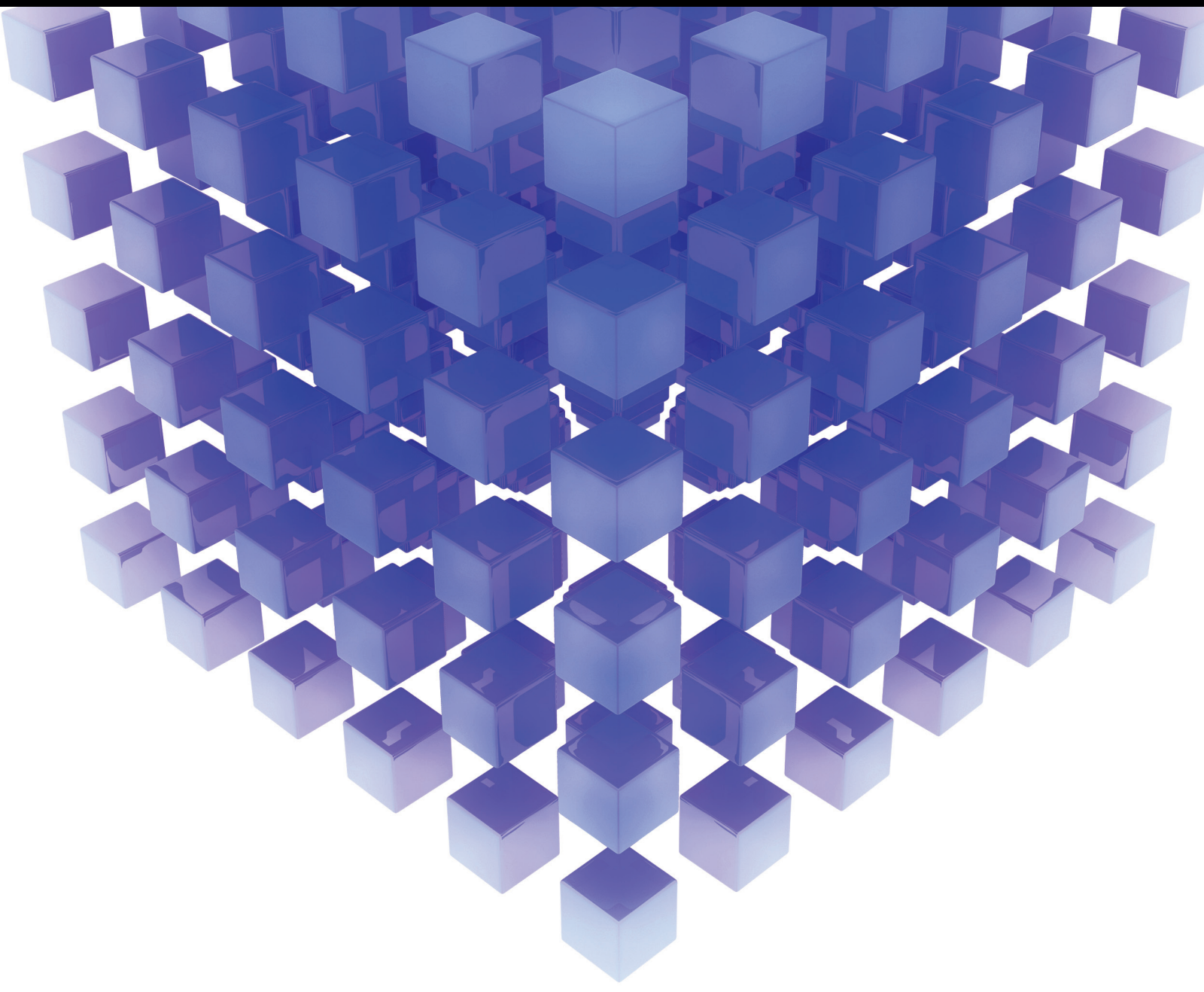
Mathematical Problems in Engineering

Recent Developments on the Stability and Control of Stochastic Systems

Guest Editors: Quanxin Zhu, Son Nguyen, Ruihua Liu, and Leonid Shaikhet



Recent Developments on the Stability and Control of Stochastic Systems

Mathematical Problems in Engineering

Recent Developments on the Stability and Control of Stochastic Systems

Guest Editors: Quanxin Zhu, Son Nguyen, Ruihua Liu, and Leonid Shaikhet

Editorial Board

Contents

Optimal Design of Stochastic Distributed Order Linear SISO Systems Using Hybrid Spectral Method,
Pham Luu Trung Duong, Ezra Kwok, and Moonyong Lee
Volume 2015, Article ID 989542, 14 pages

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 454951, 3 pages
http://dx.doi.org/10.1155/2015/454951

Editorial

Recent Developments on the Stability and Control of Stochastic Systems

Quanxin Zhu,[1] Son Nguyen,[2] Ruihua Liu,[3] and Leonid Shaikhet[4]

[1] *School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Nanjing 210023, China*
[2] *University of Puerto Rico, Rio Piedras, San Juan, PR 00936, USA*
[3] *Department of Mathematics, University of Dayton, Dayton, OH 45469, USA*
[4] *Donetsk State University of Management, Donetsk 83015, Ukraine*

Correspondence should be addressed to Quanxin Zhu; zqx22@126.com

Received 17 September 2015; Accepted 17 September 2015

It is well known that stochastic models have come to play an increasingly important role in a wide range of research and application fields including engineering, physics, biology, finance, mechanics, electronics, and mathematics. The study of stochastic systems has attracted a large number of researchers. Various problems such as the stability, stabilization, state feedback control, adaptive control, filtering design, tracking control, state estimation, passivity, and adaptive synchronization have been investigated in the literature.

The accepted papers in this special issue include stochastic stability, finance and computation, stochastic H_∞ control, state estimation, state feedback control, robust filtering, multiagent systems, networked control systems, time-delayed systems, and neural networks.

The problem of stochastic stability is one of the most important problems in the fields of stochastic systems. There are 10 papers relating to this topic in this special issue. More precisely, the paper entitled "Further Results on Stability Analysis for Markovian Jump Systems with Time-Varying Delays" by O. M. Kwon et al. presents an improved delay-dependent stability criterion for a class of Markovian jump systems with time-varying delays by constructing a newly augmented Lyapunov-Krasovskii functional and combining Wirtinger-based integral inequality. The paper entitled "Globally Asymptotic Stability of Stochastic Nonlinear Systems by the Output Feedback" by W. Cheng et al. investigates the problem of the globally asymptotic stability for a class of stochastic nonlinear systems with the output feedback control by developing a new method. The paper entitled "The Mean Stability Criteria in terms of Two Measures for Stochastic Differential Equations with Coefficient's Uncertainty" by Rui Zhang et al. studies the stochastic stability criteria of two measures to the mean stability by applying optimal control approaches. The paper entitled "Exponential Stability of Jump-Diffusion Systems with Neutral Term and Impulses" by H. Yang et al. deals with the mean square and almost surely exponential stability for a class of jump-diffusion systems with neutral term and impulses. The paper entitled "Stochastic Stability of Discrete-Time Switched Systems with a Random Switching Signal" by K. Liu et al. presents a necessary and sufficient condition for stochastic stability of discrete-time linear switched system, where the switching signal allows fixed dwell time before a Markov switch occurs. The paper entitled "The Stationary Distribution and Extinction of Generalized Multispecies Stochastic Lotka-Volterra Predator-Prey System" by F. Yin and X. Yu establishes the existence of stationary distribution and extinction for multispecies stochastic Lotka-Volterra predator-prey system by using the Lyapunov method and space decomposition technique. The paper entitled "Stability Analysis of R&D Cooperation in a Supply Chain" by L. Xu et al. investigates the opportunistic behavior in the vertical R&D cooperation and analyzes the equilibrium of the cooperation. The paper entitled "Evolutionary Game-Theoretic Solution for Virtual Routers with Padding Misbehavior in Cloud Computing" by X. Bi et al. considers a detailed solution and analysis for describing the

normal behavior and padding misbehavior of virtual routers through analyzing the stability of the equilibrium points. The paper entitled "Finite-Time Boundedness of Markov Jump System with Piecewise-Constant Transition Probabilities via Dynamic Output Feedback Control" by B. Yan et al. studies the problem of finite-time boundedness of Markovian jump system with piecewise constant transition probabilities via dynamic output feedback control, which leads to both stochastic jumps and deterministic switches. The paper entitled "Consensus of Noisy Multiagent Systems with Markovian Switching Topologies and Time-Varying Delays" by Y. Shang presents some necessary and sufficient consensus conditions for two classes of multiagent systems: delays affecting only the output of the agents' neighbors and delays affecting both the agents' own outputs and the outputs of their neighbors.

The problem of H_∞ control is studied by a number of researchers, and new results are reported in these papers. The paper entitled "Asynchronous H_∞ Estimation for Two-Dimensional Nonhomogeneous Markovian Jump Systems with Randomly Occurring Nonlocal Sensor Nonlinearities" by R. Zhang et al. discusses the problem of asynchronous H_∞ estimation for a class of two-dimensional nonhomogeneous Markovian jump systems with nonlocal sensor nonlinearity, where the nonlocal measurement nonlinearity is governed by a stochastic variable satisfying the Bernoulli distribution. The paper entitled "H_2/H_∞ Control Design of Detectable Periodic Markov Jump Systems" by T. Hou and H. Ma studies the infinite horizon H_2/H_∞ control problem for a class of discrete-time periodic Markov jump systems with (x, u, v)-dependent noise by using the spectral criterion of detectability and game theoretic approach. The paper entitled "H_∞ Gain-Scheduled Control for LPV Stochastic Systems" by C.-C. Ku and G.-W. Chen investigates the robust control problem for a class of discrete-time uncertain stochastic systems by applying the gain-scheduled control scheme and linear parameter varying modeling approach as well as linear matrix inequality approach. The paper entitled "Robust H_∞ Filtering for Uncertain Neutral Stochastic Systems with Markovian Jumping Parameters and Time Delay" by Y. Li and Z. Huang considers the robust H_∞ filter design problem for a class of uncertain neutral stochastic systems with Markovian jumping parameters and time delay by using the Lyapunov-Krasovskii theory and generalized Finsler lemma. The paper entitled "Study on $\mathscr{H}$_Index of Stochastic Linear Continuous-Time Systems" by Y. Li et al. presents a necessary and sufficient condition of $\mathscr{H}$_index larger than $\gamma > 0$ and proves that the solvability of generalized differential equation and the feasibility of the $\mathscr{H}$_index are equivalent.

In recent years, the applications of stochastic systems in finance and operations management have received much attention. The paper entitled "Dynamic Inventory and Pricing Policy in a Periodic-Review Inventory System with Finite Ordering Capacity and Price Adjustment Cost" by B. Yang et al. studies a dynamic inventory control and pricing optimization problem in a periodic-review inventory system with price adjustment cost. The paper entitled "A Random Parameter Model for Continuous-Time Mean-Variance Asset-Liability Management" by H. Ma et al. investigates a continuous-time mean-variance asset-liability management problem in a market with random market parameters based on the theories of stochastic linear-quadratic optimal control and backward stochastic differential equations. The paper entitled "Concession Period Decision Models for Public Infrastructure Projects Based on Option Games" by Z. Wang et al. seeks out concession period decision models for public infrastructure with option game theory and studies the influence of minimum government income guarantee and government investment on concession period. The paper entitled "Multivariate Time-Varying *G-H* Copula GARCH Model and Its Application in the Financial Market Risk Measurement" by Q. Chen et al. depicts the return distribution of financial asset and constructs the multivariate time-varying *G-H* Copula GARCH model which can simultaneously describe "asymmetric, leptokurtic, heavy-tail" characteristics, the time-varying volatility characteristics, and the extreme-tail dependence characteristics of financial asset return by employing the strengths of *G-H* distribution, Copula function, and GARCH model. The paper entitled "Maximum Principle for Forward-Backward Stochastic Control System Driven by Lévy Process" by X. Wang and H. Huang studies some stochastic optimal control problems including the maximum principle and the linear quadratic problem, where the controlled system is described by a forward-backward stochastic differential equation, driven by Lévy process.

In addition, stochastic systems in other areas are considered in this special issue. The paper entitled "Optimal Design of Stochastic Distributed Order Linear SISO Systems Using Hybrid Spectral Method" by D. Pham et al. studies stochastic distributed order systems by using the operational matrix technique, the existing Monte-Carlo, polynomial chaos, and frequency methods. The paper entitled "The Impact of Aging Agricultural Labor Population on Farmland Output from the Perspective of Farmer Preferences" by G. Guo et al. investigates some factors to better understand the impact of an aging agricultural labor population on agricultural production. The paper entitled "On Two-Level State-Dependent Routing Polling Systems with Mixed Service" by G. Zheng et al. discusses an $N + 1$ that queues single-server two-level polling system which consists of one key queue and N normal queues based on priority differentiation and efficiency of the system. The paper entitled "Microstructure Models with Short-Term Inertia and Stochastic Volatility" by M. A. Kouritzin shows that all the price data sets exhibit strong evidence of both inertia and Heston-type stochastic volatility for a class of partially observed microstructure models, containing stochastic volatility, dynamic trading noise, and short-term inertia. The paper entitled "The Particle Filter Sample Impoverishment Problem in the Orbit Determination Application" by P. C. P. M. Pardal et al. discusses techniques for administering one implementation issue that often arises in the application of particle filters: sample impoverishment.

Finally, we remark that the selected topics and published papers may not be a comprehensive representation of recent developments on the stability and control of stochastic systems, but we strongly hope that the reader will find the special issue very useful.

Acknowledgments

We express our great appreciation to all the authors of this special issue for their high quality contributions. All the reviewers' efforts in reviewing the papers are also very greatly acknowledged. Quanxin Zhu's work was jointly supported by the National Natural Science Foundation of China (61374080) and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

Quanxin Zhu
Son Nguyen
Ruihua Liu
Leonid Shaikhet

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 192083, 7 pages
http://dx.doi.org/10.1155/2015/192083

Research Article

Exponential Stability of Jump-Diffusion Systems with Neutral Term and Impulses

Hua Yang,[1,2] Jianguo Liu,[1] and Feng Jiang[3]

[1] *School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China*
[2] *School of Mathematics and Computer Science, Wuhan Polytechnic University, Wuhan 430023, China*
[3] *School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, China*

Correspondence should be addressed to Feng Jiang; fjiang78@gmail.com

Received 13 January 2015; Accepted 11 February 2015

Academic Editor: Leonid Shaikhet

We study jump-diffusion systems with neutral term and impulses. Under some conditions, we prove that the jump-diffusion systems with neutral term and impulses are mean square and almost surely exponentially stable. Finally, we give an example to describe the theoretical results.

1. Introduction

Recently, stochastic partial differential systems (SPDS) are often used to describe some evolution phenomena in studying pattern recognition and engineering [1, 2]. Dynamic behavior of solutions for SPDS has been discussed by many researchers [3–8].

In the practical application, there exists often impulsive disturbance under specific circumstances [9, 10]. For example, in [11, 12], Zhu et al. discussed stability behavior of stochastic impulsive systems. Sakthivel and Luo [13] discussed asymptotics of stochastic impulsive systems. Further, in [14], Jiang and Shen studied asymptotic behavior for stochastic impulsive infinite delays systems. Chen et al. [15] discussed stability of stochastic impulsive systems by inequality technique.

In addition, many models such as population models and circuits models often include the derivative terms of the current state and past state, which are often described as neutral systems [16–21]. Meanwhile, there are also a few works on jump diffusions, which are discussed extensively. For example, Zhu [22] discussed the long-time behavior of the solution including the pth moment asymptotic stability and almost sure stability for stochastic jump systems. In [23, 24], the authors established dynamical behavior of stochastic jump systems and stochastic jump biological model. Cui et al. [25–27] studied the existence, uniqueness, and some stability of stochastic jump systems. Luo and Taniguchi [28] discussed the existence of solutions of neutral stochastic jump systems under non-Lipschitz condition. Ren and Sakthivel [29, 30] discussed dynamic behavior of second-order jump-diffusion systems.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries on mild solution. Then we give some conditions to guarantee stability of mild solution by the fixed point theory in Section 3. In Section 4, an example is presented to show our conclusions.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions [31]. Let $\varrho > 0$ and $\overline{R}_+ = [0, +\infty)$. Moreover, let Ξ and $\wp$ be real separable Hilbert spaces with norms $|\cdot|_\Xi$, $|\cdot|_\wp$ and let $\pounds(\wp, \Xi)$ be the space of all bounded linear operators from $\wp$ into Ξ. In this work, $\|\cdot\|$ is the norms of operators. The notation $\mathfrak{D} = \mathfrak{D}_{\mathscr{F}_0}([-\varrho, 0], \Xi)$ denotes the family of all $\mathscr{F}_0$-measurable functions from $[-\varrho, 0]$ into Ξ with the norm $|\Psi|_{\mathfrak{D}} = \sup_{t\in[-\varrho,0]}|\Psi(t)|_\Xi$.

Let $\{B(t) : t \geq 0\}$ be a $\wp$-valued Wiener process on the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$ with a trace class operator Q on $\wp$. $\mathscr{L}_2^0(\wp, \Xi)$ being the set of all Q-Hilbert-Schmidt

operators from $\wp$ to Ξ. For the construction, the reader is referred to [19, 25, 31, 32]. Assume that $m(\nu)$, $\nu \in \mathfrak{D}_m$, is a stationary $\mathscr{F}_\nu$-Poisson point process with characteristic measure λ. $N(d\nu, d\vartheta)$ defined by $N(\nu, \mathbb{X}) = \Sigma_{\nu\in\mathfrak{D}_m, \nu\geq\nu} 1_{\mathbb{X}}(m(\nu))$ for $\mathbb{X} \in \mathscr{B}(\wp - \{0\})$. Let $\widetilde{N}(d\nu, d\vartheta) = N(d\nu, d\vartheta) - \lambda(d\vartheta)d\nu$, which is independent of $B(\cdot)$. For the Poisson measure, see [21].

Suppose that $S(t)$, $t \geq 0$, is an analytic semigroup with its infinitesimal generator A [14]. For the analytic semigroup, see Pazy [32, Page 60–75]. In the paper, assume that $0 \in \rho(A)$. According to Pazy [32], a linear closed operator $(-A)^\alpha$ ($\alpha \in (0, 1]$) can be defined on $\mathfrak{D}((-A)^\alpha)$.

Consider a jump-diffusion system with neutral term and impulses:

$$\begin{aligned} &d\left[Y(t) + D(t, Y(t-\kappa(t)))\right] \\ &\quad = \left[AY(t) + P(t, Y(t-\zeta(t)))\right] dt \\ &\qquad + R(t, Y(t-\varpi(t)))\, dB(t) \\ &\qquad + \int_{\mathbb{X}} \Theta(t, Y(t-\theta(t)), \vartheta)\, \widetilde{N}(dt, d\vartheta), \quad t \neq \tau_j,\ t \geq 0, \\ &\qquad \Delta Y(\tau_j) = H_j\left(Y\left(\tau_j^-\right)\right), \quad t = \tau_j,\ j = 1, \ldots, \iota, \end{aligned} \tag{1}$$

with the initial data $Y_0(\cdot) = \Psi \in \mathfrak{D}_{\mathscr{F}_0}([-\varrho, 0], \Xi)$. Here $D, P : \overline{R}_+ \times \Xi \to \Xi$, $R : \overline{R}_+ \times \Xi \to \mathscr{L}_2^0(\wp, \Xi)$, $\Theta : \overline{R}_+ \times \Xi \times \mathbb{X} \to \Xi$ and $\kappa(t), \varpi(t), \zeta(t), \theta(t) : \overline{R}_+ \to [0, \varrho]$ are continuous. Consider $H_j : \Xi \to \Xi$, $\Delta Y(\nu) = Y(\nu^+) - Y(\nu^-)$, where $Y(\nu^+) = \lim_{\Delta\nu\to 0^+} Y(\nu + \Delta\nu)$ and $Y(\nu^-) = \lim_{\Delta\nu\to 0^+} Y(\nu - \Delta\nu)$, $0 < \tau_1 < \cdots < \tau_\iota < \infty = \lim_{j\to\infty} \tau_j$.

Definition 1. A process $Y(t)$, $t \in [0, T]$, $T \in [0, \infty)$, is said to be the mild solution to system (1) if

(i) $Y(t)$ is a $\mathscr{F}_t$-adapted, càdlàg process and is almost surely square-integrable on $[0, T]$;

(ii) for $t \in [0, T]$ $Y(t)$ satisfies

$$\begin{aligned} Y(t) &= S(t)\Psi(0) + S(t)D(0, \Psi) \\ &\quad - D(t, Y(t-\kappa(t))) \\ &\quad - \int_0^t AS(t-\nu) D(\nu, Y(\nu-\kappa(\nu)))\, d\nu \\ &\quad + \int_0^t S(t-\nu) P(\nu, Y(\nu-\zeta(\nu)))\, d\nu \\ &\quad + \int_0^t S(t-\nu) R(\nu, Y(\nu-\varpi(\nu)))\, dB(\nu) \\ &\quad + \int_0^t \int_{\mathbb{X}} S(t-\nu) \Theta(t, Y(\nu-\theta(\nu)), \vartheta)\, \widetilde{N}(d\nu, d\vartheta) \\ &\quad + \sum_{0<\tau_j<t} S\left(t-\tau_j\right) H_j\left(Y\left(\tau_j^-\right)\right), \end{aligned} \tag{2}$$

and $Y_0 = \Psi \in \mathfrak{D}_{\mathscr{F}_0}([-\varrho, 0], \Xi)$.

To establish exponential stability [7, 10, 20] of system (1), we need the following hypotheses.

(H_1) $\|S(t)\| \leq e^{-\beta t}$, where β is a positive constant.

(H_2) There exists $\widetilde{K} > 0$ such that, for $t \geq 0$, $Y_1, Y_2 \in \Xi$,

$$\left|(-A)^\alpha D(t, Y_1) - (-A)^\alpha D(t, Y_2)\right|_\Xi^2 \leq \widetilde{K} \left|Y_1 - Y_2\right|_\Xi^2. \tag{3}$$

(H_3) There exist positive constants L_1, L_2, L_3 such that, for $t \geq 0$, $Y_1, Y_2 \in \Xi$,

$$\begin{aligned} &\left|P(t, Y_1) - P(t, Y_2)\right|_\Xi^2 \leq L_1 \left|Y_1 - Y_2\right|_\Xi^2, \\ &\left|R(t, Y_1) - R(t, Y_2)\right|_{\mathscr{L}_2^0}^2 \leq L_2 \left|Y_1 - Y_2\right|_\Xi^2, \\ &\int_{\mathbb{X}} \left|\Theta(t, Y_1, \vartheta) - \Theta(t, Y_2, \vartheta)\right|_\Xi^2 \lambda(d\vartheta) \leq L_3 \left|Y_1 - Y_2\right|_\Xi^2. \end{aligned} \tag{4}$$

(H_4) There exist constants l_j such that, for $Y_1, Y_2 \in \Xi$, $|H_j(Y_1) - H_j(Y_2)|^2 \leq l_j |Y_1 - Y_2|^2$ ($j = 1, \ldots, \iota$).

(H_5) One has $D(t, 0) = P(t, 0) = R(t, 0) = \Theta(t, 0, \vartheta) = H_j(0) = 0$ ($j = 1, 2, \ldots$) for $t \geq 0$.

Remark 2. We should point out that it is clear that system (1) has a trivial solution when $\Psi = 0$ by (H_1)–(H_5).

3. Main Results

In the section, we will state and prove our main results on mean square and almost surely exponential stability to system (1) by the fixed point theory. To prove our main results, we firstly give a useful lemma.

Lemma 3 (see [18, 32]). *Under (H_1), assume that $0 \in \rho(A)$. Then, for $\alpha \in (0, 1]$,*

(i) *for $Y \in \mathfrak{D}((-A)^\alpha)$, $S(t)(-A)^\alpha Y = (-A)^\alpha S(t) Y$;*

(ii) *there exist constants $M_\alpha > 0$ and $\beta > 0$ such that, for $t > 0$,*

$$\left\|(-A)^\alpha S(t)\right\| \leq M_\alpha t^{-\alpha} e^{-\beta t}. \tag{5}$$

Now we will state and prove the main results on stability.

Theorem 4. *Suppose that (H_1)–(H_5) hold. Then system (1) has a unique mild solution and is mean square exponentially stable, if the initial data Ψ is mean square exponentially stable and*

$$\begin{aligned} &6\left[\widetilde{K}\left\|(-A)^{-\alpha}\right\|^2 + \widetilde{K} M_{1-\alpha}^2 \beta^{-2\alpha} \Gamma^2(\alpha)\right. \\ &\quad \left. + L_1 \beta^{-2} + L_2 (2\beta)^{-1} + L_3 (2\beta)^{-1} + \widetilde{l} e^{-2\beta T}\right] < 1. \end{aligned} \tag{6}$$

Here $\widetilde{l} = \mathbb{E}(\sum_{j=1}^\iota |l_j|)$ and $M_{1-\alpha}$ and β are defined by (5).

Proof. Let Υ be the Banach space of $Y(t)$ with the norm $\|Y\|_\Upsilon := \sup_{t\geq 0} \mathbb{E}|Y(t)|_\Xi^2$ and there exist $\overline{M} > 0$ and $\delta > 0$ such that, for $t \geq 0$,

$$\mathbb{E}\left|Y(t)\right|_\Xi^2 < \overline{M} \mathbb{E}\left|\Psi\right|_{\mathfrak{D}}^2 e^{-\delta t}. \tag{7}$$

Define an operator $\Pi : \Upsilon \to \Upsilon$ by $\Pi(Y)(t) = \Psi(t)$ for $t \in [-\varrho, 0]$ and for $t \geq 0$,

$$\begin{aligned}\Pi(Y)(t) &= S(t)\Psi(0) + S(t)D(0,\Psi) \\ &\quad - D(t, Y(t-\kappa(t))) \\ &\quad - \int_0^t AS(t-v)D(v, Y(v-\kappa(v)))\,dv \\ &\quad + \int_0^t S(t-v)P(v, Y(v-\zeta(v)))\,dv \\ &\quad + \int_0^t S(t-v)R(v, Y(v-\varpi(v)))\,dB(v) \\ &\quad + \int_0^t \int_{\mathbb{X}} S(t-v)\Theta(t, Y(v-\theta(v)), \vartheta)\,\widetilde{N}(dv, d\vartheta) \\ &\quad + \sum_{0<\tau_j<t} S(t-\tau_j) H_j\left(Y\left(\tau_j^-\right)\right).\end{aligned} \tag{8}$$

Now we will prove that the operator Π has a fixed point in Υ. Without loss of generality, we suppose that $0 < \max\{\delta, \eta\} < \beta$. Let $\beta_\varsigma := (\beta - \varsigma)^{-1}$. We firstly claim that $\Pi(\Upsilon) \subset \Upsilon$. Let $Y(t) \in \Upsilon$ and we then have from (8)

$$\begin{aligned}&\mathbb{E}\,|\Pi(Y)(t)|_\Xi^2 \\ &\quad \leq 7\mathbb{E}\,|S(t)\Psi(0) + S(t)D(0,\Psi)|_\Xi^2 \\ &\qquad + 7\mathbb{E}\,|D(t, Y(t-\kappa(t)))|_\Xi^2 \\ &\qquad + 7\mathbb{E}\left|\int_0^t AS(t-v)D(v, Y(v-\kappa(v)))\,dv\right|_\Xi^2 \\ &\qquad + 7\mathbb{E}\left|\int_0^t S(t-v)P(v, Y(v-\zeta(v)))\,dv\right|_\Xi^2 \\ &\qquad + 7\mathbb{E}\left|\int_0^t S(t-v)R(v, Y(v-\varpi(v)))\,dB(v)\right|_\Xi^2 \\ &\qquad + 7\mathbb{E}\left|\int_0^t \int_{\mathbb{X}} S(t-v)\Theta(t, Y(v-\theta(v)), \vartheta)\,\widetilde{N}(dv, d\vartheta)\right|_\Xi^2 \\ &\qquad + 7\mathbb{E}\left|\sum_{0<\tau_j<t} S(t-\tau_j) H_j\left(Y\left(\tau_j^-\right)\right)\right|^2 \\ &\quad = 7\sum_{i=1}^{7} F_i.\end{aligned} \tag{9}$$

Note that the initial data Ψ is mean square exponentially stable; that is, there exist, for $\widetilde{M} > 0$, $\eta > 0$ such that $-\varrho \leq t \leq 0$, $\mathbb{E}|\Psi(t)|^2 \leq \widetilde{M}\mathbb{E}|\Psi(0)|_\Xi^2 e^{-\eta t}$. By (H_2) and (H_5), we have

$$\begin{aligned}F_2 &= \mathbb{E}\left|(-A)^{-\alpha}(-A)^{\alpha} D(t, Y(t-\kappa(t)))\right|_\Xi^2 \\ &\leq \widetilde{K}\left\|(-A)^{-\alpha}\right\|^2 \mathbb{E}\,|Y(t-\kappa(t))|_\Xi^2 \\ &\leq \widetilde{K}\left\|(-A)^{-\alpha}\right\|^2 \left(\overline{M}\mathbb{E}\,|\Psi|_{\mathfrak{D}}^2 e^{\delta\varrho-\delta t} + \widetilde{M}\mathbb{E}\,|\Psi(0)|_\Xi^2 e^{\eta\varrho-\eta t}\right).\end{aligned} \tag{10}$$

Then (5) together with (H_2) and (H_5) yields

$$\begin{aligned}F_3 &\leq M_{1-\alpha}^2 \mathbb{E}\left(\int_0^t t_v^{(1-\alpha)/2} e^{-\beta(t-v)/2} t_v^{(1-\alpha)/2} e^{-\beta(t-v)/2} \right. \\ &\qquad\qquad \left. \cdot\, |(-A)^{\alpha} D(v, Y(v-\kappa(v)))|_\Xi dv\right)^2 \\ &\leq M_{1-\alpha}^2 \int_0^t e^{\beta v-\beta t} t_v^{1-\alpha} dv \\ &\qquad \cdot \int_0^t e^{\beta v-\beta t} t_v^{1-\alpha} \mathbb{E}\left|(-A)^{\alpha} D(v, Y(v-\kappa(v)))\right|_\Xi^2 dv \\ &\leq \widetilde{K}\Gamma(\alpha)\beta^{-\alpha} M_{1-\alpha}^2 \\ &\qquad \cdot \int_0^t e^{\beta v-\beta t} t_v^{1-\alpha} \mathbb{E}\,|Y(v-\kappa(v))|_\Xi^2 dv \\ &\leq \widetilde{K}\Gamma(\alpha)\beta^{-\alpha} M_{1-\alpha}^2 \\ &\qquad \cdot \int_0^t e^{\beta v-\beta t} t_v^{1-\alpha} \\ &\qquad\qquad \cdot \left(\overline{M}\mathbb{E}\,|\Psi|_{\mathfrak{D}}^2 e^{\delta\varrho-\delta v} + \widetilde{M}\mathbb{E}\,|\Psi(0)|_\Xi^2 e^{\eta\varrho-\eta v}\right) dv \\ &\leq \widetilde{K}\Gamma^2(\alpha)\beta^{-\alpha} M_{1-\alpha}^2 \\ &\qquad \cdot \left(\beta_\delta^\alpha \overline{M}\mathbb{E}\,|\Psi|_{\mathfrak{D}}^2 e^{\delta\varrho-\delta t} + \beta_\eta^\alpha \widetilde{M}\mathbb{E}\,|\Psi|_{\mathfrak{D}}^2 e^{\eta\varrho-\eta t}\right).\end{aligned} \tag{11}$$

By (H_1), we have

$$\begin{aligned}F_4 &\leq \mathbb{E}\left(\int_0^t e^{\beta v-\beta t}\left|P(v, Y(v-\zeta(v)))\right|_\Xi dv\right)^2 \\ &\leq \int_0^t \left(e^{(\beta v-\beta t)/2}\right)^2 dv \\ &\qquad \cdot \mathbb{E}\left(\int^t \left(e^{(\beta v-\beta t)/2}\left|P(v, Y(v-\zeta(v)))\right|_\Xi\right)^2 dv\right) \\ &\leq \frac{L_1}{\beta}\int_0^t e^{\beta v-\beta t}\mathbb{E}\left|Y(v-\zeta(v))\right|_\Xi^2 dv \\ &\leq \frac{L_1}{\beta}\left(\beta_\delta \overline{M}\mathbb{E}\,|\Psi|_{\mathfrak{D}}^2 e^{\delta\varrho-\delta t} + \beta_\eta \widetilde{M}\mathbb{E}\,|\Psi|_{\mathfrak{D}}^2 e^{\eta\varrho-\eta t}\right).\end{aligned} \tag{12}$$

By the properties of the martingales, we have

$$\begin{aligned}
F_5 &\le \int_0^t \mathbb{E}\left|S(v-t)R(v,Y(v-\varpi(v)))\right|_{\mathscr{L}_2^0}^2 dv \\
&\le \int_0^t e^{2\beta v-2\beta t}\mathbb{E}\left|R(v,Y(v-\varpi(v)))\right|_{\mathscr{L}_2^0}^2 dv \\
&\le L_2\int_0^t e^{2\beta v-2\beta t}\mathbb{E}\left|Y(v-\varpi(v))\right|_{\Xi}^2 dv \\
&\le L_2\int_0^t e^{2\beta v-2\beta t}\left(\overline{M}\mathbb{E}\left|\Psi\right|_{\mathfrak{D}}^2 e^{\delta\varrho-\delta v}+\widetilde{M}\mathbb{E}\left|\Psi\right|_{\Xi}^2 e^{\eta\varrho-\eta v}\right)dv \\
&\le L_2\left(\beta_\delta\overline{M}\mathbb{E}\left|\Psi\right|_{\mathfrak{D}}^2 e^{2\delta\varrho-2\delta t}+\beta_\eta\widetilde{M}\mathbb{E}\left|\Psi\right|_{\Xi}^2 e^{\eta\varrho-\eta t}\right), \\
F_6 &\le \mathbb{E}\int_0^t\int_{\mathbb{X}} e^{2\beta v-2\beta t}\left|\Theta(v,Y(v-\theta(v)),\vartheta)\right|^2\lambda(d\vartheta)\,dv \\
&\le L_3\int_0^t e^{2\beta v-2\beta t}\mathbb{E}\left|Y(v-\theta(v))\right|_{\Xi}^2 dv \\
&\le L_3\left(\beta_\delta\overline{M}\mathbb{E}\left|\Psi\right|_{\mathfrak{D}}^2 e^{\delta\varrho-\delta t}+\beta_\eta\widetilde{M}\mathbb{E}\left|\Psi\right|_{\Xi}^2 e^{\eta\varrho-\eta t}\right).
\end{aligned} \tag{13}$$

By (H_1) and (H_4), we have

$$F_7 \le e^{-2\beta t} l_j \mathbb{E}\left|Y\left(t_j^-\right)\right|_{\Xi}^2. \tag{14}$$

From (9) to (14), we can see obviously that there exist $\widehat{M}>0$ and $\widehat{\delta}>0$ such that

$$\mathbb{E}\left|\Pi(Y)(t)\right|_{\Xi}^2 \le \widehat{M}\mathbb{E}\left|\Psi\right|_{\mathfrak{D}}^2 e^{-\widehat{\delta}t}. \tag{15}$$

Next we claim that $\Pi(Y)(t)$ is càdlàg on Υ. Let $Y\in\Upsilon$, $\widehat{t}\ge 0$, and $\Delta\widehat{t}>0$; we have from (8) that

$$\begin{aligned}
&\mathbb{E}\left|\Pi(Y)\left(\widehat{t}+\Delta\widehat{t}\right)-\Pi(Y)\left(\widehat{t}\right)\right|_{\Xi}^2 \\
&\quad\le 7\sum_{i=1}^{7}\mathbb{E}\left|F_i\left(\widehat{t}+\Delta\widehat{t}\right)-F_i\left(\widehat{t}\right)\right|_{\Xi}^2.
\end{aligned} \tag{16}$$

We can easily see that $\mathbb{E}|F_i(\widehat{t}+\Delta\widehat{t})-F_i(\widehat{t})|_{\Xi}^2 \to 0$ as $\Delta\widehat{t}\to 0$, $i=1,\ldots,4$, and $i=7$. Moreover, by the properties of the martingales, we have the fact that when $\Delta\widehat{t}\to 0$,

$$\begin{aligned}
&\mathbb{E}\left|F_5\left(\widehat{t}+\Delta\widehat{t}\right)-F_5\left(\widehat{t}\right)\right|_{\Xi}^2 \\
&\quad\le 2\int_0^{\widehat{t}}\mathbb{E}\left|\left(S\left(\widehat{t}+\Delta\widehat{t}-v\right)-S\left(\widehat{t}-v\right)\right)\right. \\
&\qquad\qquad\left.\cdot R(v,Y(v-\zeta(v)))\right|_{\Xi}^2 dv \\
&\qquad+2\int_{\widehat{t}}^{\widehat{t}+\Delta\widehat{t}}\mathbb{E}\left|S\left(\widehat{t}+\Delta\widehat{t}-v\right)R(v,Y(v-\zeta(v)))\right|_{\Xi}^2 dv \\
&\quad\longrightarrow 0,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left|F_6\left(\widehat{t}+\Delta\widehat{t}\right)-F_6\left(\widehat{t}\right)\right|_{\Xi}^2 \\
&\quad\le 2\mathbb{E}\left|\int_0^{\widehat{t}}\int_{\mathbb{X}}\left(S\left(\widehat{t}+\Delta\widehat{t}-v\right)-S\left(\widehat{t}-v\right)\right)\right. \\
&\qquad\qquad\left.\cdot\,\Theta(t,Y(v-\theta(v)),\vartheta)\,\widetilde{N}(dv,d\vartheta)\right|_{\Xi}^2 \\
&\qquad+2\mathbb{E}\left|\int_{\widehat{t}}^{\widehat{t}+\Delta\widehat{t}}\int_{\mathbb{X}}S\left(\widehat{t}+\Delta\widehat{t}-v\right)\right. \\
&\qquad\qquad\left.\cdot\,\Theta(t,Y(v-\theta(v)),\vartheta)\,\widetilde{N}(dv,d\vartheta)\right|_{\Xi}^2 \\
&\quad\le 2\left\|S\left(\Delta\widehat{t}\right)-I\right\|^2\mathbb{E} \\
&\qquad\cdot\int_0^{\widehat{t}}\int_{\mathbb{X}}\left|S\left(\widehat{t}-v\right)\Theta(t,Y(v-\theta(v)),\vartheta)\right|_{\Xi}^2 \upsilon(d\vartheta)\,dv \\
&\qquad+2\mathbb{E} \\
&\qquad\cdot\int_{\widehat{t}}^{\widehat{t}+\Delta\widehat{t}}\int_{\mathbb{X}}\left|S\left(\widehat{t}+\Delta\widehat{t}-v\right)\Theta(t,Y(v-\theta(v)),\vartheta)\right|_{\Xi}^2 \\
&\qquad\qquad\cdot\,\upsilon(d\vartheta)\,dv \\
&\quad\longrightarrow 0.
\end{aligned} \tag{17}$$

Consequently, we obtain that $\Pi(\Upsilon)\subset\Upsilon$.

We finally claim Π is contractive. From (8), $Y_1,Y_2\in\Upsilon$,

$$\sup_{0\le t\le T}\mathbb{E}\left|\Pi(Y_1)(t)-\Pi(Y_2)(t)\right|_{\Xi}^2 \le 6\sum_{i=1}^{6}G_i. \tag{18}$$

Similar to (10)–(14), we have

$$\begin{aligned}
G_1 &= \sup_{0\le t\le T}\mathbb{E}\left|D(t,Y_1(t-\kappa(t)))-D(t,Y_2(t-\kappa(t)))\right|_{\Xi}^2 \\
&\le \widetilde{K}\left\|(-A)^{-\alpha}\right\|^2\sup_{0\le v\le T}\mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2, \\
G_2 &= \sup_{0\le t\le T}\mathbb{E}\left|\int_0^t AS(t-v)\right. \\
&\qquad\qquad\cdot\left[D(v,Y_1(v-\kappa(v)))\right. \\
&\qquad\qquad\left.\left.-D(v,Y_2(v-\kappa(v)))\right]dv\right|_{\Xi}^2 \\
&\le \widetilde{K}M_{1-\alpha}^2\beta^{-2\alpha}\Gamma^2(\alpha)\sup_{0\le v\le T}\mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2, \\
G_3 &= \sup_{0\le t\le T}\mathbb{E}\left|\int_0^t S(t-v)\right. \\
&\qquad\qquad\cdot\left[P(v,Y_1(v-\zeta(v)))\right. \\
&\qquad\qquad\left.\left.-P(v,Y_2(v-\zeta(v)))\right]dv\right|_{\Xi}^2 \\
&\le L_1\beta^{-2}\sup_{0\le v\le T}\mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2,
\end{aligned}$$

$$
\begin{aligned}
G_4 = & \sup_{0\le t\le T} \mathbb{E}\left|\int_0^t S(t-v)\right. \\
& \cdot \left[R\left(v, Y_1\left(v-\varpi(v)\right)\right)\right. \\
& \left.\left. -R\left(v, Y_2\left(v-\varpi(v)\right)\right)\right] dB(v)\right|_{\Xi}^2 \\
\le & L_2 (2\beta)^{-1} \sup_{0\le v\le T} \mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2, \\
G_5 = & \sup_{0\le t\le T} \mathbb{E}\left|\int_0^t \int_{\mathbb{X}} S(v-t)\right. \\
& \cdot \left[\Theta\left(t, Y_1\left(v-\theta(v)\right), \vartheta\right)\right. \\
& \left.\left. -\Theta\left(t, Y_2\left(v-\theta(v)\right), \vartheta\right)\right]\right. \\
& \left.\cdot \widetilde{N}(dv, d\vartheta)\right|_{\Xi}^2 \\
\le & L_3 (2\beta)^{-1} \sup_{0\le v\le T} \mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
G_6 = & \sup_{0\le t\le T} \mathbb{E}\left|\sum_{0<\tau_j<t} S\left(t-\tau_j\right)\left(H_j\left(Y_1\left(\tau_j^-\right)\right)-H_j\left(Y_2\left(\tau_j^-\right)\right)\right)\right|^2 \\
\le & \widetilde{l} e^{-2\beta T} \sup_{0\le v\le T} \mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2.
\end{aligned}
\tag{20}
$$

Here $\widetilde{l} = \mathbb{E}(\sum_{j=1}^{\iota} |l_j|)$.

Consequently, we have

$$
\begin{aligned}
& \sup_{0\le t\le T} \mathbb{E}\left|\Pi\left(Y_1\right)(t)-\Pi\left(Y_2\right)(t)\right|_{\Xi}^2 \\
& \le 6\left[\widetilde{K}\left\|(-A)^{-\alpha}\right\|^2+\widetilde{K} M_{1-\alpha}^2 \beta^{-2\alpha} \Gamma^2(\alpha)\right. \\
& \quad + L_1 \beta^{-2}+L_2(2\beta)^{-1} \\
& \quad \left.+ L_3(2\beta)^{-1}+\widetilde{l} e^{-2\beta T}\right] \\
& \quad \cdot \sup_{0\le v\le T} \mathbb{E}\left|Y_1(v)-Y_2(v)\right|_{\Xi}^2.
\end{aligned}
\tag{21}
$$

Then if (6) holds, Π is contractive. Therefore, system (1) has a unique $Y(t) \in \Upsilon$ and $Y(t)$ is mean square exponentially stable if (6) holds. This proof is complete. □

According to [5], we similarly have the following.

Theorem 5. *Under the conditions in Theorem 4, system (1) is almost surely exponentially stable.*

If $\Theta = 0$, system (1) becomes

$$
\begin{aligned}
& d\left[Y(t)+D(t, Y(t-\kappa(t)))\right] \\
& \quad = \left[AY(t)+P(t, Y(t-\zeta(t)))\right] dt \\
& \qquad + R(t, Y(t-\varpi(t))) dB(t), \quad t \ge 0,\ t \ne \tau_j, \\
& \Delta Y\left(\tau_j\right) = H_j\left(Y\left(\tau_j^-\right)\right), \quad t=\tau_j,\ j=1,\ldots,\iota,
\end{aligned}
\tag{22}
$$

with the initial data $Y_0(\cdot) = \Psi \in \mathfrak{D}_{\mathcal{F}_0}([-\varrho, 0], \Xi)$.

From Theorems 4 and 5, we have the following.

Corollary 6. *Assume that the conditions in Theorem 4 hold, but (6) is replaced with the following condition:*

$$
\begin{aligned}
& 5\left[\widetilde{K}\left\|(-A)^{-\alpha}\right\|^2+M_{1-\alpha}^2 \widetilde{K} \beta^{-2\alpha} \Gamma^2(\alpha)\right. \\
& \quad \left.+ L_1 \beta^{-2}+L_2(2\beta)^{-1}+e^{-2\beta T} \mathbb{E}\left(\sum_{j=1}^{\iota}\left|l_j\right|\right)\right] < 1.
\end{aligned}
\tag{23}
$$

Then system (22) admits a unique mild solution and is mean square and almost surely exponentially stable.

If $\Theta \equiv 0$ and $D \equiv 0$, system (1) becomes

$$
\begin{aligned}
& dY(t) = \left[AY(t)+P(t, Y(t-\zeta(t)))\right] dt \\
& \qquad + R(t, Y(t-\varpi(t))) dB(t), \quad t \ge 0,\ t \ne \tau_j, \\
& \Delta Y\left(\tau_j\right) = H_j\left(Y\left(\tau_j^-\right)\right), \quad t=\tau_j,\ j=1,\ldots,\iota,
\end{aligned}
\tag{24}
$$

with the initial data $Y_0(\cdot) = \Psi \in \mathfrak{D}_{\mathcal{F}_0}([-\varrho, 0], \Xi)$.

Corollary 7. *Assume that the conditions in Theorem 4 hold, but (H_2) and (6) are replaced with the following condition:*

$$
4\left(L_1 \beta^{-2}+L_2(2\beta)^{-1}+e^{-2\beta T} \mathbb{E}\left(\sum_{j=1}^{\iota}\left|l_j\right|\right)\right) < 1. \tag{25}
$$

Then system (24) has a unique mild solution and is mean square and almost surely exponentially stable.

Remark 8. We think that the results of the paper can be generalized to infinite delay systems. Systems (22) and (24) have been discussed in [14] and [13], respectively, which focus on asymptotic stability of mild solution. Also by Theorem 4 system (1) without impulses is also mean square and almost surely exponential stability under some conditions, which has been studied in [25]. However, it is well known that there are great differences on the method between the time-delay cases, in particular when considering a problem involved in perturbation. In the paper, we mainly focus on exponential stability. In the sense, [13, 14, 25] are generalized to more extensive systems.

Remark 9. In particular, when $D \equiv 0$, $\Theta \equiv 0$, system (1) without jumps, impulses, and neutral term reduces to SPDS,

which is mean square and almost surely exponential stability if $3(L_1\beta^{-2}+L_2(2\beta)^{-1})<1$. When $L_1\beta^{-2}+L_2\beta^{-1}<1/3$, Luo [5] showed that system (1) without jumps, impulses, and neutral term is mean square exponentially stable to this system. In the sense, the result of the paper improves the result of [5].

Remark 10. Besides, it should be pointed out that the proposed method in the paper can be employed to consider the pth moment ($p\geq 2$) exponential stability to system (1).

4. Illustrative Example

Example 1. Consider a jump-diffusion system with neutral term and impulses:

$$\begin{aligned} &d\left(Y(t,\chi)+\beta_1 Y(t-\kappa(t),\chi)\right)\\ &\quad=\left(\frac{\partial^2}{\partial\chi^2}Y(t,\chi)+\beta_2 Y(t-\zeta(t),\chi)\right)dt\\ &\qquad+\beta_3 Y(t-\varpi(t),\chi)\,dB(t)\\ &\qquad+\int_{\mathbb{X}}\beta_4\vartheta Y(t-\theta(t),\chi)\,\widetilde{N}(dt,d\vartheta),\quad t\geq 0,\\ &\Delta Y(\tau_j,\chi)=b_jY(\tau_j^-,\chi),\quad t=\tau_j\ \ (j=1,2,3,\ldots,\iota), \end{aligned}\tag{26}$$

with $Y(s,\cdot)=\Psi(s,\cdot)\in\mathscr{L}^2[0,\pi]$, $Y(\cdot,0)=Y(\cdot,\pi)=0$, $s\leq 0$, where $\beta_j>0$, $b_j\geq 0$ and $\sum_{j=1}^{\iota}b_j<\infty$.

Let $\mathbb{X}=\{\mathfrak{x}\in R:0<|\mathfrak{x}|\leq\ell,\ \ell>0\}$ and $\mathfrak{H}=L^2(0,\pi)$. The operator A is defined by $A:\mathfrak{H}\to\mathfrak{H}$ with $A=\partial^2/\partial\chi^2$ and

$$\begin{aligned}\mathfrak{D}(A)=\Bigg\{&Y\in\mathfrak{H}:Y,\ \frac{\partial Y}{\partial\chi}\text{ are absolutely continuous,}\\ &\frac{\partial^2 Y}{\partial\chi^2}\in\mathfrak{H},\ Y(0)=Y(\pi)=0\Bigg\};\end{aligned}\tag{27}$$

then $(-A)^{3/5}$ is given by

$$(-A)^{3/5}Y=\sum_{n=1}^{\infty}n\left\langle Y,\sqrt{\frac{2}{n}}\sin n\chi\right\rangle_{\mathfrak{H}}\sqrt{\frac{2}{n}}\sin n\chi,\tag{28}$$

and the domain

$$\begin{aligned}&\mathfrak{D}\left((-A)^{3/5}\right)\\ &\quad=\left\{Y\in\mathfrak{H},\ \sum_{n=1}^{\infty}n\left\langle Y,\sqrt{\frac{2}{n}}\sin n\chi\right\rangle_{\mathfrak{H}}\sqrt{\frac{2}{n}}\sin n\chi\in\mathfrak{H}\right\}.\end{aligned}\tag{29}$$

Since, for $t\geq 0$, $\|S(t)\|\leq\exp(-\pi^2 t)$, from Pazy [32, Page 70], we have

$$\left\|(-A)^{-3/5}\right\|\leq\frac{1}{\Gamma(3/5)}\int_0^{\infty}v^{-2/5}\|S(v)\|\,dv\leq\frac{1}{\pi^{6/5}}.\tag{30}$$

Obviously, (H_1)–(H_5) are satisfied with

$$\begin{aligned}&\beta=\pi^2,\qquad\widetilde{K}=\left\|(-A)^{3/5}\right\|^2\beta_1^2,\qquad L_1=\beta_2^2,\\ &L_2=\beta_3^2,\qquad L_3=\beta_4^2\int_{\mathbb{X}}\vartheta^2\lambda(d\vartheta),\qquad l_j=b_j^2.\end{aligned}\tag{31}$$

Thus, by Theorems 4 and 5, system (26) is mean square and almost surely exponentially stable if

$$\begin{aligned}&\left\|(-A)^{3/5}\right\|^2\beta_1^2\left(\frac{1}{\pi^2}+\frac{M_{2/5}^2}{\pi}\right)+\frac{\beta_2^2}{\pi^4}+\frac{\beta_3^2}{2\pi^2}\\ &\quad+\frac{\beta_4^2}{2\pi^2}\int_{\mathbb{X}}\vartheta^2\lambda(d\vartheta)+e^{-2\pi T}\mathbb{E}\left(\sum_{j=1}^{\iota}\left|b_j\right|^2\right)<\frac{1}{6},\end{aligned}\tag{32}$$

where $M_{2/5}$ is defined by (5).

5. Concluding Remarks

In this paper, we have discussed jump-diffusion systems with neutral term and impulses. Some conditions on mean square and almost surely exponential stability of the mild solutions to the jump-diffusion systems with neutral term and impulses are derived by the fixed point theory. The obtained results extend some earlier results to the case of SPDS with neutral term and jump and impulses. Finally, the results of this paper are demonstrated well with an example.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (61304067 and 61071136), the Natural Science Foundation of Hubei Province of China (2013CFB443), and Research Fund for the Doctoral Program of Higher Education of China (20110142110069).

References

[1] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, vol. 119 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1996.

[2] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, Chapman & Hall, CRC Press, London, UK, 2006.

[3] G. da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1992.

[4] T. Caraballo and K. Liu, "Exponential stability of mild solutions of stochastic partial differential equations with delays," *Stochastic Analysis and Applications*, vol. 17, no. 5, pp. 743–763, 1999.

[5] J. Luo, "Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 2, pp. 753–760, 2008.

[6] J. Luo and T. Taniguchi, "Fixed points and stability of stochastic neutral partial differential equations with infinite delays," *Stochastic Analysis and Applications*, vol. 27, no. 6, pp. 1163–1173, 2009.

[7] H. Chen, "Integral inequality and exponential stability for neutral stochastic partial differential equations with delays," *Journal of Inequalities and Applications*, vol. 2009, no. 1, Article ID 297478, 2009.

[8] L. Pan, "Local stability and parameter dependence of mild solutions for stochastic differential equations," *Advances in Difference Equations*, vol. 2014, no. 1, article 276, 2014.

[9] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, New York, NY, USA, 2006.

[10] L. Zhang, Y. Ding, T. Wang, L. Hu, and K. Hao, "Stability of neutral impulsive nonlinear stochastic evolution equations with time varying delays," *Asian Journal of Control*, vol. 16, no. 5, pp. 1416–1424, 2014.

[11] Q. Zhu and B. Song, "Exponential stability of impulsive nonlinear stochastic differential equations with mixed delays," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 5, pp. 2851–2860, 2011.

[12] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic BAM neural networks with impulse control and mixed time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 3, pp. 467–479, 2012.

[13] R. Sakthivel and J. Luo, "Asymptotic stability of nonlinear impulsive stochastic differential equations," *Statistics & Probability Letters*, vol. 79, no. 9, pp. 1219–1223, 2009.

[14] F. Jiang and Y. Shen, "Stability of impulsive stochastic neutral partial differential equations with infinite delays," *Asian Journal of Control*, vol. 14, no. 6, pp. 1706–1709, 2012.

[15] H. Chen, C. Zhu, and Y. Zhang, "A note on exponential stability for impulsive neutral stochastic partial functional differential equations," *Applied Mathematics and Computation*, vol. 227, pp. 139–147, 2014.

[16] F. Jiang, "Mild solutions of neutral semilinear stochastic functional dynamic systems with local non-Lipschitz coefficients," *Advances in Mathematical Physics*, vol. 2013, Article ID 823535, 6 pages, 2013.

[17] J. Bao and Z. Hou, "Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 207–214, 2010.

[18] F. Jiang and Y. Shen, "A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients," *Computers & Mathematics with Applications*, vol. 61, no. 6, pp. 1590–1594, 2011.

[19] H. Bao and J. Cao, "Existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay," *Applied Mathematics and Computation*, vol. 215, no. 5, pp. 1732–1743, 2009.

[20] T. E. Govindan, "Almost sure exponential stability for stochastic neutral partial functional differential equations," *Stochastics*, vol. 77, no. 2, pp. 139–154, 2005.

[21] J. Bao, A. Truman, and C. Yuan, "Stability in distribution of mild solutions to stochastic partial differential delay equations with jumps," *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, vol. 465, no. 2107, pp. 2111–2134, 2009.

[22] Q. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 126–142, 2014.

[23] X. Yang and Q. Zhu, "pth moment exponential stability of stochastic partial differential equations with Poisson jumps," *Asian Journal of Control*, vol. 16, no. 5, pp. 1482–1491, 2014.

[24] L. Guo and Q. Zhu, "Stability analysis for stochastic Volterra-Levin equations with Poisson jumps: fixed point approach," *Journal of Mathematical Physics*, vol. 52, no. 4, Article ID 042702, 15 pages, 2011.

[25] J. Cui, L. Yan, and X. Sun, "Exponential stability for neutral stochastic partial differential equations with delays and Poisson jumps," *Statistics & Probability Letters*, vol. 81, no. 12, pp. 1970–1977, 2011.

[26] J. Cui and L. Yan, "Successive approximation of neutral stochastic evolution equations with infinite delay and Poisson jumps," *Applied Mathematics and Computation*, vol. 218, no. 12, pp. 6776–6784, 2012.

[27] J. Cui and L. Yan, "Existence result for fractional neutral stochastic integro-differential equations with infinite delay," *Journal of Physics. A. Mathematical and Theoretical*, vol. 44, no. 33, Article ID 335201, 2011.

[28] J. Luo and T. Taniguchi, "The existence and uniqueness for non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps," *Stochastics and Dynamics*, vol. 9, no. 1, pp. 135–152, 2009.

[29] Y. Ren and R. Sakthivel, "Existence, uniqueness, and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps," *Journal of Mathematical Physics*, vol. 53, no. 7, Article ID 073517, 2012.

[30] R. Sakthivel and Y. Ren, "Exponential stability of second-order stochastic evolution equations with Poisson jumps," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 12, pp. 4517–4523, 2012.

[31] H. Yang and F. Jiang, "Exponential stability of mild solutions to impulsive stochastic neutral partial differential equations with memory," *Advances in Difference Equations*, vol. 2013, article 148, 2013.

[32] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, NY, USA, 1992.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 702802, 12 pages
http://dx.doi.org/10.1155/2015/702802

Research Article

Maximum Principle for Forward-Backward Stochastic Control System Driven by Lévy Process

Xiangrong Wang[1] and Hong Huang[2]

[1] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*
[2] *College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Hong Huang; huangqian1201@163.com

Received 4 February 2015; Revised 26 April 2015; Accepted 12 May 2015

Academic Editor: Son Nguyen

We study a stochastic optimal control problem where the controlled system is described by a forward-backward stochastic differential equation driven by Lévy process. In order to get our main result of this paper, the maximum principle, we prove the continuity result depending on parameters about fully coupled forward-backward stochastic differential equations driven by Lévy process. Under some additional convexity conditions, the maximum principle is also proved to be sufficient. Finally, the result is applied to the linear quadratic problem.

1. Introduction

The stochastic optimal control problem is one of the central themes of modern control science. Forward-backward stochastic control systems which the controlled systems described by forward-backward stochastic differential equations (FBSDEs) are widely used in mathematics and finance. Peng and Wu [1] firstly used a probabilistic method to get the existence and uniqueness results of fully coupled FBSDEs; then Peng [2] considered one kind of forward-backward stochastic control systems with economic background when the control domain is convex and obtained the maximum principle; since then a number of developments in this direction were reported in Wu [3] and Shi and Wu [4]. Wu [5] firstly proved the existence and uniqueness results of the solutions to fully coupled FBSDEs with Brownian motion and Poisson process; then Shi and Wu [6] got the stochastic maximum principle for fully coupled FBSDEs with random jumps. More conclusions about stochastic maximum principle about forward-backward stochastic control systems driven by Brownian motion and Poisson process can be seen in [7–9].

It is natural to extend the stochastic differential equations (SDEs) with Brownian motion and Poisson process to the case of Lévy process with independent and stationary increments. Baghery et al. [10] firstly considered the following fully coupled forward-backward stochastic differential equation driven by Lévy process (FBSDEL):

$$\begin{aligned} dx_t &= b(t, x_t, y_t, z_t, r_t)\,dt + \sum_{i=1}^{d} \sigma^i(t, x_t, y_t, z_t, r_t)\,dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g^i(t, x_{t-}, y_{t-}, z_t, r_t)\,dH_t^i, \\ -dy_t &= f(t, x_t, y_t, z_t, r_t)\,dt - \sum_{i=1}^{d} z_t^i dB_t^i - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\ x_0 &= a, \\ y_T &= \Phi(x_T), \end{aligned} \tag{1}$$

and, under some monotonicity assumptions, they got the existence and uniqueness of solutions for this equation. Zhu [11] had proposed the asymptotic stability in the Pth moment for SDE with Lévy noise. Nualart and Schoutens [12] constructed a set of pairwise strongly orthonormal martingales called Teugels martingale and they also proved a martingale representation theorem for Lévy processes satisfying some

exponential moment condition. Using the martingale representation theorem they [13] had proved the existence and uniqueness of a solution for backward stochastic differential equations (BSDEs) driven by Teugels martingale. Bahlali et al. [14] extended this conclusion to the BSDEs driven by Teugels martingale and an independent Brownian motion; they got the existence, uniqueness, and comparison of solutions for these equations under Lipschitz and locally Lipschitz conditions on the coefficient. Based on these consequences, Mitsui and Tabata [15] established the closeness property of the solution of the multidimensional backward stochastic Riccati differential equation with Lévy process; then they used this solution to study a linear quadratic regulation problem with Lévy process. After the foundation of the existence and uniqueness of the solutions of SDEs and multidimensional BSDEs driven by Lévy process, Tang and Wu [16] proceed to study a stochastic linear quadratic optimal control problem with a Lévy process, where the cost weighting matrices of the state and control were allowed to be indefinite.

These consequences are important for the researching of maximum principle for forward stochastic control system driven by Lévy process, as the adjoint equation for forward stochastic control system is a BSDE. Meng and Tang [17] firstly were concerned with optimal control for forward stochastic control system driven by Teugels martingale; they got the maximum principle and verification theorem for this system. In 2012, Tang and Zhang [18] were concerned with optimal control of BSDE driven by Teugels martingale and an independent multidimensional Brownian motion; they derived the necessary and sufficient conditions for the existence of the optimal control by means of convex variation methods and duality techniques. When the control domain was nonconcave and the control variable was allowed to enter the coefficients of the Teugels martingales, Lin [19] got the necessary maximum principle for optimal control of stochastic system driven by multidimensional Teugels martingales. Zhang et al. [20] firstly studied the forward-backward stochastic control system where the system was driven by Teugels martingale and an independent multidimensional Brownian motion as follows:

$$\begin{aligned} dx_t &= b(t,x_t,u_t)\,dt + \sum_{i=1}^{d}\sigma^i(t,x_t,u_t)\,dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g^i(t,x_{t-},u_t)\,dH_t^i, \\ -dy_t &= f(t,x_t,y_t,z_t,r_t,u_t)\,dt - \sum_{i=1}^{d} z_t^i dB_t^i - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\ x_0 &= a, \\ y_T &= \Phi(x_T), \end{aligned} \tag{2}$$

and they had got the maximum principle and verification theorem in the condition of the SDE part did not contain the backward state variables; the forward-backward stochastic control system they studied was not fully coupled.

In this paper, we extend the result of Zhang et al. [20] to the fully coupled forward-backward stochastic control system. Here the state variables are described by fully coupled FBSDEs driven by Brownian motion and an independent Teugels martingale. Before applying the convex variation and duality technique to obtain the stochastic maximum principle, we use the same method in [5] to get the continuity result depending on parameters, as the continuity result is not only important for us to get the stochastic maximum principle but also important property of FBSDEL especially in practice. Different from the Wu [3] and Shi and Wu [6] about maximum principles to Brownian motions and Poisson process, we also need more general *Ito's* formula about *càdlàg* semimartingale.

This paper is organized as follows. In Section 2, we will give some preliminaries used in this paper. Section 3 presents the continuity result depending on parameters about fully coupled FBSDEs driven by Lévy process. In Section 4, we obtain the main result of this paper, the maximum principle. We also prove that, under some additional convexity conditions, the maximum principle can be a sufficient condition for optimal control. And, in Section 5, an application of our stochastic maximum principle to the linear quadratic problem which the linear control system described by fully coupled FBSDEL is proved.

2. Preliminaries and Notations

Let $(\Omega, F, P, \mathscr{F}_t, B_t, L_t)$ $(t \in [0,T])$ be a complete space driving by Brownian motion and Lévy process in $R^m \times R \setminus \{0\}$, with Lévy measure ν; that is, $\{B_t\}_{0\le t\le T}$ is a standard Brownian motion. $\{L_t\}_{0\le t\le T}$ is R-valued Lévy process of the form $L_t = bt + \ell_t$ independent of $\{B_t\}_{0\le t\le T}$, corresponding to a standard Lévy measure ν satisfying the following conditions:

(i) $\int_R (1\wedge x^2)\nu(dx) < \infty$,

(ii) $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}\nu(dx) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$, and

$$\mathscr{F}_t = \sigma(L_s,\ s\le t) \vee \sigma(B_s,\ s\le t) \vee \aleph. \tag{3}$$

Here $\aleph$ is the totality of P-null sets and $g_1 \vee g_2$ denotes the σ-field generated by $g_1 \cup g_2$.

Let x be a Lévy process and denote the left limit process by $x_{t-} = \lim_{s\to t, s\le t} x_s$ and the jump size at time t by $\Delta x_t = x_t - x_{t-}$. Set

$$x_t^i = \begin{cases} \sum_{0<s\le t} (\Delta x_s)^i, & i \ge 2 \\ x_t, & i = 1 \end{cases} \tag{4}$$

and we denote the compensated power jump process of order i by $Y_t^i = x_t^i - E[x_t^i]$; then Teugels martingale $(H_t^i)_{0\le t\le T}$ can be defined as follows:

$$H_t^i = c_{i,i}Y_t^i + c_{i,i-1}Y_t^{i-1} + c_{i,i-2}Y_t^{i-2} + \cdots + c_{i,1}Y_t^1. \tag{5}$$

Here the coefficients $c_{i,k}$ correspond to orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = \upsilon(dx) + \sigma^2\delta_0(dx)$.

Now we introduce some notations adopted in this paper as follows:

(1) H: Hilbert space,

(2) $\langle \alpha, \beta \rangle$: the inner product in R^n, $\forall \alpha, \beta \in R^n$,

(3) $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm in R^n, $\forall \alpha \in R^n$,

(4) $\langle A, B \rangle = \mathrm{tr}(AB^T)$: the inner product in $R^{n\times m}$, $\forall A, B \in R^{n\times m}$,

(5) $|A| = \sqrt{\mathrm{tr}(AA^T)}$: the inner product in $R^{n\times m}$, $\forall A \in R^{n\times m}$,

(6) l^2: the space of real valued sequences $X = (x_n)_{n\geq 0}$ such that

$$\|x\|_{l^2}^2 = \sum_{i=1}^{\infty} x_i^2 < \infty, \tag{6}$$

(7) $l^2(H)$: the space of H-valued sequences $\phi = \varphi^i{}_{i\geq 1}$ such that

$$\|\phi\|_{l^2(H)}^2 = \sum_{i=1}^{\infty} \left\|\phi^i\right\|^2 < \infty, \tag{7}$$

(8) $l^2(0,T;H)$: the corresponding spaces of $l^2(H)$ valued $\mathscr{F}_t$-measurable processes equipped with the norm

$$\|\phi\|_{l^2(0,T;H)}^2 = E\int_0^T \sum_{i=1}^{\infty} \left\|\phi_t^i\right\|_H^2 dt < \infty, \tag{8}$$

(9) $L^2(\Omega, \mathscr{F}_t, P; H)$: the space of H-valued random variable ξ with the norm

$$\|\xi\|^2 = E\|\xi\|_{L^2(\Omega,\mathscr{F}_t,P;H)}^2 < \infty, \tag{9}$$

(10) $M^2(0,T;H)$: the space of H-valued $\mathscr{F}_t$-measurable process $\phi(\cdot) = \{\phi(t,\omega) : (t,\omega) \in [0,T]\times\Omega\}$ with the norm

$$\|\phi(\cdot)\|_{M^2(0,T;H)}^2 = E\int_0^T \|\phi_t\|_{M^2(0,T;H)}^2 dt < \infty, \tag{10}$$

(11) $S^2(0,T;H)$: the space of H-valued $\mathscr{F}_t$-measurable *càdlàg* process $f(\cdot) = \{f(t,\omega) : (t,\omega) \in [0,T]\times\Omega\}$ with the norm

$$\|f(\cdot)\|_{S^2(0,T;H)}^2 = E\sup_{0\leq t\leq T} \|f_t\|_H^2 dt < \infty, \tag{11}$$

(12) for notational brevity:

$$\begin{aligned} M^2(0,T) &= M^2(0,T;R^n)\times M^2(0,T;R^m) \\ &\quad \times M^2\left(0,T;R^{m\times d}\right)\times l^2(0,T;R^m). \end{aligned} \tag{12}$$

Let us recall more general *Ito's* formula about *càdlàg* semimartingales which is important for us to get the maximum principle. Let $X = \{X_t : t \in [0,T]\}$ be *càdlàg* semimartingales, and we denoted the quadratic variation by $[X] = \{[X]_t : t \in [0,T]\}$; F is a $\mathscr{C}^2$ real valued function; then $F(X)$ is also semimartingales and following *Ito's* formula holds:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-})\, dX_s \\ &\quad + \frac{1}{2}\int_0^t F''(X_s)\, d[X]_s^{\mathscr{C}} \\ &\quad + \sum_{0<s\leq t} \left\{F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\right\}, \end{aligned} \tag{13}$$

where $[X]^{\mathscr{C}}$ is the continuous part of the quadratic variation $[X]$.

When $F(X) = X^2$ and $F(X) = X_tY_t$, where X, Y are two *càdlàg* semimartingales, we get

$$\begin{aligned} X_t^2 &= X_0^2 + \int_0^t 2X_{s-}dX_s + \int_0^t d[X]_s, \\ X_tY_t &= X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s \\ &\quad + \int_0^t d[X,Y]_s. \end{aligned} \tag{14}$$

Here $[X,Y]$ is the quadratic covariation of X, Y. We can refer to Protter [21] for a complete survey in this topic.

Next, we introduce the existence and uniqueness results for fully coupled FBSDEL (1):

$$\begin{aligned} dx_t &= b(t,x_t,y_t,z_t,r_t)\, dt + \sum_{i=1}^{d} \sigma^i(t,x_t,y_t,z_t,r_t)\, dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g^i(t,x_{t-},y_{t-},z_t,r_t)\, dH_t^i, \\ -dy_t &= f(t,x_t,y_t,z_t,r_t)\, dt - \sum_{i=1}^{d} z_t^i dB_t^i - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\ x_0 &= a, \\ y_T &= \Phi(x_T), \end{aligned} \tag{15}$$

where

$$\begin{aligned} b &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \longrightarrow R^n, \\ \sigma &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \longrightarrow R^{n\times d}, \\ g &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \\ &\quad \longrightarrow l^2(R^n), \\ f &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \longrightarrow R^m. \end{aligned} \tag{16}$$

For a given $m \times n$ full rank matrix G, we set

$$\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A(t,\lambda,r) = \begin{pmatrix} -G^T f(t,\lambda,r) \\ Gb(t,\lambda,r) \\ G\sigma(t,\lambda,r) \end{pmatrix}. \tag{17}$$

Assumption 1. Assume the following.

(i) b, σ, g, and f are uniformly Lipschitz continuous with respect to (x, y, z, r).

(ii) For each $(\omega, t) \in \Omega \times [0, T]$, $l(\omega, t, 0, 0, 0, 0) \in M^2(0, T)$ and $g(\omega, t, 0, 0, 0, 0) \in H^2(l^2)$, where $l = b, \sigma, f$, respectively.

(iii) $\Phi(\cdot)$ is uniformly Lipschitz continuous with respect to x and $\forall x, \Phi(x) \in L^2(\Omega, F_T, P)$.

Assumption 2. We also assume that

$$\begin{aligned} &\langle A(t,\lambda_1,r_1) - A(t,\lambda_2,r_2), \lambda_1 - \lambda_2 \rangle + \sum_{i=1}^{\infty} \left\langle G\widehat{g^i}, \widehat{r^i} \right\rangle \\ &\quad \le -\beta_1 |G\widehat{x}|^2 \\ &\qquad - \beta_2 \left(\left|G^T \widehat{y}\right|^2 + \left|G^T \widehat{z}\right|^2 + \sum_{i=1}^{\infty} \left\|G^T \widehat{r^i}\right\|^2 \right); \\ &\langle \Phi(x_1) - \Phi(x_2), G(x_1 - x_2) \rangle \ge \mu_1 |G\widehat{x}|^2, \end{aligned} \tag{18}$$

where $\lambda_1 = (x_1, y_1, z_1)$, $\lambda_2 = (x_2, y_2, z_2)$, $\widehat{x} = x_1 - x_2$, $\widehat{y} = y_1 - y_2$, $\widehat{z} = z_1 - z_2$, $\widehat{g^i} = g^i(t, \lambda_1, r_1) - g^i(t, \lambda_2, r_2)$, $\widehat{r^i} = r_1^i - r_2^i$, and β_1, β_2 and μ_1 are nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$. Moreover, we have $\beta_1 > 0$, $\mu_1 > 0$ (resp., $\beta_2 > 0$) when $m > n$ (resp., $n > m$). Under Assumptions 1 and 2, in [10], they have got the following lemma.

Lemma 3 (existence and uniqueness theorem of FBSDEL [10]). *Under Assumptions 1 and 2, FBSDEL (15) has a unique solution.*

3. Continuity Result Depending on Parameters about FBSDEL

Next, we are going to get the continuity result depending on parameters about FBSDEL.

Let $(b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$, $\alpha \in R$ be a family of FBSDEL:

$$\begin{aligned} dx_t^\alpha &= b_\alpha(t, x_t^\alpha, y_t^\alpha, z_t^\alpha, r_t^\alpha)\, dt \\ &\quad + \sum_{i=1}^{d} \sigma_\alpha^i (t, x_t^\alpha, y_t^\alpha, z_t^\alpha, r_t^\alpha)\, dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g_\alpha^i (t, x_{t-}^\alpha, y_{t-}^\alpha, z_t^\alpha, r_t^\alpha)\, dH_t^i, \\ -dy_t &= f_\alpha(t, x_t^\alpha, y_t^\alpha, z_t^\alpha, r_t^\alpha)\, dt - \sum_{i=1}^{d} z_t^{i,\alpha} dB_t^i \\ &\quad - \sum_{i=1}^{\infty} r_t^{i,\alpha} dH_t^i, \\ x_0^\alpha &= a, \\ y_T^\alpha &= \Phi_\alpha(x_T^\alpha). \end{aligned} \tag{19}$$

Assumption 4. (i) The family $(b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$, $\alpha \in R$ are equi-Lipschitz with respect to (x, y, z, r) and x separately.

(ii) The function $\alpha \to (b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$ is continuous in their existing space norm sense, respectively.

Then we can get the following continuity result depending on parameters of forward-backward stochastic differential equation driven by Lévy processes.

Theorem 5. *Let $(b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$, $\alpha \in R$ be a family of FBSDEL satisfying Assumptions 1, 2, and 4 with solutions denoted by $(x^\alpha, y^\alpha, z^\alpha, r^\alpha)$. Thus, the function*

$$\begin{aligned} &\alpha \longrightarrow (x^\alpha, y^\alpha, z^\alpha, r^\alpha, x_T^\alpha): \\ &R \longrightarrow M^2(0,T) \times L^2(\Omega, \mathcal{F}_t, P, R^n) \end{aligned} \tag{20}$$

is continuous.

Proof. For notational brevity, we only prove the continuity of FBSDEL (19) at $\alpha = 0$. Set $\widehat{\lambda}_t = \lambda_t^\alpha - \lambda_t^0 = (x_t^\alpha - x_t^0, y_t^\alpha - y_t^0, z_t^\alpha - z_t^0)$ and $\widehat{r}_t = r_t^\alpha - r_t^0$. From Assumptions 1, 2, and 4, applying the usual technique to $\widehat{x}_t$ of *Itô's* SDE with Lévy process, we can get

$$\begin{aligned} \sup_{0 \le t \le T} E\left|\widehat{x}_t\right|^2 &\le K_1 E \int_0^T \left(\left|\widehat{y}_t\right|^2 + \left|\widehat{z}_t\right|^2 + \left\|\widehat{r}_t^i\right\|^2 \right) dt \\ &\quad + K_1 E \int_0^T \left(\left|\widehat{b}_t\right|^2 + \left|\widehat{\sigma}_t\right|^2 + \left\|\widehat{g}_t\right\|^2 \right) dt. \end{aligned} \tag{21}$$

Applying the same technique to $\widehat{y}_t$ of BSDE with Lévy process, then

$$\begin{aligned} &E \int_0^T \left(\left|\widehat{y}_t\right|^2 + \left|\widehat{z}_t\right|^2 + \left\|\widehat{r}_t^i\right\|^2 \right) dt \\ &\quad \le K_1 \left(E \int_0^T \left|\widehat{x}_t\right|^2 dt + E\left|\widehat{x}_T\right|^2 + E \int_0^T \left|\widehat{f}_t\right|^2 dt \right. \\ &\quad \left. + E\left|\Phi_\alpha(x_T^0) - \Phi_0(x_T^0)\right|^2 \right). \end{aligned} \tag{22}$$

Here K_1, K_2 depend on the Lipschitz constants and T, and

$$\begin{aligned}\widehat{f}_t &= f_\alpha\left(t,\lambda_t^0,r_t^0\right)-f_0\left(t,\lambda_t^0,r_t^0\right),\\ \widehat{b}_t &= b_\alpha\left(t,\lambda_t^0,r_t^0\right)-b_0\left(t,\lambda_t^0,r_t^0\right),\\ \widehat{\sigma}_t &= \sigma_\alpha\left(t,\lambda_t^0,r_t^0\right)-\sigma_0\left(t,\lambda_t^0,r_t^0\right),\\ \widehat{g}_t &= g_\alpha\left(t,\lambda_t^0,r_t^0\right)-g_0\left(t,\lambda_t^0,r_t^0\right).\end{aligned} \tag{23}$$

Set

$$A_\alpha(t,\lambda,r)=\begin{pmatrix}-G^T f_\alpha(t,\lambda,r)\\ Gb_\alpha(t,\lambda,r)\\ G\sigma_\alpha(t,\lambda,r)\end{pmatrix}, \tag{24}$$

and applying *Itô's* formula to $\langle G\widehat{x}_t,\widehat{y}_t\rangle$ yields

$$\begin{aligned}&E\left\langle\Phi_\alpha\left(x_T^\alpha\right)-\Phi_\alpha\left(x_T^0\right),G\widehat{x}_T\right\rangle+E\left\langle\Phi_0\left(x_T^\alpha\right)\right.\\ &\quad\left.-\Phi_0\left(x_T^0\right),G\widehat{x}_T\right\rangle\\ &\quad=E\int_0^T\left[\left\langle A_\alpha\left(t,\lambda_t^\alpha,r_t^\alpha\right)-A_\alpha\left(t,\lambda_t^0,r_t^0\right),\widehat{\lambda}_t\right\rangle\right.\\ &\quad\left.+\sum_{i=1}^\infty\left\langle G\left(g_\alpha^i\left(t,\lambda_t^\alpha,r_t^\alpha\right)-g_\alpha^i\left(t,\lambda_t^0,r_t^0\right)\right),\widehat{r}_t^i\right\rangle\right]dt\\ &\quad+E\int_0^T\left[\left\langle\widehat{x}_t,-G^T\widehat{f}_t\right\rangle+\left\langle G^T\widehat{y}_t,\widehat{b}_t\right\rangle+\sum_{i=1}^d\left\langle\widehat{z}_t^i,G\widehat{\sigma}_t^i\right\rangle\right.\\ &\quad\left.+\sum_{i=1}^\infty\left\langle G^T\widehat{r}_t^i,\widehat{g}_t^i\right\rangle\right]dt.\end{aligned} \tag{25}$$

From the above three estimates, we get

$$\begin{aligned}&\beta_1E\int_0^T\left|G\widehat{x}_t\right|^2dt+\mu_1E\left|G\widehat{x}_T\right|^2\\ &\quad+\beta_2E\int_0^T\left[\left|G^T\widehat{y}_t\right|^2+\left|G^T\widehat{z}_t\right|^2+\left\|G^T\widehat{r}_t\right\|^2\right]dt\\ &\quad\le C_1\left[E\left|\Phi_\alpha\left(x_T^0\right)-\Phi_0\left(x_T^0\right)\right|^2\right.\\ &\quad\left.+E\int_0^T\left(\left|\widehat{f}_t\right|^2+\left|\widehat{b}_t\right|^2+\left|\widehat{\sigma}_t\right|^2+\left\|\widehat{g}_t\right\|^2\right)dt\right]\\ &\quad+\delta E\left|\widehat{x}_T\right|^2\\ &\quad+\delta E\int_0^T\left(\left|\widehat{x}_t\right|^2+\left|\widehat{y}_t\right|^2+\left|\widehat{z}_t\right|^2+\left\|\widehat{r}_t\right\|^2\right)dt,\end{aligned} \tag{26}$$

where the constant C_1 depends on the Lipschitz constants T and δ. When $m\ge n$, $\beta_1>0$, and $\mu_1>0$, then

$$\delta=\min\left(\frac{1}{3},\frac{1}{3K_1},\frac{\beta_1\left|G^TG\right|}{3},\frac{\mu_1\left|G^TG\right|}{3}\right). \tag{27}$$

If $m\le n$, $\beta_2>0$, and $\mu_1\ge 0$, then

$$\delta=\min\left(\frac{1}{3},\frac{1}{3K_1},\frac{1}{3K_1T},\frac{\beta_2\left|GG^T\right|}{3}\right). \tag{28}$$

Thus, it is clear whatever $\beta_1>0$, $\beta_2\ge 0$, and $\mu_1>0$ or $\beta_1\ge 0$, $\beta_2>0$, and $\mu_1\ge 0$ we always have

$$\begin{aligned}&E\left|\widehat{x}_T\right|^2+E\int_0^T\left(\left|\widehat{\lambda}_t\right|^2+\left\|\widehat{r}_t\right\|^2\right)dt\\ &\quad\le C\left[E\left|\Phi_\alpha\left(x_T^0\right)-\Phi_0\left(x_T^0\right)\right|^2\right.\\ &\quad\left.+E\int_0^T\left(\left|\widehat{f}_t\right|^2+\left|\widehat{b}_t\right|^2+\left|\widehat{\sigma}_t\right|^2+\left\|\widehat{g}_t\right\|^2\right)\right].\end{aligned} \tag{29}$$

The proof is completed. □

4. Maximum Principle

Let us consider the following full coupled forward-backward stochastic control system:

$$\begin{aligned}dx_t &= b\left(t,x_t,y_t,z_t,r_t,u_t\right)dt\\ &\quad+\sum_{i=1}^d\sigma^i\left(t,x_t,y_t,z_t,u_t\right)dB_t^i\\ &\quad+\sum_{i=1}^\infty g^i\left(t,x_{t-},y_{t-},r_t,u_t\right)dH_t^i,\\ -dy_t &= f\left(t,x_t,y_t,z_t,r_t,u_t\right)dt-\sum_{i=1}^d z_t^idB_t^i-\sum_{i=1}^\infty r_t^idH_t^i,\\ x_0 &= a,\\ y_T &= \Phi\left(x_T\right),\end{aligned} \tag{30}$$

where (x_t,y_t,z_t,r_t) take values in $R^n\times R^m\times R^{m\times d}\times l^2(R^m)$; $a\in R^n$ is given.

Let U be a nonempty convex subset of R^k. We define the admissible control set $U_{ad}=\{u(\cdot)\in M^2(0,T;R^k);u_t\in U,0\le t\le T,\text{a.e., a.s.}\}$ and the cost functional:

$$J(u)=E\int_0^T L\left(x_t,y_t,z_t,r_t,u_t\right)dt+h\left(x_T\right)+\gamma\left(y_0\right). \tag{31}$$

The optimal control problem is to find $\overline{u}\in U_{ad}$, such that

$$J(\overline{u}(\cdot))=\inf_{u(\cdot)\in U[0,T]}J(u(\cdot)). \tag{32}$$

Assumption 6. Now we introduce the basic assumptions of this section as follows.

(i) b, f, and L are continuously differentiable with respect to (x,y,z,r,u); σ is continuously differentiable with respect to (x,y,z,u); g is continuously differentiable with respect to (x,y,r,u); Φ and h are continuously differentiable with respect to x; γ is continuously differentiable with respect to y. And the derivatives of each function are all bounded.

(ii) For each $(x, y, z, r, u) \in R^n \times R^m \times R^{m\times d} \times l^2(R^m) \times U_{ad}$, there exists a constant $C > 0$, such that

$$\begin{aligned}
&|L| \le C\left(1+|x|^2+|y|^2+|z|^2+\|r\|^2+|u|^2\right),\\
&|h| \le C\left(1+|x|^2\right),\\
&|\gamma| \le \left(1+|y|^2\right),\\
&|L_x|+|L_y|+|L_z|+|L_r|+|L_u|\\
&\quad \le C\left(1+|x|+|y|+|z|+\|r\|+|u|\right),\\
&|h_x| \le C\left(1+|x|\right),\\
&|\gamma_y| \le C\left(1+|y|\right).
\end{aligned} \tag{33}$$

(iii) For any given admissible control $u(\cdot)$, (30) satisfies Assumptions 1 and 2.

Then, for a given admissible control, from Lemma 3, there exists a unique solution satisfying control system (30).

Let $\overline{u}_t$ be an optimal control and let $(\overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t)$ be the corresponding trajectory. For any given admissible control u_t and $0 \le \varepsilon \le 1$, we define

$$u_t^\varepsilon = \overline{u}_t + \varepsilon\left(u_t - \overline{u}_t\right). \tag{34}$$

Since U_{ad} is convex, then u_t^ε is in U_{ad}; that is, u_t^ε is an admissible control and $(x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon)$ is the corresponding trajectory.

We introduce the following variational equation:

$$\begin{aligned}
dX_t &= \left[b_x(t)X_t + b_y(t)Y_t + b_z(t)Z_t + b_r(t)R_t\right.\\
&\quad \left.+ b_u(t)\left(u_t-\overline{u}_t\right)\right]dt + \sum_{i=1}^{d}\left[\sigma_x^i(t)X_t + \sigma_y^i(t)Y_t\right.\\
&\quad \left.+ \sigma_z^i(t)Z_t + \sigma_u^i(t)\left(u_t-\overline{u}_t\right)\right]dB_t^i + \sum_{i=1}^{\infty}\left[g_x^i(t)X_{t-}\right.\\
&\quad \left.+ g_y^i(t)Y_{t-} + g_r^i(t)R_t + g_u^i(t)\left(u_t-\overline{u}_t\right)\right]dH_t^i,\\
-dy_t &= \left[f_x(t)X_t + f_y(t)Y_t + f_z(t)Z_t + f_r(t)R_t\right.\\
&\quad \left.+ f_u^i(t)\left(u_t-\overline{u}_t\right)\right]dt - \sum_{i=1}^{d}Z_t^i dB_t^i - \sum_{i=1}^{\infty}R_t^i dH_t^i,\\
X_0 &= 0,\\
Y_T &= \Phi_x\left(\overline{x}_t\right)X_T.
\end{aligned} \tag{35}$$

From Assumption 6, we can verify that variational equation (35) satisfies Lemma 3. Thus, there exists a unique solution (X_t, Y_t, Z_t, R_t) satisfying variational equation. In order to get the maximum principle, we also need the following lemma.

Lemma 7. *Assume that Assumption 6 holds. We have*

$$\begin{aligned}
&E\sup_{0\le t\le T}\left|x_t^\varepsilon - \overline{x}_t - \varepsilon X_t\right|^2 = o\left(\varepsilon^2\right),\\
&E\sup_{0\le t\le T}\left|y_t^\varepsilon - \overline{y}_t - \varepsilon Y_t\right|^2 + E\int_0^T\left|z_t^\varepsilon - \overline{z}_t - \varepsilon Z_t\right|^2 dt\\
&\quad + E\int_0^T\left\|r_t^\varepsilon - \overline{r}_t - \varepsilon R_t\right\|^2 dt = o\left(\varepsilon^2\right),
\end{aligned} \tag{36}$$

where (X_t, Y_t, Z_t, R_t) is the solution of variational equation (35).

Proof. Set

$$\begin{aligned}
\Delta x_t &= \varepsilon^{-1}\left(x_t^\varepsilon - \overline{x}_t\right),\\
\Delta y_t &= \varepsilon^{-1}\left(y_t^\varepsilon - \overline{y}_t\right),\\
\Delta z_t &= \varepsilon^{-1}\left(z_t^\varepsilon - \overline{z}_t\right),\\
\Delta r_t &= \varepsilon^{-1}\left(r_t^\varepsilon - \overline{r}_t\right),
\end{aligned} \tag{37}$$

and then

$$\begin{aligned}
d\Delta x_t &= \varepsilon^{-1}\left[b\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- b\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dt\\
&\quad + \varepsilon^{-1}\left[\sum_{i=1}^{d}\sigma^i\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- \sum_{i=1}^{d}\sigma^i\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dB_t^i\\
&\quad + \varepsilon^{-1}\left[\sum_{i=1}^{\infty}g^i\left(t, x_{t-}^\varepsilon, y_{t-}^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- \sum_{i=1}^{\infty}g^i\left(t, \overline{x}_{t-}, \overline{y}_{t-}, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dH_t^i,\\
-d\Delta y_t &= \varepsilon^{-1}\left[f\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- f\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dt - \sum_{i=1}^{d}\Delta z_t^i dB_t^i\\
&\quad - \sum_{i=1}^{\infty}\Delta r_t^i dH_t^i,\\
\Delta x_0 &= 0,\\
\Delta y_T &= \varepsilon^{-1}\left[\Phi\left(x_T^\varepsilon\right) - \Phi\left(\overline{x}_T\right)\right].
\end{aligned} \tag{38}$$

We can transform (38) into

$$\begin{aligned}
&d\Delta x_t \\
&\quad = \widehat{b}\left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dt \\
&\qquad + \sum_{i=1}^{d} \widehat{\sigma}^i \left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dB_t^i \\
&\qquad + \sum_{i=1}^{\infty} \widehat{g}^i \left(t, \Delta x_{t-}, \Delta y_{t-}, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dH_t^i, \\
&-d\Delta y_t \\
&\quad = \widehat{f}\left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dt - \sum_{i=1}^{d} \Delta z_t^i dB_t^i \\
&\qquad - \sum_{i=1}^{\infty} \Delta r_t^i dH_t^i, \\
&\Delta x_0 = 0, \\
&\Delta y_T = \varepsilon^{-1}\left[\Phi\left(x_T^\varepsilon\right) - \Phi\left(\overline{x}_T\right)\right],
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
&\widehat{l}\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, u_t - \overline{u}_t\right) \\
&\quad = A^l(t)\,\overline{x}_t + B^l(t)\,\overline{y}_t + C^l(t)\,\overline{z}_t + D^l(t)\,\overline{r}_t \\
&\qquad + E^l(t)\left(u_t - \overline{u}_t\right),
\end{aligned} \tag{40}$$

for $l = b, \sigma, g, f$, respectively, and

$$\begin{aligned}
A^l(t) &= \begin{cases} A(t), & x_t^\varepsilon - \overline{x}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
B^l(t) &= \begin{cases} B(t), & y_t^\varepsilon - \overline{y}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
C^l(t) &= \begin{cases} C(t), & z_t^\varepsilon - \overline{z}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
D^l(t) &= \begin{cases} D(t), & r_t^\varepsilon - \overline{r}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
E^l(t) &= \begin{cases} E(t), & u_t - \overline{u}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
A(t) &= \left(x_t^\varepsilon - \overline{x}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right], \\
B(t) &= \left(y_t^\varepsilon - \overline{y}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right], \\
C(t) &= \left(z_t^\varepsilon - \overline{z}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, \overline{y}_t, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, r_t^\varepsilon, u_t^\varepsilon\right)\right], \\
D(t) &= \left(r_t^\varepsilon - \overline{r}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, u_t^\varepsilon\right)\right], \\
E(t) &= \left[\varepsilon\left(u_t - \overline{u}_t\right)\right]^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right].
\end{aligned} \tag{42}$$

From the continuity result depending on parameters we have got in Section 3, we know that

$$\begin{aligned}
&E \sup_{0 \le t \le T} \left|x_t^\varepsilon - \overline{x}_t\right|^2 + E \sup_{0 \le t \le T} \left|y_t^\varepsilon - \overline{y}_t\right|^2 \\
&\qquad + E \int_0^t \left|z_t^\varepsilon - \overline{z}_t\right|^2 dt + E \int_0^t \left\|r_t^\varepsilon - \overline{r}_t\right\|^2 dt \le K\varepsilon^2;
\end{aligned} \tag{43}$$

that is, $(x_t^\varepsilon - \overline{x}_t, y_t^\varepsilon - \overline{y}_t, z_t^\varepsilon - \overline{z}_t, r_t^\varepsilon - \overline{r}_t) \to 0$ in $M^2(0, T)$, as $\varepsilon \to 0$. Together with Assumption 6, we can get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} A^l(t) = l_x\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} B^l(t) = l_y\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} C^l(t) = l_z\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} D^l(t) = l_r\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} E^l(t) = l_u\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} \widehat{l}\left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) \\
&\quad = l_x\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta x_t \\
&\qquad + l_y\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta y_t \\
&\qquad + l_z\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta z_t \\
&\qquad + l_r\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta r_t \\
&\qquad + l_u\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\left(u_t - \overline{u}_t\right).
\end{aligned} \tag{44}$$

As FBSDEL (35) has a unique solution (X_t, Y_t, Z_t, R_t) under Assumption 6, from the continuity and uniqueness result by Lemma 3, we know that $(\Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t)$ converges to (X_t, Y_t, Z_t, R_t) in $M^2([0, T])$ as $\varepsilon \to 0$.

The proof is complete. □

Notice that $\lim_{\varepsilon\to 0^+}(J(u_t^\varepsilon - J(\overline{u}_t))/\varepsilon) \geq 0$ and we can get the following variational inequality.

Lemma 8. *Assume that Assumption 6 holds; then*

$$\begin{aligned} &E\int_0^T L_x(t)X_t dt + E\int_0^T L_y(t)Y_t dt \\ &\quad + E\int_0^T L_z(t)Z_t dt + E\int_0^T L_r(t)R_t dt \\ &\quad + E\int_0^T L_u(t)(u_t - \overline{u}_t)dt + Eh_x(x_T)X_T \\ &\quad + E\gamma_y(y_0)Y_0 \geq 0. \end{aligned} \tag{45}$$

Proof. First we have

$$\begin{aligned} \frac{J(u_t^\varepsilon - J(\overline{u}_t))}{\varepsilon} &= \frac{1}{\varepsilon}\Big\{E\int_0^T [L(t,x_t^\varepsilon,y_t^\varepsilon,z_t^\varepsilon,r_t^\varepsilon,u_t^\varepsilon) \\ &\quad - L(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)]dt + E[h(x_T^\varepsilon) - h(\overline{x}_T)] \\ &\quad + E[\gamma(y_0^\varepsilon) - \gamma(\overline{y}_0)]\Big\}. \end{aligned} \tag{46}$$

Under Assumption 6, from Lemma 7, we can get

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E\int_0^T [L(t,x_t^\varepsilon,y_t^\varepsilon,z_t^\varepsilon,r_t^\varepsilon,u_t^\varepsilon) \\ &\quad - L(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)]dt \\ &\quad \longrightarrow E\int_0^T \big[L_x(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)X_t \\ &\quad + L_y(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)Y_t \\ &\quad + L_z(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)Z_t \\ &\quad + L_r(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)R_t \\ &\quad + L_u(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t)(u_t - \overline{u}_t)\big]dt, \\ &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E[h(x_T^\varepsilon) - h(\overline{x}_T)] \longrightarrow E[h_x(\overline{x}_T)X_T], \\ &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E[\gamma(y_0^\varepsilon) - \gamma(\overline{y}_0)] \longrightarrow E[\gamma_y(\overline{y}_0)Y_0]. \end{aligned} \tag{47}$$

Then (45) is proved.

The proof is complete. □

We define the Hamiltonian function H as follows:

$$\begin{aligned} &H(t,x,y,z,r,u,p,q,k,\rho) \\ &\quad = \langle p, -f(t,x,y,z,r,u)\rangle + \langle q, b(t,x,y,z,r,u)\rangle \\ &\quad + \langle k, \sigma(t,x,y,z,u)\rangle + \langle \rho, g(t,x,y,r,u)\rangle \\ &\quad + L(t,x,y,z,r,u), \end{aligned} \tag{48}$$

and the following adjoint forward-backward equation to variational equation (35):

$$\begin{aligned} &dp_t \\ &\quad = -H_y(t,x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t)dt \\ &\quad - \sum_{i=1}^d H_z^i(t,x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t)dB_t^i \\ &\quad - \sum_{i=1}^\infty H_r^i(t,x_{t-},y_{t-},z_t,r_t,u_t,p_t,q_t,k_t,\rho_t)dH_t^i, \\ &-dq_t \\ &\quad = H_x(t,x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t)dt - \sum_{i=1}^d k_t^i dB_t^i \\ &\quad - \sum_{i=1}^\infty \rho_t^i dH_t^i, \\ &p_0 = -\gamma_y(y_0), \\ &q_T = h_x(x_T) - \Phi_x(x_T)p_T. \end{aligned} \tag{49}$$

It is easy to verify that (49) satisfies Assumptions 1 and 2; then there exists a unique quarter (p_t, q_t, k_t, ρ_t) satisfying (49).

Then we have the main result of this paper which is the following theorem.

Theorem 9. *Supposing that Assumptions 1, 2, 4, and 6 hold, $(\overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)$ is an optimal pair for our optimal control problem and (p_t, q_t, k_t, ρ_t) is the solution to corresponding adjoint equation (49). Then for each admissible control $u_t \in U(0,T)$ we have*

$$\langle H_u(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t,p_t,q_t,k_t,\rho_t), (u_t - \overline{u}_t)\rangle \geq 0 \quad a.e., a.s., \tag{50}$$

where H is defined by (48).

Proof. Applying *Ito's* formula to $\langle X_t, q_t\rangle + \langle Y_t, p_t\rangle$, we can obtain

$$\begin{aligned} &E[h_x(\overline{x}_T)X_T] + E[\gamma_y(\overline{y}_0)Y_0] + E\int_0^T \big[L_x(t)X_t \\ &\quad + L_y(t)Y_t + L_z(t)Z_t + L_r(t)R_t + L_u(t) \\ &\quad \cdot (u_t - \overline{u}_t)\big]dt \\ &\quad = E\int_0^T \langle H_u(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t,p_t,q_t,k_t,\rho_t), \\ &\quad (u_t - \overline{u}_t)\rangle dt. \end{aligned} \tag{51}$$

The variational inequality implies for each $u_t \in U[0,T]$ that

$$E\int_0^T \langle H_u(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t,p_t,q_t,k_t,\rho_t),(u_t-\overline{u}_t)\rangle\,dt \geq 0. \tag{52}$$

The proof is completed. □

Next, under some additional convexity conditions, we prove that the maximum principle can be a sufficient condition for optimal control.

Theorem 10. *For stochastic control system (30) and the cost functional $J(u)$, if Assumptions 1, 2, 4, and 6 hold, and $y_T = Mx_T$, $M \in L^2(\Omega,\mathcal{F}_T,P;R^{m\times n})$, h is convex in x, and γ is convex in y. Let u_t be an admissible control and let (x_t,y_t,z_t,r_t) be the corresponding trajectory. Let (p_t,q_t,k_t,ρ_t) be the solution of corresponding adjoint equation (49). Suppose that the Hamiltonian function H is convex in (x,y,z,r,u) and inequality (50) holds; then u_t is an optimal control.*

Proof. Let v_t be an arbitrary admissible control and the corresponding trajectory is $(x_t^v,y_t^v,z_t^v,r_t^v)$; then

$$\begin{aligned} &J(v(\cdot)) - J(u(\cdot)) \\ &\quad = E\int_0^T [L(x_t^v,y_t^v,z_t^v,r_t^v,v_t) - L(x_t,y_t,z_t,r_t,u_t)]\,dt \\ &\qquad + E[h(x_T^v) - h(x_T)] + E[\gamma(y_0^v) - \gamma(y_0)] \\ &\quad = I_1 + I_2, \end{aligned} \tag{53}$$

where

$$\begin{aligned} &I_1 \\ &\quad = E\int_0^T [L(x_t^v,y_t^v,z_t^v,r_t^v,v_t) - L(x_t,y_t,z_t,r_t,u_t)]\,dt, \\ &I_2 = E[h(x_T^v) - h(x_T)] + E[\gamma(y_0^v) - \gamma(y_0)]. \end{aligned} \tag{54}$$

From the definition of Hamiltonian function H, we get

$$\begin{aligned} I_1 &= E\int_0^T [H(x_t^v,y_t^v,z_t^v,r_t^v,v_t,p_t^v,q_t^v,k_t^v,\rho_t^v) \\ &\quad - H(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t)]\,dt \\ &\quad + E\int_0^T \langle f(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - f(x_t,y_t,z_t,r_t,u_t),p_t\rangle\,dt \\ &\quad - E\int_0^T \langle b(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - b(x_t,y_t,z_t,r_t,u_t),q_t\rangle\,dt - \sum_{i=1}^d E \\ &\quad \cdot \int_0^T \langle \sigma^i(x_t^v,y_t^v,z_t^v,v_t) - \sigma^i(x_t,y_t,z_t,u_t),k_t^i\rangle\,dt \\ &\quad - \sum_{i=1}^\infty E\int_0^T \langle g^i(x_t^v,y_t^v,r_t^v,v_t) \\ &\quad - g^i(x_t,y_t,r_t,u_t),\rho_t^i\rangle\,dt. \end{aligned} \tag{55}$$

By convexity of h,γ and using *Itô's* formula to $\langle q_t, x_t^v - x_t\rangle + \langle p_t, y_t^v - y_t\rangle$ we can get

$$\begin{aligned} I_2 &\geq E\langle h_x(x_T), x_T^v - x_T\rangle + E\langle \gamma_y(y_0), y_0^v - y_0\rangle \\ &= -E\int_0^T \langle H_x(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), x_t^v \\ &\quad - x_t\rangle\,dt \\ &\quad - E\int_0^T \langle H_y(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), y_t^v \\ &\quad - y_t\rangle\,dt - \sum_{i=1}^d E \\ &\quad \cdot \int_0^T \langle H_z^i(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), z_t^{iv} - z_t^i\rangle\,dt \\ &\quad - \sum_{i=1}^\infty E\int_0^T \langle H_r^i(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), r_t^{iv} \\ &\quad - r_t^i\rangle\,dt - E\int_0^T \langle f(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - f(x_t,y_t,z_t,r_t,u_t),p_t\rangle\,dt \\ &\quad + E\int_0^T \langle b(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - b(x_t,y_t,z_t,r_t,u_t),q_t\rangle\,dt + \sum_{i=1}^d E \\ &\quad \cdot \int_0^T \langle \sigma^i(x_t^v,y_t^v,z_t^v,v_t) - \sigma^i(x_t,y_t,z_t,u_t),k_t^i\rangle\,dt \\ &\quad + \sum_{i=1}^\infty E\int_0^T \langle g^i(x_t^v,y_t^v,r_t^v,v_t) \\ &\quad - g^i(x_t,y_t,r_t,u_t),\rho_t^i\rangle\,dt, \end{aligned} \tag{56}$$

and from (53) to (56) we have

$$\begin{aligned}
&J(v(\cdot)) - J(u(\cdot)) \\
&\quad = E\int_0^T \left[H\left(x_t^v, y_t^v, z_t^v, r_t^v, v_t, p_t^v, q_t^v, k_t^v, \rho_t^v\right)\right. \\
&\quad \left. - H\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right)\right] dt \\
&\quad - E\int_0^T \left\langle H_x\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), x_t^v - x_t\right\rangle dt \\
&\quad - E\int_0^T \left\langle H_y\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), y_t^v\right. \\
&\quad \left. - y_t\right\rangle dt - \sum_{i=1}^{d} E \\
&\quad \cdot \int_0^T \left\langle H_z^i\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), z_t^{iv} - z_t^i\right\rangle dt \\
&\quad - \sum_{i=1}^{\infty} E\int_0^T \left\langle H_r^i\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), r_t^{iv}\right. \\
&\quad \left. - r_t^i\right\rangle dt.
\end{aligned} \tag{57}$$

Moreover, as the Hamiltonian function H is convex in (x, y, z, r, u), the following inequality holds:

$$\begin{aligned}
&H\left(x_t^v, y_t^v, z_t^v, r_t^v, v_t, p_t^v, q_t^v, k_t^v, \rho_t^v\right) \\
&\quad - H\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right) \\
&\quad \geq \left\langle H_x\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), x_t^v - x_t\right\rangle \\
&\qquad + \left\langle H_y\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), y_t^v - y_t\right\rangle \\
&\qquad + \sum_{i=1}^{d} \left\langle H_z^i\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), z_t^{iv} - z_t^i\right\rangle \\
&\qquad + \sum_{i=1}^{\infty} \left\langle H_r^i\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), r_t^{iv} - r_t^i\right\rangle \\
&\qquad + \left\langle H_u\left(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t\right), v_t - u_t\right\rangle,
\end{aligned} \tag{58}$$

and from (57) to (58) and together with (50) for arbitrary admissible control v_t we have

$$J(v(\cdot)) - J(u(\cdot)) \geq 0, \tag{59}$$

then admissible control u_t is an optimal control.

The proof is completed. □

5. Applications in Linear Quadratic Problem

In this section, we will apply our stochastic maximum principle to the linear quadratic problem which the linear control system described by fully coupled forward-backward stochastic differential equation driven by Lévy processes:

$$\begin{aligned}
dx_t &= \left(A_t x_t + B_t y_t + \sum_{i=1}^{d} C_t^i z_t^i + \sum_{i=1}^{\infty} D_t^i r_t^i + E_t u_t\right) dt \\
&\quad + \sum_{i=1}^{d} \left(F_t^i x_t + G_t^i y_t + V_t^i z_t^i + W_t^i u_t\right) dB_t^i \\
&\quad + \sum_{i=1}^{\infty} \left(J_t^i x_{t_-} + K_t^i y_{t_-} + L_t^i r_t^i + M_t^i u_t\right) dH_t^i, \\
-dy_t &= \left(N_t x_t + P_t y_t + \sum_{i=1}^{d} Q_t^i z_t^i + \sum_{i=1}^{\infty} R_t^i r_t^i + S_t u_t\right) dt \\
&\quad - \sum_{i=1}^{d} z_t^i dB_t - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\
x_0 &= a, \\
y_T &= h x_T + \xi
\end{aligned} \tag{60}$$

and the cost functional:

$$\begin{aligned}
&J(u) \\
&\quad = E\left\langle \widetilde{Q} x_T, x_T\right\rangle + E\left\langle \widetilde{P} y_0, y_0\right\rangle \\
&\qquad + E\int_0^T \left(\left\langle \widetilde{R}_t x_t, x_t\right\rangle + \left\langle \widetilde{N}_t u_t, u_t\right\rangle + \left\langle \widetilde{L}_t y_t, y_t\right\rangle\right) dt \\
&\qquad + \sum_{i=1}^{d} E\int_0^T \left\langle \widetilde{M}_t^i z_t^i, z_t^i\right\rangle dt + \sum_{i=1}^{\infty} E\int_0^T \left\langle \widetilde{K}_t^i r_t^i, r_t^i\right\rangle dt,
\end{aligned} \tag{61}$$

where the $\mathcal{F}_t$-predictable matrix processes

$$\begin{aligned}
A, F^i, J^j, \widetilde{R} &: [0,T]\times\Omega \longrightarrow R^{n\times n}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
B, C^i, D^j, G^i, V^i, K^j, L^j &: [0,T]\times\Omega \longrightarrow R^{n\times m}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
E, W^i, M^j &: [0,T]\times\Omega \longrightarrow R^{n\times k}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
P, Q^i, R^j, \widetilde{L}, \widetilde{M}^i, \widetilde{K}^j &: [0,T]\times\Omega \longrightarrow R^{m\times m}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
N &: [0,T]\times\Omega \longrightarrow R^{m\times n}, \\
S &: [0,T]\times\Omega \longrightarrow R^{m\times k}, \\
\widetilde{N} &: [0,T]\times\Omega \longrightarrow R^{k\times k};
\end{aligned} \tag{62}$$

the $\mathcal{F}_T$-predictable random matrix $\widetilde{Q} : \Omega \to R^{n\times n}$ and the $\mathcal{F}_0$-predictable stochastic matrix $\widetilde{P} : \Omega \to R^{m\times m}$ are all

uniformly bounded. And $a \in \mathbb{R}^{n\times n}$, $h \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^{m\times n})$, and $\xi \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^m)$.

To study this problem, we need the assumptions on the coefficients as follows.

Assumption 11. The state weighting matrix processes $\widetilde{R}, \widetilde{L}, \widetilde{M}^i$, and $\widetilde{K}^j, i = 1, 2, \ldots, d,\ j = 1, 2, \ldots$, and the control weighting matrix process $\widetilde{N}$ and random matrixes $\widetilde{Q}$ and $\widetilde{P}$ are almost everywhere almost surely symmetric and nonnegative. Furthermore, $\widetilde{N}$ is almost everywhere almost surely uniformly positive; that is, $N \geq \delta I$, for some positive constant δ almost everywhere almost surely.

Assumption 12. For the control processes there is no further constraint:

$$\begin{aligned} U_{ad} = \Big\{ & u(\cdot) : u(\cdot) \\ & \text{is } \mathscr{F}_t\text{-predictable with values in } \mathbb{R}^k, \\ & E\int_0^T |u(t)|\, dt < \infty \Big\}. \end{aligned} \tag{63}$$

If we denote the norm of U_{ad} by

$$\|u(\cdot)\|_{U_{ad}} = E\sqrt{\int_0^T |u(t)|^2\, dt}; \tag{64}$$

then U_{ad} is a Hilbert space. And by Lemma 3 we also know that the linear FBSDEL (60) has a unique solution; that is, the linear quadratic problem is well defined.

Theorem 13. *Under Assumptions 11 and 12, LQ problems (60) and (61) have a unique optimal control, and the optimal control is*

$$\begin{aligned} u_t = -\frac{1}{2}\widetilde{N}_t^{-1} \Big[& E_t^T q_t - S_t^T p_t + \sum_{i=1}^{d} \left(W_t^i\right)^T k_t^i \\ & + \sum_{i=1}^{\infty} \left(M_t^i\right)^T \rho_t^i \Big]; \end{aligned} \tag{65}$$

here p, q, k, and ρ are the solution of the following adjoint FBSDEL:

$$\begin{aligned} dp_t = \Big[& P_t p_t - B_t^T q_t - \sum_{i=1}^{d} \left(G_t^i\right)^T k_t^i - \sum_{i=1}^{\infty} \left(K_t^i\right)^T \rho_t^i \\ & - 2\widetilde{L}_t y_t \Big] dt \\ & - \sum_{i=1}^{d} \Big[Q_t^i p_t - \left(C_t^i\right)^T q_t - \left(V_t^i\right)^T k_t^i - 2\left(\widetilde{M}_t^i\right) z_t^i \Big] dB_t^i \\ & - \sum_{i=1}^{\infty} \Big[R_t^i p_t^i - \left(D_t^i\right)^T q_t - \left(L_t^i\right)^T \rho_t - 2\widetilde{K}_t^i r_t^i \Big] dH_t^i, \\ -dq_t = \Big[& N_t^T p_t - A_t q_t - \sum_{i=1}^{d} F_t^i k_t^i - \sum_{i=1}^{\infty} J_t^i \rho_t^i - 2\widetilde{R}_t x_t \Big] dt \\ & - \sum_{i=1}^{d} k_t^i dB_t - \sum_{i=1}^{\infty} \rho_t^i dH_t^i, \\ p_0 = & -2\widetilde{P} y_0, \\ q_T = & 2\widetilde{Q} x_T - h p_T. \end{aligned} \tag{66}$$

Proof. From Assumptions 11 and 12 and inequality (29), we can verify that the cost function $J(u(\cdot))$ (61) is strictly convex and continuous over U_{ad} and

$$\lim_{\|u(\cdot)\|_{U_{ad}} \to \infty} J(u(\cdot)) = \infty; \tag{67}$$

then, from Lemma 5.3 in [18], the cost function has a unique minimal value and, together with Lemma 3 (existence and uniqueness theorem of FBSDEL), the LQ problem has a unique optimal control. Next, we will prove that the optimal control u_t has an expression as (65).

Let (x_t, y_t, z_t, r_t) be the optimal state process corresponding to the optimal control u_t and let (p_t, q_t, k_t, ρ_t) be the unique solution to adjoint equation (66) corresponding to the optimal pair $(u_t; x_t, y_t, z_t, r_t)$; then the Hamiltonian function

$$\begin{aligned} H(t, x, y, z, r, u, p, q, k, \rho) = & -\Big\langle p, \\ & \Big(N_t x_t + P_t y_t + \sum_{i=1}^{d} Q_t^i z_t^i + \sum_{i=1}^{\infty} R_t^i r_t^i + S_t u_t \Big) \Big\rangle + \Big\langle q, \\ & \Big(A_t x_t + B_t y_t + \sum_{i=1}^{d} C_t^i z_t^i + \sum_{i=1}^{\infty} D_t^i r_t^i + E_t u_t \Big) \Big\rangle + \Big\langle k, \\ & \sum_{i=1}^{d} \left(F_t^i x_t + G_t^i y_t + V_t^i z_t^i + W_t^i u_t \right) \Big\rangle + \Big\langle \rho, \\ & \sum_{i=1}^{\infty} \left(J_t^i x_{t_-} + K_t^i y_{t_-} + L_t^i r_t^i + M_t^i u_t \right) \Big\rangle + \left\langle \widetilde{R}_t x_t, x_t \right\rangle \\ & + \left\langle \widetilde{N}_t u_t, u_t \right\rangle + \left\langle \widetilde{L}_t y_t, y_t \right\rangle + \sum_{i=1}^{d} \left\langle \widetilde{M}_t^i z_t^i, z_t^i \right\rangle \\ & + \sum_{i=1}^{\infty} \left\langle \widetilde{K}_t^i r_t^i, r_t^i \right\rangle. \end{aligned} \tag{68}$$

From Theorem 9 and Assumption 12 we have

$$H_u = -S_t^T p_t + E_t^T q_t + \sum_{i=1}^{d} \left(W_t^i\right)^T k_t^i + \sum_{i=1}^{\infty} \left(M_t^i\right)^T \rho_t^i + 2N_t u_t = 0; \tag{69}$$

that is

$$u_t = -\frac{1}{2}\widetilde{N}_t^{-1}\left[E_t^T q_t - S_t^T p_t + \sum_{i=1}^{d}\left(W_t^i\right)^T k_t^i + \sum_{i=1}^{\infty}\left(M_t^i\right)^T \rho_t^i\right]; \tag{70}$$

then (65) holds.

The proof is completed. □

6. Conclusion

In this paper, the continuity result depending on parameters of forward-backward stochastic differential equation driven by Lévy process is proved. Based on this result, we get the stochastic maximum principle for fully coupled forward-backward stochastic control system driven by Lévy process. And then we use the stochastic maximum principle to solve LQ problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11271007), the SDUET Research Fund (no. 2012kytd105), and the Doctoral Fund of the Ministry of Education of China (no. 20123718110010).

References

[1] S. Peng and Z. Wu, "Fully coupled forward-backward stochastic differential equations and applications to optimal control," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 825–843, 1999.

[2] S. Peng, "Backward stochastic differential equations and applications to optimal control," *Applied Mathematics and Optimization*, vol. 27, no. 2, pp. 125–144, 1993.

[3] Z. Wu, "Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems," *Systems Science and Mathematical Sciences*, vol. 11, no. 3, pp. 249–259, 1998.

[4] J.-T. Shi and Z. Wu, "The maximum principle for fully coupled forward-backward stochastic control system," *Acta Automatica Sinica*, vol. 32, no. 2, pp. 161–169, 2006.

[5] Z. Wu, "Forward-backward stochastic differential equations with Brownian motion and Poisson process," *Acta Mathematicae Applicatae Sinica*, vol. 15, no. 4, pp. 433–443, 1999.

[6] J. Shi and Z. Wu, "Maximum principle for fully coupled forward-backward stochastic control system with random jumps," in *Proceedings of the 26th Chinese Control Conference (CCC '07)*, pp. 375–380, Hunan, China, July 2007.

[7] J. Shi and Z. Wu, "Maximum principle for forward-backward stochastic control system with random jumps and applications to finance," *Journal of Systems Science and Complexity*, vol. 23, no. 2, pp. 219–231, 2010.

[8] J. Shi and Z. Wu, "Necessary condition for optimal control of fully coupled forward-backward stochastic system with random jumps," in *Proceedings of the 31st Chinese Control Conference (CCC '12)*, pp. 1620–1628, Hefei, China, July 2012.

[9] J. Shi, "Necessary conditions for optimal control of forward-backward stochastic systems with random jumps," *International Journal of Stochastic Analysis*, vol. 2012, Article ID 258674, 50 pages, 2012.

[10] F. Baghery, N. Khelfallah, B. Mezerdi, and I. Turpin, "Fully coupled forward backward stochastic dfferential wquations driven by Lévy processes and application to dfferential games," *Random Operators and Stochastic Equations*, vol. 22, no. 3, pp. 151–161, 2014.

[11] Q. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 126–142, 2014.

[12] D. Nualart and D. Schoutens, "Chaotic and predicatable resp-resentation for Lévy processes with applications in finance," *Journal of Systems Science and Complexity*, vol. 90, pp. 109–122, 2000.

[13] D. Nualart and D. Schoutens, "Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance," *Bernoulli*, vol. 7, pp. 761–776, 2001.

[14] K. Bahlali, M. Eddahbi, and E. Essaky, "BSDE associated with Lévy processes and application to PDIE," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 16, no. 1, pp. 1–17, 2003.

[15] K.-I. Mitsui and Y. Tabata, "A stochastic linear-quadratic problem with Lévy processes and its application to finance," *Stochastic Processes and Their Applications*, vol. 118, no. 1, pp. 120–152, 2008.

[16] H. Tang and Z. Wu, "Stochastic dfferential equations and stochastic linear quadratic optimal control problem with Lévy processes," *Journal of Systems Science & Complexity*, vol. 22, no. 1, pp. 122–136, 2009.

[17] Q. Meng and M. Tang, "Necessary and sufficient conditions for optimal control of stochastic systems associated with Lévy processes," *Science in China: Series F: Information Sciences*, vol. 52, no. 11, pp. 1982–1992, 2009.

[18] M. Tang and Q. Zhang, "Optimal variational principle for backward stochastic control systems associated with Lévy processes," *Science China Mathematics*, vol. 55, no. 4, pp. 745–761, 2012.

[19] J. Z. Lin, "A general stochastic maximum principle for optimal control of stochastic systems driven by multidimensional Teugel's martingales," http://arxiv.org/abs/1205.6315v1.

[20] F. Zhang, M. N. Tang, and Q. X. Meng, "Sochastic maximum principle for forward-backward stochastic control systems associated with Lévy processes," *Chinese Annals of Mathematics*, vol. 35, no. 1, pp. 83–100, 2014 (Chinese).

[21] P. Protter, *Stochastic Integration and Differential Equations*, vol. 21 of *Applications of Mathematics*, Springer, Berlin, Germany, 1990.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 837053, 10 pages
http://dx.doi.org/10.1155/2015/837053

Research Article

Study on $\mathscr{H}_-$ Index of Stochastic Linear Continuous-Time Systems

Yan Li,[1] Tianliang Zhang,[2] Xikui Liu,[3] and Xiushan Jiang[2]

[1] *College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao, Shandong 266590, China*
[2] *College of Information and Control Engineering, China University of Petroleum (East China), Qingdao, Shandong 266590, China*
[3] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China*

Correspondence should be addressed to Xikui Liu; liuxikuiqd@163.com

Received 3 February 2015; Accepted 30 March 2015

Academic Editor: Ruihua Liu

This paper studies the $\mathscr{H}_-$ index problem. We obtain a necessary and sufficient condition of $\mathscr{H}_-$ index larger than $\gamma > 0$. A generalized differential equation is introduced and it is proved that its solvability and the feasibility of the $\mathscr{H}_-$ index are equivalent. We extend the deterministic cases to the stochastic models. Our results can be used to fault detection filter analysis. Finally, the effectiveness of the proposed results is illustrated by an example.

1. Introduction

It is well known that many control and filtering problems have been discussed based on a certain performance index of a system, such as $\mathscr{H}_2$ norm, $\mathscr{H}_\infty$ norm, and $\mathscr{H}_-$ index; see [1–9]. $\mathscr{H}_\infty$ norm is the measure of the worst-case disturbance inputs on the controlled outputs [1–4]. The $\mathscr{H}_-$ index is a measure of the minimum sensitivity of system outputs to system inputs. $\mathscr{H}_\infty$ norm and $\mathscr{H}_-$ index with specific application to fault detection filter have been carried out in [10–17]. To ensure robustness, $\mathscr{H}_-$ index should be maximized and $\mathscr{H}_\infty$ norm should be minimized. Using $\mathscr{H}_-/\mathscr{H}_\infty$ performance can make certain that the residual signal is maximally sensitive to faults and highly robust to disturbance inputs; see [16, 17].

In [12], $\mathscr{H}_-$ index was defined as the minimum nonzero singular value in zero frequency. In [10], the authors extended the results of [12] to all frequency range. By means of LMIs, a necessary and sufficient condition was given for the infinite frequency range. The case for finite frequency range was concluded through frequency weighting. In recent decades, a great deal of attention has been attracted to $\mathscr{H}_-$ index in time domain. A fault residual generator was designed to maximize the fault sensitivity in the finite time domain [16–20]. Based on $\mathscr{H}_-$ index, results on optimal fault detection can be found in [17, 18] and the references. The lower bound of $\mathscr{H}_-$ index for linear time-varying systems was proposed in [19, 20]. A sliding mode observer was designed for sensor fault diagnosis of nonlinear time-delay systems; see [21]. In [22], a fault-tolerant controller was projected to compensate nonlinear faults by using a fuzzy adaptive fault observer.

Although there is much work on the $\mathscr{H}_-$ index problem, to the best of our knowledge, very little work was concerned with the $\mathscr{H}_-$ index in stochastic systems. In this paper, the $\mathscr{H}_-$ index for stochastic linear continuous-time systems is discussed. The definition of the $\mathscr{H}_-$ index is extended to the stochastic case. We present a necessary and sufficient condition of the $\mathscr{H}_-$ index. A generalized differential equation is introduced and it is proved that its solvability and the feasibility of the $\mathscr{H}_-$ index are equivalent. Comparing our results with the bounded real lemma [2, 9], it shows that the $\mathscr{H}_-$ index is not completely dual to $\mathscr{H}_\infty$ norm. The $\mathscr{H}_-$ index discussed in this paper is only for *tall* or *square* systems. The reason for this is that $\mathscr{H}_-$ index is zero for *wide* systems. But bounded real lemma for $\mathscr{H}_\infty$ is applicable to any systems. Finally, the effectiveness of the given methods is illustrated by numerical example.

The outline of the paper is organized as follows. In Section 2, some efficient criteria are given for the $\mathscr{H}_-$ index of stochastic linear systems in finite horizon. Section 3 contains an example provided to show the efficiency of the proposed results. Finally, we conclude this paper in Section 4.

Notations. R is the field of real numbers. $R^{m\times n}$ is the vector space of all $m \times n$ matrices with entries in R. $\mathcal{S}_n(R)$ is the set of all real symmetric matrices $R^{n\times n}$. A' is the transpose of matrix A. A^{-1} is the inverse of A. Given positive semidefinite (positive definite) matrix A, we denote it by $A \geq 0$ ($A > 0$). E is the mathematical expectation. I is identity matrix. 0_n is $n\times n$ zero matrix. $\mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$ is the space of nonanticipative stochastic process $y(t) \in R^p$ with respect to increasing σ-algebras $\mathscr{F}_t$ ($t \geq 0$) satisfying $\|y(t)\|^2_{[0,T]} < \infty$, where $\|y(t)\|^2_{[0,T]} = E\int_0^T y(t)'y(t)dt = E\int_0^T \|y(t)\|^2 dt$. A *square* (*wide* or *tall*) system denotes a system when the number of inputs equals (is more than or less than) the outputs number.

2. Finite Horizon Stochastic $\mathscr{H}_-$ Index

In this section, we will discuss the $\mathscr{H}_-$ index problem of stochastic linear continuous-time systems. We give a necessary and sufficient condition of the $\mathscr{H}_-$ index larger than $\gamma > 0$ in finite horizon.

Consider the following stochastic linear time-varying system $\mathscr{G}$:

$$\begin{aligned} dx(t) &= [A(t)x(t) + B(t)v(t)]\,dt \\ &\quad + [A_0(t)x(t) + B_0(t)v(t)]\,d\omega(t), \\ z(t) &= C(t)x(t) + D(t)v(t), \quad x(0) = x_0. \end{aligned} \tag{1}$$

In the above, $\omega(t)$ is the one-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$, with the natural filter $\mathscr{F}_t$ generated by $\omega(t)$ up to time t. Consider $x(t) \in R^n$, $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, and $z(t) \in R^l$ are the system state, control input, and regulated output, respectively. $A(t)$, $B(t)$, $A_0(t)$, $B_0(t)$, $C(t)$, and $D(t)$ are coefficients with appropriate dimensions. For any $0 < T < \infty$ and $(v(t), x_0) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p) \times R^n$, there exists unique solution $x(t) = x(t; v, x_0) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^n)$ of (1).

The finite horizon stochastic $\mathscr{H}_-$ index of system (1) can be stated as follows.

Definition 1. For stochastic system (1), given $0 < T < \infty$, its $\mathscr{H}_-$ index in $[0,T]$ is defined as

$$\begin{aligned} \|\mathscr{G}\|_-^{[0,T]} &= \inf_{v\neq 0, x_0=0} \frac{\|z(t)\|_{[0,T]}}{\|v(t)\|_{[0,T]}} \\ &= \inf_{v\neq 0, x_0=0} \frac{\left\{E\int_0^T z(t)'z(t)\,dt\right\}^{1/2}}{\left\{E\int_0^T v(t)'v(t)\,dt\right\}^{1/2}}, \end{aligned} \tag{2}$$

where $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$.

Remark 2. If v is fault signal and z is the residual, then the $\mathscr{H}_-$ index describes the smallest fault sensitivity of system (1). In this paper, we suppose that system (1) is *tall* or *square* because the $\mathscr{H}_-$ index is zero for *wide* system.

Given $\gamma > 0$ and $0 < T < \infty$, let

$$\begin{aligned} J_T^\gamma(x_0, v) &= \|z(t)\|^2_{[0,T]} - \gamma^2 \|v(t)\|^2_{[0,T]} \\ &= E\int_0^T \left[z(t)'z(t) - \gamma^2 v(t)'v(t)\right] dt. \end{aligned} \tag{3}$$

We will study the following optimal problem:

$$\min_{v\in\mathscr{L}^2_{\mathscr{F}}([0,T],R^p)} J_T^\gamma(x_0, v). \tag{4}$$

Remark 3. It can be shown that $\|\mathscr{G}\|_-^{[0,T]} > \gamma$ is equivalent to the following inequality

$$\begin{aligned} J_T^\gamma(0, v) &= \|z(t)\|^2_{[0,T]} - \gamma^2 \|v(t)\|^2_{[0,T]} \\ &= E\int_0^T \left[z(t)'z(t) - \gamma^2 v(t)'v(t)\right] dt > 0, \end{aligned} \tag{5}$$

$\forall v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, $v(t) \neq 0$.

Remark 4. When $T = \infty$, (2) corresponds to the infinite horizon case.

Lemma 5. *Suppose $P(t) : [0,T] \mapsto \mathcal{S}^n(R)$ is continuously differentiable, $T > 0$. Then, for every $x_0 \in R^n$, $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$,*

$$\begin{aligned} J_T^\gamma(x_0, v) &= x_0'P(0)x_0 - E\left[x(T)'P(T)x(T)\right] \\ &\quad + E\int_0^T \left[\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{M}(t, P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right] dt, \end{aligned} \tag{6}$$

where $\mathbb{M}(t, P(t)) = \begin{bmatrix} L(P(t)) + \dot{P}(t) & K(P(t)) \\ K(P(t))' & H^\gamma(P(t)) \end{bmatrix} \in \mathcal{S}^{n+l}(R)$,

$$\begin{aligned} L(P(t)) &= P(t)A(t) + A(t)'P(t) \\ &\quad + A_0(t)'P(t)A_0(t) + C(t)'C(t), \end{aligned}$$

$$K(P(t)) = P(t)B(t) + A_0(t)'P(t)B_0(t) + C(t)'D(t),$$
$$H^\gamma(P(t)) = B_0(t)'P(t)B_0(t) + D(t)'D(t) - \gamma^2 I. \tag{7}$$

Proof. Let $x_0 \in R^n$, $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, and $x(t) = x(t;v,x_0)$ denote the corresponding solution of (1). Applying Ito's formula to $x(t)'P(t)x(t)$ and taking expectations, we have that, for any $T > 0$,

$$\begin{aligned} &E\left[x(T)'P(T)x(T)\right] - \left[x(0)'P(0)x(0)\right] \\ &= E\int_0^T d\left[x(t)'P(t)x(t)\right] \\ &= E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{Q}(t,P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt, \end{aligned} \tag{8}$$

where

$$\mathbb{Q}(t,P(t)) = \begin{bmatrix} P(t)A(t) + A(t)'P(t) + A_0(t)'P(t)A_0(t) + \dot{P}(t) & P(t)B(t) + A_0(t)'P(t)B_0(t) \\ B(t)'P(t) + B_0(t)'P(t)A_0(t) & B_0(t)'P(t)B_0(t) \end{bmatrix}. \tag{9}$$

So

$$\begin{aligned} J_T^\gamma(x_0,v) &= E\int_0^T \left[\|Cx(t) + Dv(t)\|^2 - \gamma^2\|v(t)\|^2\right] dt \\ &\quad + x(0)'P(0)x(0) \\ &\quad - E\left[x(T)'P(T)x(T)\right] \\ &\quad + E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{Q}(t,P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt \\ &= x(0)'P(0)x(0) - E\left[x(T)'P(T)x(T)\right] \\ &\quad + E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{M}(t,P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt, \end{aligned} \tag{10}$$

which ends the proof. □

Below, we prove the following theorem which is necessary in this paper.

Theorem 6. *For (1) and some given $\gamma > 0$, if the following differential Riccati equation*

$$\begin{aligned} &L(P(t)) + \dot{P}(t) = K(P(t))H^\gamma(P(t))^{-1}K(P(t))', \\ &H^\gamma(P(t)) > 0, \\ &P(T) = 0 \end{aligned} \tag{11}$$

admits solution $P_T(t)$ on $[0,T]$, then $\|\mathscr{G}\|_-^{[0,T]} > \gamma$.

Proof. By Lemma 5, for every $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, $v \neq 0$, $x_0 = 0$, we conclude that

$$J_T^\gamma(0,v) = E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{M}(t,P_T(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt. \tag{12}$$

By using completion of squares argument and the first equality in (11), we have

$$\begin{aligned} J_T^\gamma(0,v) &= E\int_0^T x(t)'\left[L(P_T(t)) + \dot{P}_T(t) - K(P_T(t))\right. \\ &\quad \left.\cdot H^\gamma(P_T(t))^{-1}K(P_T(t))'\right]x(t)\,dt \\ &\quad + E\int_0^T \left\{\left[v(t) + H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t)\right]'\right. \\ &\quad \cdot H^\gamma(P_T(t)) \\ &\quad \left.\cdot\left[v(t) + H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t)\right]\right\}dt \\ &= E\int_0^T \left\{\left[v(t) - v^*(t)\right]'H^\gamma(P_T(t))\right. \\ &\quad \left.\cdot\left[v(t) - v^*(t)\right]\right\}dt, \end{aligned} \tag{13}$$

where $v^*(t) = -H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t)$.

From $H^\gamma(P_T(t)) > 0$, $J_T^\gamma(0,v) \geq 0$, to show $J_T^\gamma(0,v) > 0$, we define the operator $\mathscr{L}$: $\mathscr{L}\mathscr{V}(t) = v(t) - v^*(t)$ with its realization:

$$\begin{aligned} dx(t) &= (A(t)x(t) + B(t)v(t))\,dt \\ &\quad + [A_0(t)x(t) + B_0(t)v(t)]\,d\omega(t), \\ x(0) &= 0, \end{aligned} \tag{14}$$

$$v(t) - v^*(t) = v(t) + H^\gamma(P_T(t))^{-1} K(P_T(t))' x(t).$$

Then $\mathscr{L}^{-1}$ exists, which is determined by

$$\begin{aligned} dx(t) &= \Big[A(t) - B(t)H^\gamma(P_T(t))^{-1}\big(B(t)'P_T(t) \\ &\quad + B_0(t)'P_T(t)A_0(t) + D(t)'C(t)\big)\Big]x(t)\,dt \\ &\quad + \Big[A_0(t) - B_0(t)H^\gamma(P_T(t))^{-1}\big(B(t)'P_T(t) \\ &\quad + B_0'(t)P_T(t)A_0(t) + D(t)'C(t)\big)\Big]x(t)\,d\omega(t) \\ &\quad + B(t)(v(t) - v^*(t))\,dt + B_0(t)(v(t) - v^*(t))\,d\omega(t), \\ x_0 &= 0, \end{aligned} \tag{15}$$

where $v(t) = -H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t) + (v(t) - v^*(t))$.

