Journal of Function Spaces

Operator Methods in Approximation Theory

Lead Guest Editor: Vita Leonessa

Guest Editors: Tuncer Acar, Mirella Cappelletti Montano, and Pedro Garrancho

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Editorial Board

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Editorial

Operator Methods in Approximation Theory

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Approximation theory is an intensive research area, developed in different directions by many mathematicians.

For example, approximation and iteration processes arise in a very natural way in many problems dealing with the constructive approximation of functions as well as solutions to (partial) differential equations and integral equations. Moreover, approximation theory can be successfully applied in fixed point theory, in computer aided geometric design, in artificial neural networks, in the study of evolution problems, and in function algebras.

The goal of this special issue is to attract original research and review articles that highlight recent advances in operator methods within approximation theory and related applications.

The interest aroused by the mathematicians who work in this area is remarkable, as evidenced by the thirty-six submissions received.

The papers that have been accepted for the publication in the issue recover the following topics: *means and inequalities, approximation by positive operators, function algebras, fixed point theorems, and iteration processes*.

In what follows we give a brief description of the contents of this special issue.

In the review article titled "On Sequences of J. P. King-Type Operators," T. Acar et al. provide an essential exposition of a series of investigations developed in the last fifteen years after the release of a paper written by J. P. King, where a modification of the classical Bernstein operators was considered in order to get better approximation properties than the original ones. After a brief history devoted to the sequences of positive linear operators fixing certain (polynomial, exponential, or more general) functions obtained by applying King's approach, the authors illustrate certain King-type modifications of the well-known Bernstein and Szász-Mirakjan operators.

A. A. Bakery and M. M. Mohammed in the paper "Small Pre-Quasi Banach Operator Ideals of Type Orlicz-Cesáro Mean Sequence Spaces" deal with determining sufficient conditions on an Orlicz-Cesàro mean sequence space ces_φ in order that the class S_{ces_φ}, consisting of all bounded linear operators between arbitrary Banach spaces such that the corresponding sequence of s-numbers belongs to ces_φ, forms an operator ideal. Moreover, the authors determine some inclusion relations between pre-quasi operator ideals as well as their duals. Finally, they present sufficient conditions on ces_φ in order that the pre-quasi Banach operator ideal generated by approximation number is small.

The purpose of the paper "Asymptotic Behavior of Almost Quartic $*$-Derivations on Banach $*$-Algebras" by H.-M. Kim et al. is determining, in the context of Banach $*$-algebras, stability theorems of quartic $*$-derivations associated with the quartic functional equation $f(3x-y)+f(x+y)+6f(x-y)=4f(2x-y)+4f(y)+24f(x)$.

S. Z. Ullah et al. generalize and improve some known results concerning integral majorization type and generalized Favard's inequalities for the class of strongly convex functions in the paper titled "Integral Majorization Type Inequalities for the Functions in the Sense of Strong Convexity."

In the paper "Bivariate Chlodowsky-Stancu Variant of (p, q)-Bernstein-Schurer Operators" by T. Vedi-Dilek and E.

Gemikonakli, the bivariate Chlodowsky-Stancu variant of (p,q)-Bernstein-Schurer Operators, as well as a generalization of it, is proposed and some approximation properties are investigated. The paper ends by discussing some numerical results.

N. Özmen in the paper titled “New Generating Function Relations for the q-Generalized Cesàro Polynomials” examines a q-analogue of generalized Cesàro polynomials for which she derives bilinear and bilateral generating functions; in addition, she gets a specific linear q-generating relationship that recovers the basic analogue of certain special functions.

H. J. Lee proves that the k-homogeneous polynomial and analytic numerical index of certain X-valued function algebras, X being a complex Banach space, are the same as those of X in the paper titled “Generalized Numerical Index of Function.”

In the paper “Convergence Analysis of an Accelerated Iteration for Monotone Generalized α-Nonexpansive Mappings with a Partial Order,” a new accelerated iteration process for finding fixed points of monotone generalized α-nonexpansive mappings in ordered Banach spaces is introduced. Y.-A. Chen and D.-J. Wen establish some weak and strong convergence theorems of fixed point for monotone generalized α-nonexpansive mappings in a uniformly convex partially ordered Banach space. Moreover, they provide a numerical example that illustrates the convergence behavior and effectiveness of their method.

C. Zhang and S. Wang in the paper titled “Structure Properties for Binomial Operators” discussed some structures properties of the binomial operators, such as moments representation, derivatives representation, and binary representation. As applications, the authors prove that the binomial operators considered preserve increasing functions, convex functions, and Hölder (continuous) functions.

The purpose of the paper “On New Picard-Mann Iterative Approximations with Mixed Errors for Implicit Midpoint Rule and Applications” written by T. Li and H. Lan is to introduce and study a new class of Picard-Mann iteration processes with mixed errors for the implicit midpoint rules and to analyze the convergence and stability of the proposed method. Some numerical examples and applications to optimal control problems with elliptic boundary value constraints are presented, and they show that the Picard-Mann iteration process discussed in the article is more effective than other related iterative processes.

J.-L. Wang et al. in the paper titled “On Approximating the Toader Mean by Other Bivariate Means” provide several sharp bounds for the Toader mean by using certain combinations of the arithmetic, quadratic, contraharmonic, and Gaussian arithmetic geometric means.

The paper “ C^* -Basic Construction from the Conditional Expectation on the Drinfeld Double” by Q. Xin et al. deals with the following topics. Let G be a finite group and H a subgroup of G. Starting from the Drinfeld double $D(G)$ and the crossed product $D(G;H)$ of $C(G)$ and $\mathbb{C}H$, and considering the C^*-basic construction C^*fbffk$D(G)$, $e$fbfft from the conditional expectation E of $D(G)$ onto $D(G;H)$, the authors construct a crossed product C^*-algebra $C(G/H \times G) \rtimes \mathbb{C}G$, in such a way that C^*fbffk$D(G)$, $e$fbfft is C^*-algebra isomorphic to $C(G/H \times G) \rtimes \mathbb{C}G$.

In the paper “On a New Stability Problem of Radical nth-Degree Functional Equation by Brzdęk’s Fixed-Point Method,” D. Kang and H. B. Kim, given a positive integer n, discuss the general solutions to the radical n-th degree functional equation $f(\sqrt[n]{x^n + y^n}) = f(x) + f(y)$ and prove new Hyers-Ulam type stability results by using Brzdęk’s fixed point method.

In the context of singular Hadamard fractional boundary value problems, J. Mao et al. establish, by using an iterative algorithm, the existence and uniqueness of the exact iterative solution in the paper titled “The Unique Positive Solution for Singular Hadamard Fractional Boundary Value Problems.” Moreover, they show the iterative sequences converge uniformly to the exact solution, and they provide estimation of the approximation error and the convergence rate.

Conflicts of Interest

The Guest Editors declare that they have no conflicts of interest regarding the publication of this special issue.

Acknowledgments

We are very grateful to the mathematical community for the great interest shown in this special issue. In particular, we want to thank all the authors of the published papers for their contribution in this field. Also, our gratitude goes to the reviewers, for their precious help in handling all the submitted manuscripts, and to Hindawi who supported us throughout the development of this special issue. Finally, the Lead Guest Editor sincerely thanks her Guest Editors for agreeing to join her in this project.

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Research Article

On a New Stability Problem of Radical *n*th-Degree Functional Equation by Brzdęk's Fixed-Point Method

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In this paper, we introduce the radical nth-degree functional equation of the form $f(\sqrt[n]{x^n + y^n}) = f(x) + f(y)$ with a positive integer n, discuss its general solutions, and prove new Hyers-Ulam-type stability results for the equation by using Brzdęk's fixed-point method.

1. Introduction

In mathematical analysis, we often deal with the following question: Under what conditions should a mathematical object satisfying certain properties approximately be close to the one satisfying the properties exactly? If we consider a functional equation, then we can ask the same question: When could the approximates to a functional equation be close to the solution of the equation? Then, there would be an issue of error estimation between the approximates and the solution of a functional equation that we will investigate in this paper, not only the process of finding the solution of the equation. Such a fundamental question for functional equations led to the theory of Hyers-Ulam stability.

The Hyers-Ulam stability problem of functional equations was first raised in a talk at the University of Wisconsin. In 1940, a Polish-American mathematician called Ulam [1] proposed the stability problem of a group homomorphism: When does a linear mapping near an approximately linear mapping exist?

In 1941, Hyers [2] gave the first, affirmative, and partial solution to Ulam's question with an additive function, $f(x + y) = f(x) + f(y)$, in Banach spaces as in the following theorem.

Theorem 1. *Assume that E_1 and E_2 are Banach spaces. If a function $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$\left\| f(x + y) - f(x) - f(y) \right\| \leq \epsilon \tag{1}$$

for some $\epsilon \geq 0$ and for all $x, y \in E_1$, then the limit $a(x) = \lim_{n\longrightarrow\infty} 2^{-n} f(2^n x)$ exists for each $x \in E_1$ and $a : E_1 \longrightarrow E_2$ is the unique additive function (or the solution to Cauchy function) such that

$$\left\| f(x) - a(x) \right\| \leq \epsilon \tag{2}$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then a is linear.

In 1950, Aoki [3] provided a generalization of Hyers' theorem with a positive monotone nondecreasing symmetric function of $\|x\|$ and $\|y\|$ involving a power $0 \leq p < 0$, where Theorem 1 is a special case when $p = 0$. Also see [4, 5] for the generalization in terms of bordering transformations and an approximately linear mapping of addition and scalar multiplication, respectively.

For the last decades, stability problems of various functional equations, not only linear case, have been extensively investigated and generalized by many mathematicians (see [6–10]). One of the functional equations of the form

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y) \tag{3}$$

is called a radical quadratic functional equation and every solution to this functional equation is referred to as a radical quadratic function or mapping. The question of existence and uniqueness of the general solution of the functional equation (3) was answered by Brzdęk (see [11], p.196); i.e., a real-valued function $f:\mathbb{R}\longrightarrow\mathbb{R}$ ($\mathbb{R}$ stands for the set of real numbers) is a solution to (3) if and only if $f(x)=L(x^2)$, $x\in\mathbb{R}$, where the function $L:\mathbb{R}\longrightarrow\mathbb{R}$ is additive; i.e., it satisfies $L(x+y)=L(x)+L(y)$ for all $x,y\in\mathbb{R}$. Gordji and Parviz [12] and Cho et al. [13] investigated the Hyers-Ulam stability problems of the functional equations of the radical type as (3). In particular, Baker [14] applied for the first time a variant of Banach's fixed-point theorem to obtain the stability of a functional equation in a single variable. For the applications and surveys of this approach in detail, see [15–17] where the paper [16] is an updated version of survey [15]. Moreover, Brzdęk and Ciepliński [18] introduced the following existence theorem of the fixed point for nonlinear operator in metric spaces.

Theorem 2 (see [18]). *Let X be a nonempty set, (Y,d) a complete metric space, and $\Lambda:Y^X\longrightarrow Y^X$ a nondecreasing operator satisfying the hypothesis*

$$\begin{gathered}\lim_{n\longrightarrow\infty}\Lambda\delta_n=0\\ \text{for every sequence } \{\delta_n\}_{n\in\mathbb{N}} \text{ with } \lim_{n\longrightarrow\infty}\delta_n=0 \text{ in } Y^X.\end{gathered} \tag{4}$$

Suppose that $\mathcal{T}:Y^X\longrightarrow Y^X$ is an operator satisfying the inequality

$$\begin{gathered}d\left(\mathcal{T}\varepsilon(x),\mathcal{T}\mu(x)\right)\le\Lambda\left(\triangle(\varepsilon,\mu)\right)(x),\\ \varepsilon,\mu\in Y^X,\ x\in X\end{gathered} \tag{5}$$

where $\triangle:(Y^X)^2\longrightarrow\mathbb{R}_+^X$ is a mapping which is defined by

$$\begin{gathered}\triangle(\varepsilon,\mu)(x):=d\left(\varepsilon(x),\mu(x)\right)\\ \text{for } \varepsilon,\mu\in Y^X,\ x\in X.\end{gathered} \tag{6}$$

If there exist functions $\varepsilon:X\longrightarrow\mathbb{R}_+$ and $\phi:X\longrightarrow Y$ such that

$$d\left(\mathcal{T}\phi(x),\phi(x)\right)\le\varepsilon(x) \tag{7}$$

and

$$\varepsilon^*(x):=\sum_{n\in\mathbb{N}_0}\left(\Lambda^n\varepsilon\right)(x)<\infty \tag{8}$$

for all $x\in X$, then the limit

$$\lim_{n\longrightarrow\infty}\left(\mathcal{T}^n\phi\right)(x) \tag{9}$$

exists for each $x\in X$. Moreover, the function $\phi\in Y^X$ defined by

$$\psi(x):=\lim_{n\longrightarrow\infty}\left(\mathcal{T}^n\phi\right)(x) \tag{10}$$

is a fixed point of $\mathcal{T}$ with

$$d\left(\phi(x),\psi(x)\right)\le\varepsilon^*(x) \tag{11}$$

for all $x\in X$.

Then they used this result to prove the stability problem of functional equations in non-Archimedean metric spaces and obtained the fixed-point results in arbitrary metric spaces. Another version of the Brzdęk fixed-point method was also obtained from Theorem 2 (see [19] for details) as follows.

Theorem 3 (see [19]). *Let X be a nonempty set, (Y,d) a complete metric space, and $f_1,f_2:X\longrightarrow X$ given mappings. Suppose that $\mathcal{T}:Y^X\longrightarrow Y^X$ and $\Lambda:\mathbb{R}_+^X\longrightarrow\mathbb{R}_+^X$ are two operators satisfying the following conditions:*

$$\begin{aligned}d\left(\mathcal{T}\xi(x),\mathcal{T}\mu(x)\right)\le{}& d\left(\xi\left(f_1(x)\right),\mu\left(f_1(x)\right)\right)\\ &+d\left(\xi\left(f_2(x)\right),\mu\left(f_2(x)\right)\right)\end{aligned} \tag{12}$$

and

$$\Lambda\delta(x):=\delta\left(f_1(x)\right)+\delta\left(f_2(x)\right) \tag{13}$$

for all $\xi,\mu\in Y^X,\delta\in\mathbb{R}_+^X$, and $x\in X$. If there exist $\varepsilon:X\longrightarrow\mathbb{R}_+$ and $\phi:X\longrightarrow Y$ such that

$$\begin{gathered}d\left(\mathcal{T}\phi(x),\phi(x)\right)\le\varepsilon(x)\\ \varepsilon^*(x):=\sum_{n=0}^{\infty}\left(\Lambda^n\varepsilon\right)(x)<\infty\end{gathered} \tag{14}$$

for all $x\in X$, then the limit $\lim_{n\longrightarrow\infty}(\mathcal{T}^n\phi)(x)$ exists for each $x\in X$. Moreover, the function $\psi(x):=\lim_{n\longrightarrow\infty}(\mathcal{T}^n\phi)(x)$ is a fixed point of $\mathcal{T}$ with

$$d\left(\phi(x),\psi(x)\right)\le\varepsilon^*(x) \tag{15}$$

for all $x\in X$.

Recently, Aiemsomboon and Sintunavarat [20] used Theorem 3 above to investigate a new type of stability for the radical quadratic functional equation of the form (3). We refer to [21, 22] for more results from the Brzdęk fixed-point method in the stability problems of various functional equations such as Drygas functional equations and the general linear equations. Also Kang [23] studied the stability problem for generalized quadratic radical functional equations by using Brzdęk's fixed-point approach.

In this paper, we consider the radical nth-degree functional equation of the form

$$f\left(\sqrt[n]{x^n+y^n}\right)=f(x)+f(y) \tag{16}$$

for all positive integers $n > 0$ and give an application of Brzdęk's fixed-point method for the stability problem of the radical nth-degree functional equation (16) (see [24–26] for the cases of $n = 2, 3, 4$ of (16) with various approaches). As the radical quadratic functional equation (3), a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies (16) if and only if it is of the form

$$f(x) = a(x^n), \quad x \in \mathbb{R} \tag{17}$$

with an additive function $a : \mathbb{R} \longrightarrow \mathbb{R}$. For more extensions and generalizations of the solutions to functional equations of this radical type (16) and a different type of the radical nth-degree functional equation of the form $f(\sqrt[n]{x^n + y^n}) + f(\sqrt[n]{|x^n - y^n|}) = 2f(x) + 2f(y)$, we refer to [27, 28].

The stability results in this article will be an improvement and generalization of the stability problem of the radical quadratic, cubic, and quartic functional equations like (3). Throughout this paper, $\mathbb{N}_0, \mathbb{N}$, and $\mathbb{R}_+$ denote the set of nonnegative integers, the set of positive integers, and the set of nonnegative real numbers, respectively.

2. Stability of the Radical nth-Degree Functional Equation

In this section, we will investigate the stability problems of the radical nth-degree functional equation (16) as introduced earlier; i.e.,

$$f\left(\sqrt[n]{x^n + y^n}\right) = f(x) + f(y) \tag{18}$$

for a positive integer n by using Brzdęk's fixed-point method; see Theorem 3 in the introduction.

Theorem 4. *Let d be a complete metric in $\mathbb{R}$ which is invariant (i.e., $d(x+z, y+z) = d(x, y)$ for $x, y, z \in \mathbb{R}$). Assume that for each positive integer $n \in \mathbb{N}$, $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function such that*

$$M_0 := \{m \in \mathbb{N} : s(m^n) + s(1 + m^n) < 1\} \neq \emptyset \tag{19}$$

where

$$s(l) := \inf\{r \in \mathbb{R}_+ : h(lx^n) \leq r\, h(x^n) \ \text{ for all } x \in \mathbb{R}\}, \tag{20}$$

for $l \in \mathbb{N}$. Also suppose that a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies the inequality

$$d\left(f\left(\sqrt[n]{x^n + y^n}\right), f(x) + f(y)\right) \leq h(x^n) + h(y^n) \tag{21}$$

for all $x, y \in \mathbb{R}_+$. Then, there exists a unique radical nth-degree mapping $R : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that

$$d(f(x), R(x)) \leq s_0 h(x^n) \tag{22}$$

for all $x \in \mathbb{R}_+$, where

$$s_0 := \inf\left\{\frac{1 + s(m^n)}{1 - s(m^n) - s(1 + m^n)} : m \in M_0\right\}. \tag{23}$$

Proof. Let $m \in \mathbb{N}$ be a positive integer. On taking $y = mx$ in the inequality (21), we will see easily that

$$d\left(f\left(\sqrt[n]{(1 + m^n)x^n}\right), f(x) + f(mx)\right) \leq c_m(x) \tag{24}$$

where $c_m(x) = (1 + s(m^n))h(x^n)$ for all $x \in \mathbb{R}_+$. Now, let us define two operators $\mathcal{T}_m : \mathbb{R}^{\mathbb{R}_+} \longrightarrow \mathbb{R}^{\mathbb{R}_+}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_+} \longrightarrow \mathbb{R}_+^{\mathbb{R}_+}$ by

$$\mathcal{T}_m \varepsilon(x) := \varepsilon\left(\sqrt[n]{(1 + m^n)x^n}\right) - \varepsilon(mx) \tag{25}$$

and

$$\Lambda_m \mu(x) := \mu\left(\sqrt[n]{(1 + m^n)x^n}\right) + \mu(mx) \tag{26}$$

for all $x \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}^{\mathbb{R}_+}$ (or a function $\varepsilon : \mathbb{R}_+ \longrightarrow \mathbb{R}$), and $\mu \in \mathbb{R}_+^{\mathbb{R}_+}$, respectively. Then we note that for each $m \in \mathbb{N}$, $\Lambda = \Lambda_m$ as in Theorem 3 with

$$\begin{aligned} f_1(x) &= \sqrt[n]{(1 + m^n)x^n}, \\ f_2(x) &= mx \end{aligned} \tag{27}$$

for all $x \in \mathbb{R}_+$. Hence, (13) in Theorem 3 is satisfied by (26). With the properties of the metric d, two inequalities (24) and (25) imply that

$$\begin{aligned} &d(\mathcal{T}_m f(x), f(x)) \\ &\quad = d\left(f\left(\sqrt[n]{(1 + m^n)x^n}\right) - f(mx), f(x)\right) \leq c_m(x) \end{aligned} \tag{28}$$

and also we have

$$\begin{aligned} d(\mathcal{T}_m \varepsilon(x), \mathcal{T}_m \mu(x)) &= d\left(\varepsilon\left(\sqrt[n]{(1 + m^n)x^n}\right)\right. \\ &\quad \left. - \varepsilon(mx), \mu\left(\sqrt[n]{(1 + m^n)x^n}\right) - \mu(mx)\right) \\ &\leq d\left(\varepsilon\left(\sqrt[n]{(1 + m^n)x^n}\right), \mu\left(\sqrt[n]{(1 + m^n)x^n}\right)\right) \\ &\quad + d(\varepsilon(mx), \mu(mx)) = d(\varepsilon(f_1(x)), \mu(f_1(x))) \\ &\quad + d(\varepsilon(f_2(x)), \mu(f_2(x))) \end{aligned} \tag{29}$$

for all $x \in \mathbb{R}_+$ and $\varepsilon, \mu \in \mathbb{R}^{\mathbb{R}_+}$. Thus, the metric d satisfies (12) in Theorem 3. Let $m \in M_0$. We; then, note that

$$\begin{aligned} \Lambda_m c_m(x) &= c_m\left(\sqrt[n]{(1 + m^n)x^n}\right) + c_m(mx) \\ &= (1 + s(m^n))(h((1 + m^n)x^n) + h(m^n x^n)) \\ &\leq (1 + s(m^n))(s(1 + m^n) + s(m^n))h(x^n) \end{aligned} \tag{30}$$

for all $x \in \mathbb{R}_+$. Applying the mathematical induction, it is not hard to show that

$$\begin{aligned} &\Lambda_m^k c_m(x) \\ &\quad \leq (1 + s(m^n))[s(1 + m^n) + s(m^n)]^k h(x^n) \end{aligned} \tag{31}$$

for all $x \in \mathbb{R}_+$ and each $k \in \mathbb{N}$. Hence, for each $m \in M_0$ and $x \in \mathbb{R}_+$, we conclude that

$$\begin{aligned} c_m^*(x) &:= \sum_{j=0}^{\infty} \Lambda_m^j c_m(x) \\ &\leq \left(\frac{1 + s(m^n)}{1 - s(1 + m^n) - s(m^n)} \right) h(x^n) \end{aligned} \tag{32}$$

where $\Lambda_m^0 c_m(x) = c_m(x)$, which means it satisfies the inequalities (14) in Theorem 3. Therefore, Brzdęk's fixed-point method implies that

$$T_m(x) := \lim_{k \to \infty} \mathcal{T}_m^k f(x) \tag{33}$$

exists for each $m \in M_0$ and $x \in \mathbb{R}_+$, and we have

$$d(f(x), T_m(x)) \leq c_m^*(x) \tag{34}$$

for all $m \in M_0$ and $x \in \mathbb{R}_+$ (refer to Theorem 3 in the introduction). By using the mathematical induction on $k \in \mathbb{N}_0$, we will show that

$$\begin{aligned} d&\left(\mathcal{T}_m^k f\left(\sqrt[n]{x^n + y^n}\right), \mathcal{T}_m^k f(x) + \mathcal{T}_m^k f(y)\right) \\ &\leq \left(s(1 + m^n) + s(m^n)\right)^k \left(h(x^n) + h(y^n)\right) \end{aligned} \tag{35}$$

for all $x, y \in \mathbb{R}_+$ and $m \in M_0$, where $\mathcal{T}_m^0 f(x) = f(x)$. The case of $k = 0$ follows from the inequality (24). Assume that it holds when $k = t$. By using the properties of d, we will see that

$$\begin{aligned} d&\left(\mathcal{T}_m^{t+1} f\left(\sqrt[n]{x^n + y^n}\right), \mathcal{T}_m^{t+1} f(x) + \mathcal{T}_m^{t+1} f(y)\right) \\ &= d\left(\mathcal{T}_m^t f\left(\sqrt[n]{(1 + m^n)(x^n + y^n)}\right)\right. \\ &\quad - \mathcal{T}_m^t f\left(\sqrt[n]{m^n (x^n + y^n)}\right), \\ &\mathcal{T}_m^t f\left(\sqrt[n]{(1 + m^n) x^n}\right) - \mathcal{T}_m^t f(mx) \\ &\quad + \mathcal{T}_m^t f\left(\sqrt[n]{(1 + m^n) y^n}\right) - \left.\mathcal{T}_m^t f(my)\right) \\ &\leq d\left(\mathcal{T}_m^t f\left(\sqrt[n]{(1 + m^n)(x^n + y^n)}\right),\right. \\ &\left.\mathcal{T}_m^t f\left(\sqrt[n]{(1 + m^n) x^n}\right) + \mathcal{T}_m^t f\left(\sqrt[n]{(1 + m^n) y^n}\right)\right) \\ &\quad + d\left(\mathcal{T}_m^t f\left(\sqrt[n]{m^n (x^n + y^n)}\right),\right. \\ &\left.\mathcal{T}_m^t f(mx) + \mathcal{T}_m^t f(my)\right) \leq \left(s(1 + m^n)\right. \\ &\quad \left.+ s(m^n)\right)^t \left[h\left((1 + m^n) x^n\right) + h\left((1 + m^n) y^n\right)\right. \\ &\quad \left.+ h(m^n x^n) + h(m^n y^n)\right] \leq \left(s(1 + m^n)\right. \\ &\quad \left.+ s(m^n)\right)^{t+1} \left(h(x^n) + h(y^n)\right) \end{aligned} \tag{36}$$

for all $x, y \in \mathbb{R}_+$. Letting $k \to \infty$ in the inequality (35), we may obtain the following equality:

$$T_m\left(\sqrt[n]{x^n + y^n}\right) = T_m(x) + T_m(y) \tag{37}$$

for all $x, y \in \mathbb{R}_+$ and $m \in M_0$. For each $m \in M_0$, the mapping T_m is a solution of the radical nth-degree functional equation; that is,

$$R(x) = R\left(\sqrt[n]{(1 + m^n) x^n}\right) - R(mx) \tag{38}$$

for all $x \in \mathbb{R}_+$. Let $L > 0$ be constant. Then, the mapping $R : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$d(f(x), R(x)) \leq L h(x^n) \tag{39}$$

for all $x \in \mathbb{R}_+$ should be equal to T_m for each $m \in M_0$. Let $m_0 \in M_0$ be fixed and $R : \mathbb{R}_+ \to \mathbb{R}$ satisfy the inequality (39). We, then, note that

$$\begin{aligned} d&\left(R(x), T_{m_0}(x)\right) \\ &\leq d(R(x), f(x)) + d\left(f(x), T_{m_0}(x)\right) \\ &\leq \left(L + \frac{1 + s({m_0}^n)}{1 - s(1 + {m_0}^n) - s({m_0}^n)}\right) h(x^n) \\ &\leq h(x^n) L_0 \sum_{j=0}^{\infty} \left(s(1 + {m_0}^n) + s({m_0}^n)\right)^j \end{aligned} \tag{40}$$

where $L_0 = (1 - s(1 + {m_0}^n) - s({m_0}^n))L + (1 + s({m_0}^n))$ (the case $h(x) = 0$ is trivial, so we may exclude it here). Observe that R and T_{m_0} are solutions for (38) for each $m_0 \in M_0$. Now, we will show that for each $l \in \mathbb{N}_0$,

$$\begin{aligned} d&\left(R(x), T_{m_0}(x)\right) \\ &\leq h(x^n) L_0 \sum_{j=l}^{\infty} \left[s(m_0^n) + s(1 + m_0^n)\right]^j \end{aligned} \tag{41}$$

for all $x \in \mathbb{R}_+$. To show this, we will use the mathematical induction, again. The case $l = 0$ follows from the previous inequality. Assume that it holds when the case $l \in \mathbb{N}_0$. Now, $m, m_0 \in M_0$; (37) and (38) imply that

$$\begin{aligned}
d\left(R(x), T_{m_0}(x)\right) &= d\left(R\left(\sqrt[n]{(1+m_0{}^n)x^n}\right)\right. \\
&\quad \left. - R(m_0 x), T_{m_0}\left(\sqrt[n]{(1+m_0{}^2)x^n}\right) - T_{m_0}(m_0 x)\right) \\
&\le d\left(R\left(\sqrt[n]{(1+m_0{}^n)x^n}\right), T_{m_0}\left(\sqrt[n]{(1+m_0{}^n)x^n}\right)\right) \\
&\quad + d\left(R(m_0 x), T_{m_0}(m_0 x)\right) \le h\left((1+m_0^n)x^n\right) \\
&\quad \cdot L_0 \sum_{j=l}^{\infty} \left[s(m_0^n) + s(1+m_0^n)\right]^j + h(m_0^n x^n) \\
&\quad \cdot L_0 \sum_{j=l}^{\infty} \left[s(m_0^n) + s(1+m_0^n)\right]^j \le \left(s(m_0^n)\right. \\
&\quad \left. + s(1+m_0^n)\right) h(x^n) L_0 \sum_{j=l}^{\infty} \left[s(m_0^n) + s(1+m_0^n)\right]^j \\
&= h(x^n) L_0 \sum_{j=l+1}^{\infty} \left[s(m_0^n) + s(1+m_0^n)\right]^j .
\end{aligned} \tag{42}$$

Hence, the inequality (41) holds whenever $l \in \mathbb{N}_0$. Letting $l \longrightarrow \infty$ in the inequality (41), we have

$$R = T_{m_0} \tag{43}$$

where $m_0 \in M_0$. This means that $T_m = T_{m_0}$ for each $m \in M_0$. Hence, we get that

$$d\left(f(x), T_{m_0}(x)\right) \le \frac{1 + s(m^n)}{1 - s(1+m^n) - s(m^n)} h(x^n) \tag{44}$$

for all $m \in M_0$ and $x \in \mathbb{R}_+$. Thus, we may conclude that the inequality (22) holds with $R \coloneqq T_{m_0}$ and also the uniqueness follows from the equality (43). □

Let us give a generalized classical Cauchy-difference-type stability of the radical n-th degree functional equation (16) from Theorem 4.

Corollary 5. *Let n be a positive integer and $h : \mathbb{R}_+ \longrightarrow (0, \infty)$ a mapping such that*

$$\lim_{k\longrightarrow\infty} \inf \sup_{x\in\mathbb{R}_+} \frac{h(k^n x^n) + h((1+k^n)x^n)}{h(x^n)} = 0. \tag{45}$$

Suppose that $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies

$$d\left(\sqrt[n]{x^n + y^n}\right), f(x) + f(y)) \le h(x^n) + h(y^n) \tag{46}$$

for all $x, y \in \mathbb{R}_+$. Then, there exists a unique radical nth-degree functional equation $R : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that

$$d\left(f(x), R(x)\right) \le h(x^n) \tag{47}$$

for all $x \in \mathbb{R}_+$.

Proof. By the definition $s(m)$ as in Theorem 4, we will see that

$$\begin{aligned}
s(m^n) &\le \sup_{x\in\mathbb{R}_+} \frac{h(m^n x^n)}{h(x^n)} \\
&\le \sup_{x\in\mathbb{R}_+} \frac{h(m^n x^n) + h((1+m^n)x^n)}{h(x^n)}
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
s(1+m^n) &\le \sup_{x\in\mathbb{R}_+} \frac{h((1+m^n)x^n)}{h(x^n)} \\
&\le \sup_{x\in\mathbb{R}_+} \frac{h(m^n x^n) + h((1+m^n)x^n)}{h(x^n)}
\end{aligned} \tag{49}$$

These inequalities imply that

$$\begin{aligned}
&s(m^n) + s(1+m^n) \\
&\quad \le 2 \sup_{x\in\mathbb{R}_+} \frac{h(m^n x^n) + h((1+m^n)x^n)}{h(x^n)}
\end{aligned} \tag{50}$$

for all $x \in \mathbb{R}_+$. Now, for each $m \in \mathbb{N}$, let

$$a_m \coloneqq \sup_{x\in\mathbb{R}_+} \frac{h(m^n x^n) + h((1+m^n)x^n)}{h(x^n)} \tag{51}$$

for each $x \in \mathbb{R}_+$. By our assumption, it is a sequence $\{a_m\}$ with subsequence $\{a_{m_k}\}$ such that $\lim_{k\longrightarrow\infty} a_{m_k} = 0$; that is,

$$\lim_{k\longrightarrow\infty} \sup_{x\in\mathbb{R}_+} \frac{h(m_k{}^n x^n) + h((1+m_k{}^n)x^n)}{h(x^n)} = 0 \tag{52}$$

The inequalities (50) and (52) imply that

$$\lim_{k\longrightarrow\infty} \left[s(m_k{}^n) + s(1+m_k{}^n)\right] = 0; \tag{53}$$

that is, $\lim_{k\longrightarrow\infty} s(m_k{}^n) = 0$. Thus, we have

$$\lim_{k\longrightarrow\infty} \frac{1 + s(m_k{}^n)}{1 - s(m_k{}^n) - s(1+m_k{}^n)} = 1. \tag{54}$$

Letting $s_0 = 1$ as in Theorem 4, the inequality (47) follows from the inequality (22). □

Data Availability

There were no data used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

The Unique Positive Solution for Singular Hadamard Fractional Boundary Value Problems

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In this paper, we investigate singular Hadamard fractional boundary value problems. The existence and uniqueness of the exact iterative solution are established only by using an iterative algorithm. The iterative sequences have been proved to converge uniformly to the exact solution, and estimation of the approximation error and the convergence rate have also been derived.

This paper is dedicated to our advisors

1. Introduction

Fractional differential operators play an important role in describing phenomenons in many fields such as physics, chemistry, control, and electromagnetism [1–9]. They have many applications in fractional differential equations and fractional integral equations. In [10], authors investigate a class of fractional integral equations arising from a symmetric transition model

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{-\beta}\left(u'(t)\right)+\frac{1}{2}\,{}_tD_T^{-\beta}\left(u'(t)\right)\right)+\nabla F(t,u(t)) \\ &\quad =0, \quad \text{a.e. } t\in[0,T] \qquad (1)\\ &u(0)=u(T)=0 \end{aligned}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0\le\beta<1$, respectively, and $\nabla F(t,x)$ is the gradient of F at x.

The fractional order diffusion equation

$$\frac{\partial^\alpha\phi}{\partial t^\alpha}=K\frac{\partial^\beta\phi}{\partial|x|^\beta} \qquad (2)$$

where $0<\alpha\le1$ and $1<\beta\le2$, contains fractional differential operators which can describe different diffusion process.

In the past ten years, fractional differential equations have been considered in many papers (see [11–22]). Most of the works on the topic have been based on Riemann-Liouville type and Caputo type fractional differential equations. By means of fixed point theorems and variational methods, authors obtain at least one or multiple positive solutions for boundary value problems of fractional differential equations. Very recently, more studies have been carried out on the boundary value problems of nonlinear Hadamard fractional differential equations. An important characteristic of Hadamard fractional derivative is that it contains logarithmic function of arbitrary exponent. However, there are few results about this topic (see [23–28]).

By using the Krasnoselskii-Zabreiko fixed point theorem, Yang [26] obtained at least one positive solution for the boundary value problem

$$\mathrm{D}^q u(t)+f(t,u(t))=0, \quad t\in(1,e),$$

$$u^{(m)}(1)=0,$$

$$u(e) = \int_1^e g(t) u(t) \frac{dt}{t}, \tag{3}$$

where $f \in C([1,e] \times R^+, R)$, $g \in C([1,e], R^+)$. D^q was the Hadamard fractional derivative of order q. $0 \le m \le n-2$, $n \in N$, $n \ge 3, n-1 < q \le n$.

In [28], the authors studied the Hadamard fractional differential equation

$$\begin{gathered} {}^H D^q x(t) + \sigma(t) f(t, x(t)) = 0, \\ 2 < q \le 3, \ t \in (1, +\infty), \end{gathered} \tag{4}$$

with boundary conditions

$$\begin{gathered} x(1) = x'(1) = 0, \\ {}^H D^{q-1} x(\infty) = a\, {}^H I^\beta x(\xi) + b \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \\ 1 < \xi < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty, \end{gathered} \tag{5}$$

where ${}^H D^q$ denoted Hadamard fractional derivative of order q and ${}^H I^\beta$ denoted Hadamard fractional integral of order β. $f \in C([1,\infty) \times [0,\infty), [0,\infty))$, $\sigma : [1,\infty) \longrightarrow [0,\infty)$. By employing the complete continuity of the associated integral operator T and the monotone iterative method, the authors obtained twin positive solutions and the unique positive solution.

Most of the above works required the associated integral operators to be completely continuous because fixed point theorems could be applied. Furthermore, the uniqueness of positive solutions was rarely investigated while the existence and multiplicity of positive solutions were investigated widely.

Inspired by the above results, in this work, we study the existence and uniqueness of positive solutions for the following boundary value problem:

$$\begin{gathered} {}^H D^q x(t) + f(t, x(t)) = 0, \quad 2 < q \le 3, \ t \in (1, e), \\ x(1) = x'(1) = 0, \\ x(e) = 0, \end{gathered} \tag{6}$$

where ${}^H D^q$ denotes Hadamard fractional derivative of order q; $f : (1,e) \times [0,\infty) \longrightarrow [0,\infty)$ is continuous.

In this work, only by using the monotone iterative technique, we aim to establish the unique positive solution for problem (6). The main contributions of this work are as follows: (a) the nonlinear term $f(t,x)$ can be singular at $t = 1$ and $t = e$; (b) we do not need the continuity and complete continuity of the associated integral operator; (c) we get the unique positive solution.

Throughout this work, we assume that the following conditions hold without further mention.

(H_1) : $f : (1,e) \times [0,\infty) \longrightarrow [0,\infty)$ is continuous.

(H_2): For $(t,x) \in (1,e) \times [0,\infty)$, $f(t,x)$ is nondecreasing in x and there exists a constant $k \in (0,1)$ such that, for $\forall \sigma \in (0,1]$,

$$f(t, \sigma x) \ge \sigma^k f(t, x). \tag{7}$$

It is easy to verify that if $\sigma \in (1,+\infty)$, then $f(t,\sigma x) \le \sigma^k f(t,x)$.

2. Preliminaries

The way to attack this new problem follows a scheme similar to that used in [21], with the necessary adaptations that Hadamard fractional derivative contains logarithmic function of arbitrary exponent.

In this section, we present some basic concepts and conclusions needed in the proof of our main results.

Definition 1. The Hadamard fractional integral of order $\beta > 0$ of a function $x : (1,\infty) \longrightarrow R$ is given by

$${}^H I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \frac{x(s)}{s} ds. \tag{8}$$

Definition 2. The Hadamard fractional derivative of order q is defined by

$$\begin{aligned} &{}^H D^q x(t) \\ &= \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{x(s)}{s} ds, \end{aligned} \tag{9}$$

where $n-1 < q \le n$, $n = [q]+1$, $q > 0$.

Specifically, ${}^H D^n x(t) = x^{(n)}(t)$, $n = 1, 2, 3, \ldots$.

Lemma 3. *Suppose that $h(t) \in C(1,e)$, $0 < \int_1^e h(s)(ds/s) < +\infty$. Then the Hadamard type fractional differential equation*

$$\begin{gathered} {}^H D^q x(t) + h(t) = 0, \quad 2 < q \le 3, \ t \in (1, e), \\ x(1) = x'(1) = 0, \\ x(e) = 0 \end{gathered} \tag{10}$$

has a unique solution

$$x(t) = \int_1^e G(t,s) h(s) \frac{ds}{s} \tag{11}$$

where

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(q)} \\ &\cdot \begin{cases} (\log t)^{q-1} (1 - \log s)^{q-1} - \left(\log\left(\frac{t}{s}\right)\right)^{q-1}, & 1 \le s \le t \le e; \\ (\log t)^{q-1} (1 - \log s)^{q-1}, & 1 \le t \le s \le e. \end{cases} \end{aligned} \tag{12}$$

Proof. As argued in [9], the solution of the Hadamard differential equation in (10) can be written as the equivalent integral equation

$$\begin{aligned} x(t) &= c_1 (\log t)^{q-1} + c_2 (\log t)^{q-2} + c_3 (\log t)^{q-3} \\ &\quad - \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} h(s) \frac{ds}{s}. \end{aligned} \tag{13}$$

From $x(1) = x'(1) = 0$, we have $c_3 = c_2 = 0$. Thus (47) reduces to

$$x(t) = c_1 (\log t)^{q-1} - \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} h(s) \frac{ds}{s}. \quad (14)$$

Using $x(e) = 0$ we obtain

$$c_1 = \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} h(s) \frac{ds}{s}. \quad (15)$$

Substituting (15) into (14), we obtain

$$\begin{aligned} x(t) &= (\log t)^{q-1} \cdot \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} h(s) \frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} h(s) \frac{ds}{s} \\ &= \int_1^e G(t,s) h(s) \frac{ds}{s}. \end{aligned} \quad (16)$$

□

Take $\rho(t) = (\log t)^{q-1}(1 - \log t)$ and $\widehat{\rho}(t) = (1 - \log t)^{q-1} \log t$ for $q > 2$, $t \in [1,e]$. We can prove that $G(t,s)$ have the following properties.

Lemma 4 (see [26]). *For $t, s \in [1,e]$, Green's function $G(t,s)$ satisfies the following properties:*

(i) $G(t,s)$ is continuous on $[1,e] \times [1,e]$ and $G(t,s) \geq 0$,
(ii) $\rho(t)\widehat{\rho}(s) \leq \Gamma(q)G(t,s) \leq (q-1)\rho(t)$,
(iii) $\rho(t)\widehat{\rho}(s) \leq \Gamma(q)G(t,s) \leq (q-1)\widehat{\rho}(s)$,
(iv) $G(t,s) = G(e/t, e/s)$.

From $\Gamma(q+1) = q\Gamma(q)$, $q > 0$, and (ii) we get

$$\rho(t)\left[\frac{1}{\Gamma(q)}\widehat{\rho}(s)\right] \leq G(t,s) \leq \frac{1}{\Gamma(q-1)}\rho(t). \quad (17)$$

In this paper, we will work in the Banach space $E = C[1,e]$ with the norm $\|x\| = \max_{t\in[1,e]}|x(t)|$.

Define a set $P \subset E$ as follows.

$P = \{x \in E \mid$ there are constants $0 < l_x < 1 < L_x$ such that $l_x\rho(t) \leq x(t) \leq L_x\rho(t)$, $t \in [1,e]\}$. Evidently $\rho(t) \in P$. Therefore, P is not empty.

3. The Main Results

Theorem 5. *Assume (H_1), (H_2) hold. And*

$$0 < \int_1^e f(t, \rho(t)) \frac{dt}{t} < \infty. \quad (18)$$

Then problem (6) has at least one positive solution $x^*(t)$.

Proof. Define the operator $T : E \longrightarrow E$ by

$$Tx(t) = \int_1^e G(t,s) f(s, x(s)) \frac{ds}{s}. \quad (19)$$

We can see easily the equivalence between x is a solution of (6) and x is a fixed point of T.

Claim 1. The operator $T : P \longrightarrow P$ is nondecreasing.

In fact, for $x \in P$, it is obvious that $Tx \in E$, $Tx(1) = Tx(e) = 0$. For any $x \in P$ and $t \in [1,e]$, from (17),

$$\begin{aligned} Tx(t) &= \int_1^e G(t,s) f(s, x(s)) \frac{ds}{s}. \\ &\leq \int_1^e \frac{1}{\Gamma(q-1)} \rho(t) f(s, L_x\rho(s)) \frac{ds}{s} \\ &\leq \rho(t) L_x^k \frac{1}{\Gamma(q-1)} \int_1^e f(s, \rho(s)) \frac{ds}{s} \\ &\leq L_{Tx}\rho(t) \end{aligned} \quad (20)$$

and

$$\begin{aligned} Tx(t) &= \int_1^e G(t,s) f(s, x(s)) \frac{ds}{s}. \\ &\geq \int_1^e \frac{1}{\Gamma(q)} \widehat{\rho}(s) \rho(t) f(s, l_x\rho(s)) \frac{ds}{s} \\ &\geq \rho(t) l_x^k \frac{1}{\Gamma(q)} \int_1^e \widehat{\rho}(s) f(s, \rho(s)) \frac{ds}{s} \\ &= l_{Tx}\rho(t), \end{aligned} \quad (21)$$

where L_{Tx} and l_{Tx} are positive constants satisfying

$$\begin{aligned} L_{Tx} &> \max\left\{1, L_x^k \frac{1}{\Gamma(q-1)} \int_1^e f(s, \rho(s)) \frac{ds}{s}\right\}, \\ 0 < l_{Tx} &< \min\left\{1, l_x^k \frac{1}{\Gamma(q)} \int_1^e \widehat{\rho}(s) f(s, \rho(s)) \frac{ds}{s}\right\}. \end{aligned} \quad (22)$$

Thus, it follows that there are constants $0 < l_{Tx} < 1 < L_{Tx}$ such that, for $t \in [1,e]$,

$$l_{Tx}\rho(t) \leq Tx(t) \leq L_{Tx}\rho(t). \quad (23)$$

Therefore, for any $x \in P$, $Tx \in P$, T is the operator $P \longrightarrow P$. From (19), it is easy to see that T is nondecreasing with respect to x. Hence, Claim 1 holds.

Claim 2. There exist a nondecreasing sequence $\{u_n\}$ and a nonincreasing sequence $\{v_n\}$ and there exists $x^* \in P$ such that

$$\begin{aligned} u_n(t) &\longrightarrow x^*(t), \\ v_n(t) &\longrightarrow x^*(t), \end{aligned} \quad (24)$$

uniformly on $[1,e]$.

First, there exist two constants $l_{T\rho}, L_{T\rho}$ with $0 < l_{T\rho} < 1 < L_{T\rho}$ since $T\rho \in P$. Take δ and γ to be fixed numbers satisfying

$$\begin{aligned} 0 < \delta &\leq l_{T\rho}^{1/(1-k)}, \\ \gamma &\geq L_{T\rho}^{1/(1-k)}. \end{aligned} \quad (25)$$

Obviously, $0 < \delta < 1 < \gamma$.

We construct two iterative sequences as follows:

$$u_0(t) = \delta\rho(t), \quad v_0(t) = \gamma\rho(t), \tag{26}$$

$$u_n = Tu_{n-1}, \quad v_n = Tv_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{27}$$

Then

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots v_n \le \ldots \le v_1 \le v_0. \tag{28}$$

In fact, from (26), we have $u_0, v_0 \in P$ and $u_0 \le v_0$. Furthermore,

$$\begin{aligned} u_1(t) = Tu_0(t) &= \int_1^e G(t,s) f(s, \delta\rho(s)) \frac{ds}{s} \\ &\ge \delta^k \int_1^e G(t,s) f(s, \rho(s)) \frac{ds}{s} = \delta^k T\rho \\ &\ge \delta^k l_{T\rho}\rho(t) \ge \delta^k \delta^{1-k}\rho(t) = u_0(t), \\ v_1(t) = Tv_0(t) &= \int_1^e G(t,s) f(s, \gamma\rho(s)) \frac{ds}{s} \\ &\le \gamma^k \int_1^e G(t,s) f(s, \rho(s)) \frac{ds}{s} = \gamma^k T\rho \\ &\le \gamma^k L_{T\rho}\rho(t) \le \gamma^k \gamma^{1-k}\rho(t) = v_0(t). \end{aligned} \tag{29}$$

From $u_0 \le v_0$ and T nondecreasing, (28) holds. Let $c_0 = \delta/\gamma$; then $0 < c_0 < 1$. It follows from (7) that

$$T(c_0 u) \ge c_0^k Tu. \tag{30}$$

And, for any natural number n,

$$\begin{aligned} u_n = Tu_{n-1} = T^n u_0 = T^n(\delta\rho) = T^n(c_0\gamma\rho) \\ \ge c_0^{k^n} T^n(\gamma\rho) = c_0^{k^n} v_n. \end{aligned} \tag{31}$$

Thus, for any natural number n and p^*, we have

$$\begin{aligned} 0 \le u_{n+p^*} - u_n \le v_n - u_n \le \left(1 - c_0^{k^n}\right) v_n \\ \le \left(1 - c_0^{k^n}\right)\gamma\rho, \end{aligned} \tag{32}$$

which implies that $\{u_n\}$ is a cauchy sequence in $[u_0, v_0]$. So there exists $x^* \in [u_0, v_0] \subset P$ such that

$$u_n(t) \longrightarrow x^*(t), \tag{33}$$

and from (32)

$$v_n(t) \longrightarrow x^*(t). \tag{34}$$

From $x^* \in [u_0, v_0]$, we have $Tx^* \in [u_0, v_0]$. Combining with T nondecreasing on x,

$$u_n \le Tx^* \le v_n. \tag{35}$$

Let $n \longrightarrow \infty$,

$$x^*(t) = Tx^*(t), \tag{36}$$

which implies x^* is a positive solution of problem (6). □

Theorem 6. *Assume $(H_1), (H_2)$ hold. Then, we have the following:*

(i)Problem (6) has unique positive solution $x^*(t)$ in P and there exist constants l, L with $0 < l < 1 < L$ such that

$$l\rho(t) \le x^*(t) \le L\rho(t), \quad t \in [1, e]. \tag{37}$$

(ii)For any initial value $x_0(t) \in P$, there exists a sequence $\{x_n(t)\}$ that uniformly converges to the unique positive solution $x^*(t)$, and we have the error estimation

$$\max\left|x_n(t) - x^*(t)\right| = o\left(1 - \lambda^{k^n}\right), \tag{38}$$

where λ is a constant with $0 < \lambda < 1$ and determined by x_0; $o(1 - \lambda^{k^n})$ represents the same order infinitesimal of $(1 - \lambda^{k^n})$.

Proof. Let u_0, v_0, u_n, v_n be defined in (26) and (27).

(i) It follows from Theorem 5 that problem (6) has a positive solution $x^*(t) \in P$, which implies that there exists constants l and L with $0 < l < L < 1$ such that

$$l\rho(t) \le x^*(t) \le L\rho(t), \quad t \in [1, e]. \tag{39}$$

Let $\overline{x}(t) \in P$ be another positive solution of problem (6). Then there exist constants c_1 and c_2 with $0 < c_1 < 1 < c_2$ such that

$$c_1\rho(t) \le \overline{x}(t) \le c_2\rho(t), \quad t \in [1, e]. \tag{40}$$

Let δ defined in (25) be small enough so that $\delta < c_1$ and γ defined in (25) be large enough so that $\gamma > c_2$. Then

$$u_0(t) \le \overline{x}(t) \le v_0(t), \quad t \in [1, e]. \tag{41}$$

Note that $T\overline{x}(t) = \overline{x}(t)$ and T is nondecreasing; we have

$$u_n(t) \le \overline{x}(t) \le v_n(t), \quad t \in [1, e]. \tag{42}$$

Letting $n \longrightarrow \infty$, we obtain that $x^*(t) = \overline{x}(t)$. Hence, the positive solution of problem (6) is unique.

(ii) For any $x_0(t) \in P$, there exist constants l_0 and L_0 with $0 < l_0 < 1 < L_0$ such that

$$l_0\rho(t) \le x_0(t) \le L_0\rho(t), \quad t \in [1, e]. \tag{43}$$

Similar to (i), take δ and γ to be defined by (25) satisfying $\delta < l_0$ and $\gamma > L_0$. Then

$$u_0(t) \le x_0(t) \le v_0(t), \quad t \in [1, e]. \tag{44}$$

Let

$$x_n(t) = Tx_{n-1}(t) = \int_1^e G(t,s) f(s, x_{n-1}(s)) \frac{ds}{s}, \quad n = 1, 2, \ldots. \tag{45}$$

Note that T is nondecreasing,

$$u_n(t) \le x_n(t) \le v_n(t), \quad t \in [1, e]. \tag{46}$$

Letting $n \longrightarrow \infty$, it follows from (33) and (34) that $x_n(t) \longrightarrow x^*(t)$ uniformly on $[1, e]$.

At the same time, (38) follows from (32). □

Remark 7. We just investigate a simple form of boundary value problems for Hadamard differential equations. We can easily apply the monotone iterative technique to multipoint or multistrip boundary value problems.

Remark 8. Suppose that $\beta_i(t)(i = 0, 1, 2, \ldots m)$ are nonnegative continuous functions on $(1, e)$, which may be unbounded at the end points of $(1, e)$. Ω is the set of functions $f(t, x)$ which satisfy the conditions (H_1) and (H_2). Then we have the following conclusions:

(1) $\beta_i(t) \in \Omega$, $x^b \in \Omega$, where $0 < b < 1$.

(2) If $0 < b_i < +\infty (i = 1, 2, \ldots m)$ and $b > \max_{1\le i\le m}\{b_i\}$, then $[\beta_0(t) + \sum_{i=1}^{m} \beta_i(t)x^{b_i}]^{1/b} \in \Omega$.

(3) If $f(t, x) \in \Omega$, then $\beta_i(t) f(t, x) \in \Omega$.

(4) If $f_i(t, x) \in \Omega (i = 1, 2, \ldots m)$, then $\max_{1\le i\le m}\{f_i(t, x)\} \in \Omega$, $\min_{1\le i\le m}\{f_i(t, x)\} \in \Omega$.

The above four facts can be verified directly. This indicates that there are many kinds of functions which satisfy the conditions (H_1) and (H_2).

4. An Example

Consider the following boundary value problem:

$$
\begin{gathered}
{}^{H}\mathrm{D}^{5/2}u(t) + a(t)u^{1/4} + b(t)u^{1/3} = 0, \quad t \in (1, e), \\
u(1) = u'(1) = 0, \\
u(e) = 0,
\end{gathered} \tag{47}
$$

where $q = 5/2$, $f(t, u) = a(t)u^{1/4} + b(t)u^{1/3}$, $a(t), b(t) \in C((1, e), (0, +\infty))$.

Analysis 1. First, $f \in C((1, e) \times [0, \infty), [0, \infty))$ and so (H_1) holds.

For any $\sigma \in (0, 1)$, we take $k = 1/2$ and have

$$f(t, \sigma u) \ge \sigma^k f(t, u). \tag{48}$$

Then (H_2) holds.

Obviously

$$0 < \int_1^e f(t, \rho(t)) \frac{dt}{t} < \infty, \tag{49}$$

where $\rho(t) = (\log t)^{5/2-1}(1 - \log t)$. Hence all conditions of Theorem 5 are satisfied, and consequently we have the following corollary.

Corollary 9. *Problem (47) has unique positive solution $u^*(t)$. For any initial value $x_0 \in P$, the successive iterative sequence $\{x_n(t)\}$ generated by*

$$
\begin{gathered}
x_n(t) = \int_0^1 G(t, s)\left(a(s)u^{1/4} + b(s)u^{1/3}\right)\frac{ds}{s}, \\
n = 1, 2, \ldots
\end{gathered} \tag{50}
$$

uniformly converges to the unique positive solution $u^*(t)$ on $[1, e]$. We have the error estimation

$$\max\left|x_n(t) - u^*(t)\right| = \circ\left(1 - \lambda^{(1/2)^n}\right), \tag{51}$$

where λ is a constant with $0 < \lambda < 1$ and determined by the initial value x_0. And there are constants l, L with $0 < l < 1 < L$ such that

$$l\rho(t) \le u^*(t) \le L\rho(t), \quad t \in [1, e]. \tag{52}$$

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

C^*-Basic Construction from the Conditional Expectation on the Drinfeld Double

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Let $D(G)$ be the Drinfeld double of a finite group G and $D(G;H)$ be the crossed product of $C(G)$ and $\mathbb{C}H$, where H is a subgroup of G. Then the sets $D(G)$ and $D(G;H)$ can be made C^*-algebras naturally. Considering the C^*-basic construction $C^*\langle D(G), e\rangle$ from the conditional expectation E of $D(G)$ onto $D(G;H)$, one can construct a crossed product C^*-algebra $C(G/H \times G) \rtimes \mathbb{C}G$, such that the C^*-basic construction $C^*\langle D(G), e\rangle$ is C^*-algebra isomorphic to $C(G/H \times G) \rtimes \mathbb{C}G$.

1. Introduction

Index theory for subfactors was initiated by Jones ([1]) and has experienced rapid progress beyond the framework of operator algebras. For example, Jones' index theory has found important applications in knots theory, conformal field theory, algebraic quantum field theory, and so on ([2–8]). For a nontechnical but broad overview of the subject including a lot of important connections with other areas, the readers can refer to [9].

Let $N \subseteq M$ be factors of type II_1 with finite Jones index and tr the faithful normal tracial state on M. Denoted by $L^2(M, \mathrm{tr})$ the Hilbert space closure of M is with respect to the norm $\langle x, y\rangle = \mathrm{tr}(y^* x)$. Then M acts on $L^2(\mathrm{M}, \mathrm{tr})$ by the left multiplication. The involution $x \longrightarrow x^*$ extends to an isometric conjugate linear operator on $L^2(M, \mathrm{tr})$ denoted by J. The remarkable discovery of Jones is that the possible values for the index are $[M : N] = 4\cos^2(\pi/n), n = 3, 4, \ldots$, or $[M : N] \geq 4$. It is rather easy to construct a reducible inclusion of factors with any index value larger or equal to 4. All the values $4\cos^2(\pi/n)$ in the discrete series are realized by means of the basic construction. To be precise, let E_N be a conditional expectation from M onto N associated with the trace, such that $\mathrm{tr}(E_N(x)y) = \mathrm{tr}(xy)$ for $x \in M$ and $y \in N$. The extension of E_N to $L^2(M, \mathrm{tr})$, denoted by e_N, is the orthogonal projection of $L^2(M, \mathrm{tr})$ onto the closure of N regarded as a subspace of $L^2(M, \mathrm{tr})$. Then $M_1 \triangleq \langle M, e_N\rangle$, the von Neumann algebra generated by M and e_N on $L^2(M, \mathrm{tr})$, is called the basic construction, and $\langle M, e_N\rangle = JN'J$, which is a perfect result. Subsequently Jones used the basic construction to obtain an increasing sequence of type II_1 factors, $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ which is called the Jones tower, iteratively by adding the Jones projections $\{e_n : n \geq 1, e_1 = e_N\}$ which satisfy the Temperley-Lieb relations. Finally, Jones used this structure to construct an example for any possible index value below 4.

The Jones index theory for subfactors of type II_1 has been extended to unital C^*-algebras by Watatani ([10]), and many interesting results of C^*-index theory can be found in [11, 12]. Note that in [10] Watatani showed that if α is an outer action of a finite group G on a simple C^*-algebra A and E is the condition expectation from A onto the fixed point subalgebra A^α given by the average over G, then the basic construction is identified with the crossed product $A \rtimes G$. It was also shown in [12] that if A is a pure Hopf algebra acting outerly on a factor M, then $M \rtimes A$ is a factor and therefore the Jones basic construction $M_1 = M \rtimes A$. However, different from the basic construction for type II_1 factors, the C^*-basic construction $C^*\langle B, \gamma_A\rangle$ does not have the concrete form in general, where Γ is a conditional expectation of a C^*-algebra

B onto a C^*-subalgebra A. The reason is that any factor of type II_1 possesses the faithful trace which is a state of this kind for which the Gelfand-Naimark-Segal construction may be performed, while for general C^*-algebras, the existence of this functional is uncertain.

Letting G be a finite group and H its subgroup, denoted by $H \leq G$, then $D(G;H)$ is defined as the crossed product of $C(G)$, the algebra of complex functions on G, and group algebra $\mathbb{C}H$ with respect to the adjoint action of the latter on the former. In particular, $H = G$; then $D(G;H) \triangleq D(G)$ is the Drinfeld double of G. It was shown in [13] that there is the conditional expectation $E : D(G) \longrightarrow D(G;H)$ of index-finite type. In this paper, we prove that the C^*-basic construction from the conditional expectation E can be described as a crossed product $C(G/H \times G) \rtimes \mathbb{C}G$.

The paper is organized as follows. In Section 2, we give a brief description of the C^*-basic construction for C^*-algebras, and we also collect the necessary definitions and facts about the set $D(G;H)$. In Section 3, we define an action of $\mathbb{C}G$ on $C(G/H \times G)$, and obtain the resulting crossed product $C(G/H \times G) \rtimes \mathbb{C}G$. Theorem 10 is our main result which means that the C^*-basic construction $C^*\langle D(G), e\rangle$ is C^*-algebra isomorphic to the C^*-algebras $C(G/H\times G)\rtimes\mathbb{C}G$.

2. The C^*-Basic Construction of the Drinfeld Double

In this section, suppose that B is a C^*-algebra over $\mathbb{C}$ and A a C^*-subalgebra with a common unit 1. Let us recall these definitions and facts which can be found in the works of Watatani ([10]) and Jiang ([13]).

Definition 1. A linear map $\Gamma : B \longrightarrow A$ is called a conditional expectation if it satisfies the following conditions:

(1) $\Gamma(1) = 1$;

(2) (bimodular property) for $a_1, a_2 \in A$, $b \in B$,

$$\Gamma(a_1 b a_2) = a_1 \Gamma(b) a_2; \tag{1}$$

(3) Γ is positive; i.e., $\Gamma(b^* b)$ is a positive element in A for any $b \in B$.

Actually, if Γ is a conditional expectation from B onto A, then Γ is a projection of norm one ([14]).

Definition 2. Let $\Gamma : B \longrightarrow A$ be a conditional expectation. A finite family $\{(u_1, u_1^*), (u_2, u_2^*), \ldots, (u_n, u_n^*)\} \subseteq B \times B$ is called a quasi-basis for Γ if for all $x \in B$,

$$\sum_{i=1}^{n} u_i \Gamma(u_i^* x) = x = \sum_{i=1}^{n} \Gamma(x u_i) u_i^*. \tag{2}$$

Furthermore, if there exists a quasi-basis for Γ, we call Γ of index-finite type. In this case we define the index of Γ by

$$\text{Index } \Gamma = \sum_{i=1}^{n} u_i u_i^*. \tag{3}$$

If Γ is of index-finite type, then Index Γ is in the center of B and does not depend on the choice of quasi-basis ([10]).

In the following we recall the C^*-basic construction from the conditional expectation $\Gamma : B \longrightarrow A$.

Notice that B is an A-bimodule algebra; one can consider the module tensor product algebra $B \otimes_A B$, the product of which depends on Γ as follows:

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \Gamma(b_2 b_3) \otimes b_4 \tag{4}$$

for $b_i \in B$, $i = 1,2,3,4$. Then $B \otimes_A B$ turns out to be an algebra. And define an involution $*$ by

$$(x \otimes y)^* = y^* \otimes x^* \tag{5}$$

for $x, y \in B$. The involution $*$ is well defined by considering conjugate operation in C^*-algebras. Thus $B \otimes_A B$ becomes an $*$-algebra.

Recall that $L(\mathscr{X})$ denotes the algebra of all bounded linear operators on a Hilbert space $\mathscr{X}$.

Lemma 3. *Let $\Gamma : B \longrightarrow A$ be a conditional expectation. Let $\phi : B \otimes_A B \longrightarrow L(\mathscr{X})$ be a $*$-representation of $B \otimes_A B$ on a Hilbert space $\mathscr{X}$. Consider a conditional expectation $\Gamma \otimes id : B \otimes M_n \longrightarrow A \otimes M_n$. Then there exists a $*$-representation ϕ' : $(B \otimes M_n) \otimes_{A\otimes M_n} (B \otimes M_n) \longrightarrow L(\mathscr{X}) \otimes M_n$ such that*

$$\phi'\left((x_{ij})_{ij} \otimes (y_{jk})_{jk}\right) = \left(\sum_j \phi(x_{ij} \otimes y_{jk})\right)_{ik} \tag{6}$$

for $(x_{ij})_{ij}, (y_{jk})_{jk} \in B \otimes M_n$.

For $c = \sum_i x_i \otimes y_i \in B \otimes_A B$, set

$$\begin{aligned} &\|c\|_{max} \\ &= \sup\{\|\phi(c)\| : \phi \text{ is a } *-\text{representation of } B \otimes_A B\}, \end{aligned} \tag{7}$$

then

$$\begin{aligned} \|c\|_{max} = &\left\| \left(\left(\Gamma(x_i^* x_j)\right)_{ij}\right)^{1/2} \left(\left(\Gamma(y_i y_j^*)\right)_{ij}\right)^{1/2} \right\| \\ &< +\infty. \end{aligned} \tag{8}$$

Thus $\|\cdot\|_{max}$ is a C^*-seminorm on $B \otimes_A B$ ([10]).

Definition 4. The completion of $B \otimes_A B$ by the norm $\|\cdot\|_{\max}$ (after taking quotient by the ideal $\{c \in B \otimes_A B : \|c\|_{\max} = 0\}$ if necessary) is called the C^*-basic construction from the conditional expectation Γ. We denote this C^*-algebra by $C^*\langle B, \gamma_A\rangle$.

Now assume that G is a finite group with a subgroup H and $t_1 = u, t_2, \ldots, t_k$ is a left coset representation of H in G, namely, $G = \bigcup_{i=1}^{k} t_i H$ and $i \neq j$ induces that $t_i H \cap t_j H = \emptyset$, where u is the unit of G. Let us begin with the following definitions.

Definition 5. $D(G;H)$ is the crossed product of $C(G)$ and group algebra $\mathbb{C}H$, where $C(G)$ denotes the set of complex functions on G, with respect to the adjoint action of the latter on the former.

Using the basis elements (g,h) of $D(G;H)$, the multiplication rule is given as follows:

$$(g_1,h_1)(g_2,h_2)=\delta_{g_1h_1,h_1g_2}(g_1,h_1h_2). \tag{9}$$

The unit of $D(G;H)$ is $\sum_{g\in G}(g,u)\triangleq I$. $D(G;H)$ becomes a unital $*$-algebra by defining the $*$-operation $(g,h)^*=(h^{-1}gh,h^{-1})$ on the basis elements and extending antilinearly to $D(G;H)$.

The coproduct $\triangle$, the counit ε, and the antipode S are defined by

$$\begin{aligned}\triangle(g,h)&=\sum_{t\in H}(t,h)\otimes\left(t^{-1}g,h\right),\\ \varepsilon(g,h)&=\delta_{g,u},\\ S(g,h)&=\left(h^{-1}g^{-1}h,h^{-1}\right),\end{aligned} \tag{10}$$

where $g\in G, h\in H$, and $\delta_{g,h}=\begin{cases}1, & \text{if } h=g\\ 0, & \text{if } h\neq g.\end{cases}$ One can prove $D(G;H)$ becomes a Hopf $*$-algebra ([15]). There exists a unique element $z_H=(1/|H|)\sum_{h\in H}(u,h)$, called a cointegral, satisfying $\forall a\in D(G;H)$, $az_H=z_Ha=\varepsilon(a)z_H$, and $\varepsilon(z_H)=1$. As a result, $D(G;H)$ is semisimple with a natural $*$-structure. Consequently it can be a C^*-algebra of finite dimension.

In particular, if $H=G$, then $D(G;H)\triangleq D(G)$ is the Drinfeld double of G. For more information about $D(G)$ one can refer to [16–18]. The main difference between $D(G)$ and $D(G;H)$ is that the former is a quasitriangular Hopf algebra while the latter is not ([19]).

Considering a linear map

$$\begin{aligned}E:\quad & D(G)\longrightarrow D(G;H)\\ & \sum_{g,h\in G}k_{g,h}(g,h)\longmapsto\sum_{g\in G,h\in H}k_{g,h}(g,h)\end{aligned} \tag{11}$$

where $k_{g,h}\in\mathbb{C}$, one can show that it is the conditional expectation from $D(G)$ onto $D(G;H)$. Moreover, setting $u_i=\sum_{g\in G}(g,t_i)$, then $u_i^*=\sum_{g\in G}(g,t_i^{-1})$ and $\{(u_i,u_i^*)\}$ is a quasibasis for E. Thus Index $E=[G:H]I$. In this case, E is of index-finite type.

Definition 6. The completion of $D(G)\otimes_{D(G;H)}D(G)$ with respect to the norm $\|\cdot\|_{\max}$ (after taking quotient by the ideal $\{c\in D(G)\otimes_{D(G;H)}D(G):\|c\|_{\max}=0\}$ if necessary) is called the C^*-basic construction from the conditional expectation E of $D(G)$ onto $D(G;H)$. We denote this C^*-algebra by $C^*\langle D(G),e\rangle$.

Note that $\sum_{g\in G}\sum_{i=1}^{k}(g,t_i)\otimes(g,t_i)^*$ is the unit of $C^*\langle D(G),e\rangle$.

3. The Construction of a Crossed Product C^*-Algebra

Let us continue to suppose that G is a finite group and $H\leq G$. Denoted by G/H the set of all left cosets of H, i.e., $G/H=\{[t_1],[t_2],\dots,[t_k]\}$. Let $C(G/H\times G)$ and $\mathbb{C}G$ be the algebra of complex functions on $G/H\times G$ and the group algebra over $\mathbb{C}$, respectively.

The set $\{\chi_{[t_i]}: i=1,2,\dots,k\}$ is a linear basis of $C(G/H)$ where $\chi_{[t_i]}[t_j]=\begin{cases}1, & \text{if } j=i,\\ 0, & \text{if } j\neq i,\end{cases}$ while the set $\{\delta_g: g\in G\}$ is a linear basis of $C(G)$ where $\delta_g(h)=\begin{cases}1, & \text{if } h=g,\\ 0, & \text{if } h\neq g.\end{cases}$ Since G is a finite group and $C(G/H\times G)\cong C(G/H)\otimes C(G)$, then $\{(t_i,g): i=1,2,\dots,k;\ g\in G\}$ can be regarded as a linear basis of $C(G/H\times G)$, where we write (t_i,g) instead of $(\chi_{[t_i]},\delta_g)$ for notational convenience.

For any $h\in G$, we can define the map $\sigma_h: C(G/H\times G)\longrightarrow C(G/H\times G)$ given on the basis elements (t_i,g) of $C(G/H\times G)$ as

$$\sigma_h(t_i,g)=\left(ht_i,hgh^{-1}\right), \tag{12}$$

which can be linearly extended in $C(G/H\times G)$. One can verify that σ_h is an automorphic map, and then σ defines an automorphic action of $\mathbb{C}(G)$ on $C(G/H\times G)$.

Assume that $C(G/H\times G)$ acts on a Hilbert space $l^2(G/H\times G)$, and an action σ of $C(G)$ on $C(G/H\times G)$ is given as above. Let $\mathscr{H}=l^2(G/H\times G)\otimes l^2(G)$. We view $\mathscr{H}$ as the Hilbert space of all square summable $l^2(G/H\times G)$-valued functions on G, and define:

$$\begin{aligned}(\pi(t_i,g)\xi)(h)&=\sigma_{h^{-1}}(t_i,g)\xi(h),\\ (\lambda(g)\xi)(h)&=\xi\left(g^{-1}h\right),\end{aligned} \tag{13}$$

for $g,h\in G,\xi\in\mathscr{H}, i=1,2,\dots,k$.

Lemma 7. *π is a faithful $*$-representation of $C(G/H\times G)$ on $\mathscr{H}$, while λ is a unitary representation of G on $\mathscr{H}$. And*

$$\lambda(h)\pi(t_i,g)\lambda(h)^*=\pi(\sigma_h(t_i,g)),\qquad g,h\in G,\ i=1,2,\dots,k. \tag{14}$$

Definition 8. The associative C^*-algebra on $\mathscr{H}$ generated by $\{\pi(t_i,g),\lambda(h): g,h\in G;\ i=1,2,\dots,k\}$ is called the crossed product of $C(G/H\times G)$ by $\mathbb{C}G$ with respect to σ, and we denote it by $C(G/H\times G)\rtimes_\sigma\mathbb{C}G$ (or simply by $C(G/H\times G)\rtimes\mathbb{C}G$).

Here and from now on, by $h(t_i,g)$, (t_i,g) and h we always denote $\sigma_h(t_i,g)$, $\pi(t_i,g)$ and $\lambda(h)$, respectively.

Lemma 9.

(1) *The element $\sum_{g\in G}(t_1,g)$ in $C(G/H\times G)$ is a projection on $\mathscr{H}$, i.e.,*

$$\sum_{g\in G}(t_1,g)=\left(\sum_{g\in G}(t_1,g)\right)^2=\left(\sum_{g\in G}(t_1,g)\right)^*. \tag{15}$$

(2) *For $g,h\in G$, we have*

$$\begin{aligned}&\left(\sum_{\alpha\in G}(t_1,\alpha)\right)\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\beta\in G}(t_1,\beta)\right)\\ &=\delta_{[t_1],[h]}\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\gamma\in G}(t_1,\gamma)\right).\end{aligned} \tag{16}$$

(3) *Let $h \in G$, then $h \in H$ if and only if*

$$\left(\sum_{i=1}^{k}(t_i,g)h\right)\sum_{\gamma\in G}(t_1,\gamma)=\sum_{\gamma\in G}(t_1,\gamma)\left(\sum_{i=1}^{k}(t_i,g)h\right). \quad (17)$$

Proof. (1) For any $\xi \in \mathscr{H}$ and $h \in G$, observe that

$$\begin{aligned}\left(\sum_{g\in G}(t_1,g)\xi\right)(h)&=h^{-1}\left(\sum_{g\in G}(t_1,g)\right)\xi(h)\\&=\left(\sum_{g\in G}\left(h^{-1},h^{-1}gh\right)\right)\xi(h),\end{aligned} \quad (18)$$

and

$$\begin{aligned}&\left(\left(\sum_{g\in G}(t_1,g)\right)^2\xi\right)(h)\\&=h^{-1}\left(\sum_{f\in G}(t_1,f)\right)\left(\left(\sum_{g\in G}(t_1,g)\right)\xi\right)(h)\\&=h^{-1}\left(\sum_{f\in G}(t_1,f)\right)\left(h^{-1}\left(\sum_{g\in G}(t_1,g)\right)\xi(h)\right)\\&=h^{-1}\left(\sum_{f\in G}(t_1,f)\right)\left(h^{-1}\left(\sum_{g\in G}(t_1,g)\right)\right)\xi(h)\\&=\left(\sum_{f\in G}\left(h^{-1},h^{-1}fh\right)\right)\left(\sum_{g\in G}\left(h^{-1},h^{-1}gh\right)\right)\xi(h)\\&=\left(\sum_{g\in G}\left(h^{-1},h^{-1}gh\right)\right)\xi(h),\end{aligned} \quad (19)$$

From the above equalities, we conclude that $\sum_{g\in G}(t_1,g) = (\sum_{g\in G}(t_1,g))^2$. And

$$\begin{aligned}&\left\langle\left(\sum_{g\in G}(t_1,g)\right)\xi,\eta\right\rangle\\&=\sum_{x\in G}\sum_{g\in G}\langle((t_1,g)\xi)(x),\\&\eta(x)\rangle\\&=\sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}((t_1,g)\xi)(x)(m,n)\overline{\eta(x)(m,n)}\\&=\sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\left(x^{-1}(t_1,g)\right)\xi(x)(m,n)\overline{\eta(x)(m,n)}\\&=\sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\left(\left(x^{-1},x^{-1}gx\right)(m,n)\right)\xi(x)(m,n)\overline{\eta(x)(m,n)}\\&=\sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\xi(x)(m,n)\overline{((x^{-1},x^{-1}gx)(m,n))\eta(x)(m,n)}\\&=\sum_{x\in G}\sum_{g\in G}\sum_{m\in G/H}\sum_{n\in G}\xi(x)(m,n)\overline{(x^{-1}(t_1,g))\eta(x)(m,n)}\\&=\sum_{x\in G}\sum_{g\in G}\langle\xi(x),\\&((t_1,g)\eta)(x)\rangle=\left\langle\xi,\left(\sum_{g\in G}(t_1,g)\right)\eta\right\rangle,\end{aligned} \quad (20)$$

which shows that $\sum_{g\in G}(t_1,g) = (\sum_{g\in G}(t_1,g))^*$.

(2) Let $g, h \in G$, we can compute that

$$\begin{aligned}&\left(\sum_{\alpha\in G}(t_1,\alpha)\right)\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\beta\in G}(t_1,\beta)\right)\\&=\sum_{\alpha\in G}\sum_{\beta\in G}\sum_{i=1}^{k}(t_1,\alpha)((t_i,g)h)(t_1,\beta)\\&=\sum_{\alpha\in G}\sum_{\beta\in G}\sum_{i=1}^{k}(t_1,\alpha)(t_i,g)(h(t_1,\beta))\\&=\sum_{\alpha\in G}\sum_{\beta\in G}\sum_{i=1}^{k}(t_1,\alpha)(t_i,g)\left(h,h\beta h^{-1}\right)h\\&=\sum_{\beta\in G}(t_1,g)\left(h,h\beta h^{-1}\right)h=\delta_{[t_1],[h]}(t_1,g)h,\end{aligned} \quad (21)$$

and

$$\begin{aligned}&\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\gamma\in G}(t_1,\gamma)\right)\\&=\sum_{\gamma\in G}\sum_{i=1}^{k}((t_i,g)h)(t_1,\gamma)\\&=\sum_{\gamma\in G}\sum_{i=1}^{k}(t_i,g)\left(h,h\gamma h^{-1}\right)h\\&=\sum_{\gamma\in G}\sum_{i=1}^{k}\delta_{[t_i],[h]}\delta_{gh,h\gamma}(t_i,g)h=(t_1,g)h.\end{aligned} \quad (22)$$

(3) If $h \in H$, then $(\sum_{\alpha\in G}(t_1,\alpha))(\sum_{i=1}^{k}(t_i,g)h) = \sum_{\alpha\in G}\sum_{i=1}^{k}(t_1,\alpha)(t_i,g)h = (t_1,g)h$ and

$$\left(\sum_{i=1}^{k}(t_i,g)h\right)\left(\sum_{\alpha\in G}(t_1,\alpha)\right)$$

$$= \sum_{i=1}^{k} \sum_{\alpha \in G} ((t_i, g) h) (t_1, \alpha)$$
$$= \sum_{i=1}^{k} \sum_{\alpha \in G} (t_i, g) \left(h, h\alpha h^{-1}\right) h = (t_1, g) h. \tag{23}$$

Thus $\sum_{i=1}^{k}(t_i, g)h$ commutes with $\sum_{\alpha\in G}(t_1, \alpha)$.

Conversely, if $h \in G$ and $(\sum_{i=1}^{k}(t_i, g)h) \sum_{\gamma\in G}(t_1, \gamma) = \sum_{\gamma\in G}(t_1, \gamma)(\sum_{i=1}^{k}(t_i, g)h)$, then $(h, g)h = (t_1, g)h$ which implies that $(h, g) = (t_1, g)$. This shows that $h \in H$. □

From the proof of Lemma 9, we can obtain $(t_i, g)^* = (t_i, g)$ for any $g \in G$, $i = 1, 2, \dots, k$.

Now we give the main theorem of this paper.

Theorem 10. *There exists an isometric $*$-isomorphism of $C^*\langle D(G), e\rangle$ onto $C(G/H \times G) \rtimes \mathbb{C}G$.*

Proof. By the definition of $\|\cdot\|_{\max}$, it suffices to show that there exists an $*$-isomorphism Φ of $D(G) \otimes_{D(G;H)} D(G)$ on $\mathscr{H}$.

The map $\Phi : D(G) \otimes_{D(G;H)} D(G) \longrightarrow C(G/H \times G) \rtimes \mathbb{C}G$ is defined on the generators of $D(G) \otimes_{D(G;H)} D(G)$ by

$$\Phi((g, \alpha) \otimes (h, \beta))$$
$$= \left(\sum_{i=1}^{k} (t_i, g) \alpha\right) \left(\sum_{f\in G} (t_1, f)\right) \left(\sum_{j=1}^{k} (t_j, h) \beta\right) \tag{24}$$
$$= \delta_{g\alpha,\alpha h} (\alpha, g) \alpha\beta,$$

for (g, α), (h, β) in $D(G)$. Since $\sum_{i=1}^{k}(t_i, g)h$ and $\sum_{\gamma\in G}(t_1, \gamma)$ commute, Φ is a well defined map. Then it can be linearly extended in $D(G) \otimes_{D(G;H)} D(G)$. Again since

$$\Phi\left(((g, \alpha) \otimes (h, \beta))^*\right) = \Phi\left((h, \beta)^* \otimes (g, \alpha)^*\right)$$
$$= \Phi\left(\left(\beta^{-1}h\beta, \beta^{-1}\right) \otimes \left(\alpha^{-1}g\alpha, \alpha^{-1}\right)\right) \tag{25}$$
$$= \delta_{g\alpha,\alpha h} \left(\beta^{-1}, \beta^{-1}h\beta\right) \beta^{-1}\alpha^{-1},$$

and

$$(\Phi((g, \alpha) \otimes (h, \beta)))^* = \delta_{g\alpha,\alpha h} ((\alpha, g) \alpha\beta)^*$$
$$= \delta_{g\alpha,\alpha h}\beta^{-1}\alpha^{-1} (\alpha, g)^*$$
$$= \delta_{g\alpha,\alpha h} \left(\beta^{-1}\alpha^{-1}\alpha, \beta^{-1}\alpha^{-1}g\alpha\beta\right) \tag{26}$$
$$= \delta_{g\alpha,\alpha h} \left(\beta^{-1}, \beta^{-1}h\beta\right) \beta^{-1}\alpha^{-1}.$$

Then

$$\Phi\left(((g, \alpha) \otimes (h, \beta))^*\right) = (\Phi((g, \alpha) \otimes (h, \beta)))^*. \tag{27}$$

Hence, combining with Lemma 9, we have that the map Φ is a $*$-homomorphism.

For $(t_i, g)h \in C(G/H \times G) \rtimes \mathbb{C}G$, choose $(g, t_i) \otimes (t_i^{-1}gt_i, t_i^{-1}h) \in D(G) \otimes_{D(G;H)} D(G)$ such that $\Phi((g, t_i) \otimes (t_i^{-1}gt_i, t_i^{-1}h)) = (t_i, g)h$. This shows that Φ is surjective. Finally, we will verify that Φ is isometric.

Let $\psi : D(G; H) \longrightarrow C(G/H \times G) \rtimes \mathbb{C}G \subseteq L(\mathscr{H})$ be a map given as $\psi(g, \alpha) = (t_1, g)\alpha$. It is easy to see that ψ is a injective $*$-homomorphism. Define $\psi' : D(G; H) \otimes M_n \longrightarrow L(\mathscr{H}) \otimes M_n$ by $\psi' = \psi \otimes \mathrm{id}$. Then ψ' is also injective. Again, by Lemma 3, there exists

$$\Phi' : (D(G) \otimes M_n) \otimes_{D(G;H)\otimes M_n} (D(G) \otimes M_n)$$
$$\longrightarrow L(\mathscr{H}) \otimes M_n \tag{28}$$

such that $\Phi'((x_{ij})_{ij} \otimes (y_{jk})_{jk}) = (\sum_j \Phi(x_{ij} \otimes y_{jk}))_{ik}$. Since $\psi'((x_{ij})_{ij}) = \Phi'((x_{ij})_{ij} \otimes 1)$,

$$\|\Phi(c)\|^2 = \left\|\sum_i \Phi(x_i \otimes y_i)\right\|^2$$
$$= \left\|\begin{pmatrix} \sum_i \Phi(x_i \otimes y_i) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\right\|^2$$
$$= \left\|\Phi'\left(\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\right.\right.$$
$$\left.\left.\otimes \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ y_2 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_n & 0 & 0 & \cdots & 0 \end{pmatrix}\right)\right\|^2 = \left\|\Phi'(X \otimes Y)\right\|^2$$
$$= \left\|\Phi'(X \otimes Y)^* \Phi'(X \otimes Y)\right\| = \left\|\Phi'(Y^* \otimes X^*)\right.$$
$$\left.\cdot \Phi'(X \otimes Y)\right\| = \left\|\Phi'(Y^* F(X^*X) \otimes Y)\right\|$$
$$= \left\|\Phi'\left(Y^* F(X^*X)^{1/2} \otimes 1\right) \Phi'\left(1\right.\right.$$
$$\left.\left.\otimes F(X^*X)^{1/2} Y\right)\right\| = \left\|\Phi'\left(1 \otimes F(X^*X)^{1/2} Y\right)^*\right.$$
$$\left.\cdot \Phi'\left(1 \otimes F(X^*X)^{1/2} Y\right)\right\| = \left\|\Phi'\left(1\right.\right.$$
$$\left.\left.\otimes F(X^*X)^{1/2} Y\right) \Phi'\left(1 \otimes F(X^*X)^{1/2} Y\right)^*\right\|$$

$$
\begin{aligned}
&= \left\| \Phi' \left(F\left(X^* X\right)^{1/2} F\left(YY^*\right) F\left(X^* X\right)^{1/2} \otimes 1 \right) \right\| \\
&= \left\| F\left(X^* X\right)^{1/2} F\left(YY^*\right) F\left(X^* X\right)^{1/2} \otimes 1 \right\| \\
&= \left\| F\left(X^* X\right)^{1/2} F\left(YY^*\right)^{1/2} \right\|^2 \\
&= \left\| \left(\left(E\left(x_i^* x_j\right) \right)_{ij} \right)^{1/2} \left(\left(E\left(y_i y_j^*\right) \right)_{ij} \right)^{1/2} \right\|,
\end{aligned} \tag{29}
$$

for $c = \sum_i x_i \otimes y_i \in D(G) \otimes_{D(G;H)} D(G)$, where $F = E \otimes \mathrm{id} : D(G) \otimes M_n \longrightarrow D(G;H) \otimes M_n$ is a conditional expectation and $X = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ y_2 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_n & 0 & 0 & \cdots & 0 \end{pmatrix}$. Thus, we have that $\|\Phi(c)\| = \|c\|$. Now we have shown Φ is an isometric $*$-isomorphism. □

Example 11. (1) Let $C^*(G)$ (or $C^*(H)$) be the group C^*-algebra of G (or H), namely, the C^*-subalgebra of $L(l^2(G))$ generated by the left regular representation of G (or H). Consider the basic construction from the conditional expectation $E : C^*(G) \longrightarrow C^*(H)$ defined by

$$E\left(\sum_{g\in G} k_g \lambda_g\right) = \sum_{h\in H} k_h \lambda_h \tag{30}$$

where $k_g \in \mathbb{C}$. Let $\tau : \mathbb{C}G \longrightarrow \mathrm{Aut}\, C(G/H)$ be the action induced by translation from left, where $\mathrm{Aut}\, C(G/H)$ stands for the group of all automorphism of $C(G/H)$. In [10], Watatani showed that the C^*-basic construction $C^*\langle C^*(G), e\rangle$ is C^*-algebra isomorphic to $C(G/H) \rtimes_\tau \mathbb{C}G$. This result is a special case of Theorem 10. In fact, there is an inclusion $i : C^*(G) \longrightarrow D(G)$ given by $i(\lambda_g) = \sum_{f\in G}(f, g)$.

(2) If G is a finite abelian group, then $D(G)$ reduces to a symmetry group $\widehat{G}\times G$, where $\widehat{G}$ denotes the Pontryagin dual of G. Let $E : \widehat{G} \times G \longrightarrow \widehat{G} \times H$ be a conditional expectation defined by

$$E\left(\sum_{g,h\in G} k_{g,h}(g,h)\right) = \sum_{g\in G, h\in H} k_{g,h}(g,h). \tag{31}$$

We can define the map $\varsigma : \mathbb{C}G \times C(G/H\times G) \longrightarrow C(G/H\times G)$ by

$$\varsigma\left(h \times (t_i, g)\right) = (ht_i, g) \tag{32}$$

for all $(t_i, g) \in C(G/H \times G)$. Then ς is an automorphic action of $\mathbb{C}G$ on $C(G/H\times G)$. Then the C^*-basic construction $C^*\langle \widehat{G}\times G, e\rangle$ is C^*-algebra isomorphic to $C(G/H\times G) \rtimes_\varsigma \mathbb{C}G$. Furthermore, the C^*-basic construction $C^*\langle \widehat{G} \times G, e\rangle$ can be described as $\widehat{G} \times (C(G/H) \rtimes_\tau \mathbb{C}G)$, where τ is the action induced by translation from left.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Review Article

On Sequences of J. P. King-Type Operators

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This survey is devoted to a series of investigations developed in the last fifteen years, starting from the introduction of a sequence of positive linear operators which modify the classical Bernstein operators in order to reproduce constant functions and x^2 on $[0,1]$. Nowadays, these operators are known as King operators, in honor of J. P. King who defined them, and they have been a source of inspiration for many scholars. In this paper we try to take stock of the situation and highlight the state of the art, hoping that this will be a useful tool for all people who intend to extend King's approach to some new contents within Approximation Theory. In particular, we recall the main results concerning certain King-type modifications of two well known sequences of positive linear operators, the Bernstein operators and the Szász-Mirakyan operators.

1. Introduction

The aim of this paper is to provide a survey on a series of recent investigations which are centered around the problem of obtaining better properties by modifying properly some well known sequences of positive linear operators in the underlying Banach function spaces.

Such results are principally inspired by the pioneering work [1]. In that paper the author, J. P. King, introduces a new sequence $(V_{n,r_n})_{n\geq 1}$ of positive linear Bernstein-type operators defined, for every $f \in C[0,1]$, $n \geq 1$ and $0 \leq x \leq 1$, by

$$V_{n,r_n}(f;x) = \sum_{k=1}^{n} \binom{n}{k} (r_n(x))^k (1-r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad (1)$$

$r_n : [0,1] \longrightarrow [0,1]$ being continuous functions for every $n \geq 1$. Such operators turn into the classical Bernstein operators B_n whenever, for any $n \geq 1$ and $0 \leq x \leq 1$, $r_n(x) = x$, but unlike the B_n's, they are not in general polynomial-type operators. In fact, for every $n \geq 1$ and $0 \leq x \leq 1$,

$$V_{n,r_n}(\mathbf{1}) = \mathbf{1},$$
$$V_{n,r_n}(e_1) = r_n,$$
$$V_{n,r_n}(e_2) = r_n^2 + \frac{r_n(1-r_n)}{n}, \quad (2)$$

where, for any $t \in [0,1]$, $\mathbf{1}(t) = 1$, and $e_i(t) = t^i$ for $i = 1,2$. By applying Korovkin theorem to V_{n,r_n}, for every $f \in C[0,1]$, and $x \in [0,1]$, $\lim_{n\longrightarrow\infty} V_{n,r_n}(f;x) = f(x)$ if and only if $\lim_{n\longrightarrow\infty} r_n(x) = x$. Among all possible choices, King focuses his attention on the operators V_{n,r_n^*} that fix e_2, obtained by means of the generating functions

$$r_n^*(x) = \begin{cases} x^2 & \text{if } n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{nx^2}{n-1} + \dfrac{1}{4(n-1)^2}} & \text{if } n \geq 2. \end{cases} \quad (3)$$

He shows that $(V_{n,r_n^*})_{n\geq 1}$ is a positive approximation process in $C[0,1]$. Moreover, the operator V_{n,r_n^*} interpolates f at the end points 0 and 1, and it is not a polynomial operator, because of (2) and (3). Through a quantitative estimate in terms of the classical first-order modulus of continuity, King also proves that the order of approximation of $V_{n,r_n^*}(f;x)$ to $f(x)$ is at least as good as the order of approximation of $B_n(f;x)$ to $f(x)$ for $0 \leq x < 1/3$.

A systematic study of the operators V_{n,r_n^*} is due to Gonska and Pițul [2], who determine new estimates for the rate of convergence in terms of the first and second moduli of continuity and, among the others, the behavior of the iterates $V_{n,r_n^*}^m$ as $m \longrightarrow +\infty$.

The A-statistical convergence of operators (1) is considered in [3].

King's idea inspires many other mathematicians to construct other modifications of well-known approximation processes fixing certain functions and to study their approximation and shape preserving properties.

In this review article we try to take stock of the situations and highlight the state of the art, hoping that this will be useful for all people that work in Approximation Theory and intend to apply King's approach in some new contexts.

The paper is organized as follows: after a brief history on what has been done in this research area up to now, in Sections 3 and 4 we illustrate certain King-type modifications of the well-known Bernstein and Szász-Mirakyan operators.

2. A Brief History

From King's work to nowadays, several investigations have been devoted to sequences of positive linear operators fixing certain (polynomial, exponential, or more general) functions. In this section we try to give some essential information about the construction of King-type operators. For all details we refer the readers to the references quoted in the text and we apologize in advance for any possible omission.

We begin to recall the contents of the first papers that generalize in some sense King's idea ([4–7]). In [5] Cárdenas-Morales, Garrancho, and Muñoz-Delgado present a family of sequences of linear Bernstein-type operators $B_{n,\alpha}$ $(n > 1)$, depending on a real parameter $\alpha \geq 0$, and fixing the polynomial function $e_2 + \alpha e_1$ (note that $B_{n,0} = V_{n,r_n^*}$). Among other things, the authors prove that if f is convex and increasing on $[0,1]$, then $f(x) \leq B_{n,\alpha}(f;x) \leq B_n(f;x)$ for every $x \in [0,1]$. Section 3.1 is indeed devoted to the operators $B_{n,\alpha}$. More general results can be found in [8].

On the other hand, in [6] Duman and Özarslan apply the King's original idea to Meyer-König and Zeller operators, and they obtain a better estimation error on the interval $[1/2, 1[$.

The generalizations in [4, 7] contain a different challenge: the authors propose King-type approximation processes in spaces of continuous functions on unbounded intervals.

In particular, in [7] (see also Examples 1) Duman and Özarslan consider the modified Szász-Mirakyan operators reproducing $\mathbf{1}$ and e_2 and obtain better error estimates on the whole interval $[0,\infty)$.

A study in full generality is undertaken in [4]. In fact, in that article, Agratini indicates how to construct sequences $(L_n^*)_{n\geq 1}$ of positive linear operators of discrete type that act on a suitable weighted subspace of $C[0,\infty)$ and preserve $\mathbf{1}$ and e_2. Besides the variant of Szász-Mirakjan operators, introduced independently in [7], he also constructs a variant of Baskakov and Bernstein-Chlodovsky operators.

In [9] Agratini investigates convergence and quantitative estimates for the bivariate version of the general operators previously considered in [4]. It is worthwhile noticing that the above results seem to be the only obtained in a multidimensional setting.

Subsequently, other articles appear. First, we recall the paper due to Duman, Özarslan, and Aktuğlu [10] in which Szász-Mirakyan-Beta type operators preserving e_2 are considered. Moreover, Duman and Özarslan, jointly with Della Vecchia ([11]), study a Kantorovich modification of Szász-Mirakjan type operators preserving linear functions, and they show their operators enable better error estimation on the interval $[1/2,\infty)$ than the classical Szász-Mirakjan-Kantorovich operators.

Post Widder and Stancu operators are instead object of a modification that preserves e_2 in polynomial weighted spaces, proposed by Rempulska and Skorupka in [12]. Also in this case better approximation properties than the original operators are achieved.

Another new general approach is considered by Agratini and Tarabie in [13] (see also [14]). The authors construct classes of discrete linear positive operators, acting on $[0,1]$ or on $[0,\infty)$, and preserving both the constants and the polynomial $e_2 + \alpha e_1$ $(\alpha \geq 0)$. Those classes of operators include the ones considered in [5] and a new modification of Szász-Mirakyan operators (see also [15]).

Modifications which fix constants and linear functions, or the function e_2, have been introduced in [16–20] (see also [21, Chapter 5]). In particular, such studies are concerned with modified Bernstein-Durrmeyer operators, Phillips operators, integrated Szász-Mirakjan operators, Beta operators of the second kind, and a Durrmeyer-Stancu type variant of Jain operators.

New King-type operators which reproduce e_1 and e_2 are studied in [22] by Braica, Pop and Indrea. Subsequently, Pop's school deals with modifications of Kantorovich type operators, Durrmeyer type operators, Schurer operators, Bernstein-type operators, and Baskakov operators, fixing exactly two test functions from the set $\{\mathbf{1}, e_1, e_2\}$, (see, e.g., [23, 24]).

Another general approach deserves to be mentioned. Coming back to the classical Bernstein operators B_n, in [25] Gonska, Pițul, and Rașa construct a sequence of King-type operators V_n^τ which preserve $\mathbf{1}$ and a strictly increasing function $\tau \in C[0,1]$, such that $\tau(0) = 0$ and $\tau(1) = 1$. Such operators are defined as $V_n^\tau(f) = B_n(f) \circ (B_n\tau)^{-1} \circ \tau$, and they are a positive approximation process in $C[0,1]$. Moreover, they preserve some global smoothness properties. The authors also discuss the monotonicity of the sequence $(V_n^\tau f)_{n\geq 1}$ when f is a convex and decreasing function. They establish a Voronovskaja-type theorem, and finally they prove a recursion formula generalizing a corresponding result valid for the classical Bernstein operators. Note that the class of operators presented in [25] recovers the cases previously studied in [1, 5].

Subsequently, the study of the operators V_n^τ has been deepened by Birou in [26], where he finds some conditions under which V_n^τ's provide a lower approximation error than the classical Bernstein operators for the class of decreasing and generalized convex functions (see, also [27]). Moreover, he analyzes some shape preserving properties in the case τ is a polynomial of degree at most 2, or $\tau(x) = (e^{bx} - 1)/(e^b - 1)$ $(x \in [0,1], b < 0)$.

Very soon, the construction of the operators V_n^τ motivates other works.

In [27] the operators $B_n^\tau(f) = B_n(f \circ \tau^{-1}) \circ \tau$ which fix the function τ are studied and, among other things, they are compared with B_n's and V_n^τ's in the approximation of functions which are increasing and convex with respect to τ. The authors focus on the case for which B_n^τ and V_n^τ fix polynomials of degree m (see [28] for other generalizations of B_n's reproducing $\mathbf{1}$ and a strictly increasing polynomial). For more details about B_n^τ, see Section 3.2.

Subsequently, the above idea has been applied to other positive linear operators (see [29–33]).

In particular, in [32] the authors propose a generalization of the classical Szász-Mirakyan operators S_n by setting $S_n^\rho(f) = S_n(f \circ \rho^{-1}) \circ \rho$, where ρ is a continuously differentiable function on $[0,\infty)$ with $\rho(0) = 0$ and $\inf_{x \geq 0} \rho'(x) \geq 1$. We want to point out that this class of operators does not include the ones studied in [7]. However, very recently (see [34]; cf. Section 4.1), Aral, Ulusoy, and Deniz generalize the operators S_n^ρ, extending in this way the results contained in [7, 32]. See [35] for a modification of Baskakov-type operators in the spirit of what has been done for S_n^ρ.

We want to emphasize that the above constructions based on fixing suitable increasing functions do not recover the interesting case of linear operators fixing exponential functions, which has been a new and very popular direction in this research area in the last few years.

A sequence of Bernstein-type operators preserving $e^{\lambda_0 x}$ and $e^{\lambda_1 x}$, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 \neq \lambda_1$, was already present in the literature (see [36, 37]).

In [38] a modification of Szász-Mirakyan operators preserving constants and e^{2ax}, $a > 0$, is considered, while in [39] another modification of Szász-Mirakyan operators reproducing e^{ax} and e^{2ax} $(a > 0)$ is studied. For more details about these two different variants, see Section 4.2.

Later, the idea of preserving exponential functions of different type has been applied to some other well-known linear positive operators, for which approximation and shape preserving properties, as well as quantitative estimates and Voronovskaya-type theorems, are proven.

For papers inspired by [38, 39] we refer the readers to [40–42] and [43–46], respectively.

For modifications of linear operators preserving constants and e^{-x}, constants and e^{-2x}, or constants and e^{Ax}, $A \in \mathbb{R}$ cf. [47–51].

We end this section underlying that King's idea has been applied also to some $q-$ or $(p,q)-$ analogue operators (see, e.g., [52–56]) and to some sequences of operators involving orthogonal polynomials (see, e.g., [57]).

3. On Bernstein-Type Operators

In this section we review some results contained in [5, 27, 43], where the authors deal with different modifications of the Bernstein operators based on King's idea.

Let us start with some preliminaries. Throughout this section, $C[0,1]$ is the space of all continuous real valued functions on $[0,1]$, endowed with the sup norm $\|\cdot\|_\infty$ and the natural pointwise ordering. If $k \in \mathbb{N}$, the symbol $C^k[0,1]$ stands for the space of all continuously k-times differentiable functions on $[0,1]$.

We recall that the classical Bernstein operators are the positive linear operators $B_n : C[0,1] \longrightarrow C[0,1]$ defined by setting, for every $n \geq 1$, $f \in C[0,1]$, and $0 \leq x \leq 1$,

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{4}$$

It is very well known that the sequence $(B_n)_{n\geq 1}$ is an approximation process in $C[0,1]$; i.e., for every $f \in C[0,1]$, $\lim_{n\longrightarrow\infty} B_n(f) = f$ uniformly on $[0,1]$.

In what follows, it will be useful to recall the following inequality which is an estimate of the rate of the above approximation presented by Shisha and Mond: for any $C[0,1]$,

$$\left|B_n(f;x) - f(x)\right| \leq \left(1 + \frac{x(1-x)/n}{\delta^2}\right) \omega_1(f,\delta), \tag{5}$$

where $\omega_1(f,\delta)$ is the first-order modulus of continuity.

Besides the usual notion of convexity, other notions of convexity will be considered (see [58]; see also [59]).

Let $\{u, v\}$ be an extended complete Tchebychev system on $[0,1]$.

A function $f : (0,1) \longrightarrow \mathbb{R}$ is said to be convex with respect to $\{u\}$ (in symbols $f \in \mathscr{C}(u)$), whenever

$$\begin{vmatrix} u(x_0) & u(x_1) \\ f(x_0) & f(x_1) \end{vmatrix} \geq 0, \quad 0 < x_0 < x_1 < 1. \tag{6}$$

Moreover, a function $f : (0,1) \longrightarrow \mathbb{R}$ is said to be convex with respect to $\{u, v\}$, in symbol $f \in \mathscr{C}(u,v)$, whenever

$$\begin{vmatrix} u(x_0) & u(x_1) & u(x_2) \\ v(x_0) & v(x_1) & v(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0, \quad 0 < x_0 < x_1 < x_2 < 1. \tag{7}$$

If $f \in C[0,1]$, then (6) and (7) hold for $0 \leq x_0 < x_1 < x_2 \leq 1$.

For the convenience of the reader we split up the discussion into three subsections.

3.1. Bernstein-Type Operators Fixing Polynomials. In [5], the following Bernstein-type operators, depending on a real parameter $\alpha \geq 0$, are defined:

$$B_{n,\alpha}(f;x) := \sum_{k=0}^{n} \binom{n}{k} r_{n,\alpha}(x)^k \left(1 - r_{n,\alpha}(x)\right)^{n-k} f\left(\frac{k}{n}\right) \tag{8}$$

($n \geq 1$, $f \in C[0,1]$, $x \in [0,1]$), where $\{r_{n,\alpha} : [0,1] \longrightarrow \mathbb{R}\}_{n>1}$ is the sequence of functions defined by

$$r_{n,\alpha}(x) := -\frac{n\alpha + 1}{2(n-1)} + \sqrt{\frac{(n\alpha+1)^2}{4(n-1)^2} + \frac{n(\alpha x + x^2)}{n-1}} \qquad (0 \leq x \leq 1). \tag{9}$$

It is easy to check that $B_{n,\alpha} f = (B_n f) \circ (r_{n,\alpha})$. Note that, if $= 0 B_{n,\alpha}$'s turn into the classical King operators (1), while if α goes to infinity they become the classical Bernstein operators.

The operators $B_{n,\alpha}$ are positive and map $C[0,1]$ into itself, and they fix the functions $\mathbf{1}$ and $e_2 + \alpha e_1$. Moreover, $B_{n,\alpha}(e_1) = r_{n,\alpha}$ and $B_{n,\alpha}(e_2) = (1/n) r_{n,\alpha} + ((n-1)/n) r_{n,\alpha}^2$.

Korovkin theorem can be applied in order to conclude that, for $f \in C[0,1]$, $\lim_{n\longrightarrow\infty} B_{n,\alpha}(f;x) = f(x)$ for $0 \leq x \leq 1$ since, for all $\alpha \geq 0$, $r_{n,\alpha}(x)$ converges to x.

Considering the first and second modulus of smoothness, the following quantitative estimates can be achieved:

$$\left|B_{n,\alpha}(f;x) - f(x)\right| \leq \left(1 + \frac{2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha + 2x)}{\delta^2}\right) \omega_1(f,\delta), \tag{10}$$

$$\left|B_{n,\alpha}(f;x) - f(x)\right| \leq \frac{\left|r_{n,\alpha}(x) - x\right|}{\delta} \omega_1(f,\delta) + \left(1 + \frac{2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha+2x)}{2\delta^2}\right) \omega_2(f,\delta). \tag{11}$$

By comparing estimates (10) and (5), we have then the approximation error for the operators $B_{n,\alpha}$ is at least as good as the one for B_n's on the interval $[0, H_\alpha]$, where $H_\alpha = (1 - 2\alpha + \sqrt{1 + 8\alpha + 4\alpha^2})/6$. Indeed, we have that the inequality

$$2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha + 2x) \leq \frac{x(1-x)}{n} \tag{12}$$

holds if and only if

$$0 \leq x \leq \frac{1 + n - 2n\alpha + \sqrt{1 + 2n + n^2 + 8n^2\alpha + 4n^2\alpha^2}}{2(1+3n)}. \tag{13}$$

Note that the right-end term in the above inequalities decreases to H_α as n goes to infinity. We point out that for $H_0 = 1/3$ we recover the result due to King, while for $\alpha \longrightarrow +\infty$ we get $H_\alpha \longrightarrow 1/2$; therefore King's result is improved.

The operators $B_{n,\alpha}$ share some shape preserving properties. We begin to recall that they map continuous and increasing functions into (continuous) increasing functions. Moreover, if f is convex and increasing, then $B_{n,\alpha}(f)$ is convex. Finally, if f is convex with respect to $\{1, e_2 + \alpha e_1\}$, then $B_{n,\alpha}(f) \geq f$ on $[0,1]$.

The operators $B_{n,\alpha}$ verify the following asymptotic formula:

$$\lim_{n\longrightarrow\infty} 2n\left(B_{n,\alpha}(f;x) - f(x)\right) = x(1-x)\left(f''(x) - \frac{2}{2x+\alpha} f'(x)\right), \tag{14}$$

for all functions $f \in C[0,1]$, which are two times differentiable at $x \in (0,1)$.

We end this subsection observing that if we impose additional conditions on f, we can get tangible improvements in the approximation error. In fact, if $f \in C[0,1]$ is increasing and if the divided difference $f[x_0, x_1, x_2]$ of f on the nodes $0 \leq x_0 < x_1 < x_2 \leq 1$ satisfy $f[x_0, x_1, x_2] \geq M$, M being a real strictly positive constant, there exists $\overline{\alpha} \geq 0$ such that

$$0 \leq B_{n,\alpha}(f;x) - f(x) < B_n(f;x) - f(x), \quad \text{for } \alpha \geq \overline{\alpha} \text{ and } 0 < x < 1. \tag{15}$$

In particular, $\overline{\alpha} := \min\{\alpha \geq 0 : (f(1) - f(0))/(1+\alpha) \leq M\}$. Note that, if $f \in C^2[0,1]$ is increasing and strictly convex and M is the lower bound of f'', then $\overline{\alpha} = 2f'(1)/M$.

3.2. Polynomial Operators Fixing Increasing Functions. The operators considered in the previous section fix $\mathbf{1}, e_2 + \alpha e_1$, but they are not polynomial-type operators. The construction of polynomial-type operators fixing the above functions is presented in [27]. In that paper operators of the form $B_n^\tau f = B_n(f \circ \tau^{-1}) \circ \tau$ are considered, where τ is any infinitely times continuously differentiable function on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$. More precisely,

$$B_n^\tau(f;x) = \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right), \qquad f \in C[0,1], x \in [0,1]. \tag{16}$$

The Bernstein operators can be obtained as a particular case for $\tau = e_1$. On the other hand, if $\tau = (e_2 + \alpha e_1)/(1+\alpha)$, B_n^τ is a polynomial-type operator and $B_n^\tau(\tau) = \tau$. For a Durrmeyer variant of the operators B_n^τ we refer the readers to [29] (and for a genuine Durrmeyer variant see [33]).

We note that $B_n^\tau \tau^2 = \tau/n + ((n-1)/n)\tau^2$. From the positivity of B_n^τ, together with the fact that $\{1, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$, we easily get that $\lim_{n\longrightarrow\infty} B_n^\tau(f) = f$ uniformly on $[0,1]$.

Moreover, the operators B_n^τ map continuous and increasing functions into (continuous) and increasing functions. Finally, $B_n^\tau(f)$ is τ-convex of order k provided that f is so too (if $k \in \mathbb{N}$, we say that a function $f \in C^k[0,1]$ is τ-convex of order k whenever $D_\tau^m f = D^m(f \circ \tau^{-1}) \circ \tau$, D^k being the usual k-th differential operator).

For any function $f \in C[0,1]$, two times differentiable at $x \in (0,1)$, we have that

$$\lim_{n\longrightarrow\infty} 2n\left(B_n^\tau f(x) - f(x)\right) = \tau(x)(1 - \tau(x))\left(-\frac{\tau''(x) f'(x)}{\tau'^3} + \frac{f''(x)}{\tau'^2}\right). \tag{17}$$

We end this subsection by comparing the operators B_n^τ with B_n's.

First, if we take a positive constant K, whose existence is guaranteed by Freud [60], such that $K(t-x)^2 \leq \tau'^2$ for all

$t, x \in [0,1]$; we have the following estimate: for $f \in C[0,1]$, $\delta > 0$, and $x \in [0,1]$,

$$\begin{aligned}&\left|B_n^{\tau}(f;x) - f(x)\right| \\ &\quad \le \omega_1(f,\delta)\left(1 + \frac{\tau'(x)\,\tau(x)\,(1-\tau(x))}{nK\delta^2}\right).\end{aligned} \tag{18}$$

Moreover, the following statement holds.

Theorem 1. *Let $f \in C^2[0,1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \le B_n^{\tau}(f;x) \le B_n(f;x), \quad \forall n \ge n_0,\ x \in (0,1). \tag{19}$$

Then

$$\begin{aligned}f''(x) &\ge \frac{\tau''(x)}{\tau'(x)} f'(x) \\ &\ge \left(1 - \frac{x(1-x)\tau'^2}{\tau(x)(1-\tau(x))}\right) f''(x), \\ &\qquad\qquad x \in (0,1).\end{aligned} \tag{20}$$

In particular, $f''(x) \ge 0$.

Conversely, if (20) holds with strict inequalities at a given point $x_0 \in (0,1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$f(x_0) < B_n^{\tau}(f;x_0) < B_n(f;x_0). \tag{21}$$

The proof is based on the comparison between the expression $x(1-x)$ and $\tau(x)(1-\tau(x))(-\tau''(x)f'(x)/\tau'^3 + f''(x)/\tau'^2)$ in the asymptotic formulae for B_n's and B_n^{τ}'s, respectively.

3.3. Fixing Increasing Exponential Functions. In this section we discuss the operators defined in [43]. From now on, set $a_n(x) := (e^{\mu x/n} - 1)/(e^{\mu/n} - 1)$ and recall that $\exp_\mu(x) := e^{\mu x}$ ($\mu > 0$). We define the sequence of positive linear operators $\mathcal{G}_n$ as

$$\mathcal{G}_n(f;x) = \exp_\mu(x)\, B_n\left(\frac{f}{\exp_\mu}; a_n(x)\right), \tag{22}$$

or, equivalently,

$$\begin{aligned}&\mathcal{G}_n(f;x) \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} a_n(x)^k (1 - a_n(x))^{n-k} f\left(\frac{k}{n}\right) e^{-\mu k/n} e^{\mu x},\end{aligned} \tag{23}$$

for $f : [0,1] \longrightarrow \mathbb{R}$, $n \ge 1$, and $0 \le x \le 1$. The functions fixed by these operators are $\exp_\mu$ and $\exp_\mu^2$ ($\mu > 0$). Moreover, for any $x \in [0,1]$ and $n \ge 1$, the following identities hold:

$$\begin{aligned}&\mathcal{G}_n(\mathbf{1};x) = e^{\mu(x-1)}\left(e^{\mu/n} + 1 - e^{\mu x/n}\right)^n, \\ &\mathcal{G}_n\left(\exp_\mu^3;x\right) = e^{\mu x}\left(e^{\mu(x+1)/n} + e^{\mu x/n} - e^{\mu/n}\right)^n, \\ &\mathcal{G}_n\left(\exp_\mu^4;x\right) \\ &\quad = e^{\mu x}\left(e^{\mu(x+2)/n} + e^{\mu(x+1)/n} + e^{\mu x/n} - e^{\mu/n} - e^{2\mu/n}\right)^n.\end{aligned} \tag{24}$$

Since $\{\mathbf{1}, \exp_\mu, \exp_\mu^2\}$ is an extended complete Tchebychev system, and the operators $\mathcal{G}_n$ are positive, they are an approximation process in $C[0,1]$ (i.e., for each $f \in C[0,1]$, $\lim_{n\longrightarrow\infty}\mathcal{G}_n(f;x) = f(x)$ uniformly w.r.t. $x \in [0,1]$).

Other (shape preserving) properties that this sequence verifies are

(i) if $f/\exp_\mu$ is increasing, then it is $\mathcal{G}_n(f)/\exp_\mu$;

(ii) if $f/\exp_\mu$ is increasing and convex, then $\mathcal{G}_n(f/\exp_\mu)$ is convex;

(iii) if $f \in \mathscr{C}(\exp_\mu)$, then $\mathcal{G}_n(f) \in \mathscr{C}(\exp_\mu)$ (see (6)).

Moreover,

$$\begin{aligned}&\left|\mathcal{G}_n(f;x) - f(x)\right| \le \left|f(x)\right|\left(\mathcal{G}_n(\mathbf{1};x) - 1\right) \\ &\quad + \left(\mathcal{G}_n(\mathbf{1};x) + \frac{e^{2\mu x}\left(\mathcal{G}_n(\mathbf{1};x) - 1\right)}{\delta^2}\right) \\ &\quad \cdot \omega_1\left(f \circ \log_\mu; \delta\right),\end{aligned} \tag{25}$$

for $f \in C[0,1]$, $x \in (0,1)$, and $\delta > 0$. Here $\log_\mu$ denotes the inverse function of $\exp_\mu$. If $\mu \ge 1$, then $\omega_1(f \circ \log_\mu; \delta)$ can be replaced by $\omega_1(f;\delta)$.

For the operators $\mathcal{G}_n$, the following Voronovskaya-type result holds:

$$\begin{aligned}&\lim_{n\longrightarrow\infty} 2n\left(\mathcal{G}_n(f;x) - f(x)\right) \\ &\quad = x(1-x)\left(f''(x) - 3\mu f'^2 f(x)\right),\end{aligned} \tag{26}$$

if $f \in C[0,1]$ has second derivative at a point $x \in (0,1)$.

As in the previous subsection, by comparing the asymptotic formulae for B_n and $\mathcal{G}_n$, we are able to get an improvement in the approximation by means of operators $\mathcal{G}_n$ with respect to the operators B_n under certain conditions.

Theorem 2. *Let $f \in C^2[0,1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \le \mathcal{G}_n(f;x) \le B_n(f;x), \quad \forall n \ge n_0,\ x \in (0,1). \tag{27}$$

Then

$$f''(x) \ge 3\mu f'^2 f(x) \ge 0, \quad x \in (0,1). \tag{28}$$

In particular, $f''(x) \ge 0$.

Conversely, if (28) holds with strict inequalities at a given point $x \in (0,1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$f(x) < \mathcal{G}_n(f;x) < B_n(f;x). \tag{29}$$

We end this section by observing that if the following conjecture is true, we might obtain an even better improvement in the approximation error.

Conjecture 3. *If $f \in C[0,1]$ is such that $f \in \mathscr{C}(\exp_\mu)$ and $f \in \mathscr{C}(\exp_\mu, \exp_\mu^2)$, then for all $n \in \mathbb{N}$ and all $x \in [0,1]$, one has that $f(x) \le \mathcal{G}_n(f;x) \le B_n(f;x)$.*

4. On Szász-Mirakyan Type Operators

In the present section we pass to discuss sequences of positive linear operators acting on spaces of continuous functions on unbounded intervals. To this end, we need to fix preliminarily some notations and recall definition and main results concerning the classical Szász-Mirakyan operators.

First of all, we denote by $C[0,\infty)$ the space of all continuous real valued functions on $[0,\infty)$. We also indicate by $C_b[0,\infty)$ the subspace of all continuous bounded functions on $[0,\infty)$. The space $C_b[0,\infty)$, endowed with the sup-norm $\|\cdot\|_\infty$ and the natural pointwise ordering, is a Banach lattice. Moreover, the space of all continuous functions that converge at infinity will be denoted by $C^*[0,\infty)$.

In what follows, let φ be a weight function on $[0,\infty)$; we define

$$B_\varphi[0,\infty)=\{f:[0,\infty)\longrightarrow\mathbb{R}\mid \text{there exists } M_f \geq 0 \text{ such that } |f(x)|\leq M_f\varphi(x)\ \forall x\geq 0\}. \tag{30}$$

Clearly, $B_\varphi[0,\infty)$ is a normed space when endowed with the weighed norm

$$\|f\|_\varphi=\sup_{x\geq 0}\frac{|f(x)|}{\varphi(x)}\quad \left(f\in B_\varphi[0,\infty)\right). \tag{31}$$

Moreover, we denote by $C_\varphi[0,\infty)$ the space of all continuous functions in $B_\varphi[0,\infty)$, and by $C_\varphi^*[0,\infty)$ the space consisting of all functions in $C_\varphi[0,\infty)$ that converge at infinity. Finally, we say that $f\in U_\varphi[0,\infty)$ if f/φ is uniformly continuous.

It is well known that Szász-Mirakyan operators were introduced independently in the 1940s by J. Favard ([61]), G. M. Mirakjan ([62]), and O. Szász ([63]), and they are defined by setting

$$S_n(f;x):=\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)\quad(n\geq 1,x\geq 0), \tag{32}$$

for all functions $f:[0,\infty)\longrightarrow\mathbb{R}$ for which the series at the right-hand side is absolutely convergent. This space includes, in particular, all functions $f:[0,\infty)\longrightarrow\mathbb{R}$ such that $|f(x)|\leq M\exp(\alpha x)$ $(x\geq 0)$, for some $M\geq 0$ and $\alpha\in\mathbb{R}$.

In particular S_n's map $C_b[0,\infty)$ and $C^*[0,\infty)$ into themselves.

It might be useful for the following subsections to recall that (see [64, Lemma 3]), $S_n(\mathbf{1})=\mathbf{1}$, $S_n(e_1)=e_1$, and $S_n(e_2)=e_2+(1/n)e_1$.

Moreover, for every $x\geq 0$,

$$\begin{aligned}S_n(\psi_x(t);x)&=0,\\ S_n\left(\psi_x^2(t);x\right)&=\frac{x}{n},\end{aligned} \tag{33}$$

where, for every $y\geq 0$, $\psi_x(y)=y-x$.

It is well known that the sequence $(S_n)_{n\geq 1}$ is an approximation process in $C^*[0,\infty)$; more precisely, for every $f\in C^*[0,\infty)$, $\lim_{n\longrightarrow\infty}S_n(f;x)=f(x)$ uniformly w.r.t. $x\in[0,\infty)$.

In particular, we recall that, taking (33) into account, for every $f\in C_b[0,\infty)$, $x\geq 0$ and $n\geq 1$ (see, for example, [65, Theorem 5.1.2]),

$$\begin{aligned}\left|S_n(f;x)-f(x)\right|&\leq 2\omega_1\left(f,\sqrt{S_n\left(\psi_x^2(t);x\right)}\right)\\ &=2\omega_1\left(f,\sqrt{\frac{x}{n}}\right),\end{aligned} \tag{34}$$

where $\omega_1(f,\delta)$ denotes the classical first modulus of continuity.

This last result might be useful to compare the Szász-Myrakyan operators with suitable generalizations that fix different functions.

4.1. Generalized Szász-Mirakyan Operators. In this subsection, we examine the Szász-Mirakyan type operators studied in [34]. Let $\rho:[0,\infty)\longrightarrow\mathbb{R}$ be a function satisfying the following properties:

(a) ρ is continuously differentiable on $[0,\infty)$;

(b) $\rho(0)=0$ and $\inf_{x\geq 0}\rho'(x)\geq 1$.

From now on, we set

$$\varphi(x)=1+\rho^2(x)\quad(x\geq 0), \tag{35}$$

and we consider the weighted spaces $B_\varphi[0,\infty)$, $C_\varphi[0,\infty)$, $C_\varphi^*[0,\infty)$, and $U_\varphi[0,\infty)$.

If $\rho(x)=x$ for each $x\geq 0$ the space $C_\varphi[0,\infty)$ (resp., $C_\varphi^*[0,\infty)$) becomes the classical weighed space

$$E_2=\left\{f\in C[0,\infty):\sup_{x\geq 0}\frac{f(x)}{1+x^2}\in\mathbb{R}\right\} \tag{36}$$

(resp.,

$$\left.E_2^*=\left\{f\in C[0,\infty):\lim_{x\longrightarrow+\infty}\frac{f(x)}{1+x^2}\in\mathbb{R}\right\}\right). \tag{37}$$

The following result, proven in [66], shows that $\{\mathbf{1},\rho,\rho^2\}$ is a Korovkin set in $C_\varphi^*[0,\infty)$.

Theorem 4. *Consider a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C_\varphi[0,\infty)$ into $B_\varphi[0,\infty)$. If*

$$\lim_{n\longrightarrow\infty}\left\|L_n(\rho^\nu)-\rho^\nu\right\|_\varphi=0\quad \text{for } \nu=0,1,2, \tag{38}$$

and, then, for every $f\in C_\varphi^*[0,\infty)$,

$$\lim_{n\longrightarrow\infty}\left\|L_n(f)-f\right\|_\varphi=0. \tag{39}$$

After these preliminaries, set $\mathbb{N}_1:=\{n\in\mathbb{N}\mid n\geq n_0\}$, for a suitable $n_0\in\mathbb{N}$. Given an interval $I\subset[0,\infty)$, consider two sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ of functions on I such that, for any $n\in\mathbb{N}_1$,

(i) $\alpha_n,\beta_n:I\longrightarrow\mathbb{R}$ are positive functions on I;

(ii) $\beta_n(x)-\alpha_n(x)\geq 0$ for every $x\in I$.

In [34], the authors introduced and studied the sequence of the generalized Szász-Myrakjan operators, defined as

$$\widetilde{S}_n^{\rho}(f;x) = e^{-\alpha_n(x)} \sum_{k=0}^{\infty} \frac{(\beta_n(x))^k}{k!} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) \tag{40}$$

for every $f \in C(I), n \in \mathbb{N}_1$ and $x \in I$.

Some conditions have to be imposed in order that the sequence $(\widetilde{S}_n^{\rho})_{n \geq n_0}$ is an approximation process in $C_{\varphi}^{*}[0,\infty)$, and, in particular, in order to verify (38).

More precisely, for any $n \geq n_0$, there exist $u_n, v_n : I \longrightarrow \mathbb{R}$ such that, for every $x \in I$,

$$\begin{gathered} |u_n(x)| \leq u_n^0, \\ |v_n(x)| \leq v_n^0, \\ \lim_{n \longrightarrow \infty} u_n^0 = \lim_{n \longrightarrow \infty} v_n^0 = 0, \end{gathered} \tag{41}$$

$$\widetilde{S}_n^{\rho}(\mathbf{1};x) = 1 + u_n(x), \tag{42}$$

$$\widetilde{S}_n^{\rho}(\rho;x) = \rho(x) + v_n(x). \tag{43}$$

Evaluating the operators $\widetilde{S}_n^{\rho}$ on $\mathbf{1}$ and ρ, it is easy to connect the sequences $(\alpha_n)_{n \geq n_0}$ and $(\beta_n)_{n \geq n_0}$ with $(u_n)_{n \geq n_0}$ and $(v_n)_{n \geq n_0}$, taking (40), (42), and (43) into account. More precisely, for every $x \in I$ and $n \geq n_0$,

$$\begin{aligned} \alpha_n(x) &= n \frac{\rho(x) + v_n(x)}{1 + u_n(x)} - \log\left(1 + u_n(x)\right), \\ \beta_n(x) &= n \frac{\rho(x) + v_n(x)}{1 + u_n(x)}. \end{aligned} \tag{44}$$

Accordingly, for any $n \geq n_0$, $f \in C(I)$ and $x \in I$,

$$\begin{aligned} \widetilde{S}_n^{\rho}(f;x) &= e^{-n((\rho(x)+v_n(x))/(1+u_n(x)))}\left(1 + u_n(x)\right) \\ &\cdot \sum_{k=0}^{\infty} \frac{1}{k!} \left(n \frac{\rho(x) + v_n(x)}{1 + u_n(x)}\right)^k \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right). \end{aligned} \tag{45}$$

The operators $\widetilde{S}_n^{\rho}$ map $C_{\varphi}[0,\infty)$ into $B_{\varphi}[0,\infty)$. Moreover, since easy calculations show that, for every $x \in I$ and $n \geq n_0$,

$$\widetilde{S}_n^{\rho}\left(\rho^2;x\right) = \frac{\left(\rho(x) + v_n(x)\right)^2}{1 + u_n(x)} + \frac{\rho(x) + v_n(x)}{n}, \tag{46}$$

by applying Theorem 4 to an extension of the operators $\widetilde{S}_n^{\rho}(f)$ to $[0,\infty)$,

$$\lim_{n \longrightarrow \infty} \sup_{x \in I} \frac{\left|\widetilde{S}_n^{\rho}(f;x) - |f(x)|\right|}{\varphi(x)} = 0. \tag{47}$$

Some estimates of the rate of convergence are available, by using a suitable modulus of continuity, introduced by Holhoş in [67]. More precisely, it is defined by setting, for every $f \in C_{\varphi}[0,\infty)$ and $\delta > 0$,

$$\omega_{\rho}(f;\delta) = \sup_{\substack{x,t \geq 0 \\ |\rho(t)-\rho(x)| \leq \delta}} \frac{|f(t) - f(x)|}{\varphi(t) + \varphi(x)}. \tag{48}$$

In particular, by using the results in [67], it can be proven that, for every $f \in C_{\varphi}[0,\infty)$ and $n \geq n_0$,

$$\begin{aligned} &\left\|\widetilde{S}_n^{\rho}(f) - f\right\|_{\varphi^{3/2}} \\ &\quad \leq \left(7 + 4u_n^0 + 2\left(2v_n^0 + \left(v_n^0\right)^2 + \frac{2}{n} + \frac{2v_n^0}{n}\right)\right) \\ &\quad \cdot \omega_{\rho}\left(f;\delta_n\right), \end{aligned} \tag{49}$$

where

$$\begin{aligned} \delta_n = {} & \frac{16}{n} + \frac{4}{n^2} + 3u_n^0 + 20v_n^0 + \frac{22v_n^0}{n} + \frac{4v_n^0}{n^2} + 8\left(v_n^0\right)^2 \\ & + \frac{6\left(v_n^0\right)^2}{n} + \left(v_n^0\right)^3 \\ & + 2\sqrt{\left(1 + u_n^0\right)\left(\frac{2}{n} + u_n^0 + 4v_n^0 + \frac{2v_n^0}{n} + \left(v_n^0\right)^2\right)}. \end{aligned} \tag{50}$$

Moreover, since $\lim_{\delta \longrightarrow 0} \omega_{\rho}(f;\delta) = 0$ if $f \in U_{\varphi}[0,\infty)$, from the latter formula and (41), we get that

$$\lim_{n \longrightarrow \infty} \left\|\widetilde{S}_n^{\rho}(f) - f\right\|_{\varphi^{3/2}} = 0 \tag{51}$$

for every $f \in U_{\varphi}[0,\infty)$.

Further, under suitable assumptions, it is possible to determine a Voronovskaya-type result involving $\widetilde{S}_n^{\rho}$'s. More precisely, assume that

$$\begin{aligned} \lim_{n \longrightarrow \infty} n u_n(x) &= l_1, \\ \lim_{n \longrightarrow \infty} n v_n(x) &= l_2. \end{aligned} \tag{52}$$

Moreover, consider a function $f \in C_{\varphi}[0,\infty)$ for which the function $f \circ \rho^{-1}$ is twice differentiable. If the second derivative of $f \circ \rho^{-1}$ is bounded on $[0,\infty)$, then, for every $x \in I$,

$$\begin{aligned} &\lim_{n \longrightarrow \infty} n\left(\widetilde{S}_n^{\rho}(f;x) - f(x)\right) \\ &\quad = f(x) l_1 + \left(l_2 - \rho(x) l_1\right)\left(f \circ \rho^{-1}\right)'(\rho(x)) \\ &\quad + \frac{1}{2}\rho(x)\left(f \circ \rho^{-1}\right)''(\rho(x)). \end{aligned} \tag{53}$$

The following examples show that, for suitable choices of the sequences $(u_n(x))_{n \geq n_0}$, $(v_n(x))_{n \geq n_0}$ and of the function ρ, operators (45) turn into well known Szász-Myrakjan type operators that fix certain functions and the results in [34] can be applied to those operators. For quantitative Voronovskaya theorems and the study of a Durrmeyer-type variant of the operators (40) see [68] and [69], respectively.

Examples 1. (1) If $I = [0,\infty)$, $u_n(x) = v_n(x) = 0$, and $\rho(x) = x$ for every $x \geq 0$, the operators $\widetilde{S}_n^{\rho}$ turn into the classical Szász-Myrakjan operators (32), which, as it is well known, preserve the function e_0 and $\rho = e_1$.

(2) If $I = [0,\infty)$, $\rho(x) = x$, $u_n(x) = 0$, and $v_n(x) = -1/2n + \sqrt{4n^2x^2+1}/2n - x$, then operators $\widetilde{S_n}^{\rho}$ turn into

$$D_n^*(f;x) = e^{(1-\sqrt{4n^2x^2+1})/2}\sum_{k=0}^{\infty}\frac{\left(\sqrt{4n^2x^2+1}-1\right)^k}{2^k k!}f\left(\frac{k}{n}\right) \tag{54}$$

($f \in E_2, n \geq 1, x \geq 0$), which were object of investigation in [7] and, in the spirit of King's work, preserve the function e_0 and $\rho^2 = e_2$.

In particular, when applied to D_n^*, (53) gives the following result. If $f \in E_2$ is a function which is twice differentiable and whose second derivative is bounded on $[0,\infty)$, then, for every $x \geq 0$,

$$\lim_{n\to\infty} n\left(D_n^*(f;x) - f(x)\right) = -\frac{1}{2}f'(x) + \frac{x}{2}f''(x). \tag{55}$$

Formula (55) holds true uniformly w.r.t. $x \geq 0$, if $f', f'' \in E_2^*$. An estimate of convergence in (55) can be found in [70, Corollary 4].

By means of [65, Theorem 5.1.2], we have that, for every $f \in C_b[0,\infty)$,

$$\left|D_n^*(f;x) - f(x)\right| \leq 2\omega_1\left(f, \sqrt{D_n^*(\psi_x^2(x))}\right). \tag{56}$$

We point out that, as shown in [7], for every $x \geq 0$

$$D_n^*\left(\psi_x^2(t);x\right) = 2x^2 + \frac{x}{n} - \frac{x\sqrt{4n^2x^2+1}}{n}. \tag{57}$$

Easy calculations prove that $D_n^*(\psi_x^2(t);x) \leq S_n(\psi_x^2(t);x)$ for every $x \geq 0$, so that, at least for $f \in C_b[0,\infty)$, the operators D_n^* provide a better approximation error than the classical Szász-Myrakjan operators S_n (see (34)).

(3) If $I = [1/(n_0-1),\infty)$, $u_n(x) = 1/(nx-1)$, $v_n(x) = 0$, and $\rho(x) = x$, then $\widetilde{S_n}^{\rho}$'s are exactly the operators studied in [71], given by

$$S_n^*(f;x) = \frac{nxe^{1-nx}}{nx-1}\sum_{k=0}^{\infty}\frac{(nx-1)^k}{k!}f\left(\frac{k}{n}\right) \tag{58}$$

($f \in E_2, n \geq n_0, x \in I$). Those operators fix the functions $\rho = e_1$ and $\rho^2 = e_2$. In this case, the Voronovskaya-type formula becomes

$$\lim_{n\to\infty} n\left(S_n^*(f;x) - f(x)\right) = \frac{f(x)}{x} - f'(x) + \frac{x}{2}f''(x), \tag{59}$$

for all $x \in I$ and all $f \in E_2$ which are twice differentiable and whose second derivative is bounded.

(4) For $I = [0,\infty)$, $u_n(x) = v_n(x) = 0$ for every $x \geq 0$ and considering an arbitrary function ρ satisfying (a) and (b), the operators $\widetilde{S_n}^{\rho}$ reduce to

$$S_n^{\rho}(f;x) = e^{-n\rho(x)}\sum_{k=0}^{\infty}\frac{(n\rho(x))^k}{k!}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right) \tag{60}$$

($f \in C_{\varphi}[0,\infty), x \in I, n \geq 1$) that were introduced and studied in [32] and preserve the functions e_0 and ρ.

In particular, for every $f \in C_{\varphi}[0,\infty)$,

$$\left\|S_n^{\rho}(f) - f\right\|_{\varphi^{3/2}} \leq \left(7 + \frac{4}{n}\right)\omega_{\rho}\left(f; \frac{20}{n} + \sqrt{\frac{8}{n}}\right). \tag{61}$$

Moreover, if $f \in C_{\varphi}[0,\infty)$ is a function such that $f\circ\rho^{-1}$ is twice differentiable and the second derivative of $f\circ\rho^{-1}$ is bounded on $[0,\infty)$, then, for every $x \geq 0$,

$$\lim_{n\to\infty} n\left(S_n^{\rho}(f;x) - f(x)\right) = \frac{1}{2}\rho(x)\left(f\circ\rho^{-1}\right)''(\rho(x)). \tag{62}$$

4.2. Fixing Increasing Exponential Functions. Another recent modification of the sequence of Szász-Mirakyan operators relies on the preservation of some exponential functions.

For functions $f \in C[0,\infty)$, such that the right-hand side below is absolutely convergent, Szász-Mirakyan operators reproducing the functions $\mathbf{1}$ and e^{2ax}, $a > 0$, are introduced in [38] and defined by

$$R_n^*(f;x) := e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{(n\alpha_n(x))^k}{k!}f\left(\frac{k}{n}\right) \tag{63}$$

($x \geq 0, n \in \mathbb{N}$), in such a way that the conditions

$$R_n^*\left(e^{2at};x\right) = e^{2ax} \tag{64}$$

are satisfied for all x and all n. To provide condition (64), equality

$$\alpha_n(x) = \frac{2ax}{n\left(e^{2a/n}-1\right)} \tag{65}$$

must be held (for more details see [38]).

To investigate the approximation properties of the operators R_n^*, some preliminaries are needed. First, if $a \geq 0$, we get

$$\begin{aligned} R_n^*\left(e^{at};x\right) &= e^{n\alpha_n(x)(e^{a/n}-1)} = e^{2ax/(e^{a/n}+1)},\\ R_n^*(\mathbf{1};x) &= 1,\\ R_n^*(e_1;x) &= \alpha_n(x)\\ R_n^*(e_2;x) &= \alpha_n^2(x) + \frac{\alpha_n(x)}{n}. \end{aligned} \tag{66}$$

Then, letting $\psi_x^k(t) := (t-x)^k$, $k = 0,1,2,\ldots$, we have

$$\begin{aligned} R_n^*\left(\psi_x^0(t);x\right) &= 1,\\ R_n^*\left(\psi_x^1(t);x\right) &= \alpha_n(x) - x,\\ R_n^*\left(\psi_x^2(t);x\right) &= \left(\alpha_n(x)-x\right)^2 + \frac{\alpha_n(x)}{n}. \end{aligned} \tag{67}$$

Moreover, considering equality (65), one can find

$$\lim_{n\to\infty} n\left(\frac{2ax}{n(e^{2a/n}-1)}-x\right)=-ax,$$
$$\lim_{n\to\infty} n\left(\left(\frac{2ax}{n(e^{2a/n}-1)}-x\right)^2+\frac{2ax}{n^2(e^{2a/n}-1)}\right)=x. \tag{68}$$

In 1970, Boyanov and Veselinov [72] showed that uniform convergence of any sequence of positive linear operators acting on $C^*[0,\infty)$ can be checked as follows.

Theorem 5. *The sequence $A_n : C^*[0,\infty)\longrightarrow C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt};x\right)=e^{-kt},\quad k=0,1,2, \tag{69}$$

uniformly in $[0,\infty)$, if and only if

$$\lim_{n\to\infty} A_n(f;x)=f(x) \tag{70}$$

uniformly in $[0,\infty)$, for all $f\in C^*[0,\infty)$.

A quantitative form for Theorem 5 can be given using the modulus of continuity on $C^*[0,\infty)$ introduced in [73, Corollary 3.2] and defined as

$$\omega^*(f,\delta)=\sup_{\substack{x,t\geq 0\\ |e^{-x}-e^{-t}|\leq\delta}}|f(x)-f(t)| \tag{71}$$

$(\delta>0, f\in C^*[0,\infty))$.

Theorem 6. *For $f\in C^*[0,\infty)$, we have*

$$\|R_n^*(f)-f\|_\infty\leq 2\omega^*\left(f;\sqrt{2\beta_n+\gamma_n}\right), \tag{72}$$

where

$$\beta_n=\left\|R_n^*\left(e^{-t};x\right)-e^{-x}\right\|_\infty,$$
$$\gamma_n=\left\|R_n^*\left(e^{-2t};x\right)-e^{-2x}\right\|_\infty. \tag{73}$$

Moreover, β_n and γ_n tend to zero as n goes to infinity so that R_n^*f converges uniformly to f.

To investigate pointwise convergence of the operators R_n^* a quantitative Voronovskaya theorem is presented in [38] as well. Such a result allows establishing the rate of pointwise convergence and an upper bound for the error of approximation.

Theorem 7. *Let $f, f''\in C^*[0,\infty)$. Then the inequality*

$$\left|n\left[R_n^*(f;x)-f(x)\right]+axf'(x)-\frac{x}{2}f''(x)\right|$$
$$\leq|p_n(x)|\left|f'(x)\right|+|q_n(x)|\left|f''(x)\right|$$
$$+2\left(2q_n(x)+x+r_n(x)\right)\omega^*\left(f'';\frac{1}{\sqrt{n}}\right) \tag{74}$$

holds for any $x\in[0,\infty)$, where

$$p_n(x):=nR_n^*(\psi_x(t);x)+ax,$$
$$q_n(x):=\frac{1}{2}\left(nR_n^*\left(\psi_x^2(t);x\right)-x\right), \tag{75}$$
$$r_n(x):=n^2\sqrt{R_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{R_n^*\left(\psi_x^4(t);x\right)}.$$

As a uniform approximation result let us recall, as explained in [38], that the spaces $(C^*[0,\infty),\|\cdot\|_{[0,\infty)})$ and $(C[0,1],\|\cdot\|_{[0,1]})$ are isometrically isomorphic. Define $\psi(y):=e^{-y}$, $y\in[0,\infty)$, and let $T:C[0,1]\longrightarrow C^*[0,\infty)$ be given by

$$T(f)(y)=f^*(y)=f(\psi(y)),$$
$$f\in C[0,1],\ y\in[0,\infty). \tag{76}$$

We remark that

$$\lim_{t\to\infty}f^*(t)=\lim_{t\to\infty}f(\psi(t))=f(0). \tag{77}$$

Clearly, T is linear and bijective. Moreover, for all $f\in C[0,1]$ one has

$$\|Tf\|_{[0,\infty)}=\sup_{t\in[0,\infty)}|f(\psi(t))|=\|f\|_{[0,1]}. \tag{78}$$

Hence T is an isometric isomorphism and

$$T^{-1}(f^*)=f^*\circ\psi^{-1},\quad\text{for } f^*\in C^*[0,\infty). \tag{79}$$

Corollary 8. *For all $f^*\in C^*[0,\infty)$ $(f=f^*\circ\psi^{-1})$ and n large enough we have*

$$\|R_n^*f^*-f^*\|_{[0,\infty)}\leq\omega_1\left(f;\sqrt{\frac{1}{2}(\gamma_n+2\beta_n)}\right)_{[0,1]}$$
$$+2\omega_2\left(f;\sqrt{\frac{1}{2}(\gamma_n+2\beta_n)}\right)_{[0,1]}. \tag{80}$$

To see some of the advantages of new constructions of Szász-Mirakyan operators the following comparisons results were also presented in [38].

First, note that the definition of generalized convexity considered in $[0,1]$ (cf. (7)) can be given also in $[0,\infty)$ (see [59, 74]). More precisely, in this subsection we consider functions $f\in C[0,\infty)$ convex with respect to $\{1,\nu\}$, in short $\{1,\nu\}$-convex, where

$$\nu(x)=e^{2ax},\quad a>0. \tag{81}$$

Observe that this is equivalent to $f\circ\nu^{-1}$ being convex in the classical sense. Moreover, if function $f\in C^2[0,\infty)$ (the space of twice continuously differentiable functions), then f is $\{1,\nu\}$-convex if and only if

$$f''(x)\geq 2af'(x),\quad x>0 \tag{82}$$

(see [26]).

Theorem 9. *Let $f \in C^2[0,\infty)$ be increasing and $\{1,v\}$-convex. Then*

$$f(x) \le R_n^*(f;x) \le S_n(f;x) \quad \text{for } x \ge 0. \tag{83}$$

The above-mentioned modified sequence of Szász-Mirakyan operators reproduces the functions $\mathbf{1}$ and e^{2ax}, $a > 0$. Another modification of Szász-Mirakyan operators reproducing the functions e^{ax} and e^{2ax}, $a > 0$, was introduced in [39] as

$$\mathcal{R}_n(f;x) = e^{-n\alpha_n(x)} \sum_{k=0}^{\infty} \frac{(n\beta_n(x))^k}{k!} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, x \in [0,\infty), \tag{84}$$

where

$$\beta_n(x) = \frac{ax}{ne^{a/n}(e^{a/n}-1)}, \qquad \alpha_n(x) = \frac{ax\left(2-e^{a/n}\right)}{n(e^{a/n}-1)}. \tag{85}$$

This choice provides that

$$\mathcal{R}_n\left(e^{at};x\right) = e^{ax}, \qquad \mathcal{R}_n\left(e^{2at};x\right) = e^{2ax}. \tag{86}$$

For the operators $\mathcal{R}_n$, it can be shown that

(1) $\mathcal{R}_n(\mathbf{1};x) = e^{ax((e^{a/n}-1)/e^{a/n})}$,

(2) $\mathcal{R}_n(e_1;x) = (ax/ne^{a/n}(e^{a/n}-1))e^{ax((e^{a/n}-1)/e^{a/n})}$,

(3) $\mathcal{R}_n(e_2;x) = \{(ax/ne^{a/n}(e^{a/n}-1))^2 + ax/n^2e^{a/n}(e^{a/n}-1)\}e^{ax((e^{a/n}-1)/e^{a/n})}$,

and if one considers the central moment operator $\mu_n^s(x) = \mathcal{R}_n(\Psi_x^s;x)$ of order s ($s = 0,1,2\ldots$), the following formulae hold:

(1) $\mu_n^0(x) = e^{ax((e^{a/n}-1)/e^{a/n})}$,

(2) $\mu_n^1(x) = (ax/ne^{a/n}(e^{a/n}-1) - x)e^{ax((e^{a/n}-1)/e^{a/n})}$,

(3) $\mu_n^2(x) = \{(ax/ne^{a/n}(e^{a/n}-1) - x)^2 + ax/n^2e^{a/n}(e^{a/n}-1)\}e^{ax((e^{a/n}-1)/e^{a/n})}$.

Now set

$$\varphi(x) = 1 + e^{2ax} \quad (x \ge 0) \tag{87}$$

and consider the space $B_\varphi[0,\infty)$ (resp., $C_\varphi[0,\infty)$, $C_\varphi^*[0,\infty)$) defined by (30) and (31).

The first result on uniform convergence of sequence of the operators $\mathcal{R}_n$ was given in [39] by the following.

Theorem 10. *For each function $f \in C_\varphi^*[0,\infty)$*

$$\lim_{n\to\infty} \left\|\mathcal{R}_n(f) - f\right\|_\varphi = 0. \tag{88}$$

In order to approximate unbounded functions, the exponential weighed space $C_a[0,\infty)$ (with a fixed $a > 0$), consisting of $f \in C[0,\infty)$ satisfying the condition $|f(x)| \le Me^{ax}$, where M is a positive constant, is considered and this space is a normed space with the norm

$$\|f\|_a = \sup_{x\in[0,\infty)} \frac{|f(x)|}{e^{ax}}. \tag{89}$$

Also let $C_a^k[0,\infty)$ be subspace of all functions $f \in C_a[0,\infty)$ such that $\lim_{x\to\infty}(|f(x)|/e^{ax}) = k$, where k is a positive constant. A weighted modulus of continuity is defined by

$$\widetilde{\omega}(f;\delta) = \sup_{|t-x|\le\delta, x\ge 0} \frac{|f(t)-f(x)|}{e^{at}+e^{ax}}, \tag{90}$$

for $f \in C_a^k[0,\infty)$. We note that if $f \in C_a^k[0,\infty)$, then $\lim_{\delta\to 0}\widetilde{\omega}(f;\delta) = 0$ and $\widetilde{\omega}(f;m\delta) \le 2m\widetilde{\omega}(f;\delta)$ for any $m \in \mathbb{N}$ (for more details we refer the readers to [39, Section 5]).

Theorem 11. *For $f \in C_a^k[0,\infty)$*

$$\left\|\mathcal{R}_n(f) - f\right\|_{5a/2} \le \frac{a}{ne}\|f\|_a + C\widetilde{\omega}\left(f;\frac{1}{\sqrt{n}}\right), \tag{91}$$

where C is positive constant.

In [39, Section 6] a Voronovskaja-type result is also presented.

Theorem 12. *Let $f \in C_\varphi[0,\infty)$. If f is twice differentiable in $x \in [0,\infty)$ and f'' is continuous in x, and then*

$$\lim_{n\to\infty} n\left[\mathcal{R}_n(f,x) - f(x)\right] = a^2xf(x) - \frac{3}{2}axf'(x) + \frac{x}{2}f''(x). \tag{92}$$

Finally, the following saturation results for the sequence $(\mathcal{R}_n)_{n\ge1}$ hold (see [39, Section 7]).

Theorem 13. *Let $f \in C_\varphi[0,\infty)$ and consider a bounded open interval $J \subset [0,\infty)$. Then, for each $x \in J$*

$$n\left(\mathcal{R}_nf(x) - f(x)\right) = o(1) \quad \text{if and only if } f \in \left\langle e^{ax}, e^{2ax}\right\rangle. \tag{93}$$

Theorem 14. *Let $f \in C_\varphi[0,\infty)$, $M \ge 0$ and let $J \subset [0,\infty)$ be a bounded open interval. Then, for each $x \in J$, one has that*

$$n\left|\mathcal{R}_nf(x) - f(x)\right| \le M + o(1) \tag{94}$$

if and only if

$$\left|a^2xf(t) - \frac{3}{2}atf'(t) + \frac{t}{2}f''(t)\right| \le M, \quad \text{for almost every } t \in J. \tag{95}$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Small Pre-Quasi Banach Operator Ideals of Type Orlicz-Cesáro Mean Sequence Spaces

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In this paper, we give the sufficient conditions on Orlicz-Cesáro mean sequence spaces ces_{φ}, where φ is an Orlicz function such that the class $S_{ces_{\varphi}}$ of all bounded linear operators between arbitrary Banach spaces with its sequence of $s-$numbers which belong to ces_{φ} forms an operator ideal. The completeness and denseness of its ideal components are specified and $S_{ces_{\varphi}}$ constructs a pre-quasi Banach operator ideal. Some inclusion relations between the pre-quasi operator ideals and the inclusion relations for their duals are explained. Moreover, we have presented the sufficient conditions on ces_{φ} such that the pre-quasi Banach operator ideal generated by approximation number is small. The above results coincide with that known for ces_p $(1 < p < \infty)$.

1. Introduction

Throughout the paper, by w, we mean the space of all real sequences, $\mathbb{R}$ the real numbers, and $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathfrak{L}(X, Y)$ the space of all bounded linear operators from a normed space X into a normed space Y. The operator ideals theory takes an importance in functional analysis, since it has numerous applications in fixed point theorem, geometry of Banach spaces, spectral theory, eigenvalue distributions theorem, etc. Some of the operator ideals in the class of normed spaces or Banach spaces in functional analysis are characterized by various scalar sequence spaces. For example the ideal of compact operators is defined by kolmogorov numbers and the space c_0 of convergent to zero sequences. Pietsch [1] inspected the operator ideals framed by the approximation numbers and the classical sequence space $\ell^p (0 < p < \infty)$. He proved that the ideals of Hilbert Schmidt operators and nuclear operators between Hilbert spaces are defined by ℓ^2 and ℓ^1, respectively, and the sequence of approximation numbers. In [2], Faried and Bakery examined the operator ideals developed by generalized Cesáro, Orlicz sequence spaces ℓ_M, and the approximation numbers. In [3], Faried and Bakery studied the operator ideals constructed by $s-$ numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M and show that the operator ideal formed by the previous sequence spaces and approximation numbers is small under certain conditions. Also summation process and sequences spaces applications are closely related to Korovkin type approximation theorems and linear positive operators studied by Costarelli and Vinti [4] and Altomare [5]. The idea of this paper is to examine a generalized class $S_{ces_{\varphi}}$ by using Orlicz-Cesáro mean sequence spaces ces_{φ} and the sequence of s-numbers, for which $S_{ces_{\varphi}}$ constructs an operator ideal. The components of $S_{ces_{\varphi}}$ as a pre-quasi Banach operator ideal containing finite dimensional operators as a dense subset and its completeness are proved. The inclusion relations between the pre-quasi operator ideals and the inclusion relations for their duals are determined. Finally, we show that the pre-quasi Banach operator ideal formed by the approximation numbers and ces_{φ} is small under certain conditions. These results coincide with that known for ces_p, $(1 < p < \infty)$ in [3]. Furthermore we give some examples which support our main results.

2. Definitions and Preliminaries

Definition 1 (see [6]). The sequence $(s_n(T))_{n=0}^{\infty}$, for all $T \in \mathfrak{L}(X,Y)$ is named an s-function and the number $s_n(T)$ is called the n^{th} s- number of T if the following are satisfied:

(a) monotonicity: $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for all $T \in \mathfrak{L}(X,Y)$;

(b) additivity: $s_{m+n-1}(T_1+T_2) \leq s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in \mathfrak{L}(X,Y)$, $m, n \in \mathbb{N}$;

(c) property of ideal: $s_n(RTP) \leq \|R\| s_n(T) \|P\|$ for all $P \in \mathfrak{L}(X_0, X)$, $T \in \mathfrak{L}(X,Y)$, and $R \in \mathfrak{L}(Y, Y_0)$, where X_0 and Y_0 are normed spaces;

(d) $s_n(\beta T) = |\beta| s_n(T)$ for every $T \in \mathfrak{L}(X,Y)$, $\beta \in \mathbb{R}$;

(e) rank property: if $\operatorname{rank}(T) \leq n$ then $s_n(T) = 0$ for every $T \in \mathfrak{L}(X,Y)$;

(f) property of norming:

$$s_i\left(I_j\right) = \begin{cases} 1, & \text{if } i < j; \\ 0, & \text{if } i \geq j, \end{cases} \tag{1}$$

where I_j is the identity operator on $\mathbb{R}^j$.

There are a few instances of s-numbers; we notice the accompanying conditions:

(1) The n-th approximation number, denoted by $\alpha_n(T)$, is defined by $\alpha_n(T) = \inf\{ \|T - B\| : B \in \mathfrak{L}(X,Y) \text{ and } \operatorname{rank}(B) \leq n\}$.

(2) The n-th Hilbert number, denoted by $h_n(T)$, is defined by

$$h_n(T) = \sup\left\{\alpha_n(ATB) : \left\|A : Y \longrightarrow \ell_2\right\| \leq 1 \text{ and } \left\|B : \ell_2 \longrightarrow X\right\| \leq 1\right\}. \tag{2}$$

(3) The n-th Weyl number, denoted by $x_n(T)$, is defined by

$$x_n(T) = \inf\left\{\alpha_n(TB) : \left\|B : \ell_2 \longrightarrow X\right\| \leq 1\right\}. \tag{3}$$

(4) The n-th Kolmogorov number, denoted by $d_n(T)$, is defined by

$$d_n(T) = \inf_{\dim Y \leq n} \sup_{\|x\| \leq 1} \inf_{y \in Y} \left\|Tx - y\right\|. \tag{4}$$

(5) The n-th Gel'fand number, denoted by $c_n(T)$, is defined by $c_n(T) = \alpha_n(J_Y T)$, where J_Y is a metric injection from the space Y to a higher space $l_\infty(\Psi)$ for an adequate index set Ψ. This number is independent of the choice of the higher space $l_\infty(\Psi)$.

(6) The n-th Chang number, denoted by $y_n(T)$, is defined by

$$y_n(T) = \inf\left\{\alpha_n(AT) : \left\|A : Y \longrightarrow \ell_2\right\| \leq 1\right\}. \tag{5}$$

Remark 2 (see [6]). Among all the s-number sequences characterized above, it is easy to check that the approximation number, $\alpha_n(T)$, is the largest and the Hilbert number, $h_n(T)$, is the smallest s-number sequence, i.e., $h_n(T) \leq s_n(T) \leq \alpha_n(T)$ for any bounded linear operator T. If T is defined on a Hilbert space and compact, then all the s-numbers correspond with the eigenvalues of $|T|$, where $|T| = (T^*T)^{1/2}$.

Theorem 3 ([6], p.115). *Let $T \in \mathfrak{L}(X,Y)$. Then*

$$\begin{aligned} h_n(T) &\leq x_n(T) \leq c_n(T) \leq \alpha_n(T), \\ h_n(T) &\leq y_n(T) \leq d_n(T) \leq \alpha_n(T). \end{aligned} \tag{6}$$

Theorem 4 ([6], p.90). *An s-number sequence is called injective if, for any metric injection $K \in \mathfrak{L}(Y, Y_0)$, $s_n(T) = s_n(KT)$ for all $T \in \mathfrak{L}(X,Y)$.*

Theorem 5 ([6], p.95). *An s-number sequence is called surjective if, for any metric surjection $P \in \mathfrak{L}(X_0, X)$, $s_n(T) = s_n(TP)$ for all $T \in \mathfrak{L}(X,Y)$.*

Theorem 6 ([6], pp.90-94). *The Weyl and Gel'fand numbers are injective.*

Theorem 7 ([6], pp.95). *The Chang and Kolmogorov numbers are surjective.*

Definition 8. A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 9 ((dual s-numbers) [7]). For each s-number sequence $s = (s_n)$, a dual s-number function $s^d = (s_n^d)$ is defined by

$$s_n^d(T) = s_n\left(T'\right) \quad \text{for all } T \in \mathfrak{L}(X,Y), \tag{7}$$

where T' is the dual of T.

Definition 10 ([8], p.152)). An s-number sequence is called symmetric if $s_n(T) \geq s_n(T')$ for all $T \in \mathfrak{L}(X,Y)$. If $s_n(T) = s_n(T')$, then the s-number sequence is said to be completely symmetric.

Presently we express some known results of dual of an s-number sequence.

Theorem 11 ([8], p.152). *The approximation numbers are symmetric, i.e., $\alpha_n(T') \leq \alpha_n(T)$ for $T \in \mathfrak{L}(X,Y)$.*

Remark 12 (see [9]). $\alpha_n(T) = \alpha_n(T')$ for every compact operator T.

Theorem 13 ([8], p.153). *Let $T \in \mathfrak{L}(X,Y)$. Then*

$$\begin{aligned} c_n\left(T'\right) &\leq d_n(T), \\ c_n(T) &= d_n\left(T'\right). \end{aligned} \tag{8}$$

In addition, if T is a compact operator then $d_n(T) = c_n(T')$.

Theorem 14 ([6], p.96). *Let $T \in \mathfrak{L}(X,Y)$. Then*

$$y_n\left(T'\right) \le x_n(T),$$
$$x_n(T) = y_n\left(T'\right), \tag{9}$$

i.e., Chang numbers and Weyl numbers are dual to each other.

Theorem 15 ([8], p.153). *The Hilbert numbers are completely symmetric, i.e., $h_n(T') = h_n(T)$ for all $T \in \mathfrak{L}(X,Y)$.*

Definition 16 (see [10, 11]). The operator ideal $\mathbb{U} := \{\mathbb{U}(X,Y)$; *$X$ and Y are Banach Spaces*$\}$ is a subclass of linear bounded operators such that its components $\mathbb{U}(X,Y)$ which are subsets of $\mathfrak{L}(X,Y)$ fulfill the accompanying conditions:

(i) $I_A \in \mathbb{U}$ where A indicates one dimensional Banach space, where $\mathbb{U} \subset \mathfrak{L}$.

(ii) For $T_1, T_2 \in \mathbb{U}(X,Y)$, then $\beta_1 T_1 + \beta_2 T_2 \in \mathbb{U}(X,Y)$ for any scalars β_1, β_2.

(iii) If $T \in \mathfrak{L}(X_0, X)$, $R \in \mathbb{U}(X,Y)$, and $P \in \mathfrak{L}(Y, Y_0)$, then $PRT \in \mathbb{U}(X_0, Y_0)$.

Definition 17 (see [12, 13]). An Orlicz function is a function $\varphi : [0,\infty) \longrightarrow [0,\infty)$, which is nondecreasing, convex, and continuous with $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\longrightarrow\infty}\varphi(x) = \infty$.

Definition 18. An Orlicz function φ is said to satisfy Δ_2-condition for every values of $x \ge 0$, if there is $a > 0$, such that $\varphi(2x) \le a\varphi(x)$. The Δ_2-condition is corresponding to $\varphi(mx) \le am\varphi(x)$ for every values of $m > 1$ and x.

Lindenstrauss and Tzafriri [14] utilized the idea of an Olicz function to define Orlicz sequence space:

$$\ell_\varphi = \{x \in \omega : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}$$
$$\text{where } \rho(x) = \sum_{k=0}^{\infty} \varphi\left(\left|x_k\right|\right), \tag{10}$$

$(\ell_\varphi, \|.\|)$ is a Banach space with the Luxemburg norm:

$$\|x\|_{\ell_\varphi} = \inf\left\{\lambda > 0 : \rho\left(\lambda^{-1}x\right) \le 1\right\}. \tag{11}$$

Every Orlicz sequence space contains a subspace that is isomorphic to ℓ^p, for some $1 \le p < \infty$ or c_0 ([15], Theorem 4.a.9).

In the recent past lot of work has been done on sequence spaces defined by Orlicz functions by Altin et al. [16], Et et al. ([17, 18]), Tripathy et al. ([19–21]), and Mohiuddine et al. ([22–25]).

Given an Orlicz function φ, the Orlicz-Cesáro mean sequence spaces is defined by

$$ces_\varphi = \{u = (u_i) \in \omega : \rho(\beta u) < \infty \text{ for some } \beta > 0\},$$
$$\rho(u) = \sum_{i=0}^{\infty} \phi\left(\frac{\sum_{j=0}^{i}\left|u_j\right|}{i+1}\right). \tag{12}$$

$(ces_\varphi, \|.\|)$ is a Banach space with the Luxemburg norm given by

$$\|u\|_{ces_\varphi} = \inf\left\{\beta > 0 : \rho\left(\beta^{-1}u\right) \le 1\right\}. \tag{13}$$

It seems that Orlicz-Cesáro mean sequence spaces ces_φ appeared for the first time in 1988, when Lim and Yee found their dual spaces [26]. Recently Cui, Hudzik, Petrot, Suantai, and Szymaszkiewicz obtained important properties of spaces ces_φ [27]. In 2007 Maligranda, Petrot, and Suantai showed that ces_φ is not B-convex, if $\varphi \in \Delta_2$ and $ces_\varphi \ne 0$ [28]. The extreme points and strong X-points of ces_φ have been characterized by Foralewski, Hudzik, and Szymaszkiewicz in [29]. In the case when $\varphi(u) = u^p$, $1 \le p < \infty$, the space ces_φ is just a Cesáro sequence space ces_p, with the norm given by

$$\|u\|_{ces_p} = \left[\sum_{i=0}^{\infty}\left(\frac{\sum_{j=0}^{i}\left|u_j\right|}{i+1}\right)^p\right]^{1/p}. \tag{14}$$

It is well known that $ces_1 = \{0\}$ [30].

Definition 19 (see [31]). The Matuszewska Orlicz lower index α_φ of an Orlicz function φ is defined as follows:

$$\alpha_\varphi = \sup\left\{p > 0 : \exists_{K>0}\ \forall_{0<\lambda,t\le 1}\varphi(\lambda t) \le Kt^p\varphi(\lambda)\right\}. \tag{15}$$

Theorem 20 (see [31]). *For any Orlicz function φ, we have $\alpha_\varphi > 1$ if and only if $\ell_\varphi \subset ces_\varphi$. In particular, if $\alpha_\varphi > 1$ then $ces_\varphi \ne \{0\}$.*

Theorem 21 (see [31]). *Let φ_1 and φ_2 be Orlicz functions. If there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \le \varphi_1(bt)$ for all $t \in [0, t_0]$, then $ces_{\varphi_1} \subset ces_{\varphi_2}$.*

Theorem 22 (see [31]). *Let φ_1 and φ_2 be Orlicz functions and $\alpha_{\varphi_1} > 1$, then $ces_{\varphi_1} \subset ces_{\varphi_2}$ if and only if there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \le \varphi_1(bt)$ for all $t \in [0, t_0]$.*

Definition 23 (see [2]). A class of linear sequence spaces $\mathbb{E}$ is called a special space of sequences (sss) having three properties:

(1) $e_i \in \mathbb{E}$ for all $i \in \mathbb{N}$,

(2) if $x = (x_i) \in w$, $y = (y_i) \in \mathbb{E}$ and $|x_i| \le |y_i|$ for every $i \in \mathbb{N}$, then $x \in \mathbb{E}$, "i.e., $\mathbb{E}$ is solid",

(3) if $(x_i)_{i=0}^{\infty} \in \mathbb{E}$, then $(x_{[i/2]})_{i=0}^{\infty} \in \mathbb{E}$, wherever $[i/2]$ means the integral part of $i/2$.

Definition 24 (see [2]). A subclass of the special space of sequences is called a premodular (sss) if there is a function $\varrho : \mathbb{E} \longrightarrow [0,\infty[$ fulfilling the accompanying conditions:

(i) $\varrho(x) \ge 0$ for each $x \in \mathbb{E}$ and $\varrho(x) = 0 \Longleftrightarrow x = \theta$, where θ is the zero element of $\mathbb{E}$,

(ii) there exists $L \ge 1$ such that $\varrho(\lambda x) \le L|\lambda|\varrho(x)$ for all $x \in \mathbb{E}$, and for any scalar λ,

(iii) for some $K \ge 1$, we have $\varrho(x + y) \le K(\varrho(x) + \varrho(y))$ for every $x, y \in \mathbb{E}$,

(iv) if $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$, then $\varrho((x_i)) \leq \varrho((y_i))$,

(v) for some $K_0 \geq 1$, we have

$$\varrho\left((x_i)\right) \leq \varrho\left((x_{[i/2]})\right) \leq K_0 \varrho\left((x_i)\right), \tag{16}$$

(vi) the set of all finite sequences is ϱ-dense in $\mathbb{E}$. This means for each $x = (x_i)_{i=o}^{\infty} \in \mathbb{E}$ and for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\varrho((x_i)_{i=m}^{\infty}) < \varepsilon$,

(vii) there exists a constant $\xi > 0$ such that $\varrho(\lambda, 0, 0, 0, \ldots) \geq \xi |\lambda| \varrho(1, 0, 0, 0, \ldots)$ for any $\lambda \in \mathbb{R}$.

We denote $(\mathbb{E}_\varrho, \varrho)$ for the linear space $\mathbb{E}$ equipped with the metrizable topology generated by ϱ.

Theorem 25 (see [32]). *If X, Y are infinite dimensional Banach spaces and λ_i is a monotonic decreasing sequence to zero, then there exists a bounded linear operator T such that*

$$\frac{1}{16}\lambda_{3i} \leq \alpha_i(T) \leq 8\lambda_{i+1}. \tag{17}$$

Notations 26 (see [3]).

$S_{\mathbb{E}} := \{S_{\mathbb{E}}(X, Y);\ X \text{ and } Y \text{ are Banach Spaces}\}$, where

$S_{\mathbb{E}}(X, Y) := \{T \in \mathfrak{L}(X, Y) : ((s_i(T))_{i=0}^{\infty} \in \mathbb{E}\}$. Also

$S_{\mathbb{E}}^{app} := \{S_{\mathbb{E}}^{app}(X, Y);\ X \text{ and } Y \text{ are Banach Spaces}\}$, where

$S_{\mathbb{E}}^{app}(X, Y) := \{T \in \mathfrak{L}(X, Y) : ((\alpha_i(T))_{i=0}^{\infty} \in \mathbb{E}\}$.

Theorem 27 (see [3]). *If $\mathbb{E}$ is a (sss), then $S_{\mathbb{E}}$ is an operator ideal.*

The concept of pre-quasi operator ideal which is more general than the usual classes of operator ideal.

Definition 28 (see [3]). A function $g : \Omega \longrightarrow [0, \infty)$ is said to be a pre-quasi norm on the ideal Ω fulfilling the accompanying conditions:

(1) for all $T \in \Omega(X, Y)$, $g(T) \geq 0$ and $g(T) = 0$ if and only if $T = 0$,

(2) there exists a constant $L \geq 1$ such that $g(\beta T) \leq L|\beta| g(T)$, for all $T \in \Omega(X, Y)$ and $\beta \in \mathbb{R}$,

(3) there exists a constant $K \geq 1$ such that $g(T_1 + T_2) \leq K[g(T_1) + g(T_2)]$, for all $T_1, T_2 \in \Omega(X, Y)$,

(4) there exists a constant $C \geq 1$ such that if $P \in \mathfrak{L}(X_0, X)$, $R \in \Omega(X, Y)$, and $T \in \mathfrak{L}(Y, Y_0)$, then $g(TRP) \leq C\ \|T\| g(R)\ \|P\|$, where X_0 and Y_0 are normed spaces.

Theorem 29 (see [3]). *Every quasi norm on the ideal Ω is a pre-quasi norm on the ideal Ω.*

Here and after, we define $e_i = \{0, 0, \ldots, 1, 0, 0, \ldots\}$ where 1 appears at the i^{th} place for all $i \in \mathbb{N}$.

3. Main Results

We give here the conditions on Orlicz-Cesáro mean sequence spaces ces_φ such that the class S_{ces_φ} of all bounded linear operators between arbitrary Banach spaces with its sequence of $s-$numbers which belong to ces_φ forms an operator ideal.

Theorem 30. *If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then S_{ces_φ} is an operator ideal.*

Proof. (1-i) Let $x, y \in ces_\varphi$. Since φ is nondecreasing, convex, and satisfying Δ_2-condition, we get for some $k > 0$ that

$$\begin{aligned}&\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i + y_i|}{n+1}\right)\\ &\quad \leq k\left[\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) + \sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|y_i|}{n+1}\right)\right] < \infty,\end{aligned} \tag{18}$$

then $x + y \in ces_\varphi$.

(1-ii) Let $\lambda \in \mathbb{R}$ and $x \in ces_\varphi$, and since φ is convex and satisfying Δ_2-condition, we get for some $k > 0$ that

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|\lambda x_i|}{n+1}\right) \leq |\lambda|\, k \sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) < \infty, \tag{19}$$

then $\lambda x \in ces_\varphi$; from (1-i) and (1-ii) ces_φ is a linear space. Since $e_n \in \ell_\varphi$, for all $n \in \mathbb{N}$ and $\alpha_\varphi > 1$, then from Theorem 20, we get $e_n \in ces_\varphi$, for all $n \in \mathbb{N}$.

(2) Let $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$ and $y \in ces_\varphi$; since φ is nondecreasing, then we have

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) \leq \sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|y_i|}{n+1}\right) < \infty, \tag{20}$$

and we get $x \in ces_\varphi$.

(3) Let $(x_n) \in ces_\varphi$. Since φ is satisfying Δ_2-condition, we get for some $k > 0$ that

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_{[i/2]}|}{n+1}\right) \leq (k+1)\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) < \infty, \tag{21}$$

then $(x_{[n/2]}) \in ces_\varphi$. Then ces_φ is a (sss); hence by Theorem 27, S_{ces_φ} is an operator ideal. □

Corollary 31. *S_{ces_q} is an operator ideal, if $1 < q < \infty$.*

We give the conditions on Orlicz-Cesáro mean sequence spaces ces_φ such that the ideal of the finite rank operators is dense in $S_{ces_\varphi}(X, Y)$.

Theorem 32. *$S_{ces_\varphi}(X, Y) = \overline{F(X, Y)}$, if φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$.*

Proof. Let us define $\varrho(u) = \sum_{i=0}^{\infty}\varphi(\sum_{j=0}^{i}|u_j|/(i+1))$ on ces_φ. First, we have to show that $\overline{F(X, Y)} \subseteq S_{ces_\varphi}(X, Y)$. Since $\alpha_\varphi > 1$, we have $e_i \in ces_\varphi$ for each $i \in \mathbb{N}$ and φ is an

Orlicz function satisfying Δ_2-condition, so for each finite operator $P \in F(X,Y)$, i.e., we obtain $(s_i(P))_{i=0}^{\infty}$ which contains only finitely many terms different from zero; hence $P \in S_{ces_\varphi}(X,Y)$. Currently we prove that $S_{ces_\varphi}(X,Y) \subseteq \overline{F(X,Y)}$; let $P \in S_{ces_\varphi}(X,Y)$; we have $(s_i(P))_{i=0}^{\infty} \in ces_\varphi$; and hence $\varrho(s_i(P))_{i=0}^{\infty} < \infty$. By taking $\varepsilon \in (0,1)$, hence there exists a $i_0 \in \mathbb{N} - \{0\}$ such that $\varrho((s_i(P))_{i=i_0}^{\infty}) < \varepsilon/9\delta C^2$ for some $c \geq 1$, where $\delta = \max\{1, \sum_{i=i_0}^{\infty} \varphi(1/(i+1))\}$. As $s_i(P)$ is decreasing for every $i \in \mathbb{N}$ and φ is nondecreasing, we have

$$\begin{aligned} i_0\varphi\left(s_{2i_0}(P)\right) &\leq \sum_{i=i_0+1}^{2i_0} \varphi\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) \\ &\leq \sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) < \frac{\varepsilon}{9\delta C^2}. \end{aligned} \tag{22}$$

Hence, there exists $B \in F_{2i_0}(X,Y)$ such that rank $B \leq 2i_0$ and

$$i_0\varphi(\|P-B\|) \leq \sum_{i=i_0+1}^{2i_0} \varphi\left(\frac{\sum_{j=0}^{i}\|P-B\|}{i+1}\right) < \frac{\varepsilon}{9\delta C^2}. \tag{23}$$

Since φ is right continuous at 0 and nondecreasing, then on considering this

$$\|P-B\| < \frac{\varepsilon}{6C^2 i_0 \delta}. \tag{24}$$

Let $k_1 > 0$, $k_2 > 0$ and $C = \max\{1, k_1, k_2\}$, since φ is Orlicz function and by using (22), (23), and (24), we have

$$\begin{aligned} d(P,B) &= \varrho\left(s_i(P-B)\right)_{i=0}^{\infty} = \sum_{i=0}^{3i_0-1}\varphi \\ &\cdot\left(\frac{\sum_{j=0}^{i} s_j(P-B)}{i+1}\right) + \sum_{i=3i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{i} s_j(P-B)}{i+1}\right) \\ &\leq \sum_{i=0}^{3i_0-1}\varphi\left(\frac{\sum_{j=0}^{i}\|P-B\|}{i+1}\right) + \sum_{i=i_0}^{\infty}\varphi \\ &\cdot\left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1}\right) \leq 3i_0\varphi(\|P-B\|) + \sum_{i=i_0}^{\infty}\varphi \\ &\cdot\left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1}\right) \leq 3i_0\varphi(\|P-B\|) \\ &+ \sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{2i_0-1} s_j(P-B) + \sum_{j=2i_0}^{i+2i_0} s_j(P-B)}{i+1}\right) \\ &\leq 3i_0\varphi(\|P-B\|) + k_1\left[\sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{2i_0-1} s_j(P-B)}{i+1}\right)\right. \\ &\left.+ \sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=2i_0}^{i+2i_0} s_j(P-B)}{i+1}\right)\right] \leq 3i_0\varphi(\|P-B\|) \\ &+ k_1\left[\sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{2i_0-1}\|P-B\|}{i+1}\right)\right. \\ &\left.+ \sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{i} s_{j+2i_0}(P-B)}{i+1}\right)\right] \leq 3i_0\varphi(\|P-B\|) \\ &+ 2i_0k_1k_2\|P-B\|\sum_{i=i_0}^{\infty}\varphi\left(\frac{1}{i+1}\right) + k_1\sum_{i=i_0}^{\infty}\varphi \\ &\cdot\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) \leq 3i_0\varphi(\|P-B\|) + 2i_0C^2\|P \\ &- B\|\sum_{i=i_0}^{\infty}\varphi\left(\frac{1}{i+1}\right) + C\sum_{i=i_0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) < \varepsilon. \end{aligned} \tag{25}$$

□

Corollary 33. *$S_{ces_p}(X,Y) = \overline{F(X,Y)}$, if $1 < p < \infty$.*

We express the accompanying theorem without verification; these can be set up utilizing standard procedure.

Theorem 34. *The function $g(P) = \sum_{i=0}^{\infty}\varphi(\sum_{j=0}^{i}|s_j(P)|/(i+1))$ is a pre-quasi norm on S_{ces_φ}, if φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$.*

We give the sufficient conditions on Orlicz-Cesáro mean sequence spaces ces_φ such that the components of the pre-quasi operator ideal S_{ces_φ} are complete.

Theorem 35. *If X and Y are Banach spaces, φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then $(S_{ces_\varphi}(X,Y), g)$ is a pre-quasi Banach operator ideal.*

Proof. Since φ is an Orlicz function satisfying Δ_2-condition, then the function $g(P) = \varrho((s_n(P))_{n=0}^{\infty}) = \sum_{n=0}^{\infty}\varphi(\sum_{m=0}^{n}|s_m(P)|/(n+1))$ is a pre-quasi norm on S_{ces_φ}. Let (P_m) be a Cauchy sequence in $S_{ces_\varphi}(X,Y)$. Since $\mathfrak{L}(X,Y) \supseteq S_{ces_\varphi}(X,Y)$ and $\alpha_\varphi > 1$, we can find a constant $\xi > 0$ such that

$$\begin{aligned} g\left(P_i - P_j\right) &= \varrho\left(\left(s_n\left(P_i - P_j\right)\right)_{n=0}^{\infty}\right) \\ &\geq \varrho\left(s_0\left(P_i - P_j\right), 0, 0, 0, \ldots\right) \\ &= \varrho\left(\left\|P_i - P_j\right\|, 0, 0, 0, \ldots\right) \\ &\geq \xi\left\|P_i - P_j\right\|\varrho(1, 0, 0, 0, \ldots), \end{aligned} \tag{26}$$

then $(P_m)_{m\in\mathbb{N}}$ is also a Cauchy sequence in $\mathfrak{L}(X,Y)$. While the space $\mathfrak{L}(X,Y)$ is a Banach space, there exists $P \in \mathfrak{L}(X,Y)$

such that $\lim_{m\longrightarrow\infty} \|P_m - P\| = 0$, while $(s_n(P_m))_{n=0}^{\infty} \in ces_\varphi$ for every $m \in \mathbb{N}$. Since ϱ is continuous at θ and for some $K \geq 1$, we obtain

$$\begin{aligned} g(P) &= \varrho\left(\left(s_n(P)\right)_{n=0}^{\infty}\right) = \varrho\left(\left(s_n(P - P_m + P_m)\right)_{n=0}^{\infty}\right) \\ &\leq K\varrho\left(\left(s_{[n/2]}(P - P_m)\right)_{n=0}^{\infty}\right) \\ &\quad + K\varrho\left(\left(\alpha_{[n/2]}(P_m)_{n=0}^{\infty}\right)\right) \\ &\leq K\varrho\left(\left(\|P_m - P\|\right)_{n=0}^{\infty}\right) + K\varrho\left(\left(s_n(P_m)_{n=0}^{\infty}\right)\right) \\ &< \infty, \end{aligned} \tag{27}$$

we have $(s_n(P))_{n=0}^{\infty} \in ces_\varphi$, and then $P \in S_{ces_\varphi}(X,Y)$. □

Corollary 36. *If X and Y are Banach spaces and $1 < q < \infty$, then $(S_{ces_q}(X,Y), g)$ is quasi Banach operator ideal, where $g(P) = \varrho((s_n(P))_{n=0}^{\infty}) = [\sum_{n=0}^{\infty}(\sum_{m=0}^{n}|s_m(P)|/(n+1))^q]^{1/q}$.*

Theorem 37. *Let φ_1, φ_2 be Orlicz functions and $\alpha_{\varphi_1} > 1$. For any infinite dimensional Banach spaces X, Y and if there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$ for all $t \in [0,t_0]$, it is true that*

$$S^{app}_{ces_{\varphi_1}}(X,Y) \subsetneqq S^{app}_{ces_{\varphi_2}}(X,Y) \subsetneqq \mathfrak{L}(X,Y). \tag{28}$$

Proof. Let X and Y be infinite dimensional Banach spaces and there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$ for all $t \in [0,t_0]$; if $P \in S^{app}_{ces_{\varphi_1}}(X,Y)$, then $(\alpha_n(P)) \in ces_{\varphi_1}$. From Theorems 21, 22, and 25, we have $ces_{\varphi_1} \subset ces_{\varphi_2}$; hence $P \in S^{app}_{ces_{\varphi_2}}(X,Y)$. It is easy to see that $S^{app}_{ces_{\varphi_2}}(X,Y) \subset \mathfrak{L}(X,Y)$. □

Corollary 38. *For any infinite dimensional Banach spaces X, Y, and $1 < p < q < \infty$, then $S^{app}_{ces_p}(X,Y) \subsetneqq S^{app}_{ces_q}(X,Y) \subsetneqq \mathfrak{L}(X,Y)$.*

We now study some properties of the pre-quasi Banach operator ideal S_{ces_φ}.

Theorem 39. *The pre-quasi Banach operator ideal (S_{ces_φ}, g) is injective, if the s-number sequence is injective.*

Proof. Let $T \in \mathfrak{L}(X,Y)$ and $P \in \mathfrak{L}(Y,Y_0)$ be any metric injection. Assume that $PT \in S_{ces_\varphi}(X,Y_0)$, then $\varrho(s_n(PT)) < \infty$. Since the s-number sequence is injective, we have $s_n(PT) = s_n(T)$, for all $T \in \mathfrak{L}(X,Y)$, $n \in \mathbb{N}$. So $\varrho(s_n(T)) = \varrho(s_n(PT)) < \infty$. Hence $T \in S_{ces_\varphi}(X,Y)$ and clearly $g(T) = g(PT)$ is verified. □

Remark 40. The pre-quasi Banach operator ideal $(S^{Weyl}_{ces_\varphi}, g)$ and the pre-quasi Banach operator ideal $(S^{Gel}_{ces_\varphi}, g)$ are injective pre-quasi Banach operator ideal.

Theorem 41. *The pre-quasi Banach operator ideal (S_{ces_φ}, g) is surjective, if the s-number sequence is surjective.*

Proof. Let $T \in \mathfrak{L}(X,Y)$ and $P \in \mathfrak{L}(X_0,X)$ be any metric surjection. Suppose that $TP \in S_{ces_\varphi}(X_0,Y)$, then $\varrho(s_n(TP)) < \infty$. Since the s-number sequence is surjective, we have $s_n(TP) = s_n(T)$, for all $T \in \mathfrak{L}(X,Y)$, $n \in \mathbb{N}$. So $\varrho(s_n(T)) = \varrho(s_n(TP)) < \infty$. Hence $T \in S_{ces_\varphi}(X,Y)$ and clearly $g(T) = g(TP)$ is verified. □

Remark 42. The pre-quasi Banach operator ideal $(S^{Chang}_{ces_\varphi}, g)$ and the pre-quasi Banach operator ideal $(S^{Kol}_{ces_\varphi}, g)$ are surjective pre-quasi Banach operator ideal.

Likewise, we have the accompanying inclusion relations between the pre-quasi Banach operator ideals.

Theorem 43. (1) $S^{app}_{ces_\varphi} \subseteq S^{Kol}_{ces_\varphi} \subseteq S^{Chang}_{ces_\varphi} \subseteq S^{Hilb}_{ces_\varphi}$.
(2) $S^{app}_{ces_\varphi} \subseteq S^{Gel}_{ces_\varphi} \subseteq S^{Weyl}_{ces_\varphi} \subseteq S^{Hilb}_{ces_\varphi}$.

Proof. Since $h_n(T) \leq y_n(T) \leq d_n(T) \leq \alpha_n(T)$ and $h_n(T) \leq x_n(T) \leq c_n(T) \leq \alpha_n(T)$ for every $n \in \mathbb{N}$ and ϱ is nondecreasing, we obtain

$$\begin{aligned} &\varrho(h_n(T)) \leq \varrho(y_n(T)) \leq \varrho(d_n(T)) \leq \varrho(\alpha_n(T)), \\ &\varrho(h_n(T)) \leq \varrho(x_n(T)) \leq \varrho(c_n(T)) \leq \varrho(\alpha_n(T)). \end{aligned} \tag{29}$$

Hence the result is as follows. □

We presently express the dual of the pre-quasi operator ideal formed by different $s-$ number sequences.

Theorem 44. *The pre-quasi operator ideal $S^{Hilb}_{ces_\varphi}$ is completely symmetric and the pre-quasi operator ideal $S^{app}_{ces_\varphi}$ is symmetric.*

Proof. Since $h_n(T') = h_n(T)$ and $\alpha_n(T') \leq \alpha_n(T)$, for all $T \in \mathfrak{L}(X,Y)$, we have $S^{Hilb}_{ces_\varphi} = (S^{Hilb}_{ces_\varphi})'$ and $S^{app}_{ces_\varphi} \subseteq (S^{app}_{ces_\varphi})'$. □

In perspective on Theorem 13, we express the following result without proof.

Theorem 45. *The pre-quasi operator ideal $S^{Kol}_{ces_\varphi} \subseteq (S^{Gel}_{ces_\varphi})'$ and $S^{Gel}_{ces_\varphi} = (S^{Kol}_{ces_\varphi})'$. In addition if T is a compact operator from X to Y, then $S^{Kol}_{ces_\varphi} = (S^{Gel}_{ces_\varphi})'$.*

In perspective on Theorem 14, we express the following result without proof.

Theorem 46. *The pre-quasi operator ideal $S^{Chang}_{ces_\varphi} = (S^{Weyl}_{ces_\varphi})'$ and $S^{Weyl}_{ces_\varphi} = (S^{Chang}_{ces_\varphi})'$.*

Theorem 47. *If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then the pre-quasi Banach operator ideal $S^{app}_{ces_\varphi}$ is small.*

Proof. Since φ is an Orlicz function and $\alpha_\varphi > 1$, take $\beta = \sum_{i=0}^{\infty}\varphi(1/(i+1))$. Then $(S^{app}_{ces_\varphi}, g)$, where $g(T) =$

$\varrho((\alpha_n(T))_{n=0}^{\infty}) = (1/\beta)\sum_{n=0}^{\infty}\varphi(\sum_{m=0}^{n}\alpha_m(T)/(n+1))$ is a pre-quasi Banach operator ideal. Let X and Y be any two Banach spaces. Assume that $S_{ces_\varphi}^{app}(X,Y) = \mathfrak{L}(X,Y)$, then there exists a constant $C > 0$ such that $g(T) \le C\|T\|$ for all $T \in \mathfrak{L}(X,Y)$. Suppose that X and Y are infinite dimensional Banach spaces. Then by Dvoretzky's theorem [8] for $m \in \mathbb{N}$, we have quotient spaces X/M_m and subspaces N_m of Y which can be mapped onto ℓ_2^m by isomorphisms V_m and B_m such that $\|V_m\|\|V_m^{-1}\| \le 2$ and $\|B_m\|\|B_m^{-1}\| \le 2$. Consider I_m be the identity map on ℓ_2^m, P_m be the quotient map from X onto X/M_m, and Q_m be the natural embedding map from N_m into Y. Let v_n be the Bernstein numbers [7], then

$$\begin{aligned}1 &= v_n(I_m) = v_n(B_m B_m^{-1} I_m V_m V_m^{-1}) \\ &\le \|B_m\| v_n(B_m^{-1} I_m V_m)\|V_m^{-1}\| \\ &= \|B_m\| v_n(Q_m B_m^{-1} I_m V_m)\|V_m^{-1}\| \\ &\le \|B_m\| d_n(Q_m B_m^{-1} I_m V_m)\|V_m^{-1}\| \\ &= \|B_m\| d_n(Q_m B_m^{-1} I_m V_m Q_m)\|V_m^{-1}\| \\ &\le \|B_m\| \alpha_n(Q_m B_m^{-1} I_m V_m Q_m)\|V_m^{-1}\|,\end{aligned} \tag{30}$$

for $1 \le i \le m$. Now since φ is nondecreasing and having Δ_2-condition, we have

$$\begin{aligned}&\sum_{j=0}^{i}(1) \le \sum_{j=0}^{i}\|B_m\|\alpha_j(Q_m B_m^{-1} I_m V_m P_m)\|V_m^{-1}\| \Longrightarrow \\ &\frac{1}{i+1}(i+1) \le \|B_m\|\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_m B_m^{-1} I_m V_m P_m)\right) \\ &\quad\cdot\|V_m^{-1}\| \Longrightarrow \\ &\varphi(1) \le L(\|B_m\|\|V_m^{-1}\|) \\ &\quad\cdot\varphi\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_m B_m^{-1} I_m V_m P_m)\right).\end{aligned} \tag{31}$$

Therefore

$$\begin{aligned}&\sum_{i=0}^{m}\varphi(1) \le L\|B_m\|\|V_m^{-1}\|\sum_{i=0}^{m}\varphi \\ &\quad\cdot\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_m B_m^{-1} I_m V_m P_m)\right) \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le L\|B_m\|\|V_m^{-1}\|\frac{1}{\beta}\sum_{i=0}^{m}\varphi \\ &\quad\cdot\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_m B_m^{-1} I_m V_m P_m)\right) \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le L\|B_m\|\|V_m^{-1}\| g(Q_m B_m^{-1} I_m V_m P_m) \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le LC\|B_m\|\|V_m^{-1}\|\|Q_m B_m^{-1} I_m V_m P_m\| \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le LC\|B_m\|\|V_m^{-1}\|\|Q_m B_m^{-1}\|\|I_m\|\|V_m P_m\| \\ &\quad = LC\|B_m\|\|V_m^{-1}\|\|B_m^{-1}\|\|I_m\|\|V_m\| \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le 4LC,\end{aligned} \tag{32}$$

for some $L \ge 1$. Thus we arrive at a contradiction since m is arbitrary. Hence X and Y both cannot be infinite dimensional when $S_{ces_\varphi}^{app}(X,Y) = \mathfrak{L}(X,Y)$. □

Theorem 48. *If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then the pre-quasi Banach operator ideal $S_{ces_\varphi}^{Kol}$ is small.*

Corollary 49. *If $p \in (1,\infty)$, then the quasi Banach operator ideal $S_{ces_p}^{app}$ is small.*

Corollary 50. *If $p \in (1,\infty)$, then the quasi Banach operator ideal $S_{ces_p}^{Kol}$ is small.*

4. Examples

We give some examples which support our main results.

Example 1. Let φ be an Orlicz function; the subspace ces_φ^h of all order continuous elements of ces_φ is defined as [27]

$$ces_\varphi^h = \left\{x \in ces_\varphi : \forall_{k>0}\ \exists_{n_k\in\mathbb{N}} \sum_{n=n_k}^{\infty}\varphi\left(\frac{k}{n}\sum_{i=1}^{n}|x_i|\right) < \infty\right\}. \tag{33}$$

If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then the following conditions are satisfied:

(1) $S_{ces_\varphi^h}$ is an operator ideal.

(2) $S_{ces_\varphi^h}(X,Y) = \overline{F(X,Y)}$.

(3) If X and Y are Banach spaces, then $(S_{ces_\varphi^h}(X,Y), g)$ is pre-quasi Banach operator ideal.

(4) The pre-quasi Banach operator ideal $S_{ces_\varphi^h}^{app}$ is small.

Proof. Since φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then from Theorem (5) in [31] we have $ces_\varphi^h = ces_\varphi$ which completes the proof. □

Example 2. Let φ be defined as

$$\varphi(t) = a_l t^l + a_{l-1} t^{l-1} + \ldots + a_1 t, \quad \text{where } a_i > 0 \text{ for all } 1 \le i \le l,\ l \in \mathbb{N},\ l > 1 \text{ and } t \ge 0. \tag{34}$$

It is clear that φ is an Orlicz function and $\alpha_\varphi = l > 1$. Also φ is satisfying Δ_2-condition since

$$\limsup_{t \to 0^+} \frac{\varphi(2t)}{\varphi(t)} \le 2^l < \infty. \tag{35}$$

Then the following conditions are satisfied:

(1) S_{ces_φ} is an operator ideal.

(2) $S_{ces_\varphi}(X, Y) = \overline{F(X, Y)}$.

(3) If X and Y are Banach spaces, then $(S_{ces_\varphi}(X, Y), g)$ is pre-quasi Banach operator ideal.

(4) The pre-quasi Banach operator ideal $S^{app}_{ces_\varphi}$ is small.

In the following two examples we will explain the importance of the sufficient conditions.

Example 3. Let φ be defined as

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \dfrac{-t}{\ln t} & \text{if } t \in \left(0, \dfrac{1}{e}\right], \\ \dfrac{3}{2} e t^2 - t + \dfrac{1}{2e} & \text{if } t \in \left(\dfrac{1}{e}, \infty\right). \end{cases} \tag{36}$$

It is clear that φ is an Orlicz function. Since $\sum_{n=1}^{\infty} \varphi(1/n) = \sum_{n=1}^{\infty} (1/n \ln n) = \infty$, hence $ces_\varphi = \{0\}$. The space S_{ces_φ} is not operator ideal since $I_K \notin S_{ces_\varphi}$. Also since φ is convex function and for $p > 1$, we have

$$\begin{aligned} \lim_{t \to 0^+} \frac{\varphi(\lambda t)}{\varphi(\lambda) t^p} &= \lim_{t \to 0^+} \frac{t^{1-p} \ln \lambda}{\ln \lambda t} \\ &= \lim_{t \to 0^+} (1 - p) t^{1-p} \ln \lambda = \infty, \end{aligned} \tag{37}$$

for all $\lambda \in (0, 1]$, then $\alpha_\varphi = 1$. Although φ is satisfying Δ_2-condition since

$$\limsup_{t \to 0^+} \frac{\varphi(2t)}{\varphi(t)} = \limsup_{t \to 0^+} \frac{2 \ln t}{\ln 2t} \le 2 < \infty. \tag{38}$$

Example 4. Let $\varphi(u) = \int_0^u f(t) dt$, where $f(t)$ is defined as

$$f(t) = \begin{cases} 0 & \text{if } t = 0, \\ \dfrac{1}{n!} & \text{if } t \in \left[\dfrac{1}{(n+1)!}, \dfrac{1}{n!}\right) \text{ for } n = 1, 2, 3 \ldots, \\ t & \text{if } t \in [1, \infty). \end{cases} \tag{39}$$

It is clear that φ is an Orlicz function. Let $T \in S_{ces_\varphi}$ with $s_n(T) = 1/n!$ for all $n \in \mathbb{N}$. We have for $n > 2$ that

$$\begin{aligned} \varphi(s_n(2T)) &= \int_0^{2/n!} f(t)\, dt > \int_{1/n!}^{2/n!} f(t)\, dt \\ &> \int_{1/n!}^{1/(n-1)!} f(t)\, dt > \frac{1}{n!(n-1)!}, \\ n\varphi(s_n(T)) &= n \int_0^{1/n!} f(t)\, dt \\ &< n \sup_{0 \le t \le 1/n!} f(t) \int_0^{1/n!} 1\, dt < \frac{1}{n!(n-1)!}. \end{aligned} \tag{40}$$

Hence $2T \notin S_{ces_\varphi}$, so the space S_{ces_φ} is not operator ideal and $\varphi \notin \Delta_2$. Also since φ is convex function and for $p > 1$, we have

$$\lim_{t \to 0^+} \frac{\varphi(\lambda t)}{\varphi(\lambda) t^p} = \lim_{t \to 0^+} t^{-p} = \infty, \tag{41}$$

for all $\lambda \in (0, 1]$, then $\alpha_\varphi = 1$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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Conflicts of Interest

The authors declare that have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Asymptotic Behavior of Almost Quartic ∗-Derivations on Banach ∗-Algebras

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The purpose of this paper is to obtain the stability theorems of quartic ∗-derivations associated with the quartic functional equation $f(3x-y)+f(x+y)+6f(x-y)=4f(2x-y)+4f(y)+24f(x)$ on Banach ∗-algebras.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. In the next year, Hyers [2] gave a clear answer to this problem for additive mappings between Banach spaces. Then this theorem [2] was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. Since then, many mathematicians have come to deal with this problem and also there are many interesting results concerning this problem [5–8].

First, we recall definition of ∗-derivation.

Definition 1. Let B be a Banach ∗-algebra and let A be a Banach ∗-subalgebra of B. A $\mathbb{C}$-linear mapping $D: A \longrightarrow B$ is said to be derivation on A if $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. Moreover, if D satisfies the additional condition $D(a^*) = D(a)^*$ for all $a \in A$, then it is called a ∗-derivation.

The stability of ∗-derivations and of quadratic ∗-derivations with the Cauchy functional equation and the Jensen functional equation on Banach ∗-algebras was investigated in [9]. Jang and Park [9] proved the superstability of ∗-derivations and of quadratic ∗-derivations on C^*-algebras. The stability of ∗-derivations on Banach ∗-algebras by using fixed-point alternative was proven by Park and Bodaghi [10]. Thereafter, Yang et al. [11] obtained the stability results of cubic ∗-derivations on ∗-algebras and superstability of cubic ∗-derivations on Banach ∗-algebras which are left approximately unital. Recently, Koh and Kang [12] proved the stability of a generalized cubic ∗-derivations on Banach ∗-algebras.

In 1999, Rassias [13] treated the stability of the following quartic equation:

$$\begin{aligned} f(x+2y) + f(x-2y) + 6f(x) \\ = 4f(x+y) + 4f(x-y) + 24f(y) \end{aligned} \quad (1)$$

for a mapping $f: X \longrightarrow Y$ where X is a linear space and Y is a Banach space. Thereafter, Lee and Chung [14] studied the general solution and stability theorem of generalized quartic functional equations in the spaces of generalized functions between real vector spaces. Kang [15] has then extended the stability theorems of the following generalized quartic functional equation:

$$\begin{aligned} f(ax+by) + f(ax-by) + 2b^2\left(a^2-b^2\right)f(y) \\ = 2a^2\left(a^2-b^2\right)f(x) \\ + (ab)^2\left[f(x+y)+f(x-y)\right], \end{aligned} \quad (2)$$

where $a, b \neq 0, a \pm b \neq 0$ in quasi-β-normed spaces. Recently, Bodaghi [16] obtained the general solution of the generalized quartic functional equation

$$\begin{aligned} & f(x+my)+f(x-my)+(m-1)^2 f(2x) \\ & \quad = 2(7m-9)(m-1) f(x)+2m^2\left(m^2-1\right) f(y) \\ & \qquad + m^2[f(x+y)+f(x-y)] \end{aligned} \tag{3}$$

for a fixed positive integer m and proved the Hyers-Ulam stability for this quartic functional equation by the direct method and the fixed-point method on real Banach spaces and non-Archimedean spaces. For more information about the stability of quartic functional equations, we refer to [17–20].

Hyer's direct method used in [2] has been widely applied for studying the generalized Hyers-Ulam stability of various functional equations. Nevertheless, there exist also other approaches proving the Hyers-Ulam stability of functional equations. The most popular technique of proving stability of functional equations except for direct method is the fixed-point method. Although fixed-point method was used for the first time by J.A. Baker [21], most authors follow the alternative fixed-point approach [22, 23] using a theorem of Diaz and Magolis [24].

In this paper, we deal with the following quartic functional equation:

$$\begin{aligned} & f(3x-y)+f(x+y)+6f(x-y) \\ & \quad = 4f(2x-y)+4f(y)+24f(x) \end{aligned} \tag{4}$$

in Banach $*$-algebras. First of all, we show that (4) is equivalent to (1) and then the mapping satisfying (4) on the punctured domain at zero is quartic. In the sequel, we investigate the stability of quartic $*$-derivations associated with the given functional equation on Banach $*$-algebras by using direct method and fixed-point method, respectively.

2. Approximate Quartic $*$-Derivations

First of all, we find out the general solution of (4) in the class of mappings between vector spaces.

Lemma 2. *Let U and V be vector spaces. A mapping $f: U \longrightarrow V$ satisfies the functional equation (4) if and only if the mapping $f: U \longrightarrow V$ satisfies (1).*

Proof. The proof is obvious by taking $(x, y) := (x-y, x)$ in (1) and $(x, y) := (y, x+y)$ in (4) on the basis of evenness of f, respectively. □

Throughout this section, let B be a Banach $*$-algebra and let A be a Banach $*$-subalgebra of B. For a given mapping $f: A \longrightarrow B$, we define

$$\begin{aligned} \mathbf{Q}_\mu f(x, y) = {} & f(3\mu x-\mu y)+f(\mu x+\mu y) \\ & +6f(\mu x-\mu y)-4\mu^4 f(2x-y) \\ & -4\mu^4 f(y)-24\mu^4 f(x) \end{aligned} \tag{5}$$

$$\mathbf{D} f(a, b) = f(ab)-a^4 f(b)-f(a) b^4$$

for all $a, b, c, x, y \in A$ and all $\mu \in \mathbb{T}^1_{1/n_0} := \{e^{i\theta} : 0 \leq \theta \leq 2\pi/n_0, n_0 \in \mathbb{N}\}$.

The following proposition provides a solution of the functional equation (4) on the punctured domain at zero.

Proposition 3. *If $f: A \longrightarrow B$ is a mapping satisfying the equality $\mathbf{Q}_1 f(x, y) = 0$ for all $x, y \in A-\{0\}$ and $f(0)=0$, then $\mathbf{Q}_1 f(x, y) = 0$ for all $x, y \in A$ and hence f is quartic.*

Proof. Since $f(0)=0$, it is trivial that $\mathbf{Q}_1 f(0,0)=0$. We obtain the equalities $\mathbf{Q}_1 f(x, x) = 2f(2x)-32f(x)=0$ so $f(2x)=16f(x)$ for $x \in A-\{0\}$. And we get $\mathbf{Q}_1 f(x, 3x) - \mathbf{Q}_1 f(x, -x) = 6f(-2x)-6f(2x)=0$; then $f(-x)=f(x)$ for $x \in A-\{0\}$.

By using above properties, we can show that

$$\begin{aligned} & \mathbf{Q}_1 f(x, 0) = \mathbf{Q}_1 f(x, 2x) = 0, \\ & \mathbf{Q}_1 f(0, y) = 7f(-y)-7f(y) = 0 \end{aligned} \tag{6}$$

for $x, y \in A-\{0\}$. Thus, $\mathbf{Q}_1 f(x, y)=0$ for all $x, y \in A$ and so f is quartic. □

Definition 4. A mapping $\delta : A \longrightarrow B$ is called a quartic homogeneous mapping if δ satisfies (1) and $\delta(\mu x) = \mu^4 \delta(x)$ for all $x \in A$ and $\mu \in \mathbb{C}$. A quartic homogeneous mapping $\delta : A \longrightarrow B$ is said to be a quartic derivation if $\delta(ab) = a^4\delta(b)+\delta(a)b^4$ for all $a, b \in A$. In addition, if $\delta(a^*) = \delta(a)^*$ for all $a \in A$, then it is called a quartic $*$-derivation.

Now we present a main theorem, which is a stability of quartic functional equation (4) in Banach $*$-algebras.

Theorem 5. *Let $f: A \longrightarrow B$ be a mapping with $f(0)=0$ and let $\psi : A^5 \longrightarrow [0, \infty)$ be a function such that*

$$\begin{aligned} & \sum_{i=0}^{\infty} \frac{1}{2^{4i}} \psi\left(2^i x, 2^i y, 2^i z, 0, 0\right) < \infty \\ & \& \lim_{n \longrightarrow \infty} \frac{1}{2^{8n}} \psi\left(0,0,0,2^n a, 2^n b\right) = 0, \\ & \left(\sum_{i=1}^{\infty} 2^{4i} \psi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, 0, 0\right) < \infty \right. \\ & \quad \left. \& \lim_{n \longrightarrow \infty} 2^{8n} \psi\left(0,0,0,\frac{a}{2^n}, \frac{b}{2^n}\right) = 0, \textit{ respectively} \right) \end{aligned} \tag{7}$$

$$\left\|\mathbf{Q}_\mu f(x, y)+f\left(z^*\right)-f(z)^*\right\| \leq \psi(x, y, z, 0, 0), \tag{8}$$

$$\|\mathbf{D} f(a, b)\| \leq \psi(0,0,0,a, b) \tag{9}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x, y, z, a, b \in A$. Assume that the mapping $t \longmapsto f(ta)$ from $\mathbb{R}$ to B is continuous for each *fixed $a \in A$. Then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying*

$$\|f(x) - \delta(x)\| \le \frac{1}{32}\sum_{i=0}^{\infty}\frac{1}{2^{4i}}\psi\left(2^i x, 2^i x, 0, 0, 0\right)$$
$$\left(\|f(x) - \delta(x)\| \le \frac{1}{32}\sum_{i=1}^{\infty}2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{10}$$

for all $x \in A$.

Proof. Taking $y = x$, $z = 0$, and $\mu = 1$ in inequality (8), we get

$$\left\|\frac{1}{16}f(2x) - f(x)\right\| \le \frac{1}{32}\psi(x, x, 0, 0, 0) \tag{11}$$

for all $x \in A$. By using induction, it is implied from inequality (11) that

$$\left\|\frac{1}{2^{4m}}f(2^m x) - \frac{1}{2^{4n}}f(2^n x)\right\| \le \frac{1}{32} \cdot \sum_{i=n}^{m-1}\frac{\psi\left(2^i x, 2^i x, 0, 0, 0\right)}{2^{4i}}$$
$$\left(\left\|2^{4m}f\left(\frac{x}{2^m}\right) - 2^{4n}f\left(\frac{x}{2^n}\right)\right\| \le \frac{1}{32}\sum_{i=n+1}^{m}2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{12}$$

for $m > n \ge 0$ and $x \in A$. By (7) and (12), the sequence $\{f(2^n x)/2^{4n}\}$ ($\{2^{4n}f(x/2^n)\}$, *respectively*) is a Cauchy sequence. So define a mapping $\delta : A \longrightarrow B$ by

$$\lim_{n\longrightarrow\infty}\frac{f(2^n x)}{2^{4n}} = \delta(x)$$
$$\left(\lim_{n\longrightarrow\infty}2^{4n}f\left(\frac{x}{2^n}\right) = \delta(x), \text{ respectively}\right) \tag{13}$$

for all $x \in A$. And letting $n = 0$ in inequality (12), we get

$$\left\|\frac{1}{2^{4m}}f(2^m x) - f(x)\right\| \le \frac{1}{32}\sum_{i=0}^{m-1}\frac{\psi\left(2^i x, 2^i x, 0, 0, 0\right)}{2^{4i}}$$
$$\left(\left\|2^{4m}f\left(\frac{x}{2^m}\right) - f(x)\right\| \le \frac{1}{32}\sum_{i=1}^{m}2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{14}$$

for $m > 0$ and $x \in A$. Hence (7) and (14) show that approximate inequality (10) holds.

Next, we have to show that the mapping δ is a quartic $*$-derivation such that inequality (10) holds for all $x \in A$. Replacing x, y by $2^i x, 2^i y$ in (8), respectively, and putting $z = 0$, we have

$$\frac{1}{2^{4i}}\left\|\mathbf{Q}_\mu f\left(2^i x, 2^i y\right)\right\| \le \frac{\psi\left(2^i x, 2^i y, 0, 0, 0\right)}{2^{4i}}$$
$$\left(2^{4i}\left\|\mathbf{Q}_\mu f\left(\frac{x}{2^i}, \frac{y}{2^i}\right)\right\| \le 2^{4i}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{15}$$

and so it follows from (7) and (13) that

$$\left\|\mathbf{Q}_\mu \delta(x, y)\right\| = 0 \tag{16}$$

for all $x, y \in A$ and $\mu \in \mathbb{T}^1_{1/n_0}$. Thus, by the same argument in the proof of Theorem 3.2 in [18], $\mathbf{Q}_1\delta(x, y) = 0$ and $\delta(\mu x) = \mu^4\delta(x)$ for all $x, y \in A$ and $\mu \in \mathbb{C}$, which implies that the mapping δ is quartic homogeneous by Lemma 2.

Next, replacing a, b by $2^n a, 2^n b$ in inequality (9), we get

$$\frac{1}{2^{8n}}\left\|\mathbf{D}f(2^n a, 2^n b)\right\| \le \frac{1}{2^{8n}}\psi(0, 0, 0, 2^n a, 2^n b)$$
$$\left(2^{8n}\left\|\mathbf{D}f\left(\frac{a}{2^n}, \frac{b}{2^n}\right)\right\| \le 2^{8n}\psi\left(0, 0, 0, \frac{a}{2^n}, \frac{b}{2^n}\right), \text{ respectively}\right) \tag{17}$$

for all $a, b \in A$. By (7), we have $\mathbf{D}\delta(a, b) = 0$ for all $a, b \in A$. Letting $x = y = 0$ and replacing z by $2^n z$ in inequality (8), we have

$$\left\|\frac{f(2^n z^*)}{2^{4n}} - \frac{f(2^n z)^*}{2^{4n}}\right\| \le \frac{1}{2^{4n}}\psi(0, 0, 2^n z, 0, 0)$$
$$\left(\left\|2^{4n}f\left(\frac{z^*}{2^n}\right) - 2^{4n}f\left(\frac{z}{2^n}\right)^*\right\| \le 2^{4n}\psi\left(0, 0, \frac{z}{2^n}, 0, 0\right), \text{ respectively}\right) \tag{18}$$

for all $z \in A$. Also by (7), we have $\delta(z^*) = \delta(z)^*$ for all $z \in A$. Therefore δ is a quartic $*$-derivation.

Lastly, we should show that δ is unique. Suppose that $\delta' : A \longrightarrow B$ is another quartic $*$-derivation satisfying approximate inequality (10). So

$$\begin{aligned}
\left\|\delta(x)-\delta'(x)\right\| &= \frac{1}{2^{4n}}\left\|\delta(2^n x)-\delta'(2^n x)\right\| \\
&\le \frac{1}{2^{4n}}\left[\left\|\delta(2^n x)-f(2^n x)\right\|+\left\|f(2^n x)\right.\right. \\
&\quad \left.\left.-\delta'(2^n x)\right\|\right] \le \frac{1}{2^{4(n+1)}} \\
&\quad \cdot \sum_{i=0}^{\infty}\frac{1}{2^{4i}}\psi\left(2^{i+n}x,2^{i+n}x,0,0,0\right), \\
\Bigg(\left\|\delta(x)-\delta'(x)\right\| &= 2^{4n}\left\|\delta\left(\frac{x}{2^n}\right)-\delta'\left(\frac{x}{2^n}\right)\right\| \\
&\le 2^{4n}\left[\left\|\delta\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|\right. \\
&\quad \left.+\left\|f\left(\frac{x}{2^n}\right)-\delta'\left(\frac{x}{2^n}\right)\right\|\right] \le \frac{1}{16} \\
&\quad \cdot \sum_{i=n+1}^{\infty}2^{4i}\psi\left(\frac{x}{2^i},\frac{x}{2^i},0,0,0\right), \textit{ respectively}\Bigg)
\end{aligned} \tag{19}$$

which tends to zero as $n \longrightarrow \infty$ for all $x \in A$. Hence $\delta(x) = \delta'(x)$ for all $x \in A$. □

Corollary 6. *Let θ_j, p_i, q_j $(i = 1,2,3,\ j = 1,2)$ be nonnegative real constants with either $0 \le p_i < 4, q_j < 8$ or $p_i > 4, q_j > 8$ $(i = 1,2,3,\ j = 1,2)$ and let $f : A \longrightarrow B$ be a mapping with $f(0) = 0$ such that*

$$\begin{aligned}
&\left\|\mathbf{Q}_\mu f(x,y)+f(z^*)-f(z)^*\right\| \\
&\quad \le \theta_1\left(\|x\|^{p_1}+\|y\|^{p_2}+\|z\|^{p_3}\right) \\
&\left\|\mathbf{D}f(a,b)\right\| \le \theta_2\left(\|a\|^{q_1}+\|b\|^{q_2}\right)
\end{aligned} \tag{20}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Assume that the mapping $t \longmapsto f(ta)$ from $\mathbb{R}$ to B is continuous for each fixed $a \in A$. Then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying

$$\left\|f(x)-\delta(x)\right\| \le \frac{\theta_1\|x\|^{p_1}}{2\left|2^{p_1}-16\right|}+\frac{\theta_1\|x\|^{p_2}}{2\left|2^{p_2}-16\right|} \tag{21}$$

for all $x \in A$.

Proof. Letting $\psi(x,y,z,a,b) := \theta_1(\|x\|^{p_1}+\|y\|^{p_2}+\|z\|^{p_3})+\theta_2(\|a\|^{q_1}+\|b\|^{q_2})$ and applying Theorem 5, we obtain the desired result. □

Concerning the stability of quartic homogeneous function, the following example presents that the stability of functional equation $\mathbf{Q}_\mu f(x,y) = 0$ in Corollary 6 with $p_i = 4 (i = 1,2)$ does not hold.

Example 7. Let $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^4, & \text{if } |x| < 1, \\ 1, & \text{if } |x| \ge 1. \end{cases} \tag{22}$$

Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(x) = \sum_{m=0}^{\infty}\alpha^{-4m}\phi(\alpha^m x) \tag{23}$$

for all $x \in \mathbb{C}$, where $\alpha \ge 2$. Then, using similar way to that in [25], f satisfies

$$\left|\mathbf{Q}_\mu f(x,y)\right| \le \theta_1\left(|x|^4+|y|^4\right), \quad \theta_1 := \frac{40\alpha^8}{\alpha^4-1} \tag{24}$$

for all $x, y \in \mathbb{C}$ and $\mu \in \mathbb{T}^1_{1/n_0}$, but there do not exist a quartic mapping $\delta : \mathbb{C} \longrightarrow \mathbb{C}$ and a constant $\gamma > 0$ such that $|f(x) - \delta(x)| \le \gamma|x|^4$ for all $x \in \mathbb{C}$.

However, the stability problem of $p_i = 4(i = 1,2,3)$ and $q_j = 8(j = 1,2)$ is open in Corollary 6 concerning the stability of quartic $*$-derivations.

Corollary 8. *Let θ_j, p_i, q_j $(i = 1,2,3,\ j = 1,2)$ be nonnegative real constants with either $0 \le p_i < 2, 0 \le q_j < 4$ or $p_i > 2, q_j > 4$ $(i = 1,2,3,\ j = 1,2)$ and let $f : A \longrightarrow B$ be a mapping with $f(0) = 0$ such that*

$$\begin{aligned}
&\left\|\mathbf{Q}_\mu f(x,y)+f(z^*)-f(z)^*\right\| \\
&\quad \le \theta_1\left(\|x\|^{p_1}\|y\|^{p_2}+\|y\|^{p_2}\|z\|^{p_3}+\|z\|^{p_3}\|x\|^{p_1}\right), \\
&\left\|\mathbf{D}f(a,b)\right\| \le \theta_2\left(\|a\|^{q_1}\|b\|^{q_2}\right)
\end{aligned} \tag{25}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Assume that the mapping $t \longmapsto f(ta)$ from $\mathbb{R}$ to B is continuous for each fixed $a \in A$. Then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying

$$\left\|f(x)-\delta(x)\right\| \le \frac{\theta_1\|x\|^{p_1+p_2}}{2\left|16-2^{p_1+p_2}\right|} \tag{26}$$

for all $x \in A$.

Proof. Letting $\psi(x,y,z,a,b) := \theta_1(\|x\|^{p_1}\|y\|^{p_2}+\|y\|^{p_2}\|z\|^{p_3}+\|x\|^{p_1}\|z\|^{p_3})+\theta_2(\|a\|^{q_1}\|b\|^{q_2})$, as well as applying Theorem 5, we obtain the desired result. □

Now, we investigate the stability using the alternative fixed-point method. Before proceeding to the main result, we state the following definition and theorem which are useful for our purpose.

Definition 9. Let X be a set. A function $d : X \times X \longrightarrow [0,\infty]$ is called a generalized metric on X if d satisfies the following:

(i) $d(x,y) = 0$, if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(iii) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 10 (see [24]). *Let (Ω, d) be a complete generalized metric space and let $T : \Omega \longrightarrow \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Omega$, either $d(T^n\alpha, T^{n+1}\alpha) = \infty$ for all $n \geq 0$ or there exists a natural number n_0 such that*

(i) *$d(T^n\alpha, T^{n+1}\alpha) < \infty$ for all $n \geq n_0$,*

(ii) *the sequence $\{T^n\alpha\}$ is convergent to a fixed point β^* of T,*

(iii) *β^* is the unique fixed point of T in the set $\Lambda = \{\beta \in \Omega : d(T^{n_0}\alpha, \beta)\}$,*

(iv) *$d(\beta, \beta^*) \leq (1/(1-L))d(\beta, T\beta)$ for all $\beta \in \Lambda$.*

Theorem 11. *Let $f : A \longrightarrow B$ be a continuous mapping with $f(0) = 0$ and let $\psi : A^5 \longrightarrow [0, \infty)$ be a function such that*

$$\left\|\mathbf{Q}_\mu f(x, y) + f(z^*) - f(z)^*\right\| \leq \psi(x, y, z, 0, 0), \tag{27}$$

$$\left\|\mathbf{D} f(a, b)\right\| \leq \psi(0, 0, 0, a, b) \tag{28}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $a, b, x, y, z \in A$. If there exist constants $l_1, l_2 \in (0, 1)$ such that

$$\begin{gathered}\psi(2x, 2y, 2z, 0, 0) \leq 2^4 l_1 \psi(x, y, z, 0, 0) \\ \& \ \psi(0, 0, 0, 2a, 2b) \leq 2^8 l_2 \psi(0, 0, 0, a, b) \\ \left(\psi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, 0, 0\right) \leq \frac{l_1}{2^4}\psi(x, y, z, 0, 0)\right. \\ \& \ \psi(0, 0, 0, a, b) \\ \left. \leq \frac{l_2}{2^8}\psi(0, 0, 0, a, b), \text{ respectively}\right)\end{gathered} \tag{29}$$

for all $x, y, z, a, b \in A$, then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying

$$\|f(x) - \delta(x)\| \leq \frac{1}{32(1-l_1)}\widetilde{\psi}(x) \tag{30}$$

$$\left(\|f(x) - \delta(x)\| \leq \frac{l_1}{32(1-l_1)}\widetilde{\psi}(x), \text{ respectively}\right) \tag{31}$$

for all $x \in A$, where $\widetilde{\psi}(x) = \psi(x, x, 0, 0, 0)$.

Proof. First, we consider a set

$$\Omega = \{g : A \longrightarrow B : g(0) = 0\} \tag{32}$$

and define a mapping d on $\Omega \times \Omega$ as follows:

$$\begin{gathered}d(g_1, g_2) := \inf\{k \in (0, \infty) : \|g_1(x) - g_2(x)\| \\ \leq k\psi(x, x, 0, 0, 0)\}\end{gathered} \tag{33}$$

if there exists such constant k and $d(g_1, g_2) \equiv \infty$, if not. Then we can easily show that d is a generalized metric on Ω and the metric space (Ω, d) is complete. We define a mapping $\Psi : \Omega \longrightarrow \Omega$ by

$$\begin{gathered}\Psi g(x) = \frac{1}{2^4}g(2x) \\ \left(\Psi g(x) = 2^4 g\left(\frac{x}{2}\right), \text{ respectively}\right)\end{gathered} \tag{34}$$

where $g \in \Omega$ and for all $x \in A$.

Now we remark that Ψ is a strictly contractive mapping on Ω with the Lipschitz constant l_1 [18].

On the other hand, letting $\mu = 1, y = x, z = 0$ in inequality (27), we get

$$\begin{gathered}\left\|\frac{1}{16}f(2x) - f(x)\right\| \leq \frac{1}{32}\psi(x, x, 0, 0, 0) \\ \left(\left\|16 f\left(\frac{x}{2}\right) - f(x)\right\| \leq \frac{1}{2}\psi\left(\frac{x}{2}, \frac{x}{2}, 0, 0, 0\right)\right. \\ \left. \leq \frac{l_1}{32}\psi(x, x, 0, 0, 0), \text{ respectively}\right)\end{gathered} \tag{35}$$

for all $x \in A$. This implies that

$$\begin{gathered}d(\Psi f, f) \leq \frac{1}{32} \\ \left(d(\Psi f, f) \leq \frac{l_1}{32}, \text{ respectively}\right).\end{gathered} \tag{36}$$

It follows from Theorem 10 that $d(\Psi^n f, \Psi^{n+1} f) < \infty$ for all $n \geq 0$. So parts (iii) and (iv) of Theorem 10 hold on the whole Ω. Therefore, there exists a unique mapping $\delta : A \longrightarrow B$ such that δ is a fixed point of Ψ and

$$\begin{gathered}\delta(x) = \lim_{n \longrightarrow \infty}\frac{f(2^n x)}{2^{4n}} \\ \left(\delta(x) = \lim_{n \longrightarrow \infty} 2^{4n} f\left(\frac{x}{2^n}\right), \text{ respectively}\right)\end{gathered} \tag{37}$$

for all $x \in A$ and

$$\begin{gathered}d(f, \delta) \leq \frac{1}{1-l_1}d(\Psi f, f) \leq \frac{1}{32(1-l_1)} \\ \left(d(f, \delta) \leq \frac{l_1}{1-l_1}d(\Psi f, f)\right. \\ \left. \leq \frac{l_1}{32(1-l_1)}, \text{ respectively}\right)\end{gathered} \tag{38}$$

So the mapping δ satisfies inequality (30) that holds for all $x \in A$.

Since $l_1, l_2 \in (0,1)$, inequality (30) shows that

$$\lim_{i\longrightarrow\infty} \frac{\psi\left(2^i x, 2^i y, 2^i z, 0, 0\right)}{2^{4i}} = 0$$
$$\& \lim_{i\longrightarrow\infty} \frac{\psi\left(0,0,0,2^i a, 2^i b\right)}{2^{8i}} = 0 \tag{39}$$
$$\left(\lim_{i\longrightarrow\infty} 2^{4i}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, 0, 0\right) = 0\right.$$
$$\left.\& \lim_{i\longrightarrow\infty} 2^{8^i}\psi\left(0,0,0,\frac{a}{2^i},\frac{b}{2^i}\right) = 0, \textit{ respectively}\right)$$

for all $x, y, z, a, b \in A$. Replacing x, y by $2^i x, 2^i y$ and putting $z = 0$ in inequality (27), we have

$$\left\|\frac{1}{2^{4i}}\mathbf{Q}_\mu f\left(2^i x, 2^i y\right)\right\| \le \frac{\psi\left(2^i x, 2^i y, 0, 0, 0\right)}{2^{4i}}$$
$$\left(\left\|2^{4i}\mathbf{Q}_\mu f\left(\frac{x}{2^i}, \frac{y}{2^i}\right)\right\|\right. \tag{40}$$
$$\left.\le 2^{4i}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}, 0, 0, 0\right), \textit{ respectively}\right)$$

and taking the limit as i tends to infinity, we get $\mathbf{Q}_\mu \delta(x,y) = 0$ for all $x, y \in A$ and all $\mu \in \mathbb{T}^1_{1/n_0}$. Also, by the same argument in the proof of Theorem 5, the mapping δ is quartic homogeneous. Next, replacing a, b by $2^n a, 2^n b$ in inequality (28), we get

$$\frac{1}{2^{8i}}\left\|\mathbf{D}f\left(2^i a, 2^i b\right)\right\| \le \frac{1}{2^{8i}}\psi\left(0,0,0,2^i a, 2^i b\right)$$
$$\left(\left\|2^{8i}\mathbf{D}f\left(\frac{a}{2^i}, \frac{b}{2^i}\right)\right\|\right. \tag{41}$$
$$\left.\le 2^{8i}\psi\left(0,0,0,\frac{a}{2^i}, \frac{b}{2^i}\right), \textit{ respectively}\right)$$

for all $a, b \in A$. By (39), we have $\mathbf{D}\delta(a,b) = 0$ for all $a, b \in A$. Letting $x = y = 0$ and replacing z by $2^n z$ in inequality (27), we have

$$\left\|\frac{f\left(2^n z^*\right)}{2^{4n}} - \frac{f\left(2^n z\right)^*}{2^{4n}}\right\| \le \frac{1}{2^{4n}}\psi\left(0,0,2^n z, 0, 0\right)$$
$$\left(\left\|2^{4n} f\left(\frac{z^*}{2^n}\right) - 2^{4n} f\left(\frac{z}{2^n}\right)^*\right\|\right. \tag{42}$$
$$\left.\le 2^{4n}\psi\left(0,0,\frac{z}{2^n},0,0\right), \textit{ respectively}\right)$$

for all $z \in A$. Also by (39), we have $\delta(z^*) = \delta(z)^*$ for all $z \in A$. Therefore the mapping δ is a quartic $*$-derivation.

The rest of the proof is similar to the proof of Theorem 5. □

Corollary 12. *Let $\theta_i (i = 1, 2)$, p, q be nonnegative reals with either $0 \le p < 4, q < 8$ or $p > 4, q > 8$ and let $f : A \longrightarrow B$ be a continuous mapping with $f(0) = 0$ such that*

$$\left\|\mathbf{Q}_\mu f\left(x, y\right) + f\left(z^*\right) - f\left(z\right)^*\right\|$$
$$\le \theta_1\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{43}$$
$$\left\|\mathbf{D}f\left(a,b\right)\right\| \le \theta_2\left(\|a\|^q + \|b\|^q\right)$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Then there exists a unique quartic $*$-derivation δ on A satisfying

$$\left\|f\left(x\right) - \delta\left(x\right)\right\| \le \frac{\theta_1\|x\|^p}{\left|16 - 2^p\right|} \tag{44}$$

for all $x \in A$.

Proof. Letting $\psi(x, y, z, a, b) := \theta_1(\|x\|^p + \|y\|^p + \|z\|^p) + \theta_2(\|a\|^q + \|b\|^q)$ and applying Theorem 11 with $l_1 = 2^{-|p-4|}$, $l_2 = 2^{-|q-8|}$, we obtain the desired results. □

Corollary 13. *Let $\theta_i (i = 1, 2)$, p, q be nonnegative reals with either $0 \le p < 2, q < 4$ or $p > 2, q > 4$ and let $f : A \longrightarrow B$ be a continuous mapping with $f(0) = 0$ such that*

$$\left\|\mathbf{Q}_\mu f\left(x, y\right) + f\left(z^*\right) - f\left(z\right)^*\right\|$$
$$\le \theta_1\left(\|x\|^p\|y\|^p + \|y\|^p\|z\|^p + \|z\|^p\|x\|^p\right) \tag{45}$$
$$\left\|\mathbf{D}f\left(a,b\right)\right\| \le \theta_2\left(\|a\|^q\|b\|^q\right)$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Then there exists a unique quartic $*$-derivation δ on A satisfying

$$\left\|f\left(x\right) - \delta\left(x\right)\right\| \le \frac{\theta_1\|x\|^{2p}}{2\left|16 - 2^{2p}\right|} \tag{46}$$

for all $x \in A$.

Proof. Letting $\psi(x, y, z, a, b) := \theta_1(\|x\|^p\|y\|^p + \|y\|^p\|z\|^p + \|z\|^p\|x\|^p) + \theta_2(\|a\|^q\|b\|^q)$ and applying Theorem 11 with $l_1 = 2^{-|2p-4|}$, $l_2 = 2^{-|2q-8|}$, we obtain the desired results. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Integral Majorization Type Inequalities for the Functions in the Sense of Strong Convexity

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In this article, we establish several integral majorization type and generalized Favard's inequalities for the class of strongly convex functions. Our results generalize and improve the previous known results.

1. Introduction

It is well known that convex functions are a class of important functions in the fields of mathematics and other natural sciences; they have been studied for more than one hundred years. In recent years there is a growing interest in generalized convex functions (such as quasi-convex function [1], strongly convex function [2–4], s-convex function [5], approximately convex function [6], logarithmically convex function [7, 8], midconvex function [9], pseudo-convex function [10], φ-convex function [11], λ-convex function [12], h-convex function [13], delta-convex function [14], Schur convex function [15–21], and other convex functions [22–29]) among the researchers of applied mathematics due to the fact that mathematical models with these functions are more suitable to describe problems of the real world than models using conventional convex functions. Recently, a large number of remarkable results and applications for the generalized convex functions can be found in the literature [30–49].

In the article, our focus is on the integral majorization type inequalities for the strongly convex functions.

Definition 1. Let ϕ be a real-valued function defined on the interval $[\lambda_1, \xi_1]$ and c a positive real number. Then ϕ is said to be strongly convex with modulus c if the inequality

$$\begin{aligned}\phi(\eta u_1 + (1-\eta)v_1) \le \eta\phi(u_1) + (1-\eta)\phi(v_1) \\ - c\eta(1-\eta)(u_1 - v_1)^2\end{aligned} \tag{1}$$

holds for all $u_1, v_1 \in [\lambda_1, \xi_1]$ and $\eta \in [0,1]$. From (1) we clearly see that

$$\phi(u_1) - \phi(v_1) \ge \phi'_+(v_1)(u_1 - v_1) + c(u_1 - v_1)^2. \tag{2}$$

The following Lemma 2 for strongly convex function is given in [2] (see also [50, Proposition 1.1.2]).

Lemma 2. *A real-valued function $\phi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ is a strongly convex function with modulus c if and only if the function $\varphi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ defined by $\varphi(r) = \phi(r) - cr^2$ is a convex function.*

Every strongly convex function is convex, but the converse is not true in general. Strongly convex functions have been utilized for showing the convergence of a gradient type algorithm for minimizing a function. They play a significant role in mathematical economics, approximation theory, and optimization theory; many applications and properties for strongly functions can be found in [2–4, 13, 30].

Next we are going to present some basic theories of majorization.

There is a natural description of the indefinite notion that the entries of n-tuple $\boldsymbol{\delta}$ are more nearly equal, or less spread out than, to the entries of n-tuple $\boldsymbol{\beta}$. The applicable assertion is that $\boldsymbol{\delta}$ majorizes $\boldsymbol{\beta}$; it means that the sum of ℓ largest entries of $\boldsymbol{\beta}$ does not exceed the sum of ℓ largest entries of $\boldsymbol{\delta}$ for all $\ell = 1, 2, \ldots, n-1$ with equality for $\ell = n$. That is, let $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ be two real n−tuples and let

$$\begin{aligned} &\delta_1^{\downarrow} \geq \delta_2^{\downarrow} \geq \cdots \geq \delta_n^{\downarrow}, \\ &\beta_1^{\downarrow} \geq \beta_2^{\downarrow} \geq \cdots \geq \beta_n^{\downarrow} \end{aligned} \tag{3}$$

be their ordered entries. Then the n−tuple $\boldsymbol{\delta}$ is said to majorize $\boldsymbol{\beta}$ (or $\boldsymbol{\beta}$ is said to be majorized by $\boldsymbol{\delta}$), in symbol $\boldsymbol{\delta} \succ \boldsymbol{\beta}$, if

$$\sum_{j=1}^{\ell} \delta_j^{\downarrow} \geq \sum_{j=1}^{\ell} \beta_j^{\downarrow} \tag{4}$$

holds for $\ell = 1, 2, \ldots, n-1$ and

$$\sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \beta_j. \tag{5}$$

The theory of majorization is a very significant topic in mathematics; a remarkable and complete reference on the majorization subject is the book by Olkin and Marshall [51]. For example, the theory of majorization is an essential tool that permits us to transform nonconvex complicated constrained optimization problems that involve matrix valued variables into simple problems with scalar variables that can be easily solved [52–55].

The definition of majorization for integrable functions can be stated as follows (see [7]).

Definition 3. Let $\mathfrak{f}$ and $\mathfrak{g}$ be two decreasing real-valued integrable functions on the interval $[\lambda_1, \xi_1]$. Then $\mathfrak{f}$ is said to majorize $\mathfrak{g}$ (or $\mathfrak{g}$ is said to be majorized by $\mathfrak{f}$), in symbol, $\mathfrak{f} \succ \mathfrak{g}$, if the inequality

$$\int_{\lambda_1}^{x} \mathfrak{g}(r)\, dr \leq \int_{\lambda_1}^{x} \mathfrak{f}(r)\, dr \tag{6}$$

holds for all $x \in [\lambda_1, \xi_1)$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\, dr = \int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, dr. \tag{7}$$

Theorem 4 (See [56]). *Let $\mathfrak{f}$ and $\mathfrak{g}$ be two continuous and increasing real-valued functions defined on $[\lambda_1, \xi_1]$, and let $\Omega : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ be a bounded variation function. Then the following statements are true.*

(a) *If*

$$\int_{\lambda_1}^{x} \mathfrak{f}(r)\, d\Omega(r) \leq \int_{\lambda_1}^{x} \mathfrak{g}(r)\, d\Omega(r) \tag{8}$$

for all $x \in [\lambda_1, \xi_1]$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, d\Omega(r) = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\, d\Omega(r), \tag{9}$$

then

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\, d\Omega(r) \leq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\, d\Omega(r) \tag{10}$$

holds for every continuous convex function Ψ.

(b) *If (8) and (9) hold, then (10) holds for every continuous increasing convex function Ψ.*

Theorem 5 (See [57]). *Let $\Psi : [0, \infty) \longrightarrow \mathbb{R}$ be a convex function, $\mathfrak{f}$, $\mathfrak{g}$ and Ω be three positive and integrable functions defined on $[\lambda_1, \xi_1]$ such that*

$$\int_{\lambda_1}^{x} \mathfrak{f}(r)\, \Omega(r)\, dr \leq \int_{\lambda_1}^{x} \mathfrak{g}(r)\, \Omega(r)\, dr \tag{11}$$

for all $x \in [\lambda_1, \xi_1]$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, \Omega(r)\, dr = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\, \Omega(r)\, dr. \tag{12}$$

Then the following statements are true:

(a) *If $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\, \Omega(r)\, dr \leq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\, \Omega(r)\, dr. \tag{13}$$

(b) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$, then*

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\, \Omega(r)\, dr \leq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\, \Omega(r)\, dr. \tag{14}$$

Let $p > 1$, $\mathfrak{f}$ be a positive and continuous concave function defined on $[\lambda_1, \xi_1]$, and let Ψ be a convex function defined on $[0, 2\bar{\mathfrak{f}}_1]$ with

$$\bar{\mathfrak{f}}_1 = \frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, dr. \tag{15}$$

Then Favard [58] proved that the inequalities

$$\begin{aligned} \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_1\right) &= \frac{1}{2\bar{\mathfrak{f}}_1} \int_0^{2\bar{\mathfrak{f}}_1} \Psi(y)\, dy \\ &\geq \frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\, dr \end{aligned} \tag{16}$$

and

$$\frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{f}^p(r)\, dr \leq \frac{2^p}{p+1} \left(\frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, dr \right)^p \tag{17}$$

hold.

The main purpose of the article is to establish several integral majorization type and generalized Favard's inequalities for strongly convex functions.

2. Main Results

Theorem 6. *Let $c > 0$, $\Psi : [0, \infty) \longrightarrow \mathbb{R}$ be a continuous strongly convex function with modulus c, and let $\mathfrak{f}$, $\mathfrak{g}$, and Ω be three positive and integrable functions defined on $[\lambda_1, \xi_1]$ such that*

$$\int_{\lambda_1}^{x} \mathfrak{f}(r)\,\Omega(r)\,dr \le \int_{\lambda_1}^{x} \mathfrak{g}(r)\,\Omega(r)\,dr \tag{18}$$

for all $x \in [\lambda_1, \xi_1]$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\,\Omega(r)\,dr = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\,\Omega(r)\,dr. \tag{19}$$

Then the following statements are true.

(a) *If $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, then we have*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr &\ge \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\,\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r)-\mathfrak{g}(r)\}^2\,\Omega(r)\,dr.\end{aligned} \tag{20}$$

(b) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$, then one has*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\,\Omega(r)\,dr &\ge \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r)-\mathfrak{g}(r)\}^2\,\Omega(r)\,dr.\end{aligned} \tag{21}$$

Proof. (a) Let $v_1 = \mathfrak{f}$ and $u_1 = \mathfrak{g}$. Then it follows from (2) and the proof of Lemma 2 given in [57] that

$$\begin{aligned}&\Psi(\mathfrak{f}(r))\,\Omega(r) + \Psi'_+(\mathfrak{f}(r))\,(\mathfrak{g}(r)-\mathfrak{f}(r))\,\Omega(r) \\ &\quad + c\,(\mathfrak{g}(r)-\mathfrak{f}(r))^2\,\Omega(r) \le \Psi(\mathfrak{g}(r))\,\Omega(r).\end{aligned} \tag{22}$$

Let $\mathscr{F}(x) = \int_{\lambda_1}^{x} \{\mathfrak{f}(r)-\mathfrak{g}(r)\}\Omega(r)dr$. Then (18) and (19) lead to $\mathscr{F}(x) \le 0$ for all $x \in [\lambda_1, \xi_1]$, $\mathscr{F}(\lambda_1) = \mathscr{F}(\xi_1) = 0$, and

$$\begin{aligned}&\int_{\lambda_1}^{\xi_1} [\Psi\{\mathfrak{f}(r)\} - \Psi\{\mathfrak{g}(r)\}]\,\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r)-\mathfrak{g}(r)\}^2\,\Omega(r)\,dr \\ &\le \int_{\lambda_1}^{\xi_1} \Psi'_+\{\mathfrak{f}(r)\}\{\mathfrak{f}(r)-\mathfrak{g}(r)\}\,\Omega(r)\,dr \\ &= \int_{\lambda_1}^{\xi_1} \Psi'_+\{\mathfrak{f}(r)\}\,d\mathscr{F}(r) = \Psi'_+\{\mathfrak{f}(r)\}\,\mathscr{F}(r)\big|_{\lambda_1}^{\xi_1} \\ &\quad - \int_{\lambda_1}^{\xi_1} \mathscr{F}(r)\,d\left\{\Psi'_+\{\mathfrak{f}(r)\}\right\} \\ &= -\int_{\lambda_1}^{\xi_1} \mathscr{F}(r)\,d\left\{\Psi'_+\{\mathfrak{f}(r)\}\right\} \le 0.\end{aligned} \tag{23}$$

Since $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, therefore inequality (20) can be deduced easily from the above inequality. Similarly, we can prove part (b) for increasing function $\mathfrak{g}$ defined on $[\lambda_1, \xi_1]$. □

Theorem 7. *Suppose that all the assumptions of Theorem 6 hold. Then the following statements are true.*

(a) *If $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr &\ge \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\,\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \left\{\mathfrak{g}^2(r)-\mathfrak{f}^2(r)\right\}\Omega(r)\,dr.\end{aligned} \tag{24}$$

(b) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr &\le \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\,\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \left\{\mathfrak{g}^2(r)-\mathfrak{f}^2(r)\right\}\Omega(r)\,dr.\end{aligned} \tag{25}$$

Proof. Since Ψ is a strongly convex function with modulus c, therefore $\Psi(r) - cr^2$ is a convex function, and inequalities (24) and (25) follow easily from the convexity of the function $\Psi(r) - cr^2$ and Lemma 2 given in [57]. □

Remark 8. Inequalities (13) and (14) can be obtained by (24) and (25) immediately.

Remark 9. Generally, the assumptions of both the functions $\mathfrak{f}$ and $\mathfrak{g}$ are monotonic in majorization theorem, but in Theorems 6 and 7 we only need one of the functions $\mathfrak{f}$ and $\mathfrak{g}$ to be monotonic.

Theorem 10. *Suppose that $\Psi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ is a continuous strongly convex function with modulus c, and $\mathfrak{f}$, $\mathfrak{g}$, and Ω are three integrable functions on $[\lambda_1, \xi_1]$. If $\mathfrak{g}$ and $\mathfrak{f} - \mathfrak{g}$*

are nondecreasing (nonincreasing) functions on $[\lambda_1, \xi_1]$ *and* $\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\Omega(r)dr = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\Omega(r)dr$, *then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr &\geq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \\ &+ c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r)-\mathfrak{g}(r)\}^2\,\Omega(r)\,dr.\end{aligned} \tag{26}$$

Proof. Since Ψ is a strongly convex function, therefore using (2) for $u_1 = \mathfrak{f}$ and $v_1 = \mathfrak{g}$, we have

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} [\Psi(\mathfrak{f}(r)) - \Psi(\mathfrak{g}(r))]\,\Omega(r)\,dr &- c\int_{\lambda_1}^{\xi_1} (\mathfrak{f}(r)-\mathfrak{g}(r))^2\,\Omega(r)\,dr \\ &\geq \int_{\lambda_1}^{\xi_1} \Psi'_+(\mathfrak{g}(r))(\mathfrak{f}(r)-\mathfrak{g}(r))\,\Omega(r)\,dr.\end{aligned} \tag{27}$$

It follows from the Čebyšev inequality [59] that

$$\begin{aligned}&\frac{1}{\int_{\lambda_1}^{\xi_1}\Omega(r)\,dr}\int_{\lambda_1}^{\xi_1} \Psi'_+(\mathfrak{g}(r))(\mathfrak{f}(r)-\mathfrak{g}(r))\,\Omega(r)\,dr \\ &\geq \frac{1}{\int_{\lambda_1}^{\xi_1}\Omega(r)\,dr}\int_{\lambda_1}^{\xi_1} \Psi'_+(\mathfrak{g}(r))\,\Omega(r)\,dr\,\frac{1}{\int_{\lambda_1}^{\xi_1}\Omega(r)\,dr} \\ &\cdot\int_{\lambda_1}^{\xi_1}(\mathfrak{f}(r)-\mathfrak{g}(r))\,\Omega(r)\,dr \geq 0.\end{aligned} \tag{28}$$

Therefore, inequality (26) follows from (27) and (28). □

Making use of the similar idea as in the proof of Theorem 10, we can obtain the following Theorem 11 immediately.

Theorem 11. *Suppose that* $\Psi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ *is a continuous strongly convex function with modulus c, and* $\mathfrak{f}$, $\mathfrak{g}$, *and* Ω *are three integrable functions on* $[\lambda_1, \xi_1]$. *If* $\mathfrak{g}$ *and* $\mathfrak{f} - \mathfrak{g}$ *are nondecreasing (nonincreasing) functions on* $[\lambda_1, \xi_1]$, *and* $\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\Omega(r)dr \geq \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\Omega(r)dr$, *then inequality (26) holds.*

Theorem 12. *The inequality*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr &\geq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \\ &+ c\int_{\lambda_1}^{\xi_1} \left(\mathfrak{f}^2(r)-\mathfrak{g}^2(r)\right)\Omega(r)\,dr\end{aligned} \tag{29}$$

holds if all the assumptions of Theorem 10 are satisfied.

Proof. Since Ψ is a strongly convex function with modulus c, therefore $\Psi(r) - cr^2$ is a convex function and inequality (29) can be deduced by applying this convex function in Theorem 6 of [59]. □

Using strongly convex function we can give an extension of [60, Theorem 2] in the following form.

Theorem 13. *Let* $\varphi, \Psi : [0,\infty) \longrightarrow \mathbb{R}$ *be two functions such that* φ *is a strictly increasing and* $\Psi^\circ\varphi^{-1}$ *is strongly convex with modulus c,* $\mathfrak{f}$, $\mathfrak{g}$, *and* Ω *being three positive and integrable functions on* $[\lambda_1, \xi_1]$ *such that*

$$\int_{\lambda_1}^{x} \varphi(\mathfrak{f}(r))\,\Omega(r)\,dr \leq \int_{\lambda_1}^{x} \varphi(\mathfrak{g}(r))\,\Omega(r)\,dr \tag{30}$$

for all $x \in [\lambda_1, \xi_1]$ *and*

$$\int_{\lambda_1}^{\xi_1} \varphi(\mathfrak{f}(r))\,\Omega(r)\,dr = \int_{\lambda_1}^{\xi_1} \varphi(\mathfrak{g}(r))\,\Omega(r)\,dr. \tag{31}$$

Then the following statements are true.

(a) *If* $\mathfrak{f}$ *is decreasing on* $[\lambda_1, \xi_1]$, *then inequality (20) holds.*

(b) *If* $\mathfrak{g}$ *is increasing on* $[\lambda_1, \xi_1]$, *then inequality (21) holds.*

Proof. We clearly see that it is sufficient to prove the case of $\varphi(r) = r$, but this case is already proved in Theorem 6. □

Similarly, we have Theorem 14 as follows.

Theorem 14. *Suppose that all the assumptions of Theorem 13 are satisfied. Then the following statements are true.*

(a) *If* $\mathfrak{f}$ *is decreasing on* $[\lambda_1, \xi_1]$, *then inequality (24) holds.*

(b) *If* $\mathfrak{g}$ *is increasing on* $[\lambda_1, \xi_1]$, *then inequality (25) holds.*

The following Lemma 15 was given in [57].

Lemma 15. *Let* χ *be a positive integrable function and let* $\hbar$ *be an increasing function on* (λ_1, ξ_1); *then*

$$\begin{aligned}&\int_{\lambda_1}^{x} \hbar(r)\,\chi(r)\,dr\int_{\lambda_1}^{\xi_1}\chi(r)\,dr \\ &\leq \int_{\lambda_1}^{\xi_1} \hbar(r)\,\chi(r)\,dr\int_{\lambda_1}^{x}\chi(r)\,dr.\end{aligned} \tag{32}$$

If $\hbar$ *is a decreasing function on* (λ_1, ξ_1), *then inequality (32) holds in reverse directions.*

Lemma 16. *Let* $\mathfrak{f}$ *be a real-valued function defined on* $[\lambda_1, \xi_1]$. *Then the following statements are true.*

(i) *If* $\mathfrak{f}$ *is a strongly concave function with modulus c, then*

(a) *the function* $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ *is decreasing on* $(\lambda_1, \xi_1]$;

(b) *the function* $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ *is increasing on* $[\lambda_1, \xi_1)$.

(ii) *If $\mathfrak{f}$ is a strongly convex function with modulus c, then*

(c) *the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1)-cr$ is increasing on $(\lambda_1, \xi_1]$ if $\mathfrak{f}(\lambda_1) = 0$;*

(d) *the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1-r)+cr$ is decreasing on $[\lambda_1, \xi_1)$ if $\mathfrak{f}(\xi_1) = 0$.*

Proof. (i) Suppose that $\mathfrak{f}$ is a strongly concave function with modulus c.

(a) To show that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ is decreasing on $(\lambda_1, \xi_1]$, in fact, for $\lambda_1 < r_1 \le r_2 \le \xi_1$ we have

$$\begin{aligned}\mathfrak{f}(r_1) &= \mathfrak{f}\left(\frac{r_1-\lambda_1}{r_2-\lambda_1}r_2 + \frac{r_2-\lambda_1-(r_1-\lambda_1)}{r_2-\lambda_1}\lambda_1\right)\\ &\ge \frac{r_1-\lambda_1}{r_2-\lambda_1}\mathfrak{f}(r_2) + \left(1-\frac{r_1-\lambda_1}{r_2-\lambda_1}\right)\mathfrak{f}(\lambda_1)\\ &\quad - c\left(\frac{r_1-\lambda_1}{r_2-\lambda_1}\right)\left(1-\frac{r_1-\lambda_1}{r_2-\lambda_1}\right)(r_2-\lambda_1)^2\\ &\ge \frac{r_1-\lambda_1}{r_2-\lambda_1}\mathfrak{f}(r_2) - c(r_1-\lambda_1)(r_2-r_1),\end{aligned} \tag{33}$$

which shows that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ is decreasing on $(\lambda_1, \xi_1]$.

(b) To show that the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ is increasing on $[\lambda_1, \xi_1)$, in fact, for $\lambda_1 \le r_1 \le r_2 < \xi_1$ we have

$$\begin{aligned}\mathfrak{f}(r_2) &= \mathfrak{f}\left(\frac{\xi_1-r_2}{\xi_1-r_1}r_1 + \frac{\xi_1-r_1-(\xi_1-r_2)}{\xi_1-r_1}\xi_1\right)\\ &\ge \frac{\xi_1-r_2}{\xi_1-r_1}\mathfrak{f}(r_1) + \left(1-\frac{\xi_1-r_2}{\xi_1-r_1}\right)\mathfrak{f}(\xi_1)\\ &\quad - c_1\left(\frac{\xi_1-r_2}{\xi_1-r_1}\right)\left(1-\frac{\xi_1-r_2}{\xi_1-r_1}\right)(r_1-\xi_1)^2\\ &\ge \frac{\xi_1-r_2}{\xi_1-r_1}\mathfrak{f}(r_1) - c_1(\xi_1-r_2)(r_2-r_1),\end{aligned} \tag{34}$$

which shows that the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ is increasing on $[\lambda_1, \xi_1)$.

(ii) Suppose that $\mathfrak{f}$ is a strongly convex function with modulus c.

(c) Since $\mathfrak{f}(\lambda_1) = 0$, by similar method of (a) we can easily prove that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ is increasing on $(\lambda_1, \xi_1]$.

(d) Since $\mathfrak{f}(\xi_1) = 0$, by similar method of (b) we can prove that the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ is decreasing on $(\lambda_1, \xi_1]$. □

Next, we establish several Favard type inequalities for strongly convex functions.

Theorem 17. (a) *Let $\mathfrak{f}$ be a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ is a positive increasing function, let Ψ be a strongly convex function with modulus c_2 on $[0, 2\bar{\mathfrak{f}}_1]$, $\bar{z}_1 = \lambda_1(1-s) + \xi_1 s$, and*

$$\bar{\mathfrak{f}}_1 = \frac{(\xi_1-\lambda_1)\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}{2\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\,\Omega(r)\,dr}. \tag{35}$$

Then

$$\begin{aligned}\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr &\le \int_0^1\Psi\left(2s\bar{\mathfrak{f}}_1\right)\\ &\quad\cdot\Omega(\bar{z}_1)\,ds\\ &\quad - c_2\int_0^1\left\{\left[\mathfrak{f}(\bar{z}_1) - c_1\bar{z}_1(s\xi_1 - s\lambda_1)\right] - 2s\bar{\mathfrak{f}}_1\right\}^2\\ &\quad\cdot\Omega(\bar{z}_1)\,ds.\end{aligned} \tag{36}$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ is a positive increasing function and $\mathfrak{f}(\lambda_1) = 0$, then the reverse inequality in (36) holds.

(b) *Let $\mathfrak{f}$ be a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) + c_1 r(\xi_1 - r)$ is a positive decreasing function, let Ψ be a strongly convex function with modulus c_2 on $[0, 2\bar{\mathfrak{f}}_2]$, $\bar{z}_2 = \lambda_1 s + \xi_1(1-s)$, and*

$$\bar{\mathfrak{f}}_2 = \frac{(\xi_1-\lambda_1)\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}{2\int_{\lambda_1}^{\xi_1}(\xi_1-r)\,\Omega(r)\,dr}. \tag{37}$$

Then

$$\begin{aligned}\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr &\le \int_0^1\Psi\left(2s\bar{\mathfrak{f}}_2\right)\\ &\quad\cdot\Omega(\bar{z}_2)\,ds\\ &\quad - c_2\int_0^1\left\{\left[\mathfrak{f}(\bar{z}_2) + c_1\bar{z}_2(s\xi_1 - s\lambda_1)\right] - 2s\bar{\mathfrak{f}}_2\right\}^2\\ &\quad\cdot\Omega(\bar{z}_2)\,ds.\end{aligned} \tag{38}$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) + c_1 r(\xi_1 - r)$ is a positive decreasing function and $\mathfrak{f}(\xi_1) = 0$, then the reverse inequality in (38) holds.

Proof. (a) From Lemma 16(a) we know that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - c_1 r$ is decreasing; then using Lemma 15 to the functions $\chi(r) = (r-\lambda_1)\Omega(r)$ and $\hbar_1(r)$, we obtain

$$\begin{aligned}&\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr\int_{\lambda_1}^{x}(r-\lambda_1)\,\Omega(r)\,dr\\ &\quad\le\int_{\lambda_1}^{x}\mathfrak{g}(r)\,\Omega(r)\,dr\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\,\Omega(r)\,dr.\end{aligned} \tag{39}$$

It follows from (35) that inequality (39) can be rewritten as

$$\int_{\lambda_1}^{x}\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\Omega(r)\,dr \le \int_{\lambda_1}^{x}\mathfrak{g}(r)\,\Omega(r)\,dr \tag{40}$$

for all $x \in [\lambda_1, \xi_1]$.

As $\mathfrak{g}$ is an increasing function, and by use of Theorem 6(b), we have

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \leq \int_{\lambda_1}^{\xi_1} \Psi\left\{\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}\Omega(r)\,dr - c_2\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}^2\Omega(r)\,dr. \quad (41)$$

Note that

$$\begin{aligned}
&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\left\{\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}\Omega(r)\,dr-\frac{c_2}{\xi_1-\lambda_1}\\
&\cdot\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}^2\Omega(r)\,dr=\frac{1}{2\bar{\mathfrak{f}}_1}\\
&\cdot\int_0^{2\bar{\mathfrak{f}}_1}\Psi(y)\,\Omega\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)dy-\frac{c_2}{2\bar{\mathfrak{f}}_1}\\
&\cdot\int_0^{2\bar{\mathfrak{f}}_1}\left\{\left[\mathfrak{f}\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)\right.\right.\\
&\left.\left.-c_1\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)\left(y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)\right]-y\right\}^2\\
&\times\Omega\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)dy=\int_0^1\Psi\left(2s\bar{\mathfrak{f}}_1\right)\\
&\cdot\Omega\left[\lambda_1(1-s)+\xi_1 s\right]ds\\
&-c_2\int_0^1\left\{\left[\mathfrak{f}(\lambda_1(1-s)+\xi_1 s)\right.\right.\\
&\left.\left.-c_1(\lambda_1(1-s)+\xi_1 s)(s\xi_1-s\lambda_1)\right]-2s\bar{\mathfrak{f}}_1\right\}^2\\
&\times\Omega\left[\lambda_1(1-s)+\xi_1 s\right]ds.
\end{aligned} \quad (42)$$

Therefore, we get

$$\begin{aligned}
&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr\leq\int_0^1\Psi\left(2s\bar{\mathfrak{f}}_1\right)\\
&\cdot\Omega(\bar{z}_1)\,ds\\
&-c_2\int_0^1\left\{\left[\mathfrak{f}(\bar{z}_1)-c_1(\bar{z}_1)(s\xi_1-s\lambda_1)\right]-2s\bar{\mathfrak{f}}_1\right\}^2\\
&\cdot\Omega(\bar{z}_1)\,ds.
\end{aligned} \quad (43)$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)-c_1 r(r-\lambda_1)$ is a positive increasing function and $\mathfrak{f}(\lambda_1)=0$, then the reverse inequality in (36) can be proved by using a similar method as in the proof of part (a) and Lemma 16(c).

(b) From Lemma 16(b) we know that the function $\hbar_2(r)=\mathfrak{f}(r)/(\xi_1-r)+c_1 r$ is increasing; then using Lemma 15 to the functions $\chi(r)=(\xi_1-r)\Omega(r)$ and $\hbar_2(r)$, we obtain

$$\begin{aligned}
&\int_{\lambda_1}^{x}\mathfrak{g}(r)\,\Omega(r)\,dr\int_{\lambda_1}^{\xi_1}(\xi_1-r)\,\Omega(r)\,dr\\
&\quad\leq\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr\int_{\lambda_1}^{x}(\xi_1-r)\,\Omega(r)\,dr.
\end{aligned} \quad (44)$$

From (37) we clearly see that inequality (44) can be rewritten as

$$\int_{\lambda_1}^{x}\mathfrak{g}(r)\,\Omega(r)\,dr\leq\int_{\lambda_1}^{x}\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\Omega(r)\,dr \quad (45)$$

for all $x\in[\lambda_1,\xi_1]$.

As $\mathfrak{g}$ is decreasing function, and by using Theorem 6(a) we have

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \leq \int_{\lambda_1}^{\xi_1} \Psi\left\{\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}\Omega(r)\,dr - c_2\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}^2\Omega(r)\,dr. \quad (46)$$

Note that

$$\begin{aligned}
&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\left\{\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}\Omega(r)\,dr-\frac{c_2}{\xi_1-\lambda_1}\\
&\cdot\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}^2\Omega(r)\,dr=\frac{1}{2\bar{\mathfrak{f}}_2}\\
&\cdot\int_0^{2\bar{\mathfrak{f}}_2}\Psi(y)\,\Omega\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)dy-\frac{c_2}{2\bar{\mathfrak{f}}_2}\\
&\cdot\int_0^{2\bar{\mathfrak{f}}_2}\left\{\left[\mathfrak{f}\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)\right.\right.\\
&\left.\left.+c_1\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)\left(y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)\right]-y\right\}^2\\
&\times\Omega\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)dy=\int_0^1\Psi\left(2s\bar{\mathfrak{f}}_2\right)\Omega\left[\lambda_1 s\right.\\
&\left.+\xi_1(1-s)\right]ds-c_2\int_0^1\left\{\left[\mathfrak{f}(\lambda_1 s+\xi_1(1-s))\right.\right.\\
&\left.\left.+c_1(\lambda_1 s+\xi_1(1-s))(s\xi_1-s\lambda_1)\right]-2s\bar{\mathfrak{f}}_2\right\}^2\\
&\times\Omega\left[\lambda_1 s+\xi_1(1-s)\right]ds.
\end{aligned} \quad (47)$$

Therefore,

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_2\right)\cdot\Omega(\bar{z}_2)\,ds - c_2\int_0^1\left\{[\mathfrak{f}(\bar{z}_2)+c_1\bar{z}_2(s\xi_1-s\lambda_1)]-2s\bar{\mathfrak{f}}_2\right\}^2\cdot\Omega(\bar{z}_2)\,ds. \tag{48}$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)+c_1r(\xi_1-r)$ is a positive decreasing function and $\mathfrak{f}(\xi_1)=0$, then the reverse inequality in (38) can be proved by using a similar method as in the proof of part (b) and Lemma 16(d). □

Theorem 18. *The following statements are true under the assumptions of Theorem 17.*

(a) *If $\mathfrak{g}$ is a positive increasing function, then*

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_1\right)\cdot\Omega(\bar{z}_1)\,ds + c_2\int_0^1\left\{[\mathfrak{f}(\bar{z}_1)-c_1\bar{z}_1(s\xi_1-s\lambda_1)]^2-\left(2s\bar{\mathfrak{f}}_1\right)^2\right\}\cdot\Omega(\bar{z}_1)\,ds. \tag{49}$$

(b) *If $\mathfrak{g}$ is a positive decreasing function, then*

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_2\right)\cdot\Omega(\bar{z}_2)\,ds + c_2\int_0^1\left\{[\mathfrak{f}(\bar{z}_2)+c_1\bar{z}_2(s\xi_1-s\lambda_1)]^2-\left(2s\bar{\mathfrak{f}}_2\right)^2\right\}\cdot\Omega(\bar{z}_2)\,ds. \tag{50}$$

Proof. (a) We clearly see that $\Psi(r)-c_2r^2$ is convex function due to Ψ being a strongly convex function with modulus c_2. Therefore, inequality (49) follows easily from [57, Theorem 1(i)] and the strong convexity of $\Psi(r)-c_2r^2$ together with the fact that $\mathfrak{f}$ is a strongly concave function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)-c_1r(r-\lambda_1)$ is positive increasing function.

(b) Similarly, inequality (50) follows easily from [57, Theorem 1(ii)] and the strong convexity of $\Psi(r)-c_2r^2$ together with the fact that $\mathfrak{f}$ is a strongly concave function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)+c_1r(\xi_1-r)$ is positive decreasing function. □

Theorem 19. *Let $\mathfrak{g}(r)=\mathfrak{f}(r)-c_1r(r-\lambda_1)$ be an increasing function on $(0,1)$, let $\mathfrak{g}/h$ be a decreasing function on $(0,1)$, let $\mathfrak{g}$, h, and Ω be three positive functions on $(0,1)$, and let $\mathfrak{g}\Omega$ and $h\Omega$ be integrable on $(0,1)$ such that*

$$\phi=\frac{\int_0^1\mathfrak{g}(r)\Omega(r)\,dr}{\int_0^1 h(r)\Omega(r)\,dr}\ge 0. \tag{51}$$

And let Ψ be a strongly convex function with modulus c_2. Then the inequality holds

$$\int_0^1\Psi\{k\phi h(r)\}\Omega(r)\,dr \ge \int_0^1\Psi\{k\mathfrak{g}(r)\}\Omega(r)\,dr + c_2k^2\int_0^1\{\mathfrak{g}(r)-\phi h(r)\}^2\Omega(r)\,dr \tag{52}$$

for all $k>0$.

Proof. From $h>0$ and (51), applying Lemma 15 to the function $\chi(r)=h(r)\Omega(r)$ and the decreasing function $\hbar(r)=\mathfrak{g}(r)/h(r)$ we get

$$\int_0^x k\phi h(r)\Omega(r)\,dr \le \int_0^x k\mathfrak{g}(r)\Omega(r)\,dr. \tag{53}$$

Since $\mathfrak{g}$ is increasing, therefore by using Theorem 6 we have

$$\int_0^1\Psi\{k\phi h(r)\}\Omega(r)\,dr \ge \int_0^1\Psi\{k\mathfrak{g}(r)\}\Omega(r)\,dr + c_2k^2\int_0^1\{\mathfrak{g}(r)-\phi h(r)\}^2\Omega(r)\,dr. \tag{54}$$

□

Theorem 20. *Let Ψ be a strongly convex function with modulus c_2. Then the inequality*

$$\int_0^1\Psi\{k\phi h(r)\}\Omega(r)\,dr \ge \int_0^1\Psi\{k\mathfrak{g}(r)\}\Omega(r)\,dr + c_2k^2\int_0^1\left[(\phi h(r))^2-(\mathfrak{g}(r))^2\right]\Omega(r)\,dr. \tag{55}$$

holds for all $k>0$ if all the assumptions of Theorem 19 are satisfied.

Proof. We clearly see that $\Psi(r)-c_2r^2$ is a convex function due to Ψ being a strongly convex function with modulus c_2. Therefore, inequality (55) follows easily from [60, Theorem 3] and the convexity of the function $\Psi(r)-c_2r^2$. □

Remark 21. Clearly, [60, Theorem 3] can be deduced from (52) due to

$$\int_0^1 \{\mathfrak{g}(r) - \phi h(r)\}^2 \Omega(r)\, dr \geq 0 \tag{56}$$

or from (55) due to

$$\int_0^1 \left[(\phi h(r))^2 - (\mathfrak{g}(r))^2\right] \Omega(r)\, dr \geq 0 \tag{57}$$

for convex function $\Psi(r) = r^2$.

The following Theorem 22 is an extension of Theorem 19.

Theorem 22. *Let $\Psi : [0, \infty) \longrightarrow \mathbb{R}$ be a continuous strongly convex function with modulus c_2, $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r - \lambda_1)$ (c_1 is a nonnegative real number), h and Ω two positive integrable functions on $[\lambda_1, \xi_1]$, $z_1(r) = \mathfrak{g}(r)/\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\Omega(r)dr$, and $z_2(r) = h(r)/\int_{\lambda_1}^{\xi_1} h(r)\Omega(r)dr$. Then the following statements are true.*

(a) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_2(r))\,\Omega(r)\, dr \\ &\geq \int_{\lambda_1}^{\xi_1} \Psi(z_1(r))\,\Omega(r)\, dr \\ &\quad + c_2 \int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2\, \Omega(r)\, dr.\end{aligned} \tag{58}$$

(b) *If h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is increasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_1(r))\,\Omega(r)\, dr \\ &\geq \int_{\lambda_1}^{\xi_1} \Psi(z_2(r))\,\Omega(r)\, dr \\ &\quad + c_2 \int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2\, \Omega(r)\, dr.\end{aligned} \tag{59}$$

Proof. (a) Let $h > 0$; then applying Lemma 15 to the function $\chi(r) = h(r)\Omega(r)$ and the decreasing function $\hbar(r) = \mathfrak{g}(r)/h(r)$, we have

$$\int_{\lambda_1}^{x} z_2(r)\,\Omega(r)\, dr \leq \int_{\lambda_1}^{x} z_1(r)\,\Omega(r)\, dr. \tag{60}$$

Therefore, inequality (58) follows from Theorem 6 and the fact that $\mathfrak{g}$ is an increasing function on $[\lambda_1, \xi_1]$.

(b) Let $h > 0$; then applying Lemma 15 to the function $\chi(r) = h(r)\Omega(r)$ and the increasing function $\hbar(r) = \mathfrak{g}(r)/h(r)$, we get

$$\int_{\lambda_1}^{x} z_1(r)\,\Omega(r)\, dr \leq \int_{\lambda_1}^{x} z_2(r)\,\Omega(r)\, dr. \tag{61}$$

Therefore, inequality (59) follows from Theorem 6 and the fact that h is an increasing function on $[\lambda_1, \xi_1]$. □

Theorem 23. *The following statements are true under the assumptions of Theorem 22.*

(a) *If $\mathfrak{g}$ is an increasing function on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is a decreasing function on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_1(r))\,\Omega(r)\, dr \\ &\leq \int_{\lambda_1}^{\xi_1} \Psi(z_2(r))\,\Omega(r)\, dr \\ &\quad + c_2 \int_{\lambda_1}^{\xi_1} \left(z_1^2(r) - z_2^2(r)\right) \Omega(r)\, dr.\end{aligned} \tag{62}$$

(b) *If h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is an increasing function on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_2(r))\,\Omega(r)\, dr \\ &\leq \int_{\lambda_1}^{\xi_1} \Psi(z_1(r))\,\Omega(r)\, dr \\ &\quad + c_2 \int_{\lambda_1}^{\xi_1} \left(z_2^2(r) - z_1^2(r)\right) \Omega(r)\, dr.\end{aligned} \tag{63}$$

Proof. We clearly see that $\Psi(r) - c_2 r^2$ is a convex function due to Ψ being a strongly convex function with modulus c_2. Therefore, inequalities (62) and (63) follow from [61, Theorem 2.3] and the convexity of the function $\Psi(r) - c_2 r^2$. □

Remark 24. Clearly, Theorem 2.3(1) and Theorem 2.3(2) given in [57] can be deduced by (62) and (63), respectively.

Remark 25. Theorem 22 is an extension of Favard's inequality given in Theorem 17. Indeed, let $\Psi(r)$ be a strongly convex function with modulus c_2, then $\Psi(kr)$ is also a strongly convex function with modulus $k^2 c_2$ for any $k \in \mathbb{R}$. Substituting $h(r) = r - \lambda_1$ in (58), one has

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi\left[\mathfrak{f}(r) - c_1 r(r - \lambda_1)\right] \Omega(r)\, dr \\ &\leq \int_{\lambda_1}^{\xi_1} \Psi\left(\frac{\int_{\lambda_1}^{\xi_1}\left[\mathfrak{f}(r) - c_1 r(r - \lambda_1)\right]\Omega(r)\, dr}{\int_{\lambda_1}^{\xi_1}(r - \lambda_1)\,\Omega(r)\, dr}(r \right. \\ &\left.\left. - \lambda_1)\right)\Omega(r)\, dr - k^2 c_2 \int_{\lambda_1}^{\xi_1}\left(\left[\mathfrak{f}(r)\right.\right.\right.\end{aligned}$$

$$- c_1 r(r-\lambda_1)] - \frac{\int_{\lambda_1}^{\xi_1} [\mathfrak{f}(r) - c_1 r(r-\lambda_1)]\,\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\,\Omega(r)\,dr}(r-\lambda_1)\Bigg)^2 \cdot \Omega(r)\,dr. \tag{64}$$

Since $\mathfrak{f}$ is a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ is a positive increasing function, taking $k = 1$ and using (35) and $\overline{z}_1$ in (64), we obtain the Favard's inequalities given in Theorem 17.

Remark 26. From (64) we can easily obtain Remark 2.4 given in [61] due to

$$\int_{\lambda_1}^{\xi_1}\Bigg([\mathfrak{f}(r) - c_1 r(r-\lambda_1)] - \frac{\int_{\lambda_1}^{\xi_1}[\mathfrak{f}(r) - c_1 r(r-\lambda_1)]\,\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\,\Omega(r)\,dr}(r-\lambda_1)\Bigg)^2 \cdot \Omega(r)\,dr \geq 0. \tag{65}$$

For an application of Theorem 22, we get Corollary 27 as follows.

Corollary 27. *Let $r > 1$, $p \in (-\infty, 0) \cup (1, \infty)$, $\Psi(r) = r^p$, and Ω, $\mathfrak{g}$, h, z_1, and z_2 be stated as in Theorem 22. Then the following statements are true.*

(a) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\left(\frac{\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}h(r)\,\Omega(r)\,dr}\right)^p \geq \frac{\int_{\lambda_1}^{\xi_1}\mathfrak{g}^p(r)\,\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}h^p(r)\,\Omega(r)\,dr} + c_2 \cdot \frac{\left(\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr\right)^p}{\int_{\lambda_1}^{\xi_1}h^p(r)\,\Omega(r)\,dr} \cdot \int_{\lambda_1}^{\xi_1}(z_1(r) - z_2(r))^2\,\Omega(r)\,dr. \tag{66}$$

(b) *If h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is an increasing function on $[\lambda_1, \xi_1]$, then*

$$\frac{\int_{\lambda_1}^{\xi_1}\mathfrak{g}^p(r)\,\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}h^p(r)\,\Omega(r)\,dr} \geq \left(\frac{\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}h(r)\,\Omega(r)\,dr}\right)^p + c_2 \cdot \frac{\left(\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr\right)^p}{\int_{\lambda_1}^{\xi_1}h^p(r)\,\Omega(r)\,dr} \cdot \int_{\lambda_1}^{\xi_1}(z_1(r) - z_2(r))^2\,\Omega(r)\,dr. \tag{67}$$

Proof. (a) Since $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is a decreasing function on $[\lambda_1, \xi_1]$, using (58) given in Theorem 22 and substituting $\Psi(r) = r^p$, we get

$$\begin{aligned}
&\int_{\lambda_1}^{\xi_1} z_2^p(r)\,\Omega(r)\,dr \geq \int_{\lambda_1}^{\xi_1} z_1^p(r)\,\Omega(r)\,dr + c_2\int_{\lambda_1}^{\xi_1}(z_1(r) - z_2(r))^2\,\Omega(r)\,dr, \\
&\int_{\lambda_1}^{\xi_1}\left(\frac{h(r)}{\int_{\lambda_1}^{\xi_1}h(r)\,\Omega(r)\,dr}\right)^p\Omega(r)\,dr \geq \int_{\lambda_1}^{\xi_1}\left(\frac{\mathfrak{g}(r)}{\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}\right)^p\Omega(r)\,dr + c_2\int_{\lambda_1}^{\xi_1}(z_1(r) - z_2(r))^2\,\Omega(r)\,dr.
\end{aligned} \tag{68}$$

Therefore, inequality (66) follows from (68).

(b) Since h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is an increasing function on $[\lambda_1, \xi_1]$, using (59) given in Theorem 22 and substituting $\Psi(r) = r^p$ we get inequality (67). □

Remark 28. Let $h(r) = r - \lambda_1$, $\Omega(r) = 1$, and $\mathfrak{f}$ be a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ is a positive increasing function. Then inequality (66) leads to the classical Favard's inequality for strongly convex functions with modulus c_2:

$$\frac{2^p}{p+1}\left(\frac{1}{\xi_1 - \lambda_1}\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,dr\right)^p \geq \frac{1}{\xi_1 - \lambda_1} \cdot \int_{\lambda_1}^{\xi_1}\mathfrak{g}^p(r)\,dr + \frac{c_2}{\xi_1 - \lambda_1}\left(\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,dr\right)^p \cdot \int_{\lambda_1}^{\xi_1}\left(\frac{\mathfrak{g}(r)}{\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,dr} - \frac{2(r-\lambda_1)}{(\xi_1 - \lambda_1)^2}\right)^2 dr. \tag{69}$$

Remark 29. From (69) we get the classical Favard's inequality given in [58] due to

$$\left(\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,dr\right)^p \cdot \int_{\lambda_1}^{\xi_1}\left(\frac{\mathfrak{g}(r)}{\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,dr} - \frac{2(r-\lambda_1)}{(\xi_1 - \lambda_1)^2}\right)^2 dr \geq 0. \tag{70}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Bivariate Chlodowsky-Stancu Variant of (p,q)-Bernstein-Schurer Operators

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In this study, it is proposed to define bivariate Chlodowsky variant of (p,q)-Bernstein-Stancu-Schurer operators. Therefore, Korovkin-type approximation theorems and the error of approximation by using full modulus of continuity are presented. Beside this, we introduce a generalization of the bivariate Chlodowsky variant of (p,q)-Bernstein-Stancu-Schurer operators and investigate its approximation in more general weighted space. Moreover, the numerical results are discussed in order to validate the accuracy of the bivariate Chlodowsky variant of (p,q)-Bernstein-Schurer operators.

1. Introduction

In this area, many operators about q-integer and (p,q)-integer [1–25] were studied. It may not be possible to find the exact solution for the models developed for the engineering fields because of its mathematical intractability. Therefore, these operators can be used effectively to find an algorithm for approximating solutions [26].

In 2014, Vedi and Ozarslan introduced the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators in [27] as

$$C_{n,p}^{(\alpha,\beta)}(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}b_n\right)\begin{bmatrix} n+p \\ k \end{bmatrix}_q\left(\frac{x}{b_n}\right)^k \cdot \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right), \tag{1}$$

where $\{b_n\}$ is increasing sequence of real numbers satisfying $\lim_{n\longrightarrow\infty} b_n$, $n\in\mathbb{N}$, $p\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}$, $0\le\alpha\le\beta$, $0\le x\le b_n$ and $0<q<1$ and investigated its approximation properties.

In 2017, Vedi and Ozarslan defined the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators in [28] by

$$C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) := \sum_{k=0}^{n+p}\sum_{j=0}^{m+p} f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) \cdot \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \tag{2}$$

where $\{a_n\}$ and $\{b_m\}$ are increasing sequences of real numbers satisfying $\lim_{n\longrightarrow\infty} a_n=\lim_{m\longrightarrow\infty} b_m=\infty$ and $n\in\mathbb{N}$, $p\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}$, $0\le\alpha\le\beta$, $\Phi_{k,n,q_n}(z)=\begin{bmatrix} n+p \\ k\end{bmatrix}_{q_n} z^k\prod_{s=0}^{n+p-k-1}(1-q_n^s z)$ and investigated its approximation properties on the rectangular unbounded domain.

Moreover, Gemikonakli and Vedi-Dilek [29] introduced the Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers as

$$\overline{C}_{n,s}(f;p,q;x) := \sum_{k=0}^{n+s} f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}b_n\right) \cdot p^{k(k-1)/2-(n+s-1)(n+s)/2}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q}\left(\frac{x}{b_n}\right)^k \cdot \prod_{j=0}^{n+s-k-1}\left(p^j - q^j\frac{x}{b_n}\right), \tag{3}$$

where , $s \in \mathbb{N}$, $0 \le x \le b_n$ and $0 < q < p \le 1$.

Let us discuss some well-known basic definitions of (p,q)-calculus. For $0 < q \le p < 1$, the (p,q)-numbers are given as [30]

$$[k]_{p,q} = \frac{p^k - q^k}{p-q}. \tag{4}$$

For each $k \in \mathbb{N}_0$ the (p,q)-factorial is represented by

$$[k]_{p,q}! = \begin{cases} [k][k-1]\dots[1], & k = 1,2,3,\dots, \\ 1, & k = 0 \end{cases} \tag{5}$$

and (p,q)-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, \tag{6}$$

where $n \ge k \ge 0$.

Then, Chlodowsky variant of (p,q)-Bernstein-Stancu-Schurer operators was constructed in [31] as

$$C_{n,m}^{(\alpha,\beta)}(f;x,p,q) := \frac{1}{p^{(n+m)(n+m-1)/2}}\sum_{k=0}^{n+m}\begin{bmatrix} n+p \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}\left(\frac{x}{b_n}\right)^k \times \prod_{s=0}^{n+m-k-1}\left(p^s - q^s\frac{x}{b_n}\right) f\left(\frac{p^{n+m-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n\right), \tag{7}$$

where $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}_0$ with $\alpha/\beta \approx 1$, $0 \le x \le b_n$, $0 < q < p \le 1$ and b_n has the same properties as the operator $C_{n,p}^{(\alpha,\beta)}(f;q;x)$.

Lemma 1. *Let $C_{n,m}^{(\alpha,\beta)}(f;x,p,q)$ be defined in [31]. Then the first few moments of the operators are*

$$\begin{aligned} &(i)\ C_{n,m}^{(\alpha,\beta)}(1;x,p,q) = 1, \\ &(ii)\ C_{n,m}^{(\alpha,\beta)}(t;x,p,q) = \frac{[n+m]_{p,q}x + \alpha b_n}{[n]_{p,q}+\beta}, \\ &(iii)\ C_{n,m}^{(\alpha,\beta)}(t^2;x,p,q) \\ &\quad = \frac{q[n+m]_{p,q}[n+m-1]_{p,q}x^2}{\left([n]_{p,q}+\beta\right)^2} + \frac{[n+m]_{p,q}\left(2\alpha + p^{n+m-1}\right)b_n x + \alpha^2 b_n^2}{\left([n]_{p,q}+\beta\right)^2}. \end{aligned} \tag{8}$$

The bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p,q)-integers is defined and then the first few moments of the operator are provided in Section 2. Next, in Section 3, some Korovkin-type theorems are studied. And following this, the order of convergence of the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p,q)-integers by means of the first modulus of continuity is obtained in Section 4. Moreover, in Section 5, we study the generalization of the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p,q)-integers and seek its approximation properties in more general weighted space. Finally, in Section 6, numerical results for the operators constructed are provided in detail.

2. Construction of the Operators

Let $\mathbb{D}_{a_n,b_m}$ denote

$$\mathbb{D}_{a_n,b_m} = \{(x,y) : 0 \le x \le a_n,\ 0 \le y \le b_m\} \tag{9}$$

and $\{a_n\}$ and $\{b_m\}$ are increasing sequences of real numbers satisfying

$$\lim_{n\to\infty} a_n = \lim_{m\to\infty} b_m = \infty, \qquad \lim_{n\to\infty}\frac{a_n}{[n]_{p_n,q_n}} = \lim_{m\to\infty}\frac{b_m}{[m]_{p_m,q_m}} = 0. \tag{10}$$

For $(x,y) \in \mathbb{D}_{a_n,b_m}$, we construct the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p,q)-integers as

$$\begin{aligned} \overline{C}_{n,m,s}^{(\alpha,\beta)}(f;p_n,q_n,p_m,q_m;x,y) &:= \frac{1}{p_n^{(n+s)(n+s-1)/2}} \cdot \frac{1}{p_m^{(m+s)(m+s-1)/2}}\sum_{k=0}^{n+s}\sum_{l=0}^{m+s} f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n, \frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m\right) \times \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \\ &\quad \cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right), \end{aligned} \tag{11}$$

where $\Phi_{k_n,p_n,q_n}(z) = \left[{n+s \atop k}\right]_{p_n,q_n} p_n^{l(l-1)/2}(z/a_n)^k \prod_{j=0}^{n+s-k-1}(p_n^j - q_n^j(z/a_n))$, $0 < q_n, q_m < p_n, p_m \le 1$, $0 \le \alpha \le \beta$ and $n, m \in \mathbb{N}$, $s \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Lemma 2. *Let $\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)$ be given in (11). Then, the first few moments of the operators are*

$$\begin{aligned}
&(i)\ \overline{C}_{n,m,s}^{(\alpha,\beta)}(1; p_n, q_n, p_m, q_m; x, y) = 1,\\
&(ii)\ \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1; p_n, q_n, p_m, q_m; x, y) = \frac{[n+s]_{p_n,q_n} x + \alpha a_n}{[n]_{p_n,q_n} + \beta},\\
&(iii)\ \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_2; p_n, q_n, p_m, q_m; x, y) = \frac{[m+s]_{p_m,q_m} y + \alpha b_m}{[m]_{p_m,q_m} + \beta},\\
&(iv)\ \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1^2 + t_2^2; p_n, q_n, p_m, q_m; x, y) = \frac{\left(q_n [n+s-1]_{p_n,q_n} [n+s]_{p_n,q_n} x^2 + [n+s]_{p_n,q_n} \left(2\alpha + p_n^{n+s-1}\right) a_n x + \alpha^2 a_n^2\right)}{\left([n]_{p_n,q_n} + \beta\right)^2}\\
&\quad + \frac{\left(q_m [m+s-1]_{p_m,q_m} [m+s]_{p_m,q_m} y^2 + [m+s]_{p_m,q_m} \left(2\alpha + p_m^{m+s-1}\right) b_m y + \alpha^2 b_m^2\right)}{\left([m]_{p_m,q_m} + \beta\right)^2}.
\end{aligned} \tag{12}$$

3. Korovkin-Type Approximation Theorems

In this section, Korovkin-type approximation theorems are given for the bivariate Chlodowsky variant of Bernstein–Stancu-Schurer operators based on (p,q)-integers. For fixed $\nu \ge 0$ consider the space C_{ρ^ν}, which consists of all continuous functions f satisfying the condition

$$\begin{aligned}
&|f(x,y)| \le M_f \rho^\nu(x,y),\\
&(x,y) \in [0,\infty) \times [0,\infty) := \mathbb{R}_+^2,\ \ \rho(x,y) = 1 + x^2 + y^2.
\end{aligned} \tag{13}$$

Obviously, C_{ρ^ν} is a linear normed space with the following norm:

$$\|f\|_{\rho^\nu} = \sup_{0 \le x,y < \infty} \frac{|f(x,y)|}{\rho^\nu(x,y)}. \tag{14}$$

Theorem 3. *Let the numbers A and B be any fixed positive real numbers.*

Let $\mathbb{D}_{A,B} = \{(x,y) : 0 \le x \le A,\ 0 \le y \le B\}$ with $0 < q_n, q_m < p_n, p_m < 1$, $\lim_{n\to\infty} q_n = \lim_{m\to\infty} q_m = \lim_{n\to\infty} p_n = \lim_{m\to\infty} p_m = 1$ and $\{a_n\}$ and $\{b_m\}$ be increasing sequences of positive real numbers that satisfy the following properties:

$$\begin{aligned}
&\lim_{n\to\infty} a_n = \lim_{m\to\infty} b_m = \infty,\\
&\lim_{n\to\infty} \frac{a_n}{[n]_{p_n,q_n}} = \lim_{m\to\infty} \frac{b_m}{[m]_{p_m,q_m}} = 0.
\end{aligned} \tag{15}$$

For all $f \in C(\mathbb{D}_{A,B})$, we obtain

$$\begin{aligned}
\lim_{n,m\to\infty} \max_{(x,y)\in\mathbb{D}_{A,B}} &\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)\right.\\
&\left. - f(x,y)\right| = 0.
\end{aligned} \tag{16}$$

Proof. Using Lemma 2, we have

$$\begin{aligned}
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}(1; p_n, q_n, p_m, q_m; \cdot, \cdot) - 1\right\|_{C(\mathbb{D}_{A,B})} = 0\\
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1; p_n, q_n, p_m, q_m; \cdot, \cdot) - x\right\|_{C(\mathbb{D}_{A,B})}\\
&\quad \le A\left|\frac{[n+s]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1\right| + \frac{\alpha a_n}{[n]_{p_n,q_n} + \beta}\\
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}(t_2; p_n, q_n, p_m, q_m; \cdot, \cdot) - y\right\|_{C(\mathbb{D}_{A,B})}\\
&\quad \le B\left|\frac{[m+s]_{p_m,q_m}}{[m]_{p_m,q_m} + \beta} - 1\right| + \frac{\alpha b_m}{[m]_{p_m,q_m} + \beta}.
\end{aligned} \tag{17}$$

And again using Lemma 2 we get

$$\begin{aligned}
&\overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1^2 + t_2^2; p_n, q_n, p_m, q_m; \cdot, \cdot) - (x^2 + y^2)\\
&\quad = \frac{q_n [n+s-1]_{p_n,q_n} [n+s]_{p_n,q_n}}{\left([n]_{p_n,q_n} + \beta\right)^2} x^2\\
&\quad + \frac{[n+s]_{p_n,q_n}\left(2\alpha + p_n^{n+s-1}\right)}{\left([n]_{p_n,q_n} + \beta\right)^2} a_n x\\
&\quad + \frac{\alpha^2 a_n^2}{\left([n]_{p_n,q_n} + \beta\right)^2}\\
&\quad + \frac{q_m [m+s-1]_{p_m,q_m} [m+s]_{p_m,q_m}}{\left([m]_{p_m,q_m} + \beta\right)^2} y^2
\end{aligned}$$

$$+\frac{[m+s]_{p_m,q_m}\left(2\alpha+p_m^{m+s-1}\right)}{\left([m]_{p_m,q_m}+\beta\right)^2}b_m y$$
$$+\frac{\alpha^2 b_m^2}{\left([m]_{p_m,q_m}+\beta\right)^2}. \tag{18}$$

Finally, from the above equality, we obtain

$$\begin{aligned}
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(t_1^2+t_2^2;p_n,q_n,p_m,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(\mathbb{D}_{A,B})}\\
&\quad\le\left|\frac{q_n[n+s-1]_{p_n,q_n}[n+s]_{p_n,q_n}}{\left([n]_{p_n,q_n}+\beta\right)^2}\right|A^2\\
&\qquad+\frac{[n+s]_{p_n,q_n}\left(2\alpha+p_n^{n+s-1}\right)}{\left([n]_{p_n,q_n}+\beta\right)^2}a_n A\\
&\qquad+\frac{\alpha^2 a_n^2}{\left([n]_{p_n,q_n}+\beta\right)^2}\\
&\qquad+\left|\frac{q_m[m+s-1]_{p_m,q_m}[m+s]_{p_m,q_m}}{\left([m]_{p_m,q_m}+\beta\right)^2}\right|B^2\\
&\qquad+\frac{[m+s]_{p_m,q_m}\left(2\alpha+p_m^{m+s-1}\right)}{\left([m]_{p_m,q_m}+\beta\right)^2}b_m B\\
&\qquad+\frac{\alpha^2 b_m^2}{\left([m]_{p_m,q_m}+\beta\right)^2}.
\end{aligned}\tag{19}$$

Therefore, from the hypothesis of the theorem, we get

$$\begin{aligned}
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(t_1;p_n,q_n,p_m,q_m;\cdot,\cdot\right)-x\right\|_{C(\mathbb{D}_{A,B})}\longrightarrow 0\\
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(t_2;p_n,q_n,p_m,q_m;\cdot,\cdot\right)-y\right\|_{C(\mathbb{D}_{A,B})}\longrightarrow 0\\
&\left\|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(t_1^2+t_2^2;p_n,q_n,p_m,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(\mathbb{D}_{A,B})}\\
&\quad\longrightarrow 0
\end{aligned}\tag{20}$$

when n and $m\longrightarrow\infty$.

Hence, the proof is completed by the two dimensional Korovkin theorem. □

Theorem 4 (see [13]). *There exists a sequence of positive operators $T_{n,m}$, acting from $C_\rho(\mathbb{R}_+^2)$ to $C_\rho(\mathbb{R}_+^2)$, satisfying the conditions*

$$\begin{aligned}
\lim_{n,m\longrightarrow\infty}\left\|T_{n,m}(1;\cdot,\cdot)-1\right\|_\rho&=0\\
\lim_{n,m\longrightarrow\infty}\left\|T_{n,m}(t_1;\cdot,\cdot)-x\right\|_\rho&=0\\
\lim_{n,m\longrightarrow\infty}\left\|T_{n,m}(t_2;\cdot,\cdot)-y\right\|_\rho&=0\\
\lim_{n,m\longrightarrow\infty}\left\|T_{n,m}\left(t_1^2+t_2^2;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_\rho&=0
\end{aligned}\tag{21}$$

and there exists a function $f^*\in C_\rho$ for which

$$\lim_{n,m\longrightarrow\infty}\left\|T_{n,m}f^*-f^*\right\|_\rho\ge\frac{1}{4},\tag{22}$$

where $\rho=1+x^2+y^2$.

Now, consider the following operator:

$$T_{n,m}\left(f;p_n,q_n,p_m,q_m;x,y\right)=\begin{cases}\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(f;p_n,q_n,p_m,q_m;x,y\right), & (x,y)\in\mathbb{D}_{a_n,b_n}\\ f(x,y), & \mathbb{R}_+^2\setminus\mathbb{D}_{a_n,b_n}\end{cases}\tag{23}$$

Theorem 5. *Let $f\in C_\rho(\mathbb{R}_+^2)$ for any $\gamma>0$*

$$\lim_{n,m\longrightarrow\infty}\left\|T_{n,m}\left(f;p_n,q_n,p_m,q_m;\cdot,\cdot\right)-f(\cdot)\right\|_{C_{\rho^{1+\gamma}}}=0\tag{24}$$

where $\{a_n\}$, $\{b_m\}$, $\{q_n\}$, and $\{q_m\}$ have the same conditions as in Theorem 3.

Proof. For all $\varepsilon>0$, there exist sufficiently large positive real numbers A and B such that

$$\left(1+x^2+y^2\right)^{-\gamma}<\varepsilon\tag{25}$$

when $x>A$ and $y>B$.

Let n, m be sufficiently large so that $\mathbb{D}_{A,B}\subset\mathbb{D}_{a_n,b_m}$

$$\begin{aligned}
&\left\|T_{n,m}\left(f;p_n,q_n,p_m,q_m;\cdot,\cdot\right)-f(\cdot)\right\|_{C_{\rho^{1+\gamma}}}\\
&\quad\le\sup_{(x,y)\in D_{A,B}}\frac{\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(f;p_n,q_n,p_m,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\
&\qquad+\sup_{(x,y)\in D_{a_n,b_n}\setminus D_{A,B}}\frac{\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(f;p_n,q_n,p_m,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\
&\quad=y'_{n,m}+y''_{n,m}.
\end{aligned}\tag{26}$$

By Theorem 3, $\lim_{n,m\longrightarrow\infty} y'_{n,m} = 0$ and for the proof of the second term we have

$$y''_{n,m} \le \left(1+x^2+y^2\right)^{-\gamma} \cdot \left(\frac{\left|\overline{C}^{(\alpha,\beta)}_{n,m,s}\left(f; p_n, q_n, p_m, q_m; x, y\right)\right|}{1+x^2+y^2} + \frac{\left|f\left(x,y\right)\right|}{1+x^2+y^2} \right). \tag{27}$$

Finally, since $f \in C_\rho\ (\mathbb{R}^2_+)$, the term $|f(x,y)|/(1+x^2+y^2)$ is bounded. Furthermore, because of the fact that

$$\begin{aligned} &\left|\overline{C}^{(\alpha,\beta)}_{n,m,s}\left(f; p_n, q_n, p_m, q_m; x, y\right)\right| \\ &\quad \le \left|C^{(\alpha,\beta)}_{n,m,s}\left(1+t_1^2+t_2^2; q_n, q_m; x, y\right)\right|, \end{aligned} \tag{28}$$

in case Lemma 2 is used, the term $|\overline{C}^{(\alpha,\beta)}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)|/(1+x^2+y^2)$ is bounded for sufficiently large n and m. Hence, we get by (25) that

$$y''_{n,m} \le \varepsilon\left(1+M\right) \tag{29}$$

Since $\varepsilon > 0$ is arbitrary, then $\lim_{n,m\longrightarrow\infty} y''_{n,m} = 0$. This completes the proof. □

Now, consider the subspace C^0_ρ of C_ρ which is defined by

$$C^0_\rho := \left\{ f \in C_\rho : \lim_{x,y\longrightarrow 0} \frac{\left|f\left(x,y\right)\right|}{1+x^2+y^2} = 0 \right\}. \tag{30}$$

Theorem 6. *Let the sequences $\{q_n\}$, $\{q_m\}$, $\{a_n\}$, and $\{b_m\}$ satisfy the same properties as in Theorem 3. Then for all $f \in C^0_\rho\ (\mathbb{R}^2_+)$, we obtain*

$$\begin{aligned} &\lim_{n,m\longrightarrow\infty} \left\| T_{n,m}\left(\overline{C}^{(\alpha,\beta)}_{n,m,s}\left(f; p_n, q_n, p_m, q_m; \cdot, \cdot\right)\right) - f\left(\cdot\right) \right\|_{C_\rho} \\ &\quad = 0. \end{aligned} \tag{31}$$

Proof. For all $f \in C^0_\rho\ (\mathbb{R}^2_+)$, observe that

$$\begin{gathered} \lim_{x,y\longrightarrow\infty} \frac{\left|f\left(x,y\right)\right|}{1+x^2+y^2} = 0, \\ \lim_{n,m\longrightarrow\infty} \frac{\left| f\left(\left(\left(p_n^{n+s-k}\left[k\right]_{p_n,q_n} + \alpha\right) / \left(\left[n\right]_{p_n,q_n} + \beta\right) \right) a_n, \left(\left(p_m^{m+s-l}\left[l\right]_{p_m,q_m} + \alpha\right) / \left(\left[m\right]_{p_m,q_m} + \beta\right) \right) b_m \right) \right|}{1 + \left(\left(\left(p_n^{n+s-k}\left[k\right]_{p_n,q_n} + \alpha\right) / \left(\left[n\right]_{p_n,q_n} + \beta\right) \right) a_n \right)^2 + \left(\left(\left(p_m^{m+s-l}\left[l\right]_{p_m,q_m} + \alpha\right) / \left(\left[m\right]_{p_m,q_m} + \beta\right) \right) b_m \right)^2} = 0. \end{gathered} \tag{32}$$

Therefore, for all $\varepsilon > 0$, we can find sufficiently large numbers A and B such that

$$\left|f\left(x,y\right)\right| < \varepsilon\left(1+x^2+y^2\right) \tag{33}$$

for $x > A$ and $y > B$ and there exist natural numbers n_0 and m_0 such that

$$\begin{aligned} &\left| f\left(\frac{p_n^{n+s-k}\left[k\right]_{p_n,q_n} + \alpha}{\left[n\right]_{p_n,q_n} + \beta} a_n, \frac{p_m^{m+s-l}\left[l\right]_{p_m,q_m} + \alpha}{\left[m\right]_{p_m,q_m} + \beta} b_m \right) \right| \\ &\quad < \varepsilon \left(1 + \left(\frac{p_n^{n+s-k}\left[k\right]_{p_n,q_n} + \alpha}{\left[n\right]_{p_n,q_n} + \beta} a_n \right)^2 + \left(\frac{p_m^{m+s-l}\left[l\right]_{p_m,q_m} + \alpha}{\left[m\right]_{p_m,q_m} + \beta} b_m \right)^2 \right) \end{aligned} \tag{34}$$

for all $n > n_0$ and $m > m_0$.

Hence, for large n and m, we have

$$\begin{aligned} &\left\|T_{n,m}\left(f; p_n, q_n, p_m, q_m; \cdot, \cdot\right) - f\left(\cdot\right)\right\|_{C_\rho} \\ &\quad \le \sup_{(x,y)\in\mathbb{D}_{A,B}} \frac{\left|\overline{C}^{(\alpha,\beta)}_{n,m,s}\left(f; p_n, q_n, p_m, q_m; x, y\right) - f\left(x,y\right)\right|}{1+x^2+y^2} \\ &\quad + \sup_{(x,y)\in\mathbb{D}_{a_n,b_m}\setminus\mathbb{D}_{A,B}} \frac{\left|\overline{C}^{(\alpha,\beta)}_{n,m,s}\left(f; p_n, q_n, p_m, q_m; x, y\right) - f\left(x,y\right)\right|}{1+x^2+y^2} \\ &\quad = z'_{n,m} + z''_{n,m}. \end{aligned} \tag{35}$$

By Theorem 3, it is sufficient to show that $z''_{n,m} \longrightarrow 0$ as $n \longrightarrow \infty$.

Using (33) and (34), we get

$$\begin{aligned} &z''_{n,m} \\ &\quad \le \varepsilon + \sup_{(x,y)\in\mathbb{D}_{a_n,b_m}\setminus\mathbb{D}_{A,B}} \frac{\left|\overline{C}^{(\alpha,\beta)}_{n,m,s}\left(f; p_n, q_n, p_m, q_m; x, y\right))\right|}{1+x^2+y^2} \\ &\quad \le \varepsilon + \varepsilon \sup_{(x,y)\in\mathbb{D}_{a_n,b_m}\setminus\mathbb{D}_{A,B}} t_{n,m}\left(q_n, q_m; x, y\right) \\ &\quad = \varepsilon\left(1 + \sup_{(x,y)\in\mathbb{D}_{a_n,b_m}\setminus\mathbb{D}_{A,B}} t_{n,m}\left(q_n, q_m; x, y\right) \right) \end{aligned} \tag{36}$$

where

$$t_{n,m}\left(q_n,q_m;x,y\right):=\frac{\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(1;p_n,q_n,p_m,q_m;x,y\right)+\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(t_1^2;p_n,q_n,p_m,q_m;x,y\right)+\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(t_2^2;p_n,q_n,p_m,q_m;x,y\right)}{1+x^2+y^2}. \quad (37)$$

By Lemma 2, it is clear that there exist K independent of n and m such that

$$\sup_{(x,y)\in\mathbb{D}_{a_n,b_m}/\mathbb{D}_{A,B}} t_{n,m}\left(p_n,q_n,p_m,q_m;x,y\right)\le K. \quad (38)$$

Therefore, for $n>n_0$ and $m>m_0$ we have

$$z''_{n,m}<(1+K)\,\varepsilon. \quad (39)$$

This completes the proof. □

4. Order of Convergence

In this section, we compute the rate of convergence of the operators in terms of the full modulus of continuity and partial modulus of continuities.

Let $f\in\mathbb{D}_{A,B}$ and $x\ge 0$. Then the definition of the modulus of continuity of f is given by

$$\omega\left(f;\delta\right)=\max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\le\delta\\ x,y\in C(\mathbb{D}_{A,B})}}\left|f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right|. \quad (40)$$

It is known that [32] for any $\delta>0$ we know that

$$\left|f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right|\le\omega\left(f,\delta\right)\left(\frac{\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}}{\delta}+1\right). \quad (41)$$

Also, for any $\delta>0$ we have

$$\begin{aligned}\left|f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right|&\le\omega^{(1)}\left(f,\delta\right)\left(\frac{\left|x_1-x_2\right|}{\delta}+1\right),\\ \left|f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right|&\le\omega^{(2)}\left(f,\delta\right)\left(\frac{\left|y_1-y_2\right|}{\delta}+1\right).\end{aligned} \quad (42)$$

Theorem 7. *For any $f\in C(\mathbb{D}_{A,B})$, the following inequalities*

$$\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(f;p_n,q_n,p_m,q_m;x,y\right)-f\left(x,y\right)\right|\le 2\left[\omega^{(1)}\left(f;\delta_m\right)+\omega^{(2)}\left(f;\delta_n\right)\right] \quad (43)$$

$$\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(f;p_n,q_n,p_m,q_m;x,y\right)-f\left(x,y\right)\right|\le 2\omega\left(f;\sqrt{\delta_m^2+\delta_n^2}\right) \quad (44)$$

are satisfied, where

$$\delta_n^2:=\left|\frac{q_n\left[n+s-1\right]_{p_n,q_n}\left[n+s\right]_{p_n,q_n}}{\left(\left[n\right]_{p_n,q_n}+\beta\right)^2}\right|A^2+\frac{\left[n+s\right]_{p_n,q_n}\left(2\alpha+p_n^{n+s-1}\right)}{\left(\left[n\right]_{p_n,q_n}+\beta\right)^2}a_nA+\frac{\alpha^2a_n^2}{\left(\left[n\right]_{p_n,q_n}+\beta\right)^2}, \quad (45)$$

$$\delta_m^2:=\left|\frac{q_m\left[m+s-1\right]_{p_m,q_m}\left[m+s\right]_{p_m,q_m}}{\left(\left[m\right]_{p_m,q_m}+\beta\right)^2}\right|B^2+\frac{\left[m+s\right]_{p_m,q_m}\left(2\alpha+p_m^{m+s-1}\right)}{\left(\left[m\right]_{p_m,q_m}+\beta\right)^2}b_mB+\frac{\alpha^2b_m^2}{\left(\left[m\right]_{p_m,q_m}+\beta\right)^2}. \quad (46)$$

Proof. We directly have

$$\begin{aligned}&\overline{C}_{n,m,s}^{(\alpha,\beta)}\left(f;p_n,q_n,p_m,q_m;x,y\right)-f\left(x,y\right)\\ &=\sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\left[f\left(\frac{p_n^{n+s-k}\left[k\right]_{p_n,q_n}+\alpha}{\left[n\right]_{p_n,q_n}+\beta}a_n,\frac{p_m^{m+s-l}\left[l\right]_{p_m,q_m}+\alpha}{\left[m\right]_{p_m,q_m}+\beta}b_m\right)-f\left(x,y\right)\right]\\ &\quad\times\Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right)\\ &=\sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\left[f\left(\frac{p_n^{n+s-k}\left[k\right]_{p_n,q_n}+\alpha}{\left[n\right]_{p_n,q_n}+\beta}a_n,\frac{p_m^{m+s-l}\left[l\right]_{p_m,q_m}+\alpha}{\left[m\right]_{p_m,q_m}+\beta}b_m\right)\right.\end{aligned}$$

$$- f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n, y\right) + f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n, y\right) - f(x,y)\Bigg] \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right). \quad (47)$$

By linearity and positivity of the operators, we get

$$\begin{aligned}
&\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f;p_n,q_n,p_m,q_m;x,y)-f(x,y)\right| \\
&\le \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\left|f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n,\ \frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m\right) - f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n, y\right)\right| \times \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \\
&\cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) + \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\left|f\left(\frac{[k]_{p_n,q_n}}{[n]_{p_n,q_n}}a_n, y\right)-f(x,y)\right| \\
&\cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
&\le \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\omega^{(2)}\left(f;\left|\frac{p_m^{m+s-k}[k]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m-y\right|\right) \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\omega^{(1)}\left(f;\left|\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n-x\right|\right) \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) = \Omega_1(x,y) \\
&+ \Omega_2(x,y).
\end{aligned} \quad (48)$$

Using Lemma 1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\Omega_1(x,y) \\
&= \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\omega^{(2)}\left(f;\left|\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m-y\right|\right) \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
&= \sum_{l=0}^{m+s}\omega^{(2)}\left(f;\left|\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m-y\right|\right) \cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \le \omega^{(2)}(f;\delta_m)\Bigg\{1 \\
&+ \frac{1}{\delta_m}\left[\sum_{j=0}^{m+p}\left(\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m-y\right)^2 \cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right)\right]^{1/2}\Bigg\}.
\end{aligned} \quad (49)$$

Then, we get

$$\Omega_1(x,y) \le 2\omega^{(2)}(f;\delta_m), \quad (50)$$

where we choose δ_m as in (46).

In the same way, we obtain

$$\Omega_2(x,y) \le 2\omega^{(1)}(f;\delta_n), \quad (51)$$

where δ_n is given in (45). Combining (50) and (51), we obtain (43).

Now, by using linearity and the monotonicity of the operators, and taking into account (40), we have

$$\begin{aligned}
&\left|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f;p_n,q_n,p_m,q_m;x,y)-f(x,y)\right| \\
&\le \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\omega\left(f;\sqrt{\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n-x\right)^2+\left(\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m-y\right)^2}\right)\times\Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
&\le \sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\left|f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n,\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m\right)-f(x,y)\right|\times\Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$\leq 1+\frac{1}{\delta}\sum_{k=0}^{n+s}\sum_{l=0}^{m+s}\omega\left(f;\sqrt{\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n-x\right)^2+\left(\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m-y\right)^2}\right)$$
$$\times\Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \tag{52}$$

Using (41) and the Cauchy-Schwartz inequality, we get (44). □

5. Generalization of the Bivariate Chlodowsky Variant of Bernstein-Schurer Operators Based on (p,q)-Integers

In this section, we introduce generalization of bivariate Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers. The generalized operators help us to approximate continuous functions defined on more general weighted spaces. Note that this kind of generalization was considered earlier for the Chlodowsky-Bernstein polynomials [12].

For $x \geq 0$, consider any continuous function $\omega(x,y) \geq 1$ and define

$$G_f(t,s)=f(t,s)\frac{1+t^2+s^2}{w(t,s)}. \tag{53}$$

Let us consider the generalization of the $\overline{C}_{n,m,s}^{(\alpha,\beta)}(f;p_n,q_n,p_m,q_m;x,y)$ as follows:

$$L_{n,p}^{\alpha,\beta}\left(\overline{C}_{n,m,s}^{(\alpha,\beta)}(f;p_n,q_n,p_m,q_m;x,y)\right)$$
$$=\begin{cases}\dfrac{w(x,y)}{1+x^2+y^2}\displaystyle\sum_{k=0}^{n+s}\sum_{l=0}^{m+s}G_f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n,\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m\right)\times\Phi_{k_n,p_n,q_n}\left(\dfrac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\dfrac{y}{b_m}\right), & (x,y)\in\mathbb{D}_{a_n,b_n},\\ f(x,y), & \mathbb{R}_+^2\setminus\mathbb{D}_{a_n,b_n}\end{cases} \tag{54}$$

where $(x,y)\in\mathbb{D}_{a_nb_m}$ and $\{a_n\}$ and $\{b_m\}$ have the same properties of bivariate Chlodowsky variant of Bernstein-Schurer operators based on (p,q)-integers.

Theorem 8. *For all continuous functions f satisfying $|f(x,y)| \leq M_f w(x,y)$, $x,y \geq 0$, and $\lim_{x,y\longrightarrow\infty}(f(x,y)/w(x,y))=0$, we have*

$$\lim_{n,m\longrightarrow\infty}\left\|L_{n,p}^{\alpha,\beta}(f;p_n,q_n,p_m,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w=0 \tag{55}$$

where $w(x,y)=1+x^2+y^2$.

Proof. Clearly,

$$\left|L_{n,p}^{\alpha,\beta}(f;p_n,q_n,p_m,q_m;x,y)-f(x,y)\right|$$
$$=\frac{w(x,y)}{1+x^2+y^2}\left|\sum_{k=0}^{n+s}\sum_{l=0}^{m+s}G_f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n}+\alpha}{[n]_{p_n,q_n}+\beta}a_n,\right.\right.$$
$$\left.\frac{p_m^{m+s-l}[l]_{p_m,q_m}+\alpha}{[m]_{p_m,q_m}+\beta}b_m\right) \tag{56}$$
$$\left.\times\Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right)-G_f(x,y)\right|,$$

thus

$$\left\|L_{n,p}^{\alpha,\beta}(f;p_n,q_n,p_m,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w$$
$$=\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|L_{n,p}^{\alpha,\beta}(f;p_n,q_n,p_m,q_m;x,y)-f(x,y)\right|}{w(x,y)} \tag{57}$$
$$=\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|T_{n,m}\left(G_f;p_n,q_n,p_m,q_m;x,y\right)-G_f(x,y)\right|}{1+x^2+y^2}.$$

Since $|f(x,y)| \leq M_f w(x,y)$, then $|G_f(x,y)| \leq M_f w(x,y)$ for $x,y \geq 0$ and $G_f(x,y)$ is continuous function on R_+^2. Furthermore, from $\lim_{x,y\longrightarrow\infty}(f(x,y)/w(x,y)) = 0$, we have

$$\lim_{x,y\longrightarrow\infty}\frac{G_f(x,y)}{\rho(x,y)}=0. \tag{58}$$

Thus, from Theorem 6 we get the result. □

Finally, note that taking $w(x,y) = 1+x^2+y^2$, then the operators $L_{n,p}^{\alpha,\beta}(f;p_n,q_n,p_m,q_m;x,y)$ reduce $T_{n,m}(G_f;p_n,q_n,p_m,q_m;x,y)$.

TABLE 1: $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ parameters used for numerical analysis.

Parameters	Values
n	20
m	5
a_n	$(n^2+4)/(n^2+10n)$
b_m	$(m^2+4)(m^2+10m)$
p_n	0.989
q_n	0.99
p_m	0.989
q_m	0.99
s	1
α, β	Variable

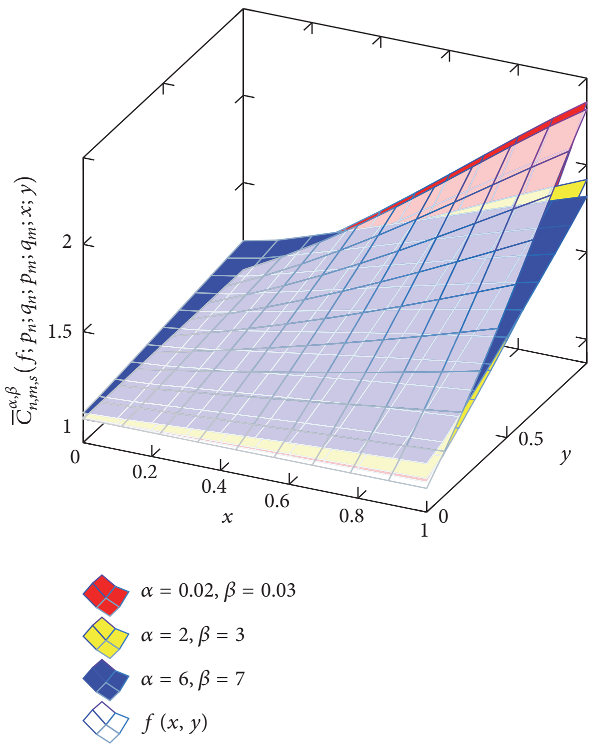

FIGURE 1: Convergence of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to multivariable function $f(x, y) = e^{\sin xy}$.

6. Numerical Results and Discussions

In this section, numerical results are presented in order to demonstrate the effectiveness and validate the accuracy of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to $f(x, y)$ with different values of parameters. The parameters used in the mathematical calculations are given in Table 1.

Figures 1 and 2 are given for $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ as a function of x and y for different α and β values considering two different multivariable functions $f(x, y) = e^{\sin xy}$ and $f(x, y) = x^2y^2$ accordingly.

Figure 1 demonstrates the convergence of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to the multivariable function $f(x, y) = e^{\sin xy}$. For different x and y values, various α and β values are taken into account. According to the results obtained, the $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ behaves like the function $f(x, y) = e^{\sin xy}$ for smaller α and β values. Therefore, the error of approximation of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to the function $f(x, y) = e^{\sin xy}$ is minimised for $\alpha = 0.02$ and $\beta = 0.03$, where s=1, n=20, and m=5 for the sequences a_n and b_m as provided in Table 1.

Figure 2 shows the convergence $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to the function $f(x, y) = x^2y^2$ but this time for different α and β values, while keeping all the other parameters constant. Unlike in Figure 1, increasing α and β values upto 0.3 and 0.5 respectively provided better approximate results. It is evident that, similar to Figure 1, α and β values have a significant effect on the convergence of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to the multivariable function $f(x, y) = x^2y^2$, which provide better approximate results.

Comparative results are given in Tables 2 and 3, for the errors of the approximation of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to the functions $f(x, y) = e^{\sin xy}$ and $f(x, y) = x^2y^2$.

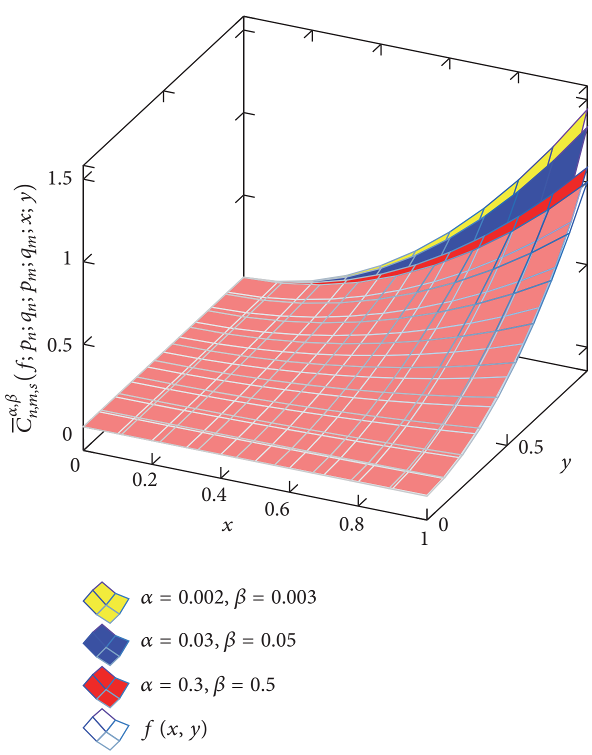

FIGURE 2: Convergence of $\overline{C}^{\alpha,\beta}_{n,m,s}(f; p_n, q_n, p_m, q_m; x, y)$ to multivariable function $f(x, y) = x^2y^2$.

Table 2: Errors of approximation $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to $f(x,y) = e^{\sin xy}$, $(|f(x,y) - \overline{C}_{n,m,s}^{0.02,0.03}(f; p_n, q_n, p_m, q_m; x, y)|)$.

x	y										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.0015	0.0039	0.0075	0.0112	0.0149	0.0186	0.0222	0.0258	0.0294	0.0329	0.0364
0.1	0.0044	0.0015	0.0022	0.0032	0.0041	0.0049	0.0056	0.0063	0.0069	0.0074	0.0078
0.2	0.0086	0.0023	0.0034	0.0049	0.0063	0.0075	0.0086	0.0096	0.0104	0.011	0.0116
0.3	0.0128	0.0033	0.0046	0.0066	0.0083	0.0099	0.0114	0.0128	0.0141	0.0153	0.0164
0.4	0.0169	0.0038	0.0051	0.0072	0.0092	0.0109	0.0127	0.0145	0.0163	0.0183	0.0206
0.5	0.0208	0.004	0.0048	0.0066	0.0082	0.0099	0.0117	0.0138	0.0164	0.0198	0.0243
0.6	0.0245	0.0035	0.0033	0.0042	0.0051	0.0061	0.0076	0.01	0.0137	0.0193	0.0274
0.7	0.028	0.0025	0.0006	0.0002	0.0009	0.0012	0.0004	0.0021	0.0074	0.0164	0.0303
0.8	0.0313	0.0007	0.0038	0.0071	0.0104	0.0128	0.0133	0.0105	0.0031	0.0108	0.0332
0.9	0.0342	0.002	0.0101	0.0171	0.0239	0.0294	0.0317	0.0288	0.0185	0.0023	0.0366
1	0.0368	0.0057	0.0184	0.0303	0.0421	0.0517	0.0566	0.0537	0.0393	0.0093	0.0407

Table 3: Errors of approximation $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to $f(x,y) = x^2 y^2$, $|f(x,y) - \overline{C}_{n,m,s}^{0.3,0.5}(f; p_n, q_n, p_m, q_m; x, y)|$.

x	y										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0	0.0001	0.0002	0.0004	0.0005	0.0007	0.001	0.0013	0.0017	0.0021
0.1	0.0001	0.0002	0.0004	0.0007	0.0012	0.0016	0.0022	0.0029	0.0037	0.0048	0.0066
0.2	0.0003	0.0005	0.001	0.0017	0.0025	0.0033	0.0043	0.0055	0.0069	0.0093	0.0134
0.3	0.0006	0.001	0.0019	0.0029	0.004	0.0052	0.0065	0.0079	0.0099	0.0136	0.0209
0.4	0.001	0.0017	0.003	0.0044	0.0058	0.0072	0.0086	0.0102	0.0127	0.0177	0.0289
0.5	0.0015	0.0024	0.0043	0.0061	0.0078	0.0093	0.0108	0.0124	0.0151	0.0217	0.0374
0.6	0.0021	0.0034	0.0058	0.0081	0.01	0.0116	0.0129	0.0143	0.0173	0.0256	0.0466
0.7	0.0028	0.0045	0.0075	0.0103	0.0124	0.014	0.0151	0.0162	0.0192	0.0292	0.0563
0.8	0.0036	0.0058	0.0094	0.0127	0.0151	0.0166	0.0173	0.0179	0.0209	0.0327	0.0665
0.9	0.0046	0.0072	0.0116	0.0154	0.018	0.0193	0.0195	0.0194	0.0222	0.0361	0.0773
1	0.0056	0.0087	0.014	0.0183	0.0211	0.0221	0.0217	0.0207	0.0233	0.0392	0.0887

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

New Generating Function Relations for the $q-$Generalized Cesàro Polynomials

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The main purpose of this paper is to examine a basic (or $q-$) analogue of the generalized Cesàro polynomials described here. We derive a bilateral $q-$generating function involving basic analogue of Fox's $H-$function and $q-$generalized Cesàro polynomials.

1. Introduction

The Cesàro polynomials $g_n^{(s)}(x)$ are defined by the generating relation ([1], p. 449, Problem 20):

$$\sum_{n=0}^{\infty} g_n^{(s)}(x)\, t^n = (1-t)^{-s-1}(1-xt)^{-1}. \tag{1}$$

It is from (1) that

$$g_n^{(s)}(x) = \binom{s+n}{n} {}_2F_1[-n, 1; -s-n; x], \tag{2}$$

where ${}_2F_1$ denotes Gauss's hypergeometric series.

Lin et al. [2] introduced the generalized Cesàro polynomials as follows:

$$g_n^{(s)}(\lambda, x) = \binom{s+n}{n} {}_2F_1[-n, \lambda; -s-n; x]. \tag{3}$$

It is noted that the special case $\lambda = 1$ of (3) reduces immediately to the Cesàro polynomials defined by (2).

Furthermore, they satisfy the generating functions [3]:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x)\, t^n = (1-t)^{-s-1}(1-xt)^{-\lambda} \tag{4}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{n+m}{n} g_{n+m}^{(s)}(\lambda, x)\, t^n \\ = (1-t)^{-s-m-1}(1-xt)^{-\lambda} g_m^{(s)}\left(\lambda, \frac{x(1-t)}{1-xt}\right),\end{aligned} \tag{5}$$

where $m = 0, 1, 2, \ldots$..

The purpose of this study is to obtain $q-$analogue of generalized Cesàro polynomials as $q-$analogue of the production functions mentioned above. The structure of this paper is as follows.

In Section 2, we give some preliminaries on $q-$calculus. In Section 3, we define some $q-$analogue of Cesàro polynomials. In Section 4, theorems are given for bilinear and bilateral generating functions for $q-$generalized Cesàro polynomials. In Section 5, the application of the theorems given in Section 4 will be given.

2. Some $q-$Calculus: The Definitions

Let $q \in \mathbb{C}$, $0 < |q| < 1$. A $q-$analogue of the hypergeometric series ${}_pF_r$ is the basic hypergeometric series [4]:

$$
{}_r\phi_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_r\\ & ;q,z\\ \beta_1,\ldots,\beta_s\end{matrix}\right]
=\sum_{k=0}^{\infty}\frac{(\alpha_1;q)_k\cdots(\alpha_r;q)_k}{(\beta_1;q)_k\cdots(\beta_s;q)_k}\left((-1)^k q^{k(k-1)/2}\right)^{1+r+s}\frac{z^k}{(q;q)_k}, \tag{6}
$$

where $q \neq 0$ when $r > s+1$, and (β_i) are such that the denominator never vanishes. We also need to define some other q−analogues, such as the q−analogue of a number $[\alpha]_q$, factorial $[\alpha]_q!$, and the Pochhammer symbol (rising factorial) $(\alpha)_n$. These q−analogues are given as follows:

$$
\begin{aligned}
[\alpha]_q &= \frac{1-q^{\alpha}}{1-q},\\
[0]_q! &= 1,\\
[n]_q! &= \prod_{k=1}^{n}[k]_q, \quad n\in\mathbb{N}.
\end{aligned} \tag{7}
$$

The number $(\mu;q)_\omega$ is given by

$$
(\mu;q)_\omega \coloneqq \frac{(\mu;q)_\infty}{(\mu q^{\omega};q)_\infty}, \tag{8}
$$

where

$$
(\mu;q)_\infty \coloneqq \prod_{s=0}^{\infty}(1-\mu q^{s}) \tag{9}
$$

and μ,ω are arbitrary parameters so that

$$
(\mu;q)_\omega \coloneqq \begin{cases}1 & \text{if } \omega=0\\ (1-\mu)(1-\mu q)\cdots(1-\mu q^{\omega-1}) & \text{if } \omega=1,2,3,\ldots;\end{cases} \tag{10}
$$

see, for instance, [5], pp. 413-414.

The q−gamma function [4] is defined by

$$
\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x\in\mathbb{C},\ x\notin\{0,-1,-2,\ldots\}. \tag{11}
$$

And it satisfies

$$
\Gamma_q(x+1)=\frac{1-q^x}{1-q}\Gamma_q(x). \tag{12}
$$

Definition 1. The q−analogue of Cesàro's polynomial is defined as follows [6]:

$$
g_n^{(s)}(x;q)=\frac{(q^{1+s};q)_n}{(q;q)_n}\,{}_2\phi_1\left[\begin{matrix}q^{-n},q\\ & ;q,x\\ q^{-s-n}\end{matrix}\right]=\sum_{k=0}^{n}\begin{bmatrix}k+s\\ s\end{bmatrix}_q (xq^s)^{n-k}, \tag{13}
$$

where ${}_2\phi_1$ denotes q−hypergeometric function and defined by [6]

$$
{}_2\phi_1\left[\begin{matrix}q^{-n},q\\ & ;q,x\\ q^{-s-n}\end{matrix}\right]=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q;q)_k}{(q^{-s-n};q)_k}\frac{x^k}{(q;q)_k}. \tag{14}
$$

Definition 2. The q−Cesàro polynomials satisfy the following generating function [6, 7]:

$$
\sum_{n=0}^{\infty}g_n^{(s)}(x;q)\,t^n=\frac{1}{(1-q^s xt)(t;q)_{s+1}}. \tag{15}
$$

Following Saxena, Modi, and Kalla [8], the basic analogue of the Fox's H−function is defined as

$$
H_{P,Q}^{M,N}\left[x;q\left|\begin{matrix}(a,\alpha)\\ \\ (b,\beta)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_C\Phi(s;q)\,x^s d_q s, \tag{16}
$$

where

$$
\Phi(s;q)=\frac{\left\{\prod_{j=1}^{M}G\left(q^{b_j-\beta_j s}\right)\right\}\left\{\prod_{j=1}^{N}G\left(q^{1-a_j+\alpha_j s}\right)\right\}\pi}{\left\{\prod_{j=M+1}^{Q}G\left(q^{1-b_j+\beta_j s}\right)\right\}\left\{\prod_{j=N+1}^{P}G\left(q^{a_j-\alpha_j s}\right)\right\}G\left(q^{1-s}\right)\sin\pi s} \tag{17}
$$

and

$$
G(q^s)=\left\{\prod_{n=0}^{\infty}\left(1-q^{a+n}\right)\right\}^{-1}=\frac{1}{(q^a;q)_\infty}. \tag{18}
$$

Also $0\le M\le Q$, $0\le N\le P$, α_i's and β_j's are positive integers. The contour C is a line parallel to $\mathrm{Re}(ws)=0$ with indentations if necessary, in such a manner that all the poles of $G(q^{b_j-\beta_j s})$, $1\le j\le M$ are to the right and those of $G(q^{1-a_j+\alpha_j s})$, $1\le j\le N$ are to the left of C. For large values of $|s|$, the integral converges if $\mathrm{Re}[s\log(x)-\log\sin\pi s]<0$ on the contour C, i.e., if $|\{\arg(x)-w_2 w_1^{-1}\log|x|\}|<\pi$, where $0<|q|<1$, $\log q=-w=-(w_1+iw_2)$, w_1 and w_2 are real.

Further, if we set $\alpha_i=\beta_j=1$, $\forall i$ and j in (16), we obtain the basic analogue of Meijer's G−function due to Saxena, Modi, and Kalla [8]:

$$
G_{P,Q}^{M,N}\left[x;q\left|\begin{matrix}a_1,\ldots,a_P\\ \\ b_1,\ldots,b_Q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_C\Phi'(s;q)\,x^s d_q s, \tag{19}
$$

where

$$
\Phi'(s;q)=\frac{\left\{\prod_{j=1}^{M}G\left(q^{b_j-s}\right)\right\}\left\{\prod_{j=1}^{N}G\left(q^{1-a_j+s}\right)\right\}\pi}{\left\{\prod_{j=M+1}^{Q}G\left(q^{1-b_j+s}\right)\right\}\left\{\prod_{j=N+1}^{P}G\left(q^{a_j-s}\right)\right\}G\left(q^{1-s}\right)\sin\pi s}. \tag{20}
$$

Detailed account of Meijer's G−function, Fox's H−function, and various functions expressed by Fox's H−function can be found in the research monographs of

Mathai and Saxena [9, 10], Srivastava, Gupta, and Goyal [11], and Mathai, Saxena, and Haubold [12]. In addition, the basic functions of a variable that can be expressed in terms of $Gq(\cdot)$ functions can be found in the works of Yadav and Purohit [13, 14]. In the last quarter of the twentieth century, the quantum calculus (also known as q−calculus) can be found on the theory of approaches of operators [15, 16].

3. Construction of the q−Generalized Cesàro Polynomials

In this section, with the help of the similar method as considered in [2, 5, 17, 18], we form the analogue of q−generalized Cesàro polynomials $g_n^{(s)}(\lambda, x; q)$ given by (3).

Definition 3. The q−generalized Cesàro polynomials $g_n^{(s)}(\lambda, x; q)$ given by (3) are written as follows:

$$\begin{aligned} g_n^{(s)}(\lambda, x; q &:= \frac{\left(q^{s+1}; q\right)_n}{(q;q)_n} {}_2\phi_1 \begin{bmatrix} q^{-n}, q^{\lambda} & \\ & ; q, x \\ q^{-s-n} & \end{bmatrix} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n-k+s \\ n-k \end{bmatrix}_q \left(q^{\lambda}; q\right)_k \frac{\left(xq^s\right)^k}{(q;q)_k}. \end{aligned} \tag{21}$$

It is noted that the special case $\lambda = 1$ of (21) reduces immediately to the generalized Cesàro polynomials defined by (4).

Theorem 4. *The q−generalized Cesàro polynomials have the following generating function relation:*

$$\frac{1}{\left(q^s xt; q\right)_{\lambda} (t;q)_{s+1}} = \sum_{k=0}^{\infty} g_n^{(s)}(\lambda, x; q)\, t^n, \tag{22}$$

where $|t| < |x_1|^{-1}$, $k \in \mathbb{N}_0$.

Proof. Using the well-known q−binomial theorem (see [19], p. 241-248, [5], p. 416) and from (21), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x; q)\, t^n \\ &\quad = \sum_{n=0}^{\infty} \frac{\left(q^{s+1}; q\right)_n}{(q;q)_n} {}_2\phi_1 \begin{bmatrix} q^{-n}, q^{\lambda} & \\ & ; q, x \\ q^{-s-n} & \end{bmatrix} t^n \\ &\quad = \sum_{n=0}^{\infty} \frac{\left(q^{s+1}; q\right)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{\left(q^{-n}; q\right)_k \left(q^{\lambda}; q\right)_k}{(q;q)_k \left(q^{-s-n}; q\right)_k} x^k t^n. \end{aligned} \tag{23}$$

Now making use of the identity

$$\left(q^{-n}; q\right)_k = \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}, \tag{24}$$

we have

$$\begin{aligned} &\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x; q)\, t^n = \sum_{n=0}^{\infty} \frac{\left(q^{s+1}; q\right)_n}{(q;q)_n} \\ &\quad \cdot \sum_{k=0}^{n} \frac{(q;q)_n \left[(-1)^k q^{\binom{k}{2}-nk}\right] (q;q)_{s+n-k} \left(q^{\lambda}; q\right)_k}{(q;q)_{n-k} (q;q)_{s+n} \left[(-1)^k q^{\binom{k}{2}-nk-sk}\right] (q;q)_k} x^k t^n \\ &\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\left(q^{s+1}; q\right)_n (q;q)_{s+n-k} \left(q^{\lambda}; q\right)_k q^{sk}}{(q;q)_{n-k} (q;q)_{s+n} (q;q)_k} x^k t^n \\ &\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(q^{s+1}; q\right)_{n+k} (q;q)_{s+n} \left(q^{\lambda}; q\right)_k q^{sk}}{(q;q)_n (q;q)_{s+n+k} (q;q)_k} x^k t^{n+k} \\ &\quad = \sum_{n=0}^{\infty} \frac{\left(q^{s+1}; q\right)_n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{\left(q^{\lambda}; q\right)_k q^{sk}}{(q;q)_k} x^k t^{n+k} \\ &\quad = \sum_{n=0}^{\infty} \frac{\left(q^{s+1}; q\right)_n}{(q;q)_n} t^n \sum_{k=0}^{\infty} \frac{\left(q^{\lambda}; q\right)_k}{(q;q)_k} \left(q^s xt\right)^k \\ &\quad = \frac{\left(1-q^{s+1}t\right)_{\infty}}{(1-t)_{\infty}} \frac{\left(1-q^{\lambda+s}xt\right)_{\infty}}{\left(1-q^s xt\right)_{\infty}} \\ &\quad = \frac{1}{(t;q)_{s+1} \left(q^s xt; q\right)_{\lambda}}, \end{aligned} \tag{25}$$

which completes the proof. □

4. The q−Generating Relations

In this section, we have obtained bilinear and bilateral generating functions of various families for the q−analogue of the generalized Cesàro polynomials $g_{n,q}^{(s)}(\lambda, x)$ given by (22). In addition, we will get a specific linear q−generating relationship that includes the basic analogue of Fox's H−function and a general class of q−hypergeometric polynomials. We begin by stating the following theorem.

Theorem 5. *For nonvanishing function $\Omega_{\mu}(y_1, \ldots, y_s)$ of complex variables $y_1, \ldots, y_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, \ldots, y_s; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \ldots, y_s)\, \zeta^k, \quad (a_k \neq 0,\ \mu, \psi \in \mathbb{C}) \tag{26}$$

and

$$\begin{aligned} &\Theta_{n,p}^{\mu,\psi}(\lambda, x; q; y_1, \ldots, y_s; \xi) \\ &\quad := \sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s)}(\lambda, x; q)\, \Omega_{\mu+\psi k}(y_1, \ldots, y_s)\, \xi^k, \end{aligned} \tag{27}$$

where $[n/p]$ denotes the integer part of n, $p \in \mathbb{R}$.

Then,

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(\lambda,x;q;y_1,\ldots,y_s;\frac{\eta}{t^p}\right)t^n = \frac{1}{(q^s xt;q)_\lambda (t;q)_{s+1}}\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\eta). \tag{28}$$

Proof. Let S denote the first member of the assertion (28) of Theorem 5. Taking $\xi \longrightarrow \eta/t^p$ and sum from $n=0$ to ∞ and also multiplying by t^n, we have

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(\lambda,x;q;y_1,\ldots,y_s;\frac{\eta}{t^p}\right)t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s)}(\lambda,x;q)\,\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\,\eta^k t^{n-pk}. \end{aligned} \tag{29}$$

Replacing n by $n+pk$, we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k g_n^{(s)}(\lambda,x;q)\,\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\,\eta^k t^n \\ &= \sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x;q)\,t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,\ldots,y_s)\,\eta^k \\ &= \frac{1}{(q^s xt;q)_\lambda (t;q)_{s+1}}\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\eta). \end{aligned} \tag{30}$$

which completes the proof. □

Theorem 6. *Let $\{S_{n,q}\}_{n=0}^{\infty}$ be an arbitrary bounded sequence, let M,N,P,Q be positive integers such that $0 \le M \le Q$, $0 \le N \le P$, let $h>0$, and let m be an arbitrary positive integer. Then the following bilateral q−generating relation holds:*

$$\begin{aligned} &\sum_{n=0}^{\infty} S_{n,q}\, g_n^{(s)}(\lambda,\rho x;q) \cdot H_{P+1,Q}^{M,N+1}\left[y;q \,\middle|\, \begin{matrix} (1-\mu-n,h),(a,\alpha) \\ (b,\beta) \end{matrix} \right] t^n \\ &= \frac{1}{2\pi i}\int_C \Phi(u;q) \cdot \frac{\Gamma_q(\mu+hu)(1-q)^{\mu+hu-1}}{(q;q)_\infty} \times \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} S_{n+k,q} \\ &\quad \cdot \frac{(q^{\mu+hu};q)_k (q^{\mu+k+hu};q)_n (q^{s+1};q)_n (q^\lambda;q)_k}{(q;q)_n (q;q)_k} (q^s \rho xt)^k \cdot y^u t^n d_q u, \end{aligned} \tag{31}$$

where $0<|q|<1$ and ρ and μ are arbitrary numbers.

Proof. Denoting, for convenience, the left-hand side of (31) by L and using the contour integral representation (16) for the q−analogue of Fox's H−function and the definition (21) for the q−generalized Cesàro polynomials, we get

$$L = \frac{1}{2\pi i}\sum_{n=0}^{\infty} S_{n,q}\left(\frac{(q^{s+1};q)_n}{(q;q)_n}\, {}_2\phi_1\left[\begin{matrix} q^{-n},q^\lambda \\ q^{-s-n} \end{matrix};q,\rho x\right]\right) \cdot \left\{\int_C \Phi(u;q)\, G\left(q^{\mu+n+hu)}\right) y^u d_q u\right\} t^n. \tag{32}$$

Changing the order of summations and integration, we obtain

$$L = \frac{1}{2\pi i}\int_C \Phi(u;q)\sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{n,q}\, G\left(q^{\mu+n+hu)}\right) l \cdot \frac{(q^{s+1};q)_n}{(q;q)_n}\frac{(q^{-n};q)_k (q^\lambda;q)_k}{(q^{-s-n};q)_k (q;q)_k}(\rho x)^k y^u t^n d_q u, \tag{33}$$

where $\Phi(u;q)$ is given by (17). Using of the relation for q−gamma function, namely,

$$G(q^\alpha) = \frac{\Gamma_q(\alpha)(1-q)^{\alpha-1}}{(q;q)_\infty}, \tag{34}$$

we obtain

$$\begin{aligned} L &= \frac{1}{2\pi i}\int_C \Phi(u;q)\frac{\Gamma_q(\mu+hu)(1-q)^{\mu+hu-1}}{(q;q)_\infty} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{n,q}\,(q^{\mu+hu};q)_n \\ &\quad \cdot \frac{(q^{s+1};q)_n}{(q;q)_n}\frac{(q^{-n};q)_k (q^\lambda;q)_k}{(q^{-s-n};q)_k (q;q)_k}(\rho x)^k y^u t^n d_q u. \end{aligned} \tag{35}$$

By using identity (24), we have

$$\begin{aligned} L &= \frac{1}{2\pi i}\int_C \Phi(u;q) \cdot \frac{\Gamma_q(\mu+hu)(1-q)^{\mu+hu-1}}{(q;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{n,q} \\ &\quad \cdot \frac{(q^{\mu+hu};q)_n (q^{s+1};q)_n (q^\lambda;q)_k (q;q)_{s+n-k}}{(q;q)_{s+n}(q;q)_{n-k}(q;q)_k} q^{ks}(\rho x)^k \cdot y^u t^n d_q u. \end{aligned} \tag{36}$$

Again, changing the order of summations and making use of the series rearrangement relation [1]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B(k,n+k), \tag{37}$$

we obtain

$$
\begin{aligned}
L &= \frac{1}{2\pi i}\int_C \Phi(u;q) \\
&\cdot \frac{\Gamma_q(\mu+hu)(1-q)^{\mu+hu-1}}{(q;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}S_{n+k,q} \\
&\cdot \frac{(q^{\mu+hu};q)_{n+k}(q^{s+1};q)_{n+k}(q^{\lambda};q)_k(q;q)_{s+n}}{(q;q)_{s+n+k}(q;q)_n(q;q)_k}q^{ks}(\rho x)^k \\
&\cdot y^u t^{n+k} d_q u.
\end{aligned}
\tag{38}
$$

Now by interchanging the order of contour integral and summation, and using the q–identities [4], namely,

$$(\alpha;q)_{n+k} = (\alpha;q)_n(\alpha q;q)_k \tag{39}$$

and

$$(\alpha;q)_n = \frac{\Gamma(\alpha+n)(1-q)^n}{\Gamma(\alpha)} \quad (n>0), \tag{40}$$

we obtain

$$
\begin{aligned}
L &= \frac{1}{2\pi i}\int_C \Phi(u;q) \\
&\cdot \frac{\Gamma_q(\mu+hu)(1-q)^{\mu+hu-1}}{(q;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}S_{n+k,q} \\
&\cdot \frac{(q^{\mu+hu};q)_k(q^{\mu+k+hu};q)_n(q^{s+1};q)_n(q^{\lambda};q)_k}{(q;q)_n(q;q)_k}(q^s\rho xt)^k \\
&\cdot y^u t^n d_q u.
\end{aligned}
\tag{41}
$$

□

5. Special Cases

As an application of the above in Theorem 5, when the multivariable function $\Omega_{\mu+\psi k}(y_1,\ldots,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give additional applications of the above theorem.

We first set

$$
\begin{gathered}
r = 1 \\
\text{and } \Omega_{\mu+\psi k}(y) = g^{(s)}_{\mu+\psi k}(\lambda, y; q)
\end{gathered}
\tag{42}
$$

in Theorem 5, where the q–generalized Cesàro polynomials are generated by (22). We thus led to the following result which provides a class of bilinear generating functions for the q–generalized Cesàro polynomials.

Corollary 1. *If*

$$
\begin{gathered}
\Lambda_{\mu,\psi}(\lambda, y; q; \zeta) := \sum_{k=0}^{\infty} a_k g^{(s)}_{\mu+\psi k}(\lambda, y; q)\zeta^k, \\
a_k \neq 0,\ \mu,\psi\in\mathbb{C},
\end{gathered}
\tag{43}
$$

then we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k g^{(s)}_{n-pk}(\lambda,x;q)\, g^{(s)}_{\mu+\psi k}(\lambda,y;q)\frac{\eta^k}{t^{pk}}t^n \\
&\quad = \frac{1}{(q^s xt;q)_\lambda (t;q)_{s+1}}\Lambda_{\mu,\psi}(\lambda,y;q;\eta).
\end{aligned}
\tag{44}
$$

Remark 2. Using the generating relation (22) for the generalized Cesàro polynomials and getting $a_k = 1$, $\mu = 0$, $\psi = 1$ in Corollary 1, we find that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} g^{(s)}_{n-pk}(\lambda,x;q)\, g^{(s)}_{k}(\lambda,y;q)\eta^k t^{n-pk} \\
&\quad = \frac{1}{(q^s xt;q)_\lambda (t;q)_{s+1}}\frac{1}{(q^s y\eta;q)_\lambda (\eta;q)_{s+1}},
\end{aligned}
\tag{45}
$$

where $|t|<1$, $|\eta|<1$.

By assigning suitable special values to the sequence $\{S_{n,q}\}_{n=0}^{\infty}$, our main result (Theorem 6) can be applied to derive certain bilateral q–generating relations for the product of orthogonal q–polynomials and the basic analogue of Fox's H–function. To illustrate this, we consider the following example.

Set $\rho = 1$ and

$$S_{n,q} = \frac{1}{(q^{\lambda};q)_k(q^{s+1};q)_{n-k}}. \tag{46}$$

Thus, in view of the above relations, Theorem 6 yields the q–generating relation involving q–generalized Cesàro polynomial and the basic Fox's H–function as below.

Corollary 3. *The following bilateral generating function holds true:*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{(q^{\lambda};q)_k(q^{s+1};q)_{n-k}}g^{(s)}_{n}(\lambda,\rho x;q) \\
&\cdot H^{M,N+1}_{P+1,Q}\left[\, y;q \,\middle|\, \begin{matrix}(1-\mu-n,h),(a,\alpha)\\ \\ (b,\beta)\end{matrix}\right] t^n = \frac{1}{(1-t)^{(\mu)}}
\end{aligned}
$$

$$\cdot \sum_{k=0}^{\infty} \frac{(q^s xt)^k}{(tq^{\mu};q)_k (q;q)_k} \frac{1}{2\pi i}$$

$$\cdot \int_C \Phi(u;q) \frac{\Gamma_q(\mu+k+hu)(1-q)^{\mu+k+hu-1}}{(tq^{\mu+k};q)_{hu}(q;q)_{\infty}} y^u d_q u$$

$$= \frac{1}{(1-t)^{(\mu)}} \sum_{k=0}^{\infty} \frac{(q^s xt)^k}{(tq^{\mu};q)_k (q;q)_k}$$

$$\cdot H_{P+1,Q}^{M,N+1} \left[\frac{y}{(1-tq^{\mu+k})^{(h)}};q \left| \begin{matrix} (1-\mu-k,h),(a,\alpha) \\ (b,\beta) \end{matrix} \right. \right], \tag{47}$$

where $|t|<1,\ 0<|q|<1$ *and* μ *is arbitrary numbers.*

If we take $\alpha_i = \beta_j = 1$ for all i and j and $m = h = 1$ and set (19) and

$$S_{n,q} = \frac{1}{(q^{\lambda};q)_k (q^{s+1};q)_{n-k}}, \tag{48}$$

in Theorem 6, we have the following bilateral generating functions for the $q-$generalized Cesàro polynomials.

Corollary 4. *Let* $\{S_{n,q}\}_{n=0}^{\infty}$ *be an arbitrary bounded sequence and let* M, N, P, Q *be positive integers satisfying* $0 \le M \le Q$, $0 \le N \le P$. *Then the following bilateral* $q-$*generating relation for the function* $G_q(\cdot)$ *holds:*

$$\sum_{n=0}^{\infty} \frac{1}{(q^{\lambda};q)_k (q^{s+1};q)_{n-k}} g_n^{(s)}(\lambda,\rho x;q) \cdot G_{P+1,Q}^{M,N+1} \left[y;q \left| \begin{matrix} 1-\mu-n, a_1,\dots,a_P \\ \\ b_1,\dots,b_Q \end{matrix} \right. \right] t^n$$
$$= \frac{1}{(1-t)^{(\mu)}} \sum_{k=0}^{\infty} \frac{(q^s \rho xt)^k}{(tq^{\mu};q)_k (q;q)_k} \cdot G_{P+1,Q}^{M,N+1} \left[\frac{y}{(1-tq^{\mu+k})^{(h)}};q \left| \begin{matrix} 1-\mu-k, a_1,\dots,a_P \\ \\ b_1,\dots,b_Q \end{matrix} \right. \right], \tag{49}$$

where $|t|<1,\ 0<|q|<1$ *and* ρ *and* μ *are arbitrary numbers.*

For every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu}(y_1,\dots,y_s)$ $(s = 2,3,\dots)$ is expressed as an appropriate product of several simpler functions, the assertion of the above Theorem 5 can be applied in order to derive various families of multilinear and multilateral generating functions for the $q-$generalized Cesàro polynomials $g_n^{(s)}(\lambda,x;q)$ defined by (22).

We conclude with the remark that by suitably assigning values to the sequence $\{S_{n,q}\}_{n=0}^{\infty}$, the $q-$generating relation (31), being of general nature, will lead to several generating relations for the product of orthogonal $q-$polynomials and the basic analogue of the Fox's H- functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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Research Article

Generalized Numerical Index of Function Algebras

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Let X be a complex Banach space and $C_b(\Omega : X)$ be the Banach space of all bounded continuous functions from a Hausdorff space Ω to X, equipped with sup norm. A closed subspace $\mathscr{A}$ of $C_b(\Omega : X)$ is said to be an X-valued function algebra if it satisfies the following three conditions: (i) $A := \{x^* \circ f : f \in \mathscr{A},\ x^* \in X^*\}$ is a closed subalgebra of $C_b(\Omega)$, the Banach space of all bounded complex-valued continuous functions; (ii) $\phi \otimes x \in \mathscr{A}$ for all $\phi \in A$ and $x \in X$; and (iii) $\phi f \in \mathscr{A}$ for every $\phi \in A$ and for every $f \in \mathscr{A}$. It is shown that k-homogeneous polynomial and analytic numerical index of certain X-valued function algebras are the same as those of X.

1. Introduction

In this paper, we consider only complex nontrivial Banach spaces. Given a Banach space X, we denote by B_X and S_X its closed unit ball and unit sphere, respectively. Let X^* be the dual space of X. If X and Y are Banach spaces, a *k-homogeneous polynomial* P from X to Y is a mapping such that there is a k-linear continuous mapping L from X to Y such that $P(x) = L(x, \ldots, x)$ for every x in X. The Banach space of all k-homogeneous polynomials from X to Y is denoted by $\mathscr{P}({}^kX : Y)$ endowed with the polynomial norm $\|P\| = \sup_{x\in B_X}\|P(x)\|$. We refer to [1] for background knowledge on polynomials.

We are mainly interested in the following spaces. For two Banach spaces X, Y and a Hausdorff topological space Ω,

$$\begin{aligned} C_b(\Omega : Y) &:= \{f : \Omega \longrightarrow Y : \\ &\quad f \text{ is a bounded continuous function on } \Omega\}, \\ \mathscr{A}_b(B_X : Y) &:= \{f \in C_b(B_X : Y) : \\ &\quad f \text{ is holomorphic on } B_X^\circ\} \\ \mathscr{A}_u(B_X : Y) &:= \{f \in \mathscr{A}_b(B_X : Y) : \\ &\quad f \text{ is uniformly continuous}\}, \end{aligned} \tag{1}$$

where B_X° is the interior of B_X. Then $C_b(\Omega : Y)$ is a Banach space under the sup norm $\|f\| := \sup\{\|f(t)\|_Y : t \in \Omega\}$ and both $\mathscr{A}_b(B_X : Y)$ and $\mathscr{A}_u(B_X : Y)$ are closed subspaces of $C_b(B_X : Y)$. In case that Y is the complex scalar field $\mathbb{C}$, we just write $C_b(B_X)$, $\mathscr{A}_b(B_X)$, and $\mathscr{A}_u(B_X)$. Let

$$\Pi(X) := \{(x, x^*) : \|x\| = \|x^*\| = 1 = x^*(x)\}. \tag{2}$$

The *spatial numerical range* of f in $C_b(B_X : X)$ is defined by

$$W(f) = \{x^*(f(x)) : (x, x^*) \in \Pi(X)\}, \tag{3}$$

and the *numerical radius* of f is defined by

$$v(f) = \sup\{|\lambda| : \lambda \in W(f)\}. \tag{4}$$

Let X be a Banach space. The *k-homogeneous polynomial numerical index* $n^{(k)}(X)$ is defined in [2] by

$$n^{(k)}(X) := \inf\left\{v(P) : P \in \mathscr{P}\left({}^kX : X\right), \|P\| = 1\right\}. \tag{5}$$

The *b-analytic numerical index* $n_{ba}(X)$ and *u-analytic index* $n_{ua}(X)$ are defined, respectively, by

$$\begin{aligned} n_{ba}(X) &:= \inf\{v(f) : f \in \mathscr{A}_b(B_X : X), \|f\| = 1\}, \\ n_{ua}(X) &:= \inf\{v(f) : f \in \mathscr{A}_u(B_X : X), \|f\| = 1\}. \end{aligned} \tag{6}$$

It is clear from the definitions that $0 \leqslant n_{ba}(X) \leqslant n_{ua}(X) \leqslant n^{(k)}(X) \leqslant 1$ for all $k \geq 1$.

Choi, García, Kim, and Maestre showed [3] that $n^{(k)}(A) = 1$ and $n_{ua}(A) = 1$ for uniform algebras A. In general, it is not difficult to see that if A is a (unital) function algebra on a Hausdorff space, then, by the Gelfand transform, A is isometric to a (unital) uniform algebra on Δ where Δ is the maximal ideal space of A. We present this fact in Proposition 2 for the completeness of the paper. In this paper, we introduce a X-valued function algebra and the Gelfand transform does not work in this case. In the proof of [3], they used a very useful Urysohn type theorem, which was obtained by Cascales, Guiro, and Kadets [4]. Recently, Kim and the author found [5] that a Urysohn type theorem holds for some function algebras. It plays an important role in the main results of this paper. For some geometric properties on k-homogeneous polynomial (analytic) numerical index, refer to [6, 7].

Let us briefly review some necessary notions. A nontrivial $\|\cdot\|_\infty$-closed subalgebra of A of $C_b(\Omega)$ is called a *function algebra* on a Hausdorff space Ω. For a Banach space X, a nontrivial subspace $\mathscr{A}$ of $C_b(\Omega : X)$ is said to be an *X-valued function algebra* if it satisfies three conditions: (i) $A := \{x^* \circ f : f \in \mathscr{A},\ x^* \in X^*\}$ is a function algebra on Ω; (ii) $A \otimes X = \{\phi \otimes x : \phi \in A, x \in X\}$, where $(\phi \otimes x)(t) = \phi(t)x$ for $t \in \Omega$; and (iii) $\phi f \in \mathscr{A}$ for every $\phi \in A$ and $f \in \mathscr{A}$, where $(\phi f)(t) = \phi(t) f(t)$ for $t \in \Omega$. A subset T of Ω is said to be *norming* for $\mathscr{A}$ if $\|f\| = \sup\{\|f(t)\| : t \in T\}$ holds for all $f \in \mathscr{A}$. By *unital function algebra*, we mean a function algebra containing all constant functions. A function algebra A on a compact Hausdorff space K is said to be a *uniform algebra* if A separates the points of K (that is, for every $x \neq y$ in K, there is $f \in A$ such that $f(x) \neq f(y)$. Note that the definition of function algebra in this paper is different from the usual one in [8].

Let f be an element of an X-valued function algebra $\mathscr{A}$. The f is said to be a *peak function* at t_0 if there exists a unique $t_0 \in \Omega$ such that $\|f\| = \|f(t_0)\|$. A peak function f is said to be a *strong peak function* at $t_0 \in \Omega$ if $\|f\| = \|f(t_0)\|$ and for every open subset V containing t_0 we get

$$\sup\{\|f(t)\| : t \in \Omega \setminus V\} < \|f\|. \tag{7}$$

The corresponding point t_0 is called a *strong peak point* for $\mathscr{A}$. We denote by $\rho\mathscr{A}$ the set of all strong peak points for $\mathscr{A}$. It is easy to see that if Ω is compact, then every peak function is a strong peak function. It is worth remarking that if A is a nontrivial separating separable subalgebra of $C(\Omega)$ on a compact Hausdorff space Ω, then ρA is a norming subset for A [9]. There is a compact Hausdorff space K such that $\rho C(K)$ is an empty set [10]. For more information about peak functions and points, refer to [8, 10].

For an X-valued function algebra $\mathscr{A}$, let $A = \{x^* \circ f : x^* \in X^*,\ f \in \mathscr{A}\}$. Then $\rho\mathscr{A} = \rho A$. Indeed, if $f \in \mathscr{A}$ is a strong peak function at t_0, then choose $x^* \in S_{X^*}$ such that $x^* f(t_0) = \|f(t_0)\| = \|f\|$ and it is clear that $x^* \circ f \in A$ is a strong peak function in A at t_0. Therefore, $\rho\mathscr{A} \subset \rho A$. Conversely, if $g \in A$ is a strong peak function at t, then choose $x \in S_X$. Therefore, $g \otimes x \in \mathscr{A}$ is a strong peak function at t. Hence we have $\rho A \subset \rho\mathscr{A}$. In addition, if $\rho\mathscr{A}$ is norming for $\mathscr{A}$, then it is also norming for A since $g \otimes x$ in $\mathscr{A}$ has the same norm as g for every $g \in A$ and $x \in S_X$.

The following lemma will be useful to get main results. In proofs of the main results, the denseness of the strong peak functions in an X-valued function algebra $\mathscr{A}$ is an important part and equivalent to the fact that the set of strong peak points is norming for $\mathscr{A}$. That means that the fact that every element in $\mathscr{A}$ can be approximated by the sequence of strong peak functions is equivalent to the fact that the norm of every element in $\mathscr{A}$ can be approximated on the set of strong peak points for $\mathscr{A}$. The approximation by strong peak functions will prove to be useful to deal with the geometric properties of function algebras especially those related to generalized numerical indices of Banach spaces.

Lemma 1 (see [5]). *Let A be a function algebra on Ω and fix $\omega_0 \in \rho A$. Then, given $0 < \epsilon < 1$ and for every open subset U containing ω_0, there exists a strong peak function $\phi \in A$ such that $\|\phi\| = 1 = |\phi(\omega_0)|$, $\sup_{\omega\in\Omega\setminus U}|\phi(\omega)| < \epsilon$, and for all $\omega \in \Omega$,*

$$|\phi(\omega)| + (1-\epsilon)|1-\phi(\omega)| \leqslant 1. \tag{8}$$

2. Main Results

The proof of [3, Theorem 2.1] shows that $n_{ba}(A) = 1$ if A is a uniform algebra. Since a function algebra is isometric to a uniform algebra by the Gelfand transform, we have the following.

Proposition 2. *Let A be a function algebra on a Hausdorff space Ω. Then it is isometric to a uniform algebra on a compact Hausdorff space and $n_{ba}(A) = 1$.*

Proof. Let A be a function algebra and Δ be the set of all nonzero algebra homomorphisms from A to $\mathbb{C}$. The maximal ideal space Δ is a compact Hausdorff space with the Gelfand topology. The Gelfand transform $\widehat{f}$ of $f \in A$ is defined by $\widehat{f}(\phi) = \phi(f)$ for $\phi \in \Delta$. For $t \in \Omega$, let δ_t be the dirac delta function by $\delta_t(f) = f(t)$ for $f \in A$. Fix a nonzero $f \in A$ and let $\Omega_f = \{t : f(t) \neq 0\}$; then $\delta_t \in \Delta$ for all $t \in \Omega_f$ and

$$\begin{aligned}\|\widehat{f}\| &= \sup\{|\widehat{f}(\phi)| : \phi \in \Delta\} \leqslant \|f\| \\ &= \sup\{|f(t)| : t \in \Omega_f\} = \sup\{|\delta_t(f)| : t \in \Omega_f\} \\ &\leqslant \|\widehat{f}\|.\end{aligned} \tag{9}$$

Since the Gelfand transform $f \longmapsto \widehat{f}$ is a homomorphism, A is isometrically isomorphic to the image $\widehat{A}$, where $\widehat{A}$ is the image of the Gelfand transform. Then $\widehat{A}$ is a closed subalgebra of $C(\Delta)$ and it is separating the points of Δ. Thus, it is a uniform algebra on the compact Hausdorff space Δ.

For the second part, the proof used in [3, Theorem 2.1] to show $n_{ua}(A) = 1$ can be applied to show that $n_{ba}(A) = 1$ for uniform algebras A. □

Proposition 2 gives a positive answer to the third question raised by Acosta and Kim [11].

Theorem 3. *Let X be a Banach space and suppose that $\mathscr{A}$ is an X-valued function algebra on a Hausdorff space Ω such that $\rho\mathscr{A}$ is a norming subset for $\mathscr{A}$. Then we have*

(i) $n^{(k)}(\mathscr{A}) \geq n^{(k)}(X)$ *for every* $k \geq 1$,

(ii) $n_{ua}(\mathscr{A}) \geq n_{ua}(X)$ *and*

(iii) $n_{ba}(\mathscr{A}) \geq n_{ba}(X)$.

Proof. We prove $n_{ba}(\mathscr{A}) \geq n_{ba}(X)$ holds. The proofs for the other two cases are exactly the same. It is well-known that $n_{ba}(X) > 0$ for all complex Banach spaces X [12].

Let $A = \{x^* \circ f : f \in \mathscr{A}\}$. Then A is a function algebra. Let $P \in \mathscr{A}_b(B_{\mathscr{A}} : \mathscr{A})$ with $\|P\| = 1$ and $0 < \epsilon < b_{ba}(X)$ be given. Choose $f_0 \in S_{\mathscr{A}}$ so that $\|P(f_0)\| > 1-\epsilon/6$. Since ρA is norming for $\mathscr{A}$, find $t_0 \in \rho A$ such that $\|P(f_0)(t_0)\| > 1-\epsilon/6$. Since P is continuous, there is $0 < \delta < 1$ such that $\|P(f_0) - P(g)\| < \epsilon/6$ for every $g \in B_{\mathscr{A}}$ with $\|f_0 - g\| < \delta$.

Let $W = \{t \in \Omega : \|f_0(t) - f_0(t_0)\| < \delta/6,\ \|P(f_0)(t) - P(f_0)(t_0)\| < \epsilon/3\}$ and W be an open subset of Ω containing t_0. Then by Lemma 1, there is a strong peak function $\phi \in A$ such that $\|\phi\| = \phi(t_0) = 1$ and $|\phi(t)| < \delta/6$ for every $t \in \Omega\backslash W$, and

$$\left|\phi(t)\right| + \left(1 - \frac{\epsilon}{6}\right)\left|1 - \phi(t)\right| \leqslant 1 \tag{10}$$

for every $t \in \Omega$.

Define $\Psi : X \longrightarrow \mathscr{A}$ by $\Psi(x) = (1-\delta/6)(1-\phi)f_0 + \phi \otimes x$ for all $x \in X$. It is easy to check that Ψ is well-defined and $\|\Psi(x)\| \leqslant 1$ for all $x \in B_X$. Then, let $x_0 = f_0(t_0)$,

$$\begin{aligned}\left\|f_0 - \Psi(x_0)\right\| &= \sup_{t\in\Omega}\bigg\|f_0(t) \\ &\quad - \left(1 - \frac{\delta}{6}\right)(1-\phi(t))\, f_0(t) - \phi(t)\, f_0(t_0)\bigg\| \\ &\leqslant \sup_{t\in\Omega}\bigg(\frac{\delta}{6}\left\|f_0(t)\right\| + \left|\phi(t)\right|\left\|f_0(t) - f_0(t_0)\right\| \\ &\quad + \frac{\delta}{6}\left|\phi(t)\right|\left\|f_0(t)\right\|\bigg) < \frac{\delta}{6} + \frac{\delta}{3} + \frac{\delta}{6} < \delta.\end{aligned} \tag{11}$$

Then we have the following.

$$\begin{aligned}\left\|P(\Psi(x_0))(t_0)\right\| &\geq \left\|P(f_0)(t_0)\right\| \\ &\quad - \left\|P(f_0)(t_0) - P(\Psi(x_0))(t_0)\right\| \\ &> 1 - \frac{\epsilon}{6} - \frac{\epsilon}{6} > 1 - \epsilon.\end{aligned} \tag{12}$$

Choose $x_0^* \in S_{X^*}$ such that $x_0^*[P(\Psi(x_0)(t_0)] > 1 - \epsilon$ and find a complex number z_0 with $|z_0| \leqslant 1$ and a proper $\tilde{x}_0 \in S_X$ satisfying $x_0 = z_0\tilde{x}_0$. Then the function $\varphi(z) = x_0^*[P(\Psi(z\tilde{x}_0))(t_0)]$ is an element of $\mathscr{A}_u(B_{\mathbb{C}})$. By the maximum modulus theorem, there exists z_1 with $|z_1| = 1$ such that φ takes its maximum modulus on $B_{\mathbb{C}}$. Hence,

$$\begin{aligned}\left\|P(\Psi(z_1\tilde{x}_0))(t_0)\right\| &\geq \left|x_0^*\left[P(\Psi(z_1\tilde{x}_0))(t_0)\right]\right| \\ &\geq \left|x_0^*\left[P(\Psi(z_0\tilde{x}_0))(t_0)\right]\right| \\ &> 1 - \epsilon.\end{aligned} \tag{13}$$

Let $x_1 = z_1\tilde{x}_0$, choose $x_1^* \in S_{X^*}$ with $x_1^*(x_1) = 1$, and define the function $\Phi : X \longrightarrow \mathscr{A}$ by

$$\Phi(x) = x_1^*(x)\left(1 - \frac{\delta}{6}\right)(1-\phi)\, f_0 + \phi \otimes x \tag{14}$$

for $x \in X$. Then $\Phi(x_1) = \Psi(x_1) = \Psi(z_1\tilde{x}_0)$, and hence $\|P(\Phi(x_1))(t_0)\| > 1 - \epsilon$. Let $Q(x) = P(\Phi(x))(t_0)$ for $x \in X$. Then $Q \in \mathscr{A}_b(B_X : X)$. Then

$$1 - \epsilon < \left\|P(\Phi(x_1))(t_0)\right\| = \left\|Q(x_1)\right\| \leqslant \|Q\| \leqslant 1. \tag{15}$$

Since $0 < \epsilon < b_{ba}(X)$, there is $(x_2, x_2^*) \in \Pi(X)$ so that

$$\left|x_2^*\left(\frac{Q(x_2)}{\|Q\|}\right)\right| > v\left(\frac{Q}{\|Q\|}\right) - \epsilon \geq b_{ba}(X) - \epsilon > 0. \tag{16}$$

Note that $(\Phi(x_2), x_2^* \circ \delta_{t_0}) \in \Pi(\mathscr{A})$ because $\Phi(x_2)(t_0) = x_2$. Hence we have

$$\begin{aligned}v(P) &\geq \left|\left(x_2^* \circ \delta_{t_0}\right)(P(\Phi(x_2)))\right| \\ &= \left|x_2^*\left[P(\Phi(x_2))(t_0)\right]\right| = \left|x_2^* Q(x_2)\right| \\ &\geq \|Q\|\left(b_{ba}(X) - \epsilon\right) \geq (1-\epsilon)\left(b_{ba}(X) - \epsilon\right).\end{aligned} \tag{17}$$

Since $0 < \epsilon < b_{ba}(X)$ is arbitrary, $v(P) \geq b_{ba}(X)$. This holds for all $P \in \mathscr{A}_b(B_{\mathscr{A}}; \mathscr{A})$ with $\|P\| = 1$. Therefore, we get $b_{ba}(\mathscr{A}) \geq b_{ba}(X)$. □

A version of the Bishop-Phelps-Bollobás type theorem for holomorphic functions has been shown [5, 13]. In the following theorem, we present a similar result. However the main focus is the denseness of the set of all strong peak functions, which is different from that of the results in [5].

Theorem 4. *Let X be a Banach space and $\mathscr{A}$ an X-valued function algebra on a Hausdorff space Ω. Then, given $\epsilon > 0$, whenever a norm-one element f in $\mathscr{A}$ and a point ω_0 in $\rho\mathscr{A}$ satisfy $\|f(\omega_0)\| > 1 - \epsilon/5$, there is a norm-one strong peak function $g \in \mathscr{A}$ at $\omega_0 \in \rho\mathscr{A}$ such that $\|f - g\| < \epsilon$.*

Proof. Suppose that f satisfies the prescribed conditions. Then

$$U_1 = \left\{\omega \in \Omega : \left\|\frac{f(\omega_0)}{\|f(\omega_0)\|} - f(\omega)\right\| < \frac{\epsilon}{5}\right\} \tag{18}$$

is an open set containing ω_0. There exists $\omega_1 \in \rho A \cap U_2$. Using Lemma 1, take a strong peak function $\phi \in A$ such that $\phi(\omega_0) = 1 = \|\phi\|$, $\sup\{|\phi(\omega)| : \omega \in \Omega \setminus U_1\} < \epsilon/5$, and

$$\left|\phi(\omega)\right| + \left(1 - \frac{\epsilon}{5}\right)\left|1 - \phi(\omega)\right| \leqslant 1 \tag{19}$$

for all $\omega \in \Omega$. Set

$$g(\omega) = \phi(\omega)\frac{f(\omega_0)}{\|f(\omega_0)\|} + \left(1 - \frac{\epsilon}{5}\right)(1 - \phi(\omega))\, f(\omega). \tag{20}$$

It is easy to check that $g \in \mathscr{A}$ and $\|g(\omega_1)\| = 1 = \|g\|$. Moreover, from the inequality

$$\begin{aligned}\|g(\omega) - f(\omega)\| \leqslant |\phi(\omega)| \left\| \frac{f(\omega_0)}{|f(\omega_0)|} - f(\omega) \right\| \\ + \frac{\epsilon}{5} |1 - \phi(\omega)| \, \|f(\omega)\|,\end{aligned} \tag{21}$$

we have that $\|g(\omega) - f(\omega)\| \leqslant \epsilon/5 + 2\epsilon/5$ for all $\omega \in \Omega$ if we consider two cases $\omega \in U_1$ and $\omega \in \Omega \setminus U_1$. Hence, we get $\|f - g\| < \epsilon$ and complete the proof since we know g is a strongly norm attaining function from the fact that ϕ is a strong peak function. □

From Theorem 4, we have the following consequence.

Corollary 5. *Let X be a Banach space and $\mathscr{A}$ be an X-valued function algebra on a Hausdorff space Ω. Then the set $\rho\mathscr{A}$ is norming if and only if the set of strong peak functions in $\mathscr{A}$ is dense.*

Proof. The necessity is proved by Theorem 4. For the converse, assume that the set of strongly norm attaining functions in $\mathscr{A}$ is dense in $\mathscr{A}$. Given $f \in \mathscr{A}$, there is a sequence $\{f_n\}$ of strong peak functions in $\mathscr{A}$ such that $\lim_{n\longrightarrow\infty}\|f_n - f\| = 0$. For each n, let t_n be the strong peak point corresponding to f_n. Then

$$\begin{aligned}0 &= \lim_{n\longrightarrow\infty} \|f_n - f\| \geq \limsup_{n\longrightarrow\infty} \|f_n(t_n) - f(t_n)\| \\ &\geq \limsup_{n\longrightarrow\infty} \left| \|f_n(t_n)\| - \|f(t_n)\| \right|.\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}0 &= \lim_{n\longrightarrow\infty} \left| \|f_n(t_n)\| - \|f(t_n)\| \right| \\ &= \lim_{n\longrightarrow\infty} \left| \|f_n\| - \|f(t_n)\| \right|.\end{aligned} \tag{23}$$

This means that $\lim_{n\longrightarrow\infty}\|f_n(t_n)\| = \lim_{n\longrightarrow\infty}\|f_n\| = \|f\|$. This shows that $\rho\mathscr{A}$ is a norming subset of $\mathscr{A}$. □

Theorem 6. *Let X be a Banach space and $\mathscr{A}$ be an X-valued function algebra on a Hausdorff space Ω such that $\rho\mathscr{A}$ is a norming subset for $\mathscr{A}$. Fix $P \in \mathscr{A}_u(B_X : X)$ and define the map $Q_P : B_{\mathscr{A}} \longrightarrow C_b(\Omega : X)$ by $Q_P(f)(t) = P(f(t))$ for $f \in B_{\mathscr{A}}$ and $t \in \Omega$. Suppose that $Q_P(f)$ is an element of $\mathscr{A}$ for every $f \in B_{\mathscr{A}}$ and for every $P \in \mathscr{A}_u(B_X : X)$. Then we have $n_{ua}(\mathscr{A}) = n_{ua}(X)$.*

Proof. By Theorem 3, we have only to show that $n_{ua}(\mathscr{A}) \leqslant n_{ua}(X)$. Consider the set

$$\begin{aligned}L = \{(f, x^* \circ \delta_t) : f \in S_{\mathscr{A}}, t \in \Omega, x^* \\ \in S_{X^*} \text{ and } x^*(f(t)) = 1\}.\end{aligned} \tag{24}$$

Let $\pi_1 : \mathscr{A} \times (\mathscr{A})^* \longrightarrow \mathscr{A}$ be the natural projection. Then since $\rho\mathscr{A}$ is norming for $\mathscr{A}$, Corollary 5 shows that $\pi_1(L)$ is dense in $S_{\mathscr{A}}$. Then, it is shown [14] that for every $Q \in \mathscr{A}_u(B_{\mathscr{A}}; \mathscr{A})$, we have

$$v(Q) = \sup\{|x^*[Q(f)(t)]| : (f, x^* \circ \delta_t) \in L\}. \tag{25}$$

Given $P \in \mathscr{A}_u(B_X : X)$ with $\|P\| = 1$, we have $Q_P \in \mathscr{A}_u(B_{\mathscr{A}}; \mathscr{A})$. Indeed, Q_P is a map from $B_{\mathscr{A}}$ to $B_{\mathscr{A}}$. Since P is uniformly continuous on B_X, given $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in B_X$ and $\|x - y\| \leqslant \delta$, then $\|P(x) - P(y)\| < \epsilon$. If $f, g \in B_{\mathscr{A}}$ and $\|f - g\| < \delta$, then $\|P(f(t)) - P(g(t))\| < \epsilon$ for all $t \in \Omega$. Hence $\|Q(f) - Q(g)\| \leqslant \epsilon$. This shows that Q_P is uniformly continuous on $B_{\mathscr{A}}$. Now it is enough to show that Q_P is G-holomorphic on $B^{\circ}_{\mathscr{A}}$ [15]. Fix $f \in B^{\circ}_{\mathscr{A}}$ and $g \in \mathscr{A}$, and let $U(f, g) = \{z \in \mathbb{C} : f + zg \in B^{\circ}_{\mathscr{A}}\}$ be an open subset in the complex plane. Let $\varphi(z) = Q_P(f + zg)$ for $z \in U(f, g)$. Then $\varphi(z)$ is a $\mathscr{A}$-valued continuous function on $U(f, g)$. For each $t \in \Omega$, $\varphi(z)(t) = Q_P(f + zg)(t) = P(f(t) + zg(t))$ is holomorphic. Fix $z_0 \in U(f, g)$ and choose $\delta > 0$ such that $z_0 + \delta B_{\mathbb{C}} \subset U(f, g)$. The Cauchy integral formula shows that, for each $t \in \Omega$,

$$\varphi(z_0)(t) = \frac{1}{2\pi i} \int_{|z - z_0| = \delta} \frac{\varphi(z)(t)}{z - z_0} dz. \tag{26}$$

As a result, we have

$$\varphi(z_0) = \frac{1}{2\pi i} \int_{|z - z_0| = \delta} \frac{\varphi(z)}{z - z_0} dz, \tag{27}$$

since the continuity of $\varphi(z)$ implies the Bochner integrability of the integral. This means that φ is holomorphic on $U(f, g)$ and Q_P is holomorphic on $B^{\circ}_{\mathscr{A}}$ [15]. We also have $\|Q_P\| = 1$ since there is a strong peak function $g \in A$ at $t_0 \in \Omega$ such that $g(t_0) = 1 = \|g\|$ and $g \otimes x$ is in $B_{\mathscr{A}}$ for each $x \in B_X$. It is clear that $v(Q_P) \geq n_{ua}(\mathscr{A})$. For every $\epsilon > 0$, there is $(f, x^* \circ \delta_t) \in L$ such that

$$\begin{aligned}n_{ua}(\mathscr{A}) - \epsilon \leqslant v(Q_P) - \epsilon < |(x^* \circ \delta_t)(Q_P(f))| \\ = |x^*(P(f(t)))| \leqslant v(P).\end{aligned} \tag{28}$$

Therefore, we get $n_{ua}(\mathscr{A}) \leqslant n_{ua}(X)$. □

The same proof shows the following.

Theorem 7. *Let X be a Banach space and $\mathscr{A}$ be an X-valued function algebra on a Hausdorff space Ω such that $\rho\mathscr{A}$ is a norming subset for $\mathscr{A}$. Fix $P \in \mathscr{P}(^k X : X)$ and define the map $Q_P : \mathscr{A} \longrightarrow C_b(\Omega : X)$ by $Q_P(f)(t) = P(f(t))$ for $f \in \mathscr{A}$ and $t \in \Omega$. Suppose that $Q_P(f)$ is an element of $\mathscr{A}$ for every f in $\mathscr{A}$ and for every $P \in \mathscr{P}(^k X : X)$). Then we have $n^{(k)}(\mathscr{A}) = n^{(k)}(X)$.*

Proof. The main difficulty in the proof of Theorem 7 is to check that Q_P is in $\mathscr{P}(^k\mathscr{A}; \mathscr{A})$. Let L be the corresponding continuous k-linear map defining P. Let $\widetilde{L} : \mathscr{A}^k \longrightarrow \mathscr{A}$ by $\widetilde{L}(f_1, \ldots, f_k)(t) = L(f_1(t), f_2(t), \ldots, f_k(t))$ for $f_i \in \mathscr{A}$ and $t \in \Omega$. Then it is easy to check that $\widetilde{L}$ is a continuous k-linear map and $Q_P(f) = \widetilde{L}(f, \ldots, f)$ for $f \in \mathscr{A}$. The other part of the proof is the same as the proof of Theorem 6. □

Let X, Y be Banach spaces and let $\mathscr{A}(B_X : Y)$ be either $\mathscr{A}_u(B_X : Y)$ or $\mathscr{A}_b(B_X : Y)$. Notice that $\mathscr{A}(B_X : Y)$ are Y-valued function algebras over B_X. If a Banach space X is finite dimensional, $\rho\mathscr{A}(B_X)$ is the set of all complex extreme points of B_X as observed in [16, 17]. A strongly exposed point of B_X is a strong peak point for $\mathscr{A}(B_X)$, so if a strongly exposed point of B_X is dense in S_X, then $\overline{\rho\mathscr{A}(B)} = S_X$ and it is norming for $\mathscr{A}(B_X)$. It is also proved in [17] that if X is locally c-uniformly convex space and it is an order continuous sequence space, then $\rho\mathscr{A}_u(B_X)$ is norming. The typical example of uniformly complex convex sequence space is ℓ_1. For the definitions related to various complex convexities and more examples, we refer to [9, 18–21].

Let C be a closed convex and bounded set in a Banach space X. The set C has the *Radon-Nikodým property* if, for every probability space $(\Omega, \mathscr{B}, \mu)$ and every X-valued countably additive measure τ on $\mathscr{B}$ such that $\tau(A)/\mu(A) \in C$ for every $A \in \mathscr{B}$ with $\mu(A) > 0$, there is a Bochner measurable $f : \Omega \longrightarrow X$ so that

$$\tau(A) = \int_A f(\omega)\, d\mu(\omega), \quad A \in \mathscr{B}. \qquad (29)$$

The space X is said to have the *Radon-Nikodým property* if its unit ball B_X has the Radon-Nikodým property [22]. For the basic properties and useful information on the Radon-Nikodým property, see also [22–25]. It has been shown [9] that if X has the Radon-Nikodým property, then $\rho\mathscr{A}(B_X)$ is norming for $\mathscr{A}(B_X)$.

Corollary 8. *Suppose that X satisfies one of the following conditions: (i) X has the Radon-Nikodým property; (ii) X is locally uniformly convex space; (iii) X is a locally c-uniformly convex order continuous sequence space. Then we have*

(i) $n^{(k)}(\mathscr{A}(B_X : Y)) = n^{(k)}(Y)$ *for every* $k \geq 1$,

(ii) $n_{ua}(\mathscr{A}(B_X : Y)) = n_{ua}(Y)$.

Proof. If X satisfies one of the three conditions, $\rho\mathscr{A}(B_X : Y) = \rho\mathscr{A}(B_X)$ and it is norming for $\mathscr{A}(B_X : Y)$. Therefore, Theorem 3 implies that $n^{(k)}(\mathscr{A}(B_X : Y)) \geq n^{(k)}(Y)$, $n_{ua}(\mathscr{A}(B_X : Y)) \geq n_{ua}(Y)$. For the case (ii), fix $P \in \mathscr{A}_u(B_Y : Y)$ and define the map $Q_P : \mathscr{A}(B_X : Y) \longrightarrow C_b(B_X : Y)$ by $Q_P(f)(x) = P(f(x))$ for $f \in \mathscr{A}(B_X : Y)$ and $x \in B_X$. Then $Q_P(f) \in \mathscr{A}(B_X : Y)$. Consequently, Theorem 6 shows that

$$n_{ua}(\mathscr{A}(B_X : Y)) = n_{ua}(Y) \qquad (30)$$

and the proof of (ii) is complete. The remaining proof (i) can be finished in the same way by Theorem 7. ☐

By Theorem 3, we get the following.

Corollary 9. *Let Ω be a Hausdorff topological space and suppose that $\rho C_b(\Omega)$ is norming for $C_b(\Omega)$. If Y is a Banach space with $n_{ba}(Y) = 1$, we have $v(f) = \|f\|$ for all $f \in \mathscr{A}_b(B_{C_b(\Omega:Y)} : C_b(\Omega : Y))$.*

As we show in the next proposition, closed bounded convex sets with the Radon-Nikodým property satisfy the condition of Corollary 9.

Proposition 10. *Suppose that Ω is a nonempty closed bounded convex subset of a Banach space and Ω has the Radon-Nikodým property. Then $\rho C_b(\Omega)$ is norming for $C_b(\Omega)$ and the set of strong peak functions of $C_b(\Omega)$ is dense.*

Proof. It is enough to show that the set of strong peak functions of $C_b(\Omega)$ is dense by Corollary 5. Given $f \in C_b(\Omega)$ and $\epsilon > 0$, the Stegall perturbed optimization theorem [25] shows that there is $x^* \in X^*$ such that the function $\varphi(x) = |f(x)| + |\operatorname{Re}(x^*(x))|$ strongly attains its norm at $x_0 \in \Omega$ and $\|x^*\| < \epsilon$. Choose a complex number $z_0 \in S_{\mathbb{C}}$ such that

$$|f(x_0)| + |\operatorname{Re}(x^*(x_0))| = |f(x_0) + z_0 \operatorname{Re}(x^*(x_0))|. \qquad (31)$$

Then it is easy to check that $g(x) = f(x) + z_0 \operatorname{Re}(x^*(x))$ is a strong peak function at x_0 and $\|f - g\| < \epsilon$. This shows the denseness of the set of strong peak functions on $C_b(\Omega)$. ☐

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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Research Article

Convergence Analysis of an Accelerated Iteration for Monotone Generalized α-Nonexpansive Mappings with a Partial Order

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In this paper, we introduce a new accelerated iteration for finding a fixed point of monotone generalized α-nonexpansive mapping in an ordered Banach space. We establish some weak and strong convergence theorems of fixed point for monotone generalized α-nonexpansive mapping in a uniformly convex Banach space with a partial order. Further, we provide a numerical example to illustrate the convergence behavior and effectiveness of the proposed iteration process.

1. Introduction

Let $(E, \leq)$ be an ordered Banach space endowed with the partial order $\leq$ and K be a nonempty closed convex subset of E. A mapping $T : K \longrightarrow K$ is called monotone if $Tx \leq Ty$ whenever $x \leq y$ for all $x, y \in K$. Moreover, T is said to be as follows:

(1) Monotone nonexpansive if T is monotone and such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x \leq y. \tag{1}$$

(2) Monotone quasinonexpansive if T is monotone with $F(T) \neq \emptyset$ such that

$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \leq x \text{ or } x \leq p, \tag{2}$$

where $p \in F(T)$, the set of fixed points of T, i.e., $F(T) = \{x \in K,\ Tx = x\}$.

(3) Monotone α-nonexpansive if T is monotone and there exists a constant $\alpha < 1$ such that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 \\ &\quad + (1 - 2\alpha) \|x - y\|^2, \quad \forall x \leq y. \end{aligned} \tag{3}$$

(4) Suzuki's generalized nonexpansive if T satisfy condition (C), i.e., $(1/2)\|x - Tx\| \leq \|x - y\|$ implies

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K, \tag{4}$$

which is an interesting generalization of nonexpansive mapping because it is weaker than nonexpansiveness and stronger than quasinonexpansiveness [1].

(5) Monotone generalized α-nonexpansive if T is monotone and exists a constant $\alpha \in [0, 1)$ such that $(1/2)\|x - Tx\| \leq \|x - y\|$ implies

$$\begin{aligned} \|Tx - Ty\| &\leq \alpha \|Tx - y\| + \alpha \|Ty - x\| \\ &\quad + (1 - 2\alpha) \|x - y\|, \quad \forall x \leq y. \end{aligned} \tag{5}$$

Obviously, a monotone α-nonexpansive mapping includes monotone nonexpansive (0-nonexpansive) mapping as special case. Every monotone mapping satisfying condition (C) is a monotone generalized α-nonexpansive mapping, but the converse is not true. Moreover, a monotone generalized α-nonexpansive mapping includes nonexpansive, firmly nonexpansive, Suzuki's generalized nonexpansive mapping as special cases and partially extends monotone α-nonexpansive mapping [2].

In 1965, Browder [3] proved that every nonexpansive self-mapping of a closed convex and bounded subset has a fixed point in a uniformly convex Banach space. Since then, a number of iteration methods have been developed to approximate fixed point of nonexpansive mappings and some other relevant problems; see [4–16] and the references therein. In these algorithms, Mann iteration is a fundamental method approximating fixed points of nonexpansive mappings, which is defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \tag{6}$$

where $\alpha_n \in (0, 1)$ and T is a nonexpansive mapping. The other important iteration widely used to approximate fixed point of nonexpansive mapping is Ishikawa iteration, which is defined by

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T x_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \end{aligned} \tag{7}$$

where $\alpha_n, \beta_n \in (0, 1)$. Note that Ishikawa iteration (7) improves the rate of convergence of Mann iteration process for an increasing function due to Ishikawa [17] and Rhoades [18].

In 2007, Agrawal et al. [19] modified (7) and considered the following two-step iteration process: for an arbitrary $x_1 \in K$, the sequence of $\{x_n\}$ is defined in the following manner:

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T x_n, \\ x_{n+1} &= (1 - \alpha_n) T x_n + \alpha_n T y_n, \end{aligned} \tag{8}$$

where $\alpha_n, \beta_n \in (0, 1)$ and T is a nearly asymptotically nonexpansive mapping. They claimed that this iteration process converges faster than the Mann iteration for some contractions.

Recently, Noor [20] modified (7) and further studied a three-step iteration process to solve the general variational inequalities: for an arbitrary $x_1 \in K$ defined a sequence $\{x_n\}$ by

$$\begin{aligned} z_n &= (1 - \gamma_n) x_n + \gamma_n T x_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T z_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \end{aligned} \tag{9}$$

where $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and T is a strong monotone mapping involved variational inequalities. Very recently, Abbas-Nazir [21] and Thakur et al. [22] modified Noor iteration (9) and introduced a new faster iteration process for solving the constrained minimization and feasibility problems and for finding the fixed point of Suzuki's generalized nonexpansive mappings, respectively.

On the other hand, in 2004, Ran-Reurings [23] firstly introduced a fixed point theorem in a partially ordered metric space and some applications to matrix equations. They developed a new field only for comparable elements instead of the nonexpansive (or Lipschitz) condition in a partially ordered metric space, which has been successfully applied to solve not only the existence of fixed points but also a positive or negative solution of ordinary differential equations [24].

In 2015, Bin Dehaish-Khamsi [25] applied the Mann iteration (6) to the case of a monotone nonexpansive mapping in a Banach space endowed with a partial order. Moreover, they proved that $\{x_n\}$ generated by (6) weakly converges to $x^* \in F(T)$ and x^* and x_1 are comparable.

In 2016, Song et al. [26] further extended the Mann iteration (6) to monotone α-nonexpansive mappings and obtained some weak and strong convergence theorems in an ordered Banach space, which complemented the fixed point results of α-nonexpansive mappings in Aoyama-Kohsaka [27]. However, in general, the monotone condition on comparable elements is a weaker assumption. In particular, the continuity property probably is not valid, which not only reduces the efficiency of numerical approach but also increases the difficulty of convergence analysis. This is also the main reason why Mann iteration has become popular in approximating the fixed point of monotone-type mappings [2, 25, 26]. Therefore, it is important and interesting to construct an iterative accelerator method for finding the fixed points problem of such class of monotone-type mappings.

Inspired and motivated by research going on in this area, we modify the iteration process (6), (8), and (9) to the case of monotone generalized α-nonexpansive mappings and introduce a new accelerated iteration: for an arbitrary $x_1 \in K$, sequence $\{x_n\}$ is defined by

$$\begin{aligned} z_n &= (1 - \gamma_n) x_n + \gamma_n T x_n, \\ y_n &= (1 - \beta_n) T x_n + \beta_n T z_n, \\ x_{n+1} &= T\left[(1 - \alpha_n) y_n + \alpha_n z_n\right], \end{aligned} \tag{10}$$

Our purpose is not only to extend Mann iteration of Bin Dehaish-Khamsi [25] and Song et al. [26] to an accelerated iteration for monotone generalized α-nonexpansive mappings, but also to establish some weak and strong convergence theorems of fixed point for monotone generalized α-nonexpansive mapping in a uniformly convex Banach space with a partial order. Furthermore, we provide a numerical example to illustrate the convergence behavior and effectiveness of the proposed iteration. The method and results presented in this paper extend and improve the corresponding results of [2, 17, 19, 20, 25, 26] and some others previously.

2. Preliminaries

Recall that a Banach space E with the norm $\|.\|$ is called uniformly convex if, for all $\varepsilon \in (0, 2]$, there exists a constant $\delta > 0$ for which $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| \ge \varepsilon$ implies

$$\frac{1}{2}\|x + y\| \le 1 - \delta. \tag{11}$$

A Banach space E is said to satisfy the Opial property [5] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x,

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\| \tag{12}$$

holds, for all $y \in E$ with $y \neq x$.

Let K be a nonempty subset of a Banach space E and $\{x_n\}$ be a bounded sequence in E. For each $x \in E$, we define the following:

(i) Asymptotic radius of $\{x_n\}$ at x by $r(x, \{x_n\}) := \limsup_{n\to\infty}\|x_n - x\|$.

(ii) Asymptotic radius of $\{x_n\}$ relative to K by $r(K,\{x_n\}) := \inf\{r(x,x_n) : x \in K\}$.

(iii) Asymptotic center of $\{x_n\}$ relative to K by $A(K,\{x_n\}) := \{x \in K : r(x,\{x_n\}) = r(K,\{x_n\})\}$.

Note that $A(K,\{x_n\})$ is nonempty. Further, if E is uniformly convex, then $A(K,\{x_n\})$ has exactly one point [28].

Recall also that an order interval $[a,b]$ is defined by

$$[a,b] = \{x \in E : a \leq x \leq b\} = [a,\longrightarrow) \cap (\longleftarrow, b], \tag{13}$$

where $[a,\longrightarrow) = \{x \in E : a \leq x\}$ and $(\longleftarrow, b] = \{x \in E : x \leq b\}$. Throughout, we assume that the order intervals are closed and convex in an ordered Banach space $(E,\leq)$.

Lemma 1 (see [1]). *Let K be a nonempty closed convex subset of an ordered Banach space $(E,\leq)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Then, for all $x, y \in K$ with $x \leq y$, the following inequalities hold.*

(i) $\|Tx - T^2x\| \leq \|x - Tx\|$,

(ii) $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|$ *or*
$\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \alpha\|Tx - Ty\| + (1-2\alpha)\|Tx - y\|$.

Lemma 2 (see [2]). *Let K be a nonempty subset of an ordered Banach space $(E,\leq)$ and $T : K \longrightarrow K$ be a generalized α-nonexpansive mapping. Then $F(T)$ is closed.*

Lemma 3 (see [2]). *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach spaces $(E,\leq)$. Let $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{T^n(x)\}$ is a bounded sequence for some $x \in K$ with $x \leq Tx$.*

Lemma 4 (see [29]). *A Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $f : [0,+\infty) \longrightarrow [0,+\infty)$ with $f(0) = 0$ such that*

$$\begin{aligned}\|\lambda x + (1-\lambda) y\|^2 &\leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 \\ &\quad - \lambda(1-\lambda) f(\|x - y\|),\end{aligned} \tag{14}$$

where $\lambda \in [0,1]$ and $x, y \in B_r(0) = \{x \in E : \|x\| \leq r,\ r > 0\}$.

Lemma 5 (see [30]). *Let E be a uniformly convex Banach space and $\{\lambda_n\}$ be a sequence with $0 < \liminf_{n\longrightarrow\infty}\lambda_n \leq \limsup_{n\longrightarrow\infty}\lambda_n < 1$. Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\longrightarrow\infty}\|x_n\| \leq r$, $\limsup_{n\longrightarrow\infty}\|y_n\| \leq r$ and $\lim_{n\longrightarrow\infty}\|\lambda_n x_n + (1-\lambda_n) y_n\| = r$. Then*

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0. \tag{15}$$

Lemma 6. *Let K be a nonempty closed convex subset of an ordered Banach space $(E,\leq)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Then*

(i) $\|Tx - Ty\| \leq (2/(1-\alpha))\|x - Tx\| + \|x - y\|$, $\forall x, y \in K$ *with $x \leq y$.*

(ii) *T is monotone quasinonexpansive if $F(T) \neq \emptyset$ and $p \in F(T)$ with $x \leq p$ or $p \leq x$.*

Proof. (i) From Lemma 1 (ii), in the first case, we have

$$\begin{aligned}\|Tx - Ty\| &\leq \alpha\|Tx - y\| + \alpha\|x - Ty\| \\ &\quad + (1-2\alpha)\|x - y\| \\ &\leq \alpha[\|Tx - x\| + \|x - y\|] \\ &\quad + \alpha[\|x - Tx\| + \|Tx - Ty\|] \\ &\quad + (1-2\alpha)\|x - y\| \\ &= 2\alpha\|x - Tx\| + \alpha\|Tx - Ty\| \\ &\quad + (1-2\alpha)\|x - y\|,\end{aligned} \tag{16}$$

which implies that

$$\|Tx - Ty\| \leq \frac{2\alpha}{1-\alpha}\|x - Tx\| + \|x - y\|. \tag{17}$$

In the other case of Lemma 1 (ii), we further have

$$\begin{aligned}\|Tx - Ty\| &\leq \|Tx - T^2x\| + \|T^2x - Ty\| \\ &\leq \|x - Tx\| + \alpha\|Tx - Ty\| + \alpha\|T^2x - y\| \\ &\quad + (1-2\alpha)\|Tx - y\| \\ &\leq \|x - Tx\| + \alpha\|Tx - Ty\| \\ &\quad + \alpha[\|T^2x - Tx\| + \|Tx - y\|] \\ &\quad + (1-2\alpha)\|Tx - y\| \\ &\leq (1+\alpha)\|x - Tx\| + \alpha\|Tx - Ty\| \\ &\quad + (1-\alpha)[\|Tx - x\| + \|x - y\|] \\ &= 2\|x - Tx\| + \alpha\|Tx - Ty\| \\ &\quad + (1-\alpha)\|x - y\|,\end{aligned} \tag{18}$$

which implies that

$$\|Tx - Ty\| \leq \frac{2}{1-\alpha}\|x - Tx\| + \|x - y\|. \tag{19}$$

The desired conclusion follows immediately from (17) and (19) for all $x, y \in K$ and $\alpha \in [0,1)$.

(ii) By the definition of monotone generalized α-nonexpansive mapping, we have

$$\begin{aligned}\|Tx - p\| &= \|Tx - Tp\| \\ &\leq \alpha\|Tx - p\| + \alpha\|Tp - x\| \\ &\quad + (1-2\alpha)\|x - p\| \\ &\leq \alpha\|Tx - p\| + (1-\alpha)\|x - p\|,\end{aligned} \tag{20}$$

where $p \in F(T)$, and so $\|Tx - p\| \leq \|x - p\|$; that is, T is monotone quasinonexpansive. □

3. Main Results

Lemma 7. *Let K be a nonempty closed convex subset of an ordered Banach space $(E, \leq)$ and $T : K \longrightarrow K$ be a monotone mapping. Assume that the sequence $\{x_n\}$ is defined by the iteration (10) and $x_1 \leq Tx_1$. Then*

(i) $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n$;

(ii) $\{x_n\}$ has at most one weak-cluster point $x \in K$. Moreover, $x_n \leq x$ for all $n \geq 1$ provided $\{x_n\}$ weakly converges to a point $x \in K$.

Proof. (i) Note that if $c_1, c_2 \in K$ are such that $c_1 \leq c_2$, then $c_1 \leq \lambda c_1 + (1-\lambda)c_2 \leq c_2$ holds from the convex property defined on order intervals. This allows us to focus only on the proof of $x_n \leq Tx_n$ for any $n \geq 1$. By $x_1 \leq Tx_1$, we suppose that $x_n \leq Tx_n$ for $n \geq 2$. From (10), we have

$$\begin{aligned} x_n &\leq (1-\gamma_n) x_n + \gamma_n Tx_n = z_n \\ &\leq (1-\gamma_n) Tx_n + \gamma_n Tx_n = Tx_n. \end{aligned} \tag{21}$$

Since T is monotone, we obtain $x_n \leq z_n \leq Tx_n \leq Tz_n$. Using (10) again, we obtain

$$\begin{aligned} Tx_n &= (1-\beta_n) Tx_n + \beta_n Tx_n \leq (1-\beta_n) Tx_n + \beta_n Tz_n \\ &= y_n \leq (1-\beta_n) Tz_n + \beta_n Tz_n = Tz_n, \end{aligned} \tag{22}$$

which implies that $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n$. Similarly, we have

$$z_n \leq (1-\alpha_n) y_n + \alpha_n z_n \leq (1-\alpha_n) y_n + \alpha_n y_n = y_n. \tag{23}$$

It follows from (23) that $Tz_n \leq x_{n+1} \leq Ty_n$. Consequently, $x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n$, which further implies that $x_{n+1} \leq Tx_{n+1}$.

(ii) The desired conclusion follows from (i) and Lemma 3.1 in Bin Dehaish-Khamsi [25]. □

Theorem 8. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping. Suppose that the sequence $\{x_n\}$ defined by (10) is bounded and* $\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. *Then $F(T) \neq \emptyset$.*

Proof. Since $\{x_n\}$ is a bounded sequence and $\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\longrightarrow\infty} \left\| x_{n_k} - Tx_{n_k} \right\| = 0. \tag{24}$$

The asymptotic center of $\{x_{n_k}\}$ with respect to K is denoted by $A(K, \{x_{n_k}\}) = \{x^*\}$ such that $x_{n_k} \leq x^*$ for all $n \in \mathbb{N}$, such x^* is unique. From the definition of asymptotic radius, we have

$$r\left(Tx^*, \left\{x_{n_k}\right\}\right) = \limsup_{k\longrightarrow\infty} \left\| x_{n_k} - Tx^* \right\|. \tag{25}$$

Using Lemma 6 (i) and (24), we further obtain

$$\begin{aligned} &r\left(Tx^*, \left\{x_{n_k}\right\}\right) \\ &\quad\leq \limsup_{k\longrightarrow\infty} \left[\left\| x_{n_k} - Tx_{n_k} \right\| + \left\| Tx_{n_k} - Tx^* \right\| \right] \\ &\quad= \limsup_{k\longrightarrow\infty} \left\| Tx_{n_k} - Tx^* \right\| \\ &\quad\leq \limsup_{k\longrightarrow\infty} \left[\frac{2}{1-\alpha} \left\| x_{n_k} - Tx_{n_k} \right\| + \left\| x_{n_k} - x^* \right\| \right] \\ &\quad= r\left(x^*, \left\{x_{n_k}\right\}\right). \end{aligned} \tag{26}$$

It follows from the uniqueness of x^* that $Tx^* = x^*$, which shows that $F(T) \neq \emptyset$. □

Theorem 9. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ with Opial property. Let $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that there exists a $x_1 \in K$ such that $x_1 \leq Tx_1$, then the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions*

(i) $\alpha_n \in [a, b] \subset (0, 1)$, $\beta_n \in (0, 1)$;

(ii) $\gamma_n \in (0, 1)$ *and* $0 < \liminf_{n\longrightarrow\infty}\gamma_n \leq \limsup_{n\longrightarrow\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ generated by (10) weakly converges to a fixed point $q \in F(T)$.

Proof. Firstly, we show $\{x_n\}$ is bounded. Taking $p \in F(T)$, without loss of generality, we assume $x_1 \leq p$. Associating with the monotone property of T, we find

$$x_1 \leq Tx_1 \leq Tp = p. \tag{27}$$

From (10) and (27), we have

$$\begin{aligned} z_1 &= (1-\gamma_1) x_1 + \gamma_1 Tx_1 \leq p, \\ Tz_1 &\leq Tp = p, \\ y_1 &= (1-\beta_1) Tx_1 + \beta_1 Tz_1 \leq p, \\ Ty_1 &\leq Tp = p, \\ x_2 &= T\left[(1-\alpha_1) y_1 + \alpha_1 z_1\right] \leq Tp = p, \\ Tx_2 &\leq Tp = p. \end{aligned} \tag{28}$$

Continuing in this way, we can assume that $x_n \leq p$, we get $Tx_n \leq Tp = p$. Similarly, we have $y_n \leq p, Ty_n \leq Tp = p$ and $z_n \leq p, Tz_n \leq Tp = p$. By Lemma 7 (i), we obtain

$$x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n \leq p, \tag{29}$$

which implies that $x_{n+1} \leq p$. Therefore, the sequence $\{x_n\}$ is bounded, and so $\{y_n\}$ and $\{z_n\}$ are also bounded.

Secondly, we prove that $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. From (10) and Lemma 6 (ii), we have

$$\begin{aligned}\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{30}$$

Similarly, from (10) and (30), we have

$$\begin{aligned}\|y_n - p\| &\le (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{31}$$

Combining (10), (30), and (31), we obtain

$$\begin{aligned}\|x_{n+1} - p\| &= \|T[(1-\alpha_n)y_n + \alpha_n z_n] - p\| \\ &\le \|(1-\alpha_n)y_n + \alpha_n z_n - p\| \\ &\le (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\le \|x_n - p\|,\end{aligned} \tag{32}$$

which implies the limit of $\{\|x_n - p\|\}$ exists, i.e., $\lim_{n\longrightarrow\infty}\|x_n - p\| = r$. Also, it follows from (30) that

$$\limsup_{n\longrightarrow\infty}\|z_n - p\| \le \limsup_{n\longrightarrow\infty}\|x_n - p\| = r. \tag{33}$$

Together (30), (31) with (32), we have

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|,\end{aligned} \tag{34}$$

which implies that

$$\begin{aligned}\|x_{n+1} - p\| - \|x_n - p\| &\le \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n} \\ &\le \|z_n - p\| - \|x_n - p\|.\end{aligned} \tag{35}$$

Hence, $\|x_{n+1} - p\| \le \|z_n - p\|$. Noting that $\alpha_n \in [a,b] \subset (0,1)$, we obtain

$$r = \liminf_{n\longrightarrow\infty}\|x_{n+1} - p\| \le \liminf_{n\longrightarrow\infty}\|z_n - p\|. \tag{36}$$

Moreover, from (33) and (36), we can get

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\|z_n - p\| \\ &\quad = \lim_{n\longrightarrow\infty}\|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| = r.\end{aligned} \tag{37}$$

On the other hand, by the nonexpansive property defined on T, we have

$$\limsup_{n\longrightarrow\infty}\|Tx_n - p\| \le \limsup_{n\longrightarrow\infty}\|x_n - p\| = r. \tag{38}$$

It follows from (37), (38) and Lemma 5 that

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0. \tag{39}$$

Finally, we show that $\{x_n\}$ weakly converges to $q \in F(T)$. By the boundness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ weakly converging $q \in C$ and $x_1 \le x_{n_k} \le q$. From Lemma 6 (i) and (39), we can obtain

$$\begin{aligned}&\limsup_{k\longrightarrow\infty}\left\|Tx_{n_k} - Tq\right\| \\ &\quad \le \limsup_{k\longrightarrow\infty}\left[\frac{2}{1-\alpha}\left\|x_{n_k} - Tx_{n_k}\right\| + \left\|x_{n_k} - q\right\|\right] \\ &\quad = \limsup_{k\longrightarrow\infty}\left\|x_{n_k} - q\right\|.\end{aligned} \tag{40}$$

Arguing by contradiction, we suppose that $q \ne Tq$. It follows from the Opial property of E that

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\left\|x_{n_k} - q\right\| &< \limsup_{k\longrightarrow\infty}\left\|x_{n_k} - Tq\right\| \\ &\le \limsup_{k\longrightarrow\infty}\left\|x_{n_k} - Tx_{n_k}\right\| \\ &\quad + \limsup_{k\longrightarrow\infty}\left\|Tx_{n_k} - Tq\right\| \\ &\le \limsup_{k\longrightarrow\infty}\left\|x_{n_k} - q\right\|.\end{aligned} \tag{41}$$

This is a contradiction. Therefore, we conclude $q = Tq$; that is, $q \in F(T)$. Moreover, if there exists another subsequence $\{x_{n_j}\} \subset \{x_n\}$ weakly converges $w \ne q$. Similarly, we have $w \in F(T)$. Note that $\lim_{n\longrightarrow\infty}\|x_n - w\|$ exists and

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - q\| &= \limsup_{k\longrightarrow\infty}\left\|x_{n_k} - q\right\| \\ &< \limsup_{k\longrightarrow\infty}\left\|x_{n_k} - w\right\| = \lim_{n\longrightarrow\infty}\|x_n - w\| \\ &= \limsup_{j\longrightarrow\infty}\left\|x_{n_j} - w\right\| \\ &< \limsup_{j\longrightarrow\infty}\left\|x_{n_j} - q\right\| = \lim_{n\longrightarrow\infty}\|x_n - q\|,\end{aligned} \tag{42}$$

This is a contradiction again. Consequently, $w = q$ and $\{x_n\}$ weakly converges to $q \in F(T)$. □

Theorem 10. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and $T : K \longrightarrow K$ be a monotone generalized α-nonexpansive mapping with $F(T) \ne \emptyset$. Suppose that there exists a $x_1 \in K$ such that $x_1 \le Tx_1$, the sequences $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions*

(i) $\alpha_n \in [a,b] \subset (0,1)$, $\beta_n \in (0,1)$;

(ii) $\gamma_n \in (0,1)$ *and* $0 < \liminf_{n\longrightarrow\infty}\gamma_n \le \limsup_{n\longrightarrow\infty}\gamma_n < 1$.

Then the sequence $\{x_n\}$ generated by (10) strong converges to a fixed point $q \in F(T)$ if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denotes the distance from x to $F(T)$.

Proof. Necessity is obvious. We only prove the sufficiency. Suppose that $\liminf_{n\longrightarrow\infty} d(x_n, F(T)) = 0$. From (31), $\lim_{n\longrightarrow\infty} d(x_n, F(T)$ exists. Thus

$$\lim_{n\longrightarrow\infty} d\left(x_n, F\left(T\right)\right) = 0. \tag{43}$$

By Theorem 9, we have that $\{x_n\}$ is bounded with $x_n \leq p$. Without loss of generality, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\|x_{n_j} - p_j\| \leq 1/2^j$ for all $j \geq 1$, where $\{p_j\}$ is a sequence in $F(T)$. Combining with (31), we have

$$\left\|x_{n_j+1} - p_j\right\| \leq \left\|x_{n_j} - p_j\right\| \leq \frac{1}{2^j}. \tag{44}$$

It follows from (44) that

$$\begin{aligned}\left\|p_{j+1} - p_j\right\| &\leq \left\|p_{j+1} - x_{n_j+1}\right\| + \left\|x_{n_j} - p_j\right\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \leq \frac{1}{2^{j-1}}.\end{aligned} \tag{45}$$

This shows that $\{p_j\}$ is a Cauchy sequence in $F(T)$. By Lemma 2, $F(T)$ is closed, so $\{p_j\}$ converges to some $q \in F(T)$. Moreover, by the triangle inequality, we have

$$\left\|x_{n_j+1} - q\right\| \leq \left\|x_{n_j} - p_j\right\| + \left\|p_j - q\right\|. \tag{46}$$

Taking $j \longrightarrow \infty$ implies that $\{x_{n_j}\}$ converges strongly to q. From (31) again, $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists, and the sequence $\{x_n\}$ converges strongly to $q \in F(T)$. □

4. Numerical Example

Example 1. Define a mapping $T : [0, 1] \longrightarrow [0, 1]$ by

$$Tx = \begin{cases} x + \dfrac{3}{5}, & x \in \left[0, \dfrac{1}{5}\right) \\ \dfrac{x+4}{5}, & x \in \left[\dfrac{1}{5}, 1\right]. \end{cases} \tag{47}$$

Note that T is not continuous. Setting $x = 18/100$, $y = 1/5$, we obtain

$$\|Tx - Ty\| = \frac{6}{100} > \frac{2}{100} = \|x - y\|, \tag{48}$$

that is, T is not a nonexpansive mapping. However, T is a monotone mapping with $x \leq Tx$ and a monotone generalized 3/8-nonexpansive mapping.

Numerical Results 4.2. To illustrate the convergence of the proposed algorithm, we provide some numerical results of Example 1 and comparison with the other iterations previously.

Firstly, we show the convergence behavior of scheme (10) with different initial points. To do this, we take $\alpha_n = n/(n+1)$, $\beta_n = 1/(n+5)$, $\gamma_n = n/\sqrt{(2n+9)^3}$ and set $\|x_n - x^*\| < 10^{-6}$ as stop criterion. From given $x_1 = 0.05, 0.50, 0.75, 0.95$, convergence behaviors of scheme (10) are displayed in Figure 1.

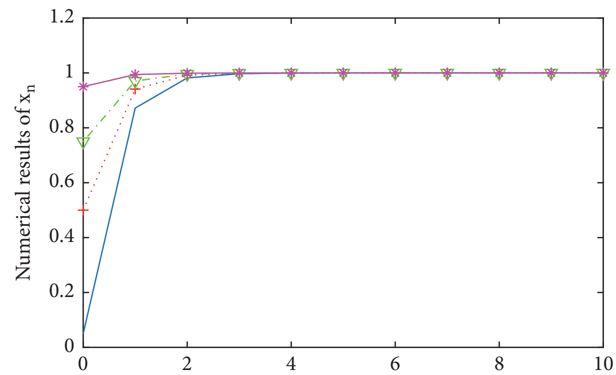

Figure 1: No. of iteration for initial $x_1 = 0.05, 0.5, 0.75, 0.95$.

Figure 1 shows that the given point x_1 has a little effect on convergence and scheme (10) is good in strong convergence and operational reliability. Moreover, numerical results show that the increasing of initial point x_1 has a little effect on the speed of convergence; that is, the sequence $\{x_n\}$ generated by (10) will converge faster to a fixed point of Example 1 when x_1 is increased.

Secondly, we further show the stability of scheme (10) based on the different iteration parameters. To complete it, we take $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ in the following manner.

(1) $\alpha_n = n/(n+1)$, $\beta_n = 1/(n+5)$ and $\gamma_n = n/\sqrt{(2n+9)^3}$.

(2) $\alpha_n = n/(n+1)$, $\beta_n = n/(n+5)$ and $\gamma_n = 1/\sqrt{2n+7}$.

(3) $\alpha_n = n/(2n+1)$, $\beta_n = n/(3n+5)$ and $\gamma_n = n/(4n+2)$.

(4) $\alpha_n = 1/\sqrt{n+5}$, $\beta_n = n/(n+1)$ and $\gamma_n = n/\sqrt{2n^2+7}$.

(5) $\alpha_n = 1/(n+1)^2$, $\beta_n = \sqrt{n}/(n+5)^3$ and $\gamma_n = n/(n+2)$.

(6) $\alpha_n = 1/(n+1)^2$, $\beta_n = \sqrt{n}/(n+5)^3$ and $\gamma_n = n/(n+2)^4$.

Moreover, we also set $\|x_n - x^*\| < 10^{-6}$ as stop criterion. From a given point $x_1 = 0.20$, computing results of scheme (10) are listed in Table 1.

Table 1 shows that the different parameters $\alpha_n, \beta_n, \gamma_n$ have an effect on iteration and scheme (10) is good in strong convergence and stability. Moreover, for the same initial point $x_1 = 0.2$, numerical results imply that the sequence $\{x_n\}$ generated by (10) will converge faster to a fixed point of Example 1 when parameter α_n is decreased or β_n is increased. In addition, $x_{n(5)}$ and $x_{n(6)}$ imply that parameters γ_n have almost no effect on convergence and iteration.

Finally, we compare the iteration numbers of new proposed method with the others known previously. To make it more obviously, we set $\|x_n - x^*\| < 10^{-10}$ as stop criterion. For given $x_1 = 0.05, 0.20, 0.50, 0.75, 0.95$, iteration numbers of scheme (10) and the known method are listed in Table 2 with some different parameters $\alpha_n, \beta_n, \gamma_n$ in Parameter 1, 3, 5.

Table 2 shows that the different parameters $\alpha_n, \beta_n, \gamma_n$ have a little effect on iteration and scheme (10) is good in strong convergence and effectiveness. Moreover, in Parameter 5, the numerical results imply that computing costs of Mann,

Table 1: Stability of iteration (10) with the different parameters $\alpha_n, \beta_n, \gamma_n$.

Iter.(n)	$x_{n(1)}$	$x_{n(2)}$	$x_{n(3)}$	$x_{n(4)}$	$x_{n(5)}$	$x_{n(6)}$
0	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000
1	0.905813	0.926044	0.932800	0.922392	0.946696	0.936396
2	0.986621	0.991600	0.993918	0.992965	0.997397	0.996334
3	0.997943	0.998943	0.999434	0.999382	0.999886	0.999817
4	0.999670	0.999858	0.999947	0.999947	0.999995	0.999992
5	0.999945	0.999980	0.999995	0.999995	1.000000	1.000000
6	0.999991	0.999997	1.000000	1.000000	1.000000	1.000000
7	0.999998	1.000000	1.000000	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table 2: Iteration numbers of scheme (10) and the others known previously.

Param.-point	Mann	Ishikawa	Agarwal	Noor	New method
Parameter 1:					
0.05	20	19	15	19	13
0.20	19	19	14	19	13
0.50	19	19	14	19	13
0.75	19	18	13	18	12
0.95	17	17	12	17	11
Parameter 3:					
0.05	48	45	14	45	10
0.20	48	44	14	44	10
0.50	47	43	14	43	10
0.75	45	42	13	42	10
0.95	42	39	12	39	9
Parameter 5:					
0.05	+	+	15	+	8
0.20	+	+	15	+	8
0.50	+	+	14	+	8
0.75	+	+	14	+	8
0.95	+	+	13	+	7

+ means the number of iterations over 1000.

Ishikawa, and Noor are too heavy. However, our scheme (10) is very advantageous for a wide range of parameters. In addition, scheme (10) requires the less number of iteration for the convergence than Agarwal's when the parameters α_n and β_n are decreased.

The computations are performed by Matlab R2016b running on a PC Desktop Intel(R) Core(TM)i5-5200U CPU @2.20GHz 2.20GHz, 8.00GB RAM.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Structure Properties for Binomial Operators

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In this paper, we discuss binomial operators structure properties, such as moments representation, derivatives representation, and binary representation and introduce some applications in preservation.

1. Introduction

As an extension to the well-known Bernstein operators, binomial operators are defined as follows (see [1], [2], or [3]):

$$\left(L_n^Q f\right)(x) = \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(1-x) f\left(\frac{k}{n}\right), \quad f \in C[0,1]; \tag{1}$$

if $b_n(1) = 0$, then those operators are substituted by

$$\left(L_n^Q f\right)(x) = \frac{1}{b_n(n)} \sum_{k=0}^{n} \binom{n}{k} b_k(nx) b_{n-k}(n-nx) f\left(\frac{k}{n}\right), \quad f \in C[0,1], \tag{2}$$

where $(b_n)_{n\geq 0}$ is a sequence of binomial polynomials; i.e., $b_n(x)$ is a polynomial of n degree satisfying

$$b_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y), \quad n = 0, 1, 2, \cdots. \tag{3}$$

Q is a delta operator (see Definition 2 below), which is determined by the sequence of binomial polynomials $(b_n)_{n\geq 0}$ uniquely.

In order to explore the binomial operators approximation and preservation, in present paper we are to investigate some structural properties, since behaviors of an operator are strongly dependent on its structure. In view of the Bernstein operators

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0,1], \tag{4}$$

with the following structural properties: endpoints interpolation, moments representation, derivatives representation, difference representation, and binary representation (see [4–10]), so our attention to the binomial operators will also focus on these aspects. On the study of Bernstein type operators, here we also want to refer to literatures [11–16].

In the next section, we introduce some primary concepts and results involved in this paper, which can be found in [1, 2, 17].

2. Notations and Preliminaries

Let Π_n be the linear space of polynomials of degree at most n and Π the space of all polynomials; i.e., $\Pi = \bigcup_{n\geq 0} \Pi_n$. We denote by I the identity operator and D the derivative. For real number a, the shift operator $E^a : \Pi \longrightarrow \Pi$ is defined by $E^a p(x) = p(x+a)$, $x \in (-\infty, +\infty)$.

Definition 1. If a linear operator $T : \Pi \longrightarrow \Pi$ commutes with all shift operators, then it is called a shift-invariant operator; i.e., $TE^a = E^a T$ for any a real number a.

From [17], we can find that if T_1 and T_2 are shift-invariant operators, then $T_1T_2 = T_2T_1$.

Definition 2. A shift-invariant operator Q is called a delta operator iff $Qe_1 = const \neq 0$, where $e_i = e_i(t) = t^i$, $i = 0, 1, \cdots$.

For delta operators, we have the following assertion.

Theorem 3. *The following statements are equivalent:*

(i) Q is a delta operator.

(ii) There exists a reversible shift-invariant operator P such that Q = DP.

(iii) There exists a power series $\phi(t) = \Sigma_{k=0}^{\infty} c_k(t^k/k!)$ *with* $c_0 = 0, c_1 \neq 0$ *such that* $Q = \phi(D)$.

Every shift-invariant operator can always be represented by any one delta operator that is so-called "First Expansion Theorem" as below.

Theorem 4. *Let T be a shift-invariant operator, and let Q be a delta operator with basic polynomials* $b_n(x)$*; then*

$$T = \sum_{k=0}^{\infty} \frac{a_k}{k!} Q^k \tag{5}$$

with $a_k = [Tb_k(x)]_{x=0}$.

Definition 5. Let $K : \Pi \longrightarrow \Pi$ be defined as $(Kh)(t) = th(t)$. For an operator $T : \Pi \longrightarrow \Pi$, its Pincherle derivative T' is defined by $T' = TK - KT$.

From [17], it is known that if T is a shift-invariant operator, then also is T'.

Definition 6. Let Q be a delta operator. A polynomial sequence $(b_n)_{n\geq 0}$ is called the sequence of basic polynomials associated with Q iff

$$\begin{aligned} b_0(x) &= 1, \\ b_n(0) &= 0, \\ Qb_n(x) &= nb_{n-1}(x), \end{aligned} \qquad n \geq 1. \tag{6}$$

It has been proved that every delta operator has a unique sequence of basic polynomials (see [17], Proposition 3]). Moreover, if $(b_n)_{n\geq 0}$ is the basic sequence of a delta operator Q, then $(b_n)_{n\geq 0}$ is a sequence of binomial polynomials, and the converse is also right.

In this paper, we also need the following assertions. If Q is a delta operator, then Q'^{-1} exists and the binomial operator L_n^Q has the following representation.

Theorem 7. *The operator* L_n^Q *can be represented in the form*

$$\left(L_n^Q f\right)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \cdots \frac{k}{n}; f\right] d_{k,n}(x), \tag{7}$$

where

$$d_{k,n}(x) = \frac{1}{b_n(1)} \left(\Theta^k E^{1-x} b_{n-k}\right)(x), \quad \Theta = KQ'^{-1} \tag{8}$$

and $[0, 1/n, \cdots k/n; f]$ *is divided difference of the function* f. *Using this theorem, we can get the moments of* L_n^Q *immediately*

$$\begin{aligned} &\left(L_n^Q e_r\right)(x) \\ &\quad = \frac{1}{b_n(1)} \sum_{k=0}^{r} \frac{k!}{n^k} \left[0, \frac{1}{n} \cdots \frac{k}{n}; e_r\right] \left(\Theta^k E^{1-x} b_{n-k}\right)(x), \\ &\qquad r = 0.1.2\cdots. \end{aligned} \tag{9}$$

In particular, when $r = 0, 1, 2$, *we have*

$$\begin{aligned} &\left(L_n^Q e_0\right)(x) = 1, \\ &\left(L_n^Q e_1\right)(x) = x, \\ &\left(L_n^Q e_2\right)(x) \\ &\quad = x^2 + \left(1 - \frac{n-1}{n} \frac{\left(Q'^{-2} b_{n-2}\right)(1)}{b_n(1)}\right) x(1-x). \end{aligned} \tag{10}$$

According to these moments and Korovkin theorem, it is easy to see that if L_n^Q *is positive and* $((n-1)/n)((Q'^{-2}b_{n-2})(1)/b_n(1)) \longrightarrow 1$, *then* L_n^Q *uniformly converges to the continuous function* f.

For convenience, we denote all the positive operators L_n^Q *by* $\mathscr{B}$. *Let* $\mathscr{F}$ *be the set of all formal power series* $\phi(t) = \sum_{k=1}^{\infty} d_k(t^k/k!)$ *with* $d_1 > 0, d_k \geq 0$*; then the positivity of* L_n^Q *can be characterized as follows.*

Theorem 8. *Let* L_n^Q *be defined as before, and* $Q = \phi(D)$*; then* $L_n^Q \in \mathscr{B}$ *iff*

$$\phi^{-1}(t) := \sum_{k=0}^{\infty} c_k \frac{t^k}{k!} \in \mathscr{F}. \tag{11}$$

Therefore, if $L_n^Q \in \mathscr{B}$, *then* $b_n(x)$ *has nonnegative coefficients, where* $(b_n)_{n\geq 0}$ *is the sequence of basic polynomials associated with Q.*

When $Q = D$, which is the simplest delta operator with the basic polynomials $b_n(x) = x^n, n = 0, 1, \cdots$, its corresponding binomial operator L_n^Q is Bernstein operator B_n; when $Q = I - E^{-1}$, which is also a delta operator and called backward difference operator with the basic polynomials $b_n(x) = x(x+1)\cdots(x+n-1)$, $n = 0, 1, \cdots$, its corresponding binomial operator is so-called Stancu operator (see [1] or [2]).

3. Some Structure Properties of the Operator L_n^Q

Similar to Bernstein operators, the binomial operators L_n^Q, on which some researches can be found in [18–24], also have the property of endpoint interpolation; i.e., $(L_n^Q f)(0) = f(0)$ and $(L_n^Q f)(1) = f(1)$.

Applying $[x_0, x_1, \ldots, x_k; f] = \triangle_h^k / k! h^k$ to Theorem 7, we have the difference representation of L_n^Q as follows:

$$\left(L_n^Q f\right)(x) = \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} \triangle_{1/n}^k f(0) \left(\Theta^k E^{1-x} b_{n-k}\right)(x). \tag{12}$$

As is known to all, the derivative of Bernstein operators can be expressed by difference operators (see [4]):

$$B_n^{(r)}(f, x) = n(n-1)\cdots(n-r+1) \sum_{k=0}^{n-r} \triangle_{1/n}^r f\left(\frac{k}{n}\right) b_{(n-r)k}(x), \tag{13}$$

but the derivatives of L_n^Q can not be expressed in this form; actually when $r = 1, 2$ they have the following representation.

Proposition 9. *Let Q be a delta operator and $(b_n)_{n\geq 0}$ be the sequence of basic polynomial associated with Q; then*

$$\begin{aligned} D\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=0}^{n-1} \binom{n}{k} b'_{n-k}(0) \\ &\cdot \sum_{l=0}^{k} \binom{k}{l} b_l(x) b_{k-l}(1-x) \cdot \left[f\left(\frac{n-k+l}{n}\right) - f\left(\frac{l}{n}\right) \right]; \\ D^2\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=1}^{n-1} \binom{n}{k} b'_{n-k}(0) \sum_{l=0}^{k-1} \binom{k}{l} b'_{k-l}(0) \\ &\cdot \sum_{j=0}^{l} \binom{l}{j} b_j(x) b_{l-j}(1-x) \cdot \left[f\left(\frac{n-l+j}{n}\right) - f\left(\frac{k-l+j}{n}\right) - f\left(\frac{n-k+j}{n}\right) + f\left(\frac{j}{n}\right) \right]. \end{aligned} \tag{14}$$

Proof. By Theorem 4, we have

$$D = \sum_{l=0}^{\infty} \frac{b'_l(0)}{l!} Q^l; \tag{15}$$

it follows that

$$b'_k(x) = D b_k(x) = \sum_{l=0}^{k} \binom{k}{l} b'_l(0) b_{k-l}(x), \tag{16}$$

so that

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} b'_k(x) b_{n-k}(1-x) f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \\ &\cdot b'_l(0) b_{k-l}(x) b_{n-k}(1-x) f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} b'_k(0) \\ &\cdot \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} b_{l-k}(x) b_{n-l}(1-x) f\left(\frac{l}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} \\ &\cdot b'_k(0) \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(x) b_{n-k-l}(1-x) f\left(\frac{k+l}{n}\right). \end{aligned} \tag{17}$$

In the same way, we have

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} b_k(x) b'_{n-k}(1-x) f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \binom{n}{k} b'_k(0) \\ &\cdot \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(x) b_{n-k-l}(1-x) f\left(\frac{l}{n}\right). \end{aligned} \tag{18}$$

Therefore, we get

$$\begin{aligned} D\left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=1}^{n} \binom{n}{k} b'_k(x) b_{n-k}(1-x) \\ &- \frac{1}{b_n(1)} \sum_{k=1}^{n} \binom{n}{k} b_k(x) b'_{n-k}(1-x) = \frac{1}{b_n(1)} \\ &\cdot \sum_{k=1}^{n} \binom{n}{k} b'_k(0) \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(x) b_{n-k-l}(1-x) \\ &\cdot \left[f\left(\frac{k+l}{n}\right) - f\left(\frac{l}{n}\right) \right] = \frac{1}{b_n(1)} \sum_{k=0}^{n-1} \binom{n}{k} b'_{n-k}(0) \\ &\cdot \sum_{l=0}^{k} \binom{k}{l} b_l(x) b_{k-l}(1-x) \\ &\cdot \left[f\left(\frac{n-k+l}{n}\right) - f\left(\frac{l}{n}\right) \right]. \end{aligned} \tag{19}$$

An argument similar to the above one, it is no difficult to obtain the expression for the second derivative of the operators $D^2(L_n^Q f)(x)$. The proof is completed. □

This proposition supplies us a new proof for the next results known (see [1]).

Theorem 10. *Let $L_n^Q \in \mathscr{B}$.*

(i) If f is increasing (decreasing) on $[0,1]$, then it also is $L_n^Q f$.

(ii) If f is convex (concave) on $[0,1]$, then it also is $L_n^Q f$.

Proof. (i) It is trivial.

(ii) Without loss of generality, we may assume f is convex. Recall that if f is convex on $[0,1]$, then for any $x_1, x_2, x_3, x_4 \in [0,1]$ and $x_1 < x_2, x_3 < x_4$ the following holds:

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq \frac{f(x_3) - f(x_4)}{x_3 - x_4}. \tag{20}$$

Obviously,

$$\begin{gathered} \frac{j}{n} < \frac{k-l+j}{n}, \\ \frac{n-k+j}{n} < \frac{n-l+j}{n}, \\ j < k < l < n, \end{gathered} \tag{21}$$

and it follows that

$$\begin{aligned} &\frac{f((n-k+j)/n) - f(j/n)}{1-k/n} \\ &\leq \frac{f((n-l+j)/n) - f((k-l+j)/n)}{1-k/n}, \end{aligned} \tag{22}$$

so that

$$\begin{aligned} &f\left(\frac{n-l+j}{n}\right) - f\left(\frac{k-l+j}{n}\right) - f\left(\frac{n-k+j}{n}\right) \\ &+ f\left(\frac{j}{n}\right) = \left(1 - \frac{k}{n}\right) \\ &\cdot \left[\frac{f((n-l+j)/n) - f((k-l+j)/n)}{1-k/n}\right. \\ &\left. - \frac{f((n-k+j)/n) - f(j/n)}{1-k/n}\right] \geq 0. \end{aligned} \tag{23}$$

According to Theorem 8, it follows that $(b_n)_{n\geq 0}$ has nonnegative coefficients, which mean $b_i(x) \geq 0$ for any $x \in [0,1]$ and $b_i'(0) \geq 0, i = 0, 1, \cdots$. These lead to $D^2(L_n^Q f) \geq 0$, which complete the proof. □

The binomial operators have binary representation similar to Bernstein operators as follows.

Proposition 11. *If L_n^Q is defined as before, then*

$$\begin{aligned} \left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{n-k} B_{nkl}(x,y) f\left(\frac{k}{n}\right); \\ \left(L_n^Q f\right)(y) &= \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{n-k} B_{nkl}(x,y) f\left(\frac{k+l}{n}\right), \end{aligned} \tag{24}$$

where $B_{nkl} = (n!/k!l!(n-k-l)!) b_k(x) b_l(y-x) b_{n-k-l}(1-y)$, $x < y$.

Proof. By definition of sequence of binomial polynomials, we have

$$\begin{aligned} b_{n-k}(1-x) &= \sum_{l=0}^{n-k} \binom{n-k}{l} b_l(y-x) b_{n-k-l}(1-y), \\ b_k(x) &= \sum_{l=0}^{k} \binom{k}{l} b_l(y-x) b_{k-l}(x), \end{aligned} \tag{25}$$

and thus

$$\begin{aligned} \left(L_n^Q f\right)(x) &= \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(1-x) f\left(\frac{k}{n}\right) \\ &= \frac{1}{b_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!l!(n-k-l)!} b_k(x) b_l(y-x) \\ &\cdot b_{n-k-l}(1-y) f\left(\frac{k}{n}\right) = \frac{1}{b_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} B_{nkl}(x,y) \\ &\cdot f\left(\frac{k}{n}\right), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \left(L_n^Q f\right)(y) &= \frac{1}{b_n(1)} \sum_{k=0}^{n} \binom{n}{k} b_k(y) b_{n-k}(1-y) f\left(\frac{k}{n}\right) \\ &= \frac{1}{b_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{k!l!(n-k-1)!} b_l(y-x) b_{k-l}(x) \\ &\cdot b_{n-k}(1-y) f\left(\frac{k}{n}\right) = \frac{1}{b_n(1)} \\ &\cdot \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!l!(n-k-l)!} b_k(x) b_l(y-x) \\ &\cdot b_{n-k-l}(1-y) f\left(\frac{k+l}{n}\right) = \frac{1}{b_n(1)} \\ &\cdot \sum_{k=0}^{n} \sum_{l=0}^{n-k} B_{nkl}(x,y) f\left(\frac{k+l}{n}\right), \end{aligned} \tag{27}$$

where

$$B_{nkl} = \frac{n!}{k!l!(n-k-l)!} b_k(x) b_l(y-x) b_{n-k-l}(1-y). \tag{28}$$

This completes the proof. □

Using this proposition, we not only can prove Theorem 2.2(i) but also show the following result, which can be found in [3].

Theorem 12. *If $L_n^Q \in \mathscr{B}$ and $f \in Lip_M \alpha (0 < \alpha \leq 1)$, then $L_n^Q f \in Lip_M \alpha$, where*

$$\begin{aligned} Lip_M \alpha = \{ f \in C[0,1] : \forall x, y \in [0,1], \exists M \\ > 0, s.t. |f(y) - f(x)| \leq M (y-x)^\alpha \}. \end{aligned} \tag{29}$$

Proof. First, we need the following two identities:

$$\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)=1,$$
$$\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)\frac{l}{n}=y-x. \quad (30)$$

The first one can be derived easily by the definition of the sequence of binomial polynomials; the rest only need to prove the second one. In fact, by the definition of $B_{nkl}(x,y)$ and exchange the order of sum, we have

$$\begin{aligned}\frac{1}{b_n(1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{n-k}B_{nkl}(x,y)\frac{l}{n}&=\frac{1}{b_n(1)}\\ \cdot\sum_{l=0}^{n}\binom{n}{l}b_l(y-x)\frac{l}{n}\\ \cdot\sum_{k=0}^{n-l}\binom{n-l}{k}b_k(x)b_{n-l-k}(1-y)&=\frac{1}{b_n(1)}\\ \cdot\sum_{l=0}^{n}\binom{n}{l}b_l(y-x)b_{n-l}(x+1-y)\frac{l}{n}&=y-x.\end{aligned} \quad (31)$$

Now, we turn to the proof of this theorem. Since, for any $x<y$ and $0<\alpha\le 1$, there is

$$\left|f(y)-f(x)\right|\le M(y-x)^{\alpha}, \quad (32)$$

by the identities above and $L_n^Q\in\mathscr{B}$; we obtain

$$\begin{aligned}&\left|\left(L_n^Qf\right)(y)-\left(L_n^Qf\right)(x)\right|\\ &\le\frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\left|f\left(\frac{k+l}{n}\right)-f\left(\frac{k}{n}\right)\right|\\ &\le M\frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\left(\frac{l}{n}\right)^{\alpha}\\ &\le M\left[\frac{1}{b_n(1)}\sum_{k=0}^{n}\sum_{l=0}^{n-k}B_{nkl}(x,y)\frac{l}{n}\right]^{\alpha}=M(y-x)^{\alpha},\end{aligned} \quad (33)$$

which means that $L_n^Qf\in Lip_M\alpha$. The proof of the theorem is now complete. □

In this paper, we discuss structure properties of the binomial operators, present the moments representation, the derivatives representation, and the binary representation of these operators, and introduce some applications in preservation. Here authors also wish to express their gratitude to the reviewers for the careful review and suggestions for improvement of this manuscript.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On New Picard-Mann Iterative Approximations with Mixed Errors for Implicit Midpoint Rule and Applications

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In order to solve (partial) differential equations, implicit midpoint rules are often employed as a powerful numerical method. The purpose of this paper is to introduce and study a class of new Picard-Mann iteration processes with mixed errors for the implicit midpoint rules, which is different from existing methods in the literature, and to analyze the convergence and stability of the proposed method. Further, some numerical examples and applications to optimal control problems with elliptic boundary value constraints are considered via the new Picard-Mann iterative approximations, which shows that the new Picard-Mann iteration process with mixed errors for the implicit midpoint rule of nonexpansive mappings is brand new and more effective than other related iterative processes.

1. Introduction

In science and engineering fields, such as Stefan-Boltzmann radiation law, Lotka-Voterra model in population dynamics, etc., more and more problems can be modeled by optimal control problems with the constraints of (partial) differential equations. Moreover, multidimensional dynamical systems can be frequently formulated by partial differential equations, which generally depend on space and time, i.e., parabolic or evolutionary type equations, and are treated with emphasis on various real-world applications in (thermo) mechanics of solids and fluids, electrical devices, engineering, chemistry, biology, etc.

Thus, it is very significative and considerable to study existence of solutions for optimal control problems constrained by (partial) differential equations. Some algorithms were provided for the following optimal control problem in [1]:

$$\min J(u), \tag{1}$$

where control variable $u \in U = L^2(\partial\Omega)$, control space, satisfies some suitable conditions or constraints; $\Omega \subset \mathbb{R}^2$ is a bounded convex region. For example, Liu and Sun [2] considered the optimal boundary control problem (1) of the following elliptic equation constraints:

$$\begin{aligned} -\Delta y &= f(x,u), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where $y(u) \in V = H^1(\Omega)$, state space, is state variable, boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ is smooth, Γ_N and Γ_D are, respectively, Neumann boundary and Dirichlet boundary, and $\Gamma_N \cap \Gamma_D = \emptyset$. Yan [3] explored the adaptive finite element methods for some optimal control problems governed by (2). However, when the adaptive finite methods are used to solve this kind of problems, the calculation will be very large if the calculation area of the problem is large. To overcome this difficulty, Yan [4] introduced iterative nonoverlapping domain decomposition method for the optimal boundary control problem (1) governed by the elliptic equations (2), which can avoid large amounts of calculation produced by traditional numerical methods. Many scholars devoted themselves to this kind of optimal control problem (1) with elliptic PDE (partial differential equation) constraints (2). For more details, we refer to [5] and references therein. Further, Pearson and

Wathen [6] presented a new Schur complement approximation for PDE-constrained optimization and designed preconditioners under some certain optimality properties, derived eigenvalue bounds to verify the effectiveness of the approximation, and presented numerical results that show that these new preconditioners work well in practice. Zeng et al. [7] developed some preconditioning techniques for reduced saddle point systems arising from linear elliptic distributed optimal control problems and obtained the bounds of these eigenvalues with respect to the mesh size. These methods are mainly based on the time required for solution scales linearly with the problem size, and the mesh size in these optimal methods is also hard to choose. Very recently, Xu and Zhang [8] explored the positive solutions for singular positone and semipositone boundary value problems by use of the Leray-Schauder nonlinear alternative and a fixed point theorem on cones.

By using the u_0-positive operator and the fixed point index theorem, Yao et al. [9] investigated the existence and uniqueness of positive solutions of the following boundary value problem:

$$\begin{aligned} -D_{0^+}^{\alpha} x(t) + bx(t) &= a(t) f(t, x(t)), \quad 0 < t < 1, \\ x(0) &= 0, \\ x(1) &= 0, \end{aligned} \tag{3}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $1 < \alpha < 2$, $b > 0$, $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous, and $a(t)$ is continuous and may be singular at $t = 0$ and $t = 1$.

In [10], Lan pointed out that "the time-dependent form of partial differential equations can be rewritten as a time-independent form under some suitable conditions". This Neumann elliptic boundary value problem of (3) has attracted widely attention to the researchers because of its existence in many fields. Ashyralyyev and Dedeturk [11] considered the inverse problem for a multidimensional elliptic equation with mixed boundary conditions and overdetermination. In this paper they constructed the first and second orders of accuracy in t and the second order of accuracy in space variables and analyzed the stability, almost coercive stability, and coercive stability for the approximate solution of this inverse problem. Furthermore, the solvability of perturbed optimal control problems, the uniqueness of their solutions, the asymptotic properties of optimal pairs as the perturbation parameter $\varepsilon > 0$ tends to zero, and deriving of optimality conditions for the perturbed optimal control problems were discussed in [12]. The sufficient conditions of the existence of weak solutions to the given class of nonlinear Neumann boundary value problems were established and a way for their approximation was also proposed. Lately, Kashiwabara and Kemmochi [13] established $O(h^2|\log h|)$ and $O(h)$ errors bounds in the L^{∞} and $W^{1,\infty}$-norms for the Neumann boundary value problems in a smooth space by combining the technique of regularized Green's function with local H^1- and L^2-estimates in dyadic annuli. And elliptic variational forms of second-order physician, physicist, and anatomist equation can also be represented by some special cases of the Dirichlet problem (2) (see [14]). As is known to all, many problems in physics and other applications cannot be formulated as equations because of some more complicated structure, usually of a so-called complementarity problem, which is equivalent to a variational inequality. And the applicability of variational inequality theory which was initially developed to cope with equilibrium problems has been extended to involve problems from engineering science, electrodynamics, optimization, economics, finance, mechanics, and game theory. So the variational method is very important in optimal control theory, and such generalization is often needed in optimal control theory of elliptic problems.

On the other hand, iteration methods of Picard, Mann, Ishikawa, and the other associated iterations are the research focus to solve optimal control problem for the constraint of partial differential equation. These iterative processes have been deeply studied and applied by many authors. Such as, Khan [15] introduced a Picard-Mann hybrid iterative process to solve equation systems which converges faster than all of Picard, Mann, and Ishikawa iterative processes for contractions. Based on this, Deng [16] introduced a modified Picard-Mann hybrid iterative algorithm for a sequence of nonexpansive mappings. He also established strong convergence and weak convergence of the iterative sequence generated by the modified hybrid iterative algorithm in a convex Banach space. After that, Picard-Krasnoselskii hybrid iterations which converge faster than Picard, Mann, Krasnoselskii, and Ishikawa iterative processes for contractive nonlinear operators were introduced by Okeke and Abbas [17]. What is more, Jiang et al. [18] studied convergence of Mann iterative sequences for approximating solutions of a higher order nonlinear neutral delay differential equation and proposed advantages of the presented results through three extraordinary examples. Very recently, Li and Lan [19] studied the convergence and stability of a class of new Picard-Mann iterative methods with mixed errors for common fixed points of two different nonexpansive and contraction operators in a normed space X as follows:

$$\begin{aligned} x_{n+1} &= T_1 y_n + h_n, \\ y_n &= (1-\alpha_n) x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \end{aligned} \tag{4}$$

where $T_1, T_2 : X \longrightarrow X$ are two nonlinear operators and $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation of the operator points. Further, we explored iterative approximation of solutions for an elliptic boundary value problem in Hilbert spaces by using the new Picard-Mann iterative methods with mixed errors. After the research of [15], Khan [20] approximated common attractive points of further generalized hybrid mappings by using iterative process without closedness assumption to the case of two mappings in Hilbert spaces. Levajković [21] presented an approximation framework for computing the solution of the stochastic linear quadratic control problem on Hilbert spaces. These brought generalizations and improvements of some results in the literature. The more expatiation of Picard, Mann, and Ishikawa process and other iterative approximation methods can be obtained by referring to [22–27] and references therein.

Moreover, as one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations (see, for example, [28–30] and the references therein), the implicit midpoint rule has been considered as the approximation methods. Luo et al. [28] introduced a viscosity iterative algorithm for the implicit midpoint rule in uniformly smooth spaces and proved some strong convergence theorems under some appropriate conditions on the parameters. Xu et al. [31] established the viscosity technique for the implicit midpoint rule in Hilbert spaces and proved the strong convergence of this technique under certain assumptions. Alghamdi [32] established implicit midpoint rule for nonexpansive mappings and analyzed the weak convergence of the algorithm in Hilbert space. As applications, the authors applied the main results in solving fixed point problems of strict pseudocontractive mappings, variational inequality problems in Banach spaces, and equilibrium problems in Hilbert spaces. Very recently, by using the generalized forward-backward splitting method and implicit midpoint rule, Chang et al. [33] introduced and proved some strong convergence of an iterative algorithm for finding a common element of solutions to quasi variational inclusions with accretive mapping and fixed points for a λ-strict pseudocontractive mapping in Banach spaces. In [34], Tang and Bao introduced a new semi-implicit midpoint rule with the general contraction for monotone mappings in Banach spaces, which converges strongly to a fixed point. To find the fixed point of nonexpansive mapping, using the implicit midpoint rule, Yao et al. [35] established an iteration algorithm, which is formed as

$$y_{n+1} = \left(1-\alpha_n\right) y_n + \alpha_n \left(\frac{T_n y_n + T_{n+1} y_{n+1}}{2}\right), \quad n \geq 0, \quad (5)$$

where $\alpha_n \in (0,1)$ for all $n \geq 0$, $T : C \longrightarrow C$ is nonexpansive operator with a nonempty closed convex bounded subset C of X. The authors proved the weak convergence of this iteration algorithm and proposed three control conditions. Moreover, as applications, the algorithm was applied in hierarchical minimization problem

$$\min_{x\in S_0} \psi_1 (x), \quad (6)$$

where $S_0 := \arg\min_{x\in H} \psi_0(x)$ and $\psi_0(x)$, $\psi_1(x)$ are two lower semicontinuous convex functions from H into $\mathbb{R}$ and their gradients satisfy the Lipschitz continuity conditions. An iteration algorithm based on (5) was presented to (6). The algorithm (5) was also explored in nonlinear time-dependent evolution equation, Fredholm integral equations, and variational inequalities and showed effectiveness in solving these problems. The technic of implicit midpoint rule and iteration algorithm are so powerful in equations that it deserves further studies.

Furthermore, Roussel [36] pointed out that "equilibria are not always stable". Being able to identify equilibrium points based on their stability is useful since stable and unstable equilibria play quite different roles in dynamics. And there are many authors and researchers who discussed stability of the iterative sequence generated by the algorithm for solving the investigated problems. See, for example, [11, 37] and the references therein. Recently, convergence and stability theorems for the Picard-Mann iterative scheme of iteration are attracting more and more attention in researches. Akewe and Okeke [38], perhaps for the first time, gave the stability theorems for the Picard-Mann hybrid iterative scheme for a general class of contractive-like operators. After that, Li and Lan [19] used different methods to analyze the stability and extended the application of the stability in iterations. But the convergence and stability theorems for the Picard-Mann iteration are often based on the iterative scheme of iteration, so the analysis of the convergence and stability is necessary when we apply the implicit midpoint rule in Picard-Mann iteration progress.

On the basis of the above studies, in this paper, it is different from existing methods in the literature that we will introduce and study a class of new Picard-Mann iteration processes with mixed errors for the implicit midpoint rules. Then, we analyze convergence and stability of the new Picard-Mann iterative approximations of the implicit midpoint rules for nonexpansive mappings in normed spaces. Finally, we give some numerical examples and applications to optimal control problems with elliptic boundary value constraints based on the new Picard-Mann iterative approximations. Finally, simulation results are provided to illustrate the effectiveness of the proposed methods.

2. New Picard-Mann Iterative Approximations

In this section, we shall introduce and study a new Picard-Mann iterative methods with mixed errors for the implicit midpoint rule of common fixed points of two different contraction and nonexpansive operators and prove convergence and stability of the new Picard-Mann iterative approximation.

We need the following definitions and lemmas for our main results.

Definition 1. Let K be a nonempty subset of a normed space X. Then a mapping $T : K \longrightarrow K$ is said to be

(i) nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in K; \quad (7)$$

(ii) contraction if

$$\|Tu - Tv\| \leq \theta \|u - v\|, \quad \forall u, v \in K, \ \theta \in [0, 1). \quad (8)$$

Here, we recall the constant θ as Lipschitz constant of T. Thus, contractive mapping is sometimes said to be Lipschitzian mapping. Further, we call the mapping T as nonexpansive when this condition hold instead for $\theta \leq 1$.

Definition 2 (see [19]). Let S be a selfmap of normed space X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\} \subset X$. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}$ converges to a fixed point x^* of S. Let $\{\omega_n\} \subset X$ and let $\varepsilon_n = \|\omega_{n+1} - h(S, w_n)\|$. If $\varepsilon_n = 0$ implies that $w_n \longrightarrow x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.

Lemma 3 (see [39]). *Let X be a normed space and C a nonempty closed convex bounded subset of X. Then each nonexpansive mapping $T : C \longrightarrow C$ has a fixed point in C.*

Lemma 4 (see [40]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n) a_n + t_n b_n + c_n, \tag{9}$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\min_{n\longrightarrow\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $a_n \longrightarrow 0$ $(n \longrightarrow \infty)$.

Based on new Picard-Mann iterative methods with mixed errors for common fixed point of two different nonexpansive and contraction operators due to Li and Lan [19], now we further consider the following new iteration process x_n (in short, (PMMDI)) with mixed errors for the implicit midpoint rule of two nonlinear mappings T_1 and T_2 in normed space X:

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 \left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n. \end{aligned} \tag{10}$$

Further, the Picard-Mann iteration process with mixed errors for the implicit midpoint rule of one of nonlinear mappings T_1 and T_2 is respectively defined as follows:

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \end{aligned} \tag{11}$$

which is called a new Picard-Mann iteration process with mixed errors for the implicit midpoint rule of Picard mapping (in short, (PMMDIP)) and that of Mann mapping (in short, (PMMDIM)) is

$$\begin{aligned} x_{n+1} &= T_1 y_n + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 \left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{12}$$

where $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation of the mapping points which satisfy the following conditions **EC**:

(i) $d_n = d'_n + d''_n$;

(ii) $\lim_{n\longrightarrow\infty} \|d'_n\| = \lim_{n\longrightarrow\infty} \|h_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|d''_n\| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Remark 5. For special choices of the operators T_1 and T_2, the space X, and errors h_n, d_n, and e_n in (10)-(12), one can obtain a large number of Picard iterative process, Mann iterative process, Picard-Mann iterative process, and other related iterations for the implicit midpoint rule. Now we list some special cases as follows.

Special Case I. If $h_n = d_n = e_n = 0$, the iterative process (10) becomes to the Picard-Mann iteration of the implicit midpoint rule in for two different operators (in short, (PMDI)). For any given $x_0 \in X$,

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 \left(\frac{x_n + y_n}{2}\right). \end{aligned} \tag{13}$$

The iterative process (11) reduces to the following implicit midpoint rule in Picard-Mann iterations for Picard type mapping (in short, (PMDIP)):

$$\begin{aligned} x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 x_n, \end{aligned} \tag{14}$$

and the iterative process (12) becomes as the following implicit midpoint rule in Picard-Mann iterations for Mann type mapping (in short, (PMDIM)):

$$\begin{aligned} x_{n+1} &= T_1 y_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T_2 \left(\frac{x_n + y_n}{2}\right). \end{aligned} \tag{15}$$

Special Case II. When $T_1 = T_2 = T$, the iterations (10), (11), and (12), respectively, reduce to the Picard-Mann iteration of implicit midpoint rule with mixed errors for one nonlinear mapping (in short, (PMMI))

$$\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T \left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{16}$$

the implicit midpoint rule in Picard-Mann iteration with mixed errors for Picard type mapping (in short, (PMMIP))

$$\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T x_n + \alpha_n d_n + e_n, \end{aligned} \tag{17}$$

and

$$\begin{aligned} x_{n+1} &= T y_n + h_n, \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T \left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \end{aligned} \tag{18}$$

which is called the implicit midpoint rule in Picard-Mann iteration with mixed errors for Mann type mapping (in short, (PMMIM)).

Special Case III. If $T_1 = T_2 = T$, then (13), (14), and (15), respectively, become as the Picard-Mann iterative process of implicit midpoint rule for one nonlinear mapping (in short, (PMI))

$$\begin{aligned} x_{n+1} &= T\left(\frac{x_n + y_n}{2}\right), \\ y_n &= (1 - \alpha_n) x_n + \alpha_n T \left(\frac{x_n + y_n}{2}\right), \end{aligned} \tag{19}$$

that for Picard mapping (in short, (PMIP))

$$x_{n+1} = T\left(\frac{x_n + y_n}{2}\right), \quad y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{20}$$

and that for Mann mapping (in short, (PMIM))

$$x_{n+1} = T y_n, \quad y_n = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right). \tag{21}$$

Special Case IV. When $T_2 = T$ and $T_1 = I$, the identity mapping, for any given $x_0 \in X$, the iterations (PMMDI) (10), (PMMDIP) (11), and (PMMDIM) (12) can be, respectively, written as Picard-Mann iterative process of the explicit and implicit midpoint rule with mixed errors (in short, (MMDI))

$$x_{n+1} = \frac{x_n + y_n}{2} + h_n, \quad y_n = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \tag{22}$$

the implicit midpoint rule for Picard iteration in Picard-Mann iterative process (in short, (MMDIP))

$$x_{n+1} = \frac{x_n + y_n}{2} + h_n, \quad y_n = (1-\alpha_n)x_n + \alpha_n T x_n + \alpha_n d_n + e_n, \tag{23}$$

and the implicit midpoint rule in Picard-Mann iteration process for Mann mapping (in short, (MMDIM)):

$$x_{n+1} = y_n + h_n, \quad y_n = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n. \tag{24}$$

Remark 6. We note that the iterations mentioned above are new and not studied in the literature yet.

Based on Lemma 3 and existence of fixed point for nonexpensive mapping, in the sequel, we will prove convergence and stability of the new Picard-Mann iterative process (PMMDI) (10).

Theorem 7. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $T_1 : C \longrightarrow C$ be nonexpansive and $T_2 : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) := \{x \in C : T_i x = x, i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then*

(i) the iterative sequence $\{x_n\}$ generated by (10) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{25}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty} \alpha_n \in (0,1]$;

(ii) if, moreover, for any sequence $\{s_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$, then

$$\lim_{n\longrightarrow\infty} s_n = x^* \quad \text{if and only if} \quad \lim_{n\longrightarrow\infty} \delta_n = 0, \tag{26}$$

where $\{\delta_n\}$ is defined by

$$\delta_n = \left\| s_{n+1} - \left[T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n\right]\right\|, \quad \eta_n = (1-\alpha_n)s_n + \alpha_n T_2\left(\frac{s_n + \eta_n}{2}\right) + \alpha_n d_n + e_n. \tag{27}$$

Proof. From the proof of [19, Theorem 2.1], iteration process of (10), and the conditions **EC**, it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \left\|\frac{x_n + y_n}{2} - x^*\right\| + \|h_n\| \leq \frac{1}{2}\Big\|(x_n - x^*) \\
&\quad + (1-\alpha_n)(x_n - x^*) + \alpha_n\left[T_2\left(\frac{x_n + y_n}{2}\right) - x^*\right]\Big\| \\
&\quad + \frac{1}{2}\|\alpha_n d_n + e_n\| + \|h_n\| \leq \frac{1}{2}\|(x_n - x^*) \\
&\quad + (1-\alpha_n)(x_n - x^*)\| + \frac{\theta\alpha_n}{2}\left\|\left(\frac{x_n + y_n}{2}\right) - x^*\right\| \\
&\quad + \frac{1}{2}\|\alpha_n d_n + e_n\| + \|h_n\| \leq \frac{2-\alpha_n}{2}\|x_n - x^*\| + \frac{\theta\alpha_n}{2} \\
&\quad \cdot \frac{2-\theta}{2-\theta\alpha_n}\|x_n - x^*\| + \frac{\theta\alpha_n}{2} \cdot \frac{1}{2-\theta\alpha_n}(\|\alpha_n d_n \\
&\quad + \|e_n\|) + \frac{1}{2}(\|\alpha_n d_n\| + \|e_n\|) + \|h_n\| \leq \Big(1 \\
&\quad - \frac{1-\theta}{2-\theta\alpha_n}\alpha_n\Big)\|x_n - x^*\| + \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \cdot \frac{1}{1-\theta}\|d'_n\| \\
&\quad + (\|d''_n\| + \|e_n\| + \|h_n\|)
\end{aligned} \tag{28}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, by Lemma 4 and (28), now we know that $\|x_n - x^*\| \longrightarrow 0$ as $n \longrightarrow \infty$. That is, the sequence $\{x_n\}$ converges to x^*.

Next, we prove stability of the new method of the implicit midpoint rule for Picard-Mann iteration with mixed errors. In fact, since $0 < \alpha \leq \alpha_n$, it follows from (27) and (10) that

$$\begin{aligned}
&\left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\| \leq \left\|\frac{s_n + \eta_n}{2} - x^*\right\| + \|h_n\| \\
&\quad \leq \frac{1}{2}\Big\|(s_n - x^*) + (1-\alpha_n)(s_n - x^*) \\
&\quad + \alpha_n\left[T_2\left(\frac{s_n + \eta_n}{2}\right) - x^*\right]\Big\| + \frac{1}{2}\|\alpha_n d_n + e_n\| \\
&\quad + \|h_n\| \leq \frac{2-\alpha_n}{2}\|s_n - x^*\| + \frac{\theta\alpha_n}{2}\left\|\frac{s_n + \eta_n}{2} - x^*\right\| \\
&\quad + \frac{1}{2}(\|\alpha_n d_n\| + \|e_n\|) + \|h_n\| \leq \frac{2-\alpha_n}{2}\|s_n - x^*\| \\
&\quad + \frac{\theta\alpha_n}{2} \cdot \frac{2-\theta}{2-\theta\alpha_n}\|s_n - x^*\| + \frac{\theta\alpha_n}{2}
\end{aligned}$$

$$\cdot \frac{1}{2-\theta\alpha_n}\left(\|\alpha_n d_n + \|e_n\|\right) + \frac{1}{2}\left(\|\alpha_n d_n\| + \|e_n\|\right) + \|h_n\| \leq \left(1 - \frac{1-\theta}{2-\theta\alpha_n}\alpha_n\right)\|s_n - x^*\| + \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \cdot \frac{1}{1-\theta}\|d_n'\| + \left(\|d_n''\| + \|e_n\| + \|h_n\|\right). \tag{29}$$

Thus, by (27), we have

$$\begin{aligned}\delta_n &= \left\|\left(s_{n+1} - x^*\right) - \left[T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right]\right\| \\ &\geq \|s_{n+1} - x^*\| - \left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\|\end{aligned} \tag{30}$$

and so it follows from (29) that

$$\begin{aligned}\|s_{n+1} - x^*\| &\leq \left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\| + \delta_n \\ &\leq \left(1 - \frac{1-\theta}{2-\theta\alpha_n}\alpha_n\right)\|s_n - x^*\| \\ &\quad + \frac{1-\theta}{2-\theta\alpha_n}\alpha_n \cdot \frac{1}{1-\theta}\|d_n'\| \\ &\quad + \left(\|d_n''\| + \|e_n\| + \|h_n\|\right) + \delta_n.\end{aligned} \tag{31}$$

Let $\lim_{n\longrightarrow\infty}\delta_n = 0$. Then by $\sum_{n=0}^{\infty}\alpha_n = \infty$, Lemma 4 and (31), we know that $\lim_{n\longrightarrow\infty}s_n = x^*$.

On the contrary, if $\lim_{n\longrightarrow\infty}s_n = x^*$, then it follows from (27), (29), $\alpha_n \leq 1$ for all $n \geq 0$, and Lemma 4 that

$$\begin{aligned}\delta_n &= \left\|s_{n+1} - \left[T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n\right]\right\| \\ &\leq \|s_{n+1} - x^*\| + \left\|T_1\left(\frac{s_n + \eta_n}{2}\right) + h_n - x^*\right\| \longrightarrow 0\end{aligned} \tag{32}$$

as $n \longrightarrow \infty$. This completes the proof. □

From Theorem 7, similarly we can obtain the following results.

Corollary 8. *Assume that X, C, and T_1 and T_2 are the same as in Theorem 7. If $F(T_1 \cap T_2) \neq \emptyset$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then*

(i) the iterative sequence $\{x_n\}$ generated by (13) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{33}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) further, if for any sequence $\{u_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty}u_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty}\aleph_n = 0, \tag{34}$$

where $\aleph_n$ is defined as follows

$$\begin{aligned}\aleph_n &= \left\|u_{n+1} - T_1\left(\frac{u_n + v_n}{2}\right)\right\|, \\ v_n &= (1-\alpha_n)u_n + \alpha_n T_2\left(\frac{u_n + v_n}{2}\right).\end{aligned} \tag{35}$$

Corollary 9. *Suppose that X and C are the same as in Theorem 7. Let $T : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$ with $F(T) := \{x \in C : Tx = x\} =\neq \emptyset$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then*

(i) the sequence $\{x_n\}$ generated by (16) converges to $x^* \in F(T)$ with convergence rate

$$\kappa = \left(1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right)\theta < 1, \tag{36}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) moreover, if for any sequence $\{w_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty}w_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty}\varsigma_n = 0, \tag{37}$$

where ς_n is defined by

$$\begin{aligned}\varsigma_n &= \left\|w_{n+1} - \left[T\left(\frac{w_n + j_n}{2}\right) + h_n\right]\right\|, \\ j_n &= (1-\alpha_n)w_n + \alpha_n T\left(\frac{w_n + j_n}{2}\right) + \alpha_n d_n + e_n.\end{aligned} \tag{38}$$

Corollary 10. *Let T, X, and C be the same as in Corollary 9. Then*

(i) the iterative sequence $\{x_n\}$ generated by (22) converges to $x^* \in F(T)$ with convergence rate

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{39}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) for any sequence $\{r_n\} \subset X$, if there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty}r_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty}\hbar_n = 0, \tag{40}$$

where

$$\begin{aligned}\hbar_n &= \left\|r_{n+1} - \left[\left(\frac{r_n + \varphi_n}{2}\right) + h_n\right]\right\|, \\ \varphi_n &= (1-\alpha_n)r_n + \alpha_n T_2\left(\frac{r_n + \varphi_n}{2}\right) + \alpha_n d_n + e_n.\end{aligned} \tag{41}$$

Further, to the Picard-Mann iteration processes (11) and (12) with mixed errors for the implicit midpoint rule of one of nonlinear mappings T_1 and T_2, we, respectively, have the following results.

Theorem 11. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $T_1 : C \longrightarrow C$ be nonexpansive and $T_2 : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) := \{x \in C : T_i x = x,\ i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMDIP) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\tau = 1 - \frac{1-\theta}{2}\widehat{\alpha} < 1, \tag{42}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty}\alpha_n \in (0,1]$;

(ii) if, moreover, for any sequence $\{p_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty} p_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty} \chi_n = 0, \tag{43}$$

where χ_n is defined as follows

$$\begin{aligned} \chi_n &= \left\| p_{n+1} - \left[T_1\left(\frac{p_n + q_n}{2}\right) + h_n \right] \right\|, \\ q_n &= (1-\alpha_n) p_n + \alpha_n T_2 p + \alpha_n d_n + e_n. \end{aligned} \tag{44}$$

Theorem 12. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $T_1 : C \longrightarrow C$ be nonexpansive and $T_2 : C \longrightarrow C$ be a contraction mapping with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) := \{x \in C : T_i x = x,\ i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMDIM) (12) converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate

$$\rho = 1 - \frac{2-2\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} < 1, \tag{45}$$

where $\widehat{\alpha} = \limsup_{n\longrightarrow\infty} \alpha_n \in (0,1]$;

(ii) if for every sequence $\{o_n\} \subset X$, there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ and

$$\lim_{n\longrightarrow\infty} o_n = x^* \quad \text{if and only if} \lim_{n\longrightarrow\infty} \mu_n = 0, \tag{46}$$

where

$$\begin{aligned} \mu_n &= \left\| o_{n+1} - (T_1 \sigma_n + h_n) \right\|, \\ \sigma_n &= (1-\alpha_n) o_n + \alpha_n T_2\left(\frac{o_n + \sigma_n}{2}\right) + \alpha_n d_n + e_n. \end{aligned} \tag{47}$$

3. Simulations and Applications

In this section, using the new Picard-Mann iterative methods with mixed errors for the implicit midpoint rule presented in the above section, we will give a numerical simulation for approximating the solution of the elliptic boundary value problem (2) and an application to an elliptic optimal control problem.

3.1. Numerical Example. In order to verify Theorem 7, we first give the following examples and their numerical simulations and to display effectiveness of the new Picard-Mann iterative methods.

Example 1. Let $X = R$, $1 < m \leq 39/16$, $C = [-1, 3 + \sqrt{23/(m-1)}]$, $T_1 x = (1/\pi)\sin(\pi x) + 5$, and $T_2 x = \sqrt{x^2 - 6x + 30}$ for all $x \in C$ and $h_n = 1/10^n$, $\alpha_n = 1/2$, $d_n = 1/n + 1/n^2$, and $e_n = -10/n^5$ for $n \geq 1$. It follows that

$$\left\| T_1 x - T_1 y \right\| \leq \frac{1}{\pi} \left\| \pi x - \pi y \right\| = \left\| x - y \right\|, \tag{48}$$

and

$$\begin{aligned} \left\| T_2 x - T_2 y \right\| &= \left\| \frac{(x-3)^2 - (y-3)^2}{\sqrt{(x-3)^2 + 21} + \sqrt{(y-3)^2 + 21}} \right\| \\ &= \left\| \frac{(x-y)\left[(x-3) + (y-3)\right]}{\|x-3\| + \|y-3\|} \right\| \\ &\quad \cdot \frac{\|x-3\| + \|y-3\|}{\sqrt{(x-3)^2 + 21} + \sqrt{(y-3)^2 + 21}} \\ &\leq \frac{1}{\sqrt{k}} \left\| x - y \right\|. \end{aligned} \tag{49}$$

It is easy to see that T_1 is nonexpansive and T_2 is contraction mapping with constant $1/\sqrt{k}$. And $F(T_1 \cap T_2) = \{5\} \neq \emptyset$. Hence, the conditions of Theorem 7 hold and the sequence $\{x_n\}$ generated by (PMMDI), (PMMDIP), and (PMMDIM) can be rewritten as follows:

(PMMDI)

$$\begin{aligned} x_{n+1} &= \frac{1}{\pi}\sin\left(\frac{x_n + y_n}{2}\pi\right) + 5 + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n \\ &\quad + \sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3(x_n + y_n) + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{50}$$

(PMMDIP)

$$\begin{aligned} x_{n+1} &= \frac{1}{\pi}\sin\left(\frac{x_n + y_n}{2}\pi\right) + 5 + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{x_n^2 - 6x_n + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{51}$$

(PMMDIP)

$$\begin{aligned} x_{n+1} &= \frac{1}{\pi}\sin(\pi y_n) + 5 + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n \\ &\quad + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3(x_n + y_n) + 30} \\ &\quad + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{52}$$

The special cases which have been discussed in the second part are listed as follows:

(PMDI)

$$x_{n+1} = \frac{1}{\pi}\sin\left(\frac{x_n + y_n}{2}\pi\right) + 5,$$

$$y_n = \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30}. \tag{53}$$

(PMMI)

$$\begin{aligned} x_{n+1} &= \sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30} + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30} + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{54}$$

(MMDI)

$$\begin{aligned} x_{n+1} &= \frac{x_n + y_n}{2} + \frac{1}{100^n}, \\ y_n &= \left(1 - \frac{1}{2}\right) x_n + \frac{1}{2}\sqrt{\left(\frac{x_n + y_n}{2}\right)^2 - 3\left(x_n + y_n\right) + 30} + \frac{1}{2^n}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{10}{n^5}. \end{aligned} \tag{55}$$

By Theorems 7, 11, and 12 and Corollaries 8, 9, and 10, one can easily know that $\{x_n\}$ generated by (PMMDI), (PMMDIP), (PMMDIM), (PMDI), (PMMI), (PMI), and (MMDI) converges to $x^* = 5$. To show the availability of the new Picard-Mann iterative methods, by using software Matlab 7.0, the numerical simulation results for the sequences $\{x_n\}$ generated by (PMMDI), (PMMDIP), and (PMMDIM) are given in Figure 1 and Table 1, the iteration process generated by (PMMDI), (PMDI), (PMMI), (PMI), and (MMDI) is given in Figure 2 and Table 2, respectively.

Remark 2. From Figure 1 and Table 1, it follows that the iterative process (10), (11), and (12) are effective and the sequences $\{x_n\}$ generated by them is convergent by using the technic of implicit midpoint rule. Further, comparing with the processes (PMMDI) and (PMMDIP), iteration process (PMMDIM) converges much faster, and the convergence rates are in accordance with the analysis in Theorems 7, 11, and 12:

$$\tau = 1 - \frac{1-\theta}{2}\widehat{\alpha} > \varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} > \rho = 1 - \frac{2-2\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}. \tag{56}$$

Table 1: A comparison of (PMMDI), (PMMDIP), and (PMMDIM).

Iteration Number	(PMMDI)	(PMMDIP)	(PMMDIM)
0	25.0000	25.0000	25.0000
5	5.1003	4.9073	4.9646
10	4.9740	5.0396	5.0010
15	5.0068	4.9825	5.0000
20	4.9982	5.0078	5.0000
25	5.0005	4.9966	5.0000
30	4.9999	5.0015	5.0000
35	5.0000	4.9993	5.0000
40	5.0000	5.0003	5.0000
45	5.0000	4.9999	5.0000
50	5.0000	5.0001	5.0000
55	5.0000	5.0000	5.0000

Remark 3. From Figure 2 and Table 2, it is easy to see that the iterative process and the special cases (PMMDI), (PMDI), (PMMI), and (MMDI) are effective and the sequences $\{x_n\}$ generated by them are also convergent with the technic of implicit midpoint rule. Iteration process (PMMI) converges much faster than the processes (PMMDI), (PMDI), and (MMDI), and the convergence rates are in accordance with the analysis in Theorem 7 and Corollaries 8, 9, and 10:

$$\varrho = 1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha} > \kappa = \left(1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right)\theta. \tag{57}$$

3.2. An Application to Elliptic Optimal Control Problems. In [19], Li and Lan studied a kind of new iterative approximation of solutions for an elliptic boundary value problem in Hilbert spaces by using the new Picard-Mann iterative methods with mixed errors for contraction operators. Here, we try to explore the new iteration methods to optimal control problem with elliptic boundary value constraint.

One can know that a weak solution of (2) is a solution of the following variational problem:

$$\begin{gathered} \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f(x,u) \cdot v dx = 0, \quad \forall v \in H_0^1(\Omega), \\ u(x) \in H_0^1(\Omega). \end{gathered} \tag{58}$$

Theorem 4. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $\phi(y) = (1/2)\|y\|^2 - \int_\Omega F(x,y)dx$, $T = I - \phi'$, $F(x,y) = \int_0^y f(x,\zeta)d\zeta$, and $C = [v,w] = u \in H_0^1(\Omega) : v(x) \le y(x) \le w(x)$ for all $x \in \Omega$, where $v, w \in H_0^1(\Omega)$ are a subsolution and a supersolution of the problem (58), respectively. Suppose that $F(T) \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMI) converges to a weak solution $x^* \in F(T_1 \cap T_2)$ of (2) with convergence rate

$$\imath = \theta\left(1 - \frac{1-\theta}{2-\theta\widehat{\alpha}}\widehat{\alpha}\right) \tag{59}$$

where $\widehat{\alpha} = \limsup_{n\to\infty} \alpha_n \in (0,1]$ and $\theta = \sup_{n\to\infty}\|I' - \varphi''\|$;

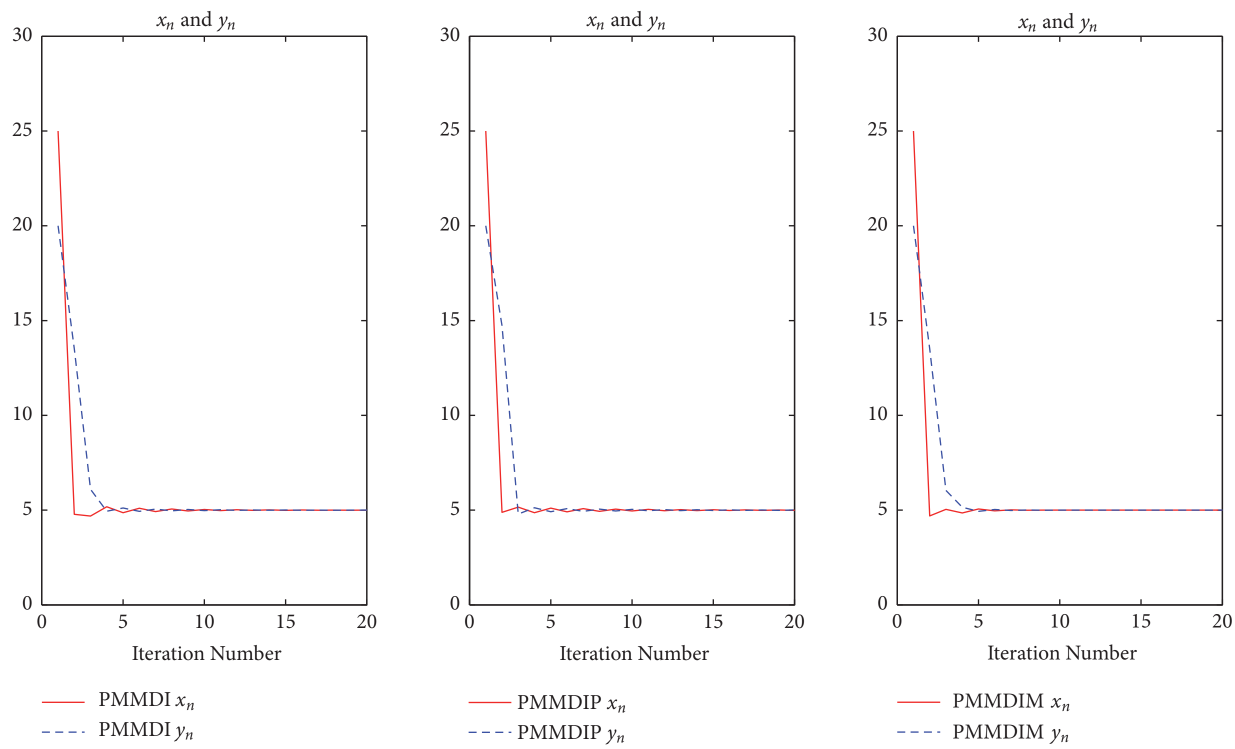

Figure 1: Iterative solutions of (PMMDI), (PMMDIP), and (PMMDIM).

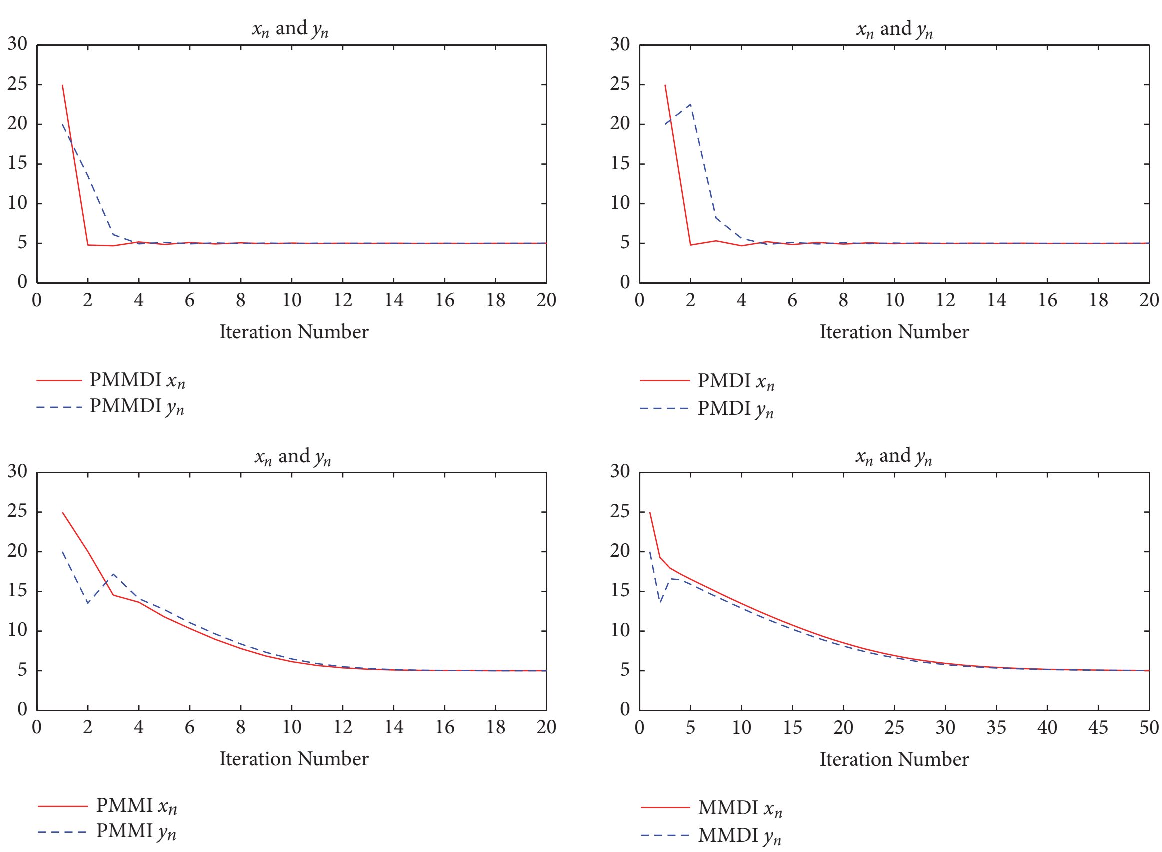

Figure 2: Iterative solutions of (PMMDI), (PMDI), (PMMI), and (MMDI).

Table 2: A comparison of (PMMDI), (PMDI), (PMMI), and (MMDI).

Iteration Number	(PMMDI)	(PMDI)	(PMMI)	(MMDI)
0	25.0000	25.0000	25.0000	25.0000
5	5.1003	4.8491	10.3264	15.9082
10	4.9740	5.0376	5.6542	12.8914
15	5.0068	4.9902	5.0246	10.2503
20	4.9982	5.0026	5.0008	8.1355
25	5.0005	4.9993	5.0000	6.6612
30	4.9999	5.0002	5.0000	5.7941
35	5.0000	5.0000	5.0000	5.3551
40	5.0000	5.0000	5.0000	5.1533
45	5.0000	5.0000	5.0000	5.0651
50	5.0000	5.0000	5.0000	5.0275
55	5.0000	5.0000	5.0000	5.0015

(ii) if there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any sequence $\{z_n\} \subset X$ and all $n \geq 0$, then

$$\lim_{n\longrightarrow\infty} z_n = u^* \quad \text{if and only if} \lim_{n\longrightarrow\infty} \kappa_n = 0, \tag{60}$$

where

$$\begin{aligned}\kappa_n &= \left\| z_{n+1} - \left[\frac{z_n + s_n}{2} - \phi'\left(\frac{z_n + s_n}{2}\right) + h_n\right]\right\|, \\ s_n &= (1-\alpha_n) z_n + \alpha_n \left[\frac{z_n + s_n}{2} - \phi'\left(\frac{z_n + s_n}{2}\right)\right] \\ &\quad + \alpha_n d_n + e_n.\end{aligned} \tag{61}$$

Proof. From the proof of [41, Theorem 6], it follows that $C \subset H_0^1(\Omega)$ is a closed convex and bounded subset and $\|(I' - \phi'')u\| < 1$ for some $u \in C$. By the proof of Theorem 4 in [41], we know that T is a contraction mapping. Since a contraction mapping shows fixed points, the results hold from Theorem 4. This completes the proof. □

Combining (17) and (18) with Theorem 4, we give the following corollaries.

Corollary 5. *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded set. Let $\phi(y) = (1/2)\|y\|^2 - \int_\Omega F(x,y)dx$, $T = I - \phi'$, $F(x,y) = \int_0^y f(x,\zeta)d\zeta$, and $C = [v,w] = u \in H_0^1(\Omega) : v(x) \leq y(x) \leq w(x)$ for all $x \in \Omega$, where $v, w \in H_0^1(\Omega)$ are a subsolution and a supersolution of the problem (58), respectively. Suppose that $F(T) \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then*

(i) the iterative sequence $\{x_n\}$ generated by (PMMIP) and (PMMIM) converges to a weak solution $x^* \in F(T_1 \cap T_2)$ of (2) with convergence rate

$$\begin{aligned}\jmath &= \theta\left(1 - \frac{1-\theta}{2}\hat{\alpha}\right), \\ \ell &= \theta\left(1 - \frac{2-2\theta}{2-\theta\hat{\alpha}}\hat{\alpha}\right),\end{aligned} \tag{62}$$

respectively;

(ii) if there exists an $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any sequence $\{z_n\} \subset X$ and all $n \geq 0$, then

$$\lim_{n\longrightarrow\infty} z_n = u^* \quad \text{if and only if} \lim_{n\longrightarrow\infty} \lambda_n = 0, \tag{63}$$

where

$$\begin{aligned}\lambda_n &= \left\| z_{n+1} - \left[\frac{z_n + t_n}{2} - \phi'\left(\frac{z_n + t_n}{2}\right) + h_n\right]\right\|, \\ t_n &= (1-\alpha_n) z_n + \alpha_n \left[z_n - \phi'(z_n)\right] + \alpha_n d_n + e_n.\end{aligned} \tag{64}$$

or

$$\begin{aligned}\lambda_n &= \left\| z_{n+1} - \left[t_n - \phi'(t_n) + h_n\right]\right\|, \\ t_n &= (1-\alpha_n) z_n + \alpha_n \left[\frac{z_n + t_n}{2} - \phi'\left(\frac{z_n + t_n}{2}\right)\right] \\ &\quad + \alpha_n d_n + e_n,\end{aligned} \tag{65}$$

which are defined by (17) and (18), respectively.

Corollary 6. *Let $\mathbb{R}^n$ be a n-dimensional bounded space and $\Omega \subset \mathbb{R}^n$ be a nonempty closed convex bounded set. If $\phi(y) = (1/2)\|y\|^2 - \int_\Omega F(x,y)dx$, $T = I - \phi'$, $F(x,y) = \int_0^y f(x,\zeta)d\zeta$,*

(i) then $T(u) = (I - \phi')u = -y'(x)$ is a constant mapping;

(ii) taking the iteration results $y^*(x_p) = p$ of (PMMI), (PMMIP), and (PMMIM) into $u(x)$, then $u(x_p)$ is the solution space of optimal control problem (1) and (2).

4. Concluding Remarks

In this paper, we introduced new Picard-Mann iteration processes with mixed errors for the implicit midpoint rule as follows:

$$\begin{aligned}x_{n+1} &= T_1\left(\frac{x_n + y_n}{2}\right) + h_n, \\ y_n &= (1-\alpha_n) x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n,\end{aligned} \tag{66}$$

and the sequence defined by

$$x_{n+1} = T_1\left(\frac{x_n + y_n}{2}\right) + h_n,$$
$$y_n = (1 - \alpha_n)\, x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \quad (67)$$

$$x_{n+1} = T_1 y_n + h_n,$$
$$y_n = (1 - \alpha_n)\, x_n + \alpha_n T_2\left(\frac{x_n + y_n}{2}\right) + \alpha_n d_n + e_n, \quad (68)$$

where $T_1, T_2 : X \longrightarrow X$ are two nonlinear operators; $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation. The iteration (66) includes Picard-Mann iterative process. What is noteworthy is that the iterative processes are not discussed in the literature.

Then, we gave convergence and stability analysis of the new Picard-Mann iterative approximation for implicit midpoint rule in normed space and proposed numerical examples to show the convergence of different iterative processes. Finally, as applications of new Picard-Mann iteration processes with mixed errors for the implicit midpoint rule of nonexpansive mappings and contractive mappings, we explored iterative approximation of solutions for the following optimal control problem with elliptic boundary value constraint:

$$\min J(u), \quad (69)$$

where state variable $y(u) \in V = H^1(\Omega)$, state space, and control variable $u \in U = L^2(\partial\Omega)$, control space, satisfy

$$-\Delta y = f(x, u), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega, \quad (70)$$

$\Omega \subset \mathbb{R}^2$ is a bounded convex region with smooth boundary $\partial\Omega$; $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function.

However, as is pointed out in [19], when T_2 is also nonexpansive in Theorems 7 and 11, what results can be obtained?

Abbreviations

PMMDI: New Picard-Mann iteration with mixed errors for the implicit midpoint rule of two different nonlinear operators
PMMDIP: New Picard-Mann iteration with mixed errors for the implicit midpoint rule of Picard mapping of two different nonlinear operators
PMMDIM: New Picard-Mann iteration with mixed errors for the implicit midpoint rule of Mann mapping of two different nonlinear operators
PMDI: Picard-Mann iterations of the implicit midpoint rule in for two different operators
PMDIP: Picard-Mann iterations of the implicit midpoint rule of Picard mapping for two different operators
PMDIM: Picard-Mann iterations of the implicit midpoint rule of Mann mapping for two different operators
PMMI: Picard-Mann iteration of implicit midpoint rule with mixed errors for one nonlinear mapping
PMMIP: Picard-Mann iteration of implicit midpoint rule of Picard mapping with mixed errors for one nonlinear mapping
PMMIM: Picard-Mann iteration of implicit midpoint rule of Mann mapping with mixed errors for one nonlinear mapping
PMI: Picard-Mann iterative process of implicit midpoint rule for one nonlinear mapping
PMIP: Picard-Mann iterative process of implicit midpoint rule of Picard mapping for one nonlinear mapping
PMIM: Picard-Mann iterative process of implicit midpoint rule of Mann mapping for one nonlinear mapping
MMDI: Picard-Mann iterative process of the explicit and implicit midpoint rule with mixed errors
MMDIP: Picard-Mann iterative process of the explicit and implicit midpoint rule of Picard mapping with mixed errors
MMDIM: Picard-Mann iterative process of the explicit and implicit midpoint rule of Mann mapping with mixed errors.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Research Article

On Approximating the Toader Mean by Other Bivariate Means

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In the article, we provide several sharp bounds for the Toader mean by use of certain combinations of the arithmetic, quadratic, contraharmonic, and Gaussian arithmetic-geometric means.

1. Introduction

Let q be a real number, $0 < u < 1$ and $x, y \in \mathbb{R}^+$ with $x \neq y$. Then the complete elliptic integrals $\mathcal{K}(u)$ and $\mathcal{E}(u)$ [1–22] of the first and second kinds, geometric mean $G(x,y)$, logarithmic mean $L(x,y)$, arithmetic mean $A(x,y)$, quadratic mean $Q(x,y)$, contraharmonic mean $C(x,y)$, second contraharmonic mean $\overline{C}(x,y)$, qth Hölder mean $H_q(x,y)$, and Toader mean $T(x,y)$ [23–27] of x and y are given by

$$\begin{aligned}
\mathcal{K}(u) &= \int_0^{\pi/2} \left(1-u^2\sin^2(v)\right)^{-1/2} dv, \\
\mathcal{E}(u) &= \int_0^{\pi/2} \sqrt{1-u^2\sin^2(v)}\,dv, \\
G(x,y) &= \sqrt{xy}, \\
L(x,y) &= \frac{x-y}{\log x - \log y}, \\
A(x,y) &= \frac{x+y}{2}, \\
Q(x,y) &= \sqrt{\frac{x^2+y^2}{2}}, \\
C(x,y) &= \frac{x^2+y^2}{x+y}, \\
\overline{C}(x,y) &= \frac{x^3+y^3}{x^2+y^2}, \\
H_q(x,y) &= \left(\frac{x^q+y^q}{2}\right)^{1/q} \quad (q \neq 0), \\
H_0(x,y) &= \sqrt{xy}
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
T(x,y) &= \frac{2}{\pi}\int_0^{\pi/2} \sqrt{x^2\cos^2(v)+y^2\sin^2(v)}\,dv \\
&= \begin{cases} \dfrac{2x}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{y}{x}\right)^2}\right), & x > y, \\ \dfrac{2y}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{x}{y}\right)^2}\right), & x < y, \end{cases}
\end{aligned} \tag{2}$$

respectively.

The classical Gaussian arithmetic-geometric mean $AGM(u,v)$ of two positive real numbers u and v is defined by the common limit of the sequences $\{u_n\}$ and $\{v_n\}$, which are given by

$$u_0 = u,$$
$$v_0 = v,$$

$$u_{n+1} = \frac{u_n + v_n}{2},$$
$$v_{n+1} = \sqrt{u_n v_n}. \tag{3}$$

The well-known Gaussian identity [10] and (2) show that

$$T(1,u) = \frac{2\mathscr{E}(u')}{\pi},$$
$$AGM(1,u) = \frac{\pi}{2\mathscr{K}(u')} \tag{4}$$

for all $0 < u < 1$, where $u' = (1-u^2)^{1/2}$.

If $x \neq y$, then it is well known that the function $q \longmapsto H_q(x,y)$ is strictly increasing on the interval $(-\infty,\infty)$ and the inequalities

$$G(x,y) = H_0(x,y) < L(x,y) < A(x,y) = H_1(x,y)$$
$$< Q(x,y) = H_2(x,y) < C(x,y) < \overline{C}(x,y) \tag{5}$$

are valid.

Recently, the bounds for the Toader mean $T(x,y)$ and Gaussian arithmetic-geometric means $AGM(x,y)$ have attracted the interest of many mathematicians. The following inequalities

$$L(x,y) < AGM(x,y) < A(x,y)$$
$$< \frac{\pi}{2}L(x,y), \tag{6}$$

$$A^{1/2}(x,y)G^{1/2}(x,y) < AGM(x,y)$$
$$< \left(\frac{\sqrt{A(x,y)} + \sqrt{G(x,y)}}{2}\right)^2 \tag{7}$$

for all $x, y \in \mathbb{R}^+$ with $x \neq y$ were established in [28–32].

Alzer and Qiu [12] and Barnard, Pearce, and Richards [33] proved that the double inequalities

$$\frac{2}{\pi}\frac{1}{L(x,y)} + \left(1 - \frac{2}{\pi}\right)\frac{1}{A(x,y)} < \frac{1}{AGM(x,y)}$$
$$< \frac{3}{4}\frac{1}{L(x,y)} + \frac{1}{4}\frac{1}{A(x,y)}, \tag{8}$$
$$H_{3/2}(x,y) < T(x,y) < H_{\log 2/(\log\pi - \log 2)}(x,y)$$

hold for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

In [23, 34], the authors proved that $\alpha = 1/2$, $\beta = (4-\pi)/[(\sqrt{2}-1)\pi] = 0.6597\cdots$, $\lambda = 1/4$ and $\mu = 4/\pi - 1 = 0.2732\cdots$ are the best possible parameters such that the double inequalities

$$\alpha Q(x,y) + (1-\alpha)A(x,y) < T(x,y)$$
$$< \beta Q(x,y) + (1-\beta)A(x,y), \tag{9}$$

$$\lambda C(x,y) + (1-\lambda)A(x,y) < T(x,y)$$
$$< \mu C(x,y) + (1-\mu)A(x,y) \tag{10}$$

hold for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

Qian, Song, Zhang, and Chu [35] proved that the two-sided inequalities

$$\lambda_1\overline{C}(x,y) + (1-\lambda_1)A(x,y) < T(x,y)$$
$$< \mu_1\overline{C}(x,y) + (1-\mu_1)A(x,y),$$
$$\overline{C}[\lambda_2 x + (1-\lambda_2)y, \lambda_2 y + (1-\lambda_2)x] < T(x,y)$$
$$< \overline{C}[\mu_2 x + (1-\mu_2)y, \mu_2 y + (1-\mu_2)x] \tag{11}$$

are valid for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\lambda_1 \leq 1/8$, $\mu_1 \geq 4/\pi - 1 = 0.2732\cdots$, $\lambda_2 \leq 1/2 + \sqrt{2}/8 = 0.6767\cdots$ and $\mu_2 \geq 1/2 + \sqrt{(4-\pi)/(3\pi-4)}/2 = 0.6988\cdots$ if $\lambda_1, \mu_1 \in (0, 1/2)$ and $\lambda_2, \mu_2 \in (1/2, 1)$.

Inequalities (5), (6), and (9) and the identity $Q(x,y) = \sqrt{A(x,y)C(x,y)}$ lead to

$$AGM(x,y) < A(x,y) < T(x,y) < Q(x,y)$$
$$< C(x,y), \tag{12}$$

$$T(x,y) < Q(x,y) = \sqrt{A(x,y)C(x,y)}, \tag{13}$$

$$T(x,y) > \frac{Q(x,y) + A(x,y)}{2}$$
$$> \sqrt{Q(x,y)A(x,y)}$$
$$> \sqrt{Q(x,y)AGM(x,y)} \tag{14}$$

for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

Motivated by inequalities (12)–(14), in this article we deal with the optimality of the parameters α_i and β_i $(i = 1,2,3,4)$ on the interval $(0,1)$ such that

$$\alpha_1 C(x,y) + (1-\alpha_1)AGM(x,y) < T(x,y)$$
$$< \beta_1 C(x,y) + (1-\beta_1)AGM(x,y),$$
$$\alpha_2 Q(x,y) + (1-\alpha_2)AGM(x,y) < T(x,y)$$
$$< \beta_2 Q(x,y) + (1-\beta_2)AGM(x,y),$$
$$\alpha_3 C(x,y) + (1-\alpha_3)AGM(x,y) < \frac{T^2(x,y)}{A(x,y)}$$
$$< \beta_3 C(x,y) + (1-\beta_3)AGM(x,y),$$
$$\alpha_4 Q(x,y) + (1-\alpha_4)AGM(x,y) < \frac{T^2(x,y)}{Q(x,y)}$$
$$< \beta_4 Q(x,y) + (1-\beta_4)AGM(x,y) \tag{15}$$

for all $x, y \in \mathbb{R}^+$ with $x \neq y$.

2. Lemmas

It is well known that $\mathscr{K}(u)$ and $\mathscr{E}(u)$ satisfy the following formulas (see [10]):

$$\frac{d\mathcal{K}(u)}{du}=\frac{\mathcal{E}(u)-u'^2\mathcal{K}(u)}{uu'^2},$$
$$\frac{d\mathcal{E}(u)}{du}=\frac{\mathcal{E}(u)-\mathcal{K}(u)}{u},$$
$$\frac{d\left[\mathcal{E}(u)-u'^2\mathcal{K}(u)\right]}{du}=u\mathcal{K}(u),$$
$$\mathcal{E}\left(\frac{2\sqrt{u}}{1+u}\right)=\frac{2\mathcal{E}(u)-u'^2\mathcal{K}(u)}{1+u}, \tag{16}$$
$$\mathcal{K}\left(\frac{2\sqrt{u}}{1+u}\right)=(1+u)\mathcal{K}(u),$$
$$\lim_{u\to 0^+}\mathcal{K}(u)=\lim_{u\to 0^+}\mathcal{E}(u)=\frac{\pi}{2},$$
$$\lim_{u\to 1^-}\mathcal{K}(u)=\infty,$$
$$\lim_{u\to 1^-}\mathcal{E}(u)=1.$$

Lemma 1 (See [10, Theorem 1.25]). *Let $a,b\in\mathbb{R}$ with $a<b$, $f,g:[a,b]\longmapsto\mathbb{R}$ be continuous real-valued functions and differentiable on (a,b) with $g'(x)\neq 0$. Then both the functions $(f(x)-f(a))/(g(x)-g(a))$ and $(f(x)-f(b))/(g(x)-g(b))$ are (strictly) increasing (decreasing) on (a,b) if the function $f'(x)/g'(x)$ is (strictly) increasing (decreasing) on (a,b).*

Lemma 2. *For the complete elliptic integrals $\mathcal{K}(u)$ and $\mathcal{E}(u)$, we have the following monotonicity results:*

(1) The function $u\longmapsto[\mathcal{E}(u)-u'^2\mathcal{K}(u)]/u^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$.

(2) The function $u\longmapsto\mathcal{K}(u)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ and the function $u\longmapsto\mathcal{E}(u)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$.

(3) The function $u\longmapsto r'^{\lambda}\mathcal{K}(u)$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$ if $\lambda\geq 1/2$.

(4) The function $u\longmapsto(1+u'^2)\mathcal{K}(u)-2\mathcal{E}(u)$ is strictly increasing from $(0,1)$ onto $(0,\infty)$.

(5) The function $u\longmapsto[\mathcal{K}(u)-\mathcal{E}(u)]/u^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,\infty)$.

(6) The function $u\longmapsto 2\mathcal{E}(u)-u'^2\mathcal{K}(u)$ is strictly increasing from $(0,1)$ onto $(\pi/2,2)$.

(7) The function $u\longmapsto\varphi_1(u)=u'^2/\mathcal{E}(u)$ is strictly decreasing from $(0,1)$ onto $(0,2/\pi)$.

(8) The function $u\longmapsto\varphi_2(u)=\sqrt{1+u^2}[\mathcal{E}(u)-u'^2\mathcal{K}(u)]/u^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,\sqrt{2})$.

(9) The function $u\longmapsto\varphi_3(u)=[2\mathcal{E}(u)-u'^2\mathcal{K}(u)]/\sqrt{1+u^2}$ is strictly decreasing from $(0,1)$ onto $(\sqrt{2},\pi/2)$.

(10) The function $u\longmapsto\varphi_4(u)=[u'^2\mathcal{K}^2(u)(\mathcal{K}(u)-\mathcal{E}(u))]/u^2$ is strictly decreasing from $(0,1)$ onto $(0,\pi^3/16)$.

(11) The function $u\longmapsto\varphi_5(u)=[(1+u^2)\mathcal{E}(u)-u'^2\mathcal{K}(u)]/(u'^2u^2)$ is strictly increasing from $(0,1)$ onto $(3\pi/4,\infty)$.

Proof. Parts (1)–(6) can be found in [10, Theorem 3.21(1), (2), (7) and (8), and Exercise 3.43(4), (11) and (13)]. For part (7), it is not difficult to verify that

$$\varphi_1(0^+)=\frac{2}{\pi},\qquad \varphi_1(1^-)=0, \tag{17}$$

$$\varphi_1'(u)=-\frac{u^2\mathcal{E}(u)+\mathcal{E}(u)-u'^2\mathcal{K}(u)}{u\mathcal{E}^2(u)}=-\frac{u}{\mathcal{E}^2(u)}\left[\mathcal{E}(u)+\frac{\mathcal{E}(u)-u'^2\mathcal{K}(u)}{u^2}\right]. \tag{18}$$

It follows from parts (1) and (2) together with (18) that

$$\varphi_1'(u)<0 \tag{19}$$

for $0<u<1$.

Therefore, part (7) follows from (17) and (19).

For part (8), simple computations lead to

$$\varphi_2(0^+)=\frac{\pi}{4},\qquad \varphi_2(1^-)=\sqrt{2}, \tag{20}$$

$$\varphi_2'(u)=\frac{2\mathcal{K}(u)-2\mathcal{E}(u)-u^2\mathcal{E}(u)}{u^3\sqrt{1+u^2}}=\frac{\left(1+u'^2\right)\mathcal{K}(u)-2\mathcal{E}(u)}{u^3\sqrt{1+u^2}}+\frac{u}{\sqrt{1+u^2}}\frac{\mathcal{K}(u)-\mathcal{E}(u)}{u^2}. \tag{21}$$

From parts (4) and (5) together with (21) we clearly see that

$$\varphi_2'(u)>0 \tag{22}$$

for $0<u<1$.

Therefore, part (8) follows from (20) and (22).

For part (9), we clearly see that

$$\varphi_3(0^+)=\frac{\pi}{2},\qquad \varphi_3(1^-)=\sqrt{2}. \tag{23}$$

Differentiating $\varphi_3(u)$ and making use of part (5) we get

$$\varphi_3'(u)=-\frac{uu'^2}{\left(1+u^2\right)^{3/2}}\left[\frac{\mathcal{K}(u)-\mathcal{E}(u)}{u^2}\right]<0 \tag{24}$$

for $0<u<1$.

Therefore, part (9) follows from (23) and (24).

For part (10), elaborated computation gives

$$\varphi_4(0^+)=\frac{\pi^3}{16},\qquad \varphi_4(1^-)=0, \tag{25}$$

$$\varphi_4'(u)=-\frac{\mathcal{K}(u)}{u}\left[\frac{\mathcal{K}(u)}{u^2}+\frac{\mathcal{K}(u)-\mathcal{E}(u)}{u^2}\right]\cdot\left[\left(1+u'^2\right)\mathcal{K}(u)-2\mathcal{E}(u)\right]. \tag{26}$$

It follows from parts (2), (4), and (5) together with (26) that

$$\varphi_4'(u) < 0 \tag{27}$$

for $0 < u < 1$.

Therefore, part (10) follows from (25) and (27).

For part (11), it is not difficult to verify that

$$\varphi_5(0^+) = \frac{3\pi}{4}, \quad \varphi_5(1^-) = \infty, \tag{28}$$

$$\varphi_5'(u) = \frac{(u^4 + 5u^2 - 2)\mathscr{E}(u) + 2(2u^4 - 3u^2 + 1)\mathscr{K}(u)}{u^3 u'^4}. \tag{29}$$

Let

$$\varphi_6(u) = (u^4 + 5u^2 - 2)\mathscr{E}(u) + 2(2u^4 - 3u^2 + 1)\mathscr{K}(u). \tag{30}$$

Then we clearly see that that

$$\varphi_6(0^+) = 0, \tag{31}$$

$$\varphi_6'(u) = 11u^3\left[\frac{5}{11}\mathscr{E}(u) + \frac{\mathscr{E}(u) - u'^2\mathscr{K}(u)}{u^2}\right]. \tag{32}$$

Therefore, part (11) follows easily from parts (1) and (2) together with (28)–(32). □

3. Main Results

Theorem 3. *The double inequality*

$$\alpha_1 C(x,y) + (1-\alpha_1) AGM(x,y) < T(x,y) < \beta_1 C(x,y) + (1-\beta_1) AGM(x,y) \tag{33}$$

holds for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\alpha_1 \leq 2/5$ and $\beta_1 \geq 2/\pi = 0.6366\cdots$.

Proof. We clearly see that $C(x,y)$, $AGM(x,y)$ and $T(x,y)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y)$. Then $0 < u < 1$ and (2) and (4) lead to

$$AGM(x,y) = \frac{\pi}{2}\frac{A(x,y)}{\mathscr{K}(u)}, \tag{34}$$

$$T(x,y) = \frac{2}{\pi}A(x,y)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]. \tag{35}$$

It follows from (34) and (35) together with $C(x,y) = A(x,y)(1+u^2)$ that

$$\frac{T(x,y) - AGM(x,y)}{C(x,y) - AGM(x,y)} = \frac{(2/\pi)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right] - \pi/2\mathscr{K}(u)}{(1+u^2) - \pi/2\mathscr{K}(u)} = \frac{(2/\pi)\mathscr{K}(u)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right] - \pi/2}{(1+u^2)\mathscr{K}(u) - \pi/2} := F(u). \tag{36}$$

Let $f_1(u) = 2\mathscr{K}(u)[2\mathscr{E}(u) - u'^2\mathscr{K}(u)]/\pi - \pi/2$ and $g_1(u) = (1+u^2)\mathscr{K}(u) - \pi/2$. Then elaborated computations lead to

$$f_1(0^+) = g_1(0^+) = 0, \quad F(u) = \frac{f_1(u)}{g_1(u)}, \tag{37}$$

$$\frac{f_1'(u)}{g_1'(u)} = \frac{4}{\pi}\frac{\mathscr{E}(u)\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]}{2u^2\mathscr{E}(u) + u'^2\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]} = \frac{4}{\pi}\frac{1}{2\left[u^2/(\mathscr{E}(u) - u'^2\mathscr{K}(u))\right] + u'^2/\mathscr{E}(u)}. \tag{38}$$

It follows from Lemma 2(1) and (7) together with (38) that $f_1'(u)/g_1'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (37) we know that $F(u)$ is strictly increasing on $(0,1)$. Moreover,

$$F(0^+) = \frac{2}{5}, \quad F(1^-) = \frac{2}{\pi}. \tag{39}$$

Therefore, Theorem 3 follows from (36) and (39) together with the monotonicity of $F(u)$. □

Theorem 4. *The double inequality*

$$\alpha_2 Q(x,y) + (1-\alpha_2) AGM(x,y) < T(x,y) < \beta_2 Q(x,y) + (1-\beta_2) AGM(x,y) \tag{40}$$

holds for all $x, y \in \mathbb{R}^+$ with $x \neq y$ if and only if $\alpha_2 \leq 2/3$ and $\beta_2 \geq 2\sqrt{2}/\pi = 0.9003\cdots$.

Proof. Since $Q(x,y)$, $AGM(x,y)$, and $T(x,y)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y)$. Then $u \in (0,1)$, and (34) and (35) together with $Q(x,y) = A(x,y)\sqrt{1+u^2}$ lead to

$$\frac{T(x,y) - AGM(x,y)}{Q(x,y) - AGM(x,y)} = \frac{(2/\pi)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right] - \pi/2\mathscr{K}(u)}{\sqrt{1+u^2} - \pi/2\mathscr{K}(u)} := G(u). \tag{41}$$

Let $f_2(u) = 2[2\mathscr{E}(u) - u'^2\mathscr{K}(u)]/\pi - \pi/[2\mathscr{K}(u)]$ and $g_2(u) = \sqrt{1+u^2} - \pi/[2\mathscr{K}(u)]$. Then simple computations lead to

$$f_2(0^+) = g_2(0^+) = 0,$$
$$G(u) = \frac{f_2(u)}{g_2(u)}, \tag{42}$$

$$\frac{f_2'(u)}{g_2'(u)} = 1 - \frac{1 - (2/\pi)\left(\sqrt{1+u^2}\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]/u^2\right)}{1 + (\pi/2)\left(\sqrt{1+u^2}\left[\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]/u^2\right)\left(1/\left[u'\mathscr{K}(u)\right]^2\right)} \tag{43}$$

It follows from Lemma 2(3) and (8) together with (43) that $f_2'(u)/g_2'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (42) we know that $G(u)$ is strictly increasing on $(0,1)$. Moreover,

$$G(0^+) = \frac{2}{3},$$
$$G(1^-) = \frac{2\sqrt{2}}{\pi}. \tag{44}$$

Therefore, Theorem 4 follows from (41) and (44) together with the monotonicity of $G(u)$. □

Theorem 5. *The double inequality*

$$\alpha_3 C(x,y) + (1-\alpha_3)\, AGM(x,y) < \frac{T^2(x,y)}{A(x,y)}$$
$$< \beta_3 C(x,y) + (1-\beta_3)\, AGM(x,y) \tag{45}$$

holds for all $x, y \in \mathbb{R}^+$ *with* $x \neq y$ *if and only if* $\alpha_3 \leq 3/5$ *and* $\beta_3 \geq 8/\pi^2 = 0.8105\cdots$.

Proof. Without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y) \in (0,1)$. Then it follows from (34), (35) and $C(x,y) = A(x,y)(1+u^2)$ that

$$\frac{T^2(x,y)/A(x,y) - AGM(x,y)}{C(x,y) - AGM(x,y)}$$
$$= \frac{\left(4/\pi^2\right)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]^2 - \pi/2\mathscr{K}(u)}{(1+u^2) - \pi/2\mathscr{K}(u)} \tag{46}$$
$$:= H(u).$$

Let $f_3(u) = 4[2\mathscr{E}(u) - u'^2\mathscr{K}(u)]^2/\pi^2 - \pi/[2\mathscr{K}(u)]$ and $g_3(u) = (1+u^2) - \pi/[2\mathscr{K}(u)]$. Then elaborated computations lead to

$$f_3(0^+) = g_3(0^+) = 0,$$
$$H(u) = \frac{f_3(u)}{g_3(u)}, \tag{47}$$

$$\frac{f_3'(u)}{g_3'(u)} = 1$$
$$- \frac{1 - \left(4/\pi^2\right)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]\left[\left(\mathscr{E}(u) - u'^2\mathscr{K}(u)\right)/u^2\right]}{1 + (\pi/4)\left[\left(\mathscr{E}(u) - u'^2\mathscr{K}(u)\right)/u^2\right]\left(1/\left[u'\mathscr{K}(u)\right]^2\right)}. \tag{48}$$

□

It follows from Lemma 2(1), (3), and (6) together with (48) that $f_3'(u)/g_3'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (47) we know that $H(u)$ is strictly increasing on $(0,1)$. Moreover,

$$H(0^+) = \frac{3}{5},$$
$$H(1^-) = \frac{8}{\pi^2}. \tag{49}$$

Therefore, Theorem 5 follows from (46) and (49) together with the monotonicity of $H(u)$.

Theorem 6. *The double inequality*

$$\alpha_4 Q(x,y) + (1-\alpha_4)\, AGM(x,y) < \frac{T^2(x,y)}{Q(x,y)}$$
$$< \beta_4 Q(x,y) + (1-\beta_4)\, AGM(x,y) \tag{50}$$

holds for all $x, y \in \mathbb{R}^+$ *with* $x \neq y$ *if and only if* $\alpha_4 \leq 1/3$ *and* $\beta_4 \geq 8/\pi^2 = 0.8105\cdots$.

Proof. Without loss of generality, we assume that $x > y > 0$. Let $u = (x-y)/(x+y) \in (0,1)$. Then it follows from (34), (35) and $Q(x,y) = A(x,y)\sqrt{1+u^2}$ that

$$\frac{T^2(x,y)/Q(x,y) - AGM(x,y)}{Q(x,y) - AGM(x,y)}$$
$$= \frac{\left(4/\pi^2\right)\left[2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right]^2 - \pi\sqrt{1+u^2}/2\mathscr{K}(u)}{(1+u^2) - \pi\sqrt{1+u^2}/2\mathscr{K}(u)}$$
$$= 1 - \frac{1 - \left(4/\pi^2\right)\left[\left(2\mathscr{E}(u) - u'^2\mathscr{K}(u)\right)/\sqrt{1+u^2}\right]^2}{1 - \pi/\left(2\sqrt{1+u^2}\mathscr{K}(u)\right)} \tag{51}$$
$$:= 1 - J(u).$$

Let $f_4(u) = 1 - 4[(2\mathscr{E}(u) - u'^2\mathscr{K}(u))/\sqrt{1+u^2}]^2/\pi^2$ and $g_4(u) = 1 - \pi/[2\sqrt{1+u^2}\mathscr{K}(u)]$. Then simple computations lead to

$$f_4(0^+) = g_4(0^+) = 0,$$
$$J(u) = \frac{f_4(u)}{g_4(u)}, \tag{52}$$

$$\begin{aligned}\frac{f_4'(u)}{g_4'(u)} &= \frac{16}{\pi^3}\left[\frac{2\mathscr{E}(u)-u'^2\mathscr{K}(u)}{\sqrt{1+u^2}}\right] \\ &\quad\cdot\frac{\left[u'^2\mathscr{K}(u)\right]^2\left[\mathscr{K}(u)-\mathscr{E}(u)\right]}{\left[\left(1+u^2\right)\mathscr{E}(u)-u'^2\mathscr{K}(u)\right]} \\ &= \frac{16}{\pi^3}\left[\frac{2\mathscr{E}(u)-u'^2\mathscr{K}(u)}{\sqrt{1+u^2}}\right] \\ &\quad\cdot\frac{\left[u'^2\mathscr{K}^2(u)\left(\mathscr{K}(u)-\mathscr{E}(u)\right)\right]/u^2}{\left[\left(1+u^2\right)\mathscr{E}(u)-u'^2\mathscr{K}(u)\right]/\left(u'^2u^2\right)}.\end{aligned} \tag{53}$$

It follows from Lemma 2(9)–(11) and (53) that $f_4'(u)/g_4'(u)$ is strictly increasing on $(0,1)$. Then from Lemma 1 and (52) we know that $J(u)$ is strictly increasing on $(0,1)$. Moreover,

$$\begin{aligned}J\left(0^+\right) &= \frac{2}{3}, \\ J\left(1^-\right) &= 1-\frac{8}{\pi^2}.\end{aligned} \tag{54}$$

Therefore, Theorem 6 follows from (51) and (54) together with the monotonicity of $J(u)$. □

Let $x = 1$ and $y = u'$. Then (2), (4) and Theorems 3–6 lead to Corollary 7 immediately.

Corollary 7. *The double inequalities*

$$\begin{aligned}&\frac{\pi\left[4\left(1+u'^2\right)\mathscr{K}(u)+3\pi\left(1+u'\right)\right]}{20\left(1+u'\right)\mathscr{K}(u)} < \mathscr{E}(u) \\ &\quad< \frac{4\left(1+u'^2\right)\mathscr{K}(u)+\pi(\pi-2)\left(1+u'\right)}{4\left(1+u'\right)\mathscr{K}(u)}, \\ &\frac{\pi\left[2\sqrt{2\left(1+u'^2\right)}\mathscr{K}(u)+\pi\right]}{12\mathscr{K}(u)} < \mathscr{E}(u) \\ &\quad< \frac{4\sqrt{1+u'^2}\mathscr{K}(u)+\pi\left(\pi-2\sqrt{2}\right)}{4\mathscr{K}(u)}, \\ &\frac{\pi}{2}\sqrt{\frac{3\left(1+u'^2\right)\mathscr{K}(u)+\pi\left(1+u'\right)}{10\mathscr{K}(u)}} < \mathscr{E}(u) \\ &\quad< \frac{1}{4}\sqrt{\frac{16\left(1+u'^2\right)\mathscr{K}(u)+\pi\left(\pi^2-8\right)\left(1+u'\right)}{\mathscr{K}(u)}}, \\ &\frac{\pi}{2}\sqrt{\frac{\left(1+u'^2\right)\mathscr{K}(u)+\pi\sqrt{2\left(1+u'^2\right)}}{6\mathscr{K}(u)}} < \mathscr{E}(u) \\ &\quad< \frac{1}{4}\sqrt{\frac{16\left(1+u'^2\right)\mathscr{K}(u)+\pi\left(\pi^2-8\right)\sqrt{2\left(1+u'^2\right)}}{\mathscr{K}(u)}}\end{aligned} \tag{55}$$

hold for all $0 < u < 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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