "H_∞ control via measurement feedback for general nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 40, no. 3, pp. 466–472, 1995.

[78] A. J. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, 1992.

[79] Z. Wang, D. Ding, H. Dong, and H. Shu, "H_∞ consensus control for multi-agent systems with missing measurements: the finite-horizon case," *Systems & Control Letters*, vol. 62, no. 10, pp. 827–836, 2013.

[80] L. Ma, Z. Wang, Y. Bo, and Z. Guo, "Robust H_∞ control of time-varying systems with stochastic non-linearities: the finite-horizon case," *Proceedings of the Institution of Mechanical Engineers I*, vol. 224, no. 5, pp. 575–585, 2010.

[81] L. Ma, Z. Wang, Y. Bo, and Z. Guo, "Robust H_∞ sliding mode control for nonlinear stochastic systems with multiple data packet losses," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 5, pp. 473–491, 2012.

[82] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed H_∞ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[83] J. Hu, Z. Wang, Y. Niu, and L. K. Stergioulas, "H_∞ sliding mode observer design for a class of nonlinear discrete time-delay systems: a delay-fractioning approach," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 16, pp. 1806–1826, 2012.

[84] Z. Wang, H. Dong, B. Shen, and H. Gao, "Finite-horizon H_∞ filtering with missing measurements and quantization effects," *IEEE Transactions on Automatic Control*, vol. 58, no. 7, pp. 1707–1718, 2013.

[85] H. Dong, Z. Wang, and H. Gao, "Distributed H_∞ filtering for a class of Markovian jump nonlinear time-delay systems over lossy sensor networks," *IEEE Transactions on Industrial Electronics*, vol. 60, no. 10, pp. 4665–4672, 2013.

[86] Z. Wang, D. W. C. Ho, Y. Liu, and X. Liu, "Robust H_∞ control for a class of nonlinear discrete time-delay stochastic systems with missing measurements," *Automatica*, vol. 45, no. 3, pp. 684–691, 2009.

[87] Z. Shu, J. Lam, and J. Xiong, "Non-fragile exponential stability assignment of discrete-time linear systems with missing data in actuators," *IEEE Transactions on Automatic Control*, vol. 54, no. 3, pp. 625–630, 2009.

[88] G. Wei, Z. Wang, and H. Shu, "Robust filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 45, no. 3, pp. 836–841, 2009.

[89] F. Yang and Y. Li, "Set-membership filtering for systems with sensor saturation," *Automatica*, vol. 45, no. 8, pp. 1896–1902, 2009.

[90] G.-H. Yang and D. Ye, "Adaptive reliable H_∞ filtering against sensor failures," *IEEE Transactions on Signal Processing*, vol. 55, no. 7, part 1, pp. 3161–3171, 2007.

[91] M. Fu and C. E. de Souza, "State estimation for linear discrete-time systems using quantized measurements," *Automatica*, vol. 45, no. 12, pp. 2937–2945, 2009.

[92] N. Xiao, L. Xie, and M. Fu, "Stabilization of Markov jump linear systems using quantized state feedback," *Automatica*, vol. 46, no. 10, pp. 1696–1702, 2010.

[93] X.-Z. Dong, "Robust strictly dissipative control for discrete singular systems," *IET Control Theory & Applications*, vol. 1, no. 4, pp. 1060–1067, 2007.

[94] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 5, pp. 708–711, 1976.

[95] Z. Li, J. Wang, and H. Shao, "Delay-dependent dissipative control for linear time-delay systems," *Journal of the Franklin Institute*, vol. 339, no. 6-7, pp. 529–542, 2002.

[96] Z. Tan, Y. C. Soh, and L. Xie, "Dissipative control for linear discrete-time systems," *Automatica*, vol. 35, no. 9, pp. 1557–1564, 1999.

[97] J. C. Willems, "Dissipative dynamical systems, part 1: general theory; part 2: linear systems with quadratic supply rate," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–393, 1972.

[98] S. Xie, L. Xie, and C. E. de Souza, "Robust dissipative control for linear systems with dissipative uncertainty," *International Journal of Control*, vol. 70, no. 2, pp. 169–191, 1998.

[99] A. Mehrsai, H. R. Karimi, and K.-D. Thoben, "Integration of supply networks for customization with modularity in cloud and make-to-upgrade strategy," *Systems Science and Control Engineering: An Open Access Journal*, vol. 1, no. 1, pp. 28–42, 2013.

[100] Z. Wang, J. Lam, L. Ma, Y. Bo, and Z. Guo, "Variance-constrained dissipative observer-based control for a class of nonlinear stochastic systems with degraded measurements," *Journal of Mathematical Analysis and Applications*, vol. 377, no. 2, pp. 645–658, 2011.

[101] H. Dong, Z. Wang, and H. Gao, "H_∞ fuzzy control for systems with repeated scalar nonlinearities and random packet losses," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 2, pp. 440–450, 2009.

[102] J. Liang, Z. Wang, and X. Liu, "On passivity and passification of stochastic fuzzy systems with delays: the discrete-time case," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 40, no. 3, pp. 964–969, 2010.

[103] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao, "Robust H_∞ fuzzy output-feedback control with multiple probabilistic delays and multiple missing measurements," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 4, pp. 712–725, 2010.

[104] H. Dong, Z. Wang, J. Lam, and H. Gao, "Fuzzy-model-based robust fault detection with stochastic mixed time delays and successive packet dropouts," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 42, no. 2, pp. 365–376, 2012.

[105] B.-K. Lee, C.-H. Chiu, and B.-S. Chen, "Adaptive minimum variance control for stochastic fuzzy T-S ARMAX model," in *Proceedings of the 7th International Conference on Machine Learning and Cybernetics (ICMLC '08)*, pp. 3811–3816, Kunming, China, July 2008.

[106] W.-J. Chang and S.-M. Wu, "State variance constrained fuzzy controller design for nonlinear TORA systems with minimizing control input energy," in *Proceedings of the IEEE International Conference on Robotics and Automation*, pp. 2616–2621, September 2003.

[107] W.-J. Chang and S.-M. Wu, "Continuous fuzzy controller design subject to minimizing control input energy with output variance constraints," *European Journal of Control*, vol. 11, no. 3, pp. 269–277, 2005.

[108] W. J. Chang and B. J. Huang, "Passive fuzzy controller design with variance constraint for nonlinear synchronous generator systems," in *Proceedings of the IEEE 10th International Conference on Power Electronics and Drive Systems*, pp. 1251–1256, 2013.

[109] V. I. Utkin, "Variable structure systems with sliding modes," *IEEE Transactions on Automatic Control*, vol. 22, no. 2, pp. 212–222, 1977.

[110] Y. Niu, D. W. C. Ho, and J. Lam, "Robust integral sliding mode control for uncertain stochastic systems with time-varying delay," *Automatica*, vol. 41, no. 5, pp. 873–880, 2005.

[111] Y. Niu and D. W. C. Ho, "Robust observer design for Itô stochastic time-delay systems via sliding mode control," *Systems & Control Letters*, vol. 55, no. 10, pp. 781–793, 2006.

[112] K. Abidi, J.-X. Xu, and X. Yu, "On the discrete-time integral sliding-mode control," *IEEE Transactions on Automatic Control*, vol. 52, no. 4, pp. 709–715, 2007.

[113] C. Bonivento, M. Sandri, and R. Zanasi, "Discrete variable-structure integral controllers," *Automatica*, vol. 34, no. 3, pp. 355–361, 1998.

[114] C. Y. Chan, "Discrete adaptive sliding-mode control of a class of stochastic systems," *Automatica*, vol. 35, no. 8, pp. 1491–1498, 1999.

[115] X. Chen and T. Fukuda, "Robust adaptive quasi-sliding mode controller for discrete-time systems," *Systems & Control Letters*, vol. 35, no. 3, pp. 165–173, 1998.

[116] C.-C. Cheng, M.-H. Lin, and J.-M. Hsiao, "Sliding mode controllers design for linear discrete-time systems with matching perturbations," *Automatica*, vol. 36, no. 8, pp. 1205–1211, 2000.

[117] C.-L. Hwang, "Robust discrete variable structure control with finite-time approach to switching surface," *Automatica*, vol. 38, no. 1, pp. 167–175, 2002.

[118] N. O. Lai, C. Edwards, and S. K. Spurgeon, "Discrete output feedback sliding-mode control with integral action," *International Journal of Robust and Nonlinear Control*, vol. 16, no. 1, pp. 21–43, 2006.

[119] N. O. Lai, C. Edwards, and S. K. Spurgeon, "On output tracking using dynamic output feedback discrete-time sliding-mode controllers," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1975–1981, 2007.

[120] W. Gao, Y. Wang, and A. Homaifa, "Discrete-time variable structure control systems," *IEEE Transactions on Industrial Electronics*, vol. 42, no. 2, pp. 117–122, 1995.

[121] Y. Xia, G. P. Liu, P. Shi, J. Chen, D. Rees, and J. Liang, "Sliding mode control of uncertain linear discrete time systems with input delay," *IET Control Theory & Applications*, vol. 1, no. 4, pp. 1169–1175, 2007.

[122] M. Yan and Y. Shi, "Robust discrete-time sliding mode control for uncertain systems with time-varying state delay," *IET Control Theory & Applications*, vol. 2, no. 8, pp. 662–674, 2008.

[123] D. S. Bernstein and W. M. Haddad, "LQG control with an H_∞ performance bound: a Riccati equation approach," *IEEE Transactions on Automatic Control*, vol. 34, no. 3, pp. 293–305, 1989.

[124] D. J. N. Limebeer, B. D. O. Anderson, and B. Hendel, "A Nash game approach to mixed H_2/H_∞ control," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 69–82, 1994.

[125] P. P. Khargonekar and M. A. Rotea, "Mixed H_2/H_∞ control: a convex optimization approach," *IEEE Transactions on Automatic Control*, vol. 36, no. 7, pp. 824–837, 1991.

[126] L. Zhang, B. Huang, and J. Lam, "LMI synthesis of H_2 and mixed H_2/H_∞ controllers for singular systems," *IEEE Transactions on Circuits and Systems II*, vol. 50, no. 9, 2003.

[127] C. Scherer, P. Gahinet, and M. Chilali, "Multi-objective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, 1997.

[128] M.-J. Khosrowjerdi, R. Nikoukhah, and N. Safari-Shad, "Fault detection in a mixed H_2/H_∞ setting," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 1461–1466, December 2003.

[129] C.-D. Yang and Y.-P. Sun, "Mixed H_2 and mixed H_2/H_∞ state-feedback design for microsatellite attitude control," *Control Engineering Practice*, vol. 10, no. 9, pp. 951–970, 2002.

[130] X. Chen and K. Zhou, "Multi-objective filtering design," in *Proceedings of the IEEE Canadian Conference on Electrical and Computer Engineering*, pp. 708–713, May 1999.

[131] Y. Theodor and U. Shaked, "A dynamic game approach to mixed H_2 and mixed H_2/H_∞ estimation," *International Journal of Robust and Nonlinear Control*, vol. 6, no. 4, pp. 331–345, 1996.

[132] Z. Wang and B. Huang, "Robust H_2/H_∞ filtering for linear systems with error variance constraints," *IEEE Transactions on Signal Processing*, vol. 48, no. 8, pp. 2463–2467, 2000.

[133] W. Lin, "Mixed H_2/H_∞ control of nonlinear systems," in *Proceedings of the 34th IEEE Conference on Decision and Control*, pp. 333–338, December 1995.

[134] B.-S. Chen, C.-S. Tseng, and H.-J. Uang, "Mixed H_2/H_∞ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 3, pp. 249–265, 2000.

[135] F. Yang, Z. Wang, and D. W. C. Ho, "Robust mixed H_2/H_∞ control for a class of nonlinear stochastic systems," *IEE Proceedings on Control Theory and Applications*, vol. 153, no. 2, pp. 175–184, 2006.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 825307, 8 pages
http://dx.doi.org/10.1155/2013/825307

Research Article

Distributed Impulsive Consensus of the Multiagent System without Velocity Measurement

Zhi-Wei Liu,[1] Hong Zhou,[1] Zhi-Hong Guan,[2] Wen-Shan Hu,[1] Li Ding,[1] and Wei Wang[1]

[1] *School of Power and Mechanical Engineering, Wuhan University, Wuhan 430072, China*
[2] *College of Automation, Huazhong University of Science and Technology, Wuhan 430074, China*

Correspondence should be addressed to Zhi-Wei Liu; liuzw@whu.edu.cn

Received 12 April 2013; Revised 27 June 2013; Accepted 27 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 Zhi-Wei Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper deals with the distributed consensus of the multiagent system. In particular, we consider the case where the velocity (second state) is unmeasurable and the communication among agents occurs at sampling instants. Based on the impulsive control theory, we propose an impulsive consensus algorithm that extends some of our previous work to account for the lack of velocity measurement. By using the stability theory of the impulsive system, some necessary and sufficient conditions are obtained to ensure the consensus of the controlled multiagent system. It is shown that the control gains, the sampled period and the eigenvalues of Laplacian matrix of communication graph play key roles in achieving consensus. Finally, a numerical simulation is provided to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Recently, distributed consensus has received great interest in the control community, due to broad applications in formation [1], flocking [2, 3], synchronization in complex network [4, 5], distributed filtering [6], distributed optimization [7], and so forth. The main idea of distributed consensus is that each agent only communicates with its neighbors while the whole system of agents can converge to a common value, which by nature is a local distributed algorithm. Vicsek et al. [8] studied a simple discrete-time model of agents moving in the plane with the same speed but with different headings via simulations. The corresponding theoretical analysis was provided in [9]. Olfati-Saber and Murray presented the framework of the distributed consensus in [10], where the distributed consensus was studied in the multiagent system with fixed/switching topology and with/without delays. From then on, much progress has been made in the studies of the distributed consensus of the multiagent system in recent years [11–14]. There is a growing interest focusing on the consensus algorithms of the second-order multiagent system. Lin and Jia [15] studied the consensus problem of the multiagent system with nonuniform timedelays and dynamically changing topologies. In [16, 17], Su et al. investigated second-order consensus of the multiagent system with nonlinear dynamics and a virtual leader in a dynamic proximity network.

Due to the application of communication, the distributed consensus with sampled communication has received much attention in recent years. Many valuable algorithms have been proposed to deal with sampled communication [18–25], where distributed algorithms regulate the velocity of each agent continuously in the sampling period. On the other hand, most consensus algorithms for the multiagent system rely on the availability of the full state, only limited works [26–29] have been done when velocity information is unmeasurable.

The main contribution of this paper is to propose an impulsive consensus algorithm for the multiagent system without velocity measurements in the presence of sampled communication. The impulsive control strategy is effective when the state can be regulated instantaneously. This kind of algorithms are reasonable for many network systems. For example, in multi vrobot systems, the velocity of each robot can be changed very quickly, and the operating time of the actuator is much smaller than the sampling time. Impulsive control strategies for the multiagent system with nonlinear (linear) dynamics were considered in [30–32], where the

impulsive controllers regulate all states of each agent in the system. We introduced impulsive algorithms for the multiagent system in [33–35], where only the velocity of each agent is regulated by the algorithms. In [33], some necessary and sufficient conditions are obtained for consensus/static consensus of the multiagent system. The consensus means that all the agents asymptotically tend to the zero-relative position (the agents may still change their positions) with a common velocity. The static consensus can ensure that all the agents tend to a common position. The leader-following case was studied in [35]. In [34], we proposed an impulsive consensus algorithm without velocity measurement for static consensus of multiagent system. How to achieve consensus without velocity measurement is still an open problem, which is the motivation of the study presented in this paper.

This paper is organized as follows. In Section 2, some necessary mathematical preliminaries are given, and the impulsive algorithm without using velocity information is also introduced. The main results of this paper, that is, the convergence of the proposed algorithm, are presented in Section 3. In Section 4, an illustrative numerical example is given. The concluding remarks are finally stated in Section 5.

Notation. Let $\mathbb{N}$ and $\mathbb{R}$ denote the natural numbers and the set of real numbers, respectively. I_n and $\mathbf{0}_{n\times m}$ are the identity matrixes of order n (or simply I if no confusion arises) and the $n\times m$ matrix with all elements equal to zero (or simply $\mathbf{0}$ if no confusion arises), respectively. $\rho(A)$ denotes the spectral radius of squares matrix A. For $\gamma\in\mathbb{C}$, $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are the real part and the imaginary part of γ.

2. Preliminary and Problem Formulation

The communication structure of the multiagent system is described by an undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with a set of agents $\mathcal{V}=\{1,2,\ldots,N\}$ and a set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ ($\mathcal{G}$ has no self-loops or repeated edges). An edge $\{i,j\}$ in $\mathcal{G}$ means that node i can receive information from node j. $\mathcal{N}_i$ denotes the set of neighbors of agents i, that is, $\mathcal{N}_i=\{j\in\mathcal{V}\mid(i,j)\in\mathcal{E}\}$. The Laplacian matrix $\mathcal{L}$ of the graph $\mathcal{G}$ is defined as

$$\mathcal{L}=\begin{cases} l_{ij}<0, & \text{if } (i,j)\in\mathcal{E},\\ -\displaystyle\sum_{j=1,j\neq i}^{N} l_{ij}, & i=j.\end{cases} \tag{1}$$

A directed path in a digraph $\mathcal{G}$ is an ordered sequence $v_1,v_2,\ldots,v_k$ of agents such that any ordered pair of vertices appearing consecutively in the sequence is an edge of the digraph, that is, $(v_i,v_{i+1})\in\mathcal{E}$, for any $i=1,2,\ldots,k-1$. A directed tree is a digraph, where there exists an agent, called the root, such that any other agent of the digraph can be reached by one and only one path starting at the root. $\mathcal{T}_{\mathcal{G}}=\{\mathcal{V}_{\mathcal{T}},\mathcal{E}_{\mathcal{T}}\}$ is a directed spanning tree of $\mathcal{G}$, if $\mathcal{T}_{\mathcal{G}}$ is a directed tree and $\mathcal{V}_{\mathcal{T}}=\mathcal{V}$.

We consider a multiagent system with N identical agents:

$$\dot{p}_i(t)=v_i(t),\qquad \dot{v}_i(t)=u_i(t), \tag{2}$$

where $i\in\mathcal{N}$, $p_i\in\mathbb{R}$ and $v_i\in\mathbb{R}$ are the position and velocity of agent i, respectively, $u_i\in\mathbb{R}$ is a control input. All results in this paper still hold for $p_i,v_i,u_i\in\mathbb{R}^n$ by using the Kronecker product operations.

Definition 1. Consensus in the multiagent system (2) is said to be achieved, if, for any initial state, $\lim_{t\to\infty}\|p_i(t)-p_j(t)\|=0$ and $\lim_{t\to\infty}\|v_i(t)-v_j(t)\|=0$, where $i,j\in\mathcal{V}$.

In this paper, we assume that both the absolute and relative velocities are unmeasurable, and the communication among agents occurs at sampling instants. The sampled sequence is given by $\{t_k|_{k=1}^{\infty}\}$, which satisfies $0<t_1<t_2<\cdots<t_k<\cdots$, $\lim_{k\to\infty}t_k=\infty$, and $t_{k+1}-t_k=h$, where sampling period h is positive constant. The following impulsive algorithm without using any velocity information is proposed and described by the following impulsive differential equations:

$$\begin{aligned}\dot{p}_i(t)&=v_i(t),\\ \dot{v}_i(t)&=0,\\ \Delta v_i(t_k)&=-\beta_1\sum_{j\in\mathcal{N}_i}l_{ij}\left(p_j(t_k)-p_i(t_k)\right)\\ &\quad-\beta_2\sum_{j\in\mathcal{N}_i}l_{ij}\left(y_j(k)-y_i(k)\right),\\ y_i(k+1)&=-\alpha\sum_{j\in\mathcal{N}_i}l_{ij}\left(p_j(t_k)-p_i(t_k)\right),\end{aligned} \tag{3}$$

where $\Delta v_i(t_k)=v_i(t_k^+)-v_i(t_k)$, $v_i(t_k^+)=\lim_{t\to t_k^+}v_i(t)$, $i\in\mathcal{V}$. We assumed that $v_i(t)$ is left-hand continuous at $t=t_k$, $k\in\mathbb{N}$, and $v(t)$ is continuous at $t_0=0$.

Remark 2. The proposed algorithm only uses sampled information of relative position (i.e. $x_i(t)-x_j(t)$) which is different from [26–29], where the continuous position information is required. It is also different from our previous work [34] which requires the sampled information of relative position to itself in previous sampling instant (i.e., $x_i(t_k)-x_i(t_{k-1})$).

The following lemmas are needed in the proof of the theorem.

Lemma 3 (see [36]). *Zero is a simple eigenvalue of $\mathcal{L}$, and all the other eigenvalues have positive real parts if and only if $\mathcal{G}$ contains a spanning tree.*

Define

$$M=\begin{pmatrix}1 & -1 & 0 & \cdots & & 0\\ 1 & 0 & -1 & 0 & \cdots & 0\\ \vdots & & & \ddots & & \vdots\\ 1 & 0 & \cdots & 0 & -1 & 0\\ 1 & 0 & \cdots & & 0 & -1\end{pmatrix}_{(N-1)\times N}, \tag{4}$$

$$G=\begin{pmatrix}0 & 0 & 0 & \cdots & 0\\ -1 & 0 & 0 & & 0\\ 0 & -1 & \ddots & & 0\\ & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & -1 & 0\\ 0 & 0 & \cdots & 0 & -1\end{pmatrix}_{N\times(N-1)}. \tag{5}$$

From [4, 37], we can get the following lemma.

Lemma 4. *Let $\mathscr{L}$ be the Laplacian matrix of the graph $\mathscr{G}$. Then the $(N-1)\times(N-1)$ matrix $\widehat{\mathscr{L}}$ defined by $\widehat{\mathscr{L}} = M\mathscr{L}G$ satisfies $M\mathscr{L} = \widehat{\mathscr{L}}M$. Furthermore,*

$$\widehat{L} = \begin{pmatrix} l_{22}-l_{12} & l_{23}-l_{13} & \cdots & l_{2N}-l_{1N} \\ l_{32}-l_{12} & l_{33}-l_{13} & \cdots & l_{3N}-l_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ l_{N2}-l_{12} & l_{N2}-l_{13} & \cdots & l_{N2}-l_{NN} \end{pmatrix}_{(N-1)\times(N-1)}. \quad (6)$$

Lemma 5 (see [29]). *The complex polynomial* $\mathrm{R}(z) = z^2 + az + b$, *where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$, is Hurwitz stable if and only if* $\mathrm{Re}(a) > 0$ *and* $\mathrm{Re}(a)\,\mathrm{Im}(a)\,\mathrm{Im}(b) + \mathrm{Re}^2(a)\,\mathrm{Re}(b) - \mathrm{Im}^2(b) > 0$.

3. Consensus in Multi-Agent System

Denote the eigenvalues of L, respectively, by $\lambda_1, \lambda_2, ..., \lambda_r$, where $\lambda_1 = 0$. According to Lemma 3, $\lambda_1 = 0$ is a simple eigenvalue if $\mathscr{G}$ contains a spanning tree. Note that when $\mathscr{G}$ is a directed graph, λ_i, for $i = 1, 2, \ldots, r$, may be complex numbers.

Theorem 6. *The controlled multiagent system* (3) *can achieve consensus if and only if the graph G contains a spanning tree and $\rho(M_l) < 1$, where λ_l are the nonzero eigenvalues of $\mathscr{L}$, $l = 2, \ldots, r$,*

$$M_l = \begin{pmatrix} 1-\lambda_l h\beta_1 & h & -\lambda_l h\beta_2 \\ -\lambda_l\beta_1 & 1 & -\lambda_l\beta_2 \\ -\lambda_l\alpha & 0 & 0 \end{pmatrix}. \quad (7)$$

Proof. Note that $p_i(t)$ is continuous at $t = t_k$. From (3), one has

$$\begin{aligned} p_i(t_{k+1}) &= p_i(t_k) + hv_i(t_k^+), \\ v_i(t_{k+1}^-) &= v_i(t_k^+), \\ v_i(t_k^+) &= v_i(t_k^-) - \beta_1 \sum_{j\in\mathcal{N}_i} l_{ij}\left(p_j(t_k) - p_i(t_k)\right) \\ &\quad - \beta_2 \sum_{j\in\mathcal{N}_i} l_{ij}\left(y_j(k) - y_i(k)\right), \\ y_i(k+1) &= -\alpha \sum_{j\in\mathcal{N}_i} l_{ij}\left(p_j(t_k) - p_i(t_k)\right). \end{aligned} \quad (8)$$

Then, one has

$$\begin{aligned} p_i(t_{k+1}) &= p_i(t_k) \\ &\quad + h\Bigg(v_i(t_k^-) - \beta_1 \sum_{j\in\mathcal{N}_i} l_{ij}\left(p_j(t_k) - p_i(t_k)\right) \\ &\quad - \beta_2 \sum_{j\in\mathcal{N}_i} l_{ij}\left(y_j(k) - y_i(k)\right)\Bigg), \\ v_i(t_{k+1}^-) &= v_i(t_k^-) \\ &\quad - \beta_1 \sum_{j\in\mathcal{N}_i} l_{ij}\left(p_j(t_k) - p_i(t_k)\right) \\ &\quad - \beta_2 \sum_{j\in\mathcal{N}_i} l_{ij}\left(y_j(k) - y_i(k)\right), \\ y_i(k+1) &= -\alpha \sum_{j\in\mathcal{N}_i} l_{ij}\left(p_j(t_k) - p_i(t_k)\right). \end{aligned} \quad (9)$$

Let $Y_i(k) = (p_i(t_k), v_i(t_k), y(k))^T$ and $Y(k) = (Y_1(k), Y_2(k), \ldots, Y_N(k))^T$; then,

$$Y(k+1) = AY(k), \quad (10)$$

where $A = I_N \otimes B - \mathscr{L} \otimes C$. Let $X(k+1) = (M \otimes I_3)Y(k+1)$, where M is defined in (4). From (10), one has

$$\begin{aligned} X(k+1) &= (M \otimes I_3)AY(k), \\ (M \otimes I_3)A &= M \otimes B - M\mathscr{L} \otimes C \\ &= I_{N-1}M \otimes B - \widehat{\mathscr{L}}M \otimes C \\ &= \left(I_{N-1} \otimes B - \widehat{\mathscr{L}} \otimes C\right)(M \otimes I_3), \end{aligned} \quad (11)$$

where $\widehat{L}$ is defined in (6). Then,

$$X(k+1) = \left(I_{N-1} \otimes B - \widehat{L} \otimes C\right)X(k). \quad (12)$$

Note that

$$E^{-1}LE = \begin{pmatrix} 0 & b \\ \mathbf{0}_{(N-1)} & \widehat{\mathscr{L}} \end{pmatrix}, \quad (13)$$

where $b = (l_{12}, l_{13}, ..., l_{1N})$ and

$$E = \begin{pmatrix} 1 & \mathbf{0}_{N-1}^T \\ \mathbf{1}_{N-1} & I_{N-1} \end{pmatrix}, \quad (14)$$

is an invertible matrix. According to Lemma 3, $\lambda_1 = 0$ is a simple eigenvalue of $\mathscr{L}$ if the $\mathscr{G}$ contains a spanning tree (it is well known that $\mathscr{G}$ contains a spanning tree which is a necessary condition for consensus). Then, $\widehat{\mathscr{L}}$ do not have zero eigenvalue. This implies that the eigenvalues of $\widehat{\mathscr{L}}$ are $\lambda_2, \lambda_3, \ldots, \lambda_r$. Then, there exists a nonsingular matrix $P \in \mathbb{R}^{(n-1)\times(n-1)}$, such that

$$P\widehat{\mathscr{L}}P^{-1} = J, \quad (15)$$

where $J = \mathrm{diag}(J_2, J_3, \ldots, J_r)$,

$$J_l = \begin{pmatrix} \lambda_l & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \lambda_l \end{pmatrix}_{N_l\times N_l}, \quad (16)$$

N_l is multiplicity of eigenvalue λ_l and $N_2 + \cdots + N_r = N - 1$.

Let $\overline{X}(k) = (P \otimes I_3)X(k) = (\overline{x}_2, \overline{x}_3, \ldots, \overline{x}_r)^T$, where $\overline{x}_l \in \mathbb{R}^{3N_l}$. Then, from (12),

$$\begin{aligned}\overline{X}(k+1) &= \left(I_{N-1} \otimes B - P\widehat{\mathscr{L}}P^{-1} \otimes C\right)\overline{X}(k) \\ &= \left(I_{N-1} \otimes B - J \otimes C\right)\overline{X}(k),\end{aligned} \tag{17}$$

where $J = \operatorname{diag}\{J_2, J_3, \ldots, J_r\}$. $\overline{X}(k+1)$ is asymptotically stable if and only if $\dot{\overline{x}} = (I_{N-1} \otimes B - J_l \otimes X)\overline{x}$. Similar to analysis in [24, 29], $\overline{X}(k+1)$ is asymptotically stable if and only if $z(k+1) = (B - \lambda_l)z(k)$ is stable. Note that $M_l = (B - \lambda_l)$ which immediately leads to the conclusion. □

Theorem 7. *The controlled multiagent system* (3) *achieves consensus asymptotically if and only if the communication graph $\mathscr{G}$ contains a spanning tree and*

$$\operatorname{Re}(\lambda_l)(\beta_1 - \alpha\beta_2 \operatorname{Re}(\lambda_l)) - \operatorname{Im}^2(\lambda_l)\alpha\beta_2 > 0, \tag{18}$$

$$abd + a^2c - d^2 > 0, \tag{19}$$

where $l = 2, 3, \ldots, r$,

$$\begin{aligned} a &= \operatorname{Re}\left(\frac{2\lambda_l^2\alpha h\beta_2}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}\right), \\ b &= \operatorname{Im}\left(\frac{2\lambda_l^2\alpha h\beta_2}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}\right), \\ c &= \operatorname{Re}\left(\frac{4 - \lambda_l^2\alpha h\beta_2 - \lambda_l h\beta_1}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}\right), \\ d &= \operatorname{Im}\left(\frac{4 - \lambda_l^2\alpha h\beta_2 - \lambda_l h\beta_1}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}\right). \end{aligned} \tag{20}$$

Proof. Let μ be an eigenvalue of matrix M_l. Then,

$$\det(\mu I_3 - M_i) = \mu^3 - (2 - \lambda_l h\beta_1)\mu^2 - \left(\lambda_l^2\alpha h\beta_2 - 1\right)\mu. \tag{21}$$

Let

$$P_l(\mu) = \mu^2 - (2 - \lambda_l h\beta_1)\mu - \left(\lambda_l^2\alpha h\beta_2 - 1\right). \tag{22}$$

It is easy to know that polynomials $P_l(\mu)$, for $l = 2, 3, \ldots, r$, are Schur stable if and only if $\rho(M_l) < 1$.

$$\begin{aligned}&(s-1)P_i\left(\frac{s+1}{s-1}\right) \\ &\quad = \left(\lambda_l h\beta_1 - \lambda_l^2\alpha h\beta_2\right)s^2 + 2\lambda_l^2\alpha h\beta_2 s \\ &\qquad + 4 - \lambda_l^2\alpha h\beta_2 - \lambda_l h\beta_1.\end{aligned} \tag{23}$$

If $\beta_1 - \lambda_i\alpha\beta_2 = 0$, 1 is a root of $P_l(\mu) = 1$. Therefore, $\lambda_i h(\beta_1 - \lambda_i\alpha\beta_2) \neq 0$, if the consensus can be achieved. Then, the consensus can be achieved if and only if the polynomials $\overline{P}_l(s)$, for $l = 2, \ldots, r$, are Hurwitz stable, where

$$\begin{aligned}\overline{P}_l(s) &= s^2 + \frac{2\lambda_l^2\alpha h\beta_2}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}s \\ &\quad + \frac{4 - \lambda_l^2\alpha h\beta_2 - \lambda_l h\beta_1}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}.\end{aligned} \tag{24}$$

It is easy to check

$$\operatorname{Re}\left(\frac{2\lambda_l^2\alpha h\beta_2}{\lambda_l h(\beta_1 - \lambda_l\alpha\beta_2)}\right) > 0, \tag{25}$$

if and only if (18) holds. By Lemma 5, the polynomials $\overline{P}_i(s)$, for $i = 1, 2, \ldots, N$, are Hurwitz stable if and only if (18) and (19) hold. The proof is thus completed. □

Remark 8. According to the previous discussion, both the real and imaginary parts of the eigenvalues of the Laplacian matrix play key roles in achieving consensus. The necessary and sufficient conditions in Theorems 6 and 7 are too complicated to directly display the relationship among consensus, control gains, and sampled period.

When it comes to undirected graph, the results will be more simple.

Corollary 9. *The controlled multiagent system* (3) *achieves consensus asymptotically if and only if the undirected communication graph $\mathscr{G}$ is connected and*

$$\lambda_i\alpha\beta_2 < \beta_1 < \frac{4 - \lambda_i^2\alpha h\beta_2}{\lambda_i h}, \tag{26}$$

where $i = 2, 3, \ldots, N$.

Proof. It is well known that $\mathscr{L}$ contains $N - 1$ positive real eigenvalues if $\mathscr{G}$ is a connected undirected graph. Then, one has $b = 0$ and $d = 0$. From Theorem 7, (18) and (19) hold if and only if (26) is satisfied. The proof is thus completed. □

Remark 10. Equation (26) is nonempty, when

$$\frac{4 - \lambda_i^2\alpha h\beta_2}{\lambda_i h} - \lambda_i\alpha\beta_2 > 0, \tag{27}$$

which implies that

$$\alpha\beta_2 < \frac{2}{\lambda_i^2 h}. \tag{28}$$

So, we can choose the control gains α and β_2 from (28) and choose β_1 from (26). Therefore, it is quite easy to find suitable control gains for any connected graph $\mathscr{G}$ and sampled period h.

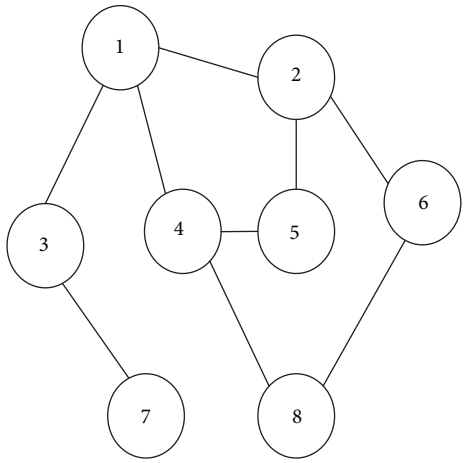


Figure 1: Communication graph.

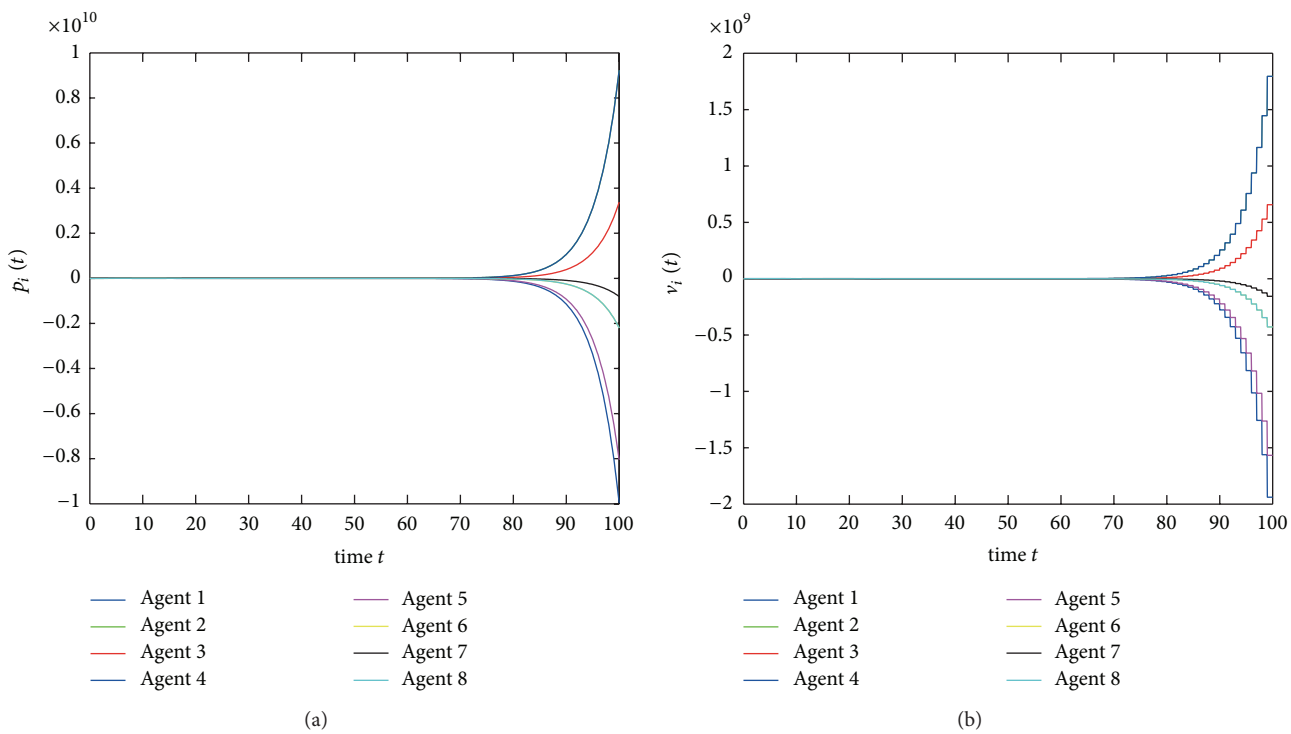


Figure 2: Trajectory of controlled multiagent system (3) with communication graph shown in Figure 1, where $\beta_1 = 0.2$, $\beta_2 = 0.05$, $\alpha = 1$. Evolution of (a) p_i and (b) v_i.

The following corollary will show, when the control gains are given, how to determine suitable control gains h.

Corollary 11. *The controlled multiagent system* (3) *can achieve consensus if and only if the undirected communication graph G is connected,*

$$h < \frac{4}{\lambda_{\max}\left(\lambda_{\max}\alpha h\beta_2 + \beta_1\right)}, \quad \beta_1 > \lambda_i \alpha \beta_2, \tag{29}$$

where $\lambda_{\max} = \max\{\lambda_2, \lambda_3, \ldots, \lambda_N\}$.

Remark 12. When $\beta_1 > \lambda_i \alpha \beta_2$ is not satisfied, the consensus will fail. The upper bound of sampled period increases as $\lambda_{\max}$, α, β_1, and β_2 decrease. The sampled period h does not have the lower bound, which is different from [34].

4. Illustrative Examples

In this section, an illustrative example is given to demonstrate the correctness of the theoretical analysis. We consider the controlled multiagent system (3) with 8 agents. The

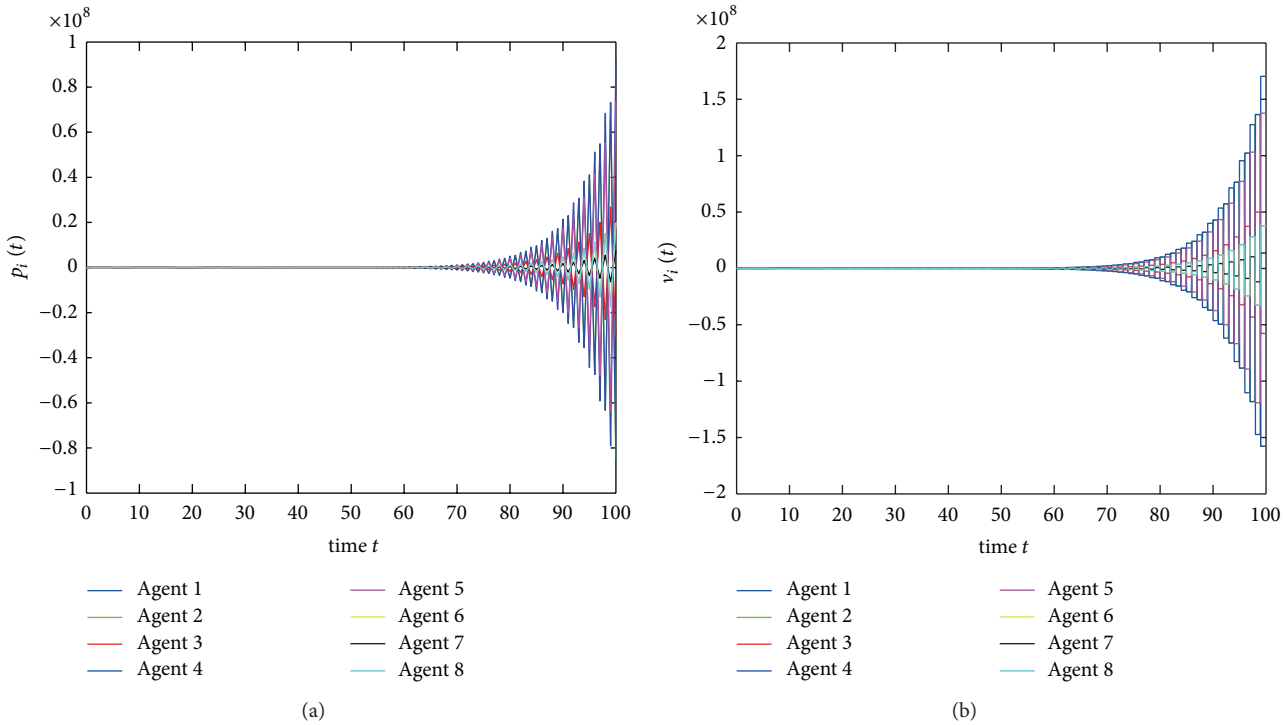


FIGURE 3: Trajectory of controlled multiagent system (3) with communication graph shown in Figure 1, where $\beta_1 = 0.55$, $\beta_2 = 0.05$, $\alpha = 1$. Evolution of (a) p_i and (b) v_i.

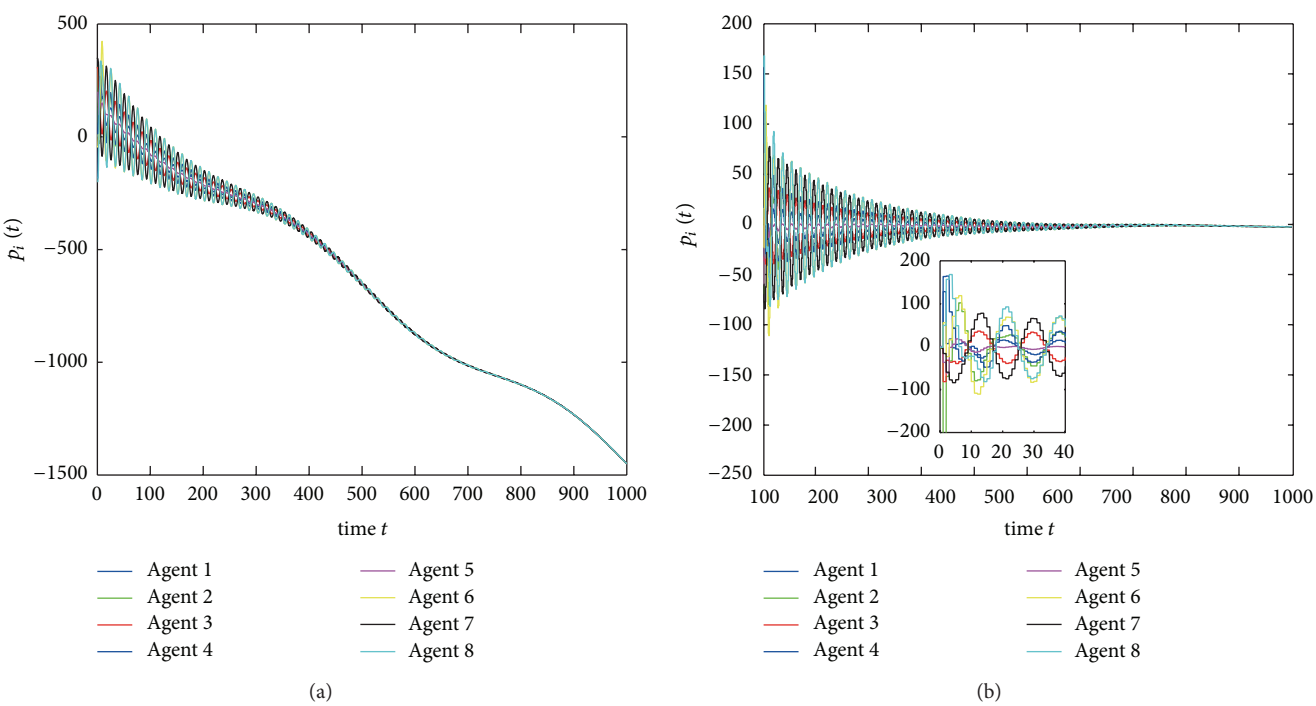


FIGURE 4: Trajectory of controlled multiagent system (3) with communication graph shown in Figure 1, where $\beta_1 = 0.3$, $\beta_2 = 0.05$, $\alpha = 1$. Evolution of (a) p_i and (b) v_i.

communication graph is shown in Figure 1. The Laplacian matrix is

$$\mathscr{L} = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 2 \end{pmatrix}. \tag{30}$$

By calculation, one has $\lambda_1 = 0$, $\lambda_2 = 0.4965$, $\lambda_3 = 1.7356$, $\lambda_4 = \lambda_5 = 2$, $\lambda_6 = 3.5767$, $\lambda_7 = 4$, and $\lambda_8 = 5.1912$.

When the sampled period $h = 1$ is given, from (28), choose $\alpha = 1$ and $\beta_2 = 0.05$ which satisfy

$$\alpha\beta_2 < \frac{2}{\lambda_{\max}^2 h}. \tag{31}$$

From Corollary 9, the consensus can be achieved if and only if $0.2596 < \beta_1 < 0.5110$. Figures 2 and 3 show that consensus cannot be reached when $\beta_1 = 0.2$ and $\beta_1 = 0.55$ but can be achieved when $\beta_1 = 0.3$ (shown in Figure 4).

5. Conclusions

In this paper, the distributed consensus problem has been considered for the continuous-time multiagent system under intermittent communication. Motivated by impulsive control strategy, an impulsive consensus algorithm has been proposed, where the local algorithm of each agent is only based on the position information. Based on the stability theory of impulsive systems and the property of graph Laplacian matrix, some necessary and sufficient conditions for consensus have been obtained. From the results, we can easily find out suitable control gains for consensus. Finally, a numerical example is given to verify the theoretical analysis. It would be interesting to further investigate the multiagent system with switching topology via impulsive control to realize consensus.

Acknowledgments

This work was supported in part by the China Postdoctoral Science Foundation funded project 2012M511258 and the National Natural Science Foundation of China under Grants 61073026, 61170031, 61272069, and 61004030.

References

[1] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605–2611, 2009.

[2] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 863–868, 2007.

[3] H. Su, X. Wang, and Z. Lin, "Flocking of multi-agents with a virtual leader," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 293–307, 2009.

[4] Z.-H. Guan, Z.-W. Liu, G. Feng, and Y.-W. Wang, "Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control," *IEEE Transactions on Circuits and Systems I*, vol. 57, no. 8, pp. 2182–2195, 2010.

[5] Z. Wang, Y. Wang, and Y. Liu, "Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 1, pp. 11–25, 2010.

[6] B. Shen, Z. Wang, and Y. Hung, "Distributed H_∞-consensus filtering in sensor networks with multiple missing measurements: the finite-horizon case," *Automatica*, vol. 46, no. 10, pp. 1682–1688, 2010.

[7] D. Yuan, S. Xu, and H. Zhao, "Distributed primal-dual subgradient method for multiagent optimization via consensus algorithms," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 41, no. 6, pp. 1715–1724, 2011.

[8] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[9] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[10] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[11] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Transactions on Industrial Informatics*, vol. 9, no. 1, pp. 427–438, 2013.

[12] Y. Liu, D. W. C. Ho, and Z. Wang, "A new framework for consensus for discrete-time directed networks of multi-agents with distributed delays," *International Journal of Control*, vol. 85, no. 11, pp. 1755–1765, 2012.

[13] Z.-W. Liu, Z.-H. Guan, T. Li, X.-H. Zhang, and J.-W. Xiao, "Quantized consensus of multi-agent systems via broadcast gossip algorithms," *Asian Journal of Control*, vol. 14, no. 6, pp. 1634–1642, 2012.

[14] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[15] P. Lin and Y. Jia, "Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies," *Automatica*, vol. 45, no. 9, pp. 2154–2158, 2009.

[16] Y. Hu, H. Su, and J. Lam, "Adaptive consensus with a virtual leader of multiple agents governed by locally Lipschitz nonlinearity," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 9, pp. 978–990, 2013.

[17] H. Su, G. Chen, X. Wang, and Z. Lin, "Adaptive second-order consensus of networked mobile agents with nonlinear dynamics," *Automatica*, vol. 47, no. 2, pp. 368–375, 2011.

[18] Y. Gao and L. Wang, "Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1226–1231, 2011.

[19] T. Li and J. Zhang, "Sampled-data based average consensus with measurement noises: convergence analysis and uncertainty principle," *Science in China F*, vol. 52, no. 11, pp. 2089–2103, 2009.

[20] H. Liu, G. Xie, and L. Wang, "Necessary and sufficient conditions for solving consensus problems of double-integrator dynamics via sampled control," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 15, pp. 1706–1722, 2010.

[21] J. Qin and H. Gao, "A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2417–2422, 2012.

[22] J. Qin, W. Zheng, and H. Gao, "Convergence analysis for multiple agents with double-integrator dynamics in a sampled-data setting," *IET Control Theory & Applications*, vol. 5, no. 18, pp. 2089–2097, 2011.

[23] J. Qin, W. X. Zheng, and H. Gao, "Sampled-data consensus for multiple agents with discrete second-order dynamics," in *Proceedings of the 49th IEEE Conference on Decision and Control (CDC '10)*, pp. 1391–1396, IEEE, December 2010.

[24] W. Yu, L. Zhou, X. Yu, J. Lu, and R. Lu, "Consensus in multi-agent systems with second-order dynamics and sampled data," *IEEE Transactions on Industrial Informatics*, 2012.

[25] Y. Zhang and Y.-P. Tian, "Consensus of data-sampled multi-agent systems with random communication delay and packet loss," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 939–943, 2010.

[26] A. Abdessameud and A. Tayebi, "On consensus algorithms for double-integrator dynamics without velocity measurements and with input constraints," *Systems & Control Letters*, vol. 59, no. 12, pp. 812–821, 2010.

[27] Y. Cao and W. Ren, "Distributed coordinated tracking with reduced interaction via a variable structure approach," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 33–48, 2012.

[28] Y. Hong, J. Hu, and L. Gao, "Tracking control for multi-agent consensus with an active leader and variable topology," *Automatica*, vol. 42, no. 7, pp. 1177–1182, 2006.

[29] W. Yu, W. X. Zheng, G. Chen, W. Ren, and J. Cao, "Second-order consensus in multi-agent dynamical systems with sampled position data," *Automatica*, vol. 47, no. 7, pp. 1496–1503, 2011.

[30] H. Jiang, Q. Bi, and S. Zheng, "Impulsive consensus in directed networks of identical nonlinear oscillators with switching topologies," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 378–387, 2012.

[31] B. Liu and D. J. Hill, "Impulsive consensus for complex dynamical networks with nonidentical nodes and coupling time-delays," *SIAM Journal on Control and Optimization*, vol. 49, no. 2, pp. 315–338, 2011.

[32] Q. Wu, J. Zhou, and L. Xiang, "Impulsive consensus seeking in directed networks of multi-agent systems with communication time delays," *International Journal of Systems Science*, vol. 43, no. 8, pp. 1479–1491, 2012.

[33] Z.-H. Guan, Z.-W. Liu, G. Feng, and M. Jian, "Impulsive consensus algorithms for second-order multi-agent networks with sampled information," *Automatica*, vol. 48, no. 7, pp. 1397–1404, 2012.

[34] Z.-W. Liu, Z.-H. Guan, X. Shen, and G. Feng, "Consensus of multi-agent networks with aperiodic sampled communication via impulsive algorithms using position-only measurements," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2639–2643, 2012.

[35] Z.-W. Liu, Z.-H. Guan, and H. Zhou, "Impulsive consensus for leader-following multiagent systems with fixed and switching topology," *Mathematical Problems in Engineering*, vol. 2013, Article ID 762861, 10 pages, 2013.

[36] F. Bullo, J. Cortés, and S. Martinez, *Distributed Control of Robotic Networks: A Mathematical Approach to Motion Coordination Algorithms*, Princeton University Press, 2009.

[37] C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems I*, vol. 42, no. 8, pp. 430–447, 1995.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 629621, 10 pages
http://dx.doi.org/10.1155/2013/629621

Research Article

Nonfragile Gain-Scheduled Control for Discrete-Time Stochastic Systems with Randomly Occurring Sensor Saturations

Wangyan Li,[1] Guoliang Wei,[1] Hamid Reza Karimi,[2] and Xiaohui Liu[3]

[1] *Department of Control Science and Engineering, University of Shanghai for Science and Technology, Shanghai 200093, China*
[2] *Department of Engineering, Faculty of Engineering and Science, University of Agder, N-4898 Grimstad, Norway*
[3] *School of Computer Science and Technology, Nanjing University of Science and Technology, Nanjing 210094, China*

Correspondence should be addressed to Guoliang Wei; guoliang.wei1973@gmail.com

Received 10 May 2013; Accepted 22 June 2013

Academic Editor: Jun Hu

Copyright © 2013 Wangyan Li et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is devoted to tackling the control problem for a class of discrete-time stochastic systems with randomly occurring sensor saturations. The considered sensor saturation phenomenon is assumed to occur in a random way based on the time-varying Bernoulli distribution with measurable probability in real time. The aim of the paper is to design a nonfragile gain-scheduled controller with probability-dependent gains which can be achieved by solving a convex optimization problem via semidefinite programming method. Subsequently, a new kind of probability-dependent Lyapunov functional is proposed in order to derive the controller with less conservatism. Finally, an illustrative example will demonstrate the effectiveness of our designed procedures.

1. Introduction

In reality, virtually almost all dynamic systems are subject to stochastic perturbation, and stochastic model has been successfully established to describe many practical systems, such as economic systems, process control systems, networked control systems (NCSs), and sensor network. For several decades, the study of stabilization, control, and filtering problem has drawn many researchers' attention; some results can be found in [1–16]. On the other hand, time delays also serve as one of the main sources for poor performance and instability. Consequently, the stochastic control issue for time-delay systems has also been intensively investigated; see, for example, [2, 4, 7, 8, 10, 11, 13, 15].

The randomly occurring phenomenon is a newly emerged research topic which has drawn many researchers' attention; see, for example, [1–3, 5, 6, 8, 9, 12–16]. It refers to these phenomena appearing in a random way based on a certain kind of probabilistic law including randomly occurring nonlinearities (RONs), missing measurements, randomly occurring actuator faults, randomly varying sensor delays (RVSDs), and randomly occurring sensor saturations (ROSSs), and so on. For more details about randomly occurring phenomena, the reader is referred to [9]. If not handled appropriately, these phenomena could cause a reduction of performance and/or launch a threat to the safety and reliability of the plant. Therefore, it is not surprising that various filtering and control techniques have been developed to deal with such randomly occurring phenomena, in addition to H_∞ control [16]/filtering [12] and H_∞ state estimation [1] methods. In [2], a robust sliding mode control has been designed for system with mixed time-delays, randomly occurring uncertainties, and RONs; while gain-constrained recursive filter approach has been used in [5] for system with probabilistic sensor delays, the extended Kalman filtering and quantized recursive filtering problem for system with missing measurements have been studied in [3, 6], respectively. Therefore, in this paper, the ROSS (one of the important randomly occurring phenomena) is studied by exploiting gain-scheduling method, which is another motivation of this paper.

Sensor saturation phenomenon is very common in practical engineering. It means that sensors cannot provide signals of unlimited amplitude due mainly to the physical or technological constraints. In another aspect, because of random occurrences of networked induced phenomena in networked control systems (NCSs), such as random sensor

failures leading to intermittent saturation and sensor aging resulting in changeable saturation level, sensor saturation may occur in a random way. We consider this phenomenon as randomly occurring sensor saturation, which has received increasing attention, for instance, [1, 12]. Reference [1] discussed the H_∞ state estimation problem for discrete-time complex networks with ROSSs and RVSDs, while [12] turned to design an H_∞ filter for system with ROSSs and missing measurements. However, to the best of authors' knowledge, rare published literature has dealt with ROSSs; therefore, this paper tries to flourish the research on this phenomenon by designing a nonfragile gain-scheduled controller.

Over the past decades, gain-scheduling method is one of the most popular methods of controller designing and has been extensively studied from theoretical and practical viewpoints; see, for example, [8, 14, 15, 17–19]. The gain-scheduling method is to design controller gains as functions of the scheduling parameters, which can update the controller with a set of tuning parameters in order to optimize the closed-loop performance when outside environment changes (e.g., the occurrences of a variety of randomly occurring phenomena). It should be noted that the designed gain-scheduling controller has not only the constant part but also time-varying part which can be scheduled online according to the corresponding time-varying parameters; see [8, 14, 15]. Therefore, it will naturally lead to less conservatism than the conventional ones with fixed gains only.

On the other hand, it is well known that in order to get better performance of the system, an accuracy controller is needed to resist the impact by the uncertainties occurring in the course of the implementation of a designed controller. Such uncertainties can be due to the existence of parameter drift, round-off errors in numerical computation during controller implementation, and the safe-tuning margins provided for engineering application. In these cases, the nonfragile controller is a good choice, as it can tolerate some level of controller parameter variations; see [7, 20–22]. However, the controller with uncertainties and outside environment changes often occur simultaneously; unfortunately, few papers have tackled this phenomenon, and therefore, we proposed a nonfragile gain-scheduled controller in this paper to fill the gap by making a few first attempts to deal with this problem.

The main contributions of this paper are summarized as follows: (1) a new nonfragile gain-scheduled control problem is addressed for a class of discrete-time nonlinear stochastic systems with randomly occurring phenomenon; (2) a sequence of stochastic variables satisfying Bernoulli distribution is introduced to describe the time-varying features of the ROSSs; (3) a time-varying Lyapunov functional dependent on the saturation probability is proposed and applied to improve the performance of system; (4) the parameters of the nonfragile gain-scheduled controller can be adjusted online according to the saturating probability estimated through statistical tests.

Notation. In this paper, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{I}^+$ denote, respectively, the n-dimensional Euclidean space, the set of all $n \times m$ real matrices, and the set of all positive integers. $|\cdot|$ refers to the Euclidean norm in $\mathbb{R}^n$. I denotes the identity matrix of compatible dimension. The notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (resp., positive definite). For a matrix M, M^T and M^{-1} represent its transpose and inverse, respectively. The shorthand $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \ldots, M_n$. In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry. Matrices, if they are not explicitly stated, are assumed to have compatible dimensions. In addition, $\mathbb{E}\{x\}$ and $\mathrm{Prob}\{y\}$ will, respectively, mean expectation of x and probability of y.

2. Problem Formulation

Consider the following discrete-time nonlinear stochastic systems:

$$\begin{aligned} x(k+1) &= Ax(k) + Dx(k-d) + Bu(k) \\ &\quad + Nf(z(k)) + Ex(k)\omega(k), \end{aligned} \tag{1}$$

$$x(k) = \rho(k), \quad k = -d, -d+1, \ldots, 0, \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, d is a constant delay and $z(k) := Gx(k) + G_d x(k-d)$, $\omega(k)$ is a one-dimensional Gaussian white noise sequence satisfying $\mathbb{E}\{\omega(k)\} = 0$ and $\mathbb{E}\{\omega^2(k)\} = \sigma^2$, and $\rho(k)$ is the initial state of the system. A, B, D, E, N, G, and G_d are constant real matrices of appropriate dimensions and B is of full column. The nonlinear function $f(\cdot)$ with $(f(0) = 0)$ is assumed as nonlinear disturbance and satisfies the following sector-bounded condition:

$$[f(z(k)) - F_1 z(k)]^T [f(z(k)) - F_2 z(k)] \leq 0, \tag{3}$$

where $f(\cdot)$ belongs to the sector $[F_1, F_2]$, F_1 and F_2 are given constant real matrices.

For the technique convenience, the nonlinear function $f(z(k))$ can be decomposed into a linear part and a nonlinear part as

$$f(z(k)) = f_s(z(k)) + F_1 z(k); \tag{4}$$

then, from (3), we have

$$f_s^T(z(k))\left(f_s(z(k)) - Fz(k)\right) \leq 0, \tag{5}$$

where $F = F_2 - F_1 > 0$.

The measurement output with sensor saturation is described as

$$y(k) = \xi(k)\varrho(Cx(k)) + (1 - \xi(k))Cx(k), \tag{6}$$

where C is a constant real matrix of appropriate dimensions and $\varrho(x) = \mathrm{sign}(x)\min\{1, |x|\}$. Here, the notation of "sign" means the signum function, and we use the notation ϱ to denote saturation functions. Note that, without loss of generality, the saturation level is taken as unity.

According to the definition of the saturation function, we can get that the nonlinear function ϱ satisfies $[\varrho(x) - ax][\varrho(x) - x] \leq 0$ and $|x| \leq a^{-1}$, where a is a positive scalar

satisfying $0 < a < 1$, so the nonlinear function $\varrho(Cx(k))$ satisfies $[\varrho(Cx(k)) - aCx(k)]^T[\varrho(Cx(k)) - Cx(k)] \leq 0$, while $|aCx(k)| \leq 1$ and a satisfies $0 < a < 1$.

The variable $\xi(k) \in \mathbb{R}$ is a random white sequence characterizing the probabilistic sensor saturation, which obeys the following time-varying Bernoulli distribution:

$$\begin{aligned} \text{Prob}\{\xi(k) = 1\} &= \mathbb{E}\{\xi(k)\} = p(k), \\ \text{Prob}\{\xi(k) = 0\} &= 1 - \mathbb{E}\{\xi(k)\} = 1 - p(k), \end{aligned} \tag{7}$$

where $p(k)$ is a time-varying positive scalar sequence and belongs to $[p_1 \ \ p_2] \subseteq [0 \ \ 1]$ with p_1 and p_2 being the lower and upper bounds of $p(k)$, respectively. Throughout the paper, for simplicity, we assume that $\xi(k)$, $\omega(k)$ and $\rho(k)$ are uncorrelated.

Remark 1. In many practical systems, especially in NCSs, the measurement output is often subject to ROSSs, and the Bernoulli distribution model has been proven to be a very flexible and effective way to model randomly occurring phenomenon; see, for example, [1–3, 5, 6, 8, 13–15]. Furthermore, in practical engineering, the occurring probability of sensor saturation phenomenon usually changes with time. Therefore, in this paper, the occurrence of sensor saturation is described by a random variable sequence $\xi(k)$ satisfying a time-varying instead of time-invariant Bernoulli distribution model, which will reduce the conservatism when used to deal with the systems with time-varying ROSSs.

In this paper, we are interested in designing the following nonfragile gain-scheduled static output feedback controller:

$$u(k) = [K(p(k)) + \Delta K]\, y(k), \tag{8}$$

where $K(p(k))$ is the controller gain sequence to be designed and assumed as the following structure:

$$K(p(k)) = K_0 + p(k) K_u; \tag{9}$$

for every time step k, $p(k)$ is the time-varying parameter of the controller gain, and K_0 and K_u are the constant parameters of the controller gain to be designed, while ΔK is an unknown matrix of appropriate dimensions and represents the uncertainty in the controller, which is assumed to be of the form

$$\Delta K = LH(k) M, \tag{10}$$

where L and M are known constant matrices with the structured information of the uncertainty, and $H(k)$ is an unknown, real, and time-varying matrix with Lebesgue-measurable elements satisfying

$$H^T(k) H(k) \leq I, \quad \forall k. \tag{11}$$

Remark 2. Instead of using the information of system states, static output feedback control directly makes use of system outputs to design controllers, which has proven to be much simpler and easier to implement and has been extensively used in various kinds of engineering fields; for more details, we recommend some papers such as [23–27].

Remark 3. Owing to the pervasive existence of the uncertainties during controller implementation, an accuracy controller is needed to resist such an impact by the uncertainties, and the nonfragile controller has been proven to be an effective one; see, for example, [7, 20–22]. In another aspect, ROSSs are ubiquitous during the process of measurement, especially in NCSs, and gain-scheduling method has been successfully utilized to tackle with randomly occurring phenomenon in [8, 14, 15]. Therefore, in this paper, we design a nonfragile gain-scheduled static output feedback controller for nonlinear stochastic systems to deal with uncertainties and ROSSs simultaneously.

From the aforementioned, the closed-loop system with the nonfragile gain-scheduled controller is

$$\begin{aligned} x(k+1) = {} & Ax(k) + Dx(k-d) + BK(p(k)) \\ & \times [\xi(k)\varrho(Cx(k)) + (1-\xi(k))Cx(k)] \\ & + Nf(z(k)) + Ex(k)\omega(k). \end{aligned} \tag{12}$$

Before formulating the problem to be investigated, we first introduce the following stability concepts.

Definition 4. The closed-loop system (12) is said to be exponentially mean-square stable if, with $\omega(k) = 0$, there exist constant $\alpha > 0$ and $\tau \in (0, 1)$ such that

$$\mathbb{E}\left\{\|x_k\|^2\right\} \leq \alpha\tau^k \sup_{-d\leq i\leq 0} \mathbb{E}\left\{\|x_i\|^2\right\}, \quad k \in \mathbb{I}^+. \tag{13}$$

In this paper, our purpose is to design a probability-dependent nonfragile gain-scheduled controller of the form (8) for the system (1) by exploiting a probability-dependent Lyapunov functional and LMI method such that, for all admissible sensor saturations and exogenous stochastic noise, the closed-loop system (12) is exponentially mean-square stable.

3. Main Results

The following lemmas will be used in the proofs of our main results in this paper.

Lemma 5 ((Schur complement) [28]). *Given constant matrices $\Sigma_1, \Sigma_2, \Sigma_3$, where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T\Sigma_2^{-1}\Sigma_3 \geq 0$ if and only if*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} \geq 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} \geq 0. \tag{14}$$

Lemma 6 (see [13]). *Let the matrix $B \in R^{n\times m}$ be of full-column rank. There always exist two orthogonal matrices $U \in R^{n\times n}$ and $V \in R^{n\times n}$ such that $B = U\left[\begin{smallmatrix}\Sigma \\ 0\end{smallmatrix}\right]V^T$ and $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$. If matrix S has the following structure: $S = U\left[\begin{smallmatrix} S_{11} & S_{12} \\ 0 & S_{22}\end{smallmatrix}\right]U^T$, where $S_{11}, S_{12} \in R^{n\times(n-m)}$ and $S_{22} \in R^{(n-m)\times(n-m)}$, then there exists a nonsingular matrix $R \in R^{m\times m}$ such that $SB = BR$.*

Lemma 7 ((S-procedure) [28]). *For given matrices $Q = Q^T$, H, and E with appropriate dimensions,*

$$Q + HF(k)E + E^T F^T(k) H^T < 0 \tag{15}$$

holds for all $F(k)$ satisfying $F^T(k)F(k) \le I$ if and only if there exists $\varepsilon > 0$ such that

$$Q + \varepsilon^{-1} HH^T + \varepsilon E^T E < 0. \tag{16}$$

For convenience of presentation, we first consider the desired controller without uncertainty (i.e., $\Delta K = 0$), and the result will be shown in Theorem 8. Then, we design the nonfragile gain-scheduled controller in Theorem 10 based on the conclusion in Theorem 8.

Theorem 8. *Consider the discrete-time nonlinear stochastic systems with ROSSs* (12). *If there exist positive-definite matrices $Q(p(k))$ and Q_τ, slack matrix S, and nonsingular matrices $Y(p(k))$ and R, such that the following LMIs hold:*

$$\Upsilon := \begin{bmatrix} Q_\tau - Q(p(k)) - 2aC^TC & * & * & * & * & * & * \\ 0 & -Q_\tau & * & * & * & * & * \\ (a+1)C & 0 & -2I & * & * & * & * \\ FG & FG_d & 0 & -2I & * & * & * \\ S^T\overline{A} + (1-p(k))BY(p(k))C & S^T\overline{D} & p(k)BY(p(k)) & S^TN & -\overline{\Lambda} & * & * \\ \sigma^2 S^T E & 0 & 0 & 0 & 0 & -\sigma^2\overline{\Lambda} & * \\ \Delta_p(k)BY(p(k))C & 0 & \Delta_p(k)BY(p(k)) & 0 & 0 & 0 & -\Delta_p(k)\overline{\Lambda} \end{bmatrix} < 0, \tag{17}$$

where

$$\begin{gathered} \overline{\Lambda} = -Q(p(k+1)) + S + S^T, \qquad \Delta_p(k) = p(k)(1-p(k)), \\ \overline{A} = A + NF_1G, \qquad \overline{D} = D + NF_1G_d, \qquad S^TB = BR, \\ RK(p(k)) = Y(p(k)), \qquad K(p(k)) = R^{-1}Y(p(k)), \\ Y(p(k)) = Y_0 + p(k)Y_u, \end{gathered} \tag{18}$$

in this case, the constant gains of the desired controller can be obtained as follows:

$$K_0 = R^{-1}Y_0, \qquad K_u = R^{-1}Y_u, \tag{19}$$

and the closed-system (12) *is then exponentially mean-square stable for all $p(k) \in [p_1 \quad p_2]$.*

Proof. Define the Lyapunov functional

$$V(k) := x^T(k)Q(p(k))x(k) + \sum_{s=k-d}^{k-1} x^T(s)Q_\tau x(s); \tag{20}$$

noting that $\mathbb{E}\{\xi(k) - p(k)\} = 0$, $\mathbb{E}\{\omega(k)\} = 0$, and $\mathbb{E}\{[\xi(k) - p(k)]^2\} = p(k)(1-p(k)) \triangleq \Delta_p(k)$, we can get

$$\begin{aligned} \mathbb{E}\{\Delta V(k)\} = \mathbb{E}\Big\{ & x^T(k+1)Q(p(k+1))x(k+1) \\ & - x^T(k)(Q(p(k)) - Q_\tau)x(k) \\ & - x^T(k-d)Q_\tau x(k-d)\Big\} \\ = \mathbb{E}\Big\{ & [Ax(k) + p(k)BK(p(k)) \\ & \times [\varrho(Cx(k)) - Cx(k)] \\ & + (\xi(k) - p(k))BK(p(k)) \\ & \times [\varrho(Cx(k)) - Cx(k)] \\ & + BK(p(k))Cx(k) + Nf(z(k)) \\ & + Dx(k-d) + Ex(k)\omega(k)]^T Q(p(k+1)) \\ & \times [Ax(k) + p(k)BK(p(k)) \\ & \times [\varrho(Cx(k)) - Cx(k)] \\ & + (\xi(k) - p(k))BK(p(k)) \\ & \times [\varrho(Cx(k)) - Cx(k)] \\ & + BK(p(k))Cx(k) + Dx(k-d) \\ & + Nf(z(k)) + Ex(k)\omega(k)] \\ & - x^T(k)Q(p(k))x(k) + x^T(k)Q_\tau x(k) \\ & - x^T(k-d)Q_\tau x(k-d)\Big\} \\ \le \mathbb{E}\Big\{ & \Big[\Big(\overline{A} + (1-p(k))BK(p(k))C\Big)x(k) \\ & + p(k)BK(p(k))\varrho(Cx(k)) + \overline{D}x(k-d) \\ & + Nf_s(z(k))\Big]^T Q(p(k+1)) \end{aligned}$$

$$
\begin{aligned}
&\times \left[\left(\overline{A} + (1 - p(k)) BK(p(k)) C\right) x(k) \right. \\
&\quad + p(k) BK(p(k)) \varrho(Cx(k)) + \overline{D} x(k-d) \\
&\quad \left. + N f_s(z(k))\right] + p(k)(1 - p(k)) \\
&\times \left[BK(p(k)) (\varrho(Cx(k)) - Cx(k))\right]^T \\
&\times Q(p(k+1)) BK(p(k)) \\
&\times \left[\varrho(Cx(k)) - Cx(k)\right] \\
&+ \sigma^2 x^T(k) E^T Q(p(k+1)) E x(k) \\
&- x^T(k) Q(p(k)) x(k) \\
&- x^T(k-d) Q_\tau x(k-d) \\
&+ x^T(k) Q_\tau x(k) + 2 f_s^T(z(k)) FGx(k) \\
&+ 2 f_s^T(z(k)) FG_d x(k-d) \\
&- 2 f_s^T(z(k)) f_s(z(k)) \\
&- 2\varrho^T(Cx(k)) \varrho(Cx(k)) \\
&+ (2 + 2a) \varrho^T(Cx(k)) Cx(k) \\
&- 2a(Cx(k))^T Cx(k) \Big\}.
\end{aligned} \tag{21}
$$

Denote the following matrix variables:

$$
\eta(k) = \left[x^T(k) \quad x^T(k-d) \quad \varrho^T(Cx(k)) \quad f_s^T(z(k))\right]^T; \tag{22}
$$

then, it is obvious that

$$
\mathbb{E}\{\Delta V(k)\} \le \mathbb{E}\left\{\eta^T(k) \Omega \eta(k)\right\}, \tag{23}
$$

where

$$
\Omega = \begin{bmatrix} \Omega_1 & * & * & * \\ \Omega_2 & \Omega_3 & * & * \\ \Omega_4 & \Omega_5 & \Omega_6 & * \\ \Omega_7 & \Omega_8 & \Omega_9 & \Omega_{10} \end{bmatrix},
$$

$$
\begin{aligned}
\Omega_1 &= \left[\overline{A} + (1 - p(k)) BK(p(k)) C\right]^T \\
&\quad \times Q(p(k+1)) \left[\overline{A} + (1 - p(k)) BK(p(k)) C\right] \\
&\quad + \sigma^2 E^T Q(p(k+1)) E \\
&\quad + \Delta_p(k) (BK(p(k)) C)^T Q(p(k+1)) \\
&\quad \times BK(p(k)) C + Q_\tau - Q(p(k)) - 2aC^T C, \\
\Omega_2 &= \overline{D}^T Q(p(k+1)) \left[\overline{A} + (1 - p(k)) BK(p(k)) C\right], \\
\Omega_3 &= \overline{D}^T Q(p(k+1)) \overline{D} - Q_\tau, \\
\Omega_4 &= p(k) (BK(p(k)))^T Q(p(k+1)) \\
&\quad \times \left[\overline{A} + (1 - p(k)) BK(p(k)) C\right] \\
&\quad + \Delta_p(k) (BK(p(k)))^T Q(p(k+1)) \\
&\quad \times BK(p(k)) C + (a+1) C, \\
\Omega_5 &= p(k) (BK(p(k)))^T Q(p(k+1)) \overline{D}, \\
\Omega_6 &= p^2(k) [BK(p(k))]^T Q(p(k+1)) \\
&\quad \times BK(p(k)) + \Delta_p(k) [BK(p(k))]^T \\
&\quad \times Q(p(k+1)) BK(p(k)) - 2I, \\
\Omega_7 &= N^T Q(p(k+1)) \\
&\quad \times \left[\overline{A} + (1 - p(k)) BK(p(k)) C\right] + FG, \\
\Omega_8 &= N^T Q(p(k+1)) \overline{D} + FG_d, \\
\Omega_9 &= p(k) N^T Q(p(k+1)) BK(p(k)), \\
\Omega_{10} &= N^T Q(p(k+1)) N - 2I.
\end{aligned} \tag{24}
$$

If $\Omega \le 0$, we can conclude the following matrix by Schur complement:

$$
\begin{bmatrix}
Q_\tau - Q(p(k)) - 2aC^T C & * & * & * & * & * & * \\
0 & -Q_\tau & * & * & * & * & * \\
(a+1) C & 0 & -2I & * & * & * & * \\
FG & FG_d & 0 & -2I & * & * & * \\
\overline{A} + (1 - p(k)) BK(p(k)) C & \overline{D} & p(k) BK(p(k)) & N & -\Lambda & * & * \\
E & 0 & 0 & 0 & 0 & -\sigma^{-2}\Lambda & * \\
BK(p(k)) C & 0 & BK(p(k)) & 0 & 0 & 0 & -\Delta_p^{-1}(k) \Lambda
\end{bmatrix} < 0, \tag{25}
$$

where

$$\Lambda = Q^{-1}\left(p\left(k+1\right)\right); \tag{26}$$

by preforming the congruence transformation diag$\{I, I, I, I, S, \sigma^2 S, \Delta_p(k)S\}$ to (25), we have

$$\begin{bmatrix} Q_\tau - Q\left(p\left(k\right)\right) - 2aC^TC & * & * & * & * & * & * \\ 0 & -Q_\tau & * & * & * & * & * \\ (a+1)C & 0 & -2I & * & * & * & * \\ FG & FG_d & 0 & -2I & * & * & * \\ S^T\overline{A} + \left(1-p\left(k\right)\right)S^TBK\left(p\left(k\right)\right)C & S^T\overline{D} & p\left(k\right)S^TBK\left(p\left(k\right)\right) & S^TN & -\widehat{\Lambda} & * & * \\ \sigma^2 S^TE & 0 & 0 & 0 & 0 & -\sigma^2\widehat{\Lambda} & * \\ \Delta_p\left(k\right)S^TBK\left(p\left(k\right)\right)C & 0 & \Delta_p\left(k\right)S^TBK\left(p\left(k\right)\right) & 0 & 0 & 0 & -\Delta_p\left(k\right)\widehat{\Lambda} \end{bmatrix} < 0, \tag{27}$$

where

$$\widehat{\Lambda} = S^TQ^{-1}\left(p\left(k+1\right)\right)S. \tag{28}$$

From inequality

$$S^TQ^{-1}\left(p\left(k+1\right)\right)S \geq S^T + S - Q\left(p\left(k+1\right)\right) \triangleq \overline{\Lambda}, \tag{29}$$

we can get

$$\begin{bmatrix} Q_\tau - Q\left(p\left(k\right)\right) - 2aC^TC & * & * & * & * & * & * \\ 0 & -Q_\tau & * & * & * & * & * \\ (a+1)C & 0 & -2I & * & * & * & * \\ FG & FG_d & 0 & -2I & * & * & * \\ S^T\overline{A} + \left(1-p\left(k\right)\right)S^TBK\left(p\left(k\right)\right)C & S^T\overline{D} & p\left(k\right)S^TBK\left(p\left(k\right)\right) & S^TN & -\overline{\Lambda} & * & * \\ \sigma^2 S^TE & 0 & 0 & 0 & 0 & -\sigma^2\overline{\Lambda} & * \\ \Delta_p\left(k\right)S^TBK\left(p\left(k\right)\right)C & 0 & \Delta_p\left(k\right)S^TBK\left(p\left(k\right)\right) & 0 & 0 & 0 & -\Delta_p\left(k\right)\overline{\Lambda} \end{bmatrix} < 0. \tag{30}$$

By using Lemma 6, we have $S^TB = BR$, and denoting that $RK(p(k)) = Y(p(k))$, then (30) can be written as (17); furthermore, we can know from Lemma 5 that $\Omega < 0$ and, subsequently,

$$\mathbb{E}\left\{\Delta V\left(k\right)\right\} < -\lambda_{\min}\left(-\Omega\right)\mathbb{E}\left|\eta\left(k\right)\right|^2, \tag{31}$$

where $\lambda_{\min}(-\Omega)$ is the minimum eigenvalue of $(-\Omega)$. Finally, we can confirm from Lemma 1 in [13] that the closed-loop system is exponentially mean-square stable; then the proof of this theorem is complete. □

Remark 9. The ROSSs have been studied in [1, 12] by constructing a concise and effective time-invariant Bernoulli distribution model; however, in many practical systems, ROSSs sometimes appear with time-varying probability. Therefore, in this case, we considered ROSSs satisfying time-varying Bernoulli distribution which is more reasonable in reality. On the other hand, unlike other time-varying parameters discussed in gain-scheduling technique or parameter-dependent Lyapunov functional; see, for example, [17–19], the parameter $p(k)$ considered in this paper is the time-varying occurrence probability of ROSSs, based on which a new kind of controller is designed and a novel probability-dependent Lyapunov functional is proposed to reduce the potential conservatism.

Next, we are in a position to consider the nonfragile gain-scheduled controller design for system (12) based on what we got in Theorem 8.

Theorem 10. *Consider the discrete-time nonlinear stochastic systems with ROSSs* (12) *and the nonfragile gain-scheduled controller* (8). *If there exist positive-definite matrices* $Q(p(k))$ *and* Q_τ, *slack matrix* S, *and nonsingular matrices* $Y(p(k))$

and R, scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, LMIs (17)*, equations* (18)*, and the following LMIs hold:*

$$\begin{bmatrix} \Upsilon & * & * & * & * \\ \varepsilon_1 \Pi_2^T & -\varepsilon_1 I & * & * & * \\ \Pi_3 & 0 & -\varepsilon_1 I & * & * \\ \varepsilon_2 \Pi_4^T & 0 & 0 & -\varepsilon_2 I & * \\ \Pi_5 & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \tag{32}$$

where

$$\begin{gathered} \Pi_2 = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; \left[(1-p(k))S^T BL\right]^T \;\; 0 \;\; \left[\Delta_p(k) S^T BL\right]^T\right]^T, \\ \Pi_3 = \left[MC \;\; 0 \;\; M \;\; 0 \;\; 0 \;\; 0 \;\; 0\right], \\ \Pi_4 = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; \left[(2p(k)-1)S^T BL\right]^T \;\; 0 \;\; 0\right]^T, \\ \Pi_5 = \left[0 \;\; 0 \;\; M \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]; \end{gathered} \tag{33}$$

in this case, the constant gains of the desired controller can be obtained as follows:

$$K_0 = R^{-1} Y_0, \qquad K_u = R^{-1} Y_u, \tag{34}$$

and the closed-system (12) *is then exponentially mean-square stable for all $p(k) \in [p_1 \;\; p_2]$.*

Proof. In order to get the nonfragile gain-scheduled controller, we replace the $K(p(k))$ with $K(p(k)) + \Delta K$; then, $RK(p(k)) = Y(p(k))$ can be written as $R[K(p(k)) + \Delta K] = Y(p(k)) + \Delta Y$, $\Delta Y = R\Delta K = RLH(k)M$. Noting that $S^T B = BR$, we can rewrite (17) as

$$\begin{gathered} \Upsilon + \Pi_2 H(k) \Pi_3 + \Pi_3^T H^T(k) \Pi_2^T \\ + \Pi_4 H(k) \Pi_5 + \Pi_5^T H^T(k) \Pi_4^T < 0. \end{gathered} \tag{35}$$

From Lemma 7, we know that a necessary and sufficient condition guaranteeing (35) is that there exist scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that

$$\begin{gathered} \Upsilon + \varepsilon_1 \Pi_2 \Pi_2^T + \varepsilon_1^{-1} \Pi_3^T \Pi_3 \\ + \varepsilon_2 \Pi_4 \Pi_4^T + \varepsilon_2^{-1} \Pi_5^T \Pi_5 < 0; \end{gathered} \tag{36}$$

by using the knowledge of Schur complement, we can find that (36) is equivalent to (32). Now, the proof is complete. □

Remark 11. In Theorem 10, a nonfragile gain-scheduled controller has been designed based on a set of LMIs. However, the LMIs are actually infinite owing to the time-varying parameter $p(k) \in [p_1 \;\; p_2]$. In this case, the desired controller cannot be obtained directly due to the infinite number of LMIs. To handle such a problem, in the next theorem, we have to convert this problem to a computationally accessible one by assigning a specific form to $p(k)$. First of all, let us set $Q(p(k)) = Q_0 + p(k) Q_u$.

Theorem 12. *Consider the discrete-time nonlinear stochastic system with ROSSs* (12)*. If there exist positive-definite matrices Q_0, Q_u and Q_τ, slack matrix S and nonsingular matrices $Y(p(k))$ and R, such that the following LMIs hold:*

$$\mathbb{M}^{ijlm} := \begin{bmatrix} \Upsilon^{ijlm} & * & * & * & * \\ \varepsilon_1 \Pi_2^{ij\,T} & -\varepsilon_1 I & * & * & * \\ \Pi_3 & 0 & -\varepsilon_1 I & * & * \\ \varepsilon_2 \Pi_4^{i\,T} & 0 & 0 & -\varepsilon_2 I & * \\ \Pi_5 & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0,$$

$$\Upsilon^{ijlm} := \begin{bmatrix} Q_\tau - Q^i(p(k)) - 2aC^T C & * & * & * & * & * & * \\ 0 & -Q_\tau & * & * & * & * & * \\ (a+1)C & 0 & -2I & * & * & * & * \\ FG & FG_d & 0 & -2I & * & * & * \\ S^T \overline{A} + (1-p_i) BY^m(p(k)) C & S^T \overline{D} & p_i BY^m(p(k)) & S^T N & -\overline{\Lambda}^l & * & * \\ \sigma^2 S^T E & 0 & 0 & 0 & 0 & -\sigma^2 \overline{\Lambda}^l & * \\ \Delta^{ij} BY^m(p(k)) C & 0 & \Delta^{ij} BY^m(p(k)) & 0 & 0 & 0 & -\Delta^{ij} \overline{\Lambda}^l \end{bmatrix}, \tag{37}$$

where

$$\begin{gathered} \overline{\Lambda} = -Q_0 - p_l Q_u + S + S^T, \qquad \Delta^{ij} = p_i (1 - p_j), \\ \overline{A} = A + NF_1 G, \qquad \overline{D} = D + NF_1 G_d, \\ S^T B = BR, \qquad RK(p(k)) = Y(p(k)), \\ K(p(k)) = R^{-1} Y(p(k)), \\ Y^m(p(k)) = Y_0 + p_m Y_u, \\ Q^i(p(k)) = Q_0 + p_i Q_u, \\ \Pi_2^{ij} = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; \left[(1-p_i) S^T BL\right]^T \;\; 0 \;\; \left[\Delta^{ij} S^T BL\right]^T\right]^T, \end{gathered}$$

$$\Pi_3 = \begin{bmatrix} MC & 0 & M & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\Pi_4^i = \begin{bmatrix} 0 & 0 & 0 & 0 & \left[(2p_i - 1) S^T BL\right]^T & 0 & 0 \end{bmatrix}^T,$$
$$\Pi_5 = \begin{bmatrix} 0 & 0 & M & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{38}$$

in this case, the constant gains of the desired controller can be obtained as follows:

$$K_0 = R^{-1} Y_0, \quad K_u = R^{-1} Y_u, \tag{39}$$

and the closed-system (12) *is then exponentially mean-square stable for all* $p(k) \in \begin{bmatrix} p_1 & p_2 \end{bmatrix}$.

Proof. Firstly, set

$$\alpha_1(k) = \frac{p_2 - p(k)}{p_2 - p_1}, \qquad \alpha_2(k) = \frac{p(k) - p_1}{p_2 - p_1}; \tag{40}$$

therefore, we have

$$p(k) = \alpha_1(k) p_1 + \alpha_2(k) p_2, \tag{41}$$

with $\alpha_i(k) \geq 0$ $(i = 1, 2)$ and $\alpha_1(k) + \alpha_2(k) = 1$. Similarly, let

$$\beta_1(k) = \frac{p_2 - p(k+1)}{p_2 - p_1}, \qquad \beta_2(k) = \frac{p(k+1) - p_1}{p_2 - p_1}, \tag{42}$$

and we have

$$p(k+1) = \beta_1(k) p_1 + \beta_2(k) p_2, \tag{43}$$

with $\beta_i(k) \geq 0$ $(i = 1, 2)$, $\beta_1(k) + \beta_2(k) = 1$. From the pervious transformation, we can easily get

$$Q(p(k)) = \sum_{i=1}^{2} \alpha_i(k) Q^i(p(k)), \qquad \overline{\Lambda} = \sum_{l=1}^{2} \beta_l(k) \overline{\Lambda}^l,$$
$$Y(p(k)) = \sum_{m=1}^{2} \alpha_m(k) Y^m(p(k)). \tag{44}$$

On the other hand, it is easy to find that

$$\sum_{i,j,l,m=1}^{2} \alpha_i(k) \alpha_j(k) \alpha_m(k) \beta_l(k) \mathbb{M}^{ijlm} < 0. \tag{45}$$

From (40)–(45), we can have that (32) in Theorem 10 is true; then the proof is now complete. □

Remark 13. By using the methods proposed in the proof of Theorem 10, we choose 4 variables; then, it is easy to calculate the number of LMIs as 2^4 depending on the upper and lower bound of $p(k)$.

Table 1: Computing results.

k	$p(k)$	$Q(p(k))$	$K(p(k))$
0	0.5068	$\begin{bmatrix} 1.7244 & 0.7450 \\ 0.7450 & 3.9564 \end{bmatrix}$	$\begin{bmatrix} 10.3093 & -12.4590 \\ -33.9040 & 40.8241 \end{bmatrix}$
1	0.5082	$\begin{bmatrix} 1.7261 & 0.7443 \\ 0.7443 & 3.9568 \end{bmatrix}$	$\begin{bmatrix} 10.3094 & -12.4590 \\ -33.9045 & 40.8242 \end{bmatrix}$
2	0.4928	$\begin{bmatrix} 1.7070 & 0.7513 \\ 0.7513 & 3.9522 \end{bmatrix}$	$\begin{bmatrix} 10.3078 & -12.4588 \\ -33.8992 & 40.8237 \end{bmatrix}$
⋮	⋮	⋮	⋮

4. An Illustrative Example

In this section, the nonfragile gain-scheduled controller is designed for the discrete-time nonlinear stochastic systems with ROSSs.

The system parameters are given as follows:

$$A = \begin{bmatrix} 0.44 & 0 \\ 0 & 0.81 \end{bmatrix}, \qquad N = \begin{bmatrix} 0.13 & 0.2 \\ 0.28 & 0.33 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.01 & 0 \\ 9.2 & 2.8 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0.19 \\ 0.6 & 2.20 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.02 & 0.14 \\ 0.15 & 0.18 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0.06 & 0 \\ 0 & 0.01 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad G = \begin{bmatrix} 0.08 & 0.12 \\ 0.08 & 0.02 \end{bmatrix}, \tag{46}$$
$$G_d = \begin{bmatrix} 0.01 & 0.09 \\ 0.18 & 0.09 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.3 & 0.19 \\ 0.1 & 0.02 \end{bmatrix},$$
$$L = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad M = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.02 \end{bmatrix},$$
$$H(k) = I, \qquad p_1 = 0.49, \qquad p_2 = 0.51,$$
$$\sigma^2 = 1, \qquad a = 0.411, \qquad \varepsilon_1 = 0.21, \qquad \varepsilon_2 = 0.2.$$

Set the time-varying Bernoulli distribution sequences as $p(k) = p_1 + (p_2 - p_1)|\sin(k)|$, and the sector nonlinear function $f(u)$ is taken by

$$f(u) = \frac{F_1 + F_2}{2} u + \frac{F_2 - F_1}{2} \sin(u), \tag{47}$$

which satisfies (3). Also, select the initial state $\rho = \begin{bmatrix} 2 & -2 \end{bmatrix}^T$.

According to Theorem 12, the constant controller parameters K_0, K_u can be obtained as follows:

$$K_0 = \begin{bmatrix} 10.2565 & -12.4544 \\ -33.7306 & 40.8091 \end{bmatrix}, \quad K_u = \begin{bmatrix} 0.1041 & -0.0090 \\ -0.3421 & 0.0296 \end{bmatrix}. \tag{48}$$

Then, according to the measured time-varying probability parameters $p(k)$, the gain-scheduled controller gain $K(p(k))$ and parameter-dependent Lyapunov matrix $Q(p(k))$ can be calculated at every time step k as in Table 1.

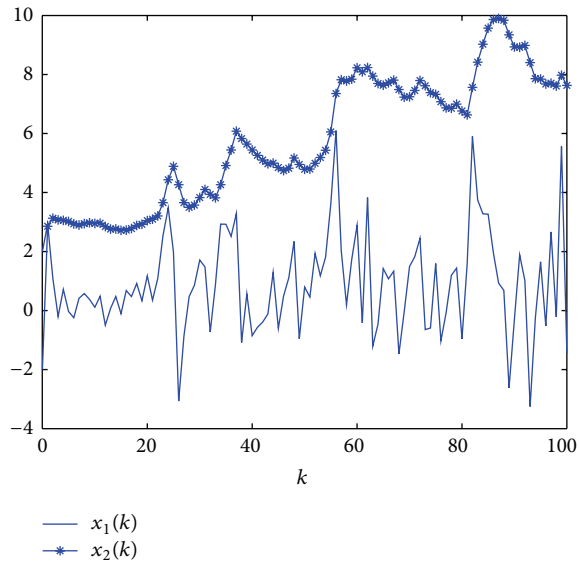


Figure 1: State evolution $x(k)$ of uncontrolled systems.

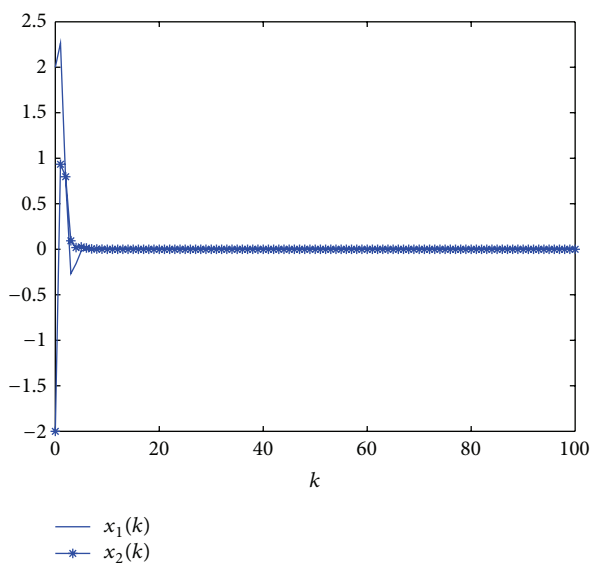


Figure 2: State evolution $x(k)$ of controlled systems.

Figure 1 gives the response curves of state $x(k)$ of uncontrolled systems. Figure 2 depicts the simulation results of state $x(k)$ of the controlled systems. The simulation results have illustrated our theoretical analysis.

5. Conclusions

In this paper, the nonfragile gain-scheduled control problem for a class of discrete stochastic systems with ROSSs is tackled, and the sensor saturation phenomenon is assumed to occur in a random way based on the time-varying Bernoulli distribution with measurable probability in real time. By employing probability-dependent Lyapunov functional, we design a nonfragile gain-scheduled controller with the gain including both constant and time-varying parameters such that, for all admissible sensor saturations, time-delays and noise disturbances, the closed-loop system is still exponentially mean-square stable. Furthermore, we can extend the main results to more complex and realistic systems, for instance, complex networks and systems with several kinds of randomly occurring phenomena simultaneously. Meanwhile, we can also consider corresponding control/filtering problems for time-varying systems with time-varying ROSSs, such as robust sliding mode control, quantized recursive filtering, or extended Kalman filtering. The related references can be found; see, for example, [2, 3, 6].

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant 61074016, the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning, the Program for New Century Excellent Talents in University under Grant NCET-11-1051, the Leverhulme Trust of the UK, the Alexander von Humboldt Foundation of Germany, and the Innovation Fund Project for Graduate Student of Shanghai under Grant JWCXSL1202.

References

[1] D. Ding, Z. Wang, B. Shen, and H. Shu, "H_∞ state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 725–736, 2012.

[2] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties and randomly occurring nonlinearities," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3008–3015, 2012.

[3] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 48, no. 9, pp. 2007–2015, 2012.

[4] J. Hu, Z. Wang, Y. Niu, and L. K. Stergioulas, "H_∞ sliding mode observer design for a class of nonlinear discrete time-delay systems: a delay-fractioning approach," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 16, pp. 1806–1826, 2012.

[5] J. Hu, Z. Wang, B. Shen, and H. Gao, "Gain-constrained recursive filtering with stochastic nonlinearities and probabilistic sensor delays," *IEEE Transactions on Signal Processing*, vol. 61, no. 5, pp. 1230–1238, 2013.

[6] J. Hu, Z. Wang, B. Shen, and H. Gao, "Quantized recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements," *International Journal of Control*, vol. 86, no. 4, pp. 650–663, 2013.

[7] L. Li and Y. Jia, "Non-fragile dynamic output feedback control for linear systems with time-varying delay," *IET Control Theory & Applications*, vol. 3, no. 8, pp. 995–1005, 2009.

[8] W. Li, G. Wei, and L. Wang, "Probability-dependent static output feedback control for discrete-time nonlinear stochastic

systems with missing measurements," *Mathematical Problems in Engineering*, vol. 2012, Article ID 696742, 15 pages, 2012.

[9] B. Shen, Z. Wang, J. Liang, and Y. Liu, "Recent advances on filtering and control for nonlinear stochastic complex systems with incomplete information: a survey," *Mathematical Problems in Engineering*, vol. 2012, Article ID 530759, 16 pages, 2012.

[10] H. Shu and G. Wei, "H_∞ analysis of nonlinear stochastic time-delay systems," *Chaos, Solitons & Fractals*, vol. 26, no. 2, pp. 637–647, 2005.

[11] H. Shu, Z. Wang, and Z. Lü, "Global asymptotic stability of uncertain stochastic bi-directional associative memory networks with discrete and distributed delays," *Mathematics and Computers in Simulation*, vol. 80, no. 3, pp. 490–505, 2009.

[12] Z. Wang, B. Shen, and X. Liu, "H_∞ filtering with randomly occurring sensor saturations and missing measurements," *Automatica*, vol. 48, no. 3, pp. 556–562, 2012.

[13] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H_∞ filtering for stochastic time-delay systems with missing measurements," *IEEE Transactions on Signal Processing*, vol. 54, no. 7, pp. 2579–2587, 2006.

[14] G. Wei, Z. Wang, and B. Shen, "Probability-dependent gain-scheduled filtering for stochastic systems with missing measurements," *IEEE Transactions on Circuits and Systems II*, vol. 58, no. 11, pp. 753–757, 2011.

[15] G. Wei, Z. Wang, and B. Shen, "Probability-dependent gain-scheduled control for discrete stochastic delayed systems with randomly occurring nonlinearities," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 7, pp. 815–826, 2013.

[16] F. Yang, Z. Wang, Y. S. Hung, and M. Gani, "H_∞ control for networked systems with random communication delays," *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 511–518, 2006.

[17] P. Apkarian and R. J. Adams, "Advanced gain-scheduling techniques for uncertain systems," *IEEE Transactions on Control Systems Technology*, vol. 6, no. 1, pp. 21–32, 1998.

[18] Y.-Y. Cao, Z. Lin, and Y. Shamash, "Set invariance analysis and gain-scheduling control for LPV systems subject to actuator saturation," *Systems & Control Letters*, vol. 46, no. 2, pp. 137–151, 2002.

[19] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.

[20] J. H. Park, "Robust non-fragile control for uncertain discrete-delay large-scale systems with a class of controller gain variations," *Applied Mathematics and Computation*, vol. 149, no. 1, pp. 147–164, 2004.

[21] D. Peter, "Non-fragile controller design: an overview," in *Proceedings of the American Control Conference*, vol. 5, pp. 2829–2831, 1998.

[22] G.-H. Yang and W.-W. Che, "Non-fragile H_∞ filter design for linear continuous-time systems," *Automatica*, vol. 44, no. 11, pp. 2849–2856, 2008.

[23] Y. Cao, J. Lam, and Y. Sun, "Static output feedback stabiliztion: an LMI approach," *Automatica*, vol. 34, no. 12, pp. 1641–1645, 1998.

[24] J. C. Geromel, C. C. de Souza, and R. E. Skelton, "Static output feedback controllers: stability and convexity," *IEEE Transactions on Automatic Control*, vol. 43, no. 1, pp. 120–125, 1998.

[25] I. N. Kar, "Design of static output feedback controller for uncertain systems," *Automatica*, vol. 35, no. 1, pp. 169–175, 1999.

[26] J. Qiu, G. Feng, and H. Gao, "Fuzzy-model-based piecewise H_∞ static-output-feedback controller design for networked nonlinear systems," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 5, pp. 919–934, 2010.

[27] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback—a survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[28] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory,*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1994.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 670878, 11 pages
http://dx.doi.org/10.1155/2013/670878

Research Article

Chaos Synchronization Based on Unknown Input Proportional Multiple-Integral Fuzzy Observer

T. Youssef,[1,2] M. Chadli,[1] H. R. Karimi,[3] and M. Zelmat[2]

[1] *Laboratory of Modeling, Information & Systems (MIS), University of Picardie Jules Verne (UPJV), 33 rue Saint Leu, 80039 Amiens Cedex 1, France*
[2] *Laboratory of Automatic Applied (LAA), M'hamed Bougara University of Boumerdès (UMBB), 35000 Boumerdès, Algeria*
[3] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*

Correspondence should be addressed to M. Chadli; mchadli@u-picardie.fr and H. R. Karimi; hamid.r.karimi@uia.no

Received 1 April 2013; Revised 17 June 2013; Accepted 18 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 T. Youssef et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents an unknown input Proportional Multiple-Integral Observer (PIO) for synchronization of chaotic systems based on Takagi-Sugeno (TS) fuzzy chaotic models subject to unmeasurable decision variables and unknown input. In a secure communication configuration, this unknown input is regarded as a message encoded in the chaotic system and recovered by the proposed PIO. Both states and outputs of the fuzzy chaotic models are subject to polynomial unknown input with kth derivative zero. Using Lyapunov stability theory, sufficient design conditions for synchronization are proposed. The PIO gains matrices are obtained by resolving linear matrix inequalities (LMIs) constraints. Simulation results show through two TS fuzzy chaotic models the validity of the proposed method.

1. Introduction

It is well known that the chaotic systems have a complex dynamical behavior and their fundamental characteristic is the chaos. The chaotic systems are highly sensitive to parameters variation and to initial conditions because the chaos is a source of oscillation and instability. Moreover, in the long term the behavior of the chaotic systems becomes difficult to predict which can lead systems to instability and undesirable performance [1]. On the other hand, the chaotic systems constitute a good platform to investigate the nonstandard control problems including synchronization and stabilization.

Since Pecora and Carroll [2, 3] have introduced in 1990 the concept of chaotic synchronization between two chaotic dynamical systems based on the Lorenz's chaotic system [4], the synchronization and control of chaotic systems attract more and more attention from various disciplines. A great deal of chaos applications have been developed in engineering fields such as secure communication, physical systems, system identification, and biological systems; see, for example, [5–10].

Recently, a particular attention has been paid to the synchronization and control problems for dynamical networks due to their extensive application in fields of science and engineering (see, e.g., [11–13]). In [11] the authors propose a novel concept of bounded H_∞ synchronization and state estimation to handle the time-varying nature of an array stochastic complex network in discrete-time domain over a finite horizon. The synchronization problem in [12] is considered for a new class of continuous-time neural networks of neutral type with parameters, discrete-time delays, and unbounded distributed time delays being all dependent on the Markovian jumping mode. The sampled-data synchronization control scheme in [13] is studied for a class of dynamical networks with stochastic sampling. In the formwork of state estimation, fault detection, and filtering for a class of nonlinear systems with sensor networks, we can mention the works of [14–17]. Indeed, in [14] the finite-horizon

distributed H_∞ state estimator design scheme is proposed for a class of discrete time-varying nonlinear systems with both stochastic parameters and stochastic nonlinearities. The problem of designing the distributed H_∞ filters in [15] is considered for a class of polynomial nonlinear stochastic systems which are represented in a state-dependent linear-like form. The distributed finite-horizon filter is proposed in [16] for a class of time-varying systems subject to randomly varying nonlinearities over lossy sensor networks that involve both the quantization errors and successive packet dropouts. The robust fuzzy fault detection filter in [17] is designed for a class of uncertain discrete-time Takagi-Sugeno fuzzy systems with successive packet dropouts which involve both the stochastic multiple time-varying discrete delays and the infinite distributed delays.

Different methods and techniques for chaos synchronization and control have been investigated including impulsive control [18], feedback control [19], adaptive control [20], lag synchronization [21], sliding mode control [22], and fuzzy control [23]. Particularly, in the chaotic secure communication problems, some messages can be masked efficiently and securely [24] by chaotic signals since the chaos has the characteristic of broadband like a noise and is consequently difficult to predict. The message in the secure communications which is recovered by the response system should synchronize with the drive system [25]. In framework of a secure communication, many approaches have been addressed such as chaos modulation [26], chaos shift key [27], and chaos masking [28]. Among different methods of synchronization and control for chaotic systems, the TS fuzzy systems have received much attention from various researches fields since the pioneering work of Takagi and Sugeno (TS) [29]. Indeed, TS fuzzy model can approximate a highly nonlinear analytical relation of chaotic system by fuzzy IF-THEN rules where the implications describe local dynamics as linear models. Then, the nonlinear behavior of the chaotic system can exactly be obtained as an aggregation of local linear models with nonlinear activation functions. TS fuzzy models are widely used as a tool of analysis and design of synchronization and control schemes because of the mathematical analysis simplicity of their simple structure with local dynamics, for example, [30–32].

Since because of, many practical control problems the states are partially or fully unavailable, the state observer methods can be used to estimate the measurements of unavailable or failed sensors. For this reason, it is important to design the observers for state estimation. In relation to that state estimation observers there are many works to that deal with stability analysis and stabilization of TS fuzzy models by applying Lyapunov theory and derive stability conditions in terms linear matrix inequalities constraints [33]; of among this works we can mention the results developed in [34]; When the decision variable is chosen as unmeasurable or unavailable state in activation functions, for example, in [35], the robust observer is designed for unknown inputs TS fuzzy models. Recently, in secure communication field the design problem of unknown input observer has been investigated in [36–39]. The authors propose in [36] a new secured transmission scheme based on smooth adaptive unknown input observers for chaotic synchronization and robust to channel noise. The unknown input observer in [37] is presented with unknown constant disturbance of parameters and unknown input to be recovered as messages in the master-slave configuration. The robust adaptive high-gain fuzzy observer is designed in [32] for chaotic systems where their parameters are assumed unknown and their states unavailable. The author deals with, in [38], the unknown input observer design for fuzzy systems with application to chaotic system reconstruction within both domains, continuous and discrete time, where the sufficient conditions have been derived in terms of linear matrix inequality constraints by using Lyapunov stability theory. Then, in [38, 39] the pole assignment in a LMI region is considered in order to improve the observer performance.

In the context of the unmeasurable decision variable, the synchronization problem for chaotic systems characterized by TS fuzzy models was not addressed by the above works. Our main contribution in this paper is to develop a synchronization procedure which takes into account the unmeasurable decision variables. The effects due to unmeasurable decision variable and the unknown input on the overall synchronization system (chaotic system and observer) are compensated with additional parameter. In addition, in the present study the estimation of the unknown input as a message to be reconstructed within a secure communication concept is considered.

In the framework of a secure communication, the design of unknown input PIO is addressed in this work for two chaotic systems, Lorenz' system and Rossler's system, characterized by chaotic TS fuzzy models. These models are subject to unmeasurable decision variable and polynomial unknown input where its kth derivative is zero. The proposed PIO estimates both the states of the considered chaotic systems and the polynomial unknown input. This latter is considered as a message to encode by the chaotic system and then to reconstruct it by the PIO. Furthermore, the integral action included in the observer structure contributes to reduce the results conservatism due to quadratic function. Indeed, this parameter allows introducing an additional degree of freedom to be determined. By utilizing Lyapunov stability theory, sufficient conditions are derived to design the polynomial unknown input PIO. Then, the PIO gains parameters are resolved in terms of linear matrix inequalities constraints. Moreover, when the decision variables are measurable we also discuss this particular case in our work. Finally, we present simulation results to illustrate the effectiveness of the proposed approach of synchronization and reconstruction.

The rest of this paper is organized as follows. In Section 2, the considered unknown input TS fuzzy model structure is described. This unknown input affects both the dynamics of the TS fuzzy model and the output signal. The unknown input PIO is designed in Section 3. In Section 4, simulation results for two chaotic systems, Lorenz' system and Rossler's system, are given. An unknown input, assumed as a message to be received in a secure communication configuration, is perfectly reconstructed. Finally, a conclusion and further works end this paper.

2. Unknown Input TS Fuzzy Model Structure

The TS fuzzy model subject to unknown input is considered as follow:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x)\left(A_i x(t) + B_i u(t) + d_i + F_i v(t)\right), \qquad y(t) = Cx(t) + Fv(t), \tag{1}$$

where $x(t) \in R^n$ represents the state vector, $u(t) \in R^{n_u}$ corresponds to known input vector, $v(t) \in R^{n_v}$ shows the unknown input, and $y(t) \in R^{n_y}$ is the output vector. $A_i \in R^{n\times n}$ are the state matrices, $B_i \in R^{n\times n_u}$ are the input matrices, $d_i \in R^n$ is a vector system dependent, $F_i \in R^{n\times n_v}$ and $F \in R^{n_y\times n_v}$ are the unknown input matrices, and $C \in R^{n_y\times n}$ is the output matrix. The activation functions $\mu_i(x)$ depend on the state $x(t)$ of the system and satisfy the following conditions:

$$\sum_{i=1}^{r} \mu_i(x) = 1, \quad \forall t \geq 0, \qquad 0 \leq \mu_i(x) \leq 1, \quad \forall i \in \{1,\ldots,r\}, \tag{2}$$

where r is the number of local models.

Hypothesis 1. Unknown input $v(t)$ has a polynomial form of $k-1$ degree in time whose kth derivative is equal to zero.

Let the following notations be introduced:

$$\begin{aligned} \dot{v}(t) &= v_1(t), \\ \dot{v}_1(t) &= v_2(t), \\ &\vdots \\ \dot{v}_{k-1}(t) &= v_k(t), \\ v_k(t) &= 0. \end{aligned} \tag{3}$$

Note that the polynomial form allows considering a wide variety of unknown inputs.

3. Unknown Input Proportional Multiple-Integral Observer Design

The unknown input PIO is considered as follow:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x})\left(A_i\hat{x}(t) + B_i u(t) + d_i + F_i\hat{v}(t) + K_{Pi}\left(y(t) - \hat{y}(t)\right)\right) + z_x(t), \\ \hat{y}(t) &= C\hat{x}(t) + F\hat{v}(t), \\ \dot{\hat{v}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}) K_{Ii}\left(y(t) - \hat{y}(t)\right) + \hat{v}_1(t) + z_v(t), \\ \dot{\hat{v}}_j(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}) K_{Ii}^{j}\left(y(t) - \hat{y}(t)\right) + \hat{v}_{j+1}(t) + z_{vj}(t), \quad \text{for } j\text{: } 1,\ldots,k-1 \\ &\text{if } j = k-1, \qquad \hat{v}_{j+1}(t) = 0, \end{aligned} \tag{4}$$

where $K_{Pi} \in R^{n\times n_y}$, $K_{Ii} \in R^{n_v\times n_y}$, and $K_{Ii}^{j} \in R^{n_v\times n_y}$ correspond to proportional and integral gains, respectively. Due the effect of the unmeasurable decision variables, the variables $z_x(t)$, $z_v(t)$, and $z_{vj}(t)$ are introduced in the PIO.

According to Hypothesis 1, TS fuzzy model (1) and unknown input PIO (4) can be rewritten, respectively, as follow:

$$\dot{x}_a(t) = \sum_{i=1}^{r} \mu_i(x)\left(\overline{A}_i x_a(t) + \overline{B}_i u(t) + \overline{d}_i\right), \qquad y(t) = \overline{C}x_a(t), \tag{5}$$

$$\begin{aligned} \dot{\hat{x}}_a(t) &= \sum_{i=1}^{r} \mu_i(\hat{x})\left(\overline{A}_i\hat{x}_a(t) + \overline{B}_i u(t) + \overline{d}_i + \overline{K}_i\left(y(t) - \hat{y}(t)\right)\right) + z(t), \\ \hat{y}(t) &= \overline{C}\hat{x}_a(t), \end{aligned} \tag{6}$$

where

$$x_a(t) = \begin{bmatrix} x(t) \\ v(t) \\ v_1(t) \\ \cdots \\ v_{k-1}(t) \end{bmatrix}, \qquad \hat{x}_a(t) = \begin{bmatrix} \hat{x}(t) \\ \hat{v}(t) \\ \hat{v}_1(t) \\ \cdots \\ \hat{v}_{k-1}(t) \end{bmatrix}, \qquad z(t) = \begin{bmatrix} z_x(t) \\ z_v(t) \\ z_{v1}(t) \\ \cdots \\ z_{vk-1}(t) \end{bmatrix} \tag{7a}$$

with

$$e_a(t) = x_a(t) - \hat{x}_a(t), \qquad e_{ay} = y(t) - \hat{y}(t), \tag{7b}$$

$$\overline{A}_i = \begin{bmatrix} A_i & F_i & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_{n_v} & 0 & \cdots & 0 \\ 0 & 0 & 0 & I_{n_v} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & I_{n_v} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \overline{B}_i = \begin{bmatrix} B_i \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \qquad \overline{d}_i = \begin{bmatrix} d_i \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \qquad \overline{K}_i = \begin{bmatrix} K_{Pi} \\ K_{Ii} \\ K_{Ii}^{1} \\ \cdots \\ K_{Ii}^{k-1} \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} C & F & 0 & \cdots & 0 \end{bmatrix}, \tag{7c}$$

where I_{n_v} is an identity matrix.

3.1. Unmeasurable Decision Varaiables. The dynamic error $e_a(t)$ of state estimation is given by

$$\dot{e}_a(t) = \sum_{i=1}^{r} \mu_i(\hat{x}) \overline{\mathscr{A}}_i e_a(t) + \overline{\Delta} A x_a(t) + \overline{\Delta} B u(t) + \overline{\Delta} d - z(t), \tag{8}$$

where

$$\overline{\mathscr{A}}_i = \overline{A}_i - \overline{K}_i \overline{C}, \qquad \overline{\Delta} A = \sum_{i=1}^{r} \overline{\mu}_i \overline{A}_i, \qquad \overline{\Delta} B = \sum_{i=1}^{r} \overline{\mu}_i \overline{B}_i,$$
$$\overline{\Delta} d = \sum_{i=1}^{r} \overline{\mu}_i \overline{d}_i, \qquad \overline{\mu}_i = \mu_i(x) - \mu_i(\hat{x}). \tag{9}$$

Remark 1. Since the activation functions satisfy the convex sum property, we can write $-1 < \overline{\mu}_i < 1$, and the variables matrices $\overline{\Delta} A$, $\overline{\Delta} B$, $\overline{\Delta} d$ are bounded and the following conditions hold:

$$\left\|\overline{\Delta} A\right\| \le \delta_1, \qquad \delta_1 = \sum_{i=1}^{r} \delta_{1i}, \qquad \left\|\overline{\Delta} B\right\| \le \delta_2,$$
$$\delta_2 = \sum_{i=1}^{r} \delta_{2i}, \qquad \left\|\overline{\Delta} d\right\| \le \delta_3, \qquad \delta_3 = \sum_{i=1}^{r} \delta_{3i} \tag{10}$$

with $\delta_{1i} > 0$, $\delta_{2i} > 0$ and $\delta_{3i} > 0$, are the Euclidian norms of $\overline{A}_i$, $\overline{B}_i$, and $\overline{d}_i$, respectively.

Theorem 2. *The dynamic error* (8) *is asymptotically stable if there exist a common positive definite matrix $P = P^T$, matrices $\overline{M}_i$ and the positive scalars α and α_0, for all $i \in \{1, \ldots, r\}$, such that:*

$$\begin{bmatrix} P\overline{A}_i + \overline{A}_i^T P - \overline{M}_i \overline{C} - \overline{C}^T \overline{M}_i^T + \alpha_0 \delta_1^2 I & P \\ P & -\alpha I \end{bmatrix} < 0, \tag{11a}$$

where the matrices and parameters $\overline{A}_i$, $\overline{C}$, δ_1 are defined in (7c), *and* (10), *respectively.*

The parameters of unknown input PIO (4) *are obtained by*

$$\overline{K}_i = P^{-1} \overline{M}_i, \tag{11b}$$

$$z = 0, \quad \text{if } \left|e_{ay}\right| < \varepsilon,$$
$$z = \sigma_1 \delta_1^2 \frac{\hat{x}_a^T \hat{x}_a}{2 e_{ay}^T e_{ay}} P^{-1} \overline{C}^T e_{ay} + \sigma_2 \delta_2^2 \frac{u^T u}{2 e_{ay}^T e_{ay}} P^{-1} \overline{C}^T e_{ay}$$
$$+ \sigma_3 \delta_3^2 \frac{1}{2 e_{ay}^T e_{ay}} P^{-1} \overline{C}^T e_{ay}, \quad \text{if } \left|e_{ay}\right| \ge \varepsilon \tag{11c}$$

with variables $\hat{x}_a$, z, e_{ay} and the parameters δ_2, δ_3 are described in (7a), (7b), (10), *respectively, and $\sigma_1 = (\alpha_0/\lambda)$, $\sigma_2 = ((\alpha\alpha_0\lambda_3)/(\alpha(\alpha_0 + \lambda_3(1+\lambda)) - \alpha_0\lambda_3))$, $\sigma_3 = \lambda_3$ where λ, λ_3 are positive scalars arbitrarily fixed and ε is a very small positive threshold.*

Proof. The proposed quadratic function of Lyapunov is $V(t) = e_a^T(t) P e_a(t)$ with $P = P^T > 0$. The conditions ((11a), (11b), and (11c)) guarantee the asymptotic stability of the dynamic error of state estimation (8). The proof is partially given in the appendix and for more details see [40]. □

3.2. Measurable Decision Variables Case. The design of unknown input PIO with measurable decision variables represents the particular case of our study developed in this section. In this condition the unknown input PIO is as follow:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(x) \left(A_i \hat{x}(t) + B_i u(t) + d_i + F_i \hat{v}(t) + K_{Pi}\left(y(t) - \hat{y}(t)\right)\right),$$
$$\hat{y}(t) = C\hat{x}(t) + F\hat{v}(t),$$
$$\dot{\hat{v}}(t) = \sum_{i=1}^{r} \mu_i(x) K_{Ii}\left(y(t) - \hat{y}(t)\right) + \hat{v}_1(t), \tag{12}$$
$$\dot{\hat{v}}_j(t) = \sum_{i=1}^{r} \mu_i(x) K_{Ii}^j \left(y(t) - \hat{y}(t)\right) + \hat{v}_{j+1}(t),$$
$$\text{for } j: \ 1, \ldots, k-1$$
$$\text{if } j = k-1, \quad \hat{v}_{j+1}(t) = 0,$$

where all variables and matrices are defined in relations ((7a), (7b), and (7c)), and the activation functions $\mu_i(x)$ depend on the measurable states.

The dynamics of the augmented state estimation error $e_a(t)$ between the TS fuzzy model (1) and PIO (12) becomes

$$\dot{e}_a(t) = \sum_{i=1}^{r} \mu_i(x) \left(\overline{A}_i - \overline{K}_i \overline{C}\right) e_a(t). \tag{13}$$

Theorem 3. *The dynamic error* (13) *is asymptotically stable if there exist a symmetric matrix $Q > 0$ and matrices $\overline{N}_i$ such that the following conditions hold, for all $i \in \{1, \ldots, r\}$:*

$$Q\overline{A}_i + \overline{A}_i^T Q - \overline{N}_i \overline{C} - \overline{C}^T \overline{N}_i^T < 0. \tag{14a}$$

The parameters of the unknown input PIO (12) *are given by:*

$$\overline{K}_i = Q^{-1} \overline{N}_i. \tag{14b}$$

Proof. Consider the Lyapunov quadratic function $V(t) = e_a^T(t) Q e_a(t)$, where $Q = Q^T > 0$. The time derivative of $V(t)$ allows writing

$$\dot{V} = \sum_{i=1}^{r} \mu_i(\hat{x}) e_a^T \left((\overline{A}_i - \overline{K}_i \overline{C})^T Q + Q(\overline{A}_i - \overline{K}_i \overline{C})_i\right) e_a. \tag{15}$$

The stability condition $\dot{V}(t) < 0$ is satisfied if

$$(\overline{A}_i - \overline{K}_i \overline{C})^T Q + Q(\overline{A}_i - \overline{K}_i \overline{C})_i < 0; \tag{16}$$

with variables change $\overline{N}_i = Q\overline{K}_i$; we obtain the linear matrix inequalities ((14a) and (14b)). The proof is completed. □

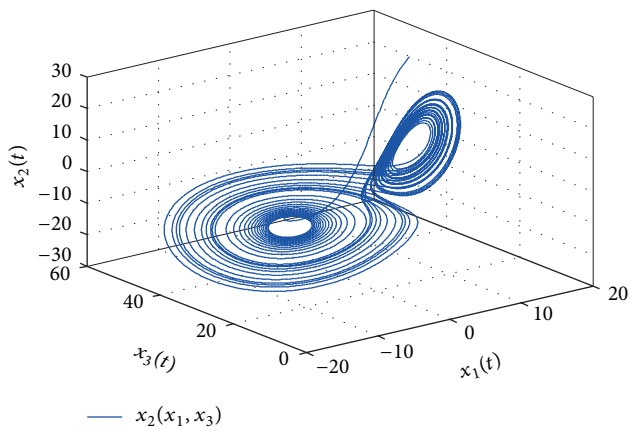


Figure 1: Chaotic behavior of Lorenz fuzzy system.

The unknown input PIO gains $\overline{K}_i = Q^{-1}\overline{N}_i$ are determined by resolving these constraints. In the following section, a simulation example is given through two chaotic systems in order to validate this proposed approach.

4. Simulation Examples

Two chaotic systems are characterized by TS fuzzy models with unmeasurable decision variables and subjected to unknown input. In a secure communication concept, the unknown input is assumed as a message to be recovered in the PIO after being encoded in the chaotic system by means of public transmission canal. These chaotic systems are used to show the good simultaneous reconstruction of states and message by the proposed unknown input PIO. The first nonlinear model is the Lorenz's system [41], and the second is the fourth Rossler's system [21].

4.1. Lorenz Chaotic System. The Lorenz chaotic system [38] is represented by following the dynamic equations:

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 10x_2, \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - \frac{8}{3}x_3. \end{aligned} \tag{17}$$

4.1.1. TS Fuzzy Model. The Lorenz's system can be exactly represented by TS fuzzy model with the decision variable $x_1(t) \in [-30, 30]$ as follow [42]:

Rule 1: $x_1(t)$ is $\mu_1(x_1(t))$, THEN $\dot{x}(t) = A_1x(t)$,

Rule 2: $x_2(t)$ is $\mu_2(x_1(t))$, THEN $\dot{x}(t) = A_2x(t)$,

where $x(t) = [x_1(t), x_2(t), x_3(t)]$, $\mu_1(x_1(t)) = (30 + x_1(t))/60$, $\mu_2(x_1(t)) = (30 - x_1(t))/60$, and

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -30 \\ 0 & 30 & \frac{-8}{3} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 30 \\ 0 & -30 & \frac{-8}{3} \end{bmatrix}. \tag{18}$$

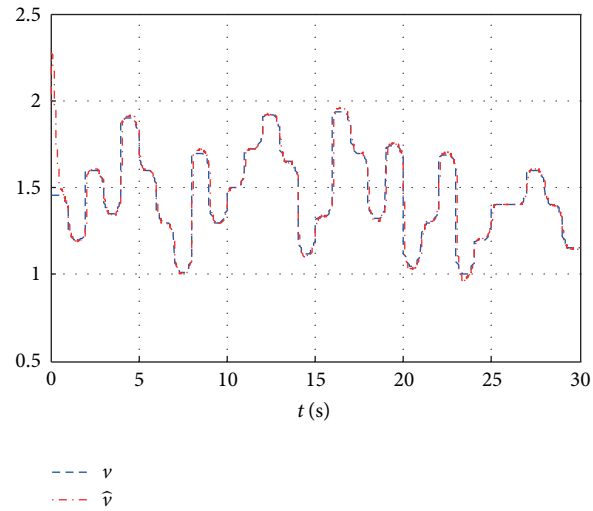


Figure 2: The unknown input and its estimated.

Table 1

$\lambda = 2 \times 10^3$	$\alpha = 5.505 \times 10^4$	$\alpha_0 = 0.001$
i	1	2
K_{pi}	$\begin{bmatrix} -11.471 & 12.445 \\ -22.084 & 10.838 \\ -94.350 & 96.029 \end{bmatrix}$	$\begin{bmatrix} -11.057 & 11.926 \\ 02.100 & 11.650 \\ 97.909 & -96.239 \end{bmatrix}$
K_{Ii}	$\begin{bmatrix} 15.123 & 03.344 \end{bmatrix}$	$\begin{bmatrix} 21.570 & -03.137 \end{bmatrix}$
K_{Ii}^1	$\begin{bmatrix} 83.889 & -02.802 \end{bmatrix}$	$\begin{bmatrix} 79.394 & 01.612 \end{bmatrix}$

The Lorenz chaotic attractor is given in Figure 1.

The TS fuzzy model of the Lorenz chaotic system (17) is

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2} \mu_i(x)\left(A_ix(t) + E_iv(t)\right), \\ y(t) &= Cx(t) + Ev(t), \end{aligned} \tag{19}$$

with $B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The unknown input $v(t)$ is assumed as a message to be recovered by the unknown input PIO (4). This observer plays the role of decoder and the chaotic system the encoder within a secure communication configuration.

4.1.2. Unknown Input PIO. The unknown input PIO gains are determined by resolving the LMIs constraints ((11a), (11b), and (11c)) of Theorem 2:

$$\overline{K}_i = \left[K_{Pi}^T \;\; K_{Ii}^T \;\; K_{Ii}^{1T}\right]^T, \qquad Z = \left[Z_x^T \;\; Z_v^T \;\; Z_{v1}^T\right]^T. \tag{20}$$

The unknown input PIO gains are given by Table 1.

The unknown input is assumed as a message to be encoded by chaotic system with the second derivative being zero, shown in Figure 2. The simulation results are obtained

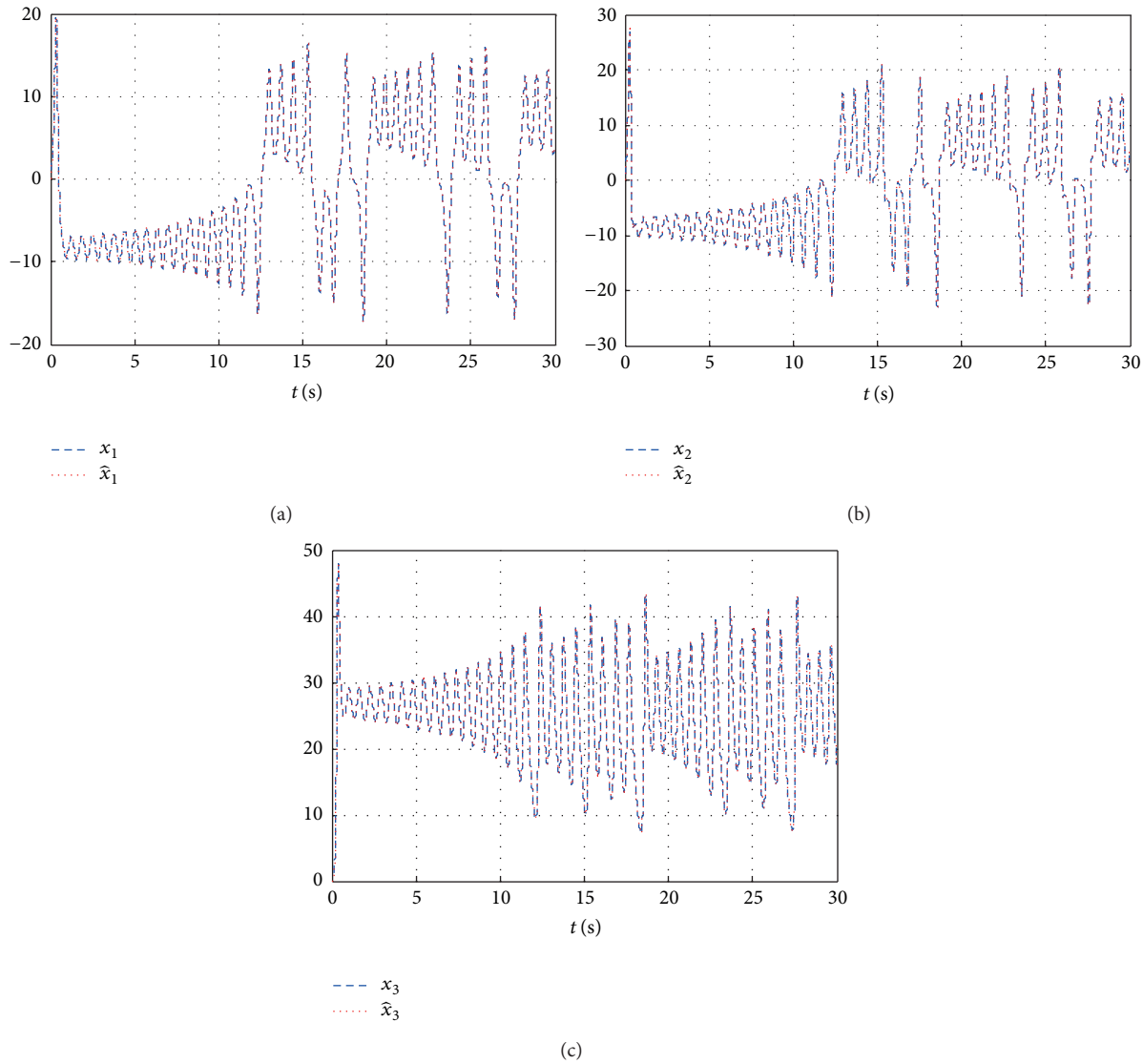


FIGURE 3: (a) The state $x_1(t)$ and its estimated $\hat{x}_1(t)$. (b) The state $x_2(t)$ and its estimated $\hat{x}_2(t)$. (c) The state $x_3(t)$ and its estimated $\hat{x}_3(t)$.

with the initial conditions $x_0 = [1 \;\; 1 \;\; 1]$ $\hat{x}_0 = [0 \;\; 0 \;\; 0]$ and with $\varepsilon = 10^{-3}$.

The unknown input and the estimated one are given in Figure 2. Excepted around the time origin, we obtained a good reconstruction of the unknown input. Figures 3(a), 3(b), and 3(c) show the states $x_1(t)$, $x_2(t)$, and $x_3(t)$ and their estimated $\hat{x}_1(t)$, $\hat{x}_2(t)$, and $\hat{x}_3(t)$, respectively.

The dynamic errors of the states estimation are given in Figure 4. The obtained simulation results show the good reconstruction of the states and the unknown input.

Remark 4. Note that applying Theorem 3 instead of Theorem 2 for this example leads to bad estimation. The simulation results for this example, carried out with the same initial conditions: $x_0 = [1 \;\; 1 \;\; 1]$ and $\hat{x}_0 = [0 \;\; 0 \;\; 0]$, are shown in Figure 5. Indeed, we see clearly that the best estimation (Figure 6) is given by Theorem 2 (dashed line) which takes into account the estimation of decision variables.

4.2. Fourth Rossler Chaotic System. The fourth Rossler chaotic system [21] is represented by the following dynamic equations:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + 0{,}254x_2 + x_4, \\ \dot{x}_3 &= x_1x_3 + 3, \\ \dot{x}_4 &= -0{,}5x_3 + 0{,}05x_4. \end{aligned} \tag{21}$$

4.2.1. TS Fuzzy Model. The fourth Rossler's system can be exactly described by TS fuzzy model with the decision variable $x_1(t) \in [-80, 20]$ as follow [42]:

Rule 1: $x_1(t)$ is $\mu_1(x_1(t))$, THEN $\dot{x}(t) = A_1x(t) + d_1$,

Rule 2: $x_2(t)$ is $\mu_2(x_1(t))$, THEN $\dot{x}(t) = A_2x(t) + d_2$,

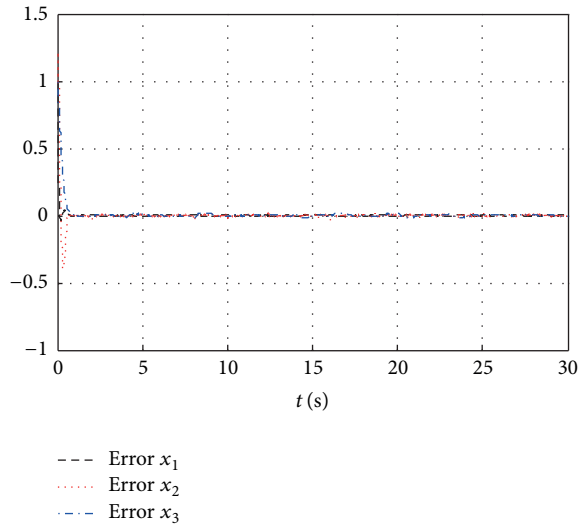


Figure 4: The errors between states and their estimated.

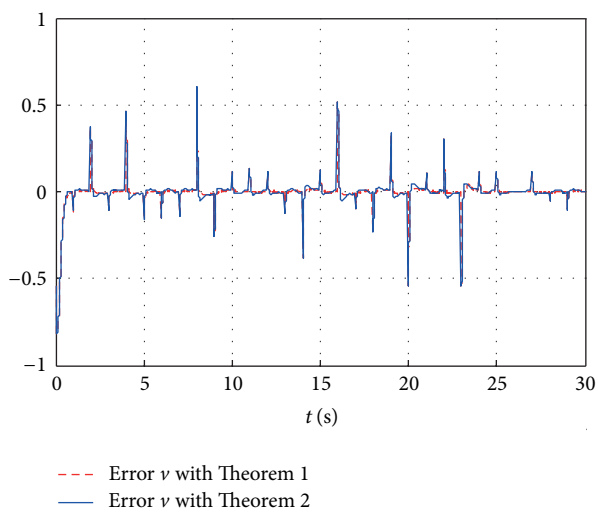


Figure 5: The errors between unknown input and its estimated.

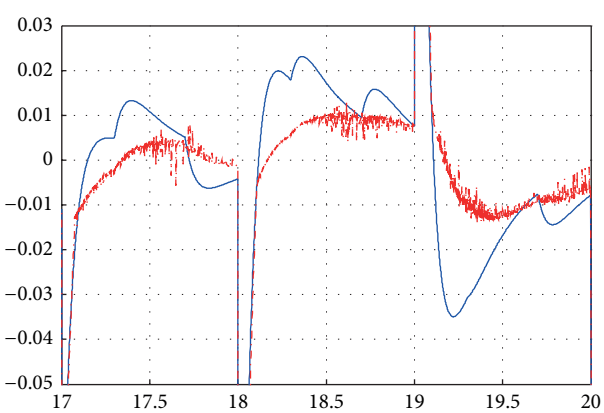


Figure 6: The zoom on the errors between unknown input and its estimated.

Table 2

$\lambda = 3 \times 10^3$	$\alpha = 6.243 \times 10^5$	$\alpha_0 = 6.467 \times 10^{-5}$
i	1	2
K_{pi}	$\begin{bmatrix} 011.018 & 001.805 \\ -000.343 & 001.344 \\ -078.779 & 109.443 \\ 011.208 & 005.138 \end{bmatrix}$	$\begin{bmatrix} 020.387 & -026.055 \\ 004.339 & -013.805 \\ 166.122 & -242.987 \\ 028.076 & -045.031 \end{bmatrix}$
K_{Ii}	$\begin{bmatrix} 052.544 & -006.911 \end{bmatrix}$	$\begin{bmatrix} 030.739 & 058.089 \end{bmatrix}$
K_{Ii}^1	$\begin{bmatrix} 097.452 & -008.136 \end{bmatrix}$	$\begin{bmatrix} 066.736 & 083.503 \end{bmatrix}$
K_{Ii}^2	$\begin{bmatrix} 091.450 & -008.807 \end{bmatrix}$	$\begin{bmatrix} 058.600 & 089.165 \end{bmatrix}$
K_{Ii}^3	$\begin{bmatrix} 034.284 & -001.174 \end{bmatrix}$	$\begin{bmatrix} 027.830 & 018.118 \end{bmatrix}$

where $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]$, $\mu_1(x_1(t)) = (80 + x_1(t))/100$, $\mu_2(x_1(t)) = (20 - x_1(t))/100$, and

$$A_1 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & -80 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \tag{22}$$
$$d_1 = d_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

The fourth Rossler chaotic attractor is given in Figure 7.

The TS fuzzy model of the fourth Rossler system (21) is

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x)\left(A_i x(t) + E_i v(t) + d_i\right),$$
$$y(t) = Cx(t) + Ev(t) \tag{23}$$

with $B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 \\ 0.5 \\ 1 \\ 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

The unknown input $v(t)$ is assumed as a message to be encoded by the TS fuzzy model (23).

4.2.2. Unknown Input PIO. The unknown PIO gains are obtained by resolving the LMIs constraints ((11a), (11b), and (11c)) of Theorem 3:

$$\overline{K}_i = \left[K_{Pi}^T \;\; K_{Ii}^T \;\; K_{Ii}^{1T} \;\; K_{Ii}^{2T} \;\; K_{Ii}^{3T}\right]^T,$$
$$Z = \left[Z_x^T \;\; Z_v^T \;\; Z_{v1}^T \;\; Z_{v2}^T \;\; Z_{v3}^T\right]^T. \tag{24}$$

The unknown PIO gains are given by Table 2.

The unknown input is assumed as a message to be encoded by chaotic system. The best results are obtained

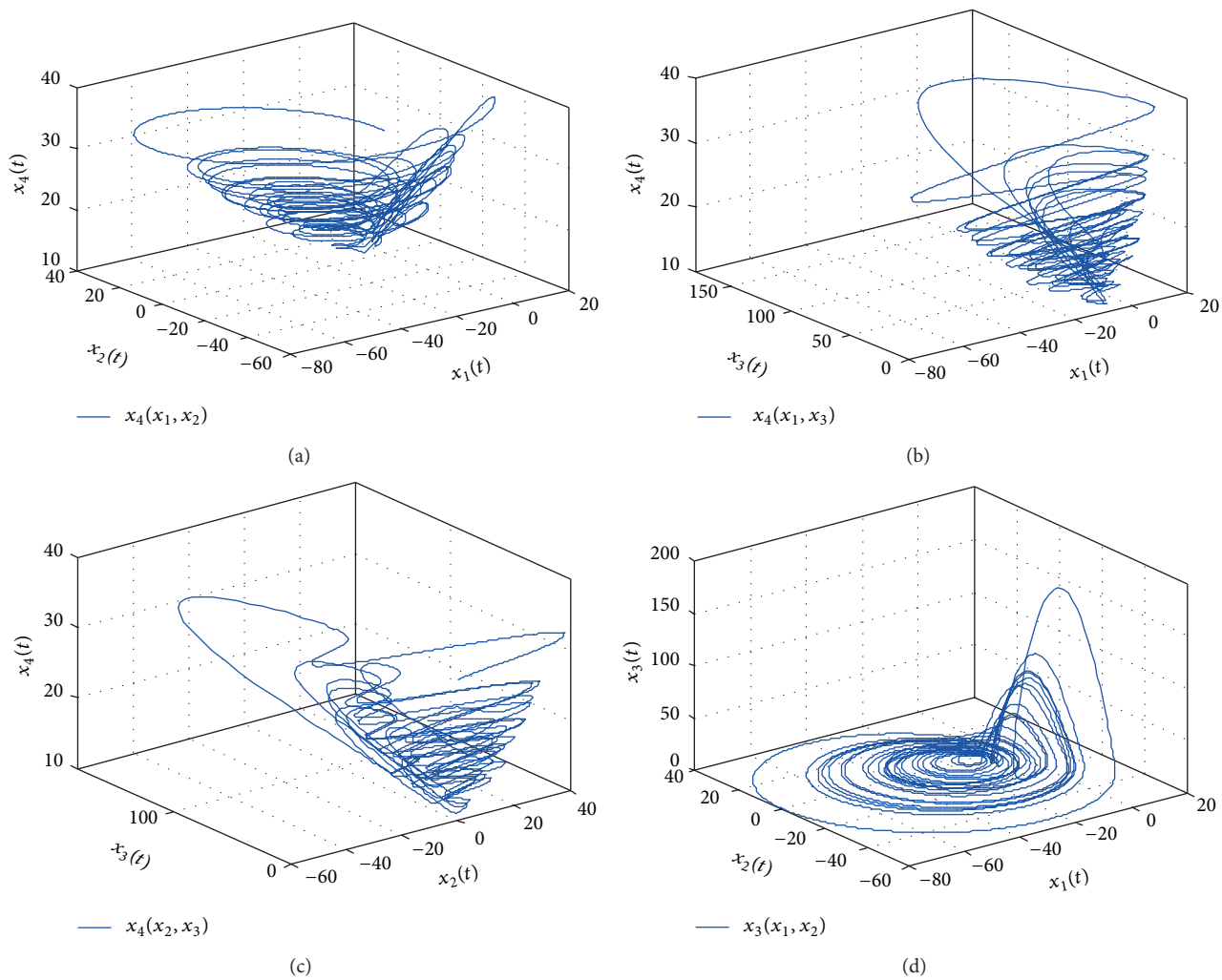


FIGURE 7: (a) Chaotic behavior $x_4(x_1(t), x_2(t))$ of fourth Rossler fuzzy system. (b) Chaotic behavior $x_4(x_1(t), x_3(t))$ of fourth Rossler fuzzy system. (c) Chaotic behavior $x_4(x_2(t), x_3(t))$ of fourth Rossler fuzzy system. (d) Chaotic behavior $x_3(x_1(t), x_2(t))$ of fourth Rossler fuzzy system.

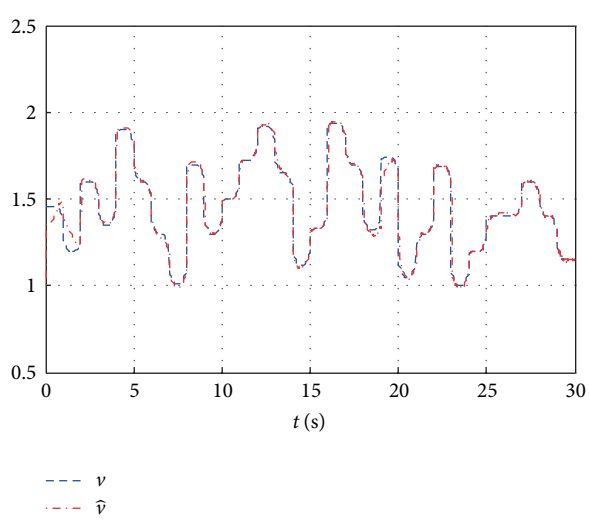


FIGURE 8: The unknown input and its estimated.

with the fourth derivative being zero, shown in Figure 8. The simulation results are obtained with the initial conditions $x_0 = [0 \;\; 0 \;\; 0 \;\; 30]$ and $\hat{x}_0 = [1 \;\; 1 \;\; 1 \;\; 29]$ and with $\varepsilon = 10^{-3}$.

The unknown input and the estimated one are given in Figure 8. Excepted around the time origin, we got a good reconstruction of the unknown input. Figures 9(a), 9(b), 9(c) and 9(d) represent the states $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ and their estimated $\hat{x}_1(t)$, $\hat{x}_2(t)$, $\hat{x}_3(t)$, and $\hat{x}_4(t)$, respectively.

The dynamic errors of the states estimation are represented in Figure 10. The obtained simulation results show the good estimation of the states and the unknown input.

5. Conclusion

In this paper, we have addressed the synchronization and reconstruction problem for chaotic systems. The TS fuzzy models subjected to unmeasurable decision variables and

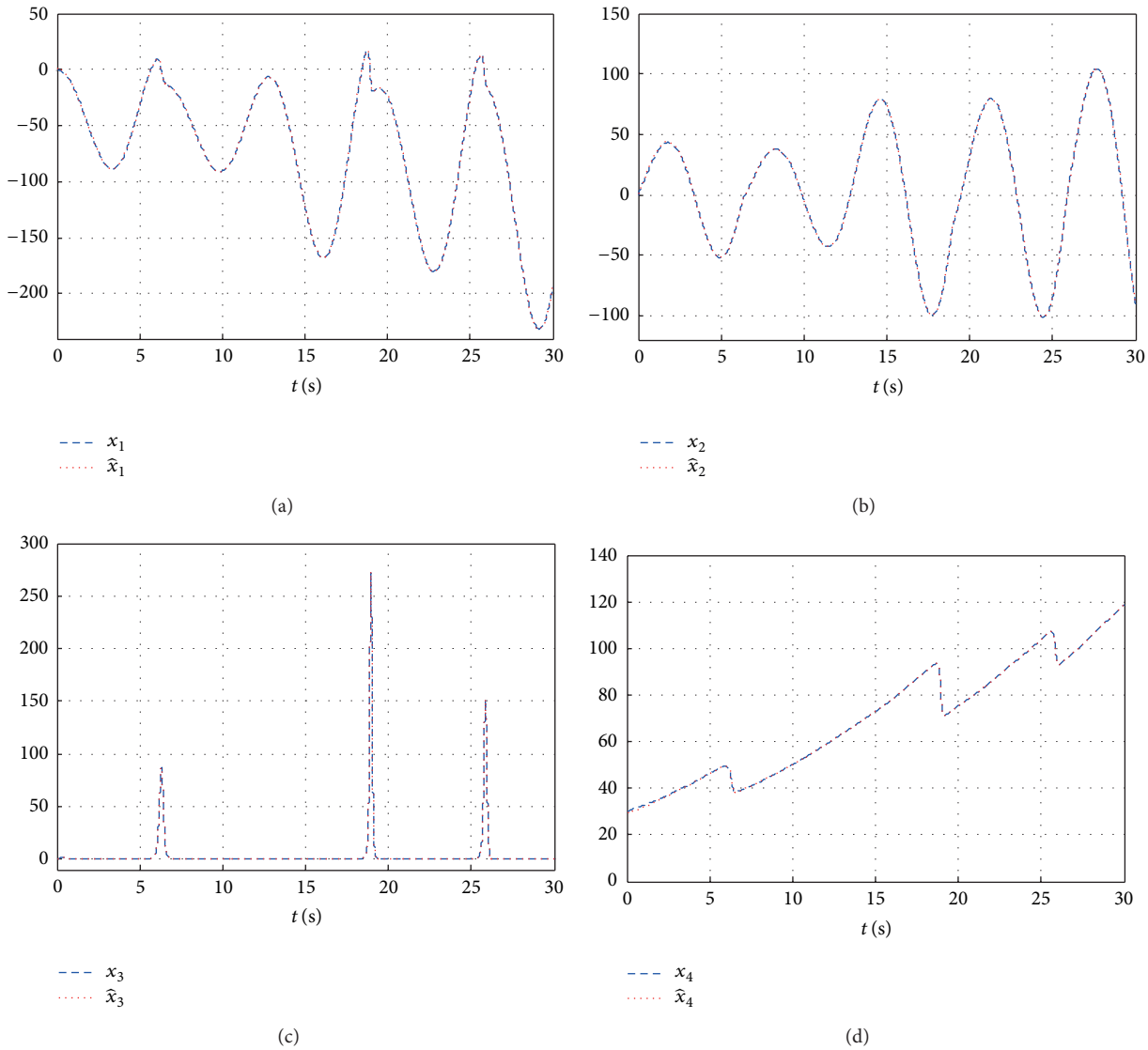


FIGURE 9: (a) The state $x_1(t)$ and its estimated $\hat{x}_1(t)$. (b) The state $x_2(t)$ and its estimated $\hat{x}_2(t)$. (c) The state $x_3(t)$ and its estimated $\hat{x}_3(t)$. (d) The state $x_4(t)$ and its estimated $\hat{x}_4(t)$.

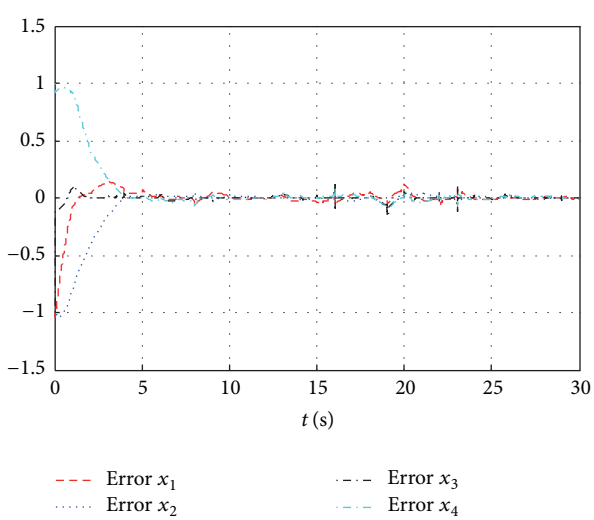


FIGURE 10: The errors between states and their estimated.

unknown inputs are employed to exactly describe the behavior of two chaotic systems, Lorenz's system and Rossler's system. Based on Lyapunov theory and LMI formulation, an unknown input proportional integral observer to achieve the synchronization and the unknown input reconstruction is designed. To take into account a wide variety of unknown inputs, a polynomial form with kth derivative zero is considered. Moreover, both the measurable and unmeasurable decision variables cases are studied. Simulation results are given to verify the effectiveness of the proposed method by reconstructing both states and unknown inputs. In the secure communication field, the proposed polynomial unknown input PIO with unmeasurable decision variables presents a good synchronization technique and messages recovering.

Motivated by the given results, the problem of diagnosis and fault tolerant control for more complex systems will be considered. Moreover, to reduce the conservatism due to the

quadratic approach, nonquadratic Lyapunov functions will be introduced in our further research.

Appendix

By using the quadratic Lyapunov function $V(t) = e_a^T(t)Pe_a(t)$ where $P = P^T > 0$ and the following lemma.

Lemma A.1. *For any matrices X and Y of appropriate dimensions, the following property is satisfied:*

$$X^TY + Y^TX \leq \lambda X^TX + \lambda^{-1}Y^TY \quad \text{with } \lambda > 0, \tag{A.1}$$

the time-derivative of $V(t)$ leads

$$\begin{aligned}\dot{V} \leq \sum_{i=1}^{r}\mu_i(\hat{x})\left(e_a^T\left(\overline{\mathscr{A}}_i^TP + P\overline{\mathscr{A}}_i + \alpha_0\delta_1^2I + \propto^{-1}P^2\right)e_a\right)\\ + \sigma_1\delta_1^2\hat{x}_a^T\hat{x}_a + \sigma_2\delta_2^2u^Tu + \sigma_3\delta_3^2I - 2e_a^TPz\end{aligned} \tag{A.2}$$

with

$$\begin{gathered}\alpha_0 = \lambda_1(1+\lambda), \qquad \propto^{-1} = \left(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}\right),\\ \sigma_1 = \lambda_1\left(1+\lambda^{-1}\right) = \left(\frac{\alpha_0}{\lambda}\right),\\ \sigma_2 = \lambda_2 = \left(\frac{\alpha\alpha_0\lambda_3}{\alpha\left(\alpha_0 + \lambda_3(1+\lambda)\right) - \alpha_0\lambda_3}\right),\\ \sigma_3 = \lambda_3.\end{gathered} \tag{A.3}$$

And taking into account (11c) we obtain

$$\begin{aligned}2e_a^TPz &= 2e_a^TP\sigma_1\delta_1^2\frac{\hat{x}_a^T\hat{x}_a}{2e_{ay}^Te_{ay}}P^{-1}\overline{C}^Te_{ay}\\ &\quad + 2e_a^TP\sigma_2\delta_2^2\frac{u^Tu}{2e_{ay}^Te_{ay}}P^{-1}\overline{C}^Te_{ay}\\ &\quad + 2e_a^TP\sigma_3\delta_3^2\frac{1}{2e_{ay}^Te_{ay}}P^{-1}\overline{C}^Te_{ay}\\ &= \sigma_1\delta_1^2\hat{x}_a^T\hat{x}_a + \sigma_2\delta_2^2u^Tu + \sigma_3\delta_3^2I.\end{aligned} \tag{A.4}$$

Then, the relation (A.2) becomes

$$\dot{V} \leq \sum_{i=1}^{r}\mu_i(\hat{x})e_a^T\left(\overline{\mathscr{A}}_i^TP + P\overline{\mathscr{A}}_i + \alpha_0\delta_1^2I + \propto^{-1}P^2\right)e_a. \tag{A.5}$$

The condition of stability $\dot{V}(t) < 0$ (for all $i = 1, \ldots, r$) is satisfied if

$$\overline{\mathscr{A}}_i^TP + P\overline{\mathscr{A}}_i + \alpha_0\delta_1^2I + \propto^{-1}P^2 + \alpha^{-1}P^2 < 0. \tag{A.6}$$

The Schur complement of condition (A.6) with variables given in (9) allows writing the LMI (11a).

References

[1] G. Jiang, G. Chen, and W. K. Tang, "Stabilizing unstable equilibria of chaotic systems from a state observer approach," *IEEE Transactions on Circuits and Systems II*, vol. 51, no. 6, pp. 281–288, 2004.

[2] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[3] T. L. Carroll and L. M. Pecora, "Synchronizing chaotic circuits," *IEEE Transactions on Circuits and Systems*, vol. 38, no. 4, pp. 453–456, 1991.

[4] E. N. Lorenz, "Deterministic nonperiodic flow," *Journal of the Atmospheric Sciences*, vol. 20, no. 2, pp. 130–141, 1963.

[5] J. Yang, G. Hu, and J. Xiao, "Chaos synchronization in coupled chaotic oscillators with multiple positive lyapunov exponents," *Physical Review Letters*, vol. 80, no. 3, pp. 496–499, 1998.

[6] S. Boccaletti, C. Grebogi, Y.-C. Lai, H. Mancini, and D. Maza, "The control of chaos: theory and applications," *Physics Reports*, vol. 329, no. 3, pp. 103–197, 2000.

[7] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, "The synchronization of chaotic systems," *Physics Reports*, vol. 366, no. 1-2, pp. 1–101, 2002.

[8] E. M. Shahverdiev, "Synchronization in systems with multiple time delays," *Physical Review E*, vol. 70, no. 6, Article ID 067202, 4 pages, 2004.

[9] J. Xu, L. Min, and G. Chen, "A chaotic communication scheme based on generalized synchronization and hash functions," *Chinese Physics Letters*, vol. 21, no. 8, pp. 1445–1448, 2004.

[10] J. W. Haefner, *Modeling Biological Systems: Principles and Applications*, Springer, New York, NY, USA, 2005.

[11] B. Shen, Z. Wang, and X. Liu, "Bounded H_∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon," *IEEE Transactions on Neural Networks*, vol. 22, no. 1, pp. 145–157, 2011.

[12] Y. Liu, Z. Wang, J. Liang, and X. Liu, "Synchronization of coupled neutral-type neural networks with jumping-mode-dependent discrete and unbounded distributed delays," *IEEE Transactions on Cybernetics*, vol. 43, no. 1, pp. 102–114, 2013.

[13] B. Shen, Z. Wang, and X. Liu, "Sampled-data synchronization control of dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[14] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed H_∞ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[15] B. Shen, Z. Wang, Y. S. Hung, and G. Chesi, "Distributed H_∞ filtering for polynomial nonlinear stochastic systems in sensor networks," *IEEE Transactions on Industrial Electronics*, vol. 58, no. 5, pp. 1971–1979, 2011.

[16] H. Dong, Z. Wang, and H. Gao, "Distributed filtering for a class of time-varying systems over sensor networks with quantization errors and successive packet dropouts," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 3164–3173, 2012.

[17] H. Dong, Z. Wang, J. Lam, and H. Gao, "Fuzzy-model-based robust fault detection with stochastic mixed time delays and successive packet dropouts," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 42, no. 2, pp. 365–376, 2012.

[18] B. Wanga, J. Wang, and S. M. Zhong, "Impulsive synchronization control for chaotic systems," *Procedia Engineering*, vol. 15, pp. 2721–2726, 2011.

[19] C. Tao, C. Yang, Y. Luo, H. Xiong, and F. Hu, "Speed feedback control of chaotic system," *Chaos, Solitons and Fractals*, vol. 23, no. 1, pp. 259–263, 2005.

[20] Y. Yu, "Adaptive synchronization of a unified chaotic system," *Chaos, Solitons and Fractals*, vol. 36, no. 2, pp. 329–333, 2008.

[21] C. Li, X. Liao, and K.-W. Wong, "Lag synchronization of hyperchaos with application to secure communications," *Chaos, Solitons and Fractals*, vol. 23, no. 1, pp. 183–193, 2005.

[22] M. S. Tavazoei and M. Haeri, "Determination of active sliding mode controller parameters in synchronizing different chaotic systems," *Chaos, Solitons and Fractals*, vol. 32, no. 2, pp. 583–591, 2007.

[23] J. G. Barajas-Ramírez, G. Chen, and L. S. Shieh, "Fuzzy chaos synchronization via sampled driving signals," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 14, no. 8, pp. 2721–2733, 2004.

[24] T. Liao and S. Tsai, "Adaptive synchronization of chaotic systems and its application to secure communications," *Chaos, solitons and fractals*, vol. 11, no. 9, pp. 1387–1396, 2000.

[25] C.-J. Cheng, "Robust synchronization of uncertain unified chaotic systems subject to noise and its application to secure communication," *Applied Mathematics and Computation*, vol. 219, no. 5, pp. 2698–2712, 2012.

[26] T. Yang, "Secure communication via chaotic parameter modulation," *IEEE Transactions on Circuits and Systems I*, vol. 43, no. 9, pp. 817–819, 1996.

[27] H. Dedieu, M. P. Kennedy, and M. Hasler, "Chaos shift keying: modulation and demodulation of a chaotic carrier using self-synchronizing Chua's circuits," *IEEE Transactions on Circuits and Systems II*, vol. 40, no. 10, pp. 634–642, 1993.

[28] K. M. Cuomo, A. V. Oppenheim, and S. H. Strogatz, "Synchronization of Lorenz-based chaotic circuits with applications to communications," *IEEE Transactions on Circuits and Systems II*, vol. 40, no. 10, pp. 626–633, 1993.

[29] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.

[30] K. Lian, T. Chiang, C. Chiu, and P. Liu, "Synthesis of fuzzy model-based designs to synchronization and secure communications for chaotic systems," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 31, no. 1, pp. 66–83, 2001.

[31] J.-H. Kim, C.-W. Park, E. Kim, and M. Park, "Adaptive synchronization of T-S fuzzy chaotic systems with unknown parameters," *Chaos, Solitons and Fractals*, vol. 24, no. 5, pp. 1353–1361, 2005.

[32] C.-H. Hyun, C.-W. Park, J.-H. Kim, and M. Park, "Synchronization and secure communication of chaotic systems via robust adaptive high-gain fuzzy observer," *Chaos, Solitons and Fractals*, vol. 40, no. 5, pp. 2200–2209, 2009.

[33] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1994.

[34] K. Tanaka and H. O. Wang, *Fuzzy Control System Design and Analysis. A Linear Matrix Inequality Approach*, John Wiley & Sons, New York, NY, USA, 2001.

[35] M. Chadli and H. R. Karimi, "Robust observer design for unknown inputs Takagi-Sugeno models," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 1, pp. 158–164, 2013.

[36] H. Dimassi, A. Loria, and S. Belghith, "A new secured transmission scheme based on chaotic synchronization via smooth adaptive unknown-input observers," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 9, pp. 3727–3739, 2012.

[37] M. Chen and W. Min, "Unknown input observer based chaotic secure communication," *Physics Letters A*, vol. 372, no. 10, pp. 1595–1600, 2008.

[38] M. Chadli and I. Zlinka, "Unknown input observer design for fuzzy systems with application to chaotic system reconstruction," *Computers and Mathematics with Applications*, vol. 66, no. 2, pp. 147–154, 2013.

[39] M. Chadli, A. Akhenak, J. Ragot, and D. Maquin, "State and unknown input estimation for discrete time multiple model," *Journal of the Franklin Institute*, vol. 346, no. 6, pp. 593–610, 2009.

[40] T. Youssef, M. Chadli, H. R. Karimi, and M. Zelmat, "Design of unknown inputs proportional integral observers for TS fuzzy models," *Neurocomputing Journal*, vol. 210, no. 163, p. 174, 2013.

[41] X. Meng, Y. Yu, G. Wen, and R. Chen, "Chaos synchronization of unified chaotic system using fuzzy logic controller," in *Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ '08)*, pp. 544–547, June 2008.

[42] K. Tanaka, T. Ikeda, and H. O. Wang, "A unified approach to controlling chaos via an LMI-based fuzzy control system design," *IEEE Transactions on Circuits and Systems I*, vol. 45, no. 10, pp. 1021–1040, 1998.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 134265, 15 pages
http://dx.doi.org/10.1155/2013/134265

Research Article

Achieving Synchronization in Arrays of Coupled Differential Systems with Time-Varying Couplings

Xinlei Yi,[1] Wenlian Lu,[1,2,3] and Tianping Chen[1,4]

[1] *School of Mathematical Sciences, Fudan University, Shanghai 200433, China*
[2] *Centre for Computational Systems Biology, Fudan University, Shanghai 200433, China*
[3] *Centre for Scientific Computing, The University of Warwick, Coventry CV4 7AL, UK*
[4] *School of Computer Science, Fudan University, Shanghai 200433, China*

Correspondence should be addressed to Wenlian Lu; wenlian@fudan.edu.cn

Received 10 April 2013; Accepted 5 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 Xinlei Yi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study complete synchronization of the complex dynamical networks described by linearly coupled ordinary differential equation systems (LCODEs). Here, the coupling is timevarying in both network structure and reaction dynamics. Inspired by our previous paper (Lu et al. (2007-2008)), the extended Hajnal diameter is introduced and used to measure the synchronization in a general differential system. Then we find that the Hajnal diameter of the linear system induced by the time-varying coupling matrix and the largest Lyapunov exponent of the synchronized system play the key roles in synchronization analysis of LCODEs with identity inner coupling matrix. As an application, we obtain a general sufficient condition guaranteeing directed time-varying graph to reach consensus. Example with numerical simulation is provided to show the effectiveness of the theoretical results.

1. Introduction

Complex networks have widely been used in theoretical analysis of complex systems, such as Internet, World Wide Web, communication networks, and social networks. A complex dynamical network is a large set of interconnected nodes, where each node possesses a (nonlinear) dynamical system and the interaction between nodes is described as diffusion. Among them, linearly coupled ordinary differential equation systems (LCODEs) are a large class of dynamical systems with continuous time and state.

The LCODEs are usually formulated as follows:

$$\dot{x}^i(t) = f\left(x^i(t)\right) + \sigma \sum_{j=1}^{m} l_{ij} B x^j(t), \quad i = 1, 2, \ldots, m, \tag{1}$$

where $t \in \mathbb{R}^+ = [0, +\infty)$ stands for the continuous time and $x^i(t) \in \mathbb{R}^n$ denotes the variable state vector of the ith node, $f : \mathbb{R}^n \to \mathbb{R}^n$ represents the node dynamic of the uncoupled system, $\sigma \in \mathbb{R}^+ = (0, +\infty)$ denotes coupling strength, $l_{ij} \geq 0$ with $i \neq j$ denotes the interaction between the two nodes, and $l_{ii} = -\sum_{j \neq i}^{m} l_{ij}$, $B \in \mathbb{R}^{n,n}$ denotes the inner coupling matrix. The LCODEs model is widely used to describe the model in nature and engineering. For example, the authors study spike-burst neural activity and the transitions to a synchronized state using a model of linearly coupled bursting neurons in [1]; the dynamics of linearly coupled Chua circuits are studied with application to image processing and many other cases in [2].

For decades, a large number of papers have focused on the dynamical behaviors of coupled systems [3–5], especially the synchronizing characteristics. The word "synchronization" comes from Greek; in this paper the concept of local complete synchronization (synchronization for simplicity) is considered (see Definition 3). For more details, we refer the readers to [6] and the references therein.

Synchronization of coupled systems have attracted a great deal of attention [7–9]. For instances, in [7], the authors considered the synchronization of a network of linearly coupled and not necessarily identical oscillators; in [8], the authors studied globally exponential synchronization for linearly coupled neural networks with time-varying delay and impulsive disturbances. Synchronization of networks with

time-varying topologies was studied in [10–16]. For example, in [10], the authors proposed the global stability of total synchronization in networks with different topologies; in [16], the authors gave a result that the network will synchronize with the time-varying topology if the time-average is achieved sufficiently fast.

Synchronization of LCODEs has also been addressed in [17–19]. In [17], mathematical analysis was presented on the synchronization phenomena of LCODEs with a single coupling delay; in [18], based on geometrical analysis of the synchronization manifold, the authors proposed a novel approach to investigate the stability of the synchronization manifold of coupled oscillators; in [19], the authors proposed new conditions on synchronization of networks of linearly coupled dynamical systems with non-Lipschitz right-handsides. The great majority of research activities mentioned above all focused on static networks whose connectivity and coupling strengths are static. In many applications, the interaction between individuals may change dynamically. For example, communication links between agents may be unreliable due to disturbances and/or subject to communication range limitations.

In this paper, we consider synchronization of LCODEs with time-varying coupling. Similar to [17–19], time-varying coupling will be used to represent the interaction between individuals. In [6, 13], they showed that the Lyapunov exponents of the synchronized system and the Hajnal diameter of the variational equation play key roles in the analysis of the synchronization in the discrete-time dynamical networks. In this paper, we extend these results to the continuous-time dynamical network systems. Different from [11, 16], where synchronization of fast-switching systems was discussed, we focus on the framework of synchronization analysis with general temporal variation of network topologies. Additional contributions of this paper are that we explicitly show that (a) the largest projection Lyapunov exponent of a system is equal to the logarithm of the Hajnal diameter, and (b) the largest Lyapunov exponent of the transverse space is equal to the largest projection Lyapunov exponent under some proper conditions.

The paper is organized as follows: in Section 2, some necessary definitions, lemmas, and hypotheses are given; in Section 3, synchronization of generalized coupled differential systems is discussed; in Section 4, criteria for the synchronization of LCODEs are obtained; in Section 5, we obtain a sufficient condition ensuring directed time-varying graph reaching consensus; in Section 6, example with numerical simulation is provided to show the effectiveness of the theoretical results; the paper is concluded in Section 7.

Notions. $e_k^n = [0, 0, \ldots, 0, 1, 0, \ldots, 0]^\top \in \mathbb{R}^n$ denotes the n-dimensional vector with all components zero except the kth component 1, $\mathbf{1}_n$ denotes the n-dimensional column vector with each component 1; for a set in some Euclidean space U, $\overline{U}$ denotes the closure of U, U^c denotes the complementary set of U, and $A \setminus B = A \cap B^c$; for $u = [u_1, \ldots, u_n]^\top \in \mathbb{R}^n$, $\|u\|$ denotes some vector norm, and for any matrix $A = (a_{ij}) \in \mathbb{R}^{n,m}$, $\|A\|$ denotes some matrix norm induced by vector norm, for example, $\|u\|_1 = \sum_{i=1}^n |u_i|$ and $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$; for a matrix $A = (a_{ij}) \in \mathbb{R}^{n,m}$, $|A|$ denotes a matrix with $|A| = (|a_{ij}|)$; for a real matrix A, $A^\top$ denotes its transpose and for a complex matrix B, B^* denotes its conjugate transpose; for a set in some Euclidean space W, $\mathcal{O}(W, \delta) = \{x : \operatorname{dist}(x, W) < \delta\}$, where $\operatorname{dist}(x, W) = \inf_{y \in W} \|x - y\|$; $\#J$ denotes the cardinality of set J; $\lfloor z \rfloor$ denotes the floor function, that is, the largest integer not more than the real number z; $\otimes$ denotes the Kronecker product; for a set in some Euclidean space W, W^m denote the Cartesian product $W \times \cdots \times W$ (m times).

2. Preliminaries

In this section we will give some necessary definitions, lemmas, and hypotheses. Consider the following general coupled differential system:

$$\dot{x}^i(t) = f^i\left(x^1(t), x^2(t), \ldots, x^m(t), t\right), \quad i = 1, 2, \ldots, m, \tag{2}$$

with initial state $x(t_0) = [x^1(t_0)^\top, \ldots, x^m(t_0)^\top]^\top \in \mathbb{R}^{nm}$, where $t_0 \in \mathbb{R}^+$ denotes the initial time, $t \in \mathbb{R}^+$ denotes the continuous time, and $x^i(t) = [x_1^i(t), \ldots, x_n^i(t)] \in \mathbb{R}^n$ denotes the variable state of the ith node, $i = 1, 2, \ldots, m$.

For the functions $f^i : \mathbb{R}^{nm} \times \mathbb{R}^+ \to \mathbb{R}^n$, $i = 1, 2, \ldots, m$, we make the following assumption.

Assumption 1. (a) There exists a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f^i(s, s, \ldots, s, t) = f(s)$ for all $i = 1, 2, \ldots, m$, $s \in \mathbb{R}^n$, and $t \geq 0$; (b) for any $t \geq 0$, $f^i(\cdot, t)$ is C^1-smooth for all $x = [x^{1\top}, \ldots, x^{m\top}]^\top \in \mathbb{R}^{nm}$, and by $DF^t(x) = ((\partial f^i/\partial x^j)(x, t))_{i,j=1}^m \in \mathbb{R}^{nm,nm}$ denotes the Jacobian matrix of $F(x, t) = [f^i(x, t)^\top, \ldots, f^m(x, t)^\top]^\top$ with respect to $x \in \mathbb{R}^{nm}$; (c) there exists a locally bounded function $\phi(x)$ such that $\|DF^t(x)\| \leq \phi(x)$ for all $(x, t) \in \mathbb{R}^{nm} \times \mathbb{R}^+$; (d) $DF^t(x)$ is *uniformly locally Lipschitz continuous*: there exists a locally bounded function $K(x, y)$ such that

$$\left\|DF^t(x) - DF^t(y)\right\| \leq K(x, y)\|x - y\| \tag{3}$$

for all $t \geq 0$ and $x, y \in \mathbb{R}^{nm}$; (e) $f^i(x, t)$ and $DF^t(x)$ are both measurable for $t \geq 0$.

We say a function $g(y) : \mathbb{R}^q \to \mathbb{R}^p$ is *locally bounded* if for any compact set $K \subset \mathbb{R}^q$, there exists $M > 0$ such that $\|g(y)\| \leq M$ holds for all $y \in K$.

The first item of Assumption 1 ensures that the diagonal synchronization manifold

$$\mathcal{S} = \left\{\left[x^{1\top}, x^{2\top}, \ldots, x^{m\top}\right]^\top \in \mathbb{R}^{nm} : x^{i\top} = x^{j\top}, \; i, j = 1, 2, \ldots, m\right\} \tag{4}$$

is an invariant manifold for (2).

If $x^1(t) = x^2(t) = \cdots = x^m(t) = s(t) \in \mathbb{R}^n$ is the synchronized state, then the synchronized state $s(t)$ satisfies

$$\dot{s}(t) = f(s(t)). \tag{5}$$

Since $f(\cdot)$ is C^1-smooth, then $s(t)$ can be denoted by the corresponding continuous semiflow $s(t) = \vartheta^{(t)}s_0$ of the intrinsic system (5). For $\vartheta^{(t)}$, we make following assumption.

Assumption 2. The system (5) has an asymptotically stable attractor: there exists a compact set $A \subset R^n$ such that (a) A is invariant through the system (5), that is, $\vartheta^{(t)}A \subset A$ for all $t \geq 0$; (b) there exists an open bounded neighborhood U of A such that $\bigcap_{t\geq 0} \vartheta^{(t)}\overline{U} = A$; (c) A is topologically transitive; that is, there exists $s_0 \in A$ such that $\omega(s_0)$, the ω limit set of the trajectory $\vartheta^{(t)}s_0$, is equal to A [3].

Definition 3. Local complete synchronization (synchronization for simplicity) is defined in the sense that the set

$$\mathcal{S} \bigcap A^m = \left\{\left[x^\top, x^\top, \ldots, x^\top\right]^\top \in \mathbb{R}^{nm} : x^\top \in A\right\} \quad (6)$$

is an asymptotically stable attractor in $\mathbb{R}^{nm}$. That is, for the coupled dynamical system (2), differences between components converge to zero if the initial states are picked sufficiently near $\mathcal{S} \bigcap A^m$, that is, if the components are all close to the attractor A and if their differences are sufficiently small.

Next we give some lemmas which will be used later, and the proofs can be seen in the appendix.

Lemma 4. *Under Assumption 1, one has*

$$\sum_{j=1}^{m} \frac{\partial f^i}{\partial x^j}(\widehat{s}, t) = \frac{\partial f}{\partial s}(s), \quad (7)$$

for all $s \in R^n$ and $t \geq 0$, where $\widehat{s} = [s^\top, s^\top, \ldots, s^\top]^\top$.

Lemma 5. *Under Assumptions 1 and 2, there exists a compact neighborhood W of A such that $\vartheta^{(t)}W \subset \vartheta^{(t')}W$ for all $t \geq t' \geq 0$ and $\bigcap_{t\geq 0} \vartheta^{(t)}W = A$.*

Let $\delta x(t) = [\delta x^1(t)^\top, \ldots, \delta x^m(t)^\top]^\top \in \mathbb{R}^{nm}$, where $\delta x^i(t) = x^i(t) - s(t) \in \mathbb{R}^n$. We have the following variational equation near the synchronized state $s(t)$:

$$\delta \dot{x}^i(t) = \sum_{j=1}^{m} \frac{\partial f^i}{\partial x^j}(\widehat{s}(t), t)\, \delta x^j(t), \quad i = 1, 2, \ldots, m, \quad (8)$$

or in matrix form:

$$\delta \dot{x}(t) = DF^t(s(t))\, \delta x(t), \quad (9)$$

where $DF^t(s(t))$ denotes the Jacobin matrix $DF^t(\widehat{s}(t))$ for simplicity.

From [20], we can give the results on the existence, uniqueness, and continuous dependence of (2) and (9).

Lemma 6. *Under Assumption 2, each of the differential equations* (2) *and* (9) *has a unique solution which is continuously dependent on the initial condition.*

Thus, the solution of the linear system (9) can be written in matrix form.

Definition 7. Solution matrix $U(t, t_0, s_0)$ of the system (9) is defined as follows. Let $U(t, t_0, s_0) = [u^1(t, t_0, s_0), \ldots, u^{nm}(t, t_0, s_0)]$, where $u^k(t, t_0, s_0)$ denotes the kth column and is the solution of the following Cauchy problem:

$$\begin{aligned} \delta \dot{x}(t) &= DF^t(s(t))\, \delta x(t), \\ s(t_0) &= s_0, \\ \delta x(t_0) &= e_k^{nm}. \end{aligned} \quad (10)$$

Immediately, according to Lemma 6, we can conclude that the solution of the following Cauchy problem

$$\begin{aligned} \delta \dot{x}(t) &= DF^t(s(t))\, \delta x(t), \\ s(t_0) &= s_0, \\ \delta x(t_0) &= \delta x_0 \end{aligned} \quad (11)$$

can be written as $\delta x(t) = U(t, t_0, s_0)\delta x_0$.

We define the time-varying Jacobin matrix DF^t by the following way:

$$\begin{aligned} D\mathcal{F} &: \mathbb{R}^+ \times R^n \longrightarrow 2^{\mathbb{R}^{nm,nm}}, \\ (t_0, s_0) &\longmapsto \left\{DF^t(s(t))\right\}_{t\geq t_0} \end{aligned} \quad (12)$$

with $s(t_0) = s_0$, where $2^{\mathbb{R}^{nm,nm}}$ is the collection of all the subsets of $\mathbb{R}^{nm,nm}$.

Definition 8. For a time varying system denoted by $D\mathcal{F}$, we can define its Hajnal diameter of the variational system (9) as follows:

$$\operatorname{diam}(D\mathcal{F}, s_0) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \left\{\operatorname{diam}(U(t, t_0, s_0))\right\}^{1/t}, \quad (13)$$

where for a $\mathbb{R}^{nm,nm}$ matrix in block matrix form: $U = (U_{ij})_{i,j=1}^m$ with $U_{ij} \in R^{n,n}$, its Hajnal diameter is defined as follows:

$$\operatorname{diam}(U) = \max_{i,j} \left\|U_i - U_j\right\|, \quad (14)$$

where $U_i = [U_{i1}, U_{i2}, \ldots, U_{im}]$.

Lemma 9 (Grounwell-Beesack's inequality). *If function $v(t)$ satisfies the following condition:*

$$v(t) \leq a(t) + b(t) \int_0^t v(\tau)\, d\tau, \quad (15)$$

where $b(t) \geq 0$ and $a(t)$ are some measurable functions, then one has

$$v(t) \leq a(t) + b(t) \int_0^t a(\tau)\, e^{\int_\tau^t b(\theta)d\theta} d\tau, \quad t \geq 0. \quad (16)$$

Based on Assumption 1, for the solution matrix U, we have the following lemma.

Lemma 10. *Under Assumption 1, one has the following:*

(1) $\sum_{j=1}^{m} U_{ij}(t,t_0,s_0) = \breve{U}(t,t_0,s_0)$, *where* $\breve{U}(t,t_0,s_0)$ *denotes the solution matrix of the following Cauchy problem:*

$$\dot{u} = \frac{\partial f}{\partial s}(s(t))\,u, \qquad s(t_0) = s_0; \tag{17}$$

(2) *for any given* $t \geq 0$ *and the compact set* W *given in Lemma 5,* $U(t+t_0,t_0,s_0)$ *is bounded for all* $t_0 \geq 0$ *and* $s_0 \in W$ *and equicontinuous with respect to* $s_0 \in W$.

Let $P = (P_{ij})_{i,j=1}^{m}$ be a $\mathbb{R}^{nm,nm}$ matrix with $P_{ij} \in \mathbb{R}^{n,n}$ satisfying (a) $P_{i1} = (1/\sqrt{m})P_0$ for some orthogonal matrix $P_0 \in \mathbb{R}^{n,n}$ and all $i = 1,2,\ldots,m$; (b) P is also an orthogonal matrix in $\mathbb{R}^{nm,nm}$. We also write P and its inverse $P^{-1} = P^{\mathsf{T}}$ in the form

$$P = [P_1, P_2], \qquad P^{\mathsf{T}} = \begin{bmatrix} P_1^{\mathsf{T}} \\ P_2^{\mathsf{T}} \end{bmatrix}, \tag{18}$$

where $P_1 = (1/\sqrt{m})\mathbf{1}_m \otimes P_0$ and $P_2 \in \mathbb{R}^{nm,n(m-1)}$. According to Lemma 10, we have

$$U(t,t_0,s_0)P_1 = \frac{1}{\sqrt{m}}\mathbf{1}_m \otimes \left[\breve{U}(t,t_0,s_0)P_0\right]. \tag{19}$$

Since $P_2^{\mathsf{T}}P_1 = 0$ which implies that each row of P_2^{T} is located in the subspace orthogonal to the subspace $\{\mathbf{1}_m \otimes \xi, \xi \in \mathbb{R}^n\}$, we can conclude that $P_2^{\mathsf{T}}U(t,t_0,s_0)P_1 = 0$. Then, we have

$$P^{-1}U(t,t_0,s_0)P = \begin{bmatrix} P_0^{\mathsf{T}}\breve{U}(t,t_0,s_0)P_0 & \alpha(t,t_0,s_0) \\ 0 & \widetilde{U}(t,t_0,s_0) \end{bmatrix}, \tag{20}$$

where $\breve{U}(t,t_0,s_0)$ denotes the common row sum of $U(t,t_0,s_0) = (U_{ij})_{i,j=1}^{m}$ as defined in Lemma 10, $\widetilde{U}(t,t_0,s_0) = P_2^{\mathsf{T}}U(t,t_0,s_0)P_2 \in \mathbb{R}^{n(m-1),n(m-1)}$, $\alpha(t,t_0,s_0) \in \mathbb{R}^{n,n(m-1)}$ denotes a matrix, and we omit its accurate expression. One can see that $\widetilde{U}(t,t_0,s_0)$ is the solution matrix of the following linear differential system.

Definition 11. We define the following linear differential system by the projection variational system of (9) along the directions P_2:

$$\dot{\phi} = D_P F^t(s(t))\,\phi, \qquad s(t_0) = s_0, \tag{21}$$

where $D_P F^t(s(t)) = P_2^{\mathsf{T}} DF^t(s(t)) P_2$.

Definition 12. For any time varying variational system $D\mathscr{F} : \mathbb{R}^+ \times \mathbb{R}^n \to 2^{\mathbb{R}^{nm,nm}}$, we define the Lyapunov exponent of the variational system (9) as follows:

$$\lambda(D\mathscr{F}, u, s_0) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \frac{1}{t} \log \left\| U(t,t_0,s_0)\,u \right\|, \tag{22}$$

where $u \in \mathbb{R}^{nm}$ and $s(t_0) = s_0$.

Similarly, we can define *the projection Lyapunov exponents* by the following projection time-varying variation:

$$D_P\mathscr{F} : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^{n(m-1),n(m-1)}}, \qquad (t_0,s_0) \longmapsto \left\{D_P F^t(s(t))\right\}_{t\geq t_0}, \tag{23}$$

that is,

$$\lambda(D_P\mathscr{F}, \tilde{u}, s_0) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \frac{1}{t} \log \left\| \widetilde{U}(t,t_0,s_0)\,\tilde{u} \right\|, \tag{24}$$

where $\tilde{u} \in \mathbb{R}^{n(m-1)}$ and $s(t_0) = s_0$. Let

$$\lambda_P(D\mathscr{F}, s_0) = \max_{\tilde{u}\in\mathbb{R}^{n(m-1)}} \lambda(D_P\mathscr{F}, \tilde{u}, s_0). \tag{25}$$

Then, we have the following lemma.

Lemma 13. $\lambda_P(D\mathscr{F}, s_0) = \log \operatorname{diam}(D\mathscr{F}, s_0)$.

Remark 14. From Lemma 13, we can see that the largest projection Lyapunov exponent is independent of the choice of matrix P.

Consider the time-varying driven by some metric dynamical system MDS$(\Omega, \mathscr{B}, \mathbb{P}, \varrho^{(t)})$, where Ω is the compact state space, $\mathscr{B}$ is the σ-algebra, $\mathbb{P}$ is the probability measure, and $\varrho^{(t)}$ is a continuous semiflow. Then, the variational equation (9) is independent of the initial time t_0 and can be rewritten as follows:

$$\dot{\phi} = DF\left(s(t), \varrho^{(t)}\omega_0\right)\phi, \qquad s(0) = s_0. \tag{26}$$

In this case, we denote *the solution matrix*, *the projection solution matrix*, and *the solution matrix* on the synchronization space by $U(t,s_0,\omega_0)$, $\widetilde{U}(t,s_0,\omega_0)$, and $\breve{U}(t,s_0,\omega_0)$, respectively. For simplicity, we write them as $U(t)$, $\widetilde{U}(t)$, and $\breve{U}(t)$, respectively. Also, we write *the Lyapunov exponents* and *the projection Lyapunov exponent* as follows:

$$\begin{aligned} \lambda(D\mathscr{F}, u, s_0, \omega_0) &= \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| U(t,s_0,\omega_0)\,u \right\|, \\ \lambda(D\mathscr{F}, s_0, \omega_0) &= \max_{u\in\mathbb{R}^{nm}} \lambda(D\mathscr{F}, u, s_0, \omega_0), \\ \lambda_P(D\mathscr{F}, u, s_0, \omega_0) &= \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \widetilde{U}(t,s_0,\omega_0)\,u \right\|, \\ \lambda_P(D\mathscr{F}, s_0, \omega_0) &= \max_{u\in\mathbb{R}^{n(m-1)}} \lambda(D_P\mathscr{F}, \tilde{u}, s_0, \omega_0). \end{aligned} \tag{27}$$

We add the following assumption.

Assumption 15. (a) $\varrho^{(t)}$ is a continuous semiflow; (b) $DF(s,\omega)$ is a continuous map for all $(s,\omega) \in \mathbb{R}^n \times \Omega$.

The following are involving linear differential systems. For more details, we refer the readers to [21]. For a continuous scalar function $u(t)$, we denote its *Lyapunov exponent* by

$$\chi[u(t)] = \overline{\lim_{t\to\infty}} \frac{1}{t} \log |u(t)|. \tag{28}$$

The following properties will be used later:

(1) $\chi[\prod_{k=1}^{n} c_k u_k(t)] \le \sum_{k=1}^{n} \chi[u_k(t)]$, where c_k, $k = 1, 2, \ldots, n$, are constants;

(2) if $\lim_{t\to\infty}(1/t)\log|u(t)| = \alpha$, which is finite, then $\chi[1/(u(t))] = -\alpha$;

(3) $\chi[u(t) + v(t)] \le \max\{\chi[u(t)], \chi[v(t)]\}$;

(4) for a vector-value or matrix-value function $U(t)$, we define $\chi[U(t)] = \chi[\|U(t)\|]$.

For the following linear differential system:

$$\dot{x}(t) = A(t)\,x(t), \tag{29}$$

where $x(t) \in \mathbb{R}^n$, a transformation $x(t) = L(t)y(t)$ is said to be a *Lyapunov transformation* if $L(t)$ satisfies

(1) $L(t) \in C^1[0, +\infty)$;

(2) $L(t)$, $\dot{L}(t)$, $L^{-1}(t)$ are bounded for all $t \ge 0$.

It can be seen that the class of Lyapunov transformations forms a group and the linear system for $y(t)$ should be

$$\dot{y}(t) = B(t)\,y(t), \tag{30}$$

where $B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\dot{L}(t)$. Then, we say system (30) is a *reducible system* of system (29). We define the *adjoint system* of (29) by

$$\dot{x}(t) = -A^*(t)\,x(t). \tag{31}$$

If letting $V(t)$ be the fundamental matrix of (29), then $[V^{-1}(t)]^*$ is the fundamental matrix of (31). Thus, we say the system (29) is a *regular system* if the adjoint systems (29) and (31) have convergent Lyapunov exponent series: $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$, respectively, which satisfy $\alpha_i + \beta_i = 0$ for $i = 1, 2, \ldots, n$, or its reducible system (30) is also regular.

Lemma 16. *Suppose that Assumptions 1, 2, and 15 are satisfied. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_{nm}\}$ be the Lyapunov exponents of the variational system (26), where $\{\sigma_1, \ldots, \sigma_n\}$ correspond to the synchronization space and the remaining correspond to the transverse space. Let $\lambda_T(D\mathcal{F}, s_0, \omega_0) = \max_{i\ge n+1}\sigma_i$ and $\lambda_S(D\mathcal{F}, s_0, \omega_0) = \max_{1\le i\le n}\sigma_i$. If (a) the linear system (17) is a regular system, (b) $\|DF(s(t), \varrho^{(t)}\omega_0)\| \le M$ for all $t \ge 0$, (c) $\lambda_P(D\mathcal{F}, s_0, \omega_0) \ne \lambda_S(D\mathcal{F}, s_0, \omega_0)$, then $\lambda_T(D\mathcal{F}, s_0, \omega_0) = \lambda_P(D\mathcal{F}, s_0, \omega_0)$.*

3. General Synchronization Analysis

In this section we provide a methodology based on the previous theoretical analysis to judge whether a general differential system can be synchronized or not.

Theorem 17. *Suppose that $W \in \mathbb{R}^n$ is the compact subset given in Lemma 5, and Assumptions 1 and 2 are satisfied. If*

$$\sup_{s_0\in W} \operatorname{diam}(D\mathcal{F}, s_0) < 1, \tag{32}$$

then the coupled system (2) is synchronized.

Proof. The main techniques of the proof come from [3, 6] with some modifications. Let $\vartheta^{(t)}$ be the semiflow of the uncoupled system (5). By the condition (32), there exist d satisfying $\sup_{s_0\in W}\operatorname{diam}(D\mathcal{F}, s_0) < d < 1$ and $T_1 \ge 0$ such that $d^{T_1} < 1/3$, and $r_0 = \inf\{r > 0, \mathcal{O}(\vartheta^{(T_1)}W, r) \subset W\} > 0$. For each $s_0 \in W$, there must exist $t(s_0) \ge T_1$ such that $\operatorname{diam}(U(t_0 + t(s_0), t_0, s_0)) < d^{t(s_0)}$ for all $t_0 \ge 0$. According to the equicontinuity of $U(t_0 + t(s_0), t_0, s_0)$, there exists $\delta > 0$ such that for any $s_0' \in \mathcal{O}(s_0, \delta)$, $\operatorname{diam}(U(t_0 + t(s_0), t_0, s_0')) < d^{t(s_0)}$ for all $t_0 \ge 0$. According to the compactness of W, there exists a finite positive number set $\mathcal{T} = \{t_1, t_2, \ldots, t_v\}$ with $t_j \ge T_1$ for all $j = 1, 2, \ldots, v$ such that for any $s_0 \in W$, there exists $t_j \in \mathcal{T}$ such that $\operatorname{diam}(U(t_0 + t_j, t_0, s_0)) < 1/3$ for all $t_0 \ge 0$. Let $x(t)$ be the collective states $\{x^1(t), \ldots, x^m(t)\}$ which is the solution of the coupled system (2) with initial condition $x^i(t_0) = x_0^i$, $i = 1, 2, \ldots, m$. And let $s(t)$ be the solution of the synchronization state equation (5) with initial condition $s(t_0) = \overline{x}_0 = (1/m)\sum_{j=1}^{m} x_0^j \in W$. Then, letting $\Delta x^i(t) = x^i(t) - s(t)$, we have

$$\begin{aligned}\Delta\dot{x}_k^i(t) &= f_k^i\left(x^1(t), \ldots, x^m(t), t\right) - f_k(s(t)) \\ &= \sum_{j=1}^{m}\sum_{l=1}^{n}\frac{\partial f_k^i}{\partial x_l^j}\left(\xi_{kl}^{ij}(t), t\right)\Delta x_l^j(t),\end{aligned} \tag{33}$$

where $\xi_{kl}^{ij}(t) \in \mathbb{R}^{mn}$, $i, j = 1, 2, \ldots, m$, $k, l = 1, 2 \ldots, n$, are obtained by the mean value principle of the differential functions. Letting $DF^t(\xi(t)) = ((\partial f_k^i/\partial x_l^i)(\xi_{kl}^{ij}(t), t))$, we can write the equations above in matrix form:

$$\Delta\dot{x}(t) = DF^t(\xi(t))\,\Delta x(t), \tag{34}$$

and denote its solution matrix by $\widehat{U}(t + t_0, t_0, x_0) = (\widehat{U}_{ij}(t + t_0, t_0, x_0))_{i,j=1}^{m}$. Then, for any $t > 0$ there exists $K_2 > 0$ such that $\|DF^{t+t_0}(\xi(t + t_0))\| \le K_2$ for all $t \in \mathcal{T}$ and $t_0 \ge 0$ according to the 3th item of Assumption 1. Then, we have

$$\begin{aligned}&\Delta x_k^i(t + t_0) \\ &\quad = x_0^i - \overline{x}_{0k} + \int_{t_0}^{t+t_0}\sum_{j=1}^{m}\sum_{l=1}^{n}\frac{\partial f_k^i}{\partial x_l^j}\left(\xi_{kl}^{ij}(\tau), \tau\right)\Delta x_l^j(\tau)\,d\tau, \\ &\sum_{j=1}^{m}\sum_{k=1}^{n}\left\|\Delta x_k^i(t + t_0)\right\| \\ &\quad \le \sum_{j=1}^{m}\sum_{k=1}^{n}\left\|x_{0k}^i - \overline{x}_{0k}\right\| + K_2\int_{t_0}^{t+t_0}\sum_{j=1}^{m}\sum_{l=1}^{n}\left\|\Delta x_l^j(\tau)\right\|d\tau.\end{aligned} \tag{35}$$

By Lemma 9, we have

$$\begin{aligned}&\sum_{j=1}^{m}\sum_{l=1}^{n}\left\|\Delta x_l^j(t + t_0)\right\| \\ &\quad \le e^{K_2 t}\sum_{j=1}^{m}\sum_{l=1}^{n}\left\|x_{0l}^j - \overline{x}_{0l}\right\|.\end{aligned} \tag{36}$$

Let

$$W_\alpha = \left\{ x = \left[x^{1\top}, \ldots, x^{m\top} \right]^\top : \overline{x} \in W, \sum_{j=1}^{m} \left\| x^j - \overline{x} \right\| \le \alpha \right\}. \tag{37}$$

Picking α sufficiently small such that for each $x_0 \in W_\alpha$, there exists $t \in \mathscr{T}$ such that $\sum_{j=1}^{m} \| \Delta x^j(t+t_0) \| < r_0/2$ and $\operatorname{diam}(\widehat{U}(t+t_0, t_0, x_0)) < 1/2$ for all $t_0 \ge 0$.

Thus, we are to prove synchronization step by step.

For any $x_0 \in W_\alpha$, there exists $t' = t(x_0) \in \mathscr{T}$ such that

$$\begin{aligned} &\left\| x^i\left(t'+t_0\right) - x^j\left(t'+t_0\right) \right\| \\ &\quad = \left\| \Delta x^i\left(t'+t_0\right) - \Delta x^j\left(t'+t_0\right) \right\| \\ &\quad \le \sum_{k=1}^{m} \left\| \widehat{U}_{ik}\left(t'+t_0, t_0, x_0\right) - \widehat{U}_{jk}\left(t'+t_0, t_0, x_0\right) \right\| \left\| \Delta x_0^k \right\| \\ &\quad \le \operatorname{diam}\left(\widehat{U}\left(t'+t_0, t_0, x_0\right)\right) \max_{i,j} \left\| x_0^i - x_0^j \right\| \\ &\quad \le \frac{1}{2} \max_{i,j} \left\| x_0^i - x_0^j \right\|. \end{aligned} \tag{38}$$

Therefore, we have $\max_{i,j} \| x^i(t'+t_0) - x^j(t'+t_0) \| \le (1/2)\max_{i,j}\|x_0^i - x_0^j\|$, which implies that $\overline{x}(t'+t_0) \in W$ and $x(t'+t_0) \in W_{\alpha/2}$.

Then, reinitiated with time $t'+t_0$ and condition $x(t'+t_0)$, continuing with the phase above, we can obtain that $\lim_{t\to\infty}\max_{i,j}\|x^i(t) - x^j(t)\| = 0$. Namely, the coupled system (2) is synchronized. Furthermore, from the proof, we can conclude that the convergence is exponential with rate $O(\delta^t)$ where $\delta = \sup_{s_0 \in W} \operatorname{diam}(D\mathscr{F}^t, s_0)$, and uniform with respect to $t_0 \ge 0$ and $x_0 \in W_\alpha$. This completes the proof. □

Remark 18. According to Assumption 2 that attractor A is asymptotically stable and the properties of the compact neighbor W given in Lemma 5, we can conclude that the quantity

$$\sup_{s_0 \in W} \operatorname{diam}\left(D\mathscr{F}, s_0\right) \tag{39}$$

is independent on the choice of W.

If the timevariation is driven by some MDS$(\Omega, \mathscr{B}, \mathbb{P}P, \varrho^{(t)})$ and there exists a metric dynamical system $\{W \times \Omega, \mathbf{F}, \mathbf{P}, \pi^{(t)}\}$, where $\mathbf{F}$ is the product σ-algebra on $W \times \Omega$, $\mathbf{P}$ is the probability measure, and $\pi^{(t)}(s_0, \omega) = (\theta^{(t)}s_0, \varrho^{(t)}\omega)$. From Theorem 17, we have the following.

Corollary 19. *Suppose that the conditions in Lemma 16 are satisfied, $W \times \Omega$ is compact in the topology defined in this MDS, the semiflow $\pi^{(t)}$ is continuous, and on $W \times \Omega$ the Jacobian matrix $DF(\theta^{(t)}s_0, \varrho^{(t)}\omega)$ is continuous. Let $\{\sigma_i\}_{i=1}^{nm}$ be the Lyapunov exponents of this MDS with multiplicity and $\{\sigma_i\}_{i=1}^{n}$ correspond to the synchronization space. If*

$$\sup_{\mathbf{P} \in Erg_\pi(W\times\Omega)} \sup_{i \ge n+1} \sigma_i < 0, \tag{40}$$

where $Erg_\pi(W \times \Omega)$ denotes the ergodic probability measure set supported in the MDS $\{W \times \Omega, \mathbf{F}, \mathbf{P}, \pi^{(t)}\}$, then the coupled system (2) is synchronized.

4. Synchronization of LCODEs with Identity Inner Coupling Matrix and Time-Varying Couplings

In this section we study synchronization in linearly coupled ordinary differential equation systems (LCODEs) with time-varying couplings. Considering the following LCODEs with identity inner coupling matrix:

$$\dot{x}^i(t) = f\left(x^i(t)\right) + \sigma \sum_{j=1}^{m} l_{ij}(t)\, x^j(t), \quad i = 1, 2, \ldots, m, \tag{41}$$

where $x^i(t) \in \mathbb{R}^n$ denotes the state variable of the ith node, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a differential map, $\sigma \in \mathbb{R}^+$ denotes coupling strength, and $l_{ij}(t)$ denotes the coupling coefficient from node j to i at time t, for all $i \ne j$, which are supposed to satisfy the following assumption. Here, we highlight that the inner coupling matrix is the identity matrix.

Assumption 20. (a) $l_{ij}(t) \ge 0$, $i \ne j$ are measurable and $l_{ii}(t) = -\sum_{j=1, j\ne i}^{m} l_{ij}(t)$; (b) there exists $M_1 > 0$ such that $|l_{ij}(t)| \le M_1$ for all $i, j = 1, 2, \ldots, m$.

Similarly, we can define the Hajnal diameter of the following linear system:

$$\dot{u}(t) = \sigma L(t)\, u(t). \tag{42}$$

Let $V(t) = (v_{ij}(t))_{i,j=1}^{m}$ be the fundamental solution matrix of the system (42). Then, its solution matrix can be written as $V(t, t_0) = V(t)V(t_0)^{-1}$. Thus, the Hajnal diameter of the system (42) can be defined as follows:

$$\operatorname{diam}(\mathscr{L}) = \overline{\lim_{t\to\infty}} \sup_{t_0 \ge 0} \left\{ \operatorname{diam}\left(V(t, t_0)\right) \right\}^{1/t}. \tag{43}$$

By Theorem 17, we have the following theorem.

Theorem 21. *Suppose Assumptions 1, 2, and 20 are satisfied. Let μ be the largest Lyapunov exponent of the synchronized system $\dot{s}(t) = f(s(t))$, that is,*

$$\mu = \sup_{s_0 \in W} \max_{u \in \mathbb{R}^n} \lambda\left(Df, u, s_0\right). \tag{44}$$

If $\log(\operatorname{diam}(\mathscr{L})) + \mu < 0$, then the LCODEs (41) is synchronized.

Proof. Considering the variational equation of (41):

$$\delta\dot{x}(t) = \left\{ I_m \otimes Df\left(s(t)\right) + \sigma L(t) \otimes I_n \right\} \delta x(t). \tag{45}$$

Let $\breve{U}(t,t_0,s_0)$ be the solution matrix of the synchronized state system (17) and $V(t,t_0)=(v_{ij}(t,t_0))_{i,j=1}^m$ be the solution matrix of the linear system (42). We can see that $V(t,t_0)\otimes\breve{U}(t,t_0,s_0)$ is the solution matrix of the variational system (45). Then,

$$\begin{aligned}&\operatorname{diam}\left(V\left(t+t_0,t_0\right)\otimes\breve{U}\left(t+t_0,t_0,s_0\right)\right)\\&\quad=\max_{i,j=1,\dots,m}\sum_{k=1}^m\left|v_{ik}\left(t+t_0,t_0\right)-v_{jk}\left(t+t_0,t_0\right)\right|\\&\qquad\times\left\|\breve{U}\left(t+t_0,t_0,s_0\right)\right\|\\&\quad=\operatorname{diam}\left(V\left(t+t_0,t_0\right)\right)\left\|\breve{U}\left(t+t_0,t_0,s_0\right)\right\|.\end{aligned}\tag{46}$$

This implies that the Hajnal diameter of the variational system (45) is less than $e^{\mu}\operatorname{diam}(\mathscr{L})$. This completes the proof according to Theorem 17. □

For the linear system (42), we firstly have the following lemma.

Lemma 22 (see [22]). *$V(t,t_0)$ is a stochastic matrix.*

From Lemmas 13 and 16, we have the following corollary.

Corollary 23. $\log\operatorname{diam}(\mathscr{L})=\lambda_P(\mathscr{L})$, *where $\lambda_P(\mathscr{L})$ denotes the largest one of all the projection Lyapunov exponents of system* (41). *Moreover, if the conditions in Lemma 16 are satisfied, then* $\log\operatorname{diam}(\mathscr{L})=\lambda_T(\mathscr{L})$, *where $\lambda_T(\mathscr{L})$ denotes the largest one of all the Lyapunov exponents corresponding to the transverse space, that is, the space orthogonal to the synchronization space.*

If $L(t)$ is periodic, we have the following.

Corollary 24. *Suppose that $L(t)$ is periodic. Let ς_i, $i=1,2,\dots,m$, are the Floquet multipliers of the linear system* (42). *Then, there exists one multiplier denoted by $\varsigma_1=1$ and* $\operatorname{diam}(\mathscr{L})=\max_{i\geq2}\varsigma_i$.

If $L(t)=L(\varrho^{(t)}\omega)$ is driven by some $\mathrm{MDS}(\Omega,\mathscr{B},P,\varrho^{(t)})$, from Corollaries 19 and 23, we have the following corollary.

Corollary 25. *Suppose $L(\omega)$ is continuous on Ω and conditions in Lemma 16 are satisfied. Let $\mu=\sup_{s_0\in W}\max_{u\in\mathbb{R}^n}\lambda(Df,u,s_0)$, ς_i, $i=1,2,\dots,m$, be the Lyapunov exponents of the linear system* (42) *with $\varsigma_1=0$, and $\varsigma=\sup_{P\in Erf_\theta(\Omega)}\max_{i\geq2}\varsigma_i$. If $\mu+\varsigma<0$, then the coupled system* (41) *is synchronized.*

Let $\mathscr{I}$ be the set consisting of all compact time intervals in $[0,+\infty)$ and $\mathscr{G}$ be the the set consisting of all graph with vertex set $\mathscr{N}=\{1,2,\dots,m\}$.

Define

$$\begin{aligned}G:\mathscr{I}\times R^+&\longrightarrow\mathscr{G},\\(I=[t_1,t_2],\delta)&\longmapsto G(I,\delta),\end{aligned}\tag{47}$$

where $G(I,\delta)=\{\mathscr{N},\mathscr{E}\}$ is a graph with vertex set $\mathscr{N}$ and its edge set $\mathscr{E}$ is defined as follows: there exists an edge from vertex j to vertex i if and only if $\int_{t_1}^{t_2}l_{ij}(\tau)d\tau>\delta$. Namely, we say that *there is a δ-edge from vertex j to i across $I=[t_1,t_2]$.*

Definition 26. We say that the LCODEs (41) has a δ-spanning tree across the time interval I if the corresponding graph $G(I,\delta)$ has a spanning tree.

For a stochastic matrix $V=(v_{ij})_{i,j=1}^m$, let

$$\eta(V)=\min_{i,j}\left\|v_i\wedge v_j\right\|_1,\tag{48}$$

where $v_i=[v_{i1},\dots,v_{im}]$, $i=1,2,\dots,m$, and $v_i\wedge v_j=[\min\{v_{i1},v_{j1}\},\dots,\min\{v_{im},v_{jm}\}]^{\mathrm{T}}$. Then, we can also define that V is *δ-scrambling* if $\eta(V)>\delta$.

Theorem 27. *Suppose Assumption 20 is satisfied.* $\operatorname{diam}(\mathscr{L})<1$ *if and only if there exist $\delta>0$ and $T>0$ such that the LCODEs* (41) *has a δ-spanning tree across any T-length time interval.*

Remark 28. Different from [16], we do not need to assume that $L(t)$ has zero column sums and the timeaverage is achieved sufficiently fast.

Before proving this theorem, we need the following lemma.

Lemma 29. *If the LCODEs* (41) *has a δ-spanning tree across any T-length time interval, then there exist $\delta_1>0$ and $T_1>0$ such that $V(t,t_0)$ is δ_1-scrambling for any T_1-length time interval.*

Proof of Theorem 27. Sufficiency. From Lemma 29, we can conclude that there exist $\delta_1>0$, $\delta'>0$, and $T_1>0$ such that $V(t,t_0)$ is δ_1-scrambling across any T_1-length time interval and $\inf_{t_0\geq0}\eta(V(T_1+t_0,t_0))>\delta'$. For any $t\geq t_0$, let $t-t_0=pT_1+T'$, where p is an integer and $0\leq T'<T_1$ and $t_l=t_0+lT_1$, $0\leq l\leq p$. Then, we have

$$\begin{aligned}\operatorname{diam}\left(V\left(t,t_0\right)\right)&=\operatorname{diam}\left(V\left(t,t_p\right)\prod_{l=1}^pV\left(t_l,t_{l-1}\right)\right)\\&\leq\operatorname{diam}\left(\prod_{l=1}^pV\left(t_l,t_{l-1}\right)\right)\\&\leq2\prod_{l=1}^p\left(1-\eta\left(V\left(t_l,t_{l-1}\right)\right)\right)\\&\leq2\left(1-\delta'\right)^{\lfloor(t-t_0)/T_1\rfloor}.\end{aligned}\tag{49}$$

For the first inequality, we use the results in [23, 24]. This implies $\operatorname{diam}(\mathscr{L})\leq(1-\delta')^{1/T_1}<1$.

Necessity. Suppose that for any $T\geq0$ and $\delta>0$, there exists $t_0=t_0(T,\delta)$, $\int_{t_0}^{T+t_0}L(\tau)d\tau$ does not have a δ-spanning tree. According to the condition, there exist $1>d>\operatorname{diam}(\mathscr{L})$,

$\epsilon > 0$, and $T' > 0$ such that $\operatorname{diam}(V(t+t_0)) < d^t$ for all $t_0 \geq 0$ and $t \geq T'$ and $d^{T'} < 1-\epsilon$. Thus, picking $T > T'$, $\delta = m^{-3}e^{-M_1 mT}\epsilon/2$, $t_1 = t_0(T,\delta)$, and $L' = (l'_{ij})_{i,j=1}^m = (\int_T^{T+t_1} l_{ij}(\tau)d\tau)_{i,j=1}^m$, there exist two vertex set J_1 and J_2 such that $l'_{ij} \leq \delta$ if $i \in J_1$ and $j \notin J_1$, or $i \in J_2$ and $j \notin J_2$. For each $i \in J_1$ and $j \notin J_1$, we have

$$\begin{aligned}\dot{v}_{ij}(t) &= l_{ii}(t)\, v_{ij}(t) + \sum_{k\in J_1}^{k\neq i} l_{ik}(t)\, v_{kj}(t) \\ &\quad + \sum_{k\notin J_1} l_{ik}(t)\, v_{kj}(t) \\ &\leq M_1 \sum_{k\in J_1}^{k\neq i} v_{kj}(t) + \sum_{k\notin J_1} l_{ik}(t).\end{aligned} \tag{50}$$

Then,

$$\begin{aligned}\sum_{i\in J_1, j\notin J_1} \dot{v}_{ij}(t) &\leq M_1 \sum_{i\in J_1, k\in J_1}^{k\neq i, j\notin J_1} v_{kj}(t) + \sum_{i\in J_1, k\notin J_1}^{j\notin J_1} l_{ik}(t) \\ &= M_1 (\#J_1 - 1) \sum_{k\in J_1}^{j\notin J_1} v_{kj}(t) + (m - \#J_1) \sum_{i\in J_1}^{k\notin J_1} l_{ik}(t).\end{aligned} \tag{51}$$

Let $v(t) = \sum_{i\in J_1, j\notin J_1} v_{ij}(t)$. According to Lemma 9, we have

$$\begin{aligned}v(T+t_1) &\leq e^{M_1(\#J_1-1)T}(m-\#J_1)\int_{t_1}^{T+t_1} \sum_{i\in J_1}^{j\notin J_1} l_{ij}(\tau)\, d\tau \\ &\leq (m-\#J_1)\, e^{M_1(\#J_1-1)T} \#J_1 (m-\#J_1)\,\delta \\ &\leq m^3 e^{mM_1T}\delta \leq \frac{\epsilon}{2}.\end{aligned} \tag{52}$$

Similarly, we can conclude that $\sum_{i\in J_l, j\notin J_l} v_{ij}(T+t_1) \leq \epsilon/2$ for all $l = 1,2$. Without loss of generality, we suppose $J_1 = \{1,2,\ldots,p\}$ and $J_2 = \{p+1, p+2, \ldots, p+q\}$, where p and q are integers with $p+q \leq m$. Then, we can write $V(T+t_1, t_1)$ in the following matrix form:

$$V(T+t_1, t_1) = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \tag{53}$$

where $X_{11} \in \mathbb{R}^{p,p}$ and $X_{22} \in R^{q,q}$ correspond to the vertex subset J_1 and J_2, respectively. Immediately, we have $\|X_{12}\|_\infty + \|X_{13}\|_\infty + \|X_{21}\|_\infty + \|X_{23}\|_\infty \leq \epsilon$. Let $v = \begin{bmatrix} \mathbf{1}_p \\ 0 \\ 0 \end{bmatrix}$. We let

$$V(t_1+T, t_1)\, v = \begin{bmatrix} X_{11}\mathbf{1}_p \\ X_{21}\mathbf{1}_p \\ X_{31}\mathbf{1}_p \end{bmatrix}. \tag{54}$$

Let $u = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix} = [u_1, \ldots, u_m]^\top$ with $u^i = [u_1^i, \ldots, u_{p_i}^i]^\top = X_{i1}\mathbf{1}_p$ and $p_1 = p$, $p_2 = q$, $p_3 = m-p-q$. Then,

$$\begin{aligned}\max_{i,j} |u_i - u_j| &\geq \max_{k,l} |u_k^1 - u_j^2| \\ &\geq 1 - \|X_{12}\|_\infty - \|X_{13}\|_\infty - \|X_{21}\|_\infty - \|X_{23}\|_\infty \\ &\geq 1-\epsilon.\end{aligned} \tag{55}$$

Also,

$$\max_{i,j} |u_i - u_j| \leq \operatorname{diam}(V(t_1+T, t_1)) \leq d^T. \tag{56}$$

This implies $d^T \geq 1-\epsilon$ which leads contradiction with $d^T < 1-\epsilon$. Therefore, we can conclude the necessity. □

5. Consensus Analysis of Multiagent System with Directed Time-Varying Graphs

If we let $n = 1$, $f \equiv 0$, and $\sigma = 1$ in system (41), then we have

$$\dot{x}^i(t) = \sum_{j=1}^m l_{ij}(t)\, x^j(t), \quad i = 1,2,\ldots,m. \tag{57}$$

In this case, if Assumption 20 is satisfied, then the synchronization analysis of system (57) becomes another important research field named consensus problems.

Definition 30. We say the differential system (57) reaches consensus if for any $x(t_0) \in \mathbb{R}^m$, $\|x^i(t) - x^j(t)\| \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{N}$.

In graph view, the coefficients matrix of (57) $L(t) = (l_{ij}(t)) \in \mathbb{R}^{m,m}$ is equal to the negative *graph Laplacian* associated with the digraph $G(t)$ at time t, where $G(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ is a weighted digraph (or directed graph) with m vertices, the set of nodes $\mathcal{V} = \{v_1, \ldots, v_m\}$, set of edges $\mathcal{E}(t) \subseteq \mathcal{V}\times\mathcal{V}$, and the weighted adjacency matrix $\mathcal{A}(t) = (a_{ij}(t))$ with nonnegative adjacency elements $a_{ij}(t)$. An edge of $G(t)$ is denoted by $e_{ij}(t) = (v_i, v_j) \in \mathcal{E}(t)$ if there is a directed edge from vertex i to vertex j at time t. The adjacency elements associated with the edges of the graph are positive, that is, $e_{ij}(t) \in \mathcal{E}(t) \Leftrightarrow a_{ij}(t) > 0$, for all $i, j \in \mathcal{N}$. It is assumed that $a_{ii}(t) = 0$ for all $i \in \mathcal{N}$. The indegree and outdegree of node v_i at time t are, respectively, defined as follows:

$$\deg_{\text{in}}(v_i(t)) = \sum_{j=1}^N a_{ji(t)}, \qquad \deg_{\text{out}}(v_i(t)) = \sum_{j=1}^N a_{ij(t)}. \tag{58}$$

The degree matrix of digraph $G(t)$ is defined as $D(t) = \operatorname{diag}(\deg_{\text{out}}(v_1(t)), \ldots, \deg_{\text{out}}(v_m(t)))$ at time t. The *graph Laplacian* associated with the digraph $G(t)$ at time t is defined as

$$-L(t) = \mathcal{L}(G(t)) = D(t) - \mathcal{A}(t). \tag{59}$$

Let $G(I,\delta)$ defined as before. We say that the digraph $G(t)$ has a δ-spanning tree across the time interval I if $G(I,\delta)$ has a spanning.

Theorem 31. *Suppose Assumption 20 is satisfied. The system* (57) *reaches consensus if and only if there exist $\delta > 0$ and $T > 0$ such that the corresponding digraph $G(t)$ has a δ-spanning tree across any T-length time interval.*

Proof. Since $f \equiv 0$, we have $\mu = 0$ in Theorem 21. This completes the proof according to Theorems 27 and 21. □

Remark 32. This theorem is a part of Theorem 17 in [25].

6. Numerical Examples

In this section, a numerical example is given to demonstrate the effectiveness of the presented results on synchronization of LCODEs with time-varying couplings. The Lyapunov exponents are computed numerically. By this way, we can verify the the synchronization criterion and analyze synchronization numerically. We use the Rössler system [16, 26] as the node dynamics

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) - x_3(t), \\ \dot{x}_2(t) &= x_1(t) + a x_2(t), \\ \dot{x}_3(t) &= b + x_3(t)\left(x_1(t) - c\right), \end{aligned} \tag{60}$$

where $a = 0.165$, $b = 0.2$, and $c = 10$. Figure 1 shows the dynamical behaviors of the Rössler system (60) with random initial value in $[0,1]$ that includes a chaotic attractor [16, 26].

The network with time-varying topology we used here is NW small-world network with a time-varying coupling, which was introduced as the blinking model in [11, 27]. The time-varying network model algorithm is presented as follows: we divide the time axis into intervals of length τ, in each interval: (a) begin with the nearest neighbor coupled network consisting of m nodes arranged in a ring, where each node i is adjacent to its $2k$-nearest neighbor nodes; (b) add a connection between each pair of nodes with probability p, which usually is a random number between $[0, 0.1]$; for more details, we refer the readers to [11]. Figure 2 shows the time-varying structure of shortcut connections in the blinking model with $m = 50$ and $k = 3$.

In this example, the parameters are taken values as $m = 50$, $k = 3$, $\tau = 1$, and $p = 0.04$. Then blinking small-world network can be generated with the coupling *graph Laplacian* $\mathscr{L}(G(t)) = -L(t)$. The dynamical network system can be described as follows:

$$\begin{aligned} \dot{x}_1^i(t) &= -x_2^i(t) - x_3^i(t) + \sigma \sum_{j=1}^{m} l_{ij}(t)\, x_1^j(t), \\ \dot{x}_2^i(t) &= x_1^i(t) + a x_2^i(t) + \sigma \sum_{j=1}^{m} l_{ij}(t)\, x_2^j(t), \\ \dot{x}_3^i(t) &= b + x_3^i(t)\left(x_1^i(t) - c\right) + \sigma \sum_{j=1}^{m} l_{ij}(t)\, x_3^j(t), \\ & \qquad i = 1, 2, \ldots, m. \end{aligned} \tag{61}$$

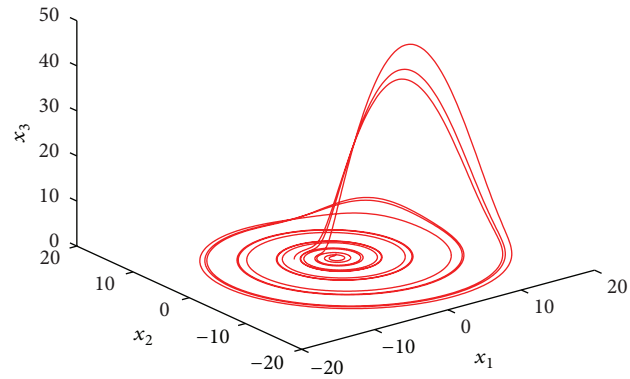


Figure 1: The dynamical behavior of the Rössler system (60) with $a = 0.165$, $b = 0.2$, and $c = 10$.

Let $e(t) = \max_{1\le i<j\le 50}\|x^i(t) - x^j(t)\|$ denotes the maximum distance between nodes at time t. Let $E = \int_T^{T+R} e(t)dt$, for some sufficiently large $T > 0$ and $R > 0$. Let $H = \mu + \varsigma$ defined in Corollary 25. As described in Corollary 25, two steps are needed for verification: (a) calculating the largest Lyapunov exponent of the uncoupled synchronized system (60), μ and (b) calculating the second largest Lyapunov exponent of the linear system (42). In detail, we use Wolf's method [28] to compute μ and the Jacobian method [29] to compute Lyapunov spectra of (42). More details can be found in [28–30]. Figure 3 shows convergence of the maximum distance between nodes during the topology evolution with a different coupling strength σ. It can be seen from Figure 3 that the dynamical network system (61) can be synchronized with $\sigma = 0.4$ and $\sigma = 0.5$.

We pick the time length 200. Let $T = 190$ and $R = 10$. And choose initial state randomly from the interval $[0,1]$. Figure 4 shows the variation of E and H with respect to the coupling strength σ. It can be seen that the parameter (coupling strength σ) region where H is negative coincides with that of synchronization, that is, where E is near zero. This verified the theoretical result (Corollary 25). In addition, we find that $\sigma \approx 0.38$ is the threshold for synchronizing the coupled systems in this case.

7. Conclusions

In this paper, we present a theoretical framework for synchronization analysis of general coupled differential dynamical systems. The extended Hajnal diameter is introduced to measure the synchronization. The coupling between nodes is timevarying in both network structure and reaction dynamics. Inspired by the approaches in [6, 13], we show that the Hajnal diameter of the linear system induced by the time-varying coupling matrix and the largest Lyapunov exponent of the synchronized system play the key roles in synchronization analysis of LCODEs. These results extend synchronization analysis of discrete-time network in [6] to continuous-time case. As an application, we obtain a very general sufficient condition ensuring directed time-varying graph reaching consensus, and the way we get this result is different from [25]. An example of numerical simulation is provided

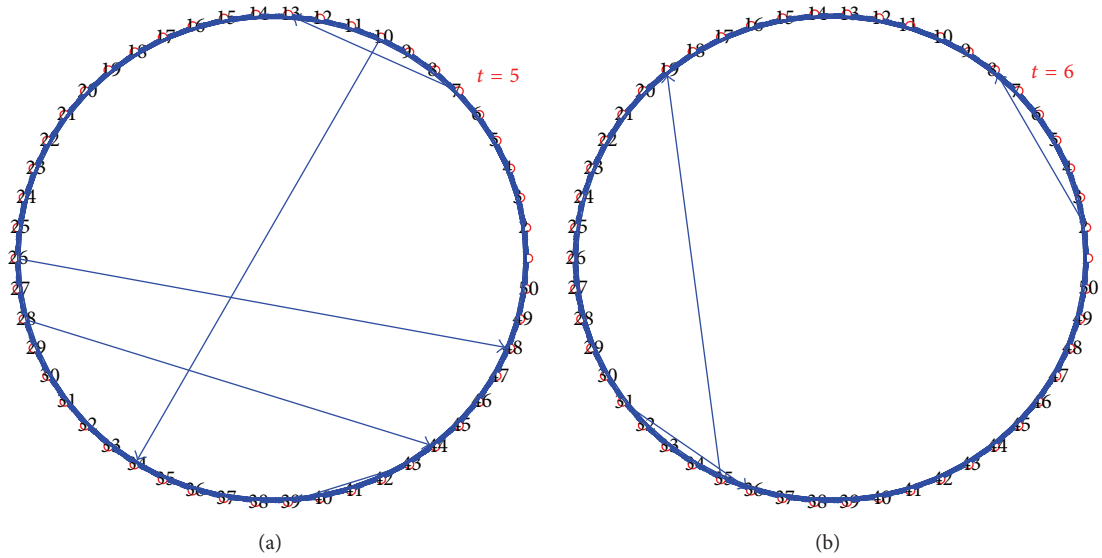


Figure 2: The blinking model of shortcuts connections. Probability of switchings $p = 0.04$, the switching time step $\tau = 1$.

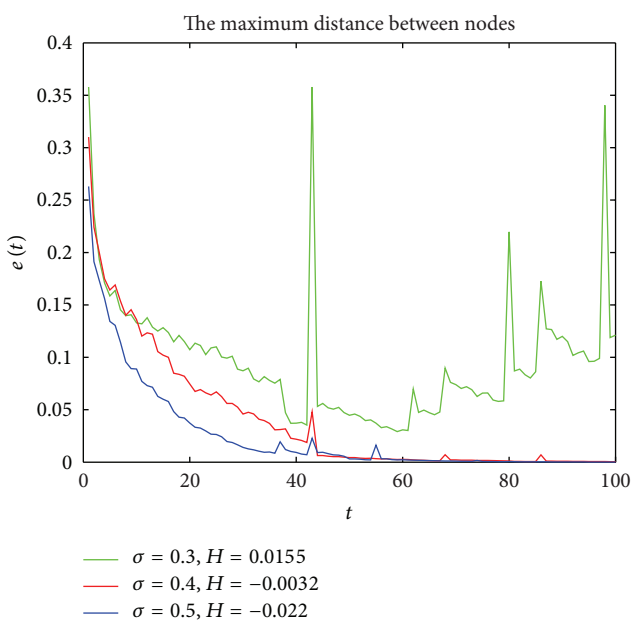


Figure 3: Convergence of the maximum distance between nodes with a different coupling strength σ.

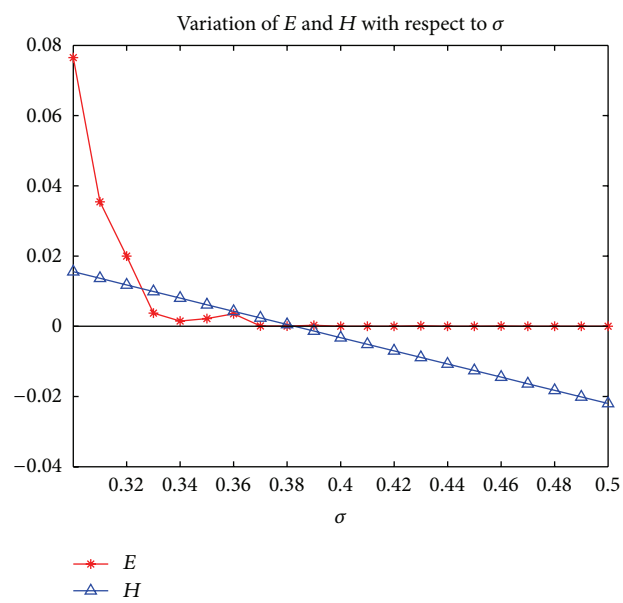


Figure 4: Variation of e and H with respect to σ for the blinking topology.

to show the effectiveness the theoretical results. Additional contributions of this paper are that we explicitly show that the largest projection Lyapunov exponent, the Hajnal diameter, and the largest Lyapunov exponent of the transverse space are equal to each other in coupled differential systems (see Lemmas 13 and 16), which was proved in [6] for couple discrete-time systems.

Appendix

Proof of Lemma 5. Let U be a bounded open neighborhood of A satisfying $\bigcap_{t\geq 0} \vartheta^{(t)}\overline{U} = A$ and $U_t = \{x \in R^n : \vartheta^{(\tau)}x \in U, 0 \leq \tau \leq t\}$. This implies $U_t \supset U_{t'}$ if $t' \geq t \geq 0$, U_t is an open set due to the continuity of the semiflow $\vartheta^{(t)}$, and $\vartheta^{(\delta)}U_t \subset U_{t-\delta}$ for all $t \geq \delta \geq 0$. Let $V = \bigcap_{t\geq 0} U_t$. We claim that there exists $t_0 \geq 0$ such that $V = U_t$ for all $t \geq t_0$.

For any $\delta > 0$, let $t_n = n\delta$ and $U_n = U_{t_n}$. We can conclude that $V = \bigcap_{n=1}^{\infty} U_n$. We will prove in the following that there exists n_0 such that $V = U_{n_0}$. Otherwise, there always exists $x_n \in U_n \setminus U_{n+1}$ for $n \geq 0$. Let $y_n = \vartheta^{(t_{n+1})}x_n$. We have (i) $y_n \in \bigcap_{k=0}^{n} \vartheta^{(t_k)}\overline{U}$ and (ii) $y_n \notin U$. For any limit point $\widehat{y}$ of y_n, $\widehat{y}$ can be either finite or infinite. For both cases, $\widehat{y} \notin U$ which implies $\widehat{y} \notin A$. However, the claim (i) implies that

$\widehat{y} \in A$, which contradicts with the claim (ii). This completes the proof by letting $W = \overline{V}$. □

Proof of Lemma 10. (a) For any initial condition with the form $\delta x_0 = \mathbf{1}_m \otimes u_0$, the solution of (11) can be $U(t, t_0, s_0)(\mathbf{1}_m \otimes u_0) = \mathbf{1}_m \otimes \breve{U}(t, t_0, s_0)u_0$ according to Lemma 4. This implies the first claim in this lemma.

(b) According to Lemma 5, there exists $K_1 > 0$ such that $s(t)$, the solution of (5), satisfies $\| s(t) \| \leq K_1$ for all $s_0 \in W$ and $t \geq 0$. So, there exists $K > 0$ such that $\| DF^t(s(t)) \| \leq K$ according to the 3th item of Assumption 1. Write the solution of (11) $\delta x(t) = U(t, t_0, s_0)\delta x_0$ as

$$\delta x(t + t_0) = \delta x_0 + \int_{t_0}^{t+t_0} DF^{\tau}(s(\tau))\, \delta x(\tau)\, d\tau. \tag{A.1}$$

Then,

$$\begin{aligned} &\|\delta x(t + t_0)\| \\ &\quad \leq \|\delta x_0\| + \int_{t_0}^{t+t_0} \|DF^{\tau}(s(\tau))\| \, \|\delta x(\tau)\| \, d\tau \\ &\quad \leq \|\delta x_0\| + K \int_0^t \|\delta x(\tau + t_0)\| \, d\tau. \end{aligned} \tag{A.2}$$

According to Lemma 9, we have $\|\delta x(t + t_0)\| \leq \|\delta x_0\| + K \int_0^t \|\delta x_0\| e^{(t-\tau)K} d\tau = e^{Kt} \|\delta x_0\|$. This implies that $\| U(t+t_0, t_0, s_0) \| \leq e^{Kt}$ for all $s_0 \in W$ and $t_0 \geq 0$.

For any $s_0, s'_0 \in W$, let $s(t)$ and $s'(t)$ be the solution of the synchronized state equation (5) with initial condition $s(t_0) = s_0$ and $s'(t_0) = s'_0$, respectively. We have

$$\begin{aligned} &s(t + t_0) - s'(t + t_0) \\ &\quad = \int_{t_0}^{t+t_0} \left[f(s(\tau)) - f\left(s'(\tau)\right) \right] d\tau + s(t_0) - s'(t_0), \\ &\|s(t + t_0) - s'(t + t_0)\| \\ &\quad \leq \|s(t_0) - s'(t_0)\| + K \int_{t_0}^{t+t_0} \|s(\tau) - s'(\tau)\| \, d\tau. \end{aligned} \tag{A.3}$$

By Lemma 9, we have $\|s(t + t_0) - s'(t + t_0)\| \leq e^{Kt} \|s_0 - s'_0\|$ for all $t_0, t \geq 0$ and $s_0, s'_0 \in W$. Also, according to the 4th item of Assumption 1, there must exist $K_2 > 0$ such that $\| DF^t(s(t)) - DF^t(s'(t)) \| \leq K_2 \| s(t) - s'(t) \|$ for all $t \geq 0$ and $s_0, s'_0 \in W$. Then, let $\delta x(t) = U(t, t_0, s_0)\delta x_0$, $\delta y(t) = U(t, t_0, s'_0)\delta x_0$, and $v(t) = \delta x(t) - \delta y(t)$. We have

$$\begin{aligned} &v(t + t_0) \\ &\quad = \int_{t_0}^{t+t_0} \left[DF^{\tau}(s(\tau))\, \delta x(\tau) - DF^{\tau}\left(s'(\tau)\right) \delta y(\tau) \right] d\tau \\ &\quad = \int_{t_0}^{t+t_0} \left[DF^{\tau}(s(\tau)) - DF^{\tau}\left(s'(\tau)\right) \right] \delta x(\tau)\, d\tau \\ &\qquad + \int_{t_0}^{t+t_0} DF^{\tau}\left(s'(\tau)\right) v(\tau)\, d\tau, \\ &\|v(t + t_0)\| \\ &\quad \leq \int_{t_0}^{t+t_0} \left[\left\| DF^{\tau}(s(\tau)) - DF^{\tau}\left(s'(\tau)\right) \right\| \|\delta x(\tau)\| \right. \\ &\qquad \left. + \left\| DF^{\tau}\left(s'(\tau)\right) \right\| \|v(\tau)\| \right] d\tau \\ &\quad \leq K_2 \int_0^t e^{2K\tau} d\tau \, \|\delta x_0\| \, \|s_0 - s'_0\| \\ &\qquad + K \int_{t_0}^{t_0+t} \|v(\tau)\| \, d\tau. \end{aligned} \tag{A.4}$$

According to Lemma 9,

$$\|v(t + t_0)\| \leq \left[\frac{K_2\left(e^{2Kt} - e^{Kt}\right)}{K} \right] \|\delta x_0\| \, \|s_0 - s'_0\|. \tag{A.5}$$

This implies

$$\begin{aligned} &\left\| U(t + t_0, t_0, s_0) - U\left(t + t_0, t_0, s'_0\right) \right\| \\ &\quad \leq \left[\frac{K_2\left(e^{2Kt} - e^{Kt}\right)}{K} \right] \|s_0 - s'_0\| \end{aligned} \tag{A.6}$$

for all $s_0, s'_0 \in W$. This completes the proof. □

Proof of Lemma 13. We define the projection joint spectral radius as follows:

$$\rho_P(D\mathscr{F}, s_0) = \overline{\lim_{t \to \infty}} \sup_{t_0 \geq 0} \left\| \widetilde{U}(t, t_0, s_0) \right\|^{1/t}. \tag{A.7}$$

First, we will prove that $\operatorname{diam}(D\mathscr{F}, s_0) = \rho_P(D\mathscr{F}, s_0)$. For any $d > \rho_P(D\mathscr{F}, s_0)$, there exists $T \geq 0$ such that $\|\widetilde{U}(t+t_0, t_0, s_0)\| \leq d^t$ for all $t_0 \geq 0$ and $t \geq T$. This implies that

$$\begin{aligned} &\left\| P^{-1} U(t + t_0, t_0, s_0) P - \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right. \\ &\qquad \left. \times \left[P_0^{\mathrm{T}} \breve{U}(t + t_0, t_0, s_0) P_0, \alpha(t + t_0, t_0, s_0) \right] \right\| \\ &\quad = \left\| \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{U}(t + t_0, t_0, s_0) \end{bmatrix} \right\| \leq C_1 d^t \end{aligned} \tag{A.8}$$

for some $C_1 > 0$, all $t_0 \geq 0$ and all $t \geq T$. Thus, there exist some $C_2 > 0$ and some matrix function $q(t) \in \mathbb{R}^{n,nm}$ such that

$$\begin{aligned}
&\left\|U\left(t+t_0,t_0,s_0\right)-\mathbf{1}_m\otimes q\left(t\right)\right\| \\
&\quad=\left\|U\left(t+t_0,t_0,s_0\right)-P\begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0\end{bmatrix}\right. \\
&\qquad\left.\times\left[P_0^{\mathsf{T}}\breve{U}\left(t+t_0,t_0,s_0\right)P_0,\alpha\left(t+t_0,t_0,s_0\right)\right]P^{-1}v\right\| \\
&\quad\leq C_2 d^t
\end{aligned} \tag{A.9}$$

for all $t_0 \geq 0$ and $t \geq T$, where $q(t) \in \mathbb{R}^{n,nm}$ denotes a matrix, and we omit its accurate expression. So, we can conclude that $\operatorname{diam}(U(t+t_0,t_0,s_0)) \leq C_3 d^t$ for some $C_3 > 0$, all $t_0 \geq 0$, and $t \geq T$. This implies that $\operatorname{diam}(D\mathcal{F},s_0) \leq d$, that is, $\operatorname{diam}(D\mathcal{F},s_0) \leq \rho_P(D\mathcal{F},s_0)$ due to the arbitrariness of $d \geq \rho_P(D\mathcal{F},s_0)$. Conversely, for any $d > \operatorname{diam}(D\mathcal{F},s_0)$, there exists $T > 0$ such that

$$\left\|U\left(t+t_0,t_0,s_0\right)-\mathbf{1}_m\otimes U_1\right\| \leq C_4 d^t \tag{A.10}$$

for some $C_4 > 0$, all $t_0 \geq 0$, and $t \geq T$, where $U_1 = [U_{11}, U_{12},\ldots,U_{1m}]$ the first n rows of $U(t+t_0,t_0,s_0)$. Then,

$$\begin{aligned}
&\left\|P^{-1}U\left(t+t_0,t_0,s_0\right)P-P^{-1}\mathbf{1}_m\otimes U_1P\right\| \\
&\quad=\left\|P^{-1}U\left(t+t_0,t_0,s_0\right)P-\begin{bmatrix}\gamma(t) & \beta(t)\\ 0 & 0\end{bmatrix}\right\| \\
&\quad=\left\|\begin{bmatrix}0 & \beta(t)\\ 0 & \widetilde{U}\left(t+t_0,t_0,s_0\right)\end{bmatrix}\right\| \leq C_5 d^t
\end{aligned} \tag{A.11}$$

for some $C_5 > 0$, all $t_0 \geq 0$, and $t \geq T$, where $\gamma(t) = P_0^{\mathsf{T}}\breve{U}(t,t_0,s_0)P_0 \in \mathbb{R}^{n,n}$ and $\beta(t) \in \mathbb{R}^{n,n(m-1)}$ denotes a matrix, and we omit its accurate expression. This implies that $\|\widetilde{U}(t+t_0,t_0,s_0)\| \leq C_6 d^t$ holds for some $C_6 > 0$, all $t_0 \geq 0$, and $t \geq T$. Therefore, we can conclude that $\rho_P(D\mathcal{F},s_0) \leq d$. So, $\rho_P(D\mathcal{F},s_0) = \operatorname{diam}(D\mathcal{F},s_0)$.

Second, it is clear that $\log \rho_P(D\mathcal{F},s_0) \geq \lambda_P(D\mathcal{F},s_0)$. We will prove that $\log \rho_P(D\mathcal{F},s_0) = \lambda_P(D\mathcal{F},s_0)$. Otherwise, there exists some $r, r_0 > 0$ satisfying $\rho_P(D\mathcal{F},s_0) > r > r_0 > e^{\lambda_P(D\mathcal{F},s_0)}$. If so, there exists a sequence $t_k \uparrow \infty$ as $k \to \infty$, $t_0^k \geq 0$, and $v_k \in \mathbb{R}^{n(m-1)}$ with $\|v_k\| = 1$ such that $\|\widetilde{U}(t_k+t_0^k,t_0^k,s_0)v_k\| > r^{t_k}$ for all $k \in \mathcal{N}$. Then, there exists a subsequence v_{k_l} with $\lim_{l\to\infty} v_{k_l} = v^*$. Let $\{e_1,e_2,\ldots,e_{n(m-1)}\}$ be a normalized orthogonal basis of $\mathbb{R}^{n(m-1)}$. And, let $v_{k_l} - v^* = \sum_{j=1}^{n(m-1)} \xi_j^{k_l} e_j$. We have $\lim_{l\to\infty}\xi_j^{k_l} = 0$ for all $j = 1,\ldots,n(m-1)$. Thus, there exists $L > 0$ such that

$$\begin{aligned}
&\left\|\widetilde{U}\left(t_{k_l}+t_0^{k_l},t_0^{k_l},s_0\right)v^*\right\| \\
&\quad\geq\left\|\widetilde{U}\left(t_{k_l}+t_0^{k_l},t_0^{k_l},s_0\right)v_{k_l}\right\| \\
&\qquad-\left\|\widetilde{U}\left(t_{k_l}+t_0^{k_l},t_0^{k_l},s_0\right)\left(v_{k_l}-v^*\right)\right\| \\
&\quad\geq r^{t_{k_l}}-\sum_{j=1}^{n(m-1)}\left|\xi_j^{k_l}\right|\left\|\widetilde{U}\left(t_{k_l}+t_0^{k_l},t_0^{k_l},s_0\right)e_j\right\| \\
&\quad\geq r^{t_{k_l}}-r_0^{t_{k_l}} > r_0^{t_{k_l}}
\end{aligned} \tag{A.12}$$

for all $l \geq L$. This implies $e^{\lambda(D_P\mathcal{F},v^*,s_0)} \geq r_0$ which contradicts with $e^{\lambda_P(D\mathcal{F},s_0)} < r_0$. This implies $\rho_P(D\mathcal{F},s_0) = e^{\lambda_P(D\mathcal{F},s_0)}$. Therefore, we can conclude $\log \operatorname{diam}(D\mathcal{F},s_0) = \lambda_P(\mathcal{F},s_0)$. The proof is completed. □

Proof of Lemma 16. Let $\widetilde{\phi} = P^{-1}\phi$. We have

$$\begin{aligned}
\dot{\widetilde{\phi}} &= P^{-1}DF\left(s(t),\varrho^{(t)}\omega_0\right)P\widetilde{\phi} \\
&= \begin{bmatrix} P_0^{\mathsf{T}}\dfrac{\partial f}{\partial s}(s(t))P_0 & \alpha(t) \\ 0 & \widetilde{D}F\left(s(t),\varrho^{(t)}\omega_0\right)\end{bmatrix}\widetilde{\phi}.
\end{aligned} \tag{A.13}$$

Write $\widetilde{\phi} = \left[\begin{smallmatrix} y(t) \\ z(t)\end{smallmatrix}\right]$, where $y(t) \in \mathbb{R}^n$. Then, we have

$$\begin{aligned}
\dot{z}(t) &= \widetilde{D}F\left(s(t),\varrho^{(t)}\omega_0\right)z(t), \\
\dot{y}(t) &= P_0^{\mathsf{T}}\frac{\partial f}{\partial s}(s(t))P_0 y(t) + \alpha(t)z(t).
\end{aligned} \tag{A.14}$$

Thus, we can write its solution by

$$\begin{aligned}
z(t) &= \widetilde{U}(t)z_0, \\
y(t) &= P_0^{\mathsf{T}}\breve{U}(t)P_0y_0 + \int_0^t P_0^{\mathsf{T}}\breve{U}(t)\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)z_0\,d\tau.
\end{aligned} \tag{A.15}$$

We write $\lambda_P(D\mathcal{F},s_0,\omega_0)$, $\lambda_S(D\mathcal{F},s_0,\omega_0)$, and $\lambda_T(D\mathcal{F},s_0,\omega_0)$ by λ_P, λ_S, and λ_T, respectively for simplicity.

Case 1 ($\lambda_P > \lambda_S$). We can conclude that $\chi[z(t)] \leq \lambda_P$ and

$$\begin{aligned}
\chi\left[y(t)\right] \leq \max\Bigg\{&\chi\left[P_0^{\mathsf{T}}\breve{U}(t)P_0y_0\right], \\
&\chi\left[\int_0^t P_0^{\mathsf{T}}\breve{U}(t)\breve{U}^{-1}(\tau)P_0\alpha(\tau)\right. \\
&\left.\times\widetilde{U}(\tau)z(0)\,d\tau\right]\Bigg\}.
\end{aligned} \tag{A.16}$$

From Cauchy-Buniakowski-Schwarz inequality, we have

$$\begin{aligned}\chi&\left[\left\|\int_0^t P_0^{\mathsf{T}}\breve{U}(t)\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)\,d\tau\right\|\right]\\&\le\chi\left[\left\{\int_0^t\left\|\breve{U}(t)\breve{U}^{-1}(\tau)\right\|^2d\tau\right\}^{1/2}\right]\\&\quad+\chi\left[\left\{\int_0^t\left\|\alpha(\tau)\widetilde{U}(\tau)\right\|d\tau\right\}^{1/2}\right].\end{aligned}\tag{A.17}$$

Claim 1 ($\chi(\int_0^t\|\breve{U}(t)\breve{U}^{-1}(\tau)\|^2d\tau)\le 0$). Considering the linear system

$$\dot{u}(t)=\frac{\partial f}{\partial s}(s(t))\,u(t),\tag{A.18}$$

due to its regularity and the boundedness of its coefficients, there exists a Lyapunov transform $L(t)$ such that letting $u(t)=L(t)v(t)$, consider the transformed linear system

$$\begin{aligned}\dot{v}(t)&=\left[L^{-1}(t)\frac{\partial f}{\partial s}(s(t))L(t)-L^{-1}(t)\dot{L}(t)\right]v(t)\\&=\breve{A}(t)v(t).\end{aligned}\tag{A.19}$$

Let solution matrix $\breve{V}(t)=(\breve{v}_{ij}(t))_{i,j=1}^n$, $\breve{A}(t)=(\breve{a}_{ij}(t))_{i,j=1}^n$ which satisfies that $\breve{A}(t)$ and $\breve{V}(t)$ are lowertriangular. And its Lyapunov exponents can be written as follows:

$$\sigma_i=\lim_{t\to\infty}\frac{1}{t}\int_0^t\breve{a}_{ii}(\tau)\,d\tau,\tag{A.20}$$

which are just the Lyapunov exponents of the regular linear system (A.18), $i=1,2,\ldots,n$. We have $\chi[\breve{v}_{ii}(t)]=\sigma_i$ and

$$\begin{aligned}&\breve{v}_{k+1,k}(t)\\&\quad=e^{\int_0^t\breve{a}_{k+1,k+1}(\tau)d\tau}\int_0^t e^{-\int_0^\tau\breve{a}_{k+1,k+1}(\vartheta)d\vartheta}\breve{a}_{k+1,k}(\tau)\,\breve{v}_{k,k}(\tau)\,d\tau.\end{aligned}\tag{A.21}$$

This implies

$$\chi[\breve{v}_{k+1,k}(t)]\le\sigma_{k+1}-\sigma_{k+1}+0+\sigma_k=\sigma_k.\tag{A.22}$$

By induction, we can conclude that $\chi[\breve{v}_{jk}(t)]\le\sigma_k$ for all $j>k$. For $j<k$, $\chi[\breve{v}_{jk}(t)]=-\infty$ due to the lower-triangularity of the matrix $\breve{V}(t)$.

Considering the lower-triangular matrix $\breve{V}^{-1}(t)=(\breve{w}_{ij})_{i,j=1}^n$, its transpose $(\breve{V}^{-1}(t))^{\mathsf{T}}$ can be regarded as the solution matrix of the adjoint system of (A.18):

$$\dot{w}(t)=-\breve{A}^{\mathsf{T}}(t)\,w(t),\tag{A.23}$$

which is also regular. By the same arguments, we can conclude that $\chi[\breve{w}_{kk}]=-\sigma_k$ for all $k=1,2,\ldots,n$, $\chi[\breve{w}_{jk}]\le-\sigma_k$ for all $k>j$, and $\chi[\breve{w}_{jk}]=-\infty$ for all $k<j$. Therefore, for each $i>j$,

$$\begin{aligned}&\max_{i,j}\chi\left[\int_0^t\left|\breve{U}(t)\breve{U}^{-1}(\tau)\right|_{ij}d\tau\right]\\&\quad\le\max_{i,j}\chi\left[\int_0^t\left|\breve{V}(t)\breve{V}^{-1}(\tau)\right|_{ij}d\tau\right]\\&\quad\le\max_{i,j}\chi\left[\int_0^t\sum_{j\le k\le i}\left|\breve{v}_{ik}(t)\,\breve{w}_{kj}(\tau)\right|d\tau\right]\\&\quad\le\max_{i,j}\max_{j\le k\le i}\chi\left[\int_0^t\left|\breve{v}_{ik}(t)\,\breve{w}_{kj}(\tau)\right|d\tau\right]\\&\quad\le\max_{i,j}\max_{j\le k\le i}(\sigma_k-\sigma_k)=0.\end{aligned}\tag{A.24}$$

This implies that $\chi[\int_0^t\|\breve{U}(t)\breve{U}^{-1}(\tau)d\tau\|^2d\tau]\le 0$.

Noting that

$$\chi\left[\int_0^t\left\|\alpha(\tau)\widetilde{U}(\tau)\right\|_2^2d\tau\right]\le\chi\left[\left\|\alpha(t)\widetilde{U}(t)\right\|^2\right]\le 2\lambda_P.\tag{A.25}$$

So, $\chi[y(t)]\le\max\{\lambda_S,\lambda_P\}=\lambda_P$. This leads to $\chi[\widetilde{\phi}(t)]\le\lambda_P$. This implies that $\lambda_P=\max\{\lambda_S,\lambda_T\}$. Thus, $\lambda_P=\lambda_T$ can be concluded due to $\lambda_P>\lambda_S$.

Case 2 ($\lambda_P<\lambda_S$). For any ϵ with $0<\epsilon<(\lambda_S-\lambda_P)/3$, there exists $T>0$ such that

$$\begin{aligned}&\left\|\breve{U}^{-1}(\tau)\right\|\le e^{(-\lambda_S+\epsilon)\tau},\qquad\|\alpha(\tau)\|\le e^{\epsilon\tau},\\&\left\|\widetilde{U}(\tau)\right\|\le e^{(\lambda_P+\varepsilon)\tau}\end{aligned}\tag{A.26}$$

for all $t\ge T$. Define the subspace of $\mathbb{R}^{nm}$:

$$V=\left\{\begin{bmatrix}y\\z\end{bmatrix}:y=-\int_0^\infty P_0^{\mathsf{T}}\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)\,d\tau z\right\},\tag{A.27}$$

which is well defined due to $\|P_0^{\mathsf{T}}\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)\|\le e^{(3\epsilon-\lambda_S+\lambda_P)\tau}\in L([T,+\infty))$. For each $\widetilde{\phi}(t)$ with initial condition $\left[\begin{smallmatrix}y\\z\end{smallmatrix}\right]\in V$, we have $\chi[z(t)]\le\lambda_P$ and

$$\begin{aligned}&\chi[y(t)]\\&\quad=\chi\left[\left\{-P_0^{-1}\breve{U}(t)P_0\int_0^\infty P_0^{\mathsf{T}}\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)\,d\tau\right.\right.\\&\qquad\left.\left.+P_0^{\mathsf{T}}\breve{U}(t)P_0\int_0^t P_0^{\mathsf{T}}\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)\,d\tau\right\}z\right]\\&\quad=\chi\left[-P_0^{\mathsf{T}}\breve{U}(t)\int_t^\infty\breve{U}^{-1}(\tau)P_0\alpha(\tau)\widetilde{U}(\tau)\,d\tau z\right]\le\lambda_P\end{aligned}\tag{A.28}$$

according to the arguments above. Thus, we have $\max_{u\in V}\lambda(D\mathscr{F},u,s_0,\omega_0)=\lambda_P$. Since $\dim(V)=n(m-1)$, V define the transverse space and $\lambda_T=\lambda_P$. This completes the proof. □

Proof of Lemma 22. Since $L(t)$ satisfies Assumption 20, if the initial condition is $u(t_0) = \mathbf{1}_m$, then the solution must be $u(t) = \mathbf{1}_m$, which implies that each row sum of $V(t, t_0)$ is one. Then, we will prove all elements in $V(t, t_0)$ are nonnegative. Consider the ith column of $V(t, t_0)$ denoted by $V^i(t, t_0)$ which can be regarded as the solution of the following equation:

$$\begin{aligned} \dot{u} &= \sigma L(t) u, \\ u(t_0) &= e_i^m. \end{aligned} \tag{A.29}$$

For any $t \geq t_0$, if $i_0 = i_0(t)$ is the index with $u_{i_0}(t) = \min_{i=1,2,\ldots,m} u_i(t)$, we have $\dot{u}_{i_0}(t) = \sum_{j=1}^m \sigma l_{i_0 j}(u_j(t) - u_{i_0}(t)) \geq 0$. This implies that $\min_{i=1,2,\ldots,m} u_i(t)$ is always nondecreasing for all $t \geq t_0$. Therefore, $u_i(t) \geq 0$ holds for all $i = 1, 2, \ldots, m$ and $t \geq t_0$. We can conclude that $V(t, t_0)$ is a stochastic matrix. The proof is completed. □

Proof of Lemma 29. Consider the following Cauchy problem:

$$\begin{aligned} \dot{u}_i(t) &= \sum_{j=1}^m \sigma l_{ij}(t) u_j(t), \\ u_i(t_0) &= \begin{cases} 1, & i = k, \\ 0, & \text{otherwise}, \end{cases} \\ & i = 1, 2, \ldots, m. \end{aligned} \tag{A.30}$$

Noting that $\dot{u}_k(t) \geq \sigma l_{kk} u_k$, we have $u_k(t) \geq e^{-M_1(t-t_0)}$. For each $i \neq k$, since $u_i(t) \geq 0$ for all $i = 1, 2, \ldots, m$ and $t \geq t_0$, we have

$$\begin{aligned} u_i(t) &= \sum_{j \neq i} \int_{t_0}^t e^{\int_\tau^t \sigma l_{ii}(\vartheta) d\vartheta} \sigma l_{ij}(\tau) u_j(\tau) d\tau \\ &\geq \int_{t_0}^t e^{\int_\tau^t \sigma l_{ii}(\vartheta) d\vartheta} \sigma l_{ik}(\tau) u_k(\tau) d\tau \\ &\geq \int_{t_0}^t e^{-M_1(t-\tau)} e^{-M_1(\tau - t_0)} \sigma l_{ik}(\tau) d\tau \\ &= e^{-M_1(t-t_0)} \int_{t_0}^t \sigma l_{ik}(\tau) d\tau. \end{aligned} \tag{A.31}$$

So, if there exists a δ-edge from vertex j to i across $[t_0, t_0 + T]$, then we have $v_{ij}(t_0 + T, t_0) \geq e^{-M_1 T}\delta$. Let $\delta_2 = \min\{e^{-M_1 T}, e^{-M_1 T}\delta\}$. We can see that $V(t, t_0)$ has a δ_2 spanning tree across any T-length time interval. Therefore, according to [31, 32], there exist $\delta_1 > 0$ and $T_1 = (m-1)T$ such that $V(t, t_0)$ is δ_1 scrambling across any T_1-length time interval. The Lemma is proved. □

Acknowledgments

This work is jointly supported by the National Key Basic Research and Development Program (no. 2010CB731403), the National Natural Sciences Foundation of China under Grant (nos. 61273211 and 61273309), the Shanghai Rising-Star Program (no. 11QA1400400), the Marie Curie International Incoming Fellowship from the European Commission under Grant FP7-PEOPLE-2011-IIF-302421, and the Laboratory of Mathematics for Nonlinear Science, Fudan University.

References

[1] M. Dhamala, V. K. Jirsa, and M. Ding, "Transitions to synchrony in coupled bursting neurons," *Physical Review Letters*, vol. 92, no. 2, pp. 281011–281014, 2004.

[2] V. Perez-Munuzuri, V. Perez-Villar, and L. O. Chua, "Autowaves for image processing on a two-dimensional CNN array of excitable nonlinear circuits: flat and wrinkled labyrinths," *IEEE Transactions on Circuits and Systems I*, vol. 40, no. 3, pp. 174–181, 1993.

[3] P. Ashwin, J. Buescu, and I. Stewart, "From attractor to chaotic saddle: a tale of transverse instability," *Nonlinearity*, vol. 9, no. 3, pp. 703–737, 1996.

[4] J. Milnor, "Correction and remarks: 'On the concept of attractor,'" *Communications in Mathematical Physics*, vol. 102, no. 3, pp. 517–519, 1985.

[5] J. Milnor, "On the concept of attractor," *Communications in Mathematical Physics*, vol. 99, no. 2, pp. 177–195, 1985.

[6] W. Lu, F. M. Atay, and J. Jost, "Synchronization of discrete-time dynamical networks with time-varying couplings," *SIAM Journal on Mathematical Analysis*, vol. 39, no. 4, pp. 1231–1259, 2007/08.

[7] A. Margheri and R. Martins, "Generalized synchronization in linearly coupled time periodic systems," *Journal of Differential Equations*, vol. 249, no. 12, pp. 3215–3232, 2010.

[8] J. Lu, D. W. C. Ho, J. Cao, and J. Kurths, "Exponential synchronization of linearly coupled neural networks with impulsive disturbances," *IEEE Transactions on Neural Networks*, vol. 22, no. 2, pp. 329–335, 2011.

[9] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, no. 3, pp. 1057–1064, 2005.

[10] V. N. Belykh, I. V. Belykh, and M. Hasler, "Connection graph stability method for synchronized coupled chaotic systems," *Physica D*, vol. 195, no. 1-2, pp. 159–187, 2004.

[11] I. V. Belykh, V. N. Belykh, and M. Hasler, "Blinking model and synchronization in small-world networks with a time-varying coupling," *Physica D*, vol. 195, no. 1-2, pp. 188–206, 2004.

[12] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criterion," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[13] W. L. Lu, F. M. Atay, and J. Jost, "Chaos synchronization in networks of coupled maps with time-varying topologies," *The European Physical Journal B*, vol. 63, no. 3, pp. 399–406, 2008.

[14] B. Shen, Z. Wang, and X. Liu, "Sampled-data synchronization control of dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[15] B. Shen, Z. Wang, and X. Liu, "Bounded H_∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon," *IEEE Transactions on Neural Networks*, vol. 22, no. 1, pp. 145–157, 2011.

[16] D. J. Stilwell, E. M. Bollt, and D. G. Roberson, "Sufficient conditions for fast switching synchronization in time-varying network topologies," *SIAM Journal on Applied Dynamical Systems*, vol. 5, no. 1, pp. 140–156, 2006.

[17] W. Lu, T. Chen, and G. Chen, "Synchronization analysis of linearly coupled systems described by differential equations with a coupling delay," *Physica D*, vol. 221, no. 2, pp. 118–134, 2006.

[18] W. Lu and T. Chen, "New approach to synchronization analysis of linearly coupled ordinary differential systems," *Physica D*, vol. 213, no. 2, pp. 214–230, 2006.

[19] B. Liu, W. Lu, and T. Chen, "New conditions on synchronization of networks of linearly coupled dynamical systems with non-Lipschitz right-hand sides," *Neural Networks*, vol. 25, pp. 5–13, 2012.

[20] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, NY, USA, 1969.

[21] L. Ya. Adrianova, *Introduction to Linear Systems of Differential Equations*, vol. 146 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1995.

[22] B. Liu, W. Lu, and T. Chen, "Consensus in networks of multiagents with switching topologies modeled as adapted stochastic processes," *SIAM Journal on Control and Optimization*, vol. 49, no. 1, pp. 227–253, 2011.

[23] J. Hajnal, "Weak ergodicity in non-homogeneous Markov chains," *Proceedings of the Cambridge Philosophical Society*, vol. 54, pp. 233–246, 1958.

[24] J. Hajnal, "The ergodic properties of non-homogeneous finite Markov chains," *Proceedings of the Cambridge Philosophical Society*, vol. 52, pp. 67–77, 1956.

[25] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *Proceedings of the 43rd IEEE Conference on Decision and Control (CDC '04)*, vol. 4, pp. 3998–4003, December 2004.

[26] O. E. Rössler, "Continuous chaos four prototype equations," *Annals of the New York Academy of Sciences*, vol. 316, pp. 376–392, 1979.

[27] M. E. J. Newman and D. J. Watts, "Scaling and percolation in the small world network model," *Physical Review E*, vol. 60, no. 6, pp. 7332–7342, 1999.

[28] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining Lyapunov exponents from a time series," *Physica D*, vol. 16, no. 3, pp. 285–317, 1985.

[29] G. Barna and I. Tsuda, "A new method for computing Lyapunov exponents," *Physics Letters A*, vol. 175, no. 6, pp. 421–427, 1993.

[30] M. Sandri, "Numerical calculation of Lyapunov exponents," *Mathematica Journal*, vol. 6, no. 3, pp. 78–84, 1996.

[31] J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices," *Proceedings of the American Mathematical Society*, vol. 14, no. 5, pp. 733–737, 1963.

[32] V. I. Oseledec, "A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems," *Transactions of the Moscow Mathematical Society*, vol. 19, pp. 197–231, 1968.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 653628, 11 pages
http://dx.doi.org/10.1155/2013/653628

Research Article

Optimal Guaranteed Cost Control of a Class of Discrete-Time Nonlinear Systems with Markovian Switching and Mode-Dependent Mixed Time Delays

Yurong Liu,[1] Hamid Reza Karimi,[2] and Xiaohui Liu[3]

[1] *Department of Mathematics, Yangzhou University, Yangzhou 225002, China*
[2] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*
[3] *School of Computer Science and Technology, Nanjing University of Science and Technology, Nanjing 210094, China*

Correspondence should be addressed to Yurong Liu; liuyurong@gmail.com

Received 8 May 2013; Accepted 8 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 Yurong Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The guaranteed cost control problem is investigated for a class of nonlinear discrete-time systems with Markovian jumping parameters and mixed time delays. The mixed time delays involved consist of both the mode-dependent discrete delay and the distributed delay with mode-dependent lower bound. The associated cost function is of a quadratic summation form over the infinite horizon. The nonlinear functions are assumed to satisfy sector-bounded conditions. By introducing new Lyapunov-Krasovskii functionals and developing some new analysis techniques, sufficient conditions for the existence of guaranteed cost controllers are derived with respect to the given cost function. Moreover, a convex optimization approach is applied to search for the optimal guaranteed cost controller by minimizing the guaranteed cost of the closed-loop system. Numerical simulation is further carried out to demonstrate the effectiveness of the proposed methods.

1. Introduction

In the past decades, the control problems for the linear or nonlinear systems have attracted considerable research interest and significant advances on this topic have been made; see, for example, [1–15] and the references therein. It is well known that the time delay in feedback control can be caused by physical properties of control equipments, measurements of system responses, and data processing, calculating and executing control forces, and so forth. The time delay in feedback control may not only deteriorate the performance of controlled systems but also destabilize the controlled systems. There have been a lot of reports on the dynamics analysis of time delay feedback controlled systems. Various sufficient conditions, either delay dependent or delay independent, have been proposed to guarantee the stability for the delayed systems; see, for example, [2, 9, 12, 13] for some recent publications.

On the other hand, a great deal of attention has recently been devoted to the study of Markovian jump systems. This class of systems can be modeled with variable structure subject to random abrupt changes resulting from the occurrence of some inner discrete events in the system such as failures and repairs of machine in manufacturing systems, random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, and so on. Recently, stability and control problems for Markovian jump systems have been extensively investigated; see, for example, [16, 17] and the references therein.

It is also noted that in practical applications, the choice of control policy depends upon the optimization of some preassigned performance criterion. When designing a controller for a real system, it is often desirable to make the controlled system not only stable but also guarantee an adequate level of performance. To deal with such control

problems, the so-called guaranteed cost control approach was first introduced by Chang and Peng [2]. The objective of this approach is to establish an upper bound on a given performance index so that the system performance degradation incurred by the uncertainties is guaranteed to be less than this bound. For guaranteed cost control, a great number of results on this topic have been reported in the literature and various approaches have been proposed. For example, in [18], notion of the quadratic guaranteed cost control was introduced to allow for a quadratic performance index and a Riccati equation approach was presented for designing quadratic guaranteed cost controllers, where the system was delay-free. The authors in [19] extended the Riccati equation approach given in [18] to uncertain delayed systems and proposed a guaranteed cost controller design method by solving a certain parameter-dependent Riccati equation. In [15], an LMI approach [20] was proposed to deal with the guaranteed cost control problem for a class of linear time delay systems with time-varying norm-bounded parameter uncertainty, and a sufficient condition for the existence of memoryless state-feedback guaranteed cost controllers was derived. In [21], the solutions to the guaranteed cost control problem via state-feedback are presented for a class of uncertain Markovian jump systems with mode-dependent delays in LMI framework, and the delay-dependent/independent sufficient conditions for the existence of guaranteed cost state-feedback controllers have been derived.

Based on LMI approach, [22] considered the robust guaranteed cost control problem for uncertain linear discrete-time systems subject to actuator saturation, where the saturation nonlinearity was transformed into a convex polytope of linear systems, and then this problem was formulated into a convex optimization problem with constraints given by a set of LMIs. Very recently, the filtering problems have been investigated for discrete-time nonlinear stochastic systems with network-induced phenomena in [7, 8, 23, 24]. As far as we know, however, little research has been focused on the guaranteed cost control problem for discrete-time systems with *distributed time delay and Markovian jumping parameters*.

In this paper, we consider the guaranteed cost control problem for a class of nonlinear discrete-time systems *with Markovian jumping parameters and mixed time delays*. The mixed time delays involved consist of both the mode-dependent discrete delay and the infinite distributed delay with mode-dependent lower bound. The relevant cost function is chosen as a quadratic summation form over the infinite horizon. The nonlinear functions are assumed to satisfy sector-bounded conditions. By constructing novel Lyapunov-Krasovskii functionals and employing some new analysis techniques, sufficient conditions for the existence of guaranteed cost controllers are derived with respect to the given cost function. In addition, a convex optimization approach is applied to search for the optimal guaranteed cost controller by minimizing the guaranteed cost of the closed-loop system. Finally, a numerical example is presented to demonstrate the effectiveness of the proposed methods.

Notations. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices; $\mathbb{N}^-$ stands for the set of all the negative integers and zero. The superscript "T" denotes matrix transposition. The notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (resp., positive definite). $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix, I is the identity matrix with compatible dimension, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. If A is a square matrix, $\lambda_{\max}(A)$ (resp., $\lambda_{\min}(A)$) denotes the largest (resp., smallest) eigenvalue of A, and $\mathrm{Tr}(A)$ denotes the trace of A. In symmetric block matrices, an asterisk "$*$" is used to represent a term that is induced by symmetry. $\mathbb{E}[x]$ and $\mathbb{E}[x \mid y]$ will, respectively, mean the expectation of x and the expectation of x conditional on y. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation

Let $r(k)$ $(k \geq 0)$ be a Markov chain taking values in a finite state space $\mathcal{N} = \{1, 2, \ldots, n_0\}$ with probability transition matrix $\Pi = (\pi_{ij})_{n_0\times n_0}$ given by

$$\Pr\{r(k+1) = j \mid r(k) = i\} = \pi_{ij}, \quad \forall i,\ j \in \mathcal{N}, \tag{1}$$

where $\pi_{ij} \geq 0$ $(i, j \in \mathcal{N})$ is the transition probability from i to j and $\sum_{j=1}^{n_0} \pi_{ij} = 1$, for all $i \in N$.

Consider a discrete-time nonlinear system with n_0 modes described by the following dynamical equation:

$$\begin{aligned} x(k+1) = {} & A(r(k))\,x(k) + B(r(k))\,f(x(k)) \\ & + C(r(k))\,g\left(x\left(k-\tau_{1,r(k)}\right)\right) + D(r(k)) \\ & \times \sum_{m=\tau_{2,r(k)}}^{+\infty} \mu_m h(x(k-m)) + E(r(k))\,u(k), \end{aligned} \tag{2a}$$

$$x(m) = \phi(m) \quad \text{for } m \in \mathbb{N}^-,\ r(0) = r_0, \tag{2b}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; for $r(k) = i \in \mathcal{N}$, $A(r(k)) \in \mathbb{R}^{n\times n}$, $B(r(k)) \in \mathbb{R}^{n\times n}$, $C(r(k)) \in \mathbb{R}^{n\times n}$, $D(r(k)) \in \mathbb{R}^{n\times n} \in \mathbb{R}^{n\times n}$, and $E(r(k)) \in \mathbb{R}^{n\times q}$ are known constant matrices; $f(\cdot)$, $g(\cdot)$ and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear vector functions; μ_m $(m = 1, 2, \ldots)$ are scalar constants; $u(k) \in \mathbb{R}^q$ is the control input; $\tau_{1,r(k)}$ stands for the mode-dependent discrete time-delay while $\tau_{2,r(k)} \geq 0$ describes a mode-dependent lower bound for the distributed time-delay; $\phi : \mathbb{N}^- \to \mathbb{R}^n$ is the initial value; and $r(0) = r_0$ is the initial mode of the Markov chain.

For nonlinear vector functions f, g, h, we assume that

$$\begin{aligned}
&\left(f(x)-F_1x\right)^T\left(f(x)-F_2x\right)\le 0, \quad \forall x\in\mathbb{R}^n,\\
&\left(g(x)-G_1x\right)^T\left(g(x)-G_2x\right)\le 0, \quad \forall x\in\mathbb{R}^n,\\
&\left(h(x)-H_1x\right)^T\left(h(x)-H_2x\right)\le 0, \quad \forall x\in\mathbb{R}^n,
\end{aligned} \tag{3}$$

where F_1, F_2, G_1, G_2, H_1, and $H_2 \in \mathbb{R}^{(n\times n)}$ are known constant matrices.

Remark 1. The conditions (2) are quite general, and such a description, compared with the usual Lipschitz condition, is very helpful for using LMI-based approach to reduce the possible conservatism. The similar form of the conditions has been used, for example, by the authors in [10].

Remark 2. It is not difficult to verify that the conditions (2) imply that $f(0)=g(0)=h(0)=0$, and $x=0$ is therefore an equilibrium point.

The cost function associated with system (2a) and (2b) is

$$\begin{aligned}
J=\mathbb{E}\Bigg[\sum_{k=0}^{\infty}\Big(&x^T(k)R_1(r(k))x(k)\\
&+u^T(t)R_2(r(k))u(k)\Big)\mid\phi(k),r_0\Bigg],
\end{aligned} \tag{4}$$

where $R_1(i)>0$ and $R_2(i)>0$, for all $i\in\mathcal{N}$.

Now, consider the following state-feedback control law $u(k)=K(r(k))x(k)$, where $K(i)\in\mathbb{R}^{q\times n}$ $(i\in\mathcal{N})$ are controller gains to be designed. Then, the closed-loop system can be given as follows:

$$\begin{aligned}
x(k+1)=&A_K(r(k))x(k)+B(r(k))f(x(k))\\
&+C(r(k))g\left(x\left(k-\tau_{1,r(k)}\right)\right)\\
&+D(r(k))\sum_{m=\tau_{2,r(k)}}^{+\infty}\mu_m h(x(k-m)),
\end{aligned} \tag{5a}$$

$$x(m)=\phi(m) \quad \text{for } m\in\mathbb{N}^-,\ r(0)=r_0, \tag{5b}$$

where $A_K(r(k))=A(r(t))+E(r(k))\,K(r(k))$.

Definition 3. System (2a) and (2b) with $u(k)\equiv 0$ is said to be asymptotically stable in mean square if, for any solution $x(k)$ of system (2a) and (2b), the following holds:

$$\lim_{k\to\infty}\mathbb{E}\left[|x(k)|^2\right]=0. \tag{6}$$

Definition 4. Consider the system (2a) and (2b). If there exists a state-feedback control law $u(k)$ and a positive number γ such that the closed-loop system (5a) and (5b) is asymptotically stable in mean square and the resulting cost function satisfies

$$J\le\gamma, \tag{7}$$

then γ is said to be a guaranteed cost and $u(k)$ is said to be a guaranteed cost controller for the system (2a) and (2b).

The objective of this paper is to develop a procedure to design a memoryless state-feedback guaranteed cost controller $u(k)=K(r(k))x(k)$, which achieves as small value of γ as possible.

Assumption 5. Constant $\mu_m \ge 0$ satisfies the following convergent conditions:

$$\sum_{m=1}^{+\infty}\mu_m<+\infty, \qquad \sum_{m=1}^{+\infty}m\mu_m<+\infty. \tag{8}$$

Remark 6. Assumption 5 makes sense as they guarantee that the term $D(r(k))\sum_{m=\tau_{2,r(k)}}^{+\infty}\mu_m h(x(k-m))$ in (2a) is convergent, which is necessary for the subsequent analysis.

3. Main Results and Proofs

We first introduce some lemmas to be used in deriving our results.

Lemma 7 (see [25]). *Let $M\in\mathbb{R}^{n\times n}$ be a positive semi definite matrix, $\mathbf{x}_i\in\mathbb{R}^n$ and $a_i\ge 0$ $(i=1,2,\ldots)$. If the series concerned are convergent, the following inequality holds:*

$$\left(\sum_{i=1}^{+\infty}a_i\mathbf{x}_i\right)^T M\left(\sum_{i=1}^{+\infty}a_i\mathbf{x}_i\right)\le\left(\sum_{i=1}^{+\infty}a_i\right)\sum_{i=1}^{+\infty}a_i\mathbf{x}_i^T M\mathbf{x}_i. \tag{9}$$

Lemma 8 (see [10]). *Assume that nonlinear function $\hbar(\cdot):\mathbb{R}^n\to\mathbb{R}^n$ satisfies*

$$\left(\hbar(x)-U_1x\right)^T\left(\hbar(x)-U_2x\right)\le 0, \quad \forall x\in\mathbb{R}^n \tag{10}$$

with U_1 and U_2 being constant matrices. Then, the following matrix inequality holds:

$$\begin{bmatrix} x\\ \hbar(x)\end{bmatrix}^T\begin{bmatrix}\breve{U}_1 & -\breve{U}_2\\ -\breve{U}_2^T & I\end{bmatrix}\begin{bmatrix} x\\ \hbar(x)\end{bmatrix}\le 0, \tag{11}$$

or

$$x^T\breve{U}_1x - 2x^T\breve{U}_2\hbar(x) + \hbar^T(x)\hbar(x) \le 0, \tag{12}$$

where $\breve{U}_1 = (U_1^TU_2 + U_2^TU_1)/2$ *and* $\breve{U}_2 = (U_1^T + U_2^T)/2$.

Proof. It can be verified by simple matrix operations. □

Lemma 9 (Schur complement [20]). *Given constant matrices* Ω_1, Ω_2, Ω_3 *where* $\Omega_1 = \Omega_1^T$ *and* $\Omega_2 > 0$, *then*

$$\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0 \tag{13}$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0. \tag{14}$$

Hereafter, one denote $\overline{\tau}_1 = \max_{1\le j\le n_0}\{\tau_{1,j}\}$, $\overline{\tau}_2 = \max_{1\le j\le n_0}\{\tau_{2,j}\}$, $\underline{\tau}_1 = \min_{1\le j\le n_0}\{\tau_{1,j}\}$, $\underline{\tau}_2 = \min_{1\le j\le n_0}\{\tau_{2,j}\}$, and $\underline{\pi} = \min_{1\le i\le n_0}\{|\pi_{ii}|\}$.

One also denotes

$$\begin{aligned} \breve{F}_1 &= \frac{\left(F_1^TF_2 + F_2^TF_1\right)}{2}, & \breve{F}_2 &= \frac{\left(F_1^T + F_2^T\right)}{2}, \\ \breve{G}_1 &= \frac{\left(G_1^TG_2 + G_2^TG_1\right)}{2}, & \breve{G}_2 &= \frac{\left(G_1^T + G_2^T\right)}{2}, \\ \breve{H}_1 &= \frac{\left(H_1^TH_2 + H_2^TH_1\right)}{2}, & \breve{H}_2 &= \frac{\left(H_1^T + H_2^T\right)}{2}, \\ \sigma_m &= \sum_{\iota=m}^{+\infty}\mu_\iota, & \widehat{\mu} &= \max\{\mu_m \mid \underline{\tau}_2 \le m \le \overline{\tau}_2 - 1\}. \end{aligned} \tag{15}$$

The following is a sufficient condition for the existence of state-feedback guaranteed cost control laws for the system (2a) and (2b).

Theorem 10. *Given a state-feedback controller* $u(k) = K(r(k))x(k)$. *If there exist a set of positive definite matrices* P_i $(i \in \mathcal{N})$, *and two positive definite matrices* Q_1 *and* Q_2 *such that the LMIs* (17) *hold, then* $u(k) = K(r(k))x(k)$ *is a guaranteed cost controller for the system* (2a) *and* (2b), *and the cost function satisfies the following bound:*

$$\begin{aligned} J \le{}& x^T(0)P_{r_0}x(0) + \sum_{\nu=-\overline{\tau}_{1,r_0}}^{-1} g^T(x(\nu))Q_1g(x(\nu)) \\ &+ (1-\underline{\pi})\sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1}\sum_{\nu=-\iota}^{-1} g^T(x(\nu))Q_1g(x(\nu)) \\ &+ \sum_{\iota=\overline{\tau}_{2,r_0}}^{+\infty}\mu_\iota\sum_{\nu=-\iota}^{-1} h^T(x(\nu))Q_2h(x(\nu)) \\ &+ (1-\underline{\pi})\widehat{\mu}\sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\iota=1}^{s-1}\sum_{\nu=-\iota}^{-1} h^T(x(\nu))Q_2h(x(\nu)), \end{aligned} \tag{16}$$

$$\begin{aligned} &\Phi(i) \\ &= \begin{bmatrix} -P_i - \Xi + R_1(i) + K^T(i)R_2K(i) & \Theta & A_K^T(i)\overline{P}_i \\ \Theta^T & \Upsilon(i) & \Sigma^T(i)\overline{P}_i \\ \overline{P}_iA_K(i) & \overline{P}_i\Sigma(i) & -\overline{P}_i \end{bmatrix} \\ &< 0, \quad i \in \mathcal{N}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Xi &= \breve{F}_1 + \breve{G}_1 + \breve{H}_1, \qquad \overline{P}_i = \sum_{j=1}^{n_0}\pi_{ij}P_j, \\ \Theta &= \left[\breve{F}_2 \;\; \breve{G}_2 \;\; 0 \;\; \breve{H}_2 \;\; 0\right], \\ \Sigma(i) &= \left[B(i) \;\; 0 \;\; C(i) \;\; 0 \;\; D(i)\right], \\ \Upsilon(i) &= \operatorname{diag}\Big\{-I,\ \alpha_1Q_1 - I, \\ &\qquad -Q_1, \alpha_2Q_2 - I,\ -\frac{1}{\sigma_{\tau_{2,i}}}Q_2\Big\} \end{aligned} \tag{18}$$

with

$$\begin{aligned} \alpha_1 &= (1-\underline{\pi})(\overline{\tau}_1 - \underline{\tau}_1) + 1, \\ \alpha_2 &= \sigma_{\underline{\tau}_2} + \frac{1}{2}\widehat{\mu}(1-\underline{\pi})(\overline{\tau}_2 - \underline{\tau}_2)(\overline{\tau}_2 + \underline{\tau}_2 - 3). \end{aligned} \tag{19}$$

Proof. For convenience, we denote

$$\begin{aligned} \mathscr{A}_K(i) &= \left[A_K(i) \;\; B(i) \;\; 0 \;\; C(i) \;\; 0 \;\; D(i)\right], \\ \xi(\mathbf{x}_k, i) &= \left[x^T(k) \;\; f^T(x(k)) \;\; g^T(x(k)) \;\; g^T(x(k-\tau_{1,r(k)})) \;\; h^T(x(k)) \;\; \sum_{m=\overline{\tau}_{2,i}}^{+\infty}\mu_m h^T(x(k-m))\right]^T. \end{aligned} \tag{20}$$

By Lemma 9, the inequality (17) is equivalent to

$$\overline{\Phi}(i) + \mathscr{A}_K^T(i)\overline{P}_i\mathscr{A}_K(i) < 0, \tag{21}$$

where

$$\overline{\Phi}(i) = \begin{bmatrix} -P_i - \Xi + R_1(i) + K^T(i)R_2(i)K(i) & \Theta \\ \Theta^T & \Upsilon(i) \end{bmatrix}. \tag{22}$$

Define $\mathbf{x}_k : \mathbb{N}^- \to \mathbb{R}^n$ by $\mathbf{x}_k(m) = x(k+m)$ for $m \in \mathbb{N}^-$. To proceed the stability analysis, we construct the following Lyapunov-Krasovskii functional for the system (5a) and (5b):

$$V(\mathbf{x}_k, k, r(k)) = \sum_{i=1}^{5} V_i(\mathbf{x}_k, k, r(k)), \tag{23}$$

where

$$\begin{aligned}
V_1(\mathbf{x}_k, k, r(k)) &= x^T(k)P_{r(k)}x(k), \\
V_2(\mathbf{x}_k, k, r(k)) &= \sum_{v=k-\tau_{1,r(k)}}^{k-1} g^T(x(v))Q_1 g(x(v)), \\
V_3(\mathbf{x}_k, k, r(k)) &= (1-\underline{\pi})\sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1}\sum_{v=k-\iota}^{k-1} g^T(x(v))Q_1 g(x(v)), \\
V_4(\mathbf{x}_k, k, r(k)) &= \sum_{\iota=\tau_{2,r(k)}}^{+\infty}\mu_\iota \sum_{v=k-\iota}^{k-1} h^T(x(v))Q_2 h(x(v)), \\
V_5(\mathbf{x}_k, k, r(k)) &= (1-\underline{\pi})\hat{\mu}\sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\iota=1}^{s-1}\sum_{v=k-\iota}^{k-1} h^T(x(v))Q_2 h(x(v)).
\end{aligned} \tag{24}$$

For $i \in \mathscr{N}$, associated with the closed-loop system (5a) and (5b) we can carry out the following computation:

$$\begin{aligned}
&\mathbb{E}[V_1(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V_1(\mathbf{x}_k, k, i) \\
&= \sum_{j=1}^{n_0}\pi_{ij}x^T(k+1)P_j x(k+1) - x^T(k)P_i x(k) \\
&= \xi^T(\mathbf{x}_k, i)\mathscr{A}_K^T(i)\overline{P}_i\mathscr{A}_K(i)\xi(\mathbf{x}_k, i) - x^T(k)P_i x(k); \\
&\mathbb{E}[V_2(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V_2(\mathbf{x}_k, k, i) \\
&= \sum_{j=1}^{n_0}\pi_{ij}\sum_{v=k-\tau_{1,j}+1}^{k} g^T(x(v))Q_1 g(x(v)) \\
&\quad - \sum_{v=k-\tau_{1,i}}^{k-1} g^T(x(v))Q_1 g(x(v)) \\
&= \sum_{j=1}^{n_0}\pi_{ij}g^T(x(k))Q_1 g(x(k)) \\
&\quad - g^T(x(k-\tau_{1,i}))Q_1 g(x(k-\tau_{1,i})) \\
&\quad + \sum_{j=1}^{n_0}\pi_{ij}\sum_{v=k-\tau_{1,j}+1}^{k-1} g^T(x(v))Q_1 g(x(v)) \\
&\quad - \sum_{v=k-\tau_{1,i}+1}^{k-1} g^T(x(v))Q_1 g(x(v)) \\
&= g^T(x(k))Q_1 g(x(k)) - g^T(x(k-\tau_{1,i})) \\
&\quad \times Q_1 g(x(k-\tau_{1,i})) \\
&\quad + \sum_{j\neq i}\pi_{ij}\left(\sum_{v=k-\tau_{1,j}+1}^{k-1} g^T(x(v))Q_1 g(x(v)) \right. \\
&\qquad \left. - \sum_{v=k-\tau_{1,i}+1}^{k-1} g^T(x(v))Q_1 g(x(v))\right) \\
&\le g^T(x(k))Q_1 g(x(k)) - g^T(x(k-\tau_{1,i}))Q_1 g(x(k-\tau_{1,i})) \\
&\quad + \sum_{j\neq i}\pi_{ij}\sum_{v=k-\overline{\tau}_1+1}^{k-\underline{\tau}_1} g^T(x(v))Q_1 g(x(v)) \\
&\le g^T(x(k))Q_1 g(x(k)) - g^T(x(k-\tau_{1,i}))Q_1 g(x(k-\tau_{1,i})) \\
&\quad + (1-\underline{\pi})\sum_{v=k-\overline{\tau}_1+1}^{k-\underline{\tau}_1} g^T(x(v))Q_1 g(x(v)); \\
&\mathbb{E}[V_3(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V_3(\mathbf{x}_k, k, i) \\
&= (1-\underline{\pi})\left(\sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1}\sum_{v=k-\iota+1}^{k} g^T(x(v))Q_1 g(x(v)) \right. \\
&\qquad \left. - \sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1}\sum_{v=k-\iota}^{k-1} g^T(x(v))Q_1 g(x(v))\right) \\
&= (1-\underline{\pi})\sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1}\left(g^T(x(k))Q_1 g(x(k)) \right. \\
&\qquad \left. - g^T(x(k-\iota))Q_1 g(x(k-\iota))\right) \\
&= (1-\underline{\pi})(\overline{\tau}_1 - \underline{\tau}_1)g^T(x(k))Q_1 g(x(k)) \\
&\quad - (1-\underline{\pi})\sum_{v=k-\overline{\tau}_1+1}^{k-\underline{\tau}_1} g^T(x(v))Q_1 g(x(v)), \\
&\mathbb{E}[V_4(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V_4(\mathbf{x}_k, k, i) \\
&= \sum_{j=1}^{n_0}\pi_{i,j}\sum_{\iota=\tau_{2,j}}^{+\infty}\mu_\iota\sum_{v=k-\iota+1}^{k} h^T(x(v))Q_2 h(x(v))
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\iota=\tau_{2,i}}^{+\infty}\mu_\iota\sum_{\nu=k-\iota}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\\
&\le\sum_{j\neq i}\pi_{i,j}\left(\sum_{\iota=\tau_{2,j}}^{+\infty}\mu_\iota\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\right.\\
&\qquad\left.-\sum_{\iota=\tau_{2,i}}^{+\infty}\mu_\iota\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\right)\\
&+\sigma_{\underline{\tau}_2}h^T(x(k))Q_2h(x(k))\\
&-\sum_{\iota=\tau_{2,i}}^{+\infty}\mu_\iota h^T(x(k-\iota))Q_2h(x(k-\iota))\\
&\le\sum_{j\neq i}\pi_{i,j}\left(\sum_{\iota=\underline{\tau}_2}^{+\infty}\mu_\iota\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\right.\\
&\qquad\left.-\sum_{\iota=\overline{\tau}_2}^{+\infty}\mu_\iota\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\right)\\
&+\sigma_{\underline{\tau}_2}h^T(x(k))Q_2h(x(k))\\
&-\sum_{\iota=\tau_{2,i}}^{+\infty}\mu_\iota h^T(x(k-\iota))Q_2h(x(k-\iota))\\
&\le(1-\underline{\pi})\sum_{\iota=\underline{\tau}_2}^{\overline{\tau}_2-1}\mu_\iota\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\\
&+\sigma_{\underline{\tau}_2}h^T(x(k))Q_2h(x(k))\\
&-\sum_{\iota=\tau_{2,i}}^{+\infty}\mu_\iota h^T(x(k-\iota))Q_2h(x(k-\iota))\\
&\le(1-\underline{\pi})\widehat{\mu}\sum_{\iota=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\\
&+\sigma_{\underline{\tau}_2}h^T(x(k))Q_2h(x(k))\\
&-\sum_{m=\tau_{2,i}}^{+\infty}\mu_m h^T(x(k-m))Q_2h(x(k-m));
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[V_5(\mathbf{x}_{k+1},k+1,r(k+1))\mid\mathbf{x}_k,r(k)=i\right]-V_5(\mathbf{x}_k,k,i)\\
&=(1-\underline{\pi})\widehat{\mu}\left[\sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\iota=1}^{s-1}\sum_{\nu=k-\iota+1}^{k}h^T(x(\nu))Q_2h(x(\nu))\right.\\
&\qquad\left.-\sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\iota=1}^{s-1}\sum_{\nu=k-\iota}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\right]\\
&=(1-\underline{\pi})\widehat{\mu}\left[\sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\iota=1}^{s-1}\left(h^T(x(k))Q_2h(x(k))\right.\right.\\
&\qquad\left.\left.-h^T(x(k-\iota))Q_2h(x(k-\iota))\right)\right]\\
&=(1-\underline{\pi})\widehat{\mu}\left[\frac{1}{2}(\overline{\tau}_2-\underline{\tau}_2)(\overline{\tau}_2+\underline{\tau}_2-3)h^T(x(k))Q_2h\right.\\
&\qquad\left.\times(x(k))-\sum_{\iota=\underline{\tau}_2}^{\overline{\tau}_2-1}\sum_{\nu=k-\iota+1}^{k-1}h^T(x(\nu))Q_2h(x(\nu))\right].
\end{aligned}
\tag{25}
$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[V(\mathbf{x}_{k+1},k+1,r(k+1))\mid\mathbf{x}_k,r(k)=i\right]-V(\mathbf{x}_k,k,i)\\
&=\sum_{j=1}^{5}\left[\mathbb{E}\left[V_j(\mathbf{x}_{k+1},k+1,r(k+1))\mid\mathbf{x}_k,r(k)=i\right]\right.\\
&\qquad\left.-V_j(\mathbf{x}_k,k,i)\right]\\
&\le\xi^T(\mathbf{x_k},i)\mathscr{A}_K^T(i)\overline{P}_i\mathscr{A}_K(i)\xi(\mathbf{x_k},i)-x^T(k)P_ix(k)\\
&+\alpha_1g^T(x(k))Q_1g(x(k))\\
&-g^T(x(k-\tau_{1,i}))Q_1g(x(k-\tau_{1,i}))\\
&+\alpha_2h^T(x(k))Q_2h(x(k))\\
&-\sum_{m=\tau_{2,i}}^{+\infty}\mu_m h^T(x(k-m))Q_2h(x(k-m)),
\end{aligned}
\tag{26}
$$

where $\alpha_1=(1-\underline{\pi})(\overline{\tau}_1-\underline{\tau}_1)+1$, $\alpha_2=\sigma_{\underline{\tau}_2}+(1/2)\widehat{\mu}(1-\underline{\pi})(\overline{\tau}_2-\underline{\tau}_2)(\overline{\tau}_2+\underline{\tau}_2-3)$.

By Lemma 7, it is clear that

$$
\begin{aligned}
&-\sum_{m=\tau_{2,i}}^{+\infty}\mu_m h^T(x(k-m))Q_2h(x(k-m))\\
&\le-\frac{1}{\sigma_{\tau_{2,i}}}\sum_{m=\tau_{2,i}}^{+\infty}\mu_m h^T(x(k-m))Q_2\sum_{m=\tau_{2,i}}^{+\infty}\mu_m h(x(k-m)).
\end{aligned}
\tag{27}
$$

Also, from the conditions (2) and Lemma 8, it follows that

$$
\begin{aligned}
&x^T(k)\breve{F}_1x(k)-2x^T(k)\breve{F}_2f(x(k))\\
&\quad+f^T(x(k))f(x(k))\le0,\\
&x^T(k)\breve{G}_1x(k)-2x^T(k)\breve{G}_2g(x(k))\\
&\quad+g^T(x(k))g(x(k))\le0,\\
&x^T(k)\breve{H}_1x(k)-2x^T(k)\breve{H}_2h(x(k))\\
&\quad+h^T(x(k))h(x(k))\le0.
\end{aligned}
\tag{28}
$$

From (26)–(28), it follows readily that

$$\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{x}_{k+1}, k+1, r(k+1)\right) \mid \mathbf{x}_k, r(k)=i\right] - V\left(\mathbf{x}_k, k, i\right) \\
&\le \xi^T\left(\mathbf{x}_{\mathbf{k}}, i\right) \mathscr{A}_K^T(i) \overline{P}_i \mathscr{A}_K(i) \xi\left(\mathbf{x}_{\mathbf{k}}, i\right) - x^T(k) P_i x(k) \\
&\quad + \alpha_1 g^T(x(k)) Q_1 g(x(k)) \\
&\quad - g^T\left(x\left(k-\tau_{1,i}\right)\right) Q_1 g\left(x\left(k-\tau_{1,i}\right)\right) \\
&\quad + \alpha_2 h^T(x(k)) Q_2 h(x(k)) \\
&\quad - \frac{1}{\sigma_{\tau_{2,i}}} \sum_{m=\tau_{2,i}}^{+\infty} \mu_m h^T(x(k-m)) Q_2 \sum_{m=\tau_{2,i}}^{+\infty} \mu_m h^T(x(k-m)) \\
&\quad - \Big(x^T(k) \breve{F}_1 x(k) - 2x^T(k) \breve{F}_2 f(x(k)) \\
&\qquad + f^T(x(k)) f(x(k))\Big) \\
&\quad - \Big(x^T(k) \breve{G}_1 x(k) - 2x^T(k) \breve{G}_2 g(x(k)) \\
&\qquad + g^T(x(k)) g(x(k))\Big) \\
&\quad - \Big(x^T(k) \breve{H}_1 x(k) - 2x^T(k) \breve{H}_2 h(x(k)) \\
&\qquad + h^T(x(k)) h(x(k))\Big) \\
&= \xi^T\left(\mathbf{x}_{\mathbf{k}}, i\right) \left[\overline{\Phi}(i) + \mathscr{A}_K^T(i) \overline{P}_i \mathscr{A}_K(i)\right] \xi\left(\mathbf{x}_{\mathbf{k}}, i\right) \\
&\quad - x^T(k) \left[R_1(i) + K^T(i) R_2 K(i)\right] x(k),
\end{aligned} \tag{29}$$

which, together with (21), implies

$$\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{x}_{k+1}, k+1, r(k+1)\right)\right] - \mathbb{E}\left[V\left(\mathbf{x}_k, k, r(k)\right)\right] \\
&\le -\mathbb{E}\left[x^T(k)\left(R_1(r(k)) + K^T(r(k)) R_2 K(r(k))\right) x(k)\right].
\end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{x}_{k+1}, k+1, r(k+1)\right)\right] - \mathbb{E}\left[V\left(\mathbf{x}_k, k, r(k)\right)\right] \\
&\quad \le -\lambda_0 \mathbb{E}|x(k)|^2,
\end{aligned} \tag{31}$$

where $\lambda_0 = \min_{i\in\mathcal{N}}\{\lambda_{\min}(R_1(i))\}$.

Let s be an arbitrary positive integer; then it can be inferred from (31) that

$$\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{x}_{s+1}, s+1, r(s+1)\right)\right] - \mathbb{E}\left[V\left(\mathbf{x}_0, 0, r(0)\right)\right] \\
&\quad \le -\lambda_0 \sum_{k=0}^{s} \mathbb{E}\left[|x(k)|^2\right],
\end{aligned} \tag{32}$$

or

$$\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{x}_{s+1}, s+1, r(s+1)\right)\right] - V\left(\mathbf{x}_0, 0, r(0)\right) \\
&\quad \le -\lambda_0 \sum_{k=0}^{s} \mathbb{E}\left[|x(k)|^2\right],
\end{aligned} \tag{33}$$

which results in

$$\sum_{k=0}^{s} \mathbb{E}\left[|x(k)|^2\right] \le \frac{1}{\lambda_0} V\left(\mathbf{x}_0, 0, r(0)\right). \tag{34}$$

It can now be concluded that the series $\sum_{k=0}^{+\infty} \mathbb{E}[|x(k)|^2]$ is convergent, and therefore

$$\lim_{k\to+\infty} \mathbb{E}\left[|x(k)|^2\right] = 0. \tag{35}$$

Therefore, the closed-loop system (5a) and (5b) is asymptotically stable in mean square.

On the other hand, for any positive integer s, from (30) we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{s}\left(x^T(k) R_1(r(k)) x(k) + u^T(k) R_2(r(k)) u(k)\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{s} x^T(k)\left(R_1(r(k)) + K^T(r(k)) R_2(r(k)) K(r(k))\right)\right. \\
&\qquad \left. \times x(k) + V\left(\mathbf{x}_{k+1}, k+1, r(k+1)\right) - V\left(\mathbf{x}_k, k, r(k)\right)\right] \\
&\quad - \mathbb{E}\left[V\left(\mathbf{x}_{s+1}, s+1, r(s+1)\right)\right] + V\left(\mathbf{x}_0, 0, r(0)\right) \\
&\le V\left(x_0, 0, r(0)\right).
\end{aligned} \tag{36}$$

Letting $s \to +\infty$, we have

$$J \le V\left(x_0, 0, r_0\right); \tag{37}$$

namely, (16) holds. This completes the proof of the theorem. □

Theorem 10 provides a sufficient condition to determine if a given controller is a guaranteed cost controller. Next, we turn to the design problem of guaranteed cost controller for the system (2a) and (2b). For this, we have the following results.

Theorem 11. *Consider the system* (2a) *and* (2b). *If there exist a set of positive definite matrices X_i ($i \in \mathcal{N}$), a set of matrices Y_i ($i \in \mathcal{N}$), and two positive definite matrices Q_1 and Q_2 such that the LMIs* (39) *hold, then $u(k) = K(r(k))x(k)$ with $K(i) = Y_i X_i^{-1}$ is a guaranteed cost controller for the system* (2a) *and* (2b), *and the cost function satisfies the following bound:*

$$J \le \gamma = x^T(0) X_{r_0}^{-1} x(0) + \sum_{\nu=-\tau_{1,r_0}}^{-1} g^T(x(\nu)) Q_1 g(x(\nu)) + (1-\underline{\pi}) \sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1} \sum_{\nu=-\iota}^{-1} g^T(x(\nu)) Q_1 g(x(\nu))$$
$$+ \sum_{\iota=\tau_{2,r_0}}^{+\infty} \mu_\iota \sum_{\nu=-\iota}^{-1} h^T(x(\nu)) Q_2 h(x(\nu)) + (1-\underline{\pi}) \widehat{\mu} \sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1} \sum_{\iota=1}^{s-1} \sum_{\nu=-\iota}^{-1} h^T(x(\nu)) Q_2 h(x(\nu)); \tag{38}$$

$$\overline{\Phi}(i) \triangleq \begin{bmatrix} -X_i & X_i\Theta & X_i & X_i & Y_i^T & (A(i)X_i + E(i)Y_i)^T W_i \\ \Theta^T X_i & \Upsilon(i) & 0 & 0 & 0 & \Sigma^T(i) W_i \\ X_i & 0 & \Xi^{-1} & 0 & 0 & 0 \\ X_i & 0 & 0 & -R_1^{-1}(i) & 0 & 0 \\ Y_i & 0 & 0 & 0 & -R_2^{-1}(i) & 0 \\ W_i^T(A(i)X_i + E(i)Y_i) & W_i^T\Sigma(i) & 0 & 0 & 0 & -\mathcal{X} \end{bmatrix} < 0, \tag{39}$$

where $W_i = [\sqrt{\pi_{i1}} I, \sqrt{\pi_{i2}} I, \ldots, \sqrt{\pi_{in_0}} I]$, $\mathcal{X} = \operatorname{diag}\{X_1, X_2, \ldots, X_{n_0}\}$, *and* Θ, $\Upsilon(i)$, Ξ *and* $\Sigma(i)$ *are defined as in Theorem 10.*

Proof. Let $P_i = X_i^{-1}$, $Y_i = K(i)P_i^{-1}$, and $\vartheta(i) = \operatorname{diag}\{P_i, \overbrace{I, \ldots, I}^{6}\}$. Then, inequality (39) is equivalent to

$$\vartheta(i) \overline{\Phi}(i) \vartheta(i) < 0; \tag{40}$$

namely,

$$\begin{bmatrix} -P_i & \Theta & I & I & K^T(i) & A_K^T(i) W_i \\ \Theta^T & \Upsilon(i) & 0 & 0 & 0 & \Sigma^T(i) W_i \\ I & 0 & \Xi^{-1} & 0 & 0 & 0 \\ I & 0 & 0 & -R_1^{-1}(i) & 0 & 0 \\ K(i) & 0 & 0 & 0 & -R_2^{-1}(i) & 0 \\ W_i^T A_K(i) & W_i^T\Sigma(i) & 0 & 0 & 0 & -\mathcal{P}^{-1} \end{bmatrix} < 0, \tag{41}$$

where $\mathcal{P} = \operatorname{diag}\{P_1, P_2, \ldots, P_{n_0}\}$.

From Lemma 9, it follows readily that (41) is equivalent to (21) and is therefore equivalent to (17).

By Theorem 10, $u(k) = K(r(k))x(k)$ with $K(i) = Y_i X_i^{-1}$ is a guaranteed cost controller for the system (2a) and (2b), and the cost function satisfies the bound as shown in (38). □

Remark 12. In Theorem 11, the bound γ of the cost function depends on the parameters X_i, Q_1, and Q_2 in addition to the initial value of the state and mode of the system. Next, we will design an optimal state-feedback guaranteed cost controller $u(k) = K(r(k))x(k)$, which minimizes the bound of the guaranteed cost function.

Theorem 13. *Consider the system* (2a) *and* (2b) *with cost function* (4). *If the following optimal problem of a linear objective*

$$\min_{\substack{\beta_0, X_i, Y_i \\ Q_1, Q_2, M_i}} \left(\beta_0 + \operatorname{Tr}(M_1 + M_2 + M_3 + M_4)\right) \tag{42}$$

subject to LMI constraints

$$\begin{aligned} &(i) \quad LMI \quad (39) \\ &(ii) \quad \begin{bmatrix} -\beta_0 & x^T(0) \\ x(0) & -X_{r_0} \end{bmatrix} < 0, \\ &(iii) \quad \Omega_k < 0, \ (k = 1, 2, 3, 4) \end{aligned} \tag{43}$$

has a set of solutions β_0, X_i $(i \in \mathcal{N})$, Y_i $(i \in \mathcal{N})$, Q_1, Q_2, M_k $(k = 1, 2, 3, 4)$, *then* $u(k) = K(r(k))x(k)$ *with* $K(i) = Y_i X_i^{-1}$ *is an optimal guaranteed cost controller for the system* (2a) *and* (2b), *which minimizes the guaranteed cost* (38).

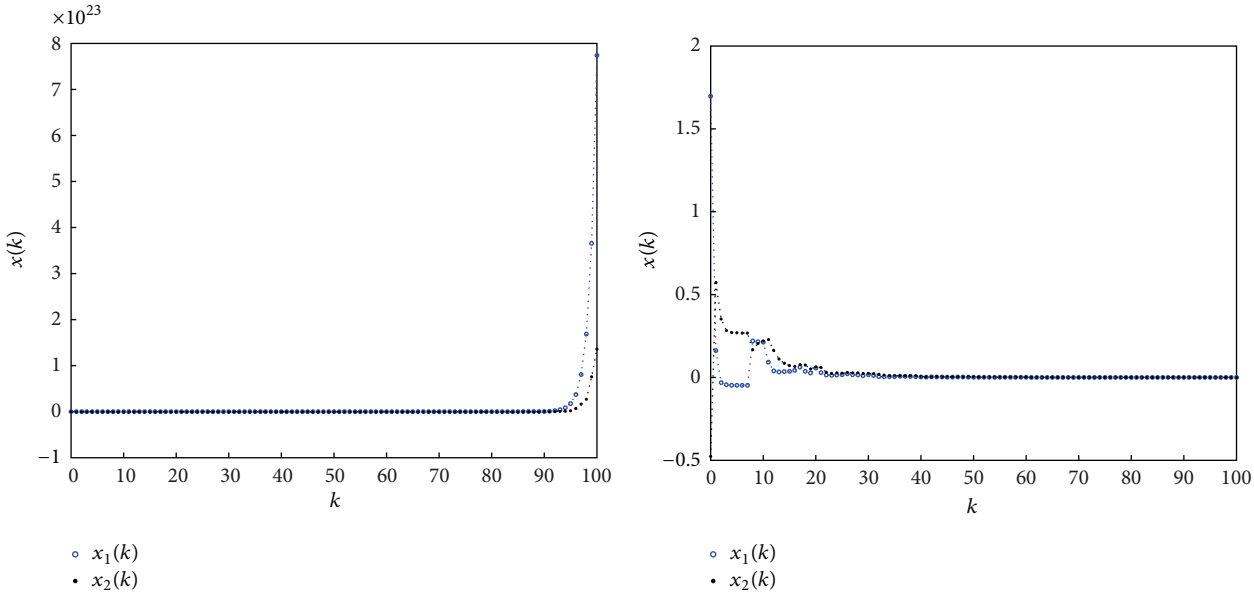
Here,

$$\Omega_1 = \begin{bmatrix} -M_1 & N_1^T Q_1 \\ Q_1 N_1 & -Q_1 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -M_2 & N_2^T Q_1 \\ Q_1 N_2 & -Q_1 \end{bmatrix},$$
$$\Omega_3 = \begin{bmatrix} -M_3 & N_3^T Q_2 \\ Q_2 N_3 & -Q_2 \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} -M_4 & N_4^T Q_2 \\ Q_2 N_4 & -Q_2 \end{bmatrix}, \tag{44}$$

where N_k $(k = 1, 2, 3, 4)$ satisfy

$$\begin{aligned} N_1 N_1^T &= \sum_{\nu=-\tau_{1,r_0}}^{-1} g(x(\nu)) g^T(x(\nu)), \\ N_2 N_2^T &= (1-\underline{\pi}) \sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1} \sum_{\nu=-\iota}^{-1} g(x(\nu)) g^T(x(\nu)), \\ N_3 N_3^T &= \sum_{\iota=\tau_{2,r_0}}^{+\infty} \mu_\iota \sum_{\nu=-\iota}^{-1} h(x(\nu)) h^T(x(\nu)), \\ N_4 N_4^T &= (1-\underline{\pi}) \widehat{\mu} \sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1} \sum_{\iota=1}^{s-1} \sum_{\nu=-\iota}^{-1} h(x(\nu)) h^T(x(\nu)). \end{aligned} \tag{45}$$

Proof. According to Theorem 11, $u(k) = K(r(k))x(k)$ with $K(i) = Y_i X_i^{-1}$ is a guaranteed cost controller for the system (2a) and (2b) if LMI (39) has a set of solutions X_i, Y_i, Q_1, and Q_2.



(a) The state evolution of the unforced system

(b) The state evolution of the closed-loop system

Figure 1: The comparison of state trajectories of the unforced and the controlled systems.

On the other hand, by Lemma 9, the inequality in (ii) is equivalent to $x^T(0)X_{r_0}^{-1}x(0) < \beta_0$. Notice that $\Omega_1 < 0$ is equivalent to

$$N_1^T Q_1 N_1 < M_1, \tag{46}$$

$$\begin{aligned}
&\sum_{\nu=-\tau_{1,r_0}}^{-1} g^T(x(\nu)) Q_1 g(x(\nu)) \\
&\quad = \sum_{\nu=-\tau_{1,r_0}}^{-1} \operatorname{Tr}\left(g^T(x(\nu)) Q_1 g(x(\nu))\right) \\
&\quad = \sum_{\nu=-\tau_{1,r_0}}^{-1} \operatorname{Tr}\left(g(x(\nu)) g^T(x(\nu)) Q_1\right) \\
&\quad = \operatorname{Tr}\left(\sum_{\nu=-\tau_{1,r_0}}^{-1} g(x(\nu)) g^T(x(\nu)) Q_1\right) \\
&\quad = \operatorname{Tr}\left(N_1 N_1^T Q_1\right) \\
&\quad = \operatorname{Tr}\left(N_1^T Q_1 N_1\right) \quad (\text{Thanks to } (46)) \\
&\quad < \operatorname{Tr}(M_1).
\end{aligned} \tag{47}$$

Similarly, from $\Omega_k < 0$ $(k = 2, 3, 4)$ it follows that

$$(1-\underline{\pi}) \sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1} \sum_{\nu=-\iota}^{-1} g(x(\nu)) g^T(x(\nu)) < \operatorname{Tr}(M_2),$$

$$\sum_{\iota=\tau_{2,r_0}}^{+\infty} \mu_\iota \sum_{\nu=-\iota}^{-1} h(x(\nu)) h^T(x(\nu)) \le \operatorname{Tr}(M_3),$$

$$(1-\underline{\pi}) \widehat{\mu} \sum_{s=\underline{\tau}_2}^{\overline{\tau}_2-1} \sum_{\iota=1}^{s-1} \sum_{\nu=-\iota}^{-1} h(x(\nu)) h^T(x(\nu)) \le \operatorname{Tr}(M_4). \tag{48}$$

Accordingly, it follows that $\gamma < \lambda_0 + \operatorname{Tr}(M_1 + M_2 + M_3 + M_4)$, where γ is defined in (38). Since the optimal problem (42) has a set of solutions, the minimization of the guaranteed cost for the system (2a) and (2b) follows from the minimization of $\lambda_0 + \operatorname{Tr}(M_1 + M_2 + M_3 + M_4)$. The proof of this theorem is completed. □

4. A Numerical Example

In this section, an example is presented to demonstrate the effectiveness of our main results.

Example 1. For simplicity, consider a two-dimensional system (2a) and (2b) with probability transition matrix $\Pi = \begin{bmatrix} 0.6 & 0.3 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$, and the following matrix parameters:

$$A(1) = \begin{bmatrix} 1.2 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}, \qquad B(1) = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & -0.2 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}, \qquad D(1) = \begin{bmatrix} 0.2 & -0.2 \\ 0.2 & 0.2 \end{bmatrix},$$

$$E(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 1.5 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}, \qquad B(2) = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & 0.2 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}, \qquad D(2) = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{bmatrix},$$

$$E(2) = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix},$$
$$A(3) = \begin{bmatrix} 2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \qquad B(3) = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{bmatrix},$$
$$C(3) = \begin{bmatrix} 0.2 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \qquad D(3) = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.3 \end{bmatrix},$$
$$E(3) = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}. \tag{49}$$

In addition, the parameters for time delays are listed as $\tau_{1,1} = 6$, $\tau_{1,2} = 8$, $\tau_{1,3} = 9$, $\tau_{2,1} = 9$, $\tau_{2,2} = 8$, $\tau_{2,3} = 10$, and the initial mode of Markov chain is $r_0 = 1$, and the initial value of the system is $x(m) = (1.6969, -0.4770)^T$ for $m \in (-\infty, 0]$, which is stochastically produced by Matlab.

Also, the nonlinear functions are taken as

$$\begin{aligned} f(x) = g(x) = h(x) &= \big(0.1x_1 + 0.2x_2 + 0.4x_1 \sin x_2, 0.3x_1 \\ &\quad + 0.1x_2 + 0.4x_2 \cos x_1\big)^T. \end{aligned} \tag{50}$$

It can also be seen from (50) that

$$\breve{F}_1 = \begin{bmatrix} -0.0600 & 0.0500 \\ 0.0500 & -0.1100 \end{bmatrix}, \qquad \breve{F}_2 = \begin{bmatrix} 0.1000 & 0.3000 \\ 0.2000 & 0.1000 \end{bmatrix}. \tag{51}$$

With the previous parameters, based on Theorem 13 and by using Matlab LMI Toolbox, we solve the linear objective minimization problem (42) and obtain the feasible solutions for X_i, Y_1, Q_1, Q_2, β_0, and M_i (the values are omitted for space saving). Here, we just give the corresponding optimal control gain matrices $K(1) = -[1.2941, 0.3499]$, $K(2) = -[1.1552, 0.3687]$, $K(3) = -[1.7817, 0.2616]$, and the minimal upper bound $\gamma = 5.3853$ of the guaranteed cost. Moreover, the dynamical comparison between the unforced system and closed-loop system is shown in Figure 1.

5. Conclusions

In this paper, we have dealt with the guaranteed cost control problem for a class of nonlinear discrete-time systems with Markovian jumping parameters and mode-dependent mixed time delays. The sufficient conditions for the existence of guaranteed cost controllers are established for the system under consideration and related cost function. Furthermore, an LMI-based approach to design the optimal guaranteed cost controller has been formulated to minimize the guaranteed cost of the closed-loop system. A numerical example is also given to illustrate the effectiveness of the proposed methods. It will be interesting to extend the present results to more general cases, for example, the case where the quantized state-feedback is used for stabilization of the system concerned, and the case when the nonlinear stochastic systems are considered with missing measurements. And it would be one of the future research topics.

Acknowledgments

This work was supported in part by the International Travel Grant 2011/R3 sponsored by the Royal Society of the U.K., the National Natural Science Foundation of China under Grant 61074129, and the Natural Science Foundation of Jiangsu Province of China under Grant BK2012682.

References

[1] Y.-Y. Cao, J. Lam, and L. Hu, "Delay-dependent stochastic stability and H_∞ analysis for time-delay systems with Markovian jumping parameters," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 340, no. 6-7, pp. 423–434, 2003.

[2] S. S. L. Chang and T. K. C. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters," *IEEE Transactions on Automatic Control*, vol. 17, no. 4, pp. 474–483, 1972.

[3] D. Ding, Z. Wang, B. Shen, and H. Shu, "H_∞ state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 725–736, 2012.

[4] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed $\mathscr{H}_\infty$ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[5] H. Dong, Z. Wang, and H. Gao, "Distributed filtering for a class of time-varying systems over sensor networks with quantization errors and successive packet dropouts," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 3164–3173, 2012.

[6] H. Dong, Z. Wang, and H. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Transactions on Circuits and Systems. I. Regular Papers*, vol. 59, no. 10, pp. 2354–2362, 2012.

[7] J. Hu, Z. Wang, B. Shen, and H. Gao, "Gain-constrained recursive filtering with stochastic nonlinearities and probabilistic sensor delays," *IEEE Transactions on Signal Processing*, vol. 61, no. 5, pp. 1230–1238, 2013.

[8] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3008–3015, 2012.

[9] Z. Lin and G. Duan, "Robust guaranteed cost control for discrete-time uncertain systems with delay," *IEE Proceedings—Control Theory and Application*, vol. 146, no. 6, pp. 598–602, 1999.

[10] Y. Liu, Z. Wang, and X. Liu, "Robust H_∞ control for a class of nonlinear stochastic systems with mixed time delay," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 16, pp. 1525–1551, 2007.

[11] B. Shen, Z. Wang, and X. Liu, "Sampled-data synchronization control of dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[12] Z. Wang, D. W. C. Ho, H. Dong, and H. Gao, "Robust $\mathscr{H}_\infty$ finite-horizon control for a class of stochastic nonlinear time-varying systems subject to sensor and actuator saturations," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1716–1722, 2010.

[13] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H_∞ control for networked systems with random packet losses," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 37, no. 4, pp. 916–924, 2007.

[14] Z. Wang, B. Shen, H. Shu, and G. Wei, "Quantized H_∞ control for nonlinear stochastic time-delay systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1431–1444, 2012.

[15] L. Yu and J. Chu, "An LMI approach to guaranteed cost control of linear uncertain time-delay systems," *Automatica*, vol. 35, no. 6, pp. 1155–1159, 1999.

[16] E. K. Boukas and Z. K. Liu, "Robust stability and stabilizability of Markov jump linear uncertain systems with mode-dependent time delays," *Journal of Optimization Theory and Applications*, vol. 109, no. 3, pp. 587–600, 2001.

[17] E.-K. Boukas and Z. Liu, *Deterministic and Stochastic Time-Delay Systems*, Springer, New York, NY, USA, 2002.

[18] I. R. Petersen and D. C. McFarlane, "Optimal guaranteed cost control and filtering for uncertain linear systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1971–1977, 1994.

[19] S. O. R. Moheimani and I. R. Petersen, "Optimal quadratic guaranteed cost control of a class of uncertain time-delay systems," *IEE Proceedings—Control Theory and Applications*, vol. 144, no. 2, pp. 183–188, 1997.

[20] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1994.

[21] W.-H. Chen, J.-X. Xu, and Z.-H. Guan, "Guaranteed cost control for uncertain Markovian jump systems with mode-dependent time-delays," *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2270–2277, 2003.

[22] Z. Zuo, Z. Jia, Y. Wang, H. Zhao, and G. Zhang, "Guaranteed cost control for discrete-time uncertain systems with saturating actuators," in *American Control Conference Westin Seattle Hotel*, Seattle, Wash, USA, June 2008.

[23] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Probability-guaranteed H_∞ finite-horizon filtering for a class of nonlinear time-varying systems with sensor saturations," *Systems & Control Letters*, vol. 61, no. 4, pp. 477–484, 2012.

[24] J. Hu, Z. Wang, B. Shen, and H. Gao, "Quantized recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements," *International Journal of Control*, vol. 86, no. 4, pp. 650–663, 2013.

[25] Y. Liu, Z. Wang, J. Liang, and X. Liu, "Synchronization and state estimation for discrete-time complex networks with distributed delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 38, no. 5, pp. 1314–1325, 2008.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 420648, 10 pages
http://dx.doi.org/10.1155/2013/420648

Research Article

Convergence Rate of Numerical Solutions for Nonlinear Stochastic Pantograph Equations with Markovian Switching and Jumps

Zhenyu Lu,[1] Tingya Yang,[2] Yanhan Hu,[3] and Junhao Hu[3]

[1] *College of Electrical and Information Engineering, Nanjing University of Information Science & Technology, Nanjing 210044, China*
[2] *Jiangsu Meteorological Observatory, Nanjing 210008, China*
[3] *College of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China*

Correspondence should be addressed to Junhao Hu; junhaomath@163.com

Received 8 April 2013; Accepted 5 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 Zhenyu Lu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The sufficient conditions of existence and uniqueness of the solutions for nonlinear stochastic pantograph equations with Markovian switching and jumps are given. It is proved that Euler-Maruyama scheme for nonlinear stochastic pantograph equations with Markovian switching and Brownian motion is of convergence with strong order 1/2. For nonlinear stochastic pantograph equations with Markovian switching and pure jumps, it is best to use the mean-square convergence, and the order of mean-square convergence is close to 1/2.

1. Introduction

Stochastic modelling has been used with great success in a variety of application areas, including control theory, biology, epidemiology, mechanic, and neural networks, economics, and finance [1–5]. In general, stochastic different equations do not have explicit solutions. Therefore, approximate schemes for stochastic differential equations with Markovian switching and Poisson jumps have been investigated by many authors [3, 6, 7]. The convergence results of numerical solutions of stochastic differential equations with Markovian switching and Poisson jumps under the Lipschitz condition and the linear growth condition are obtained by using Euler-Maruyama scheme or semi-implicit Euler scheme. However, recently, more and more convergence results have been given under weaker conditions than the Lipschitz condition and the linear growth condition. Gyöngy and Rásonyi [8] revealed the convergence rate of Euler approximations for stochastic differential equations whose diffusion coefficient is not Lipschitz but only $(1/2 + \alpha)$-Hölder continuous for some $\alpha > 0$. Mao et al. [9] discussed L^1 and L^2-convergence of the Euler-Maruyama scheme for stochastic differential equations with Markovian switching under non-Lipschitz coefficients. Wu et al. [10] proved existence of the nonnegative and the strong convergence of the Euler-Maruyama Scheme for the Cox-Ingersoll-Ross model with delay whose diffusion coefficient is nonlinear and non-Lipschitz continuous. Bao and Yuan [11] studied the convergence rate for stochastic differential delay equations whose coefficients may be highly nonlinear with respect to the delay variable.

So far, the research of the numerical solutions for stochastic pantograph equations has just begun [12–15]. Fan et al. [12] gave the strong convergence for stochastic pantograph equations under the Lipschitz condition and the linear growth condition. Ronghua et al. [14] proved that the Euler approximation solution converges to the analytic solution in probability under weaker conditions, but the convergence rate has not been given.

In this paper, we will study the convergence rate for nonlinear stochastic pantograph equations with Markovian switching and Poisson jump under weaker conditions than the Lipschitz condition and the linear growth condition. The rest of the paper is organized as follows. In Section 2, we will give the existence and uniqueness of the analytic solutions

for Markovian switching and Brownian motion case and also reveal that the convergence order of Euler-Maruyama scheme is 1/2. In Section 3, we show that it is best to use the mean-square convergence for Markovian switching and the pure jump case and that the rate of mean-square convergence is close to 1/2.

2. Convergence Rate for Markovian Switching and Brownian Motion Case

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W(t)$ be an m-dimensional Brownian motion defined on the probability space adapted to the filtration. For integer $n > 0$, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ be the Euclidean space and $\|A\| := \sqrt{\operatorname{trace}(A^*A)}$ the Hilbert-Schmidt norm for a matrix A, where A^* is its transpose. Throughout this paper, $C > 0$ denotes a generic constant whose values may change from lines to lines.

Let $r(t)$, $t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + r_{ij}\delta + o(\delta) & \text{if } i = j, \end{cases} \tag{1}$$

where $\delta > 0$. Here $\gamma_{ij} > 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}. \tag{2}$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right continuous step function with finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, +\infty)$.

For fixed $T > 0$, we consider the stochastic pantograph equation with Markovian switching of the form

$$\begin{aligned} \mathrm{d}X(t) &= b(X(t), X(qt), r(t))\,\mathrm{d}t \\ &\quad + \sigma(X(t), X(qt), r(t))\,\mathrm{d}W(t), \qquad t \in [t_0, T], \end{aligned} \tag{3}$$

with initial data $X(\theta) = \xi(\theta)$, $r(\theta) = r_0$, $\theta \in [qt_0, t_0]$, $0 < t_0$, $0 < q < 1$. $r(t)$ is a Markov chain. On the time interval $[t_0, T]$, let $q < \epsilon < \min\{1, ((T+1)/T)q\}$, and we define the partition

$$\begin{aligned} &0 < t_0 < \frac{t_0}{q}\epsilon < \frac{t_0}{q^2}\epsilon^2 < \frac{t_0}{q^3}\epsilon^3 \\ &\quad < \cdots < \frac{t_0}{q^{n-1}}\epsilon^{n-1} < \frac{t_0}{q^n}\epsilon^n, \quad n = \left[\log_{\epsilon/q}\frac{T}{t_0}\right] + 1, \\ &0 < \Delta_i = \frac{t_0}{q^i}\epsilon^i - \frac{t_0}{q^{i-1}}\epsilon^{i-1} = \frac{t_0}{q^i}\epsilon^i\left(1 - \frac{q}{\epsilon}\right) \\ &\quad < \frac{\epsilon}{q}T\left(1 - \frac{q}{\epsilon}\right) = \left(\frac{\epsilon}{q} - 1\right)T < 1. \end{aligned} \tag{4}$$

The integral version of (3) is given by the following:

$$\begin{aligned} X(t) &= \xi(\theta) + \int_{t_0}^{t} b(X(s), X(qs), r(s))\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t} \sigma(X(s), X(qs), r(s))\,\mathrm{d}W(s). \end{aligned} \tag{5}$$

To guarantee the existence and uniqueness of the solutions of (3) we introduce the following conditions:

(A1) $b : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ and there exists $L_1 > 0$ such that

$$\begin{aligned} &|b(x_1, y_1, j) - b(x_2, y_2, j)| \\ &\quad \leq L_1|x_1 - x_2| + V_1(y_1, y_2)|y_1 - y_2| \end{aligned} \tag{6}$$

for $x_i, y_i \in \mathbb{R}^n$, $i = 1, 2$, $j \in S$;

(A2) $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n\times m}$ and there exists $L_2 > 0$ such that

$$\begin{aligned} &\|\sigma(x_1, y_1, j) - \sigma(x_2, y_2, j)\| \\ &\quad \leq L_2|x_1 - x_2| + V_2(y_1, y_2)|y_1 - y_2| \end{aligned} \tag{7}$$

for $x_i, y_i \in \mathbb{R}^n$, $i = 1, 2$, $j \in S$,

where $V_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$V_i(x, y) \leq K_i\left(1 + |x|^{q_i} + |y|^{q_i}\right), \qquad i = 1, 2 \tag{8}$$

for some $K_i > 0$, $q_i \geq 1$ and arbitrary $x, y \in \mathbb{R}^n$.

Remark 1. From (A1)-(A2), we know that the coefficients of (3) are much weaker than those of the Lipschitz condition and the linear growth condition. In many examples, b and σ do not satisfy the Lipschitz condition or the linear growth condition but can be covered by (A1)-(A2).

Lemma 2. *Assume that (A1) and (A2) hold. Then, for any initial data $\xi \in C^b_{\mathcal{F}_0}([qt_0, t_0]; \mathbb{R}^n)$ and $r(0) = r_0 \in S$, $X(t)$ is a unique global strong solution of* (3). *Moreover, for any $p \geq 2$ there exists $C > 0$ such that*

$$\mathbb{E}\left(\sup_{t_0\leq t\leq T}|X(t)|^p\right) \leq C. \tag{9}$$

Proof. From (A1) and (A2), b and σ are locally Lipschitzian. So, (3) has a unique local solution [3]. In order to verify that (3) has a unique global solution on time interval $[t_0, T]$, it is sufficient to show that

$$\mathbb{E}\left(\sup_{t_0\leq t\leq T}|X(t)|^p\right) \leq C, \quad p \geq 2. \tag{10}$$

From (A1), (A2), and (8), we can obtain

$$|b(x, y, i)| \leq C\left(1 + |x| + |y| + |y|^{q_1+1}\right), \qquad x, y \in \mathbb{R}^n, \tag{11}$$

$$\|\sigma(x, y, i)\| \leq C\left(1 + |x| + |y| + |y|^{q_2+1}\right), \qquad x, y \in \mathbb{R}^n. \tag{12}$$

Substituting (11) and (12) into (5) and by the Hölder inequality and the Burkhold-Davis-Gundy inequality, we have that for any $p \geq 2$ and $t \in [t_0, T]$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|X(s)|^p\right)\\
&\quad\leq 3^{p-1}\left\{\left|\xi(\theta)\right|^p\right.\\
&\qquad+\mathbb{E}\left(\sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s} b(X(\alpha), X(q\alpha), r(\alpha))\,\mathrm{d}\alpha\right|^p\right)\\
&\qquad\left.+\mathbb{E}\left(\sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s}\sigma(X(\alpha), X(q\alpha), r(\alpha))\,\mathrm{d}W(\alpha)\right|^p\right)\right\}\\
&\quad\leq C\left\{1+\mathbb{E}\int_{t_0}^{t}\left(\left|b(X(s), X(qs), r(s))\right|^p\right.\right.\\
&\qquad\left.\left.+\left\|\sigma(X(s), X(qs), r(s))\right\|^p\right)\mathrm{d}s\right\}\\
&\quad\leq C\left\{1+\mathbb{E}\int_{t_0}^{t}|X(s)|^p\mathrm{d}s\right.\\
&\qquad\left.+\mathbb{E}\int_{t_0}^{t}\left(\left|X(qs)\right|^{p(q_1+1)}+\left|X(qs)\right|^{p(q_2+1)}\right)\mathrm{d}s\right\}.
\end{aligned}\tag{13}$$

Let $\beta := (q_1+1)\vee(q_2+1)$; then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|X(s)|^p\right)\\
&\quad\leq C\left\{1+\mathbb{E}\int_{t_0}^{t}|X(s)|^p\mathrm{d}s+\mathbb{E}\int_{t_0}^{t}\left|X(qs)\right|^{p\beta}\mathrm{d}s\right\}.
\end{aligned}\tag{14}$$

By virtue of the Gronwall inequality, we get

$$\mathbb{E}\left(\sup_{t_0\leq s\leq t}|X(s)|^p\right)\leq C\left\{1+\mathbb{E}\int_{t_0}^{t}\left|X(qs)\right|^{p\beta}\mathrm{d}s\right\}.\tag{15}$$

Let

$$\begin{aligned}
p_i &:= \left(\left[\log_{\epsilon/q}\frac{T}{t_0}\right]+2-i\right)p\beta^{[\log_{\epsilon/q}(T/t_0)]+1-i},\\
&\qquad i=1,2,\ldots,\left[\log_{\epsilon/q}\frac{T}{t_0}\right]+1,
\end{aligned}\tag{16}$$

where $[a]$ denotes the integer part of real number a; thus, for $\beta \geq 1$ and $p \geq 2$, we have

$$\begin{aligned}
&p_{i+1}\beta < p_i,\quad p_{[\log_{\epsilon/q}(T/t_0)]+1}=p,\\
&\qquad i=1,2,\ldots,\left[\log_{\epsilon/q}\frac{T}{t_0}\right].
\end{aligned}\tag{17}$$

Together with $\xi \in C^b_{\mathcal{F}_0}([qt_0, t_0]; \mathbb{R}^n)$ and $\epsilon < 1$, we obtain that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq (t_0/q)\epsilon}|X(s)|^{p_1}\right)\\
&\quad\leq C\left\{1+\mathbb{E}\int_{t_0}^{(t_0/q)\epsilon}\left|X(qs)\right|^{p_1\beta}\mathrm{d}s\right\}\\
&\quad\leq C\left\{1+\mathbb{E}\int_{qt_0}^{t_0}|X(s)|^{p_1\beta}\mathrm{d}s\right\}\\
&\quad\leq C.
\end{aligned}\tag{18}$$

In the similar way, combining (15) with the Hölder inequality further leads to

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq (t_0/q^2)\epsilon^2}|X(s)|^{p_2}\right)\\
&\quad\leq C\left\{1+\mathbb{E}\int_{t_0}^{(t_0/q^2)\epsilon^2}\left|X(qs)\right|^{p_2\beta}\mathrm{d}s\right\}\\
&\quad\leq C\left\{1+\int_{t_0}^{(t_0/q)\epsilon}\left(\mathbb{E}|X(s)|^{p_1}\right)^{p_2\beta/p_1}\mathrm{d}s\right\}\\
&\quad\leq C.
\end{aligned}\tag{19}$$

Repeating the previous procedures we then get (9). So the existence and uniqueness have been proved. □

In the following, we define the Euler-Maruyama based computational method. The method makes use of the following lemma.

Lemma 3. *Given $\Delta > 0$, then $\{r(k\Delta), k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(\Delta) = \left(P_{i,j}(\Delta)\right)_{N\times N} = e^{\Delta\Gamma}.\tag{20}$$

Given a fixed step size $\Delta > 0$ and the one-step transition probability matrix $P(\Delta)$ in (20), the discrete Markov chain $\{r(k\Delta), k = 0, 1, 2, \ldots\}$ can be simulated as follows: let $r(0) = i_0$, and compute a pseudorandom number ξ_1 from the uniform $(0, 1)$ distribution.

Define

$$r(\Delta) = \begin{cases} i, & i\in S-\{N\}\ \text{ such that} \displaystyle\sum_{j=1}^{i-1}P_{r(0),j}(\Delta)\leq\xi_1\\ & \qquad\qquad < \displaystyle\sum_{j=1}^{i}P_{r(0),j}(\Delta),\\ N, & \displaystyle\sum_{j=1}^{N-1}P_{r(0),j}(\Delta)\leq\xi_1,\end{cases}\tag{21}$$

where we set $\sum_{j=1}^{0} P_{r(0),j}(\Delta) = 0$ as usual. Having computed $r(0), r(\Delta), \ldots, r(k\Delta)$, we can compute $r((k+1)\Delta)$ by drawing a uniform $(0,1)$ pseudo-random number ξ_{k+1} and setting

$$r((k+1)\Delta) = \begin{cases} i, & i \in S - \{N\} \text{ such that } \sum_{j=1}^{i-1} P_{r(k\Delta),j}(\Delta) \\ & \qquad \le \xi_{k+1} < \sum_{j=1}^{i} P_{r(k\Delta),j}(\Delta), \\ N, & \sum_{j=1}^{N-1} P_{r(k\Delta),j}(\Delta) \le \xi_{k+1}. \end{cases} \tag{22}$$

The procedure can be carried out independently to obtain more trajectories.

Define the Euler-Maruyama approximation for (3) by

$$\begin{aligned} dY(t) &= b\left(\overline{Y}(t), \overline{Y}(qt), \overline{r}(t)\right) dt \\ &\quad + \sigma\left(\overline{Y}(t), \overline{Y}(qt), \overline{r}(t)\right) dW(t), \qquad t \in [t_0, T], \end{aligned} \tag{23}$$

where $\overline{Y}(t) := Y(t_i)$, $\overline{r}(t) := r(t_i)$ for $t \in [t_i, t_{i+1})$, $i = 0, 1, \ldots, [\log_{\epsilon/q}(T/t_0)]$, which

$$\begin{aligned} t_0 < \frac{t_0}{q}\epsilon = t_1 < \frac{t_0}{q^2}\epsilon^2 = t_2 < \cdots < \frac{t_0}{q^{n-1}}\epsilon^{n-1} = t_{n-1} \\ < \frac{t_0}{q^n}\epsilon^n = t_n \end{aligned} \tag{24}$$

and $\overline{Y}(\theta) = \xi(\theta)$, $\overline{r}(\theta) = r_0$, $\theta \in [qt_0, t_0]$.

By using the method of Lemma 2, we obtain

$$\mathbb{E}\left(\sup_{t_0 \le t \le T} |Y(t)|^p\right) \le C, \tag{25}$$

$$\mathbb{E}\left|Y(t) - \overline{Y}(t)\right|^p \le C\Delta^{p/2}, \qquad t \in [t_0, T], \tag{26}$$

where $\Delta = \max\{\Delta_1, \Delta_2, \ldots, \Delta_n\} := (\epsilon/q - 1)T$, $\Delta \to 0$, when $\epsilon \to q$.

Lemma 4. *If (A1) and (A2) hold, then*

$$\begin{aligned} &\mathbb{E}\int_{t_0}^{T} \left|b\left(\overline{Y}(s), \overline{Y}(qs), r(s)\right) - b\left(\overline{Y}(s), \overline{Y}(qs), \overline{r}(s)\right)\right|^p ds \le C\Delta, \\ &\mathbb{E}\int_{t_0}^{T} \left\|\sigma\left(\overline{Y}(s), \overline{Y}(qs), r(s)\right) - \sigma\left(\overline{Y}(s), \overline{Y}(qs), \overline{r}(s)\right)\right\|^p ds \le C\Delta. \end{aligned} \tag{27}$$

Proof. Let $n = [\log_{\epsilon/q}(T/t_0)] + 1$, then

$$\begin{aligned} &\mathbb{E}\int_{t_0}^{T} \left|b\left(\overline{Y}(s), \overline{Y}(qs), r(s)\right) - b\left(\overline{Y}(s), \overline{Y}(qs), \overline{r}(s)\right)\right|^p ds \\ &= \sum_{i=0}^{n-1} \mathbb{E}\int_{t_i}^{t_{i+1}} \left|b\left(\overline{Y}(s), \overline{Y}(qs), r(s)\right) - b\left(\overline{Y}(s), \overline{Y}(qs), r(t_i)\right)\right|^p ds. \end{aligned} \tag{28}$$

By (11), we compute

$$\begin{aligned} &\mathbb{E}\int_{t_i}^{t_{i+1}} \left|b\left(\overline{Y}(s), \overline{Y}(qs), r(s)\right) - b\left(\overline{Y}(s), \overline{Y}(qs), r(t_i)\right)\right|^p ds \\ &\le 2^{p-1}\mathbb{E}\int_{t_i}^{t_{i+1}} \left(\left|b\left(\overline{Y}(s), \overline{Y}(qs), r(s)\right)\right|^p + \left|b\left(\overline{Y}(s), \overline{Y}(qs), r(t_i)\right)\right|^p\right) \times I_{\{r(s) \ne r(t_i)\}} ds \\ &\le C\mathbb{E}\int_{t_i}^{t_{i+1}} \left(1 + \left|\overline{Y}(s)\right|^p + \left|\overline{Y}(qs)\right|^{p(q_1+1)}\right) I_{\{r(s) \ne r(t_i)\}} ds \\ &\le C\int_{t_i}^{t_{i+1}} \mathbb{E}\left[\mathbb{E}\left[\left(1 + \left|\overline{Y}(s)\right|^p + \left|\overline{Y}(qs)\right|^{p(q_1+1)}\right) I_{\{r(s) \ne r(t_i)\}} \left| r(t_i)\right.\right]\right] ds \\ &= C\int_{t_i}^{t_{i+1}} \mathbb{E}\left[\mathbb{E}\left[\left(1 + \left|\overline{Y}(s)\right|^p + \left|\overline{Y}(qs)\right|^{p(q_1+1)}\right) \left| r(t_i)\right.\right] \times \mathbb{E}\left[I_{\{r(s) \ne r(t_i)\}} \left| r(t_i)\right.\right]\right] ds. \end{aligned} \tag{29}$$

By the Markov property, we have

$$\begin{aligned} &\mathbb{E}\left[I_{\{r(s) \ne r(t_k)\}} \mid r(t_k)\right] \\ &= \sum_{i \in S} I_{\{r(t_k)=i\}} P\left(r(s) \ne i \mid r(t_k) = i\right) \\ &= \sum_{i \in S} I_{\{r(t_k)=i\}} \sum_{j \ne i} \left(\gamma_{ij}(s - t_k) + o(s - t_k)\right) \\ &\le \left(\max_{1 \le i \le n}(-\gamma_{ii})\Delta + o(\Delta)\right) \sum_{i \in S} I_{\{r(t_k)=i\}} \\ &\le C\Delta. \end{aligned} \tag{30}$$

Substituting the above inequality into (29) yields

$$\begin{aligned}\mathbb{E}\int_{t_i}^{t_{i+1}} &\left|b\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)-b\left(\overline{Y}(s),\overline{Y}(qs),r(t_i)\right)\right|^p \mathrm{d}s\\ &\le C\Delta\int_{t_i}^{t_{i+1}}\left[1+\left|\overline{Y}(t_i)\right|^p+\left|\overline{Y}(qt_i)\right|^{p(q_1+1)}\right]\mathrm{d}s\\ &\le C\Delta.\end{aligned} \tag{31}$$

So, (28) becomes

$$\begin{aligned}\mathbb{E}\int_{t_0}^{T} &\left|b\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)-b\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\right|^p \mathrm{d}s\\ &\le C\Delta.\end{aligned} \tag{32}$$

Similarly, we also obtain that

$$\begin{aligned}\mathbb{E}\int_{t_0}^{T} &\left\|\sigma\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\right\|^p \mathrm{d}s\\ &\le C\Delta.\end{aligned} \tag{33}$$

The proof is complete. □

Theorem 5. *Under (A1) and (A2), for any $p \ge 2$ there exits $C > 0$ such that*

$$\mathbb{E}\left(\sup_{t_0\le t\le T}|X(t)-Y(t)|^p\right) \le C\Delta^{p/2}; \tag{34}$$

that is, the convergence order of Euler-Maruyama scheme (23) *is* 1/2.

Proof. Let $\delta > 1$ and $\varepsilon > 0$; then $\int_{\varepsilon/\delta}^{\varepsilon}(1/x)\mathrm{d}x = \ln x|_{\varepsilon/\delta}^{\varepsilon} = \ln\delta$, and there is a continuous nonnegative function $\psi_{\delta\varepsilon}(x)(x \ge 0)$, which is zero outside $[\varepsilon/\delta, \varepsilon]$, such that

$$\int_{\varepsilon/\delta}^{\varepsilon}\psi_{\delta\varepsilon}(x)\,\mathrm{d}x = 1, \qquad \psi_{\delta\varepsilon}(x) \le \frac{2}{x\ln\delta}, \quad x > 0. \tag{35}$$

Define

$$\begin{aligned}\phi_{\delta\varepsilon}(x) &:= \int_0^x\int_0^y \psi_{\delta\varepsilon}(z)\,\mathrm{d}z\mathrm{d}y, \quad x > 0,\\ V_{\delta\varepsilon}(x) &:= \phi_{\delta\varepsilon}(|x|), \quad x \in \mathbb{R}^n.\end{aligned} \tag{36}$$

For any $t \in [t_0, T]$, let

$$\begin{aligned}Z(t) &:= X(t) - Y(t),\\ \overline{Z}(t) &:= Y(t) - \overline{Y}(t),\\ \widetilde{Z}(t) &:= \left(X(t),\overline{Y}(t)\right) \in \mathbb{R}^{2n}.\end{aligned} \tag{37}$$

Using the Itô formula, we have

$$\begin{aligned}V_{\delta\varepsilon}&(Z(t))\\ &= \int_{t_0}^{t}\Big\langle (V_{\delta\varepsilon})_x(Z(s)), b(X(s),X(qs),r(s))\\ &\qquad -b\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\rangle\,\mathrm{d}s\\ &\quad + \frac{1}{2}\int_{t_0}^{t}\operatorname{trace}\Big\{\Big(\sigma(X(s),X(qs),r(s))\\ &\qquad -\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big)^*\\ &\qquad \times (V_{\delta\epsilon})_{xx}(Z(s))\\ &\qquad \times\Big(\sigma(X(s),X(qs),r(s))\\ &\qquad -\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big)\Big\}\,\mathrm{d}s\\ &\quad + \int_{t_0}^{t}\Big\langle (V_{\delta\varepsilon})_x(Z(s)), \sigma(X(s),X(qs),r(s))\\ &\qquad -\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\rangle\,\mathrm{d}W(s)\\ &:= I_1(t) + I_2(t) + I_3(t).\end{aligned} \tag{38}$$

By virtue of condition (A1), the Hölder inequality, and Lemma 4, we deduce that

$$\begin{aligned}\mathbb{E}&\left(\sup_{t_0\le s\le t}\left|I_1(s)\right|^p\right)\\ &\le (t-t_0)^{p-1}\mathbb{E}\int_{t_0}^{t}\Big|\Big\langle (V_{\delta\varepsilon})_x(Z(s)), b(X(s),X(qs),r(s))\\ &\qquad -b\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\rangle\Big|^p\mathrm{d}s\\ &\le (t-t_0)^{p-1}\mathbb{E}\int_{t_0}^{t}\Big|b(X(s),X(qs),r(s))\\ &\qquad -b\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big|^p\mathrm{d}s\\ &\le C\Delta^{p-1}\int_{t_0}^{t}\mathbb{E}\Big(\Big|b(X(s),X(qs),r(s))\\ &\qquad -b\left(\overline{Y}(s),\overline{Y}(qs),(s)\right)\Big|^p\\ &\qquad +\Big|b\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)\\ &\qquad -b\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big|^p\Big)\,\mathrm{d}s\\ &\le C\Delta^{p-1}\int_{t_0}^{t}\Big\{\mathbb{E}|Z(s)|^p + \left(\mathbb{E}V_1^{2p}\left(\widetilde{Z}(qs)\right)\right)^{1/2}\\ &\qquad \times\left(\mathbb{E}\left|Z(qs)\right|^{2p}\right)^{1/2} + \mathbb{E}\left|\overline{Z}(s)\right|^p\end{aligned}$$

$$+\left(\mathbb{E}V_1^{2p}\left(\widetilde{Z}(qs)\right)\right)^{1/2}$$
$$\times\left(\mathbb{E}\left|\overline{Z}(qs)\right|^{2p}\right)^{1/2}\Bigg\}\,\mathrm{d}s+C\Delta^p. \tag{39}$$

By the Hölder inequality and (A2), we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le t}\left|I_2(s)\right|^p\right)\\
&\le\frac{1}{2}(t-t_0)^{p-1}\mathbb{E}\\
&\quad\times\int_{t_0}^{t}\Big|\operatorname{trace}\Big\{\Big(\sigma(X(s),X(qs),r(s))\\
&\qquad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big)^*\\
&\qquad\times(V_{\delta\varepsilon})_{xx}(Z(s))\\
&\qquad\times\Big(\sigma(X(s),X(qs),r(s))\\
&\qquad\quad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big)\Big\}\Big|^p\mathrm{d}s\\
&\le C\Delta^{p-1}\int_{t_0}^{t}\mathbb{E}\Big\{\left\|(V_{\delta\varepsilon})_{xx}(Z(s))\right\|\\
&\qquad\times\Big\|\sigma(X(s),X(qs),r(s))\\
&\qquad\quad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\|^2\Big\}^p\mathrm{d}s\\
&\le C\Delta^{p-1}\mathbb{E}\int_{t_0}^{t}\frac{1}{|Z(s)|^p}\\
&\qquad\times\Big\{\Big\|\sigma(X(s),X(qs),r(s))\\
&\qquad\quad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)\Big\|^{2p}\\
&\qquad\quad+\Big\|\sigma\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)\\
&\qquad\qquad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\|^{2p}\Big\}\\
&\le C\Delta^{p-1}\int_{t_0}^{t}\Big\{\mathbb{E}|Z(s)|^p\\
&\qquad+\frac{1}{\varepsilon^p}\left(\mathbb{E}V_2^{4p}\left(\widetilde{Z}(qs)\right)\right)^{1/2}\left(\mathbb{E}|Z(qs)|^{4p}\right)^{1/2}\\
&\qquad+\frac{1}{\varepsilon^p}\mathbb{E}\left|\overline{Z}(s)\right|^{2p}+\frac{1}{\varepsilon^p}\left(\mathbb{E}V_2^{4p}\left(\widetilde{Z}(qs)\right)\right)^{1/2}\\
&\qquad\times\left(\mathbb{E}\left|\overline{Z}(qs)\right|^{4p}\right)^{1/2}\Big\}\,\mathrm{d}s+\frac{C\Delta^p}{\varepsilon^p}.
\end{aligned}\tag{40}$$

Making use of the Burkhold-Davis-Gundy inequality yields

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le t}\left|I_3(s)\right|^p\right)\\
&\le C\Delta^{p/2-1}\mathbb{E}\int_{t_0}^{t}\Big\|\sigma(X(s),X(qs),r(s))\\
&\qquad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\|^p\mathrm{d}s\\
&\le C\Delta^{p/2-1}\mathbb{E}\int_{t_0}^{t}\Big(\Big\|\sigma(X(s),X(qs),r(s))\\
&\qquad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)\Big\|^p\\
&\qquad+\Big\|\sigma(X(s),X(qs),r(s))\\
&\qquad\quad-\sigma\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\Big\|^p\Big)\mathrm{d}s\\
&\le C\Delta^{p/2-1}\int_{t_0}^{t}\Big\{\mathbb{E}|Z(s)|^p+\left(\mathbb{E}V_2^{2p}\left(\widetilde{Z}(qs)\right)\right)^{1/2}\\
&\qquad\times\left(\mathbb{E}|Z(qs)|^{2p}\right)^{1/2}\\
&\qquad+\mathbb{E}\left|\overline{Z}(s)\right|^p+\left(\mathbb{E}V_2^{2p}\left(\widetilde{Z}(qs)\right)\right)^{1/2}\\
&\qquad\times\left(\mathbb{E}\left|\overline{Z}(qs)\right|^{2p}\right)^{1/2}\Big\}\,\mathrm{d}s+C\Delta^{p/2}.
\end{aligned}\tag{41}$$

Moreover, by (8), (9), and (25), we have

$$\begin{gathered}
\mathbb{E}V_1^{2p}\left(\widetilde{Z}(qs)\right)\vee\mathbb{E}V_2^{4p}\left(\widetilde{Z}(qs)\right)\le C,\\
\mathbb{E}\left|\overline{Z}(t)\right|^p\le C\Delta^{p/2}.
\end{gathered}\tag{42}$$

Thus, combing (39), and (40) with (41), for any $t\in[t_0,T]$ and $p\ge 2$, we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le t}|Z(s)|^p\right)\\
&\le 2^{p-1}\left\{\varepsilon^p+\mathbb{E}\left(\sup_{t_0\le s\le t}V_{\delta\varepsilon}^p(Z(s))\right)\right\}\\
&\le C\Big\{\varepsilon^p+\Delta^{3p/2-1}+\Delta^p+\frac{\Delta^{2p-1}}{\varepsilon^p}+\frac{\Delta^p}{\varepsilon^p}+\Delta^{p-1}+\Delta^{p/2}\\
&\qquad+\Delta^{p/2-1}\Big\{\int_{t_0}^{t}\mathbb{E}|Z(s)|^p\mathrm{d}s+\int_{t_0}^{t}\left(\mathbb{E}|Z(qs)|^{2p}\right)^{1/2}\mathrm{d}s\\
&\qquad\quad+\frac{1}{\varepsilon^p}\int_{t_0}^{t}\left(\mathbb{E}|Z(qs)|^{4p}\right)^{1/2}\mathrm{d}s\Big\}\Big\}.
\end{aligned}\tag{43}$$

Let $\varepsilon = \Delta^{1/2}$, and using the Gronwall inequality, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t_0\le s\le t}|Z(s)|^p\right) &\le C\left\{\Delta^{p/2}+\Delta^{(p/2)-1}\int_{t_0}^{t}\left(\mathbb{E}|Z(qs)|^{2p}\right)^{1/2}\mathrm{d}s\right.\\ &\quad\left.+\Delta^{-1}\int_{t_0}^{t}\left(\mathbb{E}|Z(qs)|^{4p}\right)^{1/2}\mathrm{d}s\right\}.\end{aligned} \tag{44}$$

Let

$$\begin{aligned}p_i &:= \left(\left[\log_{\epsilon/q}\frac{T}{t_0}\right]+2-i\right)p4^{[\log_{\epsilon/q}(T/t_0)]+1-i},\\ &\qquad i=1,2,\ldots,\left[\log_{\epsilon/q}\frac{T}{t_0}\right]+1;\end{aligned} \tag{45}$$

by $p\ge 2$, it is easy to see that $p_i\ge 2$ such that

$$\begin{aligned}&4p_{i+1}<p_i,\qquad p_{[\log_{\epsilon/q}(T/t_0)]+1}=p,\\ &\qquad i=1,2,\ldots,\left[\log_{\epsilon/q}\frac{T}{t_0}\right].\end{aligned} \tag{46}$$

Noting that $Z(s)=0$ for $s\in[qt_0,t_0]$ and substituting $\epsilon<1$ into (44) yields that

$$\begin{aligned}\mathbb{E}\left(\sup_{t_0\le s\le(t_0/q)\epsilon}|Z(s)|^{p_1}\right) &\le C\left\{\Delta^{p_1/2}+\Delta^{p_1/2-1}\int_{t_0}^{(t_0/q)\epsilon}\left(\mathbb{E}|Z(qs)|^{2p_1}\right)^{1/2}\mathrm{d}s\right.\\ &\quad\left.+\Delta^{-1}\int_{t_0}^{(t_0/q)\epsilon}\left(\mathbb{E}|Z(qs)|^{4p_1}\right)^{1/2}\mathrm{d}s\right\}\\ &\le C\left\{\Delta^{p_1/2}+\Delta^{p_1/2-1}\int_{qt_0}^{t_0}\left(\mathbb{E}|Z(s)|^{2p_1}\right)^{1/2}\mathrm{d}s\right.\\ &\quad\left.+\Delta^{-1}\int_{qt_0}^{t_0}\left(\mathbb{E}|Z(s)|^{4p_1}\right)^{1/2}\mathrm{d}s\right\}\\ &\le C\Delta^{p_1/2}.\end{aligned} \tag{47}$$

Using (46) and the Hölder inequality, further gives that

$$\begin{aligned}\mathbb{E}\left(\sup_{t_0\le s\le(t_0/q^2)\epsilon^2}|Z(s)|^{p_2}\right) &\le C\left\{\Delta^{p_2/2}+\Delta^{p_2/2-1}\int_{t_0}^{(t_0/q^2)\epsilon^2}\left(\mathbb{E}|Z(qs)|^{2p_2}\right)^{1/2}\mathrm{d}s\right.\\ &\quad\left.+\Delta^{-1}\int_{t_0}^{(t_0/q^2)\epsilon^2}\left(\mathbb{E}|Z(qs)|^{4p_2}\right)^{1/2}\mathrm{d}s\right\}\\ &\le C\left\{\Delta^{p_2/2}+\Delta^{p_2/2-1}\int_{t_0}^{(t_0/q)\epsilon}\left(\mathbb{E}|Z(s)|^{p_1}\right)^{p_2/p_1}\mathrm{d}s\right.\\ &\quad\left.+\Delta^{-1}\int_{t_0}^{(t_0/q)\epsilon}\left(\mathbb{E}|Z(s)|^{p_1}\right)^{2p_2/p_1}\mathrm{d}s\right\}\\ &\le C\Delta^{p_2/2}.\end{aligned} \tag{48}$$

Repeating the previous procedures, the desired result follows. □

In this section, under general conditions, we reveal that the convergence order of Euler-Maruyama scheme for stochastic pantograph equations with Markovian switching and Brownian motion is 1/2. In Section 3, we will discuss the convergence rate for stochastic pantograph equation with Markovian switching and pure jumps.

3. Convergence Rate for Markovian Switching and Pure Jumps Case

Let $\mathscr{B}(\mathbb{R})$ be the Borel σ-algebra on $\mathbb{R}$, and $\lambda(\mathrm{d}x)$ a σ-finite measure defined on $\mathscr{B}(\mathbb{R})$. Let $p=(p(t))$, $t\in D_p$, be a stationary $\mathscr{F}_t$-Poisson point process on $\mathbb{R}$ with characteristic measure $\lambda(\cdot)$. Denote by $N(\mathrm{d}t,\mathrm{d}u)$ the Poisson counting measure associated with p, that is, $N(t,U)=\sum_{s\in D_p,s\le t}I_U(p(s))$ for $U\in\mathscr{B}(\mathbb{R})$. Let $\widetilde{N}(\mathrm{d}t,\mathrm{d}u):=N(\mathrm{d}t,\mathrm{d}u)-\mathrm{d}t\lambda(\mathrm{d}u)$ be the compensated Poisson measure associated with $N(\mathrm{d}t,\mathrm{d}u)$. In what follows, we further assume that $\int_U|u|^p\lambda(u)<\infty$ for any $p\ge 2$.

In this section, we consider the following stochastic pantograph equation with Markovian switching and pure jumps on $\mathbb{R}^n$:

$$\begin{aligned}\mathrm{d}X(t) &= b(X(t),X(qt),r(t))\,\mathrm{d}t\\ &\quad+\int_U h(X(t),X(qt),u)\,\widetilde{N}(\mathrm{d}t,\mathrm{d}u),\qquad t\in[t_0,T]\end{aligned} \tag{49}$$

with initial data $X(\theta)=\xi(\theta)$ and $r(\theta)=r_0$, $\theta\in[qt_0,t_0]$.

We assume that

(A1) $b:\mathbb{R}^n\times\mathbb{R}^n\times S\to\mathbb{R}^n$ and there exists $L_1>0$ such that

$$\begin{aligned}&|b(x_1,y_1,j)-b(x_2,y_2,j)|\\ &\quad\le L_1|x_1-x_2|+V_1(y_1,y_2)|y_1-y_2|\end{aligned} \tag{50}$$

for $x_i,y_i\in\mathbb{R}^n$, $i=1,2$, $j\in S$;

(A3) $h:\mathbb{R}^n\times\mathbb{R}^n\times U\to\mathbb{R}^n$ and there exists $L_3>0$ such that

$$\begin{aligned}&|h(x_1,y_1,u)-h(x_2,y_2,u)|\\ &\quad\le(L_3|x_1-x_2|+V_3(y_1,y_2)|y_1-y_2|)|u|\end{aligned} \tag{51}$$

for $x_i, y_i \in \mathbb{R}^n, i = 1, 2$, and $u \in U$, where $V_3 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$V_3(x, y) \le K_3\left(1 + |x|^{q_3} + |y|^{q_3}\right) \tag{52}$$

for some $K_3 > 0$, $q_3 \ge 1$ and arbitrary $x, y \in \mathbb{R}^n$.

From (A3), the jump coefficient may be also highly nonlinear. We define the Euler-Maruyama scheme associated with (49) by

$$\begin{aligned} dY(t) &= b\left(\overline{Y}(t), \overline{Y}(qt), \overline{r}(t)\right) dt \\ &\quad + \int_U h\left(\overline{Y}(t), \overline{Y}(qt), u\right) \widetilde{N}(dt, du), \end{aligned} \tag{53}$$

where $\overline{Y}(t) := Y(t_i)$, $\overline{r}(t) := r(t_i)$ for $t \in [t_i, t_{i+1})$, $i = 0, 1, \ldots, n-1$, and $\overline{Y}(\theta) = \xi(\theta), \overline{r}(\theta) = r_0$, for $\theta \in [qt_0, t_0]$.

In order to state the main theorem, the following two lemmas are useful.

Lemma 6 (see [16]). *Let $\Phi : \mathbb{R}_+ \times U \to \mathbb{R}^n$ and assume that*

$$\int_{t_0}^t \int_U \mathbb{E}|\Phi(s, u)|^p \lambda(du)\, ds < \infty, \qquad t_0 > 0, \ p \ge 2. \tag{54}$$

Then there exists $D(p) > 0$ such that

$$\begin{aligned} &\mathbb{E}\left(\sup_{t_0 \le s \le t}\left|\int_{t_0}^s \int_U \Phi(r, u) \widetilde{N}(du, ds)\right|^p\right) \\ &\quad \le D(p)\left\{\mathbb{E}\left(\int_{t_0}^t \int_U |\Phi(s, u)|^2 \lambda(du)\, ds\right)^{p/2}\right. \\ &\qquad \left. + \mathbb{E}\int_{t_0}^t \int_U |\Phi(s, u)|^p \lambda(du)\, ds\right\}. \end{aligned} \tag{55}$$

Lemma 7. *Let (A1) and (A3) hold. Then (49) has a unique global solution $(X(t))_{t \in [t_0, T]}$. Moreover, for any $p \ge 2$ there exists $C > 0$ such that*

$$\mathbb{E}\left(\sup_{t_0 \le t \le T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{t_0 \le t \le T} |Y(t)|^p\right) \le C, \tag{56}$$

$$\mathbb{E}\left|Y(t) - \overline{Y}(t)\right|^p \le C\Delta. \tag{57}$$

Proof. The proof is very similar to that of Lemma 2 and (25). □

Now we present the main theorem in this section.

Theorem 8. *Let (A1) and (A3) hold. For any $p \ge 2$ and arbitrary $\theta, \alpha \in (0, 1)$, there exists $C > 0$, independent of Δ, such that*

$$\mathbb{E}\left(\sup_{t_0 \le t \le T} |X(t) - Y(t)|^p\right) \le C\Delta^{(1+\theta)^{1/[\log_{\varepsilon/q}(T/t_0)](1+\alpha)}}. \tag{58}$$

Proof. The proof is similar to that of Theorem 5. Set

$$\begin{aligned} Z(t) &:= X(t) - Y(t), \\ \overline{Z}(t) &:= Y(t) - \overline{Y}(t), \\ \widetilde{Z}(t) &:= \left(X(t), \overline{Y}(t)\right) \in \mathbb{R}^{2n}, \qquad t \in [t_0, T]. \end{aligned} \tag{59}$$

Define

$$\begin{aligned} \Gamma_1(t) &:= b(X(t), X(qt), r(t)) - b\left(\overline{Y}(t), \overline{Y}(qt), \overline{r}(t)\right), \\ \Gamma_2(t, u) &:= h(X(t), X(qt), u) - h\left(\overline{Y}(t), \overline{Y}(qt), u\right). \end{aligned} \tag{60}$$

Using the Itô formula and the Taylor expansion we have that for $t \in [t_0, T]$

$$\begin{aligned} V_{\delta\varepsilon}(Z(t)) &= \int_{t_0}^t \left\langle (V_{\delta\varepsilon})_x (Z(s)), \Gamma_1(s)\right\rangle ds \\ &\quad + \int_{t_0}^t \int_U \{V_{\delta\varepsilon}(Z(s) + \Gamma_2(s, u)) \\ &\qquad - V_{\delta\varepsilon}(Z(s)) \\ &\qquad - \left\langle (V_{\delta\varepsilon})_x (Z(s)), \Gamma_2(s, u)\right\rangle\} \\ &\qquad \times \lambda(du)\, ds \\ &\quad + \int_{t_0}^t \int_U \{V_{\delta\varepsilon}(Z(s) + \Gamma_2(s, u)) \\ &\qquad - V_{\delta\varepsilon}(Z(s))\} \widetilde{N}(du, ds) \\ &= \int_{t_0}^t \left\langle (V_{\delta\varepsilon})_x (Z(s)), \Gamma_1(s)\right\rangle ds \\ &\quad + \int_{t_0}^t \int_U \left\{\int_0^1 \left\langle (V_{\delta\varepsilon})_x (Z(s) + \theta\Gamma_2(s, u))\right.\right. \\ &\qquad \left.\left. - (V_{\delta\varepsilon})_x (Z(s)), \Gamma_2(s, u)\right\rangle d\theta\right\} \\ &\qquad \times \lambda(du)\, ds \\ &\quad + \int_{t_0}^t \int_U \left\{\int_0^1 \left\langle (V_{\delta\varepsilon})_x (Z(s) + \theta\Gamma_2(s, u)),\right.\right. \\ &\qquad \left.\left. \Gamma_2(s, u)\right\rangle d\theta\right\} \widetilde{N}(du, ds). \end{aligned} \tag{61}$$

By the property of $V_{\delta\varepsilon}(x)$, we deduce that

$$\begin{aligned} |Z(t)| &\le \varepsilon + V_{\delta\varepsilon}(Z(t)) \\ &\le \varepsilon + \int_{t_0}^t |\Gamma_1(s)|\, ds + 2\int_{t_0}^t \int_U |\Gamma_2(s, u)| \lambda(du)\, ds \end{aligned}$$

$$+\int_{t_0}^{t}\int_U\left\{\int_0^1\left\langle\left(V_{\delta\varepsilon}\right)_x\left(Z(s)+\theta\Gamma_2(s,u)\right),\right.\right.$$
$$\left.\left.\Gamma_2(s,u)\right\rangle\mathrm{d}\theta\right\}\widetilde{N}(\mathrm{d}u,\mathrm{d}s),$$
$$t\in[t_0,T]. \tag{62}$$

From (8), (52), and (56), we compute that for any $p\geq 2$

$$\mathbb{E}\left(\sup_{t_0\leq t\leq T}V_1^p\left(\widetilde{Z}(qs)\right)\right)\vee\mathbb{E}\left(\sup_{t_0\leq t\leq T}V_3^p\left(\widetilde{Z}(qs)\right)\right)\leq C. \tag{63}$$

Applying Lemma 6, Lemma 4, (57), and the Hölder inequality, we obtain that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq t}|Z(s)|^p\right)\\
&\leq 2^{p-1}\left\{\varepsilon^p+\mathbb{E}\left(\sup_{t_0\leq s\leq t}V_{\delta\varepsilon}^p(Z(s))\right)\right\}\\
&\leq C\left\{\varepsilon^p+\int_{t_0}^{t}\mathbb{E}\left|\Gamma_1(s)\right|^p\mathrm{d}s\right.\\
&\quad+\int_{t_0}^{t}\int_U\mathbb{E}\left|\Gamma_2(s,u)\right|^p\lambda(\mathrm{d}u)\,\mathrm{d}s\\
&\quad\left.+\mathbb{E}\left(\int_{t_0}^{t}\int_U\left|\Gamma_2(s,u)\right|^2\lambda(\mathrm{d}u)\,\mathrm{d}s\right)^{p/2}\right\}\\
&\leq C\left\{\varepsilon^p+\int_{t_0}^{t}\mathbb{E}\left|\Gamma_1(s)\right|^p\mathrm{d}s\right.\\
&\quad\left.+\int_{t_0}^{t}\int_U\mathbb{E}\left|\Gamma_2(s,u)\right|^p\lambda(\mathrm{d}u)\,\mathrm{d}s\right\}\\
&\leq C\left\{\varepsilon^p+\int_{t_0}^{t}\mathbb{E}\left(\left|X(s)-\overline{Y}(s)\right|\right.\right.\\
&\quad\left.+V_1\left(\widetilde{Z}(qs)\right)\left|X(qs)-\overline{Y}(qs)\right|\right)^p\mathrm{d}s\\
&\quad+\int_{t_0}^{t}\mathbb{E}\left|b\left(\overline{Y}(s),\overline{Y}(qs),r(s)\right)\right.\\
&\qquad\left.-b\left(\overline{Y}(s),\overline{Y}(qs),\overline{r}(s)\right)\right|^p\mathrm{d}s\\
&\quad+\int_{t_0}^{t}\mathbb{E}\left(\left|X(s)-\overline{Y}(s)\right|\right.\\
&\qquad\left.\left.+V_3\left(\widetilde{Z}(qs)\right)\left|X(qs)-\widetilde{Y}(qs)\right|\right)^p\mathrm{d}s\right\}\\
&\leq C\left\{\varepsilon^p+\Delta+\int_{t_0}^{t}\left\{\mathbb{E}|Z(s)|^p\right.\right.\\
&\qquad+\mathbb{E}\left(V_1^p\left(\widetilde{Z}(qs)\right)|Z(qs)|^p\right)\\
&\qquad+\mathbb{E}\left(V_1^p\left(\widetilde{Z}(qs)\right)\left|\overline{Z}(qs)\right|^p\right)\\
&\qquad+\mathbb{E}\left(V_3^p\left(\widetilde{Z}(qs)\right)|Z(qs)|^p\right)\\
&\qquad\left.\left.+\mathbb{E}\left(V_3^p\left(\widetilde{Z}(qs)\right)\left|\overline{Z}(qs)\right|^p\right)\right\}\mathrm{d}s\right\}\\
&\leq C\left\{\varepsilon^p+\Delta+\int_{t_0}^{t}\mathbb{E}|Z(s)|^p\mathrm{d}s+\int_{t_0}^{t}\mathbb{E}\left|Z(qs)\right|^p\mathrm{d}s\right\}.
\end{aligned} \tag{64}$$

Together with the Gronwall inequality and taking $\varepsilon=\Delta^{1/p}$, we get

$$\mathbb{E}\left(\sup_{t_0\leq s\leq t}|Z(s)|^p\right)\leq C\left\{\Delta+\int_{t_0}^{t}\mathbb{E}\left|Z(qs)\right|^p\mathrm{d}s\right\}. \tag{65}$$

For $\theta\in(0,1)$ and any $\alpha\in(0,1)$, let

$$\begin{aligned}
&p_i:=p(1+\theta)^{\left(\left[\log_{\epsilon/q}(T/t_0)\right]+1-i\right)(1+\alpha)},\\
&\qquad i=1,2,\ldots,\left[\log_{\epsilon/q}\frac{T}{t_0}\right]+1.
\end{aligned} \tag{66}$$

It is easy to see that

$$\begin{aligned}
&(1+\theta)p_{i+1}<p_i,\qquad p_{\left[\log_{\epsilon/q}(T/t_0)\right]+1}=p,\\
&\qquad i=1,2,\ldots,\left[\log_{\epsilon/q}\frac{T}{t_0}\right].
\end{aligned} \tag{67}$$

Noting that $Z(t)=\overline{Z}(t)=0$ for $t\in[qt_0,t_0]$, from (65) we obtain

$$\mathbb{E}\left(\sup_{t_0\leq s\leq(t_0/q)\epsilon}|Z(s)|^{p_1}\right)\leq C\left\{\Delta+\int_{qt_0}^{t_0}\mathbb{E}|Z(s)|^{p_1}\mathrm{d}s\right\}\leq C\Delta. \tag{68}$$

Then, together with (67) and the Hölder inequality, it further gives that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\leq s\leq(t_0/q^2)\,\epsilon^2}|Z(s)|^{p_2}\right)\\
&\leq C\left\{\Delta+\int_{t_0}^{(t_0/q)\varepsilon}\left(\mathbb{E}|Z(s)|^{p_2(1+\theta)}\right)^{1/(1+\theta)}\mathrm{d}s\right\}\\
&\leq C\left\{\Delta+\int_{t_0}^{(t_0/q)\varepsilon}\left(\mathbb{E}|Z(s)|^{p_1}\right)^{p_2/p_1}\mathrm{d}s\right\}\\
&\leq C\Delta^{p_2/p_1}.
\end{aligned} \tag{69}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le s\le (t_0/q^3)\epsilon^3}|Z(s)|^{p_3}\right)\\
&\quad\le C\left\{\Delta+\int_{t_0}^{(t_0/q^2)\epsilon^2}\left(\mathbb{E}|Z(s)|^{p_3(1+\theta)}\right)^{1/(1+\theta)}\mathrm{d}s\right\}\\
&\quad\le C\left\{\Delta+\int_{t_0}^{(t_0/q^2)\epsilon^2}\left(\mathbb{E}|Z(s)|^{p_2}\right)^{p_3/p_2}\mathrm{d}s\right\}\\
&\quad\le C\Delta^{p_3/p_1}.
\end{aligned}\tag{70}$$

Repeating the previous procedures, we have

$$\mathbb{E}\left(\sup_{t_0\le s\le T}|Z(s)|^{p}\right)\le C\Delta^{(1+\theta)^{1/[\log_{\epsilon/q}(T/t_0)](1+\alpha)}}.\tag{71}$$

The proof is complete. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (nos. 61104062 and 61174077), Jiangsu Qing Lan Project, and PAPD.

References

[1] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao, "Robust H_∞ filtering for Markovian jump systems with randomly occurring nonlinearities and sensor saturation: the finite-horizon case," *IEEE Transactions on Signal Processing*, vol. 59, no. 7, pp. 3048–3057, 2011.

[2] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, UK, Second edition, 2008.

[3] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, UK, 2006.

[4] Y. Shen and J. Wang, "An improved algebraic criterion for global exponential stability of recurrent neural networks with time-varying delays," *IEEE Transactions on Neural Networks*, vol. 19, no. 3, pp. 528–531, 2008.

[5] Z. Wang, Y. Liu, and X. Liu, "Exponential stabilization of a class of stochastic system with Markovian jump parameters and mode-dependent mixed time-delays," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1656–1662, 2010.

[6] J. Bao, B. Böttcher, X. Mao, and C. Yuan, "Convergence rate of numerical solutions to SFDEs with jumps," *Journal of Computational and Applied Mathematics*, vol. 236, no. 2, pp. 119–131, 2011.

[7] L. Ronghua and C. Zhaoguang, "Convergence of numerical solution to stochastic delay differential equation with Poisson jump and Markovian switching," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 451–463, 2007.

[8] I. Gyöngy and M. Rásonyi, "A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients," *Stochastic Processes and their Applications*, vol. 121, no. 10, pp. 2189–2200, 2011.

[9] X. Mao, C. Yuan, and G. Yin, "Approximations of Euler-Maruyama type for stochastic differential equations with Markovian switching, under non-Lipschitz conditions," *Journal of Computational and Applied Mathematics*, vol. 205, no. 2, pp. 936–948, 2007.

[10] F. Wu, X. Mao, and K. Chen, "The Cox-Ingersoll-Ross model with delay and strong convergence of its Euler-Maruyama approximate solutions," *Applied Numerical Mathematics*, vol. 59, no. 10, pp. 2641–2658, 2009.

[11] J. Bao and C. Yuan, "Convergence rate of EM scheme for SDDEs," http://arxiv.org/abs/1111.4130.

[12] Z. Fan, M. Liu, and W. Cao, "Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 1142–1159, 2007.

[13] Z. Fan, M. Song, and M. Liu, "The αth moment stability for the stochastic pantograph equation," *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 109–120, 2009.

[14] L. Ronghua, L. Min, and P. Wan-kai, "Convergence of numerical solutions to stochastic pantograph equations with Markovian switching," *Applied Mathematics and Computation*, vol. 215, no. 1, pp. 414–422, 2009.

[15] Y. Xiao and H. Y. Zhang, "A note on convergence of semi-implicit Euler methods for stochastic pantograph equations," *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1419–1424, 2010.

[16] C. Marinelli, C. Prévôt, and M. Röckner, "Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise," *Journal of Functional Analysis*, vol. 258, no. 2, pp. 616–649, 2010.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 158731, 9 pages
http://dx.doi.org/10.1155/2013/158731

Research Article

Distributed Consensus for Discrete-Time Directed Networks of Multiagents with Time-Delays and Random Communication Links

Yurong Liu,[1] Hamid Reza Karimi,[2] and Xiaohui Liu[3]

[1] *Department of Mathematics, Yangzhou University, Yangzhou 225002, China*
[2] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*
[3] *School of Computer Science and Technology, Nanjing University of Science and Technology, Nanjing 210094, China*

Correspondence should be addressed to Yurong Liu; liuyurong@gmail.com

Received 11 May 2013; Accepted 5 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 Yurong Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is concerned with the leader-following consensus problem in mean-square for a class of discrete-time multiagent systems. The multiagent systems under consideration are the directed and contain arbitrary discrete time-delays. The communication links are assumed to be time-varying and stochastic. It is also assumed that some agents in the network are well informed and act as leaders, and the others are followers. By introducing novel Lyapunov functionals and employing some new analytical techniques, sufficient conditions are derived to guarantee the leader-following consensus in mean-square for the concerned multiagent systems, so that all the agents are steered to an anticipated state target. A numerical example is presented to illustrate the main results.

1. Introduction

In recent years, the multiagent distributed coordination problem has attracted many researchers since it has broad applications in satellite formation flying, cooperative search of unmanned air vehicles, scheduling of automated highway systems, air traffic control, and distributed optimization of multiple mobile robotic systems. In many applications involving multiagent systems, one of the most fundamental problems is that groups of agents need to agree upon certain quantities of interest, which is called the consensus or agreement problem in the literature. Consensus problems have a long history in the field of computer science [1], many distributed control and estimation strategies are designed based on consensus algorithms [2–8], and consensus problems are used to model many different phenomena involving information flow among agents, including flocking, swarming, synchronization, distributed decision making, and schooling; see, for example, the survey paper [9]. Consensus problems for networked dynamic systems have been extensively studied in the last few years [10–12].

Usually, algebraic graph theory [13] acts as a good framework for analyzing consensus problems; see, for example, [10, 11, 14, 15]. In this framework, each agent is modeled as a vertex of a graph, and an edge of the graph joins node i to node j if agent j is receiving information from agent i. The models and algorithms for consensus have been recently reported by a number of investigators. In [16], Vicsek et al. proposed a simple discrete-time model to simulate a group of autonomous agents moving in the plane with the same speed but different headings. Vicsek's model in essence is a simplified version of the model introduced earlier by Reynolds [17]. Based on the algebraic graph theory [18], it has been shown that the network connectivity is a key factor in reaching consensus [11, 14, 15]. It has also been proved that consensus in a network with a dynamically changing topology can be reached if and only if the time-varying network topology contains a spanning tree frequently enough as the network evolves with time [11, 14]. Recently, stochastic-approximation-type algorithms with a decreasing step size are developed, and almost sure convergence is established for consensus seeking; see, for example, [19] and the references

therein. It has been recognized that time-delay is unavoidable in signal transmission and is also one of the main sources for causing instability and poor performances of systems [20–22]. Recently, the multiagent networks with time-delay have started to receive some initial attention [15, 23, 24].

On the other hand, in many multiagent systems, some agents are well informed and served as leaders, and the others track the leaders and act as followers. It was reported that the leader-following configuration is an energy saving mechanism [25] which was found in many biological systems, and it can also enhance the communication and orientation of the flock [26]. The leader-following consensus has been an active area of research [14, 27, 28]. Such a leader-following consensus problem is considered and proved in [14] that if all the agents were jointly connected with their leader, their states would converge to that of the leader as time goes on. Reference [28] studied a leader-following consensus problem for a multiagent system with a varying-velocity leader and time-varying delays, where the interaction graph among the followers was switching and balanced. Reference [27] investigated the leader-following consensus problem of higher-order multiagent systems. Unfortunately, so far, the delayed networks considered for the leader-following consensus problem are almost *continuous-time* multiagent systems, and the leader-following consensus problems for *discrete-time* multiagent systems with *time-delay* and *random communication links* have received little research attention. Hence, it is our intention in this paper to tackle such an important yet challenging problem.

In this paper, we will investigate the leader-following consensus problem for the *discrete-time* directed multiagent systems with time-delay and random communication links. By constructing new Lyapunov functionals and employing some analytical techniques, sufficient conditions for the leader-following consensus in mean-square are established for multiagent system, so that all the agents are steered to an anticipated state target. A numerical example is used to illustrate the proposed theory.

2. Problem Formulation

Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}_+$ stand for the natural numbers and the positive integer set, respectively; $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote, respectively, the set of real numbers, the n dimensional Euclidean space, and the set of all $n \times m$ real matrices. The superscript T represents the transpose for a matrix, and $|\cdot|$ may stand for any absolute value of real numbers or the standard Euclidean norm from the context. In an underlying probability space $(\mathbf{\Omega}, \mathscr{F}, \mathscr{P})$, $\mathbb{E}[\cdot]$ and $\mathrm{Var}[\cdot]$ denote, respectively, the *mean* and the *variance* for a random variable, and $\mathbb{E}[x \mid y]$ will mean the expectation of x conditional on y.

Consider n agents distributed according to a directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with a set of nodes $\mathscr{V} = \{1, 2, \ldots, n\}$, a set of edges $\mathscr{E} \in \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. In $\mathscr{G}$, the ith node represents ith agent, and a directed edge (simply called an edge) from node i to node j denoted as an ordered pair $(i, j) \in \mathscr{E}$ represents a unidirectional information exchange link from node i to node j; that is, agent j can receive or obtain information from agent i, but not necessarily vice versa. The set of neighbors of node i is denoted by $\mathscr{N}_i = \{j : (j, i) \in \mathscr{E}\}$. A weighted adjacency $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ of a weighted directed graph is defined such that a_{ij} is a positive weight if only $(j, i) \in \mathscr{E}$ (so there is no edge between a node and itself; that is, $a_{ii} = 0$, for all $i \in \mathscr{V}$). In other words, $a_{ij} > 0$, if $j \in \mathscr{N}_i$, otherwise $a_{ij} = 0$. A directed path (simply called a path) of length k from v_t to v_l $(t, l \in \mathscr{V})$ is a sequence of edges $(i_0, i_1), (i_1, i_2), \ldots, (i_k, i_{k+1})$ with $i_0 = t$, $i_{k+1} = l$ and $(i_s, i_{s+1}) \in \mathscr{E}$ for $s = 0, 1, \ldots, k$. A graph $\mathscr{G}$ is said to be strongly connected if there exists a path between any two distinct nodes in it. For convenience of presentation, the two names, agent and node, will be used interchangeably.

Now consider the dynamics of n agents distributed over a directed graph $\mathscr{G}$. Let $x_i(k) \in \mathbb{R}$ denote the state of node i at time k, $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T$ the state of the system accordingly, and let $A = [a_{ij}]$ be the weighted adjacency matrix associated with the graph. In general, the dynamics of discrete-time multiagent network with fixed topology are described by

$$x_i(k+1) = x_i(k) + \sum_{j\in\mathscr{N}_i} a_{ij}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right), \quad i \in \mathscr{V}, \tag{1}$$

where $\tau_{ij} \in \mathbb{Z}^+$ is the time-delay of the information transmission from node v_j to node v_i.

Remark 1. The consensus problem for the multiagent system (1) is considered in [29], and the consensus problem for its continuous-time counterpart (analogue)

$$\dot{x}_i(t) = \sum_{j\in\mathscr{N}_i} a_{ij}\left(x_j\left(t-\tau_{ij}\right) - x_i(t)\right), \quad i \in \mathscr{V} \tag{2}$$

is investigated in [24], and system (1) without time-delays is also investigated extensively; see, for example, [10, 19] and the references therein.

In multiagent network (1), it is assumed that there is no communication failure between agents. However, during signal exchange of the sensor nodes, an important uncertainty feature is signal losses, which may be caused by the temporary extreme deterioration of the link quality, for instance, due to blocking objects traveling between the transmitter or receiver [30]. Therefore, we consider the general case where each communication link is subject to some probability distribution. Assume that weighted adjacency matrix $A^{(k)} = [a_{ij}^{(k)}]$ is time-varying with $a_{ij}^{(k)}$ being random variable. Denote $\overline{a}_{ij} = \mathbb{E}[a_{ij}^{(k)}]$ and $\sigma_{ij} = \mathrm{Var}[a_{ij}^{(k)}]$. As usual, we assume that $(j, i) \in \mathscr{E}$ if and only if $\overline{a}_{ij} > 0$, and the set of neighbors of node i is denoted by $\mathscr{N}_i = \{j : (j, i) \in \mathscr{E}\}$.

Now, the dynamics of discrete-time multiagent network with random communication links are given by

$$x_i(k+1) = x_i(k) + \sum_{j\in\mathscr{N}_i} a_{ij}^{(k)}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right), \quad i \in \mathscr{V}, \tag{3}$$

where, as in the previous discussion, $\tau_{ij} \in \mathbb{Z}^+$ is the time-delay of the information transmission from node v_j to node v_i. For convenience, we let $\tau_{ii} = 0$, $(i \in \mathscr{V})$, hereafter.

In practical applications, it is often important to steer the state of each agent in a network to a fixed objective. In this paper, we will consider the regulation of the multiagent network (3) so that all agents can reach a common objective. Suppose that there are some agents acting as leaders and well informed. Specifically, let x^* be the anticipated state target. If necessary, we can relabel the agents, and without loss of generality, we assume that the first i_0 agents serve as leaders, and the other ones act as followers. Consider the following controlled multiagent network:

$$x_i(k+1) = \begin{cases} x_i(k) + \sum_{j\in\mathscr{N}_i} a_{ij}^{(k)}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right) \\ \quad -\alpha_i\left(x_i(k) - x^*\right), \quad 1 \le i \le i_0, \\ x_i(k) + \sum_{j\in\mathscr{N}_i} a_{ij}^{(k)}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right), \\ \qquad\qquad i_0 + 1 \le i \le n, \end{cases} \tag{4}$$

where $\alpha_i > 0$ are given constants.

Definition 2. The multiagent network (3) is said to reach leader-following consensus on a state target x^* in mean-square if for any solution $\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T$ of system (4), it always holds that

$$\lim_{k\to\infty} \mathbb{E}\left[x_i(k) - x^*\right]^2 = 0. \tag{5}$$

In this paper, we will investigate the leader-following consensus problem in mean-square for discrete-time multiagent system (3). By constructing novel Lyapunov functionals and employing some new analytical techniques, sufficient conditions are established to ensure the leader-following consensus in mean-square for multiagent system (3).

3. Main Results and Proofs

This section is devoted to the leader-following consensus analysis for system (3), and let us make some necessary preparations before introducing our main results.

Assume that $\{a_{ij}^{(k)} : (j,i) \in \mathscr{E}, k \in \mathbb{Z}^+\}$ are independent with respect to (i, j) and k and also independent of the initial states.

Let $B^{(k)} = [b_{ij}^{(k)}]$ with $b_{ij}^{(k)} = a_{ij}^{(k)}$ for $j \in \mathscr{N}_i$, $b_{ii}^{(k)} = 1 - \sum_{j\in\mathscr{N}_i} a_{ij}^{(k)}$, otherwise $b_{ij}^{(k)} = a_{ij}^{(k)} = 0$. Then, (4) is rewritten as

$$x_i(k+1) = \begin{cases} \sum_{j=1}^{n} b_{ij}^{(k)} x_j\left(k-\tau_{ij}\right) - \alpha_i\left(x_i(k) - x^*\right), \\ \qquad\qquad 1 \le i \le i_0, \\ \sum_{j=1}^{n} b_{ij}^{(k)} x_j\left(k-\tau_{ij}\right), \quad i_0 + 1 \le i \le n. \end{cases} \tag{6}$$

Denote $\overline{B} = (\overline{b}_{ij}) \overset{\text{def}}{=} \mathbb{E}[B^{(k)}]$, and $\Delta B^{(k)} = [\widetilde{b}_{ij}^{(k)}] \overset{\text{def}}{=} B^{(k)} - \overline{B}$. Then, all row sums of both $B^{(k)}$ and $\overline{B}$ are one, and all row sums of $\Delta B^{(k)}$ are zero; namely,

$$\sum_{j=1}^{n} b_{ij}^{(k)} = 1, \quad \sum_{j=1}^{n} \overline{b}_{ij} = 1, \quad \sum_{j=1}^{n} \widetilde{b}_{ij}^{(k)} = 0, \quad i \in \mathscr{V}. \tag{7}$$

Also denote the variance of random variable $b_{ij}^{(k)}$ by σ_{ij}^2, where σ_{ij} is its standard deviation. Notice that $\mathbb{E}[\widetilde{b}_{ij}^{(k)}]^2$ is the variance of random variable $b_{ij}^{(k)}$, and $\widetilde{b}_{ij}^{(k)}$ for $j \ne i$ is dependent with respect to (i, j) and k. Therefore, we have

$$\mathbb{E}\left[\widetilde{b}_{ij}^{(k)}\right]^2 = \operatorname{Var}\left[b_{ij}^{(k)}\right] = \sigma_{ij}^2, \quad j \ne i, \tag{8}$$

$$\mathbb{E}\left[\widetilde{b}_{ij}^{(k)}\widetilde{b}_{il}^{(k)}\right] = 0, \quad l \ne j,\ j \ne i,\ l \ne i, \tag{9}$$

$$\begin{aligned} \mathbb{E}\left[\widetilde{b}_{ii}^{(k)}\widetilde{b}_{ij}^{(k)}\right] &= \mathbb{E}\left[-\sum_{l\ne i} \widetilde{b}_{il}^{(k)}\widetilde{b}_{ij}^{(k)}\right] \\ &= -\mathbb{E}\left[\widetilde{b}_{ij}^{(k)}\right]^2 \\ &= -\sigma_{ij}^2, \quad j \ne i, \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbb{E}\left[\widetilde{b}_{ii}^{(k)}\right]^2 &= \mathbb{E}\left[-\sum_{j\ne i} \widetilde{b}_{ij}^{(k)}\right]^2 \\ &= \sum_{l\ne i} \mathbb{E}\left[\widetilde{b}_{ij}^{(k)}\right]^2 \\ &= \sum_{j\ne i} \sigma_{ij}^2. \end{aligned} \tag{11}$$

For the interaction topology of multiagent system (3), we make the following assumption.

Assumption 3. The graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is strongly connected; namely, the matrix $\overline{B}$ is irreducible.

Lemma 4 (see [31]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a nonnegative matrix; that is, $a_{ij} \ge 0$, and let $\rho(A)$ be the spectral radius (called the Perron root of A). In addition, suppose that A is strongly connected, then there is a positive vector x such that $Ax = \rho(A)x$.*

From Lemma 4, it follows readily that there exists a positive left eigenvector $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T$ of $\overline{B}$ such that

$$\xi^T\overline{B} = \xi^T, \qquad \sum_{j=1}^{n} \xi_i = 1. \tag{12}$$

In the sequel, we denote

$$\overline{\xi} = \max\{\xi_i : 1 \le i \le n\}, \tag{13}$$

$$\underline{\xi} = \min\{\xi_i : 1 \le i \le n\}, \tag{14}$$

$$\overline{\sigma} = \max\left\{\sigma_{ij} : (j,i) \in \mathscr{E}\right\}, \tag{15}$$

$$\underline{\alpha} = \min\{\overline{\alpha}_i : 1 \le i \le i_0\}, \tag{16}$$

$$\gamma = \min\left\{\overline{a}_{ij} : 1 \le i \le i_0, j \in \mathcal{N}\right\}, \tag{17}$$

$$\widehat{\mathcal{N}}_i = \mathcal{N}_i \cup \{i\}. \tag{18}$$

Also, we make the following assumption.

Assumption 5. Assume that

$$\bigcup_{i=1}^{n_0} \widehat{\mathcal{N}}_i = \mathcal{V}, \quad \alpha_i \le \overline{b}_{ii} \quad \text{for } 1 \le i \le i_0. \tag{19}$$

Remark 6. Notice that in Assumption 5 the condition $\bigcup_{i=1}^{n_0} \widehat{\mathcal{N}}_i = \mathcal{V}$ means that the set of the first i_0 nodes and their neighbors contains all the nodes of the network.

We are now in a position to introduce the main results of this paper.

Theorem 7. *Consider the multiagent systems* (3) *and* (4). *Suppose that Assumptions 3 and 5 are satisfied, and assume that* $\overline{\sigma} < \sqrt{\underline{\alpha}\underline{\xi}\gamma/(4(n-1)\overline{\xi})}$ *holds. Then, the multiagent network* (3) *reaches leader-following consensus on the state target* x^* *in mean-square.*

Proof. Let $C = [c_{ij}]$ with

$$c_{ij} = \begin{cases} \overline{b}_{ij} - \alpha_i, & \text{for } 1 \le i \le i_0,\ j = i, \\ \overline{b}_{ij}, & \text{otherwise}, \end{cases} \tag{20}$$

and denote $e_i(k) = x_i(k) - x^*$. Then, the controlled network (6) can be rewritten as

$$\begin{aligned} e_i(k+1) = {} & \sum_{j=1}^{n} c_{ij} e_j\left(k - \tau_{ij}\right) \\ & + \sum_{j=1}^{n} \widetilde{b}_{ij}^{(k)} e_j\left(k - \tau_{ij}\right), \quad i \in \mathcal{V}. \end{aligned} \tag{21}$$

Let ϕ is the initial value of network (21), and denote by $\mathcal{F}_k$ the σ-algebras consisting of all events induced by the random variables $\phi, a_{ij}^{(s)}$ with $0 \le s \le k-1$, $(j,i) \in \mathcal{E}$; that is, $\mathcal{F}_k = \sigma(\phi, a_{ij}^{(s)}, 0 \le s \le k-1, (j,i) \in \mathcal{E})$. Also denote $\tau = \max\{\tau_{ij} : (i,j) \in \mathcal{E}\}$, and $\mathbf{e}_k(s) = [e_1(k+s), e_2(k+s), \ldots, e_n(k+s)]^T$, $-\tau \le s \le 0$.

To prove that the multiagent network (3) reaches leader-following consensus on the state target x^* in mean-square, it suffices to prove the mean-square stability of (21). To this end, we construct the following Lyapunov functional:

$$V(\mathbf{e}_k) = V_1(\mathbf{e}_k) + V_2(\mathbf{e}_k) + V_3(\mathbf{e}_k), \tag{22}$$

where

$$V_1(\mathbf{e}_k) = \sum_{i=1}^{n} \xi_i e_i^2(k), \tag{23}$$

$$V_2(\mathbf{e}_k) = \sum_{i=1}^{n} \xi_i \sum_{j=1}^{n} \sum_{l \ge j} c_{ij} c_{il} V_{ijl}(\mathbf{e}_k), \tag{24}$$

$$V_3(\mathbf{e}_k) = \sum_{i=1}^{n} \xi_i \sum_{j \ne i} 2\sigma_{ij}^2 \sum_{s=k-\tau_{ij}}^{k-1} e_j^2(s) \tag{25}$$

with

$$V_{ijl}(\mathbf{e}_k) = \sum_{s=k-\tau_{ij}}^{k-1} e_j^2(s) + \sum_{s=k-\tau_{il}}^{k-1} e_l^2(s). \tag{26}$$

Then, for system (21), using (8)–(11), we conduct the following computation:

$$\begin{aligned}
& \mathbb{E}\left[V_1(\mathbf{e}_{k+1}) \mid \mathcal{F}_k\right] - V_1(\mathbf{e}_k) \\
& \quad = \mathbb{E}\left[\sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} c_{ij} e_j\left(k - \tau_{ij}\right) + \sum_{j=1}^{n} \widetilde{b}_{ij}^{(k)} e_j\left(k - \tau_{ij}\right)\right)^2\right] \\
& \qquad - \sum_{i=1}^{n} \xi_i e_i^2(k) \\
& \quad = \sum_{i=1}^{n} \xi_i \mathbb{E}\left[\left(\sum_{j=1}^{n} c_{ij} e_j\left(k - \tau_{ij}\right)\right)^2\right] \\
& \qquad + \sum_{i=1}^{n} \xi_i \mathbb{E}\left[\left(\sum_{j=1}^{n} \widetilde{b}_{ij}^{(k)} e_j\left(k - \tau_{ij}\right)\right)^2\right] \\
& \qquad - \sum_{i=1}^{n} \xi_i e_i^2(k) \\
& \quad = \sum_{i=1}^{n} \xi_i \left[\sum_{j=1}^{n} c_{ij}^2 e_j^2\left(k - \tau_{ij}\right)\right. \\
& \qquad\qquad \left. + \sum_{j=1}^{n} \sum_{l>j} 2 c_{ij} c_{il} e_j\left(k - \tau_{ij}\right) e_l\left(k - \tau_{il}\right)\right] \\
& \qquad - \sum_{i=1}^{n} \xi_i e_i^2(k) \\
& \qquad + \sum_{i=1}^{n} \xi_i \left[\sum_{j=1}^{n} \mathbb{E}\left[\widetilde{b}_{ij}^{(k)}\right]^2 e_j^2\left(k - \tau_{ij}\right)\right. \\
& \qquad\qquad \left. + \sum_{j=1}^{n} \sum_{l>j} 2\mathbb{E}\left[\widetilde{b}_{ij}^{(k)} \widetilde{b}_{il}^{(k)}\right] e_j\left(k - \tau_{ij}\right) e_l\left(k - \tau_{il}\right)\right] \\
& \quad = \sum_{i=1}^{n} \xi_i \left[\sum_{j=1}^{n} c_{ij}^2 e_j^2\left(k - \tau_{ij}\right)\right. \\
& \qquad\qquad \left. + \sum_{j=1}^{n} \sum_{l>j} 2 c_{ij} c_{il} e_j\left(k - \tau_{ij}\right) e_l\left(k - \tau_{il}\right)\right] \\
& \qquad - \sum_{i=1}^{n} \xi_i e_i^2(k)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\xi_i\left[\sum_{j\neq i}\sigma_{ij}^2 e_i^2(k)+\sum_{j\neq i}\sigma_{ij}^2 e_j^2\left(k-\tau_{ij}\right)\right.\\
&\qquad\left.+\sum_{j\neq i}2\mathbb{E}\left[\widetilde{b}_{ii}^{(k)}\widetilde{b}_{ij}^{(k)}\right]e_i\left(k-\tau_{ii}\right)e_j\left(k-\tau_{ij}\right)\right]\\
&=\sum_{i=1}^{n}\xi_i\left[\sum_{j=1}^{n}c_{ij}^2 e_j^2\left(k-\tau_{ij}\right)\right.\\
&\qquad\left.+\sum_{j=1}^{n}\sum_{l>j}2c_{ij}c_{il}e_j\left(k-\tau_{ij}\right)e_l\left(k-\tau_{il}\right)\right]\\
&-\sum_{i=1}^{n}\xi_i e_i^2(k)\\
&+\sum_{i=1}^{n}\xi_i\left[\sum_{j\neq i}\sigma_{ij}^2 e_i^2(k)+\sum_{j\neq i}\sigma_{ij}^2 e_j^2\left(k-\tau_{ij}\right)\right.\\
&\qquad\left.-\sum_{j\neq i}2\sigma_{ij}^2 e_i(k)e_j\left(k-\tau_{ij}\right)\right].
\end{aligned}
\tag{27}
$$

It is not difficult to see that

$$
\begin{aligned}
&\mathbb{E}\left[V_2\left(\mathbf{e}_{k+1}\right)\mid\mathscr{F}_k\right]-V_2\left(\mathbf{e}_k\right)\\
&\quad=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{n}\sum_{l>j}c_{ij}c_{il}\left(e_j^2(k)-e_j^2\left(k-\tau_{ij}\right)\right.\\
&\qquad\qquad\left.+e_l^2(k)-e_l^2\left(k-\tau_{il}\right)\right)\\
&\qquad+\sum_{i=1}^{n}\xi_i\sum_{j=1}^{n}c_{ij}^2\left(e_j^2(k)-e_j^2\left(k-\tau_{ij}\right)\right),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&\mathbb{E}\left[V_3\left(\mathbf{e}_{k+1}\right)\mid\mathscr{F}_k\right]-V_3\left(\mathbf{e}_k\right)\\
&\quad=\sum_{i=1}^{n}\xi_i\sum_{j\neq i}2\sigma_{ij}^2\left(e_j^2(k)-e_j^2\left(k-\tau_{ij}\right)\right).
\end{aligned}
\tag{29}
$$

From (27)–(29), it follows that

$$
\begin{aligned}
&\mathbb{E}\left[V\left(\mathbf{e}_{k+1}\right)\mid\mathscr{F}_k\right]-V\left(\mathbf{e}_k\right)\\
&\quad=\sum_{i=1}^{n}\xi_i\left[\sum_{j=1}^{n}c_{ij}^2 e_j^2(k)\right.\\
&\qquad+\sum_{j=1}^{n}\sum_{l>j}c_{ij}c_{il}\left(e_j^2(k)-e_j^2\left(k-\tau_{ij}\right)\right.\\
&\qquad\qquad+e_l^2(k)-e_l^2\left(k-\tau_{il}\right)\\
&\qquad\qquad\left.\left.+2e_j\left(k-\tau_{ij}\right)e_l\left(k-\tau_{il}\right)\right)\right]\\
&\qquad-\sum_{i=1}^{n}\xi_i e_i^2(k)\\
&\qquad+\sum_{i=1}^{n}\xi_i\left[\sum_{j\neq i}\sigma_{ij}^2 e_i^2(k)+\sum_{j\neq i}\sigma_{ij}^2 e_j^2(k)\right.\\
&\qquad\qquad+\sum_{j\neq i}\sigma_{ij}^2\left(e_j^2(k)-e_j^2\left(k-\tau_{ij}\right)\right.\\
&\qquad\qquad\left.\left.-2e_i(k)e_j\left(k-\tau_{ij}\right)\right)\right]\\
&=\sum_{i=1}^{n}\xi_i\left[\sum_{j=1}^{n}c_{ij}^2 e_j^2(k)\right.\\
&\qquad+\sum_{j=1}^{n}\sum_{l>j}c_{ij}c_{il}\left(e_j^2(k)+e_l^2(k)\right.\\
&\qquad\qquad-\left(e_j\left(k-\tau_{ij}\right)\right.\\
&\qquad\qquad\left.\left.-e_l(k-\tau_{il})\right)^2\right)\Bigg]\\
&\qquad-\sum_{i=1}^{n}\xi_i e_i^2(k)\\
&\qquad+\sum_{i=1}^{n}\xi_i\left[\sum_{j\neq i}\sigma_{ij}^2 e_i^2(k)+\sum_{j\neq i}\sigma_{ij}^2 e_j^2(k)\right.\\
&\qquad\qquad+\sum_{j\neq i}\sigma_{ij}^2\left(e_i^2(k)+e_j^2(k)\right.\\
&\qquad\qquad\left.\left.-\left(e_i(k)+e_j\left(k-\tau_{ij}\right)\right)^2\right)\right]\\
&\le\sum_{i=1}^{n}\xi_i\left[\sum_{j=1}^{n}c_{ij}^2 e_j^2(k)\right.\\
&\qquad\left.+\sum_{j=1}^{n}\sum_{l>j}c_{ij}c_{il}\left(e_j^2(k)+e_l^2(k)\right)\right]\\
&\qquad-\sum_{i=1}^{n}\xi_i e_i^2(k)\\
&\qquad+2\sum_{i=1}^{n}\xi_i\left[\sum_{j\neq i}\sigma_{ij}^2 e_i^2(k)+\sum_{j\neq i}\sigma_{ij}^2 e_j^2(k)\right].
\end{aligned}
\tag{30}
$$

A straightforward computation yields that

$$
\begin{aligned}
&\sum_{i=1}^{n}\xi_i\left[\sum_{j=1}^{n}c_{ij}^2 e_j^2(k)+\sum_{j=1}^{n}\sum_{l>j}c_{ij}c_{il}\left(e_j^2(k)+e_l^2(k)\right)\right]\\
&\quad=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}c_{ij}c_{il}e_j^2(k).
\end{aligned}
\tag{31}
$$

Noticing the equality $\sum_{j=1}^{n}\overline{b}_{ij} = 1$ (see (7)), it follows readily that

$$\begin{aligned}
&\sum_{i=1}^{n}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}c_{ij}c_{il}e_j^2(k)\\
&\quad=\sum_{i=1}^{i_0}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}c_{ij}c_{il}e_j^2(k)\\
&\qquad+\sum_{i=i_0+1}^{n}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}c_{ij}c_{il}e_j^2(k)\\
&\quad=\sum_{i=1}^{i_0}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b}_{ij}\overline{b}_{il}e_j^2(k)\\
&\qquad+\sum_{i=i_0+1}^{n}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b}_{ij}\overline{c}_{il}e_j^2(k)\\
&\qquad-\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)+(1-\alpha_i)e_i^2(k)\right]\\
&\quad=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b}_{ij}\overline{b}_{il}e_j^2(k)\\
&\qquad-\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)+(1-\alpha_i)e_i^2(k)\right]\\
&\quad=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)\\
&\qquad-\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)+(1-\alpha_i)e_i^2(k)\right]\\
&\quad=\sum_{i=1}^{n}\xi_ie_i^2(k)\\
&\qquad-\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)+(1-\alpha_i)e_i^2(k)\right].
\end{aligned}\tag{32}$$

Substituting (31) and (32) into (30) yields that

$$\begin{aligned}
&\mathbb{E}\left[V(\mathbf{e}_{k+1})\mid\mathscr{F}_k\right]-V(\mathbf{e}_k)\\
&\quad\le 2\sum_{i=1}^{n}\xi_i\left[\sum_{j\ne i}\sigma_{ij}^2e_i^2(k)+\sum_{j\ne i}\sigma_{ij}^2e_j^2(k)\right]\\
&\qquad-\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)+(1-\alpha_i)e_i^2(k)\right].
\end{aligned}\tag{33}$$

It is easy to see that

$$\begin{aligned}
&\sum_{i=1}^{n}\xi_i\left[\sum_{j\ne i}\sigma_{ij}^2e_i^2(k)+\sum_{j\ne i}\sigma_{ij}^2e_j^2(k)\right]\\
&\quad\le\overline{\xi}\sum_{i=1}^{n}\sum_{j\ne i}\sigma_{ij}^2e_i^2(k)+\overline{\xi}\sum_{i=1}^{n}\sum_{j\ne i}\sigma_{ij}^2e_j^2(k)\\
&\quad\le\overline{\xi}\sum_{i=1}^{n}\sum_{j\ne i}\sigma_{ij}^2e_i^2(k)+\overline{\xi}\sum_{j=1}^{n}\sum_{i\ne j}\sigma_{ij}^2e_j^2(k)\\
&\quad\le 2(n-1)\overline{\xi}\overline{\sigma}^2\sum_{i=1}^{n}e_i^2(k),
\end{aligned}\tag{34}$$

and

$$\begin{aligned}
&\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j=1}^{n}\overline{b}_{ij}e_j^2(k)+(1-\alpha_i)e_i^2(k)\right]\\
&\quad=\sum_{i=1}^{i_0}\alpha_i\xi_i\left[\sum_{j\in\mathscr{N}_i}\overline{b}_{ij}e_j^2(k)\right.\\
&\qquad\left.+\left(1+\overline{b}_{ii}-\alpha_i\right)e_i^2(k)\right]\\
&\quad\ge\underline{\alpha}\underline{\xi}\gamma\sum_{i=1}^{i_0}\sum_{j\in\widehat{\mathscr{N}}_i}e_j^2(k)\\
&\quad\ge\underline{\alpha}\underline{\xi}\gamma\sum_{i=1}^{n}e_i^2(k).
\end{aligned}\tag{35}$$

Substituting (34) into (33) results in

$$\begin{aligned}
&\mathbb{E}\left[V(\mathbf{e}_{k+1})\mid\mathscr{F}_k\right]-V(\mathbf{e}_k)\\
&\quad\le\left(4(n-1)\overline{\xi}\overline{\sigma}^2-\underline{\alpha}\underline{\xi}\gamma\right)|\mathbf{e}(k)|^2,
\end{aligned}\tag{36}$$

which implies that

$$\begin{aligned}
&\mathbb{E}\left[V(\mathbf{e}_{k+1})\right]-\mathbb{E}\left[V(\mathbf{e}_k)\right]\\
&\quad\le\left(4(n-1)\overline{\xi}\overline{\sigma}^2-\underline{\alpha}\underline{\xi}\gamma\right)\mathbb{E}\left[|\mathbf{e}(k)|^2\right].
\end{aligned}\tag{37}$$

Employing the Lyapunov stability theory, we can deduce that $\lim_{k\to\infty}\mathbb{E}[V(\mathbf{e}_k)] = 0$. This completes the proof of the theorem. □

Remark 8. In Theorem 7, the condition $\overline{\sigma} < \sqrt{\underline{\alpha}\underline{\xi}\gamma/4(n-1)\overline{\xi}}$ always holds when $\overline{\sigma}$ is sufficiently small. In particular, when the interaction topology of multiagent system is deterministic, the system (3) and the controlled network (4) are reduced, respectively, to

$$x_i(k+1) = x_i(k)+\sum_{j\in\mathscr{N}_i}a_{ij}\left(x_j\left(k-\tau_{ij}\right)-x_i(k)\right),\quad i\in\mathscr{V},\tag{38}$$

$$x_i(k+1)=\begin{cases}x_i(k)+\displaystyle\sum_{j\in\mathscr{N}_i}a_{ij}\left(x_j\left(k-\tau_{ij}\right)-x_i(k)\right)\\ \qquad-\alpha_i\left(x_i(k)-x^*\right),\quad 1\le i\le i_0,\\ x_i(k)+\displaystyle\sum_{j\in\mathscr{N}_i}a_{ij}\left(x_j\left(k-\tau_{ij}\right)-x_i(k)\right),\\ \qquad\qquad i_0+1\le i\le n.\end{cases}\tag{39}$$

In this case, $\overline{\sigma} = 0$; accordingly, the condition $\overline{\sigma} < \sqrt{\underline{\alpha}\underline{\xi}\gamma/4(n-1)\overline{\xi}}$ is always satisfied, and from Theorem 7, we have the following corollary.

Corollary 9. *Consider the multiagent systems* (38) *and* (39). *Under Assumptions 3 and 5, the multiagent network* (38) *reaches leader-following consensus on the state target* x^*.

In the previous discussion, we only consider scalar individual states, and it is easy to extend them to the case where the individual states are vectors. Consider the following multiagent system of n nodes with vector-valued states:

$$x_i(k+1) = x_i(k) + \sum_{j\in\mathcal{N}_i} a_{ij}^{(k)}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right), \quad i \in \mathcal{V}, \tag{40}$$

and the controlled network is given by

$$x_i(k+1) = \begin{cases} x_i(k) + \sum_{j\in\mathcal{N}_i} a_{ij}^{(k)}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right) \\ \quad -\alpha_i\left(x_i(k) - x^*\right), \quad 1 \le i \le i_0, \\ x_i(k) + \sum_{j\in\mathcal{N}_i} a_{ij}^{(k)}\left(x_j\left(k-\tau_{ij}\right) - x_i(k)\right), \\ \qquad\qquad i_0+1 \le i \le n, \end{cases} \tag{41}$$

where $x(k) \in \mathbb{R}^n$. We have the following results.

Theorem 10. *Consider the multiagent systems* (40) *and* (41). *Suppose that Assumptions 3 and 5 are satisfied, and assume that* $\overline{\sigma} < \sqrt{\underline{\alpha}\underline{\xi}\gamma/4(n-1)\overline{\xi}}$ *holds. Then, the multiagent network* (40) *reaches the leader-following consensus on the state target* x^* *in mean-square.*

Proof. The proof of this theorem is similar to that of Theorem 7. The minor modification is to replace some scalar multiplication operations by the Kronecker product of matrices, and we omit the details here. □

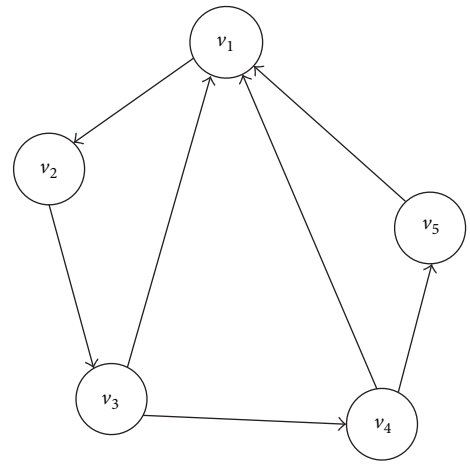
4. A Numerical Example

In this section, we present a numerical example to illustrate the proposed methods.

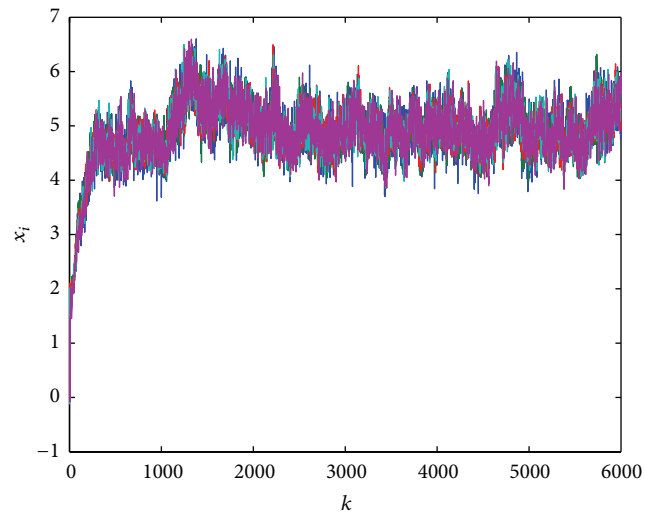
Example 1. Consider the multiagent networks (3) and (4), and for simplicity, we take $n = 5$. The interaction topology between the agents is shown in Figure 1(a), and other parameters are taken as follows:

$$\overline{B} = \left[\overline{b}_{ij}\right]_{5\times5} = \begin{bmatrix} 0.4 & 0 & 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix},$$
$$\left[\tau_{ij}\right]_{5\times5} = \begin{bmatrix} 0 & 0 & 5 & 1 & 2 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}. \tag{42}$$

Clearly, the network topology is strongly connected, and it is also obvious that $\widehat{N}_1 \bigcup \widehat{\mathcal{N}}_2 = \mathcal{V}$. Therefore, we can choose



(a) The interaction topology of multiagent system



(b) Consensus on an anticipated state target

Figure 1: Numerical simulation.

$i_0 = 2$. Assume that $\sigma_{13} = \sigma_{14} = \sigma_{15} = \sigma_{21} = \sigma_{32} = \sigma_{43} = \sigma_{54} = 0.03$, $\alpha_1 = \alpha_2 = 0.38$, and $x^* = 5$. By a straightforward computation, we can get that $\xi = [0.1923, 0.2885, 0.2885, 0.1538, 0.0769]^T$, and it is also easy to see that $\overline{\sigma} = 0.03$, $\underline{\alpha} = 0.38$, $\gamma = 0.2$, $\overline{\xi} = 0.2885$, and $\underline{\xi} = 0.0769$. In this case, $\sqrt{\underline{\alpha}\underline{\xi}\gamma/4(n-1)\overline{\xi}} = 0.0398$, and $\overline{\sigma} < \sqrt{\underline{\alpha}\underline{\xi}\gamma/4(n-1)\overline{\xi}}$ the multiagent. Therefore, by Theorem 7, network (3) reaches the leader-following consensus on an anticipated state target in mean-square. With the above parameters and a set of initial values produced in a stochastic way, the numerical simulation shown in Figure 1(b) matches well with the theoretical results.

5. Conclusions

We have investigated the leader-following consensus problem in mean-square for a class of discrete-time multiagent systems. The network under study is bidirectional and contains arbitrary time-delays and the random communication links.

Some agents in the network are well informed and serve as leaders. By employing novel Lyapunov functionals and analytical skills, sufficient conditions are established to ensure the leader-following consensus in mean-square for multiagent system. A numerical example is given to demonstrate the proposed approach.

Acknowledgments

This work was supported in part by the International Travel Grant 2011/R3 sponsored by the Royal Society of the UK, the National Natural Science Foundation of China under Grant 61074129, and the Natural Science Foundation of Jiangsu Province of China under Grant BK2012682.

References

[1] N. A. Lynch, *Distributed Algorithms*, Morgan Kaufmann, San Francisco, Calif, USA, 1997.

[2] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed H_∞ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[3] D. Ding, Z. Wang, B. Shen, and H. Shu, "H_∞ state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 725–736, 2012.

[4] H. Dong, Z. Wang, and H. Gao, "Distributed filtering for a class of time-varying systems over sensor networks with quantization errors and successive packet dropouts," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 3164–3173, 2012.

[5] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 48, no. 9, pp. 2007–2015, 2012.

[6] J. Hu, Z. Wang, Y. Niu, and L. K. Stergioulas, "H_∞ sliding mode observer design for a class of nonlinear discrete time-delay systems: a delay-fractioning approach," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 16, pp. 1806–1826, 2012.

[7] J. Liang, Z. Wang, B. Shen, and X. Liu, "Distributed state estimation in sensor networks with randomly occurring nonlinearities subject to time-delays," *ACM Transactions on Sensor Networks*, vol. 9, no. 1, article 4, 2012.

[8] R. Olfati-Saber, "Distributed Kalman filter with embedded consensus filters," in *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference (CDC-ECC '05)*, pp. 8179–8184, Seville, Spain, December 2005.

[9] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proceedings of the American Control Conference (ACC '05)*, vol. 3, pp. 1859–1864, June 2005.

[10] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1867–1872, 2005.

[11] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[12] B. Shen, Z. Wang, and X. Liu, "Sampled-data synchronization control of dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[13] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York, NY, USA, 2001.

[14] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[15] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[16] T. Vicsek, A. Czirk, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[17] C. W. Reynolds, "Flocks, herds, and schools: a distributed behavior model," *Computer Graphics*, vol. 21, no. 4, pp. 25–34, 1987.

[18] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 98, pp. 298–305, 1973.

[19] M. Huang and J. H. Manton, "Stochastic consensus seeking with noisy and directed inter-agent communication: fixed and randomly varying topologies," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 235–241, 2010.

[20] H. Dong, Z. Wang, and H. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Transactions on Circuits and Systems I*, vol. 59, no. 10, pp. 2354–2362, 2012.

[21] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3008–3015, 2012.

[22] Z. Wang, B. Shen, H. Shu, and G. Wei, "Quantized H_∞ control for nonlinear stochastic time-delay systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1431–1444, 2012.

[23] J. Hu, Z. Wang, B. Shen, and H. Gao, "Gain-constrained recursive filtering with stochastic nonlinearities and probabilistic sensor delays," *IEEE Transactions on Signal Processing*, vol. 61, no. 5, pp. 1230–1238, 2013.

[24] J. Lu, D. W. C. Ho, and J. Kurths, "Consensus over directed static networks with arbitrary finite communication delays," *Physical Review E*, vol. 80, no. 6, Article ID 066121, 2009.

[25] D. Hummel, "Formation flight as an energy-saving mechanism," *Israel Journal of Zoology*, vol. 41, no. 3, pp. 261–278, 1995.

[26] M. Andersson and J. Wallander, "Kin selection and reciprocity in flight formation," *Behavioral Ecology*, vol. 15, no. 1, pp. 158–162, 2004.

[27] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Systems & Control Letters*, vol. 59, no. 3-4, pp. 209–217, 2010.

[28] K. Peng and Y. Yang, "Leader-following consensus problem with a varying-velocity leader and time-varying delays," *Physica A*, vol. 388, no. 2-3, pp. 193–208, 2009.

[29] Y. Liu, D. W. C. Ho, and Z. Wang, "A new framework for consensus for discrete-time directed networks of multi-agents with distributed delays," *International Journal of Control*, vol. 85, no. 11, pp. 1755–1765, 2012.

[30] M. Huang, S. Dey, G. N. Nair, and J. H. Manton, "Stochastic consensus over noisy networks with Markovian and arbitrary switches," *Automatica*, vol. 46, no. 10, pp. 1571–1583, 2010.

[31] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 707686, 8 pages
http://dx.doi.org/10.1155/2013/707686

Research Article

Estimate of Number of Periodic Solutions of Second-Order Asymptotically Linear Difference System

Honghua Bin and Zhenkun Huang

School of Science, Jimei University, Xiamen 361021, China

Correspondence should be addressed to Honghua Bin; hhbin@jmu.edu.cn

Received 1 April 2013; Accepted 5 June 2013

Academic Editor: Zidong Wang

Copyright © 2013 H. Bin and Z. Huang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the number of periodic solutions of second-order asymptotically linear difference system. The main tools are Morse theory and twist number, and the discussion in this paper is divided into three cases. As the system is resonant at infinity, we use perturbation method to study the compactness condition of functional. We obtain some new results concerning the lower bounds of the nonconstant periodic solutions for discrete system.

1. Introduction

In this paper we are interested in the lower bound of the number of periodic solutions for second-order autonomous difference system

$$\Delta^2 x_{n-1} + f(x_n) = 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $x_n \in \mathbb{R}^N$, $f = (f_1, f_2, \ldots, f_N)^T \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, and N is a fixed positive integer.

Discrete systems have been investigated by many authors using various methods, and many interesting results have obtained; see [1–7] and references therein. The critical point theory [8, 9] is a useful tool to investigate differential equations, which is developed to study difference equations. Using minimax methods in critical point theory, Guo and Yu [10, 11] investigated the existence of periodic and subharmonic solutions of system (1), where nonlinearity f is either sublinear or superlinear. In this paper, we assume that

(P1) there exist a function $g \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and a $N \times N$ symmetric matrix A_∞ such that $f(x_n) = A_\infty x_n + g(x_n)$, $x_n \in \mathbb{R}^N$, and

$$|g(x_n)| = o(|x_n|) \quad \text{as } |x_n| \longrightarrow \infty, \tag{2}$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}^N$. Moreover there exist functions F, G such that $F'(x_n) = (\partial F/\partial x_{n1}, \ldots, \partial F/\partial x_{nN})^T = f(x_n) = (f_1, f_2, \ldots, f_N)^T$, $G'(x_n) = g(x_n)$, where $'$ denotes the gradient of function.

System (1) can be regarded as discrete analogous of the following differential system:

$$-\Delta u = f(u). \tag{3}$$

A great deal of research has been devoted to (3). For example, by using minimax theory, Rabinowitz [12] has given some interesting results, and Mawhin and Willem [9] obtained some results using the critical point theory. Moreover, there is a vast literature on the problems concerning periodic solutions, BVP, asymptotically behavior of solutions, and so forth.

Morse theory [8, 9, 13–16] has been used to solve the asymptotically linear problem. Chang [17], Amann and Zehnder [18] obtained the existence of three distinct solutions via Morse theory, where (3) was nonresonant at infinity. Moreover, the resonant case has been considered in [19–23]. The estimate of number of periodic solutions of (3) was established in [24]. Motivated by [24], we will use Morse theory to consider the lower bound of number of periodic solutions for system (1).

Throughout this paper we employ some standard notations. Denote by $\mathbb{R}, \mathbb{Z}$ the real number and the integer sets, respectively. $\mathbb{R}^N$ is the real space with dimension N. $Z[a, b] = \{a, a+1, \ldots, b\}$ if $a \le b$ and $a, b \in \mathbb{Z}$. A^T or x^T denotes the transpose of matrix A or vector x.

If $g(t)$ and $G(t)$ are bounded on $\mathbb{R}^N$, and system (1) is p-resonant at ∞, then functional J does not satisfy the

compactness condition of the Palais-Smale type. Therefore our discussion will be divided into three cases. Moreover, we assume that

(P2) J has a finite number of nondegenerated critical points;

(P3) all p-periodic solutions of system (1) are not p-resonant;

(P4) for $m \in Z[0,r]$, $\sigma(A_\infty) \subset (\lambda_m, \lambda_{m+1}]$, where $\lambda_m = 4\sin^2(m\pi/p)$ and $r = [p/2]$.

Now we state the main results as follows.

Theorem 1. *Assume that (P1)–(P4) hold, and system* (1) *is not p-resonant at ∞. Then*

$$n(p) \geq \frac{1}{2}\Theta p - h\left(pN + \frac{1}{2}\right) + \frac{1}{2}, \tag{4}$$

where $n(p)$ is the number of the nonconstant p-periodic solutions of system (1)*, Θ is the global twist number (see* (32)*), and h will be defined in Section 3.*

Theorem 2. *Assume that (P1)–(P4) hold, system* (1) *is p-resonant at ∞, and $g(t)$ is bounded in $\mathbb{R}^N$, $\lim_{|t|\to+\infty} G(t) = -\infty$. Then* (4) *is valid.*

Theorem 3. *Assume that (P1)–(P4) hold, system* (1) *is p-resonant at ∞, and $g(t), G(t)$ are bounded in $\mathbb{R}^N$. Then*

$$n(p) \geq \frac{1}{2}\Theta p - h(pN + 1). \tag{5}$$

Remark 4. Benci and Fortunato [24] studied asymptotically linear equation (3). Theorem 1 extends and generalizes the analogous results in [24], and Theorems 2-3 are new results.

The organization of this paper is organized as follows. In Section 2 we study the compactness condition for functional J. Some facts about Morse theory and necessary preliminaries are given in Section 3. In Section 4 the main results are proved.

2. (PS) Condition

We say that a C^1-functional ϕ on Hilbert space X satisfies the Palais-Smale (PS) condition, if every sequence $\{x^{(j)}\}$ in X such that $\{\phi(x^{(j)})\}$ is bounded and $\phi'(x^{(j)}) \to 0$ as $j \to \infty$, contains a convergent subsequence.

Here we first introduce space E_p.

Let $E_p = \{x = \{x_n\} \in S \mid x_{n+p} = x_n, n \in \mathbb{Z}\}$, where $S = \{x = \{x_n\} \mid x_n \in \mathbb{R}^N, n \in \mathbb{Z}\}$. For any $x, y \in S$, $a, b \in \mathbb{R}$, $ax + by = \{ax_n + by_n\}_{n\in\mathbb{Z}}$. Then S is a linear space. Let E_p equip with inner product and norm as follows:

$$\langle x, y\rangle = \sum_{n=1}^{p} (x_n, y_n), \qquad \|x\| = \left(\sum_{n=1}^{p} |x_n|^2\right)^{1/2}, \qquad \forall x, y \in E_p, \tag{6}$$

where $(\cdot,\cdot)$ and $|\cdot|$ are the usual inner product and norm in $\mathbb{R}^N$, respectively. Obviously, E_p is a Hilbert space with dimension pN and homeomorphism to $\mathbb{R}^{pN}$.

By the variational method, the p-periodic solutions of (1) are same as the critical points of the C^2-functional

$$J(x) = \sum_{n=1}^{p} \left[\frac{1}{2}|\Delta x_n|^2 - F(x_n)\right], \quad x \in E_p. \tag{7}$$

By assumption (P1), the functional J can be rewritten as

$$J(x) = \frac{1}{2}\sum_{n=1}^{p} \left[|\Delta x_n|^2 - (A_\infty x_n, x_n)\right] - \sum_{n=1}^{p} G(x_n), \tag{8}$$

and we write $I(x) = \sum_{n=1}^{p}[|\Delta x_n|^2 - (A_\infty x_n, x_n)]$.

Consider eigenvalue problem

$$-\Delta^2 x_{n-1} = \lambda x_n, \quad x_{n+p} = x_n, \quad x_n \in \mathbb{R}^N, \tag{9}$$

that is, $x_{n+1} + (\lambda - 2)x_n + x_{n-1} = 0$, $x_{n+p} = x_n$. By the periodicity, the difference system has complexity solution $x_n = e^{in\theta}c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/p$, $k \in \mathbb{Z}$. Moreover, $\lambda = 2 - e^{-i\theta} - e^{i\theta} = 2(1 - \cos\theta) = 4\sin^2(k\pi/p)$.

Let η_k denote the real eigenvector corresponding to the eigenvalues $\lambda_k = 4\sin^2(k\pi/p)$, $k \in Z[0,r]$, and $r = [p/2]$, where $[\cdot]$ stands for the greatest-integer function. In terms of eigenvalue $\lambda_m = 4\sin^2(m\pi/p)$ for some $m \in Z[0,r]$, we can split space E_p as follows:

$$E_p = W^- \oplus W^0 \oplus W^+, \tag{10}$$

where

$$W^- = \operatorname{span}\{\eta_k \mid k \in Z[0, m-1]\}, \qquad W^0 = \operatorname{span}\{\eta_m\},$$
$$W^+ = \operatorname{span}\{\eta_k \mid k \in Z[m+1, r]\}. \tag{11}$$

Moreover, there exists $\delta > 0$ such that

$$I(u) \geq \delta\|u\|^2 \quad \text{for } u \in W^+,$$
$$I(v) \leq -\delta\|v\|^2 \quad \text{for } v \in W^-, \quad I(w) = 0 \quad \text{for } w \in W^0. \tag{12}$$

Let us recall the definition of resonance (see [24]).

A p-periodic solution $\{x_n\}$ of (1) is called p-resonance, if there exists $\lambda_k = 4\sin^2(k\pi/p) \in \sigma(F''(x_n))$, where F'' denotes the Hessian matrix of F and $\sigma(\cdot)$ is the spectrum of matrix. We say that (1) is p-resonant at ∞, if there exists $\lambda_k = 4\sin^2(k\pi/p) \in \sigma(A_\infty)$.

Lemma 5. *Assume that (P1) and (P4) hold, and system* (1) *is not p-resonant at ∞. Then functional J (see* (8)*) satisfies the (PS) condition.*

Proof. Let $\{x^{(j)}\} \subset E_p$ be the (PS) sequence for functional J; that is, $\{J(x^{(j)})\}$ is bounded, and $J'(x^{(j)}) \to 0$ as $j \to \infty$. Therefore, for any $\varphi \in E_p$, we have

$$\left\langle J'\left(x^{(j)}\right), \varphi\right\rangle = o(\|\varphi\|) \quad \text{as } j \longrightarrow \infty. \tag{13}$$

By $W^0 = \{0\}$, we write $x^{(j)} = u^{(j)} + v^{(j)}$ with $u^{(j)} \in W^+$, $v^{(j)} \in W^-$. To show that J satisfies (PS) condition, it is enough to prove that $\{x^{(j)}\}$ is bounded in E_p. That is, we need only to prove that $\{u^{(j)}\}$ and $\{v^{(j)}\}$ are bounded in E_p. By contradiction, without loss of generality, there exists $k \in Z[1,p]$ such that

$$\begin{gathered} \left|x_n^{(j)}\right| \longrightarrow \infty \quad \text{as } j \longrightarrow \infty \text{ for } n \in Z[1,k], \\ x_n^{(j)} \text{ are bounded for } n \in Z[k+1,p]. \end{gathered} \tag{14}$$

Therefore, for all $n \in Z[1,p]$, by assumption (P1), there exist $\varepsilon > 0$ and $c_1 > 0$ such that

$$\left|G\left(x_n^{(j)}\right)\right| \le \varepsilon\left|x_n^{(j)}\right|^2 + c_1, \qquad \left|g\left(x_n^{(j)}\right)\right| \le \varepsilon\left|x_n^{(j)}\right| + c_1 \tag{15}$$

for large j. Thus there is $c > 0$, $|\sum_{n=1}^{p}(g(x_n^{(j)}), x_n^{(j)})| \le \sum_{n=1}^{p}|g(x_n^{(j)})|\,|x_n^{(j)}| \le \varepsilon\|x^{(j)}\|^2 + c\|x^{(j)}\|$. Taking $\varphi = u^{(j)} - v^{(j)}$ in (13), by previous argument,

$$\begin{aligned} &o\left(\left\|u^{(j)}\right\| + \left\|v^{(j)}\right\|\right) \\ &\quad = \left\langle J'\left(x^{(j)}\right), u^{(j)} - v^{(j)}\right\rangle \\ &\quad = I\left(u^{(j)}\right) - I\left(v^{(j)}\right) - \sum_{n=1}^{p}\left(g\left(x_n^{(j)}\right), u_n^{(j)} - v_n^{(j)}\right) \\ &\quad \ge \delta\left(\left\|u^{(j)}\right\|^2 + \left\|v^{(j)}\right\|^2\right) - \varepsilon\left(\left\|u^{(j)}\right\|^2 + \left\|v^{(j)}\right\|^2\right) \\ &\qquad - c\left(\left\|u^{(j)}\right\| + \left\|v^{(j)}\right\|\right), \end{aligned} \tag{16}$$

it follows a contradiction. Therefore $\{u^{(j)}\}$ and $\{v^{(j)}\}$ are bounded in E_p. This completes the proof. □

Here and in the sequel, the letter δ will be indiscriminately used to denote various positive constants whose exact values are irrelevant, and $\varepsilon \in (0,1)$ is arbitrarily small. Moreover we also denote by c the various positive constants in this paper.

Lemma 6. *Assume that (P1) and (P4) hold. System* (1) *is p-resonant at ∞, $g(t)$ is bounded in $\mathbb{R}^N$, and $\lim_{|t|\to+\infty}G(t) = -\infty$. Then J satisfies the (PS) condition.*

Proof. Let $\{x^{(j)}\} \subset E_p$ be the (PS) sequence for functional J; that is, $\{J(x^{(j)})\}$ is bounded, and $J'(x^{(j)}) \to 0$ as $j \to \infty$.

Since system (1) is p-resonant at ∞, $W^0 \ne \{0\}$. Similarly, let $x^{(j)} = u^{(j)} + v^{(j)} + w^{(j)}$ with $u^{(j)} \in W^+$, $v^{(j)} \in W^-$, and $w^{(j)} \in W^0$. By the same method as proof of Lemma 5, it also follows that $\{u^{(j)}\}$ and $\{v^{(j)}\}$ are bounded in E_p. Next we prove that $\{w^{(j)}\}$ is bounded in E_p.

$J(x^{(j)}) = (1/2)I(u^{(j)}) + (1/2)I(v^{(j)}) - \sum_{n=1}^{p}G(x_n^{(j)})$, by $\{u^{(j)}\}$, $\{v^{(j)}\}$, and $J(x^{(j)})$ are bounded in E_p, and it follows that $\sum_{n=1}^{p}G(x_n^{(j)})$ is bounded. On the other hand, $|\sum_{n=1}^{p}G(x_n^{(j)}) - \sum_{n=1}^{p}G(w_n^{(j)})| \le \sup_{t\in E_p}\|g(t)\|(\|u^{(j)}\| + \|v^{(j)}\|)$, so $\sum_{n=1}^{p}G(w_n^{(j)})$ is bounded. It is easy to see from assumption $\lim_{|t|\to+\infty}G(t) = -\infty$ that $\{w^{(j)}\}$ is bounded. The proof is completed. □

If we assume that $G(t)$, $g(t)$ are bounded and system (1) is p-resonant at ∞, then functional J does not satisfy the (PS) condition. In order to overcome the difficult arising from the lack of compactness condition, we use a suitable penalization technique (one can refer to [20, 24]) and add a perturbation term to the functional J. Define

$$\varphi_R(t) = \begin{cases} (t-R)^4, & \text{if } t > R, \\ 0, & \text{if } t \le R, \end{cases} \tag{17}$$

where R is a positive real number and the penalized functional is given by

$$J_R(x) = J(x) + \varphi_R\left(\|w\|^2\right), \tag{18}$$

where $x = u + v + w \in W^+ \oplus W^- \oplus W^0$. Obviously, if $x \in E_p$ is a critical point of J_R with $\|w\|^2 \le R$, then x is also the critical point of J.

Lemma 7. *Assume that (P1) and (P4) hold, $G(t), g(t)$ are bounded in $\mathbb{R}^N$, and system* (1) *is p-resonant at ∞. Then J_R satisfies the (PS) condition. Moreover, for any critical point x of J_R, there exists $M > 0$ such that $\|u + v\| \le M$, where $x = u + v + w \in E_p$, $u \in W^+$, $v \in W^-$, and $w \in W^0$.*

Proof. Let $\{x^{(j)}\} \subset E_p$ be the (PS) sequence for functional J_R; that is, $\{J_R(x^{(j)})\}$ is bounded in E_p, and for any $\varphi \in E_p$,

$$\left\langle J_R'\left(x^{(j)}\right), \varphi\right\rangle = o\left(\|\varphi\|\right) \quad \text{as } j \longrightarrow \infty. \tag{19}$$

Similarly to the proof of Lemma 5, we need only to prove that $\{w^{(j)}\}$ is bounded in E_p.

Taking $\varphi = w^{(j)}$ in (19), it follows that $o(\|w^{(j)}\|) = \langle J_R'(x^{(j)}), w^{(j)}\rangle \ge -c\|w\| + 2\|w\|^2\varphi_R'(\|w\|^2)$. By the definition of φ_R, it follows that $\{w^{(j)}\}$ is bounded. Therefore the penalized functional J_R satisfies the (PS) condition.

Let x be the critical point of J_R, then

$$\begin{aligned} 0 = \left\langle J_R'(x), u - v\right\rangle &= I(u) - I(v) \\ &\quad - \sum_{n=1}^{p}\left(g(x_n), u_n - v_n\right) \ge \delta\|u + v\|^2 \\ &\quad - \varepsilon\|u + v\|^2 - c\|u + v\|. \end{aligned} \tag{20}$$

So there is a $M > 0$ such that $\| u + v \| \le M$, and the proof is completed. □

3. Preliminaries

Let E be a real Hilbert space, and let ϕ be a C^2-functional on E. We denote by $\operatorname{crit}(\phi) = \{x \in E \mid \phi'(x) = 0\}$ the set of critical points of ϕ, $\phi^c = \{x \in E \mid \phi(x) \le c\}$ the level set of ϕ, and $\phi_a^b = \{x \in E \mid a \le \phi(x) \le b\}$. In the following we suppose that ϕ is a C^2-functional on E which satisfies the (PS) condition.

Definition 8 (see [9, 14]). Let x be a critical point of ϕ. The Morse index of x by $m(x,\phi)$ is defined as the supremum of the dimensions of the vector subspace of E on which $\phi''(x)$ is negative definite. The nullity of x by $\nu(x,\phi)$ is defined as the dimension of $\operatorname{Ker}\phi''(x)$. A critical point x will be said to be nondegenerate if $\phi''(x)$ is invertible.

Denote by m_∞, ν_∞ the Morse index and nullity of ∞ for functional J. By (10), $m_\infty = \dim W^-$, $\nu_\infty = \dim W^0$.

A set $K \subset E$ is called critical set if $K \subset \phi^{-1}(c) \cap \operatorname{crit}(\phi)$ for some $c \in \mathbb{R}$. A critical set K is called discrete nondegenerate critical manifold, if K is connected and $m(x,\phi)$ does not depend on $x \in K$.

Definition 9. The Poincare polynomial of the pair (ϕ^b, ϕ^a) is defined by $P_\lambda(\phi^b,\phi^a) = \sum_{n=0}^{\infty} \dim H_n(\phi^b,\phi^a;\Gamma)\lambda^n$, where $H_n(\phi^b,\phi^a;\Gamma)$ denotes the nth singular relative homology of the pair (ϕ^b,ϕ^a) with coefficients in field Γ. Define the topological Morse index of critical set K as $i_\lambda(K) = \sum_{n=0}^{\infty} \dim H_n(\phi^c, \phi^c \setminus K;\Gamma)\lambda^n$.

For simplicity, we write $m(x)$ and $m(K)$ instead of $m(x,\phi)$ and $m(K,\phi)$, respectively. It is well known that if x is a nondegenerate critical point and $m(x)$ is finite, then $i_\lambda(x) = \lambda^{m(x)}$. If K is a nondegenerate critical manifold and $m(K)$ is finite, then $i_\lambda(K) = \lambda^{m(K)}Q(\lambda)$, where $Q(\lambda)$ is a polynomial with nonnegative integer coefficients (see [13, 15]).

Next we investigate $P_\lambda(E,\phi^a)$ and use functional J (see (8)) or J_R (see (18)) instead of ϕ, E_p instead of E.

Lemma 10 (see [19, 24]). *Assume that (P1) and (P4) hold, and system* (1) *is not p-resonant at ∞. Then there exists $a \in \mathbb{R}$, $a < J(crit\,(J))$ such that*

$$P_\lambda\left(E_p, J^a\right) = \lambda^{m(\infty)}. \tag{21}$$

Lemma 11. *Assume that (P1) and (P4) hold, system* (1) *is p-resonant at ∞, $\lim_{|t|\to+\infty} G(t) = -\infty$, and $g(t)$ is bounded in $\mathbb{R}^N$. Then there exists $a \in \mathbb{R}$, and* (21) *is valid.*

Proof. Write $x = u + v + w \in E_p$ with $u \in W^+$, $v \in W^-$, $w \in W^0$. Then there exist $M_1 > 0$, $M_2 > 0$ such that $\langle J'(x), u\rangle \geq \delta\|u\|^2 - c\|u\| > 0$ as $\|u\| > M_1$, $\langle J'(x), v\rangle \leq -\delta\|v\|^2 + c\|v\| < 0$ as $\|v\| > M_2$. Let $B_{M_1} = \{x \in E_p \mid \|u\| \leq M_1\}$, $B_{M_2} = \{x \in E_p \mid \|v\| \leq M_2\}$. By previous argument, it follows that J has no critical points in $E_p \setminus (B_{M_1} \cup B_{M_2})$.

On the other hand, for all $x \in B_{M_1} \cup B_{M_2}$,

$$J(x) \geq c - \sum_{n=1}^{p} G\left(u_n + v_n + \|w_n\| \cdot \frac{w_n}{\|w_n\|}\right) \longrightarrow +\infty, \quad \text{as } \|w\| \longrightarrow \infty. \tag{22}$$

Therefore there exists $a_1 \in \mathbb{R}$, such that $a_1 < J(\operatorname{crit}(J))$. For $x \in B_{M_2}$, we have

$$\begin{aligned} J(x) \geq{}& \frac{1}{2}\delta\|u\|^2 - c \\ &- \sum_{n=1}^{p} G\left(\|u_n\| \cdot \frac{u_n}{\|u_n\|} + v_n + \|w_n\| \cdot \frac{w_n}{\|w_n\|}\right), \end{aligned} \tag{23}$$

hence $J(x) \to +\infty$ as $\| u + w \| \to \infty$, which implies that J is bounded from the following in B_{M_2}. Let $a < \min\{a_1, \inf_{x\in B_{M_2}} J(x)\}$, then $J^a \subset E_p \setminus B_{M_2}$, and J^a is a strong deformation retraction of $E_p \setminus B_{M_2}$. By Lemma 6, J satisfies (PS) condition, and we have

$$\begin{aligned} H_n\left(E_p, J^a\right) &\cong H_n\left(E_p, E_p \setminus B_{M_2}\right) \\ &\cong H_n\left(W^-, W^- \setminus B_{M_2}\right) \cong \delta_{n,m(\infty)}\Gamma. \end{aligned} \tag{24}$$

So we obtain (21). □

Lemma 12. *Under the assumption of Theorem 3, there exists $a \in \mathbb{R}$ such that $P_\lambda(E_p, J_R^a) = \lambda^{m(\infty)}$.*

Proof. Let $x = u + v + w \in E_p$ with $u \in W^+$, $v \in W^-$, and $w \in W^0$. Then there exist $R_1 > R + 1$ such that all critical points of J_R are in $B_{M_1} \cap B_{M_2} \cap B_{M_3}$, where B_{M_1} and B_{M_2} are the same as in proof of Lemma 11, and $B_{M_3} = \{x \in E_p \mid \|w\|^2 \leq R_1\}$. In fact,

$$\begin{aligned} \left\langle J_R'(x), u\right\rangle &= \left\langle J'(x), u\right\rangle > 0, \quad x \notin B_{M_1}, \\ \left\langle J_R'(x), v\right\rangle &= \left\langle J'(x), v\right\rangle < 0, \quad x \notin B_{M_2}, \\ \left\langle J_R'(x), w\right\rangle &\geq -c\|w\| + 8\|w\|^2\left(\|w\|^2 - R\right)^3 \\ &\geq 8\|w\|^2 - c\|w\| > 0, \quad x \notin B_{M_3}. \end{aligned} \tag{25}$$

Similarly, for $x \in B_{M_2}$, $J_R(x) \geq (1/2)\delta\|u\|^2 + \varphi_R(\|w\|^2) - c - \sum_{n=1}^{p} G(x_n)$, and $J_R(x) \to +\infty$ as $\|u + w\| \to \infty$, which implies that $J_R(x)$ is bounded from the following in B_{M_2}. Let $a_0 = \inf_{x\in B_{M_2}} J_R(x)$. If $a < \min\{a_0, J_R(\operatorname{crit}(J_R))\}$, by J_R satisfies (PS) condition, and methods of strong deformation retract, we have $P_\lambda(E_p, J_R^a) = \lambda^{m(\infty)}$. The proof is completed. □

Assume that on Hilbert space E there is an action of discrete group G, and denote by $\operatorname{fix}(G)$ the fixed points set for the G action; that is, $\operatorname{fix}(G) = \{x \in E \mid gx = x, \forall g \in G\}$. The functional ϕ is called G invariant, if $\phi(gx) = \phi(x)$, $\forall x \in E$, and $\forall g \in G$. In the following, Z_p denotes a cyclic group of p order. In terms of Proposition 8.2 and Proposition 8.5 in [13], we have following lemma.

Lemma 13. *Assume that ϕ is a C^2-functional on an Hilbert space E and satisfies (PS) condition. Let a, b (b possible ∞) be two regular values of ϕ. Assume that $crit(\phi_a^b) = crit(\phi) \cap \phi^{-1}(a,b)$ consists only of critical sets, and then the following Morse relation holds:*

$$\sum_{K\subset crit(\phi_a^b)} i_\lambda(K) = P_\lambda\left(\phi^b, \phi^a\right) + (1+\lambda)Q(\lambda), \tag{26}$$

where $Q(\lambda)$ is a polynomial with nonnegative integer coefficients. If all the critical points of ϕ in ϕ_a^b are nondegenerate and have finite Morse index, then (26) *can be written as*

$$\sum_{x\in crit(\phi_a^b)} \lambda^{m(x)} = P_\lambda\left(\phi^b, \phi^a\right) + (1+\lambda)Q(\lambda). \tag{27}$$

Now if ϕ is Z_p invariant, and $crit(\phi_a^b) \cap fix(Z_p)$ consists only of nondegenerate critical points having finite Morse index, then (26) *becomes*

$$\sum_{x\in crit(\phi_a^b)\cap fix(Z_p)} \lambda^{m(x)} + (1+\lambda) Z(\lambda) = P_\lambda\left(\phi^b, \phi^a\right) + (1+\lambda) Q(\lambda), \tag{28}$$

where $Z(\lambda)$ is a formal series with nonnegative integer coefficients. Moreover if $crit(\phi_a^b) - fix(Z_p)$ consists only of nondegenerate critical manifolds having finite Morse index, then

$$Z(\lambda) = \sum_{K\subset crit(\phi_a^b)-fix(Z_p)} \lambda^{m(K)}. \tag{29}$$

Remark 14. By (29), our main goal in this paper is to estimate $Z(1)$ which gives a lower bound of the number of the nonconstant critical points of J in E_p.

Lemma 15. *Let $z = \{z_n\}$ be a critical point of functional J. Denote by $\tau_1^2, \tau_2^2, \ldots, \tau_l^2$ the positive eigenvalues (repeated according to their multiplicity) of $F''(z_n)$, where $\tau_j > 0$, $j \in Z[1,l]$. Under the assumption (P2), we have $\sharp(z,J) = l + 2\sum_{j=1}^{l}[(p/\pi)\arcsin(\tau_j/2)]$, where $[\cdot]$ denotes the greatest-integer function and $\sharp(z,J)$ is the number of eigenvalues $\lambda < 0$ such that $\langle J''(z)u,u\rangle = \lambda\|u\|^2$.*

Proof. By $\langle J''(z)u,u\rangle = \sum_{n=1}^{p}[|\Delta u_n|^2 - (F''(z_n)u_n,u_n)] = -\sum_{n=1}^{p}[(\Delta^2 u_{n-1} + F''(z_n)u_n,u_n)]$, we consider the equation $\Delta^2 y_{n-1} + F''(z_n)y_n = -\lambda y_n$, $y_{n+p} = y_n$, where $z = \{z_n\}$ is the critical point of J. It is easy to see that $\lambda_{k,j} = 4\sin^2(k\pi/p) - \tau_j^2$ are eigenvalues of $\Delta^2 y_{n-1} + F''(z_n)y_n$ on $\mathbb{R}^N$, where $n,k \in Z[1,p]$, $j \in Z[1,l]$. Therefore the number of negative eigenvalues $\lambda_{k,j}$ is just what we are looking for; the proof is completed. □

Definition 16. For any critical point z of J, there are l positive eigenvalues (repeated according to their multiplicity) of $F''(z_n)$, which will be denoted by $\tau_1^2, \ldots, \tau_l^2$. The number $\rho(z) = (2p/\pi)\sum_{j=1}^{l}\arcsin(\tau_j/2)$ is called twist number of z. Moreover the twist number of ∞ is defined by $\rho(\infty) = (2p/\pi)\sum_{j=1}^{l(\infty)}\arcsin(\tau_j/2)$, where $l(\infty)$ is the number of the positive eigenvalues (repeated according to their multiplicity) of A_∞.

Let $z = \{z_n\}$ be a constant critical point of functional J; that is, $z_1 = z_2 = \cdots = z_p$. By Lemma 15 and Definition 16, it is easy to deduce the following relation between the Morse index and the twist number as follows:

$$\rho(z)p - pN \le m(z,J) \le \rho(z)p + pN. \tag{30}$$

In view of the number l or $l(\infty)$ of the positive eigenvalues (repeated according to their multiplicity) of $F''(z)$ or A_∞, the constant critical point z is called τ-positive (resp., τ-negative) if l is even (resp., odd). On the contrary, the virtual critical point ∞ is called τ-positive (resp., τ-negative) if $l(\infty)$ is odd (resp., even), see [24].

We denote by h_1 and h_2 the number of τ-positive and τ-negative critical points of J. If A_∞ is invertible, then $h_1 - h_2 = (-1)^{l(\infty)}$. Thus, if we consider ∞ as a virtual critical point, we have that the number of τ-positive critical points equals the number of τ-negative critical points. However, if A_∞ is singular, the result is not hold in general. If we introduce $|h_1 - h_2|$ virtual critical points having twist number zero, where they are considered as τ-positive if $h_1 < h_2$ and as τ-negative if $h_1 > h_2$, then the number of τ-positive critical points is also equal to the number of τ-negative critical points.

Let $h = \max\{h_1, h_2\}$, which has been used in (4) and (5). We denote by $x_1, \ldots, x_h$ the τ-positive critical points and by $y_1, \ldots, y_h$ the τ-negative critical points such that

$$\rho(x_1) \le \cdots \le \rho(x_h), \qquad \rho(y_1) \le \cdots \le \rho(y_h). \tag{31}$$

Then the global twist number Θ of the system (1) is defined by

$$\Theta = \sum_{i=1}^{h}\left|\rho(x_i) - \rho(y_i)\right|. \tag{32}$$

4. Proof of Main Results

Proof of Theorem 1. The argument is analogous to one used by Benci and Fortunato in [24]. Set $m(z) = m(z,J)$. Under the assumption (P2), let $z_1, \ldots, z_n$ be the nondegenerate constant critical points of J.

By Lemmas 5 and 10, functional J satisfies (PS) condition, and there exists sufficiently small $a \in \mathbb{R}$ such that $P_\lambda(J^b, J^a) = P_\lambda(E_p, J^a) = \lambda^{m(\infty)}$, where $b = \infty$. Since J is C^2 and Z_p invariant functional on E_p, then by assumption (P3), we have $\sum_{i=1}^{n}\lambda^{m(z_i)} + (1+\lambda)Z(\lambda) = \lambda^{m(\infty)} + (1+\lambda)Q(\lambda)$; that is,

$$\sum_{i=1}^{n}\lambda^{m(z_i)} - \lambda^{m(\infty)} = (1+\lambda)(Q(\lambda) - Z(\lambda)). \tag{33}$$

Let m_i, f_i ($i \in Z[1,h]$) denote the Morse indices of the τ-positive and τ-negative critical points (including ∞) of J, and without loss of generalities, assume that ∞ is τ-negative. So $m(\infty) = f_j$ for some $j \in Z[1,h]$, where h is referred to (31). Then (33) becomes

$$\sum_{i=1, i\ne j}^{h}\frac{\lambda^{m_i} + \lambda^{f_i}}{1+\lambda} + \frac{\lambda^{m_j} - \lambda^{m(\infty)}}{1+\lambda} = Q(\lambda) - Z(\lambda). \tag{34}$$

Set $Q(\lambda) = \sum_s q_s\lambda^s$, $Z(\lambda) = \sum_s z_s\lambda^s$, and $B(\lambda) = Q(\lambda) - Z(\lambda) = \sum_s b_s\lambda^s$, where q_s, z_s are nonnegative integer and $b_s = q_s - z_s$.

By Remark 14, the lower bound of the number of nonconstant p-periodic solutions for system (1) is to estimate $Z(1)$. Since $q_s \ge 0$, $z_s \ge 0$, then

$$n(p) = \sum_s z_s \ge \sum_{b_s<0} z_s = \sum_{b_s<0}(q_s - b_s) \ge -\sum_{b_s<0} b_s. \tag{35}$$

Let $B^- = -\sum_{b_s<0} b_s$. By (35), we turn our attention to estimate B^-.

If l is even (resp., odd), by Lemma 15, $m(z, J)$ is also even (resp., odd). Therefore by the definition of τ-positive and τ-negative critical points of J, m_i ($i \in Z[1,h]$) are even numbers, f_i are odd numbers for $i \neq j$, $i \in Z[1,h]$, and $f_j = m(\infty)$ is a even number.

Set $M_1 = \{r \mid m_r > f_r, r \in Z[1,h], r \neq j\}$, $M_2 = \{r \mid m_r < f_r, r \in Z[1,h], r \neq j\}$, and

$$C(\lambda) = \sum_{r\in M_1} \frac{\lambda^{m_r} + \lambda^{f_r}}{1+\lambda}, \qquad D(\lambda) = \sum_{r\in M_2} \frac{\lambda^{m_r} + \lambda^{f_r}}{1+\lambda}, \qquad E(\lambda) = \frac{\lambda^{m_j} - \lambda^{f_j}}{1+\lambda}. \tag{36}$$

By (34), we have $B(\lambda) = Q(\lambda) - Z(\lambda) = \sum_s b_s \lambda^s = C(\lambda) + D(\lambda) + E(\lambda)$, and

$$C(\lambda) = \sum_{r\in M_1} \sum_{i=f_r}^{m_r-1} c_{r,i}\lambda^i, \qquad D(\lambda) = \sum_{r\in M_2} \sum_{i=m_r}^{f_r-1} d_{r,i}\lambda^i, \tag{37}$$

where $c_{r,i} = (-1)^{i+1}$, $d_{r,i} = (-1)^i$. Meanwhile, if $m_j > f_j$, $E(\lambda) = \sum_{i=f_j}^{m_j-1} e_{i,1}\lambda^i$, $e_{i,1} = (-1)^{i+1}$. If $m_j < f_j$, $E(\lambda) = \sum_{i=m_j}^{f_j-1} e_{i,2}\lambda^i$, $e_{i,2} = (-1)^i$. Clearly $E(\lambda) = 0$ if $m_j = f_j$.

A straight analysis shows that $B^- = (1/2)\sum_{r=1}^{h} |m_r - f_r| - ((h-1)/2)$. By (30) and the definition of global twist number that refer to (32), we have $n(p) \geq (1/2)\Theta p - h(pN + (1/2)) + (1/2)$. It completes the proof of Theorem 1. □

Proof of Theorem 2. Under the assumptions of Theorem 2, by Lemmas 6 and 11, functional J satisfies (PS) condition, and there exists $a \in \mathbb{R}$ such that $P_\lambda(J^b, J^a) = P_\lambda(E_p, J^a) = \lambda^{m(\infty)}$, for $b = \infty$.

Similarly, we have $\sum_{i=1}^{n} \lambda^{m(z_i)} + (1+\lambda)Z(\lambda) = \lambda^{m(\infty)} + (1+\lambda)Q(\lambda)$, where z_i ($i \in Z[1,n]$) are nondegenerate critical points of J. The remainder is the same as that of Theorem 1. □

The following lemma is needed to prove Theorem 3.

Lemma 17. *If all assumptions in Theorem 3 hold, then there exists $Q > 0$ (independent of R) such that*

$$m(x, J_R) + \nu(x, J_R) \leq m(\infty) + \nu(\infty), \tag{38}$$

where $x = u + v + w$ with $u \in W^+$, $v \in W^-$, $w \in W^0$, $\|w\| \geq Q$, and $m(x, J_R)$, $\nu(x, J_R)$ denote the Morse index and nullity of critical point x for functional J_R, respectively.

Proof. Let $x = u + v + w \in E_p$ be a critical point of J_R. By Lemma 7, we have $\|u + v\| \leq M$. Therefore $\|x\| \to \infty$ if and only if $\|w\| \to \infty$. Since

$$\begin{aligned}\left\langle J_R''(x)u, u\right\rangle &= \sum_{n=1}^{p} \left[|\Delta u_n|^2 - (A_\infty u_n, u_n) - \left(g'(x_n)u_n, u_n\right)\right] \geq \delta\|u\|^2 \\ &\quad - \sum_{n=1}^{p} \left(g'(x_n)u_n, u_n\right),\end{aligned} \tag{39}$$

by assumption (P1), there exists $\varepsilon \in (0, \delta)$ such that

$$\begin{aligned}&\sum_{n=1}^{p} \left(g'(x_n)u_n, u_n\right) \\ &\quad = \sum_{n=1}^{p} \left(g'\left(\|w\| \cdot \frac{w_n}{\|w\|} + u_n + v_n\right)u_n, u_n\right) \\ &\quad \leq \varepsilon\|u\|^2,\end{aligned} \tag{40}$$

as $\|w\| \to \infty$. Therefore there exists $Q > 0$ such that $\langle J_R''(x)u, u\rangle > 0$ as $\|w\| \geq Q$. It follows that relation (38) is valid, and the proof is completed. □

Proof of Theorem 3. Let $L = m(\infty) + \nu(\infty)$, and denote by $z_1, \ldots, z_n$ the constant critical points of J. We assume, without loss of generalities, $m(z_i, J) \geq L + 1$ as $i \in Z[1,r]$, $m(z_i, J) < L$ as $i \in Z[r+1,n]$. Clearly $L \leq pN$.

Set $H_1 = \{x \in \mathrm{crit}(J_R) \mid \|w\| > Q\}$, $H_2 = \{x \in \mathrm{crit}(J_R) \cap \mathrm{fix}(Z_p) \mid \|w\| \leq Q\}$, and $H_3 = \mathrm{crit}(J_R) - (H_1 \cup H_2)$, where $x = u + v + w$ is the decomposition of $x \in E_p$ with $w \in W^0$ and Q is large enough.

By Lemma 17, H_1 contains only critical points of J_R which have $m(x, J_R) + \nu(x, J_R) \leq L$. $H_2 = \{z_i \mid i \in Z[1,n]\}$, since $J_R(x) = J(x)$ as $\|w\| \leq Q$. $H_3 \subset \mathrm{crit}(J) - \mathrm{fix}(Z_p)$, since $\varphi_R(x) = 0$ when $\|w\| \leq Q$. Moreover by assumption (P3), H_3 contains only nondegenerate critical manifolds.

Since J_R satisfies (PS) condition, by Lemma 13, relation (28) reads

$$\begin{aligned}&\sum_{x\in H_1} i_\lambda(x) + \sum_{x\in H_2} i_\lambda(x) + \sum_{K\subset H_3} i_\lambda(K) \\ &\quad = P_\lambda\left(E_p, J_R^a\right) + (1+\lambda)Q(\lambda),\end{aligned} \tag{41}$$

that is,

$$\begin{aligned}&\sum_{x\in H_1} i_\lambda(x) + \sum_{i=1}^{r} i_\lambda(z_i) + \sum_{i=r+1}^{n} i_\lambda(z_i) \\ &\quad + (1+\lambda)Z(\lambda) = \lambda^{m(\infty)} + (1+\lambda)Q(\lambda),\end{aligned} \tag{42}$$

where $Z(\lambda) = \sum_{K\subset H_3} \lambda^{m(K)}$. For $Z(\lambda) = \sum_{i=0}^{\infty} z_i\lambda^i$, we set $Z_l = \sum_{i=l}^{\infty} z_i\lambda^i$, where $l \in \mathbb{N}$. And analogous notation can be introduced for $Q(\lambda)$. Then, considering the terms of degree $\geq L + 1$ in (42), we have

$$\sum_{i=1}^{r} \lambda^{m(z_i,J)} + b_L\lambda^{L+1} = (1+\lambda)B(\lambda), \tag{43}$$

where $b_L = z_L - q_L$, $B(\lambda) = Q_{L+1}(\lambda) - Z_{L+1}(\lambda)$. Clearly

$$\begin{aligned}n(p) &= \sum_s z_s \geq \sum_{b_s\leq 0, s\geq L+1} z_s \\ &= \sum_{b_s\leq 0, s\geq L+1} (q_s - b_s) \geq -\sum_{b_s\leq 0, s\geq L+1} b_s = B^-,\end{aligned} \tag{44}$$

that is, B^- is the absolute value of the sum of the negative coefficients of $B(\lambda)$. Next we estimate the number B^-.

Let $x_1, \ldots, x_{h_1}$ and $y_1, \ldots, y_{h_2}$ $(h_1 + h_2 = r)$ be the τ-positive and τ-negative critical points of J with nonzero twist numbers, whose order satisfies (31), and $x_i, y_j \in \{z_1, \ldots, z_r\}$, $i \in Z[1, h_1]$, $j \in Z[1, h_2]$. Without loss of generalities, assume $h_1 \geq h_2$, and introduce $h_3 (= h_1 - h_2)$ virtual τ-negative critical points $\overline{y}_i$ $(i \in Z[1, h_3])$ having twist number 0 and Morse index 0; that is,

$$\rho(\overline{y}_i) = 0, \quad f_i = m(\overline{y}_i, J) = 0, \quad i \in Z[1, h_3]. \tag{45}$$

For $i \in Z[1, h_1]$, set $m_i = m(x_i, J)$, $f_i = m(\overline{y}_i, J)$, where $\overline{y}_{j+h_3} = y_j$, $j \in Z[1, h_2]$. Then (43) can be written as $\sum_{i=1}^{h_1} \lambda^{m_i} + \sum_{i=h_3+1}^{h_1} \lambda^{f_i} + b_L \lambda^{L+1} = (1+\lambda) B(\lambda)$. Setting $\lambda = -1$, then $b_L = -h_3$ if L is odd, and $b_L = h_3$ if L is even. So

$$B(\lambda) = \sum_{i=1}^{h_3} \frac{\lambda^{m_i} + \lambda^{L+1}}{1+\lambda} + \sum_{i=h_3+1}^{h_1} \frac{\lambda^{m_i} + \lambda^{f_i}}{1+\lambda}, \quad \text{if } L \text{ is even},$$

$$B(\lambda) = \sum_{i=1}^{h_3} \frac{\lambda^{m_i} - \lambda^{L+1}}{1+\lambda} + \sum_{i=h_3+1}^{h_1} \frac{\lambda^{m_i} + \lambda^{f_i}}{1+\lambda}, \quad \text{if } L \text{ is odd}. \tag{46}$$

A straight analysis shows that $B^- = \sum_{i=1}^{h_3} [(1/2)(m_i - L) - 1] + \sum_{i=h_3+1}^{h} (1/2)|m_i - f_i|$ if L is even, and $B^- = \sum_{i=1}^{h_3} (1/2)(m_i - L) + \sum_{i=h_3+1}^{h} (1/2)(|m_i - f_i| - 1)$ if L is odd. Therefore

$$B^- \geq \sum_{i=1}^{h_3} \frac{1}{2}(m_i - L - 2) + \sum_{i=h_3+1}^{h} \frac{1}{2}(|m_i - f_i| - 1). \tag{47}$$

By (30) and (45), we have

$$\begin{aligned} m_i = m_i - f_i &\geq (\rho(x_i) - \rho(\overline{y}_i))\, p - pN, \quad i \in Z[1, h_3], \\ |m_i - f_i| &\geq |\rho(x_i) - \rho(\overline{y}_i)|\, p - 2pN, \quad i \in Z[h_3 + 1, h]. \end{aligned} \tag{48}$$

In view of (45), (47), and (48), we have

$$\begin{aligned} B^- \geq{}& \frac{1}{2} \sum_{i=1}^{h} p\, |\rho(x_i) - \rho(y_i)| \\ &- \frac{1}{2} pNh_3 - \frac{1}{2} Lh_3 - pNh_2 - h_3 \\ &- \frac{1}{2} h_2 \geq \frac{1}{2} \Theta p - h(pN + 1). \end{aligned} \tag{49}$$

The proof is completed. □

Remark 18. Although A_∞ is invertible under the assumptions of Theorem 3, we do not make use of (42) directly, because we consider only the terms of degree $\geq L + 1$ in proof of Theorem 3.

Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants (11101187), NCETFJ (JA11144), the Excellent Youth Foundation of Fujian Province (2012J06001), and the Foundation of Education of Fujian Province (JA09152).

References

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[2] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian System: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[3] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Application*, Academic Press, Boston, Mass, USA, 1991.

[4] W.-T. Li, M.-F. Niu, and J.-P. Sun, "Existence of positive solutions of BVPs for second-order nonlinear difference systems," *Applied Mathematics and Computation*, vol. 152, no. 3, pp. 779–798, 2004.

[5] J.-P. Sun and W.-T. Li, "Multiple positive solutions of a discrete difference system," *Applied Mathematics and Computation*, vol. 143, no. 2-3, pp. 213–221, 2003.

[6] G. Wei, Z. Wang, and B. Shen, "Error-constrained filtering for a class of nonlinear time-varying delay systems with non-Gaussian noises," *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2876–2882, 2010.

[7] G. L. Wei, Z. D. Wang, and B. Shen, "Probability-dependent gain-scheduled filtering for stochastic systems with missing measurements," *IEEE Transactions on Circuits and Systems II*, vol. 58, no. 11, pp. 753–757, 2011.

[8] K. C. Chang, *Critical Point Theory and Its Applications*, Science and Technical Press, Shanghai, China, 1980.

[9] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, NY, USA, 1989.

[10] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society*, vol. 68, no. 2, pp. 419–430, 2003.

[11] Z. Guo and J. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China A*, vol. 46, no. 4, pp. 506–515, 2003.

[12] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[13] V. Benci, "A new approach to the Morse-Conley theory and some applications," *Annali di Matematica Pura ed Applicata*, vol. 158, pp. 231–305, 1991.

[14] K.-c. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser Boston, Boston, Mass, USA, 1993.

[15] C. Conley, *Isolated invariant sets and the Morse index*, vol. 38 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1978.

[16] A. Szulkin, "Cohomology and Morse theory for strongly indefinite functionals," *Mathematische Zeitschrift*, vol. 209, no. 3, pp. 375–418, 1992.

[17] K. C. Chang, "Solutions of asymptotically linear operator equations via Morse theory," *Communications on Pure and Applied Mathematics*, vol. 34, no. 5, pp. 693–712, 1981.

[18] H. Amann and E. Zehnder, "Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations," *Annali della Scuola Normale Superiore di Pisa*, vol. 7, no. 4, pp. 539–603, 1980.

[19] T. Bartsch and S. Li, "Critical point theory for asymptotically quadratic functionals and applications to problems with resonance," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 28, no. 3, pp. 419–441, 1997.

[20] M.-Y. Jiang, "Remarks on the resonant elliptic problem," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 53, no. 7-8, pp. 1193–1207, 2003.

[21] J. Su, "Multiplicity results for asymptotically linear elliptic problems at resonance," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 397–408, 2003.

[22] J. Su, "Existence and multiplicity results for classes of elliptic resonant problems," *Journal of Mathematical Analysis and Applications*, vol. 273, no. 2, pp. 565–579, 2002.

[23] W. Zou and J. Q. Liu, "Multiple solutions for resonant elliptic equations via local linking theory and Morse theory," *Journal of Differential Equations*, vol. 170, no. 1, pp. 68–95, 2001.

[24] V. Benci and D. Fortunato, "Estimate of the number of periodic solutions via the twist number," *Journal of Differential Equations*, vol. 133, no. 1, pp. 117–135, 1997.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 615947, 15 pages
http://dx.doi.org/10.1155/2013/615947

Research Article

Sliding Intermittent Control for BAM Neural Networks with Delays

Jianqiang Hu,[1] Jinling Liang,[1] Hamid Reza Karimi,[2] and Jinde Cao[1,3]

[1] *Department of Mathematics, Research Center for Complex Systems and Network Sciences, Southeast University, Nanjing 210096, China*
[2] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*
[3] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Jinling Liang; jinlliang@gmail.com

Received 9 May 2013; Accepted 29 May 2013

Academic Editor: Zidong Wang

Copyright © 2013 Jianqiang Hu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper addresses the exponential stability problem for a class of delayed bidirectional associative memory (BAM) neural networks with delays. A sliding intermittent controller which takes the advantages of the periodically intermittent control idea and the impulsive control scheme is proposed and employed to the delayed BAM system. With the adjustable parameter taking different particular values, such a sliding intermittent control method can comprise several kinds of control schemes as special cases, such as the continuous feedback control, the impulsive control, the periodically intermittent control, and the semi-impulsive control. By using analysis techniques and the Lyapunov function methods, some sufficient criteria are derived for the closed-loop delayed BAM neural networks to be globally exponentially stable. Finally, two illustrative examples are given to show the effectiveness of the proposed control scheme and the obtained theoretical results.

1. Introduction

Since the bidirectional associative memory (BAM) neural networks were first proposed by Kosko [1] which are well known as the extension of the unidirectional autoassociators such as the Hopfield neural networks, they have been widely studied due to their extensive applications such as pattern recognition, signal or image processing, solving optimization problems, and automatic control [2–7]. Later, constant delays are introduced in [8] to the BAM neural networks, and it is proved that the delayed versions of the neural networks are significant for handling certain motion-related optimization problems [9]. For more results concerning the dynamical behaviors of the BAM neural networks with delays, we refer to [10, 11].

In practice, most of the neural networks are unstable or convergent with a rate far less than the requirement. Under such cases we need to try to stabilize them or speed up the convergence rate of the neural system in order to make the system work more efficiently. Therefore, the designing of appropriate control input becomes extremely urgent. When it comes to the problem of stabilizing a nonlinear system, it is natural to consider the feedback strategies. There are two basic kinds of feedback control: the state feedback control and the output feedback control. When referring to the control methods, different kinds of schemes have been utilized to stabilize the nonlinear system such as static feedback control [12], delayed feedback control [13], adaptive control [14], fuzzy control [15], sampled control [16], sliding mode control [17], and random control [18, 19]. In terms of the control time, the controllers are classified with continuous control and discontinuous control. Compared with the continuous control, the discontinuous control including the impulsive control [20, 21] and the intermittent control [22, 23] has attracted much more attention, and it is very effective, practical, and applicable in many areas, especially in secure communication [24].

In the literature, the impulsive neural networks have been extensively studied from the following two aspects: either the system is subject to the impulsive state displacements at fixed time instants or the system is imposed by external impulsive control [25]. The main idea of periodically intermittent

control [26] is that when the system signal becomes weak to a low level, the external control will be imposed to supplement the loss of signal; after some period of time the external control is stopped; in the next control period the external control is needed again. Compared with the method of periodically intermittent control, the system with the impulsive control is activated only at some isolated time moments. Both the impulsive control and the intermittent control have their own benefits and disadvantages. The main difference between these two control techniques lies in the length of control period; the former has zero duration, while the later has a nonzero control width. Meanwhile, the cost of the intermittent control is much higher.

The aim of this paper is to design a sliding intermittent controller by combining the advantages of both the impulsive control and the periodically intermittent control. More specifically, in one control period, we will impose the continuous state feedback control at the preceding control width and the impulsive control in the latter control width. The sliding intermittent control method is very flexible and could achieve the expected control performance. The sketch of such a controller is shown in Figure 1. Motivated by the name of the slide rheostat in the physical electronic circuitry, we named such a joint controller as the sliding intermittent controller.

In this paper, we will investigate the exponential stability problem of the delayed BAM neural networks under the proposed sliding intermittent control. The closed-loop neural system becomes a switched network where the switching rules are dependent on the time index. To the best of the authors' knowledge, this is the first time in the literature to consider such a joint controller. The rest of the paper is organized as follows. In Section 2, the model discussed in this paper and the novel sliding intermittent control idea are introduced, and some preliminaries are also given. In Section 3, several sufficient criteria are established to ensure the delayed BAM neural networks to be exponentially stable under the sliding intermittent control scheme. Meanwhile, several particular cases are discussed. In Section 4, two illustrative examples are given to demonstrate the effectiveness of the proposed results. And finally, the paper is concluded in Section 5.

2. Problem Formulation and Some Preliminaries

The delayed bidirectional associative memory (BAM) neural networks have been investigated in [8, 9] as follows:

$$\begin{aligned} \dot{w}_i(t) &= -a_i w_i(t) + \sum_{j=1}^{m} p_{ij} \tilde{f}_j\left(z_j\left(t-\tau_{ij}\right)\right) + c_i, \quad i \in \mathcal{I}_n, \\ \dot{z}_j(t) &= -b_j z_j(t) + \sum_{i=1}^{n} q_{ji} \tilde{g}_i\left(w_i\left(t-\sigma_{ji}\right)\right) + d_j, \quad j \in \mathcal{I}_m, \end{aligned} \tag{1}$$

where the index set $\mathcal{I}_n = \{1, 2, \ldots, n\}$, $\mathcal{I}_m = \{1, 2, \ldots, m\}$; $w_i(t)$, $z_j(t) \in \mathbb{R}$ are the activations of the ith neuron and the jth neuron, respectively; a_i, b_j are positive constants denoting the rates with which the cells i and j reset their potential to the resting states when isolated from the other cells and inputs; time delays τ_{ij} and σ_{ji} are nonnegative constants corresponding to the finite speeds of axonal signal transmission with $\tau^* = \max_{i,j}\{\tau_{ij}, \sigma_{ji}\}$; p_{ij} and q_{ji} are the delayed connection weights denoting the strengths of connectivity between the cells j and i; c_i and d_j denote, respectively, the ith and the jth components of an external input source introduced from the network outside to the cells i and j; $\tilde{f}_j$ and $\tilde{g}_i$ are bounded nonlinear activation functions, and throughout this paper they are assumed to satisfy the following conditions:

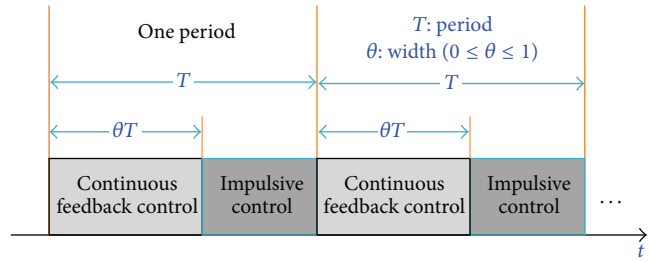


FIGURE 1: Sketch map of the sliding intermittent control.

$$\begin{aligned} \left|\tilde{f}_j(\nu_1) - \tilde{f}_j(\nu_2)\right| &\le L_j^f \left|\nu_1 - \nu_2\right|, \\ \left|\tilde{g}_i(\nu_1) - \tilde{g}_i(\nu_2)\right| &\le L_i^g \left|\nu_1 - \nu_2\right|, \end{aligned} \tag{2}$$

where $\nu_1, \nu_2 \in \mathbb{R}$ and $i \in \mathcal{I}_n$, $j \in \mathcal{I}_m$; positive scalars L_j^f, L_i^g are known.

Remark 1. The above conditions are general in the literature to study the existence and uniqueness of the equilibrium for the delayed BAM neural networks (1) without assuming the activation functions to be monotonic or differentiable [27–29].

Letting (w^*, z^*) with $w^* = (w_1^*, w_2^*, \ldots, w_n^*)^T \in \mathbb{R}^n$ and $z^* = (z_1^*, z_2^*, \ldots, z_m^*)^T \in \mathbb{R}^m$ be an equilibrium of the system (1) and denoting $x_i(t) = w_i(t) - w_i^*$, $y_j(t) = z_j(t) - z_j^*$, the equilibrium of the network (1) can be transformed to the origin of the following system:

$$\begin{aligned} \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right), \quad i \in \mathcal{I}_n, \\ \dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right), \quad j \in \mathcal{I}_m, \end{aligned} \tag{3}$$

where $f_j(y_j) = \tilde{f}_j(y_j + z_j^*) - \tilde{f}_j(z_j^*)$ and $g_i(x_i) = \tilde{g}_i(x_i + w_i^*) - \tilde{g}_i(w_i^*)$.

It is generally known that, under some specific cases such as the abrupt changes of system parameters or the occurrence of time delays, the network may present unstable dynamics such as bifurcation, oscillation, divergence, or instability. In

this paper, we will focus on the unstable delayed BAM neural system (3). To stabilize the network (3), the novel sliding intermittent control scheme is imposed as follows:

$$\begin{aligned}
\dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right) + u_i^{(1)}(t), \\
&\qquad t \in [lT, (l+\theta)T), \\
\dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right), \\
&\qquad t \in [(l+\theta)T, (l+1)T),\ t \neq t_k, \\
\Delta x_i(t) &= I_k^{(1)}\left(x_i(t)\right), \quad t = t_k, \\
\dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right) + u_j^{(2)}(t), \\
&\qquad t \in [lT, (l+\theta)T), \\
\dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right), \\
&\qquad t \in [(l+\theta)T, (l+1)T),\ t \neq t_k, \\
\Delta y_j(t) &= I_k^{(2)}\left(y_j(t)\right), \quad t = t_k,
\end{aligned} \tag{4}$$

where $i \in \mathcal{I}_n$, $j \in \mathcal{I}_m$; $l \in \mathbb{N}_0^+ \triangleq \{0, 1, 2, \ldots, \}$ and $k \in \mathbb{N}^+ \triangleq \{1, 2, \ldots\}$; constant $\theta \in [0, 1]$ and T are scalars denoting the control width and the control period, respectively; $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)$ and $x_i(t_k^+) = \lim_{h\to 0^+} x_i(t_k + h)$, $x_i(t_k) = x_i(t_k^-) = \lim_{h\to 0^-} x_i(t_k + h)$, $y_j(t_k^+) = \lim_{h\to 0^+} y_j(t_k + h)$, and $y_j(t_k) = y_j(t_k^-) = \lim_{h\to 0^-} y_j(t_k + h)$. The impulsive sequence $\{t_k\}_{k=1}^{\infty}$ satisfies

$$\begin{gathered}
0 < t_1 < t_2 < \cdots < t_k < \cdots, \ \lim_{k\to\infty} t_k = \infty, \\
t_k \in \bigcup_{l\in\mathbb{N}_0^+} ((l+\theta)T, (l+1)T].
\end{gathered} \tag{5}$$

$u_i^{(1)}(t)$, $u_j^{(2)}(t)$ are controllers imposed on the system (3) when $t \in [lT, (l+\theta)T)$, and $I_k^{(1)}(\cdot)$, $I_k^{(2)}(\cdot)$ are the impulsive operators imposed at the impulsive moments $\{t_k\}_{k=1}^{\infty}$. Here we use the linear state feedback and linear impulsive control strategies; that is,

$$\begin{aligned}
u_i^{(1)}(t) &= k_i^{(1)} x_i(t), \qquad u_j^{(2)}(t) = k_j^{(2)} y_j(t), \\
I_k^{(1)}\left(x_i(t_k)\right) &= h_{ik}^{(1)} x_i(t_k), \qquad I_k^{(2)}\left(y_j(t_k)\right) = h_{jk}^{(2)} y_j(t_k),
\end{aligned} \tag{6}$$

where $k_i^{(1)}$, $k_j^{(2)}$, $h_{ik}^{(1)}$, and $h_{jk}^{(2)}$ are gain constants.

Remark 2. The sliding intermittent control idea is illustrated in Figure 1 with the adjustable parameter θ (i.e., the control width θ shown in Figure 1). The closed-loop system (4) can be the continuous controlled neural networks ($\theta = 1$), the impulsive controlled neural networks ($\theta = 0$), or the hybrid controlled neural networks ($0 < \theta < 1$).

The system (4) is supplemented with initial function given by $x_i(s) = \varphi_i(s)$, $y_j(s) = \psi_j(s)$, $-\tau^* \le s \le 0$, where $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^T \in C([-\tau^*, 0], \mathbb{R}^n)$, $\psi(s) = (\psi_1(s), \psi_2(s), \ldots, \psi_m(s))^T \in C([-\tau^*, 0], \mathbb{R}^m)$, and $C([-\tau^*, 0], \mathbb{R}^n)$ represents the set of all n-dimensional continuous functions defined on the interval $[-\tau^*, 0]$. Obviously, the solution of (4) is piecewise left-hand continuous with possible discontinuity at $t = t_k$ for $k \in \mathbb{N}^+$.

The following definition and lemmas are introduced before we give the main results of the paper.

Definition 3. The system (4) is said to be globally exponentially stable if there exist constants $\alpha > 0$, $\beta > 0$ such that for all $t > 0$,

$$\begin{aligned}
&\sum_{i=1}^{n} \left|x_i(t)\right| + \sum_{j=1}^{m} \left|y_j(t)\right| \\
&\quad \le \beta e^{-\alpha t} \sup_{-\tau^*\le s\le 0} \left(\sum_{i=1}^{n} \left|\varphi_i(s)\right| + \sum_{j=1}^{m} \left|\psi_j(s)\right|\right)
\end{aligned} \tag{7}$$

holds for all $\varphi(\cdot) \in C([-\tau^*, 0], \mathbb{R}^n)$ and $\psi(\cdot) \in C([-\tau^*, 0], \mathbb{R}^m)$.

Lemma 4 (see [30]). *Let $V(\cdot) : [t_0 - \tau, \infty) \to [0, \infty)$ be a continuous function such that*

$$\dot{V}(t) \le -aV(t) + b\overline{V}(t) \tag{8}$$

is satisfied for $t \ge t_0$. If $a > b > 0$, then

$$V(t) \le \overline{V}(t_0) e^{-\lambda(t-t_0)}, \quad t \ge t_0, \tag{9}$$

where $\overline{V}(t) = \sup_{t-\tau\le s\le t} V(s)$, $\lambda > 0$ is the unique positive real root of the equation $-a + \lambda + be^{\lambda\tau} = 0$.

Lemma 5 (see [31]). *Let $q \ge 0$, $\tau > 0$, $\mu_k > 0$ $(k = 1, 2, \ldots)$, and p be constants, and assume that $V(t)$ is a piecewise continuous nonnegative function satisfying*

$$\begin{gathered}
D^+V(t) \le pV(t) + q\overline{V}(t), \quad t \ge t_0,\ t \neq t_k \\
V(t_k^+) \le \mu_k V(t_k), \quad k = 1, 2, \ldots.
\end{gathered} \tag{10}$$

If there exists constant β such that

$$\frac{\ln \mu_k}{t_k - t_{k-1}} \le \beta, \quad p + dq + \beta < 0, \tag{11}$$

hold for $k = 1, 2, \ldots$, then

$$V(t) \le d\overline{V}(t_0) e^{-\lambda(t-t_0)}, \tag{12}$$

where $\overline{V}(t) = \sup_{t-\tau\le s\le t} V(s)$, $d = \sup_k \{e^{\beta(t_k - t_{k-1})}, e^{-\beta(t_k - t_{k-1})}\} < \infty$, and λ is the unique positive root of the equation $\lambda + p + dqe^{\lambda\tau} + \beta = 0$.

3. Main Results

In this section, some sufficient criteria will be given based on the sliding intermittent control scheme. First, the global exponential stability of the closed-loop hybrid neural networks (4) is analyzed, and then several criteria are obtained by setting different parameters in the sliding intermittent controller.

Theorem 6. *Assume the upper bound delay $\tau^* < \min\{\theta T, (1-\theta)T\}$ and the external imposed impulsive strengths satisfy $h_{ik}^{(1)}, h_{jk}^{(2)} \neq -1$. Under the sliding intermittent control, the closed-loop control system (4) is globally exponentially stable if there exist constants β and $k_i^{(1)}, k_j^{(2)}$ such that the following conditions hold:*

(i)

$$a_i - k_i^{(1)} > L_i^g \sum_{j=1}^{m} \left|q_{ji}\right|, \quad i \in \mathcal{I}_n,$$
$$b_j - k_j^{(2)} > L_j^f \sum_{i=1}^{n} \left|p_{ij}\right|, \quad j \in \mathcal{I}_m, \tag{13}$$

(ii)

$$\frac{\ln \mu_{k+1}}{t_{k+1} - t_k} \leq \beta, \qquad \rho + d\widetilde{b}_1 + \beta < 0,$$
$$k \in \mathbb{N}^+ \setminus \{i_l \mid l = 0, 1, 2, \ldots\}, \tag{14}$$

(iii)

$$\widetilde{a}_1 > \widetilde{b}_1, \qquad \varrho \triangleq \eta + \lambda_1(\theta - \gamma) - \lambda_2(\theta + \gamma) - \frac{\ln d}{T} > 0, \tag{15}$$

where $\widetilde{a}_1 = \min_{i,j}\{a_i - k_i^{(1)} - \eta, b_j - k_j^{(2)} - \eta\}$; $\widetilde{b}_1 = \max_i\{L_i^g \sum_{j=1}^m |q_{ji}| e^{\eta\sigma_{ji}}\} + \max_j\{L_j^f \sum_{i=1}^n |p_{ij}| e^{\eta\tau_{ij}}\}$; $\mu_k = \max_{i,j}\{|1 + h_{ik}^{(1)}|, |1 + h_{jk}^{(2)}|\}$; $\rho = \max_{i,j}\{\eta - a_i, \eta - b_j\}$; $d = \sup_k\{e^{\beta(t_{k+1}-t_k)}, e^{-\beta(t_{k+1}-t_k)}\}$; t_{i_l} is used to denote the last impulsive moment on the interval $[(l+\theta)T, (l+1)T)$ with $t_{i_l} < (l+1)T - \tau^*$; $\gamma = \tau^*/T$; $0 < \eta \leq \eta^*$; $\eta^* = \min_{i,j}\{\theta_i^*, \vartheta_j^* \mid F_i(\theta_i^*) = 0, G_j(\vartheta_j^*) = 0\}$ with functions $F_i(\cdot)$, $G_j(\cdot)$ defined as

$$F_i(\theta_i) = a_i - k_i^{(1)} - \theta_i - \sum_{j=1}^{m} \left|q_{ji}\right| L_i^g e^{\theta_i \sigma_{ji}}, \quad \theta_i \in [0, \infty),$$
$$G_j(\vartheta_j) = b_j - k_j^{(2)} - \vartheta_j - \sum_{i=1}^{n} \left|p_{ij}\right| L_j^f e^{\vartheta_j \tau_{ij}}, \quad \vartheta_j \in [0, \infty), \tag{16}$$

and λ_1, λ_2 are, respectively, the unique positive real root of the equations $-\widetilde{a}_1 + \lambda_1 + \widetilde{b}_1 e^{\lambda_1 \tau^*} = 0$ and $\lambda_2 + \rho + d\widetilde{b}_1 e^{\lambda_2 \tau^*} + \beta = 0$.

More specifically, we have the following inequality:

$$\sum_{i=1}^{n} \left|x_i(t)\right| + \sum_{j=1}^{m} \left|y_j(t)\right|$$
$$\leq M e^{-\varrho t} \sup_{-\tau^* \leq s \leq 0} \left(\sum_{i=1}^{n} \left|\varphi_i(s)\right| + \sum_{j=1}^{m} \left|\psi_j(s)\right| \right), \quad t > 0, \tag{17}$$

with $M = \max\{e^{\lambda_1(\theta-\gamma)\theta T}, e^{\lambda_2(1-\theta-\gamma)\theta T + (1-\theta)\ln d}\}$.

Proof. For the functions $F_i(\cdot)$ and $G_j(\cdot)$ defined in (16), from the condition (13), it is clear that

$$F_i(0) = a_i - k_i^{(1)} - \sum_{j=1}^{m} \left|q_{ji}\right| L_i^g > 0,$$
$$G_j(0) = b_j - k_j^{(2)} - \sum_{i=1}^{n} \left|p_{ij}\right| L_j^f > 0. \tag{18}$$

Since $F_i(\cdot)$ and $G_j(\cdot)$ are continuous on $[0, \infty)$ and $\lim_{\theta_i \to \infty} F_i(\theta_i) = -\infty$, $\lim_{\vartheta_j \to \infty} G_j(\vartheta_j) = -\infty$, there must exist constants $\theta_i^* > 0$ and $\vartheta_j^* > 0$ such that $F_i(\theta_i^*) = 0$ and $G_j(\vartheta_j^*) = 0$. By setting $\eta^* = \min\{\theta_1^*, \theta_2^*, \ldots, \theta_n^*, \vartheta_1^*, \vartheta_2^*, \ldots, \vartheta_m^*\}$, one obtains that, for any $0 < \eta \leq \eta^*$,

$$F_i(\eta) = a_i - k_i^{(1)} - \eta - \sum_{j=1}^{m} \left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}} \geq 0,$$
$$G_j(\eta) = b_j - k_j^{(2)} - \eta - \sum_{i=1}^{n} \left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}} \geq 0. \tag{19}$$

Now consider the Lyapunov function defined as follows:

$$V(t) = \sum_{i=1}^{n} u_i(t) + \sum_{j=1}^{m} v_j(t), \quad t \geq 0, \tag{20}$$

where $u_i(t) = e^{\eta t}|x_i(t)|$ and $v_j(t) = e^{\eta t}|y_j(t)|$. Obviously, $V(t)$ is a positive definite function for $t \geq 0$.

(1) When $t \in [lT, (l+\theta)T)$, $l \in \mathbb{N}_0^+$, one is easy to have

$$D^+ u_i(t) \leq -\left(a_i - k_i^{(1)} - \eta\right) u_i(t) + \sum_{j=1}^{m} \left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}} v_j\left(t - \tau_{ij}\right),$$
$$D^+ v_j(t) \leq -\left(b_j - k_j^{(2)} - \eta\right) v_j(t) + \sum_{i=1}^{n} \left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}} u_i\left(t - \sigma_{ji}\right). \tag{21}$$

Calculating the upper right Dini derivative of $V(t)$ along the solutions of network (4), from the above inequality, we get

$$\begin{aligned} D^+V(t) \le & -\sum_{i=1}^{n}\left(a_i - k_i^{(1)} - \eta\right)u_i(t) \\ & -\sum_{j=1}^{m}\left(b_j - k_j^{(2)} - \eta\right)v_j(t) \\ & +\sum_{i=1}^{n}\sum_{j=1}^{m}\left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}} v_j\left(t-\tau_{ij}\right) \\ & +\sum_{j=1}^{m}\sum_{i=1}^{n}\left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}} u_i\left(t-\sigma_{ji}\right) \\ \le & -\tilde{a}_1 V(t) + \tilde{b}_1\left(\sup_{t-\tau^*\le s\le t} V(s)\right), \end{aligned} \tag{22}$$

where $\tilde{a}_1 = \min_{i,j}\{a_i - k_i^{(1)} - \eta, b_j - k_j^{(2)} - \eta\}$ and $\tilde{b}_1 = \max_i\{L_i^g \sum_{j=1}^m |q_{ji}| e^{\eta\sigma_{ji}}\} + \max_j\{L_j^f \sum_{i=1}^n |p_{ij}| e^{\eta\tau_{ij}}\}$. By Lemma 4, one has

$$\begin{aligned} V(t) \le e^{-\lambda_1(t-lT)}\left(\sup_{lT-\tau^*\le s\le lT} V(s)\right) = e^{-\lambda_1(t-lT)}\overline{V}(lT), \\ t\in[lT,(l+\theta)T), \end{aligned} \tag{23}$$

where λ_1 is the unique positive real root of the equation $-\tilde{a}_1 + \lambda_1 + \tilde{b}_1 e^{\lambda_1\tau^*} = 0$.

(2) When $t\in[(l+\theta)T,(l+1)T)$ and $t\ne t_k, l\in\mathbb{N}_0^+, k\in\mathbb{N}^+$, it is easy to have

$$\begin{aligned} D^+u_i(t) &\le -(a_i-\eta)u_i(t) + \sum_{j=1}^{m}\left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}} v_j\left(t-\tau_{ij}\right), \\ D^+v_j(t) &\le -\left(b_j-\eta\right)v_j(t) + \sum_{i=1}^{n}\left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}} u_i\left(t-\sigma_{ji}\right). \end{aligned} \tag{24}$$

Without loss of generality, suppose $t\in(t_{k-1},t_k]$, $k\le i_l$, where t_{i_l} is assumed to be the last impulsive moment on the interval $[(l+\theta)T,(l+1)T)$. It follows from the inequality (24) that

$$\begin{aligned} D^+V(t) \le & \sum_{i=1}^{n}(\eta-a_i)u_i(t) + \sum_{j=1}^{m}\left(\eta-b_j\right)v_j(t) \\ & +\sum_{i=1}^{n}\left(L_i^g\sum_{j=1}^{m}e^{\eta\sigma_{ji}}\left|q_{ji}\right|\right)u_i\left(t-\sigma_{ji}\right) \\ & +\sum_{j=1}^{m}\left(L_j^f\sum_{i=1}^{n}e^{\eta\tau_{ij}}\left|p_{ij}\right|\right)v_j\left(t-\tau_{ij}\right) \\ \le & \rho V(t) + \tilde{b}_1\left(\sup_{t-\tau^*\le s\le t} V(s)\right), \end{aligned} \tag{25}$$

where $\rho = \max_{i,j}\{\eta-a_i,\eta-b_j\}$; $\tilde{b}_1 = \max_i\{L_i^g\sum_{j=1}^m e^{\eta\sigma_{ji}}|q_{ji}|\} + \max_j\{L_j^f\sum_{i=1}^n e^{\eta\tau_{ij}}|p_{ij}|\}$.

When $t\in[(l+\theta)T,(l+1)T)$ and $t=t_k$,

$$\begin{aligned} x_i(t_k^+) &= x_i(t_k) + I_k^{(1)}(x_i(t_k)) = \left(1+h_{ik}^{(1)}\right)x_i(t_k), \\ y_j(t_k^+) &= y_j(t_k) + I_k^{(2)}\left(y_j(t_k)\right) = \left(1+h_{jk}^{(2)}\right)y_j(t_k). \end{aligned} \tag{26}$$

Considering the condition that $h_{ik}^{(1)}, h_{jk}^{(2)}\ne -1$, we have $\mu_k > 0$ and

$$\begin{aligned} u_i(t_k^+) &= e^{\eta t_k^+}\left|x_i(t_k^+)\right| \\ &\le \left|1+h_{ik}^{(1)}\right|u_i(t_k) \le \mu_k u_i(t_k), \quad i\in\mathcal{I}_n, \\ v_j(t_k^+) &= e^{\eta t_k^+}\left|y_j(t_k^+)\right| \\ &\le \left|1+h_{jk}^{(2)}\right|v_j(t_k) \le \mu_k v_j(t_k), \quad j\in\mathcal{I}_m, \end{aligned} \tag{27}$$

which infers that

$$V(t_k^+) = \sum_{i=1}^{n}u_i(t_k^+) + \sum_{j=1}^{m}v_j(t_k^+) \le \mu_k V(t_k). \tag{28}$$

From condition (14) and Lemma 5, one has

$$\begin{aligned} V(t) \le de^{-\lambda_2(t-(l+\theta)T)}\overline{V}((l+\theta)T), \\ t\in[(l+\theta)T,(l+1)T), \end{aligned} \tag{29}$$

where $d = \sup_k\{e^{\beta(t_k-t_{k-1})}, e^{-\beta(t_k-t_{k-1})}\}$ and λ_2 is the unique positive real root of the equation $\lambda_2 + \rho + d\tilde{b}_1 e^{\lambda_2\tau^*} + \beta = 0$.

(3) Now, we are ready to estimate $V(t)$ based on the inequalities (23) and (29) with the method of induction.

When $t\in[0,\theta T)$, one obtains

$$V(t) \le e^{-\lambda_1 t}\overline{V}(0). \tag{30}$$

When $t\in[\theta T,T)$, one can derive

$$\begin{aligned} V(t) &\le de^{-\lambda_2(t-\theta T)}\overline{V}(\theta T) \\ &\le de^{-[\lambda_2(t-\theta T)+\lambda_1(\theta T-\tau^*)]}\overline{V}(0). \end{aligned} \tag{31}$$

When $t\in[T,(1+\theta)T)$, we have

$$\begin{aligned} V(t) &\le e^{-\lambda_1(t-T)}\overline{V}(T) \\ &\le de^{-[\lambda_2(T-\tau^*-\theta T)+\lambda_1(t-(1-\theta)T-\tau^*)]}\overline{V}(0). \end{aligned} \tag{32}$$

When $t\in[(1+\theta)T,2T)$, one can derive

$$\begin{aligned} V(t) &\le de^{-\lambda_2(t-(1+\theta)T)}\overline{V}((1+\theta)T) \\ &\le d^2e^{-[\lambda_2(t-2\theta T-\tau^*)+2\lambda_1(\theta T-\tau^*)]}\overline{V}(0). \end{aligned} \tag{33}$$

By induction, one can derive the following estimation of $V(t)$ for any integer $l \in \mathbb{N}_0^+$:

when $t \in [lT, (l+\theta)T)$,

$$V(t) \le d^l e^{-[l\lambda_2(T-\theta T-\tau^*)+\lambda_1(t-l(T-\theta T+\tau^*))]}\overline{V}(0), \tag{34}$$

and when $t \in [(l+\theta)T, (l+1)T)$,

$$V(t) \le d^{l+1} e^{-[\lambda_2(t-(l+1)\theta T-l\tau^*)+(l+1)\lambda_1(\theta T-\tau^*)]}\overline{V}(0). \tag{35}$$

By setting $\gamma = \tau^*/T$ and substituting it to the above two inequalities, one has, for $t \in [lT, (l+\theta)T)$,

$$\begin{aligned} V(t) &\le d^l e^{-[\lambda_2 lT(1-\theta-\gamma)+\lambda_1(t-lT)+\lambda_1(\theta-\gamma)lT]}\overline{V}(0) \\ &\le d^l e^{[\lambda_2(\theta+\gamma)lT-\lambda_1(\theta-\gamma)lT]}\overline{V}(0) \\ &\le d^l e^{[\lambda_2(\theta+\gamma)t+\lambda_1(\theta-\gamma)(-t+\theta T)]}\overline{V}(0) \\ &= d^l e^{-[\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)]t} e^{\lambda_1(\theta-\gamma)\theta T}\overline{V}(0) \\ &\le e^{(\ln d/T)t} e^{-[\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)]t} e^{\lambda_1(\theta-\gamma)\theta T}\overline{V}(0) \\ &\le e^{-[\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)-((\ln d)/T)]t} M\overline{V}(0), \end{aligned} \tag{36}$$

and, for $t \in [(l+\theta)T, (l+1)T)$,

$$\begin{aligned} V(t) &\le d^{l+1} e^{-[\lambda_2(t-(l+1)\theta T-l\gamma T)+\lambda_1(\theta-\gamma)(l+1)T]}\overline{V}(0) \\ &\le d^{l+1} e^{[\lambda_2\theta T+\lambda_2 lT(\theta+\gamma)-\lambda_1(\theta-\gamma)t]}\overline{V}(0) \\ &\le d^{l+1} e^{[\lambda_2\theta T+\lambda_2(t-\theta T)(\theta+\gamma)-\lambda_1(\theta-\gamma)t]}\overline{V}(0) \\ &= d^{l+1} e^{[\lambda_2(\theta+\gamma)t-\lambda_1(\theta-\gamma)t]+\lambda_2\theta T(1-\theta-\gamma)}\overline{V}(0) \\ &\le e^{((t/T)-\theta+1)\ln d} e^{-[\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)]t} e^{\lambda_2\theta T(1-\theta-\gamma)}\overline{V}(0) \\ &\le e^{-[\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)-((\ln d)/T)]t} M\overline{V}(0). \end{aligned} \tag{37}$$

Therefore, we have

$$V(t) \le e^{-[\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)-((\ln d)/T)]t} M\overline{V}(0), \quad t > 0, \tag{38}$$

which means that, for any $t > 0$,

$$\begin{aligned} &\sum_{i=1}^{n}|x_i(t)| + \sum_{j=1}^{m}|y_j(t)| \\ &\quad \le e^{-[\eta+\lambda_1(\theta-\gamma)-\lambda_2(\theta+\gamma)-((\ln d)/T)]t} M\overline{V}(0), \end{aligned} \tag{39}$$

where $M = \max\{e^{\lambda_1(\theta-\gamma)\theta T}, e^{\lambda_2(1-\theta-\gamma)\theta T+(1-\theta)\ln d}\}$. And the conclusion that the origin of network (4) is exponentially stable. The proof is complete. □

Remark 7. In the above Theorem 6, the control width $\theta \in (0,1)$ which does not include the boundary cases. If the parameter $\theta \to 0$, the controlled system approximates to impulsive neural networks. If the parameter $\theta \to 1$, the controlled system approximates to continuous neural networks. And the controlled system can be handled, respectively, by the methods of the impulsive system and the continuous system. In order to avoid such boundary cases, usually we can take $\theta = 0.5$; that is, the periodically intermittent controller and the impulsive controller play a significant role in the process of control; this results a switched neural networks.

Remark 8. Most of the literatures [28, 32] concerning the global exponential stability of the delayed BAM neural networks with impulses have focused on the stable system. Namely, without the impulsive disturbance, the original neural networks are stable, and under the impulsive disturbance the system can still be kept stable with particular conditions. While, in this article, the impulses can be viewed as either control input or impulsive disturbance, at the same time, the original system is not required to be stable initially.

If adjustable parameter $\theta = 1$, the controlled system turns out to be the following model; that is, the original unstable system (3) is controlled with the continuous feedback controller. Under such case, the conditions in the following proposition will guarantee the closed-loop system to be globally exponentially stable:

$$\begin{aligned} \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right) \\ &\quad + u_i^{(1)}(t), \quad i \in \mathcal{I}_n, \\ \dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right) \\ &\quad + u_j^{(2)}(t), \quad j \in \mathcal{I}_m. \end{aligned} \tag{40}$$

Proposition 9. *Under the continuous feedback control scheme, the origin of the closed-loop control system* (40) *is globally exponentially stable if there exist constants* $k_i^{(1)}$ *and* $k_j^{(2)}$ *such that*

$$\begin{aligned} a_i - k_i^{(1)} &> L_i^g \sum_{j=1}^{m}\left|q_{ji}\right|, \quad i \in \mathcal{I}_n, \\ b_j - k_j^{(2)} &> L_j^f \sum_{i=1}^{n}\left|p_{ij}\right|, \quad j \in \mathcal{I}_m. \end{aligned} \tag{41}$$

Furthermore, one has the following inequality:

$$\begin{aligned} &\sum_{i=1}^{n}|x_i(t)| + \sum_{j=1}^{m}|y_j(t)| \\ &\quad \le \left(\tilde{b}_1\tau^* + 1\right) e^{-\eta t} \sup_{-\tau^*\le s\le 0}\left(\sum_{i=1}^{n}|\varphi_i(s)| \right. \\ &\qquad \left. + \sum_{j=1}^{m}|\psi_j(s)|\right), \quad t > 0, \end{aligned} \tag{42}$$

where the parameters η and $\tilde{b}_i$ are consistent with the ones in Theorem 6.

Proof. Consider the Lyapunov function defined as follows:

$$W(t) = V(t) + \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}} \int_{t-\tau_{ij}}^{t} v_j(s)\,\mathrm{d}s + \left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}} \int_{t-\sigma_{ji}}^{t} u_i(s)\,\mathrm{d}s\right), \quad t \geq 0. \tag{43}$$

The upper right Dini derivative of $W(t)$ with respect to time t along the solutions of the network (40) can be calculated as follows:

$$\begin{aligned} D^+W(t) &= \sum_{i=1}^{n} D^+u_i(t) + \sum_{j=1}^{m} D^+v_j(t) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{m}\left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}}\left[v_j(t) - v_j\left(t-\tau_{ij}\right)\right] \\ &\quad + \sum_{j=1}^{m}\sum_{i=1}^{n}\left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}}\left[u_i(t) - u_i\left(t-\sigma_{ji}\right)\right] \\ &\leq -\sum_{i=1}^{n}\left(a_i - k_i^{(1)} - \eta - \sum_{j=1}^{m}\left|q_{ji}\right| L_i^g e^{\eta\sigma_{ji}}\right) u_i(t) \\ &\quad - \sum_{j=1}^{m}\left(b_j - k_j^{(2)} - \eta - \sum_{i=1}^{n}\left|p_{ij}\right| L_j^f e^{\eta\tau_{ij}}\right) v_j(t) \\ &= -\sum_{i=1}^{n} F_i(\eta)\, u_i(t) - \sum_{j=1}^{m} G_j(\eta)\, v_j(t) \leq 0 \end{aligned} \tag{44}$$

which means that $W(t) \leq W(0)$. Hence we have $\sum_{i=1}^{n}|x_i(t)| + \sum_{j=1}^{m}|y_j(t)| \leq (\tilde{b}_1\tau^* + 1)e^{-\eta t}\overline{V}(0)$, and this completes the proof. □

If the impulsive strengths $h_{ik}^{(1)} = h_{jk}^{(2)} \equiv 0$, namely, there are no impulsive controls on the latter control interval in each control period, which means the closed-loop system is only subject to the continuous feedback control in the preceding control width of each control period. Such a case is then reduced to the pure periodically intermittent control, and the neural network system (4) turns into the following controlled neural network (45). The conditions in the following proposition will guarantee the closed-loop system to be globally exponentially stable. In order to obtain the main result, the following lemma is given firstly

$$\begin{aligned} \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right) + u_i^{(1)}(t), \\ &\qquad lT \leq t < (l+\theta)T, \\ \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right), \\ &\qquad (l+\theta)T \leq t < (l+1)T, \\ \dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right) + u_j^{(2)}(t), \\ &\qquad lT \leq t < (l+\theta)T, \\ \dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right), \\ &\qquad (l+\theta)T \leq t < (l+1)T. \end{aligned} \tag{45}$$

Lemma 10 (see [23]). *Let $V(\cdot)$: $[t_0 - \tau, \infty) \to [0, \infty)$ be a continuous function such that*

$$\dot{V}(t) \leq aV(t) + b\left(\sup_{t-\tau\leq s\leq t} V(s)\right) \tag{46}$$

is satisfied for $t \geq t_0$. If $a > 0$, $b > 0$, then

$$\begin{aligned} V(t) &\leq \sup_{t-\tau\leq s\leq t} V(s) \\ &\leq \left(\sup_{t_0-\tau\leq s\leq t_0} V(s)\right) e^{(a+b)(t-t_0)}, \quad t \geq t_0. \end{aligned} \tag{47}$$

Proposition 11. *Assuming the upper bound delay $\tau^* < \min\{\theta T, (1-\theta)T\}$, under the periodically intermittent control, the closed-loop control system (45) is globally exponentially stable if the control gains $k_i^{(1)}$ and $k_j^{(2)}$ satisfy the following conditions:*

(i)

$$\begin{aligned} a_i - k_i^{(1)} &> L_i^g \sum_{j=1}^{m}\left|q_{ji}\right|, \quad i \in \mathcal{I}_n, \\ b_j - k_j^{(2)} &> L_j^f \sum_{i=1}^{n}\left|p_{ij}\right|, \quad j \in \mathcal{I}_m, \end{aligned} \tag{48}$$

(ii)

$$\tilde{a}_1 > \tilde{b}_1, \qquad \eta + \lambda_1(\theta - \gamma) - \left(\tilde{\rho} + \tilde{b}_1\right)(1-\theta) > 0. \tag{49}$$

Moreover, we have

$$\begin{aligned}&\sum_{i=1}^{n}\left|x_i(t)\right|+\sum_{j=1}^{m}\left|y_j(t)\right|\\&\quad\le \overline{M}e^{-[\eta+\lambda_1(\theta-\gamma)-(\widetilde{\rho}+\widetilde{b}_1)(1-\theta)]t}\\&\qquad\times\sup_{-\tau^*\le s\le 0}\left(\sum_{i=1}^{n}\left|\varphi_i(s)\right|+\sum_{j=1}^{m}\left|\psi_j(s)\right|\right),\quad t>0,\end{aligned}\tag{50}$$

where $\widetilde{\rho}=\max_{i,j}\{\eta-a_i,\eta-b_j,\kappa\}$, *and* κ *is any positive constant;* $\widetilde{M}=e^{\lambda_1\theta T(\theta-\gamma)}$, *and the parameters* $\widetilde{a}_1$, $\widetilde{b}_1$, η, λ_1, *and* γ *are consistent with those in Theorem 6.*

Proof. Considering the same Lyapunov function as that in Theorem 6, by similar analytical technique, one can get the following results on the control period and the control width as follows.

(1) When $t\in[lT,(l+\theta)T)$, $l\in\mathbb{N}_0^+$, we have $D^+V(t)\le -\widetilde{a}_1V(t)+\widetilde{b}_1\overline{V}(t)$. By Lemma 4, one obtains that, for any $t\in[lT,(l+\theta)T)$, $V(t)\le e^{-\lambda_1(t-lT)}\overline{V}(lT)$.

(2) When $t\in[(l+\theta)T,(l+1)T)$, $l\in\mathbb{N}_0^+$, we have $D^+V(t)\le\widetilde{\rho}V(t)+\widetilde{b}_1\overline{V}(t)$. By Lemma 10, it is derived that, for any $t\in[(l+\theta)T,(l+1)T)$, $V(t)\le e^{(\widetilde{\rho}+\widetilde{b}_1)(t-(l+\theta)T)}\overline{V}((l+\theta)T)$.

From the above inequality relationships, by similar estimation procedure, we can get the following conclusion:

$$\begin{aligned}&V(t)\le e^{-\lambda_1(t-lT)+l(\widetilde{\rho}+\widetilde{b}_1)(1-\theta)T-\lambda_1 l(\theta-\tau^*)}\overline{V}(0),\\&\qquad t\in[lT,(l+\theta)T),\\&V(t)\le e^{-\lambda_1(l+1)(\theta T-\tau^*)+l(\widetilde{\rho}+\widetilde{b}_1)(1-\theta)T+(\widetilde{\rho}+\widetilde{b}_1)(t-lT-\theta T)}\overline{V}(0),\\&\qquad t\in[(l+\theta)T,(l+1)T).\end{aligned}\tag{51}$$

By the notational expression $\tau^*=\gamma T$, one can further obtain

$$\begin{aligned}&V(t)\le e^{[-\lambda_1(\theta-\gamma)+(\widetilde{\rho}+\widetilde{b}_1)(1-\theta)]t}e^{\lambda_1\theta T(\theta-\gamma)}\overline{V}(0),\\&\qquad t\in[lT,(l+\theta)T),\\&V(t)\le e^{[-\lambda_1(\theta-\gamma)+(\widetilde{\rho}+\widetilde{b}_1)(1-\theta)]t}\overline{V}(0),\\&\qquad t\in[(l+\theta)T,(l+1)T).\end{aligned}\tag{52}$$

Hence it follows that, for any $t>0$,

$$\begin{aligned}&\sum_{i=1}^{n}\left|x_i(t)\right|+\sum_{j=1}^{m}\left|y_j(t)\right|\\&\quad\le \overline{M}e^{-[\eta+\lambda_1(\theta-\gamma)-(\widetilde{\rho}+\widetilde{b}_1)(1-\theta)]t}\\&\qquad\times\sup_{-\tau^*\le s\le 0}\left(\sum_{i=1}^{n}\left|\varphi_i(s)\right|+\sum_{j=1}^{m}\left|\psi_j(s)\right|\right),\end{aligned}\tag{53}$$

and this completes the proof. □

If the adjustable parameter $\theta=0$, that is, the original unstable system (3) is subjected to the impulsive controller, then system (4) becomes the following impulsive neural networks (54). Under such a case, it would be natural to assume that the frequency of the impulses should not be too low; that is, some restrictions on the impulsive periods and the impulsive strengths are needed. The conditions in the following proposition will guarantee the closed-loop system (54) to be globally exponentially stable:

$$\begin{aligned}&\dot{x}_i(t)=-a_ix_i(t)+\sum_{j=1}^{m}p_{ij}f_j\left(y_j\left(t-\tau_{ij}\right)\right),\\&\qquad t\ge 0,t\ne t_k,\ k\in\mathbb{N}^+,\\&\Delta x_i(t_k)=I_k^{(1)}\left(x_i(t_k)\right),\quad k\in\mathbb{N}^+,\ i\in\mathcal{I}_n,\\&\dot{y}_j(t)=-b_jy_j(t)+\sum_{i=1}^{n}q_{ji}g_i\left(x_i\left(t-\sigma_{ji}\right)\right),\\&\qquad t\ge 0,t\ne t_k,\ k\in\mathbb{N}^+,\\&\Delta y_j(t_k)=I_k^{(2)}\left(y_j(t_k)\right),\quad k\in\mathbb{N}^+,\ j\in\mathcal{I}_m.\end{aligned}\tag{54}$$

Proposition 12. *Assume the external imposed impulsive strengths* $h_{ik}^{(1)}$, $h_{jk}^{(2)}\ne-1$. *The origin of the closed-loop control system* (54) *is globally exponentially stable if there exists constant* β *such that, for* $k=1,2,\ldots$, *the following conditions hold:*

$$\frac{\ln\mu_k}{t_k-t_{k-1}}\le\beta,\qquad \rho+d\widetilde{b}_1+\beta<0.\tag{55}$$

More specifically, we have

$$\begin{aligned}&\sum_{i=1}^{n}\left|x_i(t)\right|+\sum_{j=1}^{m}\left|y_j(t)\right|\\&\quad\le de^{-(\eta+\lambda_2)t}\sup_{-\tau^*\le s\le 0}\left(\sum_{i=1}^{n}\left|\varphi_i(s)\right|\right.\\&\qquad\left.+\sum_{j=1}^{m}\left|\psi_j(s)\right|\right),\quad t>0,\end{aligned}\tag{56}$$

where the parameters μ_k, ρ, d, $\widetilde{b}_1$, *and* η *are consistent with those in Theorem 6.*

Proof. Considering the same Lyapunov function as that in Theorem 6, by similar analytical technique, one can get the following results on the impulsive interval and the impulse moments as follows.

(1) When $t\in(t_{k-1},t_k]$, the upper right Dini derivative of $V(t)$ along the solution of (54) is depicted as $D^+V(t)\le\rho V(t)+\widetilde{b}_1(\sup_{t-\tau^*\le s\le t}V(s))$.

(2) When $t=t_k$, $k\in\mathbb{N}^+$, $V(t_k^+)\le\mu_kV(t_k)$.

From Lemma 5 and the conditions in (55), it follows that

$$V(t) \le d\left(\sup_{0-\tau^* \le s \le 0} V(s)\right) e^{-\lambda_2(t-0)}, \tag{57}$$

which means

$$\begin{aligned} &\sum_{i=1}^{n} |x_i(t)| + \sum_{j=1}^{m} |y_j(t)| \\ &\quad \le e^{-\eta t} d e^{-\lambda_2 t} \overline{V}(0) \le d e^{-(\eta+\lambda_2)t} \overline{V}(0), \quad t > 0 \end{aligned} \tag{58}$$

and the proof is completed. □

If the continuous feedback control gains $k_i^{(1)} = k_j^{(2)} \equiv 0$, that is, there is no feedback on the preceding control interval in a control period, and only impulsive control is imposed on the latter control interval, this means the system is only under the piecewise impulsive (not the uniformly distributed) control. As for such case, the impulsive system is reduced to the following one:

$$\begin{aligned} &\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{m} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right), \\ &\quad t \in [lT, (l+\theta)T) \cup \left\{[(l+\theta)T, (l+1)T) \setminus \{t \ne t_k\}_{k=1}^{\infty}\right\}, \\ &\Delta x_i(t) = I_k^{(1)}(x_i(t)), \quad t = t_k, \\ &\dot{y}_j(t) = -b_j y_j(t) + \sum_{i=1}^{n} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right), \\ &\quad t \in [lT, (l+\theta)T) \cup \left\{[(l+\theta)T, (l+1)T) \setminus \{t \ne t_k\}_{k=1}^{\infty}\right\}, \\ &\Delta y_j(t) = I_k^{(2)}(y_j(t)), \quad t = t_k. \end{aligned} \tag{59}$$

From (59), it is noticed that the occurrence of the impulses is not uniformly distributed since the impulses never occur on the interval $[lT, (l+\theta)T)$, $l \in \mathbb{N}_0^+$, whereas they frequently occur on the interval $((l+\theta)T, (l+1)T)$, $l \in \mathbb{N}_0^+$. By observing the proof of Proposition 12, the result is somewhat more conservative especially when the control period T is very large and the control width θ approaches to one. In order to describe the conservatism for such case, we will utilize the notation of average impulsive interval proposed in [21] to characterize the occurrence frequency of the impulses. The definition of average impulsive interval and the corresponding impulsive differential inequality are given firstly.

Definition 13 ([21] average impulsive interval). The average impulsive interval of the impulsive sequence $\{t_k\}_{k=1}^{\infty}$ is equal to T_a if there exist positive integer N_0 and positive number T_a such that

$$N(\kappa_2, \kappa_1) \ge \frac{\kappa_2 - \kappa_1}{T_a} - N_0, \quad \forall \kappa_2 \ge \kappa_1 \ge 0, \tag{60}$$

where $N(\kappa_2, \kappa_1)$ denotes the number of impulsive times of the impulsive sequence $\{t_k\}_{k=1}^{\infty}$ on the interval (κ_1, κ_2).

Lemma 14 (see Lakshmikantham et al., Theorem 1.4.1, [33, 34]). *Let $PC(\mathbb{R}_+, \mathbb{R})$ ($PC^1(\mathbb{R}_+, \mathbb{R})$) denote the set of piecewise continuous (piecewise continuously differentiable) functions with first kind of discontinuities at t_k, $k = 1, 2, \ldots$, from $\mathbb{R}_+$ to $\mathbb{R}$. If the following conditions hold,*

$$\begin{aligned} &m(t) \in PC^1(\mathbb{R}_+, \mathbb{R}) \text{ is left continuous at } t_k, \\ &\qquad k = 1, 2, \ldots, \\ &D^+ m(t) \le p(t) m(t) + q(t), \quad t \ge t_0, \ t \ne t_k, \\ &m(t_k^+) \le d_k m(t_k) + b_k, \quad k = 1, 2, \ldots, \end{aligned} \tag{61}$$

where $p(t)$, $q(t) \in PC(\mathbb{R}_+, \mathbb{R})$, $d_k \ge 0$ and b_k are real constants, then

$$\begin{aligned} m(t) \le\ & m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\, ds\right) \\ &+ \sum_{t_0 < t_k < t} \left(\prod_{t_0 < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)\, ds\right)\right) b_k \\ &+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\xi)\, d\xi\right) q(s)\, ds, \quad t \ge t_0. \end{aligned} \tag{62}$$

Lemma 15. *Let $q \ge 0$, $\tau > 0$, $0 < \mu < 1$, and p be constants, and assume that $V(t)$ is a piecewise continuous nonnegative function satisfying*

$$\begin{aligned} &D^+ V(t) \le p V(t) + q \overline{V}(t), \quad t \ge t_0, \ t \ne t_k, \\ &V(t_k^+) \le \mu V(t_k), \quad k = 1, 2, \ldots, \end{aligned} \tag{63}$$

and the average impulsive interval of the impulsive sequence $\{t_k\}_{k=1}^{\infty}$ is equal to T_a. If the following inequality holds,

$$p + \frac{\ln \mu}{T_a} + \mu^{-N_0} q < 0, \tag{64}$$

then one has

$$V(t) \le \mu^{-N_0} \overline{V}(t_0) e^{-\lambda(t-t_0)}, \tag{65}$$

where $\overline{V}(t) = \sup_{t-\tau \le s \le t} V(s)$, λ is the unique positive root of the equation $\lambda + p + (\ln \mu / T_a) + \mu^{-N_0} q e^{\lambda \tau} = 0$.

Proof. By Lemma 14 and the definition of average impulsive interval, it follows from (63) that, for $t \geq t_0$,

$$\begin{aligned} V(t) &\leq \overline{V}(t_0)\mu^{N(t,t_0)}e^{p(t-t_0)} + \int_{t_0}^{t} e^{p(t-s)}\mu^{N(t,s)}q\overline{V}(s)\mathrm{d}s \\ &\leq \overline{V}(t_0)\mu^{(((t-t_0)/T_a)-N_0)}e^{p(t-t_0)} \\ &\quad + \int_{t_0}^{t} e^{p(t-s)}\mu^{(((t-s)/T_a)-N_0)}q\overline{V}(s)\mathrm{d}s \\ &\leq \mu^{-N_0}\left(\overline{V}(t_0)e^{[p+(\ln\mu/T_a)](t-t_0)}\right. \\ &\qquad \left. + \int_{t_0}^{t} e^{[p+(\ln\mu/T_a)](t-s)}q\overline{V}(s)\mathrm{d}s\right). \end{aligned} \tag{66}$$

Denote $\phi(\lambda) = \lambda + p + (\ln\mu/T_a) + \mu^{-N_0}qe^{\lambda\tau}$. Since $\phi(0) = p + (\ln\mu/T_a) + \mu^{-N_0}q < 0$, and $\lim_{\lambda\to\infty}\phi(\lambda) = +\infty$ and $\phi'(\lambda) = 1 + \mu^{-N_0}q\tau e^{\lambda\tau} > 0$, one knows that $\phi(\lambda) = 0$ has a unique positive root.

Next, it is claimed that, for all $t > t_0$,

$$V(t) < \mu^{-N_0}\left(\sup_{t_0-\tau\leq s\leq t_0} V(s)\right)e^{-\lambda(t-t_0)}. \tag{67}$$

When $t \in [t_0 - \tau, t_0]$,

$$V(t) < \overline{V}(t_0) < \mu^{-N_0}\overline{V}(t_0)e^{-\lambda(t-t_0)}. \tag{68}$$

Supposing (67) is not always true for $t > t_0$, there must exist one time point $t^* > t_0$ such that

$$\begin{aligned} V(t^*) &= \mu^{-N_0}\overline{V}(t_0)e^{-\lambda(t^*-t_0)}, \\ V(t) &< \mu^{-N_0}\overline{V}(t_0)e^{-\lambda(t-t_0)}, \quad t_0 - \tau \leq t < t^*. \end{aligned} \tag{69}$$

From inequalities (66) and (69), one can obtain that

$$\begin{aligned} &V(t^*) \\ &\leq \mu^{-N_0}\left(\overline{V}(t_0)e^{[p+(\ln\mu/T_a)](t^*-t_0)}\right. \\ &\qquad \left. + \int_{t_0}^{t^*} e^{[p+(\ln\mu/T_a)](t^*-s)}q\overline{V}(s)\mathrm{d}s\right) \\ &< \mu^{-N_0}\left(\overline{V}(t_0)e^{[p+(\ln\mu/T_a)](t^*-t_0)}\right. \\ &\qquad \left. + \int_{t_0}^{t^*} e^{[p+(\ln\mu/T_a)](t^*-s)}q\mu^{-N_0}\overline{V}(t_0)e^{-\lambda(s-\tau-t_0)}\mathrm{d}s\right) \\ &= \mu^{-N_0}\overline{V}(t_0)e^{[p+(\ln\mu/T_a)](t^*-t_0)} \\ &\quad + q\mu^{-2N_0}\overline{V}(t_0)e^{\lambda\tau}\int_{t_0}^{t^*} e^{[p+(\ln\mu/T_a)](t^*-s)}e^{-\lambda(s-t_0)}\mathrm{d}s. \end{aligned} \tag{70}$$

Considering the equality $\psi(\lambda) = \lambda + p + (\ln\mu/T_a) + \mu^{-N_0}qe^{\lambda\tau} = 0$, one has

$$\begin{aligned} V(t^*) &< \mu^{-N_0}\overline{V}(t_0)e^{-\lambda(t^*-t_0)} \\ &\quad \times\left(e^{[-\mu^{-N_0}qe^{\lambda\tau}](t^*-t_0)}\right. \\ &\qquad \left. + q\mu^{-N_0}e^{\lambda\tau}\int_{t_0}^{t^*} e^{[-\mu^{-N_0}qe^{\lambda\tau}](t^*-s)}\mathrm{d}s\right) \\ &= \mu^{-N_0}\overline{V}(t_0)e^{-\lambda(t^*-t_0)}e^{-[\mu^{-N_0}qe^{\lambda\tau}]t^*} \\ &\quad \times\left(e^{[\mu^{-N_0}qe^{\lambda\tau}]t_0}\right. \\ &\qquad \left. + e^{[\mu^{-N_0}qe^{\lambda\tau}]t^*} - e^{[\mu^{-N_0}qe^{\lambda\tau}]t_0}\right) \\ &= \mu^{-N_0}\overline{V}(t_0)e^{-\lambda(t^*-t_0)} \end{aligned} \tag{71}$$

which contradicts with the first inequality of (69). Therefore, the inequality (67) holds, and this completes the proof. □

The following proposition will guarantee the system (59) to be globally exponentially stable.

Proposition 16. *Assume the external imposed impulsive strengths $h_{ik}^{(1)}, h_{jk}^{(2)} \in (-2,-1) \cup (-1,0)$, and the average impulsive interval of the impulsive sequence $\{t_k\}_{k=1}^{\infty}$ is equal to T_a. The impulsive system (59) is globally exponentially stable if the following condition holds:*

$$\rho + \frac{\ln\mu}{T_a} + \mu^{-N_0}\widetilde{b}_1 < 0. \tag{72}$$

Moreover, we have

$$\begin{aligned} &\sum_{i=1}^{n}|x_i(t)| + \sum_{j=1}^{m}|y_j(t)| \\ &\quad \leq \mu^{-N_0}e^{-(\eta+\lambda)t}\sup_{-\tau^*\leq s\leq 0}\left(\sum_{i=1}^{n}|\varphi_i(s)| + \sum_{j=1}^{m}|\psi_j(s)|\right), \quad t > 0, \end{aligned} \tag{73}$$

where $\mu = \sup_k\{\mu_k\} \in (0,1)$, and μ_k, ρ, and $\widetilde{b}_1$ are defined as in Theorem 6 and λ is the unique positive root of the equation $\lambda + p + (\ln\mu/T_a) + \mu^{-N_0}qe^{\lambda\tau} = 0$.

Proof. Considering the same Lyapunov function as that in Theorem 6, the following results on the impulsive interval and the impulse moments can be obtained.

(1) When $t \in [lT, (l+\theta)T) \setminus \{t_k\}_{k=1}^{\infty}$, $l \in \mathbb{N}_0^+$, the upper right Dini derivative of $V(t)$ along the solutions of (59) is depicted as $D^+V(t) \leq \rho V(t) + \widetilde{b}_1\overline{V}(t)$.

(2) When $t = t_k$, $k \in \mathbb{N}^+$, $V(t_k^+) \leq \mu V(t_k)$.

From the above two inequality relationships, and Lemma 15, one has

$$\sum_{i=1}^{n}\left|x_i(t)\right|+\sum_{j=1}^{m}\left|y_j(t)\right| \le \mu^{-N_0}e^{-(\eta+\lambda)t}\left(\sup_{t_0-\tau\le s\le t_0}V(s)\right), \quad t>0, \tag{74}$$

where λ is the unique positive root of the equation $\lambda + p + (\ln \mu/T_a) + \mu^{-N_0}qe^{\lambda\tau} = 0$. The proof is completed. □

Remark 17. It should be pointed out that, in the preceding control width of the control period, other kinds of continuous controllers can also be used to achieve the same performance. For example, in [35], the adaptive control scheme has been employed in the control width instead of the continuous state feedback, where the adjusting gains can be designed based on different norms. We can borrow such an idea to the sliding intermittent control design. Moreover, the sliding width parameter can be $\{\theta_i\}$ rather than the fixed width θ, and the period can be $\{T_i\}$ rather than constant T. By doing so, we might obtain more general conditions. On the other hand, the phenomena of stochastic nonlinearities are extremely ubiquitous in practical controlled systems [36–39]; hence it is more reasonable to consider neural networks with random nonlinearities, and this will be our future works.

4. Illustrative Example

In this section, we present some examples to illustrate the applicability and efficiency of the proposed control scheme.

Example 1. Considering the following extensively studied BAM neural system,

$$\begin{aligned}\dot{x}(t)&=-ax(t)+pf\left(y(t-\tau)\right),\\ \dot{y}(t)&=-by(t)+qg\left(x(t-\sigma)\right)\end{aligned} \tag{75}$$

with $a = 1.922$, $p = 9.8501$, $b = 1.1631$, $q = 8.2311$, $\tau = \sigma = 3$, and $f(x) = g(x) = 1/(1+e^{-x})-1/2$. Obviously, $L^f = L^g = 0.25$. The initial condition is given as $x(t) = -0.43$, $y(t) = 0.42$, $t \in [-3, 0]$. With the above system parameters, the phase diagram of system (75) is given in Figure 2. Obviously, the origin of system (75) is not stable. We will design the sliding intermittent controller to stabilize it.

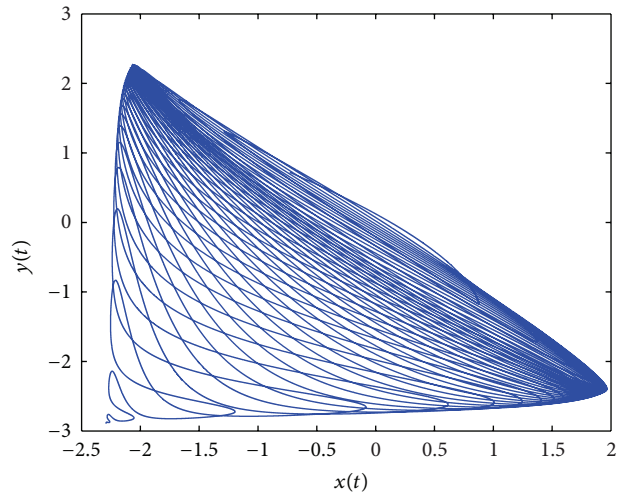


Figure 2: The phase diagram of the original system (75).

Applying the sliding intermittent controller to the unstable system (75), one can derive the following system:

$$\begin{aligned}&\dot{x}(t)=-ax(t)+pf\left(y(t-\tau)\right)+k_1x(t),\ t\in[lT,(l+\theta)T)\\ &\dot{x}(t)=-ax(t)+pf\left(y(t-\tau)\right),\\ &\quad t\in[(l+\theta)T,(l+1)T),\ t\ne t_k,\\ &\Delta x(t_k)=h_1x(t_k),\quad t=t_k,\\ &\dot{y}(t)=-by(t)+qg\left(x(t-\sigma)\right)+k_2y(t),\\ &\qquad\qquad t\in[lT,(l+\theta)T),\\ &\dot{y}(t)=-by(t)+qg\left(x(t-\sigma)\right),\\ &\quad t\in[(l+\theta)T,(l+1)T),\ t\ne t_k,\\ &\Delta y(t_k)=h_2y(t_k),\quad t=t_k.\end{aligned} \tag{76}$$

In the following, we will give the convergence results by simulating the system (76). Firstly, by setting the continuous feedback gains $k_1 = -4.8512$, $k_2 = -4.6378$ and the impulsive strengths $h_1 = h_2 = -0.15$. With the above parameters setting, calculations show that $\eta^* = 0.2679$ and $\mu = 0.85$. By setting the control period $T = 10$, we have control width $0.3 < \theta < 0.7$ and $\gamma = 0.3$. Here we take the parameter $\theta = 0.5$. If we utilize uniform distributed impulsive sequences ($t_k - t_{k-1} = 0.01$) in the latter control width of the control period, it is easy to check that when $\eta = 0.0221$, we have $\rho = -1.1410$, $\tilde{a}_1 = 5.7788$, and $\tilde{b}_1 = 4.8302$. By choosing $\beta = -4.8519$, one has $d = 1.1765$, $\ln \mu/(t_k - t_{k-1}) - \beta = -11.4000$, $\rho + d\tilde{b}_1 + \beta = -0.3104$, $\lambda_1 = 0.0565$, $\lambda_2 = 0.0168$, and $\eta + \lambda_1(\theta-\gamma) - \lambda_2(\theta+\gamma) - \ln d/T = 0.0037$, which means all the conditions in Theorem 6 hold. The simulation result under the sliding intermittent control is given in Figure 3.

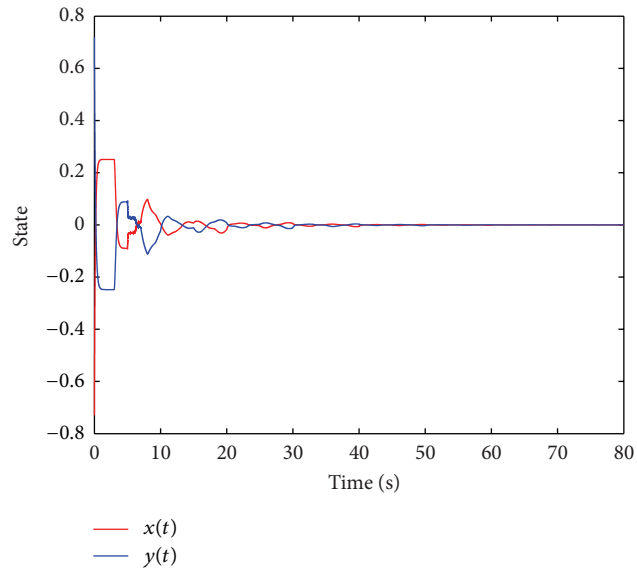


Figure 3: Simulation of the system (75) under the sliding intermittent control.

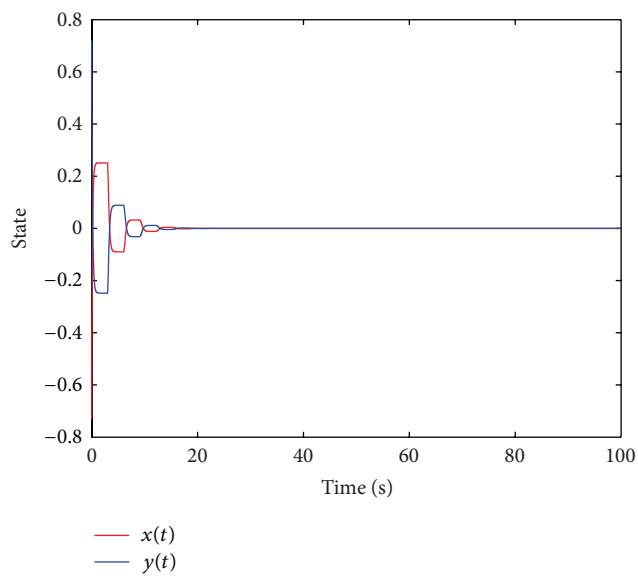


Figure 4: Simulation of the system (75) under the continuous feedback control.

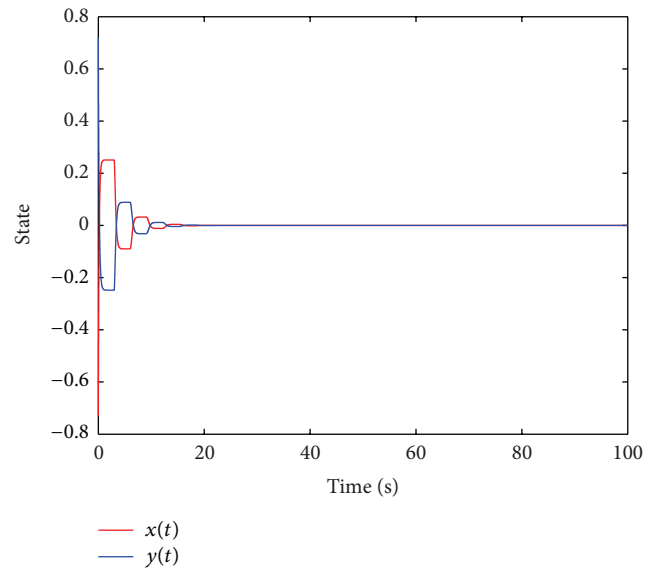


Figure 5: Simulation of the system (75) under the periodically intermittent control.

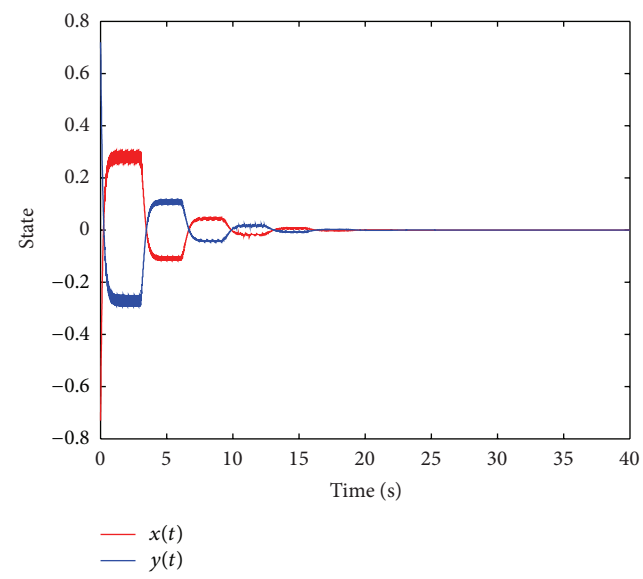


Figure 6: Simulation of the system (75) under the full impulsive control.

Next, the corresponding convergence results under the propositions obtained in this article will be illustrated.

If we utilize the continuous feedback control, by taking the same continuous feedback gains as the above sliding intermittent control, the conditions in Proposition 9 hold, and the simulation result under the continuous feedback control is given in Figure 4.

If we use pure periodically intermittent control, by setting the continuous feedback gains $k_1 = -6.8512$, $k_2 = -6.6378$, we get $\eta^* = 0.3682$, $\tilde{\rho} = \kappa = 0.0001$. It is easy to know when $\eta = 0.0002$, we have $\tilde{a}_1 = 7.8007$, $\tilde{b}_1 = 4.5230$, and $\lambda_1 = 0.1742$. In Proposition 11, by taking $T = 100$, the relationships among the upper bound delay, the control width, and the control period infer that $0.03 < \theta < 0.97$, while the last inequality in Proposition 11 means $\theta > 0.9640$. Here we set the $\theta = 0.9650$, and the simulation result under the periodically intermittent control is given in Figure 5.

If we use full impulsive control with uniform distributed impulsive sequences ($t_k - t_{k-1} = 0.04$) and the impulsive strengths $\mu_k = 0.85$, $k = 1, 2, \ldots$, it is easy to check when $\beta = -3.1262$, $\eta = 0.0002$, the parameters $d = 1.1332$, $\rho = -1.1629$, and $\tilde{b}_1 = 4.5230$, all the conditions in Proposition 12 satisfied. If we use the semi-impulsive control scheme and set the impulsive sequences satisfying the average impulsive

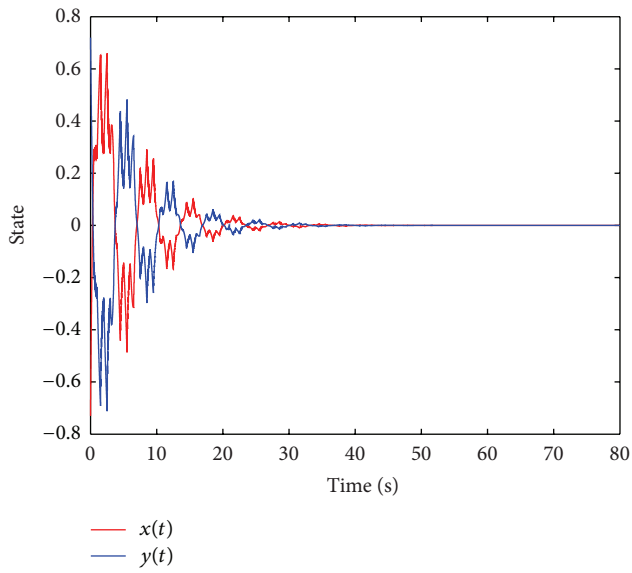


FIGURE 7: Simulation of the system (75) under the semi impulsive control.

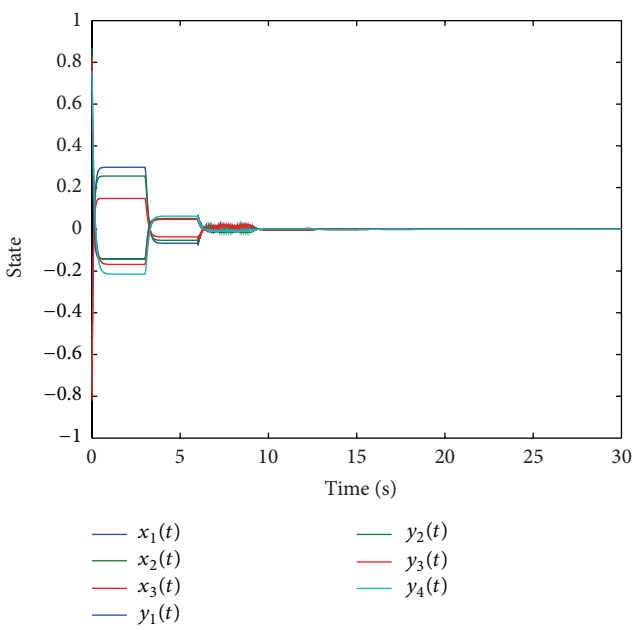


FIGURE 8: Simulation of the system (77) under the sliding intermittent controller.

interval with $T_a = 0.038$ and $N_0 = 1$; it is easy to check $\rho + (\ln \mu/T_a) + \mu^{-N_0}\tilde{b}_1 = -0.1185 < 0$, which infers that the condition in Proposition 16 holds. The simulation results for system (75) with the full impulsive control and the semi-impulsive control are given in Figures 6 and 7.

Remark 18. From the above verifying process, it can be found that the sliding intermittent control is much better than the pure periodically intermittent control. More specifically, in the periodically intermittent control, the control period $T = 100$ and the control width $\theta = 0.965$, while in the sliding intermittent control, the control period $T = 10$ and the control width $\theta = 0.5$. As for the impulsive control, the full impulsive control is better than the semi-impulsive control in that the earlier converges faster. When dealing with the nonuniformly distributed impulsive sequence, the result derived in Proposition 16 is less conservative.

Example 2. Consider the following unstable delayed BAM neural network, and we will show that the sliding intermittent control benefits the stabilization of the unstable system:

$$\begin{aligned} \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{3} p_{ij} f_j\left(y_j\left(t-\tau_{ij}\right)\right), \quad i \in \mathcal{I}_3, \\ \dot{y}_j(t) &= -b_j y_j(t) + \sum_{i=1}^{4} q_{ji} g_i\left(x_i\left(t-\sigma_{ji}\right)\right), \quad j \in \mathcal{I}_4, \end{aligned} \tag{77}$$

where $A = \mathrm{diag}\{3.1220, 2.3156, 2.2683\}$, $B = \mathrm{diag}\{2.9631, 2.3456, 2.6341, 3.0726\}$,

$$P = \begin{bmatrix} 7.8501 & 2.3070 & 3.2280 & 4.7191 \\ 3.2463 & 6.0589 & 5.3751 & 2.2609 \\ 2.0159 & 1.7803 & 2.6601 & 5.7647 \end{bmatrix},$$
$$Q = \begin{bmatrix} 4.2331 & 4.3741 & 2.2459 \\ 4.8830 & 2.3259 & 1.2857 \\ 2.4022 & 1.3377 & 3.7930 \\ 2.1351 & 2.6759 & 3.4719 \end{bmatrix}. \tag{78}$$

The activations functions $f(x) = g(x) = 1/(1 + e^{-x}) - 1/2$ with $L^f = L^g = 0.25$ and the time delays $\tau = \sigma = 3$. The system has an unstable equilibrium 0 under the above parameters with the initial functions $x(t) = [-0.73, -0.79, -0.82]^T$, $y(t) = [0.72, 0.87, 0.83, 0.76]^T$, $t \in [-3, 0]$.

In the following, we will check the convergence results for system (77) under the sliding intermittent controller. Setting the continuous feedback gains $k^{(1)} = \mathrm{diag}\{-8.0350, -10.4875, -13.1075\}$, $k^{(2)} = \mathrm{diag}\{-11.1950, -8.4125, -5.7425, -4.1475\}$, the impulsive strengths are identical with $h_{ik}^{(1)} = h_{jk}^{(2)} = -0.25$, $i \in \mathcal{I}_n$, $j \in \mathcal{I}_m$, $k \in \mathbb{N}^+$, and the control period $T = 15$; some calculations show that $\eta^* = 0.2604$, $\mu = 0.75$. We can get the control width $0.2 < \theta < 0.8$ and $\gamma = 0.2$; here we take the parameter $\theta = 0.4$. If we utilize the uniform distributed impulsive sequences ($t_k - t_{k-1} = 0.02$) in the latter control width of the control period, it is easy to know when $\eta = 0.1250$, we have $\rho = -2.1433$, $\tilde{a}_1 = 7.0951$, and $\tilde{b}_1 = 4.9664$. By choosing $\beta = -4.6841$, one has $\ln \mu/(t_k - t_{k-1}) - \beta = -9.7000$, $d = 1.333$, $\rho + d\tilde{b}_1 + \beta = -0.2055$, $\lambda_1 = 0.1135$, $\lambda_2 = 0.1924$, and $\eta + \lambda_1(\theta - \gamma) - \lambda_2(\theta + \gamma) - \ln d/T = 0.0131$, which mean all the conditions in Theorem 6 hold. The simulation result under the sliding intermittent control is given in Figure 8.

5. Conclusions

In this paper, a sliding intermittent controller has been proposed by unifying the periodically intermittent control

with the impulsive control together with continuous feedback control. More specifically, the continuous feedback control is employed as the preceding control width, and the impulsive control is resorted in the latter control width. Furthermore, the adjustable parameter $\theta \in [0, 1]$ is very flexible in that the continuous feedback control ($\theta = 1$), the impulsive control ($\theta = 0$), the periodically intermittent control, and the semi-impulsive control ($0 < \theta < 1$) are all possible cases. Based on the analysis technique and the Lyapunov function approach, the conditions have been constructed for the exponential stability of the delayed BAM neural networks under the proposed control schemes. Finally, numerical simulations are used to illustrate the effectiveness of the control technique.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant 61174136, the Natural Science Foundation of Jiangsu Province of China under Grant BK2011598, the Program for New Century Excellent Talents in University under Grant NCET-12-0117, and the Fundamental Research Funds for the Central Universities under Grant 2242012155.

References

[1] B. Kosko, "Bidirectional associative memories," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 18, no. 1, pp. 49–60, 1988.

[2] V. S. H. Rao and B. R. M. Phaneendra, "Global dynamics of bidirectional associative memory neural networks involving transmission delays and dead zones," *Neural Networks*, vol. 12, no. 3, pp. 455–465, 1999.

[3] J. Cao, J. Liang, and J. Lam, "Exponential stability of high-order bidirectional associative memory neural networks with time delays," *Physica D*, vol. 199, no. 3-4, pp. 425–436, 2004.

[4] Y. Liu, Z. Wang, J. Liang, and X. Liu, "Synchronization of coupled neutral-type neural networks with jumping-mode-dependent discrete and unbounded distributed delays," *IEEE Transactions on Cybernetics*, vol. 43, no. 1, pp. 102–114, 2012.

[5] J. Liang, Z. Wang, B. Shen, and X. Liu, "Distributed state estimation in sensor networks with randomly occurring nonlinearities subject to time delays," *ACM Transactions on Sensor Networks*, vol. 9, no. 1, article 4, 2012.

[6] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed H_∞ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[7] M. Xiao, W. X. Zheng, and J. Cao, "Hopf bifurcation of an $(n + 1)$-neuron bidirectional associative memory neural network model with delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 24, no. 1, pp. 118–132, 2013.

[8] K. Gopalsamy and X.-Z. He, "Delay-independent stability in bidirectional associative memory networks," *IEEE Transactions on Neural Networks*, vol. 5, no. 6, pp. 998–1002, 1994.

[9] S. Arik, "Global asymptotic stability analysis of bidirectional associative memory neural networks with time delays," *IEEE Transactions on Neural Networks*, vol. 16, no. 3, pp. 580–586, 2005.

[10] J. Liang and J. Cao, "Exponential stability of continuous-time and discrete-time bidirectional associative memory networks with delays," *Chaos, Solitons & Fractals*, vol. 22, no. 4, pp. 773–785, 2004.

[11] J. Cao and Q. Song, "Stability in Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays," *Nonlinearity*, vol. 19, no. 7, pp. 1601–1617, 2006.

[12] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.

[13] C. A. S. Batista, S. R. Lopes, R. L. Viana, and A. M. Batista, "Delayed feedback control of bursting synchronization in a scale-free neuronal network," *Neural Networks*, vol. 23, no. 1, pp. 114–124, 2010.

[14] J. Lu and J. Cao, "Adaptive synchronization in tree-like dynamical networks," *Nonlinear Analysis. Real World Applications*, vol. 8, no. 4, pp. 1252–1260, 2007.

[15] J. Liang, Z. Wang, and X. Liu, "On passivity and passification of stochastic fuzzy systems with delays: the discrete-time case," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 40, no. 3, pp. 964–969, 2010.

[16] B. Shen, Z. Wang, and X. Liu, "Sampled-data synchronization control of dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[17] L. Wu, P. Shi, and H. Gao, "State estimation and sliding-mode control of Markovian jump singular systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1213–1219, 2010.

[18] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 48, no. 9, pp. 2007–2015, 2012.

[19] G. Wei, Z. Wang, and B. Shen, "Probability-dependent gain-scheduled control for discrete stochastic delayed systems with randomly occurring nonlinearities," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 7, pp. 815–826, 2012.

[20] L. Pan and J. Cao, "Exponential stability of impulsive stochastic functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 382, no. 2, pp. 672–685, 2011.

[21] J. Lu, D. W. C. Ho, and J. Cao, "A unified synchronization criterion for impulsive dynamical networks," *Automatica*, vol. 46, no. 7, pp. 1215–1221, 2010.

[22] C. Li, G. Feng, and X. Liao, "Stabilization of nonlinear systems via periodically intermittent control," *IEEE Transactions on Circuits and Systems II*, vol. 54, no. 11, pp. 1019–1023, 2007.

[23] W. Xia and J. Cao, "Pinning synchronization of delayed dynamical networks via periodically intermittent control," *Chaos*, vol. 19, no. 1, Article ID 013120, 2009.

[24] D. S. Goshi, K. M. K. H. Leong, and T. Itoh, "A secure high-speed retrodirective communication link," *IEEE Transactions on Microwave Theory and Techniques*, vol. 53, no. 11, pp. 3548–3556, 2005.

[25] D. W. C. Ho, J. Liang, and J. Lam, "Global exponential stability of impulsive high-order BAM neural networks with time-varying delays," *Neural Networks*, vol. 19, no. 10, pp. 1581–1590, 2006.

[26] Q. Gan, "Exponential synchronization of stochastic Cohen-Grossberg neural networks with mixed time-varying delays and reaction-diffusion via periodically intermittent control," *Neural Networks*, vol. 31, pp. 12–21, 2012.

[27] S. Mohamad, "Global exponential stability in continuous-time and discrete-time delayed bidirectional neural networks," *Physica D*, vol. 159, no. 3-4, pp. 233–251, 2001.

[28] R. Samidurai, R. Sakthivel, and S. M. Anthoni, "Global asymptotic stability of BAM neural networks with mixed delays and impulses," *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 113–119, 2009.

[29] Y. Li and C. Wang, "Existence and global exponential stability of equilibrium for discrete-time fuzzy BAM neural networks with variable delays and impulses," *Fuzzy Sets and Systems*, vol. 217, pp. 62–79, 2013.

[30] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, NY, USA, 1966.

[31] L. Pan and J. Cao, "Anti-periodic solution for delayed cellular neural networks with impulsive effects," *Nonlinear Analysis. Real World Applications*, vol. 12, no. 6, pp. 3014–3027, 2011.

[32] Y.-T. Li and C.-B. Yang, "Global exponential stability analysis on impulsive BAM neural networks with distributed delays," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1125–1139, 2006.

[33] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of impulsive differential equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific Publishing Company, Teaneck, NJ, USA, 1989.

[34] L.-J. Pan, G.-Q. Wang, and S. S. Cheng, "Oscillation of even order nonlinear differential equations with impulses," *Funkcialaj Ekvacioj. Serio Internacia*, vol. 50, no. 1, pp. 117–131, 2007.

[35] X. Liu and T. Chen, "Cluster synchronization in directed networks via intermittent pinning control," *IEEE Transactions on Neural Networks*, vol. 22, no. 7, pp. 1009–1020, 2011.

[36] H. Dong, Z. Wang, and H. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Transactions on Circuits and Systems I*, vol. 59, no. 10, pp. 2354–2362, 2012.

[37] J. Hu, Z. Wang, B. Shen, and H. Gao, "Gain-constrained recursive filtering with stochastic nonlinearities and probabilistic sensor delays," *IEEE Transactions on Signal Processing*, vol. 61, no. 5, pp. 1230–1238, 2013.

[38] B. Shen, S. X. Ding, and Z. Wang, "Finite-horizon H_∞ fault estimation for linear discrete time-varying systems with delayed measurements," *Automatica*, vol. 49, no. 1, pp. 293–296, 2013.

[39] H. Dong, Z. Wang, J. Lam, and H. Gao, "Fuzzy-model-based robust fault detection with stochastic mixed time delays and successive packet dropouts," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 42, no. 2, pp. 365–376, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 814187, 12 pages
http://dx.doi.org/10.1155/2013/814187

Research Article

Deconvolution Filtering for Nonlinear Stochastic Systems with Randomly Occurring Sensor Delays via Probability-Dependent Method

Yuqiang Luo,[1,2] Guoliang Wei,[1] Hamid Reza Karimi,[3] and Licheng Wang[1]

[1] *Shanghai Key Lab of Modern Optical System, Department of Control Science and Engineering, University of Shanghai for Science and Technology, Shanghai 200093, China*
[2] *Informatization Office, University of Shanghai for Science and Technology, Shanghai 200093, China*
[3] *Department of Engineering, Faculty of Engineering and Science, University of Agder, N-4898 Grimstad, Norway*

Correspondence should be addressed to Guoliang Wei; guoliang.wei1973@gmail.com

Received 12 April 2013; Accepted 20 May 2013

Academic Editor: Zidong Wang

Copyright © 2013 Yuqiang Luo et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper deals with a robust H_∞ deconvolution filtering problem for discrete-time nonlinear stochastic systems with randomly occurring sensor delays. The delayed measurements are assumed to occur in a random way characterized by a random variable sequence following the Bernoulli distribution with time-varying probability. The purpose is to design an H_∞ deconvolution filter such that, for all the admissible randomly occurring sensor delays, nonlinear disturbances, and external noises, the input signal distorted by the transmission channel could be recovered to a specified extent. By utilizing the constructed Lyapunov functional relying on the time-varying probability parameters, the desired sufficient criteria are derived. The proposed H_∞ deconvolution filter parameters include not only the fixed gains obtained by solving a convex optimization problem but also the online measurable time-varying probability. When the time-varying sensor delays occur randomly with a time-varying probability sequence, the proposed gain-scheduled filtering algorithm is very effective. The obtained design algorithm is finally verified in the light of simulation examples.

1. Introduction

Filtering technology is extensively used in many domains, and an important task of filtering is to effectively restore the original signal by removing the distortion from the received signal. The deconvolution method is very useful in filtering field and has been made use of by some kinds of filters, such as self-tuning deconvolution filter and Wiener inverse filter. However, the design process of the deconvolution filter is always difficult because the delivery channels are frequently corrupted by some disturbances, such as nonlinear disturbances, external noises, and sensor delays. Therefore it is very interesting and important to solve the problem encountered in the design procedure of deconvolution filters. In the past decades, considerable attention has been paid to the analysis and design of many kinds of deconvolution filters, and some results have been published. For more details, we refer the readers to [1–3] and the references therein.

As is well known, in some practical fields, such as engineering, biological, medical, and economic systems, and health community, the sensor data is occasionally delayed before they arrive at their respective destinations [4–6]. The occurrence of sensor delay may mainly be caused by the limitations, such as the limited bandwidth of the communication channel, intermittent sensor failure in the measurement, random network congestion, and accidental loss of some collected data in a very noisy environment [6–9]. However, the majority of deconvolution filtering algorithms are based on the measurement outputs without delays. In such case, the traditional filter may fail to work. Hence, in the past decades the filtering problem for the systems with sensor delays has been attracting considerable research interests; see,

for example, [5, 7, 10–12]. Owing to the uncertainty that is widespread in the field of practical engineering, stochastic model has gained more and more attention in many fields, such as physics, economic systems, geomorphology, and gene regulatory networks, and a large number of literatures have been published; see, for example, [4, 13–18].

It should be pointed out that, so far, the Bernoulli distribution has been employed to model some randomly occurring phenomena, such as randomly occurring missing measurements [10] and randomly occurring saturation [7]. This model with time-invariant probability has become an effective model of sensor delays. Nevertheless, in the engineering environment, such as industrial automation, unmanned vehicles, real-time distributed decision-making and multiplexed data communication networks [5], and an asynchronous time-division-multiplexed network [6], the sensor delays often occur in a random way and satisfy a time-varying probability distribution. At the same time, the classical filters with fixed gains cannot adapt to the actual cases. Therefore, there is an urgent need to develop new filtering approaches for the systems with randomly occurring sensor delays (ROSDs), and some efforts have been made in this regard so far; see, for example, [6, 7]. And yet, up to now, to the best of authors' knowledge, the gain-scheduled H_∞ deconvolution filtering is still open for discrete-time stochastic systems with randomly occurring sensor delays. It is, therefore, in this paper, we aim to develop an effective gain-scheduled deconvolution filtering algorithm for the discrete-time stochastic systems with ROSDs, which is of both theoretical importance and practical significance.

The main contribution of this paper is mainly triplex: (1) for the randomly occurring sensor delay which is one kind of the information incomplete, we exploit a stochastic variable sequence satisfying time-varying Bernoulli distributions to represent the situation of the delayed measurement; (2) a time-varying Lyapunov functional dependent on the distribution probability has been developed and applied to improve the performance of the H_∞ deconvolution filters; (3) a new filtering problem with a gain-scheduling approach is addressed for a class of discrete-time nonlinear stochastic systems with randomly occurring sensor delays. It is worth mentioning that, since the considered system involves the probabilistic sensor delays, the sector-like bounded nonlinearity, and the multiplicative noises, it is comprehensive and reasonable. Thanks to the proposed time-varying H_∞ deconvolution filter which is designed by employing the gain-scheduling technique, the proposed filtering algorithm can exactly estimate the original input. On account of this merit, the proposed design scheme is more effective and practical.

The rest of this paper is organized as follows. In Section 2, we construct a gain-scheduled deconvolution filter for a class of discrete-time stochastic systems with randomly occurring sensor delays, in which the desired filter gains contain two parts, the fixed gain and time-varying one which is dependent upon the time-varying probability. In Section 3, a sufficient condition is derived to guarantee the exponential stability of the augmented system, and the proposed filter is given. By means of constructing Lyapunov functional, we make an assay of the stability for the augmented systems. With the help of the proposed probability-dependent Lyapunov functional, we derived simultaneously another condition indicating the robust ability of the deconvolution filter. A mathematical technique is used to transform the infinite number of inequalities into a finite form. The filter gains are derived from a gain-scheduling approach by resorting to solve a convex optimization problem. In Section 4, an numerical example is presented to demonstrate the reasonable structure and high reliability of the proposed filter. The last section, Section 5, sums up all the arguments in this paper.

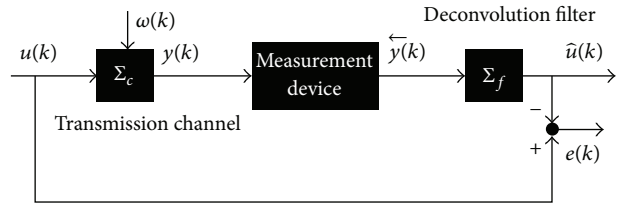


Figure 1: The deconvolution filtering system.

Notation. In this paper, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{I}^+$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, the set of all positive integers. $|\cdot|$ refers to the Euclidean norm in $\mathbb{R}^n$. I denotes the identity matrix of compatible dimension. The notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (resp., positive definite). For a matrix M, M^T and M^{-1} represent its transpose and inverse, respectively. The shorthand $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \ldots, M_n$. In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry. Matrices, if they are not explicitly stated, are assumed to have compatible dimensions.

2. Problem Formulation

In this paper, the considered stochastic deconvolution filtering system structure is shown in Figure 1. In the system, the input signal $u(t)$ is transmitted through the channel Σ_c:

$$\begin{aligned}(\Sigma_c): x_c(k+1) = {} & A_c x_c(k) \\ & + B_c u(k) + N_c f(z(k)) \\ & + M_c x_c(k)\,\omega(k),\end{aligned} \tag{1}$$

$$y(k) = C_c x_c(k), \quad x_c(0) = x_0, \tag{2}$$

where $x_c(k) \in \mathbb{R}^n$ is the state, $u(k) \in L_2[0,\infty)$ is the exogenous input signal, $y(k)$ is the actual output, and x_0 is the initial state. $\omega(k)$ is a one-dimensional Gaussian white noise sequence satisfying $\mathbb{E}\{\omega(k)\} = 0$ and $\mathbb{E}\{\omega^2(k)\} = \sigma^2$, and $z(k) = Zx_c(k)$. For the convenience, $u(k)$, $y(k)$, and $\omega(k)$ are all assumed as scalars. A_c, B_c, C_c, M_c, N_c, and Z are constant matrices with appropriate dimensions.

Remark 1. The deconvolution filtering problem involves the estimation of the signal inputted to a communication channel where the output measurements are disturbed by the channel noise. The channel can be represented by a dynamical system, such as ship roll stabilization systems. In the ideal condition, the input signal would be sent perfectly through the transmission channel without any external influences; however, in the real-world case, from the point of engineering application, most of the communication channels are of limited capacity and suffered from some uncertainties in transmission process such as large environmental noise, channel congestion, and intermittent changes of the signal intensity, and therefore, a few issues have inevitably emerged, for example, channel-induced time delay, the nonlinearity disturbances, channel fading, and so on. In order to describe the channel model $\sum_c$ closer in nature, nonlinearity disturbances and stochastic noises are both taken into consideration in the channel model (1) in this paper.

The vector-valued nonlinear disturbance $f(\cdot)$ satisfies the following sector-bounded condition with $f(0) = 0$:

$$\left[f(z(k)) - F_1 z(k)\right]^T \left[f(z(k)) - F_2 z(k)\right] \leq 0, \tag{3}$$

where F_1 and F_2 are constant real matrices of appropriate dimensions and $F = F_2 - F_1$ is a symmetric positive definite matrix. It is customary that such nonlinear function $f(\cdot)$ is called to belong to the sector $[F_1 \;\; F_2]$. In this case, the nonlinear function $f(z(k))$ can be decomposed into a linear part and a nonlinear part as

$$f(z(k)) = F_1 z(k) + f_s(z(k)), \tag{4}$$

and it is easy to follow from (3) that

$$f_s^T(z(k))\left(f_s(z(k)) - Fz(k)\right) \leq 0. \tag{5}$$

The measurement outputs $\overleftarrow{y(k)}$ with sensor delays are described by

$$\overleftarrow{y(k)} = (1 - \varpi(k))y(k) + \varpi(k)\, y(k-d), \tag{6}$$

where $d \in \mathbb{I}^+$ is the sensor delay and $\varpi(k) \in \mathbb{R}$ is a random white sequence characterizing the probabilistic sensor delays and obeys the following Bernoulli distribution with time-varying probability:

$$\begin{aligned} &\text{Prob}\{\varpi(k) = 1\} = \mathbb{E}\{\varpi(k)\} = p(k), \\ &\text{Prob}\{\varpi(k) = 0\} \\ &\quad = 1 - \mathbb{E}\{\varpi(k)\} = 1 - p(k), \end{aligned} \tag{7}$$

where $p(k)$ is a time-varying positive scalar sequence that belongs to $[p_1 \;\; p_2] \subseteq [0 \;\; 1]$ with the constants p_1 and p_2 being the lower and upper bounds of $p(k)$.

Remark 2. The sensor delays may randomly occur due to some environment reasons, and a Bernoulli distribution model has been introduced in [6, 7] to describe such random phenomenon. However, the Bernoulli distributions in this literature are assumed to be time invariant, which is apparently conservative to deal with the time-varying cases of randomly occurred sensor delays for time-varying systems. In this paper, we will utilize a stochastic variable sequence in (7) satisfying time-varying Bernoulli distributions to express the randomly intermittent phenomenon of the discussed sensor delays.

In order to recover the source signal $u(k)$, the following deconvolution filter structure is considered in this paper:

$$\Sigma_f : x_f(k+1) = A_f x_f(k) + B_f \overleftarrow{y(k)}, \tag{8}$$

$$\hat{u}(k) = C_f x_f(k) + D_f \overleftarrow{y(k)}, \tag{9}$$

where $x_f(k) \in \mathbb{R}^n$ is the filter state and the matrices A_f, B_f, C_f, and D_f are filter parameters to be determined and have the following forms:

$$\begin{aligned} A_f &= A_{f0} + p(k) A_{fp}, \qquad B_f = B_{f0} + p(k) B_{fp}, \\ C_f &= C_{f0} + p(k) C_{fp}, \qquad D_f = D_{f0} + p(k) D_{fp}, \end{aligned} \tag{10}$$

where A_{f0}, A_{fp}, B_{f0}, B_{fp}, C_{f0}, C_{fp}, D_{f0}, and D_{fp} are the constant filter gains to be designed and $p(k)$ is the time-varying probability that can be estimated/measured via statistical tests in real time.

Remark 3. Deconvolution filter is a restoration algorithm to remove a wavelet by utilizing a reverse process of convolution. Comparing with conventional filters, it not only can estimate a signal embedded in noise but also can remove the effect of any distortion in the channel systems. Furthermore, it can deal with unknown boundary problem and spatially varying blurs. It is worth mentioning that, different from many conventional deconvolution filters with only constant parameters, the proposed filter gains in (10) include two kinds of filter gains: the fixed parameters A_{f0}, A_{fp}, B_{f0}, B_{fp}, C_{f0}, C_{fp}, D_{f0}, and D_{fp} and the time-varying parameter $p(k)$. The designed filter can be scheduled with the time-varying probability, which is able to adapt to changing circumstances naturally. It can be divided into the following several steps. Firstly, compute the constant gains A_{f0}, A_{fp}, B_{f0}, B_{fp}, C_{f0}, C_{fp}, D_{f0}, and D_{fp} in terms of the main results that will be developed in this paper. Secondly, estimate/measure the time-varying probability $p(k)$ by statistical tests in real time. Lastly, the filter gains can be derived from (10). Obviously, the gain-scheduled filter is reasonable and the conservatism of which can be reduced since more information about the sensor delay phenomenon is utilized. Note that gain-scheduled technique filtering and control problems have become a hot topic and have been intensively researched in the past decades; see, for example, [19–23].

By setting $\xi(k) = [x_c^T(k) \;\; x_f^T(k)]^T$ and the signal error as $e(k) = u(k) - \hat{u}(k)$, the dynamics of the filtering process can be derived from (1)–(6) and (8)-(9) as follows:

$$\begin{aligned}\xi(k+1) = {} & A(p(k))\xi(k) \\ & + D(p(k))\xi(k-d) + Bu(k) \\ & + Nf(z(k)) + MJ\xi(k)\omega(k) \\ & + (p(k) - \varpi(k))K(p(k))J\xi(k) \\ & + (\varpi(k) - p(k))K(p(k))J\xi(k-d),\end{aligned} \tag{11}$$

$$\begin{aligned}e(k) = {} & -\left[E_{cdf}(p(k)) + (p(k) - \varpi(k))E_{df}(p(k))\right]\xi(k) \\ & - \left[p(k)E_{df}(p(k)) + (\varpi(k) - p(k))E_{df}(p(k))\right] \\ & \times \xi(k-d) + u(k),\end{aligned} \tag{12}$$

where

$$\begin{gathered} A(p(k)) = \begin{bmatrix} A_c & 0 \\ (1-p(k))B_fC_c & A_f \end{bmatrix}, \\ B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}, \qquad M = \begin{bmatrix} M_c \\ 0 \end{bmatrix}, \\ D(p(k)) = \begin{bmatrix} 0 & 0 \\ p(k)B_fC_c & 0 \end{bmatrix}, \\ K(p(k)) = \begin{bmatrix} 0 \\ B_fC_c \end{bmatrix}, \qquad N = \begin{bmatrix} N_c \\ 0 \end{bmatrix}, \\ E_{cdf}(p(k)) = [(1-p(k))D_fC_c \;\; C_f], \\ E_{df}(p(k)) = [D_fC_c \;\; 0], \qquad J = [I \;\; 0]. \end{gathered} \tag{13}$$

Definition 4. The augmented filtering system (11) is said to be *exponentially mean-square stable* if, with $u(k) \equiv 0$, there exist constant $\alpha > 0$ and $\tau \in (0,1)$ such that

$$\mathbb{E}\left\{\|\xi(k)\|^2\right\} \le \alpha\tau^k \sup_{-d\le i\le 0} \mathbb{E}\left\{\|\xi(i)\|^2\right\}, \quad k \in \mathbb{I}^+. \tag{14}$$

Definition 5. Given a scalar $\gamma > 0$, the dynamics of the augmented systems (11)-(12) are said to be *stochastically stable with disturbance attenuation level* γ if it is exponentially mean-square stable, and under zero initial condition, $\|e(k)\|_{\mathbb{E}_{l_2}} < \gamma\|u(k)\|_{l_2}$ holds for all nonzero $u(k) \in l_2[0,\infty)$, where

$$\|e(k)\|_{\mathbb{E}_{l_2}} := \left(\mathbb{E}\left\{\sum_{k=1}^{\infty}|e(k)|^2\right\}\right)^{1/2}. \tag{15}$$

This paper aims to design a H_∞ deconvolution filter to recover a input signal $u(k)$ transmitted through a noised channel (1)-(2) such that the closed-loop systems (11)-(12) are stochastically stable with disturbance attenuation level γ.

3. Main Results

In the proof procedure of theorems presented in this paper, we will use the following lemma.

Lemma 6 ((Schur complement), see [24]). *Given constant matrices Σ_1, Σ_2, and Σ_3 where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 - \Sigma_3^T\Sigma_2^{-1}\Sigma_3 \ge 0$ if and only if*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & \Sigma_2 \end{bmatrix} \ge 0 \quad \text{or} \quad \begin{bmatrix} \Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} \ge 0. \tag{16}$$

In the following theorem, a sufficient condition is derived for the augmented filtering dynamics (11)-(12) with $u(k) \equiv 0$ to ensure the exponential mean-square stability of the considered dynamics.

Theorem 7. *Consider the augmented filtering system* (11) *with $u(k) \equiv 0$. If there exist positive-definite matrix $Q_d > 0$, $Q(p(k)) > 0$, and matrix S such that the following matrix inequalities*

$$\begin{bmatrix} Q_d - Q(p(k)) & * & * & * & * & * \\ 0 & -Q_d & * & * & * & * \\ FZJ & 0 & -2I & * & * & * \\ \sigma^2 S^T MJ & 0 & 0 & -\sigma^2\widetilde{\Gamma}(k) & * & * \\ \theta(k)S^TK(p(k))J & -\theta(k)S^TK(p(k))J & 0 & 0 & -\theta(k)\widetilde{\Gamma}(k) & * \\ S^T(A(p(k)) + NF_1ZJ) & S^TD(p(k)) & S^TN & 0 & 0 & -\widetilde{\Gamma}(k) \end{bmatrix} < 0 \tag{17}$$

hold, where

$$\begin{gathered} \widetilde{\Gamma}(k) = -Q(p(k+1)) + S + S^T, \\ \theta(k) = p(k)(1-p(k)), \end{gathered} \tag{18}$$

then the augmented dynamics (11) *is exponentially stable in mean square sense.*

Proof. Now, we will show the exponential mean-square stability of the augmented system (11). To this end, define the Lyapunov functional as

$$V(k) := \xi^T(k)Q(p(k))\xi(k) + \sum_{s=k-d}^{k-1}\xi^T(s)Q_d\xi(s), \tag{19}$$

where $Q(p(k))$ is a time-varying positive definite matrix sequence dependent on the time-varying probability $p(k)$:

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\quad = \mathbb{E}\Big\{\xi^T(k+1)\,Q\big(p(k+1)\big) \\
&\qquad \times \xi(k+1) - \xi^T(k)\,Q\big(p(k)\big)\,\xi(k) \\
&\qquad + \xi^T(k)\,Q_d\xi(k) - \xi^T(k-d) \\
&\qquad \times Q_d\xi(k-d)\Big\}.
\end{aligned} \tag{20}$$

Noting that $\mathbb{E}\{\varpi(k)-p(k)\}=0$, $\mathbb{E}\{(\varpi(k)-p(k))^2\}=p(k)(1-p(k))$, $\mathbb{E}\{\omega(k)\}=0$, and $\mathbb{E}\{\omega^2(k)\}=\sigma^2$, it can be obtained from (11) that

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\quad = \mathbb{E}\Big\{\big[A\big(p(k)\big)\xi(k) + D\big(p(k)\big) \\
&\qquad \times \xi(k-d) + Bu(k) \\
&\qquad + Nf(z(k))\big]^T Q\big(p(k+1)\big) \\
&\qquad \times \big[A\big(p(k)\big)\xi(k) + D\big(p(k)\big) \\
&\qquad\quad \times \xi(k-d) + Bu(k) \\
&\qquad\quad + Nf(z(k))\big] + \sigma^2\xi^T(k)\,J^TM^T \\
&\qquad \times Q\big(p(k+1)\big)\,MJ\xi(k) \\
&\qquad + p(k)\big(1-p(k)\big) \\
&\qquad \times \big[K\big(p(k)\big)J\xi(k) \\
&\qquad\quad - K\big(p(k)\big)J\xi(k-d)\big]^T \\
&\qquad \times Q\big(p(k+1)\big) \\
&\qquad \times \big[K\big(p(k)\big)J\xi(k) \\
&\qquad\quad - K\big(p(k)\big)J\xi(k-d)\big] \\
&\qquad - \xi^T(k)\big[Q\big(p(k)\big) - Q_d\big] \\
&\qquad \times \xi(k) - \xi^T(k-d)\,Q_d\xi(k-d)\Big\}.
\end{aligned} \tag{21}$$

Besides that, by the condition of (4) and (5), (21) can be rewritten as follows:

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\quad \le \mathbb{E}\Big\{\big[\big(A\big(p(k)\big) + NF_1ZJ\big)\xi(k) \\
&\qquad + D\big(p(k)\big)\xi(k-d) + Bu(k) \\
&\qquad + Nf_s(z(k))\big]^T Q\big(p(k+1)\big) \\
&\qquad \times \big[\big(A\big(p(k)\big) + NF_1ZJ\big)\xi(k) \\
&\qquad + D\big(p(k)\big)\xi(k-d) \\
&\qquad + Bu(k) + Nf_s(z(k))\big] \\
&\qquad + p(k)\big(1-p(k)\big) \\
&\qquad \times \big[K\big(p(k)\big)J\xi(k) \\
&\qquad\quad - K\big(p(k)\big)J\xi(k-d)\big]^T \\
&\qquad \times Q\big(p(k+1)\big) \\
&\qquad \times \big[K\big(p(k)\big)J\xi(k) \\
&\qquad\quad - K\big(p(k)\big)J\xi(k-d)\big] \\
&\qquad - \xi^T(k)\big[Q\big(p(k)\big) - Q_d\big]\xi(k) \\
&\qquad - \xi^T(k-d)\,Q_d\xi(k-d) \\
&\qquad + \sigma^2\xi^T(k)\,J^TM^T \\
&\qquad \times Q\big(p(k+1)\big)\,MJ\xi(k) \\
&\qquad - 2f_s^T(z(k))\big(f_s(z(k)) - FZJ\xi(k)\big)\Big\}.
\end{aligned} \tag{22}$$

From the previous analysis and $u(k)\equiv 0$, it follows that

$$\mathbb{E}\{\Delta V(k)\} \le \mathbb{E}\Big\{\tilde{\xi}^T(k)\,\Pi\tilde{\xi}(k)\Big\}, \tag{23}$$

where $\tilde{\xi}(k) = [\xi^T(k) \;\; \xi^T(k-d) \;\; f_s^T(z(k))]^T$ and

$$\Pi = \begin{bmatrix} \Pi_1 & * & * \\ \Pi_2 & \Pi_3 & * \\ \Pi_4 & \Pi_5 & \Pi_6 \end{bmatrix}, \tag{24}$$

with

$$\begin{aligned}
\Pi_1 = {} & \big[A\big(p(k)\big) + NF_1ZJ\big]^T \\
& \times Q\big(p(k+1)\big) \\
& \times \big[A\big(p(k)\big) + NF_1ZJ\big] \\
& + \sigma^2J^TM^TQ\big(p(k+1)\big)\,MJ \\
& + p(k)\big(1-p(k)\big)\,J^TK^T \\
& \times \big(p(k)\big)\,Q\big(p(k+1)\big)\,K\big(p(k)\big)\,J \\
& - \big(Q\big(p(k)\big) - Q_d\big),
\end{aligned}$$

$$\begin{aligned}\Pi_2 &= D^T(p(k))Q(p(k+1)) \\ &\quad \times [A(p(k)) + NF_1ZJ] \\ &\quad - p(k)(1-p(k))J^TK^T \\ &\quad \times (p(k))Q(p(k+1))K(p(k))J, \\ \Pi_3 &= D^T(p(k))Q(p(k+1)) \\ &\quad \times D(p(k)) + p(k)(1-p(k)) \\ &\quad \times J^TK^T(p(k))Q(p(k+1)) \\ &\quad \times K(p(k))J - Q_d, \\ \Pi_4 &= N^TQ(p(k+1)) \\ &\quad \times [A(p(k)) + NF_1ZJ] + FZJ, \\ \Pi_5 &= N^TQ(p(k+1))D(p(k)), \\ \Pi_6 &= N^TQ(p(k+1))N - 2I.\end{aligned} \tag{25}$$

In the following, we will show that $\Pi < 0$ from (17). From the relationship $-Q(p(k+1)) + S + S^T > 0$ in (17), we can see that S is nonsingular. Performing congruence transformation $\mathrm{diag}\{I, I, I, \sigma^{-2}S^{-1}, \theta^{-1}(k)S^{-1}, S^{-1}\}$ to (17), we have

$$\begin{bmatrix} Q_d - Q(p(k)) & * & * & * & * & * \\ 0 & -Q_d & * & * & * & * \\ FZJ & 0 & -2I & * & * & * \\ MJ & 0 & 0 & -\sigma^{-2}\overline{\Gamma}(k) & * & * \\ K(p(k))J & -K(p(k))J & 0 & 0 & -\theta^{-1}(k)\overline{\Gamma}(k) & * \\ A(p(k)) + NF_1ZJ & D(p(k)) & N & 0 & 0 & -\overline{\Gamma}(k) \end{bmatrix} < 0, \tag{26}$$

where, $\overline{\Gamma}(k) = -S^{-T}Q(p(k+1))S^{-1} + S^{-1} + S^{-T}$.

Since $Q^{-1}(p(k+1))$ is positive definite, we have

$$\begin{aligned}&Q^{-1}(p(k+1)) + S^{-T}Q(p(k+1))S^{-1} - S^{-1} - S^{-T} \\ &\quad = \left[S^{-T} - Q^{-1}(p(k+1))\right] \\ &\qquad \times Q(p(k+1))S^{-1} \\ &\qquad - \left[S^{-T} - Q^{-1}(p(k+1))\right] \\ &\qquad = \left[S^{-1} - Q^{-1}(p(k+1))\right]^T \\ &\qquad \times Q(p(k+1)) \\ &\qquad \times \left[S^{-1} - Q^{-1}(p(k+1))\right] \geq 0,\end{aligned} \tag{27}$$

therefore $-Q^{-1}(p(k+1)) \leq S^{-T}Q(p(k+1))S^{-1} - S^{-1} - S^{-T}$. Then, it follows from inequality (27) that

$$\begin{bmatrix} Q_d - Q(p(k)) & * & * & * & * & * \\ 0 & -Q_d & * & * & * & * \\ FZJ & 0 & -2I & * & * & * \\ MJ & 0 & 0 & -\sigma^{-2}\Gamma(k) & * & * \\ K(p(k))J & -K(p(k))J & 0 & 0 & -\theta^{-1}(k)\Gamma(k) & * \\ A(p(k)) + NF_1ZJ & D(p(k)) & N & 0 & 0 & -\Gamma(k) \end{bmatrix} < 0, \tag{28}$$

with $\Gamma(k) = Q^{-1}(p(k+1))$. By Schur complement (Lemma 6), we can see that $\Pi < 0$. Subsequently, we have

$$\mathbb{E}\{\Delta V(k)\} < \lambda_{\min}(\Pi)\,\mathbb{E}\left|\tilde{\xi}(k)\right|^2, \tag{29}$$

where $\lambda_{\min}(\Pi)$ is the minimum eigenvalue of Π. Finally, we can confirm that the augmented system (11) is exponentially stable in mean square sense and the proof of this theorem is thus complete. □

In Theorem 7, the sufficient condition ensuring the exponential stability of augmented filtering dynamics (11) has been obtained. Now, we will consider the H_∞ performance for this dynamics under the zero initial condition.

Theorem 8. *If there exist positive-definite matrix $Q_d > 0$, $Q(p(k)) > 0$, and matrix S such that the following matrix inequalities hold:*

$$\begin{bmatrix} Q_d - Q(p(k)) & * & * & * & * & * & * & * & * \\ 0 & -Q_d & * & * & * & * & * & * & * \\ FZJ & 0 & -2I & * & * & * & * & * & * \\ 0 & 0 & 0 & -\gamma^2 & * & * & * & * & * \\ \sigma^2 S^T MJ & 0 & 0 & 0 & -\sigma^2 \widetilde{\Gamma}(k) & * & * & * & * \\ \theta(k)\Phi_1 & -\theta(k)\Phi_1 & 0 & 0 & 0 & -\theta(k)\widetilde{\Gamma}(k) & * & * & * \\ \Phi_2 & S^T D(p(k)) & S^T N & S^T B & 0 & 0 & -\widetilde{\Gamma}(k) & * & * \\ -E_{cdf}(p(k)) & -p(k)E_{df}(p(k)) & 0 & 1 & 0 & 0 & 0 & -I & * \\ -\theta(k)E_{df}(p(k)) & \theta(k)E_{df}(p(k)) & 0 & 0 & 0 & 0 & 0 & 0 & -\theta(k) \end{bmatrix} < 0, \tag{30}$$

where $\theta(k)$ has been defined in (18) *and*

$$\begin{aligned} \widetilde{\Gamma}(k) &= -Q(p(k+1)) + S + S^T, \\ \Phi_1 &= S^T K(p(k)) J, \\ \Phi_2 &= S^T (A(p(k)) + NF_1 ZJ) \end{aligned} \tag{31}$$

the dynamics of the augmented systems (11)-(12) *are stochastically stable with disturbance attenuation level γ under the zero initial condition.*

Proof. In order to investigate the H_∞ performance of the augmented systems (11)-(12), construct a functional as

$$V(k) := \overline{\xi}^T(k) Q(p(k)) \overline{\xi}(k) + \sum_{s=k-d}^{k-1} \overline{\xi}^T(s) Q_d \overline{\xi}(s). \tag{32}$$

Under the zero initial condition and (22), we have

$$\begin{aligned} J(N) &= \mathbb{E}\left\{\sum_{k=0}^{N}\left[e^T(k)e(k) - \gamma^2 u^T(k)u(k)\right]\right\} \\ &\le \mathbb{E}\left\{\sum_{k=0}^{N}\left[e^T(k)e(k) - \gamma^2 u^T(k)u(k) + \Delta V_k\right]\right\} \\ &\le \mathbb{E}\left\{\sum_{k=0}^{N}\overline{\xi}^T(k)\overline{\Pi}\overline{\xi}(k)\right\}, \end{aligned} \tag{33}$$

where $\overline{\xi}(k) = [\xi^T(k) \;\; \xi^T(k-d) \;\; f_s^T(z(k)) \;\; u^T(k)]^T$ and

$$\overline{\Pi} = \begin{bmatrix} \overline{\Pi}_1 & * & * & * \\ \overline{\Pi}_2 & \overline{\Pi}_3 & * & * \\ \overline{\Pi}_4 & \overline{\Pi}_5 & \overline{\Pi}_6 & * \\ \overline{\Pi}_7 & \overline{\Pi}_8 & \overline{\Pi}_9 & \overline{\Pi}_{10} \end{bmatrix}, \tag{34}$$

with

$$\begin{aligned} \overline{\Pi}_1 = {} & [A(p(k)) + NF_1 ZJ]^T \\ & \times Q(p(k+1)) \\ & \times [A(p(k)) + NF_1 ZJ] \\ & + \sigma^2 J^T M^T Q(p(k+1)) MJ \\ & + p(k)(1 - p(k)) J^T K^T \\ & \times (p(k)) Q(p(k+1)) \\ & \times K(p(k)) J - (Q(p(k)) - Q_d) \\ & + E_{cdf}^T(p(k)) E_{cdf}(p(k)) \\ & + p(k)(1 - p(k)) E_{df}^T \\ & \times (p(k)) E_{df}(p(k)), \end{aligned}$$

$$\begin{aligned} \overline{\Pi}_2 = {} & D(p(k))^T \\ & \times Q(p(k+1)) [A(p(k)) + NF_1 ZJ] \\ & - p(k)(1 - p(k)) J^T K^T \\ & \times (p(k)) Q(p(k+1)) \\ & \times K(p(k)) J + p(k) E_{df}^T \\ & \times (p(k)) E_{cdf}(p(k)) \\ & - p(k)(1 - p(k)) E_{df}^T \\ & \times (p(k)) E_{df}(p(k)), \end{aligned}$$

$$\begin{aligned} \overline{\Pi}_3 = {} & D(p(k))^T Q(p(k+1)) \\ & \times D(p(k)) + p(k)(1 - p(k)) \\ & \times [K(p(k)) J]^T Q(p(k+1)) \\ & \times K(p(k)) J - Q_d \\ & + [p(k) E_{df}(p(k))]^T \end{aligned}$$

$$\times p(k) E_{df}(p(k)) + p(k)$$
$$\times (1 - p(k)) E_{df}^T(p(k)) E_{df}(p(k)),$$
$$\overline{\Pi}_4 = N^T Q(p(k+1)) \times [A(p(k)) + NF_1ZJ] + FZJ,$$
$$\overline{\Pi}_5 = N^T Q(p(k+1)) D(p(k)),$$
$$\overline{\Pi}_6 = N^T Q(p(k+1)) N - 2I,$$
$$\overline{\Pi}_7 = B^T Q(p(k+1)) \times [A(p(k)) + NF_1ZJ] - E_{cdf}^T(p(k)),$$
$$\overline{\Pi}_8 = B^T Q(p(k+1)) D(p(k)) - p(k) E_{df}^T(p(k)),$$
$$\overline{\Pi}_9 = B^T Q(p(k+1)) N,$$
$$\overline{\Pi}_{10} = B^T Q(p(k+1)) B + 1 - \gamma^2. \tag{35}$$

By introducing a new slack matrix S, some mathematical techniques, and the similar proof line to the proof of Theorem 7, we can see that $\overline{\Pi} < 0$. Subsequently, letting $N \to \infty$, we have

$$\|e(k)\|_{\mathbb{E}_{l2}} < \gamma \|u(k)\|_{l2}. \tag{36}$$

From Definition 5 and the previous analysis we will safely come to the conclusion that the augmented systems (11)-(12) are stochastically stable with disturbance attenuation level γ under the zero initial condition. □

Remark 9. For the sake of the desired stability of the augmented systems (11), we propose a Lyapunov functional including the time-varying probability parameters, which reduces the conservatism of the sufficient condition in Theorems 7 and 8. At the same time, in the sufficient condition, we introduce a slack variable S to decouple the Lyapunov matrices and the filter parameters, which all contain the time-varying probability parameters. Such a technique can bypass the difficulty encountered in the filter design. In the following theorem, the filter design problem is dealt with.

Theorem 10. *Consider the augmented systems* (11)-(12). *Assume that there exist positive-definite matrix sequences* $\mathcal{Q}(p(k)) > 0$, $\mathcal{Q}_d > 0$ *matrix sequences* $\mathfrak{A}_f$, $\mathfrak{B}_f$, $\mathfrak{C}_f$, *and* D_f, *nonsingular matrices* S_{11}, R_2, *and matrix* R_1 *such that the following parameter-dependent LMIs hold:*

$$\begin{bmatrix} \mathcal{Q}_d - \mathcal{Q}(p(k)) & * & * & * & * & * & * & * & * \\ 0 & -\mathcal{Q}_d & * & * & * & * & * & * & * \\ FZJ & 0 & -2I & * & * & * & * & * & * \\ 0 & 0 & 0 & -\gamma^2 & * & * & * & * & * \\ \Psi_1 & 0 & 0 & 0 & -\sigma^{-2}\Psi_6(k) & * & * & * & * \\ \theta(k)\Psi_2(k) & -\theta(k)\Psi_2(k) & 0 & 0 & 0 & -\theta(k)\Psi_6(k) & * & * & * \\ \Psi_3(k) & p(k)\Psi_2(k) & \Psi_4 & \Psi_5 & 0 & 0 & -\Psi_6(k) & * & * \\ -\Psi_7(k) & -p(k)\Psi_8(k) & 0 & 1 & 0 & 0 & 0 & -I & * \\ -\theta(k)\Psi_8(k) & \theta(k)\Psi_8(k) & 0 & 0 & 0 & 0 & 0 & 0 & -\theta(k) \end{bmatrix} < 0, \tag{37}$$

where

$$\Psi_1 = \begin{bmatrix} S_{11}^T M_c & 0 \\ R_1^T M_c & 0 \end{bmatrix}, \qquad \Psi_2(k) = \begin{bmatrix} \mathfrak{B}_f C_c & 0 \\ \mathfrak{B}_f C_c & 0 \end{bmatrix},$$
$$\Psi_4 = \begin{bmatrix} S_{11}^T N_c \\ R_1^T N_c \end{bmatrix}, \qquad \Psi_5 = \begin{bmatrix} S_{11}^T B_c \\ R_1^T B_c \end{bmatrix},$$
$$\Psi_3(k) = \begin{bmatrix} S_{11}^T A_c + (1 - p(k)) \mathfrak{B}_f C_c + S_{11}^T N_c F_1 Z & \mathfrak{A}_f \\ R_1^T A_c + (1 - p(k)) \mathfrak{B}_f C_c + R_1^T N_c F_1 Z & \mathfrak{A}_f \end{bmatrix},$$
$$\Psi_8(k) = [D_f C_c \quad 0],$$
$$\Psi_6(k) = -\mathcal{Q}(p(k+1)) + \begin{bmatrix} S_{11} + S_{11}^T & R_1 + R_2^T \\ R_2 + R_1^T & R_2 + R_2^T \end{bmatrix},$$
$$\Psi_7(k) = [(1 - p(k)) D_f C_c \quad \mathfrak{C}_f]. \tag{38}$$

In this case, there exist nonsingular matrices S_{21} *and* S_{22} *such that* $R_2 = S_{21}^T S_{22}^{-T} S_{21}$, *and then the constant gains of the desired filter can be obtained as follows:*

$$A_f = S_{21}^{-T} \mathfrak{A}_f S_{21}^{-1} S_{22}, \qquad B_f = S_{21}^{-T} \mathfrak{B}_f,$$
$$C_f = \mathfrak{C}_f S_{21}^{-1} S_{22}. \tag{39}$$

Then, there exists a desired gain-scheduled filter in the form of (8)-(9) *such that the dynamics of the augmented systems* (11)-(12) *are stochastically stable with disturbance attenuation level* γ *under the zero initial condition.*

Proof. Let nonsingular matrix variable S in (30) be partitioned as $S = [S_{ij}]_{2\times 2}$, where S_{11}, S_{21}, and S_{22} are nonsingular matrices. Introduce matrices

$$\mathcal{T} = \begin{bmatrix} I & 0 \\ 0 & S_{22}^{-1}S_{21} \end{bmatrix}, \qquad \mathcal{Q}(p(k)) = \mathcal{T}^T Q(p(k))\mathcal{T},$$
$$\mathcal{Q}_d = \mathcal{T}^T Q_d \mathcal{T}, \tag{40}$$
$$R_1 = S_{12}S_{22}^{-1}S_{21}, \qquad R_2 = S_{21}^T S_{22}^{-T} S_{21}.$$

By performing congruence transformation diag$\{\mathcal{T}^{-1}, \mathcal{T}^{-1}, I, 1, \mathcal{T}^{-1}, \mathcal{T}^{-1}, \mathcal{T}^{-1}, 1, 1\}$ to (37), we can see that (37) is equivalent to (30). It can now be concluded from Theorem 8 that (11) and (12) are stochastically stable with disturbance attenuation level γ under the zero initial condition. □

Because of the time-varying parameter $p(k)$, the number of LMIs in Theorem 10 is infinite, and all these LMIs bring enormous difficulties to solve. In the next work, we will focus on the challenge and present an effective method to overcome this difficulty.

Theorem 11. *Consider the augmented systems* (11)-(12). *Assume that there exist positive positive-definite matrices $\mathcal{Q}_d > 0$, $\mathcal{Q}_0 > 0$, and $\mathcal{Q}_p > 0$, nonsingular matrices S_{11} and R_2, and matrices R_1, $\mathfrak{A}_{f0}$, $\mathfrak{A}_{fp}$, $\mathfrak{B}_{f0}$, $\mathfrak{B}_{fp}$, $\mathfrak{C}_{f0}$, $\mathfrak{C}_{fp}$, D_{f0}, and D_{fp}, such that the following LMIs hold:*

$$\Omega^{ijrl} = \begin{bmatrix} \mathcal{Q}_d - \mathcal{Q}^l & * & * & * & * & * & * & * & * \\ 0 & -\mathcal{Q}_d & * & * & * & * & * & * & * \\ FZJ & 0 & -2I & * & * & * & * & * & * \\ 0 & 0 & 0 & -\gamma^2 & * & * & * & * & * \\ \Psi_1 & 0 & 0 & 0 & -\delta^2\Psi_6^l & * & * & * & * \\ \Psi_2^{ijr} & -\Psi_2^{ijr} & 0 & 0 & 0 & -\theta^{ij}\Psi_6^l & * & * & * \\ \Psi_3^{jr} & \Psi_2^{ir} & \Psi_4 & \Psi_5 & 0 & 0 & -\Psi_6^l & * & * \\ -\Psi_7^{jr} & -p_i\Psi_8^r & 0 & 1 & 0 & 0 & 0 & -1 & * \\ -\theta^{ij}\Psi_8^r & \theta^{ij}\Psi_8^r & 0 & 0 & 0 & 0 & 0 & 0 & -\theta^{ij} \end{bmatrix} < 0, \tag{41}$$

for $i, j, r, l = 1, 2$,

$$\Psi_3^{jr} = \begin{bmatrix} S_{11}^T A_c + (1-p_j)\mathfrak{B}_f^r C_c + S_{11}^T N_c F_1 Z & \mathfrak{A}_f^r \\ R_1^T A_c + (1-p_j)\mathfrak{B}_f^r C_c + R_1^T N_c F_1 Z & \mathfrak{A}_f^r \end{bmatrix},$$
$$\Psi_2^{ir} = p_i \begin{bmatrix} \mathfrak{B}_f^r C_c & 0 \\ \mathfrak{B}_f^r C_c & 0 \end{bmatrix},$$
$$\Psi_6^l = -(\mathcal{Q}_0 + p_l \mathcal{Q}_p) + \begin{bmatrix} S_{11} + S_{11}^T & R_1 + R_2^T \\ R_2 + R_1^T & R_2 + R_2^T \end{bmatrix},$$
$$\Psi_2^{ijr} = p_i(1-p_j) \begin{bmatrix} \mathfrak{B}_f^r C_c & 0 \\ \mathfrak{B}_f^r C_c & 0 \end{bmatrix},$$
$$\Psi_7^{jr} = \left[(1-p_j) D_f^r C_c \quad \mathfrak{C}_f^r\right], \qquad \mathfrak{A}_f^r = \mathfrak{A}_{f0} + p_r \mathfrak{A}_{fp},$$
$$\mathfrak{B}_f^r = \mathfrak{B}_{f0} + p_r \mathfrak{B}_{fp}, \qquad \mathfrak{C}_f^r = \mathfrak{C}_{f0} + p_r \mathfrak{C}_{fp},$$
$$\mathcal{Q}^l = \mathcal{Q}_0 + p_l \mathcal{Q}_p, \qquad \theta^{ij} = p_i(1-p_j),$$
$$\Psi_8^r = \left[D_f^r C_c \quad 0\right], \tag{42}$$

and Ψ_1, Ψ_4, and Ψ_5 have been defined in (38).

In this case, there exist nonsingular matrices S_{21} and S_{22} such that $R_2 = S_{21}^T S_{22}^{-T} S_{21}$, and then the constant gains of the desired filter can be obtained as follows:

$$A_{f0} = S_{21}^{-T}\mathfrak{A}_{f0}S_{21}^{-1}S_{22}, \qquad A_{fp} = S_{21}^{-T}\mathfrak{A}_{fp}S_{21}^{-1}S_{22},$$
$$B_{f0} = S_{21}^{-T}\mathfrak{B}_{f0}, \qquad B_{fp} = S_{21}^{-T}\mathfrak{B}_{fp}, \tag{43}$$
$$C_{f0} = \mathfrak{C}_{f0}S_{21}^{-1}S_{22}, \qquad C_{fp} = \mathfrak{C}_{fp}S_{21}^{-1}S_{22}.$$

Then, a gain-scheduled filter can be obtained in the form of (8)-(9) *such that the dynamics of the augmented systems* (11)-(12) *are stochastically stable with disturbance attenuation level γ under the zero initial condition.*

Proof. Firstly, choose the probability-dependent Lyapunov matrices as

$$\mathcal{Q}(p(k)) = \mathcal{Q}_0 + p(k)\mathcal{Q}_p, \tag{44}$$

where $\mathcal{Q}_0 > 0$ and $\mathcal{Q}_p > 0$. Setting

$$\lambda_1(k) = \frac{p_2 - p(k)}{p_2 - p_1}, \qquad \lambda_2(k) = \frac{p(k) - p_1}{p_2 - p_1}, \tag{45}$$

we have

$$\lambda_1(k) + \lambda_2(k) = 1, \qquad \lambda_i(k) \geq 0 \quad (i = 1, 2),$$
$$p(k) = \lambda_1(k)p_1 + \lambda_2(k)p_2. \tag{46}$$

Similarly, letting

$$\mu_1(k) = \frac{p_2 - p(k+1)}{p_2 - p_1}, \qquad \mu_2(k) = \frac{p(k+1) - p_1}{p_2 - p_1}, \tag{47}$$

we have

$$\mu_1(k) + \mu_2(k) = 1, \qquad \mu_l(k) \geq 0 \quad (l = 1, 2),$$
$$p(k+1) = \mu_1(k)p_1 + \mu_2(k)p_2. \tag{48}$$

From the previous transformations, it can be easily derived that

$$\mathcal{Q}(p(k)) = \sum_{l=1}^{2} \lambda_l(k)\mathcal{Q}^l, \qquad \mathcal{Q}(p(k+1)) = \sum_{l=1}^{2} \mu_l(k)\mathcal{Q}^l,$$
$$\mathfrak{A}_f = \sum_{r=1}^{2} \lambda_r(k)\mathfrak{A}_f^r, \qquad \mathfrak{B}_f = \sum_{r=1}^{2} \lambda_r(k)\mathfrak{B}_f^r,$$
$$\mathfrak{C}_f = \sum_{r=1}^{2} \lambda_r(k)\mathfrak{C}_f^r, \qquad D_f = \sum_{r=1}^{2} \lambda_r(k)D_f^r. \tag{49}$$

And it follows from (41) that

$$\sum_{i,j,r,l=1}^{2} \lambda_i(k)\lambda_j(k)\lambda_r(k)\mu_l(k)\Omega^{ijrl} < 0. \tag{50}$$

Also, it follows from (46) and (48)–(50) that (37) holds, and the proof is now complete. □

Remark 12. In Theorem 11, we convert infinite LMIs to finite ones by turning the time-varying parameter $p(k)$ into the polytopic form. By such a transformation, the constant gains of the desired gain-scheduled filter can be easily derived in terms of the available LMI toolbox by using the computationally appealing gain-scheduled deconvolution filter design algorithm listed as follows.

Algorithm 13. The gain-scheduled filter design algorithm.

Step 1. Given the initial values for the positive integer N_q, the initial state ρ, the constants p_1 and p_2, and the matrices A_c, B_c, C_c, M_c, N_c, F_1, F_2, and Z select appropriate initial state estimate ρ_f, and set $k = 0$.

Step 2. Solve the LMI in (41) to obtain the positive-definite matrices $\mathcal{Q}_0$, $\mathcal{Q}_p$, and $\mathcal{Q}_d$, matrices $\mathfrak{A}_{f0}$, $\mathfrak{A}_{fp}$, $\mathfrak{B}_{f0}$, $\mathfrak{B}_{fp}$, $\mathfrak{C}_{f0}$, $\mathfrak{C}_{fp}$, R_1, R_2, and S_{11}. Choose appropriate nonsingular matrices S_{21} to derive A_{f0}, A_{fp}, B_{f0}, B_{fp}, C_{f0}, and C_{fp}.

Step 3. Based on the measured time-varying parameter $p(k)$, derive the filter gains A_f, B_f, C_f, and D_f by (10), the state $x_f(k+1)$ by (9), and the estimation of $u(k)$. Then, set $k = k + 1$.

Step 4. If $k < N$, then go to Step 3; otherwise go to Step 5.

Step 5. Stop.

Remark 14. In Algorithm 13, detailed steps have been given for the gain-scheduled deconvolution filter design problem according to Theorem 11. By employing this algorithm along with the LMI toolbox, the time-varying filter gains can be easily derived from the measured/estimated time-varying missing probability $p(k)$ in real time.

4. An Illustrative Example

In this section, an example is given to design the deconvolution filters for stochastic systems with randomly occurring sensor delays.

The system parameters are given as follows:

$$A_c = \begin{bmatrix} 0.601 & -0.065 \\ 0 & 0.420 \end{bmatrix}, \qquad M_c = \begin{bmatrix} 0.013 & 0 \\ 0 & 0.024 \end{bmatrix},$$
$$N_c = \begin{bmatrix} 0.014 & 0 \\ 0 & 0.062 \end{bmatrix}, \qquad Z = \begin{bmatrix} 0.291 & 0 \\ 0 & 0.599 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 0.159 & 0 \\ 0 & 0.311 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.409 & 0 \\ 0 & 1.501 \end{bmatrix}, \tag{51}$$
$$C_c = \begin{bmatrix} 0.22 & 0.075 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0.109 \\ 0.081 \end{bmatrix},$$
$$p_1 = 0.23, \qquad p_2 = 0.45, \qquad \sigma = 1.$$

The measurable time-varying probability sequence is assumed as $0.23e^{0.0168k}$. Then, the constant filter parameters A_{f0}, A_{fp}, B_{f0}, B_{fp}, C_{f0}, C_{fp}, D_{f0}, and D_{fp} can be obtained as follows:

$$A_{f0} = \begin{bmatrix} 0.3998 & 0.2138 \\ -0.1733 & -0.0501 \end{bmatrix}, \qquad B_{f0} = \begin{bmatrix} -0.0521 \\ 0.0364 \end{bmatrix},$$
$$C_{f0} = \begin{bmatrix} -0.0266 & -0.0249 \end{bmatrix}, \qquad D_{f0} = 16,$$
$$A_{fp} = \begin{bmatrix} -0.0109 & -0.0072 \\ -0.0246 & -0.0160 \end{bmatrix}, \qquad B_{fp} = \begin{bmatrix} -0.0230 \\ 0.0020 \end{bmatrix}, \tag{52}$$
$$C_{fp} = \begin{bmatrix} -0.0794 & -0.0520 \end{bmatrix}, \qquad D_{fp} = 17.$$

Figure 2 includes the response curves of input signal $u(k)$ and the simulation results of estimation $\hat{u}(k)$. Figure 3 gives the time-varying missing probability $p(k)$, and the corresponding filter parameters are given in Table 1. The simulation results have illustrated the rationality and effectiveness of the previous theoretical analysis.

5. Conclusions

This paper has dealt with the deconvolution filtering problem for a class of discrete-time stochastic systems with randomly occurring sensor delays, nonlinear disturbances, and external stochastic noises. We assume the sensor delays to be randomly occurring, and the occurring way is modeled by a stochastic variable sequence satisfying time-varying Bernoulli distributions. A sufficient condition has been derived to guarantee the stability of the considered stochastic systems by constructing probability-dependent Lyapunov functional and employing convex optimization method. Through some mathematical transformation, we convert the matrix inequalities into solvable form, and then a finite set of inequalities for designing the desired filter are obtained. The proposed gain-scheduled filters include both constant parameters and time-varying gains which can be updated online according to the measurable missing probabilities in real time. The desired filters can be obtained by solving a set of LMIs relying on the time-varying feature of sensor delays. By using the obtained filter, we can accurately estimate the input signal distorted by the noisy transmission channel and the delayed sensor outputs. A simulation example is exploited to illustrate the effectiveness of the proposed design scheme.

Table 1: The time-varying filter parameters.

k	$p(k)$	A_f	B_f	C_f	D_f
1	0.2339	$\begin{bmatrix} 0.3972 & 0.2121 \\ -0.1790 & -0.0539 \end{bmatrix}$	$\begin{bmatrix} -0.0574 \\ 0.0369 \end{bmatrix}$	$\begin{bmatrix} -0.0452 & -0.0370 \end{bmatrix}$	19.9762
2	0.2379	$\begin{bmatrix} 0.3972 & 0.2121 \\ -0.1791 & -0.0539 \end{bmatrix}$	$\begin{bmatrix} -0.0575 \\ 0.0369 \end{bmatrix}$	$\begin{bmatrix} -0.0455 & -0.0373 \end{bmatrix}$	20.0436
3	0.2419	$\begin{bmatrix} 0.3971 & 0.2121 \\ -0.1792 & -0.0540 \end{bmatrix}$	$\begin{bmatrix} -0.0576 \\ 0.0369 \end{bmatrix}$	$\begin{bmatrix} -0.0458 & -0.0375 \end{bmatrix}$	20.1121
4	0.2460	$\begin{bmatrix} 0.3971 & 0.2120 \\ -0.1793 & -0.0541 \end{bmatrix}$	$\begin{bmatrix} -0.0577 \\ 0.0369 \end{bmatrix}$	$\begin{bmatrix} -0.0461 & -0.0377 \end{bmatrix}$	20.1818
⋮	⋮	⋮	⋮	⋮	⋮

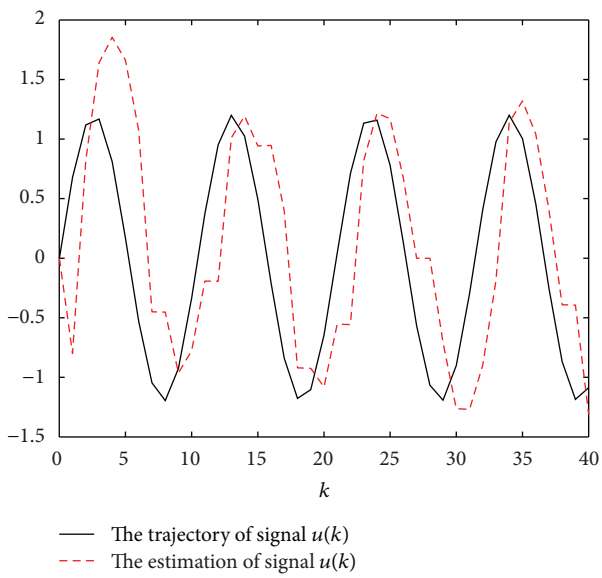


Figure 2: The input signal $u(k)$ and estimation $\hat{u}(k)$.

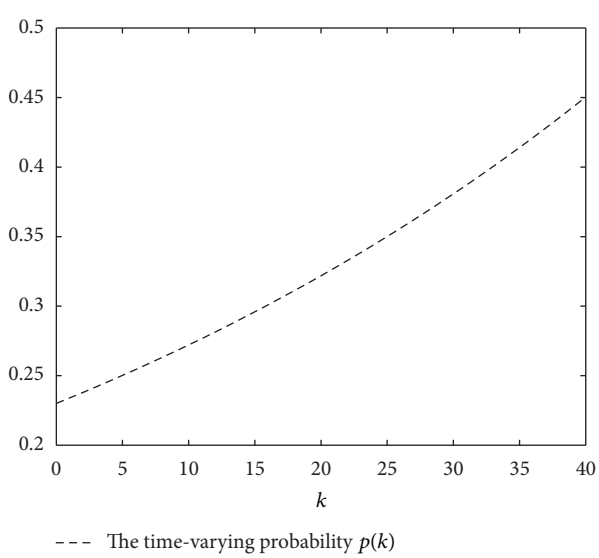


Figure 3: The time-varying probability $p(k)$.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant 61074016, the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning, the Program for New Century Excellent Talents in University under Grant NCET-11-1051, the Leverhulme Trust of the UK, and the Alexander von Humboldt Foundation of Germany.

References

[1] A. Casavola, D. Famularo, and G. Franzè, "A robust deconvolution scheme for fault detection and isolation of uncertain linear systems: an LMI approach," *Automatica*, vol. 41, no. 8, pp. 1463–1472, 2005.

[2] S. Gelfand and B. Chen, "Minimax robust deconvolution filters under stochastic parametric and noise uncertainties," *IEEE Transactions on Signal Processing*, vol. 42, no. 1, pp. 32–44, 1994.

[3] S. Peng and B. Chen, "A deconvolution filter for multichannel nonminimum phase systems via the minimax approach," *Signal Processing*, vol. 36, pp. 71–90, 1994.

[4] B. Shen, Z. Wang, H. Shu, and G. Wei, "H_∞ filtering for nonlinear discrete-time stochastic systems with randomly varying sensor delays," *Automatica*, vol. 45, no. 4, pp. 1032–1037, 2009.

[5] B. Shen, Z. Wang, J. Liang, and Y. Liu, "Recent advances on filtering and control for nonlinear stochastic complex systems with incomplete information: a survey," *Mathematical Problems in Engineering*, vol. 2012, Article ID 530759, 16 pages, 2012.

[6] Z. Wang and D. W. C. Ho, "Robust filtering under randomly varying sensor delays with variance constraints," *IEEE Transactions on Circuits and Systems*, vol. 51, no. 6, pp. 320–326, 2004.

[7] D. Ding, Z. Wang, B. Shen, and H. Shu, "H_∞ state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delayss," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 725–736, 2012.

[8] A. Ray, "Output feedback control under randomly varying distributed delays," *Journal of Guidance Control and Dynamics*, vol. 17, no. 4, pp. 701–711, 1994.

[9] F. Yang, Z. Wang, G. Feng, and X. Liu, "Robust filtering with randomly varying sensor delay: the finite-horizon case," *IEEE Transactions on Circuits and Systems I*, vol. 56, no. 3, pp. 664–672, 2009.

[10] H. Dong, Z. Wang, and H. Gao, "Robust H_∞ filtering for a class of nonlinear networked systems with multiple stochastic

communication delays and packet dropouts," *IEEE Transactions on Signal Processing*, vol. 58, no. 4, pp. 1957–1966, 2010.

[11] X. Lu, L. Xie, H. Zhang, and W. Wang, "Robust Kalman filtering for discrete-time systems with measurement delay," *IEEE Transactions on Circuits and Systems*, vol. 54, no. 6, pp. 522–526, 2007.

[12] Z. Wang, Y. Liu, and X. Liu, "H_∞ filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities," *Automatica*, vol. 44, no. 5, pp. 1268–1277, 2008.

[13] W.-H. Chen, Z.-H. Guan, and X. Lu, "Delay-dependent exponential stability of uncertain stochastic systems with multiple delays: an LMI approach," *Systems & Control Letters*, vol. 54, no. 6, pp. 547–555, 2005.

[14] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed H_∞ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[15] H. Dong, Z. Wang, and H. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Transactions on Circuits and Systems I*, vol. 59, no. 10, pp. 2354–2362, 2012.

[16] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 48, no. 9, pp. 2007–2015, 2012.

[17] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Probability-guaranteed H_∞ finite-horizon filtering for a class of nonlinear time-varying systems with sensor saturations," *Systems & Control Letters*, vol. 61, no. 4, pp. 477–484, 2012.

[18] B. Shen, Z. Wang, H. Shu, and G. Wei, "On nonlinear H_∞ filtering for discrete-time stochastic systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2170–2180, 2008.

[19] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled H_∞ controllers," *IEEE Transactions on Automatic Control*, vol. 40, no. 5, pp. 853–864, 1995.

[20] C. E. de Souza and A. Trofino, "Gain-scheduled H_2 controller synthesis for linear parameter varying systems via parameter-dependent Lyapunov functions," *International Journal of Robust and Nonlinear Control*, vol. 16, no. 5, pp. 243–257, 2006.

[21] N. T. Hoang, H. D. Tuan, P. Apkarian, and S. Hosoe, "Gain-scheduled filtering for time-varying discrete systems," *IEEE Transactions on Signal Processing*, vol. 52, no. 9, pp. 2464–2476, 2004.

[22] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.

[23] F. Wang and V. Balakrishnan, "Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 5, pp. 720–734, 2002.

[24] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15, SIAM, Philadelphia, Pa, USA, 1994.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 630623, 13 pages
http://dx.doi.org/10.1155/2013/630623

Research Article

Robust Almost Periodic Dynamics for Interval Neural Networks with Mixed Time-Varying Delays and Discontinuous Activation Functions

Huaiqin Wu, Sanbo Ding, Xueqing Guo, Lingling Wang, and Luying Zhang

Department of Applied Mathematics, Yanshan University, Qinhuangdao 066001, China

Correspondence should be addressed to Huaiqin Wu; huaiqinwu@ysu.edu.cn

Received 2 March 2013; Accepted 16 May 2013

Academic Editor: Zidong Wang

Copyright © 2013 Huaiqin Wu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The robust almost periodic dynamical behavior is investigated for interval neural networks with mixed time-varying delays and discontinuous activation functions. Firstly, based on the definition of the solution in the sense of Filippov for differential equations with discontinuous right-hand sides and the differential inclusions theory, the existence and asymptotically almost periodicity of the solution of interval network system are proved. Secondly, by constructing appropriate generalized Lyapunov functional and employing linear matrix inequality (LMI) techniques, a delay-dependent criterion is achieved to guarantee the existence, uniqueness, and global robust exponential stability of almost periodic solution in terms of LMIs. Moreover, as special cases, the obtained results can be used to check the global robust exponential stability of a unique periodic solution/equilibrium for discontinuous interval neural networks with mixed time-varying delays and periodic/constant external inputs. Finally, an illustrative example is given to demonstrate the validity of the theoretical results.

1. Introduction

In the past few decades, there was an increasing interest in different classes of neural networks such as Hopfield, cellular, Cohen-Grossberg, and bidirectional associative neural networks due to their potential applications in many areas such as classification, signal and image processing, parallel computing, associate memories, optimization, and cryptography. In the design of practical neural networks, the qualitative analysis of neural network dynamics plays an important role; for example, to solve problems of optimization, neural control, and signal processing, neural networks have to be designed in such a way that, for a given external input, they exhibit only one globally asymptotically/exponentially stable equilibrium point. Hence, exploring the global stability of neural networks is of primary importance.

In recent years, the global stability of neural networks with discontinuous activations has received extensive attention from a lot of scholars under the Filippov framework, see, for example, [1–29] and references therein. In [1], Forti and Nistri firstly dealt with the global asymptotic stability (GAS) and global convergence in finite time of a unique equilibrium point for neural networks modeled by a differential equations with discontinuous right-hand sides, and by using Lyapunov diagonally stable (LDS) matrix and constructing suitable Lyapunov function, several stability conditions were derived. In [2, 3], by applying generalized Lyapunov approach and M-matrix, Forti et al. discussed the global exponential stability (GES) of neural networks with discontinuous or non-Lipschitz activation functions. Arguing as in [1], in [4], Lu and Chen dealt with GES and GAS of Cohen-Grossberg neural networks with discontinuous activation functions. In [5–11], by using differential inclusion and Lyapunov functional approach, a series of results has been obtained for the global stability of the unique equilibrium point of neural networks with a single constant time-delay and discontinuous activations. In [12], under the framework of Filippov solutions, by using matrix measure approach, Liu et al. investigated the global dissipativity and quasi synchronization for the time-varying delayed neural networks with discontinuous

activations and parameter mismatches. In [13], similar to the method employed in [12], Liu et al. discussed the quasi-synchronization control issue of switched complex networks.

It is well known that equilibrium point can be regarded as a special case of periodic solution for a neuron system with arbitrary period or zero amplitude. Hence, through the study on periodic solution, more general results can be obtained than those of the study on equilibrium point for a neuron system. Recently, at the same time to study the global stability of the equilibrium point of neural networks with discontinuous activation functions, much attention has been paid to deal with the stability of periodic solution for various neural network systems with discontinuous activations (see [15–29]). Under the influence of Forti and Nistri, in [15], Chen et al. considered the global convergence in finite time toward a unique periodic solution for Hopfield neural networks with discontinuous activations. In [16, 17], the authors explored the periodic dynamical behavior of neural networks with time-varying delays and discontinuous activation functions; some conditions were proposed to ensure the existence and GES of the unique periodic solution. In [17–23], under the Filippov inclusion framework, by using Leray-Schauder alternative theorem and Lyapunov approach, the authors presented some conditions on the existence and GES or GAS of the unique periodic solution for Hopfield neural networks or BAM neural networks with discontinuous activation functions. In [24], take discontinuous activations as an example, Cheng et al. presented the existence of anti-periodic solutions of discontinuous neural networks. In [25, 26], Wu et al. discussed the existence and GES of the unique periodic solution for neural networks with discontinuous activation functions under impulsive control. In [28, 29], under the framework of Filippov solutions, by using Lyapunov approach and H-matrix, the authors presented the stability results of periodic solution for delayed Cohen-Grossberg neural networks with a single constant time-delay and discontinuous activation functions.

It should be pointed out that the results reported in [1–29] are concerned with the stability analysis of equilibrium point or periodic solution and neglect the effect of almost periodicity for neural networks with discontinuous activation functions. However, the almost periodicity is one of the basic properties for dynamical neural systems and appears to retrace their paths through phase space, but not exactly. Meantime, almost periodic functions, with a superior spatial structure, can be regarded as a generalization of periodic functions. In practice, as shown in [30, 31], almost periodic phenomenon is more common than periodic phenomenon, and almost periodic oscillatory behavior is more accordant with reality. Hence, exploring the global stability of almost periodic solution of dynamical neural systems is of primary importance. Very recently, under the framework of the theory of Filippov differential inclusions, Allegretto et al. proved the common asymptotic behavior of almost periodic solution for discontinuous, delayed and impulsive neural networks in [30]. In [31, 32], Lu and Chen, Qin et al. discussed the existence and uniqueness of almost periodic solution (as well as its global exponential stability) of delayed neural networks with almost periodic coefficients and discontinuous activations. In [33], Wang and Huang studied the almost periodicity for a class of delayed Cohen-Grossberg neural networks with discontinuous activations. It should be noted that the network model explored in [30–33] is a class of discontinuous neural networks with a single constant time-delay, and the stability conditions were achieved by using Lyapunov diagonally stable matrix or M-matrix. Compared with the stability conditions expressed in terms of LMIs, it is obvious that the results obtained in [30–33] are very conservative.

In hardware implementation of the neural networks, due to unavoidable factors, such as modeling error, external perturbation, and parameter fluctuation, the neural networks model certainly involves uncertainties such as perturbations and component variations, which will change the stability of neural networks. Therefore, it is of great importance to study the global robust stability of neural networks with time-varying delay. Generally speaking, two kinds of parameter uncertainty, the interval uncertainty and the norm-bounded uncertainty, are considered frequently at present. In [34, 35], based on Lyapunov stability theory and matrix inequality analysis techniques, the global robust stability of a unique equilibrium point for neural networks with norm-bounded uncertainties and discontinuous neuron activations has been discussed. In [36], Guo and Huang analyzed the global robust stability for interval neural networks with discontinuous activations. In [37], Liu and Cao discussed the robust state estimation issue for time-varying delayed neural networks with discontinuous activation functions via differential inclusions, and some criteria have been established to guarantee the existence of robust state estimator.

It should be noted that, in the above literatures [34–36], almost all results treated of the robust stability of equilibrium point for neural networks with parameter uncertainty and discontinuous neuron activations. Moreover, most of the above-mentioned results deal with only discrete time delays. Forti et al. pointed out that it would be interesting to investigate discontinuous neural networks with more general delays, such as time-varying or distributed ones. For example, in electronic implementation of analog neural networks, the delays between neurons are usually time varying and sometimes vary violently with time due to the finite switching speed of amplifiers and faults in the electrical circuit. This motivates us to consider more general types of delays, such as discrete time-varying and distributed ones, which are in general more complex and, therefore, more difficult to be dealt with. To the best of our knowledge, up to now, only a few researchers dealt with the global robust stability issue for almost periodic solution of discontinuous neural networks with mixed time-varying delays, which motivates the work of this paper.

In this paper, our aim is to study the delay-dependent robust exponential stability problem for almost periodic solution of interval neural networks with mixed time-varying delays and discontinuous activation functions. Under the framework of Filippov differential inclusions, by applying the nonsmooth Lyapunov stability theory and employing

the highly efficient LMI approach, a new delay-dependent criterion is presented to ensure the existence and global robust exponentially stability of almost periodic solution in terms of LMIs. Moreover, the obtained conclusion is applied to prove the existence and robust stability of periodic solution (or equilibrium point) for neural networks with mixed time-varying delays and discontinuous activations.

For convenience, some notation, are introduced as follows. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. For any matrix A, $A > 0$ ($A < 0$) means that A is positive definite (negative definite). A^{-1} denotes the inverse of A. A^T denotes the transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalue of A, respectively. E denotes the identity matrix with compatible dimensions. The ellipsis "$\star$" denotes the transposed elements in symmetric positions. Given the vectors $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$, $x^T y = \sum_{i=1}^n x_i y_i$. $\|A\|$ denotes the 2-norm of A; that is, $\|A\| = \sqrt{\lambda(A^T A)}$, where $\lambda(A^T A)$ denotes the spectral radius of $A^T A$. For $r > 0$, $C([-r, 0]; \mathbb{R}^n)$ denotes the family of continuous function φ from $[-r, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-r\le s\le 0}|\varphi(s)|$. $\dot{x}(t)$ denotes the derivative of $x(t)$.

Given a set $C \subset \mathbb{R}^n$, $K[C]$ denotes the closure of the convex hull of C; $P_{kc}(C)$ denotes the collection of all nonempty, closed, and convex subsets of C.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Clarke's generalized gradient [38] of V at x is defined by

$$\partial V(x) = K\left[\left\{\lim_{i\to\infty} \nabla V(x_i) : \lim_{i\to\infty} x_i = x,\ x_i \in \mathbb{R}^n \setminus \Omega_V \cup \mathscr{M}\right\}\right], \tag{1}$$

where $\Omega_V \subset \mathbb{R}^n$ is the set of Lebesgue measure zero where ∇V does not exist and $\mathscr{M} \subset \mathbb{R}^n$ is an arbitrary set with measure zero.

Let $\mathscr{N} \subset R^n$. A set-valued map $F : \mathscr{N} \hookrightarrow P_{kc}(\mathbb{R}^n)$ is said to be measurable, if, for all $y \in \mathbb{R}^n$, $\mathbb{R}^+$-valued function $x \to d(y, F(x)) = \inf\{\|y - v\|,\ v \in F(x)\}$ is measurable. This definition of measurability is equivalent to saying that

$$\text{Graph}(F) = \{(x, v) \in \mathscr{N} \times \mathbb{R}^n,\ v \in F(x)\} \in \mathscr{L}(\mathscr{N}) \times \mathscr{B}(\mathbb{R}^n) \tag{2}$$

(graph measurability), where $\mathscr{L}(\mathscr{N})$ is the Lebesgue σ-field of $\mathscr{N}$ and $\mathscr{B}(\mathbb{R}^n)$ is the Borel σ-field of $\mathbb{R}^n$.

Let Y, Z be Hausdorff topological spaces and $G(\cdot) : Y \hookrightarrow 2^Z \setminus \{\emptyset\}$. We say that the set-valued map $G(\cdot)$ is upper semicontinuous, if, for all nonempty closed subset C of Z, $G^{-1}(C) = \{y \in Y : G(y) \bigcap C \neq \emptyset\}$ is closed in Y.

The set-valued map $G(\cdot)$ is said to have a closed (convex, compact) image if, for each $x \in E$, $G(x)$ is closed (convex, compact).

The rest of this paper is organized as follows. In Section 2, the model formulation and some preliminaries are given. In Section 3, the existence and asymptotically almost periodic behavior of Filippov solutions are analyzed. Moreover, the proof of the existence of almost periodic solution is given. The global robust exponential stability is discussed, and a delay-dependent criterion is established in terms of LMIs. In Section 4, a numerical example is presented to demonstrate the validity of the proposed results. Some conclusions are drawn in Section 5.

2. Model Description and Preliminaries

Consider the following interval neural network model with discrete and distributed time delays:

$$\begin{aligned} \dot{x}(t) &= -Dx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) \\ &\quad + C\int_{t-\sigma(t)}^{t} g(x(s))\,ds + I(t), \\ D &\in \mathbb{D}, \quad A \in \mathbb{A}, \quad B \in \mathbb{B}, \quad C \in \mathbb{C}, \end{aligned} \tag{3}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is the vector of neuron states at time t, $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is an $n \times n$ diagonal matrix, $d_i > 0$, $i = 1, \ldots, n$, are the neuron self-inhibition, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, and $C = (c_{ij})_{n\times n}$ are real connection weight matrices representing the weighting coefficients of the neurons, $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T$, g_i, $i = 1, \ldots, n$, represent the neuron input-output activations, $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T$ is a real vector function representing the external inputs of the neuron at time t, and the functions and $\tau(t)$ and $\sigma(t)$ denote the discrete and distributed time-varying delays, respectively, satisfying

$$\begin{aligned} 0 \le \tau(t) \le \tau_M, &\qquad \dot{\tau}(t) \le \tau_D < 1, \\ 0 \le \sigma(t) \le \sigma_M, &\qquad \dot{\sigma}(t) \le \sigma_D < 1. \end{aligned} \tag{4}$$

We have $\mathbb{D} = [\underline{D}, \overline{D}] = \{D = \text{diag}(d_i) : 0 < \underline{d}_i \le d_i \le \overline{d}_i,\ i = 1, \ldots, n\}$, $\underline{D} = \text{diag}(\underline{d}_i)$, $\overline{D} = \text{diag}(\overline{d}_i)$, $\mathbb{A} = [\underline{A}, \overline{A}] = \{A = (a_{ij}) : \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij},\ i, j = 1, \ldots, n\}$, $\underline{A} = (\underline{a}_{ij})$, $\overline{A} = (\overline{a}_{ij})$, $\mathbb{B} = [\underline{B}, \overline{B}] = \{B = (b_{ij}) : \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij},\ i, j = 1, \ldots, n\}$, $\underline{B} = (\underline{b}_{ij})$, $\overline{B} = (\overline{b}_{ij})$, and $\mathbb{C} = [\underline{C}, \overline{C}] = \{C = (c_{ij}) : \underline{c}_{ij} \le c_{ij} \le \overline{c}_{ij},\ i, j = 1, \ldots, n\}$, $\underline{C} = (\underline{c}_{ij})$, $\overline{C} = (\overline{c}_{ij})$.

The activation function g satisfies the following assumption.

(A_1): (1) g_i, $i = 1, \ldots, n$, is piecewise continuous; that is, g_i is continuous in $\mathbb{R}$ except a countable set of jump discontinuous points and in every compact set of $\mathbb{R}$ has only a finite number of jump discontinuous points.

(2) g_i, $i = 1, 2, \ldots, n$, is nondecreasing.

System (3) can be equivalently written as

$$\begin{aligned}\dot{x}(t) = &-(D_0 + E_D F_D N_D)\, x(t) + (A_0 + E_A F_A N_A)\, g(x(t)) \\ &+ (B_0 + E_B F_B N_B)\, g(x(t-\tau(t))) \\ &+ (C_0 + E_C F_C N_C) \int_{t-\sigma(t)}^{t} g(x(s))\, ds + I(t),\end{aligned} \tag{5}$$

where $F_k \in F$, $k = D, A, B, C$,

$$\begin{aligned} F &= \left\{\operatorname{diag}(\delta_{11}, \ldots, \delta_{1n}, \ldots, \delta_{n1}, \ldots, \delta_{nn}) \in R^{n^2\times n^2} : \right. \\ &\qquad \left. \left|\delta_{ij}\right| \le 1,\ i, j = 1, 2, \ldots, n\right\}, \\ D_0 &= \frac{1}{2}\left(\underline{D} + \overline{D}\right), \qquad A_0 = \frac{1}{2}\left(\underline{A} + \overline{A}\right), \\ B_0 &= \frac{1}{2}\left(\underline{B} + \overline{B}\right), \qquad C_0 = \frac{1}{2}\left(\underline{C} + \overline{C}\right), \\ E_k &= \left(\sqrt{\beta_{11}^{(k)}} e_1, \ldots, \sqrt{\beta_{1n}^{(k)}} e_1, \ldots, \sqrt{\beta_{n1}^{(k)}} e_n, \ldots, \sqrt{\beta_{nn}^{(k)}} e_n\right)_{n\times n^2}, \\ N_k &= \left(\sqrt{\beta_{11}^{(k)}} e_1, \ldots, \sqrt{\beta_{1n}^{(k)}} e_n, \ldots, \sqrt{\beta_{n1}^{(k)}} e_1, \ldots, \sqrt{\beta_{nn}^{(k)}} e_n\right)^T_{n^2\times n}, \end{aligned} \tag{6}$$

where $e_i \in R^n$ denotes the column vector with ith element to be 1 and others to be 0.

Under assumption (A_1), $g(x)$ is undefined at the points where $g(x)$ is discontinuous, and $K[g(x)] = (K[g_1(x_1)], \ldots, K[g_n(x_n)])^T$, where $K[g_i(x_i)] = [g_i(x_i^-), g_i(x_i^+)]$, $i = 1, \ldots, n$. System (3) is a differential equation with discontinuous right-hand side. For system (3), we adopt the following definition of the solution in the sense of Filippov [39].

Definition 1. A function $x : [-\iota, T) \to \mathbb{R}^n$, $T \in (0, +\infty]$ is a solution of system (3) on $[-\iota, T)$ if

(1) $x(t)$ is continuous on $[-\iota, T)$ and absolutely continuous on $[0, T)$;

(2) $x(t)$ satisfies

$$\begin{aligned}\dot{x}(t) \in \phi(x,t) = &-(D_0 + E_D F_D N_D)\, x(t) \\ &+ (A_0 + E_A F_A N_A)\, K[g(x(t))] \\ &+ (B_0 + E_B F_B N_B)\, K[g(x(t-\tau(t)))] \\ &+ (C_0 + E_C F_C N_C) \int_{t-\sigma(t)}^{t} K[g(x(s))]\, ds \\ &+ I(t), \quad \text{for a.a. } t \in [0, T),\end{aligned} \tag{7}$$

where $\iota = \max\{\tau_M, \sigma_M\}$.

By the assumption $(A_1)(1)$, it is easy to check that $\phi(x,t)$ is an upper semicontinuous set-valued map with nonempty, compact, and convex values. Hence, $\phi(x,t)$ is measurable [40]. By the measurable selection theorem, if $x(t)$ is a solution of system (3), then there exists a measurable function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T : [-\iota, T) \to \mathbb{R}^n$ such that $\gamma(t) \in K[g(x(t))]$ and

$$\begin{aligned}\dot{x}(t) = &-(D_0 + E_D F_D N_D)\, x(t) \\ &+ (A_0 + E_A F_A N_A)\, \gamma(t) + (B_0 + E_B F_B N_B)\, \gamma(t-\tau(t)) \\ &+ (C_0 + E_C F_C N_C) \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + I(t),\end{aligned} \tag{8}$$

for a.a. $t \in [0, T)$.

The function $\gamma(t)$ in (8) is called an output solution associated with the state variable $x(t)$ and represents the vector of neural network outputs.

Definition 2. For any continuous function $\phi : [-\iota, 0] \to \mathbb{R}^n$ and any measurable selection $\psi : [-\iota, 0] \to \mathbb{R}^n$, such that $\psi(s) \in K[g(\phi(s))]$ for a.a. $s \in [-\iota, 0]$. An absolute continuous function $x(t) = x(t, \phi, \psi)$ associated with a measurable function $\gamma(t)$ is said to be a solution of the initial value problem (IVP) for system (3) on $[0, T)$ (T might be ∞) with initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$, if

$$\begin{aligned}\dot{x}(t) = &-(D_0 + E_D F_D N_D)\, x(t) + (A_0 + E_A F_A N_A)\, \gamma(t) \\ &+ (B_0 + E_B F_B N_B)\, \gamma(t-\tau(t)) \\ &+ (C_0 + E_C F_C N_C) \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + I(t), \\ x(s) &= \phi(s), \quad \forall s \in [-\iota, 0], \\ \gamma(s) &= \psi(s), \quad \text{for a.a. } s \in [-\iota, 0].\end{aligned} \tag{9}$$

Definition 3 (see [41]). A continuous function $x(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be almost periodic on $\mathbb{R}$ if, for any scalar $\varepsilon > 0$, there exist scalars $l = l(\varepsilon) > 0$ and $\omega = \omega(\varepsilon)$ in any interval with the length of l, such that $\|x(t+\omega) - x(t)\| < \varepsilon$ for all $t \in \mathbb{R}$.

Definition 4. The almost periodic solution $x^*(t)$ of interval neural network (3) is said to be global robust exponentially stable if, for any $D \in \mathbb{D}$, $A \in \mathbb{A}$, $B \in \mathbb{B}$, $C \in \mathbb{C}$, there exist scalars $\alpha > 0$ and $\delta > 0$, such that

$$\left\|x(t, \phi, \psi) - x^*(t)\right\| \le \alpha e^{-\delta t}, \quad t \ge 0, \tag{10}$$

where $x(t, \phi, \psi)$ is the solution of system (3) with initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$ and δ is called as the exponential convergence rate.

Lemma 5 (chain rule [38]). *If $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular and $x(t) : [0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact interval of $[0, +\infty)$, then $x(t)$ and $V(x(t)) : [0, +\infty) \to \mathbb{R}$ are differential for a.a. $t \in [0, +\infty)$, and*

$$\dot{V}(x(t)) = \langle \varsigma, \dot{x}(t) \rangle, \quad \forall \varsigma \in \partial V(x). \tag{11}$$

Lemma 6 (Jensen's inequality [17]). *For any constant matrix $A > 0$, any scalars a and b with $b > a$ and a vector function $x(t) : [a,b] \to \mathbb{R}^n$ such that the integrals are concerned as well defined, then*

$$\left(\int_a^b x(s)\,ds\right)^T A\left(\int_a^b x(s)\,ds\right) \le (b-a)\int_a^b x^T(s)\,Ax(s)\,ds. \tag{12}$$

Lemma 7 (see [42]). *Given any real matrices Q_1, Q_2, Q_3 of appropriate dimensions and a scalar $\varepsilon > 0$, if $Q_3 = Q_3^T > 0$, then the following inequality holds:*

$$Q_1^TQ_2 + Q_2^TQ_1 \le \varepsilon Q_1^TQ_3Q_1 + \frac{1}{\varepsilon}Q_2^TQ_3^{-1}Q_2. \tag{13}$$

Lemma 8 (see [35]). *Let U, V, and W be real matrices of appropriate dimension with M satisfying $M = M^T$, then*

$$M + UVW + W^TV^TU^T < 0, \tag{14}$$

for all $V^TV \le E$, if and only if there exists a positive constant β, such that

$$M + \beta^{-1}UU^T + \beta W^TW < 0. \tag{15}$$

Lemma 9 (see [36]). *For any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, one has*

$$\|A\| \le \|A_0\| + \|H_A\|, \qquad \|B\| \le \|B_0\| + \|H_B\|, \tag{16}$$

where $A_0 = (\overline{A} + \underline{A})/2$, $H_A = (\overline{A} - \underline{A})/2$, $B_0 = (\overline{B} + \underline{B})/2$, $H_B = (\overline{B} - \underline{B})/2$.

Lemma 10 (see [43]). *For sequence $\{f_n\} \subset L(E)$, if there exists $F(x) \in L(E)$, such that $|f_n(x)| < F(x)$, and $\lim_{n\to\infty} f_n = f$, a.e. $x \in E$, then $f \in L(E)$, and*

$$\lim_{n\to\infty}\int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx = \int_E f(x)\,dx. \tag{17}$$

Before proceeding to the main results, the following assumptions need further to be made.

(A_2): $I_i(t)$, $\tau(t)$, and $\sigma(t)$ are continuous functions and possess almost periodic property that is, for any $\varepsilon > 0$, there exist $l = l(\varepsilon) > 0$ and $\omega = \omega(\varepsilon)$ in any interval with the length of l, such that

$$\begin{aligned} &|I_i(t+\omega) - I_i(t)| < \varepsilon, \\ &|\tau(t+\omega) - \tau(t)| < \varepsilon, \\ &|\sigma(t+\omega) - \sigma(t)| < \varepsilon. \end{aligned} \tag{18}$$

(A_3): For any $\eta_i \in K[g_i(x_i)]$, $\zeta_i \in K[g_i(y_i)]$, $\zeta_i \ne \eta_i$, there exists constant $e_i > 0$, such that

$$\frac{\eta_i - \zeta_i}{x_i - y_i} \le e_i, \quad i = 1, 2, \ldots, n. \tag{19}$$

(A_4): For a given constant $\delta > 0$, there exist positive matrices P, R, and H and a positive definite diagonal matrix Q, such that

$$\Theta = \begin{pmatrix} \Pi_1 & PA_0 & PB_0 & PC_0 & PE_D & PE_A & PE_B & PE_C \\ \star & \Pi_2 & QB_0 & QC_0 & QE_D & QE_A & QE_B & QE_C \\ \star & \star & \Pi_3 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \Pi_4 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\alpha_1E & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\alpha_2E & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\alpha_3E & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\alpha_4E \end{pmatrix} < 0, \tag{20}$$

where $\Pi_1 = \Upsilon_{11} + \alpha_1N_D^TN_D$, $\Upsilon_{11} = -QD_0 - D_0^TQ + 2\delta P$, $\Pi_2 = \Upsilon_{22} + \alpha_2N_A^TN_A$, $\Upsilon_{22} = QA_0 + A_0^TQ + \delta Q + e^{\delta\tau_M}R + e^{\delta\sigma_M}\sigma_MH$, $\Pi_3 = -R + \alpha_3N_B^TN_B$, $\Pi_4 = -((1-\sigma_D)/\sigma_M)H + \alpha_4N_C^TN_C$.

3. Main Results

Theorem 11. *Suppose that assumptions (A_1), (A_2), and (A_4) are satisfied. Then interval neural network system (3) has a solution of IVP on $[0,+\infty)$ for any initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$.*

Proof. For any initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$, similar to the proof of Lemma 1 in [2], under the assumptions $(A_1)(1)$, system (3) has a local solution $x(t)$ associated with a measurable function $\gamma(t)$ with initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$ on $[0,T)$, where $T \in (0,+\infty)$ or $T = +\infty$, and $[0,T)$ is the maximal right-side existence interval of the local solution.

Consider the following Lyapunov functional candidate:

$$\begin{aligned} V(t) = {} & e^{\delta t}x^T(t)\,Px(t) + 2\sum_{i=1}^n e^{\delta t}q_i\int_0^{x_i(t)} g_i(s)\,ds \\ & + \int_{t-\tau(t)}^t e^{\delta(s+\tau_M)}\gamma^T(s)\,R\gamma(s)\,ds \\ & + \int_{-\sigma(t)}^0\int_{t+\theta}^t e^{\delta(s+\sigma_M)}\gamma^T(s)\,H\gamma(s)\,ds\,d\theta. \end{aligned} \tag{21}$$

By Lemma 5, calculating the time derivative of $V(t)$ along the local solution of system (3) on $[0,T)$, it yields

$$\begin{aligned}\dot{V}(t) &= \delta e^{\delta t} x^T(t) P x(t) \\ &\quad + 2e^{\delta t} x^T(t) P \Big[-(D_0 + E_D F_D N_D) x(t) \\ &\qquad + (A_0 + E_A F_A N_A) \gamma(t) \\ &\qquad + (B_0 + E_B F_B N_B) \gamma(t - \tau(t)) \\ &\qquad + (C_0 + E_C F_C N_C) \\ &\qquad \times \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + I(t) \Big] \\ &\quad + 2e^{\delta t} \gamma^T(t) Q \Big[-(D_0 + E_D F_D N_D) x(t) \\ &\qquad + (A_0 + E_A F_A N_A) \gamma(t) \\ &\qquad + (B_0 + E_B F_B N_B) \gamma(t - \tau(t)) \\ &\qquad + (C_0 + E_C F_C N_C) \\ &\qquad \times \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + I(t) \Big] \\ &\quad + 2\delta e^{\delta t} \sum_{i=1}^{n} q_i \int_0^{x_i(t)} g_i(s)\, ds \\ &\quad + e^{\delta(t+\tau_M)} \gamma^T(t) R \gamma(t) \\ &\quad - (1 - \dot{\tau}(t)) e^{\delta(t+\tau_M-\tau(t))} \gamma^T(t-\tau(t)) R \gamma(t-\tau(t)) \\ &\quad + \sigma(t) e^{\delta(t+\sigma_M)} \gamma^T(t) H \gamma(t) \\ &\quad - (1 - \dot{\sigma}(t)) \int_{t-\sigma(t)}^{t} e^{\delta(s+\sigma_M)} \gamma^T(s) H \gamma(s)\, ds.\end{aligned} \tag{22}$$

Without loss of generality, we can suppose that $0 \in K[g(0)]$. In fact, if this is not the case, set $G(x) = g(x) - \eta, \eta \in K[g(0)]$. Then system (8) could be equivalently changed as

$$\begin{aligned}\dot{x}(t) &= -(D_0 + E_D F_D N_D) x(t) + (A_0 + E_A F_A N_A) \overline{\gamma}(t) \\ &\quad + (B_0 + E_B F_B N_B) \overline{\gamma}(t - \tau(t)) \\ &\quad + (C_0 + E_C F_C N_C) \int_{t-\sigma(t)}^{t} \overline{\gamma}(s)\, ds + \overline{I}(t),\end{aligned} \tag{23}$$

where $\overline{\gamma}(t) = \gamma(t) - \eta \in K[G(x(t))]$, for a.a. $t \in [0,T)$, and $\overline{I}(t) = ((A_0 + E_A F_A N_A) + (B_0 + E_B F_B N_B) + \sigma(t)(C_0 + E_C F_C N_C))\eta + I(t)$. It is obvious that $0 \in K[G(0)]$. In fact, we can choose a sufficiently small constant $0 < \delta < d = \min\{\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n\}$, under the assumption $(A_1)(2)$ and $0 \in K[g_i(0)]$, such that

$$\delta \int_0^{x_i(t)} g_i(s)\, ds \le \delta x_i(t) \gamma_i(t) \le d x_i(t) \gamma_i(t). \tag{24}$$

Using Lemmas 6 and 7, we can obtain that

$$\begin{aligned}\dot{V}(t) &\le e^{\delta t} \Big\{ x^T(t) (2\delta P - 2(D_0 + E_D F_D N_D)) x(t) \\ &\qquad + 2x^T(t) P (A_0 + E_A F_A N_A) \gamma(t) \\ &\qquad + 2x^T(t) P (B_0 + E_B F_B N_B) \gamma(t - \tau(t)) \\ &\qquad + 2x^T(t) P (C_0 + E_C F_C N_C) \int_{t-\sigma(t)}^{t} \gamma(s)\, ds \\ &\qquad + 2\gamma^T(t) Q (A_0 + E_A F_A N_A) \gamma(t) \\ &\qquad + 2\gamma^T(t) Q (B_0 + E_B F_B N_B) \gamma(t - \tau(t)) \\ &\qquad + 2\gamma^T(t) Q (C_0 + E_C F_C N_C) \\ &\qquad \times \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + \delta \gamma^T(t) Q \gamma(t) \\ &\qquad + e^{\delta \tau_M} \gamma^T(t) R \gamma(t) \\ &\qquad - (1 - \tau_D) \gamma^T(t - \tau(t)) R \gamma(t - \tau(t)) \\ &\qquad + \sigma_M e^{\delta \sigma_M} \gamma^T(t) H \gamma(t) \\ &\qquad - \frac{1 - \sigma_D}{\sigma_M} \int_{t-\sigma(t)}^{t} \gamma^T(s)\, ds H \int_{t-\sigma(t)}^{t} \gamma(s)\, ds \Big\} \\ &\quad + \frac{e^{\delta t}}{\delta} \left(I^T(t) P I(t) + I^T(t) Q I(t) \right) \\ &= e^{\delta t} z^T \Theta_1 z + \frac{e^{\delta t}}{\delta} \left(I^T(t) P I(t) + I^T(t) Q I(t) \right) \\ &\le e^{\delta t} z^T \Theta_1 z + \lambda_{\max}(P + Q) \frac{e^{\delta t}}{\delta} \|I(t)\|^2,\end{aligned} \tag{25}$$

where $z = \left[x^T(t) \;\; \gamma^T(t) \;\; \gamma^T(t - \tau(t)) \;\; \int_{t-\sigma(t)}^{t} \gamma^T(s) ds \right]^T$,

$$\Theta_1 = \begin{pmatrix} \Psi_1 + \Upsilon_{11}' & P(A_0 + E_A F_A N_A) & P(B_0 + E_B F_B N_B) & P(C_0 + E_C F_C N_C) \\ \star & \Psi_2 + \Upsilon_{22}' & Q(B_0 + E_B F_B N_B) & Q(C_0 + E_C F_C N_C) \\ \star & \star & -(1 - \tau_D) R & 0 \\ \star & \star & \star & -\dfrac{1 - \sigma_D}{\sigma_M} H \end{pmatrix}, \tag{26}$$

$\Psi_1 = -Q(D_0 + E_D F_D N_D) - (D_0 + E_D F_D N_D)^T Q$, $\Psi_2 = Q(A_0 + E_A F_A N_A) + (A_0 + E_A F_A N_A)^T Q$, $\Upsilon'_{11} = 2\delta P$, $\Upsilon'_{22} = e^{\delta\tau_M} R + e^{\delta\sigma_M}\sigma_M H + \delta Q$.

Θ_1 can be rearranged as

$$\Theta_1 = \begin{pmatrix} \Upsilon_{11} & PA_0 & PB_0 & PC_0 \\ \star & \Upsilon_{22} & QB_0 & QC_0 \\ \star & \star & -R & 0 \\ \star & \star & \star & -\dfrac{1-\sigma_D}{\sigma_M} H \end{pmatrix}$$
$$+ U_D F_D W_D + W_D^T F_D^T U_D^T + U_A F_A W_A + W_A^T F_A^T U_A^T + U_B F_B W_B + W_B^T F_B^T U_B^T + U_C F_C W_C + W_C^T F_C^T U_C^T, \tag{27}$$

where $U_D = \left(E_D^T Q \;\; 0 \;\; 0 \;\; 0\right)^T$, $W_D = \left(-N_D \;\; 0 \;\; 0 \;\; 0\right)$, $U_A = (E_A^T P \;\; E_A^T Q \;\; 0 \;\; 0)^T$, $W_A = (0 \;\; N_A \;\; 0 \;\; 0)$, $U_B = (E_B^T P \;\; E_B^T Q \;\; 0 \;\; 0)^T$, $W_B = (0 \;\; 0 \;\; N_B \;\; 0)$, $U_C = (E_C^T P \;\; E_C^T Q \;\; 0 \;\; 0)^T$, and $W_C = \left(0 \;\; 0 \;\; 0 \;\; N_C\right)$.

In view of Lemma 8, $\Theta_1 < 0$ is equivalent to

$$\Theta_2 = \begin{pmatrix} \Upsilon_{11} & PA_0 & PB_0 & PC_0 \\ \star & \Upsilon_{22} & QB_0 & QC_0 \\ \star & \star & -R & 0 \\ \star & \star & \star & -\dfrac{1-\sigma_D}{\sigma_M} H \end{pmatrix} + \alpha_1^{-1} U_D U_D^T + \alpha_1 W_D^T W_D$$
$$+ \alpha_2^{-1} U_A U_A^T + \alpha_2 W_A^T W_A + \alpha_3^{-1} U_B U_B^T + \alpha_3 W_B^T W_B + \alpha_4^{-1} U_C U_C^T + \alpha_4 W_C^T W_C$$
$$= \begin{pmatrix} \Upsilon_{11} + \alpha_1 N_D^T N_D & PA_0 & PB_0 & PC_0 \\ \star & \Upsilon_{22} + \alpha_2 N_A^T N_A & QB_0 & QC_0 \\ \star & \star & -R + \alpha_3 N_B^T N_B & 0 \\ \star & \star & \star & -\dfrac{1-\sigma_D}{\sigma_M} H + \alpha_4 N_C^T N_C \end{pmatrix}$$
$$+ \alpha_1^{-1} U_D U_D^T + \alpha_2^{-1} U_A U_A^T + \alpha_3^{-1} U_B U_B^T + \alpha_4^{-1} U_C U_C^T < 0. \tag{28}$$

By the Schur complement, $\Theta < 0$ is equivalent to $\Theta_2 < 0$, so the LMI $\Theta < 0$ is also equivalent to $\Theta_1 < 0$. This implies that

$$\dot{V}(t) \le \lambda_{\max}(P + Q) \frac{e^{\delta t}}{\delta} \|I(t)\|^2. \tag{29}$$

By the assumption (A_2), $I(t)$ is bounded for $t \ge 0$. Hence, there exists a constant $M > 0$ such that

$$0 < \lambda_{\max}(P + Q) \frac{e^{\delta t}}{\delta} \|I(t)\|^2 < M, \quad t \ge 0. \tag{30}$$

It follows that

$$\dot{V}(t) \le M e^{\delta t}, \quad t \in [0, T). \tag{31}$$

Integrating both sides of (31) from 0 to t, $t \in [0, T)$, it follows that

$$V(t) \le V(0) + \int_0^t M e^{\delta s} ds = V(0) + \frac{M}{\delta}\left(e^{\delta t} - 1\right). \tag{32}$$

In view of the definition of $V(t)$ in (21) and the fact that all the terms in $V(t)$ are not negative, we have

$$V(t) \ge e^{\delta t} x^T(t) P x(t), \quad t \in [0, T). \tag{33}$$

Combining (32) and (33), it is easy to obtain

$$\lambda_{\min}(P) \|x(t)\|^2 \le e^{-\delta t} V[x, \gamma](0) + \frac{M}{\delta}\left(1 - e^{-\delta t}\right), \quad t \in [0, T). \tag{34}$$

Therefore, $\lim_{t \to T^-} \|x(t)\| < +\infty$. By the viability theorem in differential inclusions theory [40], one yields $T = +\infty$. That is, system (3) has a solution of IVP on $[0, +\infty)$ for any initial value. The proof is completed. □

Theorem 12. *Suppose that the assumptions (A_1)–(A_4) are satisfied. Then the solution of IVP of interval neural network system (3) is asymptotically almost periodic.*

Proof. Let $x(t)$ be a solution of IVP of system (3) associated with a measurable function $\gamma(t)$ with initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$. Set $y(t) = x(t + \omega) - x(t)$, we have

$$\dot{y}(t) = -\left(D_0 + E_D F_D N_D\right) x(t + \omega) + \left(A_0 + E_A F_A N_A\right) \gamma(t + \omega) + \left(B_0 + E_B F_B N_B\right) \gamma(t + \omega - \tau(t + \omega))$$

$$
\begin{aligned}
&+ \left(C_0 + E_C F_C N_C\right) \int_{t+\omega-\sigma(t+\omega)}^{t+\omega} \gamma(s)\, ds \\
&+ I(t+\omega) - \Big[-\left(D_0 + E_D F_D N_D\right) x(t) \\
&\qquad + \left(A_0 + E_A F_A N_A\right) \gamma(t) \\
&\qquad + \left(B_0 + E_B F_B N_B\right) \gamma(t-\tau(t)) \\
&\qquad + \left(C_0 + E_C F_C N_C\right) \\
&\qquad \times \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + I(t) \Big] \\
&= -\left(D_0 + E_D F_D N_D\right) y(t) \\
&\quad + \left(A_0 + E_A F_A N_A\right) \left[\gamma(t+\omega) - \gamma(t)\right] \\
&\quad + \left(B_0 + E_B F_B N_B\right) \left[\gamma(t+\omega-\tau(t)) - \gamma(t-\tau(t))\right] \\
&\quad + \left(C_0 + E_C F_C N_C\right) \\
&\quad \times \int_{t-\sigma(t)}^{t} \gamma(s+\omega) - \gamma(s)\, ds + \rho(\omega, t),
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
\rho(\omega, t) &= \left(B_0 + E_B F_B N_B\right) \\
&\quad \times \left[\gamma(t+\omega-\tau(t+\omega)) - \gamma(t+\omega-\tau(t))\right] \\
&\quad + \left(C_0 + E_C F_C N_C\right) \int_{t-\sigma(t+\omega)}^{t-\sigma(t)} \gamma(s+\omega)\, ds \\
&\quad + I(t+\omega) - I(t).
\end{aligned} \tag{36}
$$

Consider a Lyapunov functional candidate as

$$
\begin{aligned}
W(t) &= e^{\delta t} y^T(t) P y(t) \\
&\quad + 2\sum_{i=1}^{n} e^{\delta t} q_i \int_0^{y_i(t)} g_i(s)\, ds \\
&\quad + \int_{t-\tau(t)}^{t} e^{\delta(s+\tau_M)} \left(\gamma(s+\omega) - \gamma(s)\right)^T \\
&\qquad \times R\left(\gamma(s+\omega) - \gamma(s)\right) ds \\
&\quad + \int_{-\sigma(t)}^{0} \int_{t+\theta}^{t} e^{\delta(s+\sigma_M)} \left(\gamma(s+\omega) - \gamma(s)\right)^T \\
&\qquad \times H\left(\gamma(s+\omega) - \gamma(s)\right) ds\, d\theta.
\end{aligned} \tag{37}
$$

Calculating the time derivative of $W(t)$ along trajectories of system (35), similar to the proof of Theorem 11, we can get

$$
\dot{W}(t) \le \lambda_{\max}(P+Q) \frac{e^{\delta t}}{\delta} \left\|\rho(\omega, t)\right\|^2. \tag{38}
$$

From the proof of Theorem 11, we can get that $x(t)$ is bounded. Consequently, $\gamma(t)$ is also bounded. Define $H_B = (\beta_{ij}^{(B)})_{n\times n} = (1/2)(\overline{B} - \underline{B})$, $H_C = \left(\left(\beta_{ij}^{(C)}\right)\right)_{n\times n} = (1/2)(\overline{C} - \underline{C})$. By the assumption (A_3) and Lemma 9, there exist positive constants α and β, such that

$$
\begin{aligned}
\left\|\rho(\omega, t)\right\| &\le \alpha \left\|\left(B_0 + E_B F_B N_B\right)\right\| |\tau(t+\omega) - \tau(t)| \\
&\quad + \beta \left\|\left(C_0 + E_C F_C N_C\right)\right\| |\sigma(t+\omega) - \sigma(t)| \\
&\quad + \left\|I(t+\omega) - I(t)\right\| \\
&\le \alpha \left(\left\|B_0\right\| + \left\|H_B\right\|\right) |\tau(t+\omega) - \tau(t)| \\
&\quad + \beta \left(\left\|C_0\right\| + \left\|H_C\right\|\right) |\sigma(t+\omega) - \sigma(t)| \\
&\quad + \left\|I(t+\omega) - I(t)\right\|.
\end{aligned} \tag{39}
$$

Therefore, by using the assumption (A_2), it is easy to obtain that, for any $\varepsilon > 0$, there exist $l = l(\varepsilon)$ and $\omega = \omega(\varepsilon)$ in any interval with the length of l, such that

$$
\lambda_{\max}(P+Q) \left\|\rho(\omega, t)\right\|^2 \le \frac{1}{2}\delta^2\varepsilon^2, \quad t \ge 0. \tag{40}
$$

This implies that

$$
\dot{W}(t) \le \frac{1}{2}\varepsilon^2 \delta e^{\delta t}. \tag{41}
$$

By combining (37) and (41), we have

$$
\begin{aligned}
\left\|y(t)\right\|^2 &\le e^{-\delta t} W(t) \le e^{-\delta t} W(0) \\
&\quad + \frac{1}{2} e^{-\delta t} \int_0^t \varepsilon^2 \delta e^{\delta s} ds \\
&= e^{-\delta t} W(0) + \frac{1}{2}\varepsilon^2 \left(1 - e^{-\delta t}\right).
\end{aligned} \tag{42}
$$

Therefore, there exists $T > 0$, such that for any $t > T$, $\|y(t)\| < (1/\sqrt{2})\varepsilon < \varepsilon$, that is, $\|x(t+\omega) - x(t)\| < \varepsilon$. This shows that any solution of system (3) is asymptotically almost periodic. The proof is complete. □

Remark 13. In the proof of Theorem 12, the assumption (A_3) plays an important role. Under this assumption, $\|\rho(\omega, t)\| < \varepsilon$ can be ensured.

Theorem 14. *If the assumptions (A_1)–(A_4) hold, then interval neural network system (3) has a unique almost periodic solution which is global robust exponentially stable.*

Proof. Firstly, we prove the existence of the almost periodic solution for interval neural network system (3).

By Theorem 12, for any initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$, interval neural network (3) has a solution which is asymptotically almost periodic. Let $x(t)$ be any solution of system (3) associated with a measurable function $\gamma(t)$ with the initial value $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$. Then

$$
\begin{aligned}
\dot{x}(t) &= -\left(D_0 + E_D F_D N_D\right) x(t) \\
&\quad + \left(A_0 + E_A F_A N_A\right) \gamma(t) \\
&\quad + \left(B_0 + E_B F_B N_B\right) \gamma(t-\tau(t)) \\
&\quad + \left(C_0 + E_C F_C N_C\right) \int_{t-\sigma(t)}^{t} \gamma(s)\, ds + I(t),
\end{aligned} \tag{43}
$$

for a.a. $t \in [-\iota, +\infty)$.

By using (40), we can pick a sequence $\{t_k\}$ satisfying $\lim_{k\to+\infty} t_k = +\infty$ and $\|\rho(t_k, t)\| < 1/k$, for all $t \geq 0$, where $\rho(\omega, t)$ is defined in (36). In addition, the sequence $\{x(t+t_k)\}$ is equicontinuous and uniformly bounded. By Arzela-Ascoli theorem and diagonal selection principle, we can select a subsequence of $\{t_k\}$ (still denoted by $\{t_k\}$), such that $\{x(t+t_k)\}$ uniformly converges to a absolute continuous function $x^*(t)$ on any compact set of $\mathbb{R}$.

On the other hand, since $\gamma(t+t_k) \in K[g(x(t+t_k))]$ and $K[g(x(t+t_k))]$ is bounded by the boundedness of $x(t)$, the sequence $\{\gamma(t+t_k)\}$ is bounded. Hence, we can also select a subsequence of t_k (still denoted by $\{t_k\}$), such that $\{\gamma(t+t_k)\}$ converges to a measurable function $\gamma^*(t)$ for any $t \in [-\iota, +\infty)$. According to the fact that

(i) $K[g(\cdot)]$ is an upper semicontinuous set-valued map,

(ii) for $t \in [-\iota, +\infty)$, $x(t+t_k) \to x^*(t)$ as $k \to +\infty$,

we can get that for any $\epsilon > 0$, there exists $N > 0$, such that $K[g(x(t+t_k))] \subseteq K[g(x^*(t))] + \epsilon\mathscr{B}$ for $k > N$ and $t \in [-\iota, +\infty)$, where $\mathscr{B}$ is an n-dimensional unit ball. Hence, the fact $\gamma(t+t_k) \in K[g(x(t+t_k))]$ implies that $\gamma(t+t_k) \in K[g(x^*(t))] + \epsilon\mathscr{B}$. On the other hand, since $K[g(x^*(t))] + \epsilon\mathscr{B}$ is a compact subset of $\mathbb{R}^n$, we have $\gamma^*(t) = \lim_{k\to+\infty}\gamma(t+t_k) \in K[g(x^*(t))] + \epsilon\mathscr{B}$. Noting the arbitrariness of ϵ, it follows that $\gamma^*(t) \in K[g(x^*(t))]$ for a.a. $t \in [-\iota, +\infty)$.

By Lebesgue's dominated convergence theorem (Lemma 10),

$$\begin{aligned}
&x^*(t+h) - x^*(t) \\
&\quad = \lim_{k\to+\infty}\left[x(t+t_k+h) - x(t+t_k)\right] \\
&\quad = \lim_{k\to+\infty}\int_t^{t+h}\Big[-(D_0 + E_DF_DN_D)\,x(t_k+\theta) \\
&\qquad + (A_0 + E_AF_AN_A)\,\gamma(t_k+\theta) \\
&\qquad + (B_0 + E_BF_BN_B)\,\gamma(t_k+\theta-\tau(t_k+\theta)) \\
&\qquad + (C_0 + E_CF_CN_C)\int_{t_k+\theta-\sigma(t_k+\theta)}^{t_k+\theta}\gamma(s)\,ds \\
&\qquad + I(t_k+\theta)\Big]\,d\theta \\
&\quad = \lim_{k\to+\infty}\int_t^{t+h}\Big[-(D_0 + E_DF_DN_D)\,x(t_k+\theta) \\
&\qquad + (A_0 + E_AF_AN_A)\,\gamma(t_k+\theta) \\
&\qquad + (B_0 + E_BF_BN_B)\,\gamma(t_k+\theta-\tau(\theta)) \\
&\qquad + (C_0 + E_CF_CN_C)\int_{\theta-\sigma(\theta)}^{\theta}\gamma(t_k+s)\,ds \\
&\qquad + I(\theta) + \rho(t_k,\theta)\Big]\,d\theta \\
&\quad = \int_t^{t+h}\Big[-(D_0 + E_DF_DN_D)\,x^*(\theta) \\
&\qquad + (A_0 + E_AF_AN_A)\,\gamma^*(\theta) \\
&\qquad + (B_0 + E_BF_BN_B)\,\gamma^*(\theta-\tau(\theta)) \\
&\qquad + (C_0 + E_CF_CN_C)\int_{\theta-\sigma(\theta)}^{\theta}\gamma^*(s)\,ds + I(\theta)\Big]\,d\theta,
\end{aligned} \tag{44}$$

for any $t \in [-\iota, +\infty)$ and $h \in \mathbb{R}$. This implies that $x^*(t)$ is a solution of system (3).

Notice that $x(t)$ is asymptotically almost periodic. Then, for any $\varepsilon > 0$, there exist $T > 0$, $l = l(\varepsilon)$, and $\omega = \omega(\varepsilon)$ in any interval with the length of l, such that $\|x(t+\omega) - x(t)\| < \varepsilon$, for all $t > T$. Therefore, there exists a constant $N > 0$, when $k > N$, $\|x(t+t_k+\omega) - x(t+t_k)\| < \varepsilon$, for any $t \in [-\iota, +\infty)$. Let $k \to +\infty$, it follows that $\|x^*(t+\omega) - x^*(t)\| < \varepsilon$, for any $t \in [-\iota, +\infty)$. This shows that $x^*(t)$ is an almost periodic solution of system (3).

Secondly, we prove that the almost periodic solution of interval neural network system (3) is global robust exponentially stable.

Let $x(t)$ be an arbitrary, solution and let $x^*(t)$ be an almost solution of interval neural network system (3) associated with outputs $\xi(t)$ and $\gamma^*(t)$. Consider the change of variables $z(t) = x(t) - x^*(t)$, which transforms (3) into the differential equation

$$\begin{aligned}
\dot{z}(t) = &-(D_0 + E_DF_DN_D)\,z(t) \\
&+ (A_0 + E_AF_AN_A)\,\eta(t) \\
&+ (B_0 + E_BF_BN_B)\,\eta(t-\tau(t)) \\
&+ (C_0 + E_CF_CN_C)\int_{t-\sigma(t)}^{t}\eta(s)\,ds,
\end{aligned} \tag{45}$$

where $\eta(t) \in K[\widetilde{G}(z(t))]$ is measurable, $\widetilde{G}(z(t)) = (\widetilde{G}_1(z_1(t)), \widetilde{G}_2(z_2(t)), \ldots, \widetilde{G}_n(z_n(t)))^T$, and $\widetilde{G}_i(z_i(t)) = g_i(z_i(t) + x_i^*(t)) - g_i(x_i^*(t))$ $(i = 1, 2, \ldots, n)$.

Similar to $V(t)$ in (21), define a Lyapunov functional candidate as

$$\begin{aligned}
L(t) = &\,e^{\delta t}z^T(t)\,Pz(t) + 2\sum_{i=1}^{n}e^{\delta t}q_i\int_0^{z_i(t)}\widetilde{G}(s)\,ds \\
&+ \int_{t-\tau(t)}^{t}e^{\delta(s+\tau_M)}\eta^T(s)\,R\eta(s)\,ds \\
&+ \int_{-\sigma(t)}^{0}\int_{t+\theta}^{t}e^{\delta(s+\sigma_M)}\eta^T(s)\,H\eta(s)\,ds\,d\theta.
\end{aligned} \tag{46}$$

Calculating the derivative of $L(t)$ along the solution of system (45), similar to the proof of Theorem 11, we have

$$\begin{aligned}
\dot{L}(t) \leq &\,e^{\delta t}v^T\Theta_1 v - \delta e^{\delta t}z^T(t)\,Pz(t) \\
&-\delta e^{\delta t}\eta^T(t)\,Q\eta(t) < 0,
\end{aligned} \tag{47}$$

where $\upsilon = \left[z^T(t) \;\; \eta^T(t) \;\; \eta^T(t-\tau(t)) \;\; \int_{t-\sigma(t)}^{t} \eta^T(s)ds\right]^T$. Combining (46) and (47) gives

$$\|z(t)\| \le \sqrt{L(t)}e^{-(\delta/2)t} \le \sqrt{L(0)}e^{-(\delta/2)t}. \tag{48}$$

This means that the almost periodic solution $x^*(t)$ of interval neural network system (3) is global robust exponentially stable. Consequently, the almost periodic solution of system (3) is unique. The proof is complete. □

Remark 15. As far as we know, all the existing results concerning the almost periodic dynamical behaviors of neural networks with discontinuous activation functions [30–33] have not considered the global robust exponential stability performance. In this paper, by constructing appropriate generalized Lyapunov functional, we have obtained a delay-dependent criterion, which guarantee the existence, uniqueness, and global robust exponential stability of almost periodic solution. Moreover, the given result is formulated by LMIs, which can be easily verified by the existing powerful tools, such as the LMI toolbox of MATLAB. Therefore, results of this paper improve corresponding parts of those in [30–33].

Remark 16. In [34–36], some criteria on the robust stability of an equilibrium point for neural networks with discontinuous activation functions have been given. Compared to the main results in [34–36], our results make the following improvements.

(1) In [34, 35], the activation function g_i is assumed to be monotonic nondecreasing and bounded. However, from the assumption (A_1), we can see that the activation function g_i can be unbounded.

(2) Although the assumption of boundedness was dropped in [36], the monotonic nondecreasing and the growth condition were indispensable. In this paper, the activation function is only assumed to be monotonic nondecreasing.

(3) In contrast to the models in [34–36], distributed time-varying delays are considered in this paper. If we choose $\sigma(t) = 0$ and $I(t) = I$, then the models in these papers are the special cases of our model.

Notice that periodic function can be regarded as a special almost periodic function. Hence, based on Theorems 11 and 14, we can obtain the following.

Corollary 17. *Suppose that $I(t)$, $\tau(t)$, and $\sigma(t)$ are periodic functions, if the assumptions (A_1), (A_3), and (A_4) are satisfied. Then*

(1) *neural network system* (3) *has a solution of IVP on* $[0, +\infty)$ *for any initial value* $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$,

(2) *neural network system* (3) *has a unique periodic solution which is global robust exponentially stable.*

When $I_i(t)$ is a constant external input I_i, system (3) changes as

$$\begin{aligned} \dot{x}(t) &= -Dx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) \\ &\quad + C\int_{t-\sigma(t)}^{t} g(x(s))\,ds + I, \\ D \in \mathbb{D}, &\quad A \in \mathbb{A}, \quad B \in \mathbb{B}, \quad C \in \mathbb{C}. \end{aligned} \tag{49}$$

Since a constant function can be also regarded as a special almost periodic function, by applying Theorems 11 and 14, we can obtain

Corollary 18. *If the assumptions (A_1), (A_3), and (A_4) are satisfied, then*

(1) *Neural network system* (49) *has a solution of IVP on* $[0, +\infty)$ *for any initial value* $(\phi(s), \psi(s))$, $s \in [-\iota, 0]$.

(2) *Neural network system* (49) *has a unique equilibrium point which is global robust exponentially stable.*

4. Illustrative Example

Example 1. Consider the third-order interval neural network (3) with the following system parameters:

$$\begin{gathered} \overline{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \underline{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \overline{A} = \begin{pmatrix} -3 & 0.5 & 0.4 \\ 0.1 & -3 & -0.6 \\ 0.2 & 0.3 & -3 \end{pmatrix}, \qquad \underline{A} = \begin{pmatrix} -4 & 0.2 & 0.2 \\ -0.2 & -4 & -1 \\ -0.1 & 0.2 & -4 \end{pmatrix}, \\ \overline{B} = \begin{pmatrix} 0.3 & 0.3 & 0.3 \\ -0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & -0.3 \end{pmatrix}, \qquad \underline{B} = \begin{pmatrix} -0.1 & -0.1 & 0.1 \\ -0.3 & -0.3 & 0 \\ 0.2 & -0.3 & -0.5 \end{pmatrix}, \\ \overline{C} = \begin{pmatrix} 0.3 & 0.3 & -0.1 \\ 0.2 & 0.1 & -0.2 \\ 0.4 & -0.1 & 0.3 \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} -0.1 & 0.1 & -0.2 \\ -0.1 & -0.2 & -0.5 \\ 0.1 & -0.2 & 0.2 \end{pmatrix}. \end{gathered} \tag{50}$$

Set $g_1(s) = g_2(s) = g_3(s) = 5s + \operatorname{sign}(s)$, $\tau(t) = 0.5 + 0.5\cos t$, and $\sigma(t) = 0.8 - 0.2\sin t$. It is easy to check that assumptions (A_1)–(A_3) hold and $\tau_M = 1$, $\tau_D = 0.2$, $\sigma_M = 1$, and $\sigma_D = 0.5$.

Let $\delta = 0.5$. Solving the LMI in (A_4) by using appropriate LMI solver in the MATLAB, the feasible positive definite

matrices P, R, and H and positive definite diagonal matrix Q could be as

$$P = \begin{pmatrix} 148.9391 & 9.2264 & -2.4915 \\ 9.2264 & 138.5453 & -16.7765 \\ 2.4915 & -16.7765 & 178.0600 \end{pmatrix},$$
$$Q = \begin{pmatrix} 264.9438 & 0 & 0 \\ 0 & 255.8510 & 0 \\ 0 & 0 & 268.7436 \end{pmatrix},$$
$$R = \begin{pmatrix} 381.3968 & -25.5537 & -29.0966 \\ -25.5537 & 338.0614 & 54.0433 \\ -29.0966 & 54.0433 & 370.5851 \end{pmatrix}, \tag{51}$$
$$H = \begin{pmatrix} 302.0513 & -19.9994 & -4.0452 \\ -19.9994 & 319.33320 & 31.7509 \\ -4.0452 & 31.7509 & 328.8760 \end{pmatrix},$$
$$\alpha_1 = 799.7423, \qquad \alpha_2 = 723.0122,$$
$$\alpha_3 = 754.7854, \qquad \alpha_4 = 712.9184,$$

and the assumption (A_4) is also satisfied. Hence, it follows from Theorems 11–14 that system (3) with parameter ranges given above has a unique almost periodic solution which is global robust exponentially stable.

In view of Corollary 17, when the external input $I(t)$ is a periodic function, this neural network has a unique periodic solution which is global robust exponentially stable, as well as the similar result of an equilibrium for the system with constant input.

As a special case, we choose the system as follows:

$$D = \begin{pmatrix} 1.48 & 0 & 0 \\ 0 & 1.88 & 0 \\ 0 & 0 & 1.67 \end{pmatrix},$$
$$A = \begin{pmatrix} -3.69 & 0.38 & 0.27 \\ -0.13 & -3.42 & -0.64 \\ 0.16 & 0.27 & -3.53 \end{pmatrix},$$
$$B = \begin{pmatrix} 0.17 & -0.05 & 0.21 \\ -0.23 & -0.24 & 0.13 \\ 0.25 & -0.26 & -0.46 \end{pmatrix}, \tag{52}$$
$$C = \begin{pmatrix} 0.22 & 0.28 & -0.17 \\ 0.14 & -0.12 & -0.34 \\ 0.35 & -0.19 & 0.27 \end{pmatrix}.$$

Figures 1 and 2 display the state trajectories of this neural network with initial value $\phi(t) = (\sin t, -0.3\tanh t, -0.5\cos t)^T$, $t \in [-1, 0]$ when $I(t) = (15\sin t, 10\cos t, 15\cos t)^T$. It can be seen that these trajectories converge to a unique periodic. This is in accordance with the conclusion of Corollary 17. Figure 3 displays the state trajectories of this neural network with initial values $\phi(t) = (\sin t, -0.3\tanh t, -0.5\cos t)^T$, $t \in [-1, 0]$ when $I(t) = (10, 5, -10)^T$. It can be seen that these trajectories converge to a unique equilibrium point. This is in accordance with the conclusion of Corollary 18.

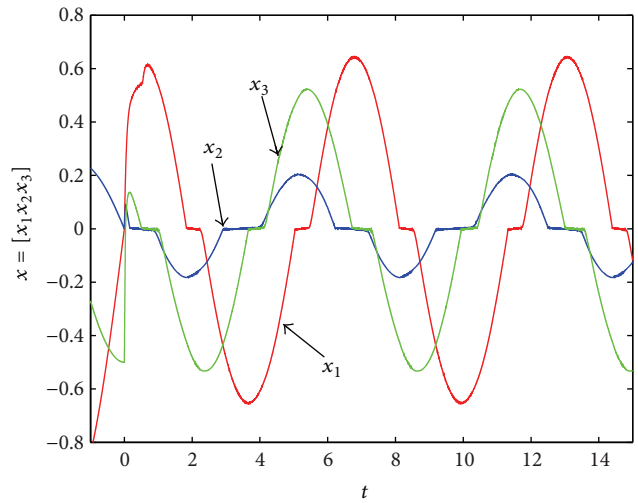


Figure 1: Time-domain behavior of the state variables x_1, x_2, and x_3 when $I(t) = (15\sin t, 10\cos t, 15\cos t)^T$.

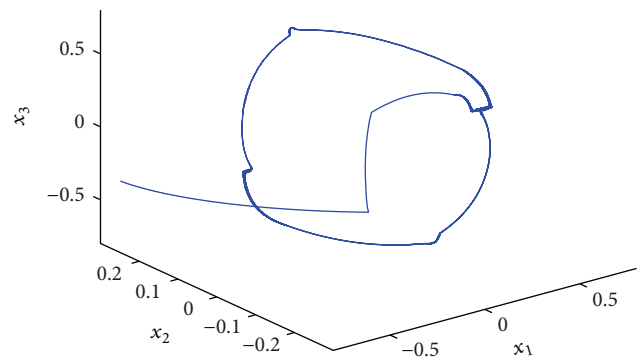


Figure 2: Phase plane behavior of the state variables x_1, x_2, and x_3 when $I(t) = (15\sin t, 10\cos t, 15\cos t)^T$.

5. Conclusion

In this paper, under the framework of Filippov differential inclusions, by constructing generalized Lyapunov-Krasovskii functional and applying LMI techniques, a sufficient condition which ensures the existence, uniqueness, and global robust exponential stability of almost periodic solution has been obtained in terms of LMIs, which is easy to be checked and applied in practice. A numerical example has been given to illustrate the validity of the theoretical results.

In [2], Forti et al. conjectured that all solutions of delayed neural networks with discontinuous neuron activations and periodic inputs converge to an asymptotically stable limit cycle. In this paper, under the assumptions (A_1)–(A_4), the results obtained conform that Forti's conjecture is true for interval neural networks with mixed time-varying delays and

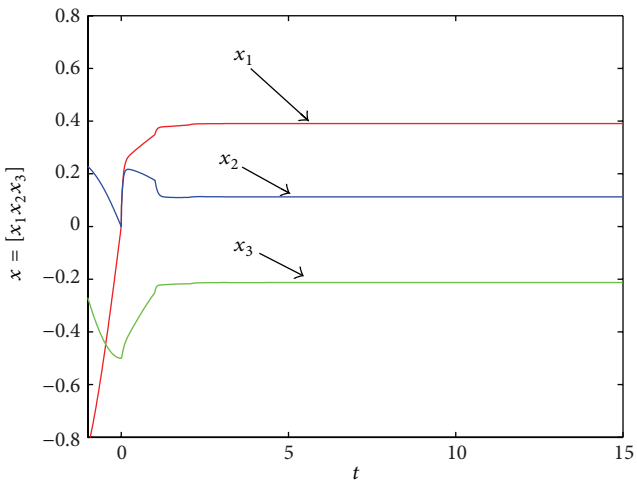


FIGURE 3: Time-domain behavior of the state variables x_1, x_2, and x_3 when $I(t) = (10, 5, -10)^T$.

discontinuous activation functions. Note that the synchronization or sliding mode control issues have been studied in [44–47] by using the delay-fractioning approach, and the obtained results have less conservative. Whether it is effective to deal with the time-delays for discontinuous neural networks via delay-fractioning approach will be the topic of our further research.

Acknowledgments

This work was supported by the Natural Science Foundation of Hebei Province of China (A2011203103) and the Hebei Province Education Foundation of China (2009157).

References

[1] M. Forti and P. Nistri, "Global convergence of neural networks with discontinuous neuron activations," *IEEE Transactions on Circuits and Systems I*, vol. 50, no. 11, pp. 1421–1435, 2003.

[2] M. Forti, M. Grazzini, P. Nistri, and L. Pancioni, "Generalized Lyapunov approach for convergence of neural networks with discontinuous or non-Lipschitz activations," *Physica D*, vol. 214, no. 1, pp. 88–99, 2006.

[3] M. Forti, "M-matrices and global convergence of discontinuous neural networks," *International Journal of Circuit Theory and Applications*, vol. 35, no. 2, pp. 105–130, 2007.

[4] W. Lu and T. Chen, "Dynamical behaviors of Cohen-Grossberg neural networks with discontinuous activation functions," *Neural Networks*, vol. 18, no. 3, pp. 231–242, 2005.

[5] M. Forti, P. Nistri, and D. Papini, "Global exponential stability and global convergence in finite time of delayed neural networks with infinite gain," *IEEE Transactions on Neural Networks*, vol. 16, no. 6, pp. 1449–1463, 2005.

[6] L. Li and L. Huang, "Global asymptotic stability of delayed neural networks with discontinuous neuron activations," *Neurocomputing*, vol. 72, no. 16-18, pp. 3726–3733, 2009.

[7] S. Qin and X. Xue, "Global exponential stability and global convergence in finite time of neural networks with discontinuous activations," *Neural Processing Letters*, vol. 29, no. 3, pp. 189–204, 2009.

[8] Y. Meng, L. Huang, Z. Guo, and Q. Hu, "Stability analysis of Cohen-Grossberg neural networks with discontinuous neuron activations," *Applied Mathematical Modelling*, vol. 34, no. 2, pp. 358–365, 2010.

[9] L. Li and L. Huang, "Dynamical behaviors of a class of recurrent neural networks with discontinuous neuron activations," *Applied Mathematical Modelling*, vol. 33, no. 12, pp. 4326–4336, 2009.

[10] X. Nie and J. Cao, "Existence and global stability of equilibrium point for delayed competitive neural networks with discontinuous activation functions," *International Journal of Systems Science*, vol. 43, no. 3, pp. 459–474, 2012.

[11] J. Liu, X. Liu, and W.-C. Xie, "Global convergence of neural networks with mixed time-varying delays and discontinuous neuron activations," *Information Sciences*, vol. 183, pp. 92–105, 2012.

[12] X. Liu, T. Chen, J. Cao, and W. Lu, "Dissipativity and quasi-synchronization for neural networks with discontinuous activations and parameter mismatches," *Neural Networks*, vol. 24, pp. 1013–1021, 2011.

[13] X. Liu, J. Cao, and W. Yu, "Filippov systems and quasi-synchronization control for switched networks," *Chaos*, vol. 22, Article ID 033110, 2012.

[14] G. Bao and Z. Zeng, "Analysis and design of associative memories based on recurrent neural network with discontinuous activation functions," *Neurocomputing*, vol. 77, pp. 101–107, 2012.

[15] X. Chen, H. Huang, and Z. Guo, "Finite time stability of periodic solution for hopfield neural networks with discontinuous activations," *Neurocomputing*, vol. 103, pp. 43–49, 2013.

[16] Z. Cai, L. Huang, Z. Guo, and X. Chen, "On the periodic dynamics of a class of time-varying delayed neural networks via differential inclusions," *Neural Networks*, vol. 33, pp. 97–113, 2012.

[17] Z. Cai and L. Huang, "Existence an global asymptotic stability of periodic solution for discrete and distributed time-varying delayed neural networks with discontinuous activations," *Neurocomputing*, vol. 74, pp. 3170–3179, 2011.

[18] X. Liu and J. Cao, "On periodic solutions of neural networks via differential inclusions," *Neural Networks*, vol. 22, no. 4, pp. 329–334, 2009.

[19] L. Huang, J. Wang, and X. Zhou, "Existence and global asymptotic stability of periodic solutions for Hopfield neural networks with discontinuous activations," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1651–1661, 2009.

[20] X. Liu, J. Cao, and G. Huang, "Complete periodic synchronization of delayed neural networks with discontinuous activations," *International Journal of Bifurcation and Chaos*, vol. 20, no. 7, pp. 2151–2164, 2010.

[21] H. Wu and Y. Li, "Existence and stability of periodic solution for BAM neural networks with discontinuous neuron activations," *Computers & Mathematics with Applications*, vol. 56, no. 8, pp. 1981–1993, 2008.

[22] H. Wu, "Stability analysis for periodic solution of neural networks with discontinuous neuron activations," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1717–1729, 2009.

[23] H. Wu, "Global stability analysis of a general class of discontinuous neural networks with linear growth activation functions," *Information Sciences*, vol. 179, no. 19, pp. 3432–3441, 2009.

[24] Y. Cheng, F. Cong, and H. Hua, "Anti-periodic solutions for nonlinear evolution equations," *Advances in Difference Equations*, vol. 2012, article 165, 15 pages, 2012.

[25] H. Wu, X. Xue, and X. Zhong, "Stability analysis for neural networks with discontinuous neuron activations and impulses," *International Journal of Innovative Computing, Information and Control*, vol. 3, no. 6B, pp. 1537–1548, 2007.

[26] H. Wu and C. Shan, "Stability analysis for periodic solution of BAM neural networks with discontinuous neuron activations and impulses," *Applied Mathematical Modelling*, vol. 33, no. 6, pp. 2564–2574, 2009.

[27] D. Papini and V. Taddei, "Global exponential stability of the periodic solution of a delayed neural network with discontinuous activations," *Physics Letters A*, vol. 343, no. 1–3, pp. 117–128, 2005.

[28] X. Chen and Q. Song, "Global exponential stability of the periodic solution of delayed Cohen-Grossberg neural networks with discontinuous activations," *Neurocomputing*, vol. 73, no. 16-18, pp. 3097–3104, 2010.

[29] X. He, W. Lu, and T. Chen, "Nonnegative periodic dynamics of delayed Cohen-Grossberg neural networks with discontinuous activations," *Neurocomputing*, vol. 73, no. 13-15, pp. 2765–2772, 2010.

[30] W. Allegretto, D. Papini, and M. Forti, "Common asymptotic behavior of solutions and almost periodicity for discontinuous, delayed, and impulsive neural networks," *IEEE Transactions on Neural Networks*, vol. 21, no. 7, pp. 1110–1125, 2010.

[31] W. Lu and T. Chen, "Almost periodic dynamics of a class of delayed neural networks with discontinuous activations," *Neural Computation*, vol. 20, no. 4, pp. 1065–1090, 2008.

[32] S. Qin, X. Xue, and P. Wang, "Global exponential stability of almost periodic solution of delayed neural networks with discontinuous activations," *Information Sciences*, vol. 220, pp. 367–378, 2013.

[33] J. Wang and L. Huang, "Almost periodicity for a class of delayed Cohen-Grossberg neural networks with discontinuous activations," *Chaos Solitons Fractals*, vol. 45, pp. 1157–1170, 2012.

[34] Y. Zuo, Y. Wang, L. Huang, Z. Wang, X. Liu, and X. Wu, "Robust stability criterion for delayed neural networks with discontinuous activation functions," *Neural Processing Letters*, vol. 29, no. 1, pp. 29–44, 2009.

[35] X. Wu, Y. Wang, L. Huang, and Y. Zuo, "Robust exponential stability criterion for uncertain neural networks with discontinuous activation functions and time-varying delays," *Neurocomputing*, vol. 73, no. 7-9, pp. 1265–1271, 2010.

[36] Z. Guo and L. Huang, "LMI conditions for global robust stability of delayed neural networks with discontinuous neuron activations," *Applied Mathematics and Computation*, vol. 215, no. 3, pp. 889–900, 2009.

[37] X. Liu and J. Cao, "Robust state estimation for neural networks with discontinuous activations.," *IEEE Transactions on Systems, Man, and Cybernetics. Part B*, vol. 40, no. 6, pp. 1425–1437, 2010.

[38] F. H. Clarke, *Oprimization and Non-Smooth Analysis*, Woley, New York, NY, USA, 1983.

[39] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Side, Mathematics and Its Applications*, Mathematics and Its Applications (Soviet Series), Kluwer Academic, Boston, Mass, USA, 1984.

[40] J.-P. Aubin and A. Cellina, *Differential Inclusions*, vol. 264 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1984.

[41] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, UK, 1982.

[42] X. Yang, C. Huang, and Q. Zhu, "Synchronization of switched neural networks with mixed delays via impulsive control," *Chaos, Solitons & Fractals*, vol. 44, no. 10, pp. 817–826, 2011.

[43] X. Ge, *Applied Functional Analysis*, Zhejiang University, Hangzhou, China, 1st edition, 1996.

[44] Z. Wang, Y. Wang, and Y. Liu, "Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 1, pp. 11–25, 2010.

[45] J. Hu, Z. Wang, Y. Niu, and L. K. Stergioulas, "H_∞ sliding mode observer design for a class of nonlinear discrete time-delay systems: a delay-fractioning approach," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 16, pp. 1806–1826, 2012.

[46] J. Hu, Z. Wang, H. Gao, and L. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities," *IEEE Transactions on Industrial Electronics*, vol. 59, pp. 3008–3015, 2012.

[47] J. Hu, Z. Wang, and H. Gao, "A delay fractioning approach to robust sliding mode control for discrete-time stochastic systems with randomly occurring non-linearities," *IMA Journal of Mathematical Control and Information*, vol. 28, no. 3, pp. 345–363, 2011.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 143585, 9 pages
http://dx.doi.org/10.1155/2013/143585

Research Article

An Analysis of Stability of a Class of Neutral-Type Neural Networks with Discrete Time Delays

Zeynep Orman[1] and Sabri Arik[2]

[1] *Department of Computer Engineering, Istanbul University, Avcilar, 34320 Istanbul, Turkey*
[2] *Department of Electrical and Electronics Engineering, Isik University, Sile, 34980 Istanbul, Turkey*

Correspondence should be addressed to Zeynep Orman; ormanz@istanbul.edu.tr

Received 17 April 2013; Accepted 21 May 2013

Academic Editor: Zidong Wang

Copyright © 2013 Z. Orman and S. Arik. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The problem of existence, uniqueness, and global asymptotic stability is considered for the class of neutral-type neural network model with discrete time delays. By employing a suitable Lyapunov functional and using the homeomorphism mapping theorem, we derive some new delay-independent sufficient conditions for the existence, uniqueness, and global asymptotic stability of the equilibrium point for this class of neutral-type systems. The obtained conditions basically establish some norm and matrix inequalities involving the network parameters of the neural system. The main advantage of the proposed results is that they can be expressed in terms of network parameters only. Some comparative examples are also given to compare our results with the previous corresponding results and demonstrate the effectiveness of the results presented.

1. Introduction

In recent years, dynamical neural networks have been employed in solving many practical engineering problems such as signal and image processing, pattern recognition, associative memories, parallel computation, and optimization and control problems [1–10]. In such applications, it is important to know the dynamics of the designed neural networks. In addition, when using delayed neural networks, time delays might affect the transmission rate and cause instability. Therefore, the analysis of stability of neural networks with time delays is indispensable for solving engineering system problems. In the recent literature, many papers have studied the problem of global stability of different classes of neural networks by exploiting various analysis techniques and methods and presented some useful stability results for delayed neural networks. In practice, in order to precisely determine the equilibrium and stability properties of neural networks, the information about time derivatives of the past states must be introduced into the state equations of neural networks. A neural network of this model is called neutral-type neural networks. Some global stability results of various classes of neural networks with time delays have been reported in [1–33]. The goal of our paper is to present some new and alternative stability results of neutral-type neural networks with discrete time delays with respect to Lipschitz continuous activation functions.

Throughout this paper we will use these notations: for any matrix $P = (p_{ij})_{n\times n}$, $P > 0$ will denote that P is symmetric and positive definite; P^T, P^{-1}, $\lambda_m(P)$, and $\lambda_M(P)$ will denote the transpose of P, the inverse of P, the minimum eigenvalue of P, and the maximum eigenvalue of P, respectively. We will use the matrix norm $\|P\|_2 = [\lambda_M(P^TP)]^{1/2}$. For any two positive definite matrices $P = (p_{ij})_{n\times n}$ and $Q = (q_{ij})_{n\times n}$. If $Q > 0$, then $P > Q$ will imply that $P > 0$. For $v = (v_1, v_2, \ldots, v_n)^T \in R^n$, we will use the vector norms $\|v\|_2 = \sqrt{\sum_{i=1}^n v_i^2}$ and $\|v\|_1 = \sum_{i=1}^n |v_i|$.

2. Problem Statement

The class of neutral-type neural network model with discrete time delays is described by the following set of nonlinear differential equations:

$$\begin{aligned}\dot{x}_i(t) + \sum_{j=1}^{n} e_{ij}\dot{x}_j(t-\tau_j) &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + u_i, \\ &\qquad i = 1,\ldots,n,\end{aligned} \tag{1}$$

where n is the number of the neurons in the network, x_i denotes the state of the ith neuron, and the parameters c_i are some constants: the constants a_{ij} denote the strengths of the neuron interconnections within the network; the constants b_{ij} denote the strengths of the neuron interconnections with time delay parameters τ_j. e_{ij} are coefficients of the time derivative of the delayed states, the functions $f_j(\cdot)$ denote the neuron activations, and the constants u_i are some external inputs. In system (1), $\tau_j \geq 0$ represents the delay parameter with $\tau = \max(\tau_j)$, $1 \leq j \leq n$. Accompanying the neutral system (1) is an initial condition of the form: $x_i(t) = \phi_i(t) \in C([-\tau, 0], R)$, where $C([-\tau, 0], R)$ denotes the set of all continuous functions from $[-\tau, 0]$ to R.

We will assume that the activation functions $f_i(\cdot)$, $i = 1, 2, \ldots, n$, are Lipschitz continuous; for example, there exist some constants $\ell_i > 0$ such that

$$\begin{aligned}|f_i(x) - f_i(y)| \leq \ell_i |x - y|, \quad i = 1,2,\ldots,n, \\ \forall x, y \in R,\ x \neq y.\end{aligned} \tag{2}$$

Neural network model (1) can be written in the vector-matrix form as follows:

$$\dot{x}(t) + E\dot{x}(t-\tau) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + u, \tag{3}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $E = (e_{ij})_{n\times n}$, $C = \operatorname{diag}(c_i > 0)$, $u = (u_1, u_2, \ldots, u_n)^T$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T$, and $f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), \ldots, f_n(x_n(t-\tau_n)))^T$.

In order to obtain our main results, the following lemma will be needed.

Lemma 1 (see [23]). *If a map $H(x) \in C^0$ satisfies the following conditions:*

(i) *$H(x) \neq H(y)$ for all $x \neq y$,*

(ii) *$\|H(x)\| \to \infty$ as $\|x\| \to \infty$,*

then, $H(x)$ is homeomorphism of R^n.

3. Existence and Uniqueness Analysis

This section deals with obtaining the sufficient conditions that ensure the existence and uniqueness of the equilibrium point for neutral-type neural network model (1). The main result is given in the following result.

Theorem 2. *For the neutral-type neural network model* (1), *let $\|E\|_2 < 1$ and the activation functions satisfy* (2). *Then, the system* (1) *has unique equilibrium point for each u if there exist positive diagonal matrices H and D and positive definite matrices P, Q, and R such that the following conditions hold:*

$$\begin{aligned}\Upsilon_1 &= C - P - Q - H - CR^{-1}C > 0, \\ \Upsilon_2 &= C\mathscr{L}^{-2} - D - A^T P^{-1} A - A^T R^{-1} A > 0, \\ \Upsilon_3 &= D - B^T Q^{-1} B - B^T R^{-1} B > 0, \\ \Upsilon_4 &= H - 3E^T RE > 0,\end{aligned} \tag{4}$$

where $\mathscr{L} = \operatorname{diag}(\ell_1, \ell_2, \ldots, \ell_n)$.

Proof. We will make use of the result of Lemma 1 for the proof of the existence and uniqueness of the equilibrium point for system (1). Let us define the following mapping associated with system (1):

$$H(x) + EH(x) = -Cx + Af(x) + Bf(x) + u. \tag{5}$$

If $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an equilibrium point of (1), then x^* satisfies the equilibrium equation:

$$H(x^*) + EH(x^*) = -Cx^* + Af(x^*) + Bf(x^*) + u = 0. \tag{6}$$

Clearly, the solution of the equation $H(x) = 0$ is an equilibrium point of (1). Therefore, in the light of Lemma 1, we can conclude that, for the system defined by (1), there exists a unique equilibrium point for every input vector u if $H(x)$ is homeomorphism of R^n. We will now show that the conditions of Theorem 2 imply that $H(x)$ is a homeomorphism of R^n. To this end, we choose any two vectors $x \in R^n$ and $y \in R^n$ such that $x \neq y$. When the activation functions satisfy (2), for $x \neq y$, we have two cases: first case is $x \neq y$ and $f(x) \neq f(y)$, and the second case is $x \neq y$ and $f(x) = f(y)$. Let us carry out the existence and uniqueness analysis for the first case where $x \neq y$ and $f(x) \neq f(y)$. In this case, for $H(x)$ defined by (5), we can write

$$\begin{aligned}&H(x) - H(y) + E(H(x) - H(y)) \\ &\quad = (I + E)(H(x) - H(y)) \\ &\quad = -C(x - y) + A(f(x) - f(y)) + B(f(x) - f(y)).\end{aligned} \tag{7}$$

If we multiply both sides of (7) by the term $2(x-y)^T(I+E)^T$, and then add the terms $(f(x)-f(y))^T D(f(x)-f(y))-(f(x)-f(y))^T D(f(x)-f(y)) = 0$ and $(x-y)^T H(x-y)-(x-y)^T H(x-y) = 0$ to the right hand side of the resulting equation, we get

$$\begin{aligned}
&2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) \\
&\quad = 2(x-y)^T(I+E)^T \\
&\qquad \times (-C(x-y)+A(f(x)-f(y))+B(f(x)-f(y))) \\
&\quad = 2(x-y)^T(I+E)^T \\
&\qquad \times (-C(x-y)+A(f(x)-f(y))+B(f(x)-f(y))) \\
&\qquad + (x-y)^T H(x-y)-(x-y)^T H(x-y) \\
&\qquad + (f(x)-f(y))^T D(f(x)-f(y)) \\
&\qquad - (f(x)-f(y))^T D(f(x)-f(y)) \\
&\quad = -2(x-y)^T C(x-y)+2(x-y)^T A(f(x)-f(y)) \\
&\qquad + 2(x-y)^T B(f(x)-f(y))-2(x-y)^T E^T C(x-y) \\
&\qquad + 2(x-y)^T E^T A(f(x)-f(y))+2(x-y)^T E^T B \\
&\qquad \times (f(x)-f(y))+(x-y)^T H(x-y)-(x-y)^T \\
&\qquad \times H(x-y)+(f(x)-f(y))^T D(f(x)-f(y)) \\
&\qquad - (f(x)-f(y))^T D(f(x)-f(y)).
\end{aligned} \tag{8}$$

We note the following inequalities:

$$\begin{aligned}
&2(x-y)^T A(f(x)-f(y)) \\
&\qquad \le (x-y)^T P(x-y)+(f(x)-f(y))^T \\
&\qquad\quad \times A^T P^{-1} A(f(x)-f(y)), \\
&2(x-y)^T B(f(x)-f(y)) \\
&\qquad \le (x-y)^T Q(x-y)+(f(x)-f(y))^T \\
&\qquad\quad \times B^T Q^{-1} B(f(x)-f(y)), \\
&-2(x-y)^T E^T C(x-y)(t) \\
&\qquad \le (x-y)^T E^T RE(x-y)+(x-y)^T \\
&\qquad\quad \times C^T R^{-1} C(x-y), \\
&2(x-y)^T E^T A(f(x)-f(y)) \\
&\qquad \le (x-y)^T E^T RE(x-y)+(f(x)-f(y))^T \\
&\qquad\quad \times A^T R^{-1} A(f(x)-f(y)), \\
&2(x-y)^T E^T B(f(x)-f(y)) \\
&\qquad \le (x-y)^T E^T RE(x-y) \\
&\qquad\quad + (f(x)-f(y))^T B^T R^{-1} B(f(x)-f(y)).
\end{aligned} \tag{9}$$

Using (9) in (8) results in

$$\begin{aligned}
&2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) \\
&\quad \le -(x-y)^T C(x-y)-(x-y)^T C(x-y) \\
&\qquad + (x-y)^T P(x-y)+(f(x)-f(y))^T A^T P^{-1} \\
&\qquad \times A(f(x)-f(y))+(x-y)^T Q(x-y) \\
&\qquad + (f(x)-f(y))^T B^T Q^{-1} B(f(x)-f(y)) \\
&\qquad + (x-y)^T E^T RE(x-y)+(x-y)^T C^T R^{-1} \\
&\qquad \times C(x-y)+(x-y)^T E^T RE(x-y) \\
&\qquad + (f(x)-f(y))^T A^T R^{-1} A(f(x)-f(y))(x-y)^T \\
&\qquad \times E^T RE(x-y)+(f(x)-f(y))^T \\
&\qquad \times B^T R^{-1} B(f(x)-f(y))+(x-y)^T H(x-y) \\
&\qquad - (x-y)^T H(x-y)+(f(x)-f(y))^T \\
&\qquad \times D(f(x)-f(y))-(f(x)-f(y))^T \\
&\qquad \times D(f(x)-f(y))
\end{aligned} \tag{10}$$

which is of the form

$$\begin{aligned}
&2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) \\
&\quad \le -(x-y)^T\left(C-P-Q-H-C^T R^{-1} C\right)(x-y)^T \\
&\qquad - (f(x)-f(y))^T\left(C\mathscr{L}^{-2}-D-A^T P^{-1} A-A^T R^{-1} A\right) \\
&\qquad \times (f(x)-f(y))-(f(x)-f(y))^T \\
&\qquad \times \left(D-B^T Q^{-1} B-B^T R^{-1} B\right)(f(x)-f(y)) \\
&\qquad - (x-y)^T\left(H-3E^T RE\right)(x-y)
\end{aligned} \tag{11}$$

or equivalently

$$\begin{aligned}
&2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) \\
&\quad \le -(x-y)^T \Upsilon_1 (x-y) \\
&\qquad - (f(x)-f(y))^T \Upsilon_2 (f(x)-f(y)) \\
&\qquad - (f(x)-f(y))^T \Upsilon_3 (f(x)-f(y)) \\
&\qquad - (x-y)^T \Upsilon_4 (x-y).
\end{aligned} \tag{12}$$

Since $x \neq y$ and $f(x) \neq f(y)$, $\Upsilon_1 > 0$, $\Upsilon_2 > 0$, $\Upsilon_3 > 0$, and $\Upsilon_4 > 0$ imply that

$$2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) < 0 \tag{13}$$

implying that

$$\left\|(x-y)^T(I+E)^T(I+E)(H(x)-H(y))\right\|_2 > 0 \tag{14}$$

from which it follows that

$$\|x-y\|_2\|I+E\|_2^2\|H(x)-H(y)\|_2 > 0. \tag{15}$$

$\|E\|_2 < 1$ implies that $\|I+E\|_2^2 > 0$, and $x \neq y$ implies that $\|x-y\|_2 > 0$. Therefore, it directly follows that $\|H(x)-H(y)\|_2 > 0$, thus implying that $\|H(x)\| \neq \|H(y)\|$. Hence, we conclude that $H(x) \neq H(y)$ for all $x \neq y$ and $f(x) \neq f(y)$.

Now consider the case where $x \neq y$ and $f(x) = f(y)$. In this case, $H(x)$ defined by (5) satisfies

$$\begin{aligned} &2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) \\ &\quad \leq -(x-y)^T\Upsilon_1(x-y) - (x-y)^T\Upsilon_4 z(x-y), \end{aligned} \tag{16}$$

$x \neq y$; $\Upsilon_1 > 0$ and $\Upsilon_4 > 0$ imply that

$$2(x-y)^T(I+E)^T(I+E)(H(x)-H(y)) < 0. \tag{17}$$

Based on the analysis carried out for the previous case, we conclude that $H(x) \neq H(y)$ for all $x \neq y$ for this case.

Now it is shown that the conditions of Theorem 2 imply that $\|H(x)\| \to \infty$ as $\|x\| \to \infty$. For $y = 0$, we can write

$$\begin{aligned} &2x^T(I+E)^T(I+E)(H(x)-H(0)) \\ &\quad \leq -x^T\Upsilon_1 x - (f(x)-f(0))^T\Upsilon_2(f(x)-f(0)) \\ &\qquad -(f(x)-f(0))^T\Upsilon_3(f(x)-f(0)) - x^T\Upsilon_4 x. \end{aligned} \tag{18}$$

Taking the absolute value of the both sides of the above inequality, we obtain

$$\begin{aligned} &\left|2x^T(I+E)^T(I+E)(H(x)-H(0))\right| \\ &\quad \geq x^T\Upsilon_1 x + (f(x)-f(0))^T\Upsilon_2(f(x)-f(0)) \\ &\qquad + (f(x)-f(0))^T\Upsilon_3(f(x)-f(0)) + x^T\Upsilon_4 x \end{aligned} \tag{19}$$

from which it follows that

$$\begin{aligned} &\|x\|_2\|I+E\|_2^2\|H(x)-H(0)\|_2 \\ &\quad \geq \lambda_m(\Upsilon_1)\|x\|_2^2 + \lambda_m(\Upsilon_2)\|f(x)-f(0)\|_2^2 \\ &\qquad + \lambda_m(\Upsilon_3)\|f(x)-f(0)\|_2^2 + \lambda_m(\Upsilon_4)\|x\|_2^2 \\ &\quad \geq \lambda_m(\Upsilon_1)\|x\|_2^2 \end{aligned} \tag{20}$$

which yields

$$\|H(x)-H(0)\|_2 \geq \frac{\lambda_m(\Upsilon_1)}{\|I+E\|_2^2}\|x\|_2. \tag{21}$$

We note that $\|H(x)-H(0)\|_2 \leq \|H(x)\|_2 + \|H(0)\|_2$. Hence, from (21), it follows that

$$\|H(x)\|_2 \geq \frac{\lambda_m(\Upsilon_1)}{\|I+E\|_2^2}\|x\|_2 - \|H(0)\|_2. \tag{22}$$

Since $\|H(0)\|_2$ is bounded, and $\|I+E\|_2^2 > 0$, then $\|H(x)\|_2 \to \infty$ as $\|x\|_2 \to \infty$. Hence, the conditions of Theorem 2 ensure that $H(x)$ is homeomorphism of R^n, proving that neutral system defined by (1) has unique equilibrium point for each u.

Choosing H, D, P, Q, and R in the conditions of Theorem 2 as $H = hI$, $D = dI$, $P = pI$, $Q = qI$, and $R = rI$, we can express some special cases of Theorem 2 as follows. □

Corollary 3. *For the neutral-type neural network model* (1), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (2). *Then, the system* (1) *has unique equilibrium point for each u if there exist some positive constants h, d, p, q, and r such that the following conditions hold:*

$$\begin{aligned} \Upsilon_1^* &= C - (p+q+h)I - \frac{1}{r}C^2 > 0, \\ \Upsilon_2^* &= C\mathscr{L}^{-2} - dI - \frac{1}{p}A^TA - \frac{1}{r}A^TA > 0, \\ \Upsilon_3^* &= dI - \frac{1}{q}B^TB - \frac{1}{r}B^TB > 0, \\ \Upsilon_4^* &= hI - 3rE^TE > 0, \end{aligned} \tag{23}$$

where $\mathscr{L} = \operatorname{diag}(\ell_1, \ell_2, \ldots, \ell_n)$.

A special case of Corollary 3 is the following result.

Corollary 4. *For the neutral-type neural network model* (1), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (2). *Then, the system* (1) *has unique equilibrium point for each u if there exist some positive constants h, d, p, q, and r such that the following conditions hold:*

$$\begin{aligned} \rho_1 &= c_m - (p+q+h) - \frac{1}{r}c_M^2 > 0, \\ \rho_2 &= c_m\ell_M^{-2} - d - \left(\frac{1}{p} + \frac{1}{r}\right)\|A\|_2^2 > 0, \\ \rho_3 &= d - \left(\frac{1}{q} + \frac{1}{r}\right)\|B\|_2^2 > 0, \\ \rho_4 &= h - 3r\|E\|_2^2 > 0, \end{aligned} \tag{24}$$

where $c_m = \min_{1\leq i\leq n}(c_i)$, $c_M = \max_{1\leq i\leq n}(c_i)$, *and* $\ell_M = \max_{1\leq i\leq n}(\ell_i)$.

4. Stability Analysis

In this section, we will prove that the conditions obtained from Theorem 2 for the existence and uniqueness of the equilibrium point are also sufficient for the global stability of the equilibrium point of neutral system defined by (1). In order to simplify the proofs, we will first shift the equilibrium point $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ of system (1) to the origin. By using the transformation $z(t) = x(t) - x^*$, the neutral-type neural network model (1) can be put in the form:

$$\begin{aligned}\dot{z}_i(t) &= -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} g_j\left(z_j(t)\right) \\ &\quad + \sum_{j=1}^{n} b_{ij} g_j\left(z_j\left(t-\tau_j\right)\right) + \sum_{j=1}^{n} e_{ij}\dot{z}_j\left(t-\tau_j\right), \\ &\qquad i = 1, \ldots, n\end{aligned} \tag{25}$$

which can be written in vector-matrix form as follows:

$$\dot{z}(t) = -Cz(t) + Ag(z(t)) + Bg(z(t-\tau)) + E\dot{z}(t-\tau), \tag{26}$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T \in R^n$ is the state vector of transformed neural system, $g(z(t)) = (g_1(z_1(t)), g_2(z_2(t)), \ldots, g_n(z_n(t)))^T$ represents the new nonlinear activation, functions, and $g(z(t-\tau)) = (g_1(z_1(t-\tau_1)), g_2(z_2(t-\tau_2)), \ldots, g_n(z_n(t-\tau_n)))^T$. The activation functions $g_i(z_i(t))$ in (25) satisfy

$$\left|g_i\left(z_i(t)\right)\right| \le \ell_i \left|z_i(t)\right|, \quad i = 1, 2, \ldots, n. \tag{27}$$

We can now state the following stability result.

Theorem 5. *For the neutral-type neural network model* (25), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (27). *Then, the origin of system* (25) *is globally asymptotically stable if there exist positive diagonal matrices H and D and positive definite matrices P, Q, and R such that the following conditions hold:*

$$\begin{aligned}\Upsilon_1 &= C - P - Q - H - C^T R^{-1} C > 0, \\ \Upsilon_2 &= C\mathscr{L}^{-2} - D - A^T P^{-1} A - A^T R^{-1} A > 0, \\ \Upsilon_3 &= D - B^T Q^{-1} B - B^T R^{-1} B > 0, \\ \Upsilon_4 &= H - 3E^T R E > 0,\end{aligned} \tag{28}$$

where $\mathscr{L} = \operatorname{diag}(\ell_1, \ell_2, \ldots, \ell_n)$.

Proof. Define the following positive definite Lyapunov functional:

$$\begin{aligned}V(z(t)) &= (z(t) + Ez(t-\tau))^T (z(t) + Ez(t-\tau)) \\ &\quad + \sum_{i=1}^{n} h_i \int_{t-\tau_i}^{t} z_i^2(s)\, ds + \sum_{i=1}^{n} d_i \int_{t-\tau_i}^{t} g_i^2\left(z_i(s)\right) ds,\end{aligned} \tag{29}$$

where p_i and d_i $i = 1, 2, \ldots, n$ are some positive constants. The time derivative of $V(z(t))$ along the trajectories of the system (25) is obtained as follows:

$$\begin{aligned}\dot{V}(z(t)) &= 2(z(t) + Ez(t-\tau))^T (\dot{z}(t) + E\dot{z}(t-\tau)) \\ &\quad + \sum_{i=1}^{n} h_i z_i^2(t) - \sum_{i=1}^{n} h_i z_i^2(t-\tau_i) \\ &\quad + \sum_{i=1}^{n} d_i g_i^2(z_i(t)) - \sum_{i=1}^{n} d_i g_i^2(z_i(t-\tau_i)) \\ &= 2(z(t) + Ez(t-\tau))^T (\dot{z}(t) + E\dot{z}(t-\tau)) \\ &\quad + z^T(t) Hz(t) - z^T(t-\tau) Hz(t-\tau) \\ &\quad + g^T(z(t)) Dg(z(t)) \\ &\quad - g^T(z(t-\tau)) Dg(z(t-\tau)).\end{aligned} \tag{30}$$

Since $\dot{z}(t) + E\dot{z}(t-\tau) = -Cz(t) + Ag(z(t)) + Bg(z(t-\tau))$, we can write

$$\begin{aligned}\dot{V}(z(t)) &= 2(z(t) + Ez(t-\tau))^T \\ &\quad \times \left(-Cz(t) + Ag(z(t)) + Bg(z(t-\tau))\right) \\ &\quad + z^T(t) Hz(t) - z^T(t-\tau) Hz(t-\tau) \\ &\quad + g^T(z(t)) Dg(z(t)) \\ &\quad - g^T(z(t-\tau)) Dg(z(t-\tau)) \\ &= -2z^T(t) Cz(t) + 2z^T(t) Ag(z(t)) \\ &\quad + 2z^T(t) Bg(z(t-\tau)) \\ &\quad - 2z^T(t-\tau) E^T Cz(t) + 2z^T(t-\tau) \\ &\quad \times E^T Ag(z(t)) + 2z^T(t-\tau) E^T Bg(z(t-\tau)) \\ &\quad + z^T(t) Hz(t) - z^T(t-\tau) Hz(t-\tau) \\ &\quad + g^T(z(t)) Dg(z(t)) \\ &\quad - g^T(z(t-\tau)) Dg(z(t-\tau)).\end{aligned} \tag{31}$$

We can write the following inequalities:

$$\begin{aligned}&2z^T(t) Ag(z(t)) \\ &\quad \le z^T(t) Pz(t) + g^T(z(t)) A^T P^{-1} Ag(z(t)), \\ &2z^T(t) Bg(z(t-\tau)) \\ &\quad \le z^T(t) Qz(t) + g^T(z(t-\tau)) B^T Q^{-1} Bg(z(t-\tau)), \\ &-2z^T(t-\tau) E^T Cz(t) \\ &\quad \le z^T(t-\tau) E^T REz(t-\tau) + z^T(t) C^T R^{-1} Cz(t),\end{aligned}$$

$$\begin{aligned}
&2z^T(t-\tau)E^TAg(z(t))\\
&\quad\le z^T(t-\tau)E^TREz(t-\tau)+g^T(z(t))A^TR^{-1}Ag(z(t)),\\
&2z^T(t-\tau)E^TBg(z(t-\tau))\\
&\quad\le z^T(t-\tau)E^TREz(t-\tau)\\
&\qquad+g^T(z(t-\tau))B^TR^{-1}Bg(z(t-\tau)),
\end{aligned}\tag{32}$$

where P, Q, and R are some positive definite matrices. Using (32) in (31) yields

$$\begin{aligned}
\dot V(z(t))\le\ &-z^T(t)Cz(t)-z^T(t)Cz(t)+z^T(t)Pz(t)\\
&+g^T(z(t))A^TP^{-1}Ag(z(t))+z^T(t)Qz(t)\\
&+g^T(z(t-\tau))B^TQ^{-1}Bg(z(t-\tau))\\
&+z^T(t-\tau)E^TREz(t-\tau)+z^T(t)C^TR^{-1}Cz(t)\\
&+z^T(t-\tau)E^TREz(t-\tau)+g^T(z(t))A^TR^{-1}\\
&\times Ag(z(t))+z^T(t-\tau)E^TREz(t-\tau)\\
&+g^T(z(t-\tau))B^TR^{-1}Bg(z(t-\tau))+z^T(t)\\
&\times Hz(t)-z^T(t-\tau)Hz(t-\tau)+g^T(z(t))\\
&\times Dg(z(t))-g^T(z(t-\tau))Dg(z(t-\tau)).
\end{aligned}\tag{33}$$

Equation (27) implies that

$$z^T(t)Cz(t)\ge g^T(z(t))C\mathscr{L}^{-2}g(z(t)).\tag{34}$$

Hence, we have

$$\begin{aligned}
\dot V(z(t))\le\ &-z^T(t)Cz(t)-g^T(z(t))C\mathscr{L}^{-2}g(z(t))\\
&+z^T(t)Pz(t)+g^T(z(t))A^TP^{-1}Ag(z(t))\\
&+z^T(t)Qz(t)+g^T(z(t-\tau))B^TQ^{-1}Bg\\
&\times(z(t-\tau))+z^T(t-\tau)E^TREz(t-\tau)\\
&+z^T(t)C^TR^{-1}Cz(t)+z^T(t-\tau)E^T\\
&\times REz(t-\tau)+g^T(z(t))A^TR^{-1}Ag(z(t))\\
&+z^T(t-\tau)E^TREz(t-\tau)+g^T(z(t-\tau))\\
&\times B^TR^{-1}Bg(z(t-\tau))\\
&+z^T(t)Hz(t)-z^T(t-\tau)\\
&\times Hz(t-\tau)+g^T(z(t))Dg(z(t))\\
&-g^T(z(t-\tau))Dg(z(t-\tau))
\end{aligned}\tag{35}$$

which can be written as

$$\begin{aligned}
\dot V(z(t))\le\ &-z^T(t)\left(C-P-Q-H-C^TR^{-1}C\right)z(t)\\
&-g^T(z(t))\\
&\times\left(C\mathscr{L}^{-2}-D-A^TP^{-1}A-A^TR^{-1}A\right)\\
&\times g(z(t))-g^T(z(t-\tau))\\
&\times\left(D-B^TQ^{-1}B-B^TR^{-1}B\right)\\
&\times g(z(t-\tau))-z^T(t-\tau)\\
&\times\left(H-3E^TRE\right)z(t-\tau)
\end{aligned}\tag{36}$$

or equivalently

$$\begin{aligned}
\dot V(z(t))\le\ &-z^T(t)\Upsilon_1z(t)-g^T(z(t))\Upsilon_2g(z(t))\\
&-g^T(z(t-\tau))\Upsilon_3g(z(t-\tau))\\
&-z^T(t-\tau)\Upsilon_4z(t-\tau).
\end{aligned}\tag{37}$$

Clearly, $\Upsilon_1>0$, $\Upsilon_2>0$, $\Upsilon_3>0$, and $\Upsilon_4>0$ imply that $\dot V(z(t))<0$ if any of the vectors $z(t)$, $g(z(t-\tau))$, $g^T(z(t))$, and $z(t-\tau)$ is nonzero, thus implying that $\dot V(z(t))=0$ if and only if $z(t)=z(t-\tau)=g(z(t-\tau))=z(t-\tau)=0$ which is the origin of system (25). On the other hand, $V(z(t))\to\infty$ as $\|z(t)\|_2\to\infty$, meaning that the Lyapunov functional used for the stability analysis is radially unbounded. Thus, it can be concluded from the standard Lyapunov theorems [34] that the origin of system (25) or equivalently the equilibrium point of system (1) is globally asymptotically stable. □

We can directly state the following corollaries.

Corollary 6. *For the neutral-type neural network model* (25), *let* $\|E\|_2<1$ *and the activation functions satisfy* (27). *Then, the origin of system* (25) *is globally asymptotically stable if there exist some positive constants* h, d, p, q, *and* r *such that the following conditions hold:*

$$\begin{aligned}
\Upsilon_1^*&=C-(p+q+h)I-\frac{1}{r}C^2>0,\\
\Upsilon_2^*&=C\mathscr{L}^{-2}-dI-\frac{1}{p}A^TA-\frac{1}{r}A^TA>0,\\
\Upsilon_3^*&=dI-\frac{1}{q}B^TB-\frac{1}{r}B^TB>0,\\
\Upsilon_4^*&=hI-3rE^TE>0,
\end{aligned}\tag{38}$$

where $\mathscr{L}=\operatorname{diag}(\ell_1,\ell_2,\ldots,\ell_n)$.

Corollary 7. *For the neutral-type neural network model* (25), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (27). *Then, the origin of system* (25) *is globally asymptotically stable if there exist some positive constants* h, d, p, q, *and* r *such that the following conditions hold:*

$$\begin{aligned}
\rho_1 &= c_m - (p+q+h) - \frac{1}{r}c_M^2 > 0,\\
\rho_2 &= c_m \ell_M^{-2} - d - \left(\frac{1}{p}+\frac{1}{r}\right)\|A\|_2^2 > 0,\\
\rho_3 &= d - \left(\frac{1}{q}+\frac{1}{r}\right)\|B\|_2^2 > 0,\\
\rho_4 &= h - 3r\|E\|_2^2 > 0,
\end{aligned} \tag{39}$$

where $c_m = \min_{1\le i\le n}(c_i)$, $c_M = \max_{1\le i\le n}(c_i)$, *and* $\ell_M = \max_{1\le i\le n}(\ell_i)$.

5. A Comparative Example

In this section, we will give a numerical example to make a comparison between our results and some previous corresponding results derived in the literature. We should point our here that the stability results regarding the neutral-type neural networks involve complicated relationships between the network parameters and some positive definite matrices to be determined, which is a difficult task to achieve. Therefore, the example we give will show that, in a particular case, our results seem to be equivalent to the previous corresponding literature results. We now state some of the previous results.

Theorem 8 (see [23]). *For the neutral-type neural network model* (1), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (2). *Then, system* (1) *is globally asymptotically stable if there exist some positive constants* k, p, q, *and* r *such that the following conditions hold:*

$$\begin{aligned}
\delta_1 &= (1-k)\gamma^2 - \left(1+\frac{1}{p}+\frac{1}{q}\right)\|A\|_2^2 > 0,\\
\delta_2 &= k\gamma^2 - \left(1+p+\frac{1}{r}\right)\|B\|_2^2 > 0,\\
\delta_3 &= 1 - (1+q+r)\|E\|_2^2 > 0,
\end{aligned} \tag{40}$$

where $\gamma = \min_{1\le i\le n}(c_i/\ell_i)$.

Theorem 9 (see [22]). *For the neutral-type neural network model* (1), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (2). *Then, system* (1) *is globally asymptotically stable if there exist positive constants* p, $\widehat{p}$, q, *and* $\widehat{q}$ *such that the following conditions hold:*

$$\begin{aligned}
\epsilon &= (2-r)c_m - (p+q) - 2c_m\|E\|_2 - (\widehat{p}+\widehat{q})\|E\|_2^2 > 0,\\
\Phi &= rc_m\ell_M^{-2} - \left(\frac{1}{p}+\frac{1}{\widehat{p}}\right)\|A\|_2^2 - \left(\frac{1}{q}+\frac{1}{\widehat{q}}\right)\|B\|_2^2 \ge 0,
\end{aligned} \tag{41}$$

where $c_m = \min_{1\le i\le n}(c_i)$, $c_M = \max_{1\le i\le n}(c_i)$, *and* $\ell_M = \max_{1\le i\le n}(\ell_i)$.

Theorem 10 (see [24]). *For the neutral-type neural network model* (1), *let* $\|E\|_2 < 1$ *and the activation functions satisfy* (2). *Then, system* (1) *is globally asymptotically stable if the following condition holds:*

$$\begin{aligned}
\delta = c_m &- \ell_M\|A\|_2(1+\|E\|_2)\\
&- \ell_M\|B\|_2(1+\|E\|_2) - c_m\|E\|_2 > 0,
\end{aligned} \tag{42}$$

where $c_m = \min_{1\le i\le n}(c_i)$, $\ell_M = \max_{1\le i\le n}(\ell_i)$.

We now consider the following example.

Example 11. Assume that the network parameters of neutral-type neural system (1) are given as follows:

$$A = B = \begin{bmatrix} a & a & a & a\\ -a & -a & a & a\\ a & -a & a & -a\\ -a & a & a & -a \end{bmatrix}, \tag{43}$$

where $a > 0$ is real number. Assume that $c_1 = c_2 = c_3 = c_4 = 1$ and $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1$. We have $\|A\|_2 = \|B\|_2 = 2a$.

For the sufficiently small values of $\|E\|_2$ and h and sufficiently large value of r, $d = 1/2$, and $p = q$, the conditions of Corollary 7 can be approximately stated as follows:

$$\begin{aligned}
\rho_1 &\cong 1 - 2p > 0,\\
\rho_2 &\cong \frac{1}{2} - \frac{1}{p}4a^2 > 0,\\
\rho_3 &\cong \frac{1}{2} - \frac{1}{p}4a^2 > 0,\\
\rho_4 &\cong h - 3r\|E\|_2^2 > 0.
\end{aligned} \tag{44}$$

The two required conditions for stability are $p < 1/2$ and $a^2 < p/8$, implying that $a < 1/4$.

In the case of Theorem 8, for the sufficiently small value of $\|E\|_2$ and sufficiently large values of r and q, $k = 1/2$, and $p = 1$, the conditions of Theorem 8 can be approximately stated as follows:

$$\begin{aligned}
\delta_1 &\cong \frac{1}{2} - 8a^2 > 0,\\
\delta_2 &\cong \frac{1}{2} - 8a^2 > 0,\\
\delta_3 &\cong 1 - (1+q+r)\|E\|_2^2 > 0.
\end{aligned} \tag{45}$$

The required condition for stability is $a < 1/4$.

In the case of Theorem 9, for the sufficiently small value of $\|E\|_2$ and sufficiently large values of $\widehat{p}$ and $\widehat{q}$, $r = 1$, and $p = q$, the conditions of Theorem 9 can be approximately stated as follows:

$$\begin{aligned}
\epsilon &\cong 1 - 2p > 0,\\
\Phi &\cong 1 - \frac{2}{p}4a^2 \ge 0.
\end{aligned} \tag{46}$$

The two required conditions for stability are $p < 1/2$ and $a^2 < p/8$, implying that $a < 1/4$.

In the case of Theorem 10, for a sufficiently small value of $\|E\|_2$, the condition of Theorem 10 can be approximately stated as follows:

$$\delta \cong 1 - 4a > 0. \tag{47}$$

The required condition for stability is $a < 1/4$.

6. Conclusions

In this paper, we have obtained some sufficient conditions for the existence, uniqueness, and global asymptotic stability of the equilibrium point for the class of neutral-type systems with discrete time delays. The results we obtained establish various relationships between the network parameters of the system. We have also given an example to show the applicability of our results and make a comparison between our results and some previous corresponding results derived in the literature. Most of the literature results express the stability conditions in terms of LMIs (linear matrix inequalities), which are then solved by some software tools. Such results may give less conservative results; however, the computational burden of this method can be high. Our results establish less complex relationships between the network parameters of the system. We should also point out that the delay-independent conditions may be more conservative than delay-dependent ones. In our paper, our stability conditions are independent of the time delays; this is due to the Lyapunov functional that we have employed in the analysis of our network model. In order to apply our techniques to obtain some delay-dependent conditions, one needs to modify the Lyapunov functional that we have used to include time delays in the conditions, which probably could be the subject of another study.

References

[1] R. Rakkiyappan and P. Balasubramaniam, "New global exponential stability results for neutral type neural networks with distributed time delays," *Neurocomputing*, vol. 71, no. 4–6, pp. 1039–1045, 2008.

[2] T. Pan, B. Shi, and J. Yuan, "Global stability of almost periodic solution of a class of neutral-type BAM neural networks," *Abstract and Applied Analysis*, vol. 2012, Article ID 482584, 18 pages, 2012.

[3] C. Bai, "Global stability of almost periodic solutions of Hopfield neural networks with neutral time-varying delays," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 72–79, 2008.

[4] J. Liu and G. Zong, "New delay-dependent asymptotic stability conditions concerning BAM neural networks of neutral type," *Neurocomputing*, vol. 72, no. 10–12, pp. 2549–2555, 2009.

[5] Y. Zhang, B. Song, J. H. Park, and Z.-G. Wu, "Global synchronization of neutral-type stochastic delayed complex networks," *Abstract and Applied Analysis*, vol. 2012, Article ID 631932, 19 pages, 2012.

[6] J. H. Park, C. H. Park, O. M. Kwon, and S. M. Lee, "A new stability criterion for bidirectional associative memory neural networks of neutral-type," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 716–722, 2008.

[7] G. Liu, S. Zhou, and H. Huang, "New LMI-based conditions on neural networks of neutral type with discrete interval delays and general activation functions," *Abstract and Applied Analysis*, vol. 2012, Article ID 306583, 14 pages, 2012.

[8] J. Feng, S. Xu, and Y. Zou, "Delay-dependent stability of neutral type neural networks with distributed delays," *Neurocomputing*, vol. 72, no. 10–12, pp. 2576–2580, 2009.

[9] H. Mai, X. Liao, and C. Li, "A semi-free weighting matrices approach for neutral-type delayed neural networks," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 44–55, 2009.

[10] C. H. Lien, K. W. Yu, Y. F. Lin, Y. J. Chung, and L. Y. Chung, "Global exponential stability for uncertain delayed neural networks of neutral type with mixed time delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 38, no. 3, pp. 709–720, 2008.

[11] L. Cheng, Z. G. Hou, and M. Tan, "A neutral-type delayed projection neural network for solving nonlinear variational inequalities," *IEEE Transactions on Circuits and Systems II*, vol. 55, no. 8, pp. 806–810, 2008.

[12] H. Zhang, Z. Liu, and G. B. Huang, "Novel delay-dependent robust stability analysis for switched neutral-type neural networks with time-varying delays via SC technique," *IEEE Transactions on Systems, Man, and Cybernetics Part B*, vol. 40, no. 6, pp. 1480–1491, 2010.

[13] Y. Liu, Z. Wang, and X. Liu, "On delay-dependent robust exponential stability of stochastic neural networks with mixed time delays and Markovian switching," *Nonlinear Dynamics*, vol. 54, no. 3, pp. 199–212, 2008.

[14] Q. Zhou, B. Chen, C. Lin, and H. Li, "Mean square exponential stability for uncertain delayed stochastic neural networks with markovian jump parameters," *Circuits, Systems, and Signal Processing*, vol. 29, no. 2, pp. 331–348, 2010.

[15] Y. Chen, A. Xue, R. Lu, and S. Zhou, "On robustly exponential stability of uncertain neutral systems with time-varying delays and nonlinear perturbations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 8, pp. 2464–2470, 2008.

[16] B. Wang, X. Liu, and S. Zhong, "New stability analysis for uncertain neutral system with time-varying delay," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 457–465, 2008.

[17] W.-A. Zhang and L. Yu, "Delay-dependent robust stability of neutral systems with mixed delays and nonlinear perturbations," *Acta Automatica Sinica*, vol. 33, no. 8, pp. 863–866, 2007.

[18] H. Li, S.-M. Zhong, and H.-B. Li, "Some new simple stability criteria of linear neutral systems with a single delay," *Journal of Computational and Applied Mathematics*, vol. 200, no. 1, pp. 441–447, 2007.

[19] F. Wang, "Exponential asymptotic stability for nonlinear neutral systems with multiple delays," *Nonlinear Analysis. Real World Applications*, vol. 8, no. 1, pp. 312–322, 2007.

[20] J. H. Park and O. M. Kwon, "Design of state estimator for neural networks of neutral-type," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 360–369, 2008.

[21] M. S. Mahmoud and Y. Xia, "Improved exponential stability analysis for delayed recurrent neural networks," *Journal of the Franklin Institute*, vol. 348, no. 2, pp. 201–211, 2011.

[22] R. Samli and S. Arik, "New results for global stability of a class of neutral-type neural systems with time delays," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 564–570, 2009.

[23] Z. Orman, "New sufficient conditions for global stability of neutral-type neural networks with time delays," *Neurocomputing*, vol. 71, pp. 141–148, 2012.

[24] C. Cheng, T. Liao, J. Yan, and C. Hwang, "Globally asymptotic stability of a class of neutral-type neural networks with delays," *IEEE Transactions on Systems, Man, and Cybernetics Part B*, vol. 36, no. 5, pp. 1191–1195, 2006.

[25] B. Shen, Z. Wang, and X. Liu, "Bounded H∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon," *IEEE Transactions on Neural Networks*, vol. 22, no. 1, pp. 145–157, 2011.

[26] Z. Wang, Y. Wang, and Y. Liu, "Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 1, pp. 11–25, 2010.

[27] L. Hu, H. Gao, and W. X. Zheng, "Novel stability of cellular neural networks with interval time-varying delay," *Neural Networks*, vol. 21, no. 10, pp. 1458–1463, 2008.

[28] S. Mou, H. Gao, J. Lam, and W. Qiang, "A new criterion of delay-dependent asymptotic stability for Hopfield neural networks with time delay," *IEEE Transactions on Neural Networks*, vol. 19, no. 3, pp. 532–535, 2008.

[29] J. Liang, Z. Wang, Y. Liu, and X. Liu, "Global synchronization control of general delayed discrete-time networks with stochastic coupling and disturbances," *IEEE Transactions on Systems, Man, and Cybernetics Part B*, vol. 38, no. 4, pp. 1073–1083, 2008.

[30] Y. Liu, Z. Wang, J. Liang, and X. Liu, "Synchronization of coupled neutral-type neural networks with jumping-mode-dependent discrete and unbounded distributed delays," *IEEE Transactions on Cybernetics*, vol. 43, no. 1, pp. 102–114, 2013.

[31] B. Shen, Z. Wang, and X. Liu, "Sampled-data synchronization control of dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[32] D. Ding, Z. Wang, H. Dong, and H. Shu, "Distributed H_∞ state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[33] H. Dong, Z. Wang, and H. Gao, "Distributed filtering for a class of time-varying systems over sensor networks with quantization errors and successive packet dropouts," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 3164–3173, 2012.

[34] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 728606, 16 pages
http://dx.doi.org/10.1155/2013/728606

Research Article

Synchronization of Switched Complex Bipartite Neural Networks with Infinite Distributed Delays and Derivative Coupling

Qiuxiang Bian,[1] Jinde Cao,[2,3] Jie Wu,[1] Hongxing Yao,[4] Tingfang Zhang,[1] and Xiaoxu Ling[5]

[1] *Department of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, China*
[2] *Department of Mathematics, Southeast University, Nanjing 210096, China*
[3] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[4] *Department of Mathematics, Jiangsu University, Zhenjiang 212003, China*
[5] *Faculty of Economics and Management, Nanjing University of Aeronautics & Astronautics, Nanjing 212096, China*

Correspondence should be addressed to Qiuxiang Bian; bianqx_1@163.com

Received 7 March 2013; Accepted 8 April 2013

Academic Editor: Hongli Dong

Copyright © 2013 Qiuxiang Bian et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A new model of switched complex bipartite neural network (SCBNN) with infinite distributed delays and derivative coupling is established. Using linear matrix inequality (LMI) approach, some synchronization criteria are proposed to ensure the synchronization between two SCBNNs by constructing effective controllers. Some numerical simulations are provided to illustrate the effectiveness of the theoretical results obtained in this paper.

1. Introduction

In recent years, neural networks have been intensively studied due to their potential applications in many different areas such as signal and image processing, content-addressable memory, optimization, and parallel computation [1–3]. Bidirectional associative memory (BAM) neural networks were first proposed by Kosko in [4, 5]. This class of networks has good applications in pattern recognition, solving optimization problems, and automatic control engineering. A large number of results on the dynamical behavior of BAM neural networks have been reported [6–9].

Switched systems, as an important kind of hybrid systems, have drawn considerable attention of researchers because of their theoretical significance and practical applications [10–12]. Switched systems are composed of a family of continuous-time or discrete-time subsystems and a rule that specifies the switching among them [13, 14]. Recently, the switched neural networks, whose individual subsystems are a set of neural networks, have found applications in the field of high speed signal processing, artificial intelligence, and biology, so there are many theoretical results about the switched neural networks [15–17].

Complex networks, which are a set of interconnected nodes with specific dynamics, have sparked the interest of many researchers from various fields of science and engineering such as the World Wide Web, electrical power grids, global economic markets, sensor networks; for example, see [18–20] and references therein. Bipartite networks are an important kind of complex networks, whose nodes can be divided into two disjoint nonempty sets such that every edge only connects a pair of nodes, which belong to different sets. Many real-world networks are naturally bipartite networks, such as the papers-scientists networks [21] and producer-consumer networks [22]. Recently, authors [23] have introduced a bipartite-graph complex dynamical network model that is only linearly coupled and has no delays. It is well known that time delays exist commonly in real-world systems. Therefore, many models of coupled networks with coupling delays are proposed, for example, constant single time delay [24], time-varying delays [25], and mix-time delays [26]. On the other hand, the coupled network often occurs

in other forms, for example, nonlinearly coupled networks [27] and linearly derivative coupled networks [28]. In [29], a general model of bipartite dynamical network (BDN) with distributed delays and nonlinear derivative coupling was introduced. Synchronization of complex networks has been intensively investigated since they can be applied in power system control, secure communication, automatic control, chemical reaction, and so on [30–32]. The study of synchronization of coupled neural networks is an important step for both understanding brain science and designing coupled neural networks for practical use. Yu et al. [33] consider the synchronization of switched linearly coupled neural networks with constant delays, but the controllers are complex and changed with the switched rule. Synchronization of two coupled BDNs was investigated by adaptive method [29], but the controllers are complicated and the model does not include infinite distributed delays coupling and switching. Extending BAM neural networks to complex networks, we get complex bipartite dynamical networks (CBDNs). The dynamics of individual node in CBDNs is switched system and the switched coupling is considered; switched complex bipartite neural network (SCBNN) can be obtained. To the best of our knowledge, up to now, there is not any work that discusses the synchronization problem in SCBNN.

Motivated by the previous discussion, we first proposed a model of SCBNN, and then investigated the synchronization between two SCBNNs with infinite distributed delays and derivative coupling. Using adaptive controllers and linear matrix inequality (LMI) approach, some synchronization criteria are proposed to ensure the synchronization between two coupled SCBNNs. In our paper, the proposed controllers are simpler and do not change with the switched rule, which can be realize more easily.

The paper is organized as follows. In Section 2, a model of SCBNN with infinite distributed delays and derivative coupling is presented, and some hypotheses and lemmas are given too. In Section 3, several synchronization criteria on the SCBNNs are deduced. In Section 4, numerical examples are given to demonstrate the effectiveness of the proposed controller design methods in Section 3. Finally, conclusions are given in Section 5.

Notations. Throughout this paper, $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ denote the maximum eigenvalue and minimum eigenvalue of a real symmetric matrix, respectively. The notation $*$ denotes the symmetric block.

2. Model Description, Assumptions, and Lemmas

Consider a complex bipartite dynamical network (CBDN) consisting of two disjoint nonempty node sets V_1 and V_2. Suppose that $V_1 = \{\mu_1, \mu_2, \ldots, \mu_l\}$ and $V_2 = \{\nu_1, \nu_2, \ldots, \nu_m\}$, l, m are integer. The coupled network is described as follows:

$$\begin{aligned}\dot{x}_i(t) = &-Dx_i + R_1 f_1(x_i(t)) + R_2 f_2(x_i(t-\tau(t))) + I \\ &+ \sum_{j=1}^{m} a_{ij} y_j(t-\tau_1(t)) + \sum_{j=1}^{m} b_{ij} g(\dot{y}_j(t-\tau_2(t))) \\ &+ \sum_{j=1}^{m} c_{ij} \int_{-\infty}^{t} h(t-s) k(y_j(s)) \mathrm{d}s, \\ &\qquad i = 1, 2, \ldots, l, \\ \dot{y}_j(t) = &-\overline{D} y_j + \overline{R}_1 \overline{f}_1(y_j(t)) + \overline{R}_2 \overline{f}_2(y_j(t-\sigma(t))) + J \\ &+ \sum_{i=1}^{l} \overline{a}_{ji} x_i(t-\sigma_1(t)) + \sum_{i=1}^{l} \overline{b}_{ji} \overline{g}(\dot{x}_i(t-\sigma_2(t))) \\ &+ \sum_{i=1}^{l} \overline{c}_{ji} \int_{-\infty}^{t} \overline{h}(t-s) \overline{k}(x_i(s)) \mathrm{d}s, \quad j = 1, 2, \ldots, m,\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T$, $y_j(t) = (y_{j1}(t), y_{j2}(t), \ldots, y_{jn}(t))^T \in \mathrm{R}^n$ denotes the state variables of nodes μ_i and ν_j, respectively. $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $\overline{D} = \mathrm{diag}(\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n)$ are diagonal matrices with $d_i, \overline{d}_i > 0$. $R_1, \overline{R}_1 \in \mathrm{R}^{n\times n}$ are weight matrices, $R_2, \overline{R}_2 \in \mathrm{R}^{n\times n}$ are delayed weight matrices, $f_k(x_i) = (f_{k1}(x_{i1}), f_{k2}(x_{i2}), \ldots, f_{kn}(x_{in}))^T$, $\overline{f}_k(y_j) = (\overline{f}_{k1}(y_{j1}), \overline{f}_{k2}(y_{j2}), \ldots, \overline{f}_{kn}(y_{jn}))^T \in \mathrm{R}^n$, $k = 1, 2$, $g(\dot{y}_j) = (g_1(\dot{y}_{j1}), g_2(\dot{y}_{j2}), \ldots, g_n(\dot{y}_{jn}))^T$, $\overline{g}(\dot{x}_i) = (\overline{g}_1(\dot{x}_{i1}), \overline{g}_2(\dot{x}_{i2}), \ldots, \overline{g}_n(\dot{x}_{in}))^T$, $k(y_j) = (k_1(y_{j1}), k_2(y_{j2}), \ldots, k_n(y_{jn}))^T$, $\overline{k}(x_i) = (\overline{k}_1(x_{i1}), \overline{k}_2(x_{i2}), \ldots, \overline{k}_n(x_{in}))^T \in \mathrm{R}^n$ corresponds to the boundedness activation functions of neurons. $h(t) = \mathrm{diag}(h_1(t), h_2(t), \ldots, h_n(t))$, $\overline{h}(t) = \mathrm{diag}(\overline{h}_1(t), \overline{h}_2(t), \ldots, \overline{h}_n(t)) \in \mathrm{R}^{n\times n}$ are the delay kernel functions. $\tau(t), \tau_1(t)$, $\tau_2(t), \sigma(t), \sigma_1(t)$, and $\sigma_2(t) > 0$ are time delays. $\int_{-\infty}^{t} h(t-s)k(y_j(s))\mathrm{d}s$ and $\int_{-\infty}^{t} \overline{h}(t-s)\overline{k}(x_i(s))\mathrm{d}s$ express infinite distributed delays. $I = (I^1, I^2, \ldots, I^n)^T$ and $J = (J^1, J^2, \ldots, J^n)^T \in \mathrm{R}^n$ are the constant external input vectors. The matrix $A = (a_{ij})_{l\times m}$ is the delayed weight coupling matrix denoting coupling strength between nodes. If there is a connection from node μ_i to ν_j, then $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$ and the matrix A satisfies the sum of every row being zero. The definitions of the other coupling matrixes $B = (b_{ij})_{l\times m}$, $C = (c_{ij})_{l\times m}$, $\overline{A} = (\overline{a}_{ji})_{m\times l}$, $\overline{B} = (\overline{b}_{ji})_{m\times l}$, and $\overline{C} = (\overline{c}_{ji})_{m\times l}$ are similar to that of matrix A; hence, they are omitted here.

In this paper, we consider a class of switched complex bipartite neural network with infinite distributed delays and derivative coupling, which is described as follows:

$$\begin{aligned}\dot{x}_i(t) = &-D_\lambda x_i + R_{\lambda 1} f_1(x_i(t)) + R_{\lambda 2} f_2(x_i(t-\tau(t))) \\ &+ I_\lambda + \sum_{j=1}^{m} a_{\lambda ij} y_j(t-\tau_1(t)) \\ &+ \sum_{j=1}^{m} b_{\lambda ij} g(\dot{y}_j(t-\tau_2(t))) \\ &+ \sum_{j=1}^{m} c_{\lambda ij} \int_{-\infty}^{t} h(t-s) k(y_j(s)) \mathrm{d}s, \\ &\qquad i = 1, 2, \ldots, l,\end{aligned}$$

$$\begin{aligned}\dot{y}_j(t) = & -\overline{D}_\lambda y_j + \overline{R}_{\lambda 1}\overline{f}_1\left(y_j(t)\right) + \overline{R}_{\lambda 2}\overline{f}_2\left(y_j(t-\sigma(t))\right) \\ & + J_\lambda + \sum_{i=1}^{l}\overline{a}_{\lambda ji}x_i\left(t-\sigma_1(t)\right) \\ & + \sum_{i=1}^{l}\overline{b}_{\lambda ji}\overline{g}\left(\dot{x}_i\left(t-\sigma_2(t)\right)\right) \\ & + \sum_{i=1}^{l}\overline{c}_{\lambda ji}\int_{-\infty}^{t}\overline{h}(t-s)\,\overline{k}\left(x_i(s)\right)\mathrm{d}s, \quad j=1,2,\ldots,m,\end{aligned} \tag{2}$$

where switching signal λ is piecewise constant functions, which is a value in the finite set $\aleph = \{1,2,\ldots,N\}$. This means that the matrices $\{D_\lambda, R_{\lambda 1}, R_{\lambda 2}, A_\lambda = (a_{\lambda ij}), B_\lambda = (b_{\lambda ij}), C_\lambda = (c_{\lambda ij}), I_\lambda, \overline{D}_\lambda, \overline{R}_{\lambda 1}, \overline{R}_{\lambda 2}, \overline{A}_\lambda = (\overline{a}_{\lambda ji}), \overline{B}_\lambda = (\overline{b}_{\lambda ji})$, and $\overline{C}_\lambda = (\overline{c}_{\lambda ji}), J_\lambda\}$ are allowed to take values at particular time, in a finite set $\{(D_r, R_{r1}, R_{r2}, A_r, B_r, C_r, I_r, \overline{D}_r, \overline{R}_{r1}, \overline{R}_{r2}, \overline{A}_r, \overline{B}_r, \overline{C}_r, J_r) \mid r = 1,2,\ldots,N\}$. We define the function as follows:

$$\xi_r(t,\lambda) = \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the } r\text{th mode, that is, } \lambda = r, \\ 0, & \text{others.}\end{cases} \tag{3}$$

It follows that under any switching rules $\sum_{r=1}^{N}\xi_r(t,\lambda) = 1$. Model (2) can be written as

$$\begin{aligned}\dot{x}_i(t) = & \sum_{r=1}^{N}\xi_r(t,\lambda) \\ & \times\Big[-D_r x_i + R_{r1}f_1\left(x_i(t)\right) \\ & + R_{r2}f_2\left(x_i(t-\tau(t))\right) \\ & + I_r + \sum_{j=1}^{m}a_{rij}y_j\left(t-\tau_1(t)\right) \\ & + \sum_{j=1}^{m}b_{rij}g\left(\dot{y}_j\left(t-\tau_2(t)\right)\right) \\ & + \sum_{j=1}^{m}c_{rij}\int_{-\infty}^{t}h(t-s)\,k\left(y_j(s)\right)\mathrm{d}s\Big], \\ & i=1,2,\ldots,l,\end{aligned}$$

$$\begin{aligned}\dot{y}_j(t) = & \sum_{r=1}^{N}\xi_r(t,\lambda) \\ & \times\Big[-\overline{D}_r y_j + \overline{R}_{r1}\overline{f}_1\left(y_j(t)\right) \\ & + \overline{R}_{r2}\overline{f}_2\left(y_j(t-\sigma(t))\right) \\ & + J_r + \sum_{i=1}^{l}\overline{a}_{rji}x_i\left(t-\sigma_1(t)\right) \\ & + \sum_{i=1}^{l}\overline{b}_{rji}\overline{g}\left(\dot{x}_i\left(t-\sigma_2(t)\right)\right) \\ & + \sum_{i=1}^{l}\overline{c}_{rji}\int_{-\infty}^{t}\overline{h}(t-s)\,\overline{k}\left(x_i(s)\right)\mathrm{d}s\Big], \\ & j=1,2,\ldots,m.\end{aligned} \tag{4}$$

The response network of the drive network (4) is

$$\begin{aligned}\dot{\widehat{x}}_i(t) = & \sum_{r=1}^{N}\xi_r(t,\lambda) \\ & \times\Big[-D_r\widehat{x}_i(t) + R_{r1}f_1\left(\widehat{x}_i(t)\right) \\ & + R_{r2}f_2\left(\widehat{x}_i(t-\tau(t))\right) \\ & + I_r + \sum_{j=1}^{m}a_{rij}\widehat{y}_j\left(t-\tau_1(t)\right) \\ & + \sum_{j=1}^{m}b_{rij}g\left(\dot{\widehat{y}}_j\left(t-\tau_2(t)\right)\right) \\ & + \sum_{j=1}^{m}c_{rij}\int_{-\infty}^{t}h(t-s)\,k\left(\widehat{y}_j(s)\right)\mathrm{d}s + u_i(t)\Big], \\ \dot{\widehat{y}}_j(t) = & \sum_{r=1}^{N}\xi_r(t,\lambda) \\ & \times\Big[-\overline{D}_r\widehat{y}_j(t) + \overline{R}_{r1}\overline{f}_1\left(\widehat{y}_j(t)\right) \\ & + \overline{R}_{r2}\overline{f}_2\left(\widehat{y}_j(t-\sigma(t))\right) \\ & + J_r + \sum_{i=1}^{l}\overline{a}_{rji}\widehat{x}_i\left(t-\sigma_1(t)\right) \\ & + \sum_{i=1}^{l}\overline{b}_{rji}\overline{g}\left(\dot{\widehat{x}}_i\left(t-\sigma_2(t)\right)\right) \\ & + \sum_{i=1}^{l}\overline{c}_{rji}\int_{-\infty}^{t}\overline{h}(t-s)\,\overline{k}\left(\widehat{x}_i(s)\right)\mathrm{d}s + v_j(t)\Big],\end{aligned} \tag{5}$$

where $u_i(t)$ and $v_j(t) \in R^n$ are the control inputs.

Let $e_i(t) = \hat{x}_i(t) - x_i(t)$, $\varepsilon_j(t) = \hat{y}_j(t) - y_j(t)$, $i = 1, 2, \ldots, l$, and $j = 1, 2, \ldots, m$. The error dynamical system of (4) and (5) is given by

$$\begin{aligned}
\dot{e}_i(t) = {} & \sum_{r=1}^{N} \xi_r(t, \lambda) \\
& \times \Bigg[-D_r e_i(t) + R_{r1} F_1(e_i(t)) \\
& + R_{r2} F_2(e_i(t - \tau(t))) \\
& + \sum_{j=1}^{m} a_{rij} \varepsilon_j(t - \tau_1(t)) \\
& + \sum_{j=1}^{m} b_{rij} G(\dot{\varepsilon}_j(t - \tau_2(t))) \\
& + \sum_{j=1}^{m} c_{rij} \int_{-\infty}^{t} h(t - s) K(\varepsilon_j(s)) \, ds + u_i(t) \Bigg], \\
& i = 1, 2, \ldots, l, \\
\dot{\varepsilon}_j(t) = {} & \sum_{r=1}^{N} \xi_r(t, \lambda) \\
& \times \Bigg[-\overline{D}_r \varepsilon_j(t) + \overline{R}_{r1} \overline{F}_1(\varepsilon_j(t)) \\
& + \overline{R}_{r2} \overline{F}_2(\varepsilon_j(t - \sigma(t))) \\
& + \sum_{i=1}^{l} \overline{a}_{rji} e_i(t - \sigma_1(t)) \\
& + \sum_{i=1}^{l} \overline{b}_{rji} \overline{G}(\dot{e}_i(t - \sigma_2(t))) \\
& + \sum_{i=1}^{l} \overline{c}_{rji} \int_{-\infty}^{t} \overline{h}(t - s) \overline{K}(e_i(s)) \, ds + v_j(t) \Bigg], \\
& j = 1, 2, \ldots, m,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
F_k(e_i(t)) &= f_k(\hat{x}_i(t)) - f_k(x_i(t)), \\
\overline{F}_k(\varepsilon_j(t)) &= \overline{f}_k(\hat{y}_j(t)) - \overline{f}_k(y_j(t)), \quad k = 1, 2, \\
G(\dot{\varepsilon}_j(t)) &= g(\dot{\hat{y}}_j(t)) - g(\dot{y}_j(t)), \\
\overline{G}(\dot{e}_i(t)) &= \overline{g}(\dot{\hat{x}}_i(t)) - \overline{g}(\dot{x}_i(t)), \\
K(\varepsilon_j(s)) &= k(\hat{y}_j(s)) - k(y_j(s)) \\
&= (K_1(\varepsilon_{j1}(s)), \ldots, K_n(\varepsilon_{jn}(s)))^T, \\
\overline{K}(e_i(s)) &= \overline{k}(\hat{x}_i(s)) - \overline{k}(x_i(s)) \\
&= (\overline{K}_1(e_{i1}(s)), \ldots, \overline{K}_n(e_{in}(s)))^T.
\end{aligned} \tag{7}$$

In this paper, the following assumptions and lemmas are needed.

(S_1) There exist diagonal matrices $L_i^- = \mathrm{diag}(l_{i1}^-, l_{i2}^-, \ldots, l_{in}^-)$ and $L_i^+ = \mathrm{diag}(l_{i1}^+, l_{i2}^+, \ldots, l_{in}^+)$, such that

$$\begin{gathered}
l_{kj}^- \le \frac{f_{kj}(x) - f_{kj}(y)}{x - y} \le l_{kj}^+, \qquad l_{3j}^- \le \frac{k_j(x) - k_j(y)}{x - y} \le l_{3j}^+, \\
l_{4j}^- \le \frac{g_j(x) - g_j(y)}{x - y} \le l_{4j}^+,
\end{gathered} \tag{8}$$

$\forall x, y \in \mathrm{R}$ and $x \ne y$, $i = 1, 2, 3, 4$, $j = 1, 2, \ldots, n$, and $k = 1, 2$.

(S_2) There exist diagonal matrices $\overline{L}_i^- = \mathrm{diag}(\overline{l}_{i1}^-, \overline{l}_{i2}^-, \ldots, \overline{l}_{in}^-)$ and $\overline{L}_i^+ = \mathrm{diag}(\overline{l}_{i1}^+, \overline{l}_{i2}^+, \ldots, \overline{l}_{in}^+)$, such that

$$\begin{gathered}
\overline{l}_{kj}^- \le \frac{\overline{f}_{kj}(x) - \overline{f}_{kj}(y)}{x - y} \le \overline{l}_{kj}^+, \qquad \overline{l}_{3j}^- \le \frac{\overline{k}_j(x) - \overline{k}_j(y)}{x - y} \le \overline{l}_{3j}^+, \\
\overline{l}_{4j}^- \le \frac{\overline{g}_j(x) - \overline{g}_j(y)}{x - y} \le \overline{l}_{4j}^+,
\end{gathered} \tag{9}$$

$\forall x, y \in \mathrm{R}$ and $x \ne y$, $i = 1, 2, 3, 4$, $j = 1, 2, \ldots, n$, and $k = 1, 2$.

(S_3) $\tau(t)$, $\tau_1(t)$, $\tau_2(t)$, $\sigma(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are differential functions with $\dot{\tau}(t) < \tau < 1$, $\dot{\sigma}(t) < \sigma < 1$, $\dot{\tau}_1(t) < \tau_1 < 1$, $\dot{\sigma}_1(t) < \sigma_1 < 1$, $\dot{\tau}_2(t) < \tau_2 < 1$, and $\dot{\sigma}_2(t) < \sigma_2 < 1$.

(S_4) $h_i(t)$, $\overline{h}_i(t)$ are real-value nonnegative continuous functions defined in $[0, \infty)$ satisfying

$$\int_0^\infty h_i(s) \, ds < \infty, \qquad \int_0^\infty \overline{h}_i(s) \, ds < \infty, \qquad i = 1, 2, \ldots, n. \tag{10}$$

Lemma 1 (see [34]). *Given any real matrices Σ_1, Σ_2, and Σ_3 of appropriate dimensions and a scalar $\varepsilon > 0$ such that $0 < \Sigma_3 = \Sigma_3^T$, then the following inequality holds:*

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 \le \varepsilon \Sigma_1^T \Sigma_3 \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_3^{-1} \Sigma_2. \tag{11}$$

Lemma 2 (see [35]). *Given a positive definite matrix $P \in R^{n \times n}$ and a symmetric matrix $Q \in R^{n \times n}$, then*

$$\rho_{\min}(P^{-1}Q) x^T P x \le x^T Q x \le \rho_{\max}(P^{-1}Q) x^T P x, \quad \forall x \in R^n. \tag{12}$$

Lemma 3 (Schur complement). *Given constant symmetric matrices Σ_1, Σ_2, and Σ_3, where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T\Sigma_2^{-1}\Sigma_3 < 0$ if and only if*

$$\begin{pmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{pmatrix} < 0. \tag{13}$$

For convenience, let

$$\begin{aligned} L_i &= \operatorname{diag}\left(\max\left\{\left|l_{i1}^-\right|, \left|l_{i1}^+\right|\right\}, \right. \\ &\qquad \left. \max\left\{\left|l_{i2}^-\right|, \left|l_{i2}^+\right|\right\}, \ldots, \max\left\{\left|l_{in}^-\right|, \left|l_{in}^+\right|\right\}\right), \\ &\qquad i = 1,2,3,4, \\ \overline{L}_i &= \operatorname{diag}\left(\max\left\{\left|\overline{l}_{i1}^-\right|, \left|\overline{l}_{i1}^+\right|\right\}, \right. \\ &\qquad \left. \max\left\{\left|\overline{l}_{i2}^-\right|, \left|\overline{l}_{i2}^+\right|\right\}, \ldots, \max\left\{\left|\overline{l}_{in}^-\right|, \left|\overline{l}_{in}^+\right|\right\}\right), \\ &\qquad i = 1,2,3,4, \\ H &= \operatorname{diag}\left(\int_0^\infty h_1(v)\,\mathrm{d}v, \int_0^\infty h_2(v)\,\mathrm{d}v, \ldots, \int_0^\infty h_n(v)\,\mathrm{d}v\right), \\ \overline{H} &= \operatorname{diag}\left(\int_0^\infty \overline{h}_1(v)\,\mathrm{d}v, \int_0^\infty \overline{h}_2(v)\,\mathrm{d}v, \ldots, \int_0^\infty \overline{h}_n(v)\,\mathrm{d}v\right). \end{aligned} \tag{14}$$

3. Main Results

Theorem 4. *Under assumptions (S_1)–(S_4), the two coupled SCBNNs (4) and (5) can be synchronized, if there exist positive constants, $\alpha, \beta, p, \overline{p}, \gamma_i, \eta_j$ $(i = 1,2,\ldots,l,\ j = 1,2,\ldots,m)$, $n \times n$ positive matrices P, Q, U, $\overline{P}$, $\overline{Q}$, $\overline{U}$ and $n \times n$ diagonal positive matrices $W = \operatorname{diag}(w_1, w_2, \ldots, w_n)$, $\overline{W} = \operatorname{diag}(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$, $M_i, \overline{M}_i$ $(i = 1,2,3)$ such that*

$$\begin{bmatrix} Z_{ri} & PR_{r1} & PR_{r2} & \left(\sum_{j=1}^m a_{rij}^2\right)^{1/2} P & \left(\sum_{j=1}^m b_{rij}^2\right)^{1/2} P & \left(l\sum_{j=1}^m c_{rij}^2\right)^{1/2} P \\ * & -M_1 & 0 & 0 & 0 & 0 \\ * & * & -M_2 & 0 & 0 & 0 \\ * & * & * & -M_3 & 0 & 0 \\ * & * & * & * & -I_n & 0 \\ * & * & * & * & * & -W \end{bmatrix} < 0, \tag{15}$$

$$\begin{bmatrix} \overline{Z}_{rj} & \overline{P}\overline{R}_{r1} & \overline{P}\overline{R}_{r2} & \left(\sum_{i=1}^l \overline{a}_{rji}^2\right)^{1/2} \overline{P} & \left(\sum_{i=1}^l \overline{b}_{rji}^2\right)^{1/2} \overline{P} & \left(m\sum_{i=1}^l \overline{c}_{rji}^2\right)^{1/2} \overline{P} \\ * & -\overline{M}_1 & 0 & 0 & 0 & 0 \\ * & * & -\overline{M}_2 & 0 & 0 & 0 \\ * & * & * & -\overline{M}_3 & 0 & 0 \\ * & * & * & * & -I_n & 0 \\ * & * & * & * & * & -\overline{W} \end{bmatrix} < 0, \tag{16}$$

$$\begin{gathered} \frac{m}{1-\sigma_2} - 2\alpha p \le 0, \qquad P \ge pI_n, \qquad \overline{P} \ge \overline{p}I_n, \\ \frac{l}{1-\tau_2} - 2\beta\overline{p} \le 0, \end{gathered} \tag{17}$$

$$L_2M_2L_2 - (1-\tau)Q \le 0, \qquad M_3 - (1-\tau_1)U \le 0, \tag{18}$$

$$\overline{L}_2\overline{M}_2\overline{L}_2 - (1-\sigma)\overline{Q} \le 0, \qquad \overline{M}_3 - (1-\sigma_1)\overline{U} \le 0, \tag{19}$$

and the adaptive feedback controllers are designed as

$$\begin{aligned} u_i(t) &= -\left[\gamma_i + \alpha_i(t)\right] e_i(t), \\ v_j(t) &= -\left[\eta_j + \beta_j(t)\right] \varepsilon_j(t), \\ \alpha_i(t) &= \begin{cases} \dfrac{\left\|\overline{G}(\dot{e}_i(t))\right\|^2}{\left\|e_i(t)\right\|^2}\alpha, & \left\|e_i(t)\right\|^2 \ne 0, \\ 0, & \left\|e_i(t)\right\|^2 = 0, \end{cases} \\ \beta_j(t) &= \begin{cases} \dfrac{\left\|G(\dot{\varepsilon}_j(t))\right\|^2}{\left\|\varepsilon_j(t)\right\|^2}\beta, & \left\|\varepsilon_j(t)\right\|^2 \ne 0, \\ 0, & \left\|\varepsilon_j(t)\right\|^2 = 0, \end{cases} \end{aligned} \tag{20}$$

where $Z_{ri} = -2PD_r + \overline{L}_3\overline{H}\,\overline{W}\,\overline{H}\,\overline{L}_3 + L_1M_1L_1 + m\overline{U} - 2\gamma_iP + Q$, $\overline{Z}_{rj} = -2\overline{P}\,\overline{D}_r + L_3HWHL_3 + \overline{L}_1\overline{M}_1\overline{L}_1 + lU + \overline{Q} - 2\eta_j\overline{P}$, $r \in \aleph$, $i = 1,2,\ldots,l$, and $j = 1,2,\ldots,m$.

Proof. For the error dynamical system (6), we design the following Lyapunov-Krasovskii function:

$$V(t) = V_1(E(t)) + V_2(E(t)), \tag{21}$$

where $E(t) = (e_1^T(t), e_2^T(t), \ldots, e_l^T(t), \varepsilon_1^T(t), \varepsilon_2^T(t), \ldots, \varepsilon_m^T(t))^T$,

$$\begin{aligned} V_1 &= \sum_{i=1}^l e_i^T(t)Pe_i(t) + \sum_{i=1}^l \int_{t-\tau(t)}^t e_i^T(s)Qe_i(s)\,\mathrm{d}s \\ &\quad + \sum_{j=1}^m\sum_{i=1}^n w_i \int_0^\infty h_i(v)\,\mathrm{d}v \int_0^\infty h_i(\theta) \\ &\quad \times \int_{t-\theta}^t K_i^2\left(\varepsilon_{ji}(s)\right)\mathrm{d}s\,\mathrm{d}\theta \\ &\quad + l\sum_{j=1}^m \int_{t-\tau_1(t)}^t \varepsilon_j^T(s)U\varepsilon_j(s)\,\mathrm{d}s \\ &\quad + \frac{l}{1-\tau_2}\sum_{j=1}^m \int_{t-\tau_2(t)}^t G^T\left(\dot{\varepsilon}_j(s)\right)G\left(\dot{\varepsilon}_j(s)\right)\mathrm{d}s, \end{aligned} \tag{22}$$

$$
\begin{aligned}
V_2 = {} & \sum_{j=1}^{m} \varepsilon_j^T(t)\,\overline{P}\varepsilon_j(t) + \sum_{j=1}^{m}\int_{t-\mu(t)}^{t} \varepsilon_j^T(s)\,\overline{Q}\varepsilon_j(s)\,\mathrm{d}s \\
& + \sum_{i=1}^{l}\sum_{j=1}^{n}\overline{w}_j \int_0^{\infty}\overline{h}_j(v)\,\mathrm{d}v \int_0^{\infty}\overline{h}_j(\theta) \\
& \times \int_{t-\theta}^{t}\overline{K}_j\left(e_{ij}(s)\right)\mathrm{d}s\,\mathrm{d}\theta \\
& + m\sum_{i=1}^{l}\int_{t-\mu_1(t)}^{t} e_i^T(s)\,\overline{U}e_i(s)\,\mathrm{d}s \\
& + \frac{m}{1-\sigma_2}\sum_{i=1}^{l}\int_{t-\mu_2(t)}^{t}\overline{G}^T\left(\dot{e}_i(s)\right)\overline{G}\left(\dot{e}_i(s)\right)\mathrm{d}s.
\end{aligned}
\tag{23}
$$

Calculating the derivative of (22) along the trajectories of (6), we have

$$
\begin{aligned}
\dot{V}_1 = {} & \sum_{r=1}^{N}\xi_r(t,\lambda) \\
& \times \Bigg\{ \sum_{i=1}^{l} 2e_i^T(t)\,P \\
& \times \Big[-D_r e_i(t) + R_{r1}F_1\left(e_i(t)\right) \\
& + R_{r2}F_2\left(e_i(t-\tau(t))\right) \\
& + \sum_{j=1}^{m} a_{rij}\varepsilon_j\left(t-\tau_1(t)\right) \\
& + \sum_{j=1}^{m} b_{rij}G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right) + \sum_{j=1}^{m} c_{rij} \\
& \times \int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)ds + u_i(t)\Big] \\
& + \sum_{i=1}^{l} e_i^T(t)\,Qe_i(t) - (1-\dot{\tau}(t)) \\
& \times \sum_{i=1}^{l} e_i^T(t-\tau(t))\,Qe_i(t-\tau(t)) \\
& + \sum_{j=1}^{m}\sum_{i=1}^{n} w_i\left(K_i\left(\varepsilon_{ji}(t)\right)\int_0^{\infty} h_i(v)\,dv\right)^2 \\
& - \sum_{j=1}^{m}\sum_{i=1}^{n} w_i\int_0^{\infty} h_i(v)\,dv \\
& \times \int_0^{\infty} h_i(\theta)\,K_i^2\left(\varepsilon_{ji}(t-\theta)\right)d\theta \\
& + l\sum_{j=1}^{m}\varepsilon_j^T(s)\,U\varepsilon_j(s) - \left(1-\dot{\tau}_1(t)\right)l \\
& \times \sum_{j=1}^{m}\varepsilon_j^T\left(t-\tau_1(t)\right)U\varepsilon_j\left(t-\tau_1(t)\right) \\
& + \frac{l}{1-\tau_2}\sum_{j=1}^{m} G^T\left(\dot{\varepsilon}_j(t)\right)G\left(\dot{\varepsilon}_j(t)\right) \\
& - \frac{1-\dot{\tau}_2(t)}{1-\tau_2}l \\
& \times \sum_{j=1}^{m} G^T\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right)G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right)\Bigg\}.
\end{aligned}
\tag{24}
$$

By Lemma 1, we can get from (S_1)

$$
\begin{aligned}
& 2e_i^T(t)\,PR_{r1}F_1\left(e_i(t)\right) \\
& \quad \le e_i^T(t)\,PR_{r1}M_1^{-1}R_{r1}^T Pe_i(t) \\
& \qquad + F_1^T\left(e_i(t)\right)M_1F_1\left(e_i(t)\right) \\
& \quad \le e_i^T(t)\left(PR_{r1}M_1^{-1}R_{r1}^T P + L_1M_1L_1\right)e_i(t), \\
& 2e_i^T(t)\,PR_{r2}F_2\left(e_i(t-\tau(t))\right) \\
& \quad \le e_i^T(t)\,PR_{r2}M_2^{-1}R_{r2}^T Pe_i(t) \\
& \qquad + F_2^T\left(e_i(t-\tau(t))\right)M_2F_2\left(e_i(t-\tau(t))\right) \\
& \quad \le e_i^T(t)\,PR_{r2}M_2^{-1}R_{r2}^T Pe_i(t) \\
& \qquad + e_i^T(t-\tau(t))\,L_2M_2L_2e_i(t-\tau(t)), \\
& 2a_{rij}e_i^T(t)\,P\varepsilon_j\left(t-\tau_1(t)\right) \\
& \quad \le a_{rij}^2 e_i^T(t)\,PM_3^{-1}Pe_i(t) \\
& \qquad + \varepsilon_j^T\left(t-\tau_1(t)\right)M_3\varepsilon_j^T\left(t-\tau_1(t)\right).
\end{aligned}
\tag{25}
$$

By assumptions (S_1) and (S_4), it is obvious that

$$
\begin{aligned}
& \sum_{j=1}^{m}\sum_{i=1}^{n} w_i\left(K_i\left(\varepsilon_{ji}(t)\right)\int_0^{\infty} h_i(v)\,\mathrm{d}v\right)^2 \\
& \quad = \sum_{j=1}^{m} K^T\left(\varepsilon_j(t)\right)HWHK\left(\varepsilon_j(t)\right) \\
& \quad \le \sum_{j=1}^{m}\varepsilon_j^T(t)\,L_3HWHL_3\varepsilon_j(t),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
& 2\sum_{i=1}^{l} c_{rij}e_i^T(t)\,P\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s \\
& \quad \le l\sum_{i=1}^{l} c_{rij}^2 e_i^T(t)\,PW^{-1}Pe_i(t) \\
& \qquad + \frac{1}{l}\sum_{i=1}^{l}\left(\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s\right)^T
\end{aligned}
$$

$$\begin{aligned}
&\times W\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s \\
&= l\sum_{i=1}^{l} c_{rij}^2 e_i^T(t)\,PW^{-1}Pe_i(t) \\
&\quad+\left(\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s\right)^T \\
&\quad\times W\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s.
\end{aligned} \tag{27}$$

Observe that

$$\begin{aligned}
&-\frac{l\left(1-\dot{\tau}_2(t)\right)}{1-\tau_2}\sum_{j=1}^{m} G^T\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right)G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right) \\
&\quad+2\sum_{i=1}^{l}\sum_{j=1}^{m} e_i^T(t)\,Pb_{rij}G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right) \\
&\le -\sum_{i=1}^{l}\sum_{j=1}^{m} G^T\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right)G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right) \\
&\quad+2\sum_{i=1}^{l}\sum_{j=1}^{m} e_i^T(t)\,Pb_{rij}G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right) \\
&= -\sum_{i=1}^{l}\sum_{j=1}^{m}\left(b_{rij}Pe_i(t)-G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right)\right)^T \\
&\qquad\times\left(b_{rij}Pe_i(t)-G\left(\dot{\varepsilon}_j\left(t-\tau_2(t)\right)\right)\right) \\
&\quad+\sum_{i=1}^{l}\sum_{j=1}^{m} b_{rij}^2 e_i^T(t)\,P^2e_i(t), \\
&\le \sum_{i=1}^{l}\sum_{j=1}^{m} b_{rij}^2 e_i^T(t)\,P^2e_i(t).
\end{aligned} \tag{28}$$

Using inequality

$$\int_0^{\infty} f^2(s)\,\mathrm{d}s\int_0^{\infty} g^2(s)\,\mathrm{d}s \ge \left(\int_0^{\infty} f(s)\,g(s)\,\mathrm{d}s\right)^2, \tag{29}$$

we have

$$\begin{aligned}
&\sum_{i=1}^{n} w_i\int_0^{\infty} h_i(v)\,\mathrm{d}v\int_0^{\infty} h_i(\theta)\,K_i^2\left(\varepsilon_{ji}(t-\theta)\right)\mathrm{d}\theta \\
&\quad\ge \sum_{i=1}^{n} w_i\left(\int_0^{\infty} h_i(\theta)\,K_i\left(\varepsilon_{ji}(t-\theta)\right)\mathrm{d}\theta\right)^2 \\
&\quad= \left(\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s\right)^T \\
&\qquad\times W\int_{-\infty}^{t} h(t-s)\,K\left(\varepsilon_j(s)\right)\mathrm{d}s.
\end{aligned} \tag{30}$$

Using Lemma 2 and condition (17), we get

$$-e^T(t)\,P\frac{\left\|\overline{G}\left(\dot{e}_i(t)\right)\right\|^2}{\left\|e_i(t)\right\|^2}e(t) \le -p\overline{G}^T\left(\dot{e}_i(t)\right)\overline{G}\left(\dot{e}_i(t)\right). \tag{31}$$

Substituting (20) into (24) and combining (24)–(31), it can be derived by condition (18) that

$$\begin{aligned}
\dot{V}_1 &\le \sum_{r=1}^{N}\xi_r(t,\lambda) \\
&\quad\times\left\{\sum_{i=1}^{l} e_i^T(t)\,\Omega_{ri}\,e_i(t)\right. \\
&\qquad+\sum_{i=1}^{l} e_i^T(t-\tau(t))\left[L_2M_2L_2-(1-\tau)Q\right] \\
&\qquad\quad\times e_i(t-\tau(t)) \\
&\qquad+\sum_{j=1}^{m}\varepsilon_j^T(s)\left(lU+L_3HWHL_3\right)\varepsilon_j(s) \\
&\qquad+\sum_{j=1}^{m}\varepsilon_j^T\left(t-\tau_1(t)\right)\left[lM_3-l\left(1-\tau_1\right)U\right] \\
&\qquad\quad\times\varepsilon_j\left(t-\tau_1(t)\right) \\
&\qquad+\frac{l}{1-\tau_2}\sum_{j=1}^{m} G^T\left(\dot{\varepsilon}_j(t)\right)G\left(\dot{\varepsilon}_j(t)\right) \\
&\qquad\left.-2\alpha p\sum_{i=1}^{l}\overline{G}^T\left(\dot{e}_i(t)\right)\overline{G}\left(\dot{e}_i(t)\right)\right\} \\
&\le \sum_{r=1}^{N}\xi_r(t,\lambda) \\
&\quad\times\left\{\sum_{i=1}^{l} e_i^T(t)\,\Omega_{ri}e_i(t)\right. \\
&\qquad+\sum_{j=1}^{m}\varepsilon_j^T(t)\left(lU+L_3HWHL_3\right)\varepsilon_j(t) \\
&\qquad+\frac{l}{1-\tau_2}\sum_{j=1}^{m} G^T\left(\dot{\varepsilon}_j(t)\right)G\left(\dot{\varepsilon}_j(t)\right) \\
&\qquad\left.-2\alpha p\sum_{i=1}^{l}\overline{G}^T\left(\dot{e}_i(t)\right)\overline{G}\left(\dot{e}_i(t)\right)\right\},
\end{aligned} \tag{32}$$

where $\Omega_i = -2PD_r + PR_{r1}M_1^{-1}R_{r1}^TP + L_1M_1L_1 + PR_{r2}M_2^{-1}R_{r2}^TP + \sum_{j=1}^{m}a_{rij}^2PM_3^{-1}P + \sum_{j=1}^{m}b_{rij}^2P^2 + l\sum_{j=1}^{m}c_{rij}^2PW^{-1}P - 2\gamma_iP + Q$.

Meanwhile, by a similar process, the following inequality can be true:

$$\begin{aligned}\dot{V}_2 \le & \sum_{r=1}^{N}\xi_r(t,\lambda) \\ & \times\left\{\sum_{j=1}^{m}\varepsilon_j^T(t)\overline{\Omega}_{rj}\varepsilon_j(t)\right. \\ & +\sum_{i=1}^{l}e_i^T(t)\left(m\overline{U}+\overline{L}_3\overline{H}\,\overline{W}\,\overline{H}\,\overline{L}_3\right)e_i(t) \\ & +\frac{m}{1-\sigma_2}\sum_{i=1}^{l}\overline{G}^T\left(\dot{e}_i(t)\right)\overline{G}\left(\dot{e}_i(t)\right) \\ & \left.-2\beta\overline{p}\sum_{j=1}^{m}G^T\left(\dot{\varepsilon}_j(t)\right)G\left(\dot{\varepsilon}_j(t)\right)\right\},\end{aligned} \tag{33}$$

where $\overline{\Omega}_j = -2\overline{P}\,\overline{D}_r + \overline{P}\,\overline{R}_{r1}\overline{M}_1^{-1}\overline{R}_{r1}^T\overline{P} + \overline{L}_1\overline{M}_1\overline{L}_1 + \overline{P}\,\overline{R}_{r2}\overline{M}_2^{-1}\overline{R}_{r2}^T\overline{P} + \sum_{i=1}^{l}\overline{a}_{rij}^2\overline{P}\,\overline{M}_3^{-1}\overline{P} + \sum_{i=1}^{l}\overline{b}_{rji}^2\overline{P}^2 + m\sum_{i=1}^{l}\overline{c}_{rji}^2\overline{P}\,\overline{W}^{-1}\overline{P} - 2\eta_j\overline{P} + \overline{Q}$.

By condition (17), we have

$$\begin{aligned}\dot{V} \le & \sum_{r=1}^{N}\xi_r(t,\lambda) \\ & \times\left[\sum_{i=1}^{l}e_i^T(t)\left(\Omega_{ri}+m\overline{U}+\overline{L}_3\overline{H}\,\overline{W}\,\overline{H}\,\overline{L}_3\right)e_i(t)\right. \\ & \left.+\sum_{j=1}^{m}\varepsilon_j^T(t)\left(\overline{\Omega}_{rj}+lU+L_3HWHL_3\right)\varepsilon_j(t)\right].\end{aligned} \tag{34}$$

By (15)-(16) and Lemma 3 (Schur complement), it can be obtained that $\Omega_{ri} + m\overline{U} + \overline{L}_3\overline{H}\,\overline{W}\,\overline{H}\,\overline{L}_3 < 0$, $\overline{\Omega}_{rj} + lU + L_3HWHL_3 < 0$. Set $\rho = \min\{\rho_1, \rho_2\}$, where

$$\begin{aligned}\rho_1 = & -\min\left\{\rho_{\min}\left(\Omega_{ri}+m\overline{U}+\overline{L}_3\overline{H}\,\overline{W}\,\overline{H}\,\overline{L}_3\right),\right. \\ & \left. r\in\aleph,\ 1\le i\le l\right\}, \\ \rho_2 = & -\min\left\{\rho_{\min}\left(\overline{\Omega}_{rj}+lU+L_3HWHL_3\right),\right. \\ & \left. r\in\aleph,\ 1\le j\le m\right\},\end{aligned} \tag{35}$$

then $\rho > 0$, and

$$\begin{aligned}\dot{V} & \le -\rho_1\sum_{i=1}^{l}e_i^T(t)e_i(t)-\rho_2\sum_{j=1}^{m}\varepsilon_j^T(t)\varepsilon_j(t) \\ & \le -\rho E^T(t)E(t).\end{aligned} \tag{36}$$

Therefore, V is nonincreasing in $t \ge 0$. One has V bounded since $0 \le V(t, E(t)) \le V(0, E(0))$, so $\lim_{t\to+\infty}V(t, E(t))$ exists and

$$\begin{aligned}\lim_{t\to+\infty}\int_0^t & E^T(s)E(s)\,\mathrm{d}s \\ & \le -\frac{1}{\rho}\lim_{t\to+\infty}\int_0^t\frac{\mathrm{d}V}{\mathrm{d}s}\mathrm{d}s \\ & = \frac{1}{\rho}V(0,E(0))-\frac{1}{\rho}\lim_{t\to+\infty}V(t,E(t)).\end{aligned} \tag{37}$$

From (22)-(23) and conditions $P \ge pI$, $\overline{P} \ge \overline{p}I$ and we have $0 \le E^T(t)E(t) \le \max\{1/p, 1/\overline{p}\}V(t, E(t))$, so $E^T(t)E(t)$ is bounded. According to error system (6), $(\mathrm{d}/\mathrm{d}t)E^T(t)E(t) = 2E^T(t)\dot{E}(t)$ is bounded for $t \ge 0$ due to the boundedness of activation functions. From the above we can see that $E(t) \in L^2 \cap L^\infty$ and $(\mathrm{d}/\mathrm{d}t)E^T(t)E(t) \in L^\infty$. By using Barbălat lemma (see [36]), one has $\lim_{t\to+\infty}E^T(t)E(t) = 0$, so the two SBNNs (4) and (5) can obtain synchronization under the controllers (20). This completes the proof. □

We take CBDN (1) as drive network. The response network of the drive network (1) is

$$\begin{aligned}\dot{\widehat{x}}_i(t) = & -D\widehat{x}_i(t)+R_1f_1\left(\widehat{x}_i(t)\right)+R_2f_2\left(\widehat{x}_i(t-\tau(t))\right) \\ & +I(t)+\sum_{j=1}^{m}a_{ij}\widehat{y}_j\left(t-\tau_1(t)\right) \\ & +\sum_{j=1}^{m}b_{ij}g\left(\dot{\widehat{y}}_j\left(t-\tau_2(t)\right)\right) \\ & +\sum_{j=1}^{m}c_{ij}\int_{-\infty}^{t}h(t-s)k\left(\widehat{y}_j(s)\right)\mathrm{d}s+u_i(t), \\ \dot{\widehat{y}}_j(t) = & -\overline{D}\widehat{y}_j(t)+\overline{R}_1\overline{f}_1\left(\widehat{y}_j(t)\right)+\overline{R}_2\overline{f}_2\left(\widehat{y}_j(t-\sigma(t))\right) \\ & +J(t)+\sum_{i=1}^{l}\overline{a}_{ji}\widehat{x}_i\left(t-\sigma_1(t)\right) \\ & +\sum_{i=1}^{l}\overline{b}_{ji}\overline{g}\left(\dot{\widehat{x}}_i\left(t-\sigma_2(t)\right)\right) \\ & +\sum_{i=1}^{l}\overline{c}_{ji}\int_{-\infty}^{t}\overline{h}(t-s)\overline{k}\left(\widehat{x}_i(s)\right)\mathrm{d}s+v_j(t),\end{aligned} \tag{38}$$

where $u_i(t), v_j(t) \in R^n$ are the control inputs.

From Theorem 4, we can get the following corollary.

Corollary 5. *Under assumptions* (S_1)–(S_4), *the two coupled CBDNs* (1) *and* (38) *can be synchronized, if there exist positive constants* $\alpha, \beta, p, \overline{p}, \gamma_i, \eta_j$ $(i = 1, 2, \ldots, l,\ j = 1, 2, \ldots, m)$, $n \times n$ *positive matrices* P, Q, U, $\overline{P}$, $\overline{Q}$, $\overline{U}$ *and* $n \times n$ *diagonal positive matrices* $W = \operatorname{diag}(w_1, w_2, \ldots, w_n)$, $\overline{W} = \operatorname{diag}(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$, M_i, $\overline{M}_i$ $(i = 1, 2, 3)$ *such that*

$$\begin{bmatrix} Z_i & PR_1 & PR_2 & \left(\sum_{j=1}^{m} a_{ij}^2\right)^{1/2} P & \left(\sum_{j=1}^{m} b_{ij}^2\right)^{1/2} P & \left(l\sum_{j=1}^{m} c_{ij}^2\right)^{1/2} P \\ * & -M_1 & 0 & 0 & 0 & 0 \\ * & * & -M_2 & 0 & 0 & 0 \\ * & * & * & -M_3 & 0 & 0 \\ * & * & * & * & -I_n & 0 \\ * & * & * & * & * & -W \end{bmatrix} < 0,$$

$$\begin{bmatrix} \overline{Z}_j & \overline{P}\,\overline{R}_1 & \overline{P}\,\overline{R}_2 & \left(\sum_{i=1}^{l} \overline{a}_{ji}^2\right)^{1/2} \overline{P} & \left(\sum_{i=1}^{l} \overline{b}_{ji}^2\right)^{1/2} \overline{P} & \left(m\sum_{i=1}^{l} \overline{c}_{ji}^2\right)^{1/2} \overline{P} \\ * & -\overline{M}_1 & 0 & 0 & 0 & 0 \\ * & * & -\overline{M}_2 & 0 & 0 & 0 \\ * & * & * & -\overline{M}_3 & 0 & 0 \\ * & * & * & * & -I_n & 0 \\ * & * & * & * & * & -\overline{W} \end{bmatrix} < 0$$

$$\frac{m}{1-\sigma_2} - 2\alpha p \le 0, \qquad P \ge pI_n, \qquad \overline{P} \ge \overline{p} I_n,$$

$$\frac{l}{1-\tau_2} - 2\beta\overline{p} \le 0,$$

$$L_2 M_2 L_2 - (1-\tau) Q \le 0, \qquad M_3 - (1-\tau_1) U \le 0,$$

$$\overline{L}_2 \overline{M}_2 \overline{L}_2 - (1-\sigma) \overline{Q} \le 0, \qquad \overline{M}_3 - (1-\sigma_1) \overline{U} \le 0, \tag{39}$$

and the adaptive feedback controllers are designed as

$$\begin{aligned} u_i(t) &= -\left[\gamma_i + \alpha_i(t)\right] e_i(t), \\ v_j(t) &= -\left[\eta_j + \beta_j(t)\right] \varepsilon_j(t), \\ \alpha_i(t) &= \begin{cases} \dfrac{\left\|\overline{G}\left(\dot{e}_i(t)\right)\right\|^2}{\left\|e_i(t)\right\|^2}\alpha, & \left\|e_i(t)\right\|^2 \ne 0, \\ 0, & \left\|e_i(t)\right\|^2 = 0, \end{cases} \\ \beta_j(t) &= \begin{cases} \dfrac{\left\|G\left(\dot{\varepsilon}_j(t)\right)\right\|^2}{\left\|\varepsilon_j(t)\right\|^2}\beta, & \left\|\varepsilon_j(t)\right\|^2 \ne 0, \\ 0, & \left\|\varepsilon_j(t)\right\|^2 = 0, \end{cases} \end{aligned} \tag{40}$$

where $Z_i = -2PD + \overline{L}_3 \overline{H}\, \overline{W}\, \overline{H}\, \overline{L}_3 + L_1 M_1 L_1 + m\overline{U} - 2\gamma_i P + Q$, $\overline{Z}_j = -2\overline{P}\,\overline{D} + L_3 HWHL_3 + \overline{L}_1 \overline{M}_1 \overline{L}_1 + lU - 2\eta_j \overline{P} + \overline{Q}$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$.

Remark 6. From Corollary 5, we can easily get that the controllers in this paper are simpler than those of Theorem 1 in [29].

Remark 7. If the coupling matrix of the SCBNN is not a diffusive matrix satisfying the sum of every row being zero, we can still obtain the same result from the proof of Theorem 4.

Theorem 8 presents another sufficient condition to ascertain that the two networks (4) and (5) can be synchronized, using the following simple adaptive feedback controllers:

$$\begin{aligned} u_i(t) &= -\gamma_i e_i(t), \\ v_j(t) &= -\overline{\gamma}_j \varepsilon_j(t), \end{aligned} \tag{41}$$

where $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$, γ_i, and $\overline{\gamma}_j$ are positive constants.

Let

$$\begin{aligned} e(t) &= \left(e_1^T(t), e_2^T(t), \ldots, e_l^T(t)\right)^T, \\ \varepsilon(t) &= \left(\varepsilon_1^T(t), \varepsilon_2^T(t), \ldots, \varepsilon_m^T(t)\right)^T, \\ \widetilde{F}_k(e(t)) &= \left(F_k^T(e_1(t)), F_k^T(e_2(t)), \ldots, F_k^T(e_l(t))\right)^T, \\ & \qquad k = 1, 2, \\ \widehat{F}_k(e(t)) &= \left(\overline{F}_k^T(\varepsilon_1(t)), \overline{F}_k^T(\varepsilon_2(t)), \ldots, \overline{F}_k^T(\varepsilon_m(t))\right)^T, \\ & \qquad k = 1, 2, \\ \widetilde{G}(\dot{\varepsilon}(t)) &= \left(G^T(\dot{\varepsilon}_1(t)), G^T(\dot{\varepsilon}_2(t)), \ldots, G^T(\dot{\varepsilon}_m(t))\right)^T, \\ \widehat{G}(\dot{e}(t)) &= \left(\overline{G}^T(\dot{e}_1(t)), \overline{G}^T(\dot{e}_2(t)), \ldots, \overline{G}^T(\dot{e}_l(t))\right)^T, \\ \widetilde{K}(\varepsilon(t)) &= \left(K^T(\varepsilon_1(t)), K^T(\varepsilon_2(t)), \ldots, K^T(\varepsilon_m(t))\right)^T, \\ \widehat{K}(e(t)) &= \left(\overline{K}^T(e_1(t)), \overline{K}^T(e_2(t)), \ldots, \overline{K}^T(e_l(t))\right)^T, \\ \widetilde{\Gamma} &= -\operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_l), \\ \widehat{\Gamma} &= -\operatorname{diag}(\overline{\gamma}_1, \overline{\gamma}_2, \ldots, \overline{\gamma}_m), \end{aligned} \tag{42}$$

then the error dynamical system of (6) becomes

$$\begin{aligned} \dot{e}(t) = \sum_{r=1}^{N} \xi_r(t, \lambda) & \\ & \times \left[-\left(I_l \otimes D_r\right) e(t) + \left(I_l \otimes R_{r1}\right) \widetilde{F}_1(e(t))\right. \\ & \quad \left. + \left(I_l \otimes R_{r2}\right) \widetilde{F}_2(e(t - \tau(t)))\right. \end{aligned}$$

$$
\begin{aligned}
&+ (A_r \otimes I_n)\, \varepsilon(t-\tau_1(t)) \\
&+ (B_r \otimes I_n)\, \widetilde{G}(\dot{\varepsilon}(t-\tau_2(t))) \\
&+ (C_r \otimes I_n) \int_{-\infty}^{t} (I_m \otimes h(t-s))\, \widetilde{K}(\varepsilon(s))\, ds \\
&+ (\widetilde{\Gamma} \otimes I_n)\, e(t)\Big], \\
&+ (\overline{B}_r \otimes I_n)\, \widehat{G}(\dot{e}(t-\sigma_2(t))) \\
&+ (\overline{C}_r \otimes I_n) \int_{-\infty}^{t} (I_l \otimes h(t-s))\, \widehat{K}(e(s))\, ds \\
&+ (\widehat{\Gamma} \otimes I_n)\, \varepsilon(t)\Big],
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\dot{\varepsilon}(t) = \sum_{r=1}^{N} \xi_r(t,\lambda) &\times \Big[-(I_m \otimes \overline{D})\, \varepsilon(t) + (I_m \otimes \overline{R}_{r1})\, \widehat{F}_1(\varepsilon(t)) \\
&+ (I_m \otimes \overline{R}_{r2})\, \widehat{F}_2(\varepsilon(t-\sigma(t))) \\
&+ (\overline{A}_r \otimes I_n)\, e(t-\sigma_1(t))
\end{aligned}
$$

Theorem 8. *Under assumptions* (S_1)–(S_4) *and using the adaptive feedback controllers* (41), *the two coupled SCBNNs* (4) *and* (5) *can be synchronized, if there exist* $n \times n$ *positive matrices* $P, U, \overline{P}, \overline{U}$ *and* $n \times n$ *diagonal positive matrices* $W = \operatorname{diag}(w_1, w_2, \ldots, w_n)$, $\overline{W} = \operatorname{diag}(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$, $Q, V, M, \overline{Q}, \overline{V}, \overline{M}$ *such that for* $r \in \aleph$, *the following matrix inequalities hold:*

$$
\Omega_r = \begin{pmatrix}
\Psi_{r1} & \Psi_{r2} & \Psi_{r3} & \Psi_{r4} & \Psi_{r5} & \Psi_{r6} \\
* & \Psi_{r7} & I_l \otimes (R_{r1}^T \overline{V} R_{r2}) & A_r \otimes (R_{r1}^T \overline{V}) & B_r \otimes (R_{r1}^T \overline{V}) & C_r \otimes (R_{r1}^T \overline{V}) \\
* & * & \Psi_{r8} & A_r \otimes (R_{r2}^T \overline{V}) & B_r \otimes (R_{r2}^T \overline{V}) & C_r \otimes (R_{r2}^T \overline{V}) \\
* & * & * & \Psi_{r9} & (A_r^T B_r) \otimes \overline{V} & (A_r^T C_r) \otimes \overline{V} \\
* & * & * & * & \Psi_{r10} & (B_r^T C_r) \otimes \overline{V} \\
* & * & * & * & * & \Psi_{r11}
\end{pmatrix} < 0,
$$

$$
\overline{\Omega}_r = \begin{pmatrix}
\overline{\Psi}_{r1} & \overline{\Psi}_{r2} & \overline{\Psi}_{r3} & \overline{\Psi}_{r4} & \overline{\Psi}_{r5} & \overline{\Psi}_{r6} \\
* & \overline{\Psi}_{r7} & I_l \otimes (\overline{R}_{r1}^T V \overline{R}_{r2}) & \overline{A}_r \otimes (\overline{R}_{r1}^T V) & \overline{B}_r \otimes (\overline{R}_{r1}^T V) & \overline{C}_r \otimes (\overline{R}_{r1}^T V) \\
* & * & \overline{\Psi}_{r8} & \overline{A}_r \otimes (\overline{R}_{r2}^T V) & \overline{B}_r \otimes (\overline{R}_{r2}^T V) & \overline{C}_r \otimes (\overline{R}_{r2}^T V) \\
* & * & * & \overline{\Psi}_{r9} & (\overline{B}_r) \otimes V & (\overline{A}_r^T \overline{C}_r) \otimes V \\
* & * & * & * & \overline{\Psi}_{r10} & \overline{B}_r^T \overline{C}_r \otimes V \\
* & * & * & * & * & \overline{\Psi}_{r11}
\end{pmatrix} < 0,
\tag{44}
$$

with

$$
\begin{aligned}
\Psi_{r1} &= I_l \otimes \Big(-PD_r - D_r^T P + Q + \overline{U} \\
&\qquad + \overline{L}_3 \overline{H}\, \overline{W}\, \overline{H}\, \overline{L}_3 + L_1 M L_1 + D_r^T \overline{V} D_r\Big) \\
&\quad + 2\Gamma \otimes P + \Gamma^2 \otimes \overline{V} - \Gamma \otimes (D_r^T \overline{V} + \overline{V} D_r), \\
\Psi_{r2} &= \Gamma \otimes (\overline{V} R_{r1}) + I_l \otimes (P R_{r1} - D_r^T \overline{V} R_{r1}), \\
\Psi_{r3} &= I_l \otimes (P R_{r2} - D_r^T \overline{V} R_{r2}) + \Gamma \otimes \overline{V} R_{r2}, \\
\Psi_{r4} &= (\Gamma A_r) \otimes \overline{V} + A_r \otimes (P - D_r^T \overline{V}), \\
\Psi_{r5} &= (\Gamma B_r) \otimes \overline{V} + B_r \otimes (P - D_r^T \overline{V}), \\
\Psi_{r6} &= (\Gamma C_r) \otimes \overline{V} + C_r \otimes (P - D_r^T \overline{V}), \\
\Psi_{r7} &= I_l \otimes (R_{r1}^T \overline{V} R_{r1} - M), \\
\Psi_{r8} &= I_l \otimes \left[R_{r2}^T \overline{V} R_{r2} - (1-\tau) L_2^{-1} Q L_2^{-1}\right], \\
\Psi_{r9} &= (A_r^T A_r) \otimes \overline{V} - (1-\tau_1)(I_m \otimes U), \\
\Psi_{r10} &= (B_r^T B_r) \otimes \overline{V} - (1-\tau_2)\left(I_m \otimes (L_4^{-1} V L_4^{-1})\right), \\
\Psi_{r11} &= (C_r^T C_r) \otimes \overline{V} - I_m \otimes \overline{W}, \\
\overline{\Psi}_{r1} &= I_m \otimes \Big(-\overline{P}\, \overline{D}_r - \overline{D}_r^T \overline{P} + \overline{Q} + U + L_3 H W H L_3 \\
&\qquad + \overline{L}_1 \overline{M}\, \overline{L}_1 + \overline{D}_r^T V \overline{D}_r\Big) \\
&\quad + 2\overline{\Gamma} \otimes \overline{P} + \overline{\Gamma}^2 \otimes V - \overline{\Gamma} \otimes (\overline{D}_r^T V + V \overline{D}_r), \\
\overline{\Psi}_{r2} &= \overline{\Gamma} \otimes (V \overline{R}_{r1}) + I_m \otimes (\overline{P}\, \overline{R}_{r1} - \overline{D}_r^T V \overline{R}_{r1}), \\
\overline{\Psi}_3 &= I_m \otimes (\overline{P}\, \overline{R}_{r2} - \overline{D}_r^T V \overline{R}_{r2}) + \overline{\Gamma} \otimes V \overline{R}_{r2}, \\
\overline{\Psi}_{r4} &= (\overline{\Gamma}\, \overline{A}_r) \otimes V + \overline{A}_r \otimes (\overline{P} - \overline{D}_r^T V), \\
\overline{\Psi}_5 &= (\overline{\Gamma}\, \overline{B}_r) \otimes V + \overline{B}_r \otimes (\overline{P} - \overline{D}_r^T V), \\
\overline{\Psi}_{r6} &= (\overline{\Gamma}\, \overline{C}_r) \otimes V + \overline{C}_r \otimes (\overline{P} - \overline{D}_r^T V),
\end{aligned}
$$

$$\begin{aligned}
\overline{\Psi}_{r7} &= I_m \otimes \left(\overline{R}_{r1}^T V \overline{R}_{r1} - \overline{M}\right),\\
\overline{\Psi}_{8} &= I_m \otimes \left[\overline{R}_{r2}^T V \overline{R}_{r2} - (1-\sigma)\,\overline{L}_2^{-1}\overline{Q}\,\overline{L}_2^{-1}\right],\\
\overline{\Psi}_{r9} &= \left(\overline{A}_r^T \overline{A}_r\right) \otimes V - (1-\sigma_1)\left(I_l \otimes \overline{U}\right),\\
\overline{\Psi}_{r10} &= \left(\overline{B}_r^{\,T} \overline{B}_r\right) \otimes V - (1-\sigma_2)\left(I_l \otimes \left(\overline{L}_4^{-1}\overline{V}\,\overline{L}_4\right)\right),\\
\overline{\Psi}_{r11} &= \left(\overline{C}_r^T \overline{C}_r\right) \otimes V - I_l \otimes W.
\end{aligned} \tag{45}$$

Proof. For the error dynamical system (43), we define the following Lyapunov-Krasovskii function:

$$\begin{aligned}
V(t, e(t), \varepsilon(t)) &= V_1(t, e(t), \varepsilon(t)) + V_2(t, e(t), \varepsilon(t)),\\
V_1 &= e^T(t)\left(I_l \otimes P\right) e(t) + \int_{t-\tau(t)}^{t} e^T(s)\left(I_l \otimes Q\right) e(s)\,\mathrm{d}s\\
&\quad + \int_{t-\tau_1(t)}^{t} \varepsilon^T(s)\left(I_m \otimes U\right)\varepsilon(s)\,\mathrm{d}s\\
&\quad + \sum_{j=1}^{m}\sum_{i=1}^{n} w_i \int_0^{\infty} h_i(v)\,\mathrm{d}v \int_0^{\infty} h_i(\theta)\\
&\quad \times \int_{t-\theta}^{t} K_i^2\left(\varepsilon_{ji}(s)\right)\mathrm{d}s\,\mathrm{d}\theta\\
&\quad + \int_{t-\tau_2(t)}^{t} \dot{\varepsilon}^T(s)\left(I_m \otimes V\right)\dot{\varepsilon}(s)\,\mathrm{d}s,\\
V_2 &= \varepsilon^T(t)\left(I_m \otimes \overline{P}\right)\varepsilon(t)\\
&\quad + \int_{t-\sigma(t)}^{t} \varepsilon^T(s)\left(I_m \otimes \overline{Q}\right)\varepsilon(s)\,\mathrm{d}s\\
&\quad + \int_{t-\sigma_1(t)}^{t} e^T(s)\left(I_l \otimes \overline{U}\right) e(s)\,\mathrm{d}s\\
&\quad + \sum_{i=1}^{l}\sum_{j=1}^{n} \overline{w}_j \int_0^{\infty} \overline{h}_j(v)\,\mathrm{d}v \int_0^{\infty} \overline{h}_j(\theta)\\
&\quad \times \int_{t-\theta}^{t} \overline{K}_j\left(e_{ij}(s)\right)\mathrm{d}s\,\mathrm{d}\theta\\
&\quad + \int_{t-\sigma_2(t)}^{t} \dot{e}^T(s)\left(I_l \otimes \overline{V}\right)\dot{e}(s)\,\mathrm{d}s,\\
\dot{V}_1 &= e^T(t)\left(I_l \otimes P\right)\dot{e}(t) + \dot{e}^T(t)\left(I_l \otimes P\right) e(t)\\
&\quad + e^T(t)\left(I_l \otimes Q\right) e(t) - (1-\dot{\tau}(t))\, e^T(t-\tau(t))\\
&\quad \times \left(I_l \otimes Q\right) e(t-\tau(t)) + \varepsilon^T(t)\left(I_m \otimes U\right)\varepsilon(t)\\
&\quad - (1-\dot{\tau}_1(t))\,\varepsilon^T(t-\tau_1(t))\left(I_m \otimes U\right)\varepsilon(t-\tau_1(t))\\
&\quad + \sum_{j=1}^{m}\sum_{i=1}^{n} w_i \left(K_i\left(\varepsilon_{ji}(t)\right)\int_0^{\infty} h_i(v)\,\mathrm{d}v\right)^2\\
&\quad - \sum_{j=1}^{m}\sum_{i=1}^{n} w_i \int_0^{\infty} h_i(v)\,\mathrm{d}v \int_0^{\infty} h_i(\theta)\, K_i^2\left(\varepsilon_{ji}(t-\theta)\right)\mathrm{d}\theta\\
&\quad + \dot{\varepsilon}^T(t)\left(I_m \otimes V\right)\dot{\varepsilon}(t)\\
&\quad - (1-\dot{\tau}_2(t))\,\dot{\varepsilon}^T(t-\tau_2(t))\left(I_m \otimes V\right)\dot{\varepsilon}(t-\tau_2(t)),\\
\dot{V}_2 &= \varepsilon^T(t)\left(I_m \otimes \overline{P}\right)\dot{\varepsilon}(t) + \dot{\varepsilon}^T(t)\left(I_m \otimes \overline{P}\right)\varepsilon(t)\\
&\quad + \varepsilon(t)\left(I_m \otimes \overline{Q}\right)\varepsilon(t) - (1-\dot{\sigma}(t))\,\varepsilon(t-\sigma(t))\\
&\quad \times \left(I_m \otimes \overline{Q}\right)\varepsilon(t-\sigma(t)) + e^T(t)\left(I_l \otimes \overline{U}\right) e(t)\\
&\quad - (1-\dot{\sigma}_1(t))\, e^T(t-\sigma_1(t))\left(I_l \otimes \overline{U}\right) e(t-\sigma_1(t))\\
&\quad + \sum_{i=1}^{l}\sum_{j=1}^{n} \overline{w}_j \left(\overline{K}_j(e_{ij}(t))\int_0^{\infty} \overline{h}_j(v)\mathrm{d}v\right)^2\\
&\quad - \sum_{i=1}^{l}\sum_{j=1}^{n} \overline{w}_j \int_0^{\infty} \overline{h}_j(v)\,\mathrm{d}v \int_0^{\infty} \overline{h}_j(\theta)\,\overline{K}_j^2\left(e_{ij}(t-\theta)\right)\mathrm{d}\theta\\
&\quad + \dot{e}^T(t)\left(I_l \otimes \overline{V}\right)\dot{e}(t) - (1-\dot{\sigma}_2(t))\,\dot{e}^T(t-\sigma_2(t))\\
&\quad \times \left(I_l \otimes \overline{V}\right)\dot{e}(t-\sigma_2(t)).
\end{aligned} \tag{46}$$

By (26), we have

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{i=1}^{n} w_i \left(K_i\left(\varepsilon_{ji}(t)\right)\int_0^{\infty} h_i(v)\,\mathrm{d}v\right)^2\\
&\quad \le \sum_{j=1}^{m} \varepsilon_j^T(t)\, L_3 H W H L_3 \varepsilon_j(t)\\
&\quad = \varepsilon^T(t)\left(I_m \otimes \left(L_3 H W H L_3\right)\right)\varepsilon(t).
\end{aligned} \tag{47}$$

Using (30), we get

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{i=1}^{n} w_i \int_0^{\infty} h_i(v)\,\mathrm{d}v \int_0^{\infty} h_i(\theta)\, K_i^2\left(\varepsilon_{ji}(s)(t-\theta)\right)\mathrm{d}\theta\\
&\quad \ge \sum_{j=1}^{m}\left(\int_{-\infty}^{t} h(t-s)\, K\left(\varepsilon_j(s)\right)\mathrm{d}s\right)^T\\
&\qquad \times W \int_{-\infty}^{t} h(t-s)\, K\left(\varepsilon_j(s)\right)\mathrm{d}s\\
&\quad = \left(\int_{-\infty}^{t}\left(I_m \otimes h(t-s)\right)\widetilde{K}(\varepsilon(s))\,\mathrm{d}s\right)^T \left(I_m \otimes W\right)\\
&\qquad \times \int_{-\infty}^{t}\left(I_m \otimes h(t-s)\right)\widetilde{K}(\varepsilon(s))\,\mathrm{d}s.
\end{aligned} \tag{48}$$

From (S_3) and (46)–(48), we have

$$\begin{aligned}\dot{V}_1 \le{}& e^T(t)(I_l\otimes P)\dot{e}(t)+\dot{e}^T(t)(I_l\otimes P)e(t)\\&+e^T(t)(I_l\otimes Q)e(t)-(1-\tau)e^T(t-\tau(t))\\&\times(I_l\otimes Q)e(t-\tau(t))\\&+\varepsilon^T(t)\left[I_m\otimes(U+L_3HWHL_3)\right]\varepsilon(t)\\&-(1-\tau_1)\varepsilon^T(t-\tau_1(t))(I_m\otimes U)\varepsilon(t-\tau_1(t))\\&-\left(\int_{-\infty}^{t}(I_m\otimes h(t-s))\widetilde{K}(\varepsilon(s))\,\mathrm{d}s\right)^T(I_m\otimes W)\\&\times\int_{-\infty}^{t}(I_m\otimes h(t-s))\widetilde{K}(\varepsilon(s))\,\mathrm{d}s\\&+\dot{\varepsilon}^T(t)(I_m\otimes V)\dot{\varepsilon}(t)-(1-\tau_2)\dot{\varepsilon}^T(t-\tau_2(t))\\&\times(I_m\otimes V)\dot{\varepsilon}(t-\tau_2(t)).\end{aligned}\tag{49}$$

In the same way, we have

$$\begin{aligned}\dot{V}_2 \le{}& \varepsilon^T(t)\left(I_m\otimes\overline{P}\right)\dot{\varepsilon}(t)+\dot{\varepsilon}^T(t)\left(I_l\otimes\overline{P}\right)\varepsilon(t)\\&+\varepsilon^T(t)\left(I_m\otimes\overline{Q}\right)\varepsilon(t)-(1-\sigma)\varepsilon^T(t-\sigma(t))\\&\times(I_m\otimes Q)\varepsilon(t-\sigma(t))\\&+e^T(t)\left[I_l\otimes\left(\overline{U}+\overline{L}_3\overline{H}\,\overline{W}\,\overline{H}\,\overline{L}_3\right)\right]e(t)\\&-(1-\sigma_1)e^T(t-\sigma_1(t))\left(I_l\otimes\overline{U}\right)e(t-\sigma_1(t))\\&-\left(\int_{-\infty}^{t}\left(I_l\otimes\overline{h}(t-s)\right)\widehat{K}(e(s))\,\mathrm{d}s\right)^T\left(I_l\otimes\overline{W}\right)\\&\times\int_{-\infty}^{t}\left(I_l\otimes\overline{h}(t-s)\right)\widehat{K}(e(s))\,\mathrm{d}s\\&+\dot{e}^T(t)\left(I_l\otimes\overline{V}\right)\dot{e}(t)-(1-\sigma_2)\dot{e}^T(t-\sigma_2(t))\\&\times\left(I_l\otimes\overline{V}\right)\dot{e}(t-\sigma_2(t)).\end{aligned}\tag{50}$$

From (S_1) and (S_2),

$$\begin{aligned}&e^T(t)\left[I_l\otimes(L_1ML_1)\right]e(t)\\&\quad-\widetilde{F}_1^T(e(t))(I_l\otimes M)\widetilde{F}_1(e(t))\ge 0,\\&\varepsilon^T(t)\left[I_m\otimes\left(\overline{L}_1\overline{M}\,\overline{L}_1\right)\right]\varepsilon(t)\\&\quad-\widehat{F}_1^T(\varepsilon(t))\left(I_m\otimes\overline{M}\right)\widehat{F}_1(\varepsilon(t))\ge 0,\\&e^T(t-\tau(t))(I_l\otimes Q)e(t-\tau(t))\\&\quad\ge\widetilde{F}_2^T(e(t-\tau(t)))\left[I_l\otimes\left(L_2^{-1}QL_2^{-1}\right)\right]\widetilde{F}_2(e(t-\tau(t))),\\&\varepsilon^T(t-\sigma(t))\left(I_m\otimes\overline{Q}\right)\varepsilon(t-\sigma(t))\\&\quad\ge\widehat{F}_2^T(\varepsilon(t-\sigma(t)))\left[I_m\otimes\left(\overline{L}_2^{-1}\overline{Q}\,\overline{L}_2^{-1}\right)\right]\widehat{F}_2(\varepsilon(t-\sigma(t))),\\&\dot{\varepsilon}^T(t-\tau_2(t))(I_m\otimes V)\dot{\varepsilon}(t-\tau_2(t))\\&\quad\ge\widetilde{G}^T(\dot{\varepsilon}(t-\tau_2(t)))\left(I_m\otimes\left(L_4^{-1}VL_4^{-1}\right)\right)\widetilde{G}(\dot{\varepsilon}(t-\tau_2(t))),\\&\dot{e}^T(t-\sigma_2(t))\left(I_l\otimes\overline{V}\right)\dot{e}(t-\sigma_2(t))\\&\quad\ge\widehat{G}^T(\dot{e}(t-\sigma_2(t)))\left(I_l\otimes\left(\overline{L}_4^{-1}\overline{V}\,\overline{L}_4^{-1}\right)\right)\widehat{G}(\dot{e}(t-\sigma_2(t))).\end{aligned}\tag{51}$$

With the aid of (43) and (51), we have

$$\dot{V}\le\sum_{r=1}^{N}\xi_r(t,\lambda)\left[\eta^T(t)\Omega_r\eta(t)+\overline{\eta}^T(t)\overline{\Omega}_r\overline{\eta}(t)\right],\tag{52}$$

where

$$\begin{aligned}\eta(t)={}&\Bigg(e^T(t),\widetilde{F}_1^T(e(t)),\widetilde{F}_2^T(e(t-\tau(t))),\varepsilon^T(t-\tau_1(t)),\\&\widetilde{G}^T(\dot{\varepsilon}(t-\tau_2(t))),\\&\left(\int_{-\infty}^{t}(I_m\otimes h(t-s))\widetilde{\mathrm{K}}(\varepsilon(s))\,\mathrm{d}s\right)^T\Bigg)^T,\\\overline{\eta}(t)={}&\Bigg(\varepsilon^T(t),\widehat{F}_1^T(\varepsilon(t)),\widehat{F}_2^T(\varepsilon(t-\sigma(t))),e^T(t-\sigma_1(t)),\\&\widehat{G}^T(\dot{e}(t-\sigma_2(t))),\\&\left(\int_{-\infty}^{t}\left(I_l\otimes\overline{h}(t-s)\right)\widehat{\mathrm{K}}(e(s))\,\mathrm{d}s\right)^T\Bigg)^T.\end{aligned}\tag{53}$$

Let $\rho=\min\{\rho_1,\rho_2\}$, where $\rho_1=-\min\{\rho_{\min}(\Omega_r),\ r\in\aleph\}$, $\rho_2=-\min\{\rho_{\min}(\overline{\Omega}_r),\ r\in\aleph\}$, then $\rho>0$ and

$$\begin{aligned}\dot{V}&\le-\rho_1\sum_{i=1}^{l}e_i^T(t)e_i(t)-\rho_2\sum_{j=1}^{m}\varepsilon_j^T(t)\varepsilon_j(t)\\&\le-\rho\left[\sum_{i=1}^{l}e_i^T(t)e_i(t)+\sum_{j=1}^{m}\varepsilon_j^T(t)\varepsilon_j(t)\right].\end{aligned}\tag{54}$$

The following proof is similar to that of Theorem 4 and is omitted here. □

4. Simulations

In this section, numerical examples are provided to demonstrate the validity of the synchronization criteria obtained

in the previous sections. Consider the following network as drive network:

$$\begin{aligned}\dot{x}_i(t) = \sum_{r=1}^{N}\xi_r(t,\lambda) &\times\Bigg[-D_r x_i + R_{r1} f_1(x_i(t)) \\ &+ R_{r2} f_2(x_i(t-\tau(t))) \\ &+ I_r + \sum_{j=1}^{m} a_{rij} y_j(t-\tau_1(t)) \\ &+ \sum_{j=1}^{m} b_{rij} g(\dot{y}_j(t-\tau_2(t))) \\ &+ \sum_{j=1}^{m} c_{rij}\int_{-\infty}^{t} h(t-s)\, k(y_j(s))\,\mathrm{d}s\Bigg], \\ &\qquad i = 12,\ldots,l,\\ \dot{y}_j(t) = \sum_{r=1}^{N}\xi_r(t,\lambda) &\times\Bigg[-\overline{D}_r y_j + \overline{R}_{r1}\overline{f}_1(y_j(t)) \\ &+ \overline{R}_{r2}\overline{f}_2(y_j(t-\sigma(t))) \\ &+ J_r + \sum_{i=1}^{l}\overline{a}_{rji} x_i(t-\sigma_1(t)) \\ &+ \sum_{i=1}^{l}\overline{b}_{rji}\overline{g}(\dot{x}_i(t-\sigma_2(t))) \\ &+ \sum_{i=1}^{l}\overline{c}_{rji}\int_{-\infty}^{t}\overline{h}(t-s)\,\overline{k}(x_i(s))\,\mathrm{d}s\Bigg], \\ &\qquad j = 1,2,\ldots,m,\end{aligned} \tag{55}$$

where $x_i(t), y_j(t) \in \mathrm{R}^2$, $l = 3$, and $m = 3$. $f_1(z(t)) = 0.1(\tanh(z_1(t)), \tanh(z_2(t)))^T$, $z(t) = (z_1(t), z_2(t))^T$, $\overline{f}_1 = g = \overline{g} = k = \overline{k} = f_2 = \overline{f}_2 = f_1$, and $h(t) = \overline{h}(t) = \mathrm{diag}(e^{-t}, e^{-t})$. Choose time delays $\tau(t) = 1 + 0.4\sin t$, $\tau_1(t) = 2 + 0.2\arctan(t)$, $\tau_2(t) = 0.6 + 0.5\cos t$, and $\sigma(t) = 1 + 0.8\sin t$, $\sigma_1(t) = 0.7 + 0.1\cos t$, $\sigma_2(t) = 0.5 + (0.3e^t/(1+e^t))$. We define a switching rule $\lambda : t \in [0,+\infty) \to \{1,2\}$, $\lambda(t) = \mathrm{int}(t) \bmod 2 + 1$. The other parameters are as follows:

$$\begin{gathered}D_1 = \begin{pmatrix}1.8 & 0\\ 0 & 4\end{pmatrix}, \qquad R_{11} = \begin{pmatrix}-1 & 1\\ 0 & 0.2\end{pmatrix},\\ R_{12} = \begin{pmatrix}1 & 0.5\\ 0.6 & -1\end{pmatrix}, \qquad I_1 = J_1 = (1,2)^T,\\ A_1 = (a_{1ij}) = \begin{pmatrix}-2 & -2 & 0\\ 0 & 2 & -2\\ 1 & 1 & -2\end{pmatrix},\\ B_1 = (b_{1ij}) = \begin{pmatrix}-0.2 & 0 & -0.2\\ 0.1 & -0.4 & 0.3\\ 0.2 & 0.1 & -0.3\end{pmatrix},\\ C_1 = (c_{1ij}) = \begin{pmatrix}1 & -1 & 0\\ 1 & 1 & -2\\ -1 & 0 & 1\end{pmatrix},\\ \overline{D}_1 = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, \qquad \overline{R}_{11} = \begin{pmatrix}-0.3 & 1\\ 0.2 & 0.3\end{pmatrix},\\ \overline{R}_{12} = \begin{pmatrix}0.3 & 0.4\\ 0.6 & -0.5\end{pmatrix}, \qquad \overline{A}_1 = (\overline{a}_{1ji}) = \begin{pmatrix}4 & 0 & -4\\ 1 & 1 & -2\\ 1 & 0 & -1\end{pmatrix},\\ \overline{B}_1 = (\overline{b}_{1ji}) = \begin{pmatrix}0.1 & 0 & -0.1\\ 0.2 & -0.3 & 0.1\\ 0.1 & 0.2 & -0.2\end{pmatrix},\\ \overline{C}_1 = (\overline{c}_{1ji}) = \begin{pmatrix}-3 & 1 & 2\\ 2 & 0 & -2\\ -4 & 0 & 4\end{pmatrix},\\ D_2 = \begin{pmatrix}1.4 & 0\\ 0 & 1.4\end{pmatrix}, \qquad R_{21} = \begin{pmatrix}1 & -1\\ -5 & 3\end{pmatrix},\\ R_{22} = \begin{pmatrix}-1.5 & -0.1\\ -3 & -1\end{pmatrix}, \qquad A_2 = (a_{2ij}) = \begin{pmatrix}-1 & 1 & 0\\ 0 & 2 & -2\\ 1.2 & 1 & -2.2\end{pmatrix},\\ B_2 = (b_{2ij}) = \begin{pmatrix}0.1 & -0.1 & 0\\ 0.1 & -0.5 & 0.4\\ 0.2 & 0 & -0.2\end{pmatrix},\\ C_2 = (c_{2ij}) = \begin{pmatrix}2 & -2 & 0\\ 3 & -1 & -2\\ 0 & -1 & 1\end{pmatrix},\\ \overline{D}_2 = \begin{pmatrix}1.2 & 0\\ 0 & 1.2\end{pmatrix}, \qquad \overline{R}_{21} = \begin{pmatrix}-0.3 & 1\\ -4 & 1\end{pmatrix},\\ \overline{R}_{22} = \begin{pmatrix}0.3 & 0.4\\ -2 & -1\end{pmatrix}, \qquad I_2 = J_2 = (3,4)^T,\\ \overline{A}_2 = (\overline{a}_{2ji}) = \begin{pmatrix}1 & -1 & 0\\ 1 & 2 & -3\\ 1 & 1 & -2\end{pmatrix},\\ \overline{B}_2 = (\overline{b}_{2ji}) = \begin{pmatrix}0.2 & -0.2 & 0\\ 0.1 & -0.2 & 0.1\\ 0.3 & 0.1 & -0.4\end{pmatrix},\\ \overline{C}_2 = (\overline{c}_{2ji}) = \begin{pmatrix}-5 & 1 & 4\\ 1 & 1 & -2\\ -1 & 0 & 1\end{pmatrix}.\end{gathered} \tag{56}$$

The response network of drive network (55) is

$$\begin{aligned}\dot{\hat{x}}_i(t) = \sum_{r=1}^{N}\xi_r(t,\lambda) &\times\Bigg[-D_r\hat{x}_i(t)+R_{r1}f_1(\hat{x}_i(t)) \\ &+R_{r2}f_2(\hat{x}_i(t-\tau(t))) \\ &+I_r+\sum_{j=1}^{m}a_{rij}\hat{y}_j(t-\tau_1(t)) \\ &+\sum_{j=1}^{m}b_{rij}g(\dot{\hat{y}}_j(t-\tau_2(t))) \\ &+\sum_{j=1}^{m}c_{rij}\int_{-\infty}^{t}h(t-s)k(\hat{y}_j(s))\,\mathrm{d}s+u_i(t)\Bigg], \\ \dot{\hat{y}}_j(t) = \sum_{r=1}^{N}\xi_r(t,\lambda) &\times\Bigg[-\overline{D}_r\hat{y}_j(t)+\overline{R}_{r1}\overline{f}_1(\hat{y}_j(t)) \\ &+\overline{R}_{r2}\overline{f}_2(\hat{y}_j(t-\sigma(t))) \\ &+J_r+\sum_{i=1}^{l}\overline{a}_{rji}\hat{x}_i(t-\sigma_1(t)) \\ &+\sum_{i=1}^{l}\overline{b}_{rji}\overline{g}(\dot{\hat{x}}_i(t-\sigma_2(t))) \\ &+\sum_{i=1}^{l}\overline{c}_{rji}\int_{-\infty}^{t}\overline{h}(t-s)\overline{k}(\hat{x}_i(s))\,\mathrm{d}s+v_j(t)\Bigg],\end{aligned} \tag{57}$$

where $u_i(t), v_j(t) \in \mathrm{R}^2$.

Let $\gamma_1 = \gamma_2 = \gamma_3 = 15, \eta_1 = \eta_2 = \eta_3 = 16, \alpha = 0.5, \beta = 0.5$, and the feasible solution of the matrix inequalities (15)–(19) by employing MATLAB LMI Toolbox be as follows:

$$\begin{gathered}p = 7.5267, \qquad \overline{p} = 7.8951, \\ P = \begin{pmatrix}12.9508 & 0.2817\\ 0.2817 & 10.7871\end{pmatrix}, \qquad Q = \begin{pmatrix}83.6405 & 3.6674\\ 3.6674 & 68.8593\end{pmatrix}, \\ U = \begin{pmatrix}119.1080 & 0.2216\\ 0.2216 & 55.7869\end{pmatrix}, \\ W = \begin{pmatrix}233.7870 & 0\\ 0 & 223.7801\end{pmatrix}, \\ M_1 = \begin{pmatrix}227.1153 & 0\\ 0 & 202.1669\end{pmatrix}, \\ M_2 = \begin{pmatrix}206.3034 & 0\\ 0 & 217.8341\end{pmatrix}, \\ M_3 = \begin{pmatrix}86.9723 & 0\\ 0 & 41.4980\end{pmatrix}, \qquad \overline{P} = \begin{pmatrix}19.4548 & 0.0336\\ 0.0336 & 11.6060\end{pmatrix}, \\ \overline{Q} = \begin{pmatrix}26.1973 & 0.5411\\ 0.5411 & 9.6902\end{pmatrix}, \qquad \overline{U} = \begin{pmatrix}30.9126 & 0.9850\\ 0.9850 & 22.6634\end{pmatrix}, \\ \overline{W} = \begin{pmatrix}316.6943 & 0\\ 0 & 287.9793\end{pmatrix}, \\ \overline{M}_1 = \begin{pmatrix}183.0419 & 0\\ 0 & 162.4080\end{pmatrix}, \\ \overline{M}_2 = \begin{pmatrix}250.2787 & 0\\ 0 & 409.0398\end{pmatrix}, \\ \overline{M}_3 = \begin{pmatrix}13.8704 & 0\\ 0 & 10.1694\end{pmatrix}.\end{gathered} \tag{58}$$

The initial values are chosen as $x_i(s) = (-5, 9)$, $y_j(s) = (-6, 7)^T$, $\hat{x}_i(s) = 2i(2, 5)^T$, $\hat{y}_j(s) = 3j(2, -1)^T$, and $s \in [-2, 0]$. Clearly, the two coupled networks (55) and (57) satisfy the conditions of Theorem 4. Figure 1 presents the synchronization errors of the state variables between the two networks. The simulation result shows that the synchronization is achieved under the proposed controllers (20). Thus, the proposed synchronization control scheme in Theorem 4 is valid.

Let $\gamma_1 = \gamma_2 = \gamma_3 = 12$, $\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}_3 = 17$, then the feasible solution of the matrix inequalities (44) in Theorem 8 by employing MATLAB LMI Toolbox is as follows:

$$\begin{gathered}P = \begin{pmatrix}0.0251 & 0.0005\\ 0.0005 & 0.0214\end{pmatrix}, \qquad U = \begin{pmatrix}3.7826 & -0.0043\\ -0.0043 & 3.5381\end{pmatrix}, \\ Q = \begin{pmatrix}4.2792 & 0\\ 0 & 3.7789\end{pmatrix}, \\ V = \begin{pmatrix}0.0116 & 0\\ 0 & 0.0079\end{pmatrix}, \qquad M = \begin{pmatrix}3.4521 & 0\\ 0 & 3.1482\end{pmatrix}, \\ W = \begin{pmatrix}3.7071 & 0\\ 0 & 3.3433\end{pmatrix}, \qquad \overline{P} = \begin{pmatrix}0.0298 & 0\\ 0 & 0.0268\end{pmatrix}, \\ \overline{U} = \begin{pmatrix}3.3712 & -0.0132\\ -0.0132 & 3.1782\end{pmatrix}, \qquad \overline{Q} = \begin{pmatrix}7.2228 & 0\\ 0 & 6.3479\end{pmatrix}, \\ \overline{V} = \begin{pmatrix}0.0077 & 0\\ 0 & 0.0052\end{pmatrix}, \qquad \overline{M} = \begin{pmatrix}3.3924 & 0\\ 0 & 3.0142\end{pmatrix}, \\ \overline{W} = \begin{pmatrix}3.4120 & 0\\ 0 & 3.2474\end{pmatrix}.\end{gathered} \tag{59}$$

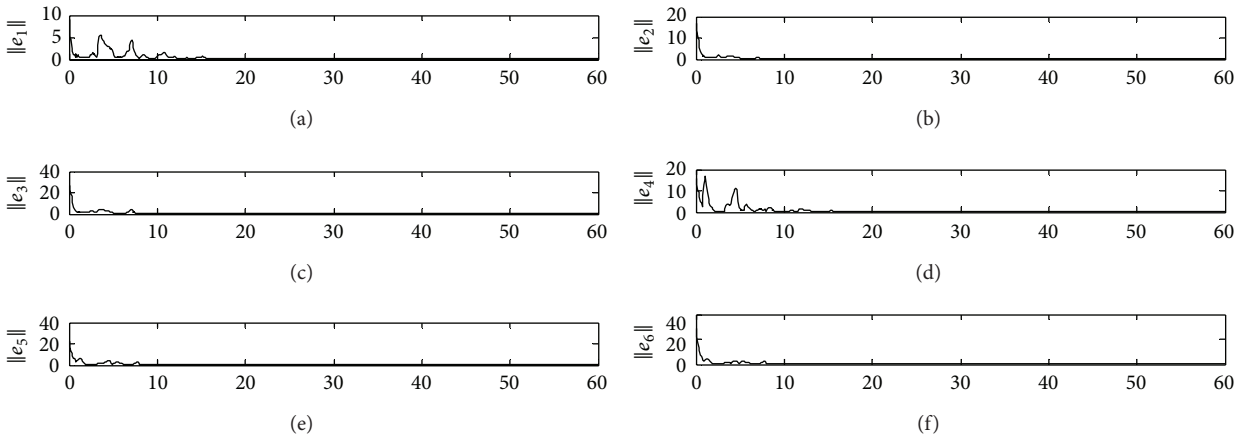


Figure 1: Synchronization errors of BDN (55) and (57) with adaptive feedback controllers (20).

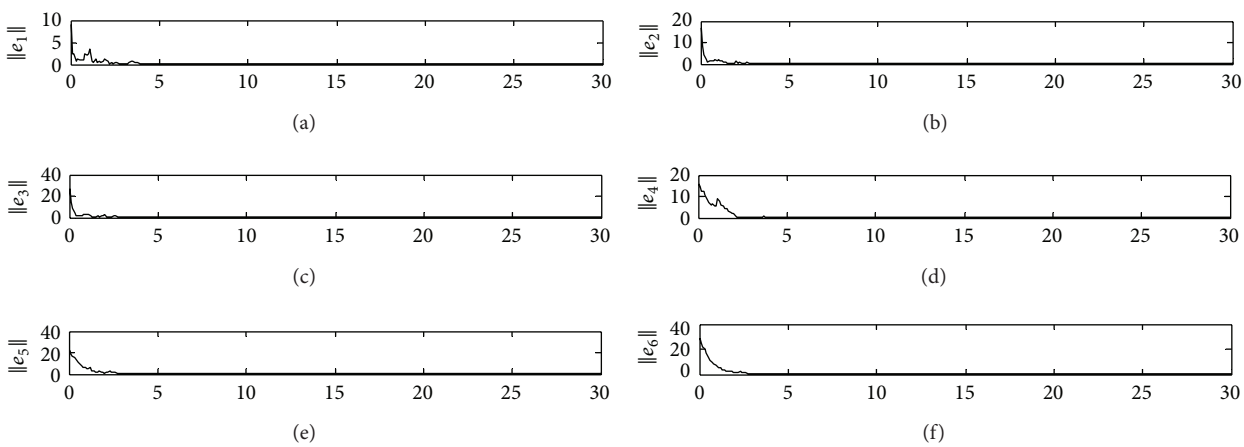


Figure 2: Synchronization errors with adaptive feedback controllers (41).

Using the controllers (41), the simulation result is given in Figure 2, which shows that the proposed synchronization control scheme in Theorem 8 is effective.

5. Conclusions

In this paper, we have proposed a general SCBNN with distributed delays and derivative coupling and investigated the synchronization problem in the two coupled SCBNNs. Using linear matrix inequality (LMI) approach and Barbălat lemma, we have deviated some useful synchronization criteria to ensure the synchronization of these two SCBNNs by constructing effective controllers. Compared with relative previous jobs, the controllers proposed by us are more simple and feasible. Some simulation results have been presented to demonstrate our theoretical results. In our future work, we will consider using pinning control to realize the synchronization of SCBNNs and identify the network topology of the unknown SCBNNs.

Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants nos. 61272530 and 11072059, the Natural Science Foundation of Jiangsu Province of China under Grants no. BK2012741, Specialized Research Fund for the Doctoral Program of Higher Education under Grants no. 20110092110017, and the National Social Science Fund of China under Grant no. 11BGL039.

References

[1] I. Tokuda, T. Nagashima, and K. Aihara, "Global bifurcation structure of chaotic neural networks and its application to traveling salesman problems," *Neural Networks*, vol. 10, no. 9, pp. 1673–1690, 1997.

[2] Y. F. Zhang and Z. Y. He, "A secure communication scheme based on cellular neural networks," *Proceedings of the IEEE International Conference on Intelligent Processing Systems*, vol. 1, pp. 521–524, 1997.

[3] T. Kwok and K. A. Smith, "Experimental analysis of chaotic neural network models for combinatorial optimization under a unifying framework," *Neural Networks*, vol. 13, no. 7, pp. 731–744, 2000.

[4] B. Kosko, "Adaptive bidirectional associative memories," *Applied Optics*, vol. 26, no. 23, pp. 4947–4960, 1987.

[5] B. Kosko, "Bi-directional associative memories," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 18, no. 1, pp. 49–60, 1988.

[6] J. Liang and J. Cao, "Exponential stability of continuous-time and discrete-time bidirectional associative memory networks with delays," *Chaos, Solitons and Fractals*, vol. 22, no. 4, pp. 773–785, 2004.

[7] S. Arik, "Global asymptotic stability analysis of bidirectional associative memory neural networks with time delays," *IEEE Transactions on Neural Networks*, vol. 16, no. 3, pp. 580–586, 2005.

[8] J. H. Ge and J. Xu, "Synchronization and synchronized periodicsolution in a simplified five-neuron BAM neural network with delays," *Neurocomputing*, vol. 74, no. 6, pp. 993–999, 2011.

[9] B. Zhou and Q. K. Song, "Stability and Hopf bifurcation analysis of a tri-neuron BAM neural network with distributed delay," *Neurocomputing*, vol. 82, no. 1, pp. 69–83, 2012.

[10] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[11] J. Zhao and M. W. Spong, "Hybrid control for global stabilization of the cart-pendulum system," *Automatica*, vol. 37, no. 12, pp. 1941–1951, 2001.

[12] K. S. Narendra, O. A. Driollet, M. Feiler, and K. George, "Adaptive control using multiple models, switching and tuning," *International Journal of Adaptive Control and Signal Processing*, vol. 17, no. 2, pp. 87–102, 2003.

[13] D. Liberzon, *Switching in Systems and Control*, Birkhauser, Boston, Mass, USA, 2003.

[14] Z. D. Sun and S. S. Ge, "Analysis and synthesis of switched linear control systems," *Automatica*, vol. 41, no. 2, pp. 181–195, 2005.

[15] Y. Tsividis, "Switched neural networks," United States Patent, Patent number 4873661, 1989.

[16] T. X. Brown, "Neural networks for switching," *IEEE Communications Magazine*, vol. 27, no. 11, pp. 72–81, 1989.

[17] M. Muselli, *Gene Selection Through Switched Neural Networks*, Contribution to NETTAB 03, Bologna, Italy, 2003.

[18] D. Ding, Z. Wang, B. Shen, and H. Shu, "H-infinity state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 725–736, 2012.

[19] V. Perez-Munuzuri, V. Perez-Villar, and L. O. Chua, "Autowaves for image processing on a two-dimensional CNN array of excitable nonlinear circuits: flat and wrinkled labyrinths," *IEEE Transactions on Circuits and Systems I*, vol. 40, no. 3, pp. 174–181, 1993.

[20] B. Hen, Z. Wang, Y. Hung, and G. YChesi, "Distributed H-infinity state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[21] M. E. J. Newman, "Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality," *Physical Review E*, vol. 64, no. 1, Article ID 016132, 2001.

[22] D. H. Wang, L. Zhou, and Z. R. Di, "Bipartite producer-consumer networks and the size distribution of firms," *Physica A*, vol. 363, no. 2, pp. 359–366, 2006.

[23] Z. Jia, H. Wang, J. Wang, and Y. Li, "The weighted identification of a bipartite-graph complex dynamicalnetwork based on adaptive synchronization," *Control Theory & Applications*, vol. 27, no. 1, pp. 107–110, 2010.

[24] W. L. Lu and T. P. Chen, "Synchronization of coupled connected neural networks with delays," *IEEE Transactions on Circuits and Systems I*, vol. 51, no. 12, pp. 2491–2503, 2004.

[25] L. Wang, H. P. Dai, H. Dong, Y. H. Shen, and Y. X. Sun, "Adaptive synchronization of weighted complex dynamical networks with coupling time-varying delays," *Physics Letters A*, vol. 372, no. 20, pp. 3632–3639, 2008.

[26] H. Dong, Z. Wang, J. Lam, and H. Gao, "Fuzzy-model-based robust fault detection with stochastic mixed time-delays and successive packet dropouts," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 42, no. 2, pp. 365–376, 2012.

[27] T. Liu, J. Zhao, and D. J. Hill, "Synchronization of complex delayed dynamical networks with nonlinearly coupled nodes," *Chaos, Solitons and Fractals*, vol. 40, no. 3, pp. 1506–1519, 2009.

[28] Y. H. Xu, W. N. Zhou, J. A. Fang, and W. Sun, "Adaptive synchronization of the complex dynamical network with non-derivative and derivative coupling," *Physics Letters A*, vol. 374, no. 15-16, pp. 1673–1677, 2010.

[29] Q. X. Bian and H. X. Yao, "Adaptive synchronization of bipartite dynamical networks with distributed delays and nonlinear derivative coupling," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 10, pp. 4089–4098, 2011.

[30] B. Shen, Z. D. Wang, and X. H. Liu, "Sampled-data synchronization control of complex dynamical networks with stochastic sampling," *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2644–2650, 2012.

[31] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. S. Zhou, "Synchronization in complex networks," *Physics Reports*, vol. 469, no. 3, pp. 93–153, 2008.

[32] G. Solís-Perales, E. Ruiz-Velázquez, and D. Valle-Rodríguez, "Synchronization in complex networks with distinct chaotic nodes," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 6, pp. 2528–2535, 2009.

[33] W. W. Yu, J. D. Cao, and W. L. Lu, "Synchronization control of switched linearly coupled neural networks with delay," *Neurocomputing*, vol. 73, no. 4–6, pp. 858–866, 2010.

[34] L. Huang, *Linear Algebra in Systems and Control Theory*, Science Press, Beijing, China, 1984.

[35] E. N. Sanchez and J. P. Perez, "Input-to-state stability analysis for dynamic neural networks," *IEEE Transactions on Circuits and Systems I*, vol. 46, no. 11, pp. 1395–1398, 1999.

[36] V. M. Popov, *Hyperstability of Control Systems*, Springer, New-York, NY, USA, 1973.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 961870, 12 pages
http://dx.doi.org/10.1155/2013/961870

Research Article

Stabilization and Controller Design of 2D Discrete Switched Systems with State Delays under Asynchronous Switching

Shipei Huang,[1] Zhengrong Xiang,[1] and Hamid Reza Karimi[2]

[1] *School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China*
[2] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*

Correspondence should be addressed to Zhengrong Xiang; xiangzr@mail.njust.edu.cn

Received 22 February 2013; Accepted 14 April 2013

Academic Editor: Jun Hu

Copyright © 2013 Shipei Huang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is concerned with the problem of robust stabilization for a class of uncertain two-dimensional (2D) discrete switched systems with state delays under asynchronous switching. The asynchronous switching here means that the switching instants of the controller experience delays with respect to those of the system. The parameter uncertainties are assumed to be norm-bounded. A state feedback controller is proposed to guarantee the exponential stability. The dwell time approach is utilized for the stability analysis and controller design. A numerical example is given to illustrate the effectiveness of the proposed method.

1. Introduction

Two-dimensional (2D) systems have attracted considerable attention for several decades due to their numerous applications in many areas, such as multidimensional digital filtering, linear image processing, signal processing and process control [1–3]. It is well known that 2D systems can be represented by different models such as Roesser model, Fornasini-Marchesini model, and Attasi model. The issues of stability analysis and control synthesis of these systems have been studied in [4–8]. Considering that time delays frequently occur in practical systems and are often the source of instability, many authors have devoted their energies to studying time-delay systems. Recently, many results on delay systems have been reported in the literature. For example, the delay-fractional approach has been utilized to deal with discrete time-delay systems in [9–11]. The stability of 2D discrete systems with state delays has been investigated in [12–16].

On the other hand, because of their wide applications in many fields, such as mechanical systems, automotive industry, aircraft and air traffic control, and switched power converters, switched systems have also received considerable attention during the past few decades. A switched system is a hybrid system which consists of a finite number of continuous-time or discrete-time subsystems and a switching signal specifying the switch between these subsystems. The stability and stabilization problems have been extensively studied in [17–25].

In many modelling problems of physical processes, a 2D switching representation is needed. One can cite a 2D physically based model for advanced power bipolar devices [26] and heat flux switching, and modulating in a thermal transistor [27]. This class of systems can correspond to 2D state space or 2D time space switched systems. Recently, there are a few reports on 2D switched systems. Benzaouia et al. firstly studied the stabilization problem of 2D discrete switched systems with arbitrary switching sequences in [28, 29]. By using the common Lyapunov function method and multiple Lyapunov functions method, two different sufficient conditions for the existence of state feedback controllers were proposed. In [30], the authors first extended the concept of average dwell time to 2D switched systems and designed a switching signal to guarantee the exponential stability of delay-free 2D switched systems. It should be pointed out that a very common assumption in [30] is that the controllers are switched synchronously with the switching of system modes, which is quite unpractical. As stated in [31, 32], there inevitably exists asynchronous switching in actual operation that is, the switching instants of the controllers exceed or lag behind those of system modes. Thus, it is necessary to

consider asynchronous switching for efficient control design. Some results on the control synthesis for switched systems under asynchronous switching have been proposed in [33–36]. However, to the best of our knowledge, the stabilization problem for 2D switched systems under asynchronous switching has not been yet investigated to date, especially for 2D switched systems with state delays, which motivates our present study.

In this paper, we are interested in designing a stabilizing controller for 2D discrete switched delayed systems represented by a model of Roesser type under asynchronous switching such that the corresponding closed-loop systems are exponentially stable. The dwell time approach is utilized for the stability analysis and controller design. The main contributions of this paper can be summarized as follows: (i) the asynchronous stabilization problem is for the first time addressed in the paper; (ii) an exponential stability criterion is established for 2D switched systems with state delays; and (iii) an asynchronous switching controller design scheme is proposed to guarantee the exponential stability of the resulting closed-loop system.

This paper is organized as follows. In Section 2, problem formulation and some necessary lemmas are given. In Section 3, based on the dwell time approach, stability and stabilization for 2D discrete switched systems with state delays are addressed. Then, a sufficient condition for the existence of a stabilizing controller for such 2D discrete switched systems under asynchronous switching is derived in terms of a set of matrix inequalities. A numerical example is provided to illustrate the effectiveness of the proposed approach in Section 4. Concluding remarks are given in Section 5.

Notations. Throughout this paper, the superscript "T" denotes the transpose, and the notation $X \geq Y$ ($X > Y$) means that the matrix $X - Y$ is positive semidefinite (positive definite, resp.). $\|\cdot\|$ denotes the Euclidean norm. I represents the identity matrix with an appropriate dimension. I_h is the identity matrix with n_1 dimension and I_v is the identity matrix with n_2 dimension. diag$\{a_i\}$ denotes the diagonal matrix with the diagonal elements a_i, and $i = 1, 2, \ldots, n$. X^{-1} denotes the inverse of X. The asterisk $*$ in a matrix is used to denote the term that is induced by symmetry. The set of all nonnegative integers is represented by Z_+.

2. Problem Formulation and Preliminaries

Consider the following uncertain 2D discrete switched systems with state delays:

$$\begin{aligned}&\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} \\ &\quad = \widehat{A}^{\sigma(i,j)} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \widehat{A}_d^{\sigma(i,j)} \begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v) \end{bmatrix} \\ &\qquad + \widehat{B}^{\sigma(i,j)} u(i,j),\end{aligned} \tag{1}$$

where $x^h(i,j) \in R^{n_1}$ and $x^v(i,j) \in R^{n_2}$ are the horizontal state and the vertical state, respectively, $x(i,j)$ is the whole state in R^n with $n = n_1 + n_2$, and $u(i,j) \in R^m$ is the control input. i and j are integers in Z_+. $\sigma(i,j) : Z_+ \times Z_+ \to \underline{N} = \{1, 2, \ldots, N\}$ is the switching signal. N is the number of subsystems, and $\sigma(i,j) = k$ means that the kth subsystem is activated. d_h and d_v are constant delays along horizontal and vertical directions, respectively. $\widehat{A}^k$ and $\widehat{A}_d^k$ ($k \in \underline{N}$) are uncertain real-valued matrices with appropriate dimensions and are assumed to be of the form

$$\begin{aligned}\widehat{A}^k &= A^k + H^k F^k(i,j) E_1^k, \\ \widehat{A}_d^k &= A_d^k + H^k F^k(i,j) E_2^k, \\ \widehat{B}^k &= B^k + H^k F^k(i,j) E_3^k,\end{aligned} \tag{2}$$

with

$$\begin{aligned}A^k &= \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \qquad A_d^k = \begin{bmatrix} A_{d11}^k & A_{d12}^k \\ A_{d21}^k & A_{d22}^k \end{bmatrix}, \\ H^k &= \begin{bmatrix} H_1^k \\ H_2^k \end{bmatrix}, \qquad E_1^k = \begin{bmatrix} E_{11}^k \\ E_{12}^k \end{bmatrix}, \\ E_2^k &= \begin{bmatrix} E_{21}^k \\ E_{22}^k \end{bmatrix}, \qquad E_3^k = \begin{bmatrix} E_{31}^k \\ E_{32}^k \end{bmatrix},\end{aligned} \tag{3}$$

where matrices $A_{11}^k \in R^{n_1 \times n_1}$, $A_{12}^k \in R^{n_1 \times n_2}$, $A_{21}^k \in R^{n_2 \times n_1}$, $A_{22}^k \in R^{n_2 \times n_2}$, $A_{d11}^k \in R^{n_1 \times n_1}$, $A_{d12}^k \in R^{n_1 \times n_2}$, $A_{d21}^k \in R^{n_2 \times n_1}$, $A_{d22}^k \in R^{n_2 \times n_2}$, $H_1^k, H_2^k, E_{11}^k, E_{12}^k, E_{21}^k, E_{22}^k, E_{31}^k, E_{32}^k$ are constant matrices. $F^k(i,j)$ ($k \in \underline{N}$) is an unknown matrix representing parameter uncertainty and satisfies

$$F^{kT}(i,j) F^k(i,j) \leq I. \tag{4}$$

The boundary conditions are given by

$$\begin{aligned}x^h(i,j) &= h_{ij}, \quad \forall 0 \leq j \leq z_1,\ -d_h \leq i \leq 0, \\ x^h(i,j) &= 0, \quad \forall j > z_1,\ -d_h \leq i \leq 0, \\ x^v(i,j) &= v_{ij}, \quad \forall 0 \leq i \leq z_2,\ -d_v \leq j \leq 0, \\ x^v(i,j) &= 0, \quad \forall i > z_2,\ -d_v \leq j \leq 0,\end{aligned} \tag{5}$$

where $z_1 < \infty$ and $z_2 < \infty$ are positive integers, and h_{ij} and v_{ij} are given vectors.

In this paper, it is assumed that (1) at each time only one subsystem is active; (2) the switching signal is not known a priori, but its value is available at each sampling period; (3) the switch occurs only at each sampling point of i or j. The switching sequence can be described as follows:

$$\begin{aligned}&((i_0, j_0), \sigma(i_0, j_0)), ((i_1, j_1), \sigma(i_1, j_1)), \ldots, \\ &((i_\kappa, j_\kappa), \sigma(i_\kappa, j_\kappa)), \ldots,\end{aligned} \tag{6}$$

where (i_κ, j_κ) denotes the κth switching instant. It should be noted that the value of $\sigma(i,j)$ only depends on $i + j$ (see [29, 30]).

However, in actual operation, there inevitably exists asynchronous switching between the controller and the system. Without loss of generality, we only consider the case where the switching instants of the controller experience delays with respect to those of the system. Let $\sigma'(i, j)$ denote the switching signal of the controller. Denoting $m_\kappa = i_\kappa + j_\kappa$, $\Delta m_\kappa = \Delta i_\kappa + \Delta j_\kappa$, $\kappa = 1,2,\ldots$, then the switching points of the controller can be described as

$$(i_0, j_0), (i_1 + \Delta i_1, j_1 + \Delta j_1), \ldots, (i_\kappa + \Delta i_\kappa, j_\kappa + \Delta j_\kappa), \ldots, \tag{7}$$

where Δi_κ and Δj_κ represent the delayed period along horizontal and vertical directions, respectively. $\Delta m_\kappa < \inf(m_{\kappa+1} - m_\kappa)$ is said to be the mismatched period.

Remark 1. Similar to the one-dimensional switched system case [33–36], the mismatched period $\Delta m_\kappa < \inf(m_{\kappa+1} - m_\kappa)$ guarantees that there always exists a period in which the controller and the system operate synchronously. This period is said to be the matched period in the later section.

Remark 2. If there is only one subsystem in system (1), it will degenerate to the following 2D system in Roesser model with state delays [12]:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \widehat{A} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \widehat{A}_d \begin{bmatrix} x^h(i-d_h, j) \\ x^v(i, j-d_v) \end{bmatrix}, \tag{8}$$

Definition 3. System (1) is said to be exponentially stable under $\sigma(i, j)$ if, for a given $z \geq 0$, there exist positive constants c and η, such that

$$\sum_{i+j=D} \|x(i, j)\|^2 \leq \eta e^{-c(D-z)} \sum_{i+j=z} \|x(i, j)\|_C^2, \tag{9}$$

holds for all $D \geq z$, where

$$\sum_{i+j=z} \|x(i, j)\|_C^2 \triangleq \sup_{\substack{-d_h \leq \theta_h \leq 0, \\ -d_v \leq \theta_v \leq 0}} \sum_{i+j=z} \left(\left\|x^h(i-\theta_h, j)\right\|^2 + \left\|x^v(i, j-\theta_v)\right\|^2 \right). \tag{10}$$

Remark 4. From Definition 3, it is easy to see that when z is given, $\sum_{i+j=z} \|x(i, j)\|_C^2$ will be bounded and $\sum_{i+j=D} \|x(i, j)\|^2$ will tend to be zero exponentially as D goes to infinity, which implies that $\|x(i, j)\|$ tends to be zero.

Definition 5. Let (i_κ, j_κ) denote the κth switching point and $(i_{\kappa+1}, j_{\kappa+1})$ denote the $(\kappa+1)$th switching point. Denote $m_\kappa = i_\kappa + j_\kappa$, $m_{\kappa+1} = i_{\kappa+1} + j_{\kappa+1}$, $\tau = \inf(m_{\kappa+1} - m_\kappa)$, then τ is called the dwell time.

Definition 6 (see [30]). For any $i + j = D \geq z = i_z + j_z$, let $N_{\sigma(i,j)}(z, D)$ denote the switching number of $\sigma(i, j)$ on the interval $[z, D)$. If

$$N_{\sigma(i,j)}(z, D) \leq N_0 + \frac{D - z}{\tau_a} \tag{11}$$

holds for given $N_0 \geq 0$ and $\tau_a \geq 0$, then the constant τ_a is called the average dwell time and N_0 is the chatter bound.

Lemma 7 (see [37]). *For a given matrix* $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, *where* S_{11} *and* S_{22} *are square matrices, the following conditions are equivalent:*

(i) $S < 0$,

(ii) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

(iii) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Lemma 8 (see [38]). *Let U, V, W, and X be real matrices of appropriate dimensions with X satisfying $X = X^T$, then for all $V^T V \leq I$,*

$$X + UVW + W^T V^T U^T < 0, \tag{12}$$

if and only if there exists a scalar ε such that

$$X + \varepsilon U U^T + \varepsilon^{-1} W^T W < 0. \tag{13}$$

3. Main results

3.1. Stability Analysis. In this section, we first focus on the stability analysis for system (8).

Lemma 9. *Consider system* (8) *with the boundary conditions* (5), *suppose that there exists a C^1 function $V : R^n \to R$. For a given positive constant α, if there exist positive definite symmetric matrices $P = \mathrm{diag}\{P_h, P_v\}$ and $Q = \mathrm{diag}\{Q_h, Q_v\}$ with appropriate dimensions, such that the following inequality holds:*

$$\begin{bmatrix} Q - \alpha P & 0 & \widehat{A}^T P \\ * & -\Lambda_1 Q & \widehat{A}_d^T P \\ * & * & -P \end{bmatrix} < 0, \tag{14}$$

where $\Lambda_1 = \mathrm{diag}\{\alpha^{d_h} I_h, \alpha^{d_v} I_v\}$, then along the trajectory of systems (8), *the following inequality holds for any $D \geq D'$:*

$$\sum_{i+j=D} V(x(i, j)) < \alpha^{D-D'} \sum_{i+j=D'} V(x(i, j)), \tag{15}$$

where $D' \geq z$ and $z = \max\{z_1, z_2\}$.

Proof. See appendix for the detailed proof, it is omitted here. □

Remark 10. Lemma 9 provides a method for the estimation of the C^1 function V which will be used to design the controller for system (1) under asynchronous switching. It is worth pointing out that when $0 < \alpha < 1$, (15) presents the decay estimation of the C^1 function V, and when $\alpha > 1$, (15) shows the growth estimation of the C^1 function V.

Remark 11. It is noted that the block diagonal matrices P and Q are often chosen as the matrices for Lyapunov functional analysis of 2D systems by the Roesser model in the existing literature (see, e.g., [12, 13, 15]). This is because 2D systems in Roesser model may be unstable when the block diagonal matrices are not chosen, which has been shown in the literature [4, 5].

3.2. Controller Design. Consider system (1), under the following asynchronous switching controller:

$$u(i,j) = K^{\sigma'(i,j)} x(i,j), \qquad K^{\sigma'(i,j)} = \left[K_1^{\sigma'(i,j)} \quad K_2^{\sigma'(i,j)}\right], \tag{16}$$

where $K_1^{\sigma'(i,j)} \in R^{m\times n_1}$ and $K_2^{\sigma'(i,j)} \in R^{m\times n_2}$, the corresponding closed-loop system is given by

$$\begin{aligned}\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1)\end{bmatrix} &= \left(\widehat{A}^{\sigma(i,j)} + \widehat{B}^{\sigma(i,j)} K^{\sigma'(i,j)}\right)\begin{bmatrix} x^h(i,j) \\ x^v(i,j)\end{bmatrix} \\ &\quad + \widehat{A}_d^{\sigma(i,j)}\begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v)\end{bmatrix}.\end{aligned} \tag{17}$$

Without loss of generality, we denote $\sigma(i_\kappa, j_\kappa) = k \in \underline{N}$ and $\sigma(i_{\kappa+1}, j_{\kappa+1}) = l \in \underline{N}$, then due to the existence of asynchronous switching, we can obtain from (7) that

$$\begin{aligned}\sigma'(i_\kappa + \Delta i_\kappa, j_\kappa + \Delta j_\kappa) &= k, \\ \sigma'(i_{\kappa+1} + \Delta i_{\kappa+1}, j_{\kappa+1} + \Delta j_{\kappa+1}) &= l.\end{aligned} \tag{18}$$

In many actual applications, it is always difficult to verify each asynchronous period in advance, but the maximal asynchronous period can be easily predicted offline. Let $\Delta = \max_{\kappa=1,2,\ldots}(\Delta m_\kappa)$ denote the maximal asynchronous period, then we can get the following result.

Theorem 12. *Consider system* (1), *for given positive constants* $\alpha < 1$ *and* $\beta > 1$, *if there exist positive definite symmetric matrices* $X^k = \mathrm{diag}\{X_h^k, X_v^k\}$, $Y^k = \mathrm{diag}\{Y_h^k, Y_v^k\}$, $X^{kl} = \mathrm{diag}\{X_h^{kl}, X_v^{kl}\}$, $Y^{kl} = \mathrm{diag}\{Y_h^{kl}, Y_v^{kl}\}$ *and* W^k *with appropriate dimensions, and positive scalars* ε_k *and* ε_{kl} *such that, for* $k, l \in \underline{N}$, $k \neq l$, *the following inequalities hold,*

$$\begin{bmatrix} -\alpha X^k & 0 & \left(A^k X^k + B^k W^k\right)^T & X^k & \left(E_1^k X^k + E_3^k W^k\right)^T \\ * & -\Lambda_1 Y^k & \left(A_d^k Y^k\right)^T & 0 & \left(E_2^k Y^k\right)^T \\ * & * & -X^k + \varepsilon_k H^k H^{kT} & 0 & 0 \\ * & * & * & -Y^k & 0 \\ * & * & * & * & -\varepsilon_k I \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} -\beta X^{kl} & 0 & \left(A^l X^{kl} + B^l K^k X^{kl}\right)^T & X^{kl} & \left(E_1^l X^{kl} + E_3^l K^k X^{kl}\right)^T \\ * & -\Lambda_2 Y^{kl} & \left(A_d^l Y^{kl}\right)^T & 0 & \left(E_2^l Y^{kl}\right)^T \\ * & * & -X^{kl} + \varepsilon_{kl} H^l H^{lT} & 0 & 0 \\ * & * & * & -Y^{kl} & 0 \\ * & * & * & * & -\varepsilon_{kl} I \end{bmatrix} < 0, \tag{20}$$

where $\Lambda_2 = \mathrm{diag}\{\beta^{d_h} I_h, \beta^{d_v} I_v\}$, *then under the following switching controller:*

$$u(i,j) = K^{\sigma'(i,j)} x(i,j), \qquad K^k = W^k\left(X^k\right)^{-1}, \tag{21}$$

and the following average dwell time scheme:

$$\tau_a > \tau_a^* = \frac{\Delta(\ln\beta - \ln\alpha) + \ln(\mu_1\mu_2)}{-\ln\alpha}, \tag{22}$$

the resulting closed-loop system (17) *is exponentially stable, where* $\mu = (\alpha/\beta)^{\overline{d}}$, $\overline{d} = \max\{d_h, d_v\}$, *and* $\mu_1\mu_2\mu \geq 1$ *satisfies*

$$\begin{aligned} X_l^{-1} &\leq \mu_1 X_{kl}^{-1}, \qquad & X_{kl}^{-1} &\leq \mu_2 X_k^{-1}, \\ Y_l^{-1} &\leq \mu_1 Y_{kl}^{-1}, \qquad & Y_{kl}^{-1} &\leq \mu_2 \mu Y_k^{-1}. \end{aligned} \tag{23}$$

Proof. See the appendix. □

Remark 13. In Theorem 12, we propose a sufficient condition for the existence of a state feedback controller such that the resulting closed-loop system (17) is exponentially stable. It is worth noting that this condition is obtained by using the average dwell time approach. Here, α plays a key role in controlling the rate of decaying of the system in the matched period, and β plays a key role in controlling the rate of increasing of the system in the mismatched period. From (22), we know that the value of τ_a^* is proportional to the value of Δ, which also means that it needs much larger dwell time to guarantee the stability of the considered system when there exists much larger asynchronous period.

Remark 14. It is noticed that (19) and (20) are mutually dependent. Therefore, we can firstly solve (19) to obtain the solution of matrices X^k, Y^k, and W^k. Then, (20) can be transformed into the LMI by substituting $K^k = W^k(X^k)^{-1}$ into it. By adjusting the parameters α and β, we can find a feasible solution of X^k, Y^k, W^k, X^{kl}, and Y^{kl} such that (19) and (20) hold.

The procedure of the controller design for system (1) can be given as follows.

Step 1. Given system matrices and positive scalar $0 < \alpha < 1$, and by solving LMI (19), we can get the feasible solution of matrices X^k, Y^k, and W^k and positive scalar ε_k then, the controller can be obtained from (21).

Step 2. Substituting K^k into (20), we can find the feasible solution of X^{kl}, Y^{kl}, and ε_{kl} such that (20) holds by adjusting the parameter β.

Step 3. From (23), we can obtain μ_1 and μ_2 satisfying $\mu_1\mu_2\mu \geq 1$.

Step 4. Taking the value of Δ, we can compute the value of τ_a^* by (22).

Remark 15. From the procedure above, it can be seen that the proposed method is feasible. We can find the desire controller and switching signal according to the procedure. However, we would like to point out that there still exists the conservatism to some extent for this method because (19) and (20) are mutually dependent, which brings about the increase of the complex computation. The result can be improved by adopting the method presented in [31, 32].

When $\widehat{A}_d^{\sigma(i,j)} = 0$, system (17) degenerates to the following delay-free system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(\widehat{A}^{\sigma(i,j)} + \widehat{B}^{\sigma(i,j)} K^{\sigma'(i,j)}\right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \tag{24}$$

Then, we can get the following result.

Corollary 16. *Consider system* (1) *with* $\widehat{A}_d^{\sigma(i,j)} = 0$, *for given positive constants* $\alpha < 1$ *and* $\beta > 1$, *if there exist positive definite symmetric matrices* $X^k = \mathrm{diag}\{X_h^k, X_v^k\}$ *and* $X^{kl} = \mathrm{diag}\{X_h^{kl}, X_v^{kl}\}$ *with appropriate dimensions, and positive scalars* ε_k *and* ε_{kl} *such that, for* $k, l \in \underline{N}$, $k \neq l$, *the following inequalities hold:*

$$\begin{bmatrix} -\alpha X^k & \left(A^k X^k + B^k W^k\right)^T & \left(E_1^k X^k + E_3^k W^k\right)^T \\ * & -X^k + \varepsilon_k H^k H^{kT} & 0 \\ * & * & -\varepsilon_k I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -\beta X^{kl} & \left(A^l X^{kl} + B^l K^k X^{kl}\right)^T & \left(E_1^l X^{kl} + E_3^l K^k X^{kl}\right)^T \\ * & -X^{kl} + \varepsilon_{kl} H^l H^{lT} & 0 \\ * & * & -\varepsilon_{kl} I \end{bmatrix} < 0, \tag{25}$$

then under the following switching controller:

$$u(i,j) = K^{\sigma'(i,j)} x(i,j), \qquad K^k = W^k\left(X^k\right)^{-1}, \tag{26}$$

and the following average dwell time scheme:

$$\tau_a > \tau_a^* = \frac{\Delta(\ln\beta - \ln\alpha) + \ln(\mu_1\mu_2)}{-\ln\alpha}, \tag{27}$$

the resulting closed-loop system (24) *is exponentially stable, where* $\mu_1\mu_2 \geq 1$ *satisfies*

$$X_l^{-1} \leq \mu_1 X_{kl}^{-1}, \qquad X_{kl}^{-1} \leq \mu_2 X_k^{-1}. \tag{28}$$

Furthermore, it should also be noted that if the criterion in Theorem 12 is satisfied when $\Delta = 0$, which means that the controller and the subsystem are synchronous, in other words, the results presented in Theorem 12 can be reduced to the synchronous case, then we can obtain the following corollary.

Corollary 17. *Consider system* (1) *under synchronous switching, for a given positive scalar* $\alpha < 1$, *if there exist positive definite symmetric matrices* $X^k = \mathrm{diag}\{X_h^k, X_v^k\}$, $Y^k = \mathrm{diag}\{Y_h^k, Y_v^k\}$, *and* W^k, *with appropriate dimensions, and a positive scalar* ε_k, *such that, for* $k \in \underline{N}$, *the following inequality holds:*

$$\begin{bmatrix} -\alpha X^k & 0 & \left(A^k X^k + B^k W^k\right)^T & X^k & \left(E_1^k X^k + E_3^k W^k\right)^T \\ * & -\Lambda_1 Y^k & \left(A_d^k Y^k\right)^T & 0 & \left(E_2^k Y^k\right)^T \\ * & * & -X^k + \varepsilon_k H^k H^{kT} & 0 & 0 \\ * & * & * & -Y^k & 0 \\ * & * & * & * & -\varepsilon_k I \end{bmatrix} < 0, \tag{29}$$

then under the following controller:

$$u(i,j) = K^k x(i,j), \quad K^k = W^k\left(X^k\right)^{-1}, \tag{30}$$

and the following average dwell time scheme:

$$\tau_a > \tau_a^* = \frac{\ln\mu_1}{-\ln\alpha}, \tag{31}$$

the resulting closed-loop system is exponentially stable, where $\mu_1 \geq 1$ *satisfies*

$$X_l^{-1} \leq \mu_1 X_k^{-1}, \qquad Y_l^{-1} \leq \mu_1 Y_k^{-1}. \tag{32}$$

Remark 18. In [30], by using the average dwell time approach, a criterion of exponential stability for a class of 2D discrete delay-free switched systems is developed. However, the focus of our work is on stability analysis and controller design under asynchronous switching, which is different from [30], and this is also the major contribution of our work. In fact, if we let $\Delta = 0$ and do not consider the uncertainties and state delays, then the closed-loop system (24) is the same as (36) in [30]. In this case, Corollary 17 can be reduced to Theorem 2 in [30].

4. Numerical Example

In this section, we present an example to illustrate the effectiveness of the proposed approach.

Consider system (1) with the following parameters:

$$A^1 = \begin{bmatrix} 1 & 1.5 \\ 1 & 0.5 \end{bmatrix}, \qquad A_d^1 = \begin{bmatrix} -0.15 & 0 \\ -0.1 & -0.12 \end{bmatrix}, \tag{33}$$

$$B^1 = \begin{bmatrix} -4.5 & 0 \\ 1 & -3 \end{bmatrix}, \qquad H^1 = \begin{bmatrix} 0.2 & 0.15 \\ 0.1 & 0.2 \end{bmatrix}, \tag{34}$$

$$
\begin{aligned}
&E_1^1 = \begin{bmatrix} -0.2 & 0 \\ -0.2 & -0.2 \end{bmatrix}, \qquad E_2^1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \\
&F^1 = \operatorname{diag}\{\sin(0.5\pi(i+j)), \sin(0.5\pi(i+j))\}, \\
&E_3^1 = \begin{bmatrix} 0.15 & 0 \\ 0.13 & 0.12 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \\
&A_d^2 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.2 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} -5 & 1 \\ -1 & -3 \end{bmatrix}, \\
&H^2 = \begin{bmatrix} 0.2 & 0.25 \\ 0.2 & 0.3 \end{bmatrix}, \qquad E_1^2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} \\
&E_2^2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \qquad E_3^2 = \begin{bmatrix} 0.12 & 0.15 \\ 0.12 & 0.1 \end{bmatrix}, \\
&F^2 = \operatorname{diag}\{\cos(0.5\pi(i+j)), \cos(0.5\pi(i+j))\}, \\
&\qquad d_h = 2, \qquad d_v = 3.
\end{aligned} \tag{35}
$$

The boundary conditions are given as follows:

$$
\begin{aligned}
x^h(0,j) &= \begin{cases} 5, & 0 \le j \le 20, \\ 0, & j > 20, \end{cases} \\
x^v(i,0) &= \begin{cases} 3, & 0 \le i \le 20, \\ 0, & i > 20, \end{cases}
\end{aligned} \tag{36}
$$

where the state dimensions are $n_1 = 1$ and $n_2 = 1$.

Take $\alpha = 0.6$ and $\beta = 1.2$, then solving (19) in Theorem 12 gives rise to

$$
\begin{aligned}
&X^1 = \begin{bmatrix} 36.3903 & 1.0247 \\ 1.0247 & 38.3909 \end{bmatrix}, \qquad X^2 = \begin{bmatrix} 40.0833 & -6.6640 \\ -6.6640 & 34.4516 \end{bmatrix}, \\
&Y^1 = \begin{bmatrix} 96.6213 & -4.5896 \\ -4.5896 & 96.9472 \end{bmatrix}, \qquad Y^2 = \begin{bmatrix} 99.6213 & -8.8472 \\ -8.8472 & 79.5315 \end{bmatrix}, \\
&W^1 = \begin{bmatrix} 8.5866 & 13.0604 \\ 15.5568 & 11.4091 \end{bmatrix}, \qquad W^2 = \begin{bmatrix} 6.9844 & 13.2930 \\ 8.3790 & 4.4234 \end{bmatrix}, \\
&\qquad \varepsilon_1 = 74.2107, \qquad \varepsilon_2 = 73.4149.
\end{aligned} \tag{37}
$$

By (21), K^1 and K^2 can be obtained as follows:

$$
K^1 = \begin{bmatrix} 0.2265 & 0.3341 \\ 0.4194 & 0.2860 \end{bmatrix}, \qquad K^2 = \begin{bmatrix} 0.2463 & 0.4335 \\ 0.2380 & 0.1744 \end{bmatrix}. \tag{38}
$$

Substituting K^1 and K^2 into (20), and solving it, we get the following solution:

$$
\begin{aligned}
&X^{12} = \begin{bmatrix} 0.5400 & -0.0934 \\ -0.0934 & 0.4811 \end{bmatrix}, \qquad X^{21} = \begin{bmatrix} 0.5805 & -0.0926 \\ -0.0926 & 0.5677 \end{bmatrix}, \\
&Y^{12} = \begin{bmatrix} 0.9398 & -0.0418 \\ -0.0418 & 0.8451 \end{bmatrix}, \qquad Y^{21} = \begin{bmatrix} 0.9436 & -0.0340 \\ -0.0340 & 0.8867 \end{bmatrix}, \\
&\qquad \varepsilon_{12} = 0.9011, \qquad \varepsilon_{21} = 0.7725.
\end{aligned} \tag{39}
$$

Then, from (23), we can get that $\mu_1 = 0.0171$ and $\mu_2 = 917.7224$. Taking $\Delta = 2$, it is easy to obtain from (22) that $\tau_a^* = 7.9$. Choosing $\tau_a = 8$, the trajectories of the states $x^h(i,j)$ and $x^v(i,j)$ are shown in Figures 1 and 2, respectively. The system switching signal $\sigma(i,j)$ and the controller switching signal $\sigma'(i,j)$ are shown in Figure 3. One can see that the states of the closed-loop system converge to zero under the asynchronous switching. This demonstrates the effectiveness of the proposed approach.

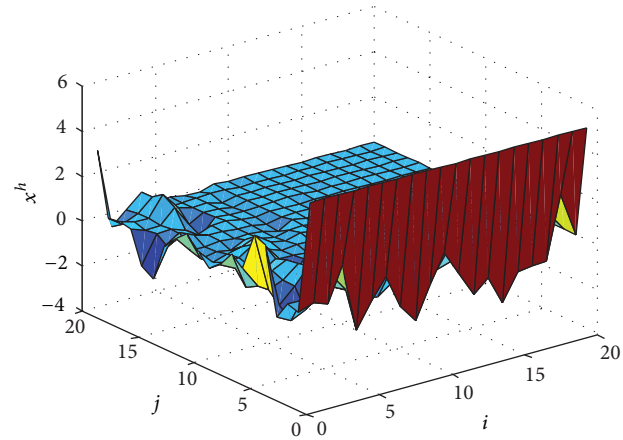


Figure 1: The trajectory of the state $x^h(i,j)$.

5. Conclusions

This paper has investigated the problem of stabilization for a class of 2D discrete switched systems with constant state delays under asynchronous switching. A state feedback controller is proposed to stabilize such system, and the dwell time approach is utilized for the stability analysis and controller design. A sufficient condition for the existence of such controller is formulated in terms of a set of LMIs. An example is also given to illustrate the applicability of the proposed approach. Our future work will focus on extending the proposed design method to other problems such as robust H_∞ control for 2D discrete switched systems with time-varying delays and fractional uncertainties under asynchronous switching.

Appendix

Proof of Lemma 9. Consider the following Lyapunov-Krasovskii functional candidate:

$$
V(x(i,j)) = V^h(x^h(i,j)) + V^v(x^v(i,j)), \tag{A.1}
$$

where

$$
\begin{aligned}
&V^h(x^h(i,j)) \\
&\quad = x^h(i,j)^T P_h x^h(i,j) \\
&\qquad + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h x^h(r,j)\, \alpha^{i-r-1}, \\
&V^v(x^v(i,j)) \\
&\quad = x^v(i,j)^T P_v x^v(i,j) \\
&\qquad + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v x^v(i,t)\, \alpha^{j-t-1}.
\end{aligned} \tag{A.2}
$$

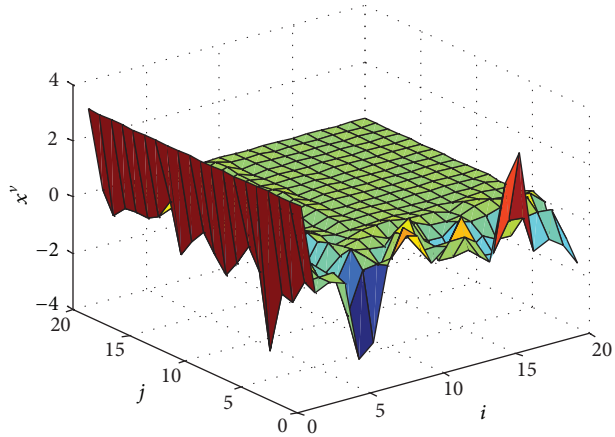


FIGURE 2: The trajectory of the state $x^v(i, j)$.

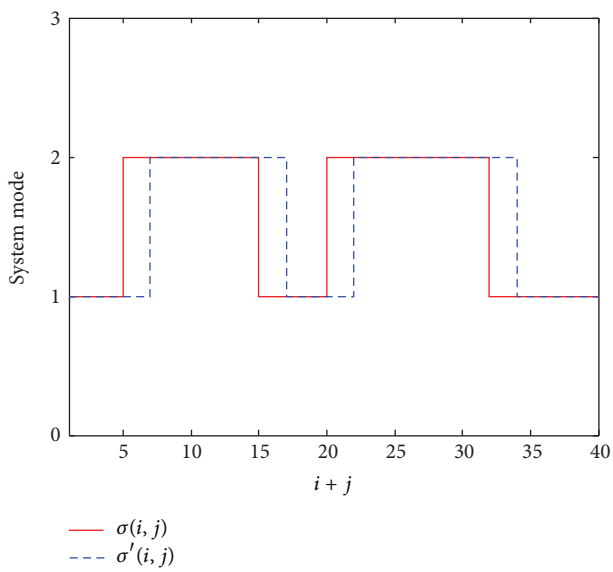


FIGURE 3: Switching signal.

Along the trajectory of system (8), we have

$$\begin{aligned} &V^h\left(x^h(i+1,j)\right) - \alpha V^h\left(x^h(i,j)\right) \\ &\quad = x^h(i+1,j)^T P_h x^h(i+1,j) - \alpha x^h(i,j)^T P_h x^h(i,j) \\ &\qquad + x^h(i,j)^T Q_h x^h(i,j) \\ &\qquad - \alpha^{d_h} x^h(i-d_h,j)^T Q_h x^h(i-d_h,j), \end{aligned} \tag{A.3}$$

$$\begin{aligned} &V^v\left(x^v(i,j+1)\right) - \alpha V^v\left(x^v(i,j)\right) \\ &\quad = x^v(i,j+1)^T P_v x^v(i,j+1) - \alpha x^v(i,j)^T P_v x^v(i,j) \\ &\qquad + x^v(i,j)^T Q_v x^v(i,j) \\ &\qquad - \alpha^{d_v} x^v(i,j-d_v)^T Q_v x^v(i,j-d_v). \end{aligned} \tag{A.4}$$

It follows that

$$\begin{aligned} &V^h\left(x^h(i+1,j)\right) - \alpha V^h\left(x^h(i,j)\right) \\ &\quad + V^v\left(x^v(i,j+1)\right) - \alpha V^v\left(x^v(i,j)\right) \\ &= \begin{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ \begin{bmatrix} x^h(i-d_h,j) \\ x^h(i,j-d_v) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ \begin{bmatrix} x^h(i-d_h,j) \\ x^h(i,j-d_v) \end{bmatrix} \end{bmatrix}, \end{aligned} \tag{A.5}$$

where

$$\begin{aligned} &\Phi_{11} = Q - \alpha P + \widehat{A}^T P \widehat{A}, \qquad \Phi_{12} = \widehat{A}^T P \widehat{A}_d, \\ &\Phi_{22} = \widehat{A}_d^T P \widehat{A}_d - \Lambda_1 Q, \qquad \Lambda_1 = \operatorname{diag}\left\{\alpha^{d_h} I_h, \alpha^{d_v} I_v\right\}. \end{aligned} \tag{A.6}$$

Applying Lemma 7, it can be obtained from (14) that

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0. \tag{A.7}$$

For simplicity, we denote

$$\begin{aligned} &V^h(i,j) = V^h\left(x^h(i,j)\right), \qquad V^v(i,j) = V^v\left(x^v(i,j)\right), \\ &V(i,j) = V\left(x(i,j)\right), \qquad V^h(i+1,j) = V^h\left(x(i+1,j)\right), \\ &\qquad V^v(i,j+1) = V^v\left(x(i,j+1)\right). \end{aligned} \tag{A.8}$$

Thus, it is easy to get that

$$V^h(i+1,j) + V^v(i,j+1) < \alpha\left(V^h(i,j) + V^v(i,j)\right). \tag{A.9}$$

Notice that for any nonnegative integer $D > z = \max(z_1, z_2)$, it holds that $V^h(0, D) = V^v(D, 0) = 0$; then summing up both sides of (A.9) from $D-1$ to 0 with respect to j and 0 to $D-1$ with respect to i, for any nonnegative integer $D > D' \geq z = \max(z_1, z_2)$, one gets

$$\begin{aligned} &\sum_{i+j=D} V(i,j) \\ &\quad = V^h(0,D) + V^h(1,D-1) + V^h(2,D-2) \\ &\qquad + \cdots + V^h(D-1,1) + V^h(D,0) \\ &\qquad + V^v(0,D) + V^v(1,D-1) + V^v(2,D-2) \\ &\qquad + \cdots + V^v(D-1,1) + V^v(D,0) \\ &\quad < \alpha\Big(V^h(0,D-1) + V^v(0,D-1) \\ &\qquad + V^h(1,D-2) + V^v(1,D-2) \\ &\qquad + \cdots + V^h(D-1,0) + V^v(D-1,0)\Big) \\ &\quad = \alpha \sum_{i+j=D-1} V(i,j) < \cdots < \alpha^{D-D'} \sum_{i+j=D'} V(i,j). \end{aligned} \tag{A.10}$$

The proof is completed. □

Proof of Theorem 12. When $D \in [m_\kappa + \Delta m_\kappa, m_{\kappa+1})$, the closed-loop system (17) can be written as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(\widehat{A}^k + \widehat{B}^k K^k\right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \widehat{A}_d^k \begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v) \end{bmatrix}. \tag{A.11}$$

For the system, we consider the following Lyapunov function candidate:

$$V_k(x(i,j)) = V_k^h\left(x^h(i,j)\right) + V_k^v\left(x^v(i,j)\right), \tag{A.12}$$

where

$$\begin{aligned} V_k^h\left(x^h(i,j)\right) &= x^h(i,j)^T P_h^k x^h(i,j) + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^k x^h(r,j)\,\alpha^{i-r-1}, \\ V_k^v\left(x^v(i,j)\right) &= x^v(i,j)^T P_v^k x^v(i,j) + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^k x^v(i,t)\,\alpha^{j-t-1}. \end{aligned} \tag{A.13}$$

By Lemma 9, one gets that if there exist positive definite symmetric matrices $P^k = \mathrm{diag}\{P_h^k, P_v^k\}$ and $Q^k = \mathrm{diag}\{Q_h^k, Q_v^k\}$ with appropriate dimensions, such that

$$\begin{bmatrix} Q^k - \alpha P^k & 0 & \left(\widehat{A}^k + \widehat{B}^k K^k\right)^T P^k \\ * & -\Lambda_1 Q^k & \widehat{A}_d^{kT} P^k \\ * & * & -P^k \end{bmatrix} < 0 \tag{A.14}$$

holds, then the following inequality holds for any $D \geq m_\kappa + \Delta m_\kappa \geq z$:

$$\sum_{i+j=D} V_k(i,j) < \alpha^{D-m_k-\Delta m_k} \sum_{i+j=m_\kappa+\Delta m_\kappa} V_k(i,j). \tag{A.15}$$

When $D \in [m_{\kappa+1}, m_{\kappa+1} + \Delta m_{\kappa+1})$, the closed-loop system (17) can be written as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(\widehat{A}^l + \widehat{B}^l K^k\right) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \widehat{A}_d^l \begin{bmatrix} x^h(i-d_h,j) \\ x^v(i,j-d_v) \end{bmatrix}. \tag{A.16}$$

Consider the following Lyapunov function candidate:

$$V_{kl}(x(i,j)) = V_{kl}^h\left(x^h(i,j)\right) + V_{kl}^v\left(x^v(i,j)\right), \tag{A.17}$$

where

$$\begin{aligned} V_{kl}^h\left(x^h(i,j)\right) &= x^h(i,j)^T P_h^{kl} x^h(i,j) + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^{kl} x^h(r,j)\,\alpha^{i-r-1}, \\ V_{kl}^v\left(x^v(i,j)\right) &= x^v(i,j)^T P_v^{kl} x^v(i,j) + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^{kl} x^v(i,t)\,\alpha^{j-t-1}. \end{aligned} \tag{A.18}$$

Similarly, by Lemma 9, we get that if there exist positive definite symmetric matrices $P^{kl} = \mathrm{diag}\{P_h^{kl}, P_v^{kl}\}$ and $Q^{kl} = \mathrm{diag}\{Q_h^{kl}, Q_v^{kl}\}$ with appropriate dimensions, such that

$$\begin{bmatrix} Q^{kl} - \beta P^{kl} & 0 & \left(\widehat{A}^l + \widehat{B}^l K^k\right)^T P^{kl} \\ * & -\Lambda_2 Q^{kl} & \widehat{A}_d^{lT} P^{kl} \\ * & * & -P^{kl} \end{bmatrix} < 0 \tag{A.19}$$

holds, then the following inequality holds for any $D \geq m_{\kappa+1} \geq z$:

$$\sum_{i+j=D} V_{kl}(i,j) < \beta^{D-m_{k+1}} \sum_{i+j=m_{\kappa+1}} V_{kl}(i,j). \tag{A.20}$$

Consider the following piecewise Lyapunov functional candidate for system (17):

$$V(i,j) = \begin{cases} x^h(i,j)^T P_h^{\sigma(i,j)} x^h(i,j) \\ \quad + x^v(i,j)^T P_v^{\sigma(i,j)} x^v(i,j) \\ \quad + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^{\sigma(i,j)} x^h(r,j)\,\alpha^{i-r-1} \\ \quad + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^{\sigma(i,j)} x^v(i,t)\,\alpha^{j-t-1}, \\ D \in [m_0, m_1) \cup [m_\pi + \Delta m_\pi, m_{\pi+1}), \\ \qquad \pi = 1,2,\ldots,\kappa\ldots, \\ x^h(i,j)^T P_h^{\sigma'(i,j)\sigma(i,j)} x^h(i,j) \\ \quad + x^v(i,j)^T P_v^{\sigma'(i,j)\sigma(i,j)} x^v(i,j) \\ \quad + \sum_{r=i-d_h}^{i-1} x^h(r,j)^T Q_h^{\sigma'(i,j)\sigma(i,j)} x^h(r,j)\,\alpha^{i-r-1} \\ \quad + \sum_{t=j-d_v}^{j-1} x^v(i,t)^T Q_v^{\sigma'(i,j)\sigma(i,j)} x^v(i,t)\,\alpha^{j-t-1}, \\ D \in [m_\pi, m_\pi + \Delta m_\pi), \quad \pi = 1,2,\ldots,\kappa\ldots. \end{cases} \tag{A.21}$$

Now, let $\upsilon = N_{\sigma(i,j)}(z, D)$ denote the switch number of $\sigma(i, j)$ on the interval $[z, D)$, and let $(i_{\kappa-\upsilon+1}, j_{\kappa-\upsilon+1})$, $(i_{\kappa-\upsilon+2}, j_{\kappa-\upsilon+2}), \ldots, (i_\kappa, j_\kappa)$ denote the switching points of $\sigma(i, j)$ over the interval $[z, D)$; then, the switching points of $\sigma'(i, j)$ can be denoted as follows:

$$\left(i_{\kappa-\upsilon+1} + \Delta i_{\kappa-\upsilon+1}, j_{\kappa-\upsilon+1} + \Delta j_{\kappa-\upsilon+1}\right),$$
$$\left(i_{\kappa-\upsilon+2} + \Delta i_{\kappa-\upsilon+2}, j_{\kappa-\upsilon+2} + \Delta j_{\kappa-\upsilon+2}\right), \ldots, \left(i_\kappa + \Delta i_\kappa, j_\kappa + \Delta j_\kappa\right). \tag{A.22}$$

Denoting $m_g = i_g + j_g$, $g = \kappa - \upsilon + 1, \ldots, \kappa$, then we can get from (23) and (A.21) that

$$\begin{aligned}
&\sum_{i+j=g+\Delta m_g} V(i, j) \le \mu_1 \sum_{i+j=(m_g+\Delta m_g)^-} V(i, j),\\
&\sum_{i+j=m_g} V(i, j) \le \mu_2 \sum_{i+j=(m_g)^-} V(i, j),
\end{aligned} \tag{A.23}$$

where $(m_g)^-$ and $(m_g + \Delta m_g)^-$ satisfy the following conditions:

$$\begin{aligned}
&0 < m_g - \left(m_g\right)^- < \lambda,\\
&0 < m_g + \Delta m_g - \left(m_g + \Delta m_g\right)^- < \lambda,
\end{aligned} \tag{A.24}$$

where λ is a sufficient small positive constant.

When $m_{\kappa-\upsilon+1} > z \ge m_{\kappa-\upsilon} + \Delta m_{\kappa-\upsilon}$, we have, for $D > m_\kappa + \Delta m_\kappa$,

$$\begin{aligned}
&\sum_{i+j=D} V(i, j) < \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=m_\kappa+\Delta m_\kappa} V(i, j) \le \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=(m_\kappa+\Delta m_\kappa)^-} V(i, j)\\
&< \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=m_\kappa} V(i, j) \le \mu_1 \mu_2 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=(m_\kappa)^-} V(i, j) < \cdots\\
&< (\mu_1\mu_2)^\upsilon \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-\upsilon+2}-m_{\kappa-\upsilon+1}-\Delta m_{\kappa-\upsilon+1}} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-\upsilon+1}} \sum_{i+j=(m_{k-\upsilon+1})^-} V(i, j)\\
&< (\mu_1\mu_2)^\upsilon \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-\upsilon+2}-m_{\kappa-\upsilon+1}-\Delta m_{\kappa-\upsilon+1}+m_{\kappa-\upsilon+1}-z} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-\upsilon+1}} \sum_{i+j=z} V(i, j)\\
&\le (\mu_1\mu_2)^\upsilon \alpha^{D-z-\upsilon\Delta} \beta^{\upsilon\Delta} \sum_{i+j=z} V(i, j) = \alpha^{D-z} e^{\upsilon \ln(\mu_1\mu_2)+\upsilon\Delta(\ln\beta-\ln\alpha)} \sum_{i+j=z} V(i, j).
\end{aligned} \tag{A.25}$$

When $m_{\kappa-\upsilon} < z < m_{\kappa-\upsilon} + \Delta m_{\kappa-\upsilon}$, we can also get that

$$\begin{aligned}
&\sum_{i+j=D} V(i, j) < \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=m_\kappa+\Delta m_\kappa} V(i, j) < \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \sum_{i+j=(m_\kappa+\Delta m_\kappa)^-} V(i, j)\\
&< \mu_1 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=m_\kappa} V(i, j) < \mu_1 \mu_2 \alpha^{D-m_\kappa-\Delta m_\kappa} \beta^{\Delta m_\kappa} \sum_{i+j=(m_\kappa)^-} V(i, j) < \cdots\\
&< (\mu_1\mu_2)^\upsilon \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-\upsilon+2}-m_{\kappa-\upsilon+1}-\Delta m_{\kappa-\upsilon+1}} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\ldots+\Delta m_{\kappa-\upsilon+1}} \sum_{i+j=(m_{k-\upsilon+1})^-} V(i, j)\\
&< (\mu_1\mu_2)^\upsilon \alpha^{D-m_\kappa-\Delta m_\kappa+m_\kappa-m_{\kappa-1}-\Delta m_{\kappa-1}+\cdots+m_{\kappa-\upsilon+2}-m_{\kappa-\upsilon+1}-\Delta m_{\kappa-\upsilon+1}+m_{\kappa-\upsilon+1}-m_{\kappa-\upsilon}-\Delta m_{\kappa-\upsilon}} \beta^{\Delta m_\kappa+\Delta m_{\kappa-1}+\cdots+\Delta m_{\kappa-\upsilon+1}+\Delta m_{\kappa-\upsilon}+m_{\kappa-\upsilon}-z} \sum_{i+j=z} V(i, j)\\
&\le \mu_1 (\mu_1\mu_2)^\upsilon \alpha^{D-z-\upsilon\Delta} \beta^{\upsilon\Delta} \sum_{i+j=z} V(i, j) = \mu_1 \alpha^{D-z} e^{\upsilon \ln(\mu_1\mu_2)+\upsilon\Delta(\ln\beta-\ln\alpha)} \sum_{i+j=z} V(i, j).
\end{aligned} \tag{A.26}$$

From (A.25) and (A.26), we can obtain

$$\begin{aligned}
&\sum_{i+j=D} V(i, j)\\
&\quad < \max\{\mu_1, 1\} \alpha^{D-z} e^{\upsilon \ln(\mu_1\mu_2)+\Delta\upsilon(\ln\beta-\ln\alpha)} \sum_{i+j=z} V(i, j).
\end{aligned} \tag{A.27}$$

According to Definition 6, one has

$$\upsilon = N_{\sigma(i,j)}(z, D) \le N_0 + \frac{D - z}{\tau_a}. \tag{A.28}$$

From condition (22), the following inequality can be obtained:

$$-\ln\alpha > \frac{\Delta(\ln\beta - \ln\alpha) + \ln(\mu_1\mu_2)}{\tau_a}. \tag{A.29}$$

Thus, it is easy to get the following inequality:

$$\begin{aligned}&\sum_{i+j=D} V(i,j)\\&\quad< \max\{\mu_1,1\}(\mu_1\mu_2)^{N_0}\\&\qquad\times\left(\frac{\beta}{\alpha}\right)^{N_0\Delta} e^{((\ln(\mu_1\mu_2)+\Delta(\ln\beta-\ln\alpha))/\tau_a+\ln\alpha)(D-z)}\sum_{i+j=z} V(i,j).\end{aligned}\tag{A.30}$$

It follows that

$$\begin{aligned}&\sum_{i+j=D}\|x(i,j)\|^2\\&\quad<\frac{\zeta_1}{\zeta_2}\max\{\mu_1,1\}(\mu_1\mu_2)^{N_0}\\&\qquad\times\left(\frac{\beta}{\alpha}\right)^{N_0\Delta} e^{((\ln(\mu_1\mu_2)+\Delta(\ln\beta-\ln\alpha))/\tau_a+\ln\alpha)(D-z)}\sum_{i+j=z}\|x(i,j)\|^2,\end{aligned}\tag{A.31}$$

where

$$\begin{aligned}\zeta_1&=\max_{k,l\in\underline{N},k\neq l}\left\{\lambda_{\max}\left(P^k\right)+\max(d_h,d_v)\lambda_{\max}\left(Q^k\right),\right.\\&\qquad\left.\lambda_{\max}\left(P^{kl}\right)+\max(d_h,d_v)\lambda_{\max}\left(Q^{kl}\right)\right\},\\\zeta_2&=\min_{k,l\in\underline{N},k\neq l}\left\{\lambda_{\min}\left(P^k\right),\lambda_{\min}\left(P^{kl}\right)\right\},\\&\sum_{i+j=z}\|x(i,j)\|_C^2\\&\quad\triangleq\sup_{\substack{-d_h\le\theta_h\le0,\\-d_v\le\theta_v\le0}}\sum_{i+j=z}\left\{\left\|x^h(i-\theta_h,j)\right\|^2+\left\|x^v(i,j-\theta_v)\right\|^2\right\},\end{aligned}\tag{A.32}$$

Thus, it can be obtained from (22) that the closed-loop system (17) is exponentially stable.

Denote $X^k=(P^k)^{-1}$ and $Y^k=(Q^k)^{-1}$, then it is easy to get $(X^k)^T=X^k$ and $(Y^k)^T=Y^k$. Using diag$\{X^k,Y^k,X^k\}$ to pre- and postmultiply the left of (A.14), respectively, and applying Lemma 7, it follows that (A.33) and (A.14) are equivalent:

$$\begin{bmatrix}-\alpha X^k & 0 & \left(\widehat{A}^kX^k+\widehat{B}^kW^k\right)^T & X^k\\ * & -\Lambda_1Y^k & \left(\widehat{A}_d^kY^k\right)^T & 0\\ * & * & -X^k & 0\\ * & * & * & -Y^k\end{bmatrix}<0,\tag{A.33}$$

where $W^k=K^kX^k$.

By substituting (2) into (A.33), we can get the following inequality

$$T=T_0+T_1<0,\tag{A.34}$$

where

$$\begin{aligned}T_0&=\begin{bmatrix}-\alpha X^k & 0 & \left(A^kX^k+B^kW^k\right)^T & X^k\\ * & -\Lambda_1Y^k & \left(A_d^kY^k\right)^T & 0\\ * & * & -X^k & 0\\ * & * & * & -Y^k\end{bmatrix},\\T_1&=\begin{bmatrix}\left(E_1^kX^k+E_3^kW^k\right)^T\\ \left(E_2^kY^k\right)^T\\0\\0\end{bmatrix}F^{kT}\begin{bmatrix}0\\0\\H^k\\0\end{bmatrix}^T\\&\quad+\begin{bmatrix}0\\0\\H^k\\0\end{bmatrix}F^k\begin{bmatrix}\left(E_1^kX^k+E_3^kW^k\right)^T\\ \left(E_2^kY^k\right)^T\\0\\0\end{bmatrix}^T.\end{aligned}\tag{A.35}$$

By Lemma 8, we get

$$\begin{aligned}&T_0+\varepsilon_k^{-1}\begin{bmatrix}\left(E_1^kX^k+E_3^kW^k\right)^T\\ \left(E_2^kY^k\right)^T\\0\\0\end{bmatrix}\begin{bmatrix}\left(E_1^kX^k+E_3^kW^k\right)^T\\ \left(E_2^kY^k\right)^T\\0\\0\end{bmatrix}^T\\&\quad+\varepsilon_k\begin{bmatrix}0\\0\\H^k\\0\end{bmatrix}\begin{bmatrix}0\\0\\H^k\\0\end{bmatrix}^T<0.\end{aligned}\tag{A.36}$$

Applying Lemma 7 again, we obtain that (A.33) holds if (19) is satisfied.

Similarly, substitute (2) into (A.19) and denote $X^{kl}=(P^{kl})^{-1}$ and $Y^{kl}=(Q^{kl})^{-1}$, then using diag$\{X^{kl},Y^{kl},X^{kl}\}$ to pre- and postmultiply the left of (A.19), respectively, and applying Lemmas 7 and 8, it is easy to get that (A.19) holds if (20) is satisfied.

The proof is completed. □

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grant no. 61273120.

References

[1] C. Du and L. Xie, *Control and Filtering of Two-Dimensional Systems*, vol. 278 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 2002.

[2] T. Kaczorek, *Two-Dimensional Linear Systems*, vol. 68 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 1985.

[3] W. S. Lu, *Two-Dimensional Digital Filters*, Marcel Dekker, New York, NY, USA, 1992.

[4] H. Mahgoub and R. Carroll, "A Lyapunov function of two-dimensional linear systems," *IEEE Transactions on Automatic Control*, vol. 31, no. 2, pp. 170–172, 1986.

[5] A. A. Kara and Z. Ünver, "Comments on 'A Lyapunov function of two-dimensional linear systems'," *IEEE Transaction on Automatic Control*, vol. 36, no. 1, pp. 125–126, 1991.

[6] T. Kaczorek, "New stability tests of positive standard and fractional linear systems," *Circuits and Systems*, vol. 2, no. 4, pp. 261–268, 2011.

[7] S. Ye and W. Wang, "Stability analysis and stabilisation for a class of 2-D nonlinear discrete systems," *International Journal of Systems Science*, vol. 42, no. 5, pp. 839–851, 2011.

[8] W. S. Lu and E. B. Lee, "Stability analysis for two-dimensional systems via a Lyapunov approach," *IEEE Transactions on Circuits and Systems*, vol. 32, no. 1, pp. 61–68, 1985.

[9] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3008–3015, 2012.

[10] J. Hu, Z. Wang, and H. Gao, "A delay fractioning approach to robust sliding mode control for discrete-time stochastic systems with randomly occurring non-linearities," *IMA Journal of Mathematical Control and Information*, vol. 28, no. 3, pp. 345–363, 2011.

[11] J. Hu, Z. Wang, Y. Niu, and L. K. Stergioulas, "H_∞ sliding mode observer design for a class of nonlinear discrete time-delay systems: a delay-fractioning approach," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 16, pp. 1806–1826, 2012.

[12] J. Xu and L. Yu, "H_∞ control of 2-D discrete state delay systems," *International Journal of Control, Automation and Systems*, vol. 4, no. 4, pp. 516–523, 2006.

[13] S.-F. Chen, "Delay-dependent stability for 2D systems with time-varying delay subject to state saturation in the Roesser model," *Applied Mathematics and Computation*, vol. 216, no. 9, pp. 2613–2622, 2010.

[14] Z.-Y. Feng, L. Xu, M. Wu, and Y. He, "Delay-dependent robust stability and stabilisation of uncertain two-dimensional discrete systems with time-varying delays," *IET Control Theory & Applications*, vol. 4, no. 10, pp. 1959–1971, 2010.

[15] S. F. Chen, "Stability analysis for 2-D systems with interval time-varying delays and saturation nonlinearities," *Signal Processing*, vol. 90, no. 7, pp. 2265–2275, 2010.

[16] S. Ye, Y. Zou, W. Wang, and J. Yao, "Delay-dependent stability analysis for two- dimensional discrete systems with shift delays by the general models," in *Proceedings of the 10th International Conference on Control, Automation, Robotics and Vision (ICARCV '08)*, pp. 973–978, December 2008.

[17] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[18] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC '99)*, pp. 2655–2660, December 1999.

[19] G. Xie and L. Wang, "Stabilization of switched linear systems with time-delay in detection of switching signal," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 277–290, 2005.

[20] Y. Wang, Z. Yao, Z. Zuo, and H. Zhao, "Delay-dependent robust H_∞ control for a class of switched systems with time delay," in *Proceedings of IEEE International Symposium on Intelligent Control (ISIC '08)*, pp. 882–887, September 2008.

[21] Y. G. Sun, L. Wang, and G. Xie, "Delay-dependent robust stability and stabilization for discrete-time switched systems with mode-dependent time-varying delays," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 428–435, 2006.

[22] Y. G. Sun, L. Wang, and G. Xie, "Delay-dependent robust stability and H_∞ control for uncertain discrete-time switched systems with mode-dependent time delays," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1228–1237, 2007.

[23] K. Hu and J. Yuan, "Improved robust H_∞ filtering for uncertain discrete-time switched systems," *IET Control Theory & Applications*, vol. 3, no. 3, pp. 315–324, 2009.

[24] D.-W. Ding and G.-H. Yang, "H_∞ static output feedback control for discrete-time switched linear systems with average dwell time," *IET Control Theory & Applications*, vol. 4, no. 3, pp. 381–390, 2010.

[25] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach," in *Proceedings of American Control Conference*, pp. 200–204, June 2000.

[26] P. M. Igic, M. S. Towers, and P. A. Mawby, "A 2D physically based compact model for advanced power bipolar devices," *Microelectronics Journal*, vol. 35, no. 7, pp. 591–594, 2004.

[27] W. C. Lo, L. Wang, and B. Li, "Thermal transistor: heat flux switching and modulating," *Journal of the Physical Society of Japan*, vol. 77, no. 1, pp. 1–4, 2008.

[28] A. Benzaouia, A. Hmamed, and F. Tadeo, "Stability conditions for discrete 2D switching systems, based on a multiple Lyapunov function," in *Proceedings of European Control Conference*, pp. 23–26, 2009.

[29] A. Benzaouia, A. Hmamed, F. Tadeo, and A. El Hajjaji, "Stabilisation of discrete 2D time switching systems by state feedback control," *International Journal of Systems Science*, vol. 42, no. 3, pp. 479–487, 2011.

[30] Z. Xiang and S. Huang, "Stability analysis and stabilization of discrete-time 2D switched systems," *Circuits, Systems, and Signal Processing*, vol. 32, no. 1, pp. 401–414, 2013.

[31] L. Zhang and H. Gao, "Asynchronously switched control of switched linear systems with average dwell time," *Automatica*, vol. 46, no. 5, pp. 953–958, 2010.

[32] L. Zhang and P. Shi, "Stability, l_2-gain and asynchronous H_∞ control of discrete-time switched systems with average dwell time," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2192–2199, 2009.

[33] Z. Xiang, C. Liang, and Q. Chen, "Robust L_2-L_∞ filtering for switched systems under asynchronous switching," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 8, pp. 3303–3318, 2011.

[34] Z. R. Xiang and R. H. Wang, "Robust stabilization of switched non-linear systems with time-varying delays under asynchronous switching," *Proceedings of the Institution of Mechanical Engineers. Part I: Journal of Systems and Control Engineering*, vol. 223, no. 8, pp. 1111–1128, 2009.

[35] Z. R. Xiang and R. H. Wang, "Robust control for uncertain switched non-linear systems with time delay under asynchronous switching," *IET Control Theory & Applications*, vol. 3, no. 8, pp. 1041–1050, 2009.

[36] Z. Xiang, Y.-N. Sun, and Q. Chen, "Robust reliable stabilization of uncertain switched neutral systems with delayed switching," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9835–9844, 2011.

[37] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1994.

[38] L. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *International Journal of Control*, vol. 63, no. 4, pp. 741–750, 1996.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 953406, 6 pages
http://dx.doi.org/10.1155/2013/953406

Research Article

Information Propagation in Online Social Network Based on Human Dynamics

Qiang Yan,[1] Lianren Wu,[1,2] Chao Liu,[2,3] and Xiye Li[1]

[1] *School of Economics and Management, Beijing University of Posts and Telecommunications, Beijing 100876, China*
[2] *Department of Electrical Engineering, Arizona State University, Tempe, AZ 85287, USA*
[3] *Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China*

Correspondence should be addressed to Lianren Wu; lianrenw@asu.edu

Received 16 January 2013; Revised 10 April 2013; Accepted 18 April 2013

Academic Editor: Bo Shen

Copyright © 2013 Qiang Yan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the impact of human dynamics on the information propagation in online social networks. First, statistical properties of the human behavior are studied using the data from "Sina Microblog," which is one of the most popular online social networks in China. We find that human activity patterns are heterogeneous and bursty and are often described by a power-law interevent time distribution $P(\tau) \sim \tau^{-\alpha}$. Second, we proposed an extended Susceptible-Infected (SI) propagation model to incorporate bursty and limited attention. We unveil how bursty human behavior and limited attention affect the information propagation in online social networks. The result in this paper can be useful for optimizing or controlling information propagation in online social networks.

1. Introduction

Rapid development of information and communication technology has increased the wide adoption of online social network in our life. Indeed, online social network such as Sina Microblog, Twitter, and Facebook had become an indispensable part of our life. Every day we sign into our homepages more than once to view and share information. These online social networks have common characteristics: instantaneity, simplicity, and universality. Taking Sina Microblog, for example, unlike the traditional blog, it allows the use of mobile devices to disseminate information by a length of 140 characters text at anytime and anywhere. Investigating the online social network is crucial in a broad range of settings from information propagation and viral marketing to political purposes.

Recent years, online social network as a platform for the empirical study of information has been widespread concern [1–4]. Despite the progresses that have been made, the empirical study of information propagation is still in its infancy. Studies in this direction have been mostly hindered by the shortcoming of available large-scale data. However, the availability of large-scale data from online social network has recently created unprecedented opportunities to explore the impact of human behaviors on the information propagation.

Firstly, information propagation in online social network is determined by rhythms and activity patterns of human [5, 6]. An increasing number of recent measurements indicate that human activity patterns are heterogeneous and bursty [7–11]. If only considering the time interval between events, these human activity patterns are often described by a power-law interevent time distribution $P(\tau) \sim \tau^{-\alpha}$, where τ is the time interval between two consecutive activities [12]. Recently, the researchers began to realize that the bursty human behavior has an important impact on the dissemination of information [13, 14].

Secondly, the wide adoption of online social network has increased the competition among information for our limited attention. Every day we receive a lot of information from various online social networks. However, we do not have enough time and attention to disseminate each message which we received. It is an interesting question that whether such a competition may affect the velocity of information propagation. The issue of limited attention has been studied through messages posted and forwarded in online social

networks [15, 16]. However, how limited attention affects velocity of information propagation is still unclear.

In this paper, we propose an extended Susceptible-Infected (SI) propagation model, incorporating bursty human activity patterns and limited attention for the first time. Then, we obtain a large number of real data to test the model. Adopting the methods of theoretical research and empirical analysis, we study the information spreading process in social networking qualitatively and quantitatively. The key contributions of this study are summarized as follows.

(1) From the empirical statistical results we find that at the group level, the interactive time (time interval between two consecutive login microblog homepage) follows power-law distribution with the slope ≈ 2.5. And the distribution of newly infected individual (calculate as the number of new forwarding per day) follows power-law with the slope ≈ 1.5. Two slope values satisfy the relationship $2.5 - 1.5 \approx 1.0$.

(2) Through both the theoretical research and simulation, we prove that (a) if the generation time distribution follows power-law with exponent β, then the decay of propagation velocity will be characterized by the same power-law distribution; (b) if bursty human behavior follows a power-law distribution with exponent α, the decay of propagation velocity also follows a power-law with exponent $\beta \approx \alpha - 1$.

In summary, although tremendous efforts have been made regarding the research about information propagation, further study based on human dynamics is still needed to unveil the role of human behaviors for the information propagation in online social network. In future studies, on the other hand, we can use other more mature theories to research the spreading dynamics, such as in the references [17, 18].

The rest of this paper is organized as follows. Section 2 gives the data description. In Section 3, we propose the extended SI model. In Section 4, we present simulation results and observations. Section 5 introduces theoretical analysis. Finally, in Section 6, we conclude the work.

2. Data Description

The dataset of this paper was collected from Sina Microblog (http://www.weibo.com/), one of the most popular microblog platforms in China at present. The dataset includes 345,095 messages from 41667 individuals during 2009/8/16 to 2011/6/4, collected by snowball sampling. These messages have been forwarded 203,997,094 times and triggered 58,617,139 comments. For each message, message ID, releasing time, times of forwarding, and number of comments were recorded. For each individual, the individual ID and the timing of individual sign in his/her microblog homepage were recorded.

The basic statistical results show that at the group level, the interactive time (time interval between two consecutive login microblog homepage) follows power-law distribution with the slope ≈ 2.5 (Figure 1(a)). And the distribution of newly infected individual (calculate as the number of new forwarding per day) follows power-law with the slope ≈ 1.5 (Figure 1(b)). If set the slope of interactive time distribution is α and the slope of newly infected individual distribution is β, we find that there is the relationship $\beta \approx \alpha - 1$ between two slopes.

3. Model

3.1. Model Description. In this paper, we use the branching processes [19, 20] in conjunction with power-law human behaviors to describe the process of information propagation. We adopt the Susceptible-Infected (SI) propagation model for the simulation of information propagation in online social networks. Similar to the classical SI model, the population is divided into two states, either susceptible (S) or infected (I). In the information propagation model, however, the susceptible individual is defined as the one who has not yet known a piece of message, and the infected individual is defined as the one who knows the message and shares the message with his/her friends. After being infected, an individual will never return to susceptible state. At time t, there are $S(t)$ susceptible individuals and $I(t)$ infected individuals, and the population $N = S(t) + I(t)$.

Initially all individuals are susceptible except for a single infected individual. Different with the traditional model, at a given time step, an infected individual can be inactive; that is to say, infected individual will not infect connected susceptible individuals at that time step. The time interval between two consequent active steps of an infected individual is defined as the interactive time, which is often characterized by a power-law distribution $P(\tau) \sim \tau^{-\alpha}$ at the group level. Meanwhile, different individuals have different active time interval and each individual i acts with an unchanged interactive time τ_i.

On the other hand, the advent of online social network has greatly lowered the cost of information generation and propagation, boosting the potential reach of each message. However, the abundance of information to which we are exposed through online social networks is exceeding our capacity to consume it. Due to the limited time and attention, the individual cannot continuously check the update of information on his/her homepage. We assume that individuals interact on a directed online social network. Each individual is equipped with two lists. One is the screen where received messages are recorded and maintained a time-ordered list of messages. The other is memory where individual interested messages are recorded. Each individual can share some of the messages from the list with his/her friends. The friends in turn pay attention to a newly received message by placing it at the top of their lists. Because of the limited attention, we allow messages to survive in an individual's screen for a finite amount of time T. Meanwhile, we assume that each individual only forwards each message once, and then the individual loses interest in the message. In addition, if the individual no forwarding the message within T, the individual will no longer be concerned about the message and delete it from the screen. Each message may attract the individual's attention with probability λ; that is to say, the individual will forward the message with probability λ.

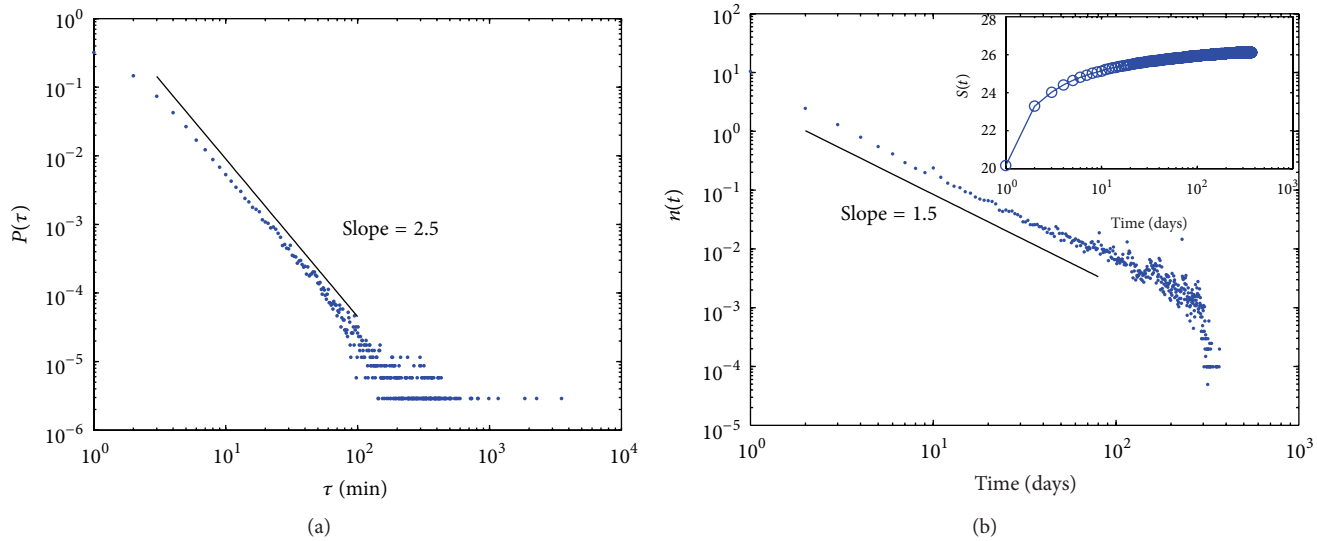


Figure 1: Empirical data. (a) The distribution of interactive time at the group level. (b) The distribution of newly infected individuals, inset: the cumulative distribution of newly infected individuals, namely, the distribution of all infected individuals. The results are the average of all messages.

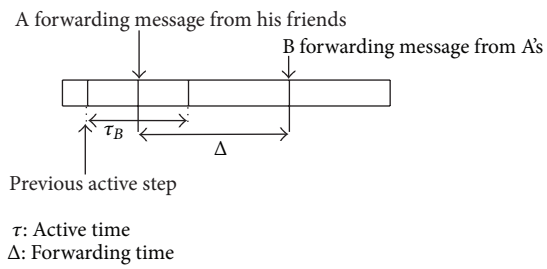


Figure 2: Schematic of individual interaction.

3.2. SI Model Based on Bursty and Limited Attention. According to the previous description, the SI model incorporating bursty and limited attention is illustrated in Figure 2. We characterize the timing of information propagation by the generation time Δ, which is defined as the time interval between the forwarding of an individual and the forwarding of his/her followers.

To sum up, the extended SI model is defined as follows.

Step 1. At time step $t = t_i$, an individual i posts a message. Meanwhile, individual j receives the message, where $j \in \delta_i$ and δ_i is the set of individual i's neighbors.

Step 2. For each individual j, the first active time step is t_{j0}, $t_{j0} \in (t_i, t_i + \tau_j)$, and individual j will be active at the time steps $t = t_{j0} + k\tau_j$, $k = 1, 2, 3, \ldots$, where τ_j is the active time interval of individual j.

Step 3. At each active time step, individual j will forward the message with the probability λ. If individual j forwards the message at the time step t_j, we obtain the generation time $\Delta = t_j - t_i$ and generation time must satisfy the condition $\Delta < T$.

Step 4. Update the time step $t = t_j$ and repeat Step 1 to Step 3 until the preset time steps.

In addition, we also introduce two indicators to characterize the velocity of information propagation:

(1) the first time step when the number of infected individuals exceeds half of the population, defined as half time T^*;

(2) the mean infection time of an individual after the outbreak, defined as mean time $T_m = \sum_{t=0}^{t_{\max}} (tn(t)/N)$, where $t_{\max}$ is the maximum simulation step, such as in our simulation $t_{\max} = 10^4$.

4. Simulation Results and Observations

In our simulations, initially all individuals are susceptible except for a single infected individual. Each individual i has an unchanged interactive time τ_i, which follows power-law distribution $P(\tau) \sim \tau^{-\alpha}$ with $2 < \alpha < 3$. We set $T = 1440$ time steps. This is because messages will survive in an individual's list one day, namely, 1440 minutes [15]. Simulations were performed on a BA network with size $N = 10^4$ and $\langle k \rangle = 10$. We set the degree of attention $\lambda = 0.5$ and randomly select an initial infected node. For detailed comparison, we also performed the same SI dynamics with exponential interactive time distribution $P(\tau) \sim e^{-\alpha\tau}$. From the numerical simulation results (Figures 3 and 4), we have the following observations of the propagation process.

Observation 1. In power-law case, the average number of newly infected individuals $n(t)$ and the generation time $g(\Delta)$ follow power-law distributions with the exponent $\beta \approx \alpha - 1$ (Figure 3).

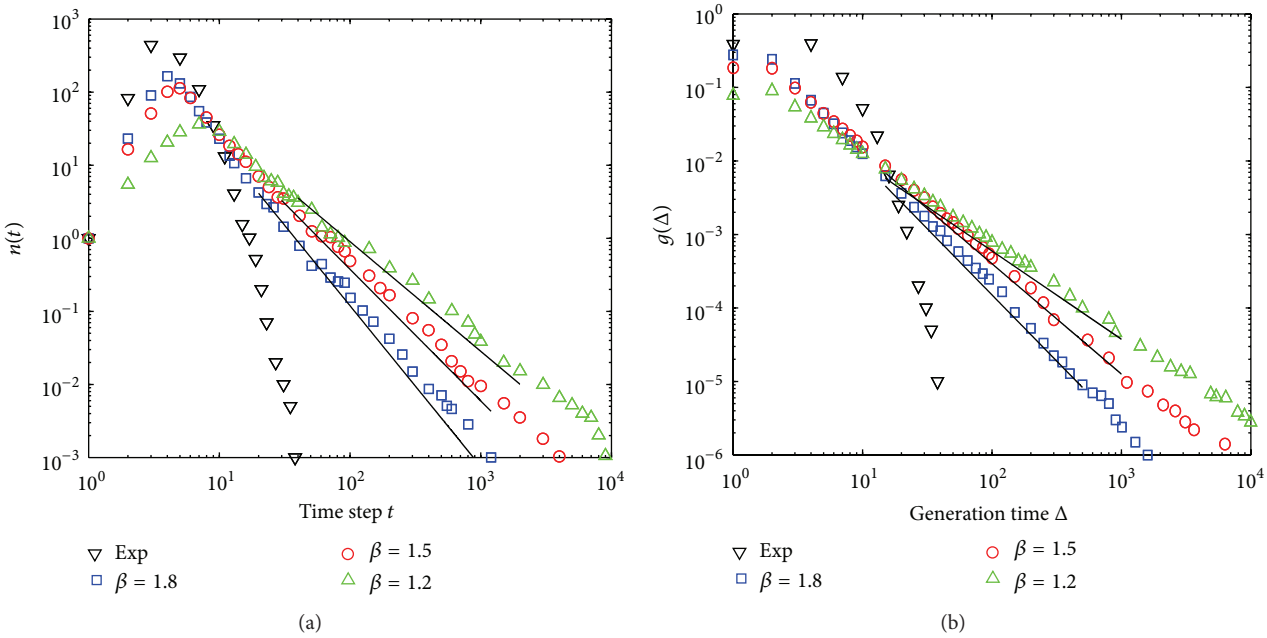


FIGURE 3: (a) The average number of newly infected individuals $n(t)$. Power-law distributions $p(\tau)$ with exponent $\alpha = 2.8$ (Squares), $\alpha = 2.5$ (circles), $\alpha = 2.2$ (up triangles), and the exponential $p(\tau)$ (down triangles). All $p(\tau)$ have the same mean interactive time $\langle\tau\rangle = 1.96$. (b) The generation time distribution $g(\Delta)$ for all $p(\tau)$. In both panels, the black lines have slopes −1.8, −1.5, and −1.2. The results show that $n(t)$ and $g(\Delta)$ decay as a power law with the exponent $\beta \approx \alpha - 1$. In the exponential case, $n(t)$ decays fast, in stark contrast to the power-law case. The results are the average over 2×10^3 independent runs.

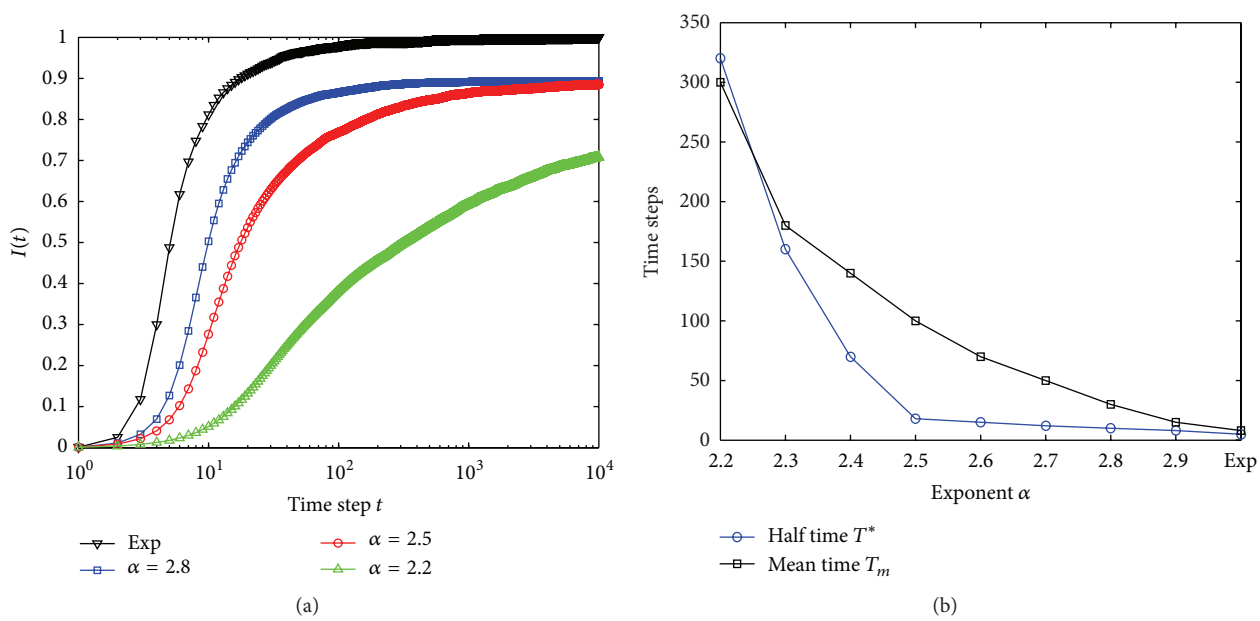


FIGURE 4: (a) The fraction of infected nodes I with different exponent α. (b) The half time T^* and the mean time T_m as the functions of exponent α.

Observation 2. The smaller the exponent α of interactive time distributions, namely, the larger heterogeneity of interactive time, resulting in the slower velocity. The half time T^* and mean time T_m monotonic decrease with the increase of exponent α (Figure 4).

In order to investigate the impact of attention on the propagation process, we fixed interactive time following power-law distribution with the exponent $\alpha = 2.5$ and randomly select an initial infected node. From other parameters $T = 1440$, simulations were also performed on a BA network with

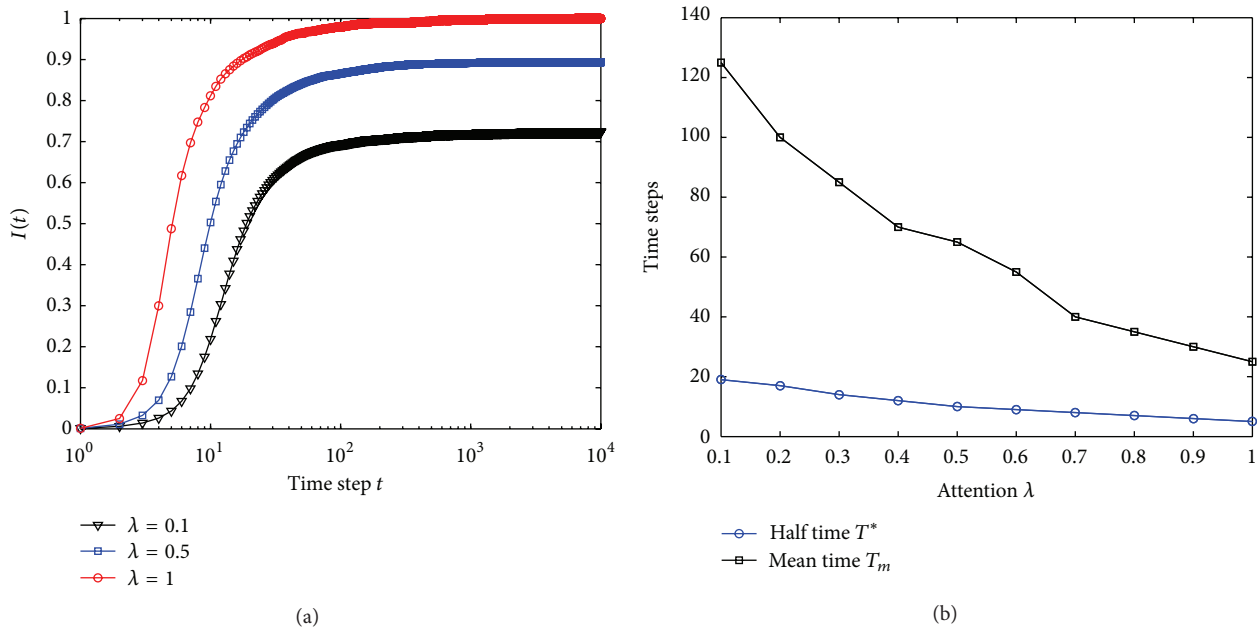


FIGURE 5: (a) The fraction of infected nodes I with different attention λ. (b) The half time T^* and mean time T_m as the functions of attention λ.

size $N = 10^4$ and $\langle k \rangle = 10$. The results are averaged over 2×10^3 independent runs. From the numerical simulation results (Figure 5), we have the following observation of the propagation process.

Observation 3. The higher the degree of attention, the faster the velocity. The half time T^* and mean time T_m monotonic decrease with the increase of attention λ (Figure 5).

5. Theoretical Analysis

In this section, the properties of propagation dynamics are analyzed. We prove that the decay exponent of propagation velocity equals that in the generation time distribution. Furthermore, we also proved that the exponent α characterizing the bursty is related to that in the decay of propagation velocity β by the relation $\beta = \alpha - 1$.

Proposition 1. *If the distribution of generation time follows power-law $g(\Delta) \sim \Delta^{-\beta}$ with $1 < \beta < 2$, the decay of propagation velocity also follows power-law $n(t) \sim t^{-\beta}$ and with the same exponent β.*

Proof. We consider a general theory of propagation process in online social networks. We assume that the propagation process outbreaks starting from a single infected individual at time $t = 0$. In this case, the average number of new infected individuals at time t is [19]

$$n(t) = \sum_{d=1}^{D} z_d \left(g^{(0)} * g^{(1)} * \cdots * g^{(d)}(t) \right), \tag{1}$$

where z_d is the average number of individuals at generation d away from the first infected individual, where $*$ denotes the convolution operation; for example,

$$g^{(0)} * g^{(1)}(t) = \int_0^t d\tau g^{(0)}(\tau) * g^{(1)}(1-\tau). \tag{2}$$

For the limited $1 \ll d$, we can obtain

$$g^{*d}(t) = g^{(0)} * g^{(1)} * \cdots * g^{(d)}(t) \sim L_{\beta-1}\left(\frac{t}{t_d}\right) t_d, \tag{3}$$

where $t_d = \Delta_0 d^{1/(\beta-1)}$, Δ_0 is some characteristic time scale, and $L_\mu(x)$ represents the Levy distribution with exponent μ.

For $1 \ll x$, the Levy distribution $L_\mu(x)$ can expressed as [21]

$$L_\mu(x) \sim x^{-(1+\mu)}. \tag{4}$$

To sum up, when $t \to \infty$, we obtain $g^{*d}(t) \sim t^{-\beta}$; namely, $n(t) \sim t^{-\beta}$. Thus, the proposition has been proved. □

This preposition means that if the generation time distribution follows a power-law with the exponent β, then the decay of propagation velocity will be characterized by the same power-law distribution.

Proposition 2. *If the distribution of interactive time follows a power-law $p(\tau) \sim \tau^{-\alpha}$ with $2 < \alpha < 3$, the decay of propagation velocity also follows a power-law distribution $n(t) \sim t^{-\beta}$ with $1 < \beta < 2$ and $\beta = \alpha - 1$.*

Proof. When the distribution of interactive time follows a power-law $p(\tau) \sim \tau^{-\alpha}$ with $2 < \alpha < 3$, the active time interval τ_i has a finite mean $\langle \tau \rangle$.

Since the generation time probability density function is related to the interactive time probability density function [21], therefore we have

$$\begin{aligned} g(\Delta) &= \frac{1}{\langle\tau\rangle}\int_{\Delta}^{\infty} P(\tau)\,d\tau = \frac{1}{\langle\tau\rangle}\int_{\Delta}^{\infty}\tau^{-\alpha}d\tau \\ &= \frac{1}{\langle\tau\rangle}\frac{1}{-(\alpha-1)}\Delta^{-(\alpha-1)} \\ &\sim \Delta^{-(\alpha-1)}. \end{aligned} \tag{5}$$

According to Proposition 1, we obtain

$$n(t) \sim t^{-(\alpha-1)}. \tag{6}$$

Namely,

$$n(t) \sim t^{-\beta}, \qquad \beta = \alpha - 1. \tag{7}$$

Thus, the proposition is proved. □

6. Conclusion

An extended SI model is proposed in this paper. Different from the analysis of the network topology, we study the information propagation in online social networks from the perspective of human dynamics. We found that human behavior affects the range and velocity of information propagation greatly.

In the future, with the development of online social systems, there may be other factors influencing information propagation in online social network. Therefore, we must improve the propagation model in order to better explain the propagation process.

Acknowledgments

The authors would like to thank Liang Huang and Byungjoon Min for helpful discussions. This work was supported by Program for New Century Excellent Talents in University (NCET-11-0597) and the Fundamental Research Funds for the Central Universities (2012RC1002).

References

[1] Q. Yan, L. Wu, and L. Yi, "Research on the human dynamics in mobile communities based on social identity," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 672756, 13 pages, 2012.

[2] Y. P. Xiao, B. Wang, Y. B. Liu et al., "Analyzing, modeling, and simulation for human dynamics in social network," *Abstract and Applied Analysis*, vol. 2012, Article ID 208791, 16 pages, 2012.

[3] Y. P. Xiao, B. Wang, B. Wu, Z. Yan, S. Jia, and Y. Liu, "A hybrid human dynamics model on analyzing hotspots in social networks," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 678286, 13 pages, 2012.

[4] Q. Yan and L. Wu, "Impact of bursty human activity patterns on the popularity of online content," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 872908, 11 pages, 2012.

[5] A. L. Barabási, "The origin of bursts and heavy tails in human dynamics," *Nature*, vol. 435, no. 7039, pp. 207–211, 2005.

[6] H. H. Jo, M. Karsai, J. Kertesz, and K. Kaski, "Circadian pattern and burstiness in mobile phone communication," *New Journal of Physics*, vol. 14, Article ID 013055, 2012.

[7] A. Vazquez, J. G. Oliveira, Z. Dezso, K. I. Goh, I. Kondor, and A. L. Barabasi, "Modeling bursts and heavy tails in human dynamics," *Physical Review E*, vol. 73, Article ID 036127, 2006.

[8] J. G. Oliveira and A. L. Barabási, "Human dynamics: Darwin and Einstein correspondence patterns," *Nature*, vol. 437, no. 7063, p. 1251, 2005.

[9] Z. D. Zhao, H. Xia, M. S. Shang, and T. Zhou, "Empirical analysis on the human dynamics of a large-scale short message communication system," *Chinese Physics Letters*, vol. 28, no. 6, Article ID 068901, 2011.

[10] W. Hong, X. P. Han, T. Zhou, and B. H. Wang, "Heavy-tailed statistics in short-message communication," *Chinese Physics Letters*, vol. 26, Article ID 028902, 2009.

[11] Y. Wu, C. Zhou, J. Xiao, J. Kurths, and H. J. Schellnhuber, "Evidence for a bimodal distribution in human communication," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 107, no. 44, pp. 18803–18808, 1880.

[12] Y. Wu, Q. Ye, L. Li, and J. Xiao, "Power-law properties of human view and reply behavior in online society," *Mathematical Problems in Engineering*, vol. 2012, Article ID 969087, 7 pages, 2012.

[13] B. Min, K. I. Goh, and A. Vazquez, "Spreading dynamics following bursty human activity patterns," *Physical Review E*, vol. 83, no. 3, Article ID 036102, 2011.

[14] A. Vazquez, B. Racz, A. Lukacs, and A. L. Barabasi, "Impact of non-Poissonian activity patterns on spreading processes," *Physical Review Letters*, vol. 98, Article ID 158702, 2007.

[15] L. Weng, A. Flammini, A. Vespignani, and F. Menczer, "Competition among memes in a world with limited attention," *Scientific Reports*, vol. 2, article 335, 2012.

[16] Q. Yan, L. L. Yi, and L. R. Wu, "Human dynamic model co-driven by interest and social identity in the MicroBlog community," *Physica A*, vol. 391, pp. 1540–1545, 2012.

[17] H. L. Dong, Z. D. Wang, and H. J. Gao, "Distributed filtering for a class of time-varying systems over sensor networks with quantization errors and successive packet dropouts," *IEEE Transactions on Signal Processing*, vol. 60, no. 6, pp. 3164–3173, 2012.

[18] H. L. Dong, Z. D. Wang, D. W. C. Ho, and H. J. Gao, "Distributed H-infinity state estimation with stochastic parameters and nonlinearities through sensor networks: the finite-horizon case," *Automatica*, vol. 48, no. 8, pp. 1575–1585, 2012.

[19] A. Vazquez, "Polynomial growth in branching processes with diverging reproductive number," *Physical Review Letters*, vol. 96, no. 3, Article ID 038702, 2006.

[20] T. E. Harris, *The Theory of Branching Processes*, Springer, Berlin, Germany, 2002.

[21] W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York, NY, USA, 1967.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 738653, 8 pages
http://dx.doi.org/10.1155/2013/738653

Research Article

BIBO Stability Analysis for Delay Switched Systems with Nonlinear Perturbation

Jincheng Wei,[1] Peng Shi,[2,3] Hamid Reza Karimi,[4] and Bo Wang[1]

[1] *School of Electrical and Information Engineering, Xihua University, Chengdu 610096, China*
[2] *College of Engineering and Science, Victoria University, Melbourne, VIC 8001, Australia*
[3] *School of Electrical and Electronic Engineering, The University of Adelaide, Adelaide, SA 5005, Australia*
[4] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*

Correspondence should be addressed to Hamid Reza Karimi; hamid.r.karimi@uia.no

Received 16 February 2013; Revised 15 April 2013; Accepted 17 April 2013

Academic Editor: Hongli Dong

Copyright © 2013 Jincheng Wei et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The problem of bounded-input bounded-output (BIBO) stability is investigated for a class of delay switched systems with mixed time-varying discrete and constant neutral delays and nonlinear perturbation. Based on the Lyapunov-Krasovskii functional theory, new BIBO stabilization criteria are established in terms of delay-dependent linear matrix inequalities. The numerical simulation is carried out to demonstrate the effectiveness of the results obtained in the paper.

1. Introduction

Time delay is a source of instability and poor performance and appears in many dynamic systems, for example, biological systems, chemical systems, metallurgical processing systems, nuclear reactor systems, and electrical networks [1]. Since the existence of time delays may lead to oscillation, divergence, or instability, considerable effort has been devoted to this area. As an important system performance index, BIBO stability means that any bounded input yields a bounded output and can be considered in many aspects, such as the free system dynamics, the basic single or double loop modulators, and the issues connected with bilinear input/output maps. Consequently, bounded-input bounded-output (BIBO) stability analysis of dynamical systems has attracted many scholars' attention. For instance, in [2], BIBO stability criterion is derived for a three-dimensional fuzzy two-term control system, in [3], the problem on BIBO stabilization for a system with nonlinear perturbations is studied by discussing the existence of the positive definite solution to an auxiliary algebraic Riccati matrix equation, in [4], based on linear matrix inequality techniques, the stabilization criterion for uncertain time-delay system is presented to guarantee that bounded input can lead to bounded output, and in [5], BIBO stability for feedback control systems with time delay is studied through investigating the boundedness of the solutions for a class of nonlinear Volterra integral equations.

Recently, switched system becomes a research hotspot. Its motivation comes from the fact that many practical systems are inherently multimodal and the fact that some of intelligent control methods are based on the idea of switching between different controllers. Up till now, many investigations about stability of multiform switched systems have been carried out; see, for instance, [6–19] and references therein. Hence, it is our intention in this paper to tackle such an important yet challenging problem for BIBO stability analysis of delay switched systems. In addition, perturbations [20–26] and time delays [27–29] exist in many kinds of systems, and this makes the practical control problem complicated and has received much attention from

scholars. Hence, in this paper, the BIBO stability for delay switched system with mixed time-varying discrete and constant neutral delays and nonlinear perturbation is concerned, and some original BIBO stability criteria are established in terms of linear matrix inequalities (LMIs). Finally, some simulation results are given to illustrate the effectiveness of our results. The main contributions of the paper are of two folds: (1) a delay-dependent technique is applied successfully into the analysis results process; (2) a Lyapunov-Krasovskii functional is constructed to derive a new form of the bounded real lemma (BRL) for the system under consideration.

The remainder of this paper is organized as follows. The model under consideration and some preliminaries are provided in Section 2. Section 3 presents the results on stability analysis. Section 4 gives an illustrative example. At last we conclude the paper in Section 5.

Notations used in this paper are fairly standard. Let R^n be the n-dimensional Euclidean space, $R^{n\times m}$ represents the set of $n \times m$ real matrices, the symbol * denotes the elements below the main diagonal of a symmetric block matrix, $A > 0$ means that A is a real symmetric positive definitive matrix, and I denotes the identity matrix with appropriate dimensions. diag$\{\cdots\}$ denotes the diagonal matrix. $E\{\cdot\}$ refers to the expectation operator with respect to some probability measure P. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. The superscript T stands for matrix transposition. $L_{n,h} = L([-h,0],R^n)$ denotes the Banach space of continuous functions mapping the interval $[-h,0]$ into R^n with the topology of uniform convergence.

2. Model Description and Preliminaries

First, consider the following delay switched system with nonlinear perturbation:

$$\begin{aligned}\dot{x}(t) - C_{\sigma(t)}\dot{x}(t-d) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-\tau(t)) \\ &\quad + f_{\sigma(t)}(t,x(t)) + H_{\sigma(t)}u(t), \\ u(t) &= L_{\sigma(t)}x(t) + r(t), \\ Y(t) &= Jx(t), \\ x(t_0+\theta) &= \varphi(\theta), \quad \theta \in [-h \ \ 0],\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, d is the neutral delay, $0 \le \tau(t) \le h$ is the time-varying discrete delay, $\varphi(\theta) \in L_{n,h}$ is the initial condition, $\sigma(t) : [0,+\infty) \to M = \{1,2,\ldots,m\}$ is the switching signal, $u(t) \in R^l$ is the control input, $Y(t) \in R^m$ is the system output, $r(t) \in R^l$ is the reference input, and $f(t) \in R^n$ is the nonlinear time-varying perturbation, which satisfies $\|f(t,x(t))\| \le \beta\|x(t)\|$, where β is a positive scalar.

Model (1) can be represented as follows:

$$\begin{aligned}\dot{x}(t) &= y(t), \\ y(t) - C_{\sigma(t)}y(t-d) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-\tau(t)) \\ &\quad + f_{\sigma(t)}(t) + H_{\sigma(t)}u(t).\end{aligned} \tag{2}$$

In this paper, the following well-known lemmas and definitions are needed.

Lemma 1 (see [30]). *For any constant matrices E, G, and F with appropriate dimensions with $F^TF \le kI$, then*

$$2x^TEFGy \le cxEE^Tx + \frac{k}{c}y^TG^TGy, \tag{3}$$

where $x \in R^n$ and $y \in R^n$ and c and k are positive scalars.

Lemma 2 (see [31]). *For any positive definite matrix $\Phi \in R^{n\times n}$, a positive scalar γ, and the vector function $w : [0,\gamma] \to R^n$ such that the integrations concerned are well defined, then*

$$\left(\int_0^\gamma w(s)\,ds\right)^T \Phi \left(\int_0^\gamma w(s)\,ds\right) \le \gamma \int_0^\gamma w^T(s)\,\Phi w(s)\,ds. \tag{4}$$

Definition 3 (see [32]). A real-valued vector $r(t) \in L_\infty^n$, if $\|r\|_\infty = \sup_{t_0\le t<\infty}\|r(t)\| < +\infty$.

Definition 4 (see [32]). The control system with reference input $r(t)$ is BIBO stable, if there exist some positive constants θ_1, θ_2, satisfying

$$\|Y(t)\| \le \theta_1\|r(t)\|_\infty + \theta_2 \tag{5}$$

for any reference input $r(t) \in L_\infty^n$.

Assumption 5. We assume that for system (1) there exist Hurwitz linear convex combinations of A_i; that is,

$$\begin{aligned}&\gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(A_1,A_2,\ldots,A_m) \\ &\quad = \{\alpha_1A_1 + \alpha_2A_2 + \cdots + \alpha_mA_m : \alpha_1,\alpha_2,\ldots\alpha_m \in [0,1], \\ &\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\}.\end{aligned} \tag{6}$$

3. Main Results

In this section, we will establish some BIBO stability criteria using Lyapunov-Krasovskii functional theory and linear matrix inequalities.

Theorem 6. *For given positive scalars h and k, switched system (1) is BIBO stable, if there exist $A \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(A_1,A_2,\ldots,A_m)$, $B \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(B_1,B_2,\ldots,B_m)$, $C \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(C_1,C_2,\ldots,C_m)$, $f \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(f_1,f_2,\ldots,f_m)$, $H \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(H_1,H_2,\ldots,H_m)$, $L \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(L_1,L_2,\ldots,L_m)$, positive scalars ε, σ, matrices P_2, P_3, U, V, W,*

and symmetric positive definite matrices P, R, M, S, Q, satisfying

$$\begin{gathered}\Sigma + \Xi + \Xi^T + he^{kh}W < 0,\\ \begin{bmatrix} W & U \\ * & S - R_{22} \end{bmatrix} > 0,\\ \begin{bmatrix} W & V \\ * & S \end{bmatrix} > 0,\end{gathered} \tag{7}$$

where

$$\begin{aligned}
\Sigma &= \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \Sigma_{1,3} & \Sigma_{1,4} & \Sigma_{1,5} & \Sigma_{1,6} & \Sigma_{1,7} \\ * & \Sigma_{2,2} & \Sigma_{2,3} & \Sigma_{2,4} & \Sigma_{2,5} & \Sigma_{2,6} & \Sigma_{2,7} \\ * & * & \Sigma_{3,3} & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{4,4} & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{5,5} & 0 & 0 \\ * & * & * & * & * & \Sigma_{6,6} & 0 \\ * & * & * & * & * & * & \Sigma_{7,7} \end{bmatrix},\\
\Sigma_{1,1} &= P_2A + A^TP_2^T + P_2HL + L^TH^TP_2^T\\
&\quad + Q + \varepsilon\beta^2 + kP + h^2e^{kh}N,\\
\Sigma_{1,2} &= P - P_2 + A^TP_3^T + L^TH^TP_3^T,\\
\Sigma_{1,3} &= P_2B + R_{12}^T,\\
\Sigma_{2,2} &= he^{kh}S - P_3 - P_3^T + M,\\
\Sigma_{2,3} &= P_3B,\\
\Sigma_{3,3} &= hR_{11} - R_{12} - R_{12}^T,\\
\Sigma_{1,4} &= P_2C,\\
\Sigma_{2,4} &= P_3C,\\
\Sigma_{4,4} &= -e^{-kd}M,\\
\Sigma_{1,5} &= P_2,\\
\Sigma_{2,5} &= P_3,\\
\Sigma_{5,5} &= -\varepsilon I,\\
\Sigma_{1,6} &= 0,\\
\Sigma_{2,6} &= 0,\\
\Sigma_{6,6} &= -e^{-kh}Q,\\
\Sigma_{1,7} &= P_2H,\\
\Sigma_{2,7} &= P_3H,\\
\Sigma_{7,7} &= -\sigma I,\\
\Xi &= \begin{bmatrix} U & 0 & -U+V & 0 & 0 & -V & 0 \end{bmatrix}.
\end{aligned} \tag{8}$$

Proof. Since

$$\begin{gathered}
A \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(A_1, A_2, \ldots, A_m),\\
B \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(B_1, B_2, \ldots, B_m),\\
C \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(C_1, C_2, \ldots, C_m),\\
L \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(L_1, L_2, \ldots, L_m),\\
H \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(H_1, H_2, \ldots, H_m),\\
f \in \gamma_{\alpha_1,\alpha_2,\ldots,\alpha_m}(f_1, f_2, \ldots, f_m),
\end{gathered} \tag{9}$$

there exist $\alpha_i \in [0,1]$, $i = 1, \ldots, m$, satisfing

$$\begin{gathered}
\sum_{i=1}^{m}\alpha_i = 1, \qquad A = \sum_{i=1}^{m}\alpha_iA_i, \qquad B = \sum_{i=1}^{m}\alpha_iB_i,\\
C = \sum_{i=1}^{m}\alpha_iC_i, \qquad L = \sum_{i=1}^{m}\alpha_iL_i,\\
H = \sum_{i=1}^{m}\alpha_iH_i, \qquad f = \sum_{i=1}^{m}\alpha_if_i.
\end{gathered} \tag{10}$$

From (7), we can obtain

$$\sum_{i=1}^{m}\alpha_i\left(\Sigma_i + \Xi + \Xi^T + he^{kh}W\right) < 0, \tag{11}$$

where

$$\Sigma_i = \begin{bmatrix} \Sigma_{i,1,1} & \Sigma_{i,1,2} & \Sigma_{i,1,3} & \Sigma_{i,1,4} & \Sigma_{i,1,5} & \Sigma_{i,1,6} & \Sigma_{i,1,7} \\ * & \Sigma_{i,2,2} & \Sigma_{i,2,3} & \Sigma_{i,2,4} & \Sigma_{i,2,5} & \Sigma_{i,2,6} & \Sigma_{i,2,7} \\ * & * & \Sigma_{i,3,3} & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{i,4,4} & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{i,5,5} & 0 & 0 \\ * & * & * & * & * & \Sigma_{i,6,6} & 0 \\ * & * & * & * & * & * & \Sigma_{i,7,7} \end{bmatrix},$$

$$\begin{aligned}
\Sigma_{i,1,1} &= P_2A_i + A_i^TP_2^T + P_2H_iL_i \\
&\quad + L_i^TH_i^TP_2^T + Q + \varepsilon\beta^2 + kP + h^2e^{kh}N, \\
\Sigma_{i,1,2} &= P - P_2 + A_i^TP_3^T + L_i^TH_i^TP_3^T, \\
\Sigma_{i,1,3} &= P_2B_i + R_{12}^T, \\
\Sigma_{i,2,2} &= he^{kh}S - P_3 - P_3^T + M, \\
\Sigma_{i,2,3} &= P_3B_i, \\
\Sigma_{i,3,3} &= hR_{11} - R_{12} - R_{12}^T, \\
\Sigma_{i,1,4} &= P_2C_i, \\
\Sigma_{i,2,4} &= P_3C_i, \\
\Sigma_{i,4,4} &= -e^{kd}M, \\
\Sigma_{i,1,5} &= P_2, \\
\Sigma_{i,2,5} &= P_3, \\
\Sigma_{i,5,5} &= -\varepsilon I, \\
\Sigma_{i,1,6} &= 0, \\
\Sigma_{i,2,6} &= 0, \\
\Sigma_{i,6,6} &= -e^{-kh}Q, \\
\Sigma_{i,1,7} &= P_2H_i, \\
\Sigma_{i,2,7} &= P_3H_i, \\
\Sigma_{i,7,7} &= -\sigma I.
\end{aligned} \tag{12}$$

Let

$$\begin{aligned}
\Omega_i = \Big\{ & q^T \mid q^T\left(\Sigma_i + \Xi + \Xi^T + hW\right)q < 0, \\
& q(t) = \left[q_1^T, \ldots, q_i^T, \ldots, q_7^T\right]^T, q_i \in R^n \Big\}.
\end{aligned} \tag{13}$$

We obtain

$$\bigcup_{i=1}^{m}\Omega_i = \frac{R^{7n}}{\{0\}}. \tag{14}$$

Construct a set as

$$\begin{aligned}
\widetilde{\Omega}_1 &= \Omega_1, \\
\widetilde{\Omega}_2 &= \Omega_2 - \widetilde{\Omega}_1, \ldots, \\
\widetilde{\Omega}_i &= \Omega_i - \bigcup_{j=1}^{i-1}\widetilde{\Omega}_j, \ldots, \\
\widetilde{\Omega}_m &= \Omega_m - \bigcup_{j=1}^{m-1}\widetilde{\Omega}_j.
\end{aligned} \tag{15}$$

We get

$$\bigcup_{i=1}^{m}\widetilde{\Omega}_i = \frac{R^{6n}}{\{0\}}, \quad \widetilde{\Omega}_i \cap \widetilde{\Omega}_j = \phi, \quad i \neq j. \tag{16}$$

Construct the switching rule (SR): $\sigma = i$, for all $q \in \widetilde{\Omega}_i$, $i = 1, \ldots, m$. The ith subsystem is activated when $q \in \widetilde{\Omega}_i$, $i = 1, \ldots, m$. Choose the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned}
V(t) &= V_1(t) + V_2(t) + V_3(t) \\
&\quad + V_4(t) + V_5(t) + V_6(t),
\end{aligned} \tag{17}$$

with

$$\begin{aligned}
V_1(t) &= \left(x^T(t) \;\; y^T(t)\right)\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P & 0 \\ P_2^T & P_3^T \end{bmatrix}\left(x^T(t) \;\; y^T(t)\right)^T, \\
V_2(t) &= \int_{-h}^{0}\int_{t+\beta}^{t} y^T(\alpha)\, e^{k(\alpha-t+h)} Sy(\alpha)\, d\alpha\, d\beta, \\
V_3(t) &= \int_{-h}^{t}\int_{\beta-\tau(\beta)}^{\beta} \eta^T e^{k(\beta-t)} R\eta\, d\alpha\, d\beta, \\
V_4(t) &= \int_{t-d}^{t} y^T(s)\, e^{k(s-t)} My(s)\, ds, \\
V_5(t) &= \int_{-h}^{0}\int_{t+\beta}^{t} \xi^T(\alpha)\, e^{k(\alpha-t+h)} W\xi(\alpha)\, d\alpha\, d\beta, \\
V_6(t) &= \int_{t-h}^{t} x^T(s)\, e^{k(s-t)} Qx(s)\, ds,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\eta &= \left[x(\beta - \tau(\beta)) \;\; y(\alpha)\right]^T, \\
\xi &= \left[x^T(t) \;\; y^T(t) \;\; x^T(t-\tau(t)) \;\; y^T(t-d) \;\; f^T(t) \;\; x(t-h) \;\; r(t)\right]^T.
\end{aligned} \tag{19}$$

The derivative of $V(t)$ along the trajectory of the ith subsystem is given by

$$\begin{aligned}
\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \\
&\quad + \dot{V}_4(t) + \dot{V}_5(t) + \dot{V}_6(t),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}\dot{V}_1(t) &= 2\left[x^T(t) \quad y^T(t)\right]\begin{bmatrix} P & P_2 \\ 0 & P_3\end{bmatrix}\begin{bmatrix} y(t) \\ 0\end{bmatrix} \\ &= 2x^T(t)Py(t) + 2\left(x^T(t)P_2 + y^T(t)P_3\right) \\ &\quad \times \left(-y(t) + \left(A_i + H_iL_i\right)x(t)\right. \\ &\quad + B_ix(t-\tau(t)) + C_iy(t-d) \\ &\quad \left. + f_i(t) + H_ir(t)\right),\end{aligned} \tag{21}$$

$$\begin{aligned}\dot{V}_2(t) &= hy^T(t)e^{kh}Sy(t) \\ &\quad - \int_{t-\tau(t)}^{t} y^T(s)e^{k(s-t+h)}Sy(s)\,ds \\ &\quad - \int_{t-h}^{t-\tau(t)} y^T(s)e^{k(s-t+h)}Sy(s)\,ds - kV_2(t) \\ &\le hy^T(t)e^{kh}Sy(t) \\ &\quad - \int_{t-\tau(t)}^{t} y^T(s)Sy(s)\,ds \\ &\quad - \int_{t-h}^{t-\tau(t)} y^T(s)Sy(s)\,ds - kV_2(t),\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{V}_3(t) &= \tau(t)x^T(t-\tau(t))R_{11}x(t-\tau(t)) \\ &\quad + 2x^T(t-\tau(t))R_{12}x(t) \\ &\quad - 2x^T(t-\tau(t))R_{12}x(t-\tau(t)) \\ &\quad + \int_{t-\tau(t)}^{t} y(s)R_{22}y(s)\,ds - kV_3(t) \\ &\le hx^T(t-\tau(t))R_{11}x(t-\tau(t)) \\ &\quad + 2x^T(t)R_{12}^Tx(t-\tau(t)) \\ &\quad - 2x^T(t-\tau(t))R_{12}x(t-\tau(t)) \\ &\quad + \int_{t-\tau(t)}^{t} y(s)R_{22}y(s)\,ds - kV_3(t),\end{aligned} \tag{23}$$

$$\dot{V}_4(t) = y^T(t)My(t) - y^T(t-d)e^{-kd}My(t-d) - kV_4(t), \tag{24}$$

$$\begin{aligned}\dot{V}_5(t) &= h\xi^T(t)e^{kh}W\xi(t) - \int_{t-\tau(t)}^{t} \xi^T(s)e^{k(s-t+h)}W\xi(s)\,ds \\ &\quad - \int_{t-h}^{t-\tau(t)} \xi^T(s)e^{k(s-t+h)}W\xi(s)\,ds - kV_5(t) \\ &\le h\xi^T(t)e^{kh}W\xi(t) - \int_{t-\tau(t)}^{t} \xi^T(s)W\xi(s)\,ds \\ &\quad - \int_{t-h}^{t-\tau(t)} \xi^T(s)W\xi(s)\,ds - kV_5(t),\end{aligned} \tag{25}$$

$$\dot{V}_6(t) = x^T(t)Qx(t) - x^T(t-h)e^{-kh}Qx(t-h) - kV_6(t). \tag{26}$$

According to Leibniz-Newton formula, we have

$$\begin{aligned}&2\xi^TU\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} y^T(s)\,ds\right] = 0, \\ &2\xi^TV\left[x(t-\tau(t)) - x(t-h) - \int_{t-h}^{t-\tau(t)} y^T(s)\,ds\right] = 0.\end{aligned} \tag{27}$$

From the obtained derivative terms in (21)–(26) and adding the left-hand side of (27) into (20), we obtain the following result:

$$\begin{aligned}\dot{V}(t) &\le \xi^T\left(\Sigma_i + \Omega + \Omega^T + he^{kh}W\right)\xi \\ &\quad - \int_{t-\tau(t)}^{t} \zeta^T\Phi_1\zeta\,ds - \int_{t-h}^{t-\tau(t)} \zeta^T\Phi_2\zeta\,ds \\ &\quad - kV(t) + \sigma\|r(t)\|_\infty^2,\end{aligned} \tag{28}$$

where

$$\begin{aligned}\zeta &= \left[\xi^T \quad y^T(s)\right]^T, \\ \Phi_1 &= \begin{bmatrix} W & U \\ * & S - R_{22}\end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} W & V \\ * & S\end{bmatrix}.\end{aligned} \tag{29}$$

When $\xi \in \bigcup_{i=1}^{m} \widetilde{\Omega}_i$, $i = 1, \ldots, m$, we can obtain

$$\begin{aligned}\dot{V}(t) &\le \sum_{i=1}^{m} a_i\left(\xi^T\left(\Sigma_i + \Omega + \Omega^T + he^{kh}W\right)\xi\right. \\ &\quad \left. - \int_{t-\tau(t)}^{t} q^T\Phi_1q\,ds - \int_{t-h}^{t-\tau(t)} q^T\Phi_2q\,ds\right) \\ &= \xi^T\left(\Sigma + \Xi + \Xi^T + he^{kh}W\right)\xi \\ &\quad - \int_{t-\tau(t)}^{t} q^T\Phi_1q\,ds - \int_{t-h}^{t-\tau(t)} q^T\Phi_2q\,ds.\end{aligned} \tag{30}$$

According to (7), we have

$$\dot{V}(t) \le -kV(t) + \sigma\|r(t)\|_\infty^2. \tag{31}$$

Hence,

$$\left(V(t)e^{kt}\right)' \le \left(\dot{V}(t) + kV(t)\right)e^{kt} \le \sigma\|r(t)\|_\infty^2 e^{kt}. \tag{32}$$

Integrating the previous inequality from t_0 to t yields

$$V(t)e^{kt} \le V(t_0)e^{kt_0} + \sigma\|r(t)\|_\infty^2 \int_{t_0}^{t} e^{ks}ds. \tag{33}$$

Then, we have

$$\begin{aligned}\lambda_{\min}(P)\|x(t)\|^2 &\le V(t) \le V(t_0)e^{-k(t-t_0)} \\ &\quad + \sigma\|r(t)\|_\infty^2 \int_{t_0}^{t} e^{-k(t-s)}ds \\ &\le V(t_0)e^{-k(t-t_0)} + \frac{\sigma\|r(t)\|_\infty^2}{k}.\end{aligned} \tag{34}$$

Define

$$\psi = \max\left\{\sup_{h\le\theta\le 0}\|\varphi(t_0+\theta)\|, \sup_{h\le\theta\le 0}\|\varphi'(t_0+\theta)\|\right\}. \quad (35)$$

According to (17), we have

$$V(t_0) \le \left[\lambda_{\max}(P) + h^2 e^{kh}\lambda_{\max}(S) + h^2\lambda_{\max}(R) + h^2 e^{kh}\lambda_{\max}(W) + h\lambda_{\max}(Q) + h\lambda_{\max}(M)\right]\psi^2. \quad (36)$$

Hence, the following inequality can be concluded:

$$\begin{aligned}\|x(t)\|^2 &\le \frac{a}{\lambda_{\min}(P)}\psi^2 + \frac{\sigma}{k\lambda_{\min}(P)}\|r(t)\|_\infty^2 \\ &\le \left(\sqrt{\frac{a}{\lambda_{\min}(P)}}\psi + \sqrt{\frac{\sigma}{k\lambda_{\min}(P)}}\|r(t)\|_\infty\right)^2,\end{aligned} \quad (37)$$

where

$$\begin{aligned}a &= \lambda_{\max}(P) + h^2 e^{kh}\lambda_{\max}(S) + h^2\lambda_{\max}(R) \\ &\quad + h^2 e^{kh}\lambda_{\max}(W) + h\lambda_{\max}(Q) + h\lambda_{\max}(M).\end{aligned} \quad (38)$$

Then,

$$\|Y\| \le \|J\|\,\|x\| \le \theta_1 + \theta_2\|r(t)\|_\infty, \quad (39)$$

with

$$\theta_1 = \|J\|\sqrt{\frac{a}{\lambda_{\min}(P)}}\psi, \qquad \theta_2 = \|J\|\sqrt{\frac{\sigma}{k\lambda_{\min}(P)}}. \quad (40)$$

Therefore, switched system (1) is BIBO stable. This completes the proof. □

4. Simulation Results

As an example, let us consider system (1) with the following parameters:

$$\begin{aligned}A_1 &= \begin{bmatrix}-1.5 & 0.5\\ 0 & -1.5\end{bmatrix}, & B_1 &= \begin{bmatrix}-0.2 & 0.3\\ 0 & -0.4\end{bmatrix},\\ C_1 &= \begin{bmatrix}0.3 & 0.2\\ 0 & -0.2\end{bmatrix}, & H_1 &= \begin{bmatrix}0.4 & 0\\ 0 & 0.4\end{bmatrix},\\ L_1 &= \begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}, & A_2 &= \begin{bmatrix}-1 & 0.5\\ 0.5 & -2\end{bmatrix},\\ B_2 &= \begin{bmatrix}-0.4 & 0\\ 0.3 & -0.4\end{bmatrix}, & C_2 &= \begin{bmatrix}-0.4 & 0.3\\ 0 & -0.3\end{bmatrix},\\ H_2 &= \begin{bmatrix}-0.3 & 0\\ 0 & -0.3\end{bmatrix}, & L_2 &= \begin{bmatrix}0.3 & 0\\ 0 & 0.3\end{bmatrix},\\ J &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, & d &= 1.5.\end{aligned} \quad (41)$$

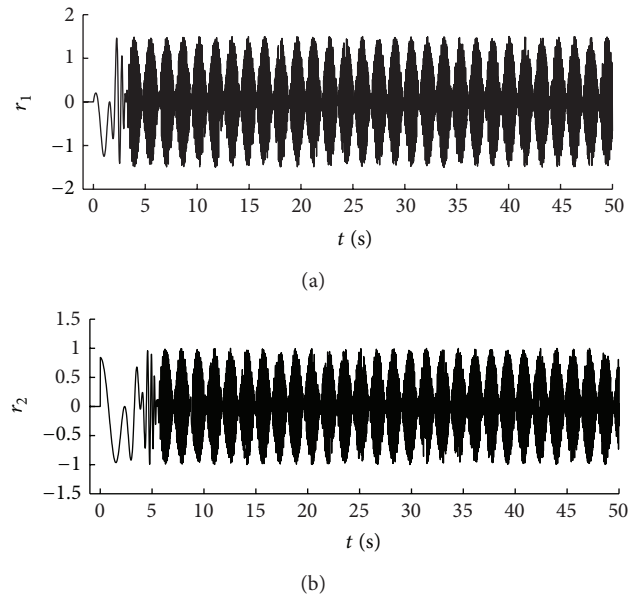


Figure 1: Time response of the reference input variable $r(t)$.

Remark 7. When the BIBO parameter k and the constant parameter β are given, the upper bound of time delay h of system (1) can be determined by solving the following optimization problem:

$$\max\ h \quad \text{when (7) is satisfied, } k \text{ and } \beta \text{ are fixed.} \quad (42)$$

Now we consider the influence of parameters k and β on the maximal allowable delay in Tables 1 and 2.

Remark 8. From Tables 1 and 2, it can be seen that the maximal allowable delay decreases with the rise of the parameter k and increases as the parameter β is reduced.

Then, we carry out some numerical simulation to verify the proposed methodology. The numerical simulation is with initial value $\varphi(\theta) = [-1.5; 1]^T$, $t \in (-1.5, 0)$, and following parameters

$$\begin{aligned}\tau(t) &= 1.1 + 0.4\cos^2(5t),\\ f(t, x(t)) &= \frac{\beta[|x(t)+1| + |x(t)-1|]}{2},\\ r(t) &= \left[1.5\sin(2t)\cos\left(e^t\right), \cos(2t)\sin\left(\frac{e^t}{t+1}\right)\right]^T,\\ k &= 0.1, \qquad \beta = 0.1.\end{aligned} \quad (43)$$

The switching signals are produced randomly with switching interval 0.2 seconds.

Remark 9. Figure 1 depicts the time response of system reference input variable $r(t)$, and Figure 2 depicts the time response of switching signals. sw_1 denotes the switching

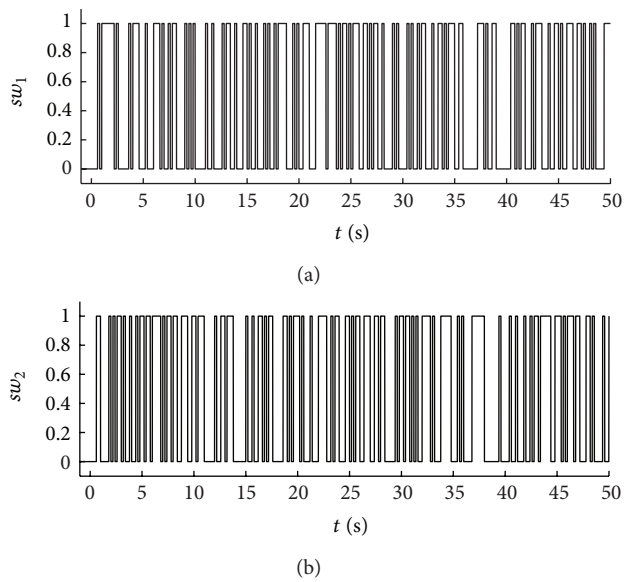


Figure 2: Time response of the switching variables sw_1 and sw_2.

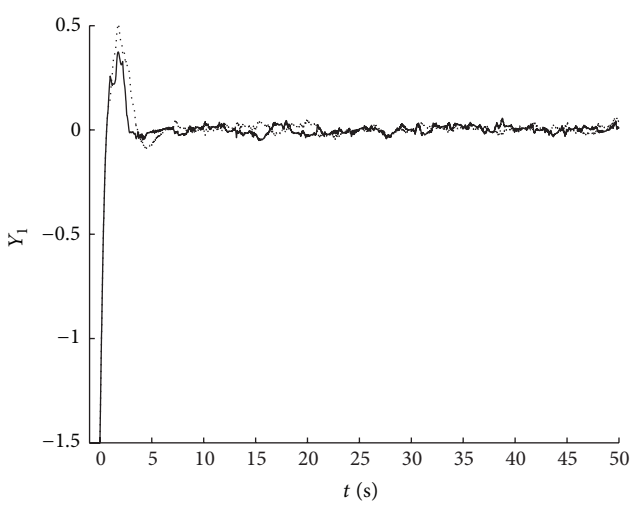


Figure 3: Time response of the output variable $Y_1(t)$.

Table 1: The maximal allowable delay for different parameters k when β is 0.1.

$k = 0$	$k = 0.1$	$k = 0.2$	$k = 0.5$
$h_{\max} = 3.3395$	$h_{\max} = 2.5482$	$h_{\max} = 2.1313$	$h_{\max} = 1.5197$

Table 2: The maximal allowable delay for different nonlinear parameters β when k is 0.1.

$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.5$
$h_{\max} = 2.6608$	$h_{\max} = 2.5482$	$h_{\max} = 2.4370$	$h_{\max} = 2.1385$

signal added to the system for the first time, and sw_2 denotes the switching signal added to the system for the second time; Figure 3 depicts the time response of system output variable $Y_1(t)$, and Figure 4 depicts the time response of system output variable $Y_2(t)$. The solid line denotes the output variable of the switched system with the switching signal sw_1, and the dashed line denotes the output variable of the switched system with the switching signal sw_2. From the figures it can be seen that the system output jitters in a range with a given bounded input after a period of time, which means that the system is BIBO stable and demonstrates the effectiveness of our theoretical results.

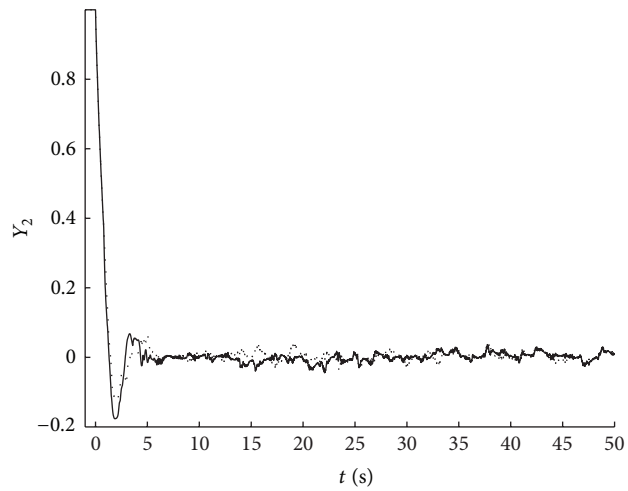


Figure 4: Time response of the output variable $Y_2(t)$.

5. Conclusions

We have studied bounded-input bounded-output stability for a class of delay switched systems with nonlinear perturbation. Based on the Lyapunov-Krasovskii functional theory, new BIBO stabilization criteria were established in terms of delay-dependent linear matrix inequalities. Some numerical simulations have been conducted to demonstrate the effectiveness of the theoretical results obtained in this paper. Future work will investigate fault detection for delay switched systems.

Acknowledgments

This work was partially supported by the Key Projects of Xihua University (Z1120946), the National Key Basic Research Program, China (2012CB215202), the 111 Project (B12018), and the National Natural Science Foundation of China (61174058 and 61134001).

References

[1] V. Kolmanovskiĭ and A. Myshkis, *Applied Theory of Functional-Differential Equations*, vol. 85, Kluwer Academic, Dordrecht, The Netherlands, 1992.

[2] X. Zhang, S. Li, and H. Li, "Structure and BIBO stability of a three-dimensional fuzzy two-term control system," *Mathematics and Computers in Simulation*, vol. 80, no. 10, pp. 1985–2004, 2010.

[3] P. Li and S.-M. Zhong, "BIBO stabilization for system with multiple mixed delays and nonlinear perturbations," *Applied Mathematics and Computation*, vol. 196, no. 1, pp. 207–213, 2008.

[4] P. Li, S.-M. Zhong, and J.-Z. Cui, "Delay-dependent robust BIBO stabilization of uncertain system via LMI approach," *Chaos, Solitons and Fractals*, vol. 40, no. 2, pp. 1021–1028, 2009.

[5] E. Awwad, I. Győri, and F. Hartung, "BIBO stabilization of feedback control systems with time dependent delays," *Applied Mathematics and Computation*, vol. 219, no. 8, pp. 3664–3676, 2012.

[6] M. Mahmoud and P. Shi, "Asynchronous H_∞ filtering of discrete-time switched systems," *Signal Processing*, vol. 92, no. 10, pp. 2356–2364, 2012.

[7] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol. 48, no. 6, pp. 1132–1137, 2012.

[8] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability and stabilization of switched linear systems with mode-dependent average dwell time," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1809–1815, 2012.

[9] D. Du, B. Jiang, and P. Shi, "Sensor fault estimation and compensation for time-delay switched systems," *International Journal of Systems Science*, vol. 43, no. 4, pp. 629–640, 2012.

[10] Z. Wu, P. Shi, H. Su, and J. Chu, "Delay-dependent stability analysis for switched neural networks with time-varying delay," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 41, no. 6, pp. 1522–1530, 2011.

[11] J. Lian, Z. Feng, and P. Shi, "Observer design for switched recurrent neural networks: an average dwell time approach," *IEEE Transactions on Neural Networks*, vol. 22, no. 10, pp. 1547–1556, 2011.

[12] D. Wang, W. Wang, and P. Shi, "Delay-dependent model reduction for continuous-time switched state-delayed systems," *International Journal of Adaptive Control and Signal Processing*, vol. 25, no. 9, pp. 843–854, 2011.

[13] D. Du, B. Jiang, and P. Shi, "Fault estimation and accommodation for switched systems with time-varying delay," *International Journal of Control, Automation and Systems*, vol. 9, no. 3, pp. 442–451, 2011.

[14] C. K. Ahn and M. K. Song, "L_2-L_∞ filtering for time-delayed switched hopfield neural networks," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 4, pp. 1831–1843, 2011.

[15] Z. G. Wu, P. Shi, H. Su, and J. Chu, "Delay-dependent exponential stability analysis for discrete-time switched neural networks with time-varying delay," *Neurocomputing*, vol. 74, no. 10, pp. 1626–1631, 2011.

[16] G. Zong, L. Hou, and J. Li, "A descriptor system approach to L_2-L_∞ filtering for uncertain discrete-time switched system with mode-dependent time-varying delays," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 5A, pp. 2213–2224, 2011.

[17] Y. Wang, W. Wang, and D. Wang, "LMI approach to design fault detection filter for discrete-time switched systems with state delays," *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 1, pp. 387–397, 2010.

[18] L. Zhang and B. Jiang, "Stability of a class of switched linear systems with uncertainties and average dwell time switching," *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 2, pp. 667–676, 2010.

[19] F. Long, C. Li, and C. Cui, "Dynamic output feedback H_∞ control for a class of switched linear systems with exponential uncertainty," *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 4, pp. 1727–1736, 2010.

[20] H. R. Karimi, "Robust synchronization and fault detection of uncertain master-slave systems with mixed time-varying delays and nonlinear perturbations," *International Journal of Control, Automation and Systems*, vol. 9, no. 4, pp. 671–680, 2011.

[21] H. R. Karimi, "Adaptive H_∞ synchronization problem of uncertain master-slave systems with mixed time-varying delays and nonlinear perturbations: an LMI approach," *International Journal of Automation and Computing*, vol. 8, no. 4, pp. 381–390, 2011.

[22] P. Shi, X. Luan, and F. Liu, "H_∞ filtering for discrete-time systems with stochastic incomplete measurement and mixed delays," *IEEE Transactions on Industrial Electronics*, vol. 59, pp. 2732–2739, 2012.

[23] H. R. Karimi, M. Zapateiro, and N. Luo, "Stability analysis and control synthesis of neutral systems with time-varying delays and nonlinear uncertainties," *Chaos, Solitons and Fractals*, vol. 42, no. 1, pp. 595–603, 2009.

[24] M. Basin, P. Shi, and P. Soto, "Central suboptimal H_∞ filtering for nonlinear polynomial systems with multiplicative noise," *Journal of the Franklin Institute*, vol. 347, no. 9, pp. 1740–1754, 2010.

[25] H. R. Karimi, N. Luo, M. Zapateiro, and L. Zhang, "H_∞ control design for building structures under seismic motion with wireless communication," *International Journal of Innovative Computing, Information and Control*, vol. 7, pp. 5269–5284, 2011.

[26] H. R. Karimi, "A sliding mode approach to H_∞ synchronization of master-slave time-delay systems with Markovian jumping parameters and nonlinear uncertainties," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 349, no. 4, pp. 1480–1496, 2012.

[27] X. Li and H. Gao, "A new model transformation of discrete-time systems with time-varying delay and its application to stability analysis," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2172–2178, 2011.

[28] X. Li, H. Gao, and X. Yu, "A unified approach to the stability of generalized static neural networks with linear fractional uncertainties and delays," *IEEE Transactions on System, Man, and Cybernetics B*, vol. 41, no. 5, pp. 1275–1286, 2011.

[29] H. Gao and X. Li, "H_∞ filtering for discrete-time state-delayed systems with finite frequency specifications," *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2935–2941, 2011.

[30] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1994.

[31] K. Gu, "An integral inequality in the stability problem of time-delay systems," in *Proceedings of the 39th IEEE Confernce on Decision and Control*, pp. 2805–2810, Sydney, Australia, December 2000.

[32] H. S. Wu and K. Mizukami, "Robust stabilization of uncertain linear dynamical systems," *International Journal of Systems Science*, vol. 24, no. 2, pp. 265–276, 1993.