We assume that $H^\gamma(P_T(t)) \geq \epsilon I$, $\epsilon > 0$, so there exists constant $C_0 > 0$, such that

$$\begin{aligned} &J_T^\gamma(0,v) \\ &\quad = E\int_0^T [v(t) - v^*(t)]' H^\gamma(P_T(t))[v(t) - v^*(t)]\,dt \\ &\quad \geq \epsilon \|v(t) - v^*(t)\|_{[0,T]}^2 = \epsilon \|\mathscr{L}\mathscr{V}(t)\|_{[0,T]}^2 \\ &\quad \geq C_0 \|v(t)\|_{[0,T]}^2 > 0. \end{aligned} \tag{16}$$

That is, $\|\mathscr{G}\|_-^{[0,T]} > \gamma$.

Now, we consider the following equation:

$$\begin{aligned} &\dot{X}(t) + L(X(t)) + K(X(t))F(t) + F(t)'K(X(t))' \\ &\quad + F(t)'H^\gamma(X(t))F(t) = 0, \quad t \in [0,T], \\ &X(T) = 0, \end{aligned} \tag{17}$$

where $F(t) \in C[0,T]$ and this equation has unique solution $X(t) = P_F^\gamma(t)$, $t \in [0,T]$.

It is easy to see that (17) satisfies the following equation:

$$\begin{aligned} &\dot{P}_F^\gamma(t) + \begin{bmatrix} I \\ F(t) \end{bmatrix}' \begin{bmatrix} L(P_F^\gamma(t)) & K(P_F^\gamma(t)) \\ K(P_F^\gamma(t))' & H^\gamma(P_F^\gamma(t)) \end{bmatrix} \begin{bmatrix} I \\ F(t) \end{bmatrix} \\ &\quad = 0, \quad t \in [0,T], \\ &P_F^\gamma(T) = 0. \end{aligned} \tag{18}$$

□

Lemma 7. *Suppose $F(t) \in C[0,T]$ and $P_F^\gamma(t)$ is the solution of (18). Then if $v(t) \in \mathscr{L}_{\mathscr{F}}^2([0,T],R^p)$, one obtains*

$$\begin{aligned} J_T^\gamma(x_0, v + Fx_F) &= x_0'P_F^\gamma(0)x_0 \\ &\quad + E\int_0^T \Big[v(t)'G(t)x_F(t) + x_F(t)'G(t)'v(t) \\ &\quad + v(t)'H^\gamma(P_F^\gamma(t))v(t)\Big]\,dt, \end{aligned} \tag{19}$$

where $x_F(t) = x(t, F(t)x_F(t) + v(t), x_0)$ is the solution of

$$\begin{aligned} dx_F(t) &= (A(t) + B(t)F(t))x_F(t)\,dt \\ &\quad + (A_0(t) + B_0(t)F(t))x_F(t)\,d\omega(t) \\ &\quad + B_0(t)v(t)\,d\omega(t) + B(t)v(t)\,dt \end{aligned} \tag{20}$$

with $x_F(0) = x_0$ and

$$G(t) = K(P_F^\gamma(t))' + H^\gamma(P_F^\gamma(t))F(t). \tag{21}$$

As $v = 0$, then

$$J_T^\gamma(x_0, Fx_F) = x_0'P_F^\gamma(0)x_0. \tag{22}$$

Proof. In terms of Lemma 5 with $P(t) = P_F^\gamma(t)$ and $F(t)x_F(t) + v(t)$ for $v(t)$,

$$\begin{aligned} J_T^\gamma(x_0, v + Fx_F) &= x_0'P_F^\gamma(0)x_0 \\ &\quad + E\int_0^T \left\{\begin{bmatrix} x_F(t) \\ F(t)x_F(t) + v(t) \end{bmatrix}' \mathbb{M}(t, P_F^\gamma(t)) \right. \\ &\quad \left. \cdot \begin{bmatrix} x_F(t) \\ F(t)x_F(t) + v(t) \end{bmatrix}\right\} dt = x_0'P_F^\gamma(0)x_0 \\ &\quad + E\int_0^T \left\{x_F(t)' \begin{bmatrix} I \\ F(t) \end{bmatrix}' \right. \\ &\quad \left. \cdot \begin{bmatrix} L(P_F^\gamma(t)) + \dot{P}_F^\gamma(t) & K(P_F^\gamma(t)) \\ K(P_F^\gamma(t))' & H^\gamma(P_F^\gamma(t)) \end{bmatrix} \begin{bmatrix} I \\ F(t) \end{bmatrix} x_F(t)\right\} dt \end{aligned}$$

$$+E\int_0^T \left\{v(t)'G(t)x_F(t)+x_F(t)'G(t)'v(t)\right.$$
$$\left.+v(t)'H^\gamma(P_F^\gamma(t))v(t)\right\}dt = x_0'P_F^\gamma(0)x_0$$
$$+E\int_0^T \left\{v(t)'G(t)x_F(t)+x_F(t)'G(t)'v(t)\right.$$
$$\left.+v(t)'H^\gamma(P_F^\gamma(t))v(t)\right\}dt. \tag{23}$$

This means that (19) holds. Let $v = 0$ in (19); we obtain (22).

Now we are in a position to prove that $H^\gamma(P_F^\gamma(t))$ is invertible for $t \in [0,T]$. □

Lemma 8. *For system (1), if $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ for some given $\gamma > 0$, $F(t) \in C[0,T]$, $T > 0$, and $P_F^\gamma(t)$ satisfies (18). Then,*

$$H^\gamma(P_F^\gamma(t)) \geq \left[\left(\|\mathcal{G}\|_-^{[0,T]}\right)^2 - \gamma^2\right] I > 0, \quad t \in [0,T]. \tag{24}$$

Proof. Let us first prove that $H^\gamma(P_F^\gamma(t)) \geq 0$. Suppose this is false; then there exists $t^* \in [0,T)$, $u \in R^l$, $\|u\| = 1$ such that $u'H^\gamma(P_F^\gamma(t^*))u \leq -\eta$ for some $\eta > 0$. Then, for sufficiently small $\delta > 0$,

$$u'H^\gamma(P_F^\gamma(t))u \leq -\frac{\eta}{2}, \quad t \in [t^*, t^*+\delta] \subset [0,T]. \tag{25}$$

Define

$$v(t) = \begin{cases} 0, & t \in [0,t^*) \cup (t^*+\delta, T], \\ u, & t \in [t^*, t^*+\delta]. \end{cases} \tag{26}$$

Using Lemma 7 with this $v(t)$ and $x_0 = 0$, we can derive that $x_F(t) = 0$ for $t \in [0,t^*]$ and

$$J_T^\gamma(0,v) = E\int_0^T \left[\|C(t)x_F(t)+D(t)v(t)\|^2\right.$$
$$\left.-\gamma^2\|v(t)\|^2\right]dt = E\int_0^T \left[v(t)'G(t)x_F(t)\right.$$
$$\left.+x_F(t)'G(t)'v(t)+v(t)'H^\gamma(P_F^\gamma(t))v(t)\right]dt$$
$$\leq E\int_{t^*}^{t^*+\delta}\left(2\|G(t)'u\|\|x_F(t)\|-\frac{\eta}{2}\right)dt. \tag{27}$$

Since $x_F(t)$ is continuous and $x_F(t^*) = 0$, (27) is negative. Moreover, the condition $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ implies $J_T^\gamma(0,v) \geq 0$. As a result, this is a contradiction. If $t^* = T$, we can replace $[t^*, t^*+\delta]$ by $[T-\delta, T]$ and use a similar proof.

Next, let $\|\mathcal{G}\|_-^{[0,T]} > (\gamma^2+\rho^2)^{1/2}$ for any $\rho > 0$ and $\lambda = (\gamma^2+\rho^2)^{1/2}$. Replacing γ with λ in (18), we obtain the corresponding solution $P_F^\lambda(t)$. Applying the previous step, we can deduce that $H^\lambda(P_F^\lambda(t)) \geq 0$. For any $t_0 \in [0,T)$, set $F_{t_0} = F(t+t_0)$, $t \in [0, T-t_0]$. Let $P_{F_{t_0}}^\lambda(t)$ be the solution of (18) with γ replaced by λ and F replaced by F_{t_0} on $[0, T-t_0]$. Then, $P_{F_{t_0}}^\lambda(t) = P_F^\lambda(t+t_0)$, $t \in [0, T-t_0]$. By (22), for any $t_0 \in [0,T)$, $x_0 \in R^n$,

$$x_0'P_F^\lambda(t_0)x_0 = x_0'P_{F_{t_0}}^\lambda(0)x_0 = J_{T-t_0}^\lambda\left(x_0, F_{t_0}x_{F_{t_0}}\right)$$
$$\leq J_{T-t_0}^\gamma\left(x_0, F_{t_0}x_{F_{t_0}}\right) = x_0'P_F^\gamma(t_0)x_0, \tag{28}$$

and so $H^\lambda(P_F^\gamma(t_0)) \geq H^\lambda(P_F^\lambda(t_0)) \geq 0$. By continuity, $H^\gamma(P_F^\gamma(t)) \geq \rho^2 I$ for all $t \in [0,T]$. As this holds for arbitrary $\rho^2 < (\|\mathcal{G}\|_-^{[0,T]})^2 - \gamma^2$, it follows that $H^\gamma(P_F^\gamma(t)) \geq [(\|\mathcal{G}\|_-^{[0,T]})^2 - \gamma^2]I > 0$. This completes the proof. □

Remark 9. When $t = T$, (24) becomes $H^\gamma(P_F^\gamma(T)) = D(T)'D(T) - \gamma^2 I > 0$. If system (1) is time-invariant, then

$$D'D - \gamma^2 I > 0. \tag{29}$$

Remark 10. By the equality $A(I-BA)^{-1} = (I-AB)^{-1}A$, we have that $C'[I - D(D'D-\gamma^2 I)^{-1}D']C = C'(I-\gamma^{-2}DD')^{-1}C$. If system (1) is time-invariant and square, by (29),

$$C'\left[I - D\left(D'D-\gamma^2 I\right)^{-1}D'\right]C$$
$$= C'\left(I-\gamma^{-2}DD'\right)^{-1}C \leq 0. \tag{30}$$

Now, we present the following theorem which is important in this paper.

Theorem 11. *Suppose system (1) is time-invariant and square and satisfies $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ for given $\gamma \leq 0$. Then (11) has a unique solution $P_T(t) \leq 0$ on $[0,T]$ for every $T > 0$. Moreover, $J_T^\gamma(x_0, v)$ is minimized by the feedback control:*

$$v^*(t) = F_T(t)x_{F_T}(t),$$
$$F_T(t) = -H^\gamma(P_T(t))^{-1}K(P_T(t))', \tag{31}$$

where $x_{F_T}(t)$ satisfies

$$dx_{F_T}(t) = \left(A+BF_T(t)\right)x_{F_T}(t)\,dt$$
$$+\left[A_0+B_0F_T(t)\right]x_{F_T}(t)\,dw(t), \tag{32}$$
$$x_{F_T}(0) = x_0$$

and the optimal cost is

$$\min_{v\in\mathcal{L}_{\mathcal{F}}^2([0,T],R^p)} J_T^\gamma(x_0,v) = x_0'P_T(0)x_0. \tag{33}$$

Proof. We prove that $\|\mathcal{G}\|_{-}^{[0,T]} > \gamma$ implies the existence of solution $P_T(t)$ of (11) on $[0,T]$. Using a contradiction argument, we suppose that (11) does not admit a solution. By the standard theory of differential equations, there exists unique solution $P_T(t)$ backward in time on maximal interval $[T_0,T]$ ($T_0 \geq 0$), and as $t \to T_0$, $P_T(t)$ becomes unbounded.

Let $0 < \delta < T - T_0$, $x(T_0+\delta) = x_{T_0,\delta} \in R^n$, $Q(t) = B'P_T(t) + B_0'P_T(t)A_0 + D'C$, $R(t) = B_0'P_T(t)B_0 + D'D - \gamma^2 I$, by completing the squares; then

$$\begin{aligned}
J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right) &= E\int_{T_0+\delta}^{T}\left[z(t)'z(t)\right.\\
&\left.-\gamma^2 v(t)'v(t)\right]dt = E\int_{T_0+\delta}^{T}\left[z(t)'z(t)\right.\\
&\left.-\gamma^2 v(t)'v(t)\right]dt + E\int_{T_0+\delta}^{T} d\left(x(t)'P_T(t)x(t)\right)dt\\
&+x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta} = E\int_{T_0+\delta}^{T} x(t)'\left[C'C\right.\\
&+A'P_T(t)+P_T(t)A+A_0'P_T(t)A_0+\dot{P}_T(t)\\
&\left.-Q(t)'R(t)^{-1}Q(t)\right]x(t)\,dt + E\int_{T_0+\delta}^{T}\left[v(t)\right.\\
&\left.+R(t)^{-1}Q(t)x(t)\right]'R(t)\left[v(t)\right.\\
&\left.+R(t)^{-1}Q(t)x(t)\right]dt + x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta}\\
&= E\int_{T_0+\delta}^{T}\left[v(t)+R(t)^{-1}Q(t)x(t)\right]'R(t)\left[v(t)\right.\\
&\left.+R(t)^{-1}Q(t)x(t)\right]dt + x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta}.
\end{aligned} \tag{34}$$

Obviously,

$$\begin{aligned}
&\min_{v\in\mathcal{L}^2_{\mathcal{F}}([T_0+\delta,T],R^p)} J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right)\\
&\quad = J^\gamma\left(x,v^*,x_{T_0,\delta},T_0+\delta\right) = x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta},
\end{aligned} \tag{35}$$

where $v^*(t) = -R(t)^{-1}Q(t)x(t)$.

Furthermore, we can see that

$$\begin{aligned}
J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right) &= E\int_{T_0+\delta}^{T}\left[z(t)'z(t)\right.\\
&\left.-\gamma^2 v(t)'v(t)\right]dt\\
&= E\int_{T_0+\delta}^{T}\left\{[Cx(t)+Dv(t)]'[Cx(t)+Dv(t)]\right.\\
&\left.-\gamma^2 v(t)'v(t)\right\}dt = E\int_{T_0+\delta}^{T} x(t)'C'\left[I\right.\\
&\left.-D\left(D'D-\gamma^2 I\right)^{-1}D'\right]Cx(t)\,dt + E\int_{T_0+\delta}^{T}\left[v(t)\right.\\
&\left.+\left(D'D-\gamma^2 I\right)^{-1}D'Cx(t)\right]'\left(D'D-\gamma^2 I\right)\left[v(t)\right.\\
&\left.+\left(D'D-\gamma^2 I\right)^{-1}D'Cx(t)\right]dt.
\end{aligned} \tag{36}$$

From Remark 10,

$$\begin{aligned}
J^\gamma\left(x,\tilde{v},x_{T_0,\delta},T_0+\delta\right) &= E\int_{T_0+\delta}^{T} x(t)'\\
&\cdot C'\left[I-D\left(D'D-\gamma^2 I\right)^{-1}D'\right]Cx(t)\,dt \leq 0,
\end{aligned} \tag{37}$$

where $\tilde{v}(t) = -(D'D-\gamma^2 I)^{-1}D'Cx(t)$. Considering (35), we get

$$x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta} \leq J^\gamma\left(x,\tilde{v},x_{T_0,\delta},T_0+\delta\right) \leq 0, \tag{38}$$

which implies that

$$P_T(T_0+\delta) \leq 0. \tag{39}$$

By linearity, the solution of (1) with initial state $x_{T_0,\delta}$ satisfies

$$\begin{aligned}
x\left(t,v,x_{T_0,\delta},T_0+\delta\right) &= x\left(t,0,x_{T_0,\delta},T_0+\delta\right)\\
&+x\left(t,v,0,T_0+\delta\right).
\end{aligned} \tag{40}$$

Suppose $\Phi(t)$ is the solution of

$$\begin{aligned}
C'C + A'\Phi(t) + \Phi(t)A + A_0'\Phi(t)A_0 + \dot{\Phi}(t) &= 0,\\
\Phi(T) &= 0,
\end{aligned} \tag{41}$$

and we have

$$\begin{aligned}
&J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right) - J^\gamma\left(x,v,0,T_0+\delta\right)\\
&\quad = x_{T_0,\delta}'\Phi(T_0+\delta)x_{T_0,\delta} + E\int_{T_0+\delta}^{T}\left[v(t)'\right.\\
&\quad\left.\cdot\left(B_0'\Phi(t)+D'C\right)x\left(t,0,x_{T_0,\delta},T_0+\delta\right)\right]dt\\
&\quad + E\int_{T_0+\delta}^{T}\left[x\left(t,0,x_{T_0,\delta},T_0+\delta\right)'\left(B_0'\Phi(t)+D'C\right)'\right.\\
&\quad\left.\cdot v(t)\right]dt.
\end{aligned} \tag{42}$$

Take $\|\mathcal{G}\|_{-}^{[0,T]} \geq (\gamma^2\epsilon^2)^{1/2}$,

$$\bar{v}(t) = \begin{cases} \bar{v}(t) = -\left(D'D-\gamma^2 I\right)^{-1} D'Cx(t), & t \in [0, T_0+\delta] \\ v(t), & t \in (T_0+\delta, T], \end{cases} \tag{43}$$

and it is easy to show that

$$\begin{aligned} J^{\gamma}(x, v, 0, T_0+\delta) &= E\int_0^T \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt \\ &- E\int_0^{T_0+\delta} \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt \\ &= E\int_0^T \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt - E\int_0^{T_0+\delta} x(t)' \\ &\cdot C'\left[I - D\left(D'D-\gamma^2 I\right)^{-1} D'\right] Cx(t)\, dt \\ &- E\int_0^{T_0+\delta} \left[\bar{v}(t) + \left(D'D-\gamma^2 I\right)^{-1} D'Cx(t)\right]' \\ &\cdot \left(D'D-\gamma^2 I\right) \\ &\cdot \left[\bar{v}(t) + \left(D'D-\gamma^2 I\right)^{-1} D'Cx(t)\right] dt \\ &\geq E\int_0^T \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt \geq \epsilon^2\|\bar{v}(t)\|_{[0,T]}^2 \\ &\geq \epsilon^2\|v(t)\|_{[T_0+\delta,T]}^2. \end{aligned} \tag{44}$$

It follows that

$$\begin{aligned} J^{\gamma}\left(x, v, x_{T_0,\delta}, T_0+\delta\right) &\geq E\int_{T_0+\delta}^T \epsilon^2\|v(t)\|^2 dt \\ &+ x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} + E\int_{T_0+\delta}^T \left[v(t)'\left(B_0'\Phi(t)\right.\right. \\ &\left.\left.+ D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right] dt \\ &+ E\int_{T_0+\delta}^T \left[x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)'\left(B_0'\Phi(t) + D'C\right)'\right. \\ &\left.\cdot v(t)\right] dt = x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} + E\int_{T_0+\delta}^T \left\|\epsilon\left[v\right.\right. \\ &\left.\left.- \epsilon^{-2}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right]\right\|^2 dt \\ &- E\int_{T_0+\delta}^T \left\|\epsilon^{-1}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0\right.\right. \\ &\left.\left.+\delta\right)\right\|^2 dt \geq x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} \\ &- E\int_{T_0+\delta}^T \left\|\epsilon^{-1}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0\right.\right. \\ &\left.\left.+\delta\right)\right\|^2 dt. \end{aligned} \tag{45}$$

It is obvious that there exists constant $C_0 > 0$ such that

$$C_2\left\|x_{T_0,\delta}\right\|^2 \geq E\int_{T_0+\delta}^T \left\|x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right\|^2 dt. \tag{46}$$

So, there is constant $C_1 > 0$ such that

$$\begin{aligned} &E\int_{T_0+\delta}^T \left\|\epsilon^{-1}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right\|^2 dt \\ &\leq C_1\left\|x_{T_0,\delta}\right\|^2. \end{aligned} \tag{47}$$

In addition,

$$\begin{aligned} &x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} \\ &= -E\int_{T_0+\delta}^T d\left(x(t)'\Phi(t)\, x(t)\right) \\ &= E\int_{T_0+\delta}^T \left\|x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right) C\right\|^2 dt \geq 0. \end{aligned} \tag{48}$$

From (45), we have

$$J^{\gamma}\left(x, v, x_{T_0,\delta}, T_0+\delta\right) \geq -C_1\left\|x_{T_0,\delta}\right\|^2. \tag{49}$$

In view of (35) and (39), it yields

$$-C_1 \leq P_T(T_0+\delta) \leq 0. \tag{50}$$

So, $P_T(T_0+\delta)$ can not become unbounded as $\delta \to 0$, which means that (11) has unique solution $P_T(t)$ on $[0, T]$.

Setting $F(t) = F_T(t)$, $t \in [0, T]$, in (17), from (31), we obtain

$$\begin{aligned} &\dot{P}_T(t) + L\left(P_T(t)\right) + K\left(P_T(t)\right) F(t) \\ &\quad + F(t)' K\left(P_T(t)\right)' + F(t)' H^{\gamma}\left(P_T(t)\right) F(t) = 0. \end{aligned} \tag{51}$$

Hence $P_T(t)$ satisfies (17), or equivalently (18). So

$$P_{F_T}^{\gamma}(t) = P_T(t), \quad t \in [0, T]. \tag{52}$$

By (31),

$$G(t) = K\left(P_T(t)\right)' + H^{\gamma}\left(P_T(t)\right) F_T(t) = 0, \tag{53}$$

and, in terms of Lemma 7,

$$J_T^\gamma(x_0, v + F_T x) = x_0' P_T(0) x_0 + E\int_0^T \left[v(t)' H^\gamma(P_T(t)) v(t)\right] dt. \tag{54}$$

But by Lemma 8,

$$H^\gamma(P_T(t)) = H^\gamma\left(P_{F_T}^\gamma(t)\right) \geq \left[\left(\|\mathcal{G}\|_-^{[0,T]}\right)^2 - \gamma^2\right] I \succ 0, \quad t \in [0,T]. \tag{55}$$

Hence, $v^*(t) = F_T(t)x(t)$ minimizes $J_T^\gamma(x_0, v)$ and $\min_{v\in\mathcal{L}^2_{\mathcal{F}}([0,T],R^p)} J_T^\gamma(x_0, v) = x_0' P_T(0) x_0$. □

According to Theorems 6 and 11, we get the following theorem.

Theorem 12. *If system (1) is time-invariant and square, for given $\gamma > 0$, the following are equivalent:*

(i) *Consider $\|\mathcal{G}\|_-^{[0,T]} > \gamma$.*

(ii) *The following equation*

$$\begin{aligned} &P(t)A + A'P(t) + A_0'P(t)A_0 + C'C + \dot{P}(t) \\ &\quad = \left(P(t)B + A_0'P(t)B_0 + C'D\right) \\ &\quad \cdot \left(B_0'P(t)B_0 + D'D - \gamma^2 I\right)^{-1} \\ &\quad \cdot \left(P(t)B + A_0'P(t)B_0 + C'D\right)', \\ &B_0'P(t)B_0 + D'D - \gamma^2 I > 0, \\ &P(T) = 0 \end{aligned} \tag{56}$$

has unique solution $P_T(t) \leq 0$ on $[0,T]$. Moreover, $\min_{v\in\mathcal{L}^2_{\mathcal{F}}([0,T],R^p)} J_T^\gamma(x_0, v) = x_0' P_T(0) x_0$.

Remark 13. For given $\gamma > 0$, if we replace B, C, D, and $v(t)$ with $B_\delta = [B \;\; 0_{n\times n}]$, $C_\delta = [C \;\; \delta I_n]'$, $D_\delta = \begin{bmatrix} D & 0_{l\times n} \\ 0_{n\times l} & 0_{n\times n}\end{bmatrix}$, and $v_\delta(t) = [v(t) \;\; 0_{n\times n}]'$, respectively, and $z(t)$ with $z_\delta(t)$ in (1), we deduce the corresponding $\mathcal{H}_-$ index $\|\mathcal{G}\|_{\delta-}^{[0,T]}$ and

$$\begin{aligned} J_{T,\delta}^\gamma(x_0, v) &= E\int_0^T \left\{\|z_\delta(t)\|^2 - \gamma^2 \|v_\delta(t)\|^2\right\} dt \\ &= E\int_0^T \left\{\|z(t)\|^2 - \gamma^2 \|v(t)\|^2 + \delta^2 I\right\} dt. \end{aligned} \tag{57}$$

When $\|\mathcal{G}\|_-^{[0,T]} > \gamma$, then $\|\mathcal{G}\|_{\delta-}^{[0,T]} > \gamma$. Using Theorem 12 to the modified data, it is easy to see that the following equation

$$\begin{aligned} &P(t)A + A'P(t) + A_0'P(t)A_0 + C'C + \delta^2 I + \dot{P}(t) \\ &\quad = \left(P(t)B + A_0'P(t)B_0 + C'D\right) \\ &\quad \cdot \left(B_0'P(t)B_0 + D'D - \gamma^2 I\right)^{-1} \\ &\quad \cdot \left(P(t)B + A_0'P(t)B_0 + C'D\right)', \\ &B_0'P(t)B_0 + D'D - \gamma^2 I > 0, \\ &P(T) = 0 \end{aligned} \tag{58}$$

has unique solution $P_{\delta,T}(t) \leq 0$ on $[0,T]$. Moreover, $\min_{v\in\mathcal{L}^2_{\mathcal{F}}([0,T],R^p)} J_{T,\delta}^\gamma(x_0, v) = x_0' P_{\delta,T}(0) x_0$.

Now, we are to show what happens as T increases.

Theorem 14. *If system (1) is time-invariant and square, $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ for some $\gamma > 0$. Then $P_T(t)$ in (56) decreases as T increases for every $t \in [0,T]$.*

Proof. Suppose $\overline{T} > T$, $t \in [0,T]$, and $x_0 \in R^n$. Let v_{T-t}^* be optimal for x_0 on $[0, T-t]$, and set

$$v(\tau) = \begin{cases} v_{T-t}^*(\tau), & \tau \in [0, T-t] \\ -\left(D'D - \gamma^2 I\right)^{-1} D'Cx(\tau), & \tau \in \left(T-t, \overline{T}-t\right]. \end{cases} \tag{59}$$

By the time invariance of $P_T(t)$, $P_{T-t}(0) = P_T(t)$. Then,

$$\begin{aligned} x_0' P_{\overline{T}}(t) x_0 &= x_0' P_{\overline{T}-t}(0) x_0 \leq J_{\overline{T}-t}^\gamma(x_0, v) \\ &= J_{T-t}^\gamma(x_0, v_{T-t}^*) + E\int_{T-t}^{\overline{T}-t} \left\{\|z(\tau)\|^2 - \gamma^2 \|v(\tau)\|^2\right\} d\tau \\ &= J_{T-t}^\gamma(x_0, v_{T-t}^*) + E\int_{T-t}^{\overline{T}-t} \Big\{x'(\tau) \\ &\quad \cdot C'\left[I - D\left(D'D - \gamma^2 I\right)^{-1} D'\right] Cx(\tau)\Big\} d\tau \\ &\quad + E\int_{T-t}^{\overline{T}-t} \Big\{\left[v(\tau) + \left(D'D - \gamma^2 I\right)^{-1} D'Cx(\tau)\right]' \\ &\quad \cdot \left(D'D - \gamma^2 I\right)\left[v(\tau) + \left(D'D - \gamma^2 I\right)^{-1} D'Cx(\tau)\right]\Big\} d\tau \end{aligned}$$

$$= J_{T-t}^{\gamma}\left(x_0, v_{T-t}^{*}\right) + E\int_{T-t}^{\overline{T}-t}\left\{x'(\tau)\right.$$

$$\left.\cdot C'\left[I - D\left(D'D - \gamma^2 I\right)^{-1} D'\right] Cx(\tau)\right\} d\tau$$

$$\leq J_{T-t}^{\gamma}\left(x_0, v_{T-t}^{*}\right) = x_0' P_T(t)\, x_0. \tag{60}$$

This means that $P_T(t)$ decreases as T increases for every $t \in [0, T]$. □

3. A Numerical Example

Below, we give a numerical example to illustrate the rightness of Theorems 12 and 14.

Example 1. In system (1), we consider a two-dimensional linear stochastic system with the following parameters:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},$$
$$B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{61}$$
$$A_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Set $\gamma = 0.5$, $T = 2, 3$; by solving (56), we can obtain the solutions of

$$P_2(t) = \begin{bmatrix} p_2^{11}(t) & p_2^{12}(t) \\ p_2^{12}(t) & p_2^{22}(t) \end{bmatrix},$$
$$P_3(t) = \begin{bmatrix} p_3^{11}(t) & p_3^{12}(t) \\ p_3^{12}(t) & p_3^{22}(t) \end{bmatrix}, \tag{62}$$

for which their trajectories are shown in Figure 1. If we set $t = 1$, then it yields

$$P_2(1) = \begin{bmatrix} -0.0892 & -0.0953 \\ -0.0953 & -0.1446 \end{bmatrix},$$
$$P_3(1) = \begin{bmatrix} -0.1044 & -0.0889 \\ -0.0889 & -0.1489 \end{bmatrix}. \tag{63}$$

It is easy to see that $P_2(1) > P_3(1)$, which verifies the rightness of Theorem 14.

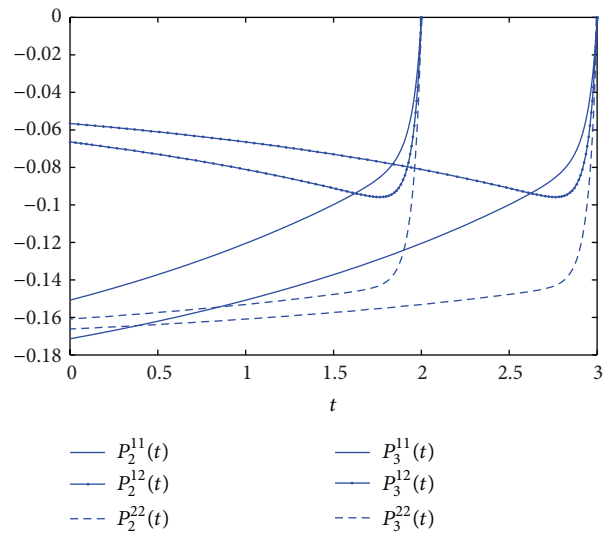


Figure 1: The trajectories of $P_2(t)$ and $P_3(t)$.

4. Conclusion

In this paper, we have solved the $\mathscr{H}_-$ index problem where both stochastic and deterministic perturbations are present. Necessary and sufficient condition for the lower bound of $\mathscr{H}_-$ index is given by means of the solvability of a generalized differential equation. The proposed results are not completely dual to $\mathscr{H}_\infty$ norm, and the effectiveness of the given methods is illustrated by numerical example.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (Grants nos. 61174078, 61170054, and 61402265) and the Research Fund for the Taishan Scholar Project of Shandong Province of China.

References

[1] K. Zhou, J. C. Doyle, and K. Clover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, USA, 1996.

[2] B.-S. Chen and W. Zhang, "Stochastic $\mathscr{H}_2/\mathscr{H}_\infty$ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[3] W. H. Zhang and B.-S. Chen, "State feedback $\mathscr{H}_\infty$ control for a class of nonlinear stochastic systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 1973–1991, 2006.

[4] W. Zhang, H. Zhang, and B.-S. Chen, "Generalized Lyapunov equation approach to state-dependent stochastic stabilization/detectability criterion," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1630–1642, 2008.

[5] Q. X. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise," *Journal of*

Mathematical Analysis and Applications, vol. 416, no. 1, pp. 126–142, 2014.

[6] Q. Zhu and J. Cao, "Stability of Markovian jump neural networks with impulse control and time varying delays," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 5, pp. 2259–2270, 2012.

[7] Q. X. Zhu, J. D. Cao, and R. Rakkiyappan, "Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays," *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1085–1098, 2015.

[8] Q. X. Zhu and J. D. Cao, "Mean-square exponential input-to-state stability of stochastic delayed neural networks," *Neurocomputing*, vol. 131, pp. 157–163, 2014.

[9] D. Hinrichsen and A. J. Pritchard, "Stochastic $\mathscr{H}_\infty$," *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pp. 1504–1538, 1998.

[10] J. Liu, J. L. Wang, and G.-H. Yang, "An LMI approach to minimum sensitivity analysis with application to fault detection," *Automatica*, vol. 41, no. 11, pp. 1995–2004, 2005.

[11] J. L. Wang, G. H. Yang, and J. Liu, "An LMI approach to $\mathscr{H}_-$ index and mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design," *Automatica*, vol. 43, pp. 1656–1665, 2007.

[12] J. Chen and R. J. Pattor, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, Boston, Mass, USA, 1999.

[13] I. Hwang, S. Kim, Y. Kim, and C. E. Seah, "A survey of fault detection, isolation, and reconfiguration methods," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 3, pp. 636–653, 2010.

[14] S. Yin, S. X. Ding, A. Haghani, H. Hao, and P. Zhang, "A comparison study of basic data-driven fault diagnosis and process monitoring methods on the benchmark Tennessee Eastman process," *Journal of Process Control*, vol. 22, no. 9, pp. 1567–1581, 2012.

[15] D. Wang, P. Shi, and W. Wang, "Robust fault detection for continuous-time switched delay systems: an linear matrix inequality approach," *IET Control Theory and Applications*, vol. 4, no. 1, pp. 100–108, 2010.

[16] S. Yin, H. Luo, and S. X. Ding, "Real-time implementation of fault-tolerant control systems with performance optimization," *IEEE Transactions on Industrial Electronics*, vol. 61, no. 5, pp. 2402–2411, 2014.

[17] K. Iftikha, P. Q. Khan, and M. Abid, "Optimal fault detection filter design for switched linear systems," *Nonlinear Analysis*, vol. 15, pp. 132–144, 2015.

[18] M. Zhong, S. X. Ding, and E. L. Ding, "Optimal fault detection for linear discrete time-varying systems," *Automatica*, vol. 46, no. 8, pp. 1395–1400, 2010.

[19] X. B. Li and H. H. T. Liu, "Characterization of $\mathscr{H}_-$ index for linear time-varying systems," *Automatica*, vol. 49, no. 3, pp. 1449–1457, 2013.

[20] X. Li and H. H. T. Liu, "Minimum system sensitivity study of linear discrete time systems for fault detection," *Mathematical Problems in Engineering*, vol. 2013, Article ID 276987, 13 pages, 2013.

[21] M. Chen, C.-S. Jiang, and Q.-X. Wu, "Sensor fault diagnosis for a class of time delay uncertain nonlinear systems using neural network," *International Journal of Automation and Computing*, vol. 5, no. 4, pp. 401–405, 2008.

[22] K. Zhang, B. Jiang, and P. Shi, "A new approach to observer-based fault-tolerant controller design for Takagi-Sugeno fuzzy systems with state delay," *Circuits, Systems, and Signal Processing*, vol. 28, no. 5, pp. 679–697, 2009.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 671764, 9 pages
http://dx.doi.org/10.1155/2015/671764

Research Article

Concession Period Decision Models for Public Infrastructure Projects Based on Option Games

Zhi Wang,[1] Huan Tan,[1] Jing Wang,[1] and Chunyan Hu[2]

[1] *Economics and Management School, Changsha University of Science and Technology, Hunan 410114, China*
[2] *School of Public Administration, Central South University, Hunan 410082, China*

Correspondence should be addressed to Chunyan Hu; mailtohcy@163.com

Received 13 May 2015; Accepted 27 August 2015

Academic Editor: Ruihua Liu

Concession period is an important decision-making variable for the investment and construction of public infrastructure projects. However, we currently have few scientific methods to exactly determine the concession period. This paper managed to seek out concession period decision models for public infrastructure with option game theory, studied the influence of minimum government income guarantee and government investment on concession period, and demonstrated those models in the formulas mentioned in the paper. The research results showed that the increase of minimum government income guarantee value would shorten the concession period, while the increase of income volatility, that is, the uncertainty, would lengthen the concession period. In terms of government investment, optimal concession period would lengthen to some extent with the increase of government investment ratio and the income and the decrease of its guarantee value. Yet, optimal concession period would shorten in case of extreme highness of the government investment ratio due to its high guarantee value. And the government would accordingly shorten the concession period in case of the unchanged government investment ratio with the increased income volatility and risks. Still, the paper put forward the argument that the government would apply various guarantee methods and implement flexible concession period in accordance with the specific circumstances of public infrastructure projects.

1. Introduction

In recent years, the rapid growth of the investment and construction of infrastructure projects both at home and abroad has become an important way to promote economic growth and structural adjustment. The government, to solve the issue of shortage of funds for the purpose of projects construction, is adopting relatively flexible policies such as encouragement of nonpublic investment in the infrastructure projects to diversify the financing channels and patterns such as PPP and BOT. Therefore, to determine a rational concession period in the concession agreement is the key to effectively urge private investment in construction of infrastructure projects which are characterised by large-scale investment, long construction period, obvious social benefits, and multiple risks (Dai and Wen [1], Liu et al. [2]). Concession period is an important decision-making variable for the investment and construction of franchised infrastructure projects. Within the concession period, private investors are responsible for projects construction and operation and collect fees to settle the debts and make profits according to the agreed modes in concession negotiation. After the concession period, private investors will transfer the projects to the government for free or at the agreed price. Therefore, the length of the concession period will profoundly affect the interests of the private investors and the government.

Currently, there are two major categories of concession period decision-making methods for infrastructure projects: one is based on the net present value (NPV) of the project; for example, scholars like Li and Shen [3] established concession period decision model for the infrastructure BOT project from the perspective of income and cash flow. Scholars like Ye, Shen, Thomas, Song, and Huang [4–8], applying net present value (NPV) and Monte Carlo simulation method, analysed design issues on the project concession period. Qin [9] adopted CAPM model to change the discount rate value of the foregoing model into the risk-adjusted discount rate which is more suitable for deciding concession period of

projects under system risks, while scholars including Wen [10–14] tried to confirm the risk income ratio of the projects on the basis of investors' risk appetite. Nevertheless, such method failed to considerate the uncertainty of the projects' future profits, the long operating period, and the influence of government guarantee and ignored the flexibility and complexity of determining the concession period under the market environment.

The second is based on game theory and real options theory. The concession period's decision of infrastructure project can be seen as a game between the investors and the government. In the process of bargaining, investors, at the price of harming the interests of the government and the public, will manage to prolong the concession period which lay a solid foundation for the construction quality of infrastructure. On the contrary, the government will try to shorten the concession period to safeguard the interests of the government and the public, and the shortened concession period will definitely damage the quality and operating life of infrastructure construction and increase the maintenance costs after retaking the project. In model PPP, Medda [15] regarded the interests allocation between the government and private investors as a bargaining game and analyzed the strategic behaviour and potential moral risks in case guarantee value is higher than finance loss. Scholars like Yang et al. [16] established concession period decision model by analysing the game features between the government and the project corporation. Gao et al. [17] adopted the "cake-sharing" model in complete information dynamic game to study how to determine BOT project concession period with relatively stable profits and known life cycle of project. Scholars like Shen et al. [18], after considering the bargaining behaviour between the government and nongovernment in the concession period negotiations, established a complete information dynamic game model to determine the concession period and explained the effect of negotiation interaction factors on confirming concession period from the angle of the negotiation. Applying game theory, Bao [19] established a dynamic game and the concession period decision model with changed investment cost. Wu et al. [20] established the game model for project corporation's investment and government's concession period and analysed the issues about the optimal strategies, respectively, with Stackelberg's game method. Zhang and Durango-Cohen [21] built a game model for concession negotiation between government and private investors and studied how the government encourages private investment in the projects by offering some preferential policies. These results laid the foundation for solving issues about concession period decision but failed to solve the problems on determining the concession period flexibly. As an option, real option bears more flexibility for the decision making of government and project corporation [22]. Gao et al. [23] pointed out that infrastructure BOT project, differing form general projects, has specific risk guarantee which is provided by the government for the purpose of attracting nongovernmental investment and it can be seen as down-and-in options. As for PPP projects, government will participate in investments and bear the risks. Scholars like Takashima et al. [24] explored the investment decisions of the government and private investors based on PPP model, applied real options method to analyze the ratio of shared investment costs and risks in project operation, and pointed out that the size of shared ratio will affect investment decisions and the value of the project. Cruz and Marques [25], taking hospital project as an example, considered the uncertainty of PPP projects as an opportunity and built a model based on real options theory to assess the benefits from flexibility of the contract and concluded that the uncertainty increased the project value.

Public infrastructure investment and financing process involves many people's interests and each has different focus on economic efficiency and social benefits. On determining the concession period of the projects, we will take the game process between the investors and the government into consideration except for the uncertainties. And the option game theory integrating real option theory with game theory explains the impact of option game characteristics on concession period under the uncertain conditions of minimum government income guarantee and government participation in the risk sharing in a better way. Smets [26] was the first to introduce the game theory to real options analysis framework. Based on this, Ottoo [27] pointed out that the government may divide BOT infrastructure projects into several phases and give the right to invest in next-phase project construction to companies which successfully won the bidding of phase I. This option can be regarded as growth option, that is, the option to grasp the growth opportunity. Scholars like Alonso-Conde et al. [28], taking financing structure and contract clauses in the PPP model as a real option, analyzed how these options affect the investment incentives, and the government transfers the project benefits to the private investors via government guarantee. Articles in China analysing the project concession period decision-making problems with option game method are few and far between. Using the option game, Guo et al. [29] studied issues about construction costs of infrastructure BOT project phase II and its concession period determination as well but failed to consider the value and effect of government guarantees. Gong et al. [30] used the real options approach and game theory and built the quantitative decision negotiation model for concession period of BOT projects with the minimum government income guarantees but did not consider the impact of other government guarantees on the concession period. In reality, the government will take various measures to attract nongovernmental investors to participate in investment of public infrastructure construction, such as offering investors the minimum income guarantee, or investing and sharing the risks. Different guarantee forms will exert different effects on concession period. Our paper contributions are as follows: beginning with projects without government guarantees, taking the elements such as minimum government income guarantee and government investment and sharing risks into consideration, building concession period decision models for public infrastructure projects with option game theory, studying the influence of different government guarantees on concession period, and determining government guarantee forms and concession period rationally.

Table 1: Complete information about bargaining game elements between two parties.

Participants	Investors	The government
The strategic space	$(0, T_x)$	(T_x, T)
Payoff function	$E[V(0, T_x) - I + E(G)]$	$E[V(T_x, T) - E(G)]$

This paper consists of 6 sections as follows. We propose basic assumption to build the models in Section 2. We establish option game decision model for the concession period of the projects without government guarantee and determine the optimal concession period under these circumstances which serves as a reference for the following analysis in Section 3. In Section 4, we establish the option game decision model for the concession period of the projects with government guarantee after considering the minimum government income guarantee and government investment and sharing risks and attain the analytical solutions to the optimal concession period. In Section 5, in order to make the analysis more intuitive and clear, we explain the optimal concession period obtained from option game decision model for the project concession period in Section 3 and Section 4 by numerical examples analysis and we make a summary of the conclusion in Section 6.

2. Model Hypotheses

Since concession negotiation comes after bid evaluation, the information is open. In order to facilitate analysis, we assume that the government and investors share the relevant information in the bid, face common project conditions, and know clearly about each other's strategies and that the other party knows theirs; both parties are inclined to pursue appetite for neutral risk and maximum benefit. Investors and the government will weigh the advantages and disadvantages in the bargaining on determining the project concession period.

We assume that the concession period negotiation is the complete information bargaining game between the two parties. Game has three elements of participants, the strategic space and payoff function; in the process of negotiation, the three elements are shown in Table 1.

Among them, E is expectations, T is the project's planning use life, T_x is the concession period, I is initial investment cost, V is the value of the project, V_t is instantaneous net cash flow, and G is government guarantee value. We assume that V_t follows geometric Brownian motion, so it satisfies the random process as follows: $dV_t = \alpha V_t dt + \sigma V_t dz$. α is the expected growth rate of V_t, σ is the volatility of dV_t/V_t, and dz is increment of standard wiener process.

In the concession negotiation stage, investors have submitted the tender, so the first round of the game is to offer price by the investors. Government may accept it or reject it: if government accepts the tender, the game ends, and the concession period is determined; if the government rejects the tender, then the game enters into the second round, and the role of two sides changes, continuing bargaining over and over again. After selecting qualified candidates through bidding, the government, according to the quality of the scheme, determines the optimal investors to negotiate and will negotiate with the next candidate in case of failure. As a dominant role in the negotiations, the government has the say in accepting or rejecting the agreement in the final phase. Overall, concession period negotiations are the bargaining game for even times of the price offered by the investors in the first round. Through backward induction, we can get the following negotiations equalization payment of investors:

$$\begin{aligned} V_I^E(R) &= E\left(\int_0^T V_t e^{-ut} dt - I\right)\left(1 - \delta + \delta^2 - \cdots \delta^{R-1}\right) \\ &= \left[\frac{V_0\left(1 - e^{-mT}\right)}{m} - I\right]\left(\frac{1 - \delta^R}{1 + \delta}\right), \end{aligned} \tag{1}$$

where $E(V)$ is the total income within project life, R is the number of negotiating rounds, δ is the discount rate of negotiation round, V_0 as the initial income flow, $u > \alpha$ is the risk-adjusted discount rate for future income, and $m = u - \alpha > 0$ is convenience income of a project. When negotiating rounds tend to be infinite, Shaked and Sutton's [31] and Cui et al.'s [32] research results show that the subgame refining Nash equilibrium of the game equals the investors' balance payment in the first round:

$$V_I^1(R) = \frac{E(V)}{1 + \delta} = \left[\frac{V_0\left(1 - e^{-mT}\right)}{m} - I\right]\left(\frac{1}{1 + \delta}\right). \tag{2}$$

In the negotiation process between government and investors, the government will determine the concession period T_x in favour of the interests of the public, and T_x will meet the following decision model:

$$\max_{T_x} \quad E\left[V_{(T_x,T)} - E(G)\right] \tag{3}$$

$$\text{s.t.} \quad E\left[V_{(0,T_x)} - I + E(G)\right] \geq \overline{u}. \tag{4}$$

Formula (3) is utility function of the government, that is, concession period under the maximum public interests. Formula (4) is the restricted conditions for government utility function, namely, the maximum expected utility the investors get from the project during the concession period. Among them, $\overline{u}$ stands for opportunity utility, that is, the maximum expected utility the investors get from other projects while being lost in the infrastructure projects.

After the government determines the concession period T_x, investors take the initial investment cost I as decision-making variables will meet the following decision model in the case of profit guarantee:

$$\max_I E\left[V\left(0, T_x\right) - I + E(G)\right]. \tag{5}$$

Formula (5) shows the cost invested by investors in the project construction in the pursuit of maximum profit after the concession period determined by the government.

3. The Option Game Model between Investors without Government Guarantees and Government

$G = 0$ in the case of no government guarantee; then, the income function in concession period investors without government guarantees is

$$E\left[V_{(0,T_x)} - I + E(G)\right] = \frac{V_0\left(1 - e^{-mT_x}\right)}{m} - I. \tag{6}$$

Then, during the period of time, that is, from the government to recover project management rights, to the end of the project period, the expected function of government revenue is:

$$E\left[V_{(T_x,T)} - E(G)\right] = \frac{V_0\left(e^{-mT_x} - e^{-mT}\right)}{m}. \tag{7}$$

According to the respective decision models of governments and investors and using backward induction, we get investors' construction costs I under the conditions of the assumed concession period, and the government determines the appropriate concession period value T_x accordingly. Thus, the government and investors reach a win-win situation by taking the interests of both sides into account.

Substituting (7) into (3) and (6) into (4), we can obtain the government's decision-making models and restricted conditions:

$$\begin{aligned} &\max_{T_x} \quad \frac{V_0\left(e^{-mT_x} - e^{-mT}\right)}{m} \\ &\text{s.t.} \quad \frac{V_0\left(1 - e^{-mT_x}\right)}{m} - I \geq \overline{u}. \end{aligned} \tag{8}$$

Lagrange multiplier method can be used to solve the maximization of formula (8). Let λ be a Lagrange multiplier, (8) combine together to form a Lagrangian function, we can know

$$\begin{aligned} L(T_x, \lambda) = &\frac{V_0\left(e^{-mT_x} - e^{-mT}\right)}{m} \\ &+ \lambda\left[\frac{V_0\left(1 - e^{-mT_x}\right)}{m} - I - \overline{u}\right]. \end{aligned} \tag{9}$$

Seeking first-order partial derivatives of T_x and λ in formula (9), respectively, we can know

$$\frac{\partial L}{\partial T_x} = (\lambda - 1) V_0 e^{-mT_x} = 0 \tag{10}$$

$$\frac{\partial L}{\partial \lambda} = \frac{V_0\left(1 - e^{-mT_x}\right)}{m} - I - \overline{u} = 0. \tag{11}$$

We can know from (11) that

$$T_x = \frac{\ln V_0 - \ln\left[V_0 - m(I + \overline{u})\right]}{m}. \tag{12}$$

4. The Option Game Models between Investors with Government Guarantees and Government

Government guarantees aim to attract investors (domestic and foreign consortiums, companies, and individuals) to invest in the construction of infrastructure projects and are government commitments to share the various risks such as investment return, franchising operation, and environmental conditions in the investment process. Generally, such guarantees have a variety of forms, such as the minimum income guarantee of project, franchising operation price guarantee, legal consistency guarantees, and risk sharing.

4.1. The Impact of Minimum Government Income Guarantee on the Concession Period. Assuming that government provides investors with a minimum income guarantee $\underline{V}$, that is to say, the guaranteed value is 0 when the income is not less than $\underline{V}$ and the guaranteed value is $\underline{V} - V_{(0,T_x)}$ when the income is less than $\underline{V}$, then the guaranteed value $G = \max[0, \underline{V} - V_{(0,T_x)}]$ is equivalent to a European put option with $V_{(0,T_x)}$ as the underlying asset, $\underline{V}$ as exercise price, and T_x as expiry date. According to real options approach, we can obtain the expected value of government guarantees with B-S option equation:

$$\begin{aligned} E(G) = &\underline{V} e^{-rT_x} N(-d_2) - \frac{V_0\left(1 - e^{-mT_x}\right)}{m} N(-d_1) \\ &\geq 0. \end{aligned} \tag{13}$$

Among them, r is risk-free income rate and $N(\cdot)$ is the accumulation normal distribution function.

Similarly, we can obtain the expected income function of the investors with government guarantee within the period of the concession:

$$\begin{aligned} &E\left[V(0, T_x) - I + E(G)\right] \\ &\quad = \frac{V_0\left(1 - e^{-mT_x}\right)}{m} - I + E(G). \end{aligned} \tag{14}$$

Then, the expected income function from the date when the government took over the operation right to the end of planned use life is

$$E\left[V(T_x, T) - E(G)\right] = \frac{V_0\left(e^{-mT_x} - e^{-mT}\right)}{m} - E(G). \tag{15}$$

Substituting (15) in (3) and (14) in (4), we can obtain the government's decision-making model and its constraints under the premise of a minimum income guarantee:

$$\begin{aligned} &\max_{T_x} \quad \frac{V_0\left(e^{-mT_x} - e^{-mT}\right)}{m} - E(G) \\ &\text{s.t.} \quad \frac{V_0\left(1 - e^{-mT_x}\right)}{m} + E(G) - I \geq \overline{u}. \end{aligned} \tag{16}$$

Lagrange multiplier method can be used to solve the maximization of formula (16). Let λ be a Lagrange multiplier, (16) combine together to form a Lagrangian function, we can know

$$L(T_x,\lambda)=\frac{V_0\left(e^{-mT_x}-e^{-mT}\right)}{m}-E(G)+\lambda\left[\frac{V_0\left(1-e^{-mT_x}\right)}{m}+E(G)-I-\overline{u}\right]. \tag{17}$$

Substituting formula (13) into (17), we can obtain

$$\begin{aligned}L(T_x,\lambda)=&\frac{V_0\left(e^{-mT_x}-e^{-mT}\right)}{m}-\underline{V}e^{-rT_x}N(-d_2)\\&+E\left[V_{(0,T_x)}\right]N(-d_1)+\lambda\left[\frac{V_0\left(1-e^{-mT_x}\right)}{m}\right.\\&\left.+\underline{V}e^{-rT_x}N(-d_2)-E\left[V_{(0,T_x)}\right]N(-d_1)-I-\overline{u}\right].\end{aligned} \tag{18}$$

Seeking first-order partial derivatives of T_x and λ in formula (18), respectively, we can know

$$\begin{aligned}\frac{\partial L}{\partial T_x}=&\left[V_0e^{-mT_x}-r\underline{V}e^{-rT_x}N(-d_2)\right.\\&\left.-V_0e^{-mT_x}N(-d_1)\right](\lambda-1)=0\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda}=&\frac{V_0\left(1-e^{-mT_x}\right)}{m}N(d_1)+\underline{V}e^{-rT_x}N(-d_2)-I\\&-\overline{u}=0.\end{aligned} \tag{20}$$

And we will work out the numerical solution to T_x and λ in (19) and (20).

4.2. The Impact on Concession Period of Government Participation in Risk Sharing. Guarantee provided by government participating in risk sharing means that the government allocates some funds into the construction of infrastructure projects. Suppose government's investment amount is G, accounting for θ of the total investment I, that is, $G=\theta I$ $(0<\theta<1)$, income sharing ratio is also θ, that is, the government guaranteed value is 0 when θ is zero, and the guaranteed value is $\theta(I-V_{(0,T_x)})$ when $0<\theta<1$, and the guaranteed value $G=\mathrm{Max}[0,\theta(I-V_{(0,T_x)})]$ is therefore equivalent to a European put option with $V_{(0,T_x)}$ as underlying asset, θI as exercise price, and T_x as expiry date. According to real options approach, we can obtain the expected value of government guarantees with B-S option equation:

$$\begin{aligned}E(G)=&\theta Ie^{-rT_x}N(-d_2)-\frac{V_0\left(1-e^{-mT_x}\right)}{m}N(-d_1)\\&\geq 0.\end{aligned} \tag{21}$$

Among them, r is risk-free income rate and $N(\cdot)$ is the accumulation normal distribution function.

Similarly, we can obtain the expected income function of the investors with government within the period of the concession:

$$\begin{aligned}&E\left[(1-\theta)V(0,T_x)-(1-\theta)I+E(G)\right]\\&=(1-\theta)\frac{V_0\left(1-e^{-mT_x}\right)}{m}-(1-\theta)I+E(G).\end{aligned} \tag{22}$$

Then, the expected income function of the government within the planned use life is

$$\begin{aligned}&E\left[V(T_x,T)+\theta V(0,T_x)-\theta I-E(G)\right]\\&=\frac{V_0\left(e^{-mT_x}-e^{-mT}\right)}{m}+\theta\frac{V_0\left(1-e^{-mT_x}\right)}{m}-\theta I\\&\quad-E(G).\end{aligned} \tag{23}$$

In the negotiation process between government and investors, the government will determine the concession period T_x in favour of the interests of the public and T_x will meet the following decision model:

$$\max_{T_x}\quad E\left[V(T_x,T)+\theta V(0,T_x)-\theta I-E(G)\right] \tag{24}$$

$$\text{s.t.}\quad E\left[(1-\theta)V(0,T_x)-(1-\theta)I+E(G)\right]\geq\overline{u}. \tag{25}$$

Lagrange multiplier method can be used to solve the maximization of formula (23) and (24). Let λ be a Lagrange multiplier, and using (23) and (24) combined together to form a Lagrangian function, we can know

$$\begin{aligned}L(T_x,\lambda)=&\frac{V_0\left(e^{-mT_x}-e^{-mT}\right)}{m}+\theta\frac{V_0\left(1-e^{-mT_x}\right)}{m}\\&-\theta I-E(G)+\lambda\left[(1-\theta)\frac{V_0\left(1-e^{-mT_x}\right)}{m}\right.\\&\left.-(1-\theta)I+E(G)-\overline{u}\right].\end{aligned} \tag{26}$$

Substituting (21) into (22),

$$\begin{aligned}L(T_x,\lambda)=&\frac{V_0\left(e^{-mT_x}-e^{-mT}\right)}{m}+\theta\frac{V_0\left(1-e^{-mT_x}\right)}{m}\\&-\theta I-\theta Ie^{-rT_x}N(-d_2)+\frac{V_0\left(1-e^{-mT_x}\right)}{m}N(-d_1)\\&+\lambda\left[(1-\theta)\frac{V_0\left(1-e^{-mT_x}\right)}{m}-(1-\theta)I\right.\\&\left.+\theta Ie^{-rT_x}N(-d_2)-\frac{V_0\left(1-e^{-mT_x}\right)}{m}N(-d_1)-\overline{u}\right].\end{aligned} \tag{27}$$

Table 2: The relationship between T_x, $E(G)$, and σ.

T_x	$\sigma = 0$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$	$\sigma = 0.6$	$\sigma = 0.7$	$\sigma = 0.8$	$\sigma = 0.9$	$\sigma = 1$
$E(G) = 0$	12	—	—	—	—	—	—	—	—	—	—
$E(G) = 1 * 10^8$	12	17	32	58	76	84	87	87	88	88	88
$E(G) = 2 * 10^8$	12	15	25	45	61	69	72	73	74	74	74
$E(G) = 3 * 10^8$	12	14	22	37	52	60	64	65	66	66	66
$E(G) = 4 * 10^8$	11	13	19	32	46	54	58	59	60	60	60
$E(G) = 5 * 10^8$	11	13	18	29	41	49	53	55	55	55	55
$E(G) = 6 * 10^8$	11	12	17	26	37	45	49	51	51	52	52
$E(G) = 7 * 10^8$	11	12	15	23	34	42	46	48	48	49	49
$E(G) = 8 * 10^8$	11	11	15	21	31	39	43	45	46	46	46
$E(G) = 9 * 10^8$	11	11	14	20	29	36	40	42	43	43	44
$E(G) = 10 * 10^8$	10	11	13	18	26	34	38	40	41	41	42

Seeking first-order partial derivatives of T_x and λ in formula (27), respectively, we can know

$$\frac{\partial L}{\partial T_x} = (\lambda - 1)\left[(1-\theta) V_0 e^{-mT_x} - r\theta I e^{-rT_x} N(-d_2) + V_0 e^{-mT_x} N(-d_1)\right] = 0 \tag{28}$$

$$\frac{\partial L}{\partial \lambda} = (1-\theta)\frac{V_0\left(1-e^{-mT_x}\right)}{m} - (1-\theta) I + \theta I e^{-rT_x} N(-d_2) - \frac{V_0\left(1-e^{-mT_x}\right)}{m} N(-d_1) - \overline{u} = 0. \tag{29}$$

And we will work out the numerical solution to T_x and λ in (28) and (29).

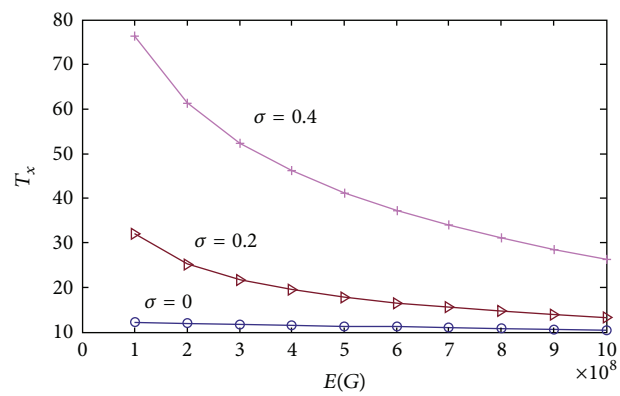


Figure 1: $\sigma = 0, 0.2, 0.4$, the relationship between $E(G)$ and T_x.

5. Numerical Examples

We will make further analysis with numerical examples to get more intuitive and clear result. Comparison results of multiple sets of data simulation show that the analysis results of this paper are not sensitive to the selection of parameter values, and the specific size of parameter value selected in the models within practical scope does not affect analysis conclusions of this paper. Therefore, this paper makes simulation analysis by selecting a set of data which best suits the actual economic situation as a basic data. Assuming that the initial investment cost of an infrastructure project I is 6 billion, and V_0, the initial income, is 400 million, then the convenience income of the project is 2% and the risk-free interest rate of financial asset pricing is computed from the average interest rates of interbank market, interbank bond repo market, and exchange repo market and so is the risk-free income rate, and we take the comparatively long retaking period of the project into account and assume that r is 5% [29], $\underline{V}$, the minimum income guarantee, is eight billion, and $\overline{u}$, the opportunity utility, is 3 billion.

Firstly, substituting the relevant data into formula (12), we can obtain concession period of projects without government guarantees $T_x = 29.8919$ years. That is to say, the initial investment cost of the investors is 6 billion, the initial income is 400 million, the convenience income is 0.02, and the opportunity utility is 3 billion. The government's optimal decision is to transfer the operation right of the infrastructure projects to investors with the concession period of 30 years which conforms to the concession operation period of most of the public infrastructure projects with concession operation right (such as highways, urban sewage treatment plants, and power plants) in China.

Secondly, using relevant data, we conduct numerical simulation of formula (19) with MATLAB; the result is shown in Figure 1 and Table 2. Figure 1 shows the relationship between the expected value of minimum government income guarantee $E(G)$ and the concession period (T_x) when the curves from the bottom to the top represent the income volatility 0, 0.2, and 0.4, respectively. We can get from Figure 1 and Table 2, under certain income volatility, that the concession period gradually shortens with the increase of expected values of the minimum income guarantee; that is to say, the increase of the government guarantee will shorten the optimal concession period, and while expected values of the minimum income guarantee are fixed, the concession period will lengthen with the increase of income volatility; that is, the uncertainties will lengthen the optimal concession period. Simultaneously, Table 2 shows that when the expected

Table 3: The relationship between T_x, θ, and σ.

T_x	$\sigma = 0$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$	$\sigma = 0.6$	$\sigma = 0.7$	$\sigma = 0.8$	$\sigma = 0.9$	$\sigma = 1$
$\theta = 0$	30	30	30	31	31	31	30	30	30	30	30
$\theta = 0.1$	31	31	32	32	32	32	31	31	31	31	31
$\theta = 0.2$	33	33	34	34	34	33	33	32	32	32	32
$\theta = 0.3$	36	36	36	36	36	35	35	34	34	34	34
$\theta = 0.4$	40	40	40	40	39	38	38	37	37	37	37
$\theta = 0.5$	46	46	46	46	45	43	42	42	42	41	41
$\theta = 0.6$	56	56	56	56	54	53	52	52	51	51	51
$\theta = 0.7$	80	80	80	80	79	78	77	77	77	77	77
$\theta = 0.8$	1	1	1	1	1	1	1	1	1	1	1
$\theta = 0.9$	3	3	3	3	3	3	3	3	4	5	6
$\theta = 1.0$	5	5	5	5	5	5	6	7	9	11	12

Table 4: The relationship between $E(G)$, θ, and σ.

$E(G)$ (10^8)	$\sigma = 0.0$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$	$\sigma = 0.6$	$\sigma = 0.7$	$\sigma = 0.8$	$\sigma = 0.9$	$\sigma = 1.0$
$\theta = 0$	—	—	—	—	—	—	—	—	—	—	—
$\theta = 0.1$	—	—	—	—	—	—	0.02	0.53	0.88	1.08	1.20
$\theta = 0.2$	—	—	—	—	—	0.32	1.07	1.64	2.01	2.23	2.34
$\theta = 0.3$	—	—	—	—	0.23	1.13	1.95	2.55	2.94	3.15	3.26
$\theta = 0.4$	—	—	—	—	0.75	1.73	2.57	3.17	3.53	3.73	3.82
$\theta = 0.5$	—	—	—	0.21	1.02	1.98	2.76	3.29	3.58	3.72	3.78
$\theta = 0.6$	—	—	—	0.26	0.94	1.68	2.23	2.56	2.71	2.77	2.79
$\theta = 0.7$	—	—	—	0.12	0.41	0.67	0.82	0.88	0.90	0.90	0.90
$\theta = 0.8$	41.11	41.11	41.11	41.11	41.11	41.11	41.11	41.11	41.12	41.13	41.10
$\theta = 0.9$	34.86	34.86	34.86	34.81	34.81	34.82	34.74	34.68	34.51	34.25	33.84
$\theta = 1.0$	30.00	30.00	29.99	30.01	30.00	30.00	29.99	29.99	30.01	30.01	30.00

value of minimum government income guarantee is relatively low, $E(G)$ varies within 0–100 million Yuan. Only when volatility rate is relatively low, government will attract the private investment by prolonging concession period. On the contrary, if the volatility rate is relatively high, the extension of the concession period is difficult to have attraction to private investors.

Thirdly, using relevant data, we conduct numerical simulation of formula (29) with MATLAB; the results are shown in Figure 2 and Table 3. When θ varies within the range of 0–0.7, the concession period gradually extends with the increase of government investment ratio under fixed income volatility. But when θ further increases, the concession period rapidly declines. That shows that the optimal concession period, with the government investment ratio within certain scope ($\theta = 0$–0.7), will extend with the increase of government investment ration and income and the decrease of its guarantee value under fixed income volatility. But optimal concession period will be shortened by the government accordingly in case of extreme highness of the government investment ($\theta > 0.7$) ratio, that is, regarding the guarantee value of government investment as put option, for the high guarantee value, which is verified in Table 4. And concession period will accordingly shorten with fixed government investment ratio and increased income volatility. That fact shows that the government will accordingly shorten the concession period in case of the unchanged government guarantee with the increased income volatility and risks.

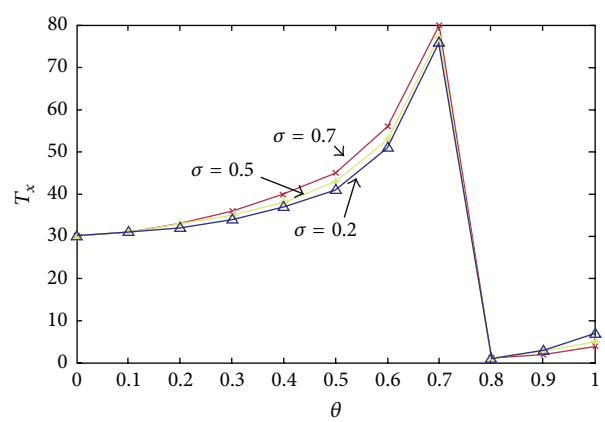


Figure 2: $\sigma = 0.2, 0.5, 0.7$, the relationship between T_x and θ.

6. Conclusions

This paper, taking the elements such as minimum government income guarantee and government investment and sharing risks into consideration, established concession

period decision models for public infrastructure with option game theory, studied the influence of minimum government income guarantee and government investment on concession period, and verified the fact that the increase of minimum government income guarantee value will shorten the concession period, while the increase of income volatility, that is, the uncertainty, will lengthen the concession period with numerical simulation. In terms of government investment, optimal concession period will lengthen to some extent with the increase of government investment ratio and the income and the decrease of its guarantee value. Yet, optimal concession period will shorten in case of extreme highness of the government investment ratio due to its high guarantee value. And the government will accordingly shorten the concession period in case of the unchanged government investment ratio with the increased income volatility and risks.

The above conclusions show that the government, in accordance with the specific circumstances of the public infrastructure projects and various guarantee modes [33], will implement flexible concession period system which will be written in the Concession Period Agreement: the concession period ends after the investors regain their investment and get the stipulated income and will not be restricted to a fixed period, hence reaching the goal to encourage private investors to participate in the construction of public infrastructure and balance the interest between investors and the public as well.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by the National Social Science Foundation of China (Grant no. 10CGL063), National Natural Science Foundation of China (no. 71303262), Program of Research Center of Modern Enterprise Management of Hunan Province (Grant no. 11jqkf07), and the Collaborative Innovation Center of Services and Urbanization of Hunan and Key Project of the Education Department of Hunan Province (no. 14A008).

References

[1] Z. Dai and F. Wen, "Robust CVaR-based portfolio optimization under a genal affine data perturbation uncertainty set," *Journal of Computational Analysis & Applications*, vol. 16, no. 1, pp. 93–103, 2014.

[2] J. Liu, M. Tao, C. Ma, and F. Wen, "Utility indifference pricing of convertible bonds," *International Journal of Information Technology & Decision Making*, vol. 13, no. 2, pp. 429–444, 2013.

[3] Q. Li and L. Shen, "Concession period decision model for infrastructure BOT projects," *Journal of Management Engineering*, vol. 14, no. 1, pp. 43–46, 2000 (Chinese).

[4] S. Ye and R. L. K. Tiong, "The effect of concession period design on completion risk management of BOT projects," *Construction Management and Economics*, vol. 21, no. 5, pp. 471–482, 2003.

[5] L. Y. Shen and Y. Z. Wu, "Risk concession model for build/operate/transfer contract projects," *Journal of Construction Engineering and Management*, vol. 131, no. 2, pp. 211–220, 2005.

[6] S. Thomas Ng, J. Xie, Y. K. Cheung, and M. Jefferies, "A simulation model for optimizing the concession period of public-private partnerships schemes," *International Journal of Project Management*, vol. 25, no. 8, pp. 791–798, 2007.

[7] J. Song, D. Song, and C. Tan, "Decision model for concession period of power generation with wastes BOT projects," *China Management Science*, no. 10, pp. 86–93, 2013 (Chinese).

[8] C. Huang, H. Kuang, X. Chen, and F. Wen, "An LMI approach for dynamics of switched cellular neural networks with mixed delays," *Abstract and Applied Analysis*, vol. 2013, Article ID 870486, 8 pages, 2013.

[9] X. Qin, "Calculation models for concession period of BOT project based on CAPM," *Journal of Management Engineering*, no. 2, pp. 60–63, 2005 (Chinese).

[10] F. Wen and X. Yang, "Skewness of return distribution and coefficient of risk premium," *Journal of Systems Science and Complexity*, vol. 22, no. 3, pp. 360–371, 2009.

[11] F. Wen, Z. He, and X. Chen, "Investors' risk preference characteristics and conditional skewness," *Mathematical Problems in Engineering*, vol. 2014, Article ID 814965, 14 pages, 2014.

[12] F. Wen, Z. He, X. Gong, and A. Liu, "Investors' risk preference characteristics based on different reference point," *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 158386, 9 pages, 2014.

[13] F. Wen, Z. He, Z. Dai, and X. Yang, "Characteristics of investors' risk preference for stock markets," *Economic Computation and Economic Cybernetics Studies and Research*, vol. 48, no. 3, pp. 235–254, 2014.

[14] F. Wen, X. Gong, Y. Chao, and X. Chen, "The effects of prior outcomes on risky choice: evidence from the stock market," *Mathematical Problems in Engineering*, vol. 2014, Article ID 272518, 8 pages, 2014.

[15] F. Medda, "A game theory approach for the allocation of risks in transport public private partnerships," *International Journal of Project Management*, vol. 25, no. 3, pp. 213–218, 2007.

[16] H. W. Yang, J. Zhou, and J. M. He, "The concession period decision model of the traffic BOT based on game theory," *Journal of Management Engineering*, vol. 17, no. 3, pp. 93–95, 2003 (Chinese).

[17] L. F. Gao, G. J. Zhang, and Y. Du, "BOT projects concession period determination with 'cake-sharing' model in dynamic game," *Business Research*, no. 2, pp. 24–26, 2006 (Chinese).

[18] L. Y. Shen, H. J. Bao, Y. Z. Wu, and W. S. Lu, "Using bargaining-game theory for negotiating concession period for BOT-type contract," *Journal of Construction Engineering and Management*, vol. 133, no. 5, pp. 385–392, 2007.

[19] H. J. Bao, "Dynamic game for concession period decision of infrastructure BOT projects," *Journal of Management Engineering*, vol. 23, no. 4, pp. 139–147, 2009 (Chinese).

[20] X. L. Wu, J. Zhou, and W. Hong, "Decision model for concession period of infrastructure BOT within effective operation period," *Journal of System Engineering*, vol. 26, no. 3, pp. 373–378, 2011 (Chinese).

[21] Z. A. Zhang and P. L. Durango-Cohen, "A strategic model of public-private partnerships in transportation: effect of taxes and cost structure on investment viability," *Research in Transportation Economics*, vol. 36, no. 1, pp. 9–18, 2012.

[22] C. Huang, C. Peng, X. Chen, and F. Wen, "Dynamics analysis of a class of delayed economic model," *Abstract and Applied Analysis*, vol. 2013, Article ID 962738, 12 pages, 2013.

[23] F. Gao, J. Guo, and Q. B. Zhao, "Government guarantee value on infrastructure projects based on barrier option," *Prediction*, no. 2, pp. 76–80, 2007 (Chinese).

[24] R. Takashima, K. Yagi, and H. Takamori, "Government guarantees and risk sharing in public-private partnerships," *Review of Financial Economics*, vol. 19, no. 2, pp. 78–83, 2010.

[25] C. O. Cruz and R. C. Marques, "Flexible contracts to cope with uncertainty in public-private partnerships," *International Journal of Project Management*, vol. 31, no. 3, pp. 473–483, 2013.

[26] F. Smets, "Exporting versus FDI: the effect of uncertainty, irreversibilities and strategic interactions," Working Paper, Yale University, New Haven, Conn, USA, 1991.

[27] R. E. Ottoo, "Valuation of internal growth opportunities: the case of a biotechnology company," *Quarterly Review of Economics and Finance*, vol. 38, no. 3, part 2, pp. 615–633, 1998.

[28] A. B. Alonso-Conde, C. Brown, and J. Rojo-Suarez, "Public private partnerships: incentives, risk transfer and real options," *Review of Financial Economics*, vol. 16, no. 4, pp. 335–349, 2007.

[29] M. L. Guo, Y. Yang, and W. X. Hu, "Decision model for concession period of infrastructure BOT projects phase II based on option game," *China Soft Science*, no. 6, pp. 81–85, 2007 (Chinese).

[30] L. Gong, J. Guo, and G. X. Zhang, "Negotiation game model for infrastructure BOT project concession period," *Statistics and Decision-Making*, no. 4, pp. 153–155, 2008 (Chinese).

[31] Shaked. A and J. Sutton, "Natural oligopolies and international trade," in *Monopolistic Competition and International Trade*, H. Kierszkowski, Ed., Oxford University Press, Oxford, UK, 1948.

[32] Q.-F. Cui, Z.-G. Wang, X.-H. Chen, and F.-H. Wen, "Sufficient conditions for non-Bazilevič functions," *Abstract and Applied Analysis*, vol. 2013, Article ID 154912, 4 pages, 2013.

[33] J. B. Song, W. Dang, and Y. Sun, "Flexible concession period decision-making models for infrastructure BOT project-multiple cases study on typical projects abroad," *Civil Engineering Journal*, no. 4, pp. 142–150, 2013 (Chinese).

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 409286, 10 pages
http://dx.doi.org/10.1155/2015/409286

Research Article

Stability Analysis of R&D Cooperation in a Supply Chain

Luyun Xu,[1] Dong Liang,[2] Zhenjie Duan,[2] and Xu Xiao[2]

[1] *College of Business Administration, Hunan University, Changsha 410082, China*
[2] *College of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China*

Correspondence should be addressed to Luyun Xu; xuluyun@hnu.edu.cn

Received 18 May 2015; Accepted 13 August 2015

Academic Editor: Leonid Shaikhet

R&D outsourcing becomes the often-adopted strategy for firms to innovate. However, R&D cooperation often ends up with failure because of its inherent quality of instability. One of the main reasons for cooperation failure is the opportunistic behavior. As the R&D contract between firms is inherently incomplete, opportunistic behavior always cannot be avoided in the collaborative process. R&D cooperation has been divided into horizontal and vertical types. This paper utilizes game theory to study opportunistic behavior in the vertical R&D cooperation and analyzes the equilibrium of the cooperation. Based on the equilibrium and numerical results, it is found that the vertical R&D cooperation is inherently unstable, and the downstream firm is more likely to break the agreement. The level of knowledge spillovers and the cost of R&D efforts have different effects on firms' payoffs. When the level of knowledge spillover is low or the cost of R&D efforts is high, mechanisms such as punishment for opportunism may be more effective to guarantee the stability of cooperation.

1. Introduction

In the knowledge economy era, the competition of technical market is increasingly fierce, and firms are forced to accelerate the process of technical innovation. However, it is more difficult for firms to accomplish knowledge creation and technological innovation in isolation [1]. R&D outsourcing becomes the often-adopted strategy for firms to innovate. R&D cooperation becomes a common phenomenon, which helps firms in sharing risk and cost, accessing knowledge and technological know-how network, and internalizing the externalities created by knowledge spillovers [2–4]. Despite these advantages, the inherent quality of instability of R&D cooperation often may not be avoided, and R&D cooperation often ends with failure [5–8]. A main reason for cooperation failure is the opportunistic behavior by one party or the other [9, 10]. Opportunism appears due to the cooperative and competitive relationship of the two collaborative firms. Opportunism is defined as selfish behavior which means seeking a firm's self-interest with deceit at the expense of its partners [11–13]. As the R&D contract between firms is inherently incomplete, firms in the cooperation are often not accessible to the detailed information about what the partners are expected to do, and it is impossible for a third party to keep watch on R&D efforts [14]. Therefore, opportunistic behavior always cannot be avoided in the collaborative process.

Based on the types of collaborative partnership, R&D cooperation has been divided into horizontal and vertical R&D cooperation. Many of research works have been done about these two types of R&D cooperation [15–20]. Although firms may arrange their R&D inputs to realize the maximization of the total profit of the two firms, opportunism may still prevail in these two types of R&D cooperation. Such noncooperative behavior may prevent a firm from losing its competitive knowledge. However, it would lead to the instability of the cooperation. Kesteloot and Veugelers study the stability of horizontal R&D cooperation in a repeated game and emphasize the important role of spillovers [21]. Cabon-Dhersin and Ramani use a noncooperative game to discuss the effect of trust on horizontal R&D cooperation, and they find that when opportunism cannot be avoided, the nature of firms, the configurations of trust, and the level of spillovers decide whether the horizontal R&D cooperation is successful or not [14]. Cassiman and Veugelers find that, in vertical cooperation, the effectiveness of strategic protection

is important to induce cooperation [22]. Lhuillery and Pfister find that vertical R&D cooperation also faces a higher risk of failures [23].

Most of the existing literatures using game theoretical approach have studied the stability of horizontal R&D cooperation. Our paper uses a game theoretical approach to analyze the stability of the vertical R&D cooperation. We focus on opportunistic behavior in the vertical R&D cooperation. The results of this paper indicate that the vertical R&D cooperation is unstable, and the downstream firm is more likely to break the agreement. When building a partnership, firms need to consider the social statue and reputation of its partner and mutual trust between the two firms. This paper also identifies the roles of knowledge spillover and the cost of R&D effort in the stability of vertical R&D cooperation. These two factors influence the firms' payoffs in different situations. And they play different roles in the decision process. It is found that when the level of knowledge spillover is low or the cost of R&D efforts is high, mechanisms such as punishment for opportunism may be more effective.

The rest of this paper is given as follows. Section 2 introduces the model of our paper. Section 3 gives the equilibrium analysis of the game and analyzes the effects of spillover level and R&D effort cost on stability of the game. Section 4 presents numerical illustration and Section 5 gives the conclusion of this paper.

2. The Model

In the part, we present the model in two subsections. Our game model is described in the first subsection. R&D expenditures and payoffs in different situations are solved in the second subsection.

2.1. Description of Game Model. Studies show that single source brings long-term benefits if used appropriately [24] and one or two suppliers are usually [25, 26] enough for a manufacturer. Following Ge et al. [20], we consider cooperative R&D in a simple supply chain with a final-good manufacturer (denoted as Firm A) and an input supplier (denoted as Firm B) in our model.

We assume that Firm A decides its production quantity based on the market demand and then submits its order to Firm B. Firm B sells inputs to Firm A in the market. The two firms in the supply chain establish a vertical strategic R&D collaboration link. The reduction of marginal cost in our study is an R&D production function following d'Aspremont and Jacquemin [5]. In the vertical R&D cooperation, firms coordinate their R&D inputs and then allocate R&D resources to reduce the production costs. The reduction of marginal production cost comes from a firm's own research. Meanwhile, R&D efforts of its cooperator also help reducing the firm's cost due to positive spillovers. Let $\beta \in [0, 1]$ be the parameter reflecting the spillover level between firms. The spillover levels of the two firms in the vertical R&D cooperation are assumed to be symmetric. We also assume that the fixed costs of firms are set to be zero. Let $\bar{c}_i$ and $\bar{c}_i > 0$ denote the original marginal cost and let e_i denote the R&D efforts of firm i. Therefore, the marginal cost c_i after R&D investment of firm i is written as follows:

$$c_i = \bar{c}_i - e_i - \beta e_j, \quad i, j = A, B, \ \ j \neq i. \tag{1}$$

We assume that R&D investment is costly. Given a level $e_i \in [0, \bar{c}_i)$ of R&D efforts, the cost of efforts is given by $(1/2)\gamma {e_i}^2$, and γ is a technological parameter and satisfies $\gamma > 0$, indicating diminishing returns to R&D.

In the production market, let us suppose that Firm B produces the inputs and sells them to Firm A at price P_s, and Firm A uses the inputs to produce final goods. Let us suppose that the inverse demand function in the market is given by $P = a - q$, where P represents the price of a final product and q represents the total production quantity of final goods produced by Firm A. At the same time, the order quantity of Firm A is equal to production quantity of final goods. Given an R&D profile (e_A, e_B) of the two firms, the quantity q of final goods and the input price P_S of Firm B and the net profits of Firm A and Firm B, (π_A, π_B), can be written as follows:

$$\pi_A = \left(a - q - \bar{c}_A + e_A + \beta e_B\right) q - P_S q - \frac{1}{2}\gamma {e_A}^2, \tag{2}$$

$$\pi_B = \left(P_S - \bar{c}_B + e_B + \beta e_A\right) q - \frac{1}{2}\gamma {e_B}^2. \tag{3}$$

As the game is dynamic, we can use the backward introduction method. We start with the quantity of final goods. Firm A chooses its output level independently to realize the maximization of its own profit. For any configuration of R&D efforts (e_A, e_B) invested by Firm A and Firm B, the optimal condition for the profit maximization of Firm A is given by

$$\frac{\partial \pi_A}{\partial q} = a - 2q - \bar{c}_A + e_A + \beta e_B - P_S = 0. \tag{4}$$

Then the production quantity of the final product can be got by

$$q = \frac{1}{2}\left(a - \bar{c}_A + e_A + \beta e_B - P_S\right). \tag{5}$$

Substituting (5) into (3), the net profit function of Firm B can be rewritten as

$$\begin{aligned}\pi_B = {} & \frac{1}{2}\left(P_S - \bar{c}_B + e_B + \beta e_A\right)\left(a - \bar{c}_A + e_A + \beta e_B - P_S\right) \\ & - \frac{1}{2}\gamma {e_B}^2.\end{aligned} \tag{6}$$

By solving partial derivatives of (6) about P_S for profit maximization, the optimal input price can be got as follows:

$$P_S = \frac{1}{2}\left[a - \bar{c}_A + \bar{c}_B + \left(1 - \beta\right) e_A - \left(1 - \beta\right) e_B\right]. \tag{7}$$

Then the production quantity of the final product can be obtained as follows:

$$q = \frac{1}{4}\left[a - \bar{c}_A - \bar{c}_B + \left(1 + \beta\right) e_A + \left(1 + \beta\right) e_B\right]. \tag{8}$$

Therefore, the net profits of Firm A and Firm B are gained as

$$\pi_A = \frac{1}{16}\left[a - c_A - c_B + (1+\beta)(e_A + e_B)\right]^2 - \frac{1}{2}\gamma {e_A}^2,$$
$$\pi_B = \frac{1}{8}\left[a - c_A - c_B + (1+\beta)(e_A + e_B)\right]^2 - \frac{1}{2}\gamma {e_B}^2. \tag{9}$$

The total profit of the supply chain profit is

$$\pi_T = \pi_A + \pi_B = \frac{3}{16}\left[a - c_A - c_B + (1+\beta)(e_A + e_B)\right]^2 - \frac{1}{2}\gamma\left({e_A}^2 + {e_B}^2\right). \tag{10}$$

2.2. R&D Expenditures and Payoffs in Different Situations. After establishing the R&D cooperation, firms choose their levels of R&D efforts. Each firm has two choices. One is to invest on R&D efforts to maximize the total profit of the two firms, which is treated as reciprocal behavior. Another is to invest on R&D efforts to maximize a firm's own profit, which is regarded as opportunistic behavior. After that, each firm invests in its R&D efforts, and this is not observable. With the above discussion, we can now solve for the R&D efforts and the corresponding net profits in four different situations of vertical R&D cooperation.

First, Firms A and B both choose reciprocal behavior, and we will define this situation as situation RR. In this situation, each firm decides its R&D efforts by the maximization of the total profit. Then the optimal condition for situation RR is obtained as follows:

$$\frac{\partial \pi_T}{\partial e_A} = \frac{3}{8}(1+\beta)\left[\theta + (1+\beta)(e_A + e_B)\right] - \gamma e_A = 0,$$
$$\frac{\partial \pi_T}{\partial e_B} = \frac{3}{8}(1+\beta)\left[\theta + (1+\beta)(e_A + e_B)\right] - \gamma e_B = 0, \tag{11}$$

where $\theta = a - c_A - c_B$.

By solving (11), we can, respectively, define the R&D efforts and net profits of Firm A and Firm B in situation RR as follows:

$$e_A = e_B = \frac{3\theta(1+\beta)}{8\gamma - 6(1+\beta)^2},$$
$$\pi_A = \frac{\gamma\theta^2\left[4\gamma - 4.5(1+\beta)^2\right]}{\left[8\gamma - 6(1+\beta)^2\right]^2},$$
$$\pi_B = \frac{\gamma\theta^2\left[8\gamma - 4.5(1+\beta)^2\right]}{\left[8\gamma - 6(1+\beta)^2\right]^2}. \tag{12}$$

Second, Firms A and B both choose opportunistic behavior, defined as situation OO. In this situation, each firm decides its R&D efforts by the maximization of its own profit. Then the optimal condition for situation OO is expressed by

$$\frac{\partial \pi_A}{\partial e_A} = \frac{1}{8}(1+\beta)\left[\theta + (1+\beta)(e_A + e_B)\right] - \gamma e_A = 0,$$
$$\frac{\partial \pi_B}{\partial e_B} = \frac{1}{4}(1+\beta)\left[\theta + (1+\beta)(e_A + e_B)\right] - \gamma e_B = 0. \tag{13}$$

And the R&D efforts and net profits of Firm A and Firm B in situation OO can be separately defined as follows:

$$e_A = \frac{\theta(1+\beta)}{8\gamma - 3(1+\beta)^2},$$
$$e_B = \frac{2\theta(1+\beta)}{8\gamma - 3(1+\beta)^2},$$
$$\pi_A = \frac{\gamma\theta^2\left[4\gamma - 0.5(1+\beta)^2\right]}{\left[8\gamma - 3(1+\beta)^2\right]^2},$$
$$\pi_B = \frac{\gamma\theta^2\left[8\gamma - 2(1+\beta)^2\right]}{\left[8\gamma - 3(1+\beta)^2\right]^2}. \tag{14}$$

Third, we will define this situation as situation RO, where Firm A chooses reciprocal behavior and Firm B chooses opportunistic behavior. At this time, Firm A decides its R&D efforts for the best interest of the cooperation, while Firm B chooses to cheat. Then the optimal condition for situation RO is written by

$$\frac{\partial \pi_T}{\partial e_A} = \frac{3}{8}(1+\beta)\left[\theta + (1+\beta)(e_A + e_B)\right] - \gamma e_A = 0,$$
$$\frac{\partial \pi_B}{\partial e_B} = \frac{1}{4}(1+\beta)\left[\theta + (1+\beta)(e_A + e_B)\right] - \gamma e_B = 0. \tag{15}$$

And the R&D efforts and net profits of Firm A and Firm B in situation RO are severally defined as follows:

$$e_A = \frac{3\theta(1+\beta)}{8\gamma - 5(1+\beta)^2},$$
$$e_B = \frac{2\theta(1+\beta)}{8\gamma - 5(1+\beta)^2},$$
$$\pi_A = \frac{\gamma\theta^2\left[4\gamma - 4.5(1+\beta)^2\right]}{\left[8\gamma - 5(1+\beta)^2\right]^2},$$
$$\pi_B = \frac{\gamma\theta^2\left[8\gamma - 2(1+\beta)^2\right]}{\left[8\gamma - 5(1+\beta)^2\right]^2}. \tag{16}$$

Table 1: R&D efforts and payoffs of Firms A and B in different situations.

Situation	RR	OO	RO	OR
e_A	$\frac{3\theta(1+\beta)}{8\gamma-6(1+\beta)^2}$	$\frac{\theta(1+\beta)}{8\gamma-3(1+\beta)^2}$	$\frac{3\theta(1+\beta)}{8\gamma-5(1+\beta)^2}$	$\frac{\theta(1+\beta)}{8\gamma-4(1+\beta)^2}$
e_B	$\frac{3\theta(1+\beta)}{8\gamma-6(1+\beta)^2}$	$\frac{2\theta(1+\beta)}{8\gamma-3(1+\beta)^2}$	$\frac{2\theta(1+\beta)}{8\gamma-5(1+\beta)^2}$	$\frac{3\theta(1+\beta)}{8\gamma-4(1+\beta)^2}$
π_A	$\frac{\theta^2(4-4.5L)}{(8-6L)^2}$	$\frac{\theta^2(4-0.5L)}{(8-3L)^2}$	$\frac{\theta^2(4-4.5L)}{(8-5L)^2}$	$\frac{\theta^2(4-0.5L)}{(8-4L)^2}$
π_B	$\frac{\theta^2(8-4.5L)}{(8-6L)^2}$	$\frac{\theta^2(8-2L)}{(8-3L)^2}$	$\frac{\theta^2(8-2L)}{(8-5L)^2}$	$\frac{\theta^2(8-4.5L)}{(8-4L)^2}$

Similarly, when A chooses opportunistic behavior and Firm B chooses reciprocal behavior, we can define R&D efforts and net profits in situation OR as follows:

$$\begin{aligned} e_A &= \frac{\theta(1+\beta)}{8\gamma-4(1+\beta)^2}, \\ e_B &= \frac{3\theta(1+\beta)}{8\gamma-4(1+\beta)^2}, \\ \pi_A &= \frac{\gamma\theta^2\left[4\gamma-0.5(1+\beta)^2\right]}{\left[8\gamma-4(1+\beta)^2\right]^2}, \\ \pi_B &= \frac{\gamma\theta^2\left[8\gamma-4.5(1+\beta)^2\right]}{\left[8\gamma-4(1+\beta)^2\right]^2}. \end{aligned} \tag{17}$$

The details in four situations are summarized in Table 1, where $L = (1+\beta)^2/\gamma$, and L increases with the spillover level between cooperative firms but decreases with the cost of R&D efforts. We will assume that $0 < L < 8/9$ to make sure that the R&D investment and production quantity exist.

3. Equilibrium Analysis

In this part, we first compare the payoffs of Firm A and Firm B in different statuses and obtain the equilibrium of the game. Second, we, respectively, discuss the effects of spillover level and R&D cost on profits of four different situations.

3.1. Comparison of Payoffs in Different Status. According to the behavior decision-making and the corresponding profit, the payoff matrix of different R&D investment profiles is given in Table 2. Given the comparison of the profits between different situations, three lemmas have been derived in the following.

Table 2: The payoff matrix of different R&D investment profiles.

	Firm B	
	Reciprocal behavior	Opportunistic behavior
Firm A		
Reciprocal behavior	(π_A^{RR}, π_B^{RR})	(π_A^{RO}, π_B^{RO})
Opportunistic behavior	(π_A^{OR}, π_B^{OR})	(π_A^{OO}, π_B^{OO})

Lemma 1. *Comparison of payoffs of Firm A in four different situations is as follows:*

(1) *If Firm A is reciprocal while its partner is opportunistic, Firm A will get a lower profit compared with the profit yielded by the R&D cooperation with two reciprocal firms*

$$\pi_A^{RR} > \pi_A^{RO}. \tag{18}$$

(2) *If Firm A is opportunistic while its partner is reciprocal, Firm A will get a higher profit compared with the profit yielded by the R&D cooperation with two opportunistic firms*

$$\pi_A^{OR} > \pi_A^{OO}. \tag{19}$$

(3) *Firm A will get a higher profit from R&D cooperation with two opportunistic firms than the R&D cooperation with two reciprocal firms*

$$\pi_A^{OO} > \pi_A^{RR}. \tag{20}$$

The particulars of the derivations are presented in the appendix. From the results of Lemma 1, the profit comparison of Firm A in four different situations is given as follows:

$$\pi_A^{OR} > \pi_A^{OO} > \pi_A^{RR} > \pi_A^{RO}. \tag{21}$$

In the vertical R&D cooperation, for any value of the cost effort γ and the spillover level β, the downstream firm always benefits more from opportunistic behavior. Cheating is an optimal strategy for Firm A in cooperation game.

Lemma 2. *Comparison of payoffs of Firm B in four different situations is as follows:*

(1) *If Firm B is reciprocal while its partner is opportunistic, Firm B will get a lower profit compared with the profit*

yielded by the R&D cooperation with two reciprocal firms

$$\pi_B^{RR} > \pi_B^{OR}. \tag{22}$$

(2) *If Firm B is opportunistic while its partner is reciprocal, Firm B will get a higher profit compared with the profit yielded by the R&D cooperation with two opportunistic firms*

$$\pi_B^{RO} > \pi_B^{OO}. \tag{23}$$

(3) *If Firm B is reciprocal while its partner is opportunistic, Firm B will get a lower profit compared with the profit yielded by the R&D cooperation with two opportunistic firms*

$$\pi_B^{OO} > \pi_B^{OR}. \tag{24}$$

(4) *Firm B will get a higher profit from R&D cooperation with two reciprocal firms than the R&D cooperation with two opportunistic firms*

$$\pi_B^{RR} > \pi_B^{OO}. \tag{25}$$

(5) *When the inequality* $0 < L < (80 - \sqrt{1216})/81$ *is satisfied, Firm B will get a higher profit from the R&D cooperation with only Firm B cheating than the cooperation with two reciprocal firms. When the inequality* $(80-\sqrt{1216})/81 < L < 8/9$ *is satisfied, Firm B will get a lower profit from the R&D cooperation with only Firm B cheating than the cooperation with two reciprocal firms*

$$\begin{gathered} \text{When } 0 < L < \frac{80-\sqrt{1216}}{81}, \\ \pi_B^{RO} > \pi_B^{RR} \\ \text{When } \frac{80-\sqrt{1216}}{81} < L < \frac{8}{9}, \\ \pi_B^{RR} > \pi_B^{RO}. \end{gathered} \tag{26}$$

The particulars of the derivations are also presented in the appendix. From the results of Lemma 2, the profit comparison of Firm B in four different situations is given as follows:

$$\begin{gathered} \text{When } 0 < L < \frac{80-\sqrt{1216}}{81}, \\ \pi_B^{RO} > \pi_B^{RR} > \pi_B^{OO} > \pi_B^{OR} \\ \text{When } \frac{80-\sqrt{1216}}{81} < L < \frac{8}{9}, \\ \pi_B^{RR} > \pi_B^{RO} > \pi_B^{OO} > \pi_B^{OR}. \end{gathered} \tag{27}$$

In the vertical R&D cooperation, the optimal strategy for Firm B is influenced by the R&D effort cost γ and the spillover level β. Lower value of L indicates that the cost of R&D efforts may be high or the spillover level may be low. Upstream Firm B benefits more from opportunistic behavior. Higher value of L indicates that the cost of R&D efforts may be low or the spillover level may be high. At this time, reciprocal behavior of two firms yields the highest profit to Firm B.

Based on the above analysis, we can obtain the equilibrium of the game. At the equilibrium, the optimal strategies for both of the two firms are choosing opportunistic behavior. Therefore, the vertical R&D cooperation is inherently unstable.

3.2. Effects of Spillover and R&D Cost on Profits. As mentioned above, the value of L depends on the level of knowledge spillover and the cost of R&D effort. In this part, we mainly analyze their effects on the profits of Firm A and Firm B in different situations.

Proposition 3. *In the vertical R&D cooperation, the profits of Firm A gained from reciprocal behavior decrease when the level of spillover β increases and the cost of R&D efforts γ declines.*

Proof. Consider

$$\begin{aligned} \frac{\partial \pi_A^{RR}}{\partial L} &= \frac{\partial \theta^2 (4-4.5L)(8-6L)^{-2}}{\partial L} \\ &= -4.5\theta^2 (8-6L)^{-2} \\ &\quad + 12\theta^2 (4-4.5L)(8-6L)^{-3} \\ &= \frac{-4.5\theta^2 (8-6L) + 12\theta^2 (4-4.5L)}{(8-6L)^3} \\ &= \frac{\theta^2 (8-27L)}{(8-6L)^3} < 0. \end{aligned} \tag{28}$$

Similarly, it is easy to obtain $\partial \pi_A^{RO}/\partial L = \theta^2(4-22.5L)/(8-5L)^3 < 0$. The values of π_A^{RR} and π_A^{RO} decrease with the increase of the value of L. As $L = (1+\beta)^2/\gamma$, we can learn that when the level of spillover β increases and the cost of R&D efforts γ declines, the profits of Firm A gotten from reciprocal behavior decrease. □

Proposition 4. *In the vertical R&D cooperation, the profits of Firm A gained from opportunistic behavior increase when the level of spillover β rises and the cost of R&D efforts γ declines.*

Proof. Consider

$$\begin{aligned} \frac{\partial \pi_A^{OO}}{\partial L} &= \frac{\partial \theta^2 (4-0.5L)(8-3L)^{-2}}{\partial L} \\ &= -0.5\theta^2 (8-3L)^{-2} \\ &\quad + 6\theta^2 (4-0.5L)(8-3L)^{-3} \end{aligned}$$

Table 3: The profits of four situations in different levels of spillovers.

β	L	π_A^{RR}	π_A^{OO}	π_A^{RO}	π_A^{OR}	π_B^{RR}	π_B^{OO}	π_B^{RO}	π_B^{OR}
0.00	0.20	167.60	178.05	158.16	188.08	383.87	346.97	387.76	342.40
0.10	0.24	169.73	183.28	157.85	196.11	402.96	355.12	407.56	349.40
0.20	0.29	171.84	189.31	157.09	205.57	426.05	364.48	431.29	357.39
0.30	0.34	173.77	196.24	155.65	216.71	454.16	375.17	459.86	366.49
0.40	0.39	175.24	204.22	153.23	229.87	488.72	387.40	494.50	376.84
0.50	0.45	175.77	213.41	149.34	245.51	531.77	401.38	536.86	388.59
0.60	0.51	174.59	224.01	143.27	264.21	586.37	417.39	589.32	401.96
0.70	0.58	170.29	236.29	133.94	286.76	657.16	435.78	655.25	417.19
0.80	0.65	160.27	250.58	119.61	314.23	751.69	456.99	739.71	434.58
0.90	0.72	139.55	267.29	97.42	348.13	882.81	481.56	850.45	454.51
1.00	0.80	97.66	286.99	62.50	390.63	1074.22	510.20	1000.00	477.43

$$= \frac{-0.5\theta^2 (8-3L) + 6\theta^2 (4-0.5L)}{(8-3L)^3}$$
$$= \frac{\theta^2 (20-1.5L)}{(8-3L)^3} > 0. \tag{29}$$

Similarly, it is easy to obtain $\partial\pi_A^{OR}/\partial L = \theta^2(28-2L)/(8-4L)^3 > 0$. The values of π_A^{OO} and π_A^{OR} increase with the value of L growing. As $L = (1+\beta)^2/\gamma$, we can learn that when the level of spillover β increases and the cost of R&D efforts γ declines, the profits of Firm A gotten from opportunistic behavior rise. □

Proposition 5. *In the vertical R&D cooperation, the profits of Firm B in four situations will increase when the level of spillover β rises and the cost of R&D efforts γ declines.*

Proof. Consider

$$\frac{\partial\pi_{\mathbf{B}}^{RR}}{\partial L} = \frac{\partial\theta^2 (8-4.5L)(8-6L)^{-2}}{\partial L}$$
$$= -4.5\theta^2 (8-6L)^{-2} + 12\theta^2 (8-4.5L)(8-6L)^{-3}$$
$$= \frac{-4.5\theta^2 (8-6L) + 12\theta^2 (8-4.5L)}{(8-3L)^3}$$
$$= \frac{\theta^2 (60-27L)}{(8-6L)^3} > 0. \tag{30}$$

Similarly, it is easy to obtain $\partial\pi_B^{RO}/\partial L = \theta^2(64-10L)/(8-5L)^3 > 0$, $\partial\pi_B^{OO}/\partial L = \theta^2(32-6L)/(8-3L)^3 > 0$, and $\partial\pi_B^{OR}/\partial L = \theta^2(28-16L)/(8-4L)^3 > 0$. The values of π_B^{RR}, π_B^{RO}, π_B^{OO}, and π_B^{OR} go up with the value of L increasing. As $L = (1+\beta)^2/\gamma$, we can learn that when the level of spillover β increases and the cost of R&D efforts γ declines, the profits of Firm B gotten from the vertical cooperation rise. □

4. Numerical Illustration

In this section, we use numerical illustration to discuss the profits in four different situations and analyze the effects of the level of knowledge spillover and the cost of R&D efforts on the stability of the R&D cooperation.

First, in order to analyze the effects of spillover levels on the two firms' profits in different situations, we assume the basic parameters to be as follows:

$$a = 100,\quad c_A = 30,\quad c_B = 20,\quad \gamma = 5. \tag{31}$$

From Table 3 it can be seen that the profits of Firm A in the four situations satisfy $\pi_A^{OR} > \pi_A^{OO} > \pi_A^{RR} > \pi_A^{RO}$. As the level of knowledge spillover goes up, the comparison of Firm A's profits remains unchanged. As for Firm B, when the level of knowledge spillover stays at a low level, choosing opportunistic behavior brings more benefits to Firm B in the game. However, as the level of knowledge spillover increases to a certain value, Firm B gets the highest profit from cooperation with two reciprocal firms.

From Figure 1 we learn that the profits of Firm A gotten from reciprocal behavior are lower than the profits gained from opportunistic behavior. The higher the level of spillover, the lower benefits Firm A get from reciprocal behavior in the vertical R&D cooperation. From Figure 2 we can learn that Firm B will get more benefits from its partner's reciprocal behavior. Therefore, Firm A is more likely to choose opportunistic behavior compared with Firm B. The vertical R&D cooperation faces a higher probability of failure. As Figure 1 shows, when the knowledge spillover stays at a low level, the differences among the profits of Firm A in four situations are small. As the level of knowledge spillover increases, the differences among the profits of Firm A in four situations grow. Therefore, when the level of knowledge spillover is low,

Table 4: The profits of four situations in different levels of R&D effort costs.

γ	L	π_A^{RR}	π_A^{OO}	π_A^{RO}	π_A^{OR}	π_B^{RR}	π_B^{OO}	π_B^{RO}	π_B^{OR}
3.00	0.75	127.55	274.10	86.51	362.50	943.88	491.49	899.65	462.50
3.50	0.64	161.27	249.48	120.85	312.07	743.91	455.36	732.90	433.26
4.00	0.56	171.66	233.31	136.45	281.19	639.15	431.33	638.70	413.52
4.50	0.50	175.00	221.89	144.63	260.42	575.00	414.20	578.51	399.31
5.00	0.45	175.77	213.41	149.34	245.51	531.77	401.38	536.86	388.59
5.50	0.41	175.52	206.86	152.23	234.31	500.71	391.42	506.38	380.23
6.00	0.38	174.86	201.65	154.10	225.59	477.32	383.47	483.13	373.52
6.50	0.35	174.04	197.41	155.35	218.61	459.08	376.97	464.83	368.02
7.00	0.32	173.18	193.90	156.21	212.91	444.46	371.56	450.05	363.43
7.50	0.30	172.35	190.93	156.80	208.15	432.49	366.99	437.87	359.54

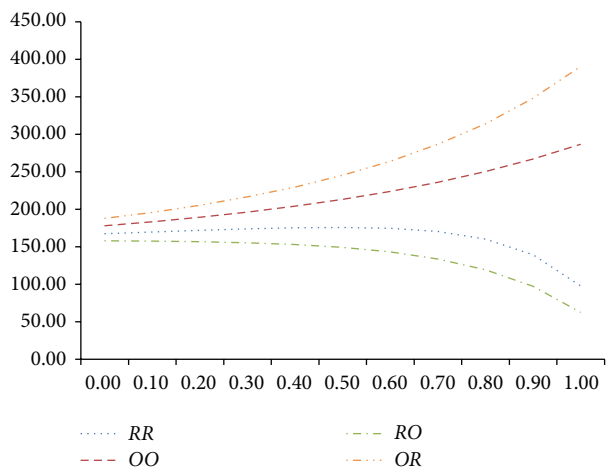


Figure 1: The profits of Firm A in different levels of spillovers.

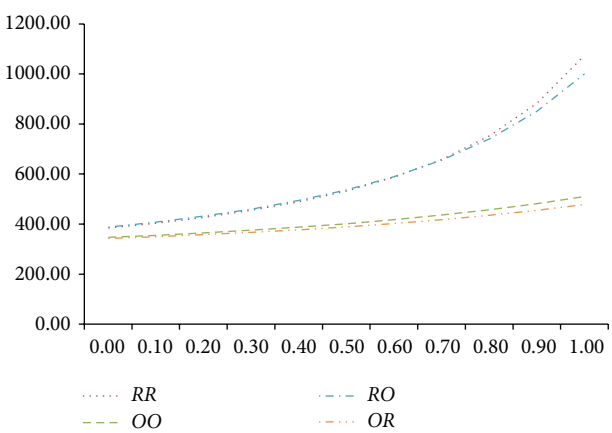


Figure 2: The profits of Firm B in different levels of spillovers.

mechanisms such as punishment for opportunism may be more effective to guarantee the stability of cooperation.

Second, in order to analyze the effects of R&D effort cost on the two firms' profits in different situations, we assume the basic parameters to be as follows:

$$\begin{aligned} a &= 100, \\ c_A &= 30, \\ c_B &= 20, \\ \beta &= 0.5. \end{aligned} \tag{32}$$

From Table 4 it can also be seen that the profits of Firm A in the four situations satisfy $\pi_A^{OR} > \pi_A^{OO} > \pi_A^{RR} > \pi_A^{RO}$. As the cost of R&D efforts goes up, the comparison of the Firm A's profits remains unchanged. As for Firm B, when the cost of R&D efforts stays at a low level, choosing reciprocal behavior brings more benefits to Firm B in the game. However, as the cost of R&D efforts increases to a certain value, Firm B gets higher profits from cooperation with two reciprocal firms.

From Figure 1 we learn that the profits of Firm A gotten from reciprocal behavior are lower than those from opportunistic behavior. But when the cost of R&D efforts increases, the profits of Firm A gotten from reciprocal behavior grow. From Figure 4 we can learn that Firm B will get more benefits from its partner's reciprocal behavior. But its profits decrease with the cost rising. Figures 3 and 4 also indicate that Firm A is more likely to choose opportunistic behavior compared with Firm B. Moreover, when the cost of R&D efforts stays at a high level, the differences among the profits of Firm A in four situations are small. Therefore, when the cost of R&D efforts is high, mechanisms such as punishment for opportunism may be more effective to guarantee the stability of cooperation.

5. Conclusion

As the R&D contract between firms is inherently incomplete, the opportunistic behavior could not be avoided, which makes the R&D cooperation unstable. Compared with the game theoretical based literature on the stability of R&D

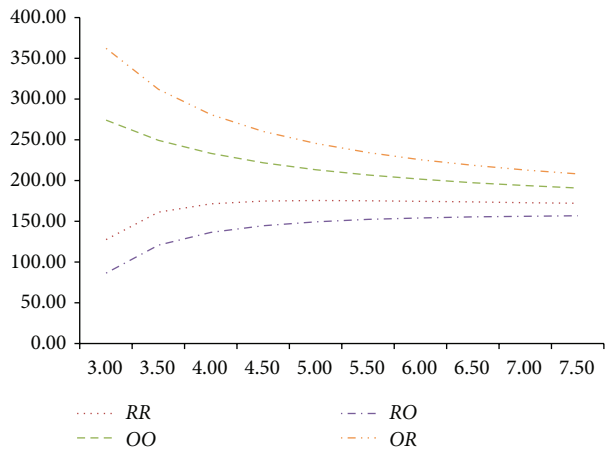


Figure 3: The profits of Firm A in different levels of R&D effort costs.

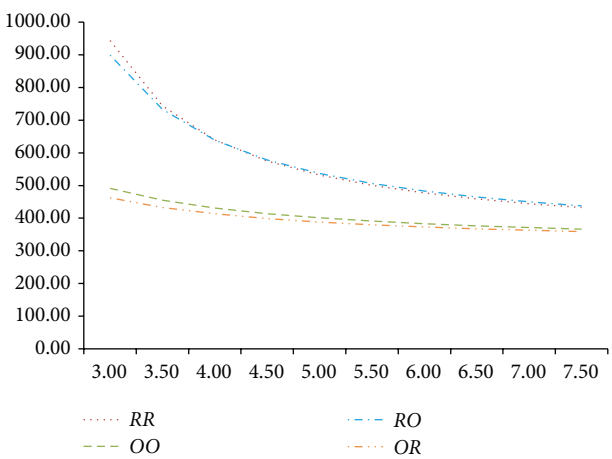


Figure 4: The profits of Firm B in different levels of R&D effort costs.

cooperation, we study the stability of vertical R&D cooperation. In this paper, we have provided a game model with two firms in the vertical R&D cooperation to discuss the stability of the cooperation. Two firms first build a partnership and then coordinate their decisions of R&D efforts. However, due to the cooperative and competitive relationship between the two collaborative firms, opportunism cannot be avoided, which makes the vertical R&D cooperation fail. We first analyze the profits of Firm A and Firm B in four different situations and then, respectively, compare the values of payoffs of Firm A and Firm B. Finally we discuss the effects of spillover level and R&D cost on profits of four different situations, and numerical illustration is presented.

Our results suggest that the vertical R&D cooperation is inherently unstable, and the downstream firm is more likely to break the agreement. When building a partnership, firms need to consider the social statue and reputation of its partner and mutual trust between the two firms. This paper also identifies the role of knowledge spillovers and the cost of R&D efforts in the stability of vertical R&D cooperation. Knowledge flow and R&D cost influence the firms' payoffs in different situations. And they play different roles in the decision process. We learn that when the level of knowledge spillover is low or the cost of R&D efforts is high, mechanisms such as punishment for opportunism may be more effective.

These results may provide a theoretical basis for the operation of vertical R&D cooperation. This paper also raises some questions for future research. First, we note that our analysis concentrates on a simple supply chain with two firms involved. In future work we hope to explore the stability of vertical R&D cooperation in a more general setting. Second, empirical work can be done to better analyze the opportunism problem in the vertical R&D cooperation. Third, other factors such as punishment and trust in the stable R&D cooperation can be considered in future studies.

Appendix

(1) Derivation of the Comparison of π_A^{OO} and π_A^{RR}. Consider

$$\begin{aligned}\pi_A^{OO} - \pi_A^{RR} &= \frac{\theta^2(4-0.5L)}{(8-3L)^2} - \frac{\theta^2(4-4.5L)}{(8-6L)^2} \\ &= \frac{\theta^2}{(8-3L)^2(8-6L)^2}\left[(4-0.5L)(8-6L)^2 - (4-4.5L)(8-3L)^2\right] \\ &= \frac{\theta^2 L}{(8-3L)^2(8-6L)^2}\left(22.5L^2 - 60L + 64\right) > 0.\end{aligned} \tag{A.1}$$

Therefore, we can get $\pi_A^{OO} > \pi_A^{RR}$.

(2) Derivation of the Comparison of π_B^{OO} and π_B^{OR}. Consider

$$\begin{aligned}\pi_B^{OO} - \pi_B^{OR} &= \frac{\theta^2(8-2L)}{(8-3L)^2} - \frac{\theta^2(8-4.5L)}{(8-4L)^2} \\ &= \frac{\theta^2}{(8-3L)^2(8-4L)^2}\left[(8-2L)(8-4L)^2 - (8-4.5L)(8-3L)^2\right] \\ &= \frac{\theta^2}{(8-3L)^2(8-4L)^2}\left(8.5L^2 - 32L + 32\right) > 0.\end{aligned} \tag{A.2}$$

Therefore, we can get $\pi_B^{OO} > \pi_B^{OR}$.

(3) Derivation of the Comparison of π_B^{RR} and π_B^{OO}. Consider

$$\begin{aligned}\pi_B^{RR} - \pi_B^{OO} &= \frac{\theta^2(8-4.5L)}{(8-6L)^2} - \frac{\theta^2(8-2L)}{(8-3L)^2} \\ &= \frac{\theta^2}{(8-3L)^2(8-6L)^2}\left[(8-4.5L)(8-3L)^2 - (8-2L)(8-6L)^2\right] \\ &= \frac{\theta^2 L}{(8-3L)^2(8-6L)^2}\left(31.5L^2 - 192L + 224\right).\end{aligned} \tag{A.3}$$

The intersections of the quadratic function $y = 31.5x^2 - 192x + 224$ with the x-axis are, respectively, $((192 - \sqrt{8640})/63, 0)$ and $((192 + \sqrt{8640})/63, 0)$. As $0 < L < 8/9$, therefore, we can get $\pi_B^{RR} > \pi_B^{OO}$.

(4) Derivation of the Comparison of π_B^{RR} and π_B^{RO}. Consider

$$\begin{aligned}\pi_B^{RR} - \pi_B^{RO} &= \frac{\theta^2 (8 - 4.5L)}{(8 - 6L)^2} - \frac{\theta^2 (8 - 2L)}{(8 - 5L)^2} \\ &= \frac{\theta^2}{(8 - 6L)^2 (8 - 5L)^2} \left[(8 - 4.5L)(8 - 5L)^2 \right. \\ &\quad \left. - (8 - 2L)(8 - 6L)^2\right] \\ &= \frac{\theta^2 L}{(8 - 3L)^2 (8 - 6L)^2} \left(-40.5L^2 + 80L - 32\right).\end{aligned} \tag{A.4}$$

The intersections of the quadratic function $y = -40.5L^2 + 80L - 32$ with the x-axis are, respectively, $((80 - \sqrt{1216})/81, 0)$ and $((80 + \sqrt{1216})/81, 0)$. As $0 < L < 8/9$, therefore, when $(80 - \sqrt{1216})/81 < L < 8/9$, we can get $\pi_B^{RR} > \pi_B^{RO}$, and, when $0 < L < (80 - \sqrt{1216})/81$, we can get $\pi_B^{RR} < \pi_B^{RO}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Science Foundation of China (no. 61502167), the Hunan Provincial Education Department Scientific Research Fund of China (no. 15C0825), Hunan Provincial Science and Technology Program Project of China (no. 2015JC3066), Scientific Research Foundation for Ph.D. Hunan Normal University (no. Math 120641), and Youth Scientific Research Fund of Hunan Normal University (no. 11301).

References

[1] T. Roediger-Schluga and M. J. Barber, "R&D collaboration networks in the European Framework Programmes: data processing, network construction and selected results," *International Journal of Foresight and Innovation Policy*, vol. 4, no. 3-4, pp. 321–347, 2008.

[2] M. Tripsas, S. Schrader, and M. Sobrero, "Discouraging opportunistic behavior in collaborative R & D: a new role for government," *Research Policy*, vol. 24, no. 3, pp. 367–389, 1995.

[3] J. E. Oxley and R. C. Sampson, "The scope and governance of international R&D alliances," *Strategic Management Journal*, vol. 25, no. 8-9, pp. 723–749, 2004.

[4] V. Mukherjee and S. V. Ramani, "R&D cooperation in emerging industries, asymmetric innovative capabilities and rationale for technology parks," *Theory and Decision*, vol. 71, no. 3, pp. 373–394, 2011.

[5] C. d'Aspremont and A. Jacquemin, "Cooperative and noncooperative R&D in duopoly with spillovers," *The American Economic Review*, vol. 78, no. 5, pp. 1133–1137, 1988.

[6] S. E. Phelan, R. J. Arend, and D. A. Seale, "Using an iterated prisoner's dilemma with exit option to study alliance behavior: results of a tournament and simulation," *Computational & Mathematical Organization Theory*, vol. 11, no. 4, pp. 339–356, 2005.

[7] S. H. Park and M. V. Russo, "When competition eclipses cooperation: an event history analysis of joint venture failure," *Management Science*, vol. 42, no. 6, pp. 875–890, 1996.

[8] M. J. Kelly, J.-L. Schaan, and H. Joncas, "Managing alliance relationships: key challenges in the early stages of collaboration," *R & D Management*, vol. 32, no. 1, pp. 11–22, 2002.

[9] J. Walter, F. W. Kellermanns, and C. Lechner, "Decision making within and between organizations: rationality, politics, and alliance performance," *Journal of Management*, vol. 38, no. 5, pp. 1582–1610, 2012.

[10] T. K. Das and B.-S. Teng, "Between trust and control: developing confidence in partner cooperation in alliances," *Academy of Management Review*, vol. 23, no. 3, pp. 491–512, 1998.

[11] K. H. Wathne and J. B. Heide, "Opportunism in interfirm relationships: forms, outcomes, and solutions," *Journal of Marketing*, vol. 64, no. 4, pp. 36–51, 2000.

[12] Y. Luo, "Opportunism in inter-firm exchanges in emerging markets," *Management and Organization Review*, vol. 2, no. 1, pp. 121–147, 2006.

[13] T. K. Das and N. Rahman, "Determinants of partner opportunism in strategic alliances: a conceptual framework," *Journal of Business and Psychology*, vol. 25, no. 1, pp. 55–74, 2010.

[14] M.-L. Cabon-Dhersin and S. V. Ramani, "Does trust matter for R&D cooperation? A game theoretic examination," *Theory and Decision*, vol. 57, no. 2, pp. 143–180, 2004.

[15] A. Ishii, "Cooperative R&D between vertically related firms with spillovers," *International Journal of Industrial Organization*, vol. 22, no. 8-9, pp. 1213–1235, 2004.

[16] G. Atallah, "Defecting from R&D cooperation," *Australian Economic Papers*, vol. 45, no. 3, pp. 204–226, 2006.

[17] A. Stepanova and A. Tesoriere, "R&D with spillovers: monopoly versus noncooperative and cooperative duopoly," *The Manchester School*, vol. 79, no. 1, pp. 125–144, 2011.

[18] C. Burr, M. Knauff, and A. Stepanova, "On the prisoner's dilemma in R&D with input spillovers and incentives for R&D cooperation," *Mathematical Social Sciences*, vol. 66, no. 3, pp. 254–261, 2013.

[19] T. Shibata, "Market structure and R&D investment spillovers," *Economic Modelling*, vol. 43, pp. 321–329, 2014.

[20] Z. Ge, Q. Hu, and Y. Xia, "Firms' R&D cooperation behavior in a supply Chain," *Production and Operations Management*, vol. 23, no. 4, pp. 599–609, 2014.

[21] K. Kesteloot and R. Veugelers, "Stable R&D cooperation with spillovers," *Journal of Economics & Management Strategy*, vol. 4, no. 4, pp. 651–672, 1995.

[22] B. Cassiman and R. Veugelers, "R&D cooperation and spillovers: some empirical evidence from Belgium," *The American Economic Review*, vol. 92, no. 4, pp. 1169–1184, 2002.

[23] S. Lhuillery and E. Pfister, "R&D cooperation and failures in innovation projects: empirical evidence from French CIS data," *Research Policy*, vol. 38, no. 1, pp. 45–57, 2009.

[24] R. Newman, "Single source qualification," *Journal of Purchasing and Materials Management*, vol. 24, no. 2, pp. 10–17, 1988.

[25] W. Presutti, "The single source issue: US and Japanese sourcing strategies," *Journal of Supply Chain Management*, vol. 28, no. 1, pp. 2–9, 1992.

[26] J. Dyer and W. Ouchi, "Japanese-style partnerships: giving companies a competitive edge," *Sloan Management Review*, vol. 35, no. 1, pp. 51–63, 1993.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 574906, 8 pages
http://dx.doi.org/10.1155/2015/574906

Research Article

Globally Asymptotic Stability of Stochastic Nonlinear Systems by the Output Feedback

Wenwen Cheng,[1] Quanxin Zhu,[1] and Zhangsong Yao[2]

[1] *School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Nanjing, Jiangsu 210023, China*
[2] *School of Information Engineering, Nanjing Xiaozhuang University, Nanjing, Jiangsu 211171, China*

Correspondence should be addressed to Zhangsong Yao; yaozhsong@163.com

Received 29 March 2015; Revised 31 May 2015; Accepted 7 June 2015

Academic Editor: Antonino Laudani

We address the problem of the globally asymptotic stability for a class of stochastic nonlinear systems with the output feedback control. By using the backstepping design method, a novel dynamic output feedback controller is designed to ensure that the stochastic nonlinear closed-loop system is globally asymptotically stable in probability. Our way is different from the traditional mathematical induction method. Indeed, we develop a new method to study the globally asymptotic stability by introducing a series of specific inequalities. Moreover, an example and its simulations are given to illustrate the theoretical result.

1. Introduction

As is well known, the stability problem of nonlinear systems with the state feedback or output feedback control has received much attention since it can be extensively applied in many fields such as engineering and finance. In the practical application, nonlinear systems with the feedback control can model many kinds of stochastic influences either natural or man-made. The output feedback control especially has been used more widely for the reason that a system by the output feedback is more flexible to respond to the information of control systems than the state feedback.

In recent years, there has been a larger number of research works on the global stability for nonlinear systems with the output feedback control [1–6]. For example, Qian and Lin [5] have considered the global stability by the output feedback for a family of triangular nonlinear systems in which the gain parameter L is dependent on the parameters of the controller. Combining the backstepping method and output feedback domination approach, M.-L. Liu and Y.-G. Liu [4] have investigated the semiglobally asymptotic stability for a class of uncertain nonlinear systems. In [1], Andrieu and Praly have applied the output feedback to study the globally asymptotic stability of nonlinear systems based on a unifying point. In [2], Du et al. have discussed the global output feedback stabilization of a class of uncertain upper-triangular systems with the input delay in which the controller with a scaling gain was used to deal with a larger input delay by a coordinate change. However, all the above works did not consider noise disturbances. Actually, the synaptic transmission in real systems can be viewed as a noisy process introduced by random fluctuations from the release of information and other probabilistic causes. Moreover, a system can be stabilized or destabilized by certain stochastic inputs. Therefore, noise disturbances should be taken into account when studying the stability of nonlinear systems.

It is worth pointing out that the problem of global output feedback stability for a class of deterministic lower-triangular systems has been solved in [5] by using the feedback domination design method and constructing a linear output compensator. Unfortunately, the noise disturbance was ignored in [5]. As discussed in the above, the noise disturbance has an important effect on the stability of a real system. So it is natural to question whether a nonlinear output feedback system is stable or not when it is affected by the noise disturbance. About this issue, the previous work on output feedback control of stochastic nonlinear systems almost combines the backstepping method and the mathematical induction to design the output feedback control. For example, Liu et al. [7, 8] have discussed the output feedback

control of a class of stochastic nonlinear systems with linearly bounded unmeasurable states and a class of stochastic nonminimum-phase nonlinear systems. Chen et al. [9] and Liu and Xie [10] have talked about the state feedback stability for stochastic nonlinear systems with time-varying delay. Guo et al. [11] have solved the output feedback stability for a class of stochastic nonlinear systems with power growth conditions. More results can be found in [12–16]. The proofs in these papers are complicated.

In the spirit of stochastic stability theorem of Khasminskii [17] and that of Mao [18] about globally asymptotic stability in probability, we construct a novel Lyapunov function directly to prove the stability of the nonlinear stochastic output feedback system. As the discussion in [5], we also abandon the separation principle paradigm and apply a recursive control algorithm to design the linear control and the Lyapunov function. Different from the mathematical induction, we use some of the ingenious distortion of inequalities to make the infinitesimal generator negative definite. To obtain more concise result, we take the dynamic gain from 2 rather than 1. In particular, a novel linear observer system is designed and the Lyapunov function is constructed by the following formula:

$$V = \frac{n+1}{2}\left(\varepsilon^T P\varepsilon\right)^2 + \sum_{j=1}^{n}\frac{1}{4L^{4j-4}}\xi_j^4. \tag{1}$$

Without using the mathematical induction, we construct some variables to achieve the multiform inequalities. As a consequence, our result has more brief frame of the linear controller than that given in [19]. Moreover, the model discussed in [19] can be regarded as the special case of ours.

The rest of this paper is arranged as follows. In Section 2, we present the preparation of globally asymptotic stability and introduce some inequalities which play an important role in the proof of our main results. In Section 3, a novel dynamic output feedback is designed by the backstepping procedure. In Section 4, we use an example to illustrate the theoretical results. Finally, in Section 5, we conclude the paper with some general remarks.

2. Preliminaries

In this section, we mainly give the definition of the globally asymptotic stability in probability and introduce several preliminary lemmas.

Consider the following stochastic nonlinear systems:

$$\begin{aligned} dx &= f(x)\,dt + g(x)\,d\omega, \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state; ω is an r-dimensional standard Brownian motion; and the Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitz and satisfy $f(0) = 0$, $g(0) = 0$.

Definition 1. The function $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be $\mathscr{K}$, if $\gamma(\cdot)$ is continuous, strictly increasing, and vanishing at zero.

Definition 2 (see [8]). The equilibrium $x = 0$ of (2) is said to be globally asymptotically stable in probability if, for any $\varepsilon > 0$, there exists $\gamma(\cdot) \in \mathscr{K}$ such that

$$P\{|x(t)| < \gamma(|x_0|)\} \geq 1-\varepsilon, \quad \forall t \geq 0,\ x_0 \in \mathbb{R}^n \setminus \{0\}, \tag{3}$$

and for any initial condition x_0,

$$P\left\{\lim_{t\to\infty} x(t) = 0\right\} = 1. \tag{4}$$

Definition 3 (see [8]). For any given $V(x) \in \mathscr{C}^2$ associated with system (2), the differential operator $\mathscr{L}$ is defined as $\mathscr{L}V = (\partial V/\partial x^T)f + (1/2)\mathrm{tr}\{g^T(\partial^2 V/\partial x^2)g\}$, where $(1/2)\mathrm{tr}\{g^T(\partial^2 V/\partial x^2)g\}$ is called Hessian term of $\mathscr{L}$.

Lemma 4 (see [8]). *Consider system (2) and suppose that there exist positive definite, radially unbounded, twice continuously differentiable function $V(x)$, and a positive definite function $W(x)$ such that $\mathscr{L}V(x) \leq -W(x)$; then*

(i) *for (2) there exists an almost surely unique strong solution on $[0,\infty)$ for each $x_0 \in \mathbb{R}^n$;*

(ii) *the equilibrium $x = 0$ of system (2) is globally asymptotically stable in probability.*

Lemma 5 (see [13]). *Let $p \geq 1$. Then for any $x, y \in \mathbb{R}$,*

$$\begin{aligned} &\left|x^{1/p} - y^{1/p}\right| \leq 2^{(p-1)/p}\left|x-y\right|^{1/p}, \\ &\quad \left|x \pm y\right|^p \leq 2^{p-1}\left|x^p \pm y^p\right| \leq 2^{p-1}\left(\left|x^p\right| + \left|y^p\right|\right), \\ &\left(|x| + |y|\right)^{1/p} \leq |x|^{1/p} + |y|^{1/p} \leq 2^{(p-1)/p}\left(|x| + |y|\right)^{1/p}. \end{aligned} \tag{5}$$

Lemma 6 (see [13]). *For any given real numbers c, d and any real-valued functions $f(x,y) > 0$, $g(x,y,z) \geq 0$, the following inequality holds:*

$$\begin{aligned} &g(x,y,z)\,|x|^c\,|y|^d \\ &\quad \leq \frac{c}{c+d} f(x,y)\,|x|^{c+d} + \frac{d}{c+d} \\ &\qquad \times \left(g(x,y,z)\right)^{(c+d)/d}\left(f(x,y)\right)^{-c/d}|y|^{c+d}, \end{aligned} \tag{6}$$

where $x, y, z \in \mathbb{R}$. Particularly when one takes $f(x,y) = g(x,y,z) = 1$, $c = 3$, and $d = 4$, then the inequality will become

$$x^3 y \leq \frac{3}{4}x^4 + \frac{1}{4}y^4. \tag{7}$$

Lemma 7. *For any constants $a > 0$ and $b \in \mathbb{R}$, one has that, for any $x, y \in \mathbb{R}$,*

$$-ax^4 + bxy^3 \leq k_1 a^{-1/3} b^{4/3} y^4, \tag{8}$$

$$-ax^4 + bx^3 y \leq k_2 b^4 a^{-3} y^4, \tag{9}$$

where $k_1 = 4^{-1/3} - 4^{-4/3} > 0$ and $k_2 = (3/4)^3 - (3/4)^4 > 0$.

Proof. We first prove (8). Let $Z_1(x) = -ax^4 + bxy^3$, where y is a parameter. Then we have

$$Z_1'(x) = -4ax^3 + by^3 = 0, \quad x = \left(\frac{b}{4a}\right)^{1/3} y. \tag{10}$$

So for any $x \in (-\infty, (b/4a)^{1/3} y)$, $Z_1'(x) > 0$, and $x \in ((b/4a)^{1/3} y, +\infty)$, $Z_1'(x) < 0$. With the sufficient condition of extreme value, $x = (b/4a)^{1/3} y$ is the maximum point of function $Z_1(x)$. Therefore, it follows that

$$\begin{aligned} Z_1(x) &\le -a\left[\left(\frac{b}{4a}\right)^{1/3} y\right]^4 + b\left[\left(\frac{b}{4a}\right)^{1/3} y\right] y^3 \\ &= \left(4^{-1/3} - 4^{-4/3}\right) a^{-1/3} b^{4/3} y^4. \end{aligned} \tag{11}$$

We now prove (9). Similarly, letting $Z_2(x) = -ax^4 + bx^3 y$, where y is a parameter, we have

$$Z_2'(x) = -4ax^3 + 3bx^2 y = 0, \quad x = \frac{3by}{4a} \text{ or } x = 0. \tag{12}$$

So for any $x \in (-\infty, 3by/4a)$, $Z_2'(x) > 0$, and $x \in (3by/4a, +\infty)$, $Z_2'(x) < 0$. With the sufficient condition of extreme value, $x = 3by/4a$ is the maximum point of function $Z_2(x)$. Thus, we get

$$\begin{aligned} Z_2(x) &\le -a\left(\frac{3by}{4a}\right)^4 + b\left(\frac{3by}{4a}\right)^3 y \\ &= \left[\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4\right] b^4 a^{-3} y^4. \end{aligned} \tag{13}$$

□

Lemma 8. *For a series of numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$, one has*

$$\begin{aligned} |a_1 + a_2 + \cdots + a_n| &\ge \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}, \\ \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} &\ge \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &\ge \sqrt[n]{a_1 a_2 \cdots a_n} \\ &\ge \frac{2}{1/a_1 + 1/a_2 + \cdots + 1/a_n}. \end{aligned} \tag{14}$$

Lemma 9 (Cauchy-Schwartz's inequality). *Let the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then*

$$\begin{aligned} &(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \\ &\quad \le \left(x_1^2 + \cdots + x_n^2\right)\left(y_1^2 + \cdots + y_n^2\right). \end{aligned} \tag{15}$$

Lemma 10 (Young's inequality). *For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, one has $x^T y \le (\varepsilon^p/p)|x|^p + (1/q\varepsilon^q)|y|^q$, where $\varepsilon > 0$, $p > 1$, $q > 1$, and $1/p + 1/q = 1$.*

3. The Output Feedback Model and Control Design

In this section, we design a novel linear observer system (18) for the stochastic nonlinear system (16) below. Using the backstepping method, a simple linear control is constructed to guarantee that the closed-loop stochastic system is globally asymptotically stable in probability.

Consider the following nonlinear stochastic system:

$$\begin{aligned} dx_1 &= (x_2 + f_1(\overline{x}_1))\, dt + g_1^T(\overline{x}_1)\, dw, \\ dx_2 &= (x_3 + f_2(\overline{x}_2))\, dt + g_2^T(\overline{x}_2)\, dw, \\ &\vdots \\ dx_{n-1} &= (x_n + f_{n-1}(\overline{x}_{n-1}))\, dt + g_{n-1}^T(\overline{x}_{n-1})\, dw, \\ dx_n &= (u + f_n(\overline{x}_n))\, dt + g_n^T(\overline{x}_n)\, dw, \\ y &= x_1, \end{aligned} \tag{16}$$

where $\overline{x}_i = (x_1, \ldots, x_i)$ is the state vector, $u \in \mathbb{R}$ is the control input, w is the r-dimensional standard Wiener process, and $y \in \mathbb{R}$ are the system output. The nonlinear functions $f_i : \mathbb{R}_+ \times \mathbb{R}^i \to \mathbb{R}$ and $g_i : \mathbb{R}_+ \times \mathbb{R}^i \to \mathbb{R}^r$ are locally Lipschitz with $f_i(0) = 0$, $g_i(0) = 0$, $i = 1, \ldots, n$.

Assumption 11. The nonlinear functions $f_i(\overline{x}_i) \in \mathbb{R}$ and $g_i(\overline{x}_i) \in \mathbb{R}^r$, $i = 1, \ldots, n$, are locally Lipschitz with $f_i(0) = 0$ and $g_i(0) = 0$ for $i = 1, \ldots, n$. Moreover, there exist two constants $l_1 \ge 0$ and $l_2 \ge 0$ such that

$$\begin{aligned} |f_i(\overline{x}_i)| &\le l_1(|x_1| + \cdots + |x_i|), \\ |g_i(\overline{x}_i)| &\le l_2(|x_1| + \cdots + |x_i|). \end{aligned} \tag{17}$$

The linear observer system is designed as

$$\begin{aligned} \dot{\widehat{x}}_1 &= \widehat{x}_2(t) + La_1(x_1 - \widehat{x}_1), \\ \dot{\widehat{x}}_2 &= \widehat{x}_3(t) + L^2 a_2(x_1 - \widehat{x}_1), \\ &\vdots \\ \dot{\widehat{x}}_{n-1} &= \widehat{x}_n(t) + L^{n-1} a_{n-1}(x_1 - \widehat{x}_1), \\ \dot{\widehat{x}}_n &= u + L^n a_n(x_1 - \widehat{x}_1), \end{aligned} \tag{18}$$

where $L \ge 2$ is an appropriate constant and $a_i > 0$, $i = 1, \ldots, n$, are coefficients of the Hurwitz polynomial:

$$p(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n. \tag{19}$$

The observation error $\varepsilon_i = (x_i - \hat{x}_i)/L^{i-1}$ satisfies

$$d\varepsilon = L\begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 0 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}\varepsilon dt + \begin{bmatrix} f_1 \\ \frac{f_2}{L} \\ \vdots \\ \frac{f_{n-1}}{L^{n-2}} \\ \frac{f_n}{L^{n-1}} \end{bmatrix} dt + \begin{bmatrix} g_1^T \\ \frac{g_2^T}{L} \\ \vdots \\ \frac{g_{n-1}^T}{L^{n-2}} \\ \frac{g_n^T}{L^{n-1}} \end{bmatrix} dw = LA\varepsilon dt + Fdt + Gdw, \tag{20}$$

where A is a Hurwitz matrix. Therefore, there is a positive-definite matrix $P = P^T > 0$ such that

$$A^T P + PA = -I. \tag{21}$$

Theorem 12. *Assume that Assumption 11 holds. Then, the equilibrium at origin of the closed-loop nonlinear stochastic system (16) and (18) with the linear controller (31) below is globally asymptotically stable in probability. Furthermore, it follows from Lemma 4 that there exists an almost surely unique strong solution on $[0, \infty)$ for each $x_0 \in \mathbb{R}^n$.*

Proof. Consider the following Lyapunov function $V_0(\varepsilon) = ((n+1)/2)(\varepsilon^T P\varepsilon)^2$. Then, we have

$$\begin{aligned} \mathscr{L}V_0 &= (n+1) \\ &\cdot\left(\left(\varepsilon^T P\varepsilon\right)\left(\varepsilon^T\left(LA^T P + LPA\right)\varepsilon + 2\varepsilon^T PF\right)\right. \\ &\left.+\frac{1}{2}\operatorname{tr}\left(G^T\left(4P\varepsilon\varepsilon^T P + 2\varepsilon^T P\varepsilon P\right)G\right)\right) \le -(n+1) \\ &\cdot\lambda_{\min} L\|\varepsilon\|^4 + 2(n+1)\varepsilon^T P\varepsilon\varepsilon^T PF + 3(n+1) \\ &\cdot r\sqrt{r}|G|^2|P|^2\|\varepsilon\|^2 \le -(n+1)\lambda_{\min} L\|\varepsilon\|^4 + 2(n \\ &+1)\lambda_{\max}^2\|\varepsilon\|^3|F| + 3(n+1)r\sqrt{r}\lambda_{\max}^2|G|^2\|\varepsilon\|^2, \end{aligned} \tag{22}$$

where $\lambda_{\min}$ denotes the minimum eigenvalue and $\lambda_{\max}$ is the maximum eigenvalue of the matrix P.

It follows from Assumption 11 that

$$\begin{aligned} |F| &= \sqrt{f_1^2 + \left(\frac{f_2}{L}\right)^2 + \cdots + \left(\frac{f_n}{L^{n-1}}\right)^2} \\ &\le 2l_1\left(|x_1| + \frac{1}{L}|x_2| + \cdots + \frac{1}{L^{n-1}}|x_n|\right), \\ |G| &= \sqrt{g_1^2 + \left(\frac{g_2}{L}\right)^2 + \cdots + \left(\frac{g_n}{L^{n-1}}\right)^2} \\ &\le 2l_2\left(|x_1| + \frac{1}{L}|x_2| + \cdots + \frac{1}{L^{n-1}}|x_n|\right). \end{aligned} \tag{23}$$

Recalling that $x_i = \hat{x}_i + L^{i-1}\varepsilon_i$, $i = 1, \ldots, n$, we get

$$\begin{aligned} 2(n+1)\lambda_{\max}^2\|\varepsilon\|^3|F| &\le \frac{3}{2}(n+1)\lambda_{\max}^2\|\varepsilon\|^4 \\ &+\frac{1}{2}(n+1)\lambda_{\max}^2|F|^4 \\ &\le\left(\frac{3}{2}(n+1) + 64(n+1)n^2c^4\right)\lambda_{\max}^2\|\varepsilon\|^4 \\ &+64(n+1) \\ &\cdot n^3c^4\lambda_{\max}^2\left(\hat{x}_1^4 + \frac{1}{L^4}\hat{x}_2^4 + \cdots + \frac{1}{L^{4n-4}}\hat{x}_n^4\right), \\ 3(n+1)r\sqrt{r}\lambda_{\max}^2|G|^2\|\varepsilon\|^2 &\le \frac{3}{2}(n+1)\lambda_{\max}^2\|\varepsilon\|^4 \\ &+\frac{3}{2}(n+1)r^3\lambda_{\max}^2|G|^4 \\ &\le\left(\frac{3}{2}(n+1) + 192(n+1)n^2r^3c^4\right)\lambda_{\max}^2\|\varepsilon\|^4 \\ &+192(n+1)n^3r^3c^4\lambda_{\max}^2\left(\hat{x}_1^4 + \cdots + \frac{1}{L^{4n-4}}\hat{x}_n^4\right), \end{aligned} \tag{24}$$

where $c = \max\{l_1, l_2\}$.

Substituting (24) into (22) yields

$$\begin{aligned} \mathscr{L}V_0 &\le -\left((n+1)\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 \\ &+c_1\left(\hat{x}_1^4 + \frac{\hat{x}_2^4}{L^4} + \cdots + \frac{\hat{x}_n^4}{L^{4n-4}}\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned} c_0 &= \left(3(n+1) + \left(64 + 192r^3\right)(n+1)n^2c^4\right)\lambda_{\max}^2, \\ c_1 &= \left(64 + 192r^3\right)(n+1)n^3c^4\lambda_{\max}^2. \end{aligned} \tag{26}$$

Now, we take the Lyapunov function as follows:

$$V = \frac{n+1}{2}\left(\varepsilon^T P\varepsilon\right)^2 + \sum_{j=1}^{n}\frac{1}{4L^{4j-4}}\xi_j^4. \tag{27}$$

And $\widehat{x}_i = \xi_i + \widehat{x}_{i-1}^*$, $\widehat{x}_{i-1}^* = -Lb_{i-1}\xi_{i-1}$, and $\widehat{x}_1 = \xi_1$, where $i = 2,\ldots,n$ and b_i is needed to be determined later. Then, a direct computation yields

$$\begin{aligned}
\mathscr{L}V \le{}& -\left((n+1)\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 + c_1\widehat{x}_1^4 + c_1\left(\frac{1}{L^4}\widehat{x}_2^4\right.\\
&\left.+\cdots+\frac{1}{L^{4n-4}}\widehat{x}_n^4\right) + \xi_1^3\left(\widehat{x}_2 + La_1\varepsilon_1\right) + \sum_{j=2}^{n-1}\frac{1}{L^{4j-4}}\xi_j^3\\
&\cdot\left(\widehat{x}_{j+1}\right.\\
&\left.+L^j a_j\varepsilon_1 - \sum_{i=1}^{j-1}\frac{\partial\widehat{x}_{j-1}^*}{\partial\widehat{x}_i}\left(\widehat{x}_{i+1} + L^i a_i\varepsilon_1\right)\right) + \frac{1}{L^{4n-4}}\xi_n^3\left(u\right.\\
&\left.+L^n a_n\varepsilon_1 - \sum_{i=1}^{n-1}\frac{\partial\widehat{x}_{n-1}^*}{\partial\widehat{x}_i}\left(\widehat{x}_{i+1} + L^i a_i\varepsilon_1\right)\right)\\
\le{}& -\left((n+1)\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 + c_1\xi_1^4\\
&+8c_1\sum_{j=2}^{n}\frac{1}{L^{4j-4}}\xi_j^4 + 8c_1\sum_{j=2}^{n}\frac{b_{j-1}^4}{L^{4j-8}}\xi_{j-1}^4 + \xi_1^3\left(-Lb_1\xi_1\right.\\
&\left.+La_1\varepsilon_1\right) + \sum_{j=1}^{n-1}\frac{1}{L^{4j-4}}\left(\xi_j^3\xi_{j+1} - L\xi_j^4\right) + \sum_{j=1}^{n-1}\frac{1}{L^{4j-5}}\xi_j^4\\
&+\sum_{j=2}^{n-1}\frac{1}{L^{4j-4}}\xi_j^3\left(-Lb_j\xi_j\right.\\
&\left.+\left(L^j a_j\varepsilon_1 - \sum_{i=1}^{j-1}\frac{\partial\widehat{x}_{j-1}^*}{\partial\widehat{x}_i}L^i a_i\varepsilon_1\right) - \sum_{i=1}^{j-1}\frac{\partial\widehat{x}_{j-1}^*}{\partial\widehat{x}_i}\widehat{x}_{i+1}\right)\\
&+\frac{1}{L^{4n-4}}\xi_n^3\left(u + L^n a_n\varepsilon_1\right.\\
&\left.-\sum_{i=1}^{n-1}\frac{\partial\widehat{x}_{n-1}^*}{\partial\widehat{x}_i}\left(\widehat{x}_{i+1} + L^i a_i\varepsilon_1\right)\right) \le -\left(\lambda_{\min}L - c_0\right)\|\varepsilon\|^4\\
&+\left(\left(-b_1 + c_1 + a_1'\right)L + 8c_1 b_1^4\right)\xi_1^4 + 8c_1\sum_{j=3}^{n}\frac{b_{j-1}^4}{L^{4j-8}}\xi_{j-1}^4\\
&+\sum_{j=2}^{n}\frac{m_{j-1}}{L^{4j-5}}\xi_j^4 + \sum_{j=1}^{n}\frac{k_2+1}{L^{4j-5}}\xi_j^4 + \sum_{j=2}^{n-1}\frac{1}{L^{4j-4}}\xi_j^3\left(-Lb_j\xi_j\right.\\
&\left.+Lb_{j-1}\xi_j + L^2 b_{j-1,j-1}\xi_{j-1} + \cdots + L^j b_{j-1,1}\xi_1\right)\\
&+\frac{1}{L^{4n-4}}\xi_n^3\left(u + Lb_{n-1}\xi_n + \cdots + L^{n-1}b_{n-1,2}\xi_2\right.\\
&\left.+L^n b_{n-1,1}\xi_1\right),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
a_1' &= k_1\lambda_{\min}^{-1/3}a_1^{4/3},\\
b_{j-1,1} &= -b_{j-1}b_{j-2}\cdots b_2 b_1^2,\\
b_{j-1,2} &= -b_{j-1}b_{j-2}\cdots b_3 b_2^2 + b_{j-1}b_{j-2}\cdots b_1,\ldots\\
b_{j-1,j-2} &= -b_{j-1}b_{j-2}^2 + b_{j-1}b_{j-2},\\
b_{j-1,j-1} &= -b_{j-1}^2 + b_{j-1}b_{j-2},\\
m_{j-1} &= k_1\lambda_{\min}^{-1/3}\left(a_j + b_{j-1}a_{j-1} + b_{j-1}b_{j-2}a_{j-2} + \cdots\right.\\
&\quad\left.+b_{j-1}\cdots b_1 a_1\right)^{4/3},\\
&\qquad j = 2,\ldots,n.
\end{aligned} \tag{29}$$

From Lemma 7, it follows that

$$\begin{aligned}
\mathscr{L}V \le{}& -\left(\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 + \left(\left(-b_1 + c_1 + a_1' + k_2\right.\right.\\
&\left.\left.+1\right)L + 8c_1 b_1^4\right)\xi_1^4 + \sum_{j=2}^{n-1}\frac{-b_j}{L^{4j-5}}\xi_j^4 + 8c_1\sum_{j=3}^{n}\frac{b_{j-1}^4}{L^{4j-8}}\xi_{j-1}^4\\
&+\sum_{j=2}^{n-1}\frac{1}{L^{4j-4}}\xi_j^3\left(\left(b_{j-1} + m_{j-1} + k_2 + 1\right)L\xi_j\right.\\
&\left.+L^2 b_{j-1,j-1}\xi_{j-1} + \cdots + L^{j-1}b_{j-1,2}\xi_2 + L^j b_{j-1,1}\xi_1\right)\\
&+\frac{1}{L^{4n-4}}\xi_n^3\left(u + L\left(b_{n-1} + m_{n-1} + k_2 + 1\right)\xi_n\right.\\
&\left.+L^2 b_{n-1,n-1}\xi_{n-1} + \cdots + L^n b_{n-1,1}\xi_1\right) \le -\left(\lambda_{\min}L\right.\\
&\left.-c_0\right)\|\varepsilon\|^4 + \left(\left(-b_1 + c_1 + a_1' + k_2 + 1\right)L + 8c_1 b_1^4\right)\xi_1^4\\
&+8c_1\sum_{j=2}^{n-1}\frac{b_j^4}{L^{4j-4}}\xi_j^4 + \sum_{j=2}^{n-1}\frac{-b_j + b_{j-1} + m_{j-1} + k_2 + 1}{L^{4j-5}}\xi_j^4\\
&+\frac{1}{L^{4n-4}}\xi_n^3 u + \sum_{j=1}^{n-1}\frac{n-j}{L^{4j-5}}\xi_j^4 + \sum_{j=2}^{n}\frac{1}{L^{4j-4}}\left(\left(-L^5\xi_{j-1}^4\right.\right.\\
&\left.+L^2 b_{j-1,j-1}\xi_{j-1}\xi_j^3\right) + \cdots + \left(-L^{4j-7}\xi_2^4\right.\\
&\left.\left.+L^{j-1}b_{j-1,2}\xi_2\xi_j^3\right) + \left(-L^{4j-3}\xi_1^4 + L^j b_{j-1,1}\xi_1\xi_j^3\right)\right)\\
&+\frac{1}{L^{4n-4}}\xi_n^3\left(u + L\left(b_{n-1} + m_{n-1} + k_2 + 1\right)\xi_n\right)\\
\le{}& -\left(\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 + \left(\left(-b_1 + c_1 + a_1' + k_2 + n\right)L\right.\\
&\left.+8c_1 b_1^4\right)\xi_1^4 + 8c_1\sum_{j=2}^{n-1}\frac{b_j^4}{L^{4j-4}}\xi_j^4 + \sum_{j=2}^{n-1}\frac{1}{L^{4j-5}}\left(-b_j + b_{j-1}\right.\\
&\left.+m_{j-1} + k_2 + 1 + n - j + d_{j-1} + d_{j-2} + \cdots + d_2\right.\\
&\left.+d_1\right)\xi_j^4 + \frac{1}{L^{4n-4}}\xi_n^3\left(u + L\left(b_{n-1} + m_{n-1} + k_2 + 1\right.\right.\\
&\left.\left.+r_{n-1} + r_{n-2} + \cdots + r_2 + r_1\right)\xi_n\right) \le -\left(\lambda_{\min}L - c_0\right)\\
&\cdot\|\varepsilon\|^4 - \sum_{j=1}^{n-1}\frac{1}{L^{4j-4}}\left(L - 8c_1 b_j^4\right)\xi_j^4 - \frac{1}{L^{4n-5}}\xi_n^4,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
d_1 &= k_1 b_{j-1,1}^{4/3},\\
d_2 &= k_1 b_{j-1,2}^{4/3}, \ldots, d_{j-2} = k_1 b_{j-1,j-2}^{4/3},\\
d_{j-1} &= k_1 b_{j-1,j-1}^{4/3},\\
r_1 &= k_1 b_{n-1,1}^{4/3},\\
r_2 &= k_1 b_{n-1,1}^{4/3}, \ldots, r_{n-2} = k_1 b_{n-1,n-2}^{4/3},\\
r_{n-1} &= k_1 b_{n-1,n-1}^{4/3},\\
b_1 &= c_1 + a_1' + k_2 + n + 1,\\
b_j &= b_{j-1} + m_{j-1} + k_2 + 1 + n - j + 1 + d_{j-1} + \cdots\\
&\quad + d_1, \quad j = 2, \ldots, n-1,\\
u &= -Lb_n\xi_n = -L\left(b_{n-1} + m_{n-1} + k_2 + 2 + r_{n-1} + r_{n-2}\right.\\
&\quad \left. + \cdots + r_2 + r_1\right)\xi_n.
\end{aligned} \tag{31}$$

Now, we choose the gain constant $L = \max\{2, c_0/\lambda_{\min}, 8c_1b_1^4, \ldots, 8c_1b_{n-1}^4\}$, and then the right-hand side of (30) becomes negative definite. Therefore, it follows from Lemma 4 that the equilibrium of the closed-loop nonlinear stochastic system is globally asymptotically stable in probability and there exists an almost surely unique strong solution on $[0, \infty)$ for each $x_0 \in \mathbb{R}^n$. □

Remark 13. Letting $g_i(x) \equiv 0$ for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, then system (16) is reduced to the deterministic system, which was studied by Qian and Lin in [5]. Therefore, we extend the conclusion for the deterministic system to the stochastic nonlinear system and construct a novel linear output feedback controller.

Remark 14. Letting $f_i(x) \equiv 0$ for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, then system (16) is reduced to that in [19], which was studied by Deng and Krstić. It is worth pointing out that the output feedback control in [19] is nonlinear, which is very complex. However, our research is based on a novel linear output feedback control. Therefore, our result extends and improves that in [19].

Remark 15. In [7], Liu and Zhang studied the stability of stochastic nonlinear systems with linearly bounded unmeasurable states by the output feedback control. It should be mentioned that the mathematical induction played a key role in the proof of the main result in [7]. However, in this paper, we construct a novel lyapunov function and prove the stability directly without using the mathematical induction, which make the proof more concise and help us construct the linear output feedback controller more easily.

4. An Example

In this section, we will use an example to illustrate our main result.

Example 1. Consider the following stochastic nonlinear system:

$$\begin{aligned}
dx_1 &= \left(x_2 + \frac{1}{16}x_1 \sin x_2^2\right)dt + \frac{1}{16}\left(2x_1^2 - x_1^3\right)dw,\\
dx_2 &= \left(u + \frac{1}{16}\left|x_1 + x_2\right|\right)dt + \frac{1}{16}\ln\left(1 + \left|x_2\right|\right)dw,\\
y &= x_1.
\end{aligned} \tag{32}$$

Obviously, the functions f_1, f_2, g_1, g_2 are locally Lipschitz such that

$$\begin{aligned}
\left|f_1(x)\right| &\le \frac{1}{16}\left|x_1\right|,\\
\left|f_2(x)\right| &\le \frac{1}{16}\left(\left|x_1\right| + \left|x_2\right|\right),\\
\left|g_1(x)\right| &\le \frac{1}{16}\left|x_1\right|,\\
\left|g_2(x)\right| &\le \frac{1}{16}\left(\left|x_1\right| + \left|x_2\right|\right),
\end{aligned} \tag{33}$$

which verifies that Assumption 11 holds. Moreover, it is easy to check that

$$\begin{aligned}
|F| &\le \frac{1}{8}\left(\left|x_1\right| + \frac{1}{L}\left|x_2\right|\right),\\
|G| &\le \frac{1}{8}\left(\left|x_1\right| + \frac{1}{L}\left|x_2\right|\right).
\end{aligned} \tag{34}$$

The linear observer system is designed as

$$\begin{aligned}
d\hat{x}_1 &= \left(\hat{x}_2 + L\left(x_1 - \hat{x}_1\right)\right)dt,\\
d\hat{x}_2 &= \left(u + L^2\left(x_1 - \hat{x}_1\right)\right)dt,
\end{aligned} \tag{35}$$

with a suitable choice of the parameter L. The observation errors $\varepsilon_1 = x_1 - \hat{x}_1$ and $\varepsilon_2 = (x_2 - \hat{x}_2)/L$ satisfy

$$\begin{aligned}
d\varepsilon &= L\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\varepsilon dt + \begin{bmatrix} f_1 \\ \dfrac{f_2}{L} \end{bmatrix}dt + \begin{bmatrix} g_1^T(y) \\ \dfrac{g_2^T(y)}{L} \end{bmatrix}dw\\
&= LA\varepsilon dt + Fdt + Gdw.
\end{aligned} \tag{36}$$

For the above matrix A, there exists a positive-definite matrix P satisfying $A^TP + PA = -I$, where

$$P = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{3}{2} \end{bmatrix}. \tag{37}$$

It is easy to get the minimum eigenvalue $\lambda_{\min} = (5 - \sqrt{5})/4$ and maximum eigenvalue $\lambda_{\max} = (5 + \sqrt{5})/4$ of the matrix P.

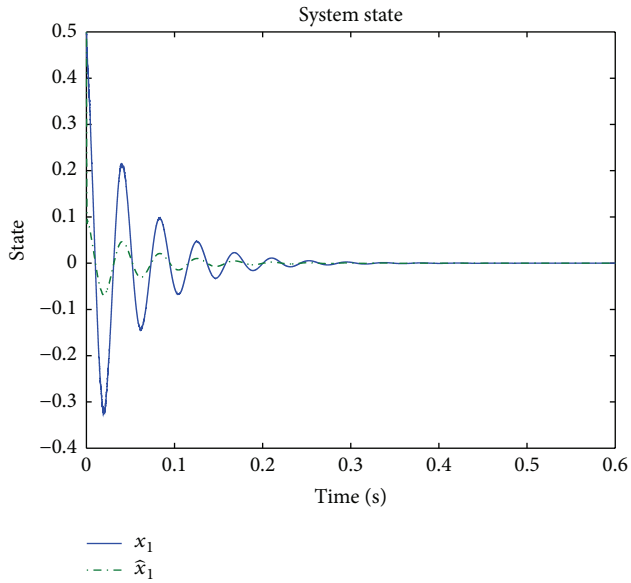


FIGURE 1: The first state response in Example 1.

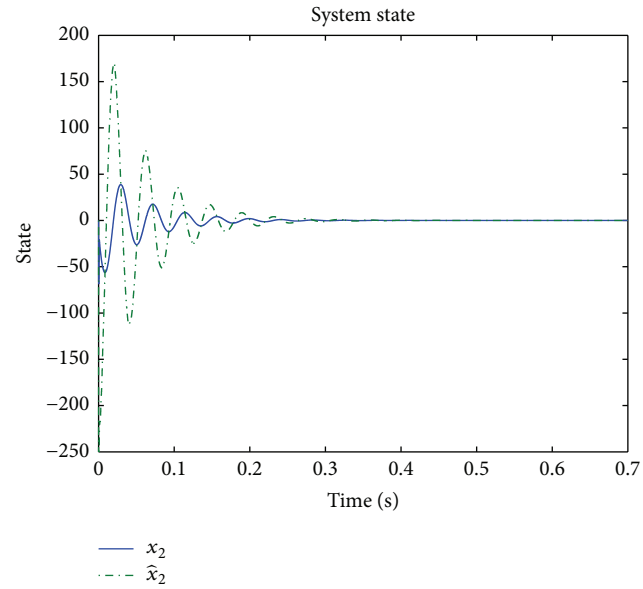


FIGURE 2: The second state response in Example 1.

Now, taking $V_0 = (3/2)(\varepsilon^T P\varepsilon)^2$, then we get

$$\mathscr{L}V_0 \le -\left(3\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 + c_1\hat{x}_1^4 + \frac{c_1}{L^4}\hat{x}_2^4, \tag{38}$$

where

$$\begin{aligned} c_0 &= 9.1\lambda_{\max}^2, \\ c_1 &= 0.1\lambda_{\max}^2. \end{aligned} \tag{39}$$

Choosing

$$V = V_0 + c_1\xi_1^4 + \frac{c_1}{L^4}\xi_2^4, \tag{40}$$

$\xi_1 = \hat{x}_1$, and $\xi_2 = \hat{x}_2 + Lb_1\hat{x}_1$, we have

$$\mathscr{L}V \le -\left(\lambda_{\min}L - c_0\right)\|\varepsilon\|^4 + \left(8c_1b_1^4 - L\right)\xi_1^4 - \frac{1}{L^3}\xi_2^4, \tag{41}$$

where $b_1 = 4$, $b_2 = 29$, $L = 664$, and $u = -Lb_2(\hat{x}_2 + Lb_1\hat{x}_1) = -Lb_2\xi_2$. Obviously,

$$V = \frac{3}{2}\left(\varepsilon^T P\varepsilon\right)^2 + \frac{1}{4}\xi_1^4 + \frac{1}{4L^4}\xi_2^4, \tag{42}$$

which is positive-definite and proper. By Theorem 12, we see that the equilibrium $x = 0$ of the nonlinear closed-loop stochastic system (32) and (35) is globally asymptotically stable in probability and there exist an almost unique strong solution on $[0, \infty)$. The state response and control input with initial conditions $x_1 = 0.5$, $x_2 = 0$, $\hat{x}_1 = 0.6$, and $\hat{x}_2 = 0$ are presented in Figures 1–3.

Figures 1–3 show that the equilibrium of the closed-loop system is unique and tends to 0 when the initial state is nonzero. In other words, for the closed-loop system, the equilibrium is globally asymptotically stable in probability and there exists an almost surely unique strong solution on $[0, \infty)$ for each $x_0 \in \mathbb{R}^n$, which verifies our theoretical results.

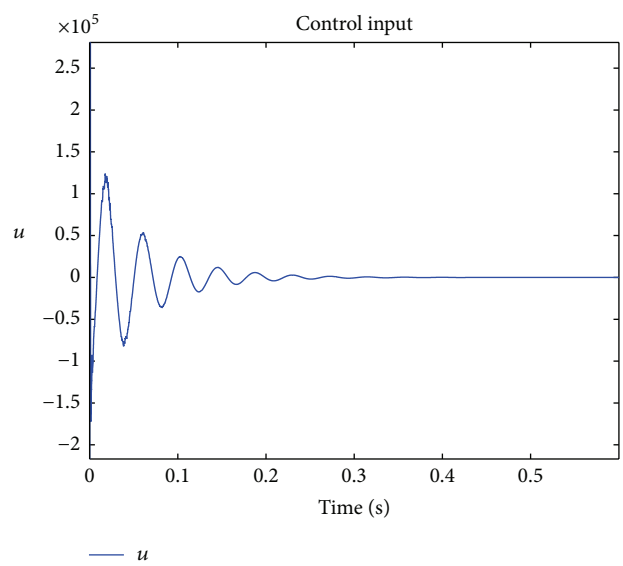


FIGURE 3: The control input in Example 1.

5. Conclusion

In this paper, we have studied the problem of globally asymptotic stability of stochastic nonlinear systems by the output feedback with a novel method. It is worth pointing out that the design of the dynamic output feedback controller plays an important role in the proof of our main result, especially that the Young inequality is a key tool. We believe that our formulation and approach can be used to analyse the stabilization problem of stochastic nonlinear systems with input delays, in which the feedback domination design will be a more complex structure.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was jointly supported by the National Natural Science Foundation of China (61374080), the Natural Science Foundation of Zhejiang Province (LY12F03010), and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] V. Andrieu and L. Praly, "A unifying point of view on output feedback designs for global asymptotic stabilization," *Automatica*, vol. 45, no. 8, pp. 1789–1798, 2009.

[2] H.-B. Du, C.-J. Qian, S.-H. Li, and M. T. Frye, "Global output feedback stabilization of a class of upper-triangular systems with input delay," in *Proceedings of the American Control Conference (ACC '12)*, pp. 5435–5439, Montréal, Canada, June 2012.

[3] D. Karagiannis, A. Astolfi, and R. Ortega, "Two results for adaptive output feedback stabilization of nonlinear systems," *Automatica*, vol. 39, no. 5, pp. 857–866, 2003.

[4] M.-L. Liu and Y.-G. Liu, "Semi-global stabilization via output-feedback for a class of uncertain nonlinear systems," *Acta Automatica Sinica*, vol. 39, no. 12, pp. 2154–2159, 2013.

[5] C.-J. Qian and W. Lin, "Output feedback control of a class of nonlinear systems: a nonseparation principle paradigm," *IEEE Transactions on Automatic Control*, vol. 47, no. 10, pp. 1710–1715, 2002.

[6] A. Teel and L. Praly, "Tools for semiglobal stabilization by partial state and output feedback," *SIAM Journal on Control and Optimization*, vol. 33, no. 5, pp. 1443–1488, 1995.

[7] S.-J. Liu and J.-F. Zhang, "Output-feedback control of a class of stochastic nonlinear systems with linearly bounded unmeasurable states," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 6, pp. 665–687, 2008.

[8] S.-J. Liu, Z.-P. Jiang, and J.-F. Zhang, "Global output-feedback stabilization for a class of stochastic non-minimum-phase nonlinear systems," *Automatica*, vol. 44, no. 8, pp. 1944–1957, 2008.

[9] W.-S. Chen, J. Wu, and L.-C. Jiao, "State-feedback stabilization for a class of stochastic time-delay nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 17, pp. 1921–1937, 2012.

[10] L. Liu and X.-J. Xie, "State feedback stabilization for stochastic feedforward nonlinear systems with time-varying delay," *Automatica*, vol. 49, no. 4, pp. 936–942, 2013.

[11] L.-C. Guo, X. Zuo, J.-W. Liu, and H.-Q. Liang, "Output-feedback control of a class of stochastic nonlinear systems with power growth conditions," *International Journal of Control, Automation, and Systems*, vol. 12, pp. 274–282, 2014.

[12] H. Deng, M. Krstić, and R. J. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1237–1253, 2001.

[13] S.-Y. Khoo, J.-L. Yin, Z.- H. Man, and X.-H. Yu, "Finite-time stabilization of stochastic nonlinear systems in strict-feedback form," *Automatica*, vol. 49, no. 5, pp. 1403–1410, 2013.

[14] Z.-J. Wu, X.-J. Xie, and S.-Y. Zhang, "Stochastic adaptive backstepping controller design by introducing dynamic signal and changing supply function," *International Journal of Control*, vol. 79, no. 12, pp. 1635–1646, 2006.

[15] Z.-J. Wu, X.-J. Xie, and S.-Y. Zhang, "Adaptive backstepping controller design using stochastic small-gain theorem," *Automatica*, vol. 43, no. 4, pp. 608–620, 2007.

[16] X.-J. Xie and L. Liu, "Further results on output feedback stabilization for stochastic high-order nonlinear systems with time-varying delay," *Automatica*, vol. 48, no. 10, pp. 2577–2586, 2012.

[17] R. Khasminskii, *Stochastic Stability of Differential Equations*, Springer, New York, NY, USA, 2011.

[18] X.-R. Mao, *Stochastic Differential Equations and Applications*, Horwood, 2008.

[19] H. Deng and M. Krstić, "Output-feedback stochastic nonlinear stabilization," *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 328–333, 1999.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 479326, 10 pages
http://dx.doi.org/10.1155/2015/479326

Research Article

The Stationary Distribution and Extinction of Generalized Multispecies Stochastic Lotka-Volterra Predator-Prey System

Fancheng Yin and Xiaoyan Yu

College of Sciences, Hohai University, Nanjing 210098, China

Correspondence should be addressed to Fancheng Yin; yfc581106@sina.com

Received 19 May 2015; Revised 21 August 2015; Accepted 30 August 2015

Academic Editor: Leonid Shaikhet

This paper is concerned with the existence of stationary distribution and extinction for multispecies stochastic Lotka-Volterra predator-prey system. The contributions of this paper are as follows. (a) By using Lyapunov methods, the sufficient conditions on existence of stationary distribution and extinction are established. (b) By using the space decomposition technique and the continuity of probability, weaker conditions on extinction of the system are obtained. Finally, a numerical experiment is conducted to validate the theoretical findings.

1. Introduction

The dynamic relationship between the predators and the preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. The classic predator-prey model is the Lotka-Volterra model, governed by the following differential equation:

$$\begin{aligned} \dot{x} &= x(a - by), \\ \dot{y} &= y(-c + fx), \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denote the prey and predator population size, respectively, at time t. For the prey component, the parameters a and b are the fixed growth and mortality rates, respectively. For the predator component, the parameters c and f are the fixed growth and mortality rates, respectively. Since then, variants of the two-species Lotka-Volterra system have been frequently investigated to describe population dynamics with predator-prey relations; see, for example, [2–4].

Recently, the multispecies predator-prey systems have received a great deal of research attention since they took the differences among individual growth and mortality into account (see [5–8]). In order to understand the nature of the competitive interactions and relationships between predator and prey, Yang and Xu [8] considered the following periodic m-prey and $n-m$-predator Lotka-Volterra differential system with periodic coefficients:

$$\begin{aligned} dx_i &= x_i\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\, x_j\right)dt, \quad i = 1, \ldots, m, \\ dx_i &= x_i\left(-b_i(t) + \sum_{j=1}^{m} a_{ij}(t)\, x_j - \sum_{j=m+1}^{n} a_{ij}(t)\, x_j\right)dt, \\ &\qquad i = m+1, \ldots, n, \end{aligned} \tag{2}$$

where $x_i(t)$, $i = 1, \ldots, m$, denotes the density of prey species at time t and $x_i(t)$, $i = m+1, \ldots, n$, denotes the density of predator species at time t. Under the assumption that $b_i(t) > 0$ $(i = 1, \ldots, m)$, $b_i(t) \geq 0$ $(i = m+1, \ldots, n)$, $a_{ii}(t) > 0$ $(i = 1, \ldots, n)$, $a_{ij} \geq 0$ $(i \neq j)$ are continuous periodic functions with a common periodic $T > 0$, a set of sufficient conditions on the existence and global attractiveness of the periodic solution to system (2) are obtained. Recently, Chen and Shi [5] further considered the almost periodic case of more complicated systems than system (2) under the almost periodic case. By constructing a suitable Lyapunov function, they obtained a set of sufficient conditions which guarantees the existence of a unique globally attractive positive almost periodic solution to the corresponding system.

On the other hand, from the biological point of view, population systems in the real world are inevitably affected by environmental noise. In the past decades, the dynamics of stochastic populations and related topic have received a great deal of research attention (see [9–21]), since they have been successfully used in a variety of application fields, including biology (see [22–28]), epidemiology (see [29, 30]), and neural networks (see [31–33]). More recently, the asymptotic properties of stochastic predator-prey systems have received a lot of attention; the readers can refer to [10, 11, 34] and the references therein. For example, the dynamics of the density dependent stochastic predator-prey system with different functional response have been studied by Ji and Jiang in [10, 11]. Vasilova [34] has investigated a stochastic Gilpin-Ayala predator-prey model with time-dependent delay, and certain asymptotic results regarding the long-time behavior of trajectories of the solution and sufficient criteria for extinction of species for a special case of the considered system are given.

In this paper, considering the effect of environmental noise, we introduce stochastic perturbation into the growth rate of the prey and the predator in system (2) and assume that parameters b_i and a_{ij} are constant. Then we obtain the following m-prey and $n-m$-predator stochastic Lotka-Volterra system with constant coefficients:

$$\begin{aligned} dx_i &= x_i\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right)dt + \sigma_i x_i dB_i(t), \\ &\qquad\qquad i = 1,\dots,m, \\ dx_i &= x_i\left(-b_i + \sum_{j=1}^{m} a_{ij}x_j - \sum_{j=m+1}^{n} a_{ij}x_j\right)dt \\ &\quad + \sigma_i x_i dB_i(t), \quad i = m+1,\dots,n, \end{aligned} \tag{3}$$

where $B(t) = (B_1(t), B_2(t), \dots, B_n(t))$ is an n-dimensional Brownian motion and σ_i^2 will be called the noise intensity. Throughout this paper, we always assume that the following hypothesis holds:

$$\begin{aligned} b_i &> 0, \quad i = 1,\dots,m, \\ b_i &\geq 0, \quad i = m+1,\dots,n, \\ a_{ii} &> 0, \quad i = 1,\dots,n, \\ a_{ij} &\geqslant 0 \quad (i \neq j). \end{aligned} \tag{4}$$

In the study of stochastic population systems, extinction and existence of stationary distribution are two important and interesting properties, respectively, meaning that the population system will die out or the distribution of the solution converges weakly to the probability measure in the future, which have received a lot of attention (see [12, 35–37]). Then one question arises naturally: under what condition can system (3) have a stationary distribution and become extinct, respectively? This issue constitutes the first motivation of this paper.

In addition, the existing literatures (see [10, 35]) show clearly that if the noise intensity of every prey species is more than twice the corresponding intrinsic growth rate, the population will become extinct exponentially. Then one interesting question is as follows: What will happen if the noise intensity equals twice the intrinsic growth rate? Thus, the second purpose of this paper is to solve this interesting problem.

The organization of the paper is as follows. Section 2 describes some preliminaries. The main results are stated in Sections 3 and 4. In Section 3, sufficient conditions are obtained under which there is a stationary distribution to system (3). By utilizing some novel stochastic analysis techniques, sufficient criteria for ensuring the extinction of system (3) are obtained in Section 4. Section 5 provides some numerical examples to check the effectiveness of the derived results. Conclusion is made in Section 6.

2. Notation

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), B_2(t), \dots, B_n(t))$ be an n-dimensional Brownian motion defined on the probability space. If $A \in R^{n\times n}$ is symmetric, its largest and smallest eigenvalues are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$. Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ be the positive equilibrium of the corresponding deterministic predator-prey system to system (3), that is, the solution to the following equation:

$$\begin{aligned} b_i - \sum_{j=1}^{n} a_{ij}x_j^* &= 0, \quad i = 1,2,\dots,m, \\ -b_i + \sum_{j=1}^{m} a_{ij}x_j^* - \sum_{j=m+1}^{n} a_{ij}x_j^* &= 0, \quad i = m+1,\dots,n. \end{aligned} \tag{5}$$

In the same way as Zhu and Yin [38] and Liu et al. [39] did, we can also show the following result on the existence of global positive solution.

Lemma 1. *Suppose that condition (4) holds; then one has the following assertions:*

(i) *For any given initial value $x_0 \in R_+^n$, there is a unique solution $x(t, x_0)$ to system (3) and the solution will remain in R_+^n with probability 1; namely,*

$$\mathbb{P}\{x(t, x_0) \in R_+^n, \ \forall t \geq 0\} = 1, \tag{6}$$

for any $x_0 \in R_+^n$.

(ii) *For any given initial value $x_0 \in R_+^n$ and any $\beta > 0$, almost every sample path of $x^\beta(t, x_0)$ is locally but uniformly Holder continuous.*

Lemma 2 (see [40]). *Let $f(t)$ be a nonnegative function defined on $[0, +\infty)$ such that $f(t)$ is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$; then $\lim_{t\to\infty} f(t) = 0$.*

3. Stationary Distribution

In this section, we mainly show that system (3) has a stationary distribution. Let us give a lemma that will be used in the following proof. Let $X(t)$ be a homogeneous Markov process in $E^n \subset R^n$ described by the following stochastic equation:

$$dX(t) = b(X)\,dt + \sum_{k=1}^{d} \sigma_k(X)\,dB_k(t). \tag{7}$$

The diffusion matrix is

$$A(x) = \left(a_{ij}(x)\right), \quad a_{ij}(x) = \sum_{k=1}^{d} \sigma_k^i(x)\,\sigma_k^j(x). \tag{8}$$

Lemma 3 (see [41]). *One assumes that there is a bounded open subset $G \subset E^n$ with a regular (i.e., smooth) boundary such that its closure $\overline{G} \subset E^n$, and*

(i) *in the domain G and some neighborhood therefore, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero;*

(ii) *if $x \in E^n \setminus G$, the mean time τ at which a path issuing from x reaches the set G is finite, and $\sup_{x\in K} E_x\tau < +\infty$ for every compact subset $K \in E^n$ and throughout this paper one sets $\inf \emptyset = \infty$.*

One then has the following assertions:

(1) *The Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E^n.*

(2) *Let $f(x)$ be a function integrable with respect to the measure $\mu(\cdot)$. Then*

$$\mathbb{P}\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t f(x(s))\,ds = \int_{E^n} f(y)\,\mu(dy)\right\} = 1. \tag{9}$$

Remark 4. The proof is given by [41] in detail. Precisely, the existence of a stationary distribution with density is obtained in Theorem 4.3 on pp. 117. The ergodic property is referred to Theorem 4.2 on pp. 110. To validate (i), we can directly show that $\lambda_{\min}\{A(x)\} > 0$. To validate (ii), it suffices to prove that there is some neighborhood U and a nonnegative C^2-function V such that, for any $x \in E^n \setminus U$, $\mathscr{L}V(x)$ is negative (for details refer to [42], pp. 1163).

Theorem 5. *Let condition (4) hold and let $x(t, x_0)$ be the global solution to system (3) with any positive initial value $x_0 \in R_+^n$. Assume that there exists $c = (c_1, c_2, \ldots, c_n) \gg 0$ such that*

$$\begin{aligned} &c_i a_{ii} - \frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij} + c_j a_{ji}\right) > 0, \quad i = 1, 2, \ldots, n, \\ &\frac{1}{2}\sum_{i=1}^{n} c_i x_i^* \sigma_i^2 \\ &\quad < \min_{1\leqslant i\leqslant n}\left\{\left(c_i a_{ii} - \frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij} + c_j a_{ji}\right)\right)\left(x_j^*\right)^2\right\}. \end{aligned} \tag{10}$$

Then there is a stationary distribution for system (3).

Proof. Let $x_i(t) = x_i(t, x_0)$ for simplicity. Applying Itô's formula to $V(x) = \sum_{i=1}^{n} c_i(x_i - x_i^* - x_i \ln(x_i/x_i^*))$ yields

$$\begin{aligned} \mathscr{L}V = &-\sum_{i=1}^{n} c_i a_{ii}\left(x_i - x_i^*\right)^2 \\ &- \sum_{i=1}^{m}\sum_{\substack{j\neq i\\ j=1}}^{m} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right) \\ &- \sum_{i=1}^{m}\sum_{j=m+1}^{n} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right) \\ &+ \sum_{i=m+1}^{n}\sum_{j=1}^{m} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right) \\ &- \sum_{i=m+1}^{n}\sum_{\substack{j\neq i\\ j=m+1}}^{n} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right) \\ &+ \frac{1}{2}\sum_{i=1}^{n} c_i x_i^* \sigma_i^2. \end{aligned} \tag{11}$$

By the inequality $2ab \leqslant (a^2 + b^2)$, we have

$$\begin{aligned} \mathscr{L}V \leqslant &-\sum_{i=1}^{n} c_i a_{ii}\left(x_i - x_i^*\right)^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{\substack{j\neq i\\ j=1}}^{m} c_i a_{ij}\left(x_i - x_i^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{\substack{j\neq i\\ j=1}}^{m} c_i a_{ij}\left(x_j - x_j^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=m+1}^{n}\sum_{\substack{j\neq i\\ j=m+1}}^{n} c_i a_{ij}\left(x_i - x_i^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=m+1}^{n}\sum_{\substack{j\neq i\\ j=m+1}}^{n} c_i a_{ij}\left(x_j - x_j^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=m+1}^{n} c_i a_{ij}\left(x_i - x_i^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=m+1}^{n} c_i a_{ij}\left(x_j - x_j^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=m+1}^{n}\sum_{j=1}^{m} c_i a_{ij}\left(x_i - x_i^*\right)^2 \\ &+ \frac{1}{2}\sum_{i=m+1}^{n}\sum_{j=1}^{m} c_i a_{ij}\left(x_j - x_j^*\right)^2 + \frac{1}{2}\sum_{i=1}^{n} c_i x_i^* \sigma_i^2 \\ = &-\sum_{i=1}^{n} c_i a_{ii}\left(x_i - x_i^*\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n} c_i a_{ij}\left(x_i - x_i^*\right)^2 \end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n}c_j a_{ij}\left(x_i-x_i^*\right)^2+\frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2$$
$$=-\sum_{i=1}^{n}\left[c_i a_{ii}-\frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij}+c_j a_{ji}\right)\right]\left(x_i-x_i^*\right)^2$$
$$+\frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2. \tag{12}$$

Note that (10); then the ellipsoid

$$\sum_{i=1}^{n}\left[c_i a_{ii}-\frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij}+c_j a_{ji}\right)\right]\left(x_i-x_i^*\right)^2 = \frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2 \tag{13}$$

lies entirely in R_+^n. Now we can take U to be a neighborhood of the ellipsoid with $\overline{U}\subseteq E_n=R_+^n$, such that, for $x\in R_+^n\setminus U$, $\mathscr{L}V<0$, which means condition (ii) of Lemma 3 is verified.

Now we begin to verify condition (i) in Lemma 3. Let us define $H(x)=\operatorname{diag}(\sigma_1x_1,\sigma_2x_2,\ldots,\sigma_nx_n)$, so the diffusion matrix is $A(x)=H^T(x)H(x)$. It is clear that $\lambda_{\min}\{H^T(x)H(x)\}\geqslant 0$. If $\lambda_{\min}\{H^T(x)H(x)\}=0$ holds, there exists $\xi=(\xi_1,\xi_2,\ldots,\xi_n)^T\in R^n$ such that $|\xi|=1$ and $\xi^TH^T(x)H(x)\xi=0$, which implies that $H(x)\xi=0$. By the definition of σ_i, $i=1,2,\ldots,n$, and $x\in R_+^n\setminus U$, we see $\xi=0$, but it contradicts with $|\xi|=1$. So $\lambda_{\min}\{H^T(x)H(x)\}>0$ for $x\in R_+^n\setminus U$ must hold. That means condition (i) of Lemma 3 is verified. Therefore, we can say that stochastic system (3) has a stationary distribution. □

Remark 6. Theorem 5 shows that system (3) has a unique stationary distribution when the perturbation is small in the sense that

$$\frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2 < \min_{1\leqslant i\leqslant n}\left\{\left(c_i a_{ii}-\frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij}+c_j a_{ji}\right)\right)\left(x_j^*\right)^2\right\}. \tag{14}$$

4. Extinction

Extinction is one of the most basic questions that can be studied in the population dynamics, which means the population system will die out. Most of the time we need to know the extinction rate of the species for which we have to make a suitable policy in advance and to make useful measures to protect them from becoming extinct.

Theorem 7. *Let condition (4) hold and let $x(t,x_0)$ be the global solution to system (3) with any initial value $x_0\in R_+^n$. Assume that there exists an integer $k\leqslant m$ such that*

$$b_i<\frac{\sigma_i^2}{2},\quad i=1,\ldots,k,$$
$$b_i=\frac{\sigma_i^2}{2},\quad i=k+1,\ldots,m. \tag{15}$$

One then has the following assertions:

(i) *For $i=1,\ldots,k$, the solution $x_i(t,x_0)$ to system (3) has the property that*

$$\lim_{t\to\infty}\frac{\log x_i(t,x_0)}{t}=b_i-\frac{\sigma_i^2}{2}\quad a.s. \tag{16}$$

That is, for each $i=1,\ldots,k$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\sigma_i/2-b_i)$.

(ii) *For $i=k+1,\ldots,m$, the solution $x_i(t,x_0)$ to system (3) has the property that*

$$\lim_{t\to\infty}x_i(t,x_0)=0,$$
$$\lim_{t\to\infty}\frac{\log x_i(t,x_0)}{t}=0\quad a.s. \tag{17}$$

That is, for each $i=k+1,\ldots,m$, the species i still becomes extinct with zero exponential extinction rate.

(iii) *For $i=m+1,\ldots,n$, the solution $x_i(t,x_0)$ to system (3) has the property that*

$$\operatorname*{Lim}_{t\to\infty}\frac{\log x_i(t,x_0)}{t}=-b_i-\frac{\sigma_i^2}{2}\quad a.s. \tag{18}$$

That is, for each $i=m+1,\ldots,n$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\sigma_i/2+b_i)$.

Proof. Let $x_i(t)=x_i(t,x_0)$ for simplicity. To make the proof clear, we are going to divide it into four steps. The first step and the third step are to show the least upper bound of exponential extinction rate for the top k preys and the predators of system (3), respectively. The second step is to show the extinction for the bottom $m-k$ preys of system (3) in the case of $\sigma_i^2=2b_i$, $i=k+1,\ldots,m$. The fourth step is to accomplish the proof of assertions (i)–(iii) based on the proof of Steps 1–3.

Step 1. Applying Itô's formula to $\log x_i(t)$ yields

$$\log x_i(t) = \log x_i(0) + \int_0^t \left(b_i - \frac{\sigma_i^2}{2}\right) ds - \int_0^t \sum_{j=1}^n a_{ij} x_j(s)\, ds + M_i(t), \quad i = 1, \ldots, m, \tag{19}$$

$$\log x_i(t) = \log x_i(0) + \int_0^t \left(\sum_{j=1}^m a_{ij} - \sum_{j=m+1}^n a_{ij}\right) x_j(s)\, ds + \int_0^t \left(-b_i - \frac{\sigma_i^2}{2}\right) ds + M_i(t), \quad i = m+1, \ldots, n, \tag{20}$$

where $M_i(t) = \int_0^t \sigma_i dB_i(s)$, $i = 1, \ldots, n$, is the real-valued continuous local martingale vanishing at $t = 0$, with the quadratic variation $\langle M_i(t), M_i(t)\rangle = \sigma_i^2 t$. Dividing both sides of (19) and (20) by t, we have that

$$\frac{1}{t}\log x_i(t) = \frac{1}{t}\log x_i(0) + \frac{1}{t}\int_0^t \left(b_i - \frac{\sigma_i^2}{2}\right) ds - \frac{1}{t}\int_0^t \sum_{j=1}^n a_{ij} x_j(s)\, ds + \frac{1}{t} M_i(t), \quad i = 1, \ldots, m, \tag{21}$$

$$\frac{1}{t}\log x_i(t) = \frac{1}{t}\log x_i(0) + \frac{1}{t}\int_0^t \left(\sum_{j=1}^m a_{ij} - \sum_{j=m+1}^n a_{ij}\right) x_j(s)\, ds + \frac{1}{t}\int_0^t \left(-b_i - \frac{\sigma_i^2}{2}\right) ds + \frac{1}{t} M_i(t), \quad i = m+1, \ldots, n. \tag{22}$$

Using the strong law of large numbers for martingales [43], we obtain that

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0 \quad \text{a.s.}, \ i = 1, \ldots, n. \tag{23}$$

For $i = 1, 2, \ldots, k$, letting $t \to \infty$ in (21) yields that

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{t} \leqslant \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(b_i - \frac{\sigma_i^2}{2}\right) ds = b_i - \frac{\sigma_i^2}{2} < 0 \quad \text{a.s.}, \ i = 1, \ldots, k. \tag{24}$$

Step 2. The main aim of this step is to show $\lim_{t\to\infty} x_i(t) = 0$ a.s. for $i = k+1, \ldots, m$. When $\sigma_i^2 = 2b_i$, (21) turns to be the following form:

$$\frac{1}{t}\log x_i(t) = \frac{1}{t}\log x_i(0) - \frac{1}{t}\int_0^t \sum_{j=1}^n a_{ij} x_j(s)\, ds + \frac{1}{t} M_i(t), \quad i = k+1, \ldots, m. \tag{25}$$

According to the convergence of $\int_0^\infty x_i(s) ds$, we can decompose the sample space into two exclusive events spaces as follows:

$$J_{i,1} = \left\{\omega : \int_0^\infty x_i(s)\, ds < \infty\right\}, \quad J_{i,2} = \left\{\omega : \int_0^\infty x_i(s)\, ds = \infty\right\}, \quad i = k+1, \ldots, m. \tag{26}$$

On the other hand we can divide the sample space into three mutually exclusive events as follows:

$$E_{i,1} = \left\{\omega : \lim_{t\to\infty} \sup x_i(t) \geqslant \lim_{t\to\infty} \inf x_i(t) = \gamma_i > 0\right\}, \quad E_{i,2} = \left\{\omega : \lim_{t\to\infty} \sup x_i(t) > \lim_{t\to\infty} \inf x_i(t) = 0\right\}, \quad E_{i,3} = \left\{\omega : \lim_{t\to\infty} x_i(t) = 0\right\}. \tag{27}$$

From the above, the proof of $\lim_{t\to\infty} x_i(t) = 0$ a.s. is equivalent to show $J_{i,1} \subset E_{i,3}$ a.s. and $J_{i,2} \subset E_{i,3}$ a.s. Now we give the process in two parts.

Part 1 of Step 2. Now we show $J_{i,1} \subset E_{i,3}$ a.s. It follows from Lemma 1 that almost every sample path of $x_i(t)$ is locally but uniformly Holder continuous and therefore almost every sample path of $x_i(t, x_0)$ must be uniformly continuous. Considering the definition of $J_{i,1}$ and Lemma 2, we obtain

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.}, \ i = k+1, \ldots, m, \tag{28}$$

which means $J_{i,1} \subset E_{i,3}$ a.s.

Part 2 of Step 2. The aim of this part is to prove that $J_{i,2} \subset E_{i,3}$ a.s. It is sufficient to show $P(J_{i,2} \cap E_{i,1}) = 0$ and $P(J_{i,2} \cap E_{i,2}) = 0$.

If this $P(J_{i,2} \cap E_{i,1}) = 0$ is not true, for any $\omega_i \in (J_{i,2} \cap E_{i,1})$ and $\varepsilon_i \in (0, \gamma_i/2)$ there exists $T(\varepsilon_i, \omega_i)$ such that $\forall t > T(\varepsilon_i, \omega_i)$

$$x_i(t) > \gamma_i - \varepsilon_i > \frac{\gamma_i}{2}, \quad i = k+1, \ldots, m \quad \text{a.s.} \tag{29}$$

Simple computations show that

$$\begin{aligned}\frac{1}{t}\int_0^t a_{ij}x_j(s)\,ds &= \frac{1}{t}\int_0^T a_{ij}x_j(s)\,ds \\ &\quad + \frac{1}{t}\int_T^t a_{ij}x_j(s)\,ds \\ &\geqslant \frac{1}{t}\int_0^T a_{ij}x_j(s)\,ds + a_{ij}\frac{t-T}{t}\frac{\gamma_i}{2} \\ &\qquad \text{a.s., } i=k+1,\ldots,m.\end{aligned} \tag{30}$$

Letting $t \to \infty$ on both sides of (30) yields

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t a_{ij}x_j(s)\,ds > \frac{1}{2}a_{ij}\gamma_i > 0 \\ \text{a.s., } i=k+1,\ldots,m.\end{aligned} \tag{31}$$

This implies that

$$\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leqslant -\frac{1}{2}\sum_{j=1}^n a_{ij}\gamma_i \quad \text{a.s., } i=k+1,\ldots,m, \tag{32}$$

which contradicts with the definition of $J_{i,2}$ and $E_{i,1}$. So $P(J_{i,2}\cap E_{i,1})=0$ must hold.

Now we proceed to show $P(J_{i,2}\cap E_{i,2})>0$ is false. Now we need more notations such as

$$\begin{aligned}B_t^\varepsilon &:= \{0\leqslant s\leqslant t : x(s)\geqslant \varepsilon\}, \\ h_t^\varepsilon &:= \frac{m(B_t^\varepsilon)}{t}, \\ h^\varepsilon &:= \limsup_{t\to\infty} h_t^\varepsilon, \\ H^\varepsilon &:= \{\omega\in J_{i,2}\cap E_{i,2} : h^\varepsilon > 0\},\end{aligned} \tag{33}$$

where $m(B_t^\varepsilon)$ means the length of B_t^ε. From the definition of H^ε, we can easily get that $H^0 = J_{i,2}\cap E_{i,2}$. The following is right for any $\varepsilon_1<\varepsilon_2$:

$$\begin{aligned}B_t^{\varepsilon_2} &\subset B_t^{\varepsilon_1}, \\ m(B_t^{\varepsilon_2}) &\leqslant m(B_t^{\varepsilon_1}), \\ h_t^{\varepsilon_2} &\leqslant h_t^{\varepsilon_1},\end{aligned} \tag{34}$$

which yields

$$\begin{aligned}h_t^{\varepsilon_2} &\leqslant h_t^{\varepsilon_1}, \\ H^{\varepsilon_2} &\subset H^{\varepsilon_1}, \\ &\qquad \forall \varepsilon_1<\varepsilon_2.\end{aligned} \tag{35}$$

From the continuity of probability, we can obviously get

$$P(H^\varepsilon) \longrightarrow P(H^0) = P(J_{i,2}\cap E_{i,2}), \quad \text{as } \varepsilon\longrightarrow 0. \tag{36}$$

Based on the hypothesis $P(J_{i,2}\cap E_{i,2})>0$, we can claim that there exists $\theta>0$ such that $P(H^\theta)>0$. It is therefore to see that, for any $\omega\in H^\theta$,

$$\begin{aligned}\frac{1}{t}\sum_{j=1}^n\int_0^t a_{ij}x_j(s)\,ds &= \frac{1}{t}\sum_{j=1}^n\int_{H_t^\theta} a_{ij}x_j(s)\,ds \\ &\quad + \frac{1}{t}\sum_{j=1}^n\int_{[0,t]\setminus H_t^\theta} a_{ij}x_j(s)\,ds \\ &\geqslant \frac{1}{t}\sum_{j=1}^n\int_{H_t^\theta} a_{ij}x_j(s)\,ds \\ &\geqslant \sum_{j=1}^n a_{ij}\theta\frac{m(B_t^\theta)}{t} \\ &\qquad \text{a.s., } i=k+1,\ldots,m.\end{aligned} \tag{37}$$

Letting $t\to\infty$ on both sides of (37) yields

$$\frac{1}{t}\sum_{j=1}^n\int_0^t a_{ij}x_j(s)\,ds \geqslant \sum_{j=1}^n a_{ij}\theta h^\theta \quad \text{a.s., } i=k+1,\ldots,m, \tag{38}$$

which means

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log x_i(t) \leqslant -\sum_{j=1}^n a_{ij}\theta h^\theta \\ \text{a.s., } i=k+1,\ldots,m.\end{aligned} \tag{39}$$

This also contradicts with the definition of $J_{i,2}$ and $E_{i,2}$. It immediately yields that the assertion $P(J_{i,2}\cap E_{i,2})=0$ must hold. Now we can claim that $P(J_{i,2}\cap E_{i,1})=0$ and $P(J_{i,2}\cap E_{i,2})=0$, which means $J_{i,2}\subset E_{i,3}$. Combining with the fact $J_{i,1}\subset E_{i,3}$ and $J_{i,2}\subset E_{i,3}$, we have

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i=k+1,\ldots,m. \tag{40}$$

Step 3. It follows from (24) and (40) that

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i=1,\ldots,m. \tag{41}$$

This implies

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^m\int_0^t a_{ij}x_j(s)\,ds = 0 \quad \text{a.s.} \tag{42}$$

Now letting $t\to\infty$ on both sides of (22) yields

$$\begin{aligned}\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leqslant -b_i - \frac{\sigma_i^2}{2} < 0 \\ \text{a.s., } i=m+1,\ldots,n.\end{aligned} \tag{43}$$

Step 4. It is immediate from (40) and (43) that

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i=1,\ldots,n. \tag{44}$$

This implies

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{n}\int_0^t a_{ij}x_j(s)\,ds = 0 \quad \text{a.s., } i=1,\dots,n. \tag{45}$$

By taking limit on both sides of (21) and (22), we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\log x_i(t)}{t} &= b_i - \frac{\sigma_i^2}{2} \quad \text{a.s., } i=1,\dots,k,\\ \lim_{t\to\infty}\frac{\log x_i(t)}{t} &= -b_i - \frac{\sigma_i^2}{2} \quad \text{a.s., } i=m+1,\dots,n.\end{aligned} \tag{46}$$

So assertions (i)–(iii) of Theorem 7 must hold. □

Remark 8. Note that, for $m = n$, system (3) becomes the following classic stochastic Lotka-Volterra competitive systems, which have received much attention (see [12, 36, 38]):

$$dx_i = x_i\left(b_i - \sum_{j=1}^{n}a_{ij}x_j\right)dt + \sigma_i x_i dB_i(t), \quad i=1,\dots,n. \tag{47}$$

And condition (4) becomes the following form:

$$\begin{aligned}&b_i > 0, \quad i=1,\dots,n,\\ &a_{ii} > 0, \quad i=1,\dots,n,\\ &a_{ij} \geqslant 0 \quad (i\neq j).\end{aligned} \tag{48}$$

Thus, by Theorem 7, we have the sufficient conditions on extinction for system (45) as Corollary 9.

Corollary 9. *Let condition (48) hold and let $x(t,x_0)$ be the global solution to system (47) with any initial value x_0. Assume that there exists an integer $k \leqslant n$ such that*

$$\begin{aligned}b_i &< \frac{\sigma_i^2}{2}, \quad i=1,\dots,k,\\ b_i &= \frac{\sigma_i^2}{2}, \quad i=k+1,\dots,n.\end{aligned} \tag{49}$$

One then has the following assertions:

(i) *For $i=1,\dots,k$, the solution $x_i(t,x_0)$ to system (47) has the property that*

$$\lim_{t\to\infty}\frac{\log x_i(t)}{t} = b_i - \frac{\sigma_i^2}{2} \quad a.s. \tag{50}$$

That is, for each $i = 1,\dots,k$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\sigma_i/2 - b_i)$.

(ii) *For $i=k+1,\dots,n$, the solution $x_i(t,x_0)$ to system (47) has the property that*

$$\begin{aligned}&\lim_{t\to\infty}x_i(t) = 0,\\ &\lim_{t\to\infty}\frac{\log x_i(t)}{t} = 0 \quad a.s.\end{aligned} \tag{51}$$

That is, for each $i=k+1,\dots,n$, the species i still becomes extinct with zero exponential extinction rate.

By using some novel stochastic analysis techniques, we point out that species i is still extinct when $\sigma^2 = 2b_i$. In comparison with Theorem 4.1 in [12], the conditions imposed on the extinction are weaker.

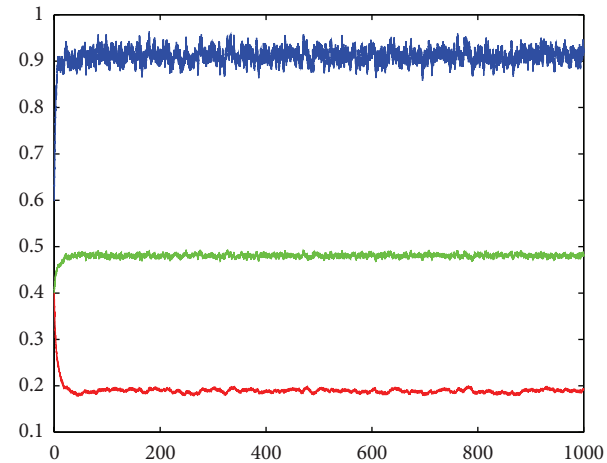


Figure 1: Computer simulation of $x_1(t)$, $x_2(t)$, and $x_3(t)$ generated by the Heun scheme for time step $\Delta = 10^{-3}$ for system (52) on $[0,1000]$, respectively.

5. Example and Simulations

In this section, we present a numerical example to illustrate the usefulness and flexibility of the theorem developed in the previous section.

Example 1. Consider a 3-dimensional stochastic Lotka-Volterra predator-prey system as follows:

$$\begin{aligned}dx_1 &= x_1(0.9 - 0.8x_1 - 0.2x_2 - 0.4x_3)\,dt + \sigma_1 x_1 dB_1(t),\\ dx_2 &= x_2(0.8 - 0.3x_1 - 0.9x_2 - 0.5x_3)\,dt + \sigma_2 x_2 dB_2(t),\\ dx_2 &= x_3(-0.1 + 0.3x_2 + 0.1x_2 - 0.6x_3)\,dt + \sigma_3 x_3 dB_3(t).\end{aligned} \tag{52}$$

System (52) is exactly system (3) with $n=3$, $m=2$, $a_{11} = 0.8 > 0$, $a_{12} = 0.2 > 0$, $a_{13} = 0.4 > 0$, $a_{21} = 0.3 > 0$, $a_{22} = 0.9 > 0$, $a_{23} = 0.5 > 0$, $a_{31} = 0.3 > 0$, $a_{32} = 0.1 > 0$, $a_{33} = 0.6 > 0$, $b_1 = 0.9 > 0$, $b_2 = 0.8 > 0$, and $b_3 = 0.1 > 0$. We compute that the equilibrium $(x_1^*, x_2^*, x_3^*)' = (0.9107, 0.4808, 0.1882)'$. The existence and uniqueness of the solution follow from Lemma 1. On the condition of the suitable parameters, we can get the simulation figures with initial value $(x_1(0), x_2(0), x_3(0)) = (0.6, 0.4, 0.4)$ by MATLAB. In Figures 1–3, the blue line represents the population of $x_1(t)$,

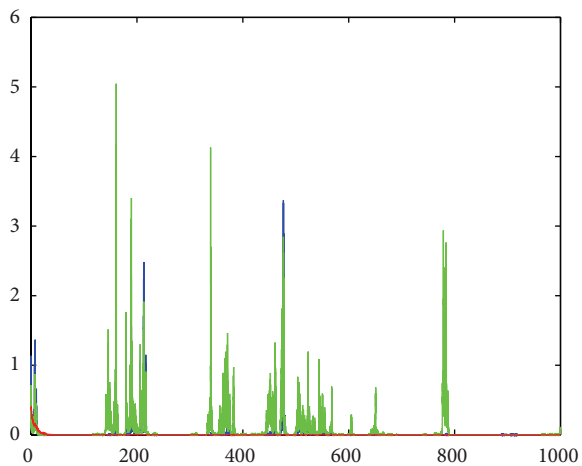


FIGURE 2: Computer simulation of a single path of $x_i(t)$, $i = 1, 2, 3$, generated by the Heun scheme for time step $\Delta = 10^{-3}$ for system (52) on $[0, 1000]$, respectively.

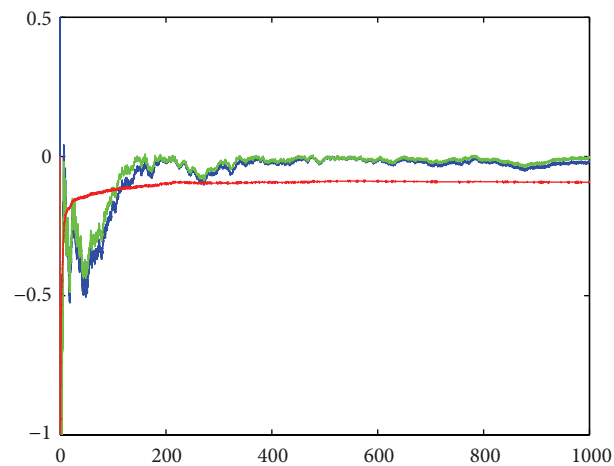


FIGURE 3: Computer simulation of $(\log x_i(t))/t$, $i = 1, 2, 3$, generated by the Heun scheme for time step $\Delta = 10^{-3}$ for system (52) on $[0, 1000]$, respectively.

the green line represents the population of $x_2(t)$, and the red line represents the population of $x_3(t)$.

(i) $(\sigma_1, \sigma_2, \sigma_3)' = (0.02, 0.01, 0.005)'$: choosing $c_1 = 1$, $c_2 = 1$, $c_3 = 1.5$, we further compute that

$$\begin{aligned}
&c_1 a_{11} - \frac{1}{2}\left(c_1 a_{12} + c_2 a_{21} + c_1 a_{13} + c_3 a_{31}\right) = 0.05 > 0,\\
&c_2 a_{22} - \frac{1}{2}\left(c_2 a_{21} + c_1 a_{12} + c_2 a_{23} + c_3 a_{32}\right) = 0.325 > 0,\\
&c_3 a_{33} - \frac{1}{2}\left(c_3 a_{31} + c_1 a_{13} + c_3 a_{32} + c_2 a_{23}\right) = 0.20 > 0,\\
&c_1 a_{11}\\
&\quad - \frac{1}{2}\left[\left(c_1 a_{12} + c_2 a_{21}\right)\left(x_2^*\right)^2 + \left(c_1 a_{13} + c_3 a_{31}\right)\left(x_3^*\right)^2\right]\\
&\quad = 0.72715466,\\
&c_2 a_{22}\\
&\quad - \frac{1}{2}\left[\left(c_2 a_{21} + c_1 a_{12}\right)\left(x_1^*\right)^2 + \left(c_2 a_{23} + c_3 a_{32}\right)\left(x_3^*\right)^2\right]\\
&\quad = 0.68114512,\\
&c_3 a_{33}\\
&\quad - \frac{1}{2}\left[\left(c_3 a_{31} + c_1 a_{13}\right)\left(x_1^*\right)^2 + \left(c_3 a_{32} + c_2 a_{23}\right)\left(x_2^*\right)^2\right]\\
&\quad = 0.47238604,\\
&\frac{1}{2}\sum_{i=1}^{3} c_i x_i^* \sigma_i^2 = 0.000209709\\
&\quad < \min_{1\leqslant i\leqslant 3}\left\{0.72715466, 0.68114512, 0.47238604\right\}.
\end{aligned} \tag{53}$$

By virtue of Theorem 5, system (52) has a unique stationary distribution.

(ii) $(\sigma_1, \sigma_2, \sigma_3)' = (1.38, \sqrt{1.6}, 0.05)'$: note that $\sigma_1^2 = 1.9044 > 2b_1 = 1.8$, $\sigma_2^2 = 1.6 = 2b_2 = 1.6$; by virtue of Theorem 7, system (52) is extinctive. From Figure 2, we can see that the predator population will die out though it suffers the small white noise when the prey population becomes extinct.

But we cannot see the value of the extinction rate of the three populations clearly. So we give Figure 3 to show $(\log x_i(t))/t$ for $i = 1, 2, 3$. According to Theorem 7, we can compute that, for $i = 1$, the growth rate is -0.05220 (it is said that the extinction rate is 0.05220). By the same method, we can know that the extinction rate is 0.00000 for $i = 2$ and the extinction rate is 0.10125 for $i = 3$.

6. Conclusion

This paper is devoted to the existence of stationary distribution and extinction for multispecies stochastic Lotka-Volterra predator-prey system. Firstly, by applying Lyapunov methods, sufficient conditions for ensuring the existence of stationary distribution of the system are obtained. Secondly, some novel techniques have been developed to derive weaker sufficient conditions under which the system is extinctive. Finally, numerical experiment is provided to illustrate the effectiveness of our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The project reported here is supported by the National Science Foundation of China (Grant nos. 61304070 and 11271146) and the Fundamental Research Funds for the Central Universities of China (Grant no. 2013B00614).

References

[1] H. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, NY, USA, 1980.

[2] A. S. Ackleh, D. F. Marshall, and H. Heatherly, "Extinction in a generalized Lotka-Volterra predator-prey model," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 13, no. 3, pp. 287–297, 2000.

[3] A. Korobeinikov and G. C. Wake, "Global properties of the three-dimensional predator-prey Lotka-Volterra systems," *Journal of Applied Mathematics and Decision Sciences*, vol. 3, no. 2, pp. 155–162, 1999.

[4] M. Zhien and W. Wendi, "Asymptotic behavior of predator-prey system with time dependent coefficients," *Applicable Analysis*, vol. 34, no. 1-2, pp. 79–90, 1989.

[5] F. Chen and C. Shi, "Global attractivity in an almost periodic multi-species nonlinear ecological model," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 376–392, 2006.

[6] C. R. Li and S. J. Lu, "The qualitative analysis of N-species prey-competition systems with nonlinear relations and periodic coefficients," *A Journal of Chinese Universities*, vol. 12, no. 2, pp. 147–156, 1997 (Chinese).

[7] X. Liao, S. Zhou, and Y. Chen, "On permanence and global stability in a general Gilpin-Ayala competition predator-prey discrete system," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 500–509, 2007.

[8] P. Yang and R. Xu, "Global attractivity of the periodic Lotka-Volterra system," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 221–232, 1999.

[9] N. Bradul and L. Shaikhet, "Stability of the positive point of equilibrium of Nicholson's blowflies equation with stochastic perturbations: numerical analysis," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 92959, 25 pages, 2007.

[10] C. Ji, D. Jiang, and N. Shi, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 2, pp. 482–498, 2009.

[11] C. Ji and D. Jiang, "Dynamics of a stochastic density dependent predator-prey system with Beddington-DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 381, no. 1, pp. 441–453, 2011.

[12] D. Jiang, C. Ji, X. Li, and D. O'Regan, "Analysis of autonomous Lotka-Volterra competition systems with random perturbation," *Journal of Mathematical Analysis and Applications*, vol. 390, no. 2, pp. 582–595, 2012.

[13] V. Kolmanovskii and A. V. Tikhonov, "Stability in probability of the Volterra-Lotka system," *Differential Equations*, vol. 32, pp. 1477–1484, 1996.

[14] A. Kovalev, V. Kolmanovskii, and L. Shaikhet, "Riccati equations in the stability of retarded stochastic linear systems," *Automation and Remote Control*, vol. 59, pp. 1379–1394, 1998.

[15] V. Kolmanovskii and L. Shaikhet, "Construction of lyapunov functionals for stochastic hereditary systems: a survey of some recent results," *Mathematical and Computer Modelling*, vol. 36, no. 6, pp. 691–716, 2002.

[16] L. Liu and Y. Shen, "Sufficient and necessary conditions on the existence of stationary distribution and extinction for stochastic generalized logistic system," *Advances in Difference Equations*, vol. 2015, article 10, 2015.

[17] L. Liu and Y. Shen, "New criteria on persistence in mean and extinction for stochastic competitive Lotka-Volterra systems with regime switching," *Journal of Mathematical Analysis and Applications*, vol. 430, no. 1, pp. 306–323, 2015.

[18] L. Shaikhet, "Stability of a positive equilibrium state for a stochastically perturbed mathematical model of glassy-winged sharpshooter population," *Mathematical Biosciences and Engineering*, vol. 11, no. 5, pp. 1167–1174, 2014.

[19] H. Wang and Q. Zhu, "Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form," *Automatica*, vol. 54, pp. 284–291, 2015.

[20] J. Zhao and W. Chen, "Global asymptotic stability of a periodic ecological model," *Applied Mathematics and Computation*, vol. 147, no. 3, pp. 881–892, 2004.

[21] Q. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Levy noise," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 126–142, 2014.

[22] G. Q. Cai and Y. K. Lin, "Stochastic analysis of predator-prey type ecosystems," *Ecological Complexity*, vol. 4, no. 4, pp. 242–249, 2007.

[23] N. H. Du, R. Kon, K. Sato, and Y. Takeuchi, "Dynamical behavior of Lotka-Volterra competition systems: non-autonomous bistable case and the effect of telegraph noise," *Journal of Computational and Applied Mathematics*, vol. 170, no. 2, pp. 399–422, 2004.

[24] L. Dong and Y. Takeuchi, "Impulsive control of multiple Lotka-Volterra systems," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 2, pp. 1144–1154, 2013.

[25] Y. Takeuchi, "Diffusion effect on stability of Lotka-Volterra models," *Bulletin of Mathematical Biology*, vol. 48, no. 5-6, pp. 585–601, 1986.

[26] Y. Takeuchi, "Diffusion-mediated persistence in two-species competition Lotka-Volterra model," *Mathematical Biosciences*, vol. 95, no. 1, pp. 65–83, 1989.

[27] Y. Takeuchi, N. H. Du, N. T. Hieu, and K. Sato, "Evolution of predator-prey systems described by a Lotka-Volterra equation under random environment," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 938–957, 2006.

[28] Y. Takeuchi, Y. Iwasa, and K. Sato, *Mathematics for Ecology and Environmental Sciences*, Springer, Berlin, Germany, 2007.

[29] A. Lahrouz and L. Omari, "Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence," *Statistics & Probability Letters*, vol. 83, no. 4, pp. 960–968, 2013.

[30] Q. Yang, D. Jiang, N. Shi, and C. Ji, "The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence," *Journal of Mathematical Analysis and Applications*, vol. 388, no. 1, pp. 248–271, 2012.

[31] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic BAM neural networks with impulse control and mixed time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 3, pp. 467–479, 2012.

[32] Q. Zhu and J. Cao, "Robust exponential stability of markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 8, pp. 1314–1325, 2010.

[33] Q. Zhu and J. Cao, "Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 2, pp. 341–353, 2011.

[34] M. Vasilova, "Asymptotic behavior of a stochastic Gilpin-Ayala predator-prey system with time-dependent delay," *Mathematical and Computer Modelling*, vol. 57, no. 3-4, pp. 764–781, 2013.

[35] X. Li, A. Gray, D. Jiang, and X. Mao, "Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 11–28, 2011.

[36] X. Li and X. Mao, "Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation," *Discrete and Continuous Dynamical Systems Series A*, vol. 24, no. 2, pp. 523–545, 2009.

[37] X. Mao, "Stationary distribution of stochastic population systems," *Systems & Control Letters*, vol. 60, no. 6, pp. 398–405, 2011.

[38] C. Zhu and G. Yin, "On hybrid competitive Lotka-Volterra ecosystems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 12, pp. 1370–1379, 2009.

[39] L. Liu, Y. Shen, and F. Jiang, "The almost sure asymptotic stability and pth moment asymptotic stability of nonlinear stochastic differential systems with polynomial growth," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1985–1990, 2011.

[40] V. Popov, *Hyperstability of Control System*, Springer, Berlin, Germany, 1973.

[41] R. Hasminskii, *Stochastic Stability of Differential Equations*, Springer, Berlin, Germany, 2011.

[42] C. Zhu and G. Yin, "Asymptotic properties of hybrid diffusion systems," *SIAM Journal on Control and Optimization*, vol. 46, no. 4, pp. 1155–1179, 2007.

[43] X. Mao, *Stochastic Differential Equations and Application*, Horwood Publishing Limited, Chichester, UK, 2nd edition, 2007.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 382756, 7 pages
http://dx.doi.org/10.1155/2015/382756

Research Article

H_2/H_∞ Control Design of Detectable Periodic Markov Jump Systems

Ting Hou[1] and Hongji Ma[2]

[1] *College of Mathematics and Systems Science, State Key Laboratory of Mining Disaster Prevention and Control Co-Founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, China*
[2] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Hongji Ma; mhj3027@sina.com

Received 22 May 2015; Accepted 6 September 2015

Academic Editor: Leonid Shaikhet

An infinite horizon H_2/H_∞ control problem is addressed for discrete-time periodic Markov jump systems with (x, u, v)-dependent noise. Above all, by use of the spectral criterion of detectability, an extended Lyapunov stability theorem is developed for the concerned dynamics. Further, based on a game theoretic approach, a state-feedback H_2/H_∞ control design is proposed. It is shown that under the condition of detectability H_2/H_∞ feedback gain can be constructed through the solution of a group of coupled periodic difference equations.

1. Introduction

H_∞ control has been one of the most active areas of modern control theory since the 1970s. Owing to the introduction of state-space approach [1], many researchers have been inspired to extend the deterministic H_∞ control theory to various stochastic systems; see [2–5]. In the development of stochastic H_∞ theory, [2] can be regarded as a pioneering work, which firstly established a stochastic version of bounded real lemma for linear Itô-type differential systems. Besides, initialed from [6], considerable progress has been made in the study of stochastic H_2/H_∞ control. By combing H_∞ index with an quadratic cost performance, the resulting multiobjective control strategy is more attractive than the sole H_∞ control in engineering applications.

The main objective of this paper is to settle an infinite horizon H_2/H_∞ control problem for periodic Markov jump systems with multiplicative noises. By now, Markov jump systems have been extensively investigated [7–9]. For example, stochastic and robust stability have been elaborately discussed in [10, 11] for networked dynamics with Markovian jump. As concerns H_∞ theory, an H_∞ estimation problem was tackled in [12] for a class of discrete homogeneous Markov jump systems. On the other hand, an infinite horizon H_∞ control problem was handled in [13] for nonlinear Itô systems with homogeneous Markov process. However, few results have been reported for H_2/H_∞ control of periodic Markov jump systems. To some extent, this study will generalize the work of [14] to the case of periodically time-varying coefficients and transition probabilities, as in [15–17].

The remainder of this paper is organized as follows. Section 2 gives basic preliminaries and problem formulations. In Section 3, the intrinsic relationship between asymptotic mean square stability and detectability is addressed. As a result, a Barbashin-Krasovskii-type theorem is established for periodic Markov jump systems with state-dependent noises. Section 4 contains an internally stabilizing control design, which can not only fulfill the prescribed disturbance attenuation level, but also minimize the output energy. To verify the effectiveness of the proposed approach, a numerical example is supplied in Section 5. Finally, Section 6 concludes this paper with a concluding remark.

Notations. R^n ($\mathscr{C}^n$) is n-dimensional real (complex) space with the usual Euclidean norm $\|\cdot\|$; $R^{n\times m}$ is the space of all $n \times m$ real matrices with the operator norm $\|\cdot\|_2$; S_n is

the set of all $n \times n$ symmetric matrices whose entries may be complex; $A > 0$ (≥ 0): $A \in S_n$ is a positive (semi)definite matrix; A' is the transpose of a matrix (vector) A; I_n is the $n \times n$ identity matrix; $\mathbb{Z}_+ := \{0, 1, \ldots\}$ and $\mathbb{Z}_{1+} := \{1, 2, \ldots\}$; $\mathcal{D} = \{1, 2, \ldots, N\}$; $\otimes$ is the operation of Kronecker product; $\mathrm{Ker}(\cdot)$ is the kernel of a matrix; $\mathrm{diag}\{\cdot\}$ is a (block) diagonal matrix.

2. Preliminaries

On a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we consider the following discrete-time periodic Markov jump system with (x, u, v)-dependent noise:

$$\begin{aligned} x(t+1) &= A_0(t,\eta_t)x(t) + B_0(t,\eta_t)v(t) + G_0(t,\eta_t) \\ &\quad \cdot u(t) + \sum_{s=1}^{r}\left[A_s(t,\eta_t)x(t) + B_s(t,\eta_t)v(t)\right. \\ &\quad \left. + G_s(t,\eta_t)u(t)\right]w_s(t), \\ z(t) &= \begin{bmatrix} C(t,\eta_t)x(t) \\ D(t,\eta_t)u(t) \end{bmatrix}, \\ D(t,\eta_t)'D(t,\eta_t) &= I_{n_u}, \\ & \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^{n_u}$, $v(t) \in R^{n_v}$, and $z(t) \in R^{n_z}$ denote the system state, control input, exogenous disturbance, and measurement output, respectively. Assume that $\{w(t) \mid w(t) = (w_1(t), \ldots, w_r(t))', t \in \mathbb{Z}_+\}$ is a sequence of independent random vectors such that $E[w(t)] = 0$ and $E[w(t)w(s)'] = I_r\delta(t-s)$ ($\delta(\cdot)$ is a Kronecker function). The Markov chain $\{\eta_t\}_{t\in\mathbb{Z}+}$ takes values in $\mathcal{D}$ with a nondegenerate transition probability matrix $\mathcal{P}_t = [p_t(i,j)]_{N\times N}(p_t(i,j) := \mathcal{P}(\eta_{t+1} = j \mid \eta_t = i))$ and the initial distribution $\pi_0(i) = \mathcal{P}(\eta_0 = i) > 0$ for all $i \in \mathcal{D}$. As usual, we set that $\{\eta_t\}_{t\in\mathbb{Z}+}$ is independent of the stochastic process $\{w(t)\}_{t\in\mathbb{Z}+}$ and its mode is measurable in real time. Moreover, the coefficients of (1) are θ-periodic (e.g., $A_s(t,i) = A_s(t+\theta, i)$) and the transition probability of η_t satisfies $p_t(i,j) = p_{t+\theta}(i,j)$ $(i, j \in \mathcal{D})$, where $\theta \in \mathbb{Z}_{1+}$. Let $\mathcal{F}_k$ be σ-algebra generated by $\{\eta_t, w(s) \mid 0 \leq t \leq k,\ 0 \leq s \leq k-1\}$. In the case of $k = 0$, $\mathcal{F}_0 = \sigma\{\eta_0\}$. Denote by $l^2(0,\infty; R^m)$ the space of R^m-valued, nonanticipative square summable stochastic processes $\{y(t,\omega) : \mathbb{Z}_+ \times \Omega \to R^m\}$ which are $\mathcal{F}_k$-measurable for all $k \in \mathbb{Z}_+$ and $\sum_{t=0}^{\infty} E\|y(t)\|^2 < +\infty$. It is clear that $l^2(0,\infty; R^m)$ is a real Hilbert space with the norm induced by the usual inner product: $\|y(\cdot)\|_{l^2(0,\infty;R^m)} = (\sum_{t=0}^{\infty} E\|y(t)\|^2)^{1/2} < +\infty$.

Definition 1 (see [18]). The zero-state equilibrium of discrete-time periodic Markov jump systems

$$x(t+1) = A(t,\eta_t)x(t) + \sum_{s=1}^{r} A_s(t,\eta_t)x(t)w_s(t), \tag{2}$$

or $(\mathbb{A}; \mathbb{P})$ is called asymptotically mean square stable (AMSS) if $\lim_{t\to\infty} E\|x(t;\xi,i,t_0)\|^2 = 0$ for all $(\xi, i) \in R^n \times \mathcal{D}$ and $t_0 \in \mathbb{Z}_+$. Here, $x(t;\xi,i,t_0)$ is the state of (2) corresponding to the initial state $x(t_0) = \xi \in R^n$ and $\eta_{t_0} = i \in \mathcal{D}$. Moreover, if there exists θ-periodic sequence $\{K(t,i)\}_{t\in\mathbb{Z}_+} \in R^{n\times n_u}$ $(i \in \mathcal{D})$ such that the zero-state equilibrium of the closed-loop system

$$\begin{aligned} &x(t+1) \\ &\quad = \left[A_0(t,\eta_t) + G_0(t,\eta_t)K(t,\eta_t)\right]x(t) \\ &\qquad + \sum_{s=1}^{r}\left[A_s(t,\eta_t) + G_s(t,\eta_t)K(t,\eta_t)\right]x(t)w_s(t) \end{aligned} \tag{3}$$

is AMSS for any $(x_0, \eta_0) \in R^n \times \mathcal{D}$, then $(\mathbb{A}, \mathbb{G}; \mathbb{P})$ is called stochastically stabilizable and $u(t) = K(t,\eta_t)x(t)$ is called a stabilizing feedback.

By Theorem 3.10 [18], we know that system (2) is asymptotically mean square stable if and only if it is exponentially mean square stable.

Definition 2 (see [19]). The periodic Markov jump system with measurement equation

$$\begin{aligned} x(t+1) &= A_0(t,\eta_t)x(t) + \sum_{s=1}^{r} A_s(t,\eta_t)x(t)w_s(t), \\ z(t) &= C(t,\eta_t)x(t), \\ & \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{4}$$

or $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is called (uniformly) detectable if for any $t_0 \in \mathbb{Z}_+$, $\xi \in R^n$, and $\eta_{t_0} \in \mathcal{D}$, there holds

$$\begin{aligned} z(t) \equiv 0 \quad (\text{a.s.}),\ t \in [t_0, T],\ \forall T > t_0 \\ \Longrightarrow \lim_{t\to\infty} E\left\|x(t;\xi,i,t_0)\right\|^2 = 0. \end{aligned} \tag{5}$$

In this paper, we will deal with the infinite horizon optimal H_2/H_∞ control problem about (1). More specifically, for a prescribed disturbance attenuation level $\gamma > 0$, we aim to find a linear, memoryless, periodic state-feedback controller $u^*(t) \in l^2(0,\infty; R^{n_u})$ such that [20]

(i) when $v(t) \equiv 0$, the closed-loop state of (1) corresponding to $u = u^*(t)$ is AMSS;

(ii) the l^2-induced norm of $L_\infty^{u^*}$ satisfies $\|L_\infty^{u^*}\| < \gamma$, where $L_\infty^{u^*}$ is the perturbation operator defined by $L_\infty^{u^*}(v) = z(t;0,v,u^*)$; it is notable that $z(t;0,v,u^*)$ is the output of (1) corresponding to $x(0) = 0$ and $u = u^*(t)$, while $v(t) \in l^2(0,\infty; R^{n_v})$ is arbitrary random disturbance;

(iii) when the worst-case disturbance $v^*(t)$, if existing, is imposed on (1), $u^*(t)$ minimizes the corresponding output energy $J_2(u, v^*) := \sum_{t=0}^{\infty} E\|z(t)\|^2$.

3. Stability and Detectability

In this section, we will focus on the detectability of periodic Markov jump system (1). This structural property will play

an essential role in the treatment of H_2/H_∞ control problem. Firstly, we present several instrumental operators.

Let S_n^N (resp., S_n^{N+}) indicate the set of all N sequences $V = (V(1), \ldots, V(N))$ with $V(i) \in S_n$ (resp., $V(i) \geq 0$). Thus, S_n^N is a Hilbert space with the inner product:

$$\langle U, V \rangle = \sum_{i=1}^{N} \operatorname{Tr}(U(i)V(i)), \quad \forall U, V \in S_n^N. \tag{6}$$

Given $U \in S_n^N$, let $\mathscr{L}_t : S_n^N \to S_n^N$ be a Lyapunov operator defined as $\mathscr{L}_t(U) = (\mathscr{L}_t(U,1), \ldots, \mathscr{L}_t(U,N))$, where

$$\mathscr{L}_t(U,i) = \sum_{s=0}^{r} \sum_{j=1}^{N} p_t(j,i) A_s(t,j) U(j) A_s(t,j)'. \tag{7}$$

Then, associated with inner product (6), the adjoint operator of $\mathscr{L}_t$ is given by $\mathscr{L}_t^*(U) = (\mathscr{L}_t^*(U,1), \ldots, \mathscr{L}_t^*(U,N))$:

$$\mathscr{L}_t^*(U,i) = \sum_{s=0}^{r} A_s(t,i)' \sum_{j=1}^{N} p_t(i,j) U(j) A_s(t,i). \tag{8}$$

In terms of $\mathscr{L}_t$, we can construct a causal evolution $\mathscr{T}_{t,s} = \mathscr{L}_{t-1} \cdots \mathscr{L}_s$ $(t > s \geq 0)$; when $t = s$, $\mathscr{T}_{t,t} = \mathscr{I}$ (i.e., the identity operator).

To proceed, let us introduce the following two linear operators (cf. [14, 21]):

$$\begin{aligned}
\psi(X) &= [\psi(X(1))' \ \cdots \ \psi(X(N))']', \\
\varphi(X) &= [\varphi(X(1))' \ \cdots \ \varphi(X(N))']', \\
\psi(X(i)) &= (x_{11}(i) \ \cdots \ x_{1n}(i) \ \cdots \ x_{n1}(i) \ \cdots \ x_{nn}(i))', \\
\varphi(X(i)) &= (x_{11}(i) \ \cdots \ x_{1n}(i) \ x_{22}(i) \ \cdots \ x_{2n}(i) \ \cdots \ x_{n-1,n-1}(i) \ x_{n-1,n}(i) \ x_{nn}(i))',
\end{aligned} \tag{9}$$

where $X \in S_n^N$ and $x_{jk}(i)$ is the entry of $X(i)$. It is easy to verify that $\psi : S_n^N \to \mathscr{C}^{n^2 N}$ and $\varphi : S_n^N \to \mathscr{C}^{(n(n+1)/2)N}$ are both invertible and satisfy

$$\begin{aligned}
\psi(\mathscr{L}_t(X)) &= M_t \psi(X), \\
\varphi(\mathscr{L}_t(X)) &= L_t \varphi(X), \\
\psi(X) &= H \varphi(X), \\
&\qquad \forall X \in S_n^N,
\end{aligned} \tag{10}$$

where $H \in R^{n^2 N \times (n(n+1)/2)N}$ is a constant matrix of full column rank and

$$\begin{aligned}
M_t &= (\mathscr{P}_t' \otimes I_{n^2}) \Delta \left\{ \sum_{s=1}^{r} A_s(t,i) \otimes A_s(t,i) \right\}, \\
L_t &= (H'H)^{-1} H M_t H.
\end{aligned} \tag{11}$$

In (11), L_t is called the induced matrix of $\mathscr{L}_t$ and $\Delta\{A(i)\} := \operatorname{diag}\{A(1), \ldots, A(N)\}$ (if $A(i) \equiv A$ for $i \in \mathscr{D}$, then $\Delta(A) := \operatorname{diag}\{\underbrace{A, \ldots, A}_{N}\}$). Repeating the above steps, the induced matrix of $\mathscr{T}_{t,s}$ is realized to be $\mathscr{A}_{t,s} = L_{t-1} \cdots L_s$. Particularly, the induced matrix of $\mathscr{T}_t^\theta$ is denoted by $\mathscr{A}_t^\theta := \mathscr{A}_{t+\theta,t}$.

Next, we will give two useful lemmas, which have been shown in [19].

Lemma 3. *$(\mathbb{A}; \mathbb{P})$ is AMSS if and only if $\Lambda(\mathscr{T}_t^\theta) = \Lambda(\mathscr{A}_t^\theta) \subset \odot$, where $\Lambda(\cdot)$ denotes the spectral set of an operator (or a matrix) and $\odot := \{\lambda \in \mathscr{C} \mid |\lambda| < 1\}$.*

Lemma 4. *$(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is detectable if and only if for some $t \in \mathbb{Z}_+$, there does not exist any nonzero $X \in S_n^N$ such that*

$$\begin{aligned}
&\mathscr{T}_t^\theta(X,i) = \lambda X(i), \quad \lambda \in \mathscr{C}, \ |\lambda| \geq 1, \ i \in \mathscr{D}, \\
&C(j,i) \mathscr{T}_{j,t}(X,i) = 0, \quad j = t, \ldots, t+\theta-1.
\end{aligned} \tag{12}$$

We are prepared to establish the following Barbashin-Krasovskii stability criterion for (4).

Theorem 5. *If $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is detectable, then $(\mathbb{A}; \mathbb{P})$ is AMSS if and only if the PLE*

$$X_t(i) = \mathscr{L}_t^*(X_{t+1}, i) + C(t,i)' C(t,i), \quad t \in \mathbb{Z}_+ \tag{13}$$

has a unique θ-periodic solution $X_t \in S_n^{N+}$.

Proof. By Theorem 2.5 [18], if $(\mathbb{A}; \mathbb{P})$ is AMSS, then the PLE (13) admits a unique θ-periodic solution $X_t \in S_n^{N+}$. Next, we will show the converse assertion. If (13) has θ-periodic solution $X_t \in S_n^{N+}$ but $(\mathbb{A}; \mathbb{P})$ is not AMSS, by Lemma 3, there must exist $\lambda \in \Lambda(\mathscr{T}_t^\theta)$ with $|\lambda| \geq 1$. Denote by $\rho(\mathscr{T}_t^\theta)$ the spectral radius of $\mathscr{T}_t^\theta$; then $\rho(\mathscr{T}_t^\theta) \geq 1$. According to the Krein-Rutman theorem, there is a positive definite $\overline{X} \in S_n^N$ such that $\mathscr{T}_t^\theta(\overline{X}) = \rho(\mathscr{T}_t^\theta)\overline{X}$. Since $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ is detectable, by Lemma 4, for some $t \in \mathbb{Z}_+$, there exists at least one $k \in \{t, \ldots, t+\theta-1\}$ such that

$$C(k) \mathscr{T}_{k,t}(\overline{X}) \neq 0. \tag{14}$$

Thus, for inner product (6), it can be computed from (13) that

$$\begin{aligned} 0 &\le \sum_{k=t}^{t+\theta-1}\sum_{i=1}^{N} \operatorname{Tr}\left(C(k,i)\mathcal{T}_{k,t}\left(\overline{X},i\right)C(k,i)'\right) \\ &= \sum_{k=t}^{t+\theta-1}\left\langle C(k)'C(k),\mathcal{T}_{k,t}\left(\overline{X}\right)\right\rangle \\ &= \left\langle X_t,\overline{X}\right\rangle - \left\langle X_{t+\theta},\mathcal{T}_t^{\theta}\left(\overline{X}\right)\right\rangle. \end{aligned} \tag{15}$$

Due to the periodicity of X_t, (15) leads to the fact that

$$\begin{aligned} 0 &\le \sum_{k=t}^{t+\theta-1}\sum_{i=1}^{N} \operatorname{Tr}\left(C(k,i)\mathcal{T}_{k,t}\left(\overline{X},i\right)C(k,i)'\right) \\ &= \left\langle X_t,\overline{X}-\mathcal{T}_t^{\theta}\left(\overline{X}\right)\right\rangle = \left\langle X_t,\left[1-\rho\left(\mathcal{T}_t^{\theta}\right)\right]\overline{X}\right\rangle \le 0, \end{aligned} \tag{16}$$

which implies $C(k,i)\mathcal{T}_{k,t}(\overline{X},i)C(k,i)'=0$ for $t\le k\le t+\theta-1$ and $i\in\mathcal{D}$. That is, $C(k)\mathcal{T}_{k,t}(\overline{X})=0$ for $t\le k\le t+\theta-1$, which contradicts (14). Hence, $(\mathbb{A};\mathbb{P})$ is AMSS. □

Remark 6. In [18], a similar result has been proven under the condition of stochastic detectability. According to [19], (uniform) detectability is a weaker prerequisite than stochastic detectability. Therefore, Theorem 5 has improved the result of Theorem 4.1 [18] within the concerned framework.

4. H_2/H_∞ Control

In this section, a game theoretic approach will be employed to deal with the infinite horizon H_2/H_∞ control problem of (1). Under the assumption of detectability, a necessary and sufficient condition can be provided for the existence of H_2/H_∞ controller.

Theorem 7. *For system (1), if the following coupled periodic difference equations (CPDEs) admit θ-periodic quaternion solution $(X_1(t),F_1(t);X_2(t),F_2(t))\in S_n^{N+}\times R_{n\times n_v}^N\times S_n^{N+}\times R_{n\times n_u}^N$ on $\mathbb{Z}_+\times\mathcal{D}$,*

$$\begin{aligned} X_1(t,i) &= \sum_{s=0}^{r}\overline{A}_s^2(t,i)'\mathcal{E}_t\left(X_1(t+1),i\right)\overline{A}_s^2(t,i) \\ &\quad + C(t,i)'C(t,i)+F_2(t,i)'F_2(t,i) \\ &\quad + F_3(t,i)H_1(t,i)^{-1}F_3(t,i)', \end{aligned} \tag{17}$$

$$\begin{aligned} &H_1(t,i)\ge\varepsilon_0 I_{n_v},\quad \varepsilon_0\in\left(0,\gamma^2-\left\|L_\infty^{u^*}\right\|^2\right), \\ &F_1(t,i)=H_1(t,i)^{-1}F_3(t,i)', \end{aligned} \tag{18}$$

$$\begin{aligned} X_2(t,i) &= \sum_{s=0}^{r}\overline{A}_s^1(t,i)'\mathcal{E}_t\left(X_2(t+1),i\right)\overline{A}_s^1(t,i) \\ &\quad + C(t,i)'C(t,i) \\ &\quad - F_4(t,i)H_2(t,i)^{-1}F_4(t,i)', \end{aligned} \tag{19}$$

$$F_2(t,i)=-H_2(t,i)^{-1}F_4(t,i)', \tag{20}$$

where

$$\begin{aligned} \overline{A}_s^1(t,i) &= A_s(t,i)+B_s(t,i)F_1(t,i), \\ \overline{A}_s^2(t,i) &= A_s(t,i)+G_s(t,i)F_2(t,i), \\ H_1(t,i) &= \gamma^2 I_{n_v} \\ &\quad - \sum_{s=0}^{r}B_s(t,i)'\mathcal{E}_t\left(X_1(t+1),i\right)B_s(t,i), \\ H_2(t,i) &= I_{n_u}+\sum_{s=0}^{r}G_s(t,i)'\mathcal{E}_t\left(X_2(t+1),i\right)G_s(t,i), \\ F_3(t,i) &= \sum_{s=0}^{r}\overline{A}_s^2(t,i)'\mathcal{E}_t\left(X_1(t+1),i\right)B_s(t,i), \\ F_4(t,i) &= \sum_{s=0}^{r}\overline{A}_s^1(t,i)'\mathcal{E}_t\left(X_2(t+1),i\right)G_s(t,i), \end{aligned} \tag{21}$$

and $(\mathbb{A},\mathbb{C};\mathbb{P})$, $(\mathbb{A}+\mathbb{B}\mathbb{F}_1,\mathbb{C};\mathbb{P})$ are detectable, then the state-feedback H_2/H_∞ control is given by $(u^*(t)=F_2(t,\eta_t)x(t),\ v^*(t)=F_1(t,\eta_t)x(t))$.

Conversely, if $(\mathbb{A}+\mathbb{B}\mathbb{F}_1,\mathbb{C};\mathbb{P})$ is detectable and the H_2/H_∞ control problem about (1) is solved by $(u^*(t)=F_2(t,\eta_t)x(t),\ v^*(t)=F_1(t,\eta_t)x(t))$, then CPDEs (17)–(20) admit a unique θ-periodic quaternion solution $(X_1(t),F_1(t);X_2(t),F_2(t))\in S_n^{N+}\times R_{n\times n_v}^N\times S_n^{N+}\times R_{n\times n_u}^N$ on $\mathbb{Z}_+\times D$.

Proof. "⇒": (a) Let us first show that u^* stabilizes system (1) internally ($v(t)\equiv 0$). To this end, we rewrite (19) as follows:

$$\begin{aligned} X_2(t,i) &= \sum_{s=0}^{r}\left[\overline{A}_s^1(t,i)+G_s(t,i)F_2(t,i)\right]' \\ &\quad \cdot\mathcal{E}_t\left(X_1(t+1),i\right)\left[\overline{A}_s^1(t,i)+G_s(t,i)F_2(t,i)\right] \\ &\quad + C_1(t,i)'C_1(t,i), \end{aligned} \tag{22}$$

where

$$C_1(t,i)=\begin{bmatrix} C(t,i) \\ H_1(t,i)^{-1/2}F_3(t,i) \\ F_2(t,i)\end{bmatrix}. \tag{23}$$

Since $(\mathbb{A}+\mathbb{B}\mathbb{F}_1,\mathbb{C};\mathbb{P})$ is detectable, by Lemma 4, we can prove that $(\mathbb{A}+\mathbb{B}\mathbb{F}_1+\mathbb{G}\mathbb{F}_2,\mathbb{C}_1;\mathbb{P})$ is also detectable. From (22) and Theorem 5, it follows that $(\mathbb{A}+\mathbb{B}\mathbb{F}_1+\mathbb{G}\mathbb{F}_2;\mathbb{P})$ is AMSS, which means $u^*(t)=F_2(r_t)x(t)\in l^2(0,\infty;R^{n_u})$ and $v^*(t)=F_1(r_t)x(t)\in l^2(0,\infty;R^{n_v})$. Similarly, we can prove that $(\mathbb{A}+\mathbb{G}\mathbb{F}_2;\mathbb{P})$ is also AMSS. Hence, $u^*(t)$ can stabilize system (1) internally.

(b) Consider the following: $\|L_{\infty}^{u^*}\| < \gamma$. Implementing $u^*(t) = F_2(t,\eta_t)x(t)$ into system (1), we get

$$\begin{aligned} x(t+1) &= \overline{A}_0^2(t,\eta_t)x(t) + B_0(t,\eta_t)v(t) \\ &\quad + \sum_{s=1}^{r}\left\{\overline{A}_s^1(t,\eta_t)x(t) + B_s(t,\eta_t)v(t)\right\}w_s(t), \\ z(t) &= \begin{bmatrix} C(t,\eta_t) \\ D(t,\eta_t)F_2(t,\eta_t) \end{bmatrix} x(t), \\ D(t,\eta_t)'D(t,\eta_t) &= I_{n_u}, \\ & \qquad t \in \mathbb{Z}_+. \end{aligned} \tag{24}$$

Noting that $(\mathbb{A} + \mathbb{B}\mathbb{F}_1; \mathbb{P})$ is AMSS, $X_1 \in S_n^{N+}$ is a stabilizing solution of (17). By use of Theorem 1 [20], we deduce that system (24) satisfies $\|L_{\infty}^{u^*}\| < \gamma$.

(c) u^* minimizes the performance $J_2(u, v^*)$. By (17) and (24), we can complete square as follows:

$$\begin{aligned} J_{\infty}(u^*, v) &:= \sum_{t=0}^{\infty}\left[E\|z(t)\|^2 - \gamma^2\|v(t)\|^2\right] = \sum_{i=1}^{N}\pi_0(i) \\ &\quad \cdot x_0'X_1(0,i)x_0 \\ &\quad - \sum_{t=0}^{\infty}E\left\{\left[v(t) - H_1(t,\eta_t)^{-1}F_3(t,\eta_t)'x(t)\right]'\right. \\ &\quad \left.\cdot H_1(t,\eta_t)\left[v(t) - H_1(t,\eta_t)^{-1}F_3(t,\eta_t)'x(t)\right]\right\} \\ &\le \sum_{i=1}^{N}\pi_0(i)x_0'X_1(0,i)x_0 = J_{\infty}(u^*, v^*), \end{aligned} \tag{25}$$

which implies that $v^*(t) = F_1(t,\eta_t)x(t)$ is the worst-case disturbance associated with u^*. Applying v^* to system (1), we have

$$\begin{aligned} x(t+1) &= \overline{A}_0^1(t,\eta_t)x(t) + G_0(t,\eta_t)u(t) \\ &\quad + \sum_{s=1}^{r}\left\{\overline{A}_s^1(t,\eta_t)x(t) + G_s(t,\eta_t)u(t)\right\}w_s(t), \\ z(t) &= \begin{bmatrix} C(t,\eta_t)x(t) \\ D(t,\eta_t)u(t) \end{bmatrix}, \\ D(t,\eta_t)'D(t,\eta_t) &= I_{n_u}, \\ & \qquad t \in \mathbb{Z}_+. \end{aligned} \tag{26}$$

It remains to show that u^* fulfills the following optimal index:

$$\begin{aligned} &\min_{u(\cdot)\in l^2(0,\infty;R^{n_u})} \quad \left\{J_2(u, v^*) = \sum_{t=0}^{\infty}E\left[\|z(t)\|^2\right]\right\}, \\ &\text{subject to} \quad (26), \end{aligned} \tag{27}$$

which is an LQ optimal control problem. Since (19) is equivalent to (22) and $(\mathbb{A} + \mathbb{B}\mathbb{F}_1 + \mathbb{G}\mathbb{F}_2, \mathbb{C}_1; \mathbb{P})$ is detectable, by Theorem 5, $X_2 \in S_n^{N+}$ is a stabilizing solution of (19). Making use of Proposition 6.3 [18], we arrive at

$$\begin{aligned} J_2(u, v^*) &= \sum_{i=1}^{N}\pi_0(i)x_0'X_2(0,i)x_0 \\ &\quad + \sum_{t=0}^{\infty}E\left\{\left[u(t) + H_2(t,\eta_t)^{-1}F_4(t,\eta_t)'x(t)\right]'\right. \\ &\quad \left.\cdot H_2(t,\eta_t)\left[u(t) + H_2(t,\eta_t)^{-1}F_4(t,\eta_t)'x(t)\right]\right\} \\ &\ge \sum_{i=1}^{N}\pi_0(i)x_0'X_2(0,i)x_0 = J_2(u^*, v^*), \end{aligned} \tag{28}$$

where $u^* = F_2(t,\eta_t)x(t)$. This justifies the sufficiency statement.

"$\Leftarrow$": Assume that $(u^*(t) = F_2(t,\eta_t)x(t)$ and $v^*(t) = F_1(t,\eta_t)x(t))$ solve the considered H_2/H_∞ control problem. Thus, u^* stabilizes system (1) internally and $\|L_{\infty}^{u^*}\| < \gamma$. By Theorem 1 [20], we conclude that (17) admits a stabilizing solution $X_1 \in S_n^{N+}$, which implies that $(\mathbb{A} + \mathbb{B}\mathbb{F}_1 + \mathbb{G}\mathbb{F}_2; \mathbb{P})$ is AMSS. Since system (24) is internally stable, by Corollary 3.9 [18], we deduce that $x(t) \in l^2(0,\infty;R^n)$ for any $v(t) \in l^2(0,\infty;R^{n_v})$. As shown in the sufficient part, by use of (17) and (24), we will come to (25), which indicates $F_1(t,\eta_t) = H_1(t,\eta_t)^{-1}F_3(t,\eta_t)'$. Further, imposing v^* on system (1) gives (26). Because u^* is the optimal H_2/H_∞ control, u^* solves LQ control problem (27). Moreover, from detectability of $(\mathbb{A} + \mathbb{B}\mathbb{F}_1, \mathbb{C}; \mathbb{P})$, we have that $(\mathbb{A} + \mathbb{B}\mathbb{F}_1 + \mathbb{G}\mathbb{F}_2, \mathbb{C}_1; \mathbb{P})$ is detectable. Recalling that $(\mathbb{A} + \mathbb{B}\mathbb{F}_1 + \mathbb{G}\mathbb{F}_2; \mathbb{P})$ is AMSS, by Theorem 5, (22) (i.e., (19)) has a stabilizing solution $X_2 \in S_n^{N+}$. Finally, by completing square in terms of (19) and (26), we obtain (28), which justifies that $F_2(t,\eta_t) = -H_2(t,\eta_t)^{-1}F_4(t,\eta_t)'$. This ends the proof. □

Remark 8. If the coefficients of (1) reduce to be time-invariant and the Markov chain $\{\eta_t\}_{t\in\mathbb{Z}_+}$ is homogeneous, then Theorem 7 is reduced to the conclusion of Theorem 3 [14]. Hence, the current study can be regarded as a periodic extension of [14]. At present, there still exists some difficulty in generalizing the above method to design H_2/H_∞ controller for time-varying Markov jump systems as in [22]. To this end, a time-varying version of PBH criterion has to be developed.

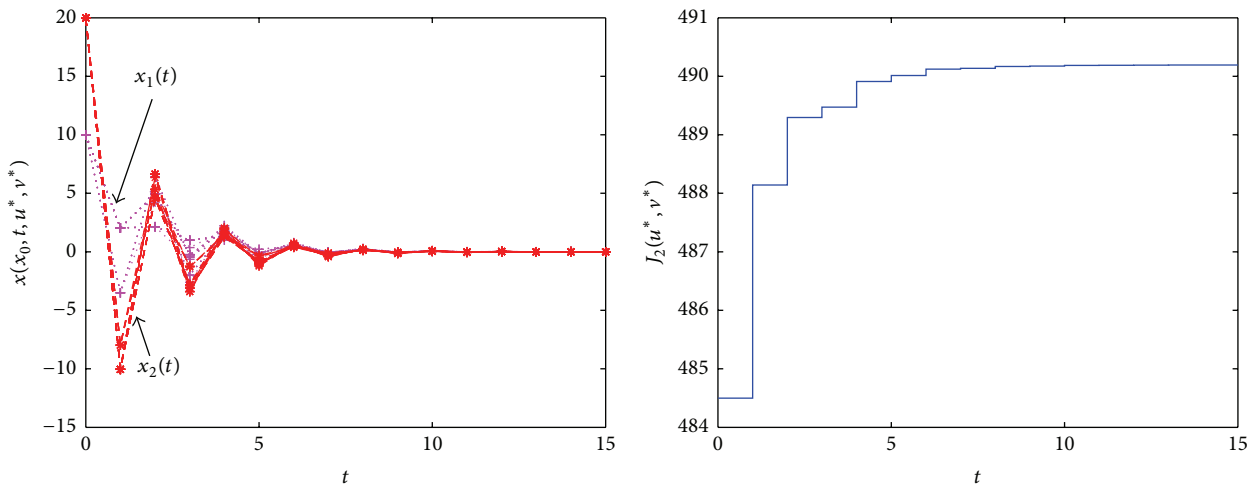


FIGURE 1: State responses and H_2 performance.

5. Numerical Example

Consider the following two-dimensional Markov jump system with the periodic coefficients listed as follows:

$$\begin{aligned}
A_0(t,1) &= \begin{bmatrix} 0.88 - 0.35\cos\pi t & 0 \\ 0 & 0.6 \end{bmatrix}, \\
A_0(t,2) &= \begin{bmatrix} 0.66 & 0 \\ 0 & 0.73 \end{bmatrix}, \\
B_0(t,1) &= \begin{bmatrix} 0.52 \\ 0.31 \end{bmatrix}, \\
B_0(t,2) &= \begin{bmatrix} 0.76 \\ 0.53 \end{bmatrix}, \\
G_0(t,1) &= \begin{bmatrix} 1.4 + 0.2\cos\pi t \\ 0.49 \end{bmatrix}, \\
G_0(t,2) &= \begin{bmatrix} 1.1 \\ 1.04 \end{bmatrix}, \\
A_1(t,1) &= \begin{bmatrix} 0.63 & 0 \\ 0 & 0.72 \end{bmatrix}, \\
A_1(t,2) &= \begin{bmatrix} 0.48 & -0.2\cos\pi t \\ 0 & 0.6 \end{bmatrix}, \\
B_1(t,1) &= \begin{bmatrix} 0.4 - 0.1\cos\pi t \\ 0.5 \end{bmatrix}, \\
B_1(t,2) &= \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, \\
G_1(t,1) &= \begin{bmatrix} 1.1 \\ 1.04 \end{bmatrix}, \\
G_1(t,2) &= \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}.
\end{aligned} \tag{29}$$

Moreover, $C(t,i) \equiv [0.9 \quad 0.56]$ and $D(t,i) \equiv 1$. It is clear that $\theta = 2$. The transition probability matrix $\mathscr{P}_t$ is determined by $p_{2k}(1,1) = p_{2k}(2,1) = 0.2$, $p_{2k}(1,2) = p_{2k}(2,2) = 0.8$, $p_{2k+1}(1,1) = p_{2k+1}(2,1) = 0.3$, and $p_{2k+1}(1,2) = p_{2k+1}(2,2) = 0.7$ ($k \in \mathbb{Z}_+$). For a prescribed disturbance attenuation level $\gamma = 2.3$, by use of the Runge-Kutta algorithm, we can solve CPDEs (17)–(20) and get the feedback gains of (u^*, v^*):

$$\begin{aligned}
F_1(2k,1) &= [0.12 \quad 0.024], \\
F_1(2k,2) &= [0.03 \quad 0.05], \\
F_2(2k,1) &= [-0.49 \quad -0.21], \\
F_2(2k,2) &= [-0.34 \quad -0.27]; \\
F_1(2k+1,1) &= [-0.06 \quad 0.05], \\
F_1(2k+1,2) &= [0.04 \quad 0.016], \\
F_2(2k+1,1) &= [-0.38 \quad -0.235], \\
F_2(2k+1,2) &= [-0.35 \quad -0.25], \\
& \qquad k \in \mathbb{Z}_+.
\end{aligned} \tag{30}$$

By Lemma 4, it can be verified that $(\mathbb{A}, \mathbb{C}; \mathbb{P})$ and $(\mathbb{A} + \mathbb{B}\mathbb{F}_1, \mathbb{C}; \mathbb{P})$ are both detectable. Applying (u^*, v^*) to the periodic Markov jump system, we get the closed-loop state trajectory and corresponding H_2 performance. Figure 1(a) has displayed 50 sampled state trajectories originating from $(x_1(0), x_2(0)) = (10, 20)$, while Figure 1(b) demonstrates the cumulative energy of the system output.

6. Conclusion

In this paper, an infinite horizon H_2/H_∞ control problem has been settled for discrete-time periodic Markov jump systems with multiplicative noise. Under the condition of (uniform) detectability, a game theoretic H_2/H_∞ control is produced by

solving a group of CPDEs. Note that there remain some open topics on this issue. For example, it is interesting as well as challenging to investigate the H_2/H_∞ control problem with input or output saturation constraint [23], which no doubt deserves a further study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 61304074), the Research Award Fund for Outstanding Young Scientists of Shandong Province (nos. BS2013 DX009 and BS2013DX012), the Research Fund for the Taishan Scholar Project of Shandong Province, the SDUST Research Fund (no. 2014JQJH103), and the Shandong Joint Innovative Center for Safe and Effective Mining Technology and Equipment of Coal Resources.

References

[1] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, 1989.

[2] D. Hinrichsen and A. J. Pritchard, "Stochastic H_∞," *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pp. 1504–1538, 1998.

[3] A. El Bouhtouri, D. Hinrichsen, and A. J. Pritchard, "H_∞-type control for discrete-time stochastic systems," *International Journal of Robust and Nonlinear Control*, vol. 9, no. 13, pp. 923–948, 1999.

[4] E. Gershon and U. Shaked, "H_∞ output-feedback control of discrete-time systems with state-multiplicative noise," *Automatica*, vol. 44, no. 2, pp. 574–579, 2008.

[5] Z. Wang, D. W. C. Ho, H. Dong, and H. Gao, "Robust H_∞ finite horizon control for a class of stochastic nonlinear timevarying systems subject to sensor and actuator saturations," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1716–1722, 2010.

[6] B.-S. Chen and W. Zhang, "Stochastic H_2/H_∞ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[7] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov, *Continuous-Time Markov Jump Linear Systems*, Springer, Berlin, Germany, 2012.

[8] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic BAM neural networks with impulse control and mixed time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 3, pp. 467–479, 2012.

[9] Q. Zhu, "pth moment exponential stability of impulsive stochastic functional differential equations with Markovian switching," *Journal of the Franklin Institute: Engineering and Applied Mathematics*, vol. 351, no. 7, pp. 3965–3986, 2014.

[10] Q. Zhu, J. Cao, T. Hayat, and F. Alsaadi, "Robust stability of Markovian jump stochastic neural networks with time delays in the leakage terms," *Neural Processing Letters*, vol. 41, no. 1, pp. 1–27, 2013.

[11] Q. Zhu, R. Rakkiyappan, and A. Chandrasekar, "Stochastic stability of Markovian jump BAM neural networks with leakage delays and impulse control," *Neurocomputing*, vol. 136, pp. 136–151, 2014.

[12] L. Zhang, "H_∞ estimation for discrete-time piecewise homogeneous Markov jump linear systems," *Automatica*, vol. 45, no. 11, pp. 2570–2576, 2009.

[13] Z. Lin, Y. Lin, and W. Zhang, "A unified design for state and output feedback H_∞ control of nonlinear stochastic Markov jump systems with state and disturbance-dependent noise," *Automatica*, vol. 45, no. 12, pp. 2955–2962, 2009.

[14] T. Hou, W. Zhang, and H. Ma, "Infinite horizon H_2/H_∞ optimal control for discrete-time Markov jump systems with (x,u,v)-dependent noise," *Journal of Global Optimization*, vol. 57, no. 4, pp. 1245–1262, 2013.

[15] S. Aberkane and V. Dragan, "H_∞ filtering of periodic Markovian jump systems: application to filtering with communication constraints," *Automatica*, vol. 48, no. 12, pp. 3151–3156, 2012.

[16] V. Dragan, T. Morozan, and A.-M. Stoica, "Output-based H_2 optimal controllers for a class of discrete-time stochastic linear systems with periodic coefficients," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 13, pp. 1897–1926, 2014.

[17] T. Morozan and V. Dragan, "An H_2-type norm of a discrete-time linear sotchastic systems with periodic coefficients simultaneously affected by an infinite Markov chain and multiplicative white noise perterbations," *Stochastic Analysis and Applications*, vol. 32, no. 5, pp. 776–801, 2014.

[18] V. Dragan, T. Morozan, and A.-M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, Springer, New York, NY, USA, 2010.

[19] T. Hou, H. Ma, and W. Zhang, "Spectral tests for observability and detectability of periodic Markov jump systems with non-homogeneous Markov chain," *Automatica*, In press.

[20] H. Ma, W. Zhang, and T. Hou, "Infinite horizon H_2/H_∞ control for discrete-time time-varying Markov jump systems with multiplicative noise," *Automatica*, vol. 48, no. 7, pp. 1447–1454, 2012.

[21] W. Zhang and B.-S. Chen, "*cal H*-representation and applications to generalized Lyapunov equations and linear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 12, pp. 3009–3022, 2012.

[22] V. Dragan and T. Morozan, "Stability and robust stabilization to linear stochastic systems described by differential equations with Markovian jumping and multiplicative white noise," *Stochastic Analysis and Applications*, vol. 20, no. 1, pp. 33–92, 2002.

[23] G. Wei, Z. Wang, H. Shu, and J. Fang, "A delay-dependent approach to H_∞ filtering for stochastic delayed jumping systems with sensor non-linearities," *International Journal of Control*, vol. 80, no. 6, pp. 885–897, 2007.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 191458, 9 pages
http://dx.doi.org/10.1155/2015/191458

Research Article

Stochastic Stability of Discrete-Time Switched Systems with a Random Switching Signal

Kai Liu,[1,2] Xiaowu Mu,[1] and Jumei Wei[1]

[1] *School of Mathematics and Statistics, Zhengzhou University, China*
[2] *College of Science, Henan Institute of Engineering, Zhengzhou, China*

Correspondence should be addressed to Xiaowu Mu; muxiaowu@zzu.edu.cn

Received 13 December 2014; Revised 15 April 2015; Accepted 4 May 2015

Academic Editor: Alexander Klimenko

Necessary and sufficient condition for stochastic stability of discrete-time linear switched system with a random switching signal is considered in this paper, assuming that the switching signal allows fixed dwell time before a Markov switch occurs. It is shown that the stochastic stability of the system is equivalent to that of an auxiliary system with state transformations at switching time, whose switching signal is a Markov chain. The stochastic stability is studied using a stochastic Lyapunov approach. The effectiveness of the proposed approach is demonstrated by a numerical example.

1. Introduction

The study of hybrid system is motivated by several real world technological processes involving the interconnection of logical and discrete dynamics. The evolution of logical variables may be modeled either within a deterministic or within stochastic framework. Among stochastic hybrid systems, a widely investigated class is given by random switched linear system, which consists of a set of linear systems and a random switching signal. If the switching signal is a Markov process or Markov chain, Markov jump linear systems are considered [1, 2]. In [1], some stable conditions for mean square stability for discrete-time jump linear system with finite state Markov chain are presented and the stochastic stability is also considered. The analysis and synthesis problems of stochastic stability in Markov jump linear system have been extensively addressed, such as Markov jump Lur'e system in [3], state and mode detection delay system in [4], antilinear system in [5], and singular system in [6], in Borel space [7]. Due to the probabilistic description of communications, Markov jump systems are well suited to model changes induced by nature, for example, unexpected events and random faults.

Among different assumptions on the switching signal, the arbitrary switching framework is described by considering the switching signal to be an exogenous perturbation [8]. The properties of stability and performance must hold for any possible switching rules [9–11]. If the switching signal could be designed or governed by a supervisor, the deterministic models are more adequate. For example, in many hybrid systems, the switching signal may be designed in order to improve some properties of the systems [8].

To the best of our knowledge, there exists a vast literature on both stochastic and deterministic hybrid systems, but fewer contributions have investigated the stability of discrete-time hybrid systems subject to both stochastic jumps and deterministic switching. For continuous-time system, [12] considers the stability analysis of linear switched systems with a random switching signal, which could be partitioned into the deterministic part and random part. In [13], the stability of a class of Markov jump linear systems characterized by piecewise-constant transition rates and system dynamics is investigated. The switching signal proposed in this paper has a wide-ranging application; for example, in general models of queuing theory, the interarrival time is not exponentially distributed and may contain a deterministic component [14].

In this paper, the stochastic stability of a discrete-time switched linear system with a random switching signal is considered. The dwell time of each mode could consist of two parts: fixed dwell time and random dwell time, which means that the system almost surely stays in each mode for a few time instants before the Markov switch occurs. Through an auxiliary Markov jump linear system with state transitions at switching time, whose stochastic stability is equivalent to that of original system, a necessary and sufficient condition is proposed by using the stochastic Lyapunov approach. When the parameters of the random switching signal are known, the system stability can be checked by solving a set of coupled linear matrix inequalities. Noting that the fixed dwell time could be designed, one can change the dwell time to affect the stochastic stability of the system, which will be introduced in the numerical example.

Compared to the previous work [1–7], a new class of random switching signal is proposed and more general views of switching signals are given. Compared to [12], the stochastic stability is considered in discrete-time system. When the system matrices contain zero eigenvalues, the proof of equivalence between the stochastic stability of system and its auxiliary system is technically difficult, which is solved by using the Jordan decompositions of the system matrices. Moreover, the results in this paper also lay a foundation for novel hybrid controller design.

The remainder of this paper is organized as follows. The mathematical model of the concerned system is formulated and some preliminaries are given in Section 2. In Section 3, a Markov jump switched linear system with state transitions at switching time is proposed, whose stochastic stability is proved to be equivalent to that of the original system. A necessary and sufficient condition is given in Section 4. A numerical example is provided in Section 5.

Notation. The notation used throughout this paper is fairly standard. The superscript T stands for matrix transposition. $\mathbb{N}$ and $\mathbb{Z}_+$ denote the set of positive integers and the set of nonnegative integers, respectively. $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, and $\mathbb{S}^+$ denote the n-dimensional Euclidean space, the set of $m \times n$ real matrices, and the set of $n \times n$ real symmetric positive definite matrices, respectively. The notation $P > 0$ means that P is real symmetric and positive definite, and $A > B$ means $A - B > 0$. For a $n \times n$ matrix A, denote $\|A\|$ and $\sigma(A)$ as the 2-norm and the set of eigenvalues of A, respectively. Let the space $(\Omega, \mathscr{F}, \Pr)$ be a complete probability space. $E\{\cdot\}$ and $\sigma\{\cdot\}$ denote the mathematical expectation and the generated σ-algebra.

2. Problem Information and Preliminaries

Consider the following discrete-time linear switched system defined on the complete probability space $(\Omega, \mathscr{F}, \Pr)$:

$$x(t+1) = A_{r(t)} x(t), \tag{1}$$

where $t \in \mathbb{Z}_+$ and $x(t) \in \mathbb{R}^n$ is the state vector. Switching signal $r(t)$, governing the switching among different system modes, takes values in the finite set $\mathscr{M} = 1, 2, \ldots, m$. Suppose that the system switches its operation mode to $i \in \mathscr{M}$ at time t_k, which means that $r(t_k - 1) \neq i$ and $r(t_k) = i$, and the switching signal $r(t)$ could be descried as follows. For the time $t \in [t_k, t_k + d_i)$, where $d_i \in \mathbb{Z}_+$, no switching is allowed almost surely; that is,

$$\Pr\{r(t) = j \mid r(t_k) = i\} = \begin{cases} 0 & \text{if } j \neq i, \\ 1 & \text{if } j = i, \end{cases} \qquad t \in [t_k, t_k + d_i). \tag{2}$$

For $t \geq t_k + d_i$, mode switching occurs according to the mode transition probabilities given by

$$\Pr\{r(t+1) = j \mid r(t) = i\} = \pi_{ij}, \tag{3}$$

where $\pi_{ij} \geq 0$ and $\sum_{s=1}^{m} \pi_{is} = 1$. The mode transition probability matrix is denoted by $\pi \triangleq [\pi_{ij}]$. If the next switching occurs at time $t_{k+1} > t_k$, we can define $\eta_i \triangleq t_{k+1} - (t_k + d_i)$. The dwell time of the system in mode i is defined as $\tau_i \triangleq t_{k+1} - t_k = d_i + \eta_i$, which indicates the total time length of the system has been in mode i.

Remark 1. The parameter d_i, which is a fixed number for every mode i, plays the roles of "dwell time" in deterministic switched systems and is called the *fixed dwell time* of system in (1). According to (3), η_i is a random variable and is called the *random dwell time* of the system. The discrete-time switching signal $r(t)$ is motivated by the continuous-time case in [12]. There are some differences between the discrete-time and the continuous-time case. In continuous-time case, at the time $t_k + d_i$, the next switching might occur after a short time interval, but in discrete-time case, when the system switches its operation mode at time t_k, the mode may not change in the time interval $[t_k, t_k + 1)$, which means $\eta_i \geq 1$, even if the parameter $d_i = 0$.

Remark 2. System (1) with the switching signal $r(t)$ is no longer a traditional Markov jump linear system because of the fixed dwell time d_i. This type of system has been studied in [1], in which the switching rule is a Markov chain. Here, a modified model setting technique, in which the fixed dwell time d_i is needed for every mode i, is proposed in order to relax the restrictions on switching signal. Obviously, if for every mode $i \in \mathscr{M}$, $d_i = 0$, which means that there does not exist the fixed dwell time for every $[t_k, t_{k+1})$, then the switching signal $r(t)$ reduces into a Markov chain.

The following Lemma is useful to study the property of random dwell time η_i.

Lemma 3. *$\forall s \in \mathbb{N}$, we have* $\Pr\{\eta_i = s\} = \pi_{ii}^{s-1}(1-\pi_{ii})$, $\forall i \in \mathscr{M}$.

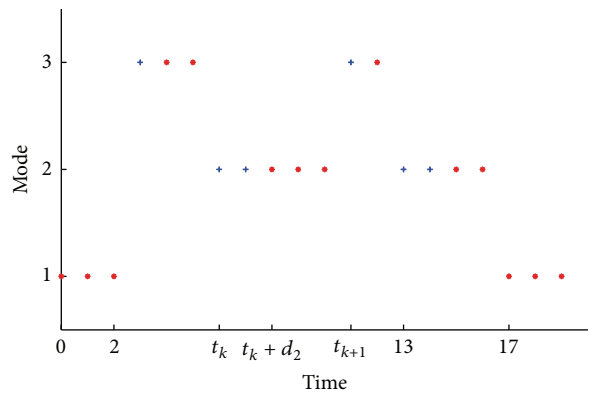


FIGURE 1: A sample path of the switching signal $r(t)$.

Proof. Consider

$$\begin{aligned}\Pr\{\eta_i = s\} &= \Pr\{r(t_k + d_i + 1) \\ &= i, \ldots, r(t_k + d_i + s - 1) = i, r(t_k + d_i + s) \\ &\neq i \mid r(t_k) = i\} = \Pr\{r(t_k + d_i + 1) = i \mid r(t_k) \\ &= i\} \cdots \Pr\{r(t_k + d_i + s - 1) = i \mid r(t_k + d_i + s - 2) \\ &= i\} \cdot \Pr\{r(t_k + d_i + s) \neq i \mid r(t_k + d_i + s - 1) = i\} \\ &= \pi_{ii}^{s-1}(1 - \pi_{ii}).\end{aligned} \tag{4}$$

The proof is completed. □

According to Lemma 3, we can get that η_i is a random variable of geometric distribution with parameter π_{ii}.

An example is given to illustrate the property of the switching signal $r(t)$.

Example 4. Suppose that the mode set $\mathcal{M} = \{1, 2, 3\}$ and the fixed dwell time $d_1 = 0$, $d_2 = 2$, and $d_3 = 1$. A sample path of the switching signal $r(t)$ is given in Figure 1, in which the symbols "+" and "$*$" denote the modes at the fixed dwell time and random dwell time, respectively. When the system switches into mode 2 at time t_k, from Figure 1, we can get that the mode switching will not happen almost surely at the time $[t_k, t_k + d_2)$, which contains d_i time intervals. After the time $t_k + d_2$, the system is allowed to switch modes and obey the switching rules (3). Although the mode may not change in the time $[t_k + d_2, t_k + d_2 + 1)$, which seems to be a part of the fixed dwell time interval, the time $t_k + d_2 + 1$ is set into the random dwell time interval.

For linear switched system (1), the following definition will be adopted in the rest of this paper.

Definition 5. The discrete-time linear switched system in (1) is said to be stochastically stable, if for any initial condition $x(0) = x_0$, $r_0 \in \mathcal{M}$, the following inequality holds

$$E\left\{\sum_{t=0}^{\infty} x^T(t)\, x(t) \mid x_0, r_0\right\} < \infty. \tag{5}$$

The above stochastic stability definition is also a uniform stability in the sense that inequality (5) is required to be true over all the switching signal defined by (2) and (3).

When the fixed dwell time $d_i = 0$, $\forall i \in \mathcal{M}$, system (1) becomes a well-known Markov jump linear system in [1] and the definition of stochastic stability becomes the definition in [1]. Some questions are naturally put forward: Is the stochastic stability of system (1) in this paper equivalent to that of the Markov jump linear system in [1]? Will the values of d_i affect the stochastic stability of the system? An example is given as follows to answer the above two questions.

Example 6. Suppose the mode set $\mathcal{M} = \{1, 2\}$ and the system matrices $A_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 2 & 0 \\ 2 & 2 \end{smallmatrix}\right]$. The mode transition probability matrix is given by $\pi = \left[\begin{smallmatrix} 0 & 1 \\ 0.5 & 0.5 \end{smallmatrix}\right]$. We consider the following two situations.

Situation 1 ($d_1 = d_2 = 0$). Easily, we can get that $E\{\sum_{t=0}^{\infty} \|x(t)\|^2\} = +\infty$. System (1) is not stochastically stable.

Situation 2 ($d_1 = 2$, $d_2 = 0$). $\pi_{21} = 0.5$, for every initial condition $x_0 \in \mathbb{R}^n$, $r_0 \in \mathcal{M}$, and it is almost sure that there exists time $\tilde{t} > 0$ such that $r(\tilde{t}) = 1$. Then, $x(\tilde{t} + d_1) = A_1^2 x(\tilde{t}) = 0$, which follows that $x(t) = 0$, $\forall t \geq \tilde{t} + d_1$. We have $E\{\sum_{t=0}^{\infty} \|x(t)\|^2\} \leq E\{\sum_{t=0}^{\tilde{t}+d_1-1} \|x(t)\|^2\} < \infty$. System (1) is stochastically stable.

Remark 7. From Example 6, we can get that the values of the fixed dwell time d_i may affect the stochastic stability of system.

3. Markov Jump Linear System with State Transitions

In this section, we will study the stochastic stability of system (1) with dwell time (2) and (3). The difficulty is how to deal with the system states in the fixed dwell time interval $[t_k, t_k + d_i)$. Thus, a Markov jump linear system with state transitions is constructed in which the state transitions at switching time are used to replace the system states in the fixed dwell time intervals. Moreover, the stochastic stability of the constructed system is shown to be equivalent to that of system (1).

For each switching time $t_k > 0$, denote $r_k = r(t_k)$ as the operation mode at time t_k. Then we have the system state $x(t)$ that will evolve from $x(t_k)$ at time t_k to $x(t_k + d_{r_k}) = A_{r_k}^{d_{r_k}} x(t_k)$ at time $t_k + d_{r_k}$ almost surely. An auxiliary system is built, in which there is a state transition from $x(t_k)$ to $A_{r_k}^{d_{r_k}} x(t_k)$ at time t_k, to squeeze the fixed dwell time interval $[t_k, t_k + d_i)$. According to this idea, the auxiliary system with state transitions can be written as

$$\begin{aligned}\zeta(\tilde{t} + 1) &= A_{\rho(\tilde{t})} \xi(\tilde{t}), \\ \xi(\tilde{t}_k) &= A_{\rho_k}^{d_{\rho_k}} \zeta(\tilde{t}_k), \quad k \in \mathbb{Z}_+, \\ \xi(\tilde{t}) &= \zeta(\tilde{t}), \quad \tilde{t} \neq \tilde{t}_k, \ k \in \mathbb{Z}_+,\end{aligned} \tag{6}$$

where $\xi(\tilde{t})$ is the system state and $\zeta(\tilde{t})$ is the auxiliary variable of the state $\xi(\tilde{t})$. $\tilde{t}_k$, $k \in \mathbb{Z}_+$, denotes the time when the

auxiliary system switches its operation modes. The Markov chain $\{\rho(\tilde{t}), \tilde{t} \in \mathbb{Z}_+\}$, which governs the switching among different system modes in $\mathscr{M}$, is described by a discrete-time homogeneous Markov chain with mode transition probabilities. Consider

$$\Pr\{\rho(\tilde{t}+1) = j \mid \rho(\tilde{t}) = i\} = \pi_{ij}, \tag{7}$$

where π_{ij} are the same as those in (3). Suppose that system (6) switches to mode $\rho_k \triangleq \rho(\tilde{t}_k)$ at time $\tilde{t}_k$, and the dwell time of the system in mode ρ_k is denoted by $\eta_{\rho_k} = \tilde{t}_{k+1} - \tilde{t}_k$. According to Lemma 3, we can get that η_{ρ_k} is a random variable of geometric distribution with parameter π_{ii}. According to term (7), it is easy to get that η_{ρ_k} in the auxiliary system equals the random dwell time τ_{r_k} in system (1), and the two system sample state paths of system (1) and auxiliary system (6) satisfy the following properties for every $k \in \mathbb{Z}_+$:

(i) $\xi(0) = A_{\rho_0}^{d_{\rho_0}} x(0), \tilde{t}_0 = t_0 = 0$.

(ii) $r_k = r(t_k) = \rho(\tilde{t}_k) = \rho_k$.

(iii) $t_{k+1} = \sum_{l=0}^{k}(d_{r_k} + \eta_{r_k}) = \sum_{l=0}^{k} d_{r_l} + \tilde{t}_{k+1}$.

(iv) $\xi(\tilde{t}_k + \tau) = x(t_k + d_{r_k} + \tau), \tau = 0, 1, \ldots, \eta_{\rho_k} - 1$.

The fourth property is a direct result of the first three properties: $\xi(\tilde{t}_k + \tau) = A_{\rho_k}^{\tau}\xi(\tilde{t}_k) = A_{\rho_k}^{\tau} A_{\rho_k}^{d_{\rho_k}} x(t_k) = A_{\rho_k}^{\tau + d_{\rho_k}} x(t_k) = x(t_k + d_{r_k} + \tau)$.

In order to get the equivalence of the stochastic stability between system (1) and its auxiliary system (6), we need the following lemma.

Lemma 8. *Given a geometrically distributed random variable η with the parameter $p \in [0, 1)$ and a constant $a > 0$ satisfied $ap < 1$, then*

$$E\left\{\sum_{\tau=0}^{\eta-1} a^{\tau}\right\} = \frac{1}{1-ap}. \tag{8}$$

Proof. η is a geometrically distributed random variable with p; by Lemma 3, we have $\Pr\{\eta = i\} = p^{i-1}(1-p), \forall i = 1, 2, \ldots$. It follows that

$$\begin{aligned} E\{a^{\eta}\} &= \sum_{i=1}^{\infty} a^i \cdot p^{i-1}(1-p) = a(1-p)\sum_{i=0}^{\infty}(ap)^i \\ &= \frac{a-ap}{1-ap}. \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} E\left\{\sum_{\tau=0}^{\eta-1} a^{\tau}\right\} &= E\left\{\frac{1-a^{\eta}}{1-a}\right\} = \frac{1}{1-a}\left(1 - E\{a^{\eta}\}\right) \\ &= \frac{1}{1-a}\left(1 - \frac{a-ap}{1-ap}\right) = \frac{1}{1-ap}. \end{aligned} \tag{10}$$

The proof is completed. □

For the systems in (1) and (6), we define the filtrations $\mathscr{F}_t \triangleq \sigma\{x(\tau), r(\tau) \mid \tau = 0, 1, \ldots, t\}$, and $\mathscr{G}_{\tilde{t}} \triangleq \sigma\{\xi(\tau), \rho(\tau) \mid \tau = 0, 1, \ldots, \tilde{t}\}$, respectively. First result of this paper which establishes the equivalence between system (1) and Markov switched system (6) with state transitions is proposed as follows.

Theorem 9. *System (1) is stochastically stable, if and only if the system (6) is stochastically stable.*

Proof.

Necessity. Suppose that system (1) is stochastically stable; that is, for every $x(0)$ and $r(0)$, we have $E\{\sum_{t=0}^{\infty}\|x(t)\|^2 \mid x_0, r_0\} < +\infty$. Then, based on the properties (i)–(iv), we have

$$\begin{aligned} &E\left\{\sum_{\tilde{t}=0}^{\infty}\left\|\xi(\tilde{t})\right\|^2 \mid \xi(0), \rho_0\right\} \\ &\quad = E\left\{\sum_{k=0}^{\infty}\left\{\sum_{\tilde{t}=\tilde{t}_k}^{\tilde{t}_{k+1}-1}\left\|\xi(\tilde{t})\right\|^2\right\} \mid \xi_0, \rho_0\right\} \\ &\quad = E\left\{\sum_{k=0}^{\infty}\left\{\sum_{t=t_k+d_{r_k}}^{t_{k+1}-1}\|x(t)\|^2\right\} \mid x_0, r_0\right\} \\ &\quad \le E\left\{\sum_{k=0}^{\infty}\left\{\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2\right\} \mid x_0, r_0\right\} \\ &\quad = E\left\{\sum_{t=0}^{\infty}\|x(t)\|^2 \mid x_0, r_0\right\} < +\infty. \end{aligned} \tag{11}$$

It follows that system (6) is stochastically stable.

Sufficiency. In order to prove the sufficiency, two cases are considered as follows, respectively.

Case 1. All of A_i have no zero eigenvalues, which means that A_i are nonsingular, $\forall i \in \mathscr{M}$.

Then, there exists a constant $0 < a < 1$, such that $\forall i \in \mathscr{M}$, $a\pi_{ii} < 1$ and $\min_{\lambda \in \sigma(A^T A)}\{\lambda\} \ge a$. It follows that $\|A_i^d x\|^2 \ge a^d\|x\|^2$, $\forall x \in \mathbb{R}^n$, $d \in \mathbb{Z}_{\ge 0}$. Thus, for any $x(t_k)$ and $r_k \in \mathscr{M}$, we have

$$\begin{aligned} &E\left\{\sum_{t=t_k+d_{r_k}}^{t_{k+1}-1}\|x(t)\|^2 \mid x(t_k), r(t_k) = r_k\right\} \\ &\quad = E\left\{\sum_{t=t_k+d_{r_k}}^{t_{k+1}-1}\left\|A_{r_k}^{t-t_k} x(t_k)\right\|^2 \mid x(t_k), r(t_k) = r_k\right\} \\ &\quad = E\left\{\sum_{\tau=d_{r_k}}^{d_{r_k}+\eta_{r_k}-1}\left\|A_{r_k}^{\tau} x(t_k)\right\|^2 \mid x(t_k), r_k\right\} \\ &\quad \ge E\left\{\sum_{\tau=d_{r_k}}^{d_{r_k}+\eta_{r_k}-1} a^{\tau}\right\}\|x(t_k)\|^2 \end{aligned}$$

$$= E\left\{\sum_{\tau=0}^{\eta_{r_k}-1} a^{\tau} \mid r_k\right\} a^{d_{r_k}} \left\|x\left(t_k\right)\right\|^2$$
$$= \frac{1}{1-a\pi_{r_k r_k}} a^{d_{r_k}} \left\|x\left(t_k\right)\right\|^2 \geq \underline{a}\left\|x\left(t_k\right)\right\|^2, \tag{12}$$

where $\underline{a} \triangleq \min\{a^{d_{r_k}}/(1-a\pi_{r_k r_k})\} > 0$. The last "=" holds because of the properties of η_{r_k}, $a\pi_{r_k r_k} < 1$, and Lemma 8. If $d_{r_k} = 0$, set $E\{\sum_{t=t_k}^{t_k+d_{r_k}-1} \|x(t)\|^2 \mid x(t_k), r_k\} = 0$. If $d_k \in \mathbb{N}$, we can obtain that

$$\begin{aligned} &E\left\{\sum_{t=t_k}^{t_k+d_{r_k}-1} \|x(t)\|^2 \mid x\left(t_k\right), r_k\right\} \\ &= E\left\{\sum_{t=t_k}^{t_k+d_{r_k}-1} \left\|A_{r_k}^{t-t_k} x\left(t_k\right)\right\|^2 \mid x\left(t_k\right), r_k\right\} \\ &= \sum_{\tau=0}^{d_{r_k}-1} \left\|A_{r_k}^{\tau} x\left(t_k\right)\right\|^2 \leq \overline{a}\overline{d}\left\|x\left(t_k\right)\right\|^2 \\ &\leq M_1 E\left\{\sum_{t=t_k+d_{r_k}}^{t_{k+1}-1} \|x(t)\|^2 \mid x\left(t_k\right), r\left(t_k\right)=r_k\right\}, \end{aligned} \tag{13}$$

where $\overline{d} = \max_{i\in\mathscr{M}} d_i$, $\overline{a} = \max_{i\in\mathscr{M}}\{\max_{j=0,1,\ldots,d_i}\|A_i^j\|^2\}$, and $M_1 = \overline{a}\overline{d}/\underline{a}$. The last inequality holds because of term (12).

Case 2. There exists A_i, whose eigenvalues contain zeros.

Suppose that system (1) switches its mode to d_{r_k} at time t_k and the eigenvalues of A_{r_k} contain zeros. Using Jordan decompositions of the matrices, there exists a nonsingular matrix T_{r_k}, such that $T_{r_k}^{-1} A_{r_k} T_{r_k} = \begin{bmatrix} J_{r_k 1} & \\ & J_{r_k 2} \end{bmatrix}$, in which $J_{r_k 1} \in \mathbb{R}^{n_{r_k}\times n_{r_k}}$ is the Jordan matrix whose diagonal blocks are Jordan blocks with eigenvalue 0 and $J_{r_k 2}$ is that with the other nonzero eigenvalues. Under the coordination transformation $\overline{x}(t) = T_{r_k}^{-1} x(t)$, denote that $\overline{x}(t) = \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \end{bmatrix}$, in which $\overline{x}_1(t) \in \mathbb{R}^{n_{r_k}}$. We have

$$\overline{x}_i(t+1) = J_{r_k i}\overline{x}_i(t) \quad i = 1,2;\ t = t_k, \ldots, t_{k+1}-1. \tag{14}$$

Similarly, denoting $\overline{\xi}(\widetilde{t}) = T_{r_k}^{-1}\xi(\widetilde{t}) \triangleq \begin{bmatrix} \overline{\xi}_1(\widetilde{t}) \\ \overline{\xi}_2(\widetilde{t}) \end{bmatrix}$, one can obtain

$$\begin{aligned} &\overline{\xi}_i(\widetilde{t}+1) = J_{r_k i}\overline{\xi}_1(\widetilde{t}) \\ &\qquad i = 1,2;\ t = \widetilde{t}_k, \widetilde{t}_k+1, \ldots, \widetilde{t}_{k+1}-1, \\ &\overline{\xi}_i(\widetilde{t}_k) = J_{r_k i}^{d_{r_k}}\overline{x}_i(t_k) \quad i = 1,2. \end{aligned} \tag{15}$$

Because $J_{r_k 2}$ contains no zero eigenvalues, together with the conclusion of Case 1, there exists a constant $\widetilde{M} > 0$, such that

$$\begin{aligned} &E\left\{\sum_{t=t_k}^{t_k+d_{r_k}-1} \left\|\overline{x}_2(t)\right\|^2 \mid \mathscr{F}_{t_k}\right\} \\ &\leq \widetilde{M} E\left\{\sum_{\widetilde{t}=\widetilde{t}_k}^{\widetilde{t}_{k+1}-1} \left\|\overline{\xi}_2(\widetilde{t})\right\|^2 \mid \mathscr{G}_{\widetilde{t}_k}\right\} \\ &\leq \widetilde{M}\underline{T} E\left\{\sum_{\widetilde{t}=\widetilde{t}_k}^{\widetilde{t}_{k+1}-1} \left\|\xi(\widetilde{t})\right\|^2 \mid \mathscr{G}_{\widetilde{t}_k}\right\}, \end{aligned} \tag{16}$$

where $\underline{T} = \max_{i\in\mathscr{M}}\|T_i^{-1}\|^2$.

$J_{r_k 1}$ is Jordan matrix whose blocks are Jordan blocks with eigenvalue 0; then we have $\|J_{r_k 1} y\| \leq \|y\|, \forall y \in \mathbb{R}^{n_{r_k}}$. It follows that

$$\begin{aligned} &E\left\{\sum_{t=t_k}^{t_k+d_{r_k}-1} \left\|\overline{x}_1(t)\right\|^2 \mid \mathscr{F}_{t_k}\right\} \\ &\leq E\left\{d_{r_k}\left\|\overline{x}_1\left(t_k\right)\right\|^2 \mid \mathscr{F}_{t_k}\right\} \leq \overline{d}\underline{T}\left\|A_{r_{r_{k-1}}} x\left(t_k-1\right)\right\|^2 \\ &\leq \overline{d}\underline{T}\overline{a}\left\|x\left(t_k-1\right)\right\|^2 \leq \overline{d}\underline{T}\overline{a}\sum_{t=t_{k-1}+d_{k-1}}^{t_k-1} \|x(t)\|^2 \\ &= \overline{d}\underline{T}\overline{a} E\left\{\sum_{\widetilde{t}=\widetilde{t}_{k-1}}^{\widetilde{t}_k-1} \left\|\xi(\widetilde{t})\right\|^2 \mid \mathscr{G}_{\widetilde{t}_k}\right\}. \end{aligned} \tag{17}$$

By (16) and (17),

$$\begin{aligned} &E\left\{\sum_{t=t_k}^{t_k+d_k-1} \|x(t)\|^2 \mid \mathscr{F}_{t_k}\right\} \\ &\leq \overline{T} E\left\{\sum_{t=t_k}^{t_k+d_k-1} \|\overline{x}(t)\|^2 \mid \mathscr{F}_{t_k}\right\} \\ &= \overline{T}\left(E\left\{\sum_{t=t_k}^{t_k+d_k-1} \left\|\overline{x}_1(t)\right\|^2 \mid \mathscr{F}_{t_k}\right\}\right. \\ &\quad \left. + E\left\{\sum_{t=t_k}^{t_k+d_k-1} \left\|\overline{x}_2(t)\right\|^2 \mid \mathscr{F}_{t_k}\right\}\right) \\ &\leq M_2 E\left\{\sum_{\widetilde{t}=\widetilde{t}_{k-1}}^{\widetilde{t}_{k+1}-1} \left\|\xi(\widetilde{t})\right\|^2 \mid \mathscr{G}_{\widetilde{t}_k}\right\}, \end{aligned} \tag{18}$$

where $\overline{T} = \max_{i\in\mathscr{M}}\|T_i\|^2$ and $M_2 = \overline{T}(\widetilde{M}\underline{T} + \overline{d}\underline{T}\overline{a})$.

Using (13) and (18), we have

$$\begin{aligned}
&E\left\{\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2 \mid \mathscr{F}_{t_k}\right\}\\
&\quad= E\left\{\sum_{t=t_k}^{t_k+d_{r_k}-1}\|x(t)\|^2 \mid \mathscr{F}_{t_k}\right\}\\
&\qquad+ E\left\{\sum_{t=t_k+d_{r_k}}^{t_{k+1}-1}\|x(t)\|^2 \mid \mathscr{F}_{t_k}\right\}\\
&\quad\le ME\left\{\sum_{\tilde{t}=\tilde{t}_{k-1}}^{\tilde{t}_{k+1}-1}\|\xi(\tilde{t})\|^2 \mid \mathscr{G}_{\tilde{t}_k}\right\}\\
&\qquad+ E\left\{\sum_{t=t_k+d_{r_k}}^{t_{k+1}-1}\|x(t)\|^2 \mid \mathscr{F}_{t_k}\right\}\\
&\quad\le (1+M)\,E\left\{\sum_{\tilde{t}=\tilde{t}_{k-1}}^{\tilde{t}_{k+1}-1}\|\xi(\tilde{t})\|^2 \mid \mathscr{G}_{\tilde{t}_k}\right\},
\end{aligned} \tag{19}$$

where $M = \max\{M_1, M_2\}$. The first "$\le$" holds because of (13) and (18), and the second "$\le$" holds because of property (iv).

Together with the fact that

$$\begin{aligned}
&E\left\{E\left\{\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2 \mid \mathscr{F}_{t_k}\right\} \mid x_0, r_0\right\}\\
&\quad= E\left\{\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2 \mid x_0, r_0\right\},\\
&E\left\{E\left\{\sum_{\tilde{t}=\tilde{t}_{k-1}}^{\tilde{t}_{k+1}-1}\|\xi(\tilde{t})\|^2 \mid \mathscr{G}_{\tilde{t}_k}\right\} \mid \xi_0, \rho_0\right\}\\
&\quad= E\left\{\sum_{\tilde{t}=\tilde{t}_{k-1}}^{\tilde{t}_{k+1}-1}\|\xi(\tilde{t})\|^2 \mid \xi_0, \rho_0\right\},
\end{aligned} \tag{20}$$

we have

$$\begin{aligned}
&E\left\{\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2 \mid x_0, r_0\right\}\\
&\quad\le (1+M)\,E\left\{\sum_{\tilde{t}=\tilde{t}_{k-1}}^{\tilde{t}_{k+1}-1}\|\xi(\tilde{t})\|^2 \mid \xi_0, \rho_0\right\}.
\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}
&E\left\{\sum_{t=0}^{\infty}\|x(t)\|^2 \mid x_0, r_0\right\}\\
&\quad= E\left\{\sum_{k=0}^{\infty}\left(\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2\right) \mid x_0, r_0\right\}\\
&\quad= \sum_{k=0}^{\infty}\left(E\left\{\sum_{t=t_k}^{t_{k+1}-1}\|x(t)\|^2 \mid x_0, r_0\right\}\right)\\
&\quad\le (1+M)\sum_{k=0}^{\infty}\left(E\left\{\sum_{\tilde{t}=\tilde{t}_{k-1}}^{\tilde{t}_{k+1}-1}\|\xi(\tilde{t})\|^2 \mid \xi_0, \rho_0\right\}\right)\\
&\quad\le 2(M+1)\,E\left\{\sum_{\tilde{t}=0}^{\infty}\|\xi(\tilde{t})\|^2 \mid \xi_0, \rho_0\right\},
\end{aligned} \tag{22}$$

which means that system (1) is stochastically stable.

The proof is completed. □

Remark 10. In the proof of sufficiency, two cases are considered. When there exist some A_i whose eigenvalues contain zero, we need to use the state information in $[t_{k-1}, t_k)$ to estimate that in $[t_k, t_{k+1})$, which is different from and more technically difficult than the continuous-time system in [12].

Remark 11. From Theorem 9, we can get that the stochastic stability of system (1) is equivalent to stochastic stability of system (6). The switching signal of system (6) is a Markov chain; thus we can use Lyapunov approach to study the stochastic stability of system (1).

4. A Necessary and Sufficient Condition of Stochastic Stability

In this section, a necessary and sufficient condition of the stochastic stability of system (1) is proposed.

Theorem 12. *System (1) is stochastically stable if and only if there exist matrices $P_i > 0$, $i \in \mathscr{M}$, such that*

$$A_i^T\left(\pi_{ii}P_i + \sum_{j\ne i}\pi_{ij}\left(A_j^{d_j}\right)^T P_j A_j^{d_j}\right)A_i - P_i < 0, \quad \forall i \in \mathscr{M}. \tag{23}$$

Proof. It follows from (6) and (7) that

$$\begin{aligned}
\Pr\{\xi(\tilde{t}+1) = A_i\xi(\tilde{t}) \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\} &= \pi_{ii},\\
\Pr\left\{\xi(\tilde{t}+1) = A_j^{d_j}A_i\xi(\tilde{t}) \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\right\} &= \pi_{ij},\\
\Pr\left\{P_{\rho(\tilde{t}+1)} = P_i \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\right\} &= \pi_{ii},\\
\Pr\left\{P_{\rho(\tilde{t}+1)} = P_j \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\right\} &= \pi_{ij}.
\end{aligned} \tag{24}$$

According to Theorem 9, the stochastic stability of systems (1) and (6) is equivalent.

Necessity. If system (6) is stochastically stable, we will show that there exist P_i such that (23) holds. Given any positive matrices $Q_i > 0$, $i \in \mathscr{M}$, define a function $P(\omega - \tilde{t}, \rho(\tilde{t}))$ such that

$$\begin{aligned} &\xi^T(\tilde{t}) P(\omega - \tilde{t}, \rho(\tilde{t})) \xi(\tilde{t}) \\ &\quad = E\left\{\sum_{\tau=\tilde{t}}^{\omega} \xi^T(\tilde{t}) Q_{\rho(\tau)} \xi(\tau) \mid \xi(\tilde{t}), \rho(\tilde{t})\right\}. \end{aligned} \tag{25}$$

Obviously, because $Q_i > 0$, $\forall i \in \mathscr{M}$, we have that the left side of (25) is nondecreasing on ω. It is also bounded as $\omega \to \infty$, because of the stochastic stability of system (6). Then the limit of the left side exists as $\omega \to \infty$. Define a new matrix-valued function $P_{\rho(\tilde{t})}$ of $\rho(\tilde{t})$, such that

$$\begin{aligned} \xi^T(\tilde{t}) P_{\rho(\tilde{t})} \xi(\tilde{t}) &= \lim_{\omega \to \infty} \xi^T(\tilde{t}) P(\omega - \tilde{t}, \rho(\tilde{t})) \xi(\tilde{t}) \\ &> 0, \end{aligned} \tag{26}$$

for any $\xi(\tilde{t})$ and $\rho(t) \in \mathscr{M}$. Then one can obtain

$$P_{\rho(\tilde{t})} = \lim_{\omega \to \infty} P(\omega - \tilde{t}, \rho(\tilde{t})) > 0. \tag{27}$$

By (25),

$$\begin{aligned} &E\left\{\xi^T(\tilde{t}+1) P(\omega - (\tilde{t}+1), \rho(\tilde{t}+1)) \xi(\tilde{t}+1) \mid \xi(\tilde{t}),\right. \\ &\quad \left.\rho(\tilde{t})\right\} = E\left\{E\left\{\sum_{\tau=\tilde{t}+1}^{\omega} \xi^T(\tau) Q_{\rho(\tau)} \xi(\tau) \mid \xi(\tilde{t}+1),\right.\right. \\ &\quad \left.\left.\rho(\tilde{t}+1)\right\} \mid \xi(\tilde{t}), \rho(\tilde{t})\right\} = E\left\{\sum_{\tau=\tilde{t}+1}^{\omega} \xi^T(\tau)\right. \\ &\quad \left.\cdot Q_{\rho(\tau)} \xi(\tau) \mid \xi(\tilde{t}), \rho(\tilde{t})\right\}, \end{aligned} \tag{28}$$

the last "=" holds because $(\xi(\tilde{t}), \rho(\tilde{t}))$ is a Markov chain. Therefore,

$$\begin{aligned} &E\left\{\xi^T(\tilde{t}+1) P(\omega - (\tilde{t}+1), \rho(\tilde{t}+1)) \xi(\tilde{t}+1)\right. \\ &\quad \left.- \xi^T(\tilde{t}) P(\omega - \tilde{t}, \rho(\tilde{t})) \xi(\tilde{t}) \mid \xi(\tilde{t}), \rho(\tilde{t})\right\} \\ &\quad = E\left\{\sum_{\tau=\tilde{t}+1}^{\omega} \xi^T(\tau) Q_{\rho(\tau)} \xi(\tau) \mid \xi(\tilde{t}), \rho(\tilde{t})\right\} \\ &\quad - E\left\{\sum_{\tau=\tilde{t}}^{\omega} \xi^T(\tau) Q_{\rho(\tau)} \xi(\tau) \mid \xi(\tilde{t}), \rho(\tilde{t})\right\} = -\xi(\tilde{t}) \\ &\quad \cdot Q_{\rho(\tilde{t})} \xi(\tilde{t}). \end{aligned} \tag{29}$$

On the other hand, we have

$$\begin{aligned} &E\left\{\xi^T(\tilde{t}+1) P(\omega - (\tilde{t}+1), \rho(\tilde{t}+1)) \xi(\tilde{t}+1)\right. \\ &\quad \left.- \xi^T(\tilde{t}) P(\omega - \tilde{t}, \rho(\tilde{t})) \xi(\tilde{t}) \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\right\} \\ &\quad = \xi^T(\tilde{t}) \left(\pi_{ii} A_i^T P(\omega - \tilde{t} - 1, i) A_i\right. \\ &\quad + \sum_{j \neq i} \pi_{ij} A_i^T \left(A_j^{d_j}\right)^T P(\omega - \tilde{t} - 1, j) A_j^{d_j} A_i \\ &\quad \left.- P(\omega - \tilde{t}, i)\right) \xi(\tilde{t}). \end{aligned} \tag{30}$$

Together with $\lim_{\omega \to \infty} P(\omega - \tilde{t} - 1, i) = P_i$, $\lim_{\omega \to \infty} P(\omega - \tilde{t} - 1, j) = P_j$, and (29), taking limit on the right side of (30), for every $i \in \mathscr{M}$, we can get

$$A_i^T \left(\pi_{ii} P_i + \sum_{j \neq i} \pi_{ij} \left(A_j^{d_j}\right)^T P_j A_j^{d_j}\right) A_i - P_i = -Q_i < 0. \tag{31}$$

Sufficiency. $\forall i \in \mathscr{M}$, denote that

$$\begin{aligned} Q_i &= -\left(A_i^T \left(\pi_{ii} P_i + \sum_{j \neq i} \pi_{ij} \left(A_j^{d_j}\right)^T P_j A_j^{d_j}\right) A_i - P_i\right) \\ &> 0. \end{aligned} \tag{32}$$

We only need to prove that system (6) is stochastically stable. Consider the following Lyapunov function:

$$V(\xi(\tilde{t}), \rho(\tilde{t})) = \xi^T(\tilde{t}) P_{\rho(\tilde{t})} \xi(\tilde{t}). \tag{33}$$

Then we have

$$\begin{aligned} &E\{V(\xi(\tilde{t}+1), \rho(\tilde{t}+1)) \\ &\quad - V(\xi(\tilde{t}), \rho(\tilde{t})) \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\} \\ &\quad = E\left\{\xi^T(\tilde{t}+1) P_{\rho(\tilde{t}+1)} \xi(\tilde{t}+1) \mid \xi(\tilde{t}), \rho(\tilde{t}) = i\right\} \\ &\quad - \xi^T(\tilde{t}) P_i \xi(\tilde{t}) = \pi\left(\xi^T(\tilde{t}) A_i^T P_i A_i \xi(\tilde{t})\right) \\ &\quad + \sum_{j \neq i} \pi_{ij} \left(\xi^T(\tilde{t}) A_i^T \left(A_j^{d_j}\right)^T P_j A_j^{d_j} A_i \xi(\tilde{t})\right) - \xi^T(\tilde{t}) \\ &\quad \cdot P_i \xi(\tilde{t}) \le -\xi^T(\tilde{t}) Q_i \xi(\tilde{t}) \le -\beta \|\xi(\tilde{t})\|^2, \end{aligned} \tag{34}$$

where $\beta = \min_{i \in \mathscr{M}}\{\min_{\lambda \in \sigma(Q_i)} \lambda\}$. For any $\xi(\tilde{t})$ and $\rho(\tilde{t}) \in \mathscr{M}$, we have

$$\begin{aligned} \|\xi(\tilde{t})\|^2 \le -\frac{1}{\beta} E\{&V(\xi(\tilde{t}+1), \rho(\tilde{t}+1)) \\ &- V(\xi(\tilde{t}), \rho(\tilde{t})) \mid \xi(\tilde{t}), \rho(\tilde{t})\}. \end{aligned} \tag{35}$$

Thus,

$$\begin{aligned} E\left\{\sum_{\tilde{t}=0}^{\infty}\|\xi(\tilde{t})\|^2 \mid \xi_0,\rho_0\right\} &\le -\frac{1}{\beta}E\left\{\sum_{\tilde{t}=0}^{\infty}E\right. \\ &\cdot\{V(\xi(\tilde{t}+1),\rho(\tilde{t}+1)) \\ &\left.-V(\xi(\tilde{t}),\rho(\tilde{t}))\mid\xi(\tilde{t}),\rho(\tilde{t})\}\mid\xi_0,\rho_0\right\} \\ &\le -\frac{1}{\beta}\left(\lim_{\tilde{t}\to\infty}E\{V(\xi(\tilde{t}),\rho(\tilde{t}))\mid\xi_0,\rho_0\}\right)+\frac{1}{\beta} \\ &\cdot V(\xi_0,\rho_0)<+\infty, \end{aligned} \tag{36}$$

which means that system (6) is stochastically stable.

The proof is completed. □

5. Numerical Example

In this section, a numerical example is given to demonstrate the validity and applicability of the developed theoretical result.

Consider a 2-dimensional discrete-time linear switched system (1) consisting of 3 operation modes with the fixed dwell time d_i, $i = 1,2,3$. The system matrices are given as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.6 & 0 \\ 0.2 & 0.7 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.4 & 0.2 \\ 0 & -0.8 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 1.4 & 1 \\ 0 & 1.4 \end{bmatrix}. \end{aligned} \tag{37}$$

The transition probability matrix is given by

$$\pi = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.5 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \tag{38}$$

and the initial condition is chosen as $x(0) = [10, 20]'$.

For the Markov jump linear system with fixed dwell time $d_0 = d_2 = d_3 = 0$, using the LMI toolbox of Matlab, we can get that there do not exist positive definite matrices P_i, such that term (23) holds, which means that the system is not stochastically stable. The switching modes and the sampled system state trajectories with fixed dwell time $d_0 = d_2 = d_3 = 0$ are shown in Figures 2 and 3, respectively.

If we set the fixed dwell time, $d_1 = 2$, $d_2 = 1$, and $d_3 = 0$, using the LMI toolbox of Matlab, we can get that there exist

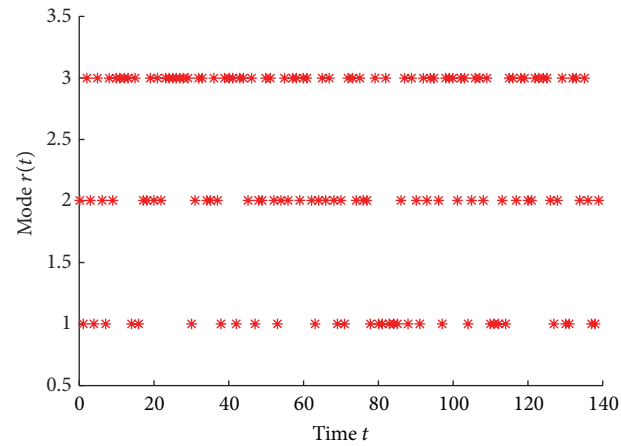


Figure 2: System switching modes with random samplings ($d_0 = d_2 = d_3 = 0$).

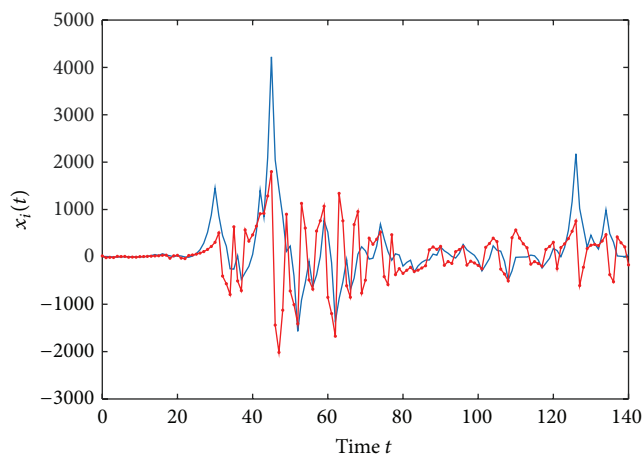


Figure 3: System state trajectories of x_1 and x_2 ($d_0 = d_2 = d_3 = 0$).

3 positive definite matrices P_i, $i = 1,2,3$, such that term (23) holds, where

$$\begin{aligned} P_1 &= \begin{bmatrix} 4.8832 & 6.8271 \\ 6.8271 & 15.8834 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 1.5705 & -1.3768 \\ -1.3768 & 16.3751 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 5.8646 & 12.8518 \\ 12.8518 & 70.7197 \end{bmatrix}. \end{aligned} \tag{39}$$

According to Theorem 12, one can obtain that system (1) is stochastically stable. Figure 4 shows the switching signal between mode 1 and mode 3, in which the symbols "+" and "$*$" denote the mode at the fixed dwell time and random dwell time, respectively. Figure 5 shows that the sampled system state trajectories of x_1 and x_2 tend to the zero equilibrium.

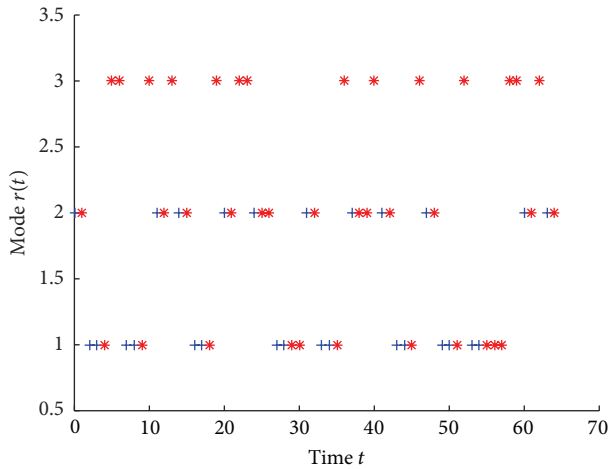


FIGURE 4: System switching modes with random samplings ($d_0 = 2$, $d_2 = 1$, and $d_3 = 0$).

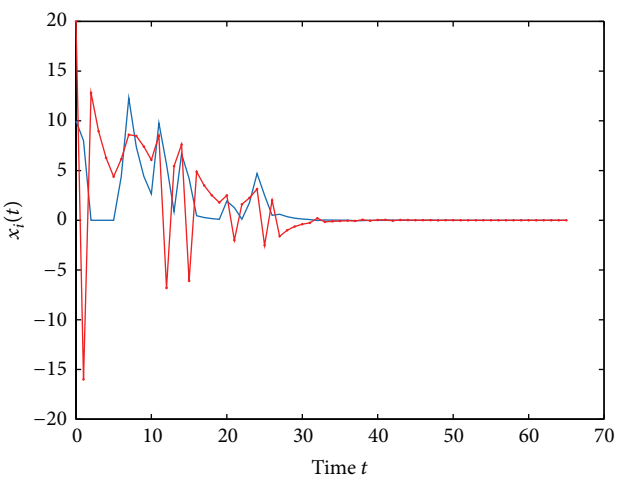


FIGURE 5: System state trajectories of x_1 and x_2 ($d_0 = 2$, $d_2 = 1$, and $d_3 = 0$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by The National Natural Science Funds of China [60874006] and [11401540], Foundation of Henan Department of Science and Technology [112300410194], and Foundation of Henan Educational Committee [12B120004].

References

[1] Y. Fang and K. A. Loparo, "Stochastic stability of jump linear systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1204–1208, 2002.

[2] O. L. V. Costa and M. D. Fragoso, "Comments on 'stochastic stability of jump linear system'," *IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1414–1416, 2004.

[3] C. A. C. Gonzaga and O. L. V. Costa, "Stochastic stability for discrete-time Markov jump Lur'e system," in *Proceedings of the 50th IEEE Conference on Decision and Control*, pp. 5993–5998, December 2013.

[4] T. Yang, L. Zhang, Y. Li, and Y. Leng, "Stabilisation of Markov jump linear systems subject to both state and mode detection delays," *IET Control Theory & Applications*, vol. 8, no. 4, pp. 260–266, 2014.

[5] Y. Qian, A. Wu, and W. Liu, "Stochastic stability of discrete-time markovian jump antilinear systems," in *Proceedings of the 26th Chinese Control And Decision Conference (CCDC '14)*, pp. 4381–4386, Changsha, China, May 2014.

[6] Y. Kao, J. Xie, and C. Wang, "Stabilization of singular Markovian jump systems with generally uncertain transition rates," *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2604–2610, 2014.

[7] O. L. Costa and D. Z. Figueiredo, "Stochastic stability of jump discrete-time linear systems with Markov chain in a general Borel space," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 223–227, 2014.

[8] D. Liberzon, *Switching in Systems and Control*, Systems & Control: Foundations & Applications, Birkhäauser, Boston, Mass, USA, 2003.

[9] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach," *IEEE Transactions on Automatic Control*, vol. 47, no. 11, pp. 1883–1887, 2002.

[10] L. Hetel, J. Daafouz, and C. Iung, "Stabilization of arbitrary switched linear systems with unknown time-varying delays," *IEEE Transactions on Automatic Control*, vol. 51, no. 10, pp. 1668–1674, 2006.

[11] G. Zhang, C. Han, Y. Guan, and L. Wu, "Exponential stability analysis and stabilization of discrete-time nonlinear switched systems with time delays," *International Journal of Innovative Computing, Information and Control*, vol. 8, no. 3, pp. 1973–1986, 2012.

[12] J. Xiong, J. Lam, Z. Shu, and X. Mao, "Stability analysis of continuous-time switched systems with a random switching signal," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 180–186, 2014.

[13] P. Bolzern, P. Colaneri, and G. de Nicolao, "Markov jump linear systems with switching transition rates: mean square stability with dwell-time," *Automatica*, vol. 46, no. 6, pp. 1081–1088, 2010.

[14] N. T. Thomopoulos, *Fundamentals of Queuing Systems: Statistical Methods for Analyzing Queuing Models*, Springer, New York, NY, USA, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 323475, 17 pages
http://dx.doi.org/10.1155/2015/323475

Research Article

Microstructure Models with Short-Term Inertia and Stochastic Volatility

Michael A. Kouritzin

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1

Correspondence should be addressed to Michael A. Kouritzin; michaelk@ualberta.ca

Received 2 March 2015; Revised 9 July 2015; Accepted 15 July 2015

Academic Editor: Quanxin Zhu

Partially observed microstructure models, containing stochastic volatility, dynamic trading noise, and short-term inertia, are introduced to address the following questions: (1) Do the observed prices exhibit statistically significant inertia? (2) Is stochastic volatility (SV) still evident in the presence of dynamical trading noise? (3) If stochastic volatility and trading noise are present, which SV model matches the observed price data best? Bayes factor methods are used to answer these questions with real data and this allows us to consider volatility models with very different structures. Nonlinear filtering techniques are utilized to compute the Bayes factor on tick-by-tick data and to estimate the unknown parameters. It is shown that our price data sets all exhibit strong evidence of both inertia and Heston-type stochastic volatility.

1. Introduction

Financial analysts list speculation, finiteness of assets, interest rates, tick size, price inertia, price clustering, belief heterogeneity, asymmetric information, greed and fear, and so forth as causes for price fluctuations over time. Yet, popular models like geometric Brownian motion (GBM) (e.g., Black and Scholes [1], Merton [2]) or the Cox-Ross-Rubinstein model [3] try to handle all these factors in an overly simple framework, resulting in unnatural phenomena like the *volatility smile*. Consequently, stochastic volatility, which has been observed in real prices, is often added to the price value evolution (e.g., Heston [4], Jachwerth and Rubinstein [5], Hull and White [6], and Nelson [7]) to avoid the volatility smile. However, which stochastic volatility model fits the market data best?

Nowadays, many authors talk about the misspecification of stochastic price-volatility models (including the Heston model which we show favorably herein) so much. It leads us to wonder whether there are missing ingredients to these very simple models. Even combined stochastic value-volatility models do not address tick size, price inertia, price clustering, hidden liquidity, and fear-greed cycles that traders, especially high frequency traders, must deal with. To handle these issues, one is drawn to tick-by-tick microstructure models and left with the perplex question: How should one model price inertia in continuous time? We are using the term price *inertia* instead of the related term price momentum because we are not weighting transaction prices by volume. Fractional Brownian motion (FBM), best known for its long memory properties, exhibits inertia and has been used to model markets (Mandelbrot [8], Shiryaev [9]) even though these models allow arbitrage strategies. We speculate that FBM's success in modeling observed data is more attributable to inertia than long memory. However, we introduce an alternative *inertia process* and show that this new process better satisfies the desired properties of inertia than FBM. We then show strong statistical evidence of price inertia that lasts for hours or days using Bayes estimates and Bayes factor on real price data. We do not consider the possibility of arbitrage nor determine derivative prices for our models but rather leave these interesting mathematical finance questions to the experts. (See Capinski and Zastawniak [10] for an excellent introduction to these types of questions and to mathematical finance in general.) Also, we leave the difficult task of obtaining theoretical error bounds for our particle filter methods to other works. (See, e.g., Kouritzin and Zeng [11] and Del Moral et al. [12] for related work on approximate

filters.) Our focus is solely on modeling observed stock price data and the methodology of determining which of a class of models best fits the observed data.

High frequency data contains complete market-participant trading activities (Engle [13]) and is modeled using microstructure (Black [14], Chan and Lakonishok [15], Hasbrouck [16, 17], Engle and Russell [18], Engle [13], and Bandi and Russell [19]). Unlike the macrostructure market, the trading noise in the microstructure market is not negligible; thus, the intrinsic asset value is not readily discernable. In this paper, we introduce a class of dynamic microstructure models, where the transaction price is formulated as a distorted and color-noise corrupted variant of the intrinsic asset value with the intrinsic asset value being a traditional stochastic value-volatility process. Indeed, we view the transaction price data as random counting-measure observations of intrinsic value corrupted by microstructure trading noise with such things as inertia and fear-greed cycles built in. However, trading noise sources themselves introduce volatility to transaction prices. This raises the question, "Do we need to model stochastic volatility explicitly in the presence of dynamic microstructure trading noise?" We will give strong evidence of the presence of stochastic volatility through Bayes factor methods and stochastic filtering theory. Moreover, we also utilize model selection to provide strong evidence of Heston-type volatility over competing stochastic volatility models based on the observed transaction data in a microstructure market. This suggests that the common viewpoint of the Heston model being highly misspecified might be better stated as overly simplistic macrostructure-only models are underspecified. Bayes factor (see, e.g., Kass and Raftery [20]) is our preferred model selection method since it provides statistical comparisons in real time as to which model best fits the market data while allowing the stochastic value-volatility (signal) models to be singular to one another. Indeed, to use the Bayes factor method, we need only to be able to transform all microstructure asset-price observation models of interest into the same canonical process via Girsanov-type measure change.

Previously, Zeng [21] studied a filtering equation for inferring the intrinsic value process in a microstructure model while Xiong and Zeng [22] proposed a branching particle approximation to this equation. Kouritzin and Zeng [23] derived a Bayes factor equation and discussed the Bayesian model selection problem to determine whether financial data, such as stock prices, display jump-type stochastic volatility. However, all these works are based on a restricted microstructure model and thus cannot be applied to our general setting. Moreover, our problems of showing statistical evidence of inertia and determining which of the classical stochastic volatility models best represents real data in the presence of microstructure noise were not considered. We also propose a new inertia process, explain its role in modeling prices, and show its statistical significance with real tick-by-tick data.

Section 2 is devoted to explaining our model. First, our five standard value-volatility models (GBM, Hull-White, Log Ornstein-Uhlenbeck, continuous GARCH, and Simplified Heston) are given followed by our microstructure inertia process and its properties and then the other components of our dynamic microstructure model. Together the value-volatility and microstructure components form our price evolution model, which, at the end of Section 2, is interpreted as a filtering model. In Section 3, we discuss model calibration and fair price/value estimation through Bayesian filter estimation. A filtering equation and a branching particle filter approximation algorithm are first given and explained. Then, their use to identify parameters and come up with initial state estimates is discussed. Finally, numeric parameter and initial state estimates for each model are given. As a byproduct, it is demonstrated that proper modeling and estimation of fair price (as is done herein) can provide information about overbought conditions and help avoid financial loss (see Figure 4). Section 4 is dedicated to Bayesian model selection. We first motivate the use of Bayes factor for model selection and explain how to estimate Bayes factor from unnormalized particle filters. Then, we establish strong statistical evidence of inertia and Heston-type volatility in all our price data through model selection using the Bayes factor method to test which fair price-volatility model and what amount of inertia best fit the observed price data.

2. The Partially Observed Market Model

In this section, we build our stochastic model that has macrostructure and microstructure components and interpret this model in terms of a *signal* that needs to be estimated in real time and *observations* which are used to form the signal estimates. The macrostructure model consists of fair price, volatility, and related parameters and will be denoted by (X, θ) in the sequel, with $X = (S, V)$ being price and volatility and θ being the parameters for this model. Unlike macrostructure models, we do not assume access to (X, θ), but rather we take it to be part of the signal to be estimated. Indeed, a model would be judged to be better if the macrostructure price S (which represents a "fair" price) is quite different than the observed price and we can use filtering to determine overbought and oversold situations.

The microstructure price construction converts the macrostructure model into the observed price. Such things as inertia (or momentum), fear-greed cycles, and whole-price clustering (or rounding), which are not part of the fair price, are incorporated into the microstructure model. A distinguishing feature in our microstructure is dynamic state: To allow the microstructure to influence price over a period of time so that the observed microstructure price can differ from fair price significantly, one needs to add and then estimate microstructure state Z. In particular, the *inertia* process, characterized by a parameter h, is introduced to capture price inertia that might be caused by hidden liquidity; various reaction and access times to information as well as momentum traders themselves. This inertia process is not Markov, so we will have to consider the historical version $\widehat{Z}^h$ of this state. Further, $\widehat{Z}^h$ is also unobservable and hence must be added to the signal along with microstructure parameters ϑ and all must be estimated as nuisance parameters.

The nondynamic part of the microstructure noise consists of rounding and clustering noise. It is widely observed in

markets that more trades occur at more even prices like whole nickel or whole dollar levels. Therefore, to match observed prices well, we should have a mechanism to convert evenly distributed raw prices into whole-price-biased observed prices. This is done by binning raw prices into sets D_1, D_2, D_3, D_4, and D_5 depending on how even they are and then randomly moving raw prices in the less even bins to close prices in the more even bins in order to match the observed prices.

The observations then become the marked counting process of the number of trades that occur at the various prices. We will later use these observations to select and calibrate models and to estimate the augmented signal:

$$\left(X, \theta, \vartheta, \widehat{Z}^h\right). \tag{1}$$

The whole point of the microstructure is to allow the macrostructure price to distinguish itself from the observations and rather to represent fair value. We then use filtering on asset prices to estimate implied value (hereafter called fair price) and thereby judge whether an asset is overbought or oversold.

2.1. General Notation. Let $[0, T]$ be a fixed time period and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space. For any stochastic process ρ, its natural filtration, defined as $\mathscr{F}_t^\rho \doteq \sigma\{\rho_u : 0 \le u \le t\}$, represents the information in ρ up to time t. $\mathbb{N}_0$ denotes the set of nonnegative integers and, for any Polish space E, $B(E)$ is the set of all bounded measurable $\mathbb{R}$-valued functions on E.

2.2. Common Macrostructure State Models. We use a macrostructure model $M = (X, \theta)$ for the unobservable fair price together with its volatility and parameters. Here, $X \in \mathbb{R}^{n_x}$ is the macrostructure financial state (fair price plus volatility) with macrostructure parameter $\theta \in \mathbb{R}^{n_\theta}$ for some $n_x, n_\theta \in \mathbb{N}_0$. We let μ be a probability distribution on $\mathbb{R}^{n_x+n_\theta}$, take $\mathbf{A}$ to be a generator with domain $\mathscr{D}(\mathbf{A}) \subset B(\mathbb{R}^{n_x+n_\theta})$, and assume (X, θ) satisfies the martingale problem.

Definition 1. (X, θ) is the unique solution of the $\mathbb{R}^{n_x+n_\theta}$-valued martingale problem for $\mathbf{A}$ with initial distribution μ. That is,

$$\begin{aligned} &\text{(i) } \mu = \mathbb{P} \circ \left(X_0, \theta\right)^{-1}, \\ &\text{(ii) } M_t^f = f\left(X_t, \theta\right) - f\left(X_0, \theta\right) - \int_0^t \mathbf{A} f\left(X_s, \theta\right) ds \end{aligned} \tag{2}$$

is $\{\mathscr{F}_t^{X,\theta}\}$-martingale for each $f \in \mathscr{D}(\mathbf{A})$. Moreover, if $(\widetilde{X}, \widetilde{\theta})$ also satisfies (i) and (ii), then (X, θ) and $(\widetilde{X}, \widetilde{\theta})$ have the same finite dimensional distributions.

Remark 2. While θ does not vary in time, we include it in our macrostructure model to be estimated because it is still unknown. Nevertheless, the operator $\mathbf{A}$ does not act on the variable θ since $d\theta_t/dt = 0$ for our fixed parameters.

The martingale problem formulation (2) (see Stroock and Varadhan [24], Ethier and Kurtz [25] for more details) is general enough to cover most interesting financial models. In this paper, the macrostructure state X consists of two components: the fair price S and the stochastic volatility V (if any). The most common example of (S, V, θ) in finance is the "geometric Brownian motion" (GBM) utilized in the classical Black-Scholes option pricing formula. Throughout this section, W and B are two independent standard Brownian motions and $(s, v, \theta) \in \mathbb{R}^{n_x+n_\theta}$.

Example 3 (GBM model; see Black and Scholes [1], Merton [2]). We have that

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3}$$

with parameters $\theta = (\mu, \sigma)$, corresponds to our martingale problem with the generator

$$\mathbf{A}^{(1)} f = \frac{1}{2}\sigma^2 s^2 \frac{d^2 f}{ds^2} + \mu s \frac{df}{ds}. \tag{4}$$

In GBM model, the volatility σ is a constant. To account for the "*volatility smile*" commonly observed in market option prices (see Jackwerth and Rubinstein [5] for a detailed survey), the GBM model is generalized to stochastic volatility (SV) models, where σ itself is replaced by a stochastic process $\{V_t^{1/2},\ t \ge 0\}$. Some of the popular SV models include the following.

Example 4 (Hull-White model; see Hull and White [6]). Consider

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + V_t^{1/2} dW_t, \\ \frac{dV_t}{V_t} &= \nu dt + \kappa dB_t, \end{aligned} \tag{5}$$

with parameters $\theta = (\mu, \nu, \kappa)$ and generator

$$\mathbf{A}^{(2)} f = \frac{1}{2}\nu s^2 \frac{\partial^2 f}{\partial s^2} + \mu s \frac{\partial f}{\partial s} + \frac{1}{2}\kappa^2 v^2 \frac{\partial^2 f}{\partial v^2} + \nu v \frac{\partial f}{\partial v}. \tag{6}$$

Example 5 (Logarithmic Ornstein-Uhlenbeck model; see Scott [26]). We have that

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + V_t^{1/2} dW_t, \\ \frac{dV_t^{1/2}}{V_t^{1/2}} &= \left(\frac{1}{2}\nu^2 - \varrho\left(\ln V_t^{1/2} - \varpi\right)\right) dt + \kappa dB_t, \end{aligned} \tag{7}$$

with parameters $\theta = (\mu, \nu, \varrho, \varpi, \kappa)$ and generator

$$\begin{aligned} \mathbf{A}^{(3)} f = &\frac{1}{2}v^2 s^2 \frac{\partial^2 f}{\partial s^2} + \mu s \frac{\partial f}{\partial s} + \frac{1}{2}\kappa^2 v^2 \frac{\partial^2 f}{\partial v^2} \\ &+ v\left(\frac{1}{2}\nu^2 - \varrho\left(\ln v - \varpi\right)\right)\frac{\partial f}{\partial v}. \end{aligned} \tag{8}$$

Example 6 (continuous GARCH model; see Nelson [7]). We have that

$$\frac{dS_t}{S_t} = \mu dt + V_t^{1/2} dW_t,$$
$$dV_t = \left(\nu - \varrho V_t\right) dt + \kappa V_t dB_t, \tag{9}$$

with parameters $\theta = (\mu, \nu, \varrho, \kappa)$ and generator

$$\mathbf{A}^{(4)} f = \frac{1}{2} \nu s^2 \frac{\partial^2 f}{\partial s^2} + \mu s \frac{\partial f}{\partial s} + \frac{1}{2} \kappa^2 \nu^2 \frac{\partial^2 f}{\partial \nu^2} + \left(\nu - \varrho \nu\right) \frac{\partial f}{\partial \nu}. \tag{10}$$

Example 7 (simplified Heston model; see Heston [4]). We have that

$$\frac{dS_t}{S_t} = \mu dt + V_t^{1/2} dW_t,$$
$$dV_t = \left(\nu - \varrho V_t\right) dt + \kappa V_t^{1/2} dB_t, \tag{11}$$

with parameters $\theta = (\mu, \nu, \varrho, \kappa)$ and generator

$$\mathbf{A}^{(5)} f = \frac{1}{2} \nu s^2 \frac{\partial^2 f}{\partial s^2} + \mu s \frac{\partial f}{\partial s} + \frac{1}{2} \kappa^2 \nu \frac{\partial^2 f}{\partial \nu^2} + \left(\nu - \varrho \nu\right) \frac{\partial f}{\partial \nu}. \tag{12}$$

We label this example as simplified because we do not allow B and W to be correlated as Heston did. There is no mathematical issue by including this correlation, but it would add a parameter to the model, which increases computation time. The Heston model already performed the best without this parameter. GBM (with microstructure) plays a special role in our study as it is our *no stochastic volatility* model. We will compare our other models against it on real data to determine whether stochastic volatility is present. In summary, refer to Table 1.

Remark 8. The continuous GARCH model is the continuous-time limit of many classical GARCH-type discrete-time processes (Nelson [7], Drost and Werker [27]). We did not consider jumping stochastic volatility models (e.g., Elliott et al. [28], Kouritzin and Zeng [23], Duffie et al. [29], Eraker et al. [30], and Eraker [31]) or models where W, B are correlated, due to our need to dedicate our limited computer resources to handling our complicated (non-Markov) microstructure with inertia. Still, we want to emphasize that the computational complexity we experienced is fundamental to the fact that we are using non-Markov (inertia) models and has little to do with our particular methods. Indeed, our Bayes factor filtering methods are what makes the computations possible on an inexpensive contemporary desktop computer.

2.3. Construction of Microstructure Price. The fair price-volatility models account for the random variances of the intrinsic asset value; thus, the selection of proper SV model is crucial for investing, derivative pricing, and hedging. On the other hand, microstructure noise (see Black [14], Hansen and Lunde [32], Duan and Fulop [33], etc.) causes random perturbations of transaction price from its intrinsic value and the disregard of such trading noise introduces severe bias into stochastic volatility estimation (see Duan and Fulop [33]). We incorporate microstructure trading noise into traditional fair price-volatility models and use statistical filtering to reveal such things as short-term inertia in the trading noise and stochastic volatility in the intrinsic value.

Table 1

Name	Model	Macrostate	Macroparameter	Generator
GBM	$M^{(1)}$	S	(μ, σ)	$\mathbf{A}^{(1)}$
Hull-White	$M^{(2)}$	(S, V)	(μ, ν, κ)	$\mathbf{A}^{(2)}$
Log O-U	$M^{(3)}$	(S, V)	$(\mu, \nu, \varrho, \varpi, \kappa)$	$\mathbf{A}^{(3)}$
GARCH	$M^{(4)}$	(S, V)	$(\mu, \nu, \varrho, \kappa)$	$\mathbf{A}^{(4)}$
Heston	$M^{(5)}$	(S, V)	$(\mu, \nu, \varrho, \kappa)$	$\mathbf{A}^{(5)}$

In microstructure markets, the price changes occur only at irregularly spaced transaction times $t_1, t_2, \ldots$ with total trading intensity $a(t)$ (see Engle [13]). Here, we assume $a(t)$ is just a time-varying measurable function as the empirical analysis illustrates that there is no need to consider more general structures. At each transaction time t_i, the transaction price Y_{t_i} is formulated as

$$Y_{t_i} = F\left(X_{t_i}, t_i\right), \tag{13}$$

where F is some nonlinear random field modeling the trading noise. Formulation (13) is similar to that of Hasbrouck [16], where X is the intrinsic and permanent component while F introduces the transitory component.

The empirical evidence reported by Hansen and Lunde [32] suggests strongly that the trading noise is serially correlated. Similar results can be found in Aït-Sahalia et al. [34]. Indeed, there exist situations in which the trading noise variance estimate is zero if the trading noise is simply assumed to be independent (see Duan and Fulop [33]). This does not mean there is no trading noise but rather that the trading noise is autocorrelated. To characterize this correlation, Hansen and Lunde [32] assume the trading noise to be some Gaussian random sequence with stationary covariance and finite dependence. However, this model is most suitable for the low-frequency data and ignores many crucial microstructure effects. We build correlation into our microstructure information noise through inertia and mean-reversion while utilizing microstructure rounding and clustering noise to explain the discreteness and whole-price biasing.

2.3.1. Inertia. The idea of momentum or inertia has been used in many studies (see Jegadeesh and Titman [35], Moskowitz and Grinblatt [36], Grundy and Martin [37], Grundy et al. [38], etc.). Basically, there is the tendency for a stock to continue to move in one direction. To illustrate our approach, we introduce the following definition.

Definition 9. A process (Z_t) is said to have stochastic inertia at time t if

$$I_t^Z \doteq \lim_{u \searrow t} \frac{\partial}{\partial t} \frac{\partial}{\partial u} \mathbb{E}\left[Z_u Z_t\right] \in (0, \infty]. \tag{14}$$

I_t^Z is called the inertia function.

The idea behind our definition is that for inertia we should expect $Z_{u+r} - Z_u$ and $Z_t - Z_{t-k}$ to have the same sign for $u > t$, but close to t and $r, k > 0$ small. We strengthen this condition to

$$\begin{aligned} &\lim_{u \searrow t} \lim_{k \to 0} \lim_{r \to 0} \frac{E\left[\left(Z_{u+r} - Z_u\right)\left(Z_t - Z_{t-k}\right)\right]}{rk} > 0 \\ &\Longleftrightarrow \lim_{u \searrow t} \lim_{k \to 0} \left[\lim_{r \to 0} \frac{E\left[\left(Z_{u+r} - Z_u\right) Z_t\right]}{rk} \right. \\ &\qquad \left. - \lim_{r \to 0} \frac{E\left[\left(Z_{u+r} - Z_u\right) Z_{t-k}\right]}{rk}\right] > 0 \\ &\Longleftrightarrow \lim_{u \searrow t} \frac{\partial}{\partial t} \frac{\partial}{\partial u} E\left[Z_u Z_t\right] > 0. \end{aligned} \tag{15}$$

Many processes have inertia. However, to model the stock price effect of the information reaching all market participants, we want the following five properties: (1) Z_t is Gaussian and driftless and $\mathrm{Var}(Z_t)$ is proportional to t so Z resembles Brownian motion; (2) I_t^Z is finite, not infinite, indicating that the influence of past values on immediate future is not too strong; (3) Z_t makes sense from informational and hidden liquidity points of view; more precisely, it can explain well the price effects due to the reactions of all market participants to information and rumor being diffused and simulated over a period of time as well as due to the purchases or sales of an agent spreading out a large change in his/her position over time; (4) Z is easy to simulate using, for example, the Gaussian property; (5) Z is easy to analyze.

Neither a Brownian motion B nor more generally a square integrable martingale has inertia. Brownian motion with drift $Z_t = Z_0 + \int_0^t m(Z_s)ds + B_t$ has inertia $I_t^Z = E[m^2(Z_t)]$ but we do not want drift. For fractional Brownian motion (FBM) B^h,

$$\mathbb{E}\left[B_t^h B_u^h\right] = \frac{1}{2}\left(|t|^{2h} + |u|^{2h} - |u-t|^{2h}\right), \tag{16}$$

where $h \in (0,1)$ is the Hurst parameter. Therefore,

$$\lim_{u \searrow t} \frac{\partial}{\partial t} \frac{\partial}{\partial u} \mathbb{E}\left[B_t^h B_u^h\right] = \lim_{u \searrow t}\left((2h-1)\, h\, (u-t)^{2h-2}\right) = \infty \quad \text{if } h > \frac{1}{2}. \tag{17}$$

Thus, the inertia function of B^h is infinity for all t if $h > 1/2$ (and is $-\infty$ if $h < 1/2$). Neither case satisfies our five properties. Still, standard representations of FBM motivate the creation of driftless inertia by convolving a Brownian motion with the desired impulse response for information dissemination. With this in mind, we consider the following inertia process.

Definition 10. Our stochastic inertia process is

$$\xi_t^h = \sqrt{h} \int_0^t \tanh\left(\frac{(t-s)}{\Delta}\right) dB_s^\xi + \sqrt{1-h}\, W_t^\xi, \tag{18}$$

where (B^ξ, W^ξ) is a 2-dimensional standard Brownian motion, $\Delta > 0$, and $0 \le h \le 1$.

Remark 11. ξ_t^h is a weighted average of the historical information (the first term) and fundamental information (the second term). In fact, $\tanh(t/\Delta)$ can be viewed as the impulse response on price created by market participants receiving and simulating the "information" dB_t^ξ and Δ determines the diffusion speed in the market. This formulation captures the idea that news or rumor and its ramifications require time to be fully disseminated and understood. When $h = 1$, it represents the case of only historical information resulting in the strongest inertia in prices. Alternatively, we can use inertia to explain "hidden liquidity." If everybody knew that an agent was going to make a big change in a position, then the price would immediately jump. However, if the agent breaks up the desired change into small transactions, then it takes time for this extra buying or selling pressure to be recognized in the market. In this case, $h = 1$ represents the case, where all changes in position are done over a period of time and Δ represents the time to effect 58% of the positional change.

Note that ξ_t^h is a centered Gaussian process such that the autocovariance

$$\begin{aligned} \mathbb{E}\left[\xi_t^h \xi_u^h\right] = h \int_0^t \tanh\left(\frac{(t-s)}{\Delta}\right) \tanh\left(\frac{(u-s)}{\Delta}\right) ds \\ + (1-h)\, t \end{aligned} \tag{19}$$

is positive for any $u \ge t$. In particular,

$$\begin{aligned} \frac{\mathrm{Var}\left(\xi_t^h\right)}{t} &= \frac{h}{t} \int_0^t \tanh^2\left(\frac{(t-s)}{\Delta}\right) ds + (1-h) \\ &= 1 - h\Delta \frac{\tanh(t/\Delta)}{t}. \end{aligned} \tag{20}$$

Thus, $\mathrm{Var}(\xi_t^h)/t$ converges to 1 as $t \to \infty$ with speed determined by Δ. (Hence, informational noise increases at the same asymptotic rate as Brownian motion.) Moreover,

$$\begin{aligned} &\frac{\partial}{\partial t} \frac{\partial}{\partial u} \mathbb{E}\left[\xi_t^h \xi_u^h\right] \\ &\quad = \frac{h}{\Delta^2} \int_0^t \mathrm{sech}^2\left(\frac{(t-s)}{\Delta}\right) \mathrm{sech}^2\left(\frac{(u-s)}{\Delta}\right) ds \end{aligned} \tag{21}$$

and, using standard antiderivatives,

$$\begin{aligned} \lim_{u \searrow t} \frac{\partial}{\partial t} \frac{\partial}{\partial u} \mathbb{E}\left[\xi_t^h \xi_u^h\right] &= \frac{h}{\Delta^2} \int_0^t \mathrm{sech}^4\left(\frac{s}{\Delta}\right) ds \\ &= \frac{h}{\Delta}\left[\tanh\left(\frac{t}{\Delta}\right) - \frac{\tanh^3(t/\Delta)}{3}\right]. \end{aligned} \tag{22}$$

Thus, the inertia function of our inertia process is $I_t^{\xi^h} = (h/\Delta)[\tanh(t/\Delta) - \tanh^3(t/\Delta)/3]$, the steady-state inertia is

$$\lim_{t\to\infty} \frac{h}{\Delta}\left[\tanh\left(\frac{t}{\Delta}\right) - \frac{\tanh^3(t/\Delta)}{3}\right] = \frac{2h}{3\Delta}, \tag{23}$$

and this happens quickly for small Δ. We can thus verify that ξ^h, defined in (18), satisfies our five desired properties. One can also look upon Δ as the time for new information to be disseminated to fifty-eight percent of the market. Below, we consider three different dissemination times: $\Delta = 40$ minutes, $\Delta = 2$ hours, and $\Delta = 1/2$ day on real stock data. Finally, the fact that ξ^h is Gaussian eases its simulation greatly.

2.3.2. Information Noise and Augmented State. Hitherto, we have focused on constructing inertia processes. Now, we include all informational noise into asset prices. Information noise is introduced to represent trading noises due to things like inertia, fear-greed cycles, belief heterogeneity, and asymmetric information. For the ith-transaction occurring at t_i, the raw price $\mathcal{Y}_{t_i}$ is defined by

$$\ln \mathcal{Y}_{t_i} = \begin{cases} \ln S_{t_i} + Z_{t_i}^{h,\Delta} + \epsilon\zeta_i, & \text{dynamical microstructure,} \\ \ln S_{t_i} + \epsilon\zeta_i, & \text{nondynamical,} \end{cases} \tag{24}$$

$$dZ_t^{h,\Delta} = -\phi Z_t^h dt + d\xi_t^h, \quad Z_0^{h,\Delta} = z_0, \tag{25}$$

where $Z^{h,\Delta}$ is the dynamical part of the microstructure through which inertia is introduced (with our inertia process ξ^h) and $X = (S, V)$. The case $Z^{h,\Delta} \equiv 0$ is of particular importance in the sequel as it represents the *nondynamical* microstructure case and is used as a calibration model.

The information noise consists of two parts: $\zeta = \{\zeta_i\}_{i=1}^{\infty}$ is a sequence of independent standard Gaussian random variables, $\epsilon > 0$; Z^h is Ornstein-Uhlenbeck- (O-U-) like inertia velocity process with mean-reverting parameter $\phi > 0$. Here, ξ^h, ζ, and X are independent and z_0 is a constant. Z^h provides an intuitive continuous-time model that accommodates the joint presence of the inertia and mean-reversion. Our information noise is more reasonable than that of Zeng [21] in that (1) we preclude the possibility of negative prices by using multiplicative noise; (2) the stochastic inertia process ξ^h captures the empirical feature of the inertia observed in transaction prices (e.g., Jegadeesh and Titman [35]); (3) the mean-reverting structure of Z^h when combined with the inertia captures the cyclic property of prices (e.g., Black [14]). Z^h is not a Markov process, so we introduce its historical process as

$$\widehat{Z}_t^h(\tau) \doteq Z_{t\wedge\tau}^h, \tag{26}$$

which is Markovian. Moreover, $\widehat{Z}_t^h \in C[0,T]$, the space of all continuous functions on $[0,T]$, since the paths of Z^h are continuous. Consequently, we augment the state vector to be

$$\left(X, \theta, \vartheta, \widehat{Z}^h\right), \tag{27}$$

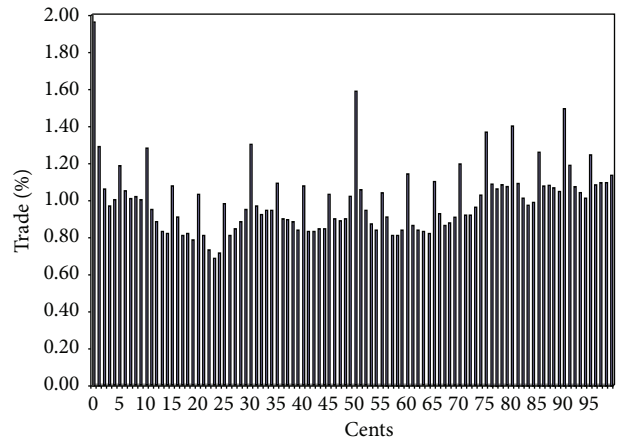


Figure 1: Clustering for 3 stocks in April 2010.

where $\vartheta = (\epsilon, \phi)$ is the microstructure noise parameter set. The advantage of this formulation is that we can estimate $\widehat{Z}^h$ and thus Z^h jointly with other components using particle filtering methods. The generalized state incorporates fair price, volatility, parameters, and the historical trading noise $\widehat{Z}^h$ while keeping the tractability of a Markovian framework.

Remark 12. We include neither h nor Δ into the model parameters but rather consider different models corresponding to different values of h and Δ as well as different SV models 1–5. Indeed, we will provide evidence of inertia in the sequel by using Bayesian methods to select a model with a large value of h based upon tick-by-tick stock data.

2.3.3. Rounding and Clustering Noise. Our final modeling goal is to convert uniform raw price into observed whole-price-biased price. While raw price $\mathcal{Y}_{t_i}$ can take any value, the trading price Y_{t_i} is restricted to multiples of the *tick*, $\{y_0 = 0, y_1 = 1/M, \ldots, y_j = j/M, \ldots\}$, for some positive integer M. The tick size in New York Stock Exchange (NYSE) was switched to \$1/16 from \$1/8 in June 24, 1997, and then further to \$0.01 from January 29, 2001. The empirical studies suggest that the tick size $1/M$ plays an important role in microstructure market analysis (e.g., Huang and Stoll [39]). Since we are concerned with price clustering for decimal pricing in stock markets, we let $M = 100$.

It is well documented that there is price clustering to more whole prices. To quantify this price clustering, we examine the price behavior for three NYSE-listed stocks over April 2010 (Figure 1 and Table 2). (In a larger study, we considered eight NYSE stocks in different sectors. However, we only report on three here to conserve space. The results for the other five were similar in nature.)

The transaction data of these stocks shows there is modest clustering at multiples of 5 cents as shown in Figure 1, plotted in terms of pennies. Supposing the raw price $\mathcal{Y}_{t_i}$ falls in the interval $[y_j - 1/2M, y_j + 1/2M)$, then if there was no clustering noise, the trading price Y_{t_i} would just be y_j. Thus,

Table 2

NYSE stock	Ticker symbol
Goldman Sachs	GS
International Business Machines Corporation	IBM
PepsiCo Inc.	PEP

the probability of trading at y_j with no clustering noise given $X_{t_i} = x$, $Z_{t_i} = z$ would be

$$R\left(y_j \mid x, z, \vartheta\right) \doteq P\left(\mathcal{Y}_{t_i} = y_j \mid X_{t_i} = x, Z_{t_i} = z, \vartheta\right)$$
$$= \begin{cases} \displaystyle\int_{\ln((y_j-1/2M)/(x\cdot e^z))}^{\ln((y_j+1/2M)/(x\cdot e^z))} \frac{1}{\sqrt{2\pi}\epsilon} e^{-u^2/2\epsilon^2} du & \text{dynamic microstructure} \\ \displaystyle\int_{\ln((y_j-1/2M)/x)}^{\ln((y_j+1/2M)/x)} \frac{1}{\sqrt{2\pi}\epsilon} e^{-u^2/2\epsilon^2} du & \text{nondynamical.} \end{cases} \quad (28)$$

Equivalently, we can write R in terms of the historical process as

$$R\left(y_j \mid X_{t_i}, \Pi_{t_i} \widehat{Z}^h_{t_i}, \vartheta\right) = \int_{\ln((y_j-1/2M)/X_{t_i} e^{\Pi_{t_i} \widehat{Z}^h_{t_i}})}^{\ln((y_j+1/2M)/X_{t_i} e^{\Pi_{t_i} \widehat{Z}^h_{t_i}})} \frac{1}{\sqrt{2\pi}\epsilon} e^{-u^2/2\epsilon^2} du, \quad (29)$$

where Π_{t_i} is the projection onto time t_i; that is,

$$\Pi_{t_i} \widehat{Z}^h_{t_i} = \widehat{Z}^h_{t_i}\left(t_i\right) = Z^h_{t_i \wedge t_i} = Z^h_{t_i}. \quad (30)$$

Clearly, $R(y_j \mid x, z, \vartheta)$ is a smooth function of (x, z, ϑ) for each fixed y_j.

To build the observed whole-price bias into our model, we introduce the following sets:

$$\begin{aligned} D_1 &= \{\text{The integers in } (0, 100] \\ &\quad \text{that are not multiples of } 5\}, \\ D_2 &= \{\text{The integers in } (0, 100] \\ &\quad \text{that are multiples of 5 but not of 25}\}, \\ D_3 &= \{25, 75\}, \\ D_4 &= \{50\}, \\ D_5 &= \{100\}. \end{aligned} \quad (31)$$

While the raw price will be uniformly distributed over $D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$ (or rather the continuous interval $(0, 100]$), the observed price model must bias D_2 over D_1, D_3 over either D_2 or D_1, and so forth. We distribute the observed price randomly over $D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$ based upon the raw price in a biased manner favoring the more whole-price ticks in $D_2 \cup D_3 \cup D_4 \cup D_5$. In particular, if the fractional part of the raw price y rounded to the nearest cent is in D_1, then the observed value will stay at the same price with probability $1 - \alpha$ or move to the closest multiple of 5 cents, that is, the closest tick level in $D_2 \cup D_3 \cup D_4 \cup D_5$ with probability α. Then, if the fractional part of the price y is in D_2, it will stay in the same level with probability $1 - \beta$ or move to the closest tick level in $D_3 \cup D_4 \cup D_5$ with probability β. Finally, if the fractional part of the price y is in D_3, then it will stay in the same level with probability $1 - \gamma_1 - \gamma_2$ or move to the closest tick level in D_4 with probability γ_1 and the closest tick level in D_5 with probability γ_2. In summary, the transition probability function is obtained iteratively by the following.

Case 1. If the fractional part of y_j belongs to D_1,

$$p\left(y_j \mid x, z, \vartheta\right) = R\left(y_j \mid x, z, \vartheta\right)(1 - \alpha). \quad (32)$$

Case 2. If the fractional part of y_j belongs to D_2,

$$p\left(y_j \mid x, z, \vartheta\right) = R^*\left(y_j \mid x, z, \vartheta\right)(1 - \beta), \quad (33)$$

where

$$\begin{aligned} R^*&\left(y_j \mid x, z, \vartheta\right) \\ &\doteq R\left(y_j \mid x, z, \vartheta\right) \\ &\quad + \alpha\left(R\left(y_{j-1} \mid x, z, \vartheta\right) + R\left(y_{j-2} \mid x, z, \vartheta\right)\right) \\ &\quad + \alpha\left(R\left(y_{j+1} \mid x, z, \vartheta\right) + R\left(y_{j+2} \mid x, z, \vartheta\right)\right). \end{aligned} \quad (34)$$

Case 3. If the fractional part of y_j belongs to D_3,

$$p\left(y_j \mid x, z, \vartheta\right) = R^{**}\left(y_j \mid x, z, \vartheta\right)(1 - \gamma_1 - \gamma_2), \quad (35)$$

where

$$\begin{aligned} R^{**}&\left(y_j \mid x, z, \vartheta\right) \\ &\doteq R^*\left(y_j \mid x, z, \vartheta\right) \\ &\quad + \beta\left(R^*\left(y_{j-5} \mid x, z, \vartheta\right) + R^*\left(y_{j-10} \mid x, z, \vartheta\right)\right) \\ &\quad + \beta\left(R^*\left(y_{j+5} \mid x, z, \vartheta\right) + R^*\left(y_{j+10} \mid x, z, \vartheta\right)\right). \end{aligned} \quad (36)$$

Case 4. If the fractional part of y_j belongs to D_4,

$$\begin{aligned} p\left(y_j \mid x, z, \vartheta\right) &= R^{**}\left(y_j \mid x, z, \vartheta\right) \\ &\quad + \gamma_1\left(R^{**}\left(y_{j-25} \mid x, z, \vartheta\right) + R^{**}\left(y_{j+25} \mid x, z, \vartheta\right)\right). \end{aligned} \quad (37)$$

Case 5. If the fractional part of y_j belongs to D_5,

$$\begin{aligned} p\left(y_j \mid x, z, \vartheta\right) &= R^{**}\left(y_j \mid x, z, \vartheta\right) \\ &\quad + \gamma_2\left(R^{**}\left(y_{j-25} \mid x, z, \vartheta\right) + R^{**}\left(y_{j+25} \mid x, z, \vartheta\right)\right). \end{aligned} \quad (38)$$

Moreover, we have to handle the case $j = 0$ separately to avoid negative prices.

Case 6. For $j = 0$,

$$\begin{aligned} p&\left(y_0 \mid x, z, \vartheta\right) \\ &= R\left(y_0 \mid x, z, \vartheta\right) \\ &\quad + \alpha\left(R\left(y_1 \mid x, z, \vartheta\right) + R\left(y_2 \mid x, z, \vartheta\right)\right) \\ &\quad + \beta\left(R^*\left(y_5 \mid x, z, \vartheta\right) + R^*\left(y_{10} \mid x, z, \vartheta\right)\right) \\ &\quad + \gamma_2 R^{**}\left(y_{25} \mid x, z, \vartheta\right). \end{aligned} \quad (39)$$

Table 3

Clustering parameters	Estimate
α	0.060475
β	0.046883
γ_1	0.03883
γ_2	0.16525

Remark 13. Our clustering setup is designed to work well for intrinsic prices over \$1. For real penny stocks, our setup would introduce positive bias and should be modified slightly.

Using relative frequency analysis on the aggregate of our three stocks, we found the values presented in Table 3.

The large degree of clustering exhibited, especially to the whole dollar, might be considered surprising. However, earlier studies of Huang and Stoll [39], Chung et al. [40], and Chung et al. [41] also showed significant clustering. Moreover, the degree of price clustering in NYSE is weaker than that of NASDAQ. For example, Barclay [42] examined 472 stocks from NASDAQ before and after their listing in NYSE or American Stock Exchange (AMEX): before the listing, the average fraction of even-eighths $(0, 1/4, 1/2, 3/4)$ is 78% while thereafter it drops to about 56%.

2.4. Nonlinear Filtering Model. Our price process can be formulated as a marked point process $\vec{Y}$: a sequence of random vectors $\vec{Y} = (t_i, Y_{t_i},\ i \geq 1)$, where $t_i \in [0, T]$ denotes the time of ith-trade and Y_{t_i} the corresponding trading price. Accordingly, the mark space of $\vec{Y}$ is $(E, \mathscr{E})$, where $E = \mathbb{N}_0$ and $\mathscr{E}$ is all its subsets. Here, $j \in E$ corresponds to the jth-tick level j/M. For each $A \in \mathscr{E}$, we associate the counting process

$$Y_t(A) \doteq \sum_{i\geq 1} 1_{\{Y_{t_i}\in A\}} 1_{\{t_i \leq t\}} \tag{40}$$

to count the trades in tick level set A up to time t. In particular, for $j \in E$,

$$Y_j(t) \doteq Y_t(\{j\}) = \sum_{i\geq 1} 1_{\{Y_{t_i}=j\}} 1_{\{t_i\leq t\}} \tag{41}$$

denotes the total trades at jth-tick level j/M until time t. Equivalently, we can introduce the random counting measure $Y(dz \times dt)$ on $\mathscr{E} \otimes \mathscr{B}[0, T]$ by

$$\begin{aligned} Y(\omega, A \times (s, t]) &\doteq Y_t(\omega, A) - Y_s(\omega, A), \\ &\forall \omega \in \Omega,\ s \leq t \in [0, T],\ A \in \mathscr{E}. \end{aligned} \tag{42}$$

The natural filtration, that is, information content, of Y is

$$\mathscr{F}_t^Y \doteq \sigma\left(Y_s(A),\ 0 \leq s \leq t,\ A \in \mathscr{E}\right). \tag{43}$$

Now, we assume the following.

(C1) The total trade process $Y_t = Y_t(E)$ admits an intensity $a(t)$ for some positive measurable function a.

Therefore, using the conditional probabilities defined in the previous subsection, we find that $Y_j(t)$ has intensity

$$\lambda_j\left(X_t, Z_t^h, \vartheta, t\right) = a(t) \cdot p\left(y_j \mid X_t, Z_t^h, \vartheta\right). \tag{44}$$

To simplify the notation, we rewrite (44) as $\lambda_j = a \cdot p_j$.

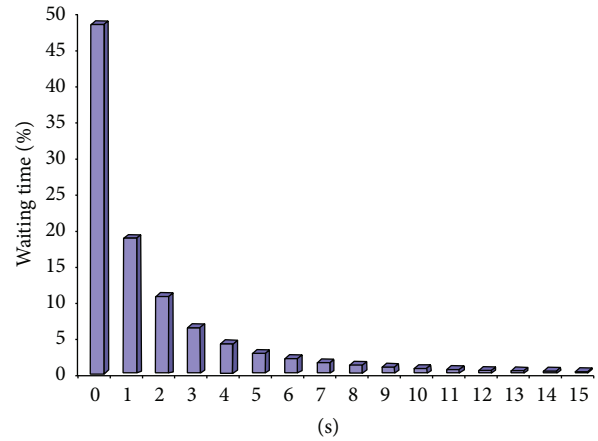


Figure 2: Intertrade duration in seconds.

For our present work, we estimated total intensity function $a(t)$ from intertrade data allowing for intraday variation. Figure 2 is the intertrade duration histogram of our 3 NYSE-listed stocks averaged over all times of the day. We divided the intertrade data into half-hour periods over the course of the day and took a to be constant over these half-hour periods:

$$a(t) = \frac{\text{Average number of trades in period}}{1800 \text{ seconds}} \tag{45}$$

for t in that daily period.

(C2) There exist some positive constants δ, C such that $\delta \leq a(t) \leq C$ for all t.

Based on representation (40), (44), $(X, \theta, \vartheta, \widehat{Z}^h; Y)$ is framed by a partial-observation model, where $(X, \theta, \vartheta, \widehat{Z}^h)$ is the state (signal), which is partially observed through the infinite dimensional counting process Y. One difficulty in calibrating these models is that their transition probability functions are usually unknown in closed form, so maximum likelihood estimation (MLE) methods are difficult to use (see Aït-Sahalia and Kimmel [43] for further details). Instead, we use Bayesian filtering because (1) Bayes estimates do not require the availability or regularity of the full likelihood functions; (2) Bayes estimates can be computed recursively for our tick-by-tick data; (3) Bayesian hypothesis tests can be conducted through Bayes factor, which is the ratio of marginal likelihoods and is easily computed even when the signals are of different dimension or, more generally, singular to each other.

3. Model Calibration

Our foremost goal is to contribute to the process of model building for financial markets both by suggesting elements to be included in the models and proposing methods to select models based on real observation data. To be able to do this effectively, we need to be able to tune each possible model effectively to get good prior (probability distribution) estimates for the complete signal $(X, \theta, \vartheta, \widehat{Z}^h)$ before the test period. We do this through nonlinear filtering and in particular through particle filtering. In this section, we first

introduce the filtering equations for our problem. Then, we introduce a branching particle filter algorithm that is an approximation to the unnormalized filter and can be implemented on a computer. Next, we explain how we did the calibration (i.e., came up with this prior distribution) and finally we give the results for the models of interest herein.

3.1. Nonlinear Filtering Equations. The available information about $(X_t, \theta, \vartheta, \widehat{Z}_t^h)$ is the observation filtration $\mathscr{F}_t^Y \subset \mathscr{F}_t$, defined in (43), and the primary goal of nonlinear filtering is to characterize the conditional distribution

$$\pi_t(\cdot) = \mathbb{P}\left[\left(X_t, \theta, \vartheta, \widehat{Z}_t^h\right) \in \cdot \mid \mathscr{F}_t^Y\right] \tag{46}$$

or, equivalently,

$$\pi_t(f) = \mathbb{E}\left[f\left(X_t, \theta, \vartheta, \widehat{Z}_t^h\right) \mid \mathscr{F}_t^Y\right] \tag{47}$$

for $f \in B(\mathbb{R}^{n_x+n_\theta+2} \times C[0,T])$. Here, $\vartheta = (\epsilon, \phi)$, $\widehat{Z}^h$ is the long memory portion of our information noise and (X, θ) is the state and parameter of our fair price-volatility martingale problem.

Remark 14. Actually, we often only want to estimate $\mathbb{P}[(X_t, \theta) \in \cdot \mid \mathscr{F}_t^Y]$, but there is no simple recursive formula for this marginal. The filter is naturally model dependent, so we can produce different filtering processes for each model, that is, for each SV choice (1–5), each value of Δ, and each value of h in our inertia process.

Suppose κ_z is a positive constant for each $z \in \mathbb{N}_0$ such that $\kappa \doteq \sum_{z=0}^{\infty} \kappa_z < \infty$, and consider the continuous-time likelihood function

$$\begin{aligned} L_t &= L_t\left(X, \vartheta, Z^h\right) \\ &= \exp\left(\int_0^t \int_E \ln\left|\frac{\lambda_z\left(X_s, \vartheta, Z_s^h, s\right)}{\kappa_z}\right| Y(dz, ds)\right. \\ &\quad \left. - \int_0^t (a(s) - \kappa)\, ds\right). \end{aligned} \tag{48}$$

L_t is a martingale under Condition (C2) and $\mathbb{Q}$, defined by

$$\begin{aligned} &\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_T} \\ &\quad = L_T^{-1}\left(\text{i.e. } \mathbb{Q}(A) = \int_A L_T^{-1} d\mathbb{P} \quad \text{for } A \in \mathscr{F}_T\right), \end{aligned} \tag{49}$$

is called the reference measure. Under $\mathbb{Q}$, the observations are just a Poisson measure, independent of the state vector $(X, \theta, \vartheta, \widehat{Z}^h)$, with mean measure $\eta(A \times (0,t]) = \sum_{z \in A} \kappa_z \times (0,t]$. To make the likelihoods more manageable in the particle filters to follow, we choose κ to be a long time average value $(1/T)\int_0^T a(s)ds$ of $a(s)$ and $z \to \kappa_z$ to be highest where the trades will be more concentrated. Bayes Theorem (see Bremaud [44], p. 165) then links the desired (real-world) conditional distribution π_t with the unnormalized filter σ_t by

$$\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}, \tag{50}$$

where the unnormalized filter σ_t is defined by

$$\sigma_t(f) \doteq \mathbb{E}^{\mathbb{Q}}\left[f\left(X_t, \theta, \vartheta, \widehat{Z}_t^h\right) L_t \mid \mathscr{F}_t^Y\right] \tag{51}$$

for all $f \in B(\mathbb{R}^{n_x+n_\theta+2} \otimes C[0,T])$. Now, we can give the evolution equation for σ_t.

Theorem 15. *Under (C1) and (C2), the unnormalized filter σ_t is the unique measure-valued solution of the stochastic filtering equation*

$$\begin{aligned} \sigma_t(f) &= \sigma_0(f) + \int_0^t \sigma_s\left(\left(\overline{\mathbf{A}} - a(s) + \kappa\right) f\right) ds \\ &\quad + \int_0^t \int_E \sigma_{s-}\left(\left(\frac{\lambda_z(s-)}{\kappa_z} - 1\right) f\right) Y(dz, ds), \end{aligned} \tag{52}$$

for $t > 0$ and $f \in \mathscr{D}(\overline{\mathbf{A}})$.

This theorem is a modest generalization of prior results and can be obtained in much the same manner as results in Kouritzin and Zeng [23] and Xiong and Zeng [22]. Here, $\overline{\mathbf{A}}$ is the generator of the joint martingale problem to $(X, \theta, \vartheta, \widehat{Z}^h)$ obtained from $\mathbf{A}$, the generator of state (X, θ) and $\mathbf{A}^Z$, the generator of the historical process $\widehat{Z}^h$. We do not need an explicit formula for $\overline{\mathbf{A}}$. Instead, we can use particle filters to approximate σ_t.

Henceforth, it is convenient to think of the reference measure $\mathbb{Q}$ as the standard measure from which we can construct the measure $\mathbb{P}^{k,\theta,\vartheta,h,\Delta}$ corresponding to model $k \in \{1, \ldots, 5\}$ with parameters θ and microstructure with parameters ϑ, h, and Δ.

3.2. Particle Filter. The weighted filter is the simplest of particle filters. The idea behind the weighted filter is that, by the independence of signal $(X, \theta, \vartheta, \widehat{Z}^h)$ from the observations Y under $\mathbb{Q}$, we can create an infinite collection of particles $\{P^k\}_{k=1}^N = \{(X^k, \theta^k, \vartheta^k, \widehat{Z}^{h,k})\}_{k=1}^N$, each having the same law as $(X, \theta, \vartheta, \widehat{Z}^h)$ that are also independent of the observations. Then, it follows from the law of large numbers that for $\mathbb{Q}$-almost all Y we have the weak convergence of finite measures

$$\sigma_t^{N,W} \doteq \frac{1}{N}\sum_{k=1}^N L_t\left(X^k, \vartheta^k, Z^{h,k}\right) \delta_{(X_t^k, \theta^k, \vartheta^k, \widehat{Z}_t^{h,k})} \Longrightarrow \sigma_t. \tag{53}$$

Unfortunately, it is well known that the weighted particle filter may not work well for a fixed number of particles N. Roughly speaking, most of the particles diffuse away, do not track the signal well, are assigned low likelihoods, and do not really affect the average $\sigma_t^{N,W}$. Meanwhile, very few particles do match the observations better and have likelihoods that are orders of magnitude higher than of the majority of particles.

$\sigma_t^{N,W}$ essentially becomes an average over too few particles to reflect σ_t well.

To fix the weighted filter particle spread problem, we add particle resampling, resulting in following novel particle filter. (See Gordon et al. [45], Del Moral et al. [46], Del Moral et al. [12], and Ballantyne et al. [47] for earlier algorithms.) For some large $N \in \mathbb{N}_0$, the particle system $\{P^k\}_{k=1}^N = \{(X^k, \theta^k, \vartheta^k, \widehat{Z}^{h,k})\}_{k=1}^N$ is constructed as follows.

3.2.1. Initialization. At the initial time $t_0 = 0$, we generate independent particles $\{P_0^k\}_{k=1}^N$ from the joint prior distribution $\pi_0(\cdot)$ of $(X_0, \theta, \vartheta, \widehat{Z}_0^h) \in \mathbb{R}^{n_x+n_\theta+2} \times C[0,T]$. The empirical measure at t_0 is

$$\sigma_0^N = \frac{1}{N}\sum_{k=1}^N \delta_{P_0^k}(\cdot), \tag{54}$$

where $\delta_x(\cdot)$ is the Dirac measure at x. By the strong law of large numbers,

$$\lim_{N\to\infty}\left(\sigma_0^N, f\right) = \sigma_0(f) \qquad \forall f \in B\left(\mathbb{R}^{n_x+n_\theta+2} \otimes C[0,T]\right), \tag{55}$$

so $\sigma_0^N \Rightarrow \sigma_0$ for almost all Y. Here, $(\mu, f) \doteq \int f(y)\mu(dy)$ for measures μ so

$$\left(\sigma_0^N, f\right) = \frac{1}{N}\sum_{k=1}^N f\left(P_0^k\right). \tag{56}$$

Remark 16. When $L_0 = 1$, $\pi_0(f) = \sigma_0(f)$. Note that $\widehat{Z}_0^h$ is a constant function defined on $[0,T]$. Whereas most particle filters approximate the filter π_t, we will approximate the unnormalized filter σ_t to facilitate Bayesian model selection without the storage of prior filter estimates.

We also initialize the number of particles to $\mathbb{N}_0 = N$ and particle likelihoods all to $\mathbb{A}_0 = 1$.

3.2.2. Evolution. Between observations at t_{i-1} and t_i, the particles, $\{(X^k, \theta^k, \vartheta^k, \widehat{Z}^{h,k})\}_{k=1}^{\mathbb{N}_{i-1}}$, move independently as samples from the transition probability of $(X, \theta, \vartheta, \widehat{Z}^h)$. In particular, we use the Euler scheme (see, e.g., Kloeden and Platen [48]) to evolve the dynamics, Examples 3–7 and (25). We let $\widehat{P}_{t_i}$ denote the evolved version of $P_{t_{i-1}}$.

3.2.3. Particle Weights and Average Weight. We simulate using the reference measure $\mathbb{Q}$ and we incorporate the observations based upon (48). At the ith observation (t_i, Y_{t_i}), the kth particle's weight is multiplied by

$$\begin{aligned}\omega_i^k &\doteq \exp\left(\int_{t_{i-1}}^{t_i}\int_E \ln\frac{\lambda_z\left(X_s^k, \vartheta^k, Z_s^{h,k}, s\right)}{\kappa_z} Y(dz, ds) - \int_{t_{i-1}}^{t_i}(a(s)-\kappa)\,ds\right)\\ &= \exp\left(\ln\frac{\lambda_{z_i}\left(X_{t_i}^k, \vartheta^k, Z_{t_i}^{h,k}, t_i\right)}{\kappa_{z_i}} - \int_{t_{i-1}}^{t_i}(a(s)-\kappa)\,ds\right) \doteq \alpha_i\left(\widehat{P}_{t_i}^k, t_i, t_{i-1}\right),\end{aligned} \tag{57}$$

where $z_i = Y_{t_i}$. Hence, the kth particle's weight becomes

$$\widehat{\mathbb{L}}_i^k = \omega_i^k \mathbb{A}_{i-1} \tag{58}$$

and the average weight is

$$\mathbb{A}_i = \frac{1}{N}\sum_{k=1}^{\mathbb{N}_{i-1}} \widehat{\mathbb{L}}_i^k. \tag{59}$$

Note that $(X_{t_i-}^k, Z_{t_i-}^{h,k}, t_i-) = (X_{t_i}^k, Z_{t_i}^{h,k}, t_i)$ in (57) by continuous paths. Here, ω_i^k depends on the observation Y and the increment of likelihood ratio of measure $\mathbb{P}$ over measure $\mathbb{Q}$ defined by (48) given the simulated particle path realized on the interval $[t_{i-1}, t_i)$. These weights do not depend upon the parameters θ directly. This is common and is why the observations are often called partial observations. We still can estimate θ and include these parameters as part of the particles' states since they do affect stock price S, which is observed in the presence of noise and distortion. The weights are stored along with the states of particles before resampling.

3.2.4. Resampling. After weighting, we resample the particles pruning the unlikely ones and duplicating the better ones in an unbiased manner. In particular, we let ρ_i^k be $(\widehat{\mathbb{L}}_i^k/\mathbb{A}_i - \lfloor \widehat{\mathbb{L}}_i^k/\mathbb{A}_i \rfloor)$-Bernoulli random variable independent of everything and produce $\lfloor \widehat{\mathbb{L}}_i^k/\mathbb{A}_i \rfloor + \rho_i^k$ particles at location $\widehat{P}_{t_i}^k$. We then give all the particles weight A_i and let

$$\mathbb{N}_i \doteq \sum_{k=1}^{\mathbb{N}_{i-1}}\left\{\left\lfloor \frac{\widehat{\mathbb{L}}_i^k}{\mathbb{A}_i}\right\rfloor + \rho_i^k\right\}. \tag{60}$$

3.2.5. Unnormalized Filter. Now, we can estimate the unnormalized filter at the ith observation time, σ_{t_i}, by

$$\sigma_{t_i}^N = A_i \sum_{k=1}^{\mathbb{N}_i} \delta_{P_{t_i}^k}. \tag{61}$$

The actual algorithm that was implemented is as follows.

Initialize. $\{P_0^k\}_{k=1}^N$ are independent samples of π_0, $\mathbb{N}_0 = N$, $\mathbb{N}_n = 0$, for all $n \in \mathbb{N}$, and $\mathbb{L}_0^k = 1$ for $k = 1, \ldots, N$.

Repeat. For $n = 0, 1, 2, \ldots$, do

(1) evolve $P^k_{t_n}$ to $\widehat{P}^k_{t_{n+1}}$ independently of other particles;

(2) weight by observation: $\widehat{\mathbb{L}}^k_{n+1} = \alpha_{n+1}(\widehat{P}^k_{t_{n+1}}, t_{n+1}, t_n)\mathbb{A}_n$ for $k = 1, 2, \ldots, \mathbb{N}_n$;

(3) estimate $\sigma_{t_{n+1}}$ by $\sigma^N_{t_{n+1}} = \dfrac{1}{N}\sum_{k=1}^{\mathbb{N}_n} \widehat{\mathbb{L}}^k_{n+1}\delta_{\widehat{P}^k_{t_{n+1}}}$;

(4) average weight: $\mathbb{A}_{n+1} = \sigma^N_{t_{n+1}}(1)$;

(5) repeat: for $k = 1, 2, \ldots, \mathbb{N}_n$ do

(a) offspring number: $\mathbb{N}^k_{n+1} = \lfloor \widehat{\mathbb{L}}^k_{n+1}/\mathbb{A}_{n+1} \rfloor + \rho^k_n$, with ρ^k_n being $((\widehat{\mathbb{L}}^k_{n+1}/\mathbb{A}_{n+1}) - \lfloor \widehat{\mathbb{L}}^k_{n+1}/\mathbb{A}_{n+1} \rfloor)$-Bernoulli independent of everything;

(b) resample: $P^{\mathbb{N}_{n+1}+j}_{t_{n+1}} = \widehat{P}^k_{t_{n+1}}$ for $j = 1, \ldots, \mathbb{N}^k_{n+1}$;

(c) add offspring number: $\mathbb{N}_{n+1} = \mathbb{N}_{n+1} + \mathbb{N}^k_{n+1}$.

Remark 17. (i) We extract our estimate before resampling to avoid excess noise. (ii) The key step is (5) that determines the new number of particles $\mathbb{N}_{n+1}$ and weights $\mathbb{L}^k_{n+1}$ in an unbiased manner. The result is zero or more particles all having the average weight at the same location as the parent. (iii) The particle evolution would typically be done via Newton's or Milstein's method.

Since the above algorithm produces unbiased resampling of the weighted particle filter, it is quite reasonable to believe the following result.

Theorem 18. *Under (C1) and (C2), $\sigma^N_{t_i} \Rightarrow \sigma_{t_i}$ for any i and almost all observation paths.*

The technicality of this result's proof would detract from our applications so it is omitted.

3.2.6. Bayesian Estimation. By Bayes rule (50), the particle approximation of the normalized filter $\pi(\cdot)$ is

$$\pi^N_{t_i}(f) = \frac{\sigma^N_{t_i}(f)}{\sigma^N_{t_i}(1)} \tag{62}$$

for all $f \in B(\mathbb{R}^{n_x+n_\theta+2} \times C[0,T])$. To get our parameter estimates, we can just set $f(X, \theta, \vartheta, \widehat{Z}^h)$ to one component of these parameters, that is, θ^i or ϑ^j.

3.3. Calibration and Historical Training. To keep the problem size manageable, we just used the clustering parameter estimates of α, β, γ_1, and γ_2 given above as the actual values throughout our simulations.

One is often faced with the problem of estimating initial distributions for fair price, volatility, and the parameters prior to filtering over the time interval of interest (April 2010 here). Our approach was to make arbitrary assignments very far in the past (January 3, 2000, to be precise) and then do an excessive amount of prior particle filtering, relying on the ability of the filter to forget its starting point and to produce reasonable distributions at a much later point, April 1, 2010. (See, e.g., Ocone and Pardoux [49], Delyon and Zeitouni [50], and Atar [51] for mathematical results regarding this phenomenon.) This had to be done for every model, namely, every combination of our three stocks, five SV models, and multiple microstructure models, characterized by inertia parameters. Our main purpose in this historical training was to get a starting joint distribution for $(X, \theta, \vartheta, \widehat{Z}^h)$ as of April 1, 2010, under each model combination. Due to the large number of cases this produced, we first display and discuss two models: the nondynamical microstructure Heston case and the *median inertia* dynamical case where $h = 1/2$ and $\Delta = 7200$ s (i.e., 2 hrs) in the inertia microstructure model. Also, to ensure that θ and ϑ did not converge to a single value, we made them vary slightly in a random manner; that is, we replaced the equation $d\theta = 0$ with $d\theta_t = d\nu_t$ for a very low variance Brownian motion ν.

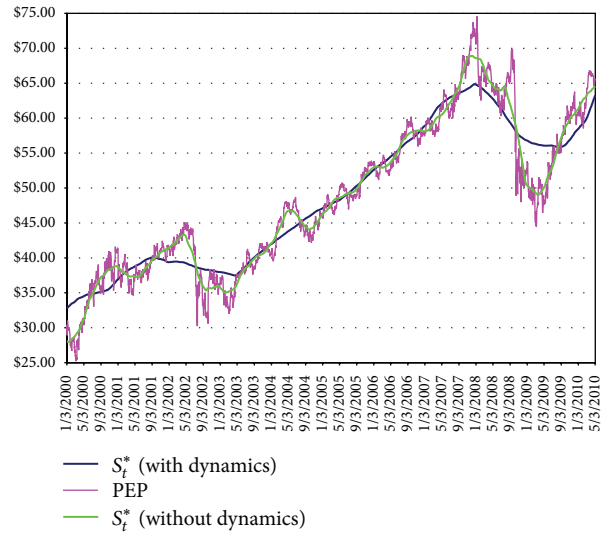


Figure 3: Long-term value estimation of PEP.

In Figure 3, we illustrate our prior filtering of PepsiCo. The choppiest curve is the actual stock price while the smoothest curve is the filter's fair price estimate $E[S_t \mid \mathcal{F}^Y_t]$ using the Heston SV model with (median) microstructure inertia. The middle curve is the filter's fair price estimate $E[S_t \mid \mathcal{F}^Y_t]$ using the Heston SV model without dynamics in the microstructure; that is, $Z^h = 0$. These curves go beyond April 1, 2010. However, the required initial distributions were taken from the filter at that point.

Notice from Figure 3 that the implied fair price process estimate is far less volatile in the presence of dynamical microstructure than without. This lower volatility for fair price is highly desirable. It does not make sense that the fair price of a stock should fluctuate dramatically from day to day or within a day in the absence of an event, but rather these short-term fluctuations are better explained by trading noise. Moreover, fair price is a mathematically more optimal version of moving averages, which are used to judge value and momentum from, and so fair price estimates should inherit the smooth nature of such moving averages.

Table 4

PEP	GBM	HW	LOU	Nelson	Heston
μ	$1.51E-06$	$1.47E-06$	$1.52E-06$	$1.44E-06$	$1.49E-06$
ν	$\sigma = 2.86E-06$	$1.17E-09$	$9.55E-06$	$1.06E-10$	$1.07E-11$
κ	—	$1.59E-03$	$1.80E-03$	$1.94E-03$	$2.58E-07$
ϱ	—	—	$4.75E-03$	$6.51E-03$	$6.02E-03$
ϖ	—	—	$4.84E-06$	—	—

Table 5

PEP	GBM	HW	LOU	Nelson	Heston
μ	$1.05E-06$	$1.02E-06$	$9.92E-07$	$1.03E-06$	$1.01E-06$
ν	$\sigma = 2.21E-06$	$5.50E-10$	$5.13E-06$	$6.32E-11$	$5.94E-12$
κ	—	$2.18E-03$	$1.87E-03$	$2.12E-03$	$2.26E-07$
ϱ	—	—	$2.25E-03$	$2.90E-03$	$3.23E-03$
ϖ	—	—	$2.60E-06$	—	—
ϵ	$2.43E-09$	$2.05E-09$	$2.33E-09$	$2.31E-09$	$2.46E-09$
ϕ	$2.13E-09$	$2.31E-09$	$2.23E-09$	$2.33E-09$	$2.31E-09$

From a modeling perspective, this fair price smoothness indicates that dynamical microstructure (with inertia) can replace much of what stochastic volatility tries to do and leads to one of our central questions addressed below. Is stochastic volatility necessary in the presence of dynamical microstructure?

3.4. Numerical Results. The data is one month (April 2010) of transaction prices of our three NYSE-listed stocks. Our filter produces Bayes estimates to the macro- and microparameter vectors θ and ϑ, respectively. These estimates in the nondynamical microstructure case (i.e., using the simpler form in (24)) for PepsiCo are as shown in Table 4.

All parameters are estimated using time in seconds. Our PepsiCo Bayes estimates in the *median* inertia case are as shown in Table 5.

While it is difficult to read much from these numbers, we can see that the main volatility parameters ν, κ, and ϱ are mostly smaller when dynamics is included in the microstructure. This further justifies our conjecture that at least some stochastic volatility is better replaced by microstructure with dynamics.

Figures 4 and 5 show the conditional expectation fair price estimation for Goldman Sachs and PepsiCo, respectively, in the cases of no dynamics and median inertia dynamics for each of our SV models. There are a total of eleven curves in both figures. The most volatile curve is the stock price itself over this month. The smoothest curves somewhat separated from the stock price are the fair price estimates using the five SV models with (median inertia) dynamical microstructure. The remaining five curves (that hug the stock price in Figures 4 and 5) are our fair price estimates for our five SV models with nondynamical microstructure. In this last case, the microstructure does not have the power to separate the fair price and actual stock price to any large degree.

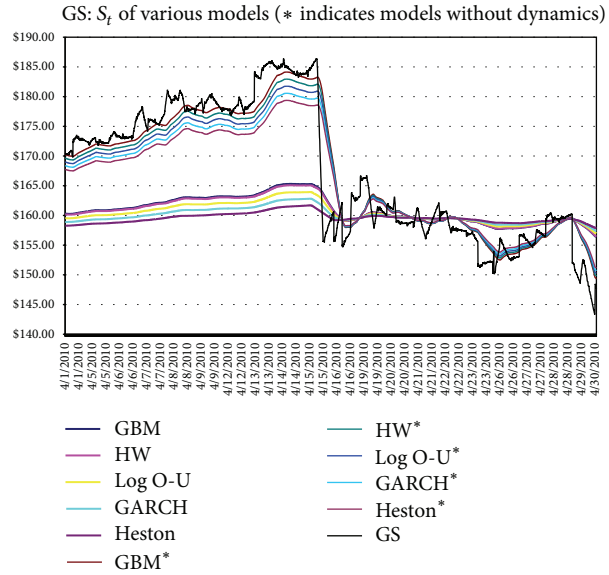


Figure 4: Value estimation of GS, April 2010.

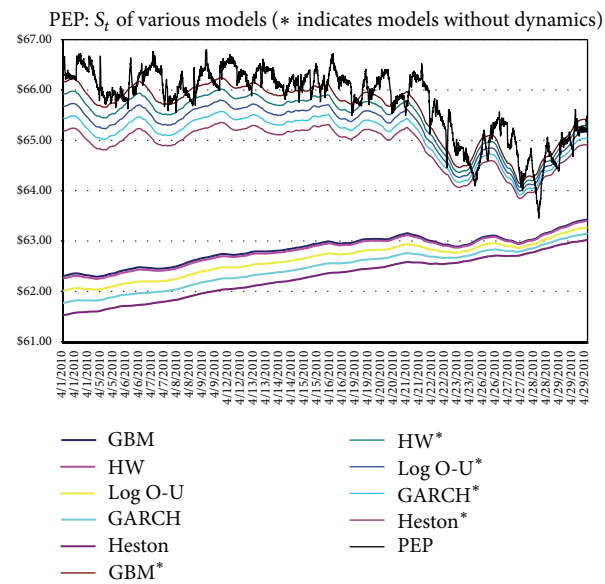


Figure 5: Value estimation of PEP, April 2010.

It is important to realize that these pictures are really just a one-month snapshot of a much bigger multiyear filtering process. This explains why many of the fair price processes are significantly different than the actual stock price on April 1, 2010: The filter is estimating that the difference is due to the microstructure. It is apparent that adding dynamics to the microstructure allows the estimated fair price process to differ significantly from the stock price. Indeed, there is a significant correction of all three stock prices (especially Goldman Sachs) towards estimated fair price of the models with (median inertia) dynamical microstructure. This produces a compelling reason to use models with microstructure dynamics. You would be estimating that the stocks were significantly overvalued before the correction if you used the model with microstructure dynamics and this could be

Table 6: Volatility estimation, April 2010.

	Without dynamics	With dynamics
PEP (2 hrs, $h = 0.6$)	$1.58416E-09$	$1.01312E-11$
GS (1/2 day, $h = 0.4$)	$4.14645E-08$	$4.3005E-10$
IBM (1/2 day, $h = 1$)	$8.21731E-10$	$4.03211E-11$

used as a warning to lessen ones exposure. You have no such warning when the microstructure does not contain (inertia) dynamics as the estimated fair price is very close to the observed price. It is interesting to ponder what this possible discrepancy would mean to option prices.

The filters provide conditional distributions and estimates for more than just fair price and parameters. Table 6 shows the average volatility estimates without microstructure dynamics (see (24)) and with (the best performing) microstructure inertia using the simplified Heston SV model. We only highlighted Heston here because (1) we will show evidence below that Heston performs the best and (2) the volatility estimates of the other SV models behave similarly. The amount of stochastic volatility estimated when there is (median inertia) dynamics in the microstructure shrank to a couple of percent of what it was without. This really suggested that by far the primary use of stochastic volatility is as a proxy for microstructure with dynamics and further raises the question about the need for stochastic volatility in the presence of microstructure dynamics.

The final and most difficult quantity the filter estimates (in the dynamical microstructure case) is the historical noise. For practical purposes, we can not let the historical path go back all the way to year 2000, but we found that there is not much loss if we just update discrete samples over the previous three years, which is still a tremendous amount of data. Also, we can not plot these historical paths so we just plot the projection onto the current time; that is, we just plot Z_t^h even though we must propagate the Markov process $\widehat{Z}_t^h$ in the filter. Figure 6 shows the noise estimate for PepsiCo. In this graph, we look at the effect of inertia. The curves where $h = 0$ represent the no-inertia case, so Z_t^0 is just an Ornstein-Uhlenbeck process. Conversely, the case $h = 1$ represents the one hundred percent inertia case and Z_t^1 is not Markov. We see from these graphs that the amount of estimated noise is very similar indicating that the amount of inertia modeled might not be that significant. However, the noise processes where $h = 1$ are far smoother due to the inertia. Below, we will produce strong evidence that inertia is important and find that the best h is in the range $[0.4, 1]$, depending upon the stock. We compare the behavior of our models in terms of the SV models and the inertia parameters h and Δ within the Bayesian model selection framework in the following section.

4. Evidence for Inertia and Stochastic Volatility

The main objective of this section is to use Bayes factor to investigate the model selection in microstructure markets. To use the Bayes factor method, we need only to be able to

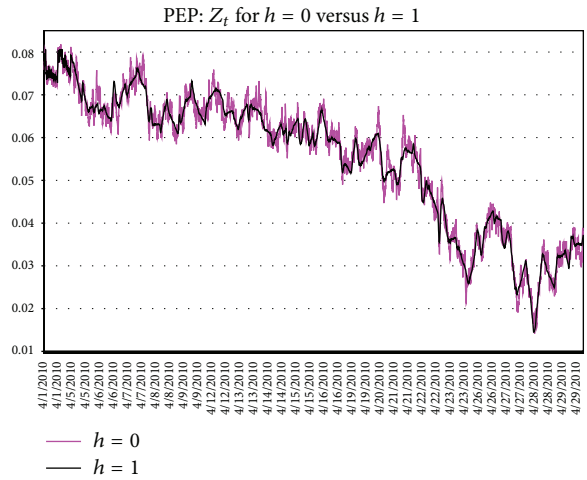


Figure 6: Noise process estimation of PEP, April 2010.

transform all observation models of interest into the same canonical process via Girsanov measure change. The signal models can be singular to one another. Kouritzin and Zeng [23] discuss the Bayesian model selection problem. However, their equations do not apply to our models.

4.1. Model Selection and Bayes Factor. Consider our five SV macrostructure fair price-volatility models

$$M^{(k)} \doteq \left(X^{(k)}, \theta^{(k)}\right) \in \mathbb{R}^{n_x^{(k)} + n_\theta^{(k)}}, \tag{63}$$

where the generators of the martingale problem to $M^{(k)}$ are, respectively, $\mathbf{A}^{(k)}$ for $k = 1, 2, 3, 4, 5$. Normally, we would have to consider a multitude of parameters θ resulting in a plethora of models. However, by our calibration process we have reduced the setting to one parameter set per martingale problem so we have a base of five models. However, we still have to consider the various choices for our inertia. For simplicity, we restrict ourselves to three distinct values for Δ, eleven choices for h, and we use the calibration process to estimate the other microstructure parameters ϑ. Therefore, we have a total of $5 \times 3 \times 11 = 165$ models to test.

The likelihood of Y being produced by model (k, h, Δ) up until time t is

$$L_t^{(k,h,\Delta)} = 1 + \int_0^t \int_E \left(\frac{\lambda_z\left(X_s^{(k)}, Z_s^{h,\Delta}, \vartheta, s\right)}{\kappa_z} - 1 \right) \cdot L_{s-}^{(k,h,\Delta)} \left(Y(dz, ds) - \kappa_z m(dz)\, ds\right). \tag{64}$$

Here, $m(dz)$ is the counting measure on $E = \mathbb{N}_0$ and the same observations and observation rate information are used for all models. One can think of $L_t^{(k,h,\Delta)}$ as the likelihood ratio of the model $M^{(k,h,\Delta)}$ with distribution $\mathbb{P}^{(k,h,\Delta)}$ characterized by (k, h, Δ) to the simple (or null) model M^0 with distribution $\mathbb{Q}$ where the observation prices just arrive according to a Poisson measure with intensity measure $\mu(A) = \int_A k_z m(dz)$, that is, with rate independent of any macrostructure model

and independent of any microstructure state. In other words, $(L_t^{(k,h,\Delta)})^{-1} = (d\mathbb{Q}/d\mathbb{P}^{(k,h,\Delta)})|_{\mathscr{F}_t}$ then transforms the observations into the same Poisson measure with intensity measure $\mu(A) = \int_A k_z m(dz)$ regardless of (k, h, Δ). Unfortunately, $L_t^{(k,h,\Delta)}$ depends upon $X_s^{(k)}$, $Z_s^{h,\Delta}$, which are unknown so we can not select models via the likelihood.

4.2. Bayes Factor. The available information in microstructure market is the observation process Y, which represents the cumulative transaction records throughout all tick price levels. The normalized filter $\pi_t^{(k,h,\Delta)}$, $k = 1, 2, 3, 4, 5$, $h \in [0, 1]$, $\Delta > 0$, satisfies

$$\pi_t^{(k,h,\Delta)}(f_k) = \frac{\sigma_t^{(k,h,\Delta)}(f_k)}{\sigma_t^{(k,h,\Delta)}(1)}, \tag{65}$$

where $f_k \in B(\mathbb{R}^{n_x^{(k)}+n_\theta^{(k)}+2} \otimes C[0,T])$ for $k = 1, 2, 3, 4, 5$, the unnormalized filter $\sigma_t^{(k,h,\Delta)}$ is

$$\begin{aligned}&\sigma_t^{(k,h,\Delta)}(f_k)\\ &\quad \doteq \mathbb{E}^{\mathbb{Q}}\left[f_k\left(X_t^{(k)}, \theta^{(k)}, \vartheta, \widehat{Z}_t^{h,\Delta}\right) L_t^{(k,h,\Delta)} \mid \mathscr{F}_t^Y\right],\end{aligned} \tag{66}$$

and $\sigma_t^{(k,h,\Delta)}(1)$ is the integrated (or marginal) likelihood of Y for model (k, h, Δ).

Now, we use Bayes factor to compare models. The Bayes factor determines which model best fits this observed data by doing pairwise comparisons. We define Bayes factor of model $M^{(k,h,\Delta)}$ to the null model by the conditional likelihood:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}^{(k,h,\Delta)}}{d\mathbb{Q}} \mid \mathscr{F}_t^Y\right] = \sigma_t^{(k,h,\Delta)}(1), \tag{67}$$

which is consistent with more basic definitions of Bayes factor. It then follows that the Bayes factors for two models, characterized by (k_1, h_1, Δ_1) and (k_2, h_2, Δ_2), are the ratios

$$\begin{aligned}B_{12}(t) &= \frac{\sigma_t^1(1)}{\sigma_t^2(1)},\\ B_{21}(t) &= \frac{\sigma_t^2(1)}{\sigma_t^1(1)},\end{aligned} \tag{68}$$

with the integrated likelihoods $\sigma_t^1(1) = \sigma_t^{(k_1,h_1,\Delta_1)}(1)$, $\sigma_t^2(1) = \sigma_t^{(k_2,h_2,\Delta_2)}(1)$ that can be approximated using the algorithm of Section 3.2.5. Kass and Raftery [20] demonstrate how to interpret Bayes factor shown in Table 7.

4.3. Numerical Results on Stochastic Volatility. First, we consider the problem of selecting the best of our five fair price-volatility models,

$$M^{(k)} \doteq \left(X^{(k)}, \theta^{(k)}\right), \tag{69}$$

and the resulting partially observed market models,

$$\left(X^{(k)}, \widehat{Z}^{h,\Delta}, \theta^{(k,h,\Delta)}, \vartheta; Y\right). \tag{70}$$

Table 7

B_{12}	Evidence against Model 2
1–3	Barely mentionable
3–12	Positive
12–150	Strong
>150	Decisive

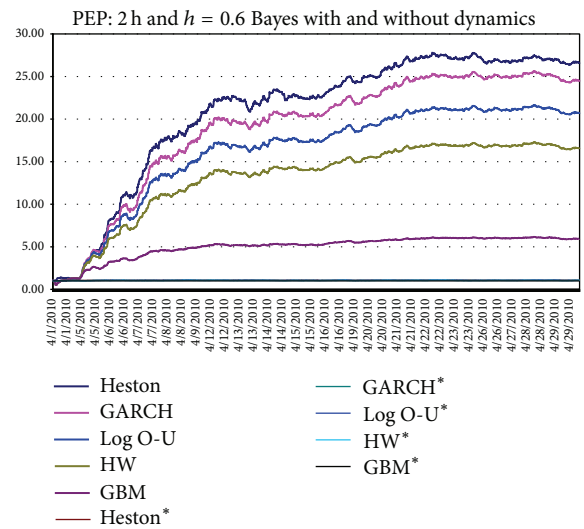


Figure 7: Bayes factor for PEP SV model determination, April 2010.

Table 8: Bayes factor for model determination, April 2010.

	Heston	GARCH	Log O-U	HW	GBM
PEP (2 hrs, $h = 0.6$)	27.30	25.23	21.31	17.04	6.08
GS (1/2 day, $h = 0.4$)	19.14	18.91	18.77	18.59	10.18
IBM (1/2 day, $h = 1$)	49.94	45.43	40.75	37.07	16.16
PEP* (2 hrs, $h = 0.6$)	1.08	1.07	1.06	1.04	1.00
GS* (1/2 day, $h = 0.4$)	1.06	1.04	1.06	1.06	1.00
IBM* (1/2 day, $h = 1$)	1.06	1.04	1.06	1.03	1.00

*Without dynamics.

We compare these five models to determine which can best represent the market data. More precisely, we run all unnormalized filters as explained in Section 3.2 with the optimal parameters discovered and reported earlier. Then, we choose Model i if $\sigma_T^{(i,h,\Delta)}$ is the largest. Naturally, this corresponds to the model whose Bayes factor ends up greater than one when compared to any other model. While we have five basic models, we also consider different market ingestion times Δ and inertia magnitude parameters h for each model.

Using GBM with nondynamic microstructure (i.e., $Z^h = 0$) as the benchmark, we determine which combination of SV model and inertia parameters outperforms GBM most. We first focus on the candidate models (Examples 3–7). In each case, we pick the inertia parameters from the sets $\Delta \in \{30\text{ mins}, 2\text{ hrs}, 1/2\text{ day}\}$ and $h \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ that would yield the highest Bayes factor against the calibration model. The data is the transaction price of PepsiCo, IBM, and Goldman Sachs, April 2010. Figure 7 and Table 8 summarize

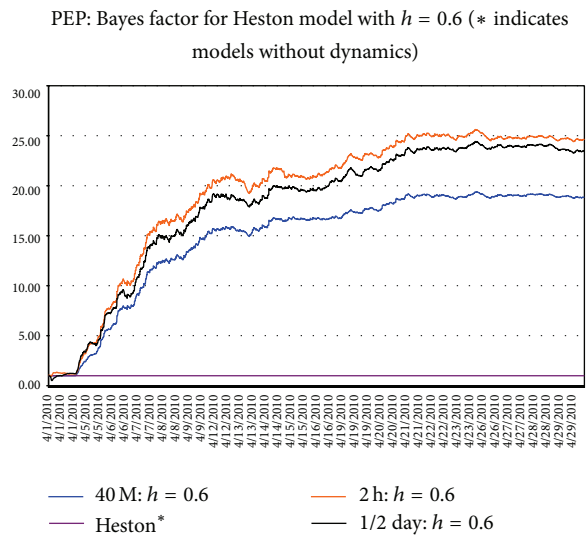


Figure 8: Bayes factor for PEP ingestion time determination, April 2010.

Table 9: Bayes factor for ingestion time determination, April 2010.

	Heston*	40 mins	2 hrs	1/2 day
GS ($h = 0.4$)	1.000	15.083	17.578	18.100
PEP ($h = 0.6$)	1.000	19.066	25.259	24.187
IBM ($h = 1$)	1.00	31.152	42.744	46.988

*Without dynamics.

the Bayes factor performance. The Bayes factors computed in this table give strong evidence (based upon the Kass and Raftery criterion mentioned before) for the Heston model based on a full month of real tick-by-tick stock price data. Indeed, as we will see below, there would still be strong evidence supporting Heston if we used different values of h and Δ. It is also interesting that the order of the models did not change over our three stock selections, with Heston always being preferred and GBM always performing the worst. Recall that all models are tuned to have their best parameters θ and ϑ.

4.4. Numerical Results on Inertia. Next, we look at the ingestion time Δ using nondynamic microstructure Heston as the calibration model. Figure 8 and Table 9 show the effect of varying Δ over $\{30\,\text{mins}, 2\,\text{hrs}, 1/2\ \text{day}\}$ for $h \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ fixed to give the highest Bayes factor. There is a drop in the Bayes factor values from the model determination experiment which is entirely due to the change of calibration model from GBM with nondynamic microstructure to Heston with nondynamic microstructure. Our results show that the best ingestion times for Goldman Sachs, PepsiCo, and International Business Machines stocks are, respectively, 1/2 day, 2 hours, and 1/2 day. The fact that the data supports long-time ingestion might add merit to the case of the momentum trader.

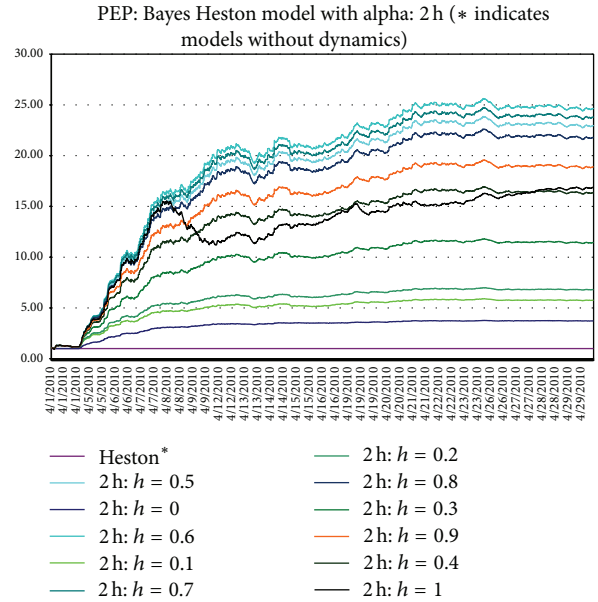


Figure 9: Bayes factor for PEP inertia determination, April 2010.

Table 10: Bayes factor for inertia determination, April 2010.

h	∗	0	0.1	0.2	0.3	0.4
PEP (2 hrs)	1.00	3.745	5.875	6.950	11.693	16.733
GS (1/2 day)	1.00	11.578	13.507	16.194	17.746	18.100
IBM (1/2 day)	1.00	3.822	7.100	8.816	10.927	13.522
h	0.5	0.6	0.7	0.8	0.9	1.0
PEP (2 hrs)	23.524	25.259	24.386	22.322	19.347	17.548
GS (1/2 day)	17.878	17.184	16.515	16.225	16.008	15.612
IBM (1/2 day)	16.707	20.611	25.388	31.225	38.345	46.988

*Without dynamics.

Finally, we investigate the optimal amount of inertia. Figure 9 and Table 10 show the effect of varying the amount of inertia h over $\{0, 0.1, 0.2, \ldots, 0.9, 1\}$ for $\Delta \in \{30\,\text{mins}, 2\,\text{hrs}, 1/2\ \text{day}\}$ fixed to give the highest Bayes factor. The table shows that inertia is important. In fact, the best h was always at least $h = 0.4$ and was even $h = 1$ in the case of IBM so all microstructure dynamics should be driven by the inertia process.

5. Conclusions

Herein, we considered five popular SV models to represent intrinsic or fair price and stochastic volatility of this price. These SV models are free of inertia or momentum. We then added microstructure noise with possible dynamics and inertia to these SV models to accommodate trading noise, trend following, information dispersion, and the slow unwinding of big positions. We used Bayesian model selection techniques to determine which of these combined models fits real NYSE data best. In the process of selecting the best model we also investigated characteristics like microstructure

dynamics, inertia, and stochastic volatility. For the stock data considered, we can conclude the following:

(1) Bayesian model selection through particle filtering provides a computationally effective means to identify the best finance models on real tick-by-tick data.

(2) The SV and inertia components of the financial models compared can be singular to each other as long as the microstructure can be changed into the same canonical Poisson measure process for all models considered.

(3) There is strong evidence of *dynamical* microstructure noise.

(4) Adding dynamics to the microstructure allowed much greater deviations of price from intrinsic value, which can be detected by filtering and used as a warning sign to investors and traders.

(5) The simplified Heston stochastic volatility model with microstructure dynamics and significant inertia performed the best in all cases.

(6) There is strong statistical evidence that such simplified Heston stochastic volatility models with microstructure dynamics and inertia match the data better than the classical geometrical Brownian motion.

(7) The amount of inertia h and the time it lasted Δ varied a little from stock to stock but in all cases there was significant inertia that lasted for hours.

More complicated SV models can be investigated in our future work. One could also postulate more complicated microstructure dynamics and consider additional real data analysis.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[2] R. C. Merton, "Theory of rational option pricing," *The Bell Journal of Economics and Management Science*, vol. 4, no. 1, pp. 141–183, 1973.

[3] J. C. Cox, S. A. Ross, and M. Rubinstein, "Option pricing: a simplified approach," *Journal of Financial Economics*, vol. 7, no. 3, pp. 229–263, 1979.

[4] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of Financial Studies*, vol. 6, no. 2, pp. 327–343, 1993.

[5] J. C. Jackwerth and M. Rubinstein, "Recovering probability distributions from option prices," *The Journal of Finance*, vol. 51, no. 5, pp. 1611–1631, 1996.

[6] J. Hull and A. White, "The pricing of options on assets with stochastic volatilities," *The Journal of Finance*, vol. 42, no. 2, pp. 281–300, 1987.

[7] D. B. Nelson, "ARCH models as diffusion approximations," *Journal of Econometrics*, vol. 45, no. 1-2, pp. 7–38, 1990.

[8] B. B. Mandelbrot, *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk*, Springer, 1997.

[9] A. Shiryaev, *Essentials of Stochastic Finance*, World Scientific, 1999.

[10] M. Capinski and T. Zastawniak, *Mathematics for Finance. An Introduction to Financial Engineering*, Springer, London, UK, 2nd edition, 2011.

[11] M. A. Kouritzin and Y. Zeng, "Weak convergence for a type of conditional expectation: application to the inference for a class of asset price models," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 60, no. 2, pp. 231–239, 2005.

[12] P. Del Moral, M. A. Kouritzin, and L. Miclo, "On a class of discrete generation interacting particle systems," *Electronic Journal of Probability*, vol. 6, pp. 1–26, 2001.

[13] R. F. Engle, "The econometrics of ultra-high-frequency data," *Econometrica*, vol. 68, no. 1, pp. 1–22, 2000.

[14] F. Black, "Noise," *Journal of Finance*, vol. 41, pp. 529–543, 1986.

[15] L. K. C. Chan and J. Lakonishok, "Institutional trades and intraday stock price behavior," *Journal of Financial Economics*, vol. 33, no. 2, pp. 173–199, 1993.

[16] J. Hasbrouck, *Handbook of Statistics*, vol. 14 of *Modelling Market Microstructure Time Series*, North-Holland, Amsterdam, The Netherlands, 1996.

[17] J. Hasbrouck, "Security bid/ask dynamics with discreteness and clustering: simple strategies for modeling and estimation," *Journal of Financial Markets*, vol. 2, no. 1, pp. 1–28, 1999.

[18] R. F. Engle and J. R. Russell, "Autoregressive conditional duration: a new model for irregularly spaced transaction data," *Econometrica*, vol. 66, no. 5, pp. 1127–1162, 1998.

[19] F. M. Bandi and J. R. Russell, "Separating microstructure noise from volatility," *Journal of Financial Economics*, vol. 79, no. 3, pp. 655–692, 2006.

[20] R. E. Kass and A. E. Raftery, "Bayes factor and model uncertainty," *Journal of the American Statistical Association*, vol. 90, pp. 773–795, 1995.

[21] Y. Zeng, "A partially observed model for micromovement of asset prices with bayes estimation via filtering," *Mathematical Finance*, vol. 13, no. 3, pp. 411–444, 2003.

[22] J. Xiong and Y. Zeng, "A branching particle approximation to a filtering micromovement model of asset price," *Statistical Inference for Stochastic Processes*, vol. 14, no. 2, pp. 111–140, 2011.

[23] M. A. Kouritzin and Y. Zeng, "Bayesian model selection via filtering for a class of micro-movement models of asset price," *International Journal of Theoretical and Applied Finance*, vol. 8, no. 1, pp. 97–121, 2005.

[24] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer, Berlin, Germany, 1979.

[25] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, John Wiley & Sons, New York, NY, USA, 1986.

[26] L. O. Scott, "Option pricing when the variance changes randomly: theory, estimation, and an application," *The Journal of Financial and Quantitative Analysis*, vol. 22, no. 4, pp. 419–438, 1987.

[27] F. C. Drost and B. J. M. Werker, "Closing the GARCH gap: continuous time GARCH modeling," *Journal of Econometrics*, vol. 74, no. 1, pp. 31–57, 1996.

[28] R. J. Elliott, W. P. Malcolm, and A. H. Tsoi, "Robust parameter estimation for asset price models with Markov modulated

volatilities," *Journal of Economic Dynamics and Control*, vol. 27, no. 8, pp. 1391–1409, 2003.

[29] D. Duffie, J. Pan, and K. Singleton, "Transform analysis and asset pricing for affine jump-diffusions," *Econometrica*, vol. 68, no. 6, pp. 1343–1376, 2000.

[30] B. Eraker, M. Johannes, and N. Polson, "The impact of jumps in volatility and returns," *The Journal of Finance*, vol. 58, no. 3, pp. 1269–1300, 2003.

[31] B. Eraker, "Do stock prices and volatility jump? Reconciling evidence from spot and option prices," *The Journal of Finance*, vol. 59, no. 3, pp. 1367–1403, 2004.

[32] P. R. Hansen and A. Lunde, "Realized variance and market microstructure noise," *Journal of Business and Economic Statistics*, vol. 24, no. 2, pp. 127–161, 2006.

[33] J.-C. Duan and A. Fulop, "Estimating the structural credit risk model when equity prices are contaminated by trading noises," *Journal of Econometrics*, vol. 150, no. 2, pp. 288–296, 2009.

[34] Y. Aït-Sahalia, P. Mykland, and L. Zhang, "How often to sample a continuous-time process in the presence of market microstructure noise," *Review of Financial Studies*, vol. 18, pp. 351–416, 2005.

[35] N. Jegadeesh and S. Titman, "Returns to buying winners and selling losers: implications for stock market efficiency," *The Journal of Finance*, vol. 48, no. 1, pp. 65–91, 1993.

[36] T. J. Moskowitz and M. Grinblatt, "Do industries explain momentum?" *The Journal of Finance*, vol. 54, no. 4, pp. 1249–1290, 1999.

[37] B. D. Grundy and J. S. Martin, "Understanding the nature of the risks and the source of the rewards to momentum investing," *Review of Financial Studies*, vol. 14, no. 1, pp. 29–78, 2001.

[38] J. M. Griffin, X. Ji, and J. S. Martin, "Momentum investing and business cycle risk: evidence from Pole to Pole," *Journal of Finance*, vol. 58, no. 6, pp. 2515–2547, 2003.

[39] R. D. Huang and H. R. Stoll, "Tick size, bid-ask spreads, and market structure," *Journal of Financial and Quantitative Analysis*, vol. 36, no. 4, pp. 503–522, 2001.

[40] K. H. Chung, B. F. Van Ness, and R. A. Van Ness, "Trading costs and quote clustering on the NYSE and NASDAQ after decimalization," *Journal of Financial Research*, vol. 27, no. 3, pp. 309–328, 2004.

[41] K. H. Chung, K. A. Kim, and P. Kitsabunnarat, "Liquidity and quote clustering in a market with multiple tick sizes," *Journal of Financial Research*, vol. 28, no. 2, pp. 177–195, 2005.

[42] M. J. Barclay, "Bid-ask spreads and the avoidance of odd-eighth quotes on nasdaq: an examination of exchange listings," *Journal of Financial Economics*, vol. 45, no. 1, pp. 35–60, 1997.

[43] Y. Aït-Sahalia and R. Kimmel, "Maximum likelihood estimation of stochastic volatility models," *Journal of Financial Economics*, vol. 83, no. 2, pp. 413–452, 2007.

[44] P. Bremaud, *Point Process and Queues: Martingale Dynamics*, Springer, New York, NY, USA, 1981.

[45] N. J. Gordon, D. J. Salmond, and A. F. M. Smith, "Novel approach to nonlinear/non-gaussian Bayesian state estimation," *IEE Proceedings, Part F: Radar and Signal Processing*, vol. 140, no. 2, pp. 107–113, 1993.

[46] P. Del Moral, J.-C. Noyer, and G. Salut, "Maslov optimisation theory: stochastic interpretation, particle resolution," in *11th International Conference on Analysis and Optimization of Systems Discrete Event Systems*, vol. 199 of *Lecture Notes in Control and Information Sciences*, pp. 312–318, Springer, Berlin, Germany, 1994.

[47] D. J. Ballantyne, H. Y. Chan, and M. A. Kouritzin, "Novel branching particle method for tracking," in *Signal and Data Processing of Small Targets*, O. E. Drummond, Ed., vol. 4048 of *Proceedings of SPIE*, pp. 277–287, July 2000.

[48] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, 1992.

[49] D. Ocone and E. Pardoux, "Asymptotic stability of the optimal filter with respect to its initial condition," *SIAM Journal on Control and Optimization*, vol. 34, no. 1, pp. 226–243, 1996.

[50] B. Delyon and O. Zeitouni, "Lyapunov exponents for filtering problems," in *Applied Stochastic Analysis*, M. H. A. Davis and R. J. Elliott, Eds., pp. 511–521, Gordon and Breach Science Publishers, London, UK, 1991.

[51] R. Atar, "Exponential stability for nonlinear filtering of diffusion processes in a noncompact domain," *Annals of Probability*, vol. 26, no. 4, pp. 1552–1574, 1998.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 148565, 13 pages
http://dx.doi.org/10.1155/2015/148565

Research Article

Further Results on Stability Analysis for Markovian Jump Systems with Time-Varying Delays

O. M. Kwon,[1] R. Sakthivel,[2,3] and M. J. Park[1]

[1] *School of Electrical Engineering, Chungbuk National University, 1 Chungdae-ro, Seowon-gu, Cheongju 362-763, Republic of Korea*
[2] *Department of Mathematics, Sri Ramakrishna Institute of Technology, Coimbatore, Tamilnadu 641010, India*
[3] *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea*

Correspondence should be addressed to O. M. Kwon; madwind@chungbuk.ac.kr and R. Sakthivel; krsakthivel@yahoo.com

Received 15 May 2015; Revised 1 July 2015; Accepted 1 July 2015

Academic Editor: Leonid Shaikhet

This paper is concerned with the problem of stability analysis for Markovian jump systems with time-varying delays. By constructing a newly augmented Lyapunov-Krasovskii functional and combining Wirtinger-based integral inequality, an improved delay-dependent stability criterion within the framework of linear matrix inequalities (LMIs) is introduced. Based on the result of delay-dependent stability criterion, when linear systems have fast time-varying delays, a corresponding stability condition is given. Via three numerical examples, the improvements of the proposed criteria are shown by comparing maximum delay bounds provided by our theorems with the recent results.

1. Introduction

Stability analysis of dynamic systems is a prerequisite and essential job before designing a controller to achieve the prescribed specifications. In particular, a great concern of stability analysis for systems with time-delays has been received due to the fact that time-delay naturally occurs in many practical systems such as networked control system, chemical processing, hot rolling mill, synchronization between chaotic systems, neural networks, and multiagent systems. For instance, see [1, 2] and references therein.

The main issue in delay-dependent stability analysis for time-delay systems with the framework of LMIs is how to increase maximum delay bounds for guaranteeing the asymptotic stability of systems. Thus, the choosing of Lyapunov-Krasovskii functional (LKF) and some techniques in estimating an upper bound of time-derivative value of the constructed LKF are the most important factors in enhancing the stability feasible region. In the LKF aspect, quadratic form, single integral, and double integral of quadratic form are the most utilized functionals. Recently, since the triple integral form of LKF was introduced in [3], this form of LKF has been utilized in many works such as [4–6]. Moreover, in [4, 5], it was shown that some augmented LKFs can increase the feasible region of stability criteria. In estimating an upper bound of time-derivative value of LKF, Jensen's inequality [7], free-weighting matrix technique [8], and reciprocally convex optimization theory [9] make big impacts on the enhancement of delay-dependent stability and stabilization. Seuret and Gouaisbaut [10] proposed the Wirtinger-based integral inequality which provides more tight lower bounds than Jensen's inequality and showed that the utilization of Wiritinger-based integral inequality can improve maximum delay bounds in many systems such as systems with constant and known delay, systems with a time-varying delay, systems with a constant distributed delay, and sampled-date systems. Cheng and Xiong [11] reduced conservative condition of stabilization criteria for continuous-time systems with time-varying input by introducing a new integral inequality. Recently, in [12, 13], for neural network with time-varying delay, it can be confirmed that the utilization of Wirtinger-based integral inequality in obtaining an upper bound of time-derivative values of some augmented LKFs can provide larger delay bounds than some other literatures. Very recently, in [14], it was shown that the results obtained by [10] can be further improved by choosing some new augmented

LKFs. From the statements mentioned above, one can see that the choosing of LKF and some techniques play key roles to reduce the conservatism of stability criteria.

On the other hand, increasing attention has been paid to Markovian jumping systems (MJSs) which are a special sort of hybrid systems and driven by Markov chain. MJSs may undergo unexpected changes in their structure and parameters including economic systems, aerospace systems, power systems, and networked control system [15, 16]. Very recently, a survey on recent developments of modeling, analysis, and design of MJSs was reported in Shi and Li [17].

In this regard, many researchers put their times and efforts into stability and stabilization of Markovian jumping systems with time-delays. In [18], the problems of robust $\mathscr{H}_\infty$ control and $\mathscr{H}_\infty$ filtering for uncertain MJSs with time-varying delays were investigated by utilizing bounded real lemma. In [19], some new results on stabilization of MJSs with time-delays were proposed based on a delay-partitioning approach. Wu et al. [20] investigated the problem of stability and $\mathscr{H}_\infty$ filtering for singular Markovian jump systems with time-delay via a delay-dependent bounded real lemma. Li et al. [21] utilized an input-output approach to stability and stabilization of MJSs with time-varying delays and showed the reduction of conservatism of the concerned criteria by a precise approximation of time-varying delay. By constructing new LKFs having distinct Lyapunov matrices for different modes, the mean square exponential stability and stabilization problems were studied in [22] for MJSs with constant time-delays. In [23], improved delay-dependent stability and $\mathscr{H}_\infty$ control for singular Markovian jump systems with time-delay by utilizing delay-partitioning technique with a tuning parameter. Zhu [24] derived some new conditions for ensuring the asymptotic stability of singular nonlinear MJSs with unknown parameters and continuously distributed delays. Recently, some new augmented LKFs and techniques in estimating upper bounds of time-derivative of LKFs were introduced in [25] in studying stability and $\mathscr{H}_\infty$ performance analysis of MJSs with time-varying delays. Very recently, in [26], an input-output approach to the delay-dependent stability analysis and $\mathscr{H}_\infty$ control for MJSs with time-varying delays and deficient transition descriptions. The problem of finite-time $\mathscr{H}_\infty$ estimation for a class of discrete-time Markov jump systems with time-varying transition probabilities subject to average dwell time switching was investigated in [27]. However, as mentioned in [17], the results on stability have still some conservativeness. Thus, there are rooms for further reduction of conservativeness caused by time-delays with the construction of a newly augmented Lyapunov-Krasovskii functional and utilization of a Wirtinger-based integral inequality [10].

Motivated by [17] and based on the result of [25], the goal of this paper is to propose a further improved result of delay-dependent stability for MJSs with time-varying delays. In Theorem 5, a new and improved stability criterion will be proposed based on the results of [25]. To derive less conservative results, Wirtinger-based integral inequality is applied to the augmented LKFs and some new techniques are introduced. When an upper bound of time-derivative value of time-varying delay is larger than one or unknown, a corresponding result will be presented in Corollary 6 by constructing some part of LKF utilized in Theorem 5. Comparing with the result of [25], the constructed Lyapunov-Krasovskii functionals in Theorem 5 and Corollary 6 are simple since the triple and quadruple integral form of Lyapunov-Krasovskii functionals will not be utilized. Via three numerical examples, the advantage and effectiveness of the proposed results will be explained by comparing maximum delay bounds with some recent results presented in other literatures.

Notation. Throughout this paper, the following notations will be used. $X > 0$ ($X \geq 0$) means that X is a real symmetric positive definitive matrix (positive semidefinite). The subscript "T" represents the transpose. $X^\perp$ denotes a basis for the null-space of X. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrix. $\mathscr{C}_{n,h} = \mathscr{C}([-h, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\mathscr{L}_2[0, \infty)$ means the space of square-integrable vector functions over $[0, \infty)$. $\mathscr{E}\{\cdot\}$ denotes the expectation operator with respect to some measure $\mathscr{P}$. I_n, 0_n, and $0_{m\cdot n}$ denote $n \times n$ identity matrix and $n \times n$ and $m \times n$ zero matrices, respectively. $\|\cdot\|$ refers to the induced matrix 2-norm. $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$. For any matrix M, $\mathrm{Sym}\{M\}$ means $M + M^T$. $\mathrm{col}\{x_1, x_2, \ldots, x_n\}$ means $[x_1^T, x_2^T, \ldots, x_n^T]^T$.

2. Problem Statement and Preliminaries

Consider the Markovian jump system with time-varying delays:

$$\begin{aligned} \dot{x}(t) &= A(r(t))\,x(t) + A_d(r(t))\,x(t-h(t)), \\ x(t) &= \phi(t), \quad \forall t \in [-h_U, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\phi(t)$ which belongs to $\mathscr{C}_{n,h_U}$ means the initial function, $A(r(t))$ and $A_d(r(t))$ are known system matrices with appropriate dimensions, and $r(t)$ denotes a finite state Markovian jump process representing the system mode. That is, $r(t)$ takes values in the finite discrete set $\mathcal{S} = \{1, 2, \ldots, N\}$ with transition probability matrix $\Pi = [\pi_{ij}]$.

The transition probability is described as

$$\begin{aligned} &\Pr\{r(t+\delta) = j \mid r(t) = i\} \\ &\quad = \begin{cases} \pi_{ij}\delta + o(\delta), & j \neq i, \\ 1 + \pi_{ij}\delta + o(\delta), & j = i, \end{cases} \end{aligned} \tag{2}$$

where $\delta > 0$, $\lim_{\delta\to 0^+(o(\delta)/\delta)=0}$, $\pi_{ij} \geq 0$ for $j \neq i$ and $\pi_{ii} = -\sum_{j\neq i}\pi_{ij}$.

The delay in states, $h(t)$, is a time-varying and continuous function satisfying

$$\begin{aligned} 0 \leq h(t) &\leq h_U, \\ \dot{h}(t) &\leq h_D, \end{aligned} \tag{3}$$

where h_U is a known positive scalar and h_D is any constant one.

For simplicity, a matrix $M(r(t))$ of ith node is denoted by M_i for each possible $r(t) = i$, $i \in \mathcal{S}$ in the rest of this paper. For example, $A(r(t))$ and $A_d(r(t))$ of ith node will be represented as A_i and A_{di}, respectively. Let $x_t = x(t+s)$ for $s \in [-h_U, 0]$. From [28], it should be noted that $\{(x_t, r(t))\}$ is a Markov process for $t \geq 0$. Then, its weak infinitesimal operator $\mathscr{L}$ acting on a functional $V(x_t, i)$ is defined by

$$\begin{aligned}\mathscr{L}V(x_t, i) &= \lim_{\delta \to 0^+} \frac{1}{\delta}\left[\mathscr{E}\left\{V\left(x_{t+\delta}, r(t+\delta) \mid x_t, r(t) = i\right)\right\}\right. \\ &\quad \left. - V(x_t, i)\right].\end{aligned} \tag{4}$$

In stability analysis of system (1), the following definition will be utilized.

Definition 1 (see [29]). For any finite $\phi(t) \in \mathscr{C}_{n,h_U}$, and the initial condition of the mode $r_0 \in \mathcal{S}$, the system $\dot{x}(t) = A(r(t))x(t) + A_d(r(t))x(t-h(t))$ is said to

(a) be stochastically stable if there exists a constant $T(r_0, \phi(t))$ such that

$$\mathscr{E}\left\{\int_0^\infty \|x(t)\|^2 \mid r_0, \phi(t)\right\} \leq T(r_0, \phi(t)), \tag{5}$$

(b) be mean square stable if

$$\lim_{t \to \infty} \mathscr{E}\|x(t)\|^2 = 0, \tag{6}$$

hold for any initial condition $(r_0, \phi(t))$,

(c) be mean exponentially stable if there exist constants $\alpha > 0$ and $\beta > 0$ such that the following holds for any initial condition $(r_0, \phi(t))$:

$$\mathscr{E}\left\{\|x(t)\|^2 \mid r_0, \phi(t)\right\} \leq \alpha \|\phi(t)\| e^{-\beta t}. \tag{7}$$

Based on the results of [25], the objective of this paper is to develop further improved delay-dependent stability criteria of system (1) which will be conducted in next section.

The following lemmas will be utilized in deriving main results.

Lemma 2. *Consider a given matrix $M > 0$. Then, for all continuous function η in $[a,b] \to \mathbb{R}^n$, the following inequality holds:*

$$\begin{aligned}\int_a^b \eta^T(s) M \eta(s)\, ds &\geq \frac{1}{b-a}\left(\int_a^b \eta(s)\, ds\right)^T \\ &\quad \cdot M\left(\int_a^b \eta(s)\, ds\right) \\ &\quad + \frac{3}{b-a}\left(\int_a^b \eta(s)\, ds - \frac{2}{b-a}\int_a^b\int_s^b \eta(u)\, du\, ds\right)^T \\ &\quad \cdot M\left(\int_a^b \eta(s)\, ds - \frac{2}{b-a}\int_a^b\int_s^b \eta(u)\, du\, ds\right).\end{aligned} \tag{8}$$

Proof. From the original Wirtinger-based integral inequality [10], since

$$\begin{aligned}&\int_a^b \eta(s)\, ds - \frac{2}{b-a}\int_a^b\int_a^s \eta(u)\, du\, ds \\ &= \int_a^b \eta(s)\, ds \\ &\quad - \frac{2}{b-a}\int_a^b\left(\int_a^b \eta(u)\, du - \int_s^b \eta(u)\, du\right) ds \\ &= \int_a^b \eta(s)\, ds - \frac{2}{b-a}\underbrace{\int_a^b\int_a^b \eta(u)\, du\, ds}_{(1/(b-a))\int_a^b \eta(s) ds} \\ &\quad + \frac{2}{b-a}\int_a^b\int_s^b \eta(u)\, du\, ds \\ &= -\int_a^b \eta(s)\, ds + \frac{2}{b-a}\int_a^b\int_s^b \eta(u)\, du\, ds,\end{aligned} \tag{9}$$

inequality (8) holds. □

Lemma 3 (see [30]). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that* $\operatorname{rank}(B) < n$. *Then, the following two statements are equivalent:*

(a) $\zeta^T \Phi \zeta < 0$, $B\zeta = 0$, $\zeta \neq 0$,

(b) $(B^\perp)^T \Phi B^\perp < 0$, *where $B^\perp$ is a right orthogonal complement of B.*

Lemma 4 (see [31]). *For the symmetric appropriately dimensional matrices $\Omega > 0$, Ξ, an any matrix Λ, the following two statements are equivalent:*

(a) $\Xi - \Lambda^T \Omega \Lambda < 0$,

(b) *there exists a matrix of appropriate dimension Ψ such that*

$$\begin{bmatrix} \Xi + \Lambda^T\Psi + \Psi^T\Lambda & \Psi^T \\ \Psi & -\Omega \end{bmatrix} < 0. \tag{10}$$

3. Main Results

In this section, improved delay-dependent stability criteria for MJSs (1) will be proposed. To express vectors and matrices in simple forms, block entry matrices e_i $(i = 1, \ldots, 9) \in \mathbb{R}^{9n\times n}$ will be used. For example, e_3 means $[0_{n\cdot 2n}, I_n, 0_{n\cdot 6n}]^T$. And some of scalars, vectors, and matrices are defined as

$$\begin{aligned}\zeta(t) = \operatorname{col}\Big\{&x(t), x(t-h(t)), x(t-h_U), \dot{x}(t), \\ &\dot{x}(t-h_U), \int_{t-h(t)}^t x(s)\, ds, \int_{t-h_U}^{t-h(t)} x(s)\, ds,\end{aligned}$$

$$\int_{t-h(t)}^{t}\int_{s}^{t} x(u)\,du\,ds, \int_{t-h_U}^{t-h(t)}\int_{s}^{t} x(u)\,du\,ds\Big\},$$

$$\eta(t) = \operatorname{col}\Big\{x(t), x(t-h_U), \int_{t-h_U}^{t} x(s)\,ds, \int_{t-h_U}^{t}\int_{s}^{t} x(u)\,du\,ds\Big\},$$

$$\alpha(t,s) = \operatorname{col}\Big\{\dot{x}(s), x(s), \int_{s}^{t}\dot{x}(u)\,du\Big\},$$

$$\beta(t,s) = \operatorname{col}\Big\{x(s), \int_{s}^{t}\dot{x}(u)\,du\Big\},$$

$$\mathcal{Q}_{\text{aug1}} = \mathcal{Q} + \begin{bmatrix} 0_n & P_1 & 0_n \\ P_1 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix},$$

$$\mathcal{Q}_{\text{aug2}} = \mathcal{Q} + \begin{bmatrix} 0_n & P_2 & 0_n \\ P_2 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix},$$

$$\begin{aligned}
\Xi_{1i} &= \operatorname{Sym}\Big\{[e_1, e_3, e_6+e_7, e_8+e_9] \cdot \mathcal{R}_i\,[e_4, e_5, e_1-e_3, h_U e_1 - e_6 - e_7]^T\Big\}, \\
&+ [e_1, e_3, e_6+e_7, e_8+e_9]\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{R}_j\right)[e_1, e_3, e_6+e_7, e_8 + e_9]^T, \\
\Xi_{2i} &= [e_4, e_1, 0_{9n\cdot n}]\,\mathcal{N}_i\,[e_4, e_1, 0_{9n\cdot n}]^T - [e_5, e_3, e_1-e_3] \cdot \mathcal{N}_i\,[e_5, e_3, e_1-e_3]^T \\
&+ \operatorname{Sym}\Big\{[e_1-e_3, e_6+e_7, h_U e_1 - e_6 - e_7] \cdot \mathcal{N}_i\,[0_{9n\cdot n}, 0_{9n\cdot n}, e_4]^T\Big\}, \\
\Xi_{3i[h(t)]} &= [e_1, 0_{9n\cdot n}]\,\mathcal{G}_i\,[e_1, 0_{9n\cdot n}]^T - (1-h_D)\,[e_2, e_1 - e_2]\,\mathcal{G}_i\,[e_2, e_1-e_2]^T + \operatorname{Sym}\Big\{[e_6, h(t)e_1 - e_6] \cdot \mathcal{G}_i\,[0_{9n\cdot n}, e_4]^T\Big\}, \\
\Xi_4 &= [e_4, e_1, 0_{9n\cdot n}]\left(h_U^2\mathcal{Q}\right)[e_4, e_1, 0_{9n\cdot n}]^T + 2h_U\left[h_U e_1 - e_6 - e_7, e_8 + e_9, \left(\frac{h_U^2}{2}\right)e_1 - e_8 - e_9\right]\mathcal{Q}\,[0_{9n\cdot n}, 0_{9n\cdot n}, e_4]^T, \\
\Xi_5 &= h_U\left(e_1P_1e_1^T - e_2P_1e_2^T + e_2P_2e_2^T - e_3P_2e_3^T\right), \\
\Lambda_{1[h(t)]} &= [e_1-e_2, e_6, h(t)e_1 - e_6, h(t)(-e_1-e_2) + 2e_6, h(t)e_6 - 2e_8, 2e_8 - h(t)e_6]^T, \\
\Lambda_{2[h(t)]} &= [e_2-e_3, e_7, (h_U - h(t))e_1 - e_7, (h_U - h(t)) \cdot (-e_2-e_3) + 2e_7, (h_U - h(t))e_7 - 2e_9, \\
&\quad -(h_U - h(t))(2e_6+e_7) + 2e_9]^T, \\
\Lambda_{3[h(t)]} &= \left[\Lambda_{1[h(t)]}^T, \Lambda_{2[h(t)]}^T\right]^T, \\
\Theta_{1i} &= \mathcal{Q}_{\text{aug1}} - \left(\frac{1}{h_U}\right)\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j - \left(\frac{1}{h_U}\right) \cdot \left[\begin{array}{c|c} 0_n & 0_{n\cdot 2n} \\ \hline 0_{2n\cdot n} & \sum_{j=1}^{s}\pi_{ij}\mathcal{G}_j \end{array}\right], \\
\Theta_{2i} &= \mathcal{Q}_{\text{aug2}} - \left(\frac{1}{h_U}\right)\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j, \\
\Omega_{1i} &= \begin{bmatrix} \Theta_{1i} & 0_{3n} \\ 0_{3n} & \dfrac{(3\Theta_{1i})}{h_U^2} \end{bmatrix}, \\
\Omega_{2i} &= \begin{bmatrix} \Theta_{2i} & 0_{3n} \\ 0_{3n} & \dfrac{(3\Theta_{2i})}{h_U^2} \end{bmatrix}, \\
\Omega_{3i} &= \begin{bmatrix} \Omega_{1i} & \mathcal{S} \\ \mathcal{S}^T & \Omega_{2i} \end{bmatrix}, \\
\Upsilon_i &= [A_i, A_{di}, 0_n, -I_n, 0_n, 0_n, 0_n, 0_n, 0_n], \\
\Sigma_{i[h(t)]} &= \Xi_{1i} + \Xi_{2i} + \Xi_{3i[h(t)]} + \Xi_4 + \Xi_5, \\
\Phi_{i[h(t)]} &= \left(\Upsilon_i^{\perp}\right)^T\left(\Sigma_{i[h(t)]}\right)\left(\Upsilon_i^{\perp}\right) + \operatorname{Sym}\Big\{\left(\Upsilon_i^{\perp}\right)^T \cdot \left(\Lambda_{3i[h(t)]}\right)^T\Psi\Big\}.
\end{aligned} \tag{11}$$

Now, we have the following theorem.

Theorem 5. *For given scalars $h_U > 0$ and h_D, system (1) is stochastically stable for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$ if there exist positive definite matrices $\mathcal{R}_i \in \mathbb{R}^{4n\times 4n}$, $\mathcal{N}_i \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G}_i \in \mathbb{R}^{2n\times 2n}$, and $\mathcal{Q} \in \mathbb{R}^{3n\times 3n}$, any matrices $\mathcal{S} \in \mathbb{R}^{6n\times 6n}$ and $\Psi \in \mathbb{R}^{12n\times 8n}$, and any symmetric matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ satisfying the following LMIs for all $r(t) = i$, $i \in \mathcal{S}$:*

$$\begin{bmatrix} \Phi_{i,k} & \Psi^T \\ \Psi & -\Omega_{3i} \end{bmatrix} < 0, \quad k = 1, 2, \tag{12}$$

where $\{\Phi_{i,k}\}_{k=1}^{2}$ means the two vertices of $\Phi_{i[h(t)]}$ with the bounds of $0 \le h(t) \le h_M$. That is, $\Phi_{i,1} = \Phi_{i[h(t)=0]}$ and $\Phi_{i,2} = \Phi_{i[h(t)=h_U]}$.

Proof. For each $r(t) = i, i \in \mathcal{S}$, let us consider the Lyapunov-Krasovskii functional candidate:

$$V(x_t, i) = \sum_{j=1}^{4} V_j(x_t, i), \tag{13}$$

where

$$\begin{aligned} V_1(x_t, i) &= \eta^T(t)\mathcal{R}_i\eta(t), \\ V_2(x_t, i) &= \int_{t-h_U}^{t} \alpha^T(t,s)\mathcal{N}_i\alpha(t,s)\,ds, \\ V_3(x_t, i) &= \int_{t-h(t)}^{t} \beta^T(t,s)\mathcal{G}_i\beta^T(t,s)\,ds, \\ V_4(x_t, i) &= \int_{t-h_U}^{t}\int_{s}^{t} \alpha^T(t,u)\mathcal{Q}\alpha(t,u)\,du\,ds. \end{aligned} \tag{14}$$

From the following relationship:

$$\begin{aligned} \eta(t) &= \begin{bmatrix} x(t) \\ \hline x(t-h_U) \\ \hline \displaystyle\int_{t-h(t)}^{t} x(s)\,ds + \int_{t-h_U}^{t-h(t)} x(s)\,ds \\ \hline \displaystyle\int_{t-h(t)}^{t}\int_{s}^{t} x(u)\,du\,ds + \int_{t-h_U}^{t-h(t)}\int_{s}^{t} x(u)\,du\,ds \end{bmatrix} \\ &= [e_1, e_3, e_6+e_7, e_8+e_9]^T\zeta(t), \\ \dot{\eta}(t) &= \begin{bmatrix} \dot{x}(t) \\ \hline \dot{x}(t-h_U) \\ \hline x(t) - x(t-h_U) \\ \hline h_U x(t) - \underbrace{\displaystyle\int_{t-h_U}^{t} x(s)\,ds}_{\int_{t-h(t)}^{t} x(s)ds + \int_{t-h_U}^{t-h(t)} x(s)ds} \end{bmatrix} \\ &= [e_4, e_5, e_1-e_3, h_U e_1 - e_6 - e_7]^T\zeta(t), \end{aligned} \tag{15}$$

$\mathcal{L}V_1(x_t, i)$ can be represented as

$$\begin{aligned} \mathcal{L}V_1(x_t, i) &= 2\eta^T(t)\mathcal{R}\dot{\eta}(t) \\ &\quad + \eta^T(t)\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{R}_j\right)\eta(t) \\ &= \zeta^T(t)\Xi_{1i}\zeta(t), \end{aligned} \tag{16}$$

where Ξ_{1i} is defined in (11).

Note that

$$\begin{aligned} \int_{t-h_U}^{t} \alpha(t,s)\,ds &= \int_{t-h_U}^{t} \begin{bmatrix} \dot{x}(s) \\ \hline x(s) \\ \hline \displaystyle\int_{s}^{t} \dot{x}(u)\,du \end{bmatrix} ds \\ &= \begin{bmatrix} \displaystyle\int_{t-h_U}^{t} \dot{x}(s)\,ds \\ \hline \displaystyle\int_{t-h_U}^{t} x(s)\,ds \\ \hline \displaystyle\int_{t-h_U}^{t}\int_{s}^{t} \dot{x}(u)\,du \end{bmatrix} \\ &= \begin{bmatrix} x(t) - x(t-h_U) \\ \hline \displaystyle\int_{t-h(t)}^{t} x(s)\,ds + \int_{t-h_U}^{t-h(t)} x(s)\,ds \\ \hline h_U x(t) - \displaystyle\int_{t-h(t)}^{t} x(s)\,ds - \int_{t-h_U}^{t-h(t)} x(s)\,ds \end{bmatrix} \\ &= [e_1 - e_3, e_6 + e_7, h_U e_1 - e_6 - e_7]^T\zeta(t). \end{aligned} \tag{17}$$

From (17), calculation of $\mathcal{L}V_2(x_t, i)$ leads to

$$\begin{aligned} \mathcal{L}V_2(x_t, i) &= \alpha^T(t,t)\mathcal{N}_i\alpha(t,t)\cdot\frac{d}{dt}(t) - \alpha^T(t, t \\ &\quad - h_U)\mathcal{N}_i\alpha(t, t-h_U)^T\cdot\frac{d}{dt}(t-h_U) \\ &\quad + \int_{t-h_U}^{t} \underbrace{\frac{d}{dt}\left(\alpha^T(t,s)\mathcal{N}_i\alpha(t,s)\right)}_{2\alpha^T(t,s)\mathcal{N}_i(d\alpha(t,s)/dt)} ds + \int_{t-h_U}^{t} \alpha^T(t,s) \\ &\quad \cdot\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j\right)\alpha(t,s)\,ds = \zeta^T(t)\left\{[e_4, e_1, 0_{9n\cdot n}]\right. \\ &\quad \cdot\mathcal{N}_i[e_4, e_1, 0_{9n\cdot n}]^T - [e_5, e_3, e_1 - e_3]\mathcal{N}_i[e_5, e_3, e_1 \\ &\quad - e_3]^T + \mathrm{Sym}\left\{[e_1 - e_3, e_6 + e_7, h_U e_1 - e_6 - e_7]\right. \\ &\quad \left.\left.\cdot\mathcal{N}_i[0_{9n\cdot n}, 0_{9n\cdot n}, e_4]^T\right\}\right\}\zeta(t) + \int_{t-h_U}^{t} \alpha^T(t,s) \\ &\quad \cdot\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j\right)\alpha(t,s)\,ds = \zeta^T(t)\Xi_{2i}\zeta(t) \\ &\quad + \int_{t-h_U}^{t} \alpha^T(t,s)\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j\right)\alpha(t,s)\,ds. \end{aligned} \tag{18}$$

An upper bound of $\mathcal{L}V_3(x_t, i)$ can be obtained as

$$\begin{aligned} \mathcal{L}V_3(x_t, i) &\le \begin{bmatrix} x(t) \\ 0_{n\cdot 1} \end{bmatrix}^T \mathcal{G}_i \begin{bmatrix} x(t) \\ 0_{n\cdot 1} \end{bmatrix} - (1 - h_D) \\ &\quad \cdot\begin{bmatrix} x(t-h(t)) \\ \displaystyle\int_{t-h(t)}^{t} \dot{x}(s)\,ds \end{bmatrix}^T \mathcal{G}_i \begin{bmatrix} x(t-h(t)) \\ \displaystyle\int_{t-h(t)}^{t} \dot{x}(s)\,ds \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&+2\int_{t-h(t)}^{t}\beta^{T}(t,s)\,\mathscr{G}\frac{\partial\beta(t,s)}{\partial t}ds+\int_{t-h(t)}^{t}\beta^{T}(t,s)\\
&\cdot\left(\sum_{j=1}^{s}\pi_{ij}\mathscr{G}_{j}\right)\beta(t,s)\,ds=\zeta^{T}(t)\\
&\cdot\left\{[e_1,0_{9n\cdot n}]\,\mathscr{G}_i\,[e_1,0_{9n\cdot n}]^{T}\right.\\
&-(1-h_D)\,[e_2,e_1-e_2]\,\mathscr{G}_i\,[e_2,e_1-e_2]^{T}\\
&\left.+\operatorname{Sym}\left\{[e_6,h(t)\,e_1-e_6]\,\mathscr{G}_i\,[0_{9n\cdot n},e_4]^{T}\right\}\right\}\zeta(t)\\
&+\int_{t-h(t)}^{t}\beta^{T}(t,s)\left(\sum_{j=1}^{s}\pi_{ij}\mathscr{G}_{j}\right)\beta(t,s)\,ds=\zeta^{T}(t)\\
&\cdot\Xi_{3i[h(t)]}\zeta(t)\\
&+\int_{t-h(t)}^{t}\beta^{T}(t,s)\left(\sum_{j=1}^{s}\pi_{ij}\mathscr{G}_{j}\right)\beta(t,s)\,ds.
\end{aligned}
\tag{19}
$$

Inspired by the work of [32], for any symmetric matrices $P_i\in\mathbb{R}^{n\times n}$ $(i=1,2)$, the following two zero equalities are satisfied:

$$
\begin{aligned}
0&=h_U\left\{x^{T}(t)\,P_1x(t)-x^{T}(t-h(t))\,P_1x(t-h(t))\right.\\
&\quad\left.-2\int_{t-h(t)}^{t}x^{T}(s)\,P_1\dot{x}(s)\,ds\right\},\\
0&=h_U\left\{x^{T}(t-h(t))\,P_2x(t-h(t))\right.\\
&\quad-x^{T}(t-h_U)\,P_2x(t-h_U)\\
&\quad\left.-2\int_{t-h_U}^{t-h(t)}x^{T}(s)\,P_2\dot{x}(s)\,ds\right\}.
\end{aligned}
\tag{20}
$$

By summing the two zero equalities in (20), we have

$$
\begin{aligned}
0&=\zeta^{T}(t)\,\Xi_5\zeta(t)-2h_U\int_{t-h(t)}^{t}x^{T}(s)\,P_1\dot{x}(s)\,ds\\
&\quad-2h_U\int_{t-h_U}^{t-h(t)}x^{T}(s)\,P_2\dot{x}(s)\,ds.
\end{aligned}
\tag{21}
$$

Let $\varphi(t,s)=\int_s^t\alpha^T(t,u)\mathscr{Q}\alpha(t,u)du$. By using the similar methods presented in (18) to (19), the calculation of $\mathscr{L}V_4(x_t,i)$ can be represented as

$$
\begin{aligned}
\mathscr{L}V_4(x_t,i)&=\underbrace{h_U\varphi(t,s)\big|_{s=t}}_{0}-h_U\varphi(t,s)\big|_{s=t-h_U}\\
&\quad+h_U\int_{t-h_U}^{t}\frac{\partial\varphi(t,s)}{\partial t}ds=-h_U\varphi(t,s)\big|_{s=t-h_U}\\
&\quad+h_U\int_{t-h_U}^{t}\left\{\alpha^{T}(t,t)\,\mathscr{Q}\alpha(t,t)\right.\\
&\quad\left.+2\int_{s}^{t}\alpha^{T}(t,u)\,\mathscr{Q}\frac{\partial\alpha(t,u)}{\partial t}du\right\}ds=h_U^2\alpha^{T}(t,t)\\
&\quad\cdot\mathscr{Q}\alpha(t,t)-h_U\int_{t-h_U}^{t}\alpha^{T}(t,s)\,\mathscr{Q}\alpha(t,s)\,ds\\
&\quad+2h_U\int_{t-h_U}^{t}\int_{s}^{t}\alpha^{T}(t,u)\,\mathscr{Q}\frac{\partial\alpha(t,u)}{\partial t}du\,ds=\zeta^{T}(t)\\
&\quad\cdot\left\{[e_4,e_1,0_{9n\cdot n}]\left(h_U^2\mathscr{Q}\right)[e_4,e_1,0_{9n\cdot n}]^{T}+2h_U\left[h_Ue_1\right.\right.\\
&\quad\left.-e_6-e_7,e_8+e_9,\left(\frac{h_U^2}{2}\right)e_1-e_8-e_9\right]\\
&\quad\left.\cdot\mathscr{Q}\,[0_{9n\cdot n},0_{9n\cdot n},e_4]^{T}\right\}\zeta(t)-h_U\int_{t-h_U}^{t}\alpha^{T}(t,s)\\
&\quad\cdot\mathscr{Q}\alpha(t,s)\,ds=\zeta^{T}(t)\,\Xi_4\zeta(t)-h_U\int_{t-h_U}^{t}\alpha^{T}(t,s)\\
&\quad\cdot\mathscr{Q}\alpha(t,s)\,ds.
\end{aligned}
\tag{22}
$$

Here, the following equations are utilized in (22):

$$
\begin{aligned}
2h_U\int_{t-h_U}^{t}\int_{s}^{t}\alpha^{T}(t,u)\,\mathscr{Q}\frac{\partial\alpha(t,u)}{\partial t}du\,ds&=2h_U\int_{t-h_U}^{t}\int_{s}^{t}\begin{bmatrix}\dot{x}(u)\\ x(u)\\ \int_{u}^{t}\dot{x}(u)\,du\end{bmatrix}^{T}\mathscr{Q}\begin{bmatrix}0_{n\cdot1}\\ 0_{n\cdot1}\\ \dot{x}(t)\end{bmatrix}du\,ds\\
&=2h_U\begin{bmatrix}\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}(u)\,du\,ds\\ \int_{t-h_U}^{t}\int_{s}^{t}x(u)\,du\,ds\\ \int_{t-h_U}^{t}\int_{s}^{t}\int_{u}^{t}\dot{x}(v)\,dv\,du\,ds\end{bmatrix}^{T}\mathscr{Q}\begin{bmatrix}0_{n\cdot1}\\ 0_{n\cdot1}\\ \dot{x}(t)\end{bmatrix}
\end{aligned}
$$

$$= 2h_U \begin{bmatrix} h_U x(t) - \int_{t-h(t)}^{t} x(s)\,ds - \int_{t-h_U}^{t-h(t)} x(s)\,ds \\ \hline \int_{t-h(t)}^{t}\int_{s}^{t} x(u)\,du\,ds + \int_{t-h_U}^{t-h(t)}\int_{s}^{t} x(u)\,du\,ds \\ \hline \left(\frac{h_U^2}{2}\right) x(t) - \int_{t-h(t)}^{t}\int_{s}^{t} x(u)\,du\,ds - \int_{t-h_U}^{t-h(t)}\int_{s}^{t} x(u)\,du\,ds \end{bmatrix}^T \mathcal{Q} \begin{bmatrix} 0_{n\cdot 1} \\ \hline 0_{n\cdot 1} \\ \hline \dot{x}(t) \end{bmatrix}. \tag{23}$$

With the consideration of $\int_{t-h_U}^{t} \alpha^T(t,s)(\sum_{j=1}^{s} \pi_{ij}\mathcal{N}_j)\alpha(t,s)ds$ in (18), $\int_{t-h(t)}^{t} \beta^T(t,s)(\sum_{j=1}^{s} \pi_{ij}\mathcal{G}_j)\beta(t,s)ds$ in (19), and the two integral terms $-2h_U \int_{t-h(t)}^{t} x^T(s)P_1\dot{x}(s)ds$ and $-2h_U \int_{t-h_U}^{t-h(t)} x^T(s)P_2\dot{x}(s)ds$ in (21), the last integral term $-h_U \int_{t-h_U}^{t} \alpha_2^T(t,s)\mathcal{Q}_1\alpha_2(t,s)ds$ at (22) with the addition of integral terms mentioned above can be estimated by the use of (a) in Lemma 2 and reciprocally convex optimization approach [9] as

$$\begin{aligned} &-h_U \int_{t-h_U}^{t} \alpha^T(t,s)\,\mathcal{Q}\alpha(t,s)\,ds - 2h_U \int_{t-h(t)}^{t} x^T(s)\,P_1\dot{x}(s)\,ds \\ &-2h_U \int_{t-h_U}^{t-h(t)} x^T(s)\,P_2\dot{x}(s)\,ds \\ &+\int_{t-h_U}^{t} \alpha^T(t,s)\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j\right)\alpha(t,s)\,ds \\ &+\int_{t-h(t)}^{t} \beta^T(t,s)\left(\sum_{j=1}^{s}\pi_{ij}\mathcal{G}_j\right)\beta(t,s)\,ds = -h_U \int_{t-h(t)}^{t} \alpha^T(t,s) \\ &\cdot\left(\underbrace{\mathcal{Q}_{\text{aug1}} - \left(\frac{1}{h_U}\right)\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j - \left(\frac{1}{h_U}\right)\left[\begin{array}{c|c} 0_n & 0_{n\cdot 2n} \\ \hline 0_{2n\cdot n} & \sum_{j=1}^{s}\pi_{ij}\mathcal{G}_j \end{array}\right]}_{\Theta_{1i}}\right) \\ &\cdot\alpha(t,s)\,ds \\ &-h_U \int_{t-h_U}^{t-h(t)} \alpha^T(t,s)\left(\underbrace{\mathcal{Q}_{\text{aug2}} - \left(\frac{1}{h_U}\right)\sum_{j=1}^{s}\pi_{ij}\mathcal{N}_j}_{\Theta_{2i}}\right)\alpha(t,s)\,ds, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \mathcal{Q}_{\text{aug1}} &= \mathcal{Q} + \begin{bmatrix} 0_n & P_1 & 0_n \\ P_1 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}, \\ \mathcal{Q}_{\text{aug2}} &= \mathcal{Q} + \begin{bmatrix} 0_n & P_2 & 0_n \\ P_2 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}, \end{aligned} \tag{25}$$

which were defined in (11).

With the use of Lemma 2, the integral term $-h_U \int_{t-h(t)}^{t} \alpha^T(t,s)\Theta_{1i}\alpha(t,s)ds$ can be bounded as

$$\begin{aligned} &-h_U \int_{t-h(t)}^{t} \alpha^T(t,s)\,\Theta_{1i}\alpha(t,s)\,ds \le -\frac{h_U}{h(t)}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds\right)^T \\ &\cdot\Theta_{1i}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds\right) \\ &-\frac{3h_U}{h(t)}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds - \frac{2}{h(t)}\int_{t-h(t)}^{t}\int_{s}^{t} \alpha(t,u)\,du\,ds\right)^T \\ &\cdot\Theta_{1i}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds - \frac{2}{h(t)}\int_{t-h(t)}^{t}\int_{s}^{t} \alpha(t,u)\,du\,ds\right) \\ &= -\frac{h_U}{h(t)}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds\right)^T \Theta_{1i}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds\right) \\ &-\frac{3h_U}{h(t)}\left(h(t)\int_{t-h(t)}^{t} \alpha(t,s)\,ds - 2\int_{t-h(t)}^{t}\int_{s}^{t} \alpha(t,u)\,du\,ds\right)^T \\ &\cdot\left(\frac{\Theta_{1i}}{h^2(t)}\right) \\ &\cdot\left(h(t)\int_{t-h(t)}^{t} \alpha(t,s)\,ds - 2\int_{t-h(t)}^{t}\int_{s}^{t} \alpha(t,u)\,du\,ds\right) \\ &\le -\frac{h_U}{h(t)}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds\right)^T \Theta_{1i}\left(\int_{t-h(t)}^{t} \alpha(t,s)\,ds\right) \\ &-\frac{3h_U}{h(t)}\left(h(t)\int_{t-h(t)}^{t} \alpha(t,s)\,ds - 2\int_{t-h(t)}^{t}\int_{s}^{t} \alpha(t,u)\,du\,ds\right)^T \\ &\cdot\left(\frac{\Theta_{1i}}{h_U^2}\right) \\ &\cdot\left(h(t)\int_{t-h(t)}^{t} \alpha(t,s)\,ds - 2\int_{t-h(t)}^{t}\int_{s}^{t} \alpha(t,u)\,du\,ds\right) \\ &= -\frac{h_U}{h(t)}\begin{bmatrix} x(t) - x(t-h(t)) \\ \hline \int_{t-h(t)}^{t} x(s)\,ds \\ \hline h(t)\,x(t) - \int_{t-h(t)}^{t} x(s)\,ds \end{bmatrix}^T \\ &\cdot\Theta_{1i}\begin{bmatrix} x(t) - x(t-h(t)) \\ \hline \int_{t-h(t)}^{t} x(s)\,ds \\ \hline h(t)\,x(t) - \int_{t-h(t)}^{t} x(s)\,ds \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&-\frac{3h_U}{h(t)}\begin{bmatrix} h(t)(-x(t)-x(t-h(t)))+2\int_{t-h(t)}^{t}x(s)\,ds \\ \hline h(t)\int_{t-h(t)}^{t}x(s)\,ds-2\int_{t-h(t)}^{t}\int_{s}^{t}x(u)\,du\,ds \\ \hline 2\int_{t-h(t)}^{t}\int_{s}^{t}x(u)\,du\,ds-h(t)\int_{t-h(t)}^{t}x(s)\,ds \end{bmatrix}^T \\
&\cdot\left(\frac{\Theta_{1i}}{h_U^2}\right)\begin{bmatrix} h(t)(-x(t)-x(t-h(t)))+2\int_{t-h(t)}^{t}x(s)\,ds \\ \hline h(t)\int_{t-h(t)}^{t}x(s)\,ds-2\int_{t-h(t)}^{t}\int_{s}^{t}x(u)\,du\,ds \\ \hline 2\int_{t-h(t)}^{t}\int_{s}^{t}x(u)\,du\,ds-h(t)\int_{t-h(t)}^{t}x(s)\,ds \end{bmatrix} \\
&=-\frac{h_U}{h(t)}\zeta^T(t)\,\Lambda_{1[h(t)]}^T\Omega_{1i}\Lambda_{1[h(t)]}\zeta(t),
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
\Lambda_{1[h(t)]} &= [e_1-e_2, e_6, h(t)e_1-e_6, h(t)(-e_1-e_2) \\
&\quad +2e_6, h(t)e_6-2e_8, 2e_8-h(t)e_6]^T, \\
\Omega_{1i} &= \begin{bmatrix} \Theta_{1i} & 0_{3n} \\ 0_{3n} & \dfrac{(3\Theta_{1i})}{h_U^2} \end{bmatrix}.
\end{aligned}
\tag{27}
$$

The other integral term $-h_U\int_{t-h_U}^{t-h(t)}\alpha^T(t,s)\Theta_{1i}\alpha(t,s)ds$ can be estimated as

$$
\begin{aligned}
&-h_U\int_{t-h_U}^{t-h(t)}\alpha^T(t,s)\,\Theta_{2i}\alpha(t,s)\,ds \\
&\le -\frac{h_U}{h_U-h(t)}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right)^T \\
&\cdot\Theta_{2i}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right) \\
&-\frac{3h_U}{h_U-h(t)}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right. \\
&\left.-\frac{2}{h_U-h(t)}\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\alpha(t,u)\,du\,ds\right)^T \\
&\cdot\Theta_{2i}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right. \\
&\left.-\frac{2}{h_U-h(t)}\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\alpha(t,u)\,du\,ds\right) \\
&=-\frac{h_U}{h_U-h(t)}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right)^T \\
&\cdot\Theta_{2i}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right) \\
&-\frac{3h_U}{h_U-h(t)}\left((h_U-h(t))\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right. \\
&\left.-2\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\alpha(t,u)\,du\,ds\right)^T\left(\frac{\Theta_{2i}}{(h_U-h(t))^2}\right) \\
&\cdot\left((h_U-h(t))\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right. \\
&\left.-2\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\alpha(t,u)\,du\,ds\right) \\
&\le -\frac{h_U}{h_U-h(t)}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right)^T \\
&\cdot\Theta_{2i}\left(\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right) \\
&-\frac{3h_U}{h_U-h(t)}\left((h_U-h(t))\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right. \\
&\left.-2\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\alpha(t,u)\,du\,ds\right)^T\left(\frac{\Theta_{2i}}{h_U^2}\right) \\
&\cdot\left((h_U-h(t))\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds\right. \\
&\left.-2\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\alpha(t,u)\,du\,ds\right).
\end{aligned}
\tag{28}
$$

Since

$$
\begin{aligned}
\int_{t-h_U}^{t-h(t)}\alpha(t,s)\,ds &= \begin{bmatrix} x(t-h(t))-x(t-h_U) \\ \hline \int_{t-h_U}^{t-h(t)}x(s)\,ds \\ \hline \int_{t-h_U}^{t-h(t)}\int_{s}^{t}\dot{x}(u)\,du\,ds \end{bmatrix} \\
&= [e_2-e_3, e_7, (h_U-h(t))e_1-e_7]^T\zeta(t), \\
2\int_{t-h_U}^{t-h(t)}\int_{s}^{th(t)}\int_{u}^{t}\dot{x}(v)\,dv\,du\,ds &= 2\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}(x(t)-x(u))\,du\,ds = 2 \\
&\cdot\frac{(h_U-h(t))^2}{2}x(t)-2\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}x(u)\,du\,ds \\
&\underbrace{-2\int_{t-h_U}^{t-h(t)}\int_{t-h(t)}^{t}x(u)\,du\,ds+2\int_{t-h_U}^{t-h(t)}\int_{t-h(t)}^{t}x(u)\,du\,ds}_{=0} \\
&=(h_U-h(t))^2x(t)-2\int_{t-h_U}^{t-h(t)}\int_{s}^{t}x(u)\,du\,ds \\
&\underbrace{+2\int_{t-h_U}^{t-h(t)}\int_{t-h(t)}^{t}x(u)\,du\,ds,}_{=2\cdot(h_U-h(t))\int_{t-h(t)}^{t}x(s)ds}
\end{aligned}
\tag{29}
$$

from (28) to (29), we have

$$
\begin{aligned}
&-h_U\int_{t-h_U}^{t-h(t)}\alpha^T(t,s)\,\Theta_{2i}\alpha(t,s)\,ds \le -\frac{h_U}{h_U-h(t)}\begin{bmatrix} x(t-h(t))-x(t-h_U) \\ \dfrac{\int_{t-h_U}^{t-h(t)} x(s)\,ds}{(h_U-h(t))\,x(t)-\int_{t-h_U}^{t-h(t)} x(s)\,ds}\end{bmatrix}^T \\
&\cdot\Theta_{2i}\begin{bmatrix} x(t-h(t))-x(t-h_U) \\ \dfrac{\int_{t-h_U}^{t-h(t)} x(s)\,ds}{(h_U-h(t))\,x(t)-\int_{t-h_U}^{t-h(t)} x(s)\,ds}\end{bmatrix} \\
&-\frac{3h_U}{h_U-h(t)}\begin{bmatrix} (h_U-h(t))\left(-x(t-h(t))-x(t-h_U)\right)+2\int_{t-h_U}^{t-h(t)} x(s)\,ds \\ \dfrac{(h_U-h(t))\int_{t-h_U}^{t-h(t)} x(s)\,ds-2\int_{t-h_U}^{t-h(t)}\int_s^t x(u)\,du\,ds+2(h_U-h(t))\int_{t-h(t)}^t x(s)\,ds}{-(h_U-h(t))\left(2\int_{t-h(t)}^t x(s)\,ds+\int_{t-h_U}^{t-h(t)} x(s)\,ds\right)+2\int_{t-h(t)}^t\int_s^t x(u)\,du\,ds}\end{bmatrix}^T\begin{pmatrix}\Theta_{2i} \\ h_U^2\end{pmatrix} \\
&\cdot\begin{bmatrix} (h_U-h(t))\left(-x(t-h(t))-x(t-h_U)\right)+2\int_{t-h_U}^{t-h(t)} x(s)\,ds \\ \dfrac{(h_U-h(t))\int_{t-h_U}^{t-h(t)} x(s)\,ds-2\int_{t-h_U}^{t-h(t)}\int_s^t x(u)\,du\,ds+2(h_U-h(t))\int_{t-h(t)}^t x(s)\,ds}{-(h_U-h(t))\left(2\int_{t-h(t)}^t x(s)\,ds+\int_{t-h_U}^{t-h(t)} x(s)\,ds\right)+2\int_{t-h(t)}^t\int_s^t x(u)\,du\,ds}\end{bmatrix} = -\frac{h_U}{h_U-h(t)}\zeta^T(t) \\
&\cdot\Lambda_{2[h(t)]}^T\Omega_{2i}\Lambda_{2[h(t)]}\zeta(t),
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
\Lambda_{2[h(t)]} = \big[&e_2-e_3, e_7, (h_U-h(t))\,e_1 \\
&-e_7, (h_U-h(t))(-e_2-e_3)+2e_7, (h_U-h(t))\,e_7 \\
&-2e_9, -(h_U-h(t))(2e_6+e_7)+2e_9\big]^T,
\end{aligned}
\tag{31}
$$

$$
\Omega_{2i} = \begin{bmatrix} \Theta_{2i} & 0_{3n} \\ 0_{3n} & \dfrac{(3\Theta_{2i})}{h_U^2}\end{bmatrix}.
$$

From (22) to (30), by utilizing reciprocally convex optimization approach [9], it can be confirmed that

$$
\begin{aligned}
&\mathscr{L}V_4(x_t,i)-2h_U\int_{t-h(t)}^t x^T(s)\,P_1\dot{x}(s)\,ds \\
&-2h_U\int_{t-h_U}^{t-h(t)} x^T(s)\,P_2\dot{x}(s)\,ds \\
&+\int_{t-h_U}^t \alpha^T(t,s)\left(\sum_{j=1}^s \pi_{ij}\mathcal{N}_j\right)\alpha(t,s)\,ds \\
&+\int_{t-h(t)}^t \beta^T(t,s)\left(\sum_{j=1}^s \pi_{ij}\mathcal{G}_j\right)\beta(t,s)\,ds \\
&\le \zeta^T(t)\left(\Xi_4-\Lambda_{3[h(t)]}^T\Omega_{3i}\Lambda_{3[h(t)]}\right)\zeta(t),
\end{aligned}
\tag{32}
$$

where $\Lambda_{3[h(t)]} = [\Lambda_{1[h(t)]}^T, \Lambda_{2[h(t)]}^T]^T$ and $\Omega_{3i} = \begin{bmatrix}\Omega_{1i} & \mathcal{S} \\ \mathcal{S}^T & \Omega_{2i}\end{bmatrix} > 0$.

From (13) to (32), an upper bound of $\mathscr{L}V(x_t,i)$ with the addition of (21) can be represented as

$$
\begin{aligned}
&\mathscr{L}V(x_t,i)+\zeta^T(t)\,\Xi_5\zeta(t) \\
&-2h_U\int_{t-h(t)}^t x^T(s)\,P_1\dot{x}(s)\,ds \\
&-2h_U\int_{t-h_U}^{t-h(t)} x^T(s)\,P_2\dot{x}(s)\,ds \le \zeta^T(t)
\end{aligned}
$$

$$\cdot \left\{ \underbrace{\Xi_{1i} + \Xi_{2i} + \Xi_{3i[h(t)]} + \Xi_4 + \Xi_5}_{\Sigma_{i[h(t)]}} - \Lambda_{3[h(t)]}^T \Omega_{3i} \Lambda_{3[h(t)]} \right\} \zeta(t). \tag{33}$$

By utilizing Lemma 3, the following inequality

$$\zeta^T(t) \left(\Sigma_{i[h(t)]} - \Lambda_{3[h(t)]}^T \Omega_{3i} \Lambda_{3[h(t)]} \right) \zeta(t) < 0 \tag{34}$$

subject to $0 = \Upsilon_i \zeta(t)$ is equivalent to

$$\left(\Upsilon_i^{\perp}\right)^T \left(\Sigma_{i[h(t)]} - \Lambda_{3[h(t)]}^T \Omega_{3i} \Lambda_{3[h(t)]} \right) \Upsilon_i^{\perp} < 0. \tag{35}$$

By Lemma 4, condition (35) can be casted into the following inequality with an appropriate dimension Ψ:

$$\left[\begin{array}{c|c} \left(\Upsilon_i^{\perp}\right)^T \left(\Sigma_{i[h(t)]}\right) \Upsilon_i^{\perp} + \mathrm{Sym}\left\{ \left(\Upsilon_i^{\perp}\right)^T \left(\Lambda_{3[h(t)]}\right)^T \Psi \right\} & \Psi^T \\ \hline \Psi & -\Omega_{3i} \end{array} \right] < 0. \tag{36}$$

It should be noted that inequality (36) is affinely dependent on $h(t)$. Therefore, if inequalities (12) hold for $k = 1, 2$, then inequality (36) is satisfied for $0 \le h(t) \le h_U$. Furthermore, one can see that $\Omega_{3i} > 0$ holds if inequalities (12) are satisfied. Therefore, if condition (12) holds, then there exists a sufficiently small positive scalar ε such that $\mathscr{L}V(x_t, i) < -\varepsilon \|x(t)\|^2$. Thus, by using the similar method in [33] and Definition 1, system (1) is stochastically stable. This completes our proof. □

In many cases, the information about an upper bound of $\dot{h}(t)$ is unknown. For this case, based on the result of Theorem 5, the corresponding stability condition will be presented in Corollary 6. In Corollary 6, for simplicity of matrix notations, some of vectors and matrices are redefined as

$$\begin{aligned} \widetilde{\Theta}_{1i} &= \mathcal{Q}_{\mathrm{aug1}} - \left(\frac{1}{h_U}\right) \sum_{j=1}^{s} \pi_{ij} \mathcal{N}_j, \\ \widetilde{\Omega}_{1i} &= \begin{bmatrix} \widetilde{\Theta}_{1i} & 0_{3n} \\ 0_{3n} & \dfrac{\left(3\widetilde{\Theta}_{1i}\right)}{h_U^2} \end{bmatrix}, \\ \widetilde{\Omega}_{3i} &= \begin{bmatrix} \widetilde{\Omega}_{1i} & \mathcal{S} \\ \mathcal{S}^T & \Omega_{2i} \end{bmatrix}, \\ \widetilde{\Sigma}_{i[h(t)]} &= \Xi_{1i} + \Xi_{2i} + \Xi_4 + \Xi_5, \\ \widetilde{\Phi}_{i[h(t)]} &= \left(\Upsilon_i^{\perp}\right)^T \left(\widetilde{\Sigma}_{i[h(t)]}\right) \left(\Upsilon_i^{\perp}\right) \\ &\quad + \mathrm{Sym}\left\{ \left(\Upsilon_i^{\perp}\right)^T \left(\Lambda_{3i[h(t)]}\right)^T \Psi \right\}. \end{aligned} \tag{37}$$

Except the above notations, all the notations defined in (11) will be used in Corollary 6. Now, the following result is given by Corollary 6.

Corollary 6. *For a given scalar $h_U > 0$, system (1) is stochastically stable for $0 \le h(t) \le h_U$ if there exist positive definite matrices $\mathcal{R}_i \in \mathbb{R}^{4n\times 4n}$, $\mathcal{N}_i \in \mathbb{R}^{3n\times 3n}$, and $\mathcal{Q} \in \mathbb{R}^{3n\times 3n}$, any matrices $\mathcal{S} \in \mathbb{R}^{6n\times 6n}$ and $\Psi \in \mathbb{R}^{12n\times 8n}$, and any symmetric matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ satisfying the following LMIs for all $r(t) = i$, $i \in \mathcal{S}$:*

$$\begin{bmatrix} \widetilde{\Phi}_{i,k} & \Psi^T \\ \Psi & -\widetilde{\Omega}_{3i} \end{bmatrix} < 0, \quad k = 1, 2, \tag{38}$$

where $\{\widetilde{\Phi}_{i,k}\}_{k=1}^2$ means the two vertices of $\widetilde{\Phi}_{i[h(t)]}$ with the bounds of $0 \le h(t) \le h_M$. That is, $\widetilde{\Phi}_{i,1} = \widetilde{\Phi}_{i[h(t)=0]}$ and $\widetilde{\Phi}_{i,2} = \widetilde{\Phi}_{i[h(t)=h_U]}$.

Proof. Let us choose LKF as

$$\begin{aligned} V(x_t, i) &= \eta^T(t) \mathcal{R}_i \eta(t) + \int_{t-h_U}^{t} \alpha^T(t,s) \mathcal{N}_i \alpha(t,s)\, ds \\ &\quad + \int_{t-h_U}^{t} \int_{s}^{t} \alpha^T(t,u) \mathcal{Q} \alpha(t,u)\, du\, ds. \end{aligned} \tag{39}$$

The proof of Corollary 6 is very similar to the proof of Theorem 5. Thus, it is omitted. This completes our proof. □

Remark 7. Theorem 5 and Corollary 6 are derived based on the result of [25]. LKFs $V_5(x_t, i)$, $V_6(x_t, i)$ of Theorem 1 in [25] are not included in this paper. Instead, an upper bound of $\mathscr{L}V_4(x_t, i)$ is derived by utilizing Wirtinger-based integral inequality.

Remark 8. Unlike the previous results [12–14], the integral terms $\int_{t-h(t)}^{t} x(s)ds$, $\int_{t-h_U}^{t-h(t)} x(s)ds$, $\int_{t-h(t)}^{t} \int_{s}^{t} x(u)du\, ds$, and $\int_{t-h_U}^{t-h(t)} \int_{s}^{t} x(u)du\, ds$ which were utilized as elements of augmented vector $\zeta(t)$ are not multiplied by $1/h(t)$ or $1/(h_U - h(t))$. As shown in [10], the appearance of the terms $(1/h(t)) \int_{t-h(t)}^{t} x(s)ds$ and $(1/(h_U - h(t))) \int_{t-h(t)}^{t} x(s)ds$ is unavoidable in utilizing Wirtinger-based integral inequality. However, with the terms $(1/h(t)) \int_{t-h(t)}^{t} x(s)ds$ and $(1/(h_U - h(t))) \int_{t-h_U}^{t-h(t)} x(s)ds$ as elements of augmented vector, the derivation of $\mathscr{L}\{\eta^T(t)\mathcal{R}_i\eta(t)\} + \mathscr{L}\{\int_{t-h_U}^{t} \alpha^T(t,s)\mathcal{N}_i\alpha(t,s)ds\}$ is more difficult than the case of the terms $\int_{t-h(t)}^{t} x(s)ds$ and $\int_{t-h(t)}^{t} x(s)ds$ as elements of augmented vector. In this paper, with the process shown in (26), the utilized integral terms in augmented vector are not multiplied by $1/h(t)$ or $1/(h_U - h(t))$.

4. Numerical Examples

In this section, three numerical examples are introduced to show the improvements of the proposed methods. In the

Table 1: Maximum delay bounds h_U with $h_D = 0$ and various π_{11} (Example 1).

Methods	$\pi_{11} = -0.1$	$\pi_{11} = -0.5$	$\pi_{11} = -0.8$	$\pi_{11} = -1$
[18]	0.6797	0.5794	0.5562	0.5465
[19] ($m = 5$)	0.8232	0.7327	0.7039	0.6934
[22] ($m = 2$)	1.2550	0.8816	0.8065	0.7783
[25]	1.2132	0.9797	0.9345	0.8986
Theorem 5	1.3954	1.1138	1.0566	1.0367

*m is delay-partitioning number.

examples, MATLAB, YALMIP, and SeDuMi 1.3 are used to solve LMI problems.

Example 1. Consider Markovian jump system (1) with the parameters

$$A_1 = \begin{bmatrix} -3.4888 & 0.8057 \\ -0.6451 & -3.2684 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2.4898 & 0.2895 \\ 1.3396 & -0.0211 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.8620 & -1.2919 \\ -0.6841 & -2.0729 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} -2.8306 & 0.4978 \\ -0.8436 & -1.0115 \end{bmatrix}. \tag{40}$$

In Table 1, when $\pi_{22} = -0.8$ and $h_D = 0$, the obtained maximum delay bounds by Theorem 5 are compared with some recent results and [25] under some various π_{11}. From Table 1, one can see that Theorem 5 significantly improves the feasible region of stability, which shows the advantages of the proposed Theorem 5.

Example 2. Consider Markovian jump system (1) where

$$A_1 = \begin{bmatrix} -2.3 & 0.8 \\ 1.0 & -2.9 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.9 & 0.2 \\ 0.6 & -0.8 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} 0.8 & 1.2 \\ 0.7 & -3.5 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} -1.3 & -2.6 \\ 0.5 & -1.4 \end{bmatrix}. \tag{41}$$

When h_D is unknown and $\pi_{22} = -0.8$, in Table 2, maximum delay bounds obtained by Corollary 6 are compared with those of [18, 21, 25]. Table 2 shows the less conservatism of Corollary 6.

Table 2: Maximum delay bounds h_U with unknown h_D and various π_{11} (Example 2).

Methods	$\pi_{11} = -0.1$	$\pi_{11} = -0.5$	$\pi_{11} = -0.8$	$\pi_{11} = -1$
[18]	0.271	0.271	0.271	0.271
[21]	0.500	0.496	0.493	0.492
[25]	0.6003	0.5909	0.5862	0.5836
Corollary 6	0.6209	0.6166	0.6152	0.6146

Table 3: Maximum delay bounds h_U with $h_D = 0.9$ and various π_{11} (Example 3).

Methods	$\pi_{11} = -0.1$	$\pi_{11} = -0.5$	$\pi_{11} = -0.8$	$\pi_{11} = -1$
[18]	1.0224	1.0148	1.0141	1.0130
[26]	1.3671	1.3565	1.3541	1.3535
[25]	1.7858	1.7006	1.6803	1.6713
Theorem 5	1.8270	1.7320	1.7093	1.6999

Example 3. Consider Markovian jump system (1) with the parameters

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -1 & 0.5 \\ 0.1 & -1 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}. \tag{42}$$

In Table 3, the results of maximum delay bounds h_U obtained by Theorem 5 with $h_D = 0.9$, $\pi_{22} = -0.8$, and various π_{11} are listed and some recent results [18, 25, 26] are also listed. The results in Table 3 also show that Theorem 5 provides larger delay bound than those of very recent results such as [26].

Example 4. Consider Markovian jump system (1) with the parameters

$$A_1 = \begin{bmatrix} -3.5 & 0.8 \\ -0.6 & -3.3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2.5 & 0.3 \\ 1.4 & -0.1 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.9 & -1.3 \\ -0.7 & -2.1 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} -2.8 & 0.5 \\ -0.8 & -1.0 \end{bmatrix}. \tag{43}$$

In Table 4, when $h_D = 0$ and $\pi_{22} = -0.8$, maximum delay bounds h_U obtained by Theorem 5 are listed and compared

Table 4: Maximum delay bounds h_U with $h_D = 0$ and various π_{11} (Example 4).

Methods	$\pi_{11} = -0.4$	$\pi_{11} = -0.55$	$\pi_{11} = -0.7$	$\pi_{11} = -0.85$	$\pi_{11} = -1.00$
[20]	0.6708	0.5894	0.5768	0.5675	0.5603
[23]	0.6322	0.6120	0.5981	0.5881	0.5805
[25]	1.0328	0.9933	0.9681	0.9523	0.9429
Theorem 5	1.1826	1.1335	1.1016	1.0799	1.0650

with those of [20, 23, 26] for various π_{11}. From the result of Table 4, the superiority of Theorem 5 can be verified.

5. Conclusion

In this paper, further improved results on stability for Markovian jump systems with time-varying delays were proposed in Theorem 5 and Corollary 6. With simple LKFs comparing with [25], it was shown that from three numerical examples, all the results obtained by Theorem 5 and Corollary 6 are larger than those of [25] by applying Wirtinger-based integral inequality and some new techniques to $\mathscr{L}V_4(x_t, i)$. With the ideas proposed in this paper, stability and stabilization for various systems such as multiagent systems, complex networks, and neural networks will be conducted in future works.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0009273). This work was also supported by the Human Resources Development of the Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea government Ministry of Trade, Industry & Energy (no. 20144030200450).

References

[1] J.-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.

[2] S. Xu and J. Lam, "A survey of linear matrix inequality techniques in stability analysis of delay systems," *International Journal of Systems Science. Principles and Applications of Systems and Integration*, vol. 39, no. 12, pp. 1095–1113, 2008.

[3] J. Sun, G. P. Liu, and J. Chen, "Delay-dependent stability and stabilization of neutral time-delay systems," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 12, pp. 1364–1375, 2009.

[4] W. Qian, S. Cong, T. Li, and S. Fei, "Improved stability conditions for systems with interval time-varying delay," *International Journal of Control, Automation and Systems*, vol. 10, no. 6, pp. 1146–1152, 2012.

[5] O.-M. Kwon, M.-J. Park, S.-M. Lee, J. H. Park, and E.-J. Cha, "Stability for neural networks with time-varying delays via some new approaches," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 24, no. 2, pp. 181–193, 2013.

[6] R. Rakkiyappan, Q. Zhu, and A. Chandrasekar, "Stability of stochastic neural networks of neutral type with Markovian jumping parameters: a delay-fractioning approach," *Journal of the Franklin Institute*, vol. 351, no. 3, pp. 1553–1570, 2014.

[7] K. Gu, "A further refinement of discretized Lyapunov functional method for the stability of time-delay systems," *International Journal of Control*, vol. 74, no. 10, pp. 967–976, 2001.

[8] Y. He, M. Wu, J.-H. She, and G.-P. Liu, "Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays," *Systems and Control Letters*, vol. 51, no. 1, pp. 57–65, 2004.

[9] P. Park, J. W. Ko, and C. Jeong, "Reciprocally convex approach to stability of systems with time-varying delays," *Automatica*, vol. 47, no. 1, pp. 235–238, 2011.

[10] A. Seuret and F. Gouaisbaut, "Wirtinger-based integral inequality: application to time-delay systems," *Automatica*, vol. 49, no. 9, pp. 2860–2866, 2013.

[11] J. Cheng and L. Xiong, "Improved integral inequality approach on stabilization for continuous-time systems with time-varying input delay," *Neurocomputing*, vol. 160, pp. 274–280, 2015.

[12] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, and E. J. Cha, "Analysis on passivity for uncertain neural networks with time-varying delays," *Mathematical Problems in Engineering*, vol. 2014, Article ID 602828, 10 pages, 2014.

[13] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, and E. J. Cha, "On less conservative stability criteria for neural networks with time-varying delays utilizing Wirtinger-based integral inequality," *Mathematical Problems in Engineering*, vol. 2014, Article ID 859736, 13 pages, 2014.

[14] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, and E. J. Cha, "Improved results on stability of linear systems with time-varying delays via Wirtinger-based integral inequality," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 351, no. 12, pp. 5386–5398, 2014.

[15] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 1, pp. 38–53, 1992.

[16] X.-D. Zhao and Q.-S. Zeng, "Delay-dependent stability analysis for Markovian jump systems with interval time-varying-delays," *International Journal of Automation and Computing*, vol. 7, no. 2, pp. 224–229, 2010.

[17] P. Shi and F. Li, "A survey on Markovian jump systems: modeling and design," *International Journal of Control, Automation and Systems*, vol. 13, no. 1, pp. 1–16, 2015.

[18] S. Xu, J. Lam, and X. Mao, "Delay-dependent H_∞ control and filtering for uncertain Markovian jump systems with time-varying delays," *IEEE Transactions on Circuits and Systems. I. Regular Papers*, vol. 54, no. 9, pp. 2070–2077, 2007.

[19] Z. Fei, H. Gao, and P. Shi, "New results on stabilization of Markovian jump systems with time delay," *Automatica*, vol. 45, no. 10, pp. 2300–2306, 2009.

[20] Z. Wu, H. Su, and J. Chu, "H_∞ filtering for singular Markovian jump systems with time delay," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 8, pp. 939–957, 2010.

[21] Z. Li, Z. Fei, and H. Gao, "Stability and stabilisation of Markovian jump systems with time-varying delay: an input-output approach," *IET Control Theory and Applications*, vol. 6, no. 17, pp. 2601–2610, 2012.

[22] H. Huang, G. Feng, and X. Chen, "Stability and stabilization of Markovian jump systems with time delay via new Lyapunov functionals," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 59, no. 10, pp. 2413–2421, 2012.

[23] J. Wang, H. Wang, A. Xue, and R. Lu, "Delay-dependent H_∞ control for singular Markovian jump systems with time delay," *Nonlinear Analysis: Hybrid Systems*, vol. 8, pp. 1–12, 2013.

[24] Q. Zhu, "Stabilization of stochastically singular nonlinear jump systems with unknown parameters and continuously distributed delays," *International Journal of Control, Automation and Systems*, vol. 11, no. 4, pp. 683–691, 2013.

[25] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, and E. J. Cha, "Stability and H_∞ performance analysis for Markovian jump systems with time-varying delays," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 351, no. 10, pp. 4724–4748, 2014.

[26] J. Qiu, Y. Wei, and H. R. Karimi, "New approach to delay-dependent H_∞ control for continuoustime Markovian jump systems with time-varying delay and deficient transition descriptions," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 352, no. 1, pp. 189–215, 2015.

[27] J. Cheng, H. Zhu, S. Zhong, Q. Zhong, and Y. Zeng, "Finite-time H_∞ estimation for discretetime Markov jump systems with time-varying transition probabilities subject to average dwell time switching," *Communications in Nonlinear Science and Numerical Simulation*, vol. 20, no. 2, pp. 571–582, 2015.

[28] Z. Shu, J. Lam, and S. Xu, "Robust stabilization of Markovian delay systems with delay-dependent exponential estimates," *Automatica*, vol. 42, no. 11, pp. 2001–2008, 2006.

[29] E. K. Boukas and Z. K. Liu, *Deterministic and Stochastic Time Delay Systems*, Birkhäuser, Boston, Mass, USA, 2002.

[30] R. E. Skelton, T. Iwasaki, and K. M. Grigoradis, *A Unified Algebraic Approach to Linear Control Design*, Taylor and Francis, New York, NY, USA, 1997.

[31] C. Briat, "Convergence and equivalence results for the Jensen's inequality—application to time-delay and sampled-data systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1660–1665, 2011.

[32] S. H. Kim, P. Park, and C. Jeong, "Robust H_∞ stabilisation of networked control systems with packet analyser," *IET Control Theory and Applications*, vol. 4, no. 9, pp. 1828–1837, 2010.

[33] M. J. Park, O. M. Kwon, J. H. Park, and S. M. Lee, "Simplified stability criteria for fuzzy Markovian jumping Hopfield neural networks of neutral type with interval time-varying delays," *Expert Systems with Applications*, vol. 39, no. 5, pp. 5625–5633, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 854957, 14 pages
http://dx.doi.org/10.1155/2015/854957

Research Article

H_∞ Gain-Scheduled Control for LPV Stochastic Systems

Cheung-Chieh Ku and Guan-Wei Chen

Department of Marine Engineering, National Taiwan Ocean University, Keelung 202, Taiwan

Correspondence should be addressed to Cheung-Chieh Ku; ccku@mail.ntou.edu.tw

Received 15 April 2015; Revised 11 June 2015; Accepted 16 June 2015

Academic Editor: Quanxin Zhu

A robust control problem for discrete-time uncertain stochastic systems is discussed via gain-scheduled control scheme subject to H_∞ attenuation performance. Applying Linear Parameter Varying (LPV) modeling approach and stochastic difference equation, the uncertain stochastic systems can be described by combining time-varying weighting function and linear systems with multiplicative noise terms. Due to the consideration of stochastic behavior, the stability in the sense of mean square is applied for the system. Furthermore, two kinds of Lyapunov functions are employed to derive their corresponding sufficient conditions to solve the stabilization problems of this paper. In order to use convex optimization algorithm, the derived conditions are converted into Linear Matrix Inequality (LMI) form. Via solving those conditions, the gain-scheduled controller can be established such that the robust asymptotical stability and H_∞ performance of the disturbed uncertain stochastic system can be achieved in the sense of mean square. Finally, two numerical examples are applied to demonstrate the effectiveness and applicability of the proposed design method.

1. Introduction

In control problems, accurate parameters of dynamic system are always important premised assumption. Unfortunately, the accurate parameters are hardly to be obtained in practical applications due to modeling errors and natural perturbations. For this reason, robust control schemes [1–12] were proposed to guarantee stability of dynamic system with admissible uncertainties. Through [1–3], the uncertainty of system is described by norm bounded time-varying function. On the other hand, based on LPV modeling approach [4–6], the uncertain systems can be interpreted by combining several subsystems and chosen time-varying weighting function. Referring to [4–6, 9], LPV system can be established to completely represent the uncertain systems by using LPV modeling approach. Furthermore, Lyapunov stability theory has been widely applied for stability analysis and synthesis of LPV systems. In the Lyapunov stability theory, the choice of Lyapunov function to present the system energy is an important issue that will influence conservatism of the derived stability criterion. Generally, Parameter Independent Lyapunov Function (PILF) [4, 7] and Parameter Dependent Lyapunov Function (PDLF) [10, 11] are applied to propose their stability criteria for LPV systems. In this paper, both PILF and PDLF are, respectively, applied to derive the corresponding stability criterion for the considered LPV system.

Referring to the literature [13, 14], gain-scheduled design scheme provides powerful tool to deal with stabilization problems of the LPV systems. Moreover, based on the gain-scheduled design scheme, robustness of LPV systems can be increased due to a gain-table designed by numerous operation points. Besides, it is well known that external disturbance often causes poor control performance and unstable source of controlled systems. Therefore, H_∞ gain-scheduled controller design methods have been proposed by [8, 14–16] to constrain the effect of external disturbance on LPV systems. With H_∞ control theory, the performance index can cope with the worst case as the effect of external disturbance.

Practically, stochastic behavior of dynamic systems often appears around operating environment. Due to unmeasurable and unpredictable property, stability of stochastic system is difficult to be analyzed and discussed. Referring

to [17, 18], the stochastic behavior is considered as external disturbance or unknown perturbation. On the other hand, stochastic difference equation has been proposed by [19] to formulate the stochastic behavior into multiplicative noise term expressed by multiplication of states and noises. Via the multiplicative noise term, the stochastic behavior of system is more representative and understandable than that described by disturbance or perturbation. Therefore, many efforts [20–27] have been developed to discuss stability analysis and synthesis of stochastic systems. Referring to [26, 27], the uncertainty is described by specific norm bounded time-varying function that limits the description of uncertain stochastic system. In order to extend stability criterion to uncertain stochastic systems, the LPV stochastic system is proposed and considered in this paper.

To the best of our knowledge, there have been less works on discussing robust stabilization problems of the LPV stochastic systems subject to H_∞ performance. The main purpose of this paper is thus to develop the gain-scheduled controller design methods for the LPV stochastic systems. According to the consideration of stochastic behavior, the robust stability criterion proposed in this paper is more general than the one in [4, 8, 14]. And both PILF and PDLF are applied to derive their corresponding sufficient conditions that are converted into the LMI form. Via solving those conditions, the feasible solutions can be obtained to establish the corresponding gain-scheduled controller to guarantee the asymptotical stability and H_∞ performance of the LPV stochastic system in the sense of mean square. For discussing the conservatism of proposed design methods, a numerical example is proposed to find the minimum performance index of the derived conditions. In addition, a ship autopilot servosystem is proposed to show the effectiveness and applicability of the proposed design methods.

The paper is organized as follows. In Section 2, disturbed discrete-time LPV stochastic systems and its stabilization problems are described. The gain-scheduled H_∞ controller design method is proposed in Section 3. And less conservative stability criterion is proposed in Section 4. Finally, two numerical examples are employed to demonstrate effectiveness and application of the proposed design methods in Section 5. Some conclusions are stated in Section 6.

2. Systems Description and Problem Formulation

In this section, the following discrete-time disturbed uncertain stochastic system is proposed:

$$\begin{aligned} x(t+1) &= \mathbf{A}(\alpha(t))\,x(t) + \mathbf{B}(\alpha(t))\,u(t) + \mathbf{E}(\alpha(t)) \\ &\quad \cdot w(t) \\ &\quad + \left(\overline{\mathbf{A}}(\alpha(t))\,x(t) + \overline{\mathbf{B}}(\alpha(t))\,u(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right) \\ &\quad \cdot \beta(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $w(t) \in \mathbb{R}^p$ is the exogenous disturbance input, and $\beta(t)$ is a discrete type scalar Brownian motion satisfying the independent increment property [19]; that is, $E\{x(t)\beta(t)\} = 0$. And the covariance of $\beta(t)$ can be assumed as $E\{\beta^T(t)\beta(t)\} = \tau^2$, where the τ is intensity level of the motion. $E\{\cdot\}$ denotes the expected value of $\cdot$. $\mathbf{A}(\alpha(t)) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(\alpha(t)) \in \mathbb{R}^{n\times m}$, $\mathbf{E}(\alpha(t)) \in \mathbb{R}^{n\times p}$, $\overline{\mathbf{A}}(\alpha(t)) \in \mathbb{R}^{n\times n}$, $\overline{\mathbf{B}}(\alpha(t)) \in \mathbb{R}^{n\times m}$, and $\overline{\mathbf{E}}(\alpha(t)) \in \mathbb{R}^{n\times p}$ which are matrices depending on time-varying parameters vector $\alpha(t) = [\alpha_1(t)\ \alpha_2(t)\ \cdots\ \alpha_r(t)]$. Referring to [12], the time-varying parameter $\alpha(t)$ can be expressed as a convex combination. Thus, the matrices of system (1) depending on the $\alpha(t)$ can be reconstructed by the following equation:

$$\begin{aligned} &\begin{bmatrix} \mathbf{A}(\alpha(t)) & \mathbf{B}(\alpha(t)) & \mathbf{E}(\alpha(t)) \\ \overline{\mathbf{A}}(\alpha(t)) & \overline{\mathbf{B}}(\alpha(t)) & \overline{\mathbf{E}}(\alpha(t)) \end{bmatrix} \\ &\quad = \sum_{i=1}^{N} \vartheta_i(t) \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i & \mathbf{E}_i \\ \overline{\mathbf{A}}_i & \overline{\mathbf{B}}_i & \overline{\mathbf{E}}_i \end{bmatrix}, \end{aligned} \tag{2}$$

where $N = 2^r$ and $\vartheta_i(t)$ is measurable at each time instant. Moreover, $\vartheta_i(t)$ satisfies $\sum_{i=1}^{N} \vartheta_i(t) = 1$ and $0 \le \vartheta_i(t) \le 1$. The $\mathbf{A}_i, \mathbf{B}_i, \mathbf{E}_i, \overline{\mathbf{A}}_i, \overline{\mathbf{B}}_i$, and $\overline{\mathbf{E}}_i$ are constant matrices with appropriate dimensions. Based on (2), system (1) can be rewritten as follows:

$$\begin{aligned} x(t+1) &= \sum_{i=1}^{N} \vartheta_i(t) \Big(\mathbf{A}_i x(t) + \mathbf{B}_i u(t) + \mathbf{E}_i w(t) \\ &\quad + \left(\overline{\mathbf{A}}_i x(t) + \overline{\mathbf{B}}_i u(t) + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\Big). \end{aligned} \tag{3}$$

Based on the LPV modeling approach and stochastic difference equation, the LPV stochastic system (3) is structured to substitute the uncertain stochastic system (1) to develop gain-scheduled controller design methods. Moreover, the design methods are proposed to satisfy the H_∞ performance such as

$$E\left\{\sum_{0}^{t_f} x^T(t)\,\mathbf{S}x(t)\right\} < E\left\{\eta^2 \sum_{0}^{t_f} w^T(t)\,w(t)\right\} \tag{4}$$

for $w(t) \neq 0$ and $x(0) = 0$, in which t_f is the terminal time of control, η is a prescribed value which denotes the worst case effect of $w(t)$ on $x(t)$, and $\mathbf{S}$ is a positive definite weighting matrix. Besides, in case such as $w(t) = 0$, the robust stability of (3) is an important issue. The concerned stability of (3) is thus provided as the following definition by the sense of mean square [20, 23].

Definition 1. For LPV stochastic system (3) with zero external disturbance $w(t) = 0$, the solution with admissible robust uncertainties is asymptotically mean square stable if $E\{x(t)\}$ and state correlation matrix $E\{x^T(t)x(t)\}$ are converged to zero as $t \to \infty$.

In next section, both PILF and PDLF are applied to derive their corresponding sufficient condition into LMI problem for applying the convex optimization algorithm [28, 29]. Through solving the condition, the gain-scheduled controller can be established to achieve robust asymptotical stability and H_∞ performance of (3) in the sense of mean square.

3. Stability Criterion for Disturbed LPV Stochastic Systems

In this section, the gain-scheduled control scheme [14] is employed to discuss the stabilization problem of (3). Thus, the following gain-scheduled controller is proposed:

$$u(t) = -\mathbf{F}(\alpha(t))\,x(t) \tag{5a}$$

or

$$u(t) = \sum_{j=1}^{N} \vartheta_j(t)\left(-\mathbf{F}_j x(t)\right). \tag{5b}$$

Substituting (5a)-(5b) into (1), the following closed-loop system can be inferred:

$$\begin{aligned} x(t+1) &= (\mathbf{A}(\alpha(t)) - \mathbf{B}(\alpha(t))\,\mathbf{F}(\alpha(t)))\,x(t) \\ &\quad + \mathbf{E}(\alpha(t))\,w(t) + \left(\left(\overline{\mathbf{A}}(\alpha(t)) - \overline{\mathbf{B}}(\alpha(t))\,\mathbf{F}(\alpha(t))\right)\right. \\ &\quad \left.\cdot\, x(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right)\beta(t) = \mathbf{R}(\alpha(t))\,x(t) \\ &\quad + \mathbf{E}(\alpha(t))\,w(t) + \left(\overline{\mathbf{R}}(\alpha(t))\,x(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right) \\ &\quad \cdot\,\beta(t) = \sum_{i=1}^{N}\vartheta_i(t)\left(\left(\mathbf{A}_i - \mathbf{B}_i\sum_{j=1}^{N}\vartheta_j(t)\,\mathbf{F}_j\right)x(t)\right. \\ &\quad + \mathbf{E}_i w(t) \\ &\quad \left. + \left(\left(\overline{\mathbf{A}}_i - \overline{\mathbf{B}}_i\sum_{j=1}^{N}\vartheta_j(t)\,\mathbf{F}_j\right)x(t) + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\vartheta_i(t)\,\vartheta_j(t)\left(\mathbf{R}_{ij}x(t) + \mathbf{E}_i w(t) + \left(\overline{\mathbf{R}}_{ij}x(t)\right.\right. \\ &\quad \left.\left. + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\right), \end{aligned} \tag{6}$$

where $\mathbf{R}(\alpha(t)) = \mathbf{A}(\alpha(t)) - \mathbf{B}(\alpha(t))\mathbf{F}(\alpha(t))$, $\overline{\mathbf{R}}(\alpha(t)) = \overline{\mathbf{A}}(\alpha(t)) - \overline{\mathbf{B}}(\alpha(t))\mathbf{F}(\alpha(t))$, $\mathbf{R}_{ij} = \mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j$, and $\overline{\mathbf{R}}_{ij} = \overline{\mathbf{A}}_i - \overline{\mathbf{B}}_i\mathbf{F}_j$. For closed-loop system (6), the following sufficient condition is derived via the PILF.

Theorem 2. *With given positive scalars τ and η, if there exist gains $\mathbf{F}_j$, positive definite matrices $\mathbf{P}$ and $\mathbf{S}$, and value $\eta > 0$ satisfying the following inequality, the robust asymptotical stability and H_∞ performance of the closed-loop system (6) are achieved in the sense of mean square:*

$$\begin{bmatrix} -\mathbf{Q} & * & * & * & * \\ 0 & -\eta^2\mathbf{I} & * & * & * \\ \mathbf{A}_i\mathbf{Q} - \mathbf{B}_i\mathbf{Y}_j & \mathbf{E}_i & -\mathbf{Q} & * & * \\ \tau\left(\overline{\mathbf{A}}_i\mathbf{Q} - \overline{\mathbf{B}}_i\mathbf{Y}_j\right) & \tau\left(\overline{\mathbf{E}}_i\right) & 0 & -\mathbf{Q} & * \\ \mathbf{Q} & 0 & 0 & 0 & -\mathbf{U} \end{bmatrix} < 0, \quad \text{for } i, j = 1, 2, \ldots, N, \tag{7}$$

where $\mathbf{Q} = \mathbf{P}^{-1}$, $\mathbf{Y}_j = \mathbf{F}_j\mathbf{Q}$, $\mathbf{U} = \mathbf{S}^{-1}$, the $*$ denotes the transposed elements of the symmetric position, and $\mathbf{I}$ denotes identity matrix.

Proof. Choosing a Lyapunov function as $V(x(t)) = x^T(t)\mathbf{P}x(t)$, one can obtain first forward difference of the $V(x(t))$

$$\begin{aligned} \Delta V(x(t)) &= V(x(t+1)) - V(x(t)) = \left(\mathbf{R}(\alpha(t))\,x(t)\right. \\ &\quad + \mathbf{E}(\alpha(t))\,w(t) + \left(\overline{\mathbf{R}}(\alpha(t))\,x(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right) \\ &\quad \left.\cdot\,\beta(t)\right)^T \mathbf{P}\left(\mathbf{R}(\alpha(t))\,x(t) + \mathbf{E}(\alpha(t))\,w(t)\right. \\ &\quad \left. + \left(\overline{\mathbf{R}}(\alpha(t))\,x(t) + \mathbf{E}(\alpha(t))\,w(t)\right)\beta(t)\right) - x^T(t) \\ &\quad \cdot\,\mathbf{P}x(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\vartheta_i(t)\,\vartheta_j(t) \\ &\quad \cdot\left(\left(\mathbf{R}_{ij}x(t) + \mathbf{E}_i w(t) + \left(\overline{\mathbf{R}}_{ij}x(t) + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\right)^T\right. \\ &\quad \left.\cdot\,\mathbf{P}\left(\mathbf{R}_{ij}x(t) + \mathbf{E}_i w(t) + \left(\overline{\mathbf{R}}_{ij}x(t) + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\right)\right) \\ &\quad - x^T(t)\,\mathbf{P}x(t), \end{aligned} \tag{8}$$

where $\mathbf{R}(\alpha(t))$, $\overline{\mathbf{R}}(\alpha(t))$, $\mathbf{R}_{ij}$, and $\overline{\mathbf{R}}_{ij}$ are defined in (6). Taking expectation of (8), the following equation can be obtained with the independent increment property of Brownian motion; that is, $E\{x(t)\beta(t)\} = 0$ and $E\{\beta(t)\beta(t)\} = \tau^2$. Consider

$$\begin{aligned} E\{\Delta V(x(t))\} &= E\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\vartheta_i(t)\,\vartheta_j(t)\right. \\ &\quad \cdot\left(x^T(t)\left(\mathbf{R}_{ij}^T\mathbf{P}\mathbf{R}_{ij} + \tau^2\overline{\mathbf{R}}_{ij}^T\mathbf{P}\overline{\mathbf{R}}_{ij} - \mathbf{P}\right)x(t)\right. \\ &\quad + w^T(t)\left(\mathbf{E}_i^T\mathbf{P}\mathbf{R}_{ij} + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{R}}_{ij}\right)x(t) \\ &\quad + x^T(t)\left(\mathbf{R}_{ij}^T\mathbf{P}\mathbf{E}_i + \tau^2\overline{\mathbf{R}}_{ij}^T\mathbf{P}\overline{\mathbf{E}}_i\right)w(t) \\ &\quad \left.\left. + w^T(t)\left(\mathbf{E}_i^T\mathbf{P}\mathbf{E}_i + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{E}}_i\right)w(t)\right)\right\} = E\{\Psi\}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Psi &= \sum_{i=1}^{N}\sum_{j=1}^{N}\vartheta_i(t)\,\vartheta_j(t)\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \\ &\quad \cdot\begin{bmatrix} \mathbf{R}_{ij}^T\mathbf{P}\mathbf{R}_{ij} + \tau^2\overline{\mathbf{R}}_{ij}^T\mathbf{P}\overline{\mathbf{R}}_{ij} - \mathbf{P} & * \\ \mathbf{E}_i^T\mathbf{P}\mathbf{R}_{ij} + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{R}}_{ij} & \mathbf{E}_i^T\mathbf{P}\mathbf{E}_i + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{E}}_i \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \end{aligned} \tag{10}$$

Let us define the following performance function:

$$J_D = \sum_{0}^{t_f}\left(x^T(t)\,\mathbf{S}x(t) - \eta^2 w^T(t)\,w(t)\right). \tag{11}$$

Then, one has the following relations with zero initial conditions:

$$\begin{aligned} J_D &= E\left\{\sum_0^{t_f}\left(x^T(t)\mathbf{S}x(t)-\eta^2 w^T(t)w(t)\right)\right. \\ &\quad \left.+\sum_0^{t_f}(\Delta V(x(t)))-V(x(t))\right\} \\ &\le E\left\{\sum_0^{t_f}\left(x^T(t)\mathbf{S}x(t)-\eta^2 w^T(t)w(t)\right.\right. \\ &\quad \left.\left.+\Delta V(x(t))\right)\right\} = E\left\{\sum_0^{t_f}\left(x^T(t)\mathbf{S}x(t)\right.\right. \\ &\quad \left.\left.-\eta^2 w^T(t)w(t)+\Psi\right)\right\} = E\left\{\sum_0^{t_f}L(x,w,t)\right\}. \end{aligned} \tag{12}$$

According to (9), one has

$$\begin{aligned} &L(x,w,t) \\ &\quad = E\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\vartheta_i(t)\vartheta_j(t)\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}^T \Lambda \begin{bmatrix}x(t)\\ w(t)\end{bmatrix}\right\}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\Lambda \\ &= \begin{bmatrix} \mathbf{R}_{ij}^T\mathbf{P}\mathbf{R}_{ij}+\tau^2\overline{\mathbf{R}}_{ij}^T\mathbf{P}\overline{\mathbf{R}}_{ij}-\mathbf{P}+\mathbf{S} & * \\ \mathbf{E}_i^T\mathbf{P}\mathbf{R}_{ij}+\tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{R}}_{ij} & \mathbf{E}_i^T\mathbf{P}\mathbf{E}_i+\tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{E}}_i-\eta^2\mathbf{I} \end{bmatrix}. \end{aligned} \tag{14}$$

Applying Schur complement [28], the following inequality can be obtained from (7):

$$\left[\begin{array}{c|c} \left(\mathbf{A}_i\mathbf{Q}-\mathbf{B}_i\mathbf{Y}_j\right)^T\mathbf{Q}^{-1}\left(\mathbf{A}_i\mathbf{Q}-\mathbf{B}_i\mathbf{Y}_j\right)+\tau^2\left(\overline{\mathbf{A}}_i\mathbf{Q}-\overline{\mathbf{B}}_i\mathbf{Y}_j\right)^T\mathbf{Q}^{-1}\left(\overline{\mathbf{A}}_i\mathbf{Q}-\overline{\mathbf{B}}_i\mathbf{Y}_j\right)-\mathbf{Q}+\mathbf{Q}\mathbf{U}^{-1}\mathbf{Q} & * \\ \hline \mathbf{E}_i^T\mathbf{Q}^{-1}\left(\mathbf{A}_i\mathbf{Q}-\mathbf{B}_i\mathbf{Y}_j\right)+\tau^2\overline{\mathbf{E}}_i^T\mathbf{Q}^{-1}\left(\overline{\mathbf{A}}_i\mathbf{Q}-\overline{\mathbf{B}}_i\mathbf{Y}_j\right) & \mathbf{E}_i^T\mathbf{Q}^{-1}\mathbf{E}_i+\tau^2\overline{\mathbf{E}}_i^T\mathbf{Q}^{-1}\overline{\mathbf{E}}_i-\eta^2\mathbf{I} \end{array}\right] < 0. \tag{15}$$

Due to definitions as $\mathbf{Q}=\mathbf{P}^{-1}$, $\mathbf{Y}_j=\mathbf{F}_j\mathbf{Q}$, and $\mathbf{U}=\mathbf{S}^{-1}$, inequality (15) can be rewritten as follows:

$$\begin{bmatrix} \mathbf{P}^{-1}\mathbf{R}_{ij}^T\mathbf{P}\mathbf{R}_{ij}\mathbf{P}^{-1}+\tau^2\mathbf{P}^{-1}\overline{\mathbf{R}}_{ij}^T\mathbf{P}\overline{\mathbf{R}}_{ij}\mathbf{P}^{-1}-\mathbf{P}^{-1}+\mathbf{P}^{-1}\mathbf{S}\mathbf{P}^{-1} & * \\ \mathbf{E}_i^T\mathbf{P}\mathbf{R}_{ij}\mathbf{P}^{-1}+\tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{R}}_{ij}\mathbf{P}^{-1} & \mathbf{E}_i^T\mathbf{P}\mathbf{E}_i+\tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{E}}_i-\eta^2\mathbf{I} \end{bmatrix} < 0. \tag{16}$$

Multiplying both sides of (16) with $\mathrm{diag}\{\mathbf{P},\mathbf{I}\}$, where the $\mathrm{diag}\{\cdot,\cdot\}$ denotes a block-diagonal matrix with element $\cdot$, one can obtain the following inequality:

$$\begin{aligned} &\begin{bmatrix} \mathbf{R}_{ij}^T\mathbf{P}\mathbf{R}_{ij}+\tau^2\overline{\mathbf{R}}_{ij}^T\mathbf{P}\overline{\mathbf{R}}_{ij}-\mathbf{P}+\mathbf{S} & * \\ \mathbf{E}_i^T\mathbf{P}\mathbf{R}_{ij}+\tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{R}}_{ij} & \mathbf{E}_i^T\mathbf{P}\mathbf{E}_i+\tau^2\overline{\mathbf{E}}_i^T\mathbf{P}\overline{\mathbf{E}}_i-\eta^2\mathbf{I} \end{bmatrix} \\ &< 0. \end{aligned} \tag{17}$$

Obviously, the left-hand side of inequality (17) is equal to Λ in (13). Thus, $\Lambda < 0$ is found if condition (7) holds. And $L(x,w,t) < 0$ can be obtained from (13) with $\Lambda < 0$. According to $L(x,w,t) < 0$, the following inequalities can be inferred from (12) as follows:

$$J_D < 0 \tag{18}$$

or

$$E\left\{\sum_0^{t_f}x^T(t)\mathbf{S}x(t)\right\} < E\left\{\eta^2\sum_0^{t_f}w^T(t)w(t)\right\}. \tag{19}$$

Because (19) is equivalent to (4), it is easy to show that the closed-loop system (6) with controller (5a)-(5b) satisfies H_∞ performance when the condition (7) holds. Next, the asymptotical stability is necessary to be proven. By assuming $w(t) = 0$, the following inequality can be found from $L(x,w,t) < 0$ if the condition in Theorem 2 holds:

$$E\left\{\Psi+x^T(t)\mathbf{S}x(t)\right\} < 0 \tag{20a}$$

or

$$E\{\Psi\} < E\left\{-x^T(t)\mathbf{S}x(t)\right\}. \tag{20b}$$

According to $\mathbf{S} > 0$, one can find $E\{\Psi\} < 0$. From (9), $E\{\Psi\} < 0$ implies $E\{\Delta V(x(t))\} < 0$. According to Definition 1,

the asymptotical stability of the closed-loop system (6) can be achieved via controller (5a)-(5b) in the sense of mean square due to $E\{\Delta V(x(t))\} < 0$. Thus, the proof of this theorem is complete. □

Based on the PILF, the sufficient conditions are derived in Theorem 2. Via finding the feasible solutions, the controller (5a)-(5b) is designed to guarantee the asymptotical stability and H_∞ performance of the closed-loop system (6) in the sense of mean square. However, Theorem 2 processes conservatism in finding a common matrix $\mathbf{P}$ to satisfy sufficient condition (7) for $i, j = 1, 2, \ldots, N$. For this reason, the less conservative sufficient conditions than the ones in Theorem 2 are proposed in the next section.

4. Relaxed Stability Criterion for Disturbed LPV Stochastic Systems

Referring to [10, 11], the PDLF is proposed to derive relaxed stability criterion for LPV systems. The reason for reducing conservatism in solving stabilization problem of the system (1) is that the PDLF consists of state and multiple positive definite matrices. Based on the PDLF, a relaxed stability criterion for system (1) is proposed in this section. Besides, arbitrary matrices $\mathbf{G}_i$ are introduced to reduce conservatism of the proposed stability criterion in this section. Thus, the following gain-scheduled controller is proposed:

$$u(t) = -\mathbf{F}(\alpha(t))\,\mathbf{G}^{-1}(\alpha(t))\,x(t), \tag{21a}$$

or

$$u(t) = -\left(\sum_{j=1}^{N}\vartheta_j(t)\,\mathbf{F}_j\right)\left(\sum_{j=1}^{N}\vartheta_j(t)\,\mathbf{G}_j\right)^{-1} x(t). \tag{21b}$$

Remark 3. According to the arbitrary matrices $\mathbf{G}_i$, the freedom of searching feasible solutions of Theorem 4 is increased. Moreover, the sufficient conditions of Theorem 4 can be converted into extended LMI form by using the arbitrary matrices $\mathbf{G}_i$. Referring to [16], the extended form possesses less conservatism than standard LMI form as in (7). Thus, the structure of (21a)-(21b) is applied to the proposed relaxed gain-scheduled controller design method for disturbed uncertain stochastic systems (1).

Substituting (21a)-(21b) into system (1), the corresponding closed-loop system can be represented as follows:

$$\begin{aligned} x(t+1) &= \mathbf{X}(\alpha(t))\,x(t) + \mathbf{E}(\alpha(t))\,w(t) \\ &\quad + \left(\overline{\mathbf{X}}(\alpha(t))\,x(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right)\beta(t) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\vartheta_i(t)\,\vartheta_j(t) \\ &\quad \cdot \left(\mathbf{X}_{ij}x(t) + \mathbf{E}_i w(t) + \left(\overline{\mathbf{X}}_{ij}x(t) + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \mathbf{X}(\alpha(t)) &= \mathbf{A}(\alpha(t))\,x(t) \\ &\quad - \mathbf{B}(\alpha(t))\,\mathbf{F}(\alpha(t))\,\mathbf{G}^{-1}(\alpha(t)), \\ \overline{\mathbf{X}}(\alpha(t)) &= \overline{\mathbf{A}}(\alpha(t))\,x(t) \\ &\quad - \overline{\mathbf{B}}(\alpha(t))\,\mathbf{F}(\alpha(t))\,\mathbf{G}^{-1}(\alpha(t)), \\ \mathbf{X}_{ij} &= \mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j\left(\sum_{j=1}^{N}\vartheta_j(t)\,\mathbf{G}_j\right)^{-1}, \\ \overline{\mathbf{X}}_{ij} &= \overline{\mathbf{A}}_i - \overline{\mathbf{B}}_i\mathbf{F}_j\left(\sum_{j=1}^{N}\vartheta_j(t)\,\mathbf{G}_j\right)^{-1}. \end{aligned} \tag{23}$$

For stability problem of closed-loop system (22), the sufficient conditions are derived by H_∞ performance definition and PDLF.

Theorem 4. *With given positive scalars τ and η, if there exist feedback gains $\mathbf{F}_i$, positive definite matrices $\mathbf{P}_i$ and $\mathbf{S}$, and arbitrary matrices $\mathbf{G}_i$ to satisfy the following conditions, then the asymptotical stability and H_∞ performance of the closed-loop system (22) are guaranteed in the sense of mean square. Consider*

$$\begin{bmatrix} \mathbf{Q}_i - \mathbf{G}_i^T - \mathbf{G}_i & * & * & * & * \\ 0 & -\eta^2\mathbf{I} & * & * & * \\ \mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j & \mathbf{E}_i & -\mathbf{Q}_k & * & * \\ \tau\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & \tau\overline{\mathbf{E}}_i & 0 & -\mathbf{Q}_k & * \\ \mathbf{G}_i & 0 & 0 & 0 & -\mathbf{U} \end{bmatrix} < 0, \quad \text{for } i, j, k = 1, 2, \ldots, N, \tag{24}$$

where $\mathbf{Q}_k = \mathbf{P}_k^{-1}$ and $\mathbf{U} = \mathbf{S}^{-1}$.

Proof. Choosing a Lyapunov function as $V(x(t)) = x^T(t)\mathbf{P}(\alpha(t))x(t)$, the first forward difference of the $V(x(t))$ can be obtained, such as

$$\begin{aligned} \Delta V(x(t)) &= V(x(t+1)) - V(x(t)) \\ &= \left(\mathbf{X}(\alpha(t))\,x(t) + \mathbf{E}(\alpha(t))\,w(t)\right. \\ &\quad + \left(\overline{\mathbf{X}}(\alpha(t))\,x(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right)\beta(t)\Big)^T \\ &\quad \cdot \mathbf{P}(\alpha(t+1))\left(\mathbf{X}(\alpha(t))\,x(t) + \mathbf{E}(\alpha(t))\,w(t)\right. \\ &\quad + \left(\overline{\mathbf{X}}(\alpha(t))\,x(t) + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right)\beta(t)\Big) - x^T(t) \\ &\quad \cdot \mathbf{P}(\alpha(t))\,x(t). \end{aligned} \tag{25}$$

In this paper, $\mathbf{P}(\alpha(t+1))$ is defined by the following equation:

$$\mathbf{P}(\alpha(t+1)) = \sum_{j=1}^{N}\vartheta_j(t+1)\,\mathbf{P}_j = \left(\sum_{k=1}^{N}\varepsilon_k(t)\,\mathbf{P}_k\right), \tag{26}$$

where $\varepsilon(t)$ is the time-varying parameter satisfying $\sum_{k=1}^{N}\varepsilon_k(t) = 1$ and $0 \le \varepsilon_k(t) \le 1$. Due to (26), (25) can be rewritten as in the following equation:

$$\begin{aligned}\Delta V(x(t)) = {} & \Big(\mathbf{X}(\alpha(t))x(t) + \mathbf{E}(\alpha(t))w(t) \\ & + \left(\overline{\mathbf{X}}(\alpha(t))x(t) + \overline{\mathbf{E}}(\alpha(t))w(t)\right)\beta(t)\Big)^T \mathbf{P}(\varepsilon(t)) \\ & \cdot \Big(\mathbf{X}(\alpha(t))x(t) \\ & + \mathbf{E}(\alpha(t))w(t) \\ & + \left(\overline{\mathbf{X}}(\alpha(t))x(t) + \overline{\mathbf{E}}(\alpha(t))w(t)\right)\beta(t)\Big) - x^T(t) \\ & \cdot \mathbf{P}(\alpha(t))x(t).\end{aligned} \tag{27}$$

Taking expectation of (27), the following equation can be found with the independent increment property of Brownian motion:

$$\begin{aligned}E\{\Delta V(x(t))\} = {} & E\Big\{x^T(t) \\ & \cdot \Big(\mathbf{X}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{X}(\alpha(t)) \\ & + \tau^2\overline{\mathbf{X}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{X}}(\alpha(t))\Big)x(t) + w^T(t) \\ & \cdot \Big(\mathbf{E}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{X}(\alpha(t)) \\ & + \tau^2\overline{\mathbf{E}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{X}}(\alpha(t))\Big)x(t) + x^T(t) \\ & \cdot \Big(\mathbf{X}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{E}(\alpha(t)) \\ & + \tau^2\overline{\mathbf{X}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{E}}(\alpha(t))\Big)w(t) + w^T(t) \\ & \cdot \Big(\mathbf{E}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{E}(\alpha(t)) \\ & + \tau^2\overline{\mathbf{E}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{E}}(\alpha(t))\Big)w(t) - x^T(t) \\ & \cdot \mathbf{P}(\alpha(t))x(t)\Big\}.\end{aligned} \tag{28}$$

Applying the cost function (11), one can find the following relations:

$$\begin{aligned}J_D = {} & E\left\{\sum_0^{t_f}\left(x^T(t)\mathbf{S}x(t) - \eta^2 w^T(t)w(t)\right) + \sum_0^{t_f}\Delta V(x(t)) - V\left(x\left(t_f\right)\right)\right\} \\ \le {} & E\left\{\sum_0^{t_f}\left(x^T(t)\mathbf{S}x(t) - \eta^2 w^T(t)w(t) + \Delta V(x(t))\right)\right\} = E\left\{\sum_0^{t_f}\Phi(x,w,t)\right\}.\end{aligned} \tag{29}$$

According to (28), one has

$$\Phi(x,w,t) = \begin{bmatrix} x(t) \\ w(t)\end{bmatrix}^T \Xi \begin{bmatrix} x(t) \\ w(t)\end{bmatrix}, \tag{30}$$

where

$$\Xi = \left[\begin{array}{c|c} \mathbf{X}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{X}(\alpha(t)) + \tau^2\overline{\mathbf{X}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{X}}(\alpha(t)) - \mathbf{P}(\alpha(t)) + \mathbf{S} & * \\ \hline \mathbf{E}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{X}(\alpha(t)) + \tau^2\overline{\mathbf{E}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{X}}(\alpha(t)) & \mathbf{E}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\mathbf{E}(\alpha(t)) + \tau^2\overline{\mathbf{E}}^T(\alpha(t))\mathbf{P}(\varepsilon(t))\overline{\mathbf{E}}(\alpha(t)) - \eta^2\mathbf{I}\end{array}\right]. \tag{31}$$

Applying the Schur complement, one has the following inequality from (24):

$$\left[\begin{array}{c|c} \mathbf{Q}_i - \mathbf{G}_i^T - \mathbf{G}_i + \mathbf{G}_i^T\mathbf{U}^{-1}\mathbf{G}_i + \left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right)^T\mathbf{Q}_k^{-1}\left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right) + \tau^2\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right)^T\mathbf{Q}_k^{-1}\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & * \\ \hline \mathbf{E}_i^T\mathbf{Q}_k^{-1}\left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right) + \tau^2\overline{\mathbf{E}}_i^T\mathbf{Q}_k^{-1}\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & \mathbf{E}_i^T\mathbf{Q}_k^{-1}\mathbf{E}_i + \tau^2\overline{\mathbf{E}}_i^T\mathbf{Q}_k^{-1}\overline{\mathbf{E}}_i - \eta^2\mathbf{I}\end{array}\right] < 0. \tag{32}$$

According to the fact that $\mathbf{P}_i^{-1} - \mathbf{G}_i^T - \mathbf{G}_i \ge -\mathbf{G}_i^T\mathbf{P}_i\mathbf{G}_i$, the following inequality holds from (32) with definition $\mathbf{Q}_k = \mathbf{P}_k^{-1}$ and $\mathbf{U} = \mathbf{S}^{-1}$

$$\left[\begin{array}{c|c} -\mathbf{G}_i^T\mathbf{P}_i\mathbf{G}_i + \mathbf{G}_i^T\mathbf{S}\mathbf{G}_i + \left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right)^T\mathbf{P}_k\left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right) + \tau^2\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right)^T\mathbf{P}_k\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & * \\ \hline \mathbf{E}_i^T\mathbf{P}_k\left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right) + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}_k\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & \mathbf{E}_i^T\mathbf{P}_k\mathbf{E}_i + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}_k\overline{\mathbf{E}}_i - \eta^2\mathbf{I}\end{array}\right] < 0. \tag{33}$$

Since $\vartheta_i \geq 0$ and $\sum_{i=1}^{N} \vartheta_i = 1$, the following inequality can be inferred from (33):

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\vartheta_i(t)\,\vartheta_j(t)\,\varepsilon_k(t) \cdot \left[\begin{array}{c|c} -\mathbf{G}_i^T\mathbf{P}_i\mathbf{G}_i + \mathbf{G}_i^T\mathbf{S}\mathbf{G}_i + \left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right)^T\mathbf{P}_k\left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right) + \tau^2\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right)^T\mathbf{P}_k\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & * \\ \hline \mathbf{E}_i^T\mathbf{P}_k\left(\mathbf{A}_i\mathbf{G}_j - \mathbf{B}_i\mathbf{F}_j\right) + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}_k\left(\overline{\mathbf{A}}_i\mathbf{G}_j - \overline{\mathbf{B}}_i\mathbf{F}_j\right) & \mathbf{E}_i^T\mathbf{P}_k\mathbf{E}_i + \tau^2\overline{\mathbf{E}}_i^T\mathbf{P}_k\overline{\mathbf{E}}_i - \eta^2\mathbf{I} \end{array}\right] < 0. \tag{34}$$

And inequality (34) can be rewritten as follows:

$$\left[\begin{array}{c|c} \begin{array}{c} -\mathbf{G}^T(\alpha(t))\,\mathbf{P}(\alpha(t))\,\mathbf{G}(\alpha(t)) + \mathbf{G}^T(\alpha(t))\,\mathbf{S}\mathbf{G}(\alpha(t)) \\ +\left(\mathbf{A}(\alpha(t))\,\mathbf{G}(\alpha(t)) - \mathbf{B}(\alpha(t))\,\mathbf{F}(\alpha(t))\right)^T\mathbf{P}(\varepsilon(t)) \\ \cdot\left(\mathbf{A}(\alpha(t))\,\mathbf{G}(\alpha(t)) - \mathbf{B}(\alpha(t))\,\mathbf{F}(\alpha(t))\right) \\ +\tau^2\left(\overline{\mathbf{A}}(\alpha(t))\,\mathbf{G}(\alpha(t)) - \overline{\mathbf{B}}(\alpha(t))\,\mathbf{F}(\alpha(t))\right)^T\mathbf{P}(\varepsilon(t)) \\ \cdot\left(\overline{\mathbf{A}}(\alpha(t))\,\mathbf{G}(\alpha(t)) - \overline{\mathbf{B}}(\alpha(t))\,\mathbf{F}(\alpha(t))\right) \end{array} & * \\ \hline \begin{array}{c} \mathbf{E}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\left(\mathbf{A}(\alpha(t))\,\mathbf{G}(\alpha(t)) - \mathbf{B}(\alpha(t))\,\mathbf{F}(\alpha(t))\right) \\ +\tau^2\overline{\mathbf{E}}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\left(\overline{\mathbf{A}}(\alpha(t))\,\mathbf{G}(\alpha(t)) - \overline{\mathbf{B}}(\alpha(t))\,\mathbf{F}(\alpha(t))\right) \end{array} & \begin{array}{c} \mathbf{E}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\mathbf{E}(\alpha(t)) \\ +\tau^2\overline{\mathbf{E}}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\overline{\mathbf{E}}(\alpha(t)) - \eta^2\mathbf{I} \end{array} \end{array}\right] < 0. \tag{35}$$

Before and after multiplying (35) by $\mathrm{diag}\{\mathbf{G}^{-T}(\alpha(t)), \mathbf{I}\}$ and $\mathrm{diag}\{\mathbf{G}^{-1}(\alpha(t)), \mathbf{I}\}$, one has

$$\left[\begin{array}{c|c} \mathbf{X}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\mathbf{X}(\alpha(t)) + \tau^2\overline{\mathbf{X}}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\overline{\mathbf{X}}(\alpha(t)) - \mathbf{P}(\alpha(t)) + \mathbf{S} & * \\ \hline \mathbf{E}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\mathbf{X}(\alpha(t)) + \tau^2\overline{\mathbf{E}}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\overline{\mathbf{X}}(\alpha(t)) & \mathbf{E}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\mathbf{E}(\alpha(t)) + \tau^2\overline{\mathbf{E}}^T(\alpha(t))\,\mathbf{P}(\varepsilon(t))\,\overline{\mathbf{E}}(\alpha(t)) - \eta^2\mathbf{I} \end{array}\right] < 0. \tag{36}$$

Obviously, if condition (24) holds, then (36) can be obtained. And $\Xi < 0$ can be also found from (31) due to (36). According to $\Xi < 0$, $\Phi(x, w, t) < 0$ can be inferred from (30). Due to $\Phi(x, w, t) < 0$ and (29), the following inequalities can be obtained:

$$J_D < 0 \tag{37}$$

or

$$E\left\{\sum_{0}^{t_f} x^T(t)\,\mathbf{S}x(t)\right\} < E\left\{\eta^2\sum_{0}^{t_f} w^T(t)\,w(t)\right\}. \tag{38}$$

Because (38) is equivalent to (4), it is easy to show that the closed-loop system (22) driven by (21a)-(21b) satisfies H_∞ performance for all nonzero external disturbances. Next, the asymptotical stability of the closed-loop system (22) is proven. If the condition of this theorem is satisfied, then $\Phi(x, w, t) < 0$ is held. By assuming $w(t) = 0$, the following inequality can be found from (29):

$$E\left\{\Delta V(x(t)) + x^T(t)\,\mathbf{S}x(t)\right\} < 0 \tag{39}$$

or

$$E\{\Delta V(x(t))\} < E\left\{-x^T(t)\,\mathbf{S}x(t)\right\}. \tag{40}$$

According to $\mathbf{S} > 0$, one can deduce that $E\{\Delta V(x(t))\} < 0$. And then the closed-loop system (22) is asymptotically stable in the sense of mean square according to $E\{\Delta V(x(t))\} < 0$ and Definition 1. The proof of this theorem is complete. □

In this section, the sufficient conditions are derived by PDLF for discussing the stabilization problems of the closed-loop system (22). Through the several positive definite matrices and arbitrary matrices $\mathbf{G}_i$, the conservatism of Theorem 4 can be reduced in finding the feasible solutions

of conditions (24). In the following section, two numerical examples are proposed to demonstrate the effectiveness and application of the proposed design method.

5. Simulation Results

In this section, two numerical examples are proposed. The first example is employed to discuss the conservatism of the proposed design methods. Another example is to discuss the stabilization problem of disturbed ship autopilot servosystem with multiplicative noise to show the application of the proposed design methods. Moreover, the design method of [14] is employed to compare with the proposed design methods of this paper.

Example 5. Consider the following disturbed stochastic LPV system:

$$\begin{aligned} x(t+1) &= \mathbf{A}(\alpha(t))\,x(t) + \mathbf{B}(\alpha(t))\,u(t) + \mathbf{E}(\alpha(t)) \\ &\quad \cdot w(t) + \left(\overline{\mathbf{A}}(\alpha(t))\,x(t) + \overline{\mathbf{B}}(\alpha(t))\,u(t)\right. \\ &\quad \left. + \overline{\mathbf{E}}(\alpha(t))\,w(t)\right)\beta(t) = \sum_{i=1}^{2}\vartheta_i(t)\left(\mathbf{A}_i x(t) + \mathbf{B}_i u(t)\right. \\ &\quad \left. + \mathbf{E}_i w(t) + \left(\overline{\mathbf{A}}_i x(t) + \overline{\mathbf{B}}_i u(t) + \overline{\mathbf{E}}_i w(t)\right)\beta(t)\right), \end{aligned} \tag{41}$$

where

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} 2 & -0.1 \\ 0.5 & 1.65 \end{bmatrix}, \\ \mathbf{A}_2 &= \begin{bmatrix} 2 & -0.1 \\ 0.5 & 0.35 \end{bmatrix}, \\ \mathbf{B}_1 &= \begin{bmatrix} 1 \\ -0.95 \end{bmatrix}, \\ \mathbf{B}_2 &= \begin{bmatrix} 1 \\ 0.35 \end{bmatrix}, \\ \mathbf{E}_1 &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \\ \mathbf{E}_2 &= \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \\ \overline{\mathbf{A}}_1 &= \begin{bmatrix} 0.03 & 0 \\ 0.004 & 0.0165 \end{bmatrix}, \\ \overline{\mathbf{A}}_2 &= \begin{bmatrix} 0.03 & 0 \\ 0.004 & 0.0035 \end{bmatrix}, \\ \overline{\mathbf{B}}_1 &= \begin{bmatrix} 0.01 \\ -0.0075 \end{bmatrix}, \\ \overline{\mathbf{B}}_2 &= \begin{bmatrix} 0.01 \\ 0.0055 \end{bmatrix}, \\ \overline{\mathbf{E}}_1 &= \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}, \\ \overline{\mathbf{E}}_2 &= \begin{bmatrix} 0.002 \\ 0 \end{bmatrix}, \\ \vartheta_1(t) &= |\sin(t)|, \\ \vartheta_2(t) &= 1 - |\sin(t)|. \end{aligned} \tag{42}$$

In this numerical example, the intensity level is given as $\tau = 1$. For discussing the conservatism of Theorems 2 and 4, the positive definite matrix $\mathbf{S}$ is determined as identity matrix to find the minimum available value of η. Applying the convex optimization algorithm [29], the minimum available value of η for the sufficient condition of the theorems is shown in Table 1. From Table 1, the minimum available value of η to satisfy Theorem 2 is 1.5166. In case such as $\eta = 1.5166$, the following feasible solutions of condition (7) can be obtained:

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} 57.1827 & 34.8967 \\ 34.8967 & 22.9765 \end{bmatrix}, \\ \mathbf{F}_1 &= \begin{bmatrix} 4.2194 & 1.4732 \end{bmatrix}, \\ \mathbf{F}_2 &= \begin{bmatrix} 1.8602 & 0.0691 \end{bmatrix}. \end{aligned} \tag{43}$$

Based on (43), the gain-scheduled controller can be designed such as

$$u(t) = -\sum_{j=1}^{2}\vartheta_j(t)\,\mathbf{F}_j x(t). \tag{44}$$

Applying (44), the responses of (41) are stated in Figure 1 with initial condition $x(t) = \begin{bmatrix} 5 & -3 \end{bmatrix}^T$. And the external disturbance $w(t)$ is chosen as zero-mean white noise with unit variance. For checking satisfaction of (4), the following ratio is obtained via using the simulation results:

$$\frac{E\left\{\sum_0^{t_f=5} x^T(t)\,\mathbf{S}x(t)\right\}}{E\left\{\sum_0^{t_f=5} w^T(t)\,w(t)\right\}} = 1.696. \tag{45}$$

Obviously, the ratio in (45) is smaller than the obtained value $\eta^2 = 2.3$ with $\eta = 1.5166$. From Figure 1 and (45), system (41) driven by (44) is robust asymptotically stable with attenuation η in the sense of mean square.

Besides, from Table 1, the minimum available value of η for satisfying Theorem 4 is 1.4. In the case such as $\eta = 1.4$,

Table 1: Comparing results for Theorems 2 and 4.

η	$\cdots$	1.5166	1.5133	1.4	$\cdots$
Theorem 2	Feasible	Feasible	Infeasible	Infeasible	Infeasible
Theorem 4	Feasible	Feasible	Feasible	Feasible	Infeasible

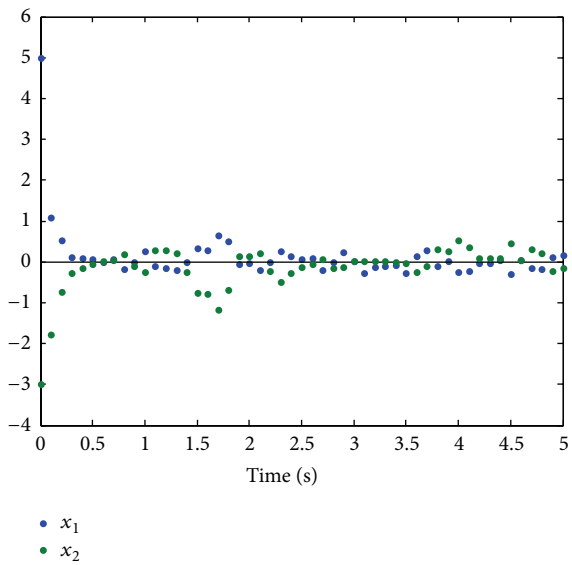


Figure 1: Responses of Example 5 with controller (44).

the feasible solutions of conditions (24) can be obtained such as

$$\mathbf{P}_1 = \begin{bmatrix} 48.989 & 30.9426 \\ 30.9426 & 21.0296 \end{bmatrix},$$
$$\mathbf{P}_2 = \begin{bmatrix} 12.7423 & 7.8087 \\ 7.8087 & 6.5371 \end{bmatrix},$$
$$\mathbf{G}_1 = \begin{bmatrix} 0.2637 & -0.3887 \\ -0.4104 & 0.6518 \end{bmatrix}, \tag{46}$$
$$\mathbf{G}_2 = \begin{bmatrix} 0.2650 & -0.3890 \\ -0.3259 & 0.5924 \end{bmatrix},$$
$$\mathbf{F}_1 = \begin{bmatrix} 0.5621 & -0.7445 \end{bmatrix},$$
$$\mathbf{F}_2 = \begin{bmatrix} 0.4703 & -0.6790 \end{bmatrix}.$$

With the above feasible solutions, gain-scheduled controller (21a)-(21b) is established as follows:

$$u(t) = -\left(\sum_{j=1}^{2} \vartheta_j(t)\mathbf{F}_j\right)\left(\sum_{j=1}^{2} \vartheta_j(t)\mathbf{G}_j\right)^{-1} x(t). \tag{47}$$

Based on controller (47), the responses of (41) are stated in Figure 2 with the same initial condition and $w(t)$ of the above case. Based on the simulation results, the following ratio value can be obtained:

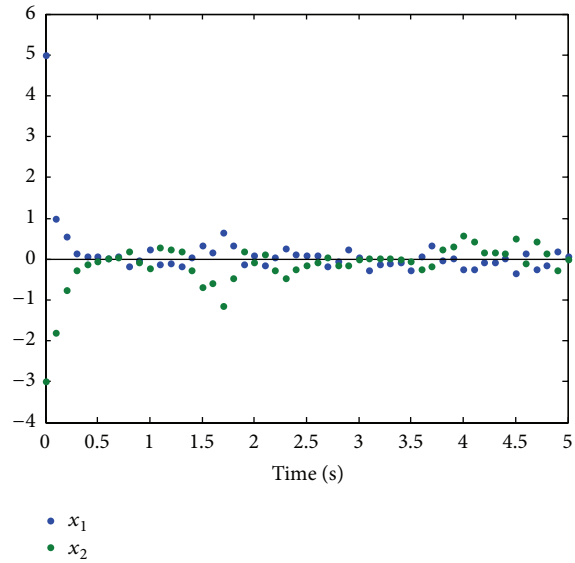


Figure 2: Responses of Example 5 with controller (47).

$$\frac{E\left\{\sum_0^{t_f=5} x^T(t)\,\mathbf{S}x(t)\right\}}{E\left\{\sum_0^{t_f=5} w^T(t)\,w(t)\right\}} = 1.675. \tag{48}$$

Obviously, the value of (48) is smaller than $\eta^2 = 1.96$ with $\eta = 1.4$. From Figure 2 and (48), the asymptotical stability and H_∞ performance of system (41) can be achieved via the controller (46).

From the simulation results of this example, the proposed design methods are useful tools to design gain-scheduled controller for stabilizing the LPV stochastic system (41). Besides, from Table 1, it is obvious to show that the minimum available value of Theorem 2 is bigger than the one of Theorem 4. Thus, the sufficient conditions of Theorem 4 are less conservative than the one in Theorem 2 for discussing stability issue of LPV systems.

Example 6. In this example, the ship autopilot servosystem is applied to show applicability of the proposed controller design methods. Referring to [30], the discretization differential equation of ship motion is proposed. Considering the practical operations, the parameter T_1 in the system is assumed as time-varying parameter $T_1(t)$ in this section. According to $T_1(t)$, the ship autopilot system belongs to uncertain system. Moreover, a multiplicative noise term is added to describe the stochastic behavior of the system. And an external disturbance is added to simulate random force from outside. Thus, the disturbed ship autopilot servosystem with multiplicative noise is considered as follows:

$$x_1(t+1) = x_1(t) + x_2(t) \times \Delta t, \tag{49a}$$

$$x_2(t+1) = x_2(t) + x_3(t) \times \Delta t + 0.0002\left(1 + 0.1\beta(t)\right) w(t), \tag{49b}$$

$$\begin{aligned} x_3(t+1) = {} & \frac{-K \times \Delta t}{T_1(t) T_2} x_2(t) \\ & + \left(\frac{-(T_1(t) + T_2) \times \Delta t}{T_1(t) T_2} + 1 \right) x_3(t) \\ & + \frac{K(T_E - T_3) \times \Delta t}{T_1(t) T_2 T_E} x_4(t) \\ & + \frac{K T_3 \times \Delta t}{T_1(t) T_2 T_E} u(t), \end{aligned} \tag{49c}$$

$$\begin{aligned} x_4(t+1) = {} & 0.2\beta(t) x_2(t) + 0.1\beta(t) x_3(t) \\ & + \left(\frac{-1 \times \Delta t}{T_E} + 1 \right) x_4(t) \\ & + \frac{1 \times \Delta t}{T_E} (1 + 0.6\beta(t)) u(t), \end{aligned} \tag{49d}$$

where $x_1(t)$ represents the difference of the heading angle and desires heading angle of ship; $x_2(t)$ represents the navigational angle velocity; $x_3(t)$ represents the navigational angle acceleration; $x_4(t)$ represents the actual rudder angle of ship; $u(t)$ represents the steering angle; and $w(t)$ is chosen as zero-mean white noise with unit variance. In order to achieve all possible values of variation of the parameter $T_1(t)$, the time-varying range of $T_1(t)$ is determined as follows:

$$T_1(t) \in [36.25 \quad 108.75]. \tag{50}$$

Besides, the constant parameters $T_2 = 8.54$, $T_3 = 17.61$, and $T_E = 2.5$, rudder gain $K = 0.1141$, and sampling time $\Delta t = 0.4$ are given in this section. According to the LPV modeling approach, system (49a)–(49d) can be described as the following disturbed LPV stochastic system:

$$\begin{aligned} x(t+1) = {} & \mathbf{A}(\alpha(t)) x(t) + \mathbf{B}(\alpha(t)) u(t) + \mathbf{E}(\alpha(t)) \\ & \cdot w(t) + \left(\overline{\mathbf{A}}(\alpha(t)) x(t) + \overline{\mathbf{B}}(\alpha(t)) u(t) \right. \\ & \left. + \overline{\mathbf{E}}(\alpha(t)) w(t) \right) \beta(t) = \sum_{i=1}^{2} \vartheta_i(t) \left(\mathbf{A}_i x(t) + \mathbf{B}_i u(t) \right. \\ & \left. + \mathbf{E}_i w(t) + \left(\overline{\mathbf{A}}_i x(t) + \overline{\mathbf{B}}_i u(t) + \overline{\mathbf{E}}_i w(t) \right) \beta(t) \right), \end{aligned} \tag{51}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0 & 1 & 0.4 & 0 \\ 0 & -0.000064 & 0.9508 & -0.000296 \\ 0 & 0 & 0 & 0.8521 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0 & 1 & 0.4 & 0 \\ 0 & -0.00013 & 0.9438 & -0.00088 \\ 0 & 0 & 0 & 0.8521 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.0003 \\ 0.1363 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0.0010 \\ 0.1363 \end{bmatrix},$$

$$\mathbf{E}_1 = \begin{bmatrix} 0 \\ 0.0002 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{E}_2 = \mathbf{E}_1,$$

$$\overline{\mathbf{A}}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.1 & 0 \end{bmatrix},$$

$$\overline{\mathbf{A}}_2 = \overline{\mathbf{A}}_1,$$

$$\overline{\mathbf{B}}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \end{bmatrix},$$

$$\overline{\mathbf{B}}_2 = \overline{\mathbf{B}}_1,$$

$$\overline{\mathbf{E}}_1 = \begin{bmatrix} 0 \\ 0.00002 \\ 0 \\ 0 \end{bmatrix},$$

$$\overline{\mathbf{E}}_2 = \overline{\mathbf{E}}_1,$$

$$\vartheta_1(t) = |\sin(t)|,$$

$$\vartheta_2(t) = 1 - |\sin(t)|. \tag{52}$$

For (51), the performance index η is given by 0.0032 and the intensity level $\tau = 1$ is given. Employing the convex optimization algorithm, the feasible solutions of Theorem 2 can be obtained as follows:

$$\mathbf{P} = \begin{bmatrix} 0.0033 & 0.0723 & 0.4088 & -0.0004 \\ 0.0723 & 3.1263 & 21.1886 & -0.0308 \\ 0.4088 & 21.1886 & 148.4819 & -0.2240 \\ -0.0004 & -0.0308 & -0.2240 & 0.0006 \end{bmatrix} \times 10^{-3},$$

$$\mathbf{S} = \begin{bmatrix} 0.2227 & 0.0452 & -0.0042 & -0.0030 \\ 0.0452 & 0.8313 & 0.0099 & 0.0059 \\ -0.0042 & 0.0099 & 0.9017 & -0.0012 \\ -0.0030 & 0.0059 & -0.0012 & 0.1516 \end{bmatrix} \times 10^{-7},$$

$$\mathbf{F}_1 = \begin{bmatrix} 0.5172 & 23.2092 & 150.4117 & -0.9578 \end{bmatrix},$$

$$\mathbf{F}_2 = \begin{bmatrix} 1.2392 & 57.8210 & 390.6686 & -1.2176 \end{bmatrix}. \tag{53}$$

According to the above feasible solutions, the following gain-scheduled controller can be designed:

$$u(t) = -\sum_{j=1}^{2} \vartheta_j(t)\, \mathbf{F}_i x(t). \tag{54}$$

Based on the gain-scheduled controller (54), the responses of system (51) are stated in Figures 3–6 via initial condition $x(0) = \begin{bmatrix} \pi/2 & 0 & 0 & 0 \end{bmatrix}^T$. For checking the achievement of (4), one can find the following values by substituting the simulated responses into the following ratio function:

$$\frac{E\left\{\sum_0^{t_{f=100}} x^T(t)\, \mathbf{S} x(t)\right\}}{E\left\{\sum_0^{t_{f=100}} w^T(t)\, w(t)\right\}} = 7.199 \times 10^{-7}. \tag{55}$$

It is easy to know that the ratio value in (55) is smaller than the given $\eta^2 = 1\times10^{-6}$ with $\eta = 0.001$. Thus, the H_∞ performance of system (49a)–(49d) can be achieved via controller (54). And, from Figures 3–6, one can find that system (49a)–(49d) driven by (54) is asymptotically stable in the sense of mean square.

Besides, applying Theorem 4, the following feasible solutions of condition (24) are obtained:

$$\mathbf{P}_1 = \begin{bmatrix} 0.0029 & 0.0655 & 0.3755 & -0.0004 \\ 0.0655 & 3.1063 & 21.4553 & -0.0321 \\ 0.3755 & 21.4553 & 152.8154 & -0.2360 \\ -0.0004 & -0.0321 & -0.2360 & 0.0006 \end{bmatrix} \times 10^{-4},$$

$$\mathbf{P}_2 = \begin{bmatrix} 0.0028 & 0.0585 & 0.3261 & -0.0003 \\ 0.0585 & 2.7418 & 18.8854 & -0.0292 \\ 0.3261 & 18.8854 & 134.7003 & -0.2158 \\ -0.0003 & -0.0292 & -0.2158 & 0.0006 \end{bmatrix} \times 10^{-4},$$

$$\mathbf{S} = \begin{bmatrix} 0.1817 & 0.0219 & -0.0022 & -0.0032 \\ 0.0219 & 0.4025 & 0.0026 & 0.0015 \\ -0.0022 & 0.0026 & 0.4152 & -0.0003 \\ -0.0032 & 0.0015 & -0.0003 & 0.1365 \end{bmatrix} \times 10^{-8},$$

$$\mathbf{G}_1 = \begin{bmatrix} 4.3444 & -0.5847 & 0.0703 & -0.7245 \\ -0.5848 & 0.0899 & -0.0111 & 0.0508 \\ 0.0703 & -0.0111 & 0.0014 & 0.0012 \\ -0.7242 & 0.0506 & 0.0013 & 4.2746 \end{bmatrix} \times 10^{7},$$

$$\mathbf{G}_2 = \begin{bmatrix} 4.3445 & -0.5847 & 0.0703 & -0.7229 \\ -0.5848 & 0.0900 & -0.0111 & 0.0503 \\ 0.0703 & -0.0111 & 0.0014 & 0.0020 \\ -0.7233 & 0.0503 & 0.0018 & 4.3617 \end{bmatrix} \times 10^{7}, \tag{56}$$

$$\mathbf{F}_1 = \begin{bmatrix} -0.0393 & 0.0594 & -0.0121 & -3.1880 \end{bmatrix} \times 10^{7},$$

$$\mathbf{F}_2 = \begin{bmatrix} -0.0370 & 0.0585 & -0.0099 & -2.7806 \end{bmatrix} \times 10^{7}.$$

And the following gain-scheduled controller can be designed with the feedback gains in (56):

$$u(t) = -\left(\sum_{j=1}^{2} \vartheta_j(t)\, \mathbf{F}_i\right)\left(\sum_{j=1}^{2} \vartheta_j(t)\, \mathbf{G}_i\right)^{-1} x(t). \tag{57}$$

Applying the controller (57), the responses of system (51) are stated in Figures 3–6 via the same initial condition. From the simulation results, the effect of the disturbance on the system driven by (57) can be criticized as follows:

$$\frac{E\left\{\sum_0^{t_{f=100}} x^T(t)\, \mathbf{S} x(t)\right\}}{E\left\{\sum_0^{t_{f=100}} w^T(t)\, w(t)\right\}} = 6.587 \times 10^{-8}. \tag{58}$$

It is easy to know that the ratio value in (58) is smaller than the given $\eta^2 = 1\times10^{-6}$ with $\eta = 0.001$. Thus, the H_∞ performance of system (49a)–(49d) can be achieved via controller (57). And, from Figures 3–6, one can find that system (49a)–(49d) driven by (57) is asymptotically stable in the sense of mean square.

In order to emphasize the advantages of this paper, the design method of [14] is applied to compare with the proposed methods in this paper. Referring to [14], the H_∞ gain-scheduled controller design method was proposed for LPV systems without consideration of stochastic behavior. On the other hand, the same PDLF was used to derive the sufficient condition in Theorem 8 of [14]. Applying the design method of [14], the corresponding controller can be established as follows:

$$u(t) = \sum_{j=1}^{2} \vartheta_j(t)\, \mathbf{K}_i x(t), \tag{59}$$

where $\mathbf{K}_1 = \begin{bmatrix} -0.0188 & -1.5204 & -15.2101 & 0.5341 \end{bmatrix}$ and $\mathbf{K}_2 = \begin{bmatrix} 1.9481 & 30.4800 & 145.7313 & 0.5315 \end{bmatrix}$. With the same initial

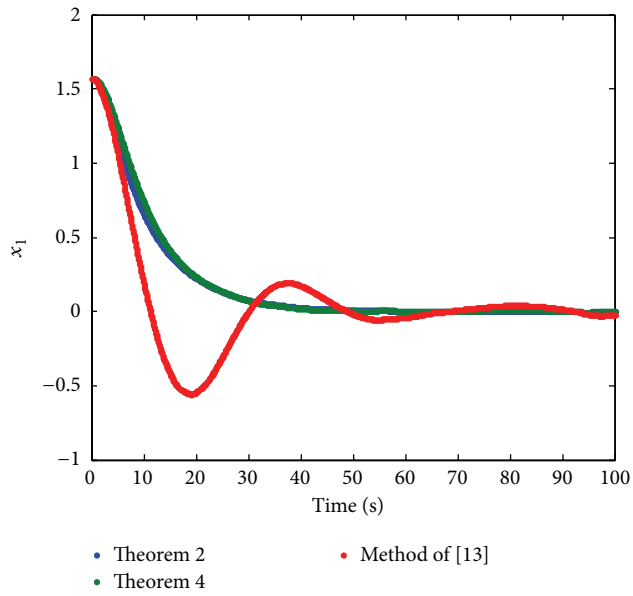


FIGURE 3: Responses for $x_1(t)$ of Example 6.

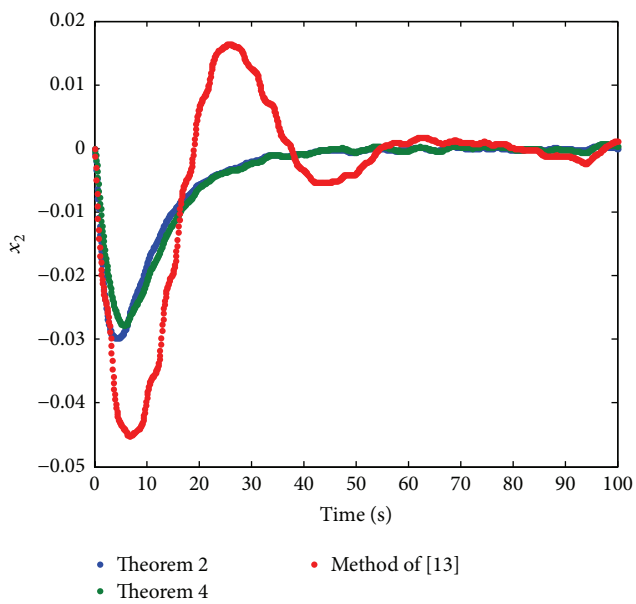


FIGURE 4: Responses for $x_2(t)$ of Example 6.

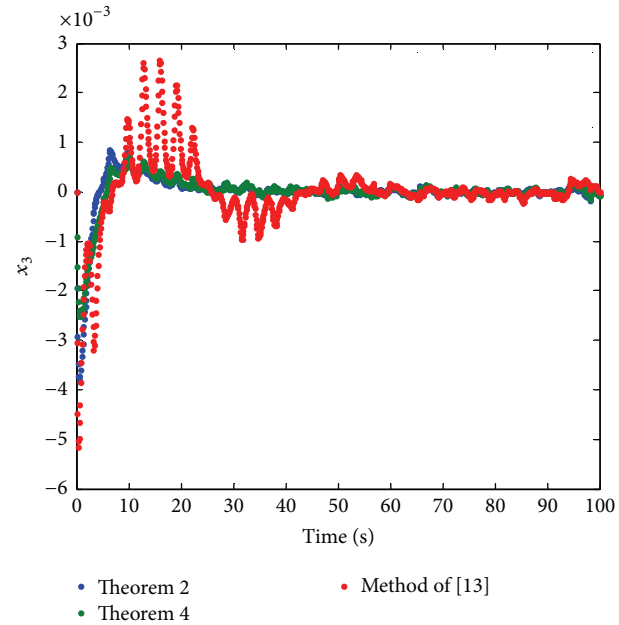


FIGURE 5: Responses for $x_3(t)$ of Example 6.

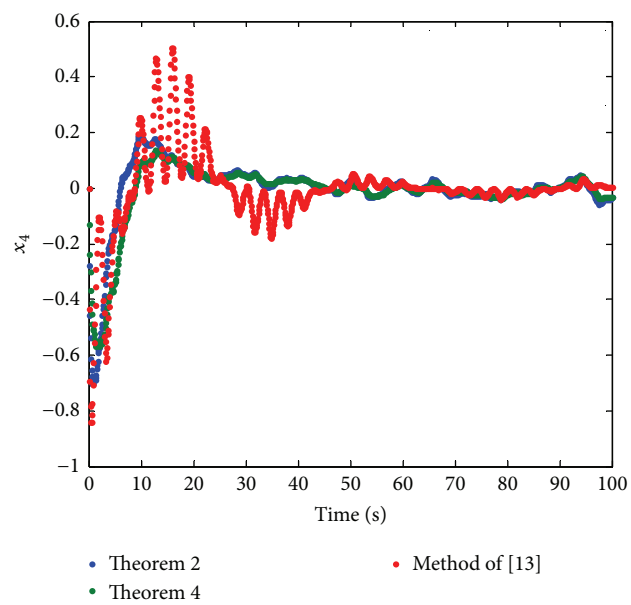


FIGURE 6: Responses for $x_4(t)$ of Example 6.

condition, the responses of (49a)–(49d) driven by (59) are stated in Figures 3–6. From Figures 3–6, one can find that controllers (57) provide better performance in both short term and long term characteristics than others. Besides, the overshoot and setting time of system (49a)–(49d) driven by the controller designed by this paper are smaller than those driven by controller (59). Therefore, the controller designed by [14] provides the worst control performance to stabilize system (49a)–(49d) due to stochastic behavior. Through the simulation results, the proposed design methods provide some improvements for [14] in stabilizing the disturbed uncertain stochastic system (49a)–(49d).

6. Conclusion

The H_∞ gain-scheduled controller design methods have been proposed in this paper for discrete-time disturbed uncertain stochastic systems described by LPV stochastic system. By choosing the Lyapunov functions, the sufficient conditions were derived to establish the corresponding gain-scheduled controller. And the H_∞ attenuation performance has been considered to constrain the effect of external disturbance on the considered systems. Applying the proposed design methods, the simulation results have been proposed to show

the effectiveness and applicability of this paper. From the simulation results, the robust asymptotical stability and H_∞ performance of uncertain stochastic systems can achieve the designed gain-scheduled controller in the sense of mean square.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their sincere gratitude to the anonymous reviewers who gave us many constructive comments and suggestions. This work was supported by the Ministry of Science and Technology, Taiwan, under Contract MOST103-2221-E-019-047.

References

[1] M. Corless, M. S. Mahmoud, and A. A. Bahnasawi, "Comments on asymptotic stability for a class of linear discrete systems with bounded uncertainties," *IEEE Transactions on Automatic Control*, vol. 35, no. 3, p. 384, 1990.

[2] W.-C. Yang and M. Tomizuka, "Discrete time robust control via state feedback for single input systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 5, pp. 590–598, 1990.

[3] J. Miguchi, H. Wu, and K. Mizukami, "Robust linear controller design for discrete time systems with norm-bounded time-varying uncertainty," in *Proceedings of the 34th SICE Annual Conference*, pp. 1399–1402, July 1995.

[4] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems & Control Letters*, vol. 37, no. 4, pp. 261–265, 1999.

[5] R. C. L. F. Oliveira and P. L. D. Peres, "Robust stability analysis and control design for time-varying discrete-time polytopic systems with bounded parameter variation," in *Proceedings of the American Control Conference*, pp. 3094–3099, 2008.

[6] G. Franze, D. Famularo, E. Garone, and A. Casavola, "Dilated model predictive control strategy for linear parameter-varying systems with a time-varying terminal set," *IET Control Theory and Applications*, vol. 3, no. 1, pp. 110–120, 2009.

[7] E. Garone, A. Casavola, G. Franze, and D. Famularo, "New stabilizability conditions for discrete-time linear parameter varying systems," in *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 2755–2760, IEEE, New Orleans, La, USA, December 2007.

[8] J. Daafouz and J. Bernussou, "Poly-quadratic stability and H_∞ performance for discrete systems with time varying uncertainties," in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 1, pp. 267–272, Orlando, Fla, USA, 2001.

[9] F. D. Adegas and J. Stoustrup, "Structured control of affine linear parameter varying systems," in *Proceedings of the American Control Conference (ACC' 11)*, pp. 739–744, July 2011.

[10] J. Daafouz and J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties," *Systems and Control Letters*, vol. 43, no. 5, pp. 355–359, 2001.

[11] N. Wang and K.-Y. Zhao, "Parameter-dependent Lyapunov function approach to stability analysis for discrete-time LPV systems," in *Proceedings of the IEEE International Conference on Automation and Logistics (ICAL '07)*, pp. 724–728, IEEE, Jinan, China, August 2007.

[12] İ. E. Köse, F. Jabbari, and W. E. Schmitendorf, "A direct characterization of L_2-gain controllers for LPV systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 9, pp. 1302–1307, 1998.

[13] J. D. Caigny, J. F. Camino, R. C. L. F. Oliveira, P. L. D. Peres, and J. Swevers, "Gain-scheduled H_∞-control of discrete-time polytopic time-varying systems," in *Proceedings of the 47th IEEE Conference on Decision and Control (CDC '08)*, pp. 3872–3877, Cancun, Mexico, December 2008.

[14] J. De Caigny, J. F. Camino, R. C. L. F. Oliveira, P. L. D. Peres, and J. Swevers, "Gain-scheduled H_2 and H_∞ control of discrete-time polytopic time-varying systems," *IET Control Theory & Applications*, vol. 4, no. 3, pp. 362–380, 2010.

[15] K. Y. Chang and W. J. Chang, "Variable structure controller design with H_∞ horm and variance constraints for stochastic model reference systems," *IEE Proceedings Part D: Control Theory and Applications*, vol. 146, no. 6, pp. 511–516, 1999.

[16] G. Pipeleers, B. Demeulenaere, J. Swevers, and L. Vandenberghe, "Extended LMI characterizations for stability and performance of linear systems," *Systems& Control Letters*, vol. 58, no. 7, pp. 510–518, 2009.

[17] L. Jetto and V. Orsini, "Efficient LMI-based quadratic stabilization of interval LPV systems with noisy parameter measures," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 993–998, 2010.

[18] T. Luspay, B. Kulcsar, and K. Grigoriadis, "Mean-square optimal control of linear parameter varying systems with noisy scheduling parameter measurements," in *Proceedings of the 1st American Control Conference (ACC '13)*, pp. 6084–6089, June 2013.

[19] E. Gershon, U. Shaked, and I. Yaesh, *H_∞ Control and Estimation of State-Multiplicative Linear Systems*, Springer, London, UK, 2005.

[20] L. El Ghaoui, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Systems & Control Letters*, vol. 24, no. 3, pp. 223–228, 1995.

[21] Q. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 126–142, 2014.

[22] H. Wang and Q. Zhu, "Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form," *Automatica*, vol. 54, pp. 284–291, 2015.

[23] Q. Zhu and J. Cao, "Mean-square exponential input-to-state stability of stochastic delayed neural networks," *Neurocomputing*, vol. 131, pp. 157–163, 2014.

[24] T. Hou, W. Zhang, and H. Ma, "Finite horizon H_2/H_∞ control for discrete-time stochastic systems with Markovian jumps and multiplicative noise," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1185–1191, 2010.

[25] T. Hou, W. Zhang, and H. Ma, "A game-based control design for discrete-time Markov jump systems with multiplicative noise," *IET Control Theory & Applications*, vol. 7, no. 5, pp. 773–783, 2013.

[26] X. Zhang, D. W. C. Ho, and G. Lu, "Robust stabilization and state estimation for uncertain stochastic discrete-delay large-scale systems," in *Proceedings of the 8th International Conference on Control, Automation, Robotics and Vision (ICARCV '04)*, vol. 1, pp. 42–47, December 2004.

[27] H. Xu and S. Jagannathan, "Stochastic optimal controller design for uncertain nonlinear networked control system via neuro dynamic programming," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 24, no. 3, pp. 471–484, 2013.

[28] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1994.

[29] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*, The MathWorks, Natick, Mass, USA, 1995.

[30] L. Song, Q. Yao, and S. Yan, "Discrete variable structure control design and its application to a ship autopilot servo system," in *Proceedings of the 8th International Conference on Electrical Machines and Systems*, vol. 2, pp. 1621–1624, 2005.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 901461, 7 pages
http://dx.doi.org/10.1155/2015/901461

Research Article

Evolutionary Game-Theoretic Solution for Virtual Routers with Padding Misbehavior in Cloud Computing

Xia-an Bi, Xiaohui Wang, and Hong Liu

College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China

Correspondence should be addressed to Xia-an Bi; bixiaan@hnu.edu.cn

Received 12 May 2015; Accepted 21 July 2015

Academic Editor: Son Nguyen

With the development of cloud computing and virtualization, a physical router can be multiplexed as a large number of virtual routers. TCP-based interactive applications have an incentive to improve their performance by padding "junk packets" into the network among real communication packets. This padding misbehavior will upgrade short TCP flows from "mice" to "elephants" and consequently lead to network congestion and breakdown. This paper presents a detailed solution and analysis for describing the normal behavior and padding misbehavior of virtual routers. In particular, a system model for analyzing behavior of virtual routers is based on evolutionary game model, and, through analyzing the stability of the equilibrium points, the stable point is the solution to the problem. The clear evolutionary path of network applications with the normal behavior and padding misbehavior is analyzed by the corresponding graph. Then this paper gives the behavior control suggestions to effectively restrain the padding misbehavior and maintain stable high-throughputs of the router. The simulation results demonstrate that our solution can effectively restrain the padding misbehavior and maintain stable high-throughputs of the router simultaneously compared with the classical queue management.

1. Introduction

During the past decade, cloud computing has been significantly developed, which provides a large number of cloud services. Lots of cloud computing service providers appear, such as Amazon and Google. The basis of cloud computing is to build virtual networks to meet the needs of the cloud services, and the deployment of virtual network requires virtual routers [1]. An ideal goal of virtualization is that large numbers of virtual routers can simultaneously run on the same physical platform. Scaling to virtual routers poses serious challenges to the performance of routers, such as low memory access, low false positive probability, flexibility of update, and high scalability. With the rapid development of processor architecture and virtual networks, the operational mode has changed dramatically. First, the physical routing platform has been transforming from single-core processors to multicore processors. Second, a physical router needs to be multiplexed as a large number of virtual routers, as Figure 1 shows. Therefore, it is significant to study the efficient algorithm of a physical router to support scalable virtual routers [2].

However, with the increasing demands of network bandwidth, network applications are also facing challenges. The challenges are to solve the problem of bandwidth competition in data transmission [3]. It is a more serious problem that a cloud computing service provider synchronously provides services for large numbers of different applications. In traditional network, it is well known that short TCP flows may experience significant performance degradations when they multiplex with long-lived TCP flows and UDP flows. The reason is that they tend to occupy more bandwidth resources for a long time and usually seize bandwidth in an unfriendly way [4]. Consequently, a new phenomenon appears. In order to obtain enough bandwidth resources, TCP-based interactive applications, such as gaming, telnet, or persistent HTTP, have an incentive to improve their performance. In particular, they send "junk packets" into the network among real communication packets, which is called padding misbehavior. This kind of misbehavior may

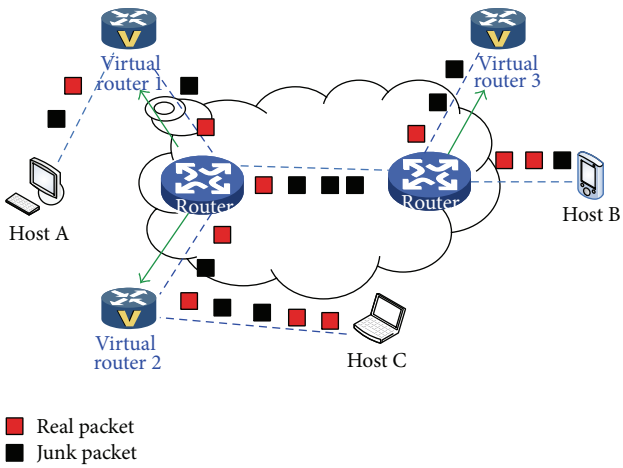


Figure 1: Padding misbehavior of network applications.

guarantee the bandwidth benefits of short TCP flows, but it is a tragedy to the cloud network, because the misbehavior will upgrade short TCP flows from "mice" to "elephants," leading to network congestion eventually [5].

Unfortunately, whether at the transportation or the application levels, it is easy to upgrade an interactive application to a fully backlogged flow. The TCP-based interactive applications realize the padding misbehavior by utilizing virtual routers on one physical router. As shown in Figure 1, in a card game application, there are three terminals, Host A, Host B, and Host C, and they play cards together. The card game applications activate three virtual routers in the cloud computing network for the hosts playing the game. The red blocks are the real packets for their card game, while the black blocks are the junk packets. The three hosts apply this kind of padding misbehavior in order to continuously occupy the network bandwidth and improve their performance. On account of consuming huge bandwidth resources of the network, such misbehavior easily lead to network congestion [6].

In recent years, the studies of the above problems focus on the queue management schemes which are generally based on two packet processing types, Drop-tail and RED [7], for instance, ARED [8], BLUE [9], PI [10], and PAQM [11] algorithms. However, both network-based and endpoint-based mechanisms that check for TCP-friendliness are incapable of detecting any violation, simply because all flows are TCP friendly. In these methods several packets would be dropped without considering the padding misbehavior. So the exiting schemes could not effectively avoid such misbehavior and the terminals could not apply appropriate communication behavior.

Game theory has been a mature and hot topic with many applications in various research fields. In order to solve the above mentioned problems, the evolutionary game theory which extends traditional game analysis is utilized in this paper. Evolutionary game theory which was first developed by Fisher to explain the approximate equality of the sex ratio in mammals in 1930 has been used to explain a number of aspects of human behavior [12]. In the evolutionary game theory, the decision-making behavior is based on the interaction among agents and helps the system to realize utility maximization. Affected by internal and external factors, participants' strategies and game's results will be presented in different states [13]. These features enable the game theory to well model the interaction of short TCP flows and obtain the solution of the padding misbehavior of virtual routers.

The main contributions of this paper can be summarized as follows:

(1) We built a novel game model which is based on the evolutionary game for describing the normal behavior and padding misbehavior of virtual routers.

(2) This paper analyses the stability of the equilibrium points which depends on the value of $\Delta J \cdot \mathrm{tr} J$, and the stable point is the solution to the problem.

(3) This paper analyses the corresponding evolutionary path of network applications with the normal behavior and padding misbehavior and gives the behavior control suggestions of padding misbehavior.

The rest of the paper is organized as follows. In Section 2, we build a game model of virtual routers for describing behavior. Section 3 gives the stability analysis of our evolutionary game model. Section 4 is further discussions and control suggestions. Section 5 is Simulation Experiment and Result Analysis. Finally, Section 6 concludes the paper.

2. System Model and Problem Analysis

2.1. Motivation. This paper, respectively, proposes a model and an algorithm on the basis of the evolutionary game theory. Comparing with the other game theories, the evolutionary game theory has four characteristics which can be described by a dynamic process for virtual routers to select normal behavior and padding misbehavior in our problem. In the stable equilibrium of the evolutionary game, none of the virtual routers is able to obtain the global utility information in our problem, so it is an appropriate choice to model our problem based on the evolutionary game.

(1) Dynamics selection: in evolutionary games, players can observe other players' behavior, learn from observation, and make the best decisions according to their knowledge. Furthermore, dynamic equations can be utilized to analyze how players adjust their behavior to achieve the desired solution.

(2) Bounded rationality: in the evolutionary game, players slowly change their strategies in order to achieve ideal solution comparing with the players in a classical single-play and noncooperative game; this is due to the lack of global information, which is leading the player's inadequate rationality.

(3) Efficient solution: in the evolutionary game theory, evolutionary equilibrium provides such an efficient solution, and it ensures the stability which means all the players will not change their chosen strategies over time.

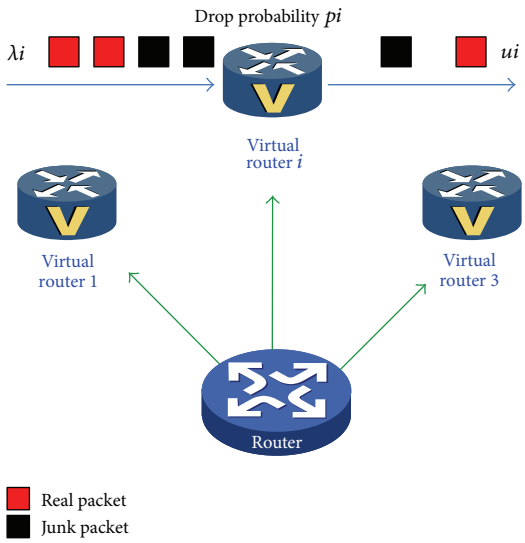


FIGURE 2: The obtained throughput of virtual routers.

(4) Stability: in Nash equilibrium, no player will change his or her strategy. Because they could not get benefits by changing his or her chosen strategy while other players maintain theirs unchanged [14].

2.2. System Model for Describing Behavior of Virtual Routers. This paper considers an evolutionary game model for virtual routers as Figure 2 shows, and there are a set of virtual routers VRi, $i \in \{1, 2, \ldots, n\}$. The virtual routers can be activated or inactivated by different TCP-based interactive applications. Usually, the applications adopt normal behavior and a single packet loss can force a short-lived TCP flow to experience long retransmission, so some interactive applications adopt padding misbehavior by virtual routers for data transmission. The problems we want to solve are how to restrain the padding misbehavior and what the condition of the selection convergence is. So we model the problem by using a dynamic evolutionary game.

The evolutionary game model can be described by the elements as follows.

(1) Players: virtual routers VRi, $i \in \{1, 2, \ldots, n\}$ are the players of the evolutionary game.

(2) Strategy: the strategies of the players are the two kinds of behavior: normal behavior and padding misbehavior.

(3) Input packet rate: it is assumed that the average packet arrival rate of the virtual routers obey the Poisson distribution. As shown in Figure 2, λi is the input rate of VRi.

(4) Drop packet probability: the algorithm is designed to use drop probability to manage the packet flows which pass through the ports of the physical router. When network congestion occurs, the algorithm will drop some packets with drop probability. Different behavior has different drop probabilities. It is assumed that the drop probability of virtual routers which adopt normal behavior is p, and the drop probability of virtual routers which adopt padding behavior is q. Obviously, the values' relationship of the two behavior is $q > p$.

(5) Output packet rate: it is the throughput of the virtual routers and is also the player's departure rate of passing through the physical router port. As shown in Figure 2, ui is the input rate of VRi.

(6) Payoff: the payoff is the obtained throughput of a flow when it departs from the virtual router. It is also the characteristic value of the noncooperative TCP-based interactive applications, and the Nash equilibrium of the whole system could be reached by these values.

In cloud computing network, there are lots of applications. They belong to two kinds: one kind is UDP-based applications and the other kind is TCP-based interactive applications. An application choosing normal behavior concentrates on mutual benefits, which is willing to share the stability networking in the data transmission process. An application choosing padding misbehavior tends to obtain more and more network bandwidth and this will cause network congestion [15]. During data's transmission, we assume that any application in the network has the opportunity to adopt padding misbehavior, which is used to achieve the goal of maximizing the benefits of the application.

Let X denote the proportion of applications which choose normal behavior in the TCP-based interactive applications; then $1-X$ denote the proportion of applications which choose padding misbehavior in the TCP-based interactive applications. Let Y denote the proportion of applications which choose normal behavior in the UDP-based applications; then $1 - Y$ denote the proportion of applications which choose padding misbehavior in the UDP-based applications.

We assume the following. (1) Whether it is TCP-based interactive applications or UDP-based applications, as long as both game participants adopt normal behavior, then they both gain $ui = \lambda i(1 - p)$. The gain obeys the linear pricing model, and the reason is that, with the drop package probability increasing, more packages will be dropped, and the number of nondropped packages will decrease. The linear pricing model is beneficial to avoid congestion and improve the system performance. (2) Whether it is TCP-based interactive applications or UDP-based applications, as long as both game participants adopt padding misbehavior, then they both gain $ui = 0$. In this case, the whole network will go into congestion state and breakdown. (3) Whether it is TCP-based interactive applications or UDP-based applications, as long as one chooses padding misbehavior and the other chooses normal behavior; then the participants who choose normal behavior gain $ui = \lambda i(1 - p) + C$, where C denotes the benefit because of the penalty of padding misbehavior. The participants who choose padding misbehavior gain $ui = (1 - Ki)\lambda i(1 - q)$, where Ki is the padding factor which equals the percentage of padding packets in the whole packets, and $q > p$, which means that this misbehavior will obtain punishment. So we can get the payoff matrix of TCP-based interactive applications or UDP-based applications as shown in Table 1.

Table 1: The payoff matrix of TCP-based interactive applications or UDP-based applications.

	TCP-based interactive applications	
	Normal behavior (X)	Padding misbehavior ($1-X$)
UDP-based applications		
Normal behavior (Y)	$(\lambda i(1-p), \lambda j(1-p))$	$(\lambda i(1-p)+C, (1-Kj)\lambda j(1-q))$
Padding misbehavior ($1-Y$)	$((1-Ki)\lambda i(1-q), \lambda j(1-p)+C)$	$(0,0)$

3. Solution and Stability Analysis of the Model

It is assumed that an evolutionary game has m strategies. At time t, the number of individuals that choose strategy i ($i = 1, 2, \ldots, m$) is denoted by $n_i(t)$, so the total number of individuals is denoted by $N = \sum_{i=1}^{m} n_i(t)$. Then the state of each strategy is given by $X(t) = \{x_1(t), x_2(t), \ldots, x_k(t)\}$, where $x_i = n_i/N$ is the proportion of strategy i. In our game model, the replicator dynamics can be defined as follows [16]:

$$\frac{d(x_i(t))}{dt} = x_i(t) * \left[U_i(t) - \overline{U}(t)\right], \quad i \in \{1, 2, \ldots, I\}. \quad (1)$$

By the utility $U_i(t)$, the average utility $\overline{U}(t)$, and $d(x_i(t))/dt = 0$, we can obtain the fixed points of the differential equations. The stable fixed points form a proportion distribution which is the evolutionary equilibrium, and the dynamic evolutionary game will converge to the equilibrium.

Based on the replicator equation (1), we can get the dynamic equations for the evolution of the padding misbehavior and normal behavior in the network

$$\begin{aligned} \frac{dX_R}{dt} &= X_R \cdot \left(E_R - \overline{E}\right) \\ &= X_R \cdot \left[E_R - X_R \cdot E_R - (1 - X_R) \cdot E_O\right] \\ &= X_R \cdot (1 - X_R) \cdot (E_R - E_O), \end{aligned} \quad (2)$$

where X_R denotes the proportion of applications which choose normal behavior, $0 \le X_R \le 1$, $1 - X_R$ denotes the proportion of applications which choose padding misbehavior, E_R denotes the expected revenue of the applications which choose normal behavior, E_O denotes the expected revenue of the applications which choose padding misbehavior, and $\overline{E}$ denotes the average expected revenue of all applications.

Based on the payoff matrix in Table 1, we can get

$$\begin{aligned} \frac{dX}{dt} &= X(1-X)\left[Y\lambda i(1-p) \right. \\ &\quad \left. + (1-Y)(1-Ki)\lambda i(1-q) - Y\lambda i(1-p) - YC\right] \\ &= X(1-X)\left[(1-Y)(1-Ki)\lambda i(1-q) - YC\right], \\ \frac{dY}{dt} &= Y(1-Y)\left[X\lambda j(1-p) \right. \\ &\quad \left. + (1-X)(1-Kj)\lambda j(1-q) - X\lambda j(1-p) - XC\right] \\ &= Y(1-Y)\left[(1-X)(1-Kj)\lambda j(1-q) - XC\right]. \end{aligned} \quad (3)$$

Let $dX/dt = 0$, $dY/dt = 0$, and we can get five equilibrium points of the model: $O(0,0)$, $A(0,1)$, $B(1,0)$, $C(1,1)$, and $D(1 - C/((1 - Kj)\lambda j(1 - q) + C), 1 - C/((1 - Ki)\lambda i(1 - q) + C))$.

The stability of the equilibrium point is obtained by local stability analysis of the Jacobian matrix of the system [17]. When $\Delta J \cdot \mathrm{tr}J < 0$, the equilibrium point is stable; when $\Delta J \cdot \mathrm{tr}J > 0$, the equilibrium point is not stable; when $\Delta J \cdot \mathrm{tr}J = 0$, the equilibrium point is a saddle point. Based on formulas (3), we can get the Jacobian J:

$$\begin{aligned} J &= \begin{bmatrix} \dfrac{\partial(dX/dt)}{\partial X} & \dfrac{\partial(dX/dt)}{\partial Y} \\ \dfrac{\partial(dY/dt)}{\partial X} & \dfrac{\partial(dY/dt)}{\partial Y} \end{bmatrix} \\ &= \begin{bmatrix} (1-2X)\left[(1-Y)(1-Ki)\lambda i(1-q) - YC\right] & -X(1-X)\left[(1-Ki)\lambda i(1-q) + C\right] \\ -Y(1-Y)\left((1-Kj)\lambda j(1-q) + C\right) & (1-2Y)\left[(1-X)(1-Kj)\lambda j(1-q) - XC\right] \end{bmatrix}. \end{aligned} \quad (4)$$

So we can get

$$\begin{aligned} \Delta J &= (1-2X)\left[(1-Y)(1-Ki)\lambda i(1-q) - YC\right] \\ &\quad \cdot (1-2Y)\left[(1-X)(1-Kj)\lambda j(1-q) - XC\right] \\ &\quad - Y(1-Y)\left((1-Kj)\lambda j(1-q) + C\right) \\ &\quad \cdot X(1-X)\left[(1-Ki)\lambda i(1-q) + C\right], \\ \mathrm{tr}J &= (1-2X)\left[(1-Y)\left((1-Ki)\lambda i(1-q) - YC\right]\right. \\ &\quad + (1-2Y)\left[(1-X)(1-Kj)\lambda j(1-q) - XC\right]. \end{aligned} \quad (5)$$

Then we can get the stability of the five equilibrium points.

(1) To the first equilibrium point $O(0,0)$: $\Delta J > 0$, $\mathrm{tr}J > 0$, so $\Delta J \cdot \mathrm{tr}J > 0$; the equilibrium point is not stable. When the initial state is at point $O(0,0)$, the whole network will go into another stable state.

(2) To the second equilibrium point $A(0,1)$: $\Delta J > 0$, $\mathrm{tr}J < 0$, so $\Delta J \cdot \mathrm{tr}J < 0$; the equilibrium point is stable. When the initial state is at point $A(0,1)$, the evolutionary game will converge.

(3) To the third equilibrium point $B(1,0)$: $\Delta J > 0$, $\mathrm{tr}J < 0$, so $\Delta J \cdot \mathrm{tr}J < 0$; the equilibrium point is stable. When the initial state is at point $B(1,0)$, the evolutionary game will converge.

(4) To the fourth equilibrium point $C(1,1)$: $\Delta J > 0$, $\mathrm{tr}J > 0$, so $\Delta J \cdot \mathrm{tr}J > 0$; the equilibrium point is not stable. When the initial state is at point $C(1,1)$, the whole network will go into another stable state.

(5) To the fifth equilibrium point $D(1-C/((1-Kj)\lambda j(1-q)+C), 1-C/((1-Ki)\lambda i(1-q)+C))$: $\Delta J < 0$, $\mathrm{tr}J = 0$, so $\Delta J \cdot \mathrm{tr}J = 0$; the equilibrium point is not stable, and it is a saddle point.

4. Further Analysis and Behavior Control Suggestions

The active queue management schemes, for instance, RED [7], ARED [8], BLUE [9], PI [10], and PAQM [11], are generally based on two packet processing types, Drop-tail and RED. But in these methods several packets would be dropped without considering the padding misbehavior. Therefore, network congestion could not be effectively avoided and the terminals could not apply appropriate communication behavior. So it is necessary to design a new scheme to avoid congestion caused by padding misbehavior and maintain stable high-throughputs of the router system.

From the above analysis, we can learn that penalty cost is a good way to avoid padding misbehavior. The larger the penalty cost, the more the willingness of the applications to choose normal behavior. On the other hand, there are also other factors that will affect the padding misbehavior in the cloud computing network. Figure 3 shows the corresponding evolutionary path of network applications with the normal behavior and padding misbehavior. The four fixed points A, B, C, and D are the vertices of a square. To the saddle point $D(1-C/((1-Kj)\lambda j(1-q)+C), 1-C/((1-Ki)\lambda i(1-q)+C))$, it is clear that there are four curves, and they divide the square into four parts: I, II, III, and IV. And the position of the point D depend on the parameters C, Kj, Ki, λi, and q. It is clear that the different initial states of the game can lead to different results. When the initial state is in the areas I and II, the game will converge to the stable equilibrium point $B(1,0)$. When the initial state is in the areas III and IV, the game will converge to the stable equilibrium point $A(0,1)$.

To avoid network congestion, the queue management algorithms could restrain padding misbehavior effectively.

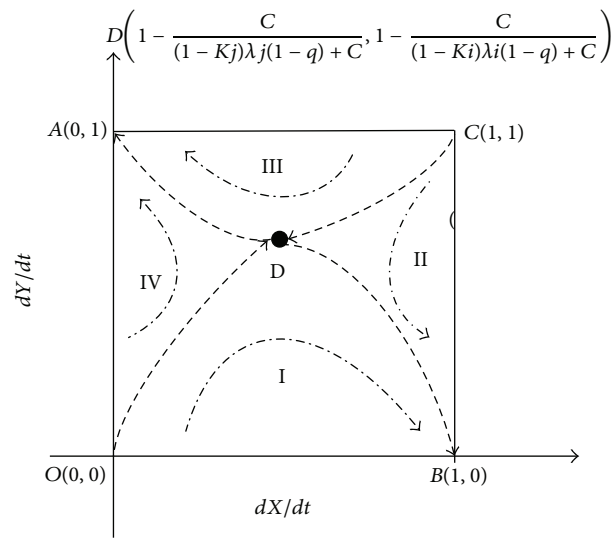


Figure 3: Evolutionary path of network applications.

Therefore, we provide the suggestions aiming at each application to adopt normal behavior in the real network. For TCP-based interactive applications and UDP-based applications, if they both adopt normal behavior, the whole network will run effectively by the network congestion control mechanism; if one of them adopts padding misbehavior, the input packet rate λi or λj is bigger than the normal case, and in order to make the state close to the stable point $A(0,1)$ or $B(1,0)$, the queue management algorithms should drop all the input packets of the applications with padding misbehavior. This way is different from the traditional queue management algorithms dropping packets randomly.

5. Simulation Experiment and Result Analysis

In this section, we present the software simulation experiments on NS-2 (Network Simulator version 2) to test the performance of our solution [18]. NS2 is a discrete event simulator targeted at networking research. It provides substantial support for simulation of TCP, routing, and multicast protocols over wired and wireless (local and satellite) networks. It consists of the following modules: flow generation module, event queue manager module, statistics collection and query module, and so on. The simulation topology is shown in Figure 1, and multiple TCP-based interactive applications and UDP-based applications share links between two physical routers. We compare the performance and effectiveness of our solution with RED solution introduced in Section 1.

In the simulation, it is assumed that the throughputs of Host A, Host B, and Host C are, respectively, set to 100 Kbp, 200 Kbps, and 300 Kbps. The link capacity is 700 Kbps, and each host has the intention to pad "junk packets" in its TCP connection which means everyone applies the padding misbehavior in order to seize more bandwidth. The hosts realized the padding misbehavior by activating the corresponding virtual routers, and the behavior of virtual routers may dynamically transfer from normal behavior to padding

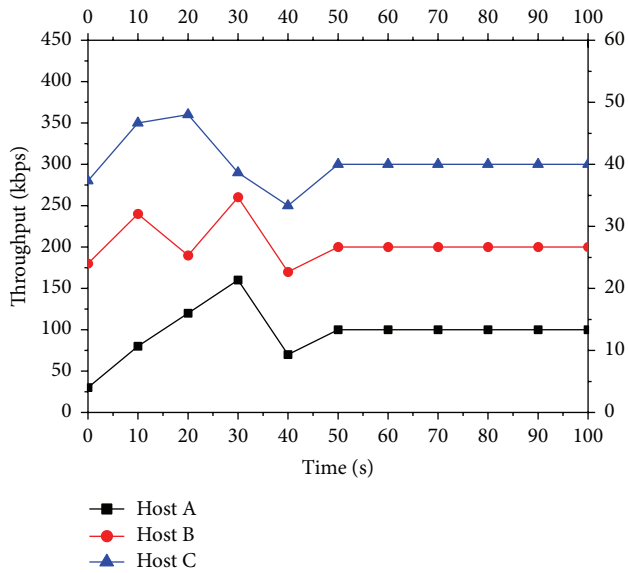


Figure 4: The hosts' throughputs of our solution.

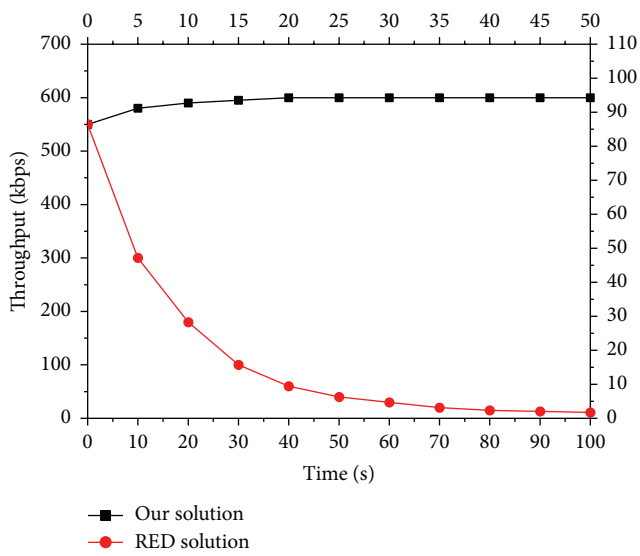


Figure 5: Comparison of the router throughput.

misbehavior. In order to verify that our solution is able to handle this complex scenario, the experimental results show the four hosts' changes of behavior and the whole throughput of the router.

Figures 4 and 5, respectively, show the three hosts' throughputs and the total throughput of RED solution and our solution. As shown in Figure 4, each host's throughput is not stable from 0 s to 50 s, and every host wants to seize bandwidth through padding "junk packets." However, any host which used the padding misbehavior will be punished, which means that the corresponding drop probability will increase. Finally, the hosts' throughputs will remain unchanged. As shown in Figure 5, in our solution the total throughput is stable and the congestion is successfully avoided. By contrast, the total throughput of RED solution obviously decreases as the time increases which means that congestion occurs, and RED solution could not restrain the padding misbehavior.

The experiment results show that our solution is able to effectively restrain the padding misbehavior and maintain stable high-throughputs of the router simultaneously compared with the classical queue management.

6. Conclusion

The core issue of network virtualization is to run multiple virtual router instances on a common physical router hardware platform, and this poses serious challenges to the performance of routers. In this paper, we built a novel game model which is based on the evolutionary game for describing the normal behavior and padding misbehavior of virtual routers and discusses the stability of the equilibrium points which depends on the value of $\Delta J \cdot \mathrm{tr}J$, and the stable point is the solution to the problem. Then we analyze the corresponding evolutionary path of network applications with the normal behavior and padding misbehavior and give the control suggestions of padding misbehavior. The study shows that the behavior control suggestions are able to effectively restrain the padding misbehavior and maintain stable high-throughputs of the router.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the Hunan Provincial Education Department Scientific Research Fund of China (no. 15C0825); Hunan Provincial Science and Technology Program Project of China (no. 2015JC3066); Scientific Research Foundation for Ph.D. Hunan Normal University (no. Math 120641); and Youth Scientific Research Fund of Hunan Normal University (no. 11301).

References

[1] J. Shamsi and M. Brockmeyer, "QoSMap: QoS aware mapping of virtual networks for resiliency and efficiency," in *Proceedings of the IEEE Global Telecommunications Conference (GLOBECOM '07)*, November 2007.

[2] N. Feamster, L. Gao, and J. Rexford, "How to lease the internet in your spare time," *SIGCOMM Computer Communication Reiew*, vol. 37, no. 1, pp. 61–64, 2007.

[3] Y. Sun, F. Zhang, and J. Qi, "Cognitive networks congestion prediction method based on bayesian networks," *International Journal of Advancements in Computing Technology*, vol. 4, no. 7, pp. 329–338, 2012.

[4] A. Mondal and A. Kuzmanovic, "When TCP friendliness becomes harmful," in *Proceedings of the 26th IEEE International Conference on Computer Communications (INFOCOM '07)*, pp. 152–160, Anchorage, Alaska, USA, May 2007.

[5] A. Mondal and A. Kuzmanovic, "Upgrading mice to elephants: effects and end-point solutions," *IEEE/ACM Transactions on Networking*, vol. 18, no. 2, pp. 367–378, 2010.

[6] J. S. Miller, A. Mondal, R. Potharaju, P. A. Dinda, and A. Kuzmanovic, "Understanding end-user perception of network problems," in *Proceedings of the 1st ACM SIGCOMM Workshop on Measurements Up the Stack (SIGCOMM '11)*, pp. 43–48, ACM, Toronto, Canada, August 2011.

[7] S. Floyd and V. Jacobson, "Random early detection gateways for congestion avoidance," *IEEE/ACM Transactions on Networking*, vol. 1, no. 4, pp. 397–413, 1993.

[8] W.-C. Feng, D. D. Kandlur, D. Saha, and K. G. Shin, "A self-configuring RED gateway," in *Proceedings of the 18th International Conference on Computer Communications (INFOCOM '99)*, pp. 1320–1328, New York, NY, USA, March 1999.

[9] J. Wu, J. Liu, and Z. Guo, "A class of active queue management algorithm based on BP neural network," in *Proceedings of the 21th Chinese Control and Decision Conference (CCDC '09)*, pp. 1580–1583, Guilin, China, June 2009.

[10] S. Mascolo, "Congestion control in high-speed communication networks using the Smith principle," *Automatica*, vol. 35, no. 12, pp. 1921–1935, 1999.

[11] V. Firoiu and M. Borden, "Study of active queue management for congestion control," in *Proceedings of the 19th IEEE International Conference on Computer Communications (INFOCOM '00)*, pp. 1435–1445, Tel-Aviv, Israel, March 2000.

[12] R. A. Fisher, "The distribution of gene ratios for rare mutation," *Proceedings of the Royal Society of Edinburgh*, vol. 50, pp. 205–220, 1930.

[13] J. M. Smith, *Evolution and the Theory of Games*, Cambridege University Press, Cambridge, UK, 1982.

[14] X.-A. Bi, D.-F. Zhang, W.-W. Li, and S.-M. He, "A population evolution algorithm for supporting scalable virtual routers," *Journal of Convergence Information Technology*, vol. 7, no. 4, pp. 255–263, 2012.

[15] D. Crisan, R. Birke, K. Barabash, R. Cohen, and M. Gusat, "Datacenter applications in virtualized networks: a cross-layer performance study," *IEEE Journal on Selected Areas in Communications*, vol. 32, no. 1, pp. 77–87, 2014.

[16] P. D. Taylor and L. B. Jonker, "Evolutionarily stable strategies and game dynamics," *Mathematical Biosciences*, vol. 40, no. 1-2, pp. 145–156, 1978.

[17] D. Friedman, "Evolutionary games in economics," *Econometrica. Journal of the Econometric Society*, vol. 59, no. 3, pp. 637–666, 1991.

[18] D. Kumar, S. Ahmed, and Y. Ryu, "Quality of Service (QoS) of voice over MAC protocol 802.11 using NS-2," *Journal of Convergence Information Technology*, vol. 3, no. 4, pp. 76–83, 2008.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 327947, 10 pages
http://dx.doi.org/10.1155/2015/327947

Research Article

Finite-Time Boundedness of Markov Jump System with Piecewise-Constant Transition Probabilities via Dynamic Output Feedback Control

Bin Yan,[1] Xiaojia Zhou,[1] Jun Cheng,[2] and Fangnian Lang[3]

[1] *School of Automation Engineering, University of Electronic Sciences and Technology of China, Chengdu 611731, China*
[2] *School of Information, Hubei University for Nationalities, Enshi 445000, China*
[3] *School of Computer, Chengdu University, Chengdu 610106, China*

Correspondence should be addressed to Bin Yan; uestyan@uestc.edu.cn

Received 14 March 2015; Accepted 28 April 2015

Academic Editor: Leonid Shaikhet

This paper first investigates the problem of finite-time boundedness of Markovian jump system with piecewise-constant transition probabilities via dynamic output feedback control, which leads to both stochastic jumps and deterministic switches. Based on stochastic Lyapunov functional, the concept of finite-time boundedness, average dwell time, and the coupling relationship among time delays, several sufficient conditions are established for finite-time boundedness and H_∞ filtering finite-time boundedness. The system trajectory stays within a prescribed bound. Finally, an example is given to illustrate the efficiency of the proposed method.

1. Introduction

Markovian jump systems were introduced as a class of stochastic switched systems, which can be governed by a Markov chain in a finite mode set of linear dynamics. In recent years, because it is appropriate to model many physical systems with economics, random failures, and networked control systems, more and more people draw their attention to Markovian jump systems [1–4]. As a special class of stochastic systems in the finite operation modes, Markovian jump systems can switch from one to another at different time. Up to now, many important results in the literature are based on the assumption that the complete knowledge of transition probabilities is available in the jump process. However, at mode observation instants, the Markovian jump modes of the systems cannot be accurately obtained, and to get the ideal information on all transition rates is hard or generally expensive in reality, and the obtained results are not accurate. Therefore, it is very important to consider systems based on the assumption that transition probabilities are completely unknown. Recently, the Markovian jump systems subject to partially known transition probabilities have been reported [5–10]. However, the Markov processes are time-invariant in most of aforementioned obtained results.

Nowadays, piecewise-homogeneous (namely, time-varying transition probabilities) Markovian jump systems are developed for practical applications, affecting not only the time-varying transition probabilities but also the state dynamics. The evolution between two operating modes with time-varying transition probabilities was proposed in economy systems [11, 12]. Because of the important issue of the possibility in measuring the variations, up till now, a few people in view of stochastic Markovian jump systems with time-varying transition probabilities except in [13–19]. In [14], there is a bounded real lemma for Markovian jump linear systems with time-varying transition probabilities in discrete-time domain. The Markov switching is employed for sustainability of US external debt in [15]. The linear matrix inequalities are used for control theory in Markov switching [16]. In [19], newly Lyapunov functional is proposed with piecewise-constant transition probabilities. It should be noted that average dwell time switching is very important in dynamic systems [20–23]. In [20], the average dwell time switching and uncertainties are considered. Correspondingly,

a dependent average dwell time approach is proposed in [21]. The piecewise-homogeneous is taken into account which makes the considering dynamic of the Markovian jump systems more controllable and optimizes the performance of systems.

Furthermore, in a finite horizon, the practical application problems tend to care about the described systems' transient characteristics state, especially the transient performances of control systems. It is necessary to consider the state in a fixed region; therefore, the concept of finite-time stability was introduced [22, 23]. Some research results in finite-time case for Markovian jump systems can be found in [24–30]. For example, the finite-time stabilization with output feedback control is introduced in [24]. Finite-time boundedness is considered with state-dependent switching strategy in [26]. In [27], finite-time H_∞ control is proposed for nonlinear jump systems. In [29], the partially unknown transition rates are introduced for finite-time filtering of stochastic systems. It is noted that, in the engineering area, there are still some problems related to stochastic systems to be solved. In order to make the finite-time behaviour of stochastic Markovian jump systems more reasonable and satisfy the requirements, the finite-time boundedness of Markov jump systems with piecewise-constant transition probabilities via dynamic output feedback control has not been studied. The problem is interesting but also challenging, which motivates us to conduct this study.

The main contribution of this paper is that we present a novel approach for finite-time boundedness of Markovian jump system with piecewise-constant transition probabilities via dynamic output feedback control. We establish a more general model to extend the existing results into more feedback control systems. The deterministic switches and stochastic jumps are taken into account at the same times. The finite-time stability is an independent concept, which is different from Lyapunov stability and can be determined by switching. By selecting the appropriate Lyapunov-Krasovskii functional, under average dwell time constraint on switching signals, the sufficient conditions among average dwell times, transition probabilities, and time-varying delay are derived to guarantee finite-time boundedness of the Markovian jump systems.

2. Preliminaries

In this paper, fixing the probability space $(\Omega, \mathscr{F}, \mathscr{P})$, we consider the following Markovian jump system described by

$$\begin{aligned} \dot{x}(t) &= A_{r_t} x(t) + A_{\tau r_t} x(t-\tau) + B_{r_t} u(t) + D_{r_t} \omega(t), \\ y(t) &= C_{r_t} x(t), \\ x(t) &= \phi(t), \\ & \qquad\qquad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathscr{R}^n$ is the state vector of the system, $y(t) \in \mathscr{R}^l$ is the measured output, and $\omega(t) \in L_2^q[0,\infty]$ is the exogenous noise signal. A_{r_t}, $A_{\tau r_t}$, B_{r_t}, D_{r_t}, and C_{r_t} are constant real matrices with appropriate dimension. τ represent the constant delay and $\phi(t)$ is the differentiable vector-valued initial function on $[-\tau, 0]$. Let the random form process r_t, $t \geq 0$ be the Markov stochastic process taking values on a finite set $\mathscr{N} = \{1, 2, \ldots, N\}$, governing the switching from mode i at time t to mode j at time $t + \Delta t$ with the following transition probabilities:

$$\begin{aligned} P_{ij} &= \Pr\left(r_{t+\Delta t} = j \mid r_t = i\right) \\ &= \begin{cases} \mu_{ij}^{(\sigma_t)} \Delta t + o(\Delta t), & i \neq j, \\ 1 + \mu_{ii}^{(\sigma_t)} \Delta t + o(\Delta t), & i = j, \end{cases} \end{aligned} \tag{2}$$

with transition rates $\mu_{ij}^{(\sigma_t)} \geq 0$, $\forall i, j \in \mathscr{N}$, $\sum_{j=1}^{N} \mu_{ij}^{(\sigma_t)} = 0$, $\Delta t > 0$, and $\lim_{\Delta t \to 0}(o(\Delta t)/\Delta t) \to 0$. Here, $\mu_{ij}^{(\sigma_t)}$ is now a function of σ_t. By σ_t, we mean that the transition rates are time-varying. Moreover, σ_t is assumed to be piecewise-constant function of time t, and transition rates Π^{σ_t} can be defined by

$$\Pi^{\sigma_t} = \begin{bmatrix} \mu_{11}^{(\sigma_t)} & \mu_{12}^{(\sigma_t)} & \cdots & \mu_{1N}^{(\sigma_t)} \\ \mu_{21}^{(\sigma_t)} & \mu_{22}^{(\sigma_t)} & \cdots & \mu_{2N}^{(\sigma_t)} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{N1}^{(\sigma_t)} & \mu_{N2}^{(\sigma_t)} & \cdots & \mu_{NN}^{(\sigma_t)} \end{bmatrix}. \tag{3}$$

Furthermore, to determine the time-varying property, σ_t represents a high-level average dwell time switching signal. σ_t is a given initial condition sequence. For simplicity, let m represent σ_t as a piecewise-constant function of time, which takes values in the finite set $\mathscr{M} \equiv \{1, 2, \ldots, M\}$. At an arbitrary time t, σ may be dependent on t or $x(t)$, or both, or other logic rules. For a switching sequence $t_0 < t_1 < t_2 < \cdots$, σ is continuous from right everywhere and maybe either autonomous or controlled. When $k \in [t_l, t_{l+1})$, we say that the σ_{t_l}th transition probabilities matrix is active and therefore the trajectory x_t of system (1) is trajectory of system (1) with the σ_{t_l}th transition probabilities matrix.

In this paper, our goal is to design the following dynamic output feedback controller, which can guarantee the system is finite-time boundness:

$$\begin{aligned} \dot{x}_f(t) &= A_{fr_t,\sigma_t} x_f(t) + B_{fr_t,\sigma_t} y(t), \\ u(t) &= C_{fr_t,\sigma_t} x_f(t) + D_{fr_t,\sigma_t} y(t), \\ x_f(t) &= 0, \\ & \qquad\qquad t \leq 0, \end{aligned} \tag{4}$$

where A_{fr_t,σ_t}, B_{fr_t,σ_t}, C_{fr_t,σ_t}, and D_{fr_t,σ_t} are matrices to be determined.

Substituting (4) into (1) and $\forall r_t = i, i \in \mathcal{N}, \sigma_t = m$, and $m \in \mathcal{M}$, we have

$$\begin{aligned} \dot{\eta}(t) &= \overline{A}_{i,m}\eta(t) + \overline{A}_{\tau i}\eta(t-\tau) + \overline{B}_i\omega(t), \\ \eta(t) &= \psi(t), \\ & \qquad t \in [-\tau, 0], \end{aligned} \tag{5}$$

where

$$\begin{aligned} \overline{A}_{i,m} &= \begin{bmatrix} A_i + B_i D_{fi,m} C_i & B_i C_{fi,m} \\ B_{fi,m} C_i & A_{fi,m} \end{bmatrix}, \\ \overline{A}_{\tau i} &= \begin{bmatrix} A_{\tau i} & 0 \\ 0 & 0 \end{bmatrix}, \\ \overline{B}_i &= \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \\ \eta(t) &= \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}. \end{aligned} \tag{6}$$

Throughout the paper, suppose that the matrices C_{r_t} have full row rank, in other words, $\operatorname{rank}(C_{r_t}) = q$. Then we have the singular decomposition of C_i as

$$C_i = U_i [S_i, 0] V_i^{\mathsf{T}}, \tag{7}$$

where $S_i \in R^{q\times q}$ is a diagonal positive matrix and $U_i \in R^{q\times q}$ and $V_i \in R^{n\times n}$ are unitary matrices.

Remark 1. In this paper, matrices C_i are singular decomposition as unitary matrices, which reduce the conservatism.

First of all, we will give definitions and lemmas about system (5), which plays an important role in the derivation of our result.

Definition 2 (see [29]). System (5) is said to be finite-time bounded with respect to $(c_1, c_2, T, R, d, \sigma)$, where $d \geq 0$, R is positive define matrix, and σ_t is a switching signal. We have

$$\begin{aligned} &\sup_{-\tau\leq t_0\leq 0} \{x^{\mathsf{T}}(t_0) R x(t_0)\} \\ &\quad \leq c_1 \Longrightarrow \sup_{-\tau\leq t\leq 0} \{x^{\mathsf{T}}(t) R x(t)\} < c_2, \quad \forall t \in [0, T], \end{aligned} \tag{8}$$

where $c_2 > c_1 \geq 0$, $\forall \omega(t) : \int_0^T \omega^{\mathsf{T}}(s)\omega(s)\, ds \leq d$.

Definition 3 (see [21]). For any $T_2 > T_1 \geq 0$, let $N_\sigma(T_1, T_2)$ denote the switching number of $\sigma(t)$ during (T_1, T_2). If $N_\sigma(T_1, T_2) \leq N_0 + (T_2 - T_1)/T_a$ holds for $N_0 \geq 0$ and $T_a > 0$, then N_0 and T_a are called chattering bound and average dwell time, respectively. Here we assume $N_0 = 0$ for simplicity as commonly used in the literature.

Definition 4 (see [31]). Consider $V(\eta_t, r_t, \sigma_t)$ as the stochastic Lyapunov function of the resulting system (4); its weak infinitesimal operator is defined as

$$\begin{aligned} &\pounds V(\eta_t, r_t, \sigma_t) \\ &\quad = \lim_{\Delta t\to 0^+} \frac{1}{\Delta t} [\mathbb{E}\{V(\eta_{t+\Delta t}, r_{t+\Delta t}, \sigma_t) \mid r_t = i,\ \sigma_t = m\} \\ &\quad - V(\eta_t, i, m)] = \frac{\partial}{\partial t} V(\eta_t, i, m) + \frac{\partial}{\partial x} V(\eta_t, i, m)\dot{\eta}_t \\ &\quad + \sum_{j=1}^{N} \mu_{ij}^{(m)} V(\eta_t, j, m). \end{aligned} \tag{9}$$

Definition 5 (see [32]). The jump rates of the visited modes from a given mode i are assumed to satisfy

$$0 < \min \mu_i \leq \mu_{ij} \leq \max \mu_i, \quad \forall i, j \in \mathbb{R},\ i \neq j, \tag{10}$$

where $\min \mu_i$ and $\max \mu_i$ are known parameters for a given mode i and represent the lower and upper bounds when all the jump rates are known; that is, $0 < \min \mu_i = \min\{\mu_{ij} \neq 0,\ i \neq j,\ j \in \mathbb{R}\}$ and $\min \mu_i \leq \max \mu_i$. Meanwhile, the number of the visited modes from a given mode i is denoted by N_i including the mode itself.

Lemma 6 (Schur complement [14]). *Given constant matrices X, Y, and Z, where $X = X^{\mathsf{T}}$ and $0 < Y = Y^{\mathsf{T}}$, then $X + Z^{\mathsf{T}}Y^{-1}Z < 0$ if and only if*

$$\begin{aligned} &\begin{bmatrix} X & Z^{\mathsf{T}} \\ * & -Y \end{bmatrix} < 0 \\ \text{or } &\begin{bmatrix} -Y & Z \\ * & X \end{bmatrix} < 0. \end{aligned} \tag{11}$$

3. Finite-Time Boundedness Analysis

Theorem 7. *System (5) is finite-time stochastic boundedness (FTSB) with respect to (c_1, c_2, R, d, T) if there exist matrices $P_{i,m}$, H, and Q_m and constants $\alpha \geq 0$, $\mu > 1$, and $\lambda_s > 0$ $(s = 1, 2, \ldots, 4)$, such that we have the following linear matrix inequalities:*

$$\begin{bmatrix} \overline{A}_{i,m}^{\mathsf{T}} P_{i,m}^{-1} + P_{i,m}^{-1}\overline{A}_{i,m} + Q_m^{-1} + \overline{P}_{i,m} - \alpha P_{i,m}^{-1} & P_{i,m}^{-1}\overline{A}_{\tau i} & P_{i,m}^{-1}\overline{B}_i \\ * & -e^{\alpha\tau}Q_m^{-1} & 0 \\ * & * & -\alpha H \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} -\lambda_1 c_2 e^{-\alpha T} + d\lambda_4\left(1 - e^{-\alpha T}\right) & \lambda_2 c_1 & \lambda_3 c_1 \\ * & -\lambda_2 c_1 & 0 \\ * & * & -\frac{1}{\tau}\lambda_3 c_1 e^{-\alpha\tau} \end{bmatrix} < 0, \tag{13}$$

$$\begin{bmatrix} -P_{i,m}^{-1} & \lambda_1 P_{i,m}^{-1} R \\ * & -\lambda_1 R \end{bmatrix} < 0,$$
$$\begin{bmatrix} -\lambda_2 R & I \\ * & -P_{i,m}^{-1} \end{bmatrix} < 0, \quad \forall i \in \mathcal{N} \tag{14}$$

$$P_{i,m} < \mu P_{i,n},$$
$$Q_m \le Q_n, \quad \forall i, j \in \mathcal{N},\ m, n \in \mathcal{M}, \tag{15}$$

$$\lambda_1 c_2 e^{-\alpha T} > (\lambda_2 + \tau e^{\alpha\tau}\lambda_3) c_1 + d\lambda_4 \left(1 - e^{-\alpha T}\right), \tag{16}$$

with the average dwell time of the switching signal σ satisfying

$$\tau_a > \tau_a^* = \frac{T \ln \mu}{\ln(\lambda_1 c_2) - \ln\left[(\lambda_2 + \tau e^{\alpha\tau}\lambda_3) c_1 + d\lambda_4 (1 - e^{-\alpha T})\right] - \alpha T}, \tag{17}$$

where

$$\begin{aligned} \overline{P}_{i,m} &= -(N_i - 1)(\min \mu_i^m) P_{i,m} + (\max \mu_i^m) \sum_{j=1, j\ne i}^{N} P_{j,m}, \\ \widetilde{P}_{i,m} &= R^{-1/2} P_{i,m} R^{-1/2}, \\ \widetilde{Q}_m &= R^{-1/2} Q_m R^{-1/2}, \\ \lambda_3 &= \lambda_{\max}(Q_m), \\ \lambda_4 &= \lambda_{\max}(H). \end{aligned} \tag{18}$$

Proof. We consider the following Lyapunov-Krasovskii functional:

$$V(\eta_t, r_t, \sigma_t) = \eta_t^{\mathsf{T}} P_{r_t,\sigma_t}^{-1} \eta_t + \int_{t-\tau}^{t} \eta_s^{\mathsf{T}} e^{\alpha(t-s)} Q_{\sigma_s}^{-1} \eta_s \, ds. \tag{19}$$

Taking the time derivative of $V(\eta_t, r_t, \sigma(t))$ along the trajectory of the system (5), one has

$$\begin{aligned} \pounds V(\eta_t, r_t, \sigma_t) &= \eta_t^{\mathsf{T}} \left(\overline{A}_{i,m}^{\mathsf{T}} P_{i,m}^{-1} + P_{i,m}^{-1} \overline{A}_{i,m} + Q_m^{-1} + \sum_{j=1}^{N} \mu_{ij}^{(m)} P_{j,m} \right) \eta_t \\ &\quad + 2\eta_t^{\mathsf{T}} P_{i,m}^{-1} \overline{A}_{\tau i,m} \eta_{t-\tau} + 2\eta_t^{\mathsf{T}} P_{i,m}^{-1} \overline{B}_i \omega_t \\ &\quad - e^{\alpha\tau} \eta_{t-\tau}^{\mathsf{T}} Q_m^{-1} \eta_{t-\tau} + \alpha \int_{t-\tau}^{t} \eta_s^{\mathsf{T}} e^{\alpha(t-s)} Q_{\sigma_s}^{-1} \eta_s \, ds. \end{aligned} \tag{20}$$

Moreover, we have

$$\begin{aligned} \sum_{j=1}^{N} \mu_{ij}^{(m)} P_{j,m}^{-1} &= \mu_{ii}^m P_{i,m}^{-1} + \sum_{j=1, j\ne i}^{N} \mu_{ij}^{(m)} P_{j,m}^{-1} \\ &= -\sum_{j=1, j\ne i}^{N} \mu_{ij}^{(m)} P_{i,m}^{-1} + \sum_{j=1, j\ne i}^{N} \mu_{ij}^{(m)} P_{j,m}^{-1} \\ &\le -(N_i - 1)(\min \mu_i^m) P_{i,m}^{-1} \\ &\quad + (\max \mu_i^m) \sum_{j=1, j\ne i}^{N} P_{j,m}^{-1}. \end{aligned} \tag{21}$$

Assuming that condition (12) is satisfied, we obtain

$$\pounds V(\eta_t, r_t, \sigma_t) - \alpha V(\eta_t, r_t, \sigma_t) < \alpha \omega_t^{\mathsf{T}} H \omega_t. \tag{22}$$

Notice that

$$\frac{d}{dt}\left(e^{-\alpha t} V(\eta_t, r_t, \sigma_t)\right) < \alpha e^{-\alpha t} \omega_t^{\mathsf{T}} H \omega_t. \tag{23}$$

Integrate (23) from t_k to t, from which we can get that

$$V(\eta_t, r_t, \sigma_t) < e^{\alpha(t-t_k)} V\left(\eta_{t_k}, r_{t_k}, \sigma_{t_k}\right) + \alpha \int_{t_k}^{t} e^{\alpha(t-s)} \omega_s^{\mathsf{T}} H \omega_s \, ds. \tag{24}$$

Noting that $\forall t \in [t_k, t_{k+1}]$, where t_k is the kth switching instant and $x_{t_k} = x_{t_k^-}$, from condition (15) it yields

$$V\left(\eta_{t_k}, r_{t_k}, \sigma_{t_k}\right) \le \mu V\left(\eta_{t_k^-}, r_{t_k^-}, \sigma_{t_k^-}\right). \tag{25}$$

From condition (24) and (25), we can easily have

$$V\left(\eta_{t_k}, r_{t_k}, \sigma_{t_k}\right) < \mu e^{\alpha(t-t_k)} V\left(\eta_{t_k^-}, r_{t_k^-}, \sigma_{t_k^-}\right) + \alpha\mu \int_{t_k}^{t} e^{\alpha(t-s)} \omega_s^{\mathsf{T}} H \omega_s \, ds. \tag{26}$$

Thus, from (24)–(26), it yields

$$V(\eta_t, r_t, \sigma_t) \le e^{\alpha(t-t_k)} V\left(\eta_{t_k}, r_{t_k}, \sigma_{t_k}\right) + \alpha \int_{t_k}^{t} e^{\alpha(t-s)} \omega_s^{\mathsf{T}} H \omega_s \, ds$$

$$
\begin{aligned}
&\le \mu e^{\alpha(t-t_{k-1})} V\left(\eta_{t_{k-1}^-}, r_{t_{k-1}^-}, \sigma_{t_{k-1}^-}\right) \\
&+ \alpha\mu \int_{t_{k-1}}^{t_k} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds + \alpha \int_{t_k}^{t} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds \\
&\le \mu^2 e^{\alpha(t_k - t_{k-2})} V\left(\eta_{t_{k-2}^-}, r_{t_{k-2}^-}, \sigma_{t_{k-2}^-}\right) \\
&+ \alpha\mu^2 \int_{t_{k-2}}^{t_{k-1}} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds \\
&+ \alpha\mu \int_{t_{k-1}}^{t_k} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds + \alpha \int_{t_k}^{t} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds \\
&\le \cdots \le \mu^{N_\sigma(0,t)} e^{\alpha t} V\left(\eta_{t_0}, r_{t_0}, \sigma_{t_0}\right) \\
&+ \alpha\mu^{N_\sigma(0,t)} \int_0^{t_1} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds \\
&+ \alpha\mu^{N_\sigma(t_1,t)} \int_{t_1}^{t_2} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds + \cdots \\
&+ \alpha \int_{t_k}^{t} e^{\alpha(t-s)} \omega_s^{\mathrm{T}} H \omega_s \, ds = \mu^{N_\sigma(0,t)} e^{\alpha t} V\left(\eta_{t_0}, r_{t_0}, \sigma_{t_0}\right) \\
&+ \alpha \int_0^{t} e^{\alpha(t-s)} \mu^{N_\sigma(s,t)} \omega_s^{\mathrm{T}} H \omega_s \, ds \\
&\le \mu^{N_\sigma(0,t)} e^{\alpha t} V\left(\eta_{t_0}, r_{t_0}, \sigma_{t_0}\right) + \alpha\mu^{N_\sigma(0,t)} d\lambda_{\max}(H) \\
&\cdot e^{\alpha t} \int_0^{t} e^{-\alpha s} \, ds \le \mu^{N_\sigma(0,T)} e^{\alpha T} \left\{ V\left(\eta_{t_0}, r_{t_0}, \sigma_{t_0}\right) \right. \\
&\left. + d\lambda_{\max}(H)\, \alpha \int_0^{T} e^{-\alpha s} \, ds \right\} \\
&\le \mu^{T/\tau_a} e^{\alpha T} \left\{ V\left(\eta_{t_0}, r_{t_0}, \sigma_{t_0}\right) \right. \\
&\left. + d\lambda_{\max}(H) \left(1 - e^{-\alpha T}\right) \right\}.
\end{aligned}
\tag{27}
$$

Note that

$$
\begin{aligned}
&V\left(\eta_{t_0}, r_{t_0}, \sigma_{t_0}\right) \\
&= \eta_{t_0}^{\mathrm{T}} P_{r_{t_0}, \sigma(t_0)} \eta_{t_0} + \int_{-\tau}^{0} \eta_s^{\mathrm{T}} e^{-\alpha s} Q_{\sigma_s} \eta_s \, ds \\
&\le \lambda_{\max}\left(\widetilde{P}_{i,m}\right) \eta_{t_0}^{\mathrm{T}} R \eta_{t_0} \\
&\quad + \tau e^{\alpha\tau} \lambda_{\max}\left(\widetilde{Q}_m\right) \sup_{-\tau \le \theta \le 0} \left\{ \eta_\theta^{\mathrm{T}} R \eta_\theta \right\} \\
&\le \left(\lambda_{\max}\left(\widetilde{P}_{i,m}\right) + \tau e^{\alpha\tau} \lambda_{\max}\left(\widetilde{Q}_m\right)\right) \sup_{-\tau \le \theta \le 0} \left\{ \eta_\theta^{\mathrm{T}} R \eta_\theta \right\} \\
&\le \left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) \sup_{-\tau \le \theta \le 0} \left\{ \eta_\theta^{\mathrm{T}} R \eta_\theta \right\} \\
&\le \left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) c_1.
\end{aligned}
\tag{28}
$$

Thus

$$
\begin{aligned}
V\left(\eta_t, r_t, \sigma_t\right) &\le \mu^{T/\tau_a} e^{\alpha T} \left\{ \left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) c_1 \right. \\
&\quad \left. + d\lambda_4 \left(1 - e^{-\alpha T}\right) \right\} \\
&= e^{(\alpha + ((\ln \mu)/\tau_a))T} \left\{ \left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) c_1 \right. \\
&\quad \left. + d\lambda_4 \left(1 - e^{-\alpha T}\right) \right\}.
\end{aligned}
\tag{29}
$$

On the other hand

$$
V\left(\eta_t, r_t, \sigma_t\right) \ge \lambda_{\max}\left(\widetilde{P}_{r_t, \sigma_t}\right) \eta_t^{\mathrm{T}} R \eta_t = \lambda_1 \eta_t^{\mathrm{T}} R \eta_t. \tag{30}
$$

Substituting (29) and (30) into (19), one obtains

$$
\begin{aligned}
&x^{\mathrm{T}}(t) R x(t) \\
&\quad \le \frac{\left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) c_1 + d\lambda_4 \left(1 - e^{-\alpha T}\right)}{\lambda_1} e^{(\alpha + ((\ln \mu)/\tau_a))T}.
\end{aligned}
\tag{31}
$$

When $\mu = 1$, which is the trivial case, from (17), $\eta_t^{\mathrm{T}} R x_t < c_2 e^{-\alpha T} e^{\alpha T} = c_2$. When $\mu > 1$, from (17), $\ln(\lambda_1 c_2) - \ln[(\lambda_2 + \tau e^{\alpha\tau} \lambda_3) c_1 + d\lambda_4 (1 - e^{-\alpha T})] - \alpha T > 0$ we have

$$
\begin{aligned}
&\frac{T_f}{\tau_a} \\
&< \frac{\ln(\lambda_1 c_2) - \ln\left[\left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) c_1 + d\lambda_4 \left(1 - e^{-\alpha T}\right)\right] - \alpha T}{\ln \mu} \\
&= \frac{\ln\left(\lambda_1 c_2 e^{-\alpha T} / \left(\left(\lambda_2 + \tau e^{\alpha\tau} \lambda_3\right) c_1 + d\lambda_4 \left(1 - e^{-\alpha T}\right)\right)\right)}{\ln \mu}.
\end{aligned}
\tag{32}
$$

Substituting (32) into (31) yields

$$
\eta_t^{\mathrm{T}} R \eta_t < c_2. \tag{33}
$$

The proof is completed. □

Remark 8. It should be noted that the linear feedback control subject to piecewise constant transition probability is first considered in the paper, and it is classical and effective to stabilize the Markov jump system.

4. Finite-Time H_∞ Performance Analysis

Theorem 9. *For a given constant $T > 0$, $\alpha > 0$, system (5) is robustly finite-time stochastic boundedness with respect to $(0, c_2, N, R, d, \sigma)$, if there exist positive definite matrices $X_{i,m}$, Q_{1m}, Q_{2m}, H, and $\mu > 1$, such that the following linear matrix inequalities*

$$\begin{bmatrix} \Sigma_{11i,m} & \Sigma_{12i,m} & A_{\tau i}Q_{1m} & 0 & D_i & X_{i,m} & 0 & \Sigma_{18i} & \Sigma_{19i} \\ * & \Sigma_{22i,m} & 0 & 0 & 0 & 0 & X_{i,m} & 0 & 0 \\ * & * & -e^{\alpha\tau}Q_{1m} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -e^{\alpha\tau}Q_{2m} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\alpha H & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{1m} & 0 & 0 & 0 \\ * & * & * & * & * & 0 & -Q_{2m} & 0 & 0 \\ * & * & * & * & * & * & * & \Sigma_{88i} & 0 \\ * & * & * & * & * & * & * & * & \Sigma_{99i} \end{bmatrix} < 0, \tag{34}$$

$$\forall i, j \in \mathcal{N},\ m, n \in \mathcal{M},$$

$$\begin{aligned} X_{i,m} &\le \mu X_{i,n}, \\ Q_{km} &\le Q_{kn}, \end{aligned} \tag{35}$$

$$(k = 1, 2)\ \ \forall i, j \in \mathcal{N},\ m, n \in \mathcal{M},$$

$$\alpha c_2 > d\lambda_4 e^{\alpha T}\left(1 - e^{-\alpha T}\right), \tag{36}$$

where

$$\begin{aligned} &\Sigma_{11i} \\ &\quad = A_i X_{i,m} + X_{i,m}^{\mathsf{T}} A_i + B_i Y_{4i,m} C_i + C_i^{\mathsf{T}} Y_{4i,m}^{\mathsf{T}} B_i^{\mathsf{T}} - \alpha X_{i,m} \\ &\qquad - (N_i - 1)\max \mu_i X_{i,m}, \\ &\Sigma_{12i} = C_i^{\mathsf{T}} Y_{2i,m}^{\mathsf{T}} + B_i Y_{3i,m}, \\ &\Sigma_{22i} = Y_{1i,m} + Y_{1i,m}^{\mathsf{T}} - \alpha X_{i,m}, \\ &\Sigma_{18i} \\ &\quad = \Sigma_{19i} \\ &\quad = \left[\overbrace{\sqrt{\max \mu_i} X_{i,m}, \sqrt{\max \mu_i} X_{i,m}, \ldots, \sqrt{\max \mu_i} X_{i,m}}^{N_i - 1}\right], \\ &\Sigma_{88i} \\ &\quad = \Sigma_{99i} \\ &\quad = -\operatorname{diag}\left\{X_{1,m}, \ldots, X_{i-1,m}, X_{i+1,m}, \ldots, X_{N,m}\right\}. \end{aligned} \tag{37}$$

with the average dwell time of the switching signal σ satisfying

$$\tau_a > \tau_a^* = \frac{T \ln \mu}{\ln(\alpha c_2) - \ln\left[d\lambda_4\left(1 - e^{-\alpha T}\right)\right] - \alpha T}, \tag{38}$$

and the feasible solutions are given as follows:

$$\begin{aligned} A_{fi,m} &= Y_{1i,m} X_{i,m}^{-1}, \\ B_{fi,m} &= Y_{2i,m} U_i S_i X_{i,m} S_i^{-1} U_i^{\mathsf{T}}, \\ C_{fi,m} &= Y_{3i,m} X_{i,m}^{-1}, \\ D_{fi,m} &= Y_{2i,m} U_i S_i X_{i,m} S_i^{-1} U_i^{\mathsf{T}}. \end{aligned} \tag{39}$$

Then the closed-loop systems (5) are finite-time boundedness with respect to $(0, c_2, T, d, R, \sigma)$.

Proof. Pre- and postmultiply inequality (12) by $\operatorname{diag}\{P_{i,m}^{-1}, I, I\}$, it yields that

$$\begin{bmatrix} P_{i,m}\overline{A}_{i,m}^{\mathsf{T}} + \overline{A}_{i,m}P_{i,m} + P_{i,m}\overline{P}_{i,m}P_{i,m} - \alpha P_{i,m} + P_{i,m}Q_m^{-1}P_{i,m} & \overline{A}_{\tau i} & \overline{B}_i \\ * & -e^{\alpha\tau}Q_m^{-1} & 0 \\ * & * & -\alpha H \end{bmatrix} < 0. \tag{40}$$

Denote $P_{i,m} = \mathrm{diag}\{X_{i,m}, X_{i,m}\}$, $Q_m = \mathrm{diag}\{Q_{1m}, Q_{2m}\}$; using Schur complement, we can obtain

$$\begin{bmatrix} \Sigma_{11i,m} & \Sigma_{12i,m} & A_{\tau i}Q_{1m} & 0 & D_i & X_{i,m} & 0 & \Sigma_{18i} & \Sigma_{19i} \\ * & \Sigma_{22i,m} & 0 & 0 & 0 & 0 & X_{i,m} & 0 & 0 \\ * & * & -e^{\alpha\tau}Q_{1m} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -e^{\alpha\tau}Q_{2m} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\alpha H & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{1m} & 0 & 0 & 0 \\ * & * & * & * & * & 0 & -Q_{2m} & 0 & 0 \\ * & * & * & * & * & * & * & \Sigma_{88i} & 0 \\ * & * & * & * & * & * & * & * & \Sigma_{99i} \end{bmatrix} < 0, \tag{41}$$

where

$$\begin{aligned} \Sigma_{11i} &= A_i X_{i,m} + X_{i,m}^{\mathsf{T}} A_i \\ &\quad + B_i D_{fi,m} U_i^{-\mathsf{T}} S_i X_{1i,m} S_i^{-1} U_i^{-1} C_i \\ &\quad + C_i^{\mathsf{T}} U_i^{-\mathsf{T}} S_i^{-\mathsf{T}} X_{1i,m}^{\mathsf{T}} S_i^{\mathsf{T}} U_i^{-1} B_i D_{fi,m}^{\mathsf{T}} - \alpha X_{i,m} \\ &\quad - (N_i - 1) \max \mu_i X_{i,m}, \\ \Sigma_{12i} &= C_i^{\mathsf{T}} U_i^{-\mathsf{T}} S_i^{-\mathsf{T}} X_{i,m} S_i^{\mathsf{T}} U_i^{-1} B_{fi,m}^{\mathsf{T}} + B_i C_{fi,m} X_{i,m}, \\ \Sigma_{22i} &= A_{fi,m} X_{i,m} + X_{i,m} A_{fi,m}^{\mathsf{T}} - \alpha X_{i,m}. \end{aligned} \tag{42}$$

Define

$$\begin{aligned} Y_{1i,m} &= A_{fi,m} X_{i,m}, \\ Y_{2i,m} &= B_{fi,m} U_i^{-\mathsf{T}} S_i X_{i,m} S_i^{-1} U_i^{-1}, \\ Y_{3i,m} &= C_{fi,m} X_{i,m}, \\ Y_{4i,m} &= D_{fi,m} U_i^{-\mathsf{T}} S_i X_{i,m} S_i^{-1} U_i^{-1}. \end{aligned} \tag{43}$$

And $X_{i,m} \le \mu X_{i,n}$ can guarantee that $P_{i,m} \le \mu P_{i,n}$; then we can obtain (34). □

5. Illustrative Example

Consider the system as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -2.0 & -1.5 & -1.2 \\ 0.7 & -1.6 & 0.5 \\ -1.3 & 0.5 & -1.1 \end{bmatrix}, \\ A_{\tau 1} &= \begin{bmatrix} 0.2 & 0.0 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.3 & 0.1 & 0.2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} -1.2 & 0.5 & 0.9 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -1.5 & -1.2 & -1.5 \\ 0.2 & -1.5 & 0.4 \\ -0.7 & 1.1 & -1.2 \end{bmatrix}, \\ A_{\tau 2} &= \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0.5 \\ 0.7 \\ 1.5 \end{bmatrix}, \\ D_2 &= \begin{bmatrix} 0.4 \\ 0.2 \\ 0.3 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} -1.0 & 1.2 & 0.5 \end{bmatrix}. \end{aligned} \tag{44}$$

The piecewise-constant transition probabilities matrices are given as

$$\begin{aligned} \Pi^1 &= \begin{bmatrix} 0.1 & -0.1 \\ -0.9 & 0.9 \end{bmatrix}, \\ \Pi^2 &= \begin{bmatrix} 0.2 & -0.2 \\ -0.6 & 0.6 \end{bmatrix}. \end{aligned} \tag{45}$$

Choosing $\alpha = 0.05, \tau = 0.2, c_2 = 30, T = 10$, and $d = 0.01$, by solving the matrix equalities in Theorem 9, we have the following filter parameters:

$$A_{f1,1} = \begin{bmatrix} 2.6498 & 0.1514 & -3.48944 \\ -1.4622 & -3.1536 & 3.9281 \\ -5.2184 & -3.2024 & -8.9715 \end{bmatrix},$$

$$B_{f1,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{f1,1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$D_{f1,1} = -46.3725,$$

$$A_{f2,1} = \begin{bmatrix} -8.1617 & 10.4852 & 14.5961 \\ -19.9581 & 38.1415 & 64.6288 \\ -10.6012 & 19.4270 & 28.2844 \end{bmatrix},$$

$$B_{f2,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{f2,1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$D_{f2,1} = -48.6350, \tag{46}$$

$$A_{f1,2} = \begin{bmatrix} -2.8214 & 1.9349 & 5.6510 \\ 6.9619 & -6.9841 & -9.9841 \\ 4.3016 & -10.9846 & -17.9894 \end{bmatrix},$$

$$B_{f1,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{f1,2} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$D_{f1,2} = -48.3364,$$

$$A_{f2,2} = \begin{bmatrix} -7.2081 & 4.0201 & 11.3047 \\ -6.3204 & 8.6193 & 16.2141 \\ -4.3612 & 17.7841 & 12.9564 \end{bmatrix},$$

$$B_{f2,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{f2,2} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$D_{f2,2} = -46.0053.$$

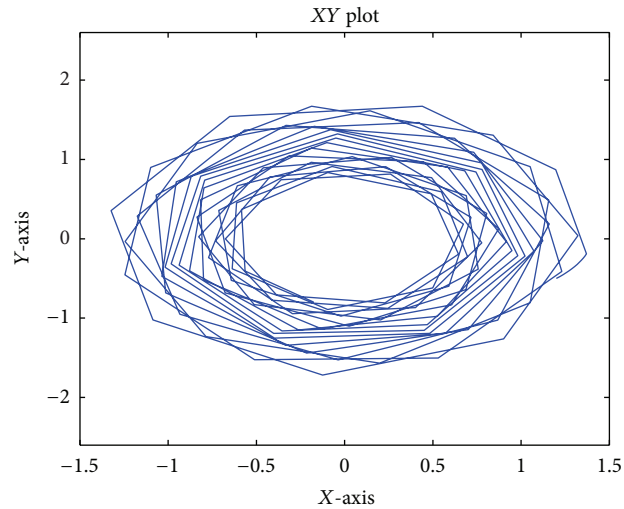


Figure 1: $\eta_t^{\mathsf{T}} T \eta_t$.

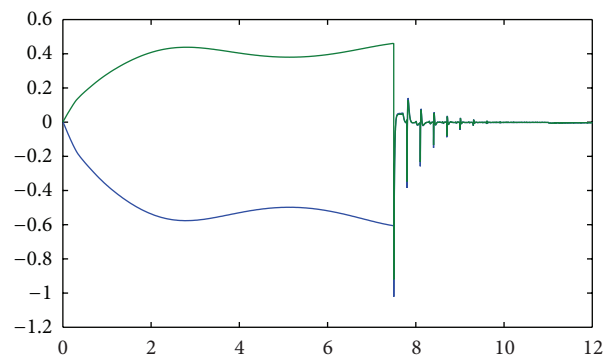


Figure 2: State trajectories of subsystem 1.

From (38), we have $\mu = 5.0216$. Moreover, we can obtain the average dwell time

$$\tau_a > \tau_a^* = 8.1023. \tag{47}$$

By Theorem 9, through the program $f$minsearch in the optimization toolbox of MATLAB, the optimal bound with minimum value of c_2 relies on the parameter α. We can find feasible solution when $\alpha \in [0, 0.05]$. Figure 1 shows the solution trajectory of the system. The state trajectory of the closed-loop system is shown in Figures 2–4, where the initial state $\eta_0 = [0, 0]^{\mathsf{T}}$. From Figures 2–4, it is easy to see that the system is finite-time boundedness.

6. Conclusions

In this paper, the problems of finite-time boundedness of Markovian jump system with piecewise-constant transition probabilities via dynamic output feedback control is concerned. By allowing new Lyapunov-Krasovskii functional, the switching signal is constraint by average dwell time, and a numerical example is also given to demonstrate the effectiveness of the proposed approach.

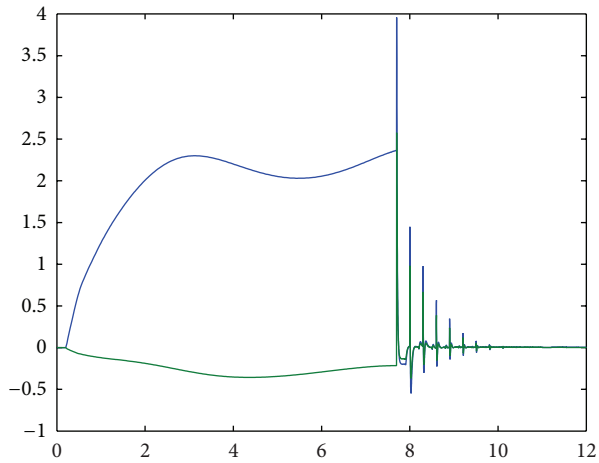


Figure 3: State trajectories of subsystem 2.

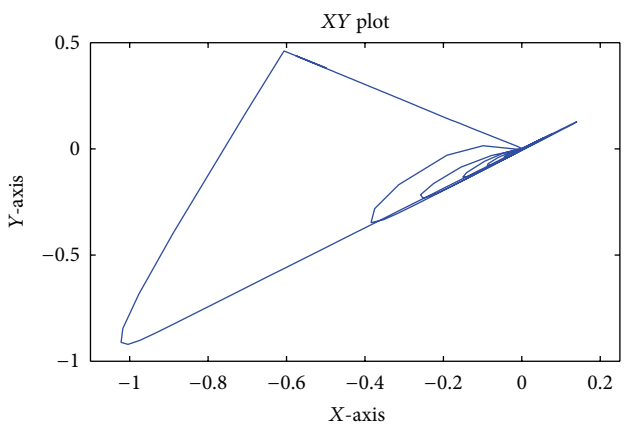


Figure 4: The closed-loop system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Nature Science Foundation of China (Grant no. 61173121), the Postdoctoral Foundation of China (Grant no. 20100471665), the Youth Science Foundation of Sichuan Province, in China (Grant no. 2010JQ0032), and the Science Foundation of Chengdu City in China (Grant no. 60873092).

References

[1] N. N. Krasovskii and È. A. Lidskiĭ, "Analytical design of controllers in systems with random attributes. I. Statement of the problem, method of solving," *Automation and Remote Control*, vol. 22, pp. 1021–1025, 1961.

[2] D. Zhang, W. Cai, L. Xie, and Q. Wang, "Non-fragile distributed filtering for T-S fuzzy systems in sensor networks," *IEEE Transactions on Fuzzy Systems*, no. 99, 2014.

[3] Q. Zhu, R. Rakkiyappan, and A. Chandrasekar, "Stochastic stability of Markovian jump BAM neural networks with leakage delays and impulse control," *Neurocomputing*, vol. 136, pp. 136–151, 2014.

[4] Q. Zhu, "pth moment exponential stability of impulsive stochastic functional differential equations with Markovian switching," *Journal of the Franklin Institute*, vol. 351, no. 7, pp. 3965–3986, 2014.

[5] Q. Zhu and J. Cao, "Robust exponential stability of markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 8, pp. 1314–1325, 2010.

[6] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic BAM neural networks with impulse control and mixed time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 3, pp. 467–479, 2012.

[7] Q. Zhu, J. Cao, T. Hayat, and F. Alsaadi, "Robust stability of Markovian jump stochastic neural networks with time delays in the leakage terms," *Neural Processing Letters*, vol. 41, pp. 1–27, 2015.

[8] Q. Zhu and J. Cao, "Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 2, pp. 341–353, 2011.

[9] L. Zhang and E.-K. Boukas, "Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 2, pp. 463–468, 2009.

[10] L. Zhang and E.-K. Boukas, "H_∞ control for discrete-time Markovian jump linear systems with partly unknown transition probabilities," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 8, pp. 868–883, 2009.

[11] M. Misas and M. T. Ramirez, "Colombian economic growth under Markov switching regimes with endogenous transition probabilities," *Borradores de Economía*, vol. 425, pp. 1–21, 2006.

[12] T. C. Cheng, C. Gao, and H. Shen, "Production and inventory rationing in a make-to-stock system with a failure-prone machine and lost sales," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1176–1180, 2011.

[13] A. J. Filardo, "Business-cycle phases and their transitional dynamics," *Journal of Business and Economic Statistics*, vol. 12, no. 3, pp. 299–308, 1994.

[14] L. Zhang, Y. Leng, L. Chen, and Y. Zhao, "A BRL for a class of discrete-time Markov jump linear system with piecewise-constant TPs," in *Proceedings of the 18th IFAC World Congress*, pp. 8699–8704, Milano, Italy, September 2011.

[15] F. Takeuchi, "US external debt sustainability revisited: bayesian analysis of extended Markov switching unit root test," *Japan and the World Economy*, vol. 22, no. 2, pp. 98–106, 2010.

[16] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1994.

[17] Z. Wu, H. Su, and J. Chu, "State estimation for discrete Markovian jumping neural networks with time delay," *Neurocomputing*, vol. 73, no. 10–12, pp. 2247–2254, 2010.

[18] Q. Zhu and J. Cao, "Adaptive synchronization under almost every initial data for stochastic neural networks with time-varying delays and distributed delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 2139–2159, 2011.

[19] L. Chen, Y. Leng, H. Guo, P. Shi, and L. Zhang, "H_∞ control of a class of discrete-time Markov jump linear systems with piecewise-constant TPs subject to average dwell time switching," *Journal of the Franklin Institute*, vol. 349, no. 6, pp. 1989–2003, 2012.

[20] L. Zhang and B. Jiang, "Stability of a class of switched linear systems with uncertainties and average dwell time switching," *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 2, pp. 667–676, 2010.

[21] L. Zhang, E.-K. Boukas, and P. Shi, "Exponential H_∞ filtering for uncertain discrete-time switched linear systems with average dwell time: a μ-dependent approach," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 11, pp. 1188–1207, 2008.

[22] F. Amato, M. Ariola, and P. Dorato, "Finite-time control of linear systems subject to parametric uncertainties and disturbances," *Automatica*, vol. 37, no. 9, pp. 1459–1463, 2001.

[23] F. Amato and M. Ariola, "Finite-time control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 724–729, 2005.

[24] Y. Shen and H. Liu, "Finite-time stabilization of switched time-delay system via dynamic output feedback control," *Advances in Intelligent and Soft Computing*, vol. 125, pp. 523–528, 2012.

[25] Y. Hong, "Finite-time stabilization and stabilizability of a class of controllable systems," *Systems and Control Letters*, vol. 46, no. 4, pp. 231–236, 2002.

[26] H. Liu, Y. Shen, and X. Zhao, "Finite-time stabilization and boundedness of switched linear system under state-dependent switching," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 350, no. 3, pp. 541–555, 2013.

[27] S. He and F. Liu, "Finite-time H_∞ fuzzy control of nonlinear jump systems with time delays via dynamic observer-based state feedback," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 4, pp. 605–614, 2012.

[28] J. Cheng, H. Zhu, S. Zhong, Q. Zhong, and Y. Zeng, "Finite-time H_∞ estimation for discrete-time Markov jump systems with time-varying transition probabilities subject to average dwell time switching," *Communications in Nonlinear Science and Numerical Simulation*, vol. 20, no. 2, pp. 571–582, 2015.

[29] X. Luan, F. Liu, and P. Shi, "Finite-time filtering for non-linear stochastic systems with partially known transition jump rates," *IET Control Theory and Applications*, vol. 4, no. 5, pp. 735–745, 2010.

[30] J. Cheng, L. Xiong, B. Wang, and J. Yang, "Robust finite-time boundedness of H_∞ filtering for switched systems with time-varying delay," *Optimal Control Applications and Methods*, 2015.

[31] X. Luan, F. Liu, and P. Shi, "Robust finite-time H_∞ control for nonlinear jump systems via neural networks," *Circuits, Systems, and Signal Processing*, vol. 29, no. 3, pp. 481–498, 2010.

[32] W. Xiang and J. Xiao, "H_∞ finite-time control for switched nonlinear discrete-time systems with norm-bounded disturbance," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 348, no. 2, pp. 331–352, 2011.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 195921, 10 pages
http://dx.doi.org/10.1155/2015/195921

Research Article

Asynchronous $\mathscr{H}_\infty$ Estimation for Two-Dimensional Nonhomogeneous Markovian Jump Systems with Randomly Occurring Nonlocal Sensor Nonlinearities

Rui Zhang,[1,2] Ying Zhang,[3] and Victor Sreeram[2]

[1] *Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences, Shenzhen 518055, China*
[2] *School of Electrical, Electronic and Computer Engineering, The University of Western Australia, Perth, WA 6009, Australia*
[3] *Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, China*

Correspondence should be addressed to Ying Zhang; zhangyinghit@126.com

Received 22 May 2015; Accepted 19 August 2015

Academic Editor: Leonid Shaikhet

This paper is devoted to the problem of asynchronous $\mathscr{H}_\infty$ estimation for a class of two-dimensional (2D) nonhomogeneous Markovian jump systems with nonlocal sensor nonlinearity, where the nonlocal measurement nonlinearity is governed by a stochastic variable satisfying the Bernoulli distribution. The asynchronous estimation means that the switching of candidate filters may have a lag to the switching of system modes, and the varying character of transition probabilities is considered to reside in a convex polytope. The jumping process of the error system is modeled as a two-component Markov chain with extended varying transition probabilities. A stochastic parameter-dependent approach is provided for the design of $\mathscr{H}_\infty$ filter such that, for randomly occurring nonlocal sensor nonlinearity, the corresponding error system is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance index. Finally, a numerical example is used to illustrate the effectiveness of the developed estimation method.

1. Introduction

Two-dimensional (2D) systems have recently received considerable attention from scientific communities for their potential applications in various areas, such as image data processing and transmission, multidimensional digital filtering, and process control [1]. The analysis and synthesis of 2D systems are much more complicated than those of one-dimensional systems due primarily to their structural complexity. As a result, a great amount of effort has been invested in such systems, and many interesting and important results have been derived so far [2–6]. In the context of state estimation, filtering problems for 2D systems have also been deeply studied. To mention a few, the minimum mean-square state estimation has been addressed in [7, 8], the $\mathscr{H}_\infty$ filtering problem has been tackled in [9–19], and the $l_2 - l_\infty$ filtering problem has been considered in [20].

As is well known, Markovian jump linear systems, which were first introduced in [21], have been widely used to model a large variety of physical systems that experience abrupt changes in their structure and parameters. The transition probabilities play a crucial role in determining the behavior and performance of Markovian jump systems. In the case of time-varying transition probabilities, the Markov chain is viewed as nonhomogeneous. Most recently, a number of outstanding analysis and design results have been obtained for Markovian jump systems with nonhomogeneous transition probabilities [22, 23].

In particular, the filter design for Markovian jump systems has also been extensively investigated [23–25]. A popular solution is to find less conservative mode-dependent filters such that the resulting filtering error system is stable and satisfies certain performance. Most of the mode-dependent methods available rely on the ideal assumption that the switches of filters are strictly synchronized with those of the system modes. However, perfect synchronization is not always possible in practical situations owing to operations related to identifying the system mode and specifying the matched filter [26]. Thus, it seems more practicable and significant to design asynchronous filters for Markovian jump

systems, especially for nonhomogeneous Markovian jump systems.

Moreover, sensor nonlinearities arise frequently under harsh filtering environments including both uncontrollable elements and aggressive conditions. It is worth pointing out that filtering techniques concerning the sensor nonlinearities usually provide a relatively reliable solution. State estimation related to Markov jump systems with sensor nonlinearities has been developed in terms of many sorts of methods [27]. However, to the best of the authors' knowledge, the problem of asynchronous $\mathscr{H}_\infty$ estimation for 2D nonhomogeneous Markovian jump systems with nonlocal sensor nonlinearity has not been fully resolved, despite its deep practical implications.

Therefore, in this paper, we will handle the problem of asynchronous $\mathscr{H}_\infty$ estimation for 2D nonhomogeneous Markovian jump systems with randomly occurring nonlocal sensor nonlinearity. A stochastic variable satisfying the Bernoulli binary distribution is employed to characterize the nonlocal nonlinear measurement behavior. The varying character of transition probabilities is described by means of a specified polytope, and the jumping process of the error system is represented by a two-component Markov chain with extended varying transition probabilities. The $\mathscr{H}_\infty$ analysis result is derived by a stochastic parameter-dependent approach. The existence condition of the desired filter is then obtained such that, for randomly occurring nonlocal sensor nonlinearity, the corresponding error system is mean-square asymptotically stable and has a guaranteed $\mathscr{H}_\infty$ performance level. A numerical example is provided to show the effectiveness of the proposed design method.

Notations. The notation used throughout the paper is fairly standard. The superscript T stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $P > 0$ means that P is real symmetrical and positive definite. $\mathbb{E}\{\cdot\}$ stands for the expectation operation. $l_2\{[0,\infty),[0,\infty)\}$ is the space of square summable sequences on $\{[0,\infty),[0,\infty)\}$. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix, for a vector x the diagonal matrix defined by the entries of x is denoted by $\mathrm{diag}(x)$. For any symmetric matrix, $(*)$ represents a symmetric term. Moreover, for each integer q, $\mathbf{1}_q$ denotes the vector in $\mathbb{R}^q$ defined by $\mathbf{1}_q = [1\cdots1]^T$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation

Consider the 2D Markovian jump system with sensor nonlinearity in the Roesser model:

$$\begin{aligned}
\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} &= A\left(r_{i,j}\right)\begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + B\left(r_{i,j}\right)w_{i,j}, \\
y_{i,j} &= \left(1-\delta_k\right)C\left(r_{i,j}\right)\begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} \\
&\quad + \delta_k\varphi\left(C\left(r_{i,j}\right)\begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}\right) + D\left(r_{i,j}\right)w_{i,j}, \\
z_{i,j} &= H\left(r_{i,j}\right)\begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + L\left(r_{i,j}\right)w_{i,j},
\end{aligned} \tag{1}$$

where $x^h_{i,j} \in \mathbb{R}^{n_1}$ and $x^v_{i,j} \in \mathbb{R}^{n_2}$ represent the horizontal and vertical states, respectively, $y_{i,j} \in \mathbb{R}^r$ is the measured output, $z_{i,j} \in \mathbb{R}^p$ is the objective signal to be estimated, and $w_{i,j} \in \mathbb{R}^q$ is the noise signal which belongs to $l_2\{[0,\infty),[0,\infty)\}$. The system matrices are functions of $r_{i,j}$, which is a discrete-time, discrete-state Markov chain taking values in a finite set $\mathscr{I} = \{1,2,\ldots,s\}$ with mode transition probabilities:

$$\begin{aligned}
\pi^{ij}_{mn} &= \Pr\left\{r_{i+1,j} = n \mid r_{i,j} = m\right\} \\
&= \Pr\left\{r_{i,j+1} = n \mid r_{i,j} = m\right\},
\end{aligned} \tag{2}$$

where $\pi^{ij}_{mn} \geq 0$ and $\sum_{n=1}^{s}\pi^{ij}_{mn} = 1$. To simplify the notation, the system matrices are denoted by $\mathscr{S}_m = (A_m, B_m, C_m, D_m, H_m, L_m)$, when $r_{i,j} = m \in \mathscr{I}$. Here, π^{ij}_{mn} are the entries of the transition matrix Π^{ij}, which is assumed to be of the form

$$\Pi^{ij} = \sum_{g=1}^{N}\alpha^{ij}_g\Pi^g, \tag{3}$$

where $\alpha^{ij}_g \geq 0$, $\sum_{g=1}^{N}\alpha^{ij}_g = 1$ and Π^g denote the vertices of the polytope.

Remark 1. It is worth stressing that, different from the homogeneous Markov chain concerned in the recently developed techniques for asynchronous filter design [18, 19, 27], the polytopic time-varying model of Markov chain under consideration is more practicable in the Markovian jump systems field. The homogeneous Markov chain has a critical assumption that the transition probabilities need to be known exactly. This ideal requirement sometimes inevitably limits the application of the derived results, since it can be hard or costly to obtain the precise information about the transition probabilities in practice. In view of that, it seems more reasonable to consider that the transition probabilities are affected by varying parameters.

The function φ represents the nonlocal sensor nonlinearity satisfying the following nonlocal sector condition [28]:

$$\begin{aligned}
&\left(\varphi(\theta) - K_1\theta - F_1\right)^T\left(\varphi(\theta) - K_2\theta - F_2\right) \leq 0, \\
&\qquad\qquad \theta \in \mathbb{R}^n,\ \|\theta\| \geq \nu_c,
\end{aligned} \tag{4}$$

where K_1 and K_2 are matrices, F_1 and F_2 are vectors, and ν_c is a positive real number. Moreover, the nonlinear function $\varphi(\theta)$ can be decomposed into a linear and a nonlinear part as

$$\varphi(\theta) = \varphi_p(\theta) + K_1\theta + F_1, \tag{5}$$

where the nonlinearity $\varphi_p(\theta)$ satisfies

$$\varphi_p^T(\theta)\left(\varphi_p(\theta) - K\theta - F\right) \leq 0 \tag{6}$$

with $K = K_2 - K_1$ and $F = F_2 - F_1$. The stochastic variable δ_k, which is introduced to account for the phenomena of randomly occurring nonlocal sensor nonlinearity, is Bernoulli sequence taking the values of 1 and 0 with

$$\begin{aligned} \Pr\{\delta_k = 1\} &= \mathbb{E}\{\delta_k\} = \mu, \\ \Pr\{\delta_k = 0\} &= 1 - \mathbb{E}\{\delta_k\} = 1 - \mu. \end{aligned} \tag{7}$$

Remark 2. It can be found that when $F_1 = 0$, $F_2 = 0$, and $v_c = 0$, the nonlocal sector nonlinearity in (4) reduces to the conventional sector nonlinearity [27], which implies that the nonlocal sector nonlinearity in (4) covers the conventional sector nonlinearity as a special case. Hence, the case considered in the sequel is more general.

For the asynchronous phenomenon considered, we are interested in estimating the objective signal $z_{i,j}$ by a filter as follows:

$$\begin{aligned} \begin{bmatrix} \hat{x}^h_{i+1,j} \\ \hat{x}^v_{i,j+1} \end{bmatrix} &= \left[(1-\sigma_k) A_{fm} + \sigma_k A_{fz}\right] \begin{bmatrix} \hat{x}^h_{i,j} \\ \hat{x}^v_{i,j} \end{bmatrix} \\ &\quad + \left[(1-\sigma_k) B_{fm} + \sigma_k B_{fz}\right] y_{i,j}, \\ \hat{z}_{i,j} &= \left[(1-\sigma_k) C_{fm} + \sigma_k C_{fz}\right] \begin{bmatrix} \hat{x}^h_{i,j} \\ \hat{x}^v_{i,j} \end{bmatrix} \\ &\quad + \left[(1-\sigma_k) D_{fm} + \sigma_k D_{fz}\right] y_{i,j}, \end{aligned} \tag{8}$$

where $\hat{x}^h_{i,j} \in \mathbb{R}^{n_1}$ and $\hat{x}^v_{i,j} \in \mathbb{R}^{n_2}$ are the filter states, $\hat{z}_{i,j} \in \mathbb{R}^p$ is the estimation of $z_{i,j}$, $\mathcal{F}_m = (A_{fm}, B_{fm}, C_{fm}, D_{fm})$ and $\mathcal{F}_z = (A_{fz}, B_{fz}, C_{fz}, D_{fz})$ are the filter gains corresponding to the current and previous stages, respectively, $\forall m, z \in \mathcal{I}$, and σ_k is a Bernoulli distributed white sequence specified by

$$\begin{aligned} \Pr\{\sigma_k = 1\} &= \mathbb{E}\{\sigma_k\} = \eta, \\ \Pr\{\sigma_k = 0\} &= 1 - \mathbb{E}\{\sigma_k\} = 1 - \eta. \end{aligned} \tag{9}$$

In addition, σ_k, δ_k, and $r_{i,j}$ are mutually independent.

In view of (1) and (8), the estimation error $\bar{z}_{i,j} = z_{i,j} - \hat{z}_{i,j}$ can be described by the following model:

$$\begin{aligned} \begin{bmatrix} \bar{x}^h_{i+1,j} \\ \bar{x}^v_{i,j+1} \end{bmatrix} &= \bar{A}_{m,l} \begin{bmatrix} \bar{x}^h_{i,j} \\ \bar{x}^v_{i,j} \end{bmatrix} + \bar{M}_l \varphi_p\left(C_m \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}\right) + \bar{M}_l F_1 \\ &\quad + \bar{B}_{m,l} w_{i,j} + (\mu - \delta_k) \\ &\quad \cdot \left(\hat{A}_{m,l} \begin{bmatrix} \bar{x}^h_{i,j} \\ \bar{x}^v_{i,j} \end{bmatrix} + \hat{M}_l \varphi_p\left(C_m \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}\right) + \hat{M}_l F_1\right), \\ \bar{z}_{i,j} &= \bar{C}_{m,l} \begin{bmatrix} \bar{x}^h_{i,j} \\ \bar{x}^v_{i,j} \end{bmatrix} + \bar{N}_l \varphi_p\left(C_m \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}\right) + \bar{N}_l F_1 \\ &\quad + \bar{D}_{m,l} w_{i,j} + (\mu - \delta_k) \\ &\quad \cdot \left(\hat{C}_{m,l} \begin{bmatrix} \bar{x}^h_{i,j} \\ \bar{x}^v_{i,j} \end{bmatrix} + \hat{N}_l \varphi_p\left(C_m \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}\right) + \hat{N}_l F_1\right), \end{aligned} \tag{10}$$

where $\bar{x}^h_{i,j} = \begin{bmatrix} x^{hT}_{i,j} & \hat{x}^{hT}_{i,j} \end{bmatrix}^T$, $\bar{x}^v_{i,j} = \begin{bmatrix} x^{vT}_{i,j} & \hat{x}^{vT}_{i,j} \end{bmatrix}^T$ and

$$\begin{aligned} \bar{A}_{m,l} &= \Lambda^T \begin{bmatrix} A_m & 0 \\ (1-\mu) B_{fl} C_m + \mu B_{fl} K_1 C_m & A_{fl} \end{bmatrix} \Lambda, \\ \bar{M}_l &= \Lambda^T \begin{bmatrix} 0 \\ \mu B_{fl} \end{bmatrix}, \\ \hat{A}_{m,l} &= \Lambda^T \begin{bmatrix} 0 & 0 \\ B_{fl} C_m - B_{fl} K_1 C_m & 0 \end{bmatrix} \Lambda, \\ \hat{M}_l &= \Lambda^T \begin{bmatrix} 0 \\ -B_{fl} \end{bmatrix}, \\ \bar{B}_{m,l} &= \Lambda^T \begin{bmatrix} B_m \\ B_{fl} D_m \end{bmatrix}, \\ \bar{C}_{m,l} &= \left[H_m - (1-\mu) D_{fl} C_m - \mu D_{fl} K_1 C_m \quad -C_{fl}\right] \Lambda, \\ \bar{N}_l &= -\mu D_{fl}, \\ \hat{C}_{m,l} &= \left[D_{fl} K_1 C_m - D_{fl} C_m \quad 0\right] \Lambda, \\ \hat{N}_l &= D_{fl}, \\ \bar{D}_{m,l} &= L_m - D_{fl} D_m, \\ \Lambda &= \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{array}\right], \\ &\qquad \forall m, l \in \mathcal{I}. \end{aligned} \tag{11}$$

Interestingly, the jumping process $\{\mathcal{S}_m, \mathcal{F}_l\}$ of the error system in (10) forms a two-component Markov chain $\bar{r}_{i,j}$ on $\mathcal{I} \times \mathcal{I}$ with the extended varying transition probabilities $\bar{\pi}^{ij}_{(m,l)(n,u)} = \Pr\{\mathcal{S}_n, \mathcal{F}_u \mid \mathcal{S}_m, \mathcal{F}_l\}$ given by

$$\bar{\pi}^{ij}_{(m,l)(n,u)} = \begin{cases} \sum_{g=1}^{N} \alpha^{ij}_g \pi^g_{mn}, & u = n,\ u = l, \\ \sum_{g=1}^{N} \alpha^{ij}_g \eta \pi^g_{mn}, & u \neq n,\ u = l, \\ \sum_{g=1}^{N} \alpha^{ij}_g (1-\eta) \pi^g_{mn}, & u = n,\ u \neq l, \\ 0, & u \neq n,\ u \neq l. \end{cases} \tag{12}$$

The following definitions for the error system in (10) are necessary to formulate the considered problem. For more details, refer to [15] and the references therein.

Definition 3. System (10) is said to be mean-square asymptotically stable if for $w_{i,j} = 0$ and bounded boundary conditions, the following holds:

$$\lim_{i+j \to \infty} \mathbb{E}\left\{\left\|\bar{x}_{i,j}\right\|^2\right\} = 0, \tag{13}$$

where $\bar{x}_{i,j} = \begin{bmatrix} \bar{x}^{hT}_{i,j} & \bar{x}^{vT}_{i,j} \end{bmatrix}^T$.

Definition 4. Given a scalar $\gamma > 0$, system (10) is said to be mean-square asymptotically stable with an $\mathscr{H}_\infty$ disturbance attenuation level γ, if it is mean-square asymptotically stable, and under zero initial conditions satisfies

$$\left\|\overline{z}_{i,j}\right\|_E < \gamma \left\|w_{i,j}\right\|_2, \tag{14}$$

for all nonzero $w_{i,j}$, where

$$\begin{aligned} \left\|\overline{z}_{i,j}\right\|_E &= \sqrt{\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\overline{z}_{i,j}\right\|^2\right\}}, \\ \left\|w_{i,j}\right\|_2 &= \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|w_{i,j}\right\|^2}. \end{aligned} \tag{15}$$

Then, the estimation problem of interest is stated as follows: given $\gamma > 0$, design a filter of the form in (8) such that the error system in (10) with extended varying transition probabilities (12) is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance level γ.

3. $\mathscr{H}_\infty$ Filter Design

This section provides a procedure for designing the asynchronous $\mathscr{H}_\infty$ filter, which guarantees that the error system with extended varying transition probabilities is mean-square asymptotically stable and has a prescribed disturbance attenuation level in the $\mathscr{H}_\infty$ sense. First, we derive an $\mathscr{H}_\infty$ analysis criterion to check if the $\mathscr{H}_\infty$ norm of the error system in (10) is bounded by the asynchronous filter. The corresponding analysis result is summarized in the following theorem.

Theorem 5. *Consider system (10) with extended varying transition probabilities (12) and let $\gamma > 0$ be a given constant. If there exist matrices $P_{m,l}^g = \operatorname{diag}\{P_{m,l}^{hg}, P_{m,l}^{vg}\} > 0$ and scalars $\tau_{m,l}^g > 0$, $\rho_{m,l}^g > 0$, $g = 1,\ldots,N$, $\forall (m,l) \in \mathscr{I}\times\mathscr{I}$, such that*

$$\begin{bmatrix} -\mathscr{P}_{n,u}^q & 0 & 0 & 0 & 0 & \Omega_{16} & \Omega_{17} & \Omega_{18} & \Omega_{19} \\ * & -\mathscr{P}_{n,u}^q & 0 & 0 & 0 & \Omega_{26} & \Omega_{27} & \Omega_{28} & 0 \\ * & * & -I & 0 & 0 & \overline{C}_{m,l} & \overline{N}_l & \Omega_{38} & \overline{D}_{m,l} \\ * & * & * & -I & 0 & \Omega_{46} & \Omega_{47} & \Omega_{48} & 0 \\ * & * & * & * & \Omega_{55} & \Omega_{56} & 0 & 0 & 0 \\ * & * & * & * & * & -\sum_{g=1}^{N}\alpha_g^{ij}P_{m,l}^g & \Omega_{67} & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77} & \Omega_{78} & 0 \\ * & * & * & * & * & * & * & \Omega_{88} & 0 \\ * & * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{aligned} \mathscr{P}_{n,u}^q &= \sum_{(n,u)\in\mathscr{I}\times\mathscr{I}}\sum_{g=1}^{N}\sum_{q=1}^{N}\alpha_g^{ij}\beta_q^{ij}\overline{\pi}_{(m,l)(n,u)}^g P_{n,u}^q, \\ &\qquad \alpha_g^{ij} \geq 0,\ \sum_{g=1}^{N}\alpha_g^{ij} = 1,\ \beta_q^{ij} \geq 0,\ \sum_{q=1}^{N}\beta_q^{ij} = 1, \\ \Omega_{16} &= \mathscr{P}_{n,u}^q\overline{A}_{m,l}, \\ \Omega_{17} &= \mathscr{P}_{n,u}^q\overline{M}_l, \\ \Omega_{18} &= \mathscr{P}_{n,u}^q\overline{M}_l \operatorname{diag}(F_1), \\ \Omega_{19} &= \mathscr{P}_{n,u}^q\overline{B}_{m,l}, \\ \Omega_{26} &= \sqrt{\lambda}\mathscr{P}_{n,u}^q\widehat{A}_{m,l}, \\ \Omega_{27} &= \sqrt{\lambda}\mathscr{P}_{n,u}^q\widehat{M}_l, \\ \Omega_{28} &= \sqrt{\lambda}\mathscr{P}_{n,u}^q\widehat{M}_l \operatorname{diag}(F_1), \\ \Omega_{38} &= \overline{N}_l \operatorname{diag}(F_1), \\ \Omega_{46} &= \sqrt{\lambda}\widehat{C}_{m,l}, \\ \Omega_{47} &= \sqrt{\lambda}\widehat{N}_l, \\ \Omega_{48} &= \sqrt{\lambda}\widehat{N}_l \operatorname{diag}(F_1), \\ \Omega_{55} &= -\mu\sum_{g=1}^{N}\alpha_g^{ij}\tau_{m,l}^g I, \\ \Omega_{56} &= \mu\sum_{g=1}^{N}\alpha_g^{ij}\tau_{m,l}^g C_m\Lambda_1, \\ \Omega_{67} &= \mu\sum_{g=1}^{N}\alpha_g^{ij}\rho_{m,l}^g\Lambda_1^T C_m^T K^T, \\ \Omega_{77} &= -2\mu\sum_{g=1}^{N}\alpha_g^{ij}\rho_{m,l}^g I, \\ \Omega_{78} &= \mu\sum_{g=1}^{N}\alpha_g^{ij}\rho_{m,l}^g \operatorname{diag}(F), \\ \Omega_{88} &= -\mu\frac{\nu_c^2}{r}\sum_{g=1}^{N}\alpha_g^{ij}\tau_{m,l}^g I, \end{aligned} \tag{17}$$

and $\lambda = \mu(1-\mu)$, then the system in (10) is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance index γ.

Proof. First, we handle the stochastic stability of system (10) with $w_{i,j} \equiv 0$. Construct the following index:

$$\begin{aligned}\mathscr{L}_{i,j} &= \mathbb{E}\left\{\begin{bmatrix}\overline{x}^h_{i+1,j}\\ \overline{x}^v_{i,j+1}\end{bmatrix}^T \begin{bmatrix}P^h\left(\overline{r}_{i+1,j}\right) & 0\\ 0 & P^v\left(\overline{r}_{i,j+1}\right)\end{bmatrix}\begin{bmatrix}\overline{x}^h_{i+1,j}\\ \overline{x}^v_{i,j+1}\end{bmatrix}\right.\\ &\quad \left. - \overline{x}^T_{i,j}P\left(\overline{r}_{i,j}\right)\overline{x}_{i,j} \mid \overline{x}_{i,j}, \overline{r}_{i,j} = (m,l)\right\},\end{aligned} \tag{18}$$

where $P(\overline{r}_{i,j}) = \operatorname{diag}\{P^h(\overline{r}_{i,j}), P^v(\overline{r}_{i,j})\}$, which is represented as $P_{m,l}$ when $\overline{r}_{i,j} = (m,l)$. It follows from expression (18) that

$$\begin{aligned}\mathscr{L}_{i,j} &= \begin{bmatrix}\overline{x}^h_{i+1,j}\\ \overline{x}^v_{i,j+1}\end{bmatrix}^T \sum_{(n,u)\in\mathscr{I}\times\mathscr{I}}\left(\sum_{g=1}^N \alpha^{ij}_g \overline{\pi}^g_{(m,l)(n,u)}\right)\\ &\quad\cdot\left(\sum_{q=1}^N \beta^{ij}_q P^q_{n,u}\right)\begin{bmatrix}\overline{x}^h_{i+1,j}\\ \overline{x}^v_{i,j+1}\end{bmatrix} - \overline{x}^T_{i,j}\sum_{g=1}^N \alpha^{ij}_g P^g_{m,l}\overline{x}_{i,j}.\end{aligned} \tag{19}$$

Collectively considering expression (19) and system (10) with $w_{i,j} \equiv 0$, we have

$$\begin{aligned}\mathscr{L}_{i,j} &= \left[\overline{A}_{m,l}\overline{x}_{i,j} + \overline{M}_l\varphi_p + \overline{M}_l F_1\right]^T\\ &\quad\cdot \mathscr{P}^q_{n,u}\left[\overline{A}_{m,l}\overline{x}_{i,j} + \overline{M}_l\varphi_p + \overline{M}_l F_1\right]\\ &\quad + \lambda\left[\widehat{A}_{m,l}\overline{x}_{i,j} + \widehat{M}_l\varphi_p + \widehat{M}_l F_1\right]^T\\ &\quad\cdot \mathscr{P}^q_{n,u}\left[\widehat{A}_{m,l}\overline{x}_{i,j} + \widehat{M}_l\varphi_p + \widehat{M}_l F_1\right]\\ &\quad - \overline{x}^T_{i,j}\sum_{g=1}^N \alpha^{ij}_g P^g_{m,l}\overline{x}_{i,j}.\end{aligned} \tag{20}$$

It is inferred from the sensor nonlinearity constraint in (6) that

$$\begin{aligned}\xi_1 &= 2\mu\sum_{g=1}^N \alpha^{ij}_g \rho^g_{m,l}\varphi^T_p\left(C_m\begin{bmatrix}x^h_{i,j}\\ x^v_{i,j}\end{bmatrix}\right)\\ &\quad\cdot\left(\varphi_p\left(C_m\begin{bmatrix}x^h_{i,j}\\ x^v_{i,j}\end{bmatrix}\right) - KC_m\begin{bmatrix}x^h_{i,j}\\ x^v_{i,j}\end{bmatrix} - F\right) \le 0,\\ \xi_2 &= 2\mu\sum_{g=1}^N \alpha^{ij}_g \tau^g_{m,l}\left(v^2_c - y^T_{i,j}y_{i,j}\right) \le 0.\end{aligned} \tag{21}$$

By the Schur complement property, the inequality in (16) yields $\mathscr{L}_{i,j} - \xi_1 - \xi_2 < 0$, which means that

$$\lim_{i+j\to\infty}\mathbb{E}\left\{\left\|\overline{x}_{i,j}\right\|^2\right\} = 0. \tag{22}$$

Therefore, the system is mean-square asymptotically stable.

Next, to establish the $\mathscr{H}_\infty$ performance for the system, we introduce the following index:

$$\begin{aligned}\mathscr{J}_{i,j} &= \mathbb{E}\left\{\begin{bmatrix}\overline{x}^h_{i+1,j}\\ \overline{x}^v_{i,j+1}\end{bmatrix}^T\right.\\ &\quad\cdot\begin{bmatrix}P^h\left(\overline{r}_{i+1,j}\right) & 0\\ 0 & P^v\left(\overline{r}_{i,j+1}\right)\end{bmatrix}\begin{bmatrix}\overline{x}^h_{i+1,j}\\ \overline{x}^v_{i,j+1}\end{bmatrix} - \overline{x}^T_{i,j}P\left(\overline{r}_{i,j}\right)\\ &\quad\left.\cdot\overline{x}_{i,j} + \overline{z}^T_{i,j}\overline{z}_{i,j} - \gamma^2 w^T_{i,j}w_{i,j} \mid \overline{x}_{i,j}, \overline{r}_{i,j} = (m,l)\right\}.\end{aligned} \tag{23}$$

It is shown from (23) that

$$\mathscr{J}_{i,j} = \zeta^T_{i,j}\Phi_{m,l}\zeta_{i,j}, \tag{24}$$

where $\zeta_{i,j} = \left[\overline{x}^T_{i,j}\ \ \varphi^T_p\ \ \mathbf{1}_r\ \ w^T_{i,j}\right]^T$ and

$$\Phi_{m,l} = \Sigma^T_1\mathscr{P}^q_{n,u}\Sigma_1 + \Sigma^T_2\mathscr{P}^q_{n,u}\Sigma_2 + \Sigma^T_3\Sigma_3 + \Sigma^T_4\Sigma_4 - \Sigma_5, \tag{25}$$

with

$$\begin{aligned}\Sigma_1 &= \left[\overline{A}_{m,l}\ \ \overline{M}_l\ \ \overline{M}_l\operatorname{diag}(F_1)\ \ \overline{B}_{m,l}\right],\\ \Sigma_2 &= \left[\sqrt{\lambda}\widehat{A}_{m,l}\ \ \sqrt{\lambda}\widehat{M}_l\ \ \sqrt{\lambda}\widehat{M}_l\operatorname{diag}(F_1)\ \ 0\right],\\ \Sigma_3 &= \left[\overline{C}_{m,l}\ \ \overline{N}_l\ \ \overline{N}_l\operatorname{diag}(F_1)\ \ \overline{D}_{m,l}\right],\\ \Sigma_4 &= \left[\sqrt{\lambda}\widehat{C}_{m,l}\ \ \sqrt{\lambda}\widehat{N}_l\ \ \sqrt{\lambda}\widehat{N}_l\operatorname{diag}(F_1)\ \ 0\right],\\ \Sigma_5 &= \begin{bmatrix}\sum_{g=1}^N \alpha^{ij}_g P^g_{m,l} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma^2 I\end{bmatrix}.\end{aligned} \tag{26}$$

By applying Schur's complement again, the inequality in (16) guarantees $\mathscr{J}_{i,j} - \xi_1 - \xi_2 < 0$, which implies that $\|\overline{z}_{i,j}\|_E < \gamma\|w_{i,j}\|_2$. Thus, the system is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance level. The proof is completed. □

Remark 6. In Theorem 5, an $\mathscr{H}_\infty$ analysis criterion for the underlying system is established by considering a nonhomogeneous Markovian process under asynchronous switching. By a closer inspection, it is found that the obtained condition reveals the relationship between the asynchronous switching and the $\mathscr{H}_\infty$ performance level.

The developments above lead to the asynchronous filtering result in the next theorem.

Theorem 7. *The system in (10) with extended varying transition probabilities (12) is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance index $\gamma > 0$, if there exist matrices $P^g_{1m,l} = \operatorname{diag}\{P^{hg}_{1m,l}, P^{vg}_{1m,l}\} > 0$, $P^g_{3m,l} = \operatorname{diag}\{P^{hg}_{3m,l}, P^{vg}_{3m,l}\} > 0$, $P^g_{2m,l} = \operatorname{diag}\{P^{hg}_{2m,l}, P^{vg}_{2m,l}\}$, X_l, S_l, U_l, $\overline{A}_{fl}$, $\overline{B}_{fl}$, $\overline{C}_{fl}$,*

$\overline{D}_{fl}$, *and scalars* $\tau_{m,l}^{g} > 0$, $\rho_{m,l}^{g} > 0$, $g = 1, \ldots, N$, $\forall (m,l) \in \mathcal{I} \times \mathcal{I}$, *such that*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 & 0 & 0 & 0 & 0 & \Psi_{18} & \mu\overline{B}_{fl} & \mu\overline{B}_{fl}\mathrm{diag}(F_1) & \Psi_{19} \\ * & \Psi_{22} & 0 & 0 & 0 & 0 & 0 & \Psi_{28} & \mu\overline{B}_{fl} & \mu\overline{B}_{fl}\mathrm{diag}(F_1) & \Psi_{29} \\ * & * & \Psi_{11} & \Psi_{12} & 0 & 0 & 0 & \Psi_{38} & -\sqrt{\lambda}\,\overline{B}_{fl} & -\sqrt{\lambda}\,\overline{B}_{fl}\mathrm{diag}(F_1) & 0 \\ * & * & * & \Psi_{22} & 0 & 0 & 0 & \Psi_{38} & -\sqrt{\lambda}\,\overline{B}_{fl} & -\sqrt{\lambda}\,\overline{B}_{fl}\mathrm{diag}(F_1) & 0 \\ * & * & * & * & -I & 0 & 0 & \Psi_{58} & -\mu\overline{D}_{fl} & -\mu\overline{D}_{fl}\mathrm{diag}(F_1) & \Psi_{59} \\ * & * & * & * & * & -I & 0 & \Psi_{68} & \sqrt{\lambda}\,\overline{D}_{fl} & \sqrt{\lambda}\,\overline{D}_{fl}\mathrm{diag}(F_1) & 0 \\ * & * & * & * & * & * & -\mu\tau_{m,l}^{g}I & \Psi_{78} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Psi_{88} & \Psi_{89} & 0 & 0 \\ * & * & * & * & * & * & * & * & -2\mu\rho_{m,l}^{g}I & \mu\rho_{m,l}^{g}\mathrm{diag}(F) & 0 \\ * & * & * & * & * & * & * & * & * & -\mu\dfrac{v_c^2}{r}\tau_{m,l}^{g}I & 0 \\ * & * & * & * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{27}$$

where

$$\begin{aligned}
\Psi_{11} &= \sum_{(n,u)\in\mathcal{I}\times\mathcal{I}} \overline{\pi}_{(m,l)(n,u)}^{g} P_{1n,u}^{q} - X_l^T - X_l, \\
\Psi_{12} &= \sum_{(n,u)\in\mathcal{I}\times\mathcal{I}} \overline{\pi}_{(m,l)(n,u)}^{g} P_{2n,u}^{q} - U_l^T - S_l, \\
\Psi_{22} &= \sum_{(n,u)\in\mathcal{I}\times\mathcal{I}} \overline{\pi}_{(m,l)(n,u)}^{g} P_{3n,u}^{q} - U_l^T - U_l, \\
\Psi_{18} &= \left[X_l^T A_m + (1-\mu)\overline{B}_{fl}C_m + \mu\overline{B}_{fl}K_1C_m \quad \overline{A}_{fl}\right], \\
\Psi_{19} &= X_l^T B_m + \overline{B}_{fl}D_m, \\
\Psi_{28} &= \left[S_l^T A_m + (1-\mu)\overline{B}_{fl}C_m + \mu\overline{B}_{fl}K_1C_m \quad \overline{A}_{fl}\right], \\
\Psi_{29} &= S_l^T B_m + \overline{B}_{fl}D_m, \\
\Psi_{38} &= \left[\sqrt{\lambda}\,\overline{B}_{fl}C_m - \sqrt{\lambda}\,\overline{B}_{fl}K_1C_m \quad 0\right], \\
\Psi_{58} &= \left[H_m - (1-\mu)\overline{D}_{fl}C_m - \mu\overline{D}_{fl}K_1C_m \quad -\overline{C}_{fl}\right], \\
\Psi_{59} &= L_m - \overline{D}_{fl}D_m, \\
\Psi_{68} &= \left[\sqrt{\lambda}\,\overline{D}_{fl}K_1C_m - \sqrt{\lambda}\,\overline{D}_{fl}C_m \quad 0\right], \\
\Psi_{78} &= \left[\mu\tau_{m,l}^{g}C_m \quad 0\right], \\
\Psi_{88} &= \begin{bmatrix} -P_{1m,l}^{g} & -P_{2m,l}^{g} \\ * & -P_{3m,l}^{g} \end{bmatrix}, \\
\Psi_{89} &= \begin{bmatrix} \mu\rho_{m,l}^{g}C_m^T K^T \\ 0 \end{bmatrix}.
\end{aligned} \tag{28}$$

In this case, the admissible filter gains are given by

$$\begin{aligned}
A_{fl} &= U_l^{-T}\overline{A}_{fl}, \\
B_{fl} &= U_l^{-T}\overline{B}_{fl}, \\
C_{fl} &= \overline{C}_{fl}, \\
D_{fl} &= \overline{D}_{fl}.
\end{aligned} \tag{29}$$

Proof. First, consider the system in (10) and define the matrices

$$\begin{aligned}
P_{m,l}^{hg} &= \begin{bmatrix} P_{1m,l}^{hg} & P_{2m,l}^{hg} \\ * & P_{3m,l}^{hg} \end{bmatrix}, \\
P_{m,l}^{vg} &= \begin{bmatrix} P_{1m,l}^{vg} & P_{2m,l}^{vg} \\ * & P_{3m,l}^{vg} \end{bmatrix}, \\
G_l &= \begin{bmatrix} G_{1l} & G_{2l} \\ G_{4l} & G_{3l} \end{bmatrix}, \\
G_{il} &= \begin{bmatrix} X_{il} & S_{il} \\ U_{il} & U_{il} \end{bmatrix}, \quad i = 1, \ldots, 4.
\end{aligned} \tag{30}$$

Notice that

$$\begin{aligned}
\Lambda P_{m,l}^{g}\Lambda^T &= \begin{bmatrix} P_{1m,l}^{g} & P_{2m,l}^{g} \\ * & P_{3m,l}^{g} \end{bmatrix}, \\
\Lambda G_l \Lambda^T &= \begin{bmatrix} X_l & S_l \\ U_l & U_l \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
&\Lambda G_l^T \overline{A}_{m,l} \Lambda^T \\
&\quad = \begin{bmatrix} X_l^T A_m + (1-\mu)\overline{B}_{fl} C_m + \mu \overline{B}_{fl} K_1 C_m & \overline{A}_{fl} \\ S_l^T A_m + (1-\mu)\overline{B}_{fl} C_m + \mu \overline{B}_{fl} K_1 C_m & \overline{A}_{fl} \end{bmatrix}, \\
&\Lambda \sqrt{\lambda} G_l^T \widehat{A}_{m,l} \Lambda^T = \begin{bmatrix} \sqrt{\lambda}\,\overline{B}_{fl} C_m - \sqrt{\lambda}\,\overline{B}_{fl} K_1 C_m & 0 \\ \sqrt{\lambda}\,\overline{B}_{fl} C_m - \sqrt{\lambda}\,\overline{B}_{fl} K_1 C_m & 0 \end{bmatrix}, \\
&\Lambda G_l^T \overline{M}_l = \begin{bmatrix} \mu \overline{B}_{fl} \\ \mu \overline{B}_{fl} \end{bmatrix}, \\
&\Lambda G_l^T \overline{M}_l \operatorname{diag}(F_1) = \begin{bmatrix} \mu \overline{B}_{fl} \operatorname{diag}(F_1) \\ \mu \overline{B}_{fl} \operatorname{diag}(F_1) \end{bmatrix}, \\
&\Lambda \sqrt{\lambda} G_l^T \widehat{M}_l = \begin{bmatrix} -\sqrt{\lambda}\,\overline{B}_{fl} \\ -\sqrt{\lambda}\,\overline{B}_{fl} \end{bmatrix}, \\
&\Lambda \sqrt{\lambda} G_l^T \widehat{M}_l \operatorname{diag}(F_1) = \begin{bmatrix} -\sqrt{\lambda}\,\overline{B}_{fl} \operatorname{diag}(F_1) \\ -\sqrt{\lambda}\,\overline{B}_{fl} \operatorname{diag}(F_1) \end{bmatrix}, \\
&\Lambda G_l^T \overline{B}_{m,l} = \begin{bmatrix} X_l^T B_m + \overline{B}_{fl} D_m \\ S_l^T B_m + \overline{B}_{fl} D_m \end{bmatrix}, \\
&\overline{C}_{m,l} \Lambda^T \\
&\quad = \left[H_m - (1-\mu)\overline{D}_{fl} C_m - \mu \overline{D}_{fl} K_1 C_m \quad -\overline{C}_{fl} \right], \\
&\sqrt{\lambda} \widehat{C}_{m,l} \Lambda^T = \left[\sqrt{\lambda}\,\overline{D}_{fl} K_1 C_m - \sqrt{\lambda}\,\overline{D}_{fl} C_m \quad 0 \right], \\
&\overline{N}_l = -\mu \overline{D}_{fl}, \\
&\overline{N}_l \operatorname{diag}(F_1) = -\mu \overline{D}_{fl} \operatorname{diag}(F_1), \\
&\sqrt{\lambda} \widehat{N}_l = \sqrt{\lambda}\,\overline{D}_{fl}, \\
&\sqrt{\lambda} \widehat{N}_l \operatorname{diag}(F_1) = \sqrt{\lambda}\,\overline{D}_{fl} \operatorname{diag}(F_1) \\
&\mu \tau_{m,l}^g C_m \Lambda_1 \Lambda^T = \left[\mu \tau_{m,l}^g C_m \quad 0 \right], \\
&\mu \rho_{m,l}^g K C_m \Lambda_1 \Lambda^T = \left[\mu \rho_{m,l}^g K C_m \quad 0 \right], \\
&\overline{D}_{m,l} = L_m - \overline{D}_{fl} D_m.
\end{aligned}
\tag{31}
$$

Further let the following matrices of the filter

$$
\begin{aligned}
\overline{A}_{fl} &= U_l^T A_{fl}, \\
\overline{B}_{fl} &= U_l^T B_{fl}, \\
\overline{C}_{fl} &= C_{fl}, \\
\overline{D}_{fl} &= D_{fl}.
\end{aligned}
\tag{32}
$$

Then, in terms of the condition in (27), we get

$$
\begin{bmatrix}
\overline{\Omega}_{11} & 0 & 0 & 0 & 0 & \overline{\Omega}_{16} & \overline{\Omega}_{17} & \overline{\Omega}_{18} & \overline{\Omega}_{19} \\
* & \overline{\Omega}_{11} & 0 & 0 & 0 & \overline{\Omega}_{26} & \overline{\Omega}_{27} & \overline{\Omega}_{28} & 0 \\
* & * & -I & 0 & 0 & \overline{C}_{m,l} & \overline{N}_l & \Omega_{38} & \overline{D}_{m,l} \\
* & * & * & -I & 0 & \Omega_{46} & \Omega_{47} & \Omega_{48} & 0 \\
* & * & * & * & \overline{\Omega}_{55} & \overline{\Omega}_{56} & 0 & 0 & 0 \\
* & * & * & * & * & -P_{m,l}^g & \overline{\Omega}_{67} & 0 & 0 \\
* & * & * & * & * & * & \overline{\Omega}_{77} & \overline{\Omega}_{78} & 0 \\
* & * & * & * & * & * & * & \overline{\Omega}_{88} & 0 \\
* & * & * & * & * & * & * & * & -\gamma^2 I
\end{bmatrix}
< 0,
\tag{33}
$$

where

$$
\begin{aligned}
\overline{\Omega}_{11} &= \overline{\mathscr{P}}_{n,u}^q - G_l^T - G_l, \\
\overline{\Omega}_{16} &= G_l^T \overline{A}_{m,l}, \\
\overline{\Omega}_{17} &= G_l^T \overline{M}_l, \\
\overline{\Omega}_{18} &= G_l^T \overline{M}_l \operatorname{diag}(F_1), \\
\overline{\Omega}_{19} &= G_l^T \overline{B}_{m,l}, \\
\overline{\Omega}_{26} &= \sqrt{\lambda} G_l^T \widehat{A}_{m,l}, \\
\overline{\Omega}_{27} &= \sqrt{\lambda} G_l^T \widehat{M}_l, \\
\overline{\Omega}_{28} &= \sqrt{\lambda} G_l^T \widehat{M}_l \operatorname{diag}(F_1), \\
\overline{\Omega}_{55} &= -\mu \tau_{m,l}^g I, \\
\overline{\Omega}_{56} &= \mu \tau_{m,l}^g C_m \Lambda_1, \\
\overline{\Omega}_{67} &= \mu \rho_{m,l}^g \Lambda_1^T C_m^T K^T, \\
\overline{\Omega}_{77} &= -2\mu \rho_{m,l}^g I, \\
\overline{\Omega}_{78} &= \mu \rho_{m,l}^g \operatorname{diag}(F), \\
\overline{\Omega}_{88} &= -\mu \frac{v_c^2}{r} \tau_{m,l}^g I, \\
\overline{\mathscr{P}}_{n,u}^q &= \sum_{(n,u)\in\mathscr{I}\times\mathscr{I}} \overline{\pi}_{(m,l)(n,u)}^g P_{n,u}^q.
\end{aligned}
\tag{34}
$$

Multiplying inequality (33) by the adequate coefficients and adding the resulting inequalities, together with the consideration of the fact that $\mathscr{P}_{n,u}^q - G_l^T - G_l \geq -G_l^T (\mathscr{P}_{n,u}^q)^{-1} G_l$, yield inequality (16) in Theorem 5. Therefore, it follows from Theorem 5 that system (10) with extended varying transition probabilities (12) is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance level. Meanwhile, the filter gains in (29) follow immediately from (32). This completes the proof. □

Remark 8. According to Theorem 7, the asynchronous $\mathscr{H}_\infty$ filter can be designed for the addressed Markovian jump system with varying transition probabilities in the presence of randomly occurring nonlocal sensor nonlinearity. By solving the convex problem contained in Theorem 7, the $\mathscr{H}_\infty$ performance γ can be optimized in terms of the feasibility of the corresponding condition. The result in Theorem 7 indicates that the different η in (12) sparks off the different optimal γ achieved for the system in (10). Thus, the effect of the asynchronous behavior can be readily comprehended by comparing the $\mathscr{H}_\infty$ performance indexes.

4. Numerical Example

In this section, an example is presented to demonstrate the merits of the proposed approach. Consider a 2D Markovian jump system with two operation modes and the parameters as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.12 & 0 \\ 1 & 0.29 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.83 & 0 \\ 1 & 0.92 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0.12 & 1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.83 & 1 \end{bmatrix}, \\ B_1 &= B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ D_1 &= D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ H_1 &= H_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ K_1 &= 0, \\ K_2 &= 5, \\ F_1 &= 0, \\ F_2 &= -1, \\ v_c &= 0.25. \end{aligned} \tag{35}$$

The varying transition probability matrix is assumed to be included in a polytope defined by its vertices:

$$\begin{aligned} \Pi^1 &= \begin{bmatrix} 0.27 & 0.73 \\ 0.35 & 0.65 \end{bmatrix}, \\ \Pi^2 &= \begin{bmatrix} 0.59 & 0.41 \\ 0.85 & 0.15 \end{bmatrix}. \end{aligned} \tag{36}$$

Moreover, the probabilities μ and η are set as 0.2 and 0.35, respectively.

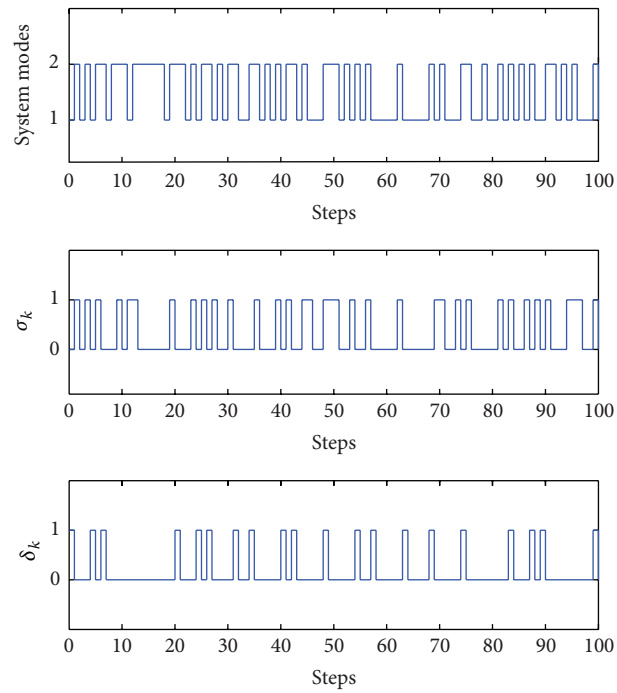


Figure 1: Paths of $r_{i,j}$, σ_k, and δ_k.

By applying the filter design method in Theorem 7, the minimum $\mathscr{H}_\infty$ cost is obtained $\gamma^* = 10.7273$ as well as the resulting filter gain matrices:

$$\begin{aligned} A_{f1} &= \begin{bmatrix} 0.0602 & 0.0163 \\ 0.2327 & 0.3538 \end{bmatrix}, \\ B_{f1} &= \begin{bmatrix} -0.0069 \\ -0.0681 \end{bmatrix}, \\ C_{f1} &= \begin{bmatrix} -0.7647 & -0.8920 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} 0.1724 & 0.0834 \\ 1.0948 & 0.4952 \end{bmatrix}, \\ B_{f2} &= \begin{bmatrix} -0.0238 \\ -0.0982 \end{bmatrix}, \\ C_{f2} &= \begin{bmatrix} -1.0738 & -0.7933 \end{bmatrix}. \end{aligned} \tag{37}$$

Given paths of $r_{i,j}$, σ_k, and δ_k demonstrated in Figure 1, the system initial condition $x_{i,0}^v = 1$, $0 \le i \le 29$, the disturbance signal

$$w_{i,j} = \begin{cases} 0.3e^{-0.3j} \sin(0.5\pi j), & 1 \le i \le 15,\ 1 \le j \le 15, \\ 0, & \text{otherwise}, \end{cases} \tag{38}$$

and the sensor nonlinearity $\varphi(\theta) = K_2\theta + F_2$, by using the achieved filter, it is seen that $\overline{z}_{i,j}$ of the error system converges

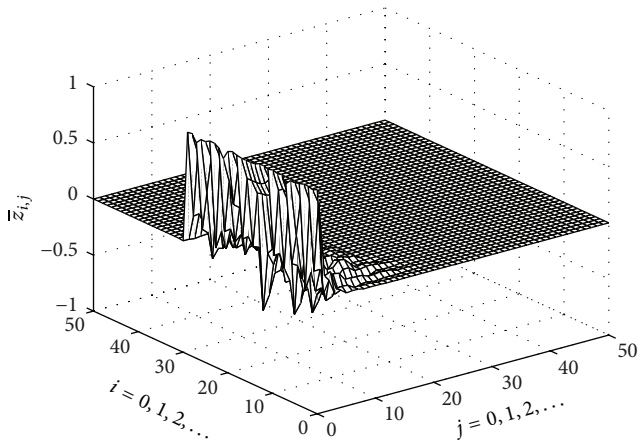


Figure 2: Filtering error response $\bar{z}_{i,j}$.

to zero, shown in Figure 2, which means that the obtained filter is effective in spite of the asynchronous switching and randomly occurring nonlocal sensor nonlinearity.

5. Conclusion

This paper has addressed the problem of asynchronous $\mathscr{H}_\infty$ estimation for a class of 2D Markovian jump systems with varying transition probabilities in the presence of randomly occurring nonlocal sensor nonlinearity. The jumping process of the estimation error system is modeled by the Markov chain with extended varying transition probabilities. The existence condition of asynchronous $\mathscr{H}_\infty$ filters has been derived to ensure the mean-square asymptotic stability and $\mathscr{H}_\infty$ performance level of the error system. A numerical example has been provided that highlights the effectiveness of the developed estimation approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by the Innovative Team Program of the National Natural Science Foundation of China (61321062), by the Specialized Research Fund for the Doctoral Program of Higher Education (20122302120069), by the Natural Science Foundation of Heilongjiang Province (QC2010064), by the Basic Research Plan in Shenzhen City (JCYJ20120613135212389), and by the Fundamental Research Funds for the Central Universities (HIT.NSRIF.2011129).

References

[1] C. Du, L. Xie, and C. Zhang, H_∞ *Control and Filtering of Two-Dimensional Systems*, Springer, Berlin, Germany, 2002.

[2] L. Wu and H. Gao, "Sliding mode control of two-dimensional systems in Roesser model," *IET Control Theory and Applications*, vol. 2, no. 4, pp. 352–364, 2008.

[3] L. Ntogramatzidis and M. Cantoni, "LQ optimal control for 2D Roesser models of finite extent," *Systems & Control Letters*, vol. 58, no. 7, pp. 482–490, 2009.

[4] T. Kaczorek, "Practical stability and asymptotic stability of positive fractional 2D linear systems," *Asian Journal of Control*, vol. 12, no. 2, pp. 200–207, 2010.

[5] Z. Feng, L. Xu, M. Wu, and Y. He, "Delay-dependent robust stability and stabilisation of uncertain two-dimensional discrete systems with time-varying delays," *IET Control Theory and Applications*, vol. 4, no. 10, pp. 1959–1971, 2010.

[6] Q. Zhu and G. Hu, "Stability and absolute stability of a general 2-D non-linear FM second model," *IET Control Theory and Applications*, vol. 5, no. 1, pp. 239–246, 2011.

[7] M. Sebek, "Polynomial solution of 2-D Kalman-Bucy filtering problem," *IEEE Transactions on Automatic Control*, vol. 37, no. 10, pp. 1530–1533, 1992.

[8] J. Woods and C. Radewan, "Kalman filtering in two dimensions," *IEEE Transactions on Information Theory*, vol. 23, no. 4, pp. 473–482, 1977.

[9] H. Gao, J. Lam, C. Wang, and S. Xu, "Robust H_∞ filtering for 2D stochastic systems," *Circuits, Systems and Signal Processing*, vol. 23, no. 6, pp. 479–505, 2004.

[10] H. Gao, X. Meng, and T. Chen, "New design of robust H_∞ filters for 2-D systems," *IEEE Signal Processing Letters*, vol. 15, pp. 217–220, 2008.

[11] S. Chen and I. Fong, "Robust filtering for 2-D state-delayed systems with NFT uncertainties," *IEEE Transactions on Signal Processing*, vol. 54, no. 1, pp. 274–285, 2006.

[12] D. Peng and X. Guan, "H_∞ filtering of 2-D discrete state-delayed systems," *Multidimensional Systems and Signal Processing*, vol. 20, no. 3, pp. 265–284, 2009.

[13] N. Hoang, H. Tuan, T. Nguyen, and S. Hosoe, "Robust mixed generalized H_2/H_∞ filtering of 2-D nonlinear fractional transformation systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 12, pp. 4697–4706, 2005.

[14] S. Xu, J. Lam, Y. Zou, Z. Lin, and W. Paszke, "Robust H_∞ filtering for uncertain 2-D continuous systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 5, pp. 1731–1738, 2005.

[15] L. Wu, P. Shi, H. Gao, and C. Wang, "H_∞ filtering for 2D Markovian jump systems," *Automatica*, vol. 44, no. 7, pp. 1849–1858, 2008.

[16] C. Souza, L. Xie, and D. Coutinho, "Robust filtering for 2-D discrete-time linear systems with convex-bounded parameter uncertainty," *Automatica*, vol. 46, no. 4, pp. 673–681, 2010.

[17] H. Xu and Y. Zou, "Robust H_∞ filtering for uncertain two-dimensional discrete systems with state-varying delays," *International Journal of Control, Automation and Systems*, vol. 8, no. 4, pp. 720–726, 2010.

[18] R. Zhang, Y. Zhang, C. Hu, M. Meng, and Q. He, "Asynchronous H_∞ filtering for a class of two-dimensional Markov jump systems," *IET Control Theory and Applications*, vol. 6, no. 7, pp. 979–984, 2012.

[19] R. Zhang, Y. Zhang, Y. Zhao, J. Liao, and B. Li, "Extended $\mathscr{H}_\infty$ estimation for two-dimensional Markov jump systems under asynchronous switching," *Mathematical Problems in Engineering*, vol. 2013, Article ID 734271, 7 pages, 2013.

[20] L. Wu, Z. Wang, H. Gao, and C. Wang, "H_∞ and $l_2 - l_\infty$ filtering for two-dimensional linear parameter-varying systems," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 12, pp. 1129–1154, 2007.

[21] N. Krasovskii and È. Lidskiĭ, "Analytical design of controllers in systems with random attributes. I. Statement of the problem, method of solving," *Automation and Remote Control*, vol. 22, pp. 1021–1294, 1961.

[22] S. Aberkane, "Stochastic stabilization of a class of nonhomogeneous Markovian jump linear systems," *Systems and Control Letters*, vol. 60, no. 3, pp. 156–160, 2011.

[23] L. Zhang, "H_∞ estimation for discrete-time piecewise homogeneous Markov jump linear systems," *Automatica*, vol. 45, no. 11, pp. 2570–2576, 2009.

[24] C. Souza, A. Trofino, and K. Barbosa, "Mode-independent H_∞ filters for Markovian jump linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 11, pp. 1837–1841, 2006.

[25] L. Zhang and E. Boukas, "Mode-dependent H_∞ filtering for discrete-time Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 6, pp. 1462–1467, 2009.

[26] L. Zhang, N. Cui, M. Liu, and Y. Zhao, "Asynchronous filtering of discrete-time switched linear systems with average dwell time," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 58, no. 5, pp. 1109–1118, 2011.

[27] Z. Wu, P. Shi, H. Su, and J. Chu, "Asynchronous $l_2 - l_\infty$ filtering for discrete-time stochastic Markov jump systems with randomly occurred sensor nonlinearities," *Automatica*, vol. 50, no. 1, pp. 180–186, 2014.

[28] V. Andrieu, C. Prieur, S. Tarbouriech, and D. Arzelier, "Global asymptotic stabilization of systems satisfying two different sector conditions," *Systems & Control Letters*, vol. 60, no. 8, pp. 570–578, 2011.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 808973, 16 pages
http://dx.doi.org/10.1155/2015/808973

Research Article

Robust H_∞ Filtering for Uncertain Neutral Stochastic Systems with Markovian Jumping Parameters and Time Delay

Yajun Li and Zhaowen Huang

Institute of Electronic and Information Engineering, Shunde Polytechnic, Foshan 528300, China

Correspondence should be addressed to Yajun Li; lyjfirst@163.com

Received 15 March 2015; Revised 4 April 2015; Accepted 12 May 2015

Academic Editor: Son Nguyen

This paper deals with the robust H_∞ filter design problem for a class of uncertain neutral stochastic systems with Markovian jumping parameters and time delay. Based on the Lyapunov-Krasovskii theory and generalized Finsler Lemma, a delay-dependent stability condition is proposed to ensure not only that the filter error system is robustly stochastically stable but also that a prescribed H_∞ performance level is satisfied for all admissible uncertainties. All obtained results are expressed in terms of linear matrix inequalities which can be easily solved by MATLAB LMI toolbox. Numerical examples are given to show that the results obtained are both less conservative and less complicated in computation.

1. Introduction

Time delay exists extensively in chemical process systems, communication systems, economic systems, microwave oscillator, and networked control systems. Meanwhile, because of the modeling inaccuracies and changes in the environment of the model parameter, uncertainties are unavoidable in the process of modeling. The appearance of time delay and uncertainties in many systems can often bring instability, oscillation, and poor performance; considerable attention has been focused on the stability analysis of uncertain time delays systems; see [1–6] and the references therein. On the other hand, a real system is often affected by external disturbances such as stochastic perturbations. The stochastic effects can also lead to oscillation, divergence, or other poor performances. Therefore, the stability study of stochastic systems with time delay has been paid great attention and a lot of significant results have been reported in the literature; see [7–12] and the references therein.

In the past few decades, filtering problem has been a hot issue in the fields of signal processing. Kalman filtering has been successfully applied in many fields such as manufacturing systems, economic systems, and maneuvered target tracking. However, the exact requirement of known dynamics system and precise noise statistics have restricted its practical application. In such a case, we can resort to H_∞ filtering [13–22] and L_2-L_∞ filtering [23–25]; see the references therein.

Markovian jump systems, originally raised by Krasovskii and Lidskii [26], are famous for the description of many dynamical practical systems whose structure and parameters are subject to random changes. Therefore, the stability analysis and filtering problem for Markovian jump systems have been studied [27–34]. For example, the stability analysis of impulsive stochastic neural networks with Markovian jump are studied in [27, 29, 31, 34]; H_∞ control and mode-dependent H_∞ filtering for discrete-time Markovian jump linear systems with partly unknown transition probabilities are, respectively, investigated in [28, 30, 32]. The design of reduced-order H_∞ filter for Markovian jumping systems with time delay is studied in [15]. The L_2-L_∞ filter design for stochastic time-varying delay systems with Markovian jumping parameters is considered in [33].

Many methods are proposed in the process of robust stochastic stability analysis and filtering design, which develops from the early solving Riccati equation to model transformation method and cross-terms bounding technique [5], free-weighting matrices method [11, 32], slack matrix variables [17–19, 24, 28], and delay-partitioning method [20]. However, model transformations may give rise to additional dynamics of the original systems [13], and cross-terms bounding techniques can bring conservatism. Moreover, as

pointed out in [33], under certain circumstance, free matrix variables may not lessen the conservatism. In recent years, another popular method called Finsler Lemma is carried out so as to decrease the computational cost as well [5], and the Finsler Lemma in deterministic setting is extended to generalized Finsler Lemma in stochastic systems in [5, 33, 34].

On the other hand, many dynamical systems can be modeled by neutral systems which are organized by neutral functional differential equations. Other than retarded time delay systems containing delays only in states, a neutral time delay system contains delays in both its state and its derivatives of state. Recently, for neutral stochastic time delay systems, the stability analysis and filter design problem are mainly addressed in [4, 9, 19]. It is necessary to point out that the delay-dependent robust H_∞ filtering for uncertain neutral stochastic time delay system is studied in [19, 24]. Robust H_∞ filter design for neutral stochastic uncertain systems with time-varying delay is studied in [13]. To the best of the authors' knowledge, the H_∞ filtering problem has not been reported about uncertain neutral stochastic systems with Markovian jumping parameters and time delay, which motivates the present study.

Motivated by the works in [5, 13, 17, 18], the robust H_∞ filtering for uncertain neutral stochastic systems with Markovian jumping parameters and time delay is considered in this paper. By generalized Finsler Lemma, the robust stochastic stability condition is obtained. The presented condition is simple and efficient. Finally, the effectiveness of the approach is verified by illustrative examples including a comparison with some recent results.

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. I is the identity matrix. $|\cdot|$ denotes Euclidean norm for vectors and $\|\cdot\|$ denotes the spectral norm of matrices. N denotes the set of all natural numbers; that is, $N = \{0, 1, 2, \ldots\}$. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$ is a complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. M^T stands for the transpose of the matrix M. For symmetric matrices X and Y, the notation $X > Y$ (resp., $X \geq Y$) means that the $X - Y$ is positive definite (resp., positive semidefinite). $*$ denotes a block that is readily inferred by symmetry. $\mathbf{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathscr{P}$.

2. Problem Description

Consider the following uncertain neutral stochastic systems with Markovian jumping parameters and time delay:

$$\begin{aligned}
d\left[x(t) - G(r_t)\,x(t-h)\right] &= \left[A(t,r_t)(t)\,x(t)\right.\\
&\quad \left.+ A_1(t,r_t)(t)\,x(t-h) + B(r_t)\,v(t)\right]dt\\
&\quad + \left[D(t,r_t)(t)\,x(t) + D_1(t,r_t)(t)\,x(t-h)\right.\\
&\quad \left.+ D_2(r_t)\,v(t)\right]dw(t),\\
dy(t) &= \left[C(r_t)\,x(t) + C_1(r_t)\,x(t-h) + C_2(r_t)\,v(t)\right]dt\\
&\quad + \left[E(r_t)(t)\,x(t) + E_1(r_t)\,x(t-h) + E_2(r_t)\,v(t)\right]dw(t),\\
z(t) &= L_1(r_t)\,x(t) + L_2(r_t)\,x(t-h) + L_3(r_t)\,v(t),\\
x(\theta) &= \psi(\tau),\\
&\qquad r_t = r_0 \in S,\ \forall \tau \in [-h, 0],
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the measured output, $v(t) \in \mathbb{R}^q$ is the disturbance input in $L_2[0,\infty)$, and $z(t) \in \mathbb{R}^p$ is the signal to be estimated. $A(t,r_t)(t)$, $A_1(t,r_t)(t)$, $B(r_t)$, $D(t,r_t)(t)$, $D_1(t,r_t)(t)$, $D_2(r_t)$, $C(r_t)$, $C_1(r_t)$, $C_2(r_t)$, $E(r_t)$, $E_1(r_t)$, $E_2(r_t)$, $L_1(r_t)$, $L_2(r_t)$, and $L_3(r_t)$ are the matrix functions of the random jumping process $r(t)$, where $r(t)$ is a finite-state Markovian jump process representing the system mode; that is, $r(t)$ takes discrete values in given finite set $\mathcal{S} = 1, 2, \ldots, N$. Here $\psi(\cdot)$ is the initial condition and is assumed to be continuously differentiable on $[-h, 0]$. Consider $h > 0$ indicates the time delay. $w(t)$ is a scalar Brownian motion (Wiener process) defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$ satisfying

$$\begin{aligned}
\mathbf{E}\{dw(t)\} &= 0,\\
\mathbf{E}\left\{dw^2(t)\right\} &= dt.
\end{aligned} \tag{2}$$

The transition probability matrix of systems (1) is given by

$$P_r\left(r_{t+\Delta} = j \mid r_t = i\right) = \begin{cases} \pi_{ij}\Delta + o(\Delta), & j \neq i\\ 1 + \pi_{ii}\Delta + o(\Delta), & j = i, \end{cases} \tag{3}$$

where $\Delta > 0$, $\lim_{\Delta\to 0}(o(\Delta)/\Delta) = 0$, $\pi_{ij} \geq 0$, $\forall j \neq i$, is the transition rate from mode i at time t to mode j at time $t + \Delta$, and

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{j=N} \pi_{ij} < 0. \tag{4}$$

For the purpose of simplicity, in this paper, for each $r(t) = i \in \mathcal{S}$, $A(t,r_t)(t)$, $A_1(t,r_t)(t)$, and $B(r_t)$ are denoted by $A_i(t)$, $A_{1i}(t)$, B_i, and so on. In systems (1),

$$\begin{aligned}
A_i(t) &= A_i + \Delta A_i(t),\\
A_{1i}(t) &= A_{1i} + \Delta A_{1i}(t),\\
D_i(t) &= D_i + \Delta D_i(t),\\
D_{1i}(t) &= D_{1i} + \Delta D_{1i}(t),
\end{aligned} \tag{5}$$

and A_i, A_{1i}, B_i, D_i, D_{1i}, D_{2i}, C_i, C_{1i}, C_{2i}, E_i, E_{1i}, E_{2i}, L_{1i}, L_{2i}, and L_{3i} are known real constant matrices with appropriate dimensions. $\Delta A_i(t)$, $\Delta A_{1i}(t)$, $\Delta D_i(t)$, and $\Delta D_{1i}(t)$ are unknown matrices representing norm-bounded parameter uncertainties, which are assumed to satisfy

$$\begin{bmatrix} \Delta A_i(t) & \Delta A_{1i}(t)\\ \Delta D_i(t) & \Delta D_{1i}(t) \end{bmatrix} = \begin{bmatrix} M_{1i}\\ M_{2i} \end{bmatrix} F_i(t) \begin{bmatrix} N_{1i} & N_{2i} \end{bmatrix}, \tag{6}$$

where M_{1i}, M_{2i}, N_{1i}, and N_{2i} are known real constant matrices and $F_i(\cdot) : \mathbb{R} \to \mathbb{R}^{k\times l}$ is an unknown time-varying matrix function satisfying

$$F_i^T(t)\,F_i(t) \leq I, \quad \forall t. \tag{7}$$

The parameter uncertainties $\Delta A_i(t)$, $\Delta A_{1i}(t)$, $\Delta D_i(t)$, and $\Delta D_{1i}(t)$ are said to be admissible if both (6) and (7) hold.

In this paper, we make the following assumption on the matrix G_i in systems (1).

Assumption 1. The matrix G_i in systems (1) satisfies

$$\rho(G_i) < 1, \tag{8}$$

where the notation $\rho(G_i)$ denotes the spectral radius of G_i.

We now consider a full-order filter for systems (1) with the following form:

$$\begin{aligned} (\Sigma_f) : d\hat{x}(t) &= A_{fi}\hat{x}(t)\,dt + B_{fi}\,dy(t), \\ \hat{z}(t) &= C_{fi}\hat{x}(t), \end{aligned} \tag{9}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the filter state, $\hat{z}(t) \in \mathbb{R}^p$ is the estimation of $z(t)$, and A_{fi}, B_{fi}, and C_{fi} $(i \in \mathcal{S})$ are the filter parameters with compatible dimensions to be determined.

Define

$$\begin{aligned} \xi(t) &= \left[x^T(t) \quad \hat{x}^T(t)\right]^T, \\ e(t) &= z(t) - \hat{z}(t). \end{aligned} \tag{10}$$

Then the filtering error systems can be obtained as follows:

$$\begin{aligned} d\left[\xi(t) - \overline{G}_i K\xi(t-h)\right] &= \left[\overline{A}_i(t)\xi(t) \right. \\ &\quad \left. + \overline{A}_{1i}(t)K\xi(t-h) + \overline{B}_i v(t)\right]dt + \left[\overline{D}_i(t)\xi(t)\right. \\ &\quad \left. + \overline{D}_{1i}(t)K\xi(t-h) + \overline{D}_{2i}v(t)\right]dw(t), \\ e(t) &= \overline{L}_{1i}\xi(t) + L_{2i}K\xi(t-h) + L_{3i}v(t), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \overline{A}_i(t) &= \overline{A}_i + \Delta\overline{A}_i(t), \\ \overline{A}_{1i}(t) &= \overline{A}_{1i} + \Delta\overline{A}_{1i}(t), \\ \overline{D}_i(t) &= \overline{D}_i + \Delta\overline{D}_i(t), \\ \overline{D}_{1i}(t) &= \overline{D}_{1i} + \Delta\overline{D}_{1i}(t) \end{aligned} \tag{12}$$

with

$$\begin{aligned} \overline{A}_i &= \begin{bmatrix} A_i & 0 \\ B_{fi}C_i & A_{fi} \end{bmatrix}, \\ \Delta\overline{A}_i(t) &= \begin{bmatrix} \Delta A_i(t) & 0 \\ 0 & 0 \end{bmatrix}, \\ \overline{A}_{1i} &= \begin{bmatrix} A_{1i} \\ B_{fi}C_{1i} \end{bmatrix}, \\ \Delta\overline{A}_{1i}(t) &= \begin{bmatrix} \Delta A_{1i}(t) \\ 0 \end{bmatrix}, \\ \overline{B}_i &= \begin{bmatrix} B_i \\ B_{fi}C_{2i} \end{bmatrix}, \\ \overline{D}_i &= \begin{bmatrix} D_i & 0 \\ B_{fi}E_i & 0 \end{bmatrix}, \\ \Delta\overline{D}_i(t) &= \begin{bmatrix} \Delta D_i(t) & 0 \\ 0 & 0 \end{bmatrix}, \\ \overline{D}_{1i} &= \begin{bmatrix} D_{1i} \\ B_{fi}E_{1i} \end{bmatrix}, \\ \Delta\overline{D}_{1i}(t) &= \begin{bmatrix} \Delta D_{1i}(t) \\ 0 \end{bmatrix}, \\ \overline{D}_{2i} &= \begin{bmatrix} D_{2i} \\ B_{fi}E_{2i} \end{bmatrix} K = \begin{bmatrix} I & 0 \end{bmatrix}, \\ \overline{L}_{1i} &= \begin{bmatrix} L_{1i} & -C_{fi} \end{bmatrix}, \\ \overline{G}_i &= \begin{bmatrix} G_i \\ 0 \end{bmatrix}. \end{aligned} \tag{13}$$

Then the problem of robust H_∞ filtering to be addressed in this paper is formulated as follows: given the uncertain stochastic delay systems (1) and a prescribed attenuation level $\gamma > 0$, design linear stochastic filter (Σ_f) as the form of (9) such that the filtering error systems (11) are robustly stochastically stable and under zero initial conditions, the following inequality holds:

$$\|e(t)\|_2 < \gamma\|v(t)\|_2 \tag{14}$$

for all nonzero $v(t) \in L_2[0,\infty)$ and all admissible uncertainties.

Before concluding this section, we introduce the following Lemmas, which will be used in the derivation of our main results in the next section.

Lemma 2. *For any vectors $x, y \in \mathbf{R}^n$ and any scalar $\epsilon > 0$, matrices D, F, E are real matrices of appropriate dimensions with $F^TF \le I$, then the following inequality hold:*

$$2x^TDFEy \le \epsilon^{-1}x^TDD^Tx + \epsilon y^TEE^Ty. \tag{15}$$

Proposition 3 ([5], generalized Finsler Lemma (GFL)). *Consider stochastic vector $\theta \in \mathbb{R}^n$, symmetric and positive matrix $\Theta \in \mathbb{R}^{n\times n}$, and matrix $\mathscr{B} \in \mathbb{R}^{m\times n}$ with rank $(\mathscr{B}) = r < n$. Let $\mathscr{B}^\perp$ represent the right orthogonal complement of $\mathscr{B}$, that is, $\mathscr{B}\mathscr{B}^\perp = 0$, then the following four statements are equivalent:*

$(T1)$ $\mathbf{E}\{\theta^T\Theta\theta\} < 0$, $\forall\theta \ne 0$, $t > t_0$, $\mathbf{E}\{\mathscr{B}\theta\} = 0$;

$(T2)$ $\mathscr{B}^{\perp T}\Theta\mathscr{B}^\perp < 0$;

$(T3)$ $\exists\epsilon \in \mathbb{R} : \Theta - \epsilon\mathscr{B}^T\mathscr{B} < 0$;

$(T4)$ $\exists\Lambda \in \mathbb{R}^{n\times m} : \Theta + \Lambda\mathscr{B} + \mathscr{B}^T\Lambda^T < 0$.

Remark 4. Based on generalized Finsler Lemma, the stability of neutral stochastic systems with time delay has been studied in [5, 34]. In addition, it should be noted that the stochastic systems in [5, 34] are not Markovian jump systems. And the filtering problem for stochastic time delay systems with Markovian jumping parameters is considered in [20]. However, it should be pointed out that the systems in [20] do not include any analysis of neutral phenomena. So H_∞ filtering for neutral stochastic systems with Markovian jumping parameters and time delay is considered in this paper.

3. Main Results

Theorem 5. *Consider the uncertain neutral stochastic Markovian jump systems (1). For given scalars $\gamma > 0$, $h > 0$, systems (1) are robustly stochastically stable for all admissible uncertainties satisfying (6) and (7), if there exist symmetric positive matrices $P_i > 0$, $R > 0$, $Q = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0$, and scalar $\epsilon_i > 0$ satisfying*

$$\Omega_i = \begin{bmatrix} \Omega_{11i} & \Omega_{12i} & -K^T R G_i & P_i \overline{B}_i & h\overline{A}_i^T K^T R & \overline{D}_i^T P_i & \overline{L}_{1i}^T & P_i \overline{M}_{1i} \\ * & \Omega_{22i} & \Omega_{23i} & -\overline{G}_i^T P_i \overline{B}_i & h\overline{A}_{1i}^T K^T R & \overline{D}_{1i}^T P_i & L_{2i}^T & -\overline{G}_i^T P_i \overline{M}_{1i} \\ * & * & \Omega_{33i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & h\overline{B}_i^T K^T R & \overline{D}_{2i}^T P_i & L_{3i}^T & 0 \\ * & * & * & * & -R & 0 & 0 & hRK\overline{M}_{1i} \\ * & * & * & * & * & -P_i & 0 & P_i \overline{M}_{2i} \\ * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & -\epsilon_i \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{aligned} \Omega_{11i} &= P_i \overline{A}_i + \overline{A}_i^T P_i + \sum_{j=1}^{N} \pi_{ij} P_j + K^T Q_1 K + \epsilon_i \widetilde{N}_{1i}^T \widetilde{N}_{1i} \\ &\quad - K^T R K, \\ \Omega_{12i} &= P_i \overline{A}_{1i} - \overline{A}_i^T P_i \overline{G}_i - \sum_{j=1}^{N} \pi_{ij} P_j \overline{G}_i + K^T Q_2 \\ &\quad + \epsilon_i \widetilde{N}_{1i}^T N_{2i} + K^T R (G_i + I), \\ \Omega_{22i} &= -\overline{G}_i^T P_i \overline{A}_{1i} - \overline{A}_{1i}^T P_i \overline{G}_i + \overline{G}_i^T \sum_{j=1}^{N} \pi_{ij} P_j \overline{G}_i + Q_3 - Q_1 \\ &\quad + \epsilon_i N_{2i}^T N_{2i} - (G_i + I)^T R (G_i + I), \\ \Omega_{23i} &= -Q_2 + (G_i + I)^T R G_i, \\ \Omega_{33i} &= -Q_3 - G_i^T R G_i, \\ \widetilde{N}_{1i} &= N_{1i} K = [N_{1i} \quad 0], \\ \overline{M}_{1i} &= \begin{bmatrix} M_{1i} \\ 0 \end{bmatrix}, \\ \overline{M}_{2i} &= \begin{bmatrix} M_{2i} \\ 0 \end{bmatrix}. \end{aligned} \tag{17}$$

Proof. For the purpose of convenience, the following notations are adopted:

$$\begin{aligned} \overline{z}(t) &= \xi(t) - \overline{G}_i K \xi(t-h), \\ f_v(t, \xi(t), i) &= \overline{A}_i(t) \xi(t) + \overline{A}_{1i}(t) K \xi(t-h) \\ &\quad + \overline{B}_i v(t), \\ g_v(t, \xi(t), i) &= \overline{D}_i(t) \xi(t) + \overline{D}_{1i}(t) K \xi(t-h) \\ &\quad + \overline{D}_{2i} v(t), \end{aligned} \tag{18}$$

and then the filtering error systems (11) become

$$d\overline{z}(t) = f_v(t, \xi(t), i)\, dt + g_v(t, \xi(t), i)\, dw(t). \tag{19}$$

Choose the Lyapunov-Krasovskii functional candidate as follows:

$$\begin{aligned} V(t, \xi(t), i) &= \overline{z}^T(t) P_i \overline{z}(t) + \int_{t-h}^{t} \eta^T(s) K^T Q K \eta(s)\, ds \\ &\quad + h \int_{-h}^{0} \int_{t+\beta}^{t} f_v^T(s, \xi(s), i) K^T R K f_v(s, \xi(s), i)\, ds\, d\beta, \end{aligned} \tag{20}$$

where $\eta(t) = \left[\xi^T(t) \quad \xi^T(t-h)\right]^T$. According to Itô's differential formula, the stochastic differential along systems (11) is

$$\begin{aligned} dV(t, \xi(t), i) &= \mathscr{L}V(t, \xi(t), i)\, dt \\ &\quad + 2\overline{z}^T(t) P_i g_v(t, \xi(t), i)\, dw(t), \end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\mathscr{L}V(t,\xi(t),i)\\
&\quad= 2\overline{z}^T(t)P_i f_v(t,\xi(t),i)\\
&\qquad+ g_v^T(t,\xi(t),i)P_i g_v(t,\xi(t),i)\\
&\qquad+ \overline{z}^T(t)\sum_{j=1}^{N}\pi_{ij}P_j\overline{z}(t)+\eta^T(t)K^TQK\eta(t)\\
&\qquad- \eta^T(t-h)K^TQK\eta(t-h)\\
&\qquad+ h^2 f_v^T(t,\xi(t),i)K^TRKf_v(t,\xi(t),i)\\
&\qquad- \int_{t-h}^{t} f_v^T(s,\xi(s),i)K^ThRKf_v(s,\xi(s),i)\,ds.
\end{aligned} \tag{22}$$

Firstly, we show that the filtering error systems (11) with $v(t)=0$ are robustly stochastically stable.

Taking mathematical expectation on both sides of system (19) and by virtue of $\mathbf{E}\{dw(t)=0\}$, we obtain

$$\mathbf{E}\{d\overline{z}(t)\} = \mathbf{E}\{f_v(t,\xi(t),i)\}\,dt. \tag{23}$$

Integrating both sides of (23) from $t-h$ to t, we have

$$\begin{aligned}
&\int_{t-h}^{t}\mathbf{E}\left\{-I\xi(t)+\left(\overline{G}_iK+I\right)\xi(t-h)-\overline{G}_iK\xi(t-2h)\right.\\
&\qquad\left.+hf_v(s,\xi(s),i)\right\}ds=0.
\end{aligned} \tag{24}$$

From the definition of $f_v(t,\xi(t),i)$ with $v(t)=0$ and (24), it is easy to obtain

$$\int_{t-h}^{t}\mathbf{E}\{\mathscr{B}_i\varsigma_i\}\,ds=0, \tag{25}$$

where

$$\begin{aligned}
\mathscr{B}_i &= \begin{bmatrix} -I & \overline{A}_i(t) & \overline{A}_{1i}(t) & 0 & 0\\ 0 & -K & K\overline{G}_i+I & -K\overline{G}_i & hI\end{bmatrix},\\
\varsigma_i &= \left[f_v^T(t,\xi(t),i)\quad \xi^T(t)\quad \xi^T(t-h)K^T\quad \xi^T(t-2h)K^T\quad f_v^T(s,\xi(s),i)K^T\right]^T.
\end{aligned} \tag{26}$$

The right orthogonal complements of $\mathscr{B}_i$ is

$$\mathscr{B}_i^{\perp} = \begin{bmatrix} \overline{A}_i^T(t) & I & 0 & 0 & \dfrac{K^T}{h}\\ \overline{A}_{1i}^T(t) & 0 & I & 0 & -\dfrac{\left(K\overline{G}_i+I\right)^T}{h}\\ 0 & 0 & 0 & I & \dfrac{\left(K\overline{G}_i\right)^T}{h}\end{bmatrix}^T. \tag{27}$$

Taking mathematical expectation on both sides of (21) and then substituting (22) into (21), we have

$$\begin{aligned}
\frac{\left[d\mathbf{E}V(t,\xi(t),i)\right]}{dt} &= \mathbf{E}\{\mathscr{L}V(t,\xi(t),i)\}\\
&\le \mathbf{E}\left\{\frac{1}{h}\int_{t-h}^{t}\varsigma_i^T\Gamma_i\varsigma_i ds\right\}\\
&= \frac{1}{h}\int_{t-h}^{t}\mathbf{E}\left\{\varsigma_i^T\Gamma_i\varsigma_i\right\}ds,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\Gamma_i &= \begin{bmatrix} h^2K^TRK & P_i & -P_i\overline{G}_i & 0 & 0\\ * & \Gamma_{22i} & \Gamma_{23i} & 0 & 0\\ * & * & \Gamma_{33i} & -Q_2 & 0\\ * & * & * & -Q_3 & 0\\ * & * & * & * & -h^2R\end{bmatrix},\\
\Gamma_{22i} &= \sum_{j=1}^{N}\pi_{ij}P_j+\overline{D}_i^T(t)P_i\overline{D}_i(t)+K^TQ_1K,\\
\Gamma_{23i} &= -\sum_{j=1}^{N}\pi_{ij}P_j\overline{G}_i+\overline{D}_i^T(t)P_i\overline{D}_{1i}(t)+K^TQ_2,\\
\Gamma_{33i} &= \overline{G}_i^T\sum_{j=1}^{N}\pi_{ij}P_j\overline{G}_i+\overline{D}_{1i}^T(t)P_i\overline{D}_{1i}(t)+Q_3-Q_1.
\end{aligned} \tag{29}$$

In order to prove the robust stochastic stability of the filtering error systems (11) with $v(t)=0$, it suffices to show

$$\mathbf{E}\left\{\varsigma_i^T\Gamma_i\varsigma_i\right\}<0. \tag{30}$$

By virtue of Proposition 3, (30) is equivalent to

$$\mathbf{E}\left\{\widehat{\Gamma}_i\right\} = \mathbf{E}\left\{\mathscr{B}_i^{\perp T}\Gamma_i\mathscr{B}_i^{\perp}\right\}<0, \tag{31}$$

where

$$\widehat{\Gamma}_i = \begin{bmatrix} \widehat{\Gamma}_{11i} & \widehat{\Gamma}_{12i} & -K^T RK\overline{G}_i \\ * & \widehat{\Gamma}_{22i} & \widehat{\Gamma}_{23i} \\ * & * & \widehat{\Gamma}_{33i} \end{bmatrix},$$

$$\begin{aligned}
\widehat{\Gamma}_{11i} &= h^2\overline{A}_i^T(t)K^T RK\overline{A}_i(t) + P_i\overline{A}_i(t) + \overline{A}_i^T(t)P_i \\
&\quad + \sum_{j=1}^{N}\pi_{ij}P_j + \overline{D}_i^T(t)P_i\overline{D}_i(t) + k^T Q_1 K \\
&\quad - K^T RK, \\
\widehat{\Gamma}_{12i} &= h^2\overline{A}_i^T(t)K^T RK\overline{A}_{1i}(t) + P_i\overline{A}_{1i}(t) - \overline{A}_i^T(t)P_i\overline{G}_i \\
&\quad - \sum_{j=1}^{N}\pi_{ij}P_j\overline{G}_i + \overline{D}_i^T(t)P_i\overline{D}_{1i}(t) + K^T Q_2 \\
&\quad + K^T R\left(K\overline{G}_i + I\right), \\
\widehat{\Gamma}_{22i} &= h^2\overline{A}_{1i}^T(t)K^T RK\overline{A}_{1i}(t) - \overline{G}_i^T P_i\overline{A}_{1i}(t) \\
&\quad - \overline{A}_{1i}^T(t)P_i\overline{G}_i + \overline{G}_i^T\sum_{j=1}^{N}\pi_{ij}P_j\overline{G}_i \\
&\quad + \overline{D}_{1i}^T(t)P_i\overline{D}_{1i}(t) + Q_3 - Q_1 \\
&\quad - \left(K\overline{G}_i + I\right)^T R\left(K\overline{G}_i + I\right), \\
\widehat{\Gamma}_{23i} &= -Q_2 + \left(K\overline{G}_i + I\right)^T RK\overline{G}_i, \\
\widehat{\Gamma}_{33i} &= -Q_3 - \left(K\overline{G}_i\right)^T RK\overline{G}_i.
\end{aligned} \tag{32}$$

It is obvious that $\widehat{\Gamma}_i < 0$ are implied by (16) according to Schur complements. Therefore, if (16) is feasible, then filtering error systems (11) with $v(t) = 0$ are robustly stochastically stable.

Next, we will establish the H_∞ performance for the filtering error systems (11) under the zero initial condition.

From the definition of $f_v(t, \xi(t), i)$ and together with (24), it implies

$$\int_{t-h}^{t}\mathbf{E}\left\{\mathscr{B}_{vi}\varsigma_{vi}\right\}ds = 0, \tag{33}$$

where

$$\begin{aligned}
\mathscr{B}_{vi} &= \begin{bmatrix} -I & \overline{A}_i(t) & \overline{A}_{1i}(t) & 0 & 0 & \overline{B}_i \\ 0 & -K & K\overline{G}_i + I & -K\overline{G}_i & hI & 0 \end{bmatrix}, \\
\varsigma_{vi} &= \left[f_v^T(t,\xi(t),i) \quad \xi^T(t) \quad \xi^T(t-h)K^T \quad \xi^T(t-2h)K^T \quad f_v^T(s,\xi(s),i)K^T \quad v^T(t)\right]^T.
\end{aligned} \tag{34}$$

The right orthogonal complements of $\mathscr{B}_{vi}$ are

$$\mathscr{B}_{vi}^{\perp} = \begin{bmatrix} \overline{A}_i^T(t) & I & 0 & 0 & \dfrac{K^T}{h} & 0 \\ \overline{A}_{1i}^T(t) & 0 & I & 0 & -\dfrac{\left(K\overline{G}_i + I\right)^T}{h} & 0 \\ 0 & 0 & 0 & I & \dfrac{\left(K\overline{G}_i\right)^T}{h} & 0 \\ \overline{B}_i^T & 0 & 0 & 0 & 0 & I \end{bmatrix}^T. \tag{35}$$

Noticing (21)-(22) together with $\mathbf{E}\{dw(t)\} = 0$, we can obtain

$$\begin{aligned}
\frac{\left[d\mathbf{E}V(t,\xi(t),i)\right]}{dt} &= \mathbf{E}\left\{\mathscr{L}V(t,\xi(t),i)\right\} \\
&\le \mathbf{E}\left\{\frac{1}{h}\int_{t-h}^{t}\varsigma_{vi}^T\Gamma_{vi}\varsigma_{vi}ds\right\} \\
&= \frac{1}{h}\int_{t-h}^{t}\mathbf{E}\left\{\varsigma_{vi}^T\Gamma_{vi}\varsigma_{vi}\right\}ds,
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\Gamma_{vi} &= \begin{bmatrix} \Gamma_i & \Gamma_v \\ * & \Gamma_{66i} \end{bmatrix}, \\
\Gamma_v &= \begin{bmatrix} 0 & \Gamma_{26i}^T & \Gamma_{36i}^T & 0 & 0 \end{bmatrix}^T, \\
\Gamma_{26i} &= \overline{D}_i^T(t)P_i\overline{D}_{2i}, \\
\Gamma_{36i} &= \overline{D}_{1i}^T(t)P_i\overline{D}_{2i}, \\
\Gamma_{66i} &= \overline{D}_{2i}^T P_i\overline{D}_{2i},
\end{aligned} \tag{37}$$

and set

$$J = \mathbf{E}\left\{\int_0^\infty\left[e^T(t)e(t) - \gamma^2 v^T(t)v(t)\right]dt\right\}. \tag{38}$$

Adding the right side of (38) to both sides of (36) and integrating both sides of (36) from 0 to ∞ and then taking the zero initial condition into account, we can acquire

$$J \le \frac{1}{h}\int_0^\infty\int_{t-h}^{t}\mathbf{E}\left\{\varsigma_{vi}^T\mathscr{H}_{vi}\varsigma_{vi}\right\}ds\,dt, \tag{39}$$

where

$$\mathscr{H}_{vi} = \Gamma_{vi} + \Pi^T \Pi + \operatorname{diag}\left\{0, 0, 0, 0, 0, -\gamma^2 I\right\}, \tag{40}$$

and $\Pi = \begin{bmatrix} 0 & \overline{L}_{1i} & L_{2i} & K & 0 & 0 & L_{3i} \end{bmatrix}$.

According to Proposition 3, $\mathbf{E}\{\varsigma_{vi}^T \mathscr{H}_{vi} \varsigma_{vi}\} < 0$ is equivalent to

$$\Theta_i = \mathscr{B}_{vi}^{\perp T} \mathscr{H}_{vi} \mathscr{B}_{vi}^{\perp} = \begin{bmatrix} \Theta_{11i} & \Theta_{12i} & -K^T RK\overline{G}_i & \Theta_{14i} \\ * & \Theta_{22i} & \Theta_{23i} & \Theta_{24i} \\ * & * & \Theta_{33i} & 0 \\ * & * & * & \Theta_{44i} \end{bmatrix} < 0, \tag{41}$$

where

$$\begin{aligned}
\Theta_{11i} &= \widehat{\Gamma}_{11i} + \overline{L}_{1i}^T \overline{L}_{1i}, \\
\Theta_{12i} &= \widehat{\Gamma}_{12i} + \overline{L}_{1i}^T L_{2i}, \\
\Theta_{22i} &= \widehat{\Gamma}_{22i} + L_{2i}^T L_{2i}, \\
\Theta_{23i} &= \widehat{\Gamma}_{23i}, \\
\Theta_{33i} &= \widehat{\Gamma}_{33i}, \\
\Theta_{14i} &= h^2 \overline{A}_i^T(t) K^T RK\overline{B}_i + P_i\overline{B}_i + \overline{D}_i^T(t) P_i \overline{D}_{2i} \\
&\quad + \overline{L}_{1i}^T L_{3i}, \\
\Theta_{24i} &= h^2 \overline{A}_{1i}^T(t) K^T RK\overline{B}_i - \overline{G}_i^T P_i \overline{B}_i + \overline{D}_{1i}^T(t) P_i \overline{D}_{2i} \\
&\quad + L_{2i}^T L_{3i}, \\
\Theta_{44i} &= h^2 \overline{B}_i^T K^T RK\overline{B}_i + \overline{D}_{2i}^T P_i \overline{D}_{2i} + L_{3i}^T L_{3i} - \gamma^2 I.
\end{aligned} \tag{42}$$

According to Schur complement and (12), we can obtain

$$\Theta_i = \begin{bmatrix} \Pi_{11i} & \Pi_{12i} & -K^T RG_i & P_i\overline{B}_i & h\overline{A}_i^T K^T R & \overline{D}_i^T P_i & \overline{L}_{1i}^T \\ * & \Pi_{22i} & \Pi_{23i} & -G_i^T P_i \overline{B}_i & h\overline{A}_{1i}^T K^T R & \overline{D}_{1i}^T P_i & L_{2i}^T \\ * & * & \Pi_{33i} & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & h\overline{B}_i^T K^T R & \overline{D}_{2i}^T P_i & L_{3i}^T \\ * & * & * & * & -R & 0 & 0 \\ * & * & * & * & * & -P_i & 0 \\ * & * & * & * & * & * & -I \end{bmatrix}$$
$$+ \begin{bmatrix} \Theta_{11i} & \Theta_{12i} & 0 & 0 & h\Delta\overline{A}_i^T(t) K^T R & \Delta\overline{D}_i^T(t) P_i & 0 \\ \Theta_{21i} & \Theta_{22i} & 0 & 0 & h\Delta\overline{A}_{1i}^T(t) K^T R & \Delta\overline{D}_{1i}^T(t) P_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ hRK\Delta\overline{A}_i(t) & hRK\Delta\overline{A}_{1i}(t) & 0 & 0 & 0 & 0 & 0 \\ P_i\Delta\overline{D}_i^T(t) & P_i\Delta\overline{D}_{1i}(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{43}$$

where

$$\begin{aligned}
\Pi_{11i} &= P_i\overline{A}_i + \overline{A}_i^T P_i + \sum_{j=1}^{N} \pi_{ij} P_j + k^T Q_1 K - K^T RK, \\
\Pi_{12i} &= P_i\overline{A}_{1i} - \overline{A}_i^T P_i \overline{G}_i - \sum_{j=1}^{N} \pi_{ij} P_j \overline{G}_i + K^T Q_2 \\
&\quad + K^T R\left(K\overline{G}_i + I\right), \\
\Pi_{22i} &= -\overline{G}_i^T P_i \overline{A}_{1i} - \overline{A}_{1i}^T P_i \overline{G}_i + \overline{G}_i^T \sum_{j=1}^{N} \pi_{ij} P_j \overline{G}_i + Q_3 \\
&\quad - Q_1 - \left(K\overline{G}_i + I\right)^T R\left(K\overline{G}_i + I\right), \\
\Pi_{23i} &= -Q_2 + \left(K\overline{G}_i + I\right)^T RK\overline{G}_i, \\
\Pi_{33i} &= -Q_3 - \left(K\overline{G}_i\right)^T RK\overline{G}_i, \\
\Theta_{11i} &= P_i\Delta\overline{A}_i(t) + \Delta\overline{A}_i^T(t) P_i, \\
\Theta_{12i} &= P_i\Delta\overline{A}_{1i}(t) - \Delta\overline{A}_i^T(t) P_i \overline{G}_i, \\
\Theta_{21i} &= \Delta\overline{A}_{1i}^T(t) P_i - \overline{G}_i^T P_i \Delta\overline{A}_i(t), \\
\Theta_{22i} &= -\overline{G}_i^T P_i \Delta\overline{A}_{1i}(t) - \Delta\overline{A}_{1i}^T(t) P_i \overline{G}_i.
\end{aligned} \tag{44}$$

Then by Lemma 2, it can be seen that

$$\begin{bmatrix} \Theta_{11i} & \Theta_{12i} & 0 & 0 & h\Delta\overline{A}_i^T(t)K^TR & \Delta\overline{D}_i^T(t)P_i & 0 \\ \Theta_{21i} & \Theta_{22i} & 0 & 0 & h\Delta\overline{A}_{1i}^T(t)K^TR & \Delta\overline{D}_{1i}^T(t)P_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ hRK\Delta\overline{A}_i(t) & hRK\Delta\overline{A}_{1i}(t) & 0 & 0 & 0 & 0 & 0 \\ P_i\Delta\overline{D}_i^T(t) & P_i\Delta\overline{D}_{1i}(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \Upsilon F_i(t)\Omega + \Omega^T F_i(t)^T\Upsilon^T \le \frac{1}{\epsilon_i}\Upsilon\Upsilon^T + \epsilon_i\Omega^T\Omega, \tag{45}$$

where $\Upsilon = \left[\overline{M}_{1i}^T P_i \;\; -\overline{M}_{1i}^T P_i\overline{G}_i \;\; 0 \;\; 0 \;\; h\overline{M}_{1i}K^TR \;\; \overline{M}_{2i}^T P_i \;\; 0\right]^T$, $\Omega = \left[\widetilde{N}_{1i} \;\; N_{2i} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]$.

By (41), (46), and Schur complements, $\Theta_i < 0$ holds if and only if $\Omega_i < 0$. This completes the proof. □

Remark 6. Theorem 5 is established based on GFL. For the sake of reducing the computational complexity, similar to [6, 8, 10], the first two equivalent conditions of Proposition 3 are adopted in this paper.

Now we are in a position to present the H_∞ filter design for uncertain neutral stochastic system with Markovian jumping parameters and time delay based on Theorem 5.

Theorem 7. *Consider systems (1), for given scalars $h > 0$, $\gamma > 0$; then there exists a linear stochastic full-order filter with the form (9), such that filter error systems (11) are robustly stochastically stable and satisfy prescribed H_∞ disturbance attenuation level γ for all admissible uncertainties (6) and (7) if there exist symmetric positive matrices $X_i > 0$, $F_i > 0$, $Q = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0$, and $R > 0$ and matrices A_{Fi}, B_{Fi}, C_{Fi}, scalars $\epsilon_i > 0$, such that the following LMI holds:*

$$\begin{bmatrix} \Upsilon_{11i} & \Upsilon_{12i} & \Upsilon_{13i} & -RG_i & \Upsilon_{15i} & hA_i^TR & \Upsilon_{17i} & \Upsilon_{18i} & L_{1i}^T - C_{Fi}^T & \Upsilon_{110i} \\ * & \Upsilon_{22i} & \Upsilon_{23i} & 0 & \Upsilon_{25i} & 0 & 0 & 0 & -C_{Fi}^T & -F_iM_{1i} \\ * & * & \Upsilon_{33i} & \Upsilon_{34i} & \Upsilon_{35i} & hA_{1i}^TR & \Upsilon_{37i} & \Upsilon_{38i} & L_{2i}^T & -G_i^TX_iM_{1i} \\ * & * & * & \Upsilon_{44i} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & hB_i^TR & \Upsilon_{57i} & \Upsilon_{58i} & L_{3i}^T & 0 \\ * & * & * & * & * & -R & 0 & 0 & 0 & hRM_{1i} \\ * & * & * & * & * & * & -X_i + F_i & 0 & 0 & \Upsilon_{710i} \\ * & * & * & * & * & * & * & -F_i & 0 & -F_iM_{2i} \\ * & * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0, \tag{46}$$

where

$$\begin{aligned}
\Upsilon_{11i} &= (X_i - F_i)A_i + A_i^T(X_i - F_i) + \sum_{j=1}^{N}\pi_{ij}(X_j - F_j) \\
&\quad + Q_1 - R + \epsilon_i N_{1i}^T N_{1i}, \\
\Upsilon_{12i} &= -A_i^TF_i + C_i^TB_{Fi}^T + A_{Fi}^T, \\
\Upsilon_{22i} &= A_{Fi} + A_{Fi}^T + \sum_{j=1}^{N}\pi_{ij}F_j, \\
\Upsilon_{13i} &= (X_i - F_i)A_{1i} - A_i^TX_iG_i + C_i^TB_{Fi}^TG_i + A_{Fi}^TG_i \\
&\quad + \sum_{j=1}^{N}\pi_{ij}(F_j - X_j)G_i + Q_2 + R(G_i + I) + \epsilon_i N_{1i}^T N_{2i}, \\
\Upsilon_{23i} &= -F_iA_{1i} + B_{Fi}C_{1i} + A_{Fi}^TG_i + \sum_{j=1}^{N}\pi_{ij}F_jG_i, \\
\Upsilon_{33i} &= -G_i^TX_iA_{1i} - A_{1i}^TX_i^TG_i + G_i^TB_{Fi}C_{1i} + C_{1i}^TB_{Fi}^TG_i \\
&\quad + G_i^T\sum_{j=1}^{N}\pi_{ij}X_jG_i + Q_3 - Q_1 - (G_i + I)^TR(G_i + I) \\
&\quad + \epsilon_i N_{2i}^T N_{2i},
\end{aligned}$$

$$\begin{aligned}
\Upsilon_{34i} &= -Q_2 + (G_i + I)^T RG_i, \\
\Upsilon_{44i} &= -Q_3 - G_i^T RG_i, \\
\Upsilon_{15i} &= (X_i - F_i) B_i, \\
\Upsilon_{25i} &= -F_i B_i + B_{Fi} C_{2i}, \\
\Upsilon_{35i} &= -G_i^T X_i B_i + G_i^T B_{Fi} C_{2i}, \\
\Upsilon_{17i} &= D_i^T (X_i - F_i), \\
\Upsilon_{37i} &= D_{1i}^T (X_i - F_i), \\
\Upsilon_{57i} &= D_{2i}^T (X_i - F_i), \\
\Upsilon_{18i} &= -D_i^T F_i + E_i^T B_{Fi}^T, \\
\Upsilon_{38i} &= -D_{1i}^T F_i + E_{1i}^T B_{Fi}^T, \\
\Upsilon_{58i} &= -D_{2i}^T F_i + E_{2i}^T B_{Fi}^T, \\
\Upsilon_{110i} &= (X_i - F_i) M_{1i}, \\
\Upsilon_{710i} &= (X_i - F_i) M_{2i}.
\end{aligned} \tag{47}$$

Meanwhile, the filter parameters are given by

$$\begin{aligned}
A_{fi} &= F_i^{-1} A_{Fi}, \\
B_{fi} &= F_i^{-1} B_{Fi}, \\
C_{fi} &= C_{Fi}.
\end{aligned} \tag{48}$$

Proof. We note that, from (48), it is easy to see $\begin{bmatrix} X_i & -F_i \\ -F_i & F_i \end{bmatrix} > 0$, and $X_i > F_i > 0$. Define

$$P_i = \begin{bmatrix} X_i & -F_i \\ -F_i & F_i \end{bmatrix}; \tag{49}$$

then applying Schur complement, $X_i - F_i F_i^{-1} F_i = X_i - F_i > 0$ guarantees $P_i > 0$. Let

$$\mathcal{J} = \operatorname{diag}\{T, I, I, I, I, T, I, I, I\}, \tag{50}$$

where $T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$. Substituting P_i and (12) into (16), then pre- and postmultiplying (16) by $\mathcal{J}^T$ and $\mathcal{J}$, respectively, and using (51), the desired result (48) follows immediately. This completes the proof. □

Remark 8. Theorem 7 considers the H_∞ filtering problem for uncertain neutral stochastic time delay systems with Markovian jumping parameters. It should be noted that the proposed conditions are formulated in terms of LMIs. Therefore, by MATLAB LMI toolbox, for given different h or γ, the lower bound of performance index γ and the upper bound of h can be efficiently obtained by solving a generalized eigenvalue problem.

Now we would like to proceed to present the H_∞ filtering for uncertain neutral stochastic time delay systems without Markovian jumping parameters. Considering the system (Σ) without the Markovian jumping parameters, the following systems can be obtained:

$$\begin{aligned}
(\Sigma_D) : d[x(t) - Gx(t-h)] \\
= [A(t)x(t) + A_1(t)x(t-h) + Bv(t)]\,dt \\
+ [D(t)x(t) + D_1(t)x(t-h) + D_2 v(t)]\,dw(t),
\end{aligned} \tag{51}$$

$$\begin{aligned}
dy(t) \\
= [Cx(t) + C_1 x(t-h) + C_2 v(t)]\,dt \\
+ [Ex(t) + E_1 x(t-h) + E_2 v(t)]\,dw(t),
\end{aligned} \tag{52}$$

$$z(t) = L_1 x(t) + L_2 x(t-h) + L_3 v(t), \tag{53}$$

$$\begin{aligned}
x(\theta) = \psi(\tau), \\
r_t = r_0 \in S, \ \forall \tau \in [-h, 0],
\end{aligned} \tag{54}$$

where

$$\begin{aligned}
A(t) &= A + \Delta A(t), \\
A_1(t) &= A_1 + \Delta A_1(t), \\
D(t) &= D + \Delta D(t), \\
D_1(t) &= D_1 + \Delta D_1(t),
\end{aligned} \tag{55}$$

and $\Delta A(t)$, $\Delta A_1(t)$, $\Delta D(t)$, and $\Delta D_1(t)$ are unknown matrices satisfying

$$\begin{bmatrix} \Delta A(t) & \Delta A_1(t) \\ \Delta D(t) & \Delta D_1(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(t) \begin{bmatrix} N_1 & N_2 \end{bmatrix}. \tag{56}$$

We can obtain the following Corollary 9 for system (Σ_D).

Corollary 9. *Consider the system (Σ_D), for given scalars $h > 0$, $\gamma > 0$; then there exists a linear stochastic full-order filter with the form (9), if there exist symmetric positive matrices $F > 0$, $X > 0$, $R > 0$, $Q = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0$, matrices A_F, B_F, C_F, and scalar $\epsilon > 0$, such that the following LMI holds:*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & -RG & \Xi_{15} & hA^TR & \Xi_{17} & \Xi_{18} & L_1^T - C_F^T & (X-F)M_1 \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 & 0 & 0 & -C_F^T & -FM_1 \\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & hA_1^TR & \Xi_{37} & \Xi_{38} & L_2^T & -G^TXM_1 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & hB^TR & \Xi_{57} & \Xi_{58} & L_3^T & 0 \\ * & * & * & * & * & -R & 0 & 0 & 0 & hRM_1 \\ * & * & * & * & * & * & -X+F & 0 & 0 & (X-F)M_2 \\ * & * & * & * & * & * & * & -F & 0 & -FM_2 \\ * & * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{57}$$

where

$$\begin{aligned}
\Xi_{11} &= (X-F)A + A^T(X-F) + Q_1 - R + \epsilon N_1^T N_1, \\
\Xi_{12} &= -A^TF + C^TB_F^T + A_F^T, \\
\Xi_{13} &= (X-F)A_1 - A^TXG + C^TB_F^TG + A_F^TG \\
&\quad + \epsilon N_1^T N_2 + Q_2 + R(G+I), \\
\Xi_{15} &= (X-F)B, \\
\Xi_{17} &= D^T(X-F), \\
\Xi_{18} &= -D^TF + E^TB_F^T, \\
\Xi_{22} &= A_F + A_F^T, \\
\Xi_{23} &= -FA_1 + B_FC_1 + A_F^TG, \\
\Xi_{25} &= -FB + B_FC_2, \\
\Xi_{33} &= -G^TXA_1 - A_1^TXG + G^TB_FC_1 + C_1^TB_F^TG + Q_3 \\
&\quad - Q_1 - (G+I)^TR(G+I) + \epsilon N_2^T N_2, \\
\Xi_{34} &= -Q_2 + (G+I)^TRG, \\
\Xi_{35} &= -G^TXB + G^TB_FC_2, \\
\Xi_{37} &= D_1^T(X-F), \\
\Xi_{38} &= -D_1^TF + E_1^TB_F^T, \\
\Xi_{44} &= -Q_3 - G^TRG, \\
\Xi_{57} &= D_2^T(X-F), \\
\Xi_{58} &= -D_2^TF + E_2^TB_F^T;
\end{aligned} \tag{58}$$

then the robust H_∞ filtering problem is solvable. Furthermore, the parameters of the desired robust H_∞ filter can be given as

$$\begin{aligned}
A_f &= F^{-1}A_F, \\
B_f &= F^{-1}B_F, \\
C_f &= C_F.
\end{aligned} \tag{59}$$

Remark 10. The proof of Corollary 9 follows the same lines as that in the proof of Theorem 7, so the detailed procedure is omitted here. When $D_2 = 0$, $E = 0$, $E_1 = 0$, and $E_2 = 0$, systems (52)–(55) in this paper reduces to systems (1) in [18]. It is noticed that the filtering problem studied in [18] is a special case of this paper. For comparisons of our results with that in [18], see Example 1 in detail.

4. Numerical Examples

In this section, numerical examples and simulations are given to illustrate the validity and benefits of the proposed approach.

Example 1. Consider systems (52)–(55) without Markovian jumping parameters as follows:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -0.9 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2 & 0 \\ -0.1 & 0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.1 & 0 \\ 0.3 & -0.2 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

Table 1: Upper bounds of h for different γ (Example 1).

γ	0.3	0.8	1.2	1.8	2.4	3.0
h by Corollary 9	1.079	1.786	1.954	2.089	2.167	2.219

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 0.15 \\ -0.1 & 0.1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.5 & 0.5 \\ 1.0 & 0 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 1.0 & 0.2 \\ 0.3 & -1 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} 0.2 & 0.5 \\ 0.3 & 0.4 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} -0.1 & 0.1 \\ 0 & -0.1 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} 0.1 & -0.15 \\ 0 & 0.15 \end{bmatrix},$$
$$L_3 = \begin{bmatrix} 0.2 & 0 \\ -0.15 & 0.1 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix},$$
$$G = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$
$$N_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}. \tag{60}$$

For different given noise attenuation levels γ, the upper bounds of delay for systems (52)–(55) in Corollary 9 of this paper are presented in Table 1. For different given time delays, the lower bounds of noise attenuation level γ for systems (52)–(55) in Corollary 9 of this paper are provided in Table 2.

In particular, when $D_2 = 0$, $E = 0$, $E_1 = 0$, and $E_2 = 0$, systems (52)–(55) in this paper reduce to systems (1) in [18]. For different given noise attenuation levels γ, the comparison of the upper bounds of delay in [18] with our results is presented in Table 3, and, for different given time delays, the lower bounds of noise attenuation level γ for systems (1) in [18] and the same systems in this paper are provided in Table 4.

Table 2: Lower bounds of γ for different h (Example 1).

h	0.5	1.0	1.5	2.0	2.2	2.4
γ by Corollary 9	0.275	0.291	0.473	1.364	2.745	10.940

Table 3: Upper bounds of h for different γ (Example 1).

γ	0.3	0.8	1.2	1.8	2.4	3.0
h by Theorem 3 [18]	1.157	1.808	1.926	1.991	2.025	2.046
h by Corollary 9	1.409	2.138	2.263	2.335	2.371	2.394

Table 4: Lower bounds of γ for different h (Example 1).

h	0.5	1	1.5	2	2.1	2.2
γ by Theorem 3 [18]	0.264	0.270	0.630	1.919	8.235	infeasible
γ by Corollary 9	0.264	0.266	0.340	0.712	0.766	0.925

Besides, Theorem 2 in [13] fails to give a feasible solution. The number of decision variables of Theorem 3 in [18] is $(13n^2 + 5n + 2)/2$, which is the same as that in Corollary 9 of this paper. From Tables 3 and 4, we can see that our proposed method is less conservative than that in [18].

Now in the case when $\gamma = 2.4$, $h_{\max} = 2.167$, we resort to the MATLAB LMI control toolbox to solve the LMI (59), and the feasible solution can be obtained as follows:

$$\begin{aligned} d\hat{x}(t) &= \begin{bmatrix} -2.6806 & -0.1892 \\ -8.4386 & -2.0417 \end{bmatrix} \hat{x}(t)\, d(t) \\ &\quad + \begin{bmatrix} 0.0232 & 0.0476 \\ -0.1573 & 0.3199 \end{bmatrix} dy(t), \\ \hat{z}(t) &= \begin{bmatrix} -0.3127 & 0.0474 \\ 0.1536 & -0.0621 \end{bmatrix} \hat{x}(t). \end{aligned} \tag{61}$$

The initial conditions are also taken as $x(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\hat{x}(0) = \begin{bmatrix} -2 & 2 \end{bmatrix}^T$. The simulation results of the state response of the system are plotted in Figures 1–3. The filter state $x_1(t)$ and its estimation $\hat{x}_1(t)$ and state $x_2(t)$ and its estimation $\hat{x}_2(t)$ are given in Figures 1 and 2, respectively. Figure 3 depicts the estimation error $e(t) = z(t) - \hat{z}(t)$. The simulation results demonstrate that the designed H_∞ filters are feasible, effective and the stochastic stability of the error systems is ensured.

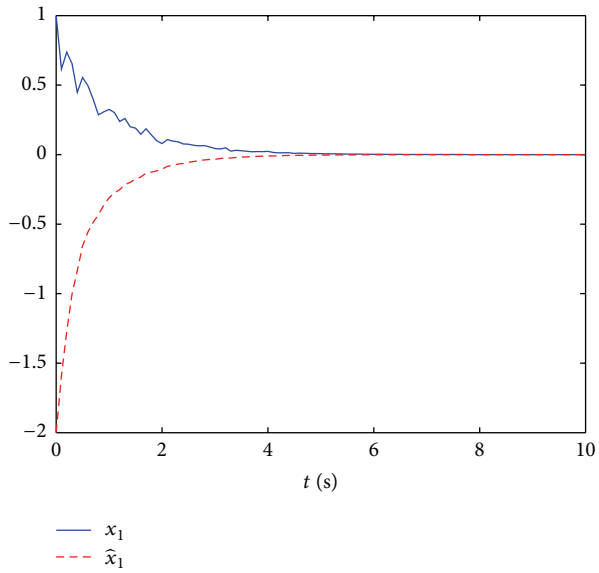


FIGURE 1: State $x_1(t)$ and its estimation $\hat{x}_1(t)$ in Example 1.

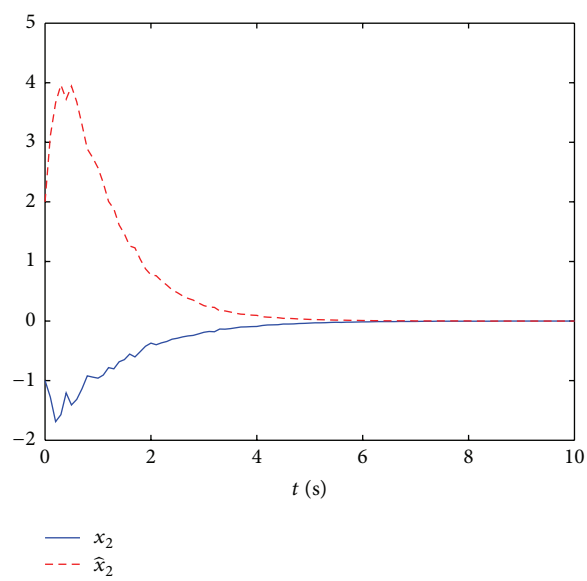


FIGURE 2: State $x_2(t)$ and its estimation $\hat{x}_2(t)$ in Example 1.

Example 2. Consider systems (1) with Markovian jumping parameters as follows.

Mode 1. Consider

$$A_1 = \begin{bmatrix} -2.0 & 0 \\ 0 & -1.9 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.15 & 0.4 \\ 0.2 & 0.3 \end{bmatrix},$$

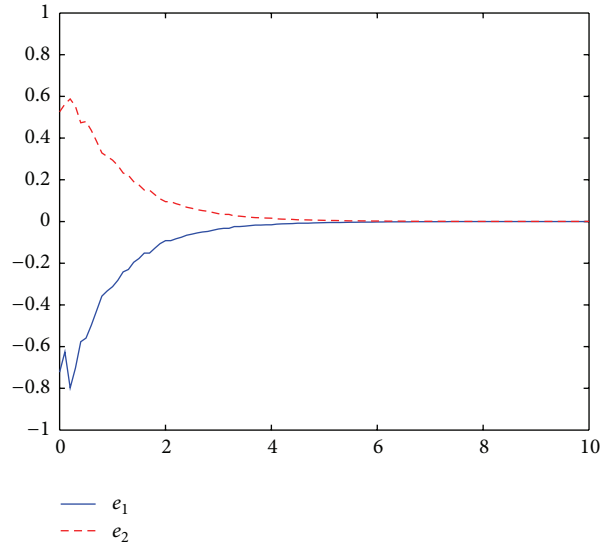


FIGURE 3: The error responses $e_1(t)$ and $e_2(t)$ in Example 1.

$$D_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$C_{11} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} 0.3 & 0.9 \\ 0.2 & 0.1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1.0 & 0.2 \\ 0.3 & -1 \end{bmatrix},$$

$$E_{11} = \begin{bmatrix} 0.2 & 0.5 \\ 0.3 & 0.4 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0.5 & 0.5 \\ 1.0 & 0 \end{bmatrix},$$

$$L_{11} = \begin{bmatrix} 0.8 & 0.6 \\ 0.4 & 0.3 \end{bmatrix},$$

$$L_{21} = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.8 \end{bmatrix},$$

$$L_{31} = \begin{bmatrix} 0.6 & 0.5 \\ 0.3 & 0.4 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$N_{11} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$N_{21} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}. \tag{62}$$

Mode 2. Consider

$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -1.0 & 0 \\ -1.0 & -0.5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.15 & 0.4 \\ 0.2 & 0.31 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.6 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_{12} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} 0.3 & 0.9 \\ 0.2 & 0.15 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.5 & 0.6 \\ 1.2 & 0 \end{bmatrix},$$

$$E_{12} = \begin{bmatrix} 1.0 & 0.3 \\ 0.4 & -1.0 \end{bmatrix},$$

$$E_{22} = \begin{bmatrix} 0.25 & 0.5 \\ 0.3 & 0.4 \end{bmatrix},$$

$$L_{12} = \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & 0.8 \end{bmatrix},$$

$$L_{22} = \begin{bmatrix} 0.4 & 0.8 \\ 0.3 & 0.5 \end{bmatrix},$$

$$L_{32} = \begin{bmatrix} 0.5 & 1.0 \\ 0.6 & 0.4 \end{bmatrix},$$

$$M_{12} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$N_{12} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$N_{22} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$\pi_{11} = -1,$$

$$\pi_{22} = -2,$$

$$\pi_{12} = 1,$$

$$\pi_{21} = 2. \tag{63}$$

Table 5: Upper bounds of h for different γ (Example 2).

γ	0.5	2	3	4	5
h by Theorem 7	0.104	0.232	0.311	0.343	0.361

Table 6: Lower bounds of γ for different h (Example 2).

h	0.1	0.2	0.3	0.4
γ by Theorem 7	1.493	1.806	2.774	15.047

In this section, the purpose of this example is to design a full-order filter in the form of (9) such that the filtering error system is robustly stochastically stable for all admissible uncertainties and satisfies the required H_∞ performance level.

By Theorem 7, for different given noise attenuation levels γ, the upper bounds of delay for systems (1) are presented in Table 5. For different given time delays, the lower bounds of noise attenuation level γ for systems (1) are provided in Table 6.

Now in the case when $\gamma = 2$, $h_{\max} = 0.232$, we resort to the MATLAB LMI control toolbox to solve the LMI (48), and we obtain the solution as follows:

$$Q_1 = \begin{bmatrix} 9.6525 & -1.9471 \\ -1.9471 & 1.6603 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -10.1587 & -0.2297 \\ 2.2216 & -1.6233 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 10.8069 & -0.0873 \\ -0.0873 & 2.3599 \end{bmatrix},$$

$$R = \begin{bmatrix} 39.1583 & -0.3727 \\ -0.3727 & 8.8082 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 2.8376 & 0.1553 \\ 0.1553 & 2.4043 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 4.3724 & 0.4730 \\ 0.4730 & 5.8205 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.2942 & 0.2896 \\ 0.2896 & 1.1395 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 1.2783 & 0.9020 \\ 0.9020 & 2.8727 \end{bmatrix},$$

$$\epsilon_1 = 1.1107,$$

$$\epsilon_2 = 8.1439,$$

$$A_{F1} = \begin{bmatrix} -0.9169 & -0.6283 \\ -1.6111 & -2.5633 \end{bmatrix},$$

$$B_{F1} = \begin{bmatrix} 0.1262 & 0.0730 \\ 0.2392 & 0.3601 \end{bmatrix},$$

$$C_{F1} = \begin{bmatrix} 0.5690 & 0.9257 \\ 0.4004 & 1.0310 \end{bmatrix},$$

$$A_{F2} = \begin{bmatrix} -1.4295 & -1.3599 \\ -2.9406 & -2.5256 \end{bmatrix},$$

$$B_{F2} = \begin{bmatrix} 1.8563 & -0.7369 \\ 3.0463 & -1.0766 \end{bmatrix},$$

$$C_{F2} = \begin{bmatrix} 1.1602 & 1.1086 \\ 0.7364 & 0.9807 \end{bmatrix}. \tag{64}$$

Therefore, the full-order stochastic filter parameters are given as follows:

$$A_{f1} = \begin{bmatrix} -2.3001 & 0.1251 \\ -0.8294 & -2.2813 \end{bmatrix},$$

$$B_{f1} = \begin{bmatrix} 0.2963 & -0.0837 \\ 0.1346 & 0.3373 \end{bmatrix},$$

$$C_{f1} = \begin{bmatrix} 0.5690 & 0.9257 \\ 0.4004 & 1.0310 \end{bmatrix},$$

$$A_{f2} = \begin{bmatrix} -0.5087 & -0.5697 \\ -0.8639 & -0.7003 \end{bmatrix},$$

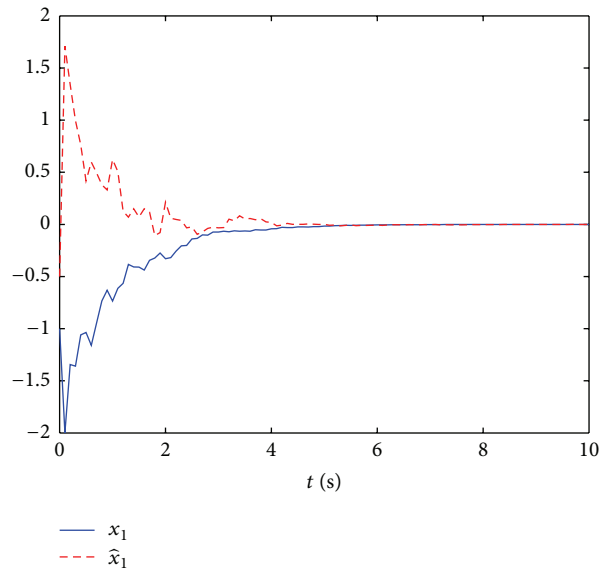


FIGURE 4: State $x_1(t)$ and its estimation $\hat{x}_1(t)$ in Example 2.

$$B_{f2} = \begin{bmatrix} 0.9043 & -0.4008 \\ 0.7765 & -0.2489 \end{bmatrix},$$

$$C_{f2} = \begin{bmatrix} 1.1602 & 1.1086 \\ 0.7364 & 0.9807 \end{bmatrix}. \tag{65}$$

The initial conditions are also taken as $x(0) = [-1 \quad 1.5]^T$ and $\hat{x}(0) = [-0.5 \quad 1]^T$. The simulation results of the state response of the system are plotted in Figures 4–7. The filter state $x_1(t)$ and its estimation $\hat{x}_1(t)$ and state $x_2(t)$ and its estimation $\hat{x}_2(t)$ are, respectively, given in Figures 4 and 5. Assuming the simulation step size $\Delta = 0.1$, one of the possible realizations of the Markovian jumping mode is plotted in Figure 6. Figure 7 depicts the estimation error $e(t) = z(t) - \hat{z}(t)$. It is clearly observed from the simulation results that the designed H_∞ filter satisfies the specified requirements and the expected objectives are well achieved.

Remark 3. In order to reduce conservatism, many methods such as cross-terms bounding techniques, free-weighting matrices method, and cross-terms bounding techniques are often adopted in the stability analysis of stochastic systems. In this paper, the generalized Finsler Lemma is utilized in uncertain neutral stochastic systems, which can bring the low conservatism and less computational cost.

5. Conclusions

In this paper, the robust filtering problem for a class of uncertain neutral stochastic systems with Markovian jumping parameters and time delay has been considered. Based on the Lyapunov-Krasovskii functional theory and generalized Finsler Lemma, a delay-dependent sufficient condition is

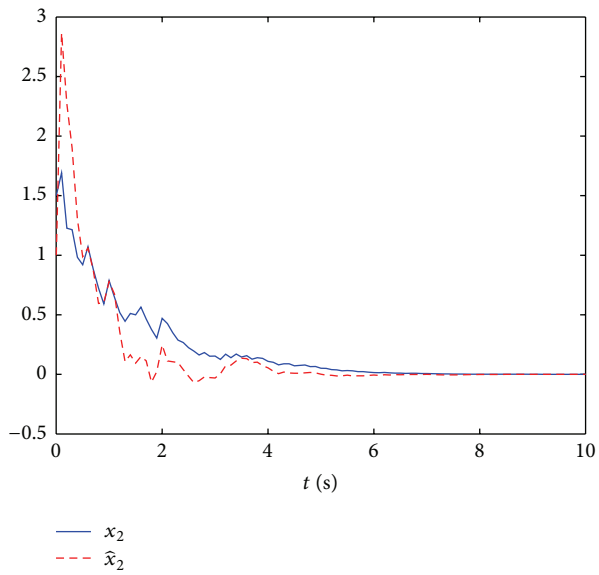


FIGURE 5: State $x_2(t)$ and its estimation $\hat{x}_2(t)$ in Example 2.

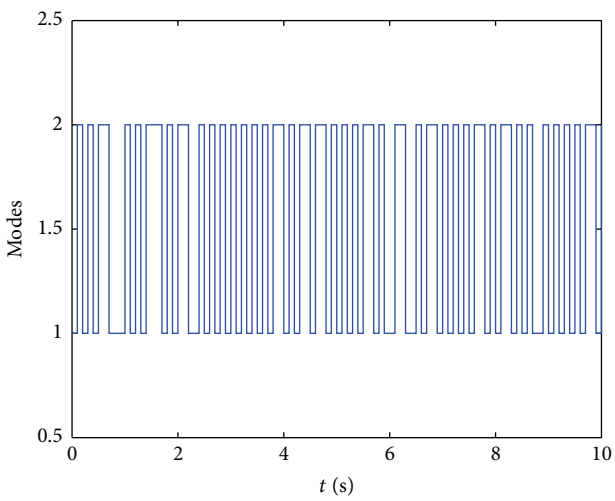


FIGURE 6: Markovian jumping mode in Example 2.

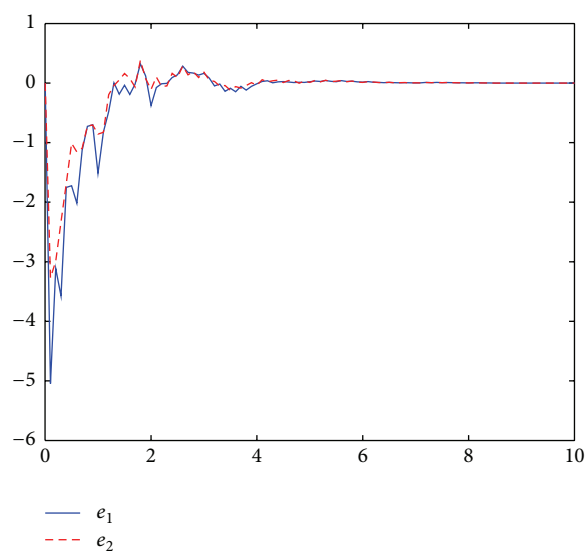


FIGURE 7: The error responses $e_1(t)$ and $e_2(t)$ in Example 2.

proposed for the existence of H_∞ filters which reduces the conservatism. The obtained result ensures the robust stability and a prescribed H_∞ performance level of the filtering error system for all admissible uncertainties. Two numerical examples and simulations have been presented to demonstrate the usefulness and effectiveness of the proposed filter design method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] X.-G. Liu, M. Wu, R. Martin, and M.-L. Tang, "Delay-dependent stability analysis for uncertain neutral systems with time-varying delays," *Mathematics and Computers in Simulation*, vol. 75, no. 1-2, pp. 15–27, 2007.

[2] O. M. Kwon, J. H. Park, and S. M. Lee, "On robust stability criterion for dynamic systems with time-varying delays and nonlinear perturbations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 937–942, 2008.

[3] W.-H. Chen, Z.-H. Guan, and X. Lu, "Delay-dependent exponential stability of uncertain stochastic systems with multiple delays: an LMI approach," *Systems & Control Letters*, vol. 54, no. 6, pp. 547–555, 2005.

[4] O. M. Kwon, J. H. Park, and S. M. Lee, "On stability criteria for uncertain delay-differential systems of neutral type with time-varying delays," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 864–873, 2008.

[5] Q. Zhu and J. Cao, "Robust exponential stability of markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 8, pp. 1314–1325, 2010.

[6] Q. Zhu and J. Cao, "Mean-square exponential input-to-state stability of stochastic delayed neural networks," *Neurocomputing*, vol. 131, no. 6, pp. 157–163, 2014.

[7] Y. Chen, A. Xue, S. Zhou, and R. Lu, "Delay-dependent robust control for uncertain stochastic time-delay systems," *Circuits, Systems, and Signal Processing*, vol. 27, no. 4, pp. 447–460, 2008.

[8] Q. Zhu, J. Cao, and R. Rakkiyappan, "Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays," *Nonlinear DynamicsSystems*, vol. 79, no. 2, pp. 1085–1098, 2015.

[9] M. Hua, H. Tan, J. Chen, and J. Fei, "Robust delay-range-dependent non-fragile H_∞ filtering for uncertain neutral stochastic systems with Markovian switching and mode-dependent time delays," *Journal of the Franklin Institute*, vol. 352, no. 3, pp. 1318–1341, 2015.

[10] Q. Zhu and J. Cao, "Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 2, pp. 341–353, 2011.

[11] M. Hua, F. Deng, X. Liu, and Y. Peng, "Robust delay-dependent exponential stability of uncertain stochastic system with time-varying delay," *Circuits, Systems, and Signal Processing*, vol. 29, no. 3, pp. 515–526, 2010.

[12] Q. Zhu and B. Song, "Exponential stability of impulsive nonlinear stochastic differential equations with mixed delays,"

Nonlinear Analysis: Real World Applications, vol. 12, no. 5, pp. 2851–2860, 2011.

[13] G. Chen and Y. Shen, "Robust H_∞ filter design for neutral stochastic uncertain systems with time-varying delay," *Journal of Mathematical Analysis and Applications*, vol. 353, no. 1, pp. 196–204, 2009.

[14] H. Li and M. Fu, "A linear matrix inequality approach to robust H_∞ filtering," *IEEE Transactions on Signal Processing*, vol. 45, no. 9, pp. 2338–2350, 1997.

[15] H. Liu, F. Sun, K. He, and Z. Sun, "Design of reduced-order H_∞ filter for Markovian jumping systems with time delay," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 51, no. 11, pp. 607–612, 2004.

[16] J. Lian, Z. Feng, and P. Shi, "Robust H_∞ filtering for a class of uncertain stochastic hybrid neutral systems with timevarying delay," *International Journal of Adaptive Control and Signal Processing*, vol. 27, no. 6, pp. 462–477, 2013.

[17] L. Ma and F. Da, "Exponential H_∞ filter design for stochastic time-varying delay systems with Markovian jumping parameters," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 7, pp. 802–817, 2010.

[18] B. Song, S. Xu, and Y. Zou, "Delay-dependent robust H_∞ filtering for uncertain neutral stochastic time-delay systems," *Circuits, Systems, and Signal Processing*, vol. 28, no. 2, pp. 241–256, 2009.

[19] B. Song, S. Xu, J. Xia, Y. Zou, and Q. Chen, "Design of robust H_∞ filters for a class of uncertain nonlinear neutral stochastic systems with time delays," *International Journal of Systems Science. Principles and Applications of Systems and Integration*, vol. 42, no. 4, pp. 633–642, 2011.

[20] Y. Chen and W. X. Zheng, "Exponential H_∞ filtering for stochastic Markovian jump systems with time delays," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 4, pp. 625–643, 2014.

[21] S. Xu, J. Lam, H. Gao, and Y. Zou, "Robust H_∞ filtering for uncertain discrete stochastic systems with time delays," *Circuits, Systems, and Signal Processing*, vol. 24, no. 6, pp. 753–770, 2005.

[22] S. Xu and T. Chen, "Robust H_∞ filtering for uncertain stochastic time-delay systems," *Asian Journal of Control*, vol. 5, no. 3, pp. 364–373, 2003.

[23] H. Gao, J. Lam, and C. Wang, "Robust energy-to-peak filter design for stochastic time-delay systems," *Systems & Control Letters*, vol. 55, no. 2, pp. 101–111, 2006.

[24] L. Li, Y. Jia, J. Du, and H. Kokame, "L_2-L_∞ filter design for neutral stochastic time-delay systems," in *Proceedings of the 49th IEEE Conference Decision and Control*, pp. 2644–2649, Atlanta, Ga, USA, 2010.

[25] J. Xia, S. Xu, and B. Song, "Delay-dependent L_2-L_∞ filter design for stochastic time-delay systems," *Systems & Control Letters*, vol. 56, no. 9-10, pp. 579–587, 2007.

[26] N. Krasovskii and E. Lidskii, "Analytical design of controllers in systems with random attributes I," *Automation and Remote Control*, vol. 22, no. 1, pp. 1021–1025, 1961.

[27] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic BAM neural networks with impulse control and mixed time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 3, pp. 467–479, 2012.

[28] L. Zhang and E.-K. Boukas, "Mode-dependent H_∞ filtering for discrete-time Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 6, pp. 1462–1467, 2009.

[29] Q. Zhu and J. Cao, "Stability of Markovian jump neural networks with impulse control and time varying delays," *Nonlinear Analysis. Real World Applications*, vol. 13, no. 5, pp. 2259–2270, 2012.

[30] L. Zhang and E.-K. Boukas, "H_∞ control for discrete-time Markovian jump linear systems with partly unknown transition probabilities," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 8, pp. 868–883, 2009.

[31] Q. Zhu, "pth moment exponential stability of impulsive stochastic functional differential equations with Markovian switching," *Journal of the Franklin Institute*, vol. 351, no. 7, pp. 3965–3986, 2014.

[32] X. Yao, L. Wu, W. X. Zheng, and C. Wang, "Robust H_∞ filtering of Markovian jump stochastic systems with uncertain transition probabilities," *International Journal of Systems Science*, vol. 42, no. 7, pp. 1219–1230, 2011.

[33] Y. Chen and W. X. Zheng, "L_2-L_∞ filtering for stochastic Markovian jump delay systems with nonlinear perturbations," *Signal Processing*, vol. 109, no. 3, pp. 154–164, 2015.

[34] Y. Chen, W. X. Zheng, and A. Xue, "A new result on stability analysis for stochastic neutral systems," *Automatica*, vol. 46, no. 12, pp. 2100–2104, 2010.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 286014, 9 pages
http://dx.doi.org/10.1155/2015/286014

Research Article

Multivariate Time-Varying *G-H* Copula GARCH Model and Its Application in the Financial Market Risk Measurement

Qi-an Chen,[1] Dan Wang,[1] and Mingyong Pan[2]

[1] *School of Economics and Business Administration, Chongqing University, Chongqing 400030, China*
[2] *School of Mathematics and Statistics, Chongqing University, Chongqing 400030, China*

Correspondence should be addressed to Qi-an Chen; chenqi_an33@163.com and Mingyong Pan; panmingyong@aliyun.com

Received 23 March 2015; Accepted 11 May 2015

Academic Editor: Ruihua Liu

Taking full advantage of the strengths of *G-H* distribution, Copula function, and GARCH model in depicting the return distribution of financial asset, we construct the multivariate time-varying *G-H* Copula GARCH model which can comprehensively describe "asymmetric, leptokurtic, and heavy-tail" characteristics, the time-varying volatility characteristics, and the extreme-tail dependence characteristics of financial asset return. Based on the conditional maximum likelihood estimator and IFM method, we propose the estimation algorithm of model parameters. Using the quantile function and simulation method, we propose the calculation algorithm of VaR on the basis of this model. To apply this model on studying a real financial market risk, we select the SSCI (China), HSI (Hong Kong, China), TAIEX (Taiwan, China), and SP500 (USA) from January 3, 2000, to June 18, 2010, as the samples to estimate the model parameters and to measure the VaRs of various index risk portfolios under different confidence levels empirically. The results of the application example are in line with the actual situation and the risk diversification theory of portfolio. To a certain extent, these results also justify the feasibility and effectiveness of the multivariate time-varying *G-H* Copula GARCH model in depicting the return distribution of financial assets.

1. Introduction

Financial market risk has always been one of the hottest topics in the field of financial investment, and many financial researchers put forward many different financial market risk measurement methods. Among them, Value-at-Risk (VaR) management technology is an assessment and measurement method of financial risk that has risen in recent years, playing an increasingly important role in the risk management and investment decision. It has been widely adopted by the major banks, nonbank financial intermediaries, corporations, and financial regulators in the world and has become the standard of risk measurement and risk management in financial industry. Accurate calculation of VaR is one of the keys to estimate the probability distribution of future return on financial assets. Usually, it is assumed that financial asset returns are independent of each other and obey the normal distribution in the calculation of VaR, but the movement of financial asset return in the financial market is extremely complex. The return of all kinds of financial assets usually does not satisfy the normal distribution hypothesis. However, it often shows "asymmetric, leptokurtic, and heavy-tail" characteristics [1–4]. At the same time, various financial asset returns do not satisfy the multivariate normal distribution hypothesis and present the extreme-tail dependence. On this occasion, a large error would be made by using normal distribution to fit financial asset return, and the estimation of the VaR may be overestimated or underestimated. To solve this problem, many scholars have proposed a lot of leptokurtic and heavy-tail distributions in recent years, such as the logistic distribution, Student's t-distribution, and the *G-H* distribution. The logistic distribution and Student's t-distribution can comprehensively describe the leptokurtic characteristics of financial asset return series, but they could not make a good explanation for the heavy-tail characteristics of financial asset return series [5]. The *G-H* distribution can comprehensively describe the asymmetric, leptokurtic, and heavy-tail characteristics of financial asset return series and it has a good fitting of the univariate unconditional return distribution of some financial assets; however, it could not

reflect the time-varying volatility characteristics of financial asset return and the extreme-tail dependence characteristics of various financial assets return [6, 7]. Meanwhile, Copula function can connect the joint distribution and the marginal distribution of multiple random variables to construct flexible multivariate distribution functions, which can be used to measure the extreme-tail dependence of multiple financial assets return. GARCH model can comprehensively describe the time-varying volatility characteristics of financial asset return. Therefore, building the multivariate time-varying *G-H* Copula GARCH Model by combining the *G-H* distribution with Copula function and GARCH model can not only comprehensively describe the "asymmetric, leptokurtic, and heavy-tail" characteristics, the time-varying volatility, and extreme-tail dependence characteristics of the financial asset return and make measurement of VaR more accurately, but also enrich and expand the risk measurement theory and method of financial market theoretically and improve the risk control ability of investors, corporations, financial institutions, and policy authorities and reduce their unnecessary losses in practice.

The *G-H* distribution, Copula function, and GARCH model, as well as the financial risk measurement model based on them, have been researched in the existing relevant literatures. A lot of innovative research results with reference value have been brought out. Zhu and Pan [8] proposed three kinds of *G-H* VaR methods based on the portfolio gains, losses, and extreme losses according to the statistical characteristics of *G-H* distribution. Their empirical results showed that this method is superior to the commonly used delta-normal method. Kuester et al. [9] believed that the *G-H* distribution can describe skewness and kurtosis of the financial asset return simultaneously and it plays a very important role of VaR measurement of financial asset return. Degen et al. [10] discussed the application of *G-H* distribution in operational risk measurement. Jondeau and Rockinger [11], Rodriguez [12], Fischer et al. [13], and Sun et al. [14] combined time series model with various Copulas functions by using the Sklar theorem to build a lot of highly flexible multivariate time-varying models for risk measurement of portfolios. Liu et al. [15] proposed a GARCH-*M* model with a time-varying coefficient of the risk premium. Their study indicated that the coefficient of the risk premium varies with the time, and even in a mature market the conditional skewness in the return distribution is negatively correlated with the time-varying coefficient of the risk premium. Wen et al. [16] built a *D*-GARCH-*M* model by separating investors' return into gains and losses on the basis of the characteristics of investors' risk preference. They found that investors become risk averse when they gain and risk-seeking when they lose, which effectively explains the inconsistent risk-return relationship. And the degrees of investors' risk aversion and risk-seeking are both in direct proportion to the value of gains and losses, respectively. Wen et al. [17] adopted aggregative indices of 14 representative stocks around the world as samples and established a TVRA-GARCH-*M* model to investigate the influence of prior gains and losses on current risk attitude. The empirical results indicated that the prior gains increase people's current willingness to take risk asset at the whole market level. Huang et al. [18] combined Student's t-marginal distribution with Archimedean Copula functions to build the conditional Copula GARCH model. They used this model to estimate the VaR of portfolios. Ghorbel and Trabelsi [19] built the conditional extremum Copula GARCH model by using extreme value theory (EVT) and measured the risk of financial asset according to this model. Chollete et al. [20] used multivariate regime-switching Copula function to build international financial asset return model and accordingly put forward the VaR calculation method. Huggenberger and Klett [21] proposed a measurement model of multivariate risk asset return VaR based on *G-H* distribution and Copula function. They used DAX30 (Germany), FTSE100 (UK), and CAC40 (French) from January 2000 to May 2010 as samples to test empirically. Wang et al. [22] applied the Gumbel Copula function in multivariate Archimedean Copula functions family to construct the joint distribution function which can describe the actual distribution and the correlation of various financial asset returns. They also used the Monte Carlo simulation technology to analyze the portfolios VaR and its composition under different confidence levels. The result showed that using the multidimensional Gumbel Copula function to construct the risk measurement model of financial asset can make the assets chosen by investors more robust, and it can also help investors to diversify and control the overall risk of the portfolios. Dai and Wen [23] proposed a computationally tractable robust optimization method for minimizing the CVaR of a portfolio under a general affine data perturbation uncertainty set. And they presented some numerical experiments with real market data to illustrate the behavior of robust optimization model. Liu et al. [24] proposed a pricing model for convertible bonds based on the utility-indifference method and got access to the empirical results by use of Information Technology. Furthermore, using the proposed theoretical model, they presented an empirical pricing study of China's market. They found that the theoretical prices are higher than the actual market prices 0.24–4.58% and the utility-indifference prices are better than the Black-Scholes (B-S) prices.

Based on the aforementioned analyses, the VaR is still the mainstream measurement method of financial market risk. In order to achieve the purpose of measuring VaR more precisely, it has been the hot issue of existing research literatures to construct the distribution functions as comprehensive as possible to describe the "asymmetric, leptokurtic, and heavy-tail" characteristics, the time-varying volatility characteristics, and the extreme-tail dependence characteristics of financial asset return through a variety of mathematical methods. However, in the process of constructing the return distribution model and measuring VaR of financial asset, existing results only grasp some characteristics of financial asset return distribution. They are not able to comprehensively describe the "asymmetric, leptokurtic, and heavy-tail" characteristics, the time-varying volatility characteristics, and the extreme-tail dependence characteristics of the financial asset return. The rationality and accuracy of VaR calculated based on the existing distribution models have a large space for further improvement.

In this paper, we would take full advantage of the strengths of G-H distribution, Copula function, and GARCH model in depicting the return distribution of financial asset to build multivariate time-varying G-H Copula GARCH model which can simultaneously describe "asymmetric, leptokurtic, and heavy-tail" characteristics, the time-varying volatility characteristics, and the extreme-tail dependence characteristics of financial asset return and propose the estimation method of model parameters and the calculation algorithm of VaR. Then, this paper selects the SSCI (China), HSI (Hong Kong, China), TAIEX (Taiwan, China), and SP500 (USA) from January 3, 2000, to June 18, 2010, as samples to estimate the parameters and calculate the VaRs of various index portfolios under different confidence levels.

2. G-H Distribution, Copula Function, and Its Tail Dependence Index

2.1. G-H Distribution

2.1.1. G Distribution. Assuming that random variable Z obeys the standard normal distribution and g is a real number, then the random variable $Y_g = G_g(z)$ obeys G distribution. Consider

$$G_g(z) = \frac{e^{gz} - 1}{g}, \quad z \sim N(0,1), \tag{1}$$

where g controls the skewness of G distribution. When $g \to 0$, $G_g(z) \to z$ and G distribution tends to be symmetric. With the increase of the absolute value of g, the degree of asymmetry increases. Changing the sign of g can change the asymmetric direction of G distribution, but it does not change its degree of asymmetry.

2.1.2. H Distribution. Assuming that random variable Z obeys the standard normal distribution and h is a real number, then the random variable $Y_h = H_h(z)$ obeys H distribution. Consider

$$H_h(z) = e^{hz^2/2}, \quad z \sim N(0,1). \tag{2}$$

H distribution stretches the tail of the standard normal distribution. h controls the tail heaviness of H distribution. The larger the h is, the heavier the tail is. Because $H_h(z)$ is an even function, H distribution is symmetric. But the heaviness of its tail changes compared to the standard normal distribution.

2.1.3. G-H Distribution. The random variable $Y_{g,h}$ can be obtained by introducing both functions $G_g(z)$ and $H_h(z)$ to revise standard normal random variable Z. Consider

$$Y_{g,h} = G_g(z) H_h(z) = \frac{e^{gz} - 1}{g} e^{hz^2/2}. \tag{3}$$

Then, $X_{g,h}$ can be obtained through linear transformation of $Y_{g,h}$. Consider

$$X_{g,h} = A + B\frac{e^{gz} - 1}{g} e^{hz^2/2}, \quad z \sim N(0,1). \tag{4}$$

The distribution of the random variable $X_{g,h}$ obeys the G-H distribution. A, B, g, and h are real numbers. g describes the asymmetry of G-H distribution, and h describes the heavy-tail characteristics of G-H distribution. Obviously, (3) is a special form of (4). The random variable $Y_{g,h}$ in (3) is the random variable of G-H distribution after central standardization.

2.2. Copula Function and Its Tail Dependence Index. Assuming that marginal distribution of random vector $u_i = F_i(x_i)$ $(i = 1, 2, \ldots, p)$ obeys uniform distribution $U(0,1)$, according to the Sklar theorem, the joint distribution function of P-dimensional random vectors $F(x_1, \ldots, x_p)$ can be represented as the following formula:

$$F(x_1, \ldots, x_p) = C(F_1(x_1), \ldots, F_p(x_p)), \tag{5}$$

where C is the Copula function of F, which is a hypercube $[0,1]^p$ multivariate density function defined on P-dimensional space $\mathbf{R}^p$. If the marginal distribution is continuous, there is a unique Copula function C. Then

$$C(u_1, \ldots, u_p) = F(F_1^{-1}(u_1), \ldots, F_p^{-1}(u_p)). \tag{6}$$

On the contrary, given P-dimensional Copula function $C(u_1, \ldots, u_p)$ and its marginal distribution function $F_1(x_1), \ldots, F_p(x_p)$, the density function of P-dimensional joint distribution function is

$$f(x_1, \ldots, x_p) = c(F_1(u_1), \ldots, F_p(u_p)) \prod_{i=1}^{p} f_i(x_i). \tag{7}$$

If $f_i(x_i)$ is the edge density, $c(u_1, \ldots, u_p)$ denotes Copula density derived from (6). Thus,

$$c(u_1, \ldots, u_p) = \frac{f(F_1^{-1}(u_1), \ldots, F_p^{-1}(u_p))}{\prod_{i=1}^{p} f_i(F_i^{-1}(u_i))}. \tag{8}$$

Since the joint distribution function of random variables defines the correlation among its components, Copula function determines the dependent structure among random variables uniquely. The upper tail index λ_u and lower tail index λ_l of tail dependence indicators can be defined as follows:

$$\begin{aligned} \lambda_u &= \lim_{q \to 1} \frac{1 - 2q + c(q,q)}{1 - q}, \\ \lambda_l &= \lim_{q \to 0} \frac{c(q,q)}{q}. \end{aligned} \tag{9}$$

According to Nelsen [25], Gauss Copula function generated by multivariate normal distribution function whose correlation matrix is $\mathbf{R}$ can be represented as follows:

$$\begin{aligned} &C_{\mathbf{R}}^{Gu}(u_1, \ldots, u_p) \\ &= \int_{-\infty}^{\Phi_1^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi_p^{-1}(u_p)} \frac{1}{\sqrt{(2\pi)^p |\mathbf{R}|}} \exp\left\{\frac{-\mathbf{u}'\mathbf{R}^{-1}\mathbf{u}}{2}\right\} d\mathbf{u}, \end{aligned} \tag{10}$$

where $\mathbf{u} = (u_1, \ldots, u_p)$ and Φ^{-1} is the inverse function of single normal distribution. Because the Gauss Copula function does not have the characteristics of tail dependence, we often use the T-Copula function whose degree of freedom is η and correlation matrix is $\mathbf{R}$ to measure tail dependence structure of risk asset in empirical analysis; that is,

$$C^t_{\eta,\mathbf{R}}(u_1, \ldots, u_p) = \int_{-\infty}^{t_\eta^{-1}(u_1)} \cdots \int_{-\infty}^{t_\eta^{-1}(u_p)} \frac{\Gamma((\eta+p)/2)\left(1+\mathbf{u}'\mathbf{R}^{-1}\mathbf{u}/2\right)^{-(\eta+p)/2}}{\Gamma(\eta/2)\sqrt{(\pi\eta)^p|\mathbf{R}|}} d\mathbf{u}, \tag{11}$$

where t_η^{-1} is the inverse function of simple standard Student's t-distribution whose degree of freedom is η. When $\eta \to \infty$, T-Copula function degenerates to Gauss Copula function. Its tail index $\lambda_u = \lambda_l = 0$; that is, the tail is independent. The tail index of T-Copula function is

$$\lambda_u = \lambda_l = 2t_{\eta+1}\left(-\frac{\sqrt{(\eta+1)(1-\rho)}}{\sqrt{1+\rho}}\right), \tag{12}$$

where $t_{\eta+1}$ is simple standard Student's t-distribution whose degree of freedom is $\eta+1$. Considering that the innovation impacts on the price of risk asset in varying degrees at different times, η and ρ should have time-varying characteristics. For this reason, tail index also has the same characteristics.

3. Multivariate Time-Varying G-H Copula GARCH Model

Let $\mathbf{r}_t = (r_{1,t}, \ldots, r_{p,t})$ denote return time series of p risk assets. The prior information set before time t is

$$\mathbf{I}_{t-1} = \{\mathbf{r}_{t-1}, \mathbf{h}_{t-1}, \mathbf{r}_{t-2}, \mathbf{h}_{t-2}, \ldots\} = \prod_{i=1}^{p} \mathbf{I}_{i,t-1}, \tag{13}$$

where $\mathbf{I}_{i,t-1} = \{r_{i,t-1}, h_{i,t-1}, r_{i,t-2}, h_{i,t-2}, \ldots\}$. $h_{i,t}$ is conditional volatility of $r_{i,t}$ about single asset prior information set $\mathbf{I}_{i,t-1}$. Let $C(\cdot \mid \mathbf{I}_{t-1})$ denote P-dimensional conditional Copula function and $F_i(r_{i,t} \mid \mathbf{I}_{i,t-1})$ be the conditional distribution of the ith component. According to Sklar theorem, the conditional joint distribution of p risk assets return is

$$F(\mathbf{r}_t \mid \mathbf{I}_{t-1}) = C\left(F_1(r_{1,t} \mid \mathbf{I}_{1,t-1}), \ldots, F_p(r_{p,t} \mid \mathbf{I}_{p,t-1}) \mid \mathbf{I}_{t-1}\right). \tag{14}$$

Numerous empirical studies show that the risk asset return series obey GARCH $(1,1)$ model. Based on this, assuming that $r_{i,t}$ satisfies the GARCH $(1,1)$ model, we can get the following G-H Copula GARCH $(1,1)$ model which describes the time-varying dependence structure of p risk assets return after filtering the time-varying characteristics of single series:

$$\begin{aligned} r_{i,t} &= \mu_i + \varepsilon_{i,t}, \quad i = 1, 2, \ldots, p, \\ \varepsilon_{i,t} &= \sqrt{h_{i,t}} z_{i,t}, \\ h_{i,t} &= \varpi_i + \alpha_i \varepsilon^2_{i,t-1} + \beta_i h_{i,t-1}, \\ F(\mathbf{z}_t \mid \mathbf{I}_{t-1}) &= C\left(F_1(z_{1,t} \mid \mathbf{I}_{1,t-1}), \ldots, F_p(z_{p,t} \mid \mathbf{I}_{p,t-1}) \mid \mathbf{I}_{t-1}\right), \end{aligned} \tag{15}$$

where the parameters satisfy the conditions $\varpi_i, \alpha_i, \beta_i > 0$ and $\alpha_i + \beta_i < 1$. These parameters can ensure the stability of conditional volatility series. The innovation series $\{\mathbf{z}_t\}$ obey G-H distribution whose parameter is (g, h) in (4). But in order to simplify the analysis, we only consider G-H distribution after central standardization given by (3) and its density function is written as $f_{Y_i}(y_i)$. The Copula function $C(\cdot \mid \mathbf{I}_{t-1})$ is given by (11) and its density $c(\cdot)$ can be represented as the following time-varying T-Copula function whose degree of freedom is η:

$$c^t_{\eta,\boldsymbol{\rho}_t}(u_{1,t}, \ldots, u_{p,t}) = \frac{f^t_{\eta,\boldsymbol{\rho}_t}\left(f^{-1}_{v_1}(u_{1,t}), \ldots, f^{-1}_{v_p}(u_{p,t})\right)}{\prod_{i=1}^{p} f_\eta\left(f^{-1}_{v_i}(u_{i,t})\right)}, \tag{16}$$

where $f^t_{\eta,\boldsymbol{\rho}_t}$ denotes the multivariate Student's t-distribution whose degree of freedom is η and time-varying correlation matrix is $\boldsymbol{\rho}_t = (\rho_{i,j,t})_{p\times p}$, and

$$\begin{aligned} \rho_{i,i,t} &= 1, \\ f_{v_i}(u_{i,t}) &= \frac{\Gamma((v_i+1)/2)}{\Gamma(v_i/2)\sqrt{v_i\pi}}\left(1+\frac{u^2_{i,t}}{v_i}\right)^{-(v_i+1)/2}. \end{aligned} \tag{17}$$

The joint density function of p risk assets return is

$$\begin{aligned} f(\mathbf{y} \mid \mathbf{u}, \mathbf{h}_t) &= c^t_{\eta,\boldsymbol{\rho}_t}(x_{1,t}, \ldots, x_{p,t}) \prod_{i=1}^{p} f_{Y_i}(y_i) \\ &= \Gamma\left(\frac{\eta+p}{2}\right)\Gamma\left(\frac{\eta}{2}\right)^{p-1}\left(1+\frac{\mathbf{x}'_t\boldsymbol{\rho}_t\mathbf{x}_t}{\eta}\right)^{-(\eta+p)/2} \\ &\quad \cdot (|\boldsymbol{\rho}_t|)^{-1/2} \prod_{i=1}^{p}\left(1+\frac{x^2_{i,t}}{\eta}\right)^{(\eta+1)/2} \prod_{i=1}^{p} f_{Y_i}(y_i), \end{aligned} \tag{18}$$

where $\mathbf{x}_t = (x_{1,t}, \ldots, x_{p,t})$ and $x_{i,t} = t_\eta^{-1}(t_{v_i}(\varepsilon_{i,t}))$. Then the likelihood function of overall samples is

$$l(\boldsymbol{\theta} \mid \mathbf{y}) = \prod_{t=1}^{T} \Gamma\left(\frac{\eta+p}{2}\right)\Gamma\left(\frac{\eta}{2}\right)^{p-1} \cdot \left(1+\frac{\mathbf{x}_t'\boldsymbol{\rho}_t\mathbf{x}_t}{\eta}\right)^{-(\eta+p)/2} (|\boldsymbol{\rho}_t|)^{-1/2}\Gamma\left(\frac{\eta+1}{2}\right)^{-p} \cdot \prod_{i=1}^{p}\left(1+\frac{x_{i,t}^2}{\eta}\right)^{(\eta+1)/2} \prod_{i=1}^{p} f_{Y_i}(y_i), \tag{19}$$

where $\boldsymbol{\theta} = \{(\mu_i, \varpi_i, \alpha_i, \beta_i, v_i)_{i=1}^{p}, a, b, \boldsymbol{\rho}_t, \eta\}$ and $\mathbf{x}_t = (x_{1,t}, \ldots, x_{p,t})$. The value of correlation matrix $\boldsymbol{\rho}_t$ is similar to the time-varying correlation matrix of multivariate Copula GARCH model proposed by Jondeau and Rockinger [11]. That is, $\boldsymbol{\rho}_t$ satisfies the following evolution equation

$$\boldsymbol{\rho}_t = (1-a-b)\,\boldsymbol{\rho} + a\boldsymbol{\Psi}_{t-1} + b\boldsymbol{\rho}_{t-1}, \tag{20}$$

where $0 \le a, b \le 1$, $a+b \le 1$. $\boldsymbol{\rho}$ is a positive definite matrix whose main diagonal elements are 1 and other elements are static correlation coefficients. $\boldsymbol{\Psi}_{t-1}$ is a $p \times p$ matrix, in which every element

$$\psi_{i,j,t-1} = \frac{\sum_{l=1}^{m} x_{i,t-l}x_{j,t-l}}{\sqrt{\sum_{l=1}^{m} x_{i,t-l}^2}\sqrt{\sum_{l=1}^{m} x_{j,t-l}^2}}, \quad i, j = 1, 2, \ldots, p \tag{21}$$

denotes the correlation coefficients of p risk asset returns, $(m \ge p+2)$, $x_t = (x_{1,t}, \ldots, x_{p,t}) = (t_{v_1}^{-1}(f_{v_1}(z_{1,t})), \ldots, t_{v_p}^{-1}(f_{v_p}(z_{p,t})))$. Each element $\rho_{i,j,t}$ of $\boldsymbol{\rho}_t$ satisfies $-1 \le \rho_{i,j,t} \le 1$.

4. Parameter Estimation Algorithm of the Multivariate Time-Varying G-H Copula GARCH Model

On the basis of Huggenberger and Klett [21], this section will use dynamic correlation matrix $\boldsymbol{\rho}_t$ instead of static correlation matrix in the multidimensional discrete-time stochastic process to estimate the parameters of multivariate time-varying G-H Copula GARCH model established in Section 3. Assuming that $\boldsymbol{\Theta}$ denotes the parameter space defined by the model and $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_T)$ denotes the log return samples of P-dimensional risk asset which is generated by multivariate conditional density function $f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}(\mathbf{r}_t \mid \mathfrak{I}_{t-1}, \boldsymbol{\theta}_0)$, where $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, $\mathfrak{I}_{t-1}$ is σ algebra of time $t-1$ and before, the maximum likelihood estimation of parameter vector $\boldsymbol{\theta}$ can be calculated by the following equation:

$$\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sum_{t=1}^{T} \log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}(\mathbf{r}_t \mid \mathfrak{I}_{t-1}, \boldsymbol{\theta}), \tag{22}$$

where $f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}(\mathbf{r}_t \mid \mathfrak{I}_{t-1}, \boldsymbol{\theta})$ can be obtained by calculating the derivative of (5). Let $c_{\boldsymbol{\theta}}$ denote Copula density function. Thus

$$f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}(r_{1,t}, \ldots, r_{p,t} \mid \mathfrak{I}_{t-1}, \boldsymbol{\theta}) = c_{\boldsymbol{\theta}}\left(F_{1,t}(r_{1,t}, \boldsymbol{\theta}), \ldots, F_{p,t}(r_{p,t}, \boldsymbol{\theta})\right) \cdot \prod_{i=1}^{p} f_{i,t}(r_{t,i}, \boldsymbol{\theta}). \tag{23}$$

The probability density function and distribution function can be obtained in the process of model built in Section 3. Using the IFM method proposed by Joe [26], we can convert (22) into an optimization problem. Therefore, we need to divide the parameter vector $\boldsymbol{\theta}$ into two subparameter vectors $\boldsymbol{\theta}_c$ and $\boldsymbol{\theta}_r$: that is $\boldsymbol{\theta} = (\boldsymbol{\theta}_c, \boldsymbol{\theta}_r)$, where $\boldsymbol{\theta}_r = (\boldsymbol{\theta}_{r_1}, \ldots, \boldsymbol{\theta}_{r_p})$, $\boldsymbol{\theta}_{r_i}$ is the parameter vector of ith marginal distribution, and $\boldsymbol{\theta}_c$ is the parameter vector of Copula function. Because IFM method is a two-step likelihood estimation method, the model parameters should be estimated through the following two steps.

Step 1. Solving the maximum likelihood estimator of the parameter vector of each risk asset return,

$$\widehat{\boldsymbol{\theta}}_{r_i} = \arg\max_{\boldsymbol{\theta}_{r_i}} \sum_{t=1}^{T} \log f_{i,t}(r_{i,t} \mid \boldsymbol{\theta}_{r_i}) \quad i = 1, 2, \ldots, p. \tag{24}$$

This means that we need to estimate parameters vector $\widehat{\boldsymbol{\theta}}_{r_i}$ of p distributions continuously.

Step 2. Taking each $\widehat{\boldsymbol{\theta}}_{r_i}$ into the likelihood equation (22), we can obtain the parameter vector $\boldsymbol{\theta}_c$ of Copula function and its maximum likelihood estimator $\widehat{\boldsymbol{\theta}}_c$. Consider

$$\widehat{\boldsymbol{\theta}}_c = \arg\max_{\boldsymbol{\theta}_c} \sum_{t=1}^{T} \log c_{\boldsymbol{\theta}_c}\left(F_{1,t}\left(|r_{1,t}|\widehat{\boldsymbol{\theta}}_{r_1}\right), \ldots, F_{p,t}\left(|r_{p,t}|\widehat{\boldsymbol{\theta}}_{r_p}\right)\right). \tag{25}$$

In the maximum likelihood estimation, we need to use the derivative function of the density function of G-H marginal distribution with respect to the component of parameter vector. Because the density function of G-H marginal distribution is very complex, this paper uses the implicit function differentiation rule to take its derivative. The estimator $\widehat{\boldsymbol{\theta}}_{2s}$ of parameter vector $\boldsymbol{\theta}$ obtained by the above-mentioned two-step method obeys normal distribution consistently and asymptotically under the standard regularity conditions proposed in Huggenberger and Klett [21], Joe [26], and Patton [27]; that is,

$$\sqrt{T}\left(\widehat{\boldsymbol{\theta}}_{2s,T} - \boldsymbol{\theta}_0\right) \xrightarrow[T\to\infty]{d} N\left(0, \boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Omega}\right), \tag{26}$$

where

$$\begin{aligned}\boldsymbol{\Omega} &= -E\left(\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}\left(\boldsymbol{\rho}_t \mid \mathfrak{I}_{t-1},\boldsymbol{\theta}_0\right)\right),\\ \boldsymbol{\Sigma} &= E\left(\frac{\partial}{\partial\boldsymbol{\theta}}\left(\log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}\left(\boldsymbol{\rho}_t \mid \mathfrak{I}_{t-1},\boldsymbol{\theta}_0\right)\right)\right.\\ &\quad\left.\cdot\frac{\partial}{\partial\boldsymbol{\theta}}\left(\log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}\left(\boldsymbol{\rho}_t \mid \mathfrak{I}_{t-1},\boldsymbol{\theta}_0\right)\right)'\right).\end{aligned} \tag{27}$$

Because the matrixes $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ can be estimated by the estimated parameter vector consistently,

$$\begin{aligned}\widehat{\boldsymbol{\Omega}}_T &= -T^{-1}\sum_{t=1}^{T}\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}\left(\mathbf{r}_t \mid \mathfrak{I}_{t-1},\widehat{\boldsymbol{\theta}}_{2s,T}\right),\\ \widehat{\boldsymbol{\Sigma}}_T &= T^{-1}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{\theta}}\left(\log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}\left(\mathbf{r}_t \mid \mathfrak{I}_{t-1},\widehat{\boldsymbol{\theta}}_{2s,T}\right)\right)\\ &\quad\cdot\frac{\partial}{\partial\boldsymbol{\theta}}\left(\log f_{\boldsymbol{\rho}_t|\mathfrak{I}_{t-1}}\left(\mathbf{r}_t \mid \mathfrak{I}_{t-1},\widehat{\boldsymbol{\theta}}_{2s,T}\right)\right)'.\end{aligned} \tag{28}$$

Thus (26) can be used to calculate the standard deviation of the estimator $\widehat{\boldsymbol{\theta}}_{2s,T}$.

5. VaR Algorithm Based on the Multivariate Time-Varying *G-H* Copula GARCH Model

After estimating the parameters of multivariate time-varying *G-H* Copula GARCH model, VaR of the risk portfolio can be measured. VaR of risk portfolio indicates the expected maximum losses of risk portfolio held by investors within a given confidence level and a certain period of time. Assuming that $\mathbf{r}_t = (r_{1,t},\ldots,r_{p,t})$ $(t = 1,2,\ldots,T)$ are the return samples of p risk assets which satisfy the multivariate time-varying *G-H* Copula GARCH (1, 1) model in Section 3 and $\sum_{i=1}^{p}\lambda_i r_{i,t}$ is the portfolio of p risk assets in which the weight of the risk asset i is λ_i $(i = 1,2,\ldots,p)$ that can be less than 0 because of permitting short-purchasing and short-selling the risk assets and meet $\sum_{i=1}^{p}\lambda_i = 1$, the VaR of risk portfolio under confidence level q at time t should satisfy $\Pr(\sum_{i=1}^{p}\lambda_i r_{i,t} \le \mathrm{VaR}_t) = q$. The confidence level q can reflect the different risk preferences of investors or financial institutions to a certain extent. Choosing a larger confidence level means that investors or financial institutions have greater risk aversion, and they hope to get a forecast result with larger probability.

Although the conditional distributions of $r_{1,T+1}, r_{2,T+1}, \ldots, r_{p,T+1}$ can be calculated through the known marginal distributions, it is very difficult to calculate quantile from time-varying Copula density function, and it is adverse to measure and calculate the VaR of risk portfolio. Therefore, this paper measures the dynamic risk of portfolio and its estimation value approximately through simulating *G-H* Copula GARCH model. Based on the parameters $\theta^{(n)}$ of the sample, the return series of risk assets $\{[r_{1,1+T}^{(n,m)},\ldots,r_{p,1+T}^{(n,m)}],\ m = 1,\ldots,M\}$ and the one-step measurement and estimation values of VaR of their portfolios can be obtained through estimating $(h_{1,T+1}^{(n)},\ldots,h_{p,T+1}^{(n+1)})$ according to the volatility equation of *G-H* Copula GARCH model and calculating $\rho_{T+1}^{(n)}$ by using Copula dynamic evolution equation and then repeating the following algorithm for M times $(M \ge 3p)$.

Table 1: Moment estimation results of the daily log return of SSCI, HSI, TAIEX, and SP500.

Types of stock index	Mean	Std.	Skewness	Kurtosis
SSCI	$2.5078e-004$	0.0181	−0.2144	7.4467
HSI	$7.4505e-005$	0.0177	−0.2765	11.6866
TAIEX	$3.9511e-005$	0.0140	−0.9091	17.2823
SP500	$-9.7173e-005$	0.0148	−0.3701	12.0177

Step 1. It is simulating M groups of random vectors $[u_{1,T+1}^{(n,m)},\ldots,u_{p,T+1}^{(n,m)}]$ according to the multivariate T-Copula density function whose degree of freedom is $\eta^{(n)}$ and correlation matrix is $\rho_{T+1}^{(n)}$.

Step 2. Calculating $r_{i,T+1}^{(n,m)} = \mu_i^{(n)} + z_{i,T+1}^{(n,m)}\sqrt{h_{i,T+1}^{(n)}}$, $i = 1,\ldots,p$.

Step 3. Firstly, one calculates the return rate of risk portfolio that is equal to $\sum_{i=1}^{p}\lambda_i r_{i,T+1}^{(n,m)}$, $m = 1,2,\ldots,M$. Secondly, one evaluates its q-quantile $\mathrm{VaR}_{T+1}^{(n)}$. Thirdly, one measures the VaR of the risk portfolio by $\mathrm{VaR}_{T+1} = (1/N)\sum_{n=1}^{N}\mathrm{VaR}_{T+1}^{(n)}$.

6. Application of the Multivariate Time-Varying *G-H* Copula GARCH Model

6.1. Date Sample and Moment Estimation. USA and China, as the most developed capitalism country and the largest developing country in the world, respectively, rank top two of the world economy. Their stock markets should have strong representation in the world. At the same time, due to the historical reasons, there exist several regions with different political systems such as Mainland China, Hong Kong, Taiwan, and Macau in Greater China. Macau is similar to Hong Kong on the whole. For the above-mentioned reasons, this paper selects the SSCI (China), HSI (Hong Kong, China), TAIEX (Taiwan, China), and SP500 (USA) from January 3, 2000, to June 18, 2010, as data samples to estimate the VaR of various index portfolios under different confidence levels by using the multivariate time-varying *G-H* Copula GARCH model. The data comes from Yahoo Finance website: http://finance.yahoo.com/.

The moment estimation results of the daily log return of SSCI, HSI, TAIEX, and SP500 are shown in Table 1.

Table 1 shows that the skewness of daily log returns of SSCI, HSI, TAIEX, and SP500 is less than 0 and their kurtosis is much larger than that of standard normal distribution which is equal to 3. These results demonstrate that the daily log returns of these indices have the right skew and leptokurtic characteristics. Therefore, it is appropriate to fit the daily log return of SSCI, HSI, TAIEX, and SP500 by applying *G-H* distribution which has leptokurtic, heavy-tail characteristics, and it is reasonable to apply the multivariate

Table 2: Parameter estimates of the four-variate time-varying G-H Copula GARCH (1, 1) model based on SSCI index risk asset.

Model parameters	μ_1	ω_1	α_1	β_1
Estimate	0.00025^{***}	0.0551^{***}	0.0617^{***}	0.8583^{***}
T-statistic	4.8653	4.6851	6.5764	4.4009

Note: *** in the table denotes that the parameter is significant at 1% level.

Table 3: Parameter estimates of the four-variate time-varying G-H Copula GARCH (1, 1) model based on HSI index risk asset.

Model parameters	μ_2	ω_2	α_2	β_2
Estimate	0.000075^{***}	0.0342^{***}	0.0586^{***}	0.8987^{***}
T-statistic	4.6539	4.8518	6.1796	6.6079

Note: *** in the table denotes that the parameter is significant at 1% level.

Table 4: Parameter estimates of the four-variate time-varying G-H Copula GARCH (1, 1) model based on TAIEX index risk asset.

Model parameters	μ_3	ω_3	α_3	β_3
Estimate	0.00040^{***}	0.0343^{***}	0.0517^{***}	0.9054^{***}
T-statistic	−6.4538	4.3523	5.9817	6.5935

Note: *** in the table denotes that the parameter is significant at 1% level.

Table 5: Parameter estimates of the four-variate time-varying G-H Copula GARCH (1, 1) model based on SP500 index risk asset.

Model parameters	μ_4	ω_4	α_4	β_4
Estimate	-0.0001^{***}	0.0251^{***}	0.0861^{***}	0.8738^{***}
T-statistic	−3.3873	5.2684	7.6324	6.3786

Note: ***in the table denotes that the parameter is significant at 1% level.

time-varying *G-H* Copula GARCH model to measure their VaR.

6.2. Parameter Estimates of the Multivariate Time-Varying G-H Copula GARCH Model. Based on the parameter estimation algorithm proposed in Section 4, the parameters of the multivariate time-varying *G-H* Copula GARCH model with SSCI, HSI, TAIEX, and SP500 can be estimated. The parameter estimation results are shown in Tables 2, 3, 4, 5, and 6.

From Tables 2 to 6, the following results can be obtained:

(1) Consider $\alpha_1 + \beta_1 = 0.92$, $\alpha_2 + \beta_2 = 0.9573$, $\alpha_3 + \beta_3 = 0.9571$, and $\alpha_4 + \beta_4 = 0.9599$. This shows that the volatility persistence of Shanghai stock market is the strongest, Taiwan and Hong Kong stock market rank second and third, and the volatility persistence of USA stock market is minimum. It indicates that the investors' expectation of risk compensation in the emerging markets represented by China's stock market is stronger than that in the mature markets represented by the USA's stock market and the price discovery efficiency of innovation in the emerging markets represented by China's stock market is lower than that in the mature markets represented by USA's stock market. In addition, the sum of the coefficients α and β is very close to 1, which indicates that the impact and shock of innovation on the index volatility of each stock market has a long memory.

(2) The degree of freedom $\eta = 14.57$ and the correlation coefficients ρ_{ij} of T-Copula in Table 6 show that there exists the strongest correlation between Hong Kong stock market and Taiwan stock market, and the correlation between Shanghai stock market and Hong Kong stock market is also relatively large. The above-mentioned facts indicate that the extreme events probably result in the phenomena that Hong Kong stock market and Taiwan stock market are up and down synchronously, and there exist comoving behaviors between Shanghai stock market and Hong Kong stock market.

(3) The time-varying coefficient $b = 0.987$ indicates that the time-varying correlation coefficient of *G-H* Copula GARCH model has a long memory; that is, the impact of historical values of each other's correlation coefficient among SSCI, HSI, TAIEX, and SP500 on the expected correlation is relatively large.

6.3. VaR Measurement Based on the Multivariate Time-Varying G-H Copula GARCH Model. Based on the multivariate time-varying *G-H* Copula GARCH model with SSCI, HSI, TAIEX, and SP500 whose parameters have been estimated, the VaRs of various index portfolios under different confidence levels can be measured. The measurement results are shown in Table 7.

From Table 7, the following results can be obtained:

(1) The inequalities *VaR (SSCI) < VaR (HSI) < VaR (SP500) < VaR (TAIEX)* can be satisfied for any confidence level. It shows that the risk of extreme losses in Shanghai stock market is higher than that in Hong Kong stock market, Taiwan stock market, and USA stock market. This measurement result is in line with the actual situation that the maturity of Shanghai stock market is far lower than that of Hong Kong stock market, Taiwan stock market, and USA stock market.

(2) For any confidence level, the extreme losses risk of the investors who equally allocate their total assets among SSCI, HSI, TAIEX, and SP500 is lower than that of the investors who put their total assets into one index asset. The extreme losses risk of the investors increases with the concentration of risk asset in the index portfolios. This measurement result is consistent with the risk diversification theory of portfolio.

7. Conclusion

Considering the "asymmetric, leptokurtic, and heavy-tail" characteristics, the time-varying volatility characteristics, and extreme-tail dependence characteristics of financial asset return, this paper combined the *G-H* distribution, Copula function, and GARCH model to construct a multivariate time-varying *G-H* Copula GARCH model which can comprehensively describe the "asymmetric, leptokurtic,

Table 6: Estimates of correlation coefficients, time-varying parameters, and degree of freedom of four-variate time-varying G-H Copula GARCH (1, 1) model based on SSCI, HSI, TAIEX, and SP500.

Model parameters	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	a	b	η
Estimate	0.256^{***}	0.139^{***}	0.021^{***}	0.528^{***}	0.193^{***}	0.128^{**}	0.008^{***}	0.987^{***}	14.57^{***}
T-statistic	13.51	11.02	3.431	5.763	9.564	2.529	5.179	3.561	6.871

Note: *** and ** in the table denote that the parameter is significant at 1% and 5% level, respectively.

Table 7: VaR estimation results based on the four-variate time-varying G-H Copula GARCH model with SSCI, HSI, TAIEX, and SP500.

Ratio of index portfolio	1%	5%	10%	15%	Ratio of index portfolio	1%	5%	10%	15%
0.25 : 0.25 : 0.25 : 0.25	**−0.0303**	**−0.0165**	**−0.0116**	**−0.0086**	**0.25 : 0.25 : 0.25 : 0.25**	**−0.0303**	**−0.0165**	**−0.0116**	**−0.0086**
0.50 : 0.00 : 0.25 : 0.25	−0.0329	−0.0177	−0.0128	−0.0093	0.25 : 0.50 : 0.00 : 0.25	−0.0361	−0.0189	−0.0132	−0.0100
0.75 : 0.00 : 0.00 : 0.25	−0.0418	−0.0226	−0.0157	−0.0116	0.25 : 0.75 : 0.00 : 0.00	−0.0426	−0.0236	−0.0165	−0.0123
1.00 : 0.00 : 0.00 : 0.00	−0.0536	−0.0285	−0.0199	−0.0150	0.00 : 1.00 : 0.00 : 0.00	−0.0491	−0.0269	−0.0190	−0.0138
0.25 : 0.25 : 0.25 : 0.25	**−0.0303**	**−0.0165**	**−0.0116**	**−0.0086**	**0.25 : 0.25 : 0.25 : 0.25**	**−0.0303**	**−0.0165**	**−0.0116**	**−0.0086**
0.25 : 0.25 : 0.50 : 0.00	−0.332	−0.0176	−0.0119	−0.0093	0.00 : 0.25 : 0.50 : 0.50	−0.0359	−0.0181	−0.0128	−0.0094
0.00 : 0.25 : 0.75 : 0.00	−0.0403	−0.0196	−0.0127	−0.0097	0.00 : 0.00 : 0.25 : 0.75	−0.0402	−0.0206	−0.0146	−0.0108
0.00 : 0.00 : 1.00 : 0.00	−0.0420	−0.0211	−0.0139	−0.0101	0.00 : 0.00 : 0.00 : 1.00	−0.0425	−0.0230	−0.0157	−0.0119

Note: the ratio of index portfolio is ranked by the sequence of SSCI : HSI : TAIEX : SP500 in Table 7.

and heavy-tail" characteristics, the time-varying volatility characteristics, and extreme-tail dependence characteristics of financial asset return. It proposed the parameter estimation algorithm of the multivariate time-varying *G-H* Copula GARCH model by using condition maximum likelihood method and IFM two-step method. An algorithm was constructed to calculate VaR by using the quantile function and the simulation method based on *G-H* Copula GARCH model. In addition, this paper selected the daily log return of SSCI (China), HSI (Hong Kong, China), TAIEX (Taiwan, China), and SP500 (USA) from January 3, 2000, to June 18, 2010, as samples to estimate the parameters of the multivariate time-varying *G-H* Copula GARCH model, and it also estimated the VaR for various index risk asset portfolios under different confidence levels. The research results showed that the multivariate time-varying *G-H* Copula GARCH model constructed in this paper could reasonably estimate and measure the extreme losses of risk portfolios in financial market, and the measurement results were in line with the actual situation of stock market and the risk diversification theory of portfolio. The achievement of this paper provided a practical and effective method for measuring the extreme losses of financial market.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the referees for all helpful comments and suggestions. This work is supported by Natural Science Foundation Project of CQ CSTC (Grant no. CSTC, 2011BB2088) and Key Project of National Natural Science Foundation (Grant no. 71232004).

References

[1] S. J. Kon, "Models of stock returns—a comparison," *The Journal of Finance*, vol. 39, no. 1, pp. 147–165, 1984.

[2] J. B. Gray and D. W. French, "Empirical comparisons of distributional models for stock index returns," *Journal of Business Finance & Accounting*, vol. 17, no. 3, pp. 451–459, 1990.

[3] E. Jondeau and M. Rockinger, "Testing for differences in the tails of stock-market returns," *Journal of Empirical Finance*, vol. 10, no. 5, pp. 559–581, 2003.

[4] C. Jiang, S. Li, and S. Liang, "Empirical investigation of distribution feature of return in china stock market," *Application of Statistics and Management*, vol. 26, no. 4, pp. 710–717, 2007.

[5] D. Dong and W. Jin, "A subjective model of the distribution of returns and empirical analysis," *Chinese Journal of Management Science*, vol. 15, no. 1, pp. 112–120, 2007.

[6] J. Martinez and B. Iglewicz, "Some properties of the tukey g and h family of distributions," *Communications in Statistics—Theory and Methods*, vol. 13, no. 3, pp. 353–369, 1984.

[7] T. C. Mills, "Modelling skewness and kurtosis in the London stock exchange FT-SE index return distributions," *The Statistician*, vol. 44, no. 3, pp. 323–332, 1995.

[8] H. Zhu and Z. Pan, "A study of portfolios VaR method based on g-h distribution," *Chinese Journal of Management Science*, vol. 13, no. 4, pp. 7–12, 2005.

[9] K. Kuester, S. Mittnik, and M. S. Paolella, "Value-at-risk prediction: a comparison of alternative strategies," *Journal of Financial Econometrics*, vol. 4, no. 1, pp. 53–89, 2006.

[10] M. Degen, P. Embrechts, and D. Lambrigger, "The quantitative modeling of operational risk: between g-and-h and EVT," *Astin Bulletin*, vol. 37, no. 2, pp. 265–291, 2007.

[11] E. Jondeau and M. Rockinger, "The Copula-GARCH model of conditional dependencies: an international stock market application," *Journal of International Money and Finance*, vol. 25, no. 5, pp. 827–853, 2006.

[12] J. C. Rodriguez, "Measuring financial contagion: a Copula approach," *Journal of Empirical Finance*, vol. 14, no. 3, pp. 401–423, 2007.

[13] M. Fischer, C. Kock, S. Schluter, and F. Weigert, "An empirical analysis of multivariate copula models," *Quantitative Finance*, vol. 9, no. 7, pp. 839–854, 2009.

[14] W. Sun, S. Rachev, F. J. Fabozzi, and S. S. Kalev, "A new approach to modeling co-movement of international equity markets: evidence of unconditional copula-based simulation of tail dependence," *Empirical Economics*, vol. 36, no. 1, pp. 201–229, 2009.

[15] Z. Liu, T. Zhang, and F. Wen, "Time-varying risk attitude and conditional skewness," *Abstract and Applied Analysis*, vol. 2014, Article ID 174848, 11 pages, 2014.

[16] F. Wen, Z. He, Z. Dai, and X. Yang, "Characteristics of investors' risk preference for stock markets," *Economic Computation and Economic Cybernetics Studies and Research*, vol. 3, no. 48, pp. 235–254, 2014.

[17] F. Wen, X. Gong, Y. Chao, and X. Chen, "The effects of prior outcomes on risky choice: evidence from the stock market," *Mathematical Problems in Engineering*, vol. 2014, Article ID 272518, 8 pages, 2014.

[18] J.-J. Huang, K.-J. Lee, H. Liang, and W.-F. Lin, "Estimating value at risk of portfolio by conditional copula-GARCH method," *Insurance: Mathematics and Economics*, vol. 45, no. 3, pp. 315–324, 2009.

[19] A. Ghorbel and A. Trabelsi, "Measure of financial risk using conditional extreme value copulas with EVT margins," *Journal of Risk*, vol. 11, no. 4, pp. 51–85, 2009.

[20] L. Chollete, A. Heinen, and A. Valdesogo, "Modeling international financial returns with a multivariate regime-switching copula," *Journal of Financial Econometrics*, vol. 7, no. 4, pp. 437–480, 2009.

[21] M. Huggenberger and T. Klett, "A g-and-h copula approach to risk measurement in multivariate financial models," Working Paper, 2010, http://ssrn.com/abstract=1677431.

[22] J. Wang, W. Bao, and J. Hu, "Portfolios VaR analyses based on the multi-dimensional gumbel copula function," *Journal of Applied Statistics and Management*, vol. 29, no. 1, pp. 137–143, 2010.

[23] Z. Dai and F. Wen, "Robust CVaR-based portfolio optimization under a genal affine data perturbation uncertainty set," *Journal of Computational Analysis and Applications*, vol. 16, no. 1, pp. 93–103, 2014.

[24] J. Liu, M. Tao, C. Ma, and F. Wen, "Utility indifference pricing of convertible bonds," *International Journal of Information Technology and Decision Making*, vol. 13, no. 2, pp. 429–444, 2014.

[25] R. B. Nelsen, *An Introduction to Copulas*, Springer Science & Business Media, New York, NY, USA, 2007.

[26] H. Joe, *Multivariate Models and Dependence Concepts*, vol. 73 of *Monographs on Statistics and Applied Probability*, Chapman & Hall, London, UK, 1997.

[27] A. J. Patton, "Estimation of multivariate models for time series of possibly different lengths," *Journal of Applied Econometrics*, vol. 21, no. 2, pp. 147–173, 2006.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 168045, 9 pages
http://dx.doi.org/10.1155/2015/168045

Research Article

The Particle Filter Sample Impoverishment Problem in the Orbit Determination Application

Paula Cristiane Pinto Mesquita Pardal,[1] Helio Koiti Kuga,[2] and Rodolpho Vilhena de Moraes[3]

[1] *University of São Paulo (USP)-EEL/LOB, Estrada Municipal do Campinho, s/n, 12602-810 Lorena, SP, Brazil*
[2] *National Institute for Space Research (INPE)-DMC, Avenida dos Astronautas, 1758 Jardim da Granja, 12227-010 São José dos Campos, SP, Brazil*
[3] *São Paulo Federal University (UNIFESP)-ICT/UNIFESP, Rua Talim, 330 Vila Nair, 12231-280 São José dos Campos, SP, Brazil*

Correspondence should be addressed to Paula Cristiane Pinto Mesquita Pardal; paulapardal@usp.br

Received 11 February 2015; Accepted 6 May 2015

Academic Editor: Ruihua Liu

The paper aims at discussing techniques for administering one implementation issue that often arises in the application of particle filters: sample impoverishment. Dealing with such problem can significantly improve the performance of particle filters and can make the difference between success and failure. Sample impoverishment occurs because of the reduction in the number of truly distinct sample values. A simple solution can be to increase the number of particles, which can quickly lead to unreasonable computational demands, which only delays the inevitable sample impoverishment. There are more intelligent ways of dealing with this problem, such as roughening and prior editing, procedures to be discussed herein. The nonlinear particle filter is based on the bootstrap filter for implementing recursive Bayesian filters. The application consists of determining the orbit of an artificial satellite using real data from the GPS receivers. The standard differential equations describing the orbital motion and the GPS measurements equations are adapted for the nonlinear particle filter, so that the bootstrap algorithm is also used for estimating the orbital state. The evaluation will be done through convergence speed and computational implementation complexity, comparing the bootstrap algorithm results obtained for each technique that deals with sample impoverishment.

1. Introduction

The orbit of an artificial satellite is determined using real data from the Global Positioning System (GPS) receivers. In the orbit determination process of artificial satellites, the nature of the dynamic system and the measurements equations are nonlinear. As a result, it is necessary to manage a fully nonlinear problem in which the disturbing forces as well as the measurements are not easily modelled. In this orbit determination problem, the variables that completely specify a satellite trajectory in the space are estimated, with the processing of a set of pseudorange measurements related to the body.

A spaceborne GPS receiver is a powerful resource to determine orbits of artificial Earth satellites by providing many redundant measurements, which ultimately yields high degree of the observability to the problem. The Jason satellite is a nice example of using GPS for space positioning. Through an on-board GPS receiver, the pseudoranges (error corrupted distance from satellite to each of the tracked GPS satellites) can be measured and used to estimate the full orbital state.

The bootstrap filter is a particle filter whose central idea is to express the required probability density function (PDF) as a set of random samples, instead of a function over state space [1–3].

Numerous strategies have been developed for solving the particles degeneracy (or sample impoverishment) problem that often arises in particle filter applications like introduction of a risk-sensitive particle filter as an alternative approach to mitigate sample impoverishment based on constructing explicit risk functions from a general class of factorizable

functions [4]; incorporation of genetic algorithms into a particle filter [5, 6]; and many others [7–9]. All these strategies, although extremely interesting and suitable for the orbit determination problem, are not in the scope of this work. Here, the option was done for studying two classical methods to solve (or try to solve) the degeneracy problem: roughening and prior editing.

Herein, the main goal is to analyze the bootstrap filter behavior for the highly nonlinear orbit determination problem. Its simulation results are compared taking into account the sample impoverishment. A reference solution is a bootstrap particle filter (BPF) applied to orbit determination that has already been compared to the unscented Kalman filter solution for the same problem and works well for the analysis of the sample impoverishment issue [10].

2. Particle Filter

The particle filter was designed to numerically implement the Bayesian estimator [2]. The Bayesian approach consists of constructing the PDF of the state based on all the available information, and, for nonlinear or non-Gaussian problem, the required PDF has no closed form. The bootstrap filter represents the required PDF as a set of random samples, which works as an alternative to the function over state space. This filter is a recursive algorithm for propagation and update of these samples for the discrete time problem. The Bayes rule, the key update stage of the method, is implemented as a weighted bootstrap [1].

The main idea of the BPF is intuitive and direct. At the beginning, N particles $x_{0,i}^{+}$ $(i = 1, \ldots, N)$ are randomly generated, based on the known initial PDF $p(x_0)$. At each step of time k, the particles are propagated to the next step using the dynamics equation [2]. After receiving the measurement at time k, the PDF $p(y_k \mid x_{k-1}^{i})$ is evaluated. That is, the conditional relative likelihood of each particle $x_{k,i}^{-}$ is calculated. If an m-dimensional measurement equation is given as $y_k = h(x_k) + v_k$ and v_k is a Gaussian random variable with a mean of zero and a variance of R, $v_k \sim N(0, R)$, then a relative likelihood q_i that the measurement is equal to a specific measurement y^*, given the premise that x_k is equal to the particle x_{k-1}^{i}, can be computed as follows [2]:

$$q_i = p\left(y_k = y^* \mid x_k = x_{k-1}^{i}\right) = p\left[v_k = y^* - h\left(x_{k-1}^{i}\right)\right] \sim \frac{1}{(2\pi)^{m/2}\,|R|^{1/2}} \cdot \exp\left(\frac{-\left[y^* - h\left(x_{k-1}^{i}\right)\right]^{\mathrm{T}} R^{-1}\left[y^* - h\left(x_{k-1}^{i}\right)\right]}{2}\right). \tag{1}$$

In (1), the symbol ~ means that the probability is directly proportional to the right side. So if the equation is used for all particles $x_{k,i}^{-}$, then the relative likelihood that the state is equal to each particle is correct. The relative likelihood values are normalized to ensure that the sum of all likelihood values is equal to one. Next, a new set of randomly generated particles $x_{k,i}^{+}$ is computed from the relative likelihood q_i. In the resampling step, roughening was used, in order to prevent sample impoverishment. At this point, there is a set of particles $x_{k,i}^{+}$ that are distributed according to the PDF $p(x_k \mid y_k)$, and any desired statistical measure of it can be computed [2].

The particle filter, adjusted to the orbit determination problem, can be summarized as follows.

(1) The dynamic and the measurement equations are given as

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k\right) + \mathbf{w}_k, \qquad \mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k\right) + \boldsymbol{\nu}_k, \tag{2}$$

where $\mathbf{w}_k$ and $\boldsymbol{\nu}_k$ are independent white noise processes with known PDFs.

(2) N initial particles $\mathbf{x}_{0,i}^{+}$ $(i = 1, \ldots, N)$ are randomly generated on the basis of the known initial state PDF $p(\mathbf{x}_0)$. N is a parameter chosen as a trade-off between computational cost and estimation accuracy [2].

(3) For $k = 1, 2, \ldots$,

(a) in the time propagation step, obtain a priori (predicted) particles $\mathbf{x}_{k,i}^{-}$, using the dynamics equation and the PDF of the process noise, both known:

$$\mathbf{x}_{k,i}^{-} = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1,i}^{+}\right) + \mathbf{w}_{k-1}^{i}, \quad i = 1, \ldots, N, \tag{3}$$

where each noise vector, $\mathbf{w}_{k-1}^{i}$, is randomly generated on the basis of the known PDF of $\mathbf{w}_{k-1}$;

(b) compute the relative likelihood q_i of each particle $x_{k,i}^{-}$, conditioned on the measurement $\mathbf{y}_k$, using the nonlinear measurement equation and the PDF of the measurement noise, as in (1);

(c) normalize the relative likelihood values:

$$q_i = \frac{q_i}{\sum_{j=1}^{N} q_j}; \tag{4}$$

(d) in the resampling step, generate a set of a posteriori (resampled) particles $\mathbf{x}_{k,i}^{+}$, on the basis of the relative likelihood q_i;

(e) now, there is a set of particles $\mathbf{x}_{k,i}^{+}$ distributed according to the PDF $p(\mathbf{x}_k \mid \mathbf{y}_k)$, and mean and covariance statistical measures can be computed.

In the implementation of the bootstrap filter, there is only a small overlap between the prior and the likelihood.

There are some procedures that may be implemented for combating the consequent reduction in the number of truly distinct sample values, such as increasing the number of particles, roughening, and prior editing [1]. Here, they were implemented: a bootstrap particle filter with resampling (PF); a PF with roughening (PFR); and a PFR with prior editing (PFPE), in order to evaluate roughening and prior editing strategies for dealing with sample impoverishment.

2.1. Roughening. Roughening will be the first remedy for sample impoverishment to be discussed. It restrains the resampled particles spread (a posteriori particles) by adding random noise to them, which is similar to adding artificial process noise to the Kalman filter [2]. In roughening approach, the a posteriori particles are modified, after the resampling step, as follows:

$$\mathbf{x}_{k,i}^{+}(m) = \mathbf{x}_{k,i}^{+}(m) + \Delta\mathbf{x}(m), \quad m = 1, \ldots, n,$$
$$\Delta\mathbf{x}(m) \sim \left(0, K\mathbf{M}(m) N^{-1/n}\right). \tag{5}$$

$\Delta\mathbf{x}(m)$ is a zero-mean random variable (usually Gaussian); K is a constant tuning parameter; N is the number of particles; n is the state space dimension; and $\mathbf{M}$ is a vector of the maximum difference between the particle elements before roughening. The mth element of the $\mathbf{M}$ vector is given as

$$\mathbf{M}(m) = \max_{i,j} \left|\mathbf{x}_{k,i}^{+}(m) - \mathbf{x}_{k,j}^{+}(m)\right|, \quad m = 1, \ldots, n, \tag{6}$$

where k is the step time and i and j are particle numbers.

The tuning parameter K choice is a compromise. Being too large, a value would blur the distribution, but being too small, it would produce tight clusters of points around the original particles [1]. In this paper, $K = 0.1$.

2.2. Prior Editing. Prior editing can be tried if roughening does not prevent sample impoverishment. Such approach edits the a posteriori particles from the prior time instant, $\mathbf{x}_{k-1,i}^{+}$ (after roughening), if the a priori particle from actual instant, $\mathbf{x}_{k,i}^{-}$, does not satisfy a coarse, pragmatic acceptance test [1]. Therefore, this procedure artificially boosts the number of samples of the prior editing in the neighborhood of the likelihood, for if an a priori particle is in a region of state space with small q_i, it is rejected. Then, the a priori rejected particle can be roughened as many times as required, according to (5), until it is in a region of significant q_i [2]. The prior editing was implemented as follows [1]:

(a) Pass the resampled sample from previous instant, $\mathbf{x}_{k-1}^{+}$, through roughening and system model to generate the predicted sample from current instant, $\mathbf{x}_{k}^{-}$.

(b) Calculate $\boldsymbol{v}_{k,i} = z_k - \mathbf{h}(\mathbf{x}_{k,i}^{-})$, the residual between the true and the predicted measurements, for the ith particle of the sample, considering that the actual instant observation is available.

(c) If the magnitude of $\boldsymbol{v}_{k,i}$ is higher than six standard deviations of the measurement noise, then it is highly unlikely that $\mathbf{x}_{k,i}^{-}$ is chosen as an a posteriori particle. In this case, $\mathbf{x}_{k,i}^{-}$ is rejected, and $\mathbf{x}_{k-1,i}^{+}$ is roughened again and passes one more time through dynamic model to generate a new a priori particle $\mathbf{x}_{k,i}^{-}$. As $\mathbf{x}_{k-1,i}^{+}$ has already passed through roughening and generated a rejected predicted particle, this procedure may be repeated while $\mathbf{x}_{k,i}^{-}$ is in a region of no negligible probability.

Due to the high computational cost involving prior editing, such approach was done only once. It is important to make it clear that, here, the ith particle is, in fact, an n-dimensional vector, while a sample is a $n \times N$ matrix, where the nth state variable is represented by N particles.

The accommodation of roughening and prior editing in the bootstrap particle filter algorithm can be schematized as Figure 1 shows.

3. Orbit Determination

The orbit determination is a process for obtaining values of the parameters that completely specify the motion of an orbiting body (as an artificial satellite), based on a set of observations of the body. It involves nonlinear dynamical and nonlinear measurement systems, which depends on the tracking system and the estimation technique [11, 12]. The dynamical system model consists of describing satellite orbital motion, which includes Earth's rotation effects and perturbation models and measurements models. These models depend on the state variables initial conditions, as well as a variety of parameters which affect both the dynamic motions as the measurement process [13]. Due to the complexity of the applied models, usually it is not possible to solve such models equations directly for any of these parameters from a given set of observations.

The observation may be obtained from the ground station networks using laser, radar, Doppler, or space navigation systems, as the GPS. The choice of the tracking system depends on a compromise between the goals of the mission and the available tools. In the case of the GPS, the advantages are global coverage, high precision, low cost, and autonomous navigation resources. The GPS may provide orbit determination with accuracy at least as good as the methods using ground tracking networks. The latter provides standard precision around tens of meters and the former can provide precision as tight as some centimetres. The GPS provides, at a given instant, a set of many redundant measurements, which makes the orbit position observable geometrically.

After some advances of technology, the single frequency GPS receivers provide a good basis to achieve fair precision at relatively low cost, still attaining the accuracy requirements of the mission operation. The GPS allows the receiver to determine its position and time, geometrically, anywhere at any instant, with data from at least four satellites. The principle of navigation by satellites is based on sending signals and data from the GPS satellites to a receiver located on board the satellite whose orbit needs to be determined. This receiver measures the travel time of the signal and then calculates the distance between the receiver and the GPS satellite. Those measurements of distances are called pseudoranges.

The instantaneous orbit determination using GPS satellites is based on the geometric method. In such method, the observer knows the set of GPS satellites position in a reference frame, obtaining its own position in the same reference frame.

3.1. Dynamics Model. In the case of orbit determination via GPS, the ordinary differential equations which represent the

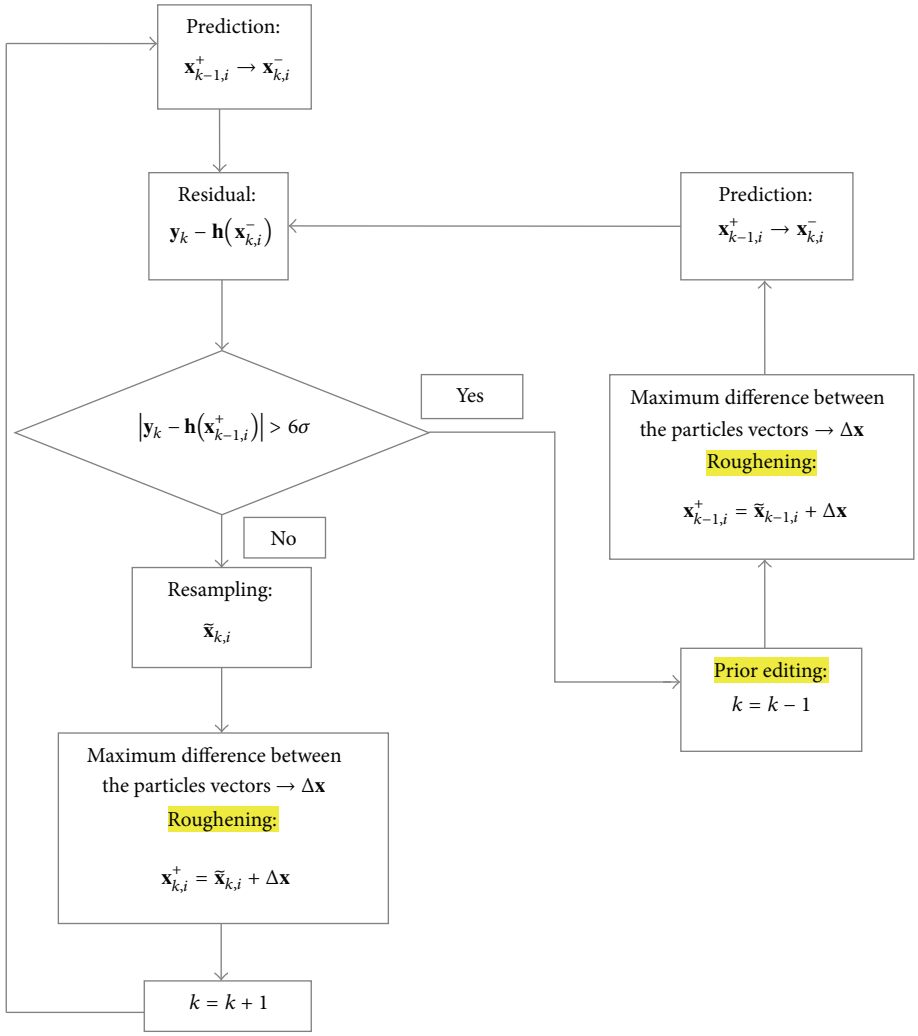


Figure 1: Roughening and prior editing accommodation in the BPF algorithm.

dynamic model are, in its simplest form, given traditionally as follows:

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v}, \\ \dot{\mathbf{v}} &= -\mu \frac{\mathbf{r}}{r^3} + \mathbf{a} + \mathbf{w}_v, \\ \dot{b} &= d, \\ \dot{d} &= w_d, \end{aligned} \tag{7}$$

wherein the variables are placed in the inertial reference frame. In (7), $\mathbf{r}$ is the vector of the position components (x, y, z); $\mathbf{v}$ is the velocity vector; $\mathbf{a}$ represents the modelled perturbing accelerations; $\mathbf{w}_v$ is the white noise vector with covariance $\mathbf{Q}$; b is the user satellite GPS clock bias; d is the user satellite GPS clock drift; and w_d is the noise associated with the GPS clock. The GPS receiver clock offset was not taken into account, so as not to obscure the conclusions drawn in this paper due to introduction of clock offset models in the filters. Indeed, the receiver clock offset was beforehand obtained and used to correct the GPS measurements, so that the measurements are free from the error derived from receiver clock offset.

3.2. Forces Model. There are gravitational and nongravitational forces that affect the orbit of an Earth's artificial satellite. The main disturbing forces of gravitational nature are the nonuniform distribution of Earth's mass; ocean and terrestrial tides; and the gravitational attraction of the Sun and the Moon. And the principal nongravitational effects are Earth atmospheric drag; direct and reflected solar radiation pressure; electric drag; emissivity effects; relativistic effects; and meteorites impacts.

The disturbing effects are, in general, included according to the physical situation presented and to the accuracy that is intended for the orbit determination. Here, only a minimum set of perturbations was included which enable analyzing the performance of the particle filter [14]: geopotential [15]; direct solar radiation pressure [16, 17]; and third body point mass effect of the Sun and the Moon [18, 19].

3.3. Observations Model. The nonlinear equation of the observation model is

$$\mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k, t_k\right) + \boldsymbol{\nu}_k, \tag{8}$$

where, at instant t_k, $\mathbf{y}_k$ is the vector of m observations; $\mathbf{h}_k(\mathbf{x}_k)$ is the nonlinear function of state $\mathbf{x}_k$, with dimension m; and $\boldsymbol{\nu}_k$ is the observation errors vector, with dimension m and covariance $\mathbf{R}_k$. For the present application, the ion-free pseudorange measurements from the Jason-2 GPS receiver are used. Also, the receiver clock offset was computed before and used to correct the pseudorange measurements. Additionally, the nonlinear pseudorange measurement was modelled according to [20].

4. Results

The tests and the analysis for the bootstrap particle filter and two procedures for avoiding sample impoverishment (roughening and prior editing) are presented. To validate and to analyze the methods, real GPS data from the Jason-2 satellite are used. Ocean Surface Topography Mission (OSTM)/Jason-2 is a follow-on altimetry mission to the very successful TOPEX/Poseidon mission and Jason-1. It is a joint mission between NASA and CNES (French space agency), launched June 20, 2008. Jason-2 has a repeat period of approximately 10 days with 254 passes per cycle. Its nodal period is 6,745.72 sec (near 1.87 hours). Sometimes there may be anomalous or missing data. Occasionally Jason-2 must perform maneuvers to maintain orbit. When the satellite detects something abnormal, it will go into safe hold and will turn off all instruments and no data will be collected [21].

The filters estimated position and velocity are compared with Jason-2 precise orbit ephemeris (POE) from JPL/NASA. The test conditions consider real ion-free pseudorange data, collected by the GPS receiver on-board Jason-2, on October 22, 2010, presenting up to 12 GPS satellites tracked. The tests were limited to 5.5 hours of GPS data spaced 10 s. After that, there was an undesirable data gap which could spoil the test case. Such 5.5-hour arc (near 3 Jason-2 orbital periods) was considered sufficient for evaluating the bootstrap particle filter and roughening and prior editing approaches, in this orbit determination application.

The force model comprises perturbations due to geopotential up to order and degree 50×50; direct solar radiation pressure; and Sun-Moon gravitational attraction. This model of forces is suitable for implementation in orbit determination because of the low computational cost added compared to the improvement in the results accuracy [22–24]. The pseudorange measurements were corrected to the first order with respect to ionosphere.

This work is not a search for results accuracy. It aims at analyzing the application of a bootstrap filter to the orbit determination problem. The analysis is based on comparing the errors in position (translated to the orbital radial, normal, and along-track RNT components) among three solutions:

(1) The bootstrap particle filter with resampling, applied without any sample impoverishment procedure (named "PF" in the results).

(2) The bootstrap particle filter applied with roughening (named "PFR" in the results).

(3) The bootstrap particle filter applied with roughening and prior editing (named "PFPE"in the results).

The RNT system interpretation is straightforward: the radial component "R" points to the nadir direction; the normal "N" is perpendicular to orbital plane; and the transversal (along-track) "T" is orthogonal to "R" and "N," and it is also the velocity component. Thus, it is possible to analyze what happens with the orbital RNT components and the orbit evolution as well. There is also interest for processing time, in order to analyze the computational efforts face to the accuracy achieved by each algorithm.

Regarding time of processing, t_{CPU}, it was expected that the time required for the bootstrap particle filter algorithm was increasing in two scenarios: if the number of particles rises and if roughening and prior editing were added to the algorithm. According to the estimator nature, each element of the state was replaced by an array of N particles, where N is a trade-off between computational cost and estimation accuracy. Here, tests were done for 17, 100, 300, and 700 particles.

As said before, prior editing recomputes any particle that does not match a specific criterion. Therefore, a relevant test is to observe the instant when each algorithm starts rejecting particles which will be presented. The goal is to verify whether, as a procedure is included, it delays the rejection process. It will also be analyzed whether the number of particles affects the particle rejection, that is, whether its increase may work as an approach for avoiding sample impoverishment.

If N is set (so the analysis is per line in Table 1), it is noticeable that t_{CPU}, CPU time, measured for PF and PFR is very similar, with a maximum 3% of difference. However, the algorithm that includes prior editing, PFPE, is significantly more costly, reaching 39% of increase. This was expected for PFPE, because of the particles that do not pass the acceptance test and need to return to prior instant. Regarding the effects of increasing the number of particle, for the same algorithm (so the analysis is per column in Table 1), the raise in processing time was 83% from 17 to 100 particles, 67% from 100 to 300, and 59% from 300 to 700. Considering that, for 17, 100, 300, and 700 particles, the increase in N is 6, 3, and 2.3 times, respectively; then, t_{CPU} increase is not directly proportional to the number of particles used. Therefore, the CPU time and the increase in the number of particles show that the time of processing is related to the chosen number of particles only when the number of particles changes, but it has no relation with the estimator implemented.

For computing time of processing and for all the simulations shown, a computer Intel Core i5-2430M processor of 2.40 GHz, with 2.70 GB of RAM, was used. The program was coded in FORTRAN 77 within operating system Windows XP and compiler Compaq Visual Fortran version 6.1.

When some particles do not reach prior editing criterion (i.e., the magnitude of $\boldsymbol{v}_{k,i}$ is higher than six standard deviation of the measurement noise), they need to be edited in the prior instant of time. The first instant when any particle

Table 1: Time of processing.

	N	PF	PFR	PFPE
t_{CPU}	17	44 s	45 s	1 min 23 s
	100	4 min 16 s	4 min 19 s	7 min 07 s
	300	12 min 49 s	13 min 05 s	20 min 56 s
	700	31 min 09 s	31 min 27 s	50 min 31 s

Table 2: Instant of the first rejection of particles occurrence.

	N	PF	PFR	PFPE
t_{PE}	17	16 min 00 s	26 min 40 s	26 min 40 s
	100	10 min 00 s	45 min 50 s	45 min 50 s
	300	9 min 20 s	45 min 50 s	45 min 50 s
	700	9 min 00 s	45 min 50 s	45 min 50 s

needed to be edited, t_{PE}, was computed, for each algorithm and N, as can be verified in Table 2.

In Table 2, the rejection of particles was first detected in the PF algorithm, despite the number of particles used. This was expected, since no procedure in order to combat the number of truly distinct samples reduction was used in this algorithm. And the instant when the first rejection occurred in PFR and PFPE concurred for all changing in the number of particles. This is considerably consistent, since prior editing procedure depends first on roughening implementation. Regarding the increase in the number of particles (from 17 to 100), it is noticeable that, when procedures for avoiding sample impoverishment are adopted (in PFR and PFPE cases), the instant when the first rejection of particles is detected is delayed in 42%. Nevertheless, for the other cases analyzed (300 and 700 particles), such instant remains the same. This suggests that increasing N as an approach for minimizing the sample impoverishment issue is a little efficient, with a very high computational burden, as seen in Table 1. For PF, the results were not conclusive. It seems that the higher the number of particles, the faster their impoverishment if nothing is done for avoiding sample impoverishment.

As said before, the number of particles is chosen as a trade-off between computational cost and estimation accuracy. The results in Table 3 present mean and standard deviation of the errors in RNT components evaluated for $N = 17$; 100; 300; and 700. If only these statistics are analyzed, it is clear that the estimation accuracy improves as the number of particles increases. The largest standard deviation occurred for PFPE ($N = 17$). In the along-track component, a divergence occurrence in all algorithms ($N = 17$) was detected, which can be verified by the high mean and standard deviation values obtained in the three cases. Such divergence disappears when N is increased. When $N = 700$, the statistics and the errors behavior were very similar in the results obtained by PFR and PFPE versions. By the information presented, it is possible to conclude that if the number of particles is very small, any sample impoverishment avoidance procedure will not be effective. Even more, the PFPE approach as used here (particles edition executed only once) does not present any significant improvement facing the PFR procedure, no matter the number of particles used. And taking into account the higher computational cost for only one edition of particles, if the PFPE is implemented as many times as necessary, the computation burden is enough to contraindicate this procedure use, even if the results are improved.

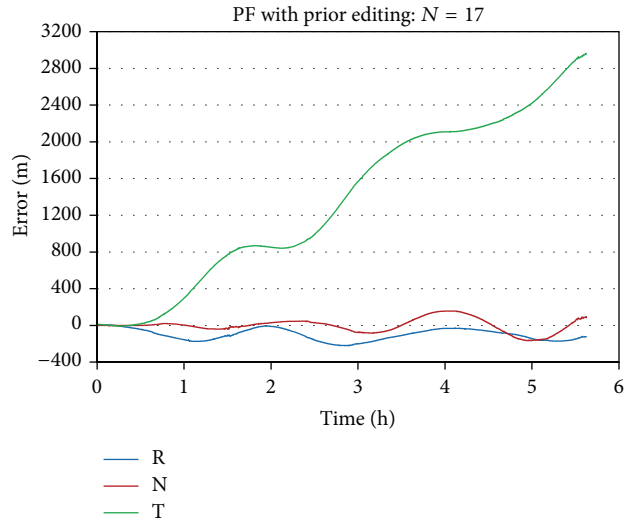


Figure 2: Errors in RNT coordinates for PFPE simulation ($N = 17$).

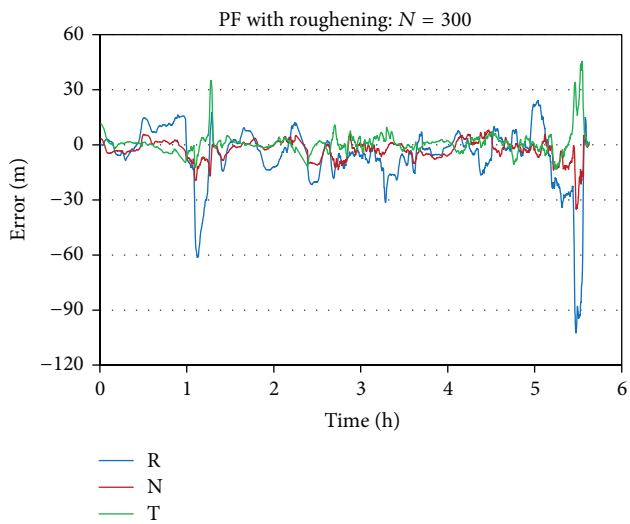


Figure 3: Errors in RNT coordinates for PFR simulation ($N = 300$).

In order to show the behavior of the errors in terms of RNT coordinates, Figures 2, 3, and 4 are presented. Figure 2 shows the worst result, the solution obtained for PFPE algorithm considering $N = 17$, where the divergence in along-track coordinate is shown. Figure 3 shows a PFR solution for $N = 300$, while Figure 4 presents the $N = 700$ solution. It was chosen to introduce PFR solutions because this is the algorithm with better performance during orbit determination process. The errors decrease considerably in all implementations, for 100 or more particles (see Table 3),

Table 3: Mean and standard deviation statistics.

Estimator	N	Mean ± standard deviation (m)		
		R	N	T
PF	17	−5.940 ± 36.276	−1.821 ± 17.869	120.504 ± 212.955
	100	−10.289 ± 14.711	−0.238 ± 9.961	94.695 ± 119.586
	300	−1.065 ± 9.367	0.139 ± 7.333	−9.177 ± 50.723
	700	**−1.031 ± 8.089**	**0.038 ± 7.047**	**−1.792 ± 34.949**
PFR	17	27.666 ± 64.094	6.011 ± 31.725	37.537 ± 343.973
	100	−6.876 ± 19.706	−2.935 ± 12.435	2.562 ± 11.078
	300	−6.105 ± 16.806	−2.335 ± 5.478	1.066 ± 6.198
	700	**−5.246 ± 12.361**	**−1.322 ± 4.574**	**−0.051 ± 3.486**
PFPE	17	−101.686 ± 63.379	−1.322 ± 75.400	1394.298 ± 902.146
	100	−9.833 ± 15.272	−1.813 ± 7.972	1.290 ± 9.093
	300	−5.839 ± 15.008	−1.890 ± 7.615	0.602 ± 3.953
	700	**−5.250 ± 12.675**	**−1.669 ± 4.164**	**−0.243 ± 4.043**

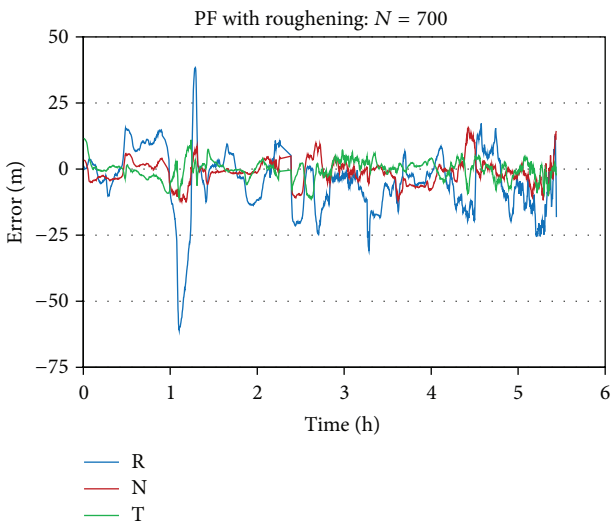


Figure 4: Errors in RNT coordinates for PFR simulation ($N = 700$).

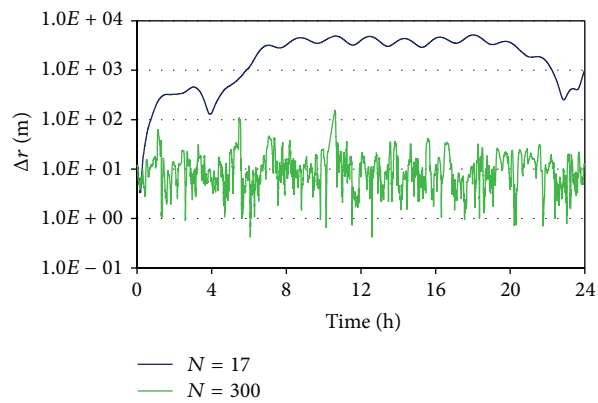


Figure 5: Δr (m) obtained in the PFR simulation.

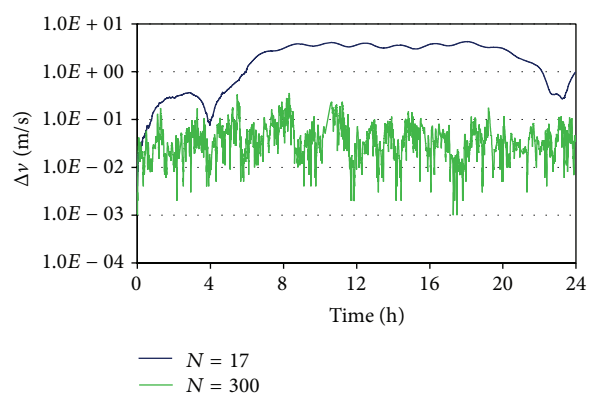


Figure 6: Δv (m/s) obtained in the PFR simulation.

despite an incongruous behavior close to 1 and 5 hours of processing, from which the filters recover later. This anomalous behavior is detected in all simulations results. In the graphics, blue curves correspond to radial component (R); red to normal (N); and green to along-track (T). Increasing N from 300 to 700 was more meaningful to along-track component, as its mean and standard deviation had higher improvement (decreasing in values) and less significant to normal coordinate.

According to Figures 3 and 4, the results for $N = 300$ are as competitive as for $N = 700$. And if computational burden of implementing 700 particles is taken into account, 300 particles are sufficient for the evaluation proposed in this paper.

Despite the undesirable data gap near 5.5 hours of GPS data, in order to properly evaluate the fittest algorithm (PFR) regarding number of particles, two other graphics were generated. Figures 5 and 6 present Δr and Δv, the errors in position and velocity, respectively, for 24 hours of implementation. The results obtained for $N = 300$ (green curves) are compared with $N = 17$ (dark blue curves). According to the results, it is clear that a higher number of particles are important for improving the results, since the amplitude of errors diminished considerably when $N = 300$

was used. And despite the troublesome data, for 300 particles case, PFR can recover and continue the estimation process, without divergence, which is an indication of the particle filter robustness. In each figure, Δr and Δv represent, respectively, the absolute value of the errors in position and in velocity, in the ECEF (Earth-Centered Earth-Fixed) reference frame, also known as ECR (Earth-Centered Rotational).

5. Conclusions

A Bayesian bootstrap particle filter was applied for the satellite orbit determination problem, using a set of GPS measurements. The development was evaluated taking into account performance and computational burden. The bootstrap filter results, implemented with resampling, were compared with two other versions, which include roughening and prior editing, aiming at avoiding sample impoverishment.

With regard to time of processing, results showed that PFPE algorithm requires greater time than PF, which is as competitive as PFR. Since the number of particles is chosen as a trade-off between computational cost and estimation accuracy, the CPU time is related to the variation in number of particles chosen.

When the number of particles used is analyzed, it is also possible to conclude that if it is very small, any sample impoverishment avoidance procedure will not be effectual. And since PFPE approach executes particles edition only once here, it does not present any significant improvement facing the PFR procedure, no matter the number of particles used. Additionally, if the computational costs between PFR and PFPE (as implemented) are compared, the computation burden is enough to contraindicate PFPE implementation as many times as necessary, even if the results are improved.

Results confirm that the greater the number of particles, the better the estimation accuracy. The best result was achieved for $N = 700$, in the three versions of the estimator, although a higher computational effort was demanded in this case. Therefore, when results are compared, it is possible to assure that 300 particles are enough to achieve the accuracy level aimed in this paper.

In order to obtain a better bootstrap particle filter performance, especially in terms of estimation accuracy, adjustments in the many filter variants might be done for improving its efficiency. Such adjustments are directly related to the knowledge about the filter. Other strategies can also be tried for solving implementation issues such as sample impoverishment. Another approach for improving particle filtering is to combine it with another filter such as extended Kalman filter or unscented Kalman filter. In this approach, each particle is updated at the measurement time using the extended or the unscented filter, and then resampling is performed using the measurements.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors wish to express their consideration to the National Institute for Space Research (INPE) that kindly provided everything necessary for this paper to be developed. The authors are also grateful to São Paulo Research Foundation (FAPESP), for the support under Contract no. 2012/21023-6, and to National Council for Scientific and Technological Development (CNPq), for the support under Contract no. 303119/2010-1.

References

[1] N. J. Gordon, D. J. Salmond, and A. F. M. Smith, "Novel approach to nonlinear/non-gaussian Bayesian state estimation," *IEE Proceedings, Part F: Radar and Signal Processing*, vol. 140, no. 2, pp. 107–113, 1993.

[2] D. Simon, *Optimal State Estimation*, Wiley-Interscience/John Wiley & Sons, Hoboken, NJ, USA, 2006.

[3] A. Doucet, N. de Freitas, and N. Gordon, Eds., *Sequential Monte Carlo Methods in Practice*, Springer, New York, NY, USA, 2001.

[4] U. Orguner and F. Gustafsson, "Risk-sensitive particle filters for mitigating sample impoverishment," *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 5001–5012, 2008.

[5] S. Park, J. P. Hwang, E. Kim, and H.-J. Kang, "A new evolutionary particle filter for the prevention of sample impoverishment," *IEEE Transactions on Evolutionary Computation*, vol. 13, no. 4, pp. 801–809, 2009.

[6] N. M. Kwok, G. Fang, and W. Zhou, "Evolutionary particle filter: re-sampling from the genetic algorithm perspective," in *IEEE IRS/RSJ International Conference on Intelligent Robots and Systems (IROS '05)*, pp. 2935–2940, IEEE, August 2005.

[7] T. Li, T. P. Sattar, and S. Sun, "Deterministic resampling: unbiased sampling to avoid sample impoverishment in particle filters," *Signal Processing*, vol. 92, no. 7, pp. 1637–1645, 2012.

[8] F. Daum and J. Huang, "Particle degeneracy: root cause and solution," in *Signal Processing, Sensor Fusion, and Target Recognition XX*, vol. 8050 of *Proceedings of SPIE*, Orlando, Fla, USA, April 2011.

[9] H. Driessen and Y. Boers, "Efficient particle filter for jump Markov nonlinear systems," *IEE Proceedings—Radar, Sonar and Navigation*, vol. 152, no. 5, pp. 323–326, 2005.

[10] P. C. P. M. Pardal, H. K. Kuga, and R. V. Moraes, "Orbit Determination Using Nonlinear Particle Filter and GPS Measurements," in *Proceedings of the AAS/AIAA Astrodynamics Specialist Conference*, vol. 150 of *Advances in the Astronautical Sciences*, pp. 1077–1092, August 2014.

[11] P. S. Maybeck, *Stochastic Models, Estimation, and Control*, vol. 2, Academic Press, New York, NY, USA, 1982.

[12] R. G. Brown and P. Y. C. Hwang, "A Kalman filter approach to precision GPS geodesy," *Navigation*, vol. 30, no. 4, pp. 338–349, 1983.

[13] O. Montenbruck and E. Gill, *Satellite Orbits: Models, Methods, and Applications*, Springer, Berlin, Germany, 2001.

[14] P. C. P. M. Pardal, H. K. Kuga, and R. Vilhena de Moraes, "Non linear sigma point kalman filter applied to orbit determination using GPS measurements," in *Proceedings of the 22nd International Technical Meeting of the Satellite Division of the Institute of Navigation (ION GNSS 2009)*, pp. 2301–2308, Savannah, Ga, USA, September 2009.

[15] W. M. Kaula, *Theory of Satellite Geodesy*, Blasdell, Walthmam, Mass, USA, 1966.

[16] P. W. Binning, "GPS, dual frequency, SA free satellite navigation. Navigation technology for the 3rd millennium," in *Proceedings of the 52nd Annual Meeting of the Institute of Navigation*, pp. 803–812, Cambridge, Mass, USA, June 1996.

[17] J. A. Marshall and S. B. Luthcke, "Modeling radiation forces acting on Topex/Poseidon for precision orbit determination," *Journal of Spacecraft and Rockets*, vol. 31, no. 1, pp. 99–105, 1994.

[18] A. F. B. A. Prado and H. K. Kuga, Eds., *Fundamentos de Tecnologia Espacial*, INPE, São José dos Campos, Brazil, 2001.

[19] T. Guan, Special cases of the three body problem, http://inside.mines.edu/fs_home/tohno/teaching/PH505_2011/Paper_Tianyu-anGuan.pdf.

[20] A. P. M. Chiaradia, H. K. Kuga, and A. F. B. A. Prado, "Single frequency GPS measurements in real-time artificial satellite orbit determination," *Acta Astronautica*, vol. 53, no. 2, pp. 123–133, 2003.

[21] "OSTM/Jason-2 Products Handbook," 2009, ftp://podaac.jpl.nasa.gov/allData/ostm/preview/L2/GPS-OGDR/docs/userhandbook.pdf.

[22] P. C. P. M. Pardal, H. K. Kuga, and R. V. de Moraes, "Comparing the extended and the sigma point kalman filters for orbit determination modeling using GPS measurements," in *Proceedings of the 23rd International Meeting of the Satellite Division of the Institute of Navigation (ION GNSS '10)*, Portland, Ore, USA, September 2010.

[23] P. C. P. M. Pardal, H. K. Kuga, and R. Vilhena de Moraes, "Unscented Kalman filter robustness assessment for orbit determination using GPS signals," *Advances in the Astronautical Sciences*, vol. 148, pp. 117–134, 2013.

[24] P. C. P. M. Pardal, H. K. Kuga, and R. V. de Moraes, "Analyzing the unscented kalman filter robustness for orbit determination through global positioning system signals," *Journal of Aerospace Technology and Management*, vol. 5, no. 4, pp. 395–408, 2013.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 109325, 9 pages
http://dx.doi.org/10.1155/2015/109325

Research Article

On Two-Level State-Dependent Routing Polling Systems with Mixed Service

Guan Zheng,[1] Yang Zhijun,[2] Qian Wenhua,[1] and He Min[1]

[1] *School of Information Science and Technology, Yunnan University, Kunming 650091, China*
[2] *Educational and Scientific Institute, Educational Department of Yunnan Province, Kunming 650223, China*

Correspondence should be addressed to Guan Zheng; gz_627@sina.com

Received 19 March 2015; Accepted 1 September 2015

Academic Editor: Ruihua Liu

Based on priority differentiation and efficiency of the system, we consider an $N + 1$ queues' single-server two-level polling system which consists of one key queue and N normal queues. The novel contribution of the present paper is that we consider that the server just polls active queues with customers waiting in the queue. Furthermore, key queue is served with exhaustive service and normal queues are served with 1-limited service in a parallel scheduling. For this model, we derive an expression for the probability generating function of the joint queue length distribution at polling epochs. Based on these results, we derive the explicit closed-form expressions for the mean waiting time. Numerical examples demonstrate that theoretical and simulation results are identical and the new system is efficient both at key queue and normal queues.

1. Introduction

In this paper, we study a class of $N+1$ queues' polling systems that consists of one key queue, Q_h, and N normal queues, $Q_1, Q_2, \ldots, Q_N$, which are attended by a single server. Studies on the polling systems have attracted extensive attentions in the last years due to their vast area of applications in communication network, production, and transportation. Excellent surveys on polling systems analysis and their applications may be found in [1–4]. However, many studies in the literatures assume that the server visit the queues in a fixed, cyclic order. This might not be a realistic assumption, as queues might have different priority level; queues with high priority should be visit more frequently than the lower ones; sometime queues might be empty and then there is no need to visit. As such, we study the case where the server just visits active queues with customers. Note that as a consequence, after skipping the empty queues, server could provide more visit opportunity to active queues with customers. Furthermore, parallel process of service period and switch-over period allows a successive service between two active queues without the duration of switch-over time. To provide priority differentiation service, queues are separated as one key queue and N normal queues. Two-level route order and mixed service scheme are used to provide high priority to key queue.

It is observed that in the wide body of literature on polling system hardly can any studies be found that take the consideration of queue state-dependent routing and service priority simultaneously. The reason for this may lie in the fact that the analysis of state-dependent routing polling model is much more complex than that of cyclic polling model, especially in priority differentiated model. In particular, waiting time and queue length analysis of two-level priorities polling systems can be found in [5–7], in which the server visits queues in a two-level route; that is, the server polls key queue with exhaustive scheme after each gated service to normal queue [5]. This work is extended in [6] with assigning 1-limited service discipline to normal queues. More recently, Yang et al. set the exhaustive service for normal queue and gated service for key queue to ensure fairness but just acquire the first moment performance of the system as mean queue length at the polling epoch and the mean cyclic time [7]. The parallel discipline is used to improve the delay performance in [8], in which when the current polling queue has customers in storage the server will process service while switching to

the successive queue simultaneously and begins to serve the successor once it finishes the service of the current one. This scheme could improve polling efficiency in high traffic cases. However, the parallel mechanism will be invalid when there is no customer in the queue. In low traffic cases, useless polling to idle queue becomes an obvious liability in cyclic polling model. Routing depends on the event whether a queue is empty or it is not helpful to this problem [9]. In this paper, we consider the special setting to a two-level mixed service polling model, where the key queue is served exhaustively while normal queues are served in 1-limited mechanism. Furthermore, the server no longer checks all the stations in a fixed order; only active stations with transfer requirements could be served and then the switch-over period and service period are processed paralleled. This mechanism increases the system utilization and reduces the mean waiting time.

Although the exhaustive service discipline in principle fits the branching property, the present model involves 1-limited service discipline, which does not satisfy the above-mentioned branching property. The explicit analysis of non-branching service disciplines is mostly in special setting, such as [10, 11] studied on two-queue polling systems and [12] studied on symmetric 1-limited model. In this paper, we follow the special setting in [8] and analyze the mean waiting time of the present model under the assumption on the symmetrical characteristic among normal queues, as will be described in greater detail in Section 2.

Initially, we follow an approach similar to the analysis of [5], which uses a recursive iteration of a functional equation, for the probability generating function (PGF) of the joint queue-length distribution at moments the server starts a visit period.

The main contributions of this paper can be summarized as follows. Firstly, we extend the parallel two-level poling system in [8] by using queue state-dependent routing, in which only active queues with customers could be visited by server. This scheme is helpful to avoid the consumptions induced by idle visit. Secondly, under the assumption of a stable system, we obtain the explicit expressions for the PGF for the joint queue length distribution at polling epochs as a starting point of key queue and normal queue separately. Thirdly, we achieve the exact closed-form expression of the mean waiting time under the assumption on the symmetrical characteristic of normal queue.

The rest of the paper is structured as follows. In Section 2, we give a formal description of the polling model that we study and we introduce the necessary notation. Based on this, in Section 3, we derive the expressions for the mean waiting time of the present model under the assumption of a semisymmetric (symmetrical characteristic of normal queue) stable system, by taking a functional equation for the PGF for the joint queue length distribution at polling epochs as a starting point. In Section 4, numerical results obtained with the proposed analytical models are shown and their very good agreement with realistic simulation results is discussed. Finally, concluding remarks and directions for future research are given in the end.

2. Model Description

Consider a discrete time (timeline is divided into time slot) polling system consisting of N ($N \geq 2$) infinite-buffer queues $Q_1, Q_2, \ldots, Q_N$, and Q_h. The single server visits active queues in a two-level state-dependent routing order and serves the customers with mixed service discipline.

In the arrival process, type-j ($j = 1, 2, \ldots, N, h$) customers arrive at Q_j according to an independent Poisson arrival process. The generating function of arrival process in queue j is $A_j(z_j)$, with the variance of $\sigma_{\lambda j}^2 = A_j''(1) + \lambda_j - \lambda_j^2$ and the arrival rate of $\lambda_j = A_j'(1)$. The total arrival rate is $\sum_{i=1}^{N} \lambda_i + \lambda_h$.

In the *service process*, we assume that customers in queue j ($j = 1, 2, \ldots, N, h$) receive individual service. The service time of a customer at each queue is independent of each other. Their generating function is $B_j(z_j)$, with the variance of $\sigma_{\beta j}^2 = B_j''(1) + \beta_j - \beta_j^2$ and the mean value $\beta_j = B_j'(1)$. We propose a two-level server routing make the high priority queue be visited more frequently than others and add mix-service discipline to ensure the high priority of Q_h. The load offered to Q_j is $\rho_j = \lambda_j \beta_j$, and the total offered load is equal to $\sum_{i=1}^{N} \rho_i + \rho_h$.

State-Dependent Routing. Queues are partitioned as active queue and idle queue by their buffer condition. Only active queues with customers waiting in the buffer could be visited by the server in order. Idle queue with empty buffer would be skipped in the current polling round.

Two-Level Polling. The server visits queues governed by a two-level routing. In the first polling level, the server polls between the high priority queue Q_h and an active normal queue; in the second level, for each time after the exhaustive service at Q_h, one normal active queue is visited in a cyclic order; that is, the server routing in this model is $1 \to h \to \cdots \to i \to h \to i+1 \to \cdots \to h \to N$.

In the *switch-over process*, a parallel mechanism is used. When the server polls an active queue at time with customers in its buffer, the server will provide service and inquire the next active queue simultaneously and then switch to serve the successor immediately without the switch-over time once it has finished the current service. Combined with the state-dependent routing scheme, over the course of a visit period, the server serves the active queues and normal queue in sequence continuously until the entire system is empty; there will be no consumption of switch-over time anymore in the present model. More especially, we assume the server consume one time slot to confirm the system state when the system is entirely empty.

Mix-Service Discipline. Exhaustive discipline is specified for the key queue and 1-limited discipline for normal queues, so that the entire customers in the key queue could be served in the present server round, while those who are in normal queues might need several rounds when there are more than one customer in the buffer. Let F_h denote the duration of a service period for the customers arrive during arbitrary time slot in Q_h. This service period consists of the services of its

ancestral customers arriving during the exact slot and the services of the offspring line of the ancestral customers [13]. The generating function of F_h is denoted by $F_h(z_h) = E[z_h^{F_h}]$. Such a functional equation has already been derived in [14] as $F_h(z_h) = A_h(B_h(z_h F(z_h)))$.

In the remainder of this paper, we are interested in the queue length distributions at the polling epoch of Q_i and Q_h. Let $\xi_j(n)$ denote the number of customers present at Q_j at t_n when the server starts a visit period at Q_i, and let $\xi_j(n^*)$ denote the number of customers present at Q_j at t_n^* when the server starts a visit period at Q_h successively with the service of Q_i. The joint distribution of $\xi_j(n+1)$ and $\xi_j(n^*)$ is represented by the N-dimensional PGF $G_{i+1}(z_1, \ldots, z_N, z_h)$ and $G_{ih}(z_1, \ldots, z_N, z_h)$.

We analyze the system under stability conditions ($\sum_{i=1}^{N} \rho_i + \rho_h < 1$) [12]. Normal queues in the present model are served in a 1-limited manner, which does not satisfy the well-known branching property in polling systems. Therefore, more specifically, in the analyses of mean waiting time, we assume the normal queues are symmetric; that is, normal queues have the same customer arrival rate and service rate.

3. Analysis for Steady-State Systems

In this section, we derive explicit expression for the joint queue length distribution. In Section 3.1, we first obtain expressions for $G_{i+1}(z_1, \ldots, z_N, z_h)$ and $G_{ih}(z_1, \ldots, z_N, z_h)$, the joint queue length PGF at the polling epoch at Q_{i+1} and Q_h. These results ultimately lead in Section 3.2 to the first and second moment of the PGF, and obtain the expressions for $E[W_i]$ and $E[W_h]$, the mean waiting time of type-i and type-h customers that arrive at an arbitrary point in time.

3.1. Joint Queue Length Distribution at Polling Epoch. Assuming that the server begin the service of Q_i at t_n, define a random variable $\xi_j(n)$ as the number of type-j ($j = 1, 2, \ldots, N, h$) customers at time t_n. Then the status of the entire polling model at time t_n can be represented as $\{\xi_1(n), \ldots, \xi_N(n), \xi_h(n)\}$. Denote $\xi_j(n+k)$ as the number of type-j customers at t_{n+k}, the polling epoch of Q_{i+k}. The status of the entire polling model at time t_{n+k} can be represented as $\{\xi_1(n+k), \ldots, \xi_N(n+k), \xi_h(n+k)\}$ while $\xi_i(n^*)$ is the number of type-j customers in at time t_n^*, at which the server begins providing service to Q_h and the status of the entire polling model at time t_n^* can be represented as $\{\xi_1(n^*), \ldots, \xi_N(n^*), \xi_h(n^*)\}$. Under the necessary and sufficient condition for the stability of the system $\sum_{i=1}^{N} \rho_i + \rho_h < 1$, the probability distribution is defined as

$$\begin{aligned} &\lim_{n\to\infty} P\left[\xi_j(n) = x_j;\ j = 1, \ldots, N, h\right] \\ &\quad = \pi_i(x_1, \ldots, x_N, x_h), \\ &\lim_{n\to\infty} p\left[\xi_j(n^*) = y_j;\ j = 1, \ldots, N, h\right] \\ &\quad = \pi_{ih}(y_1, \ldots, y_N, y_h). \end{aligned} \tag{1}$$

The generating functions at t_n and t_n^* are

$$\begin{aligned} &G_i(z_1, \ldots, z_N, z_h) \\ &\quad = \sum_{x_1=0}^{\infty} \cdots \sum_{x_N=0}^{\infty} \sum_{x_h=0}^{\infty} z_1^{x_1} \cdots z_N^{x_n} z_h^{x_h} \pi_i(x_1, \ldots, x_N, x_h) \\ &\qquad\qquad i = 1, 2, \ldots, N, \\ &G_{ih}(z_1, \ldots, z_N, z_h) \\ &\quad = \sum_{y_1=0}^{\infty} \cdots \sum_{y_N=0}^{\infty} \sum_{y_h=0}^{\infty} z_1^{y_1} \cdots z_N^{y_n} z_h^{y_h} \pi_{ih}(y_1, \ldots, y_N, y_h) \\ &\qquad\qquad i = 1, 2, \ldots, N. \end{aligned} \tag{2}$$

According to the proposed mechanism, the system variables have the following equations. When the server begins the service on Q_{i+1} at t_{n+1}, we have

$$\xi_j(n+1) = \begin{cases} \xi_j(n^*) + \eta_j(\nu_h) & j \neq h \\ 0 & j = h. \end{cases} \tag{3}$$

$\nu_j(n)$ is the service time in Q_j and $\eta_k(\nu_j)$ is the number of arrivals to Q_k during $\nu_j(n)$.

The server just finishes the service of Q_h in an exhaustive manner and starts the polling on Q_{i+1} at t_{n+1}. Such a functional equation of exhaustive service has already been derived in [12]. Applying these results to our case, we obtain

$$\begin{aligned} G_{i+1}(z_1, z_2, \ldots, z_N, z_h) &= \lim_{n\to\infty} E\left[\prod_{j=1}^{N} z_j^{\xi_i(n+1)} z_h^{\xi_h(n+1)}\right] \\ &= G_{ih}\Bigg(z_1, z_2, \ldots, z_N, \\ &\quad B_h\left(\prod_{j=1}^{N} A_j(z_j) F_h\left(\prod_{j=1}^{N} A_j(z_j)\right)\right)\Bigg). \end{aligned} \tag{4}$$

The expression can be interpreted as follows. At the start of the visit period at Q_{i+1}, type-i customers are those at the polling epoch of Q_h plus the new customers arriving at each queue during the service period of the Q_h in exhaustive scheme, and no type-h customer resumes at that moment.

When the server begins the service on Q_h at t_n^*, we have

$$\begin{aligned} \xi_j(n^*) &= \begin{cases} \xi_j(n) + \eta_j(\nu_i), & j \neq i \neq h, \\ \xi_i(n) + \eta_i(\nu_i) - 1, & j = i \quad \xi_i(n) \neq 0, \\ \eta_j(\nu_i), & j = h, \end{cases} \\ \xi_j(n^*) &= \begin{cases} \xi_j(n), & j \neq i \neq h, \\ 0, & j = i \quad \xi_i(n) = 0, \\ 0, & j = h \end{cases} \end{aligned} \tag{5}$$

$\nu_j(n)$ is the service time in Q_j, and $\eta_k(\nu_j)$ is the number of arrivals to Q_k during $\nu_j(n)$.

In our case, for normal queues, the server just polls the active queues with customers in parallel 1-limited manner. To gain more insight in the state-dependent service discipline, let P_i denote the queue length at the service epoch in an M/G/1 queue with the same arrival process and service-time distribution as Q_i. We assume that the k customers have waited in Q_i at the start of the busy period with probability $p_k \in [0,1)$, $\sum_{k=0}^{\infty} p_k = 1$. Then we can acquire the queue length generating function at the service epoch as $P_i(z_i) = A_i(z_i)\sum_{k=0}^{\infty} p_k z_i^k$, where $A_i(z_i)$ is the PGF of the arrival process as defined in Section 2. Specifically, the server does not provide service when the queue length is zero, so we assume that k^* customers resumed after the end of the busy time in 1-limited service with the probability of $p_k^* \in [0,1)$, and $p_k^* = p_k + 1$ for $k = 0, 1, \ldots$. Consequently, the probability space could be rebuilt as

$$\begin{aligned} P_i^*(z_i) &= B_i(A_i(z_i))\, A_i(z_i)\left(p_0 + \sum_{k=0}^{\infty} p_k^* z_i^k\right) \\ &= B_i(A_i(z_i))\left(\frac{\sum_{k=0}^{\infty} p_k z_i^k - p_0 z_i^0}{z_i} + p_0 z_i^0\right). \end{aligned} \tag{6}$$

With the definition of $P_i(z_i)$, we have

$$\begin{aligned} P_i^*(z_i) = B_i(A_i(z_i))\, &\frac{\left(P_i(z_i) - P_i(z_i)|_{z_i=0}\right)}{z_i} \\ &+ P_i(z_i)|_{z_i=0}. \end{aligned} \tag{7}$$

Applying these results to our case, we obtain

$$\begin{aligned} G_{ih}(z_1,\ldots,z_N,z_h) &= \lim_{n\to\infty} E\left[\prod_{j=1}^{N} z_j^{\xi_i(n^*)} z_h^{\xi_h(n^*)}\right] = \frac{1}{z_i} \\ &\cdot B_i\left(\prod_{j=1}^{N} A_j(z_j)\, A_h(z_h)\right) \\ &\cdot \left[G_i(z_1,\ldots,z_N,z_h) - G_i(z_1,\ldots,z_N,z_h)|_{z_i=0}\right] \\ &+ G_i(z_1,\ldots,z_N,z_h)|_{z_i=0} \\ &- G_i(z_1,\ldots,z_N,z_h)|_{z_1,\ldots,z_N,z_h=0} \\ &+ \prod_{j=1}^{N} A_j(z_j)\, A_h(z_h)\, G_i(z_1,\ldots,z_N,z_h)|_{z_1,\ldots,z_N,z_h=0}. \end{aligned} \tag{8}$$

The expression can be interpreted as follows. At the start of the visit period at Q_h, in the case that the former Q_i is active, one type-i customer would have been served at t_n^* and new customers arrived at each queue during the service period of the exact type-i customer. The server would skip Q_i to Q_{i+1} when Q_i is empty; in that case, the distribution of the number of customers in the systems is represented by the generating function $G_i(z_1,\ldots,z_i,\ldots,z_{N-1},z_h)|_{z_i=0}$, with the exception as the system is entirely empty, which is represented by the generating function $G_i(z_1,\ldots,z_i,\ldots,z_{N-1},z_h)|_{z_1,\ldots,z_{N-1},z_h=0}$. When the system is entirely empty, the server will stop providing service for one time slot until new customers arrive during this time slot, and this number of customers is represented by the last partition of the addition formula.

3.2. Expression for the Mean Waiting Time. Now we have derived expressions for the PGF $G_{i+1}(z_1,\ldots,z_i,\ldots,z_N,z_h)$ and $G_{ih}(z_1,\ldots,z_i,\ldots,z_N,z_h)$ pertaining to the queue length at polling epoch of Q_{i+1} and Q_h, we use these results to obtain $E[W_i]$, the mean waiting time of type-i normal customers, and $E[W_h]$, the mean waiting time of type-h high priority customers.

3.2.1. The First and Second Moment of $G_{i+1}(z_1,\ldots,z_i,\ldots,z_N,z_h)$ and $G_{ih}(z_1,\ldots,z_i,\ldots,z_N,z_h)$. To start the analysis of mean waiting time of type-j customers, we need to calculate the generating functions and its derivation at the point $\mathbf{z} = \mathbf{1}$, $\mathbf{z}$ is the abbreviation of the $(1 \times N + 1)$ vector of $(z_1,\ldots,z_i,\ldots,z_N,z_h)$, and $\mathbf{1}$ is the $(1 \times N + 1)$ vector with 1.

$G_{i+1}(\mathbf{z})$ is the PGF of the joint queue length at the polling epoch of Q_i, so we have

$$\begin{aligned} &G_i(z_1,\ldots,z_i,\ldots,z_N,z_h) \\ &= \sum_{x_1=0}^{\infty}\cdots\sum_{x_i=0}^{\infty}\cdots\sum_{x_N=0}^{\infty}\sum_{x_h=0}^{\infty} z_1^{x_1}\cdots z_i^{x_i}\cdots z_N^{x_n} z_h^{x_h} P(\xi_1(n)=x_1,\ldots,\xi_i(n)=x_i,\ldots,\xi_N(n)=x_N,\xi_h(n)=x_h) \\ &= \sum_{x_1=0}^{\infty}\cdots\sum_{x_i=0}^{\infty}\cdots\sum_{x_N=0}^{\infty}\sum_{x_h=0}^{\infty} z_1^{x_1}\cdots z_i^{x_i}\cdots z_N^{x_n} z_h^{x_h} P(\xi_1(n)=x_1,\ldots,\xi_N(n)=x_N,\xi_h(n)=x_h \mid \xi_i(n)=x_i)\, P(\xi_i(n)=x_i). \end{aligned} \tag{9}$$

Taking the kth derivative with respect to z_i yields

$$\begin{aligned} \frac{\partial^k G_i(z_1,z_2,\ldots,z_i,\ldots,z_N,z_h)}{\partial z_i^k} &= \sum_{x_1=0}^{\infty}\cdots\sum_{x_N=0}^{\infty}\sum_{x_h=0}^{\infty} z_1^{x_1}\cdots z_i^{x_i-k}\cdots z_N^{x_N} z_h^{x_h} \\ &\cdot \frac{x_i!}{(x_i-k)!} P(\xi_1(n)=x_1,\ldots,\xi_N(n)=x_N,\xi_h(n)=x_h \mid \xi_i(n)=x_i)\, P(\xi_i(n)=x_i). \end{aligned} \tag{10}$$

Setting $z_i = 0$ yields

$$\begin{aligned}
&\left.\frac{\partial^k G_i(z_1, z_2, \ldots, z_i, \ldots, z_N, z_h)}{\partial z_i^k}\right|_{z_i=0} \\
&= \sum_{x_1=0}^{\infty} \cdots \sum_{x_N=0}^{\infty} \sum_{x_h=0}^{\infty} z_1^{x_1} \cdots 1 \cdots z_N^{x_N} z_h^{x_h} k! P(\xi_1(n) = x_1, \ldots, \xi_N(n) = x_N, \xi_h(n) = x_h \mid \xi_i(n) = k) P(\xi_i(n) = k) \\
&= k! P(\xi_i(n) = k) \sum_{x_1=0}^{\infty} \cdots \sum_{x_N=0}^{\infty} \sum_{x_h=0}^{\infty} z_1^{x_1} \cdots 1 \cdots z_N^{x_N} z_h^{x_h} P(\xi_1(n) = x_1, \ldots, \xi_N(n) = x_N, \xi_h(n) = x_h \mid \xi_i(n) = k) \\
&= k! P(\xi_i(n) = k) E\left[z_1^{\xi_1(n)} \cdots 1 \cdots z_N^{\xi_N(n)} z_h^{\xi_h(n)} \mid \xi_i(n) = k\right].
\end{aligned} \tag{11}$$

Rearranging terms and setting $k = 0$, we have

$$\begin{aligned}
G_i(z_1, z_2, \ldots, z_i, \ldots, z_N, z_h)|_{z_i=0} &= P(\xi_i(n) = 0) \\
&\cdot E\left[z_1^{\xi_1(n)} \cdots 1 \cdots z_N^{\xi_N(n)} z_h^{\xi_h(n)} \mid \xi_i(n) = 0\right], \\
G_i(\mathbf{1}_i) &= P(\xi_i(n) = 0).
\end{aligned} \tag{12}$$

Extending this result we have

$$\begin{aligned}
G_i(\mathbf{0}) &= P\{\xi_1(n) = 0, \ldots, \xi_i(n) = 0, \ldots, \xi_N(n) \\
&= 0, \ \xi_h(n) = 0\}.
\end{aligned} \tag{13}$$

$\mathbf{0}$ is the $(1 \times N + 1)$ vector with 0, and $\mathbf{1}_j$ is the $(1 \times N + 1)$ vector with 0 in jth position and 1 in all other entries.

Define the first derivative of $G_i(\mathbf{z})$ and $G_{ih}(\mathbf{z})$ at $\mathbf{z} = \mathbf{1}$ as

$$\begin{aligned}
g_i(j) &= \lim_{z_1, \ldots, z_i, \ldots, z_N, z_h \to 1} \frac{\partial G_i(\mathbf{z})}{\partial z_j}, \\
g_{ih}(j) &= \lim_{z_1, \ldots, z_i, \ldots, z_N, z_h \to 1} \frac{\partial G_{ih}(\mathbf{z})}{\partial z_j}, \\
&\qquad j, k = 1, 2, \ldots, N, h.
\end{aligned} \tag{14}$$

$$g_{ih}(j) = \beta_i \lambda_j [1 - G_i(\mathbf{1}_i)] + g_i(j) + \lambda_j G_i(\mathbf{0}) \tag{15}$$

$$g_{ih}(i) = (\beta_i \lambda_i - 1)[1 - G_i(\mathbf{1}_i)] + g_i(i) + \lambda_i G_i(\mathbf{0}) \tag{16}$$

$$g_{ih}(h) = \beta_i \lambda_h [1 - G_i(\mathbf{1}_i)] + \lambda_h G_i(\mathbf{0}) \tag{17}$$

$$g_{i+1}(i) = g_{ih}(i) + g_{ih}(h) \beta_h \lambda_i \left(1 + F_h'(1)\right) \tag{18}$$

$$g_{i+1}(j) = g_{ih}(j) + g_{ih}(h) \beta_h \lambda_j \left(1 + F_h'(1)\right). \tag{19}$$

Calculate $\sum_{j=1}^{N} g_{j+1}(k)$ yields

$$1 - G_i(\mathbf{1}_i) = \frac{N \lambda_i G_i(\mathbf{0})}{1 - \rho_h - N\rho}. \tag{20}$$

Define the second derivative of $G_i(\mathbf{z})$ and $G_{ih}(\mathbf{z})$ at $\mathbf{z} = \mathbf{1}$ as

$$\begin{aligned}
g_i(j, k) &= \lim_{z_1, \ldots, z_i, \ldots, z_N, z_h \to 1} \frac{\partial^2 G_i(\mathbf{z})}{\partial z_j \partial z_k} \\
g_{i0}(j, k) &= \lim_{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N, z_h \to 1} \frac{\partial^2 G_i(\mathbf{z})|_{z_i=0}}{\partial z_j \partial z_k} \\
g_{i00}(j, k) &= \lim_{z_1, \ldots, z_N, z_h \to 0} \frac{\partial^2 G_i(\mathbf{z})|_{z_1, \ldots, z_N, z_h=0}}{\partial z_j \partial z_k} \\
g_{ih}(j, k) &= \lim_{z_1, \ldots, z_N, z_h \to 1} \frac{\partial^2 G_{ih}(\mathbf{z})}{\partial z_j \partial z_k} \\
&\qquad i = 1, 2, \ldots, N \ \ j, k = 1, 2, \ldots, N, h.
\end{aligned} \tag{21}$$

Substitute (4) and (8) into the above second derivative formulas.

We assume the N normal queues are symmetrical; that is, $\lambda_i = \lambda$, $\beta_i = \beta$, $i = 1, 2, \ldots, N$. Then simplifying these we get the second derivative of $G_i(\mathbf{z})$ and $G_{ih}(\mathbf{z})$ at $\mathbf{z} = \mathbf{1}$ as follows:

$$\begin{aligned}
g_{ih}(h, h) &= B''(1) \lambda_h^2 (1 - G_i(\mathbf{1}_i)) + \beta A_h''(1)(1 \\
&- G_i(\mathbf{1}_i)) + A_h''(1) G_i(\mathbf{0}).
\end{aligned} \tag{22}$$

$$\begin{aligned}
g_i(i) &= \frac{1 - G_i(\mathbf{1}_i)}{2} \left\{ \frac{1}{(1 - \rho_h - N\rho)(1 - \rho_h)} \left[\rho_h \right.\right. \\
&\cdot \frac{A''(1)}{\lambda} \left(1 - \rho_h^2 + A_h'' \beta_h^2 + \lambda_h B_h''\right) + N B''(1) \lambda^2 \\
&\left.\left. + N\beta A''(1) \right] + \frac{\lambda}{(1 - \rho_h)} + \frac{\lambda_h \lambda B_h''(1)}{1 - \rho_h - N\rho} \right\} + 1 \\
&- G_i(\mathbf{1}_i).
\end{aligned} \tag{23}$$

Remark 1. Though $g_i(i)$ is the first derivative at $\mathbf{z} = \mathbf{1}$ $G_i(\mathbf{z})$ in definition, it is clear that it contains the second moment parameter as $A_j''(1)$ and $B_j''(1)$. So, $g_i(i)$ is a second moment parameter for the system performance.

3.2.2. Analysis of $E[W_h]$ and $E[W_i]$. Define W_h and W_i as the waiting time of type-h and type-i customers, which denotes the time from the epoch when a customer arrives at the queue to the time it is served. In the present model, high priority type-h customers are served in the exhaustive service and normal type-i customers are served in 1-limited service. Based on the related research works in [14], the mean waiting time of type-h customers $E[W_h]$ and the type-i customers $E[W_i]$ can be calculated as follows:

$$E[W_h] = \frac{g_{ih}(h,h)}{2\lambda_h g_{ih}(h)} - \frac{A_h''(1)}{2\lambda_h^2(1+\rho_h)} + \frac{\lambda_h B_h''(1)}{2(1-\rho_h)}, \quad (24)$$

$$E[W_i] = \frac{1}{\lambda(1-G_i(\mathbf{1}_i))} g_i(i) - \frac{1}{\lambda} - \frac{A''(1)}{2\lambda^2}. \quad (25)$$

Taking (17), (22) in (24) in the above expressions, we have

$$E[W_h] = \frac{1}{2(1-\rho_h)}\left(1-\rho_h + N\lambda B''(1) + \lambda_h B_h''(1)\right) - \frac{1}{2{\lambda_h}^2(1+\rho_h)} A_h''(1). \quad (26)$$

Taking (17), (22), and (23) in (25) in the above expressions, we have

$$E[W_i] = \frac{1}{2\lambda}\left\{\frac{1}{(1-\rho_h-N\rho)(1-\rho_h)}\left[\rho_h \cdot \frac{A''(1)}{\lambda_i}\left(1-\rho_h^2 + A_h''\beta_h^2 + \lambda_h B_h''\right) + NB''(1)\lambda^2 + N\beta A''(1)\right] + \frac{\lambda}{(1-\rho_h)} + \frac{\lambda_h \lambda B_h''(1)}{1-\rho_h-N\rho}\right\} - \frac{A''(1)}{2\lambda^2}. \quad (27)$$

4. Numerical Study

In this section we study the accuracy of the theoretical analysis and compare the mean waiting time of the present model with two existing two-level polling models. Consider an $N+1$ queues' model with one high priority queue Q_h and N normal queues Q_i ($i = 1, \ldots, N$) defined as follows: the service times of all customers are exponentially distributed with mean β in Q_i and β_h in Q_h. The arrival processes are Poisson process with rate λ in Q_i and λ_h in Q_h. The relative parameter values are listed in Table 1, in which $\{a : k : b\}$ means the parameter is varied between a and b in steps of k.

From Figure 1, we can clearly see that, firstly, the theoretical value and the simulation result coincided with each other. Secondly, when the total offered load grew with the arrival rate, service time, and the number of queues, with the mean waiting time increasing distinctly in Q_i, while the performances in Q_h are much better, both queue and mean waiting time are much lower than normal queues, and the growth in Q_h with the total offered load presents much more smoothly.

It is worth considering whether the state-dependent mechanism improves the performance of the system comparing with the existing two-level polling systems. In order to answer this question, we compare a classical two-level system with switch-over time [6], abbreviated as classical system and a parallel two-level system [8], abbreviated as parallel system in Figure 2. The service discipline in the comparisons is 1-limited service for normal queues and exhaustive service for the key queue. Overall models have the same test bed as shown in Table 1. We just vary the working mechanism.

Figure 2 shows the mean waiting time of normal queues in (a) and mean waiting time of key queue in (b). Comparing with the forgoing, the state-dependent system achieves a better performance in delay guarantee and stability. It is clear in Figure 2(a), for lower load, in most of the cases, that there is no customer in the buffers; thus a switch-over time is necessary when the server switches between Q_i and Q_h in the classical and parallel system, while the empty queues would be skipped in the present model. Therefore, customers in the state-dependent system achieve a lower mean waiting time, which is under 20% of the forgoing. In the heavy traffic, the server could not provide service in the necessary switch-over time for the classical system; consequently, it becomes unstable when the arrival rate of Q_i grows over 0.06 in this case. The parallel system and the state-dependent system have better performance in system stability; especially in state-dependent system, the mean waiting time of the normal customers has less than 50% of which in the parallel system. A conclusion can be drawn from a comparison between Figures 2(a) and 2(b), which is that for all three two-level models the mean waiting time of the customers in key queue is significantly lower than that in normal queues, and as illustrated in Figure 2(b), the mean waiting time for h-type customers in state-dependent system is lower than that of the others.

5. Conclusion

When comparing the model of the present paper with the existing literature, the contribution of the present paper is twofold. One of the most striking differences is the queues which are partitioned as active queue and idle queue by their buffer condition, and only active queues with customers waiting in the buffer could be visited by the server in a two-level order. As illustrated in the numerical example, both i-type customers in normal queues and h-type customers in key queue acquire better delay performance than those in systems without queue-stated differentiation. Another notable contribution of the paper is that we achieve the closed-form exact expressions of the mean waiting time for customers in normal queues and key queue, under the assumption of the symmetric of normal queues. The total unknowns in these equations are all first moments of random variables and, thus, no correlation terms are required.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

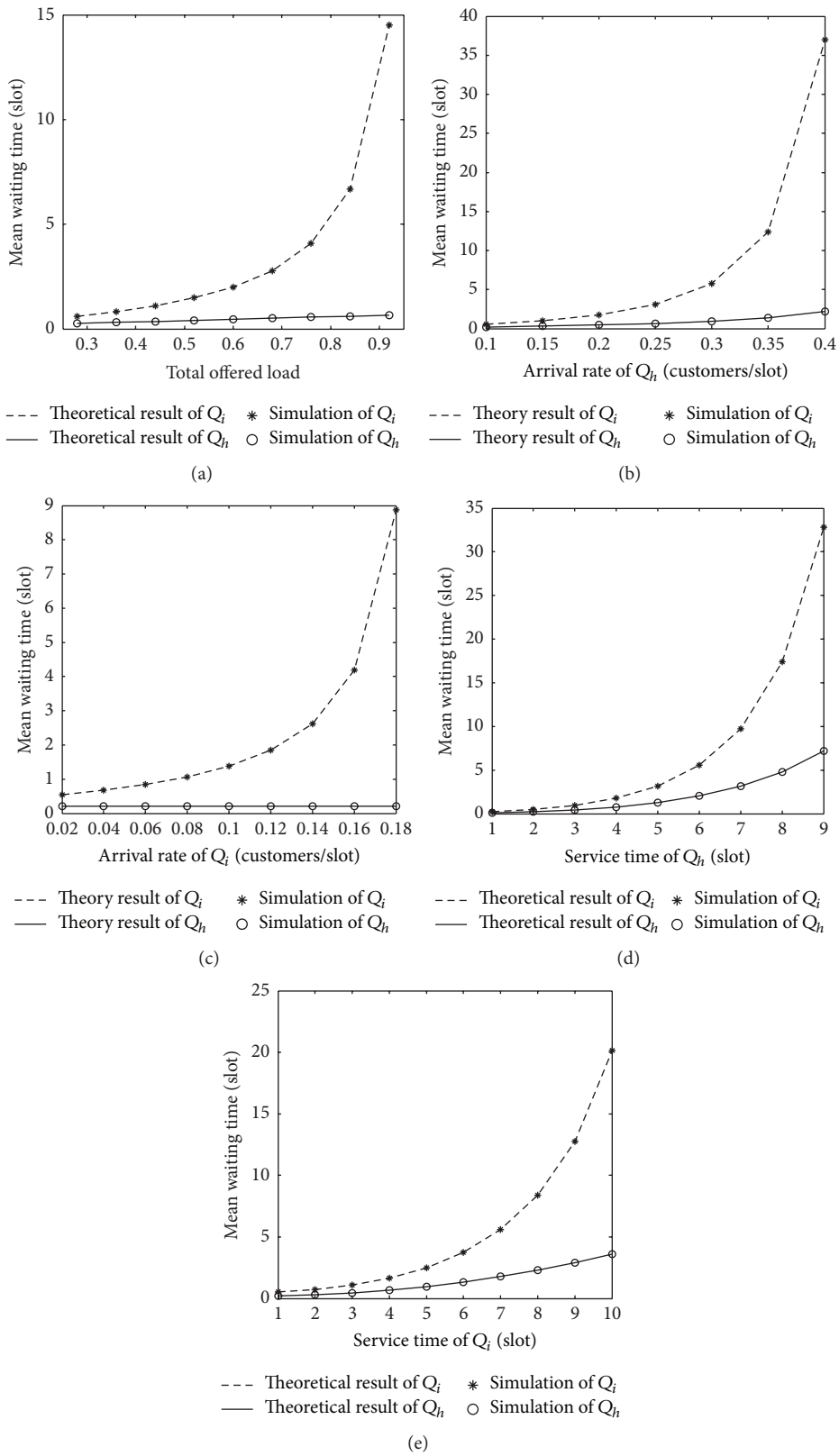


FIGURE 1: Theoretical and simulation values of $E[W_h]$ and $E[W_i]$ from different values of the load increasing with the increasing of the number of normal queues. (a) is the total offered load increasing with the growth of the number of normal queues. (b) is the total offered load increasing with the growth of the arrival rate of Q_h. (c) is the total offered load increasing with the growth of the arrival rate of Q_i. (d) is the total offered load increasing with the growth of the service time of Q_h. (e) is the total offered load increasing with the growth of the service time of Q_i.

Table 1: Test bed used to compare the mean waiting time.

Parameter	Number of normal queues	Arrival rate		Service time		Switch over time
Notation value	N	λ	λ_h	β	β_h	γ
Figure 1(a)	$\{1:1:9\}$	0.04	0.1	2	2	
Figure 1(b)	4	0.02	$\{0.1:0.05:0.4\}$	1	2	—
Figure 1(c)	4	$\{0.02:0.02:0.18\}$	0.1	1	2	—
Figure 1(d)	4	0.02	0.1	2	$\{1:1:9\}$	—
Figure 1(e)	4	0.02	0.1	$\{1:1:10\}$	2	—
Figure 2	4	$\{0.01:0.01:0.09\}$	0.1	2	2	1

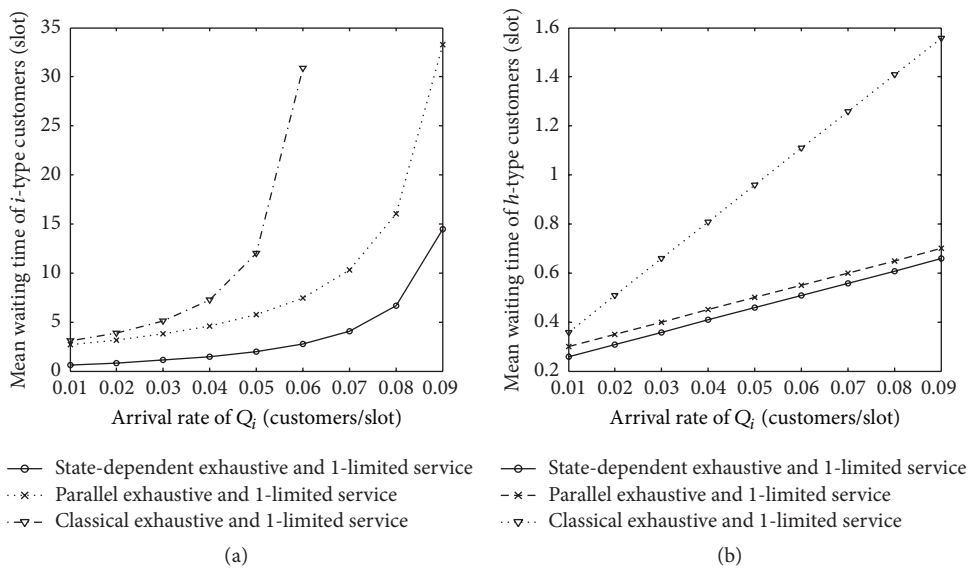


Figure 2: Comparing of mean waiting time among the classical two-level system [6], the parallel two-level system [8], and the state-dependent two-level system. (a) is the theoretical value comparison of $E[W_i]$ with the growth of the arrival rate in Q_i. (b) is the theoretical value comparison of $E[W_h]$ with the growth of the arrival rate in Q_i.

Acknowledgments

This work was supported by a Grant from the National Science Foundation of China (nos. 61463051 and 61463054), the National Science Foundation of Yunnan Province (no. 2012FD002), and the Science Foundation of Yunnan Provincial Department (no. 2014Z010).

References

[1] H. Takagi, "Analysis of polling systems," *Performance Evaluation*, vol. 5, no. 3, p. 206, 1985.

[2] H. Levy and M. Sidi, "Polling systems: applications, modeling, and optimization," *IEEE Transactions on Communications*, vol. 38, no. 10, pp. 1750–1760, 1990.

[3] M. A. A. Boon, R. D. van der Mei, and E. M. M. Winands, "Applications of polling systems," *Surveys in Operations Research and Management Science*, vol. 16, no. 2, pp. 67–82, 2011.

[4] V. M. Vishnevskii and O. M. Semenova, "Mathematical models to study the polling systems," *Automation and Remote Control*, vol. 67, no. 2, pp. 173–220, 2006.

[5] Z.-J. Yang, D.-F. Zhao, H.-W. Ding, and Y.-F. Zhao, "Research on two-class priority based polling system," *Acta Electronica Sinica*, vol. 37, no. 7, pp. 1452–1456, 2009.

[6] Q. Liu, D. Zhao, and D. Zhou, "An analytic model for enhancing IEEE 802.11 point coordination function media access control protocol," *European Transactions on Telecommunications*, vol. 22, no. 6, pp. 332–338, 2011.

[7] Z.-J. Yang, H.-W. Ding, and C.-L. Chen, "Research on E(x) characteristics of two-class polling system of exhaustive-gated service," *Tien Tzu Hsueh Pao/Acta Electronica Sinica*, vol. 42, no. 4, pp. 774–778, 2014 (Chinese).

[8] Z. Guan, D. Zhao, and Y. Zhao, "A discrete time two-level mixed service parallel polling model," *Journal of Electronics*, vol. 29, no. 1-2, pp. 103–110, 2012.

[9] J. P. Dorsman, O. J. Boxma, and R. D. van der Mei, "On two-queue Markovian polling systems with exhaustive service," *Queueing Systems*, vol. 78, no. 4, pp. 287–311, 2014.

[10] M. A. Boon, I. J. Adan, and O. J. Boxma, "A two-queue polling model with two priority levels in the first queue," *Discrete Event Dynamic Systems*, vol. 20, no. 4, pp. 511–536, 2010.

[11] Z. Dongfeng, L. Bihai, and Z. sumin, "Study of a polling systems with limited service," *Journal of Electronics*, vol. 19, no. 1, pp. 44–49, 1997 (Chinese).

[12] O. C. Ibe and X. Cheng, "Stability conditions for multi-queue systems with cyclic service," *IEEE Transactions on Automatic Control*, vol. 33, no. 1, pp. 102–103, 1988.

[13] O. J. Boxma and W. P. Groenendijk, "Pseudoconservation laws in cyclic-service systems," *Journal of Applied Probability*, vol. 24, no. 4, pp. 949–964, 1987.

[14] D. Zhao and S. Zheng, "Analysis on a polling model with exhaustive service," *Acta Electronica Sinica*, vol. 22, no. 5, pp. 102–107, 1994 (Chinese).

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 730618, 7 pages
http://dx.doi.org/10.1155/2015/730618

Research Article

The Impact of Aging Agricultural Labor Population on Farmland Output: From the Perspective of Farmer Preferences

Guancheng Guo, Qiyu Wen, and Jingjuan Zhu

College of Public Administration, Nanjing Agricultural University, Nanjing 210095, China

Correspondence should be addressed to Jingjuan Zhu; zhujingjuan@njau.edu.cn

Received 9 April 2015; Accepted 30 May 2015

Academic Editor: Ruihua Liu

Chinese agriculture is facing an aging workforce which could negatively impact the industry. In this context, research is needed on how work preferences and age of farmers affect agricultural output. This paper attempts to investigate these factors to more fully understand the impact of an aging agricultural labor population on agricultural production. The results show that, in this context of aging, changes in the working-age households have a significant impact on agricultural output. Despite the fact that the impacts of intention to abandon land management were not significant, we can ignore this preference in the workforce. The combination of changes in the composition of the working-age households indicates that 58.53 percent of the agricultural producers will likely quit. This is a potential threat for the future of agricultural development. We also found that elderly farmers who do not intend to abandon farming had higher agricultural output compared to other farmers. This indicates that the adverse effects of changes in the agricultural population age result more from the agricultural output of older farmers who intend to give up farming. This intention adversely affected other elements and reduced investment. Therefore, various forms of training should increase efforts to cultivate modern professional farmers and policies should be simultaneously developed to increase agricultural production levels.

1. Introduction

Aging agricultural labor forces are the trend in many parts of the world including China. Both the second national agricultural census and preliminary research have shown that Chinese agriculture is facing an aging workforce; the share of older population in the total labour force reaches 32.5%. In this context, people are increasingly worried about how the aging of the agricultural labor force will affect the output of agricultural land and whether aging agricultural producers will continue to engage in agricultural production [1]. There is a substantial body of scholarly literature on aging agricultural labor forces. Tang and MacLeod suggested that older workers are, on average, less productive than younger workers and that labor force aging has a modest negative direct impact on productivity growth in Canada [2]. The research of Li and Zhao showed that the agricultural labor force in Liaoning Province of China exhibited an "aging" phenomenon and that agricultural labor "aging" is not conducive to the overall development of agricultural production [3]. In particular, Siliverstovs et al. found that an increase in aging exerts a statistically significant adverse effect on the employment shares in agriculture, manufacturing, construction, and mining and quarrying industries [4]. Yang et al. have studied the impact of agricultural labor force age on agricultural land use efficiency in regions with different levels of economic development, and their conclusion was that the households with primarily young labors have lower land use efficiency than the households where the labor is mainly done by older individuals [5]. Woodsong found that in Jamaica, where agriculture occupies an important place in the life course of many elders, the rural concentration of elders may have negative consequences for agricultural production [6]. Zhang et al. found, surprisingly, that the household proportion of males among agricultural laborers did not significantly influence the occurrence of land abandonment at the parcel level, probably due to the male agricultural laborers being overwhelmingly old (average age greater than 56 years) [7]. These scholars believe that the aging of the existing agricultural labor force has an impact on agricultural production. However, a study by Hu and Zhong using rural fixed observation point data to quantify the planting

decisions and investment levels of elderly farmers and young farmers concluded that rural aging at this stage does not have a negative impact on China's grain production [8]. Another study by these two authors concluded that crops with less collective decision making and a lower degree of mechanization experience a greater impact from aging [9]. In previous studies, the potential spatial spillover effect of transportation infrastructure on economic output in the US agricultural sector has not been properly taken into account given the sector's importance in the economy and dependence on transportation. The results of Tong and other researchers suggest that road disbursement in a given state has positive direct effects on its agricultural output [10]. Based upon the China Health and Nutrition Survey, a multivariate analysis demonstrated that the migration of household members increased the time spent on farm work and domestic work by the remaining elderly and children [11]. Chen et al. showed that the combined effect of rural household population aging and the transfer of rural labor force is a premature aging of the rural population and excessive reduction in the primary sector employees which will affect industry growth [12]. He hypothesizes that the aging agricultural labor force has a negative impact on effective use of land resources, food security agricultural modernization, the reproduction of agricultural emotions, and the rural grass-roots support system [13]. The research of L. Li and Y. Li reached a similar conclusion [14]. Today, the European Union is consequently faced with a dual problem: the scarcity of new and young farmers and the rapid aging of the farmer population. Given the context of an aging agricultural labor force, the future of the farmers' profession must be ensured. Manton examined sociodemographic and health conditions in Brazil, Russia, India, and China and the potential effects of population and labor force aging on economic growth [15].

Few studies explicitly look at how farmer preferences (give up farmland or not) affect agricultural output. Research in this area has mainly focused on macroscopic factors, including agricultural subsidies, agricultural production prices, food prices, and industrial management [16–18]. Clark et al. hypothesize that industrial changes such as a decline in the proportion of the labor force employed in agriculture will lower the proportion of older persons in the labor force [19]. Given the background of aging labor forces, it is necessary to conduct research on the impact of an aging agricultural labor population on farmland output as this is related to the sustainable development of agriculture and food security issues in China. Roberts conjectured that village-based networks are important in channeling migrants into particular occupations and destinations, undermining the notion of a "blind" migration from rural areas to coastal cities during China's rapid economic transition [20]. D'Antoni et al. found that, from 1939 to 2007, increased direct government payments resulted in greater migration of labor from agriculture [21]. As Goletti and Chabot report, more research is called for on input and output market efficiency, private sector development, the effects of reform on farmers, sequencing issues, comparative advantage, water management, land tenure, and farm size [22]. From 1939 to 2007, increased direct government payments resulted in greater migration of labor from agriculture [23]. Government policy appears to have had limited success at sustaining the agricultural labor force.

In summary, the effect of agricultural labor force aging on agricultural production, as well as the influence of farmer preferences on agricultural production, has been rarely considered. This paper attempts to develop an analytical framework to more comprehensively study the agricultural labor force in the context of aging farmers and the impact of farmer preferences. The rest of the paper is organized as follows. In Section 2, this paper attempts to give an theoretical analysis, and Section 3 introduces data and does some analysis, and finally conclusions and discussion are done in Section 4.

2. Theoretical Analysis

Agricultural production depends on natural conditions, factor input, and the prevailing level of technology. Natural conditions for agricultural production are not controllable, so the analysis of influencing factors of agricultural real estate is concentrated mainly on factor input, technical level, and other indirect factors. The agricultural population aging concerns in this paper are among the indirect factors.

Furthermore, agricultural production requires the joint participation of labor, machinery, fertilizers, pesticides, and land and among elements. Agricultural producers will adjust these elements depending on the conditions of rational expectations and judgment experience. In theory, farmers that expect to continue production in the future will have significantly different factor input compared to farmers who do not intend to continue to engage in agricultural production. Furthermore, agricultural producers of different ages make different choices regarding input elements.

Physical strength is required during the process of agricultural production. For adult producers, there is first an increase in physical strength that culminates in middle age. The decline in physical strength after middle age necessitates a greater investment of labor for the same production activities. However, the experience of older farmers leads to more efficient combinations of input, which makes a unit of labor more effective.

Agricultural production requires not only labor input, but also technological development. On the one hand an aging agricultural production needs technology to compensate for physical deficiency. On the other hand, nonfarm payrolls make the opportunity cost of agricultural labor input large. This may incline them to invest in the use of machinery instead of labor input.

Agricultural knowledge and skills in agriculture, such as production, operation, and management, increase with age. The accumulated knowledge and skills help farmers maximize the efficient use of agricultural input, such as pesticides and fertilizers, as well as labor input.

Land is the basis for agricultural production. Currently, rural land is implemented under the Household Contract Responsibility System, which is a basic land institution in rural China. From the farmers' perspective, a different family will manage different agricultural land area caused by its population. Although many Chinese rural villages stopped

Table 1: Study region.

Region	Municipal city	County	Number of samples	Effective sample size
Northern Jiangsu	Yancheng	Yandu, Dongtai, Sheyang	261	211
	Suqian	Suyu, Siyang, Sihong	261	183
	Xuzhou	Tongshan, Xinyi, Fengxian	261	191
Central Jiangsu	Nantong	Gangzha, Haimen, Rugao	261	140
	Yangzhou	Jiangdu, Yizheng, Baoying	261	159
Southern Jiangsu	Suzhou	Wuzhong, Changshu, Kunshan	261	25
	Zhenjiang	Zhenjiang New Area, Dantu, Danyang	261	70

farmland adjustments, the agricultural producers can still select planting acreage, and the agricultural producers of different age groups will select modest acreage according to their own conditions.

Individual preferences and the overall labor force are employed to analyze the effect of age, but combining the two to analyze the situation of agricultural real estate will yield new insights. We separated the farmers into four categories: old farmers who do not intend to abandon farming (ONA), old farmers who intend to abandon farming (OA), young farmers who intend to give up farming (YA), and young farmers who do not intend to give up farming (YNA). For YA farmers, interest in the agricultural production is not high and incentives to increase output from the agricultural land are not high; plus they lack experience in agricultural production. Thus, YA farmers would be less concerned about agricultural land, and input will be insufficient. For ONA farmers, they lack physical strength, but they are still reluctant to give up agricultural production. Thus, they will try to compensate for their lack of physical strength through additional supplies, and so forth. Thus, ONA farmers desire production input more than other groups and they possess long-term production experience. These factors should lead to higher levels of input and outputs compared to other groups. For YNA, they are willing to engage in agricultural production and usually benefit from family experience as most agriculture is the same as that done traditionally. So they will not reduce the input, and their output can be higher than YA. For OA farmers, their interest to continue in agricultural production is not high, so they do not focus on improving the agricultural land and pursue more short-term benefits. Their advantage lies in the long-term accumulation of experience in agricultural production, input configuration is relatively reasonable, and the output of agricultural land will be relatively moderate.

In addition, based on previous studies, there are also many other factors which can affect farmer input and agricultural output. Farmer's cultural level and skills [24], the degree of farmland fragmentation [25, 26], and farmland transfer cases [26] also affect the situation of farmers. In China, farmland transfer usually means that farmland moves from one farmer to another farmer which increases the total farmland of the latter. Therefore, a study from the perspective of farmer preferences needs to examine the impact of the cultural level of farmers and the degree of land fragmentation on agricultural output.

Based on the above analysis, this study uses econometric models to analyze the impact of farmer preferences on agricultural output.

3. Data and Analysis

3.1. Study Area. This study uses the data from the Jiangsu Provincial Department and was conducted in cooperation with Nanjing Agricultural University, Jiangsu province, rural land issues hundred villages research. The survey involved a total of seven municipal cities in Jiangsu; a total of 1827 questionnaires were collected by a random sampling. And 3 counties are selected in every municipal city which represent various economic development level. We excluded questionnaires that were invalid for purposes of this study. This left 979 household level questionnaires which were used to meet the research needs of this paper (Table 1).

Jiangsu province is located on the eastern coast of China. It spans $116^\circ 18' – 121^\circ 57'$ longitude and $30^\circ 45' – 35^\circ 20'$ latitude with an area of 102,600 km^2. It accounts for at least 1.06 percent of China's per capita land area in Chinese provinces and autonomous regions. Jiangsu has become the province with the highest level of development as China has entered the "upper-middle" level of developed countries. Different cities in Jiangsu differ in economic development. Generally, the economic development is best in the southern cities, including Nanjing, Zhenjiang, Changzhou, Wuxi, and Suzhou; moderate in the central cities, including Yangzhou, Nantong, and Taizhou; and lowest in northern cities, including Xuzhou, Lianyungang, Suqian, Huai'an, and Yancheng.

3.2. Descriptive Analysis

3.2.1. The Basic Characteristics of Household Age. The age of farmers used herein refers to the average age of the main labor force engaged in agricultural production. A study by Burton supports the reasonableness of this treatment method [27]. Figure 1 shows the age distribution of the main labor force in agricultural production in the northern part of Jiangsu, central Jiangsu, and southern Jiangsu classification.

The total average age was 56.6 years. The average in the area with the highest degree of economic development in the southern region was 54.8 years old, the average in the least developed economies in the north was 55.8 years old, and the average in the central region was 58.6 years old. We conjecture that the economically developed southern portion

Table 2: Agricultural factor input of different types of farmers.

	API (RMB/mu)	AA (years)	EL	LI (person)	AOA (mu)	OFB (blocks)	SI (RMB/mu)	PI (RMB/mu)	FI (RMB/mu)
OA	1924.31	66.69	2.29	1.54	4.79	3.62	122.97	131.26	262.35
ONA	2215.90	66.14	2.38	1.57	6.53	3.89	149.64	156.29	310.58
YA	1631.56	49.33	3.19	1.27	7.94	3.76	133.64	97.52	243.26
YNA	1883.34	49.19	3.24	1.40	13.14	5.28	137.84	125.79	261.62
GF	1776.91	58.01	2.74	1.40	6.37	3.69	128.30	114.39	252.80
NGF	2048.61	57.67	2.81	1.48	9.84	4.59	143.74	141.04	286.10
OF	2068.78	66.41	2.34	1.55	5.66	3.76	136.30	143.77	286.46
YF	1756.74	49.26	3.22	1.33	10.54	4.52	135.74	111.66	252.44

Note: GF refers to the farmers who want to give up farmland and NGF refers to the farmers who do not want to give up farmland. OF signifies old farmers and YF signifies young farmers. API refers to agriculture operating income in 2012. AA refers to average age. EL refers to education level. LI refers to labor input. AOA refers to actual operating area. OFB refers to operating farmland blocks. SI refers to seed input. PI refers to pesticide input. FI refers to fertilizer input. mu is a popular area unit in China which equals 0.067 ha. The age and education level are the averages of the major agricultural labor households. Less than 30 years old is considered young in this research, and the age more than 60 is considered old.

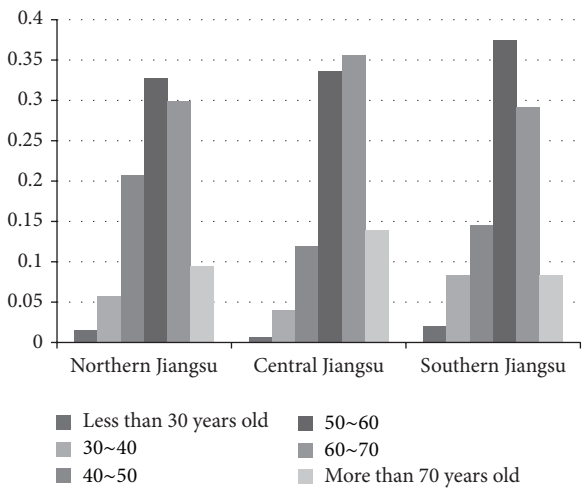


Figure 1: The age proportion distribution in different regions of Jiangsu province.

of Jiangsu attracted workers away from the central region, thereby exacerbating the aging of the rural workforce in the central region. This may help explain why the central region has a higher average farmer age compared to the less economically developed northern region.

We can see from Figure 1 that the phenomenon of an aging agricultural labor force is an objective reality. The peaks were mainly in the 50 to 60 and the 60 to 70 age categories. The agricultural producers in northern and southern Jiangsu have age distribution peaks between 50 and 60 years, but the central region has its peak in the 60 to 70 age category. This also confirms that the agricultural labor force aging in the Jiangsu central region is more serious than the other regions.

3.2.2. Different Types of Farmers in Agricultural Production Factor Input Feature. To further examine the impact of age on agricultural production, we investigated old farmers who do not intend to abandon farming (ONA), old farmers who intend to abandon farming (OA), young farmers who intend to give up farming (YA), and young farmers who do not intend to give up farming (YNA) to discover potential differences in input and output.

After comparing the different types of farmers in terms of agricultural production and input-output (see Table 2), we found that the differences were not great. ONA farmers had the highest operating income in agriculture, in terms of average per mu of view. This may partly reflect greater experience in production but also may indicate additional investment. YA farmers had the lowest operating income; it may be that YA farmers are more inclined to engage in nonagricultural industries and pursuit income from nonagricultural sectors. However, YNA have a much higher total agricultural operating income compared to the other three groups. This is mainly because they have expanded their producing land area, and the proportion of the YNA group reached 41.47% in the whole sample.

From the investment point of view, most of the operating area is attributable to YNA farmers, followed by YA, and relatively little attributable to elderly farmers. This may be due to a physical decline in elderly farmers. ONA farmers had the greatest investment in seeds, fertilizers, and pesticides. This might be to compensate for a decrease in labor input. OA and YNA farmers had comparable values for seed, fertilizer, and pesticide input. However, the total values for OA farmers were far below other groups. YA farmers invested the least amount of labor.

Input and outputs were lower for those intending to abandon farming regardless of age, indicating that this intention is not conducive to improving agricultural production. However, the influence of this negative effect requires further analysis. Although older farmers have higher input than the young farmers do, they also have higher output than young farmers. Thus, we cannot conclude from this statistical data how the aging agricultural population will affect the use of agricultural land.

Simply examining the descriptive statistics is not sufficiently comprehensive. However, the descriptive statistics appear to support our analytical framework.

From the general description of the above simple analysis, we cannot accurately separate the impacts of farmer preferences. Therefore, we used an econometric model to analyze the above conclusions more fully.

3.3. Econometric Analysis

3.3.1. Model Selection. In this paper, an extended production function is used to study the problems associated with the aging of the agriculture labor force. The model is built on the C-D production function, but its form is altered for more flexibility. The basic form of the model is as follows:

$$\ln Y_i = \beta_0 + \sum_j \beta_j \ln x_{ij} + \frac{1}{2}\sum_j \sum_m \beta_{jm} \ln x_{ij} \ln x_{im} + \delta, \quad (1)$$

where Y represents the operating income of farmers, x represents farmers' factor input, and i indicates individual farmers. j and m represent input element number. We modified the basic equation (1) with the introduction of relevant variables. In this way, we obtained the model shown in

$$\begin{aligned}\ln Y_i = {} & c + \beta_1 \ln x_{it} + \beta_2 \ln x_{il} + \beta_3 \ln x_{iz} + \beta_4 \ln x_{if} \\ & + \beta_5 \ln x_{in} + \frac{1}{2}\beta_{11} (\ln x_{it})^2 + \frac{1}{2}\beta_{22} (\ln x_{il})^2 \\ & + \frac{1}{2}\beta_{33} (\ln x_{iz})^2 + \frac{1}{2}\beta_{44} (\ln x_{if})^2 \\ & + \frac{1}{2}\beta_{55} (\ln x_{in})^2 + \beta_{12} \ln x_{it} \ln x_{il} \\ & + \beta_{13} \ln x_{it} \ln x_{iz} + \beta_{14} \ln x_{it} \ln x_{if} \\ & + \beta_{15} \ln x_{it} \ln x_{in} + \beta_{23} \ln x_{il} \ln x_{iz} \\ & + \beta_{24} \ln x_{il} \ln x_{if} + \beta_{25} \ln x_{il} \ln x_{in} \\ & + \beta_{34} \ln x_{iz} \ln x_{if} + \beta_{35} \ln x_{iz} \ln x_{in} \\ & + \beta_{45} \ln x_{if} \ln x_{in} + \beta_6 d_{ia} + \beta_7 d_{ib} + \beta_8 o_i \\ & + \beta_9 w_i + \beta_{10} s_i + \beta_{16} lz_i + \beta_{17} ds_i + \delta,\end{aligned} \quad (2)$$

where c represents the constant term, t indicates land investment, l represents labor input, z represents the seed investment, f represents fertilizer input, n represents pesticide input, d represents the type of farmers (a refers to ONA farmers and b refers to YA farmers), o represents age, w represents education level, s indicates the degree of land fragmentation represented by the number of blocks, lz indicates whether the land is transferred (1 indicates transfer and 0 indicates no transfer), ds indicates whether they intend to give up land management (1 signifies yes and 0 signifies no), and the other symbols are the same as above.

3.3.2. Model Estimation Results. The investigation was related to farmers in agricultural production data into (2); derived model estimation results are shown in Table 3.

The R-squared of the model was 0.634456. The ONA farmers, labor input, seed investment, land investment, farmers age, education level of farmers, land fragmentation degree, fertilizer input secondary effect, secondary effects of land investment, secondary effects of seed investment, and land and fertilizer interaction, as well as interaction of land and seeds show a significant impact on agricultural production. The impacts of other factors on agricultural output were not significant.

Table 3: Extended production function parameter estimates.

The independent variables	Parameters	t value
ds	−0.090977	−1.043367
$\ln(f)$	−0.253916	−0.793467
$\ln(f) \times \ln(f)$	0.145951**	2.020194
$\ln(l)$	0.967351*	1.872886
$\ln(l) \times \ln(f)$	−0.063184	−0.557116
$\ln(l) \times \ln(l)$	−0.143250	−0.402932
$\ln(l) \times \ln(n)$	0.042504	0.418499
$\ln(l) \times \ln(t)$	0.052057	0.527355
$\ln(l) \times \ln(z)$	−0.106429	−1.110889
$\ln(n)$	−0.163946	−0.576910
$\ln(f) \times \ln(n)$	−0.032321	−0.562985
$\ln(n) \times \ln(n)$	0.073402	1.271503
s	−0.007494*	−1.665416
db	−0.120524	−1.256796
$\ln(t)$	1.691591***	7.545998
$\ln(t) \times \ln(f)$	−0.183415***	−2.955062
$\ln(t) \times \ln(n)$	0.076468	1.263808
$\ln(t) \times \ln(t)$	0.170023***	3.318696
$\ln(t) \times \ln(z)$	−0.111609**	−2.257393
w	0.048475*	1.695400
$\ln(z)$	−0.374512*	−1.716832
$\ln(z) \times \ln(f)$	0.002678	0.057721
$\ln(z) \times \ln(n)$	−0.016682	−0.383357
$\ln(z) \times \ln(z)$	0.122659***	2.919681
da	0.138489*	1.840943
o	−0.288300*	−1.794592
lz	−0.106429	−1.110889
c	$1.04E-10$	$4.60E-09$

Note: the symbols $***$, $**$, and $*$, respectively, indicate 1%, 5%, and 10% confidence levels via significant testing, Log-likelihood = −1031.225.

The regression coefficient for the age of individual farmers is −0.288300 and is significant at the 10% confidence level. This indicates that older farmers negatively affected agricultural. Increased age is not conducive to improving agricultural output. The additional input and experience of the older farmers are not enough to compensate for the adverse effect of their age. In addition, elderly farmers appear to resist new technologies in agricultural production. When asked whether they make use of machinery, young farmers (84.331%) answered that they mainly used machinery at a higher rate than older farmers (77.566%). The regression coefficient of "intention to abandon land management" variable was −0.090977, indicating that this intention has

a negative effect on agricultural output. We can imagine that, sometime in the future, those who intend to give up farming will reduce spending on agriculture. The data in Table 2 also show that farmers who intended to abandon the profession had lower investment in seed, fertilizer, and pesticides input. Objectively speaking, we can presume that both those who intend to abandon farming and older farmers who do not intend to abandon farming will gradually withdraw from active or passive agricultural production. Although YNA farmers make up 41.47% of the total, nearly 58.53% of the agricultural producers will likely exit agricultural production in the near future. On the one hand, YNA farmers may take the initiative to seek another income. On the other hand, older farmers are faced with the situation of retirement regardless of intentions. Thus, although our analysis indicates that this effect is not significant at present, the problem of agricultural production successors still requires attention. However, older farmers who do not intend to abandon the profession significantly improve agricultural output at the 10% level. We conjecture that this is because of their greater experience and higher levels of investment in factor input. Young people who intend to abandon farming do not help improve agricultural output, but this effect is not significant.

We also found that labor, seed, and land investment have a significant impact on agricultural output. In addition, a higher educational level and reducing the degree of fragmentation of land are conducive to improving agricultural productivity. The secondary effects of fertilizer input, the secondary effects of land investment, and the secondary effects of seed investment, land, and fertilizer interaction, as well as the interaction of land and seeds also show a significant impact on agricultural output, which indicates that a reasonable mix of factor input can increase agricultural output.

4. Conclusions and Discussion

These results show that in the context of an aging agricultural labor force, changes in the working-age households have a significant impact on agricultural output. Concern is needed for such adverse effects. Although the influence of intention to abandon land management was not significant, we cannot ignore this effect because changes in the composition of the working-age households suggest that 58.53 percent of the agricultural producers will likely exit agricultural production. This represents a potential threat to the future development of agriculture. We also found that the elderly who do not intend to abandon the farmers compared to other farmers more conducive to agricultural output, indicating that the adverse effects of changes in the agricultural population age more agricultural output from older farmers intend to give up, and this negative impact can be reduced by an additional investment in other elements.

The aging agricultural labor force trend has become widely accepted. To reduce the adverse impact on agricultural output caused by aging in the context of food security is the current problem we are facing. We believe policy proposal should start with the following considerations.

First, increase the intensity of various forms of training to nurture professional farmers. In the next few years, the trend of urbanization will transfer more young people from rural areas and exacerbate the aging problem. Various training should be conducted to improve scientific and technological level as well as cultural attitudes. This training should equip them with more modern agricultural production skills to make up for the problems caused by aging.

Secondly, develop different policies to increase agricultural production levels for different types of farmers. For example, farmers intending to abandon the profession should be encouraged to carry out the land transfer and split the land contract and management rights, thus promoting large-scale production agriculture. We should help young people who do not intend to abandon the farming to improve their skills, increase financial support, expand operation scale, and ultimately modernize and industrialize agricultural production.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (Grant nos. 71233004 and 71003052), the Project Guide Research Fund of Philosophy and Social Science in Colleges and Universities in Jiangsu (Grant no. 2014SJD076), and the Fundamental Research Funds for the Central Universities of Nanjing Agricultural University (Grant nos. SK2013005 and SKPT2014046).

References

[1] B. Lin and H. Deng, "An empirical analysis of the agricultural labor force aging on the impact of land use efficiency—based in Zhejiang Province rural fixed observation point data," *Chinese Rural Economy*, no. 4, pp. 15–25, 2012.

[2] J. Tang and C. MacLeod, "Labour force ageing and productivity performance in Canada," *Canadian Journal of Economics*, vol. 39, no. 2, pp. 582–603, 2006.

[3] M. Li and L. Zhao, "Agricultural labor 'aging' phenomenon and its impact on agricultural production—based on empirical analysis of Liaoning Province," *Issues in Agricultural Economy*, no. 10, pp. 12–18, 2009.

[4] B. Siliverstovs, K. A. Kholodilin, and U. Thiessen, "Does aging influence structural change? Evidence from panel data," *Economic Systems*, vol. 35, no. 2, pp. 244–260, 2011.

[5] J. Yang, G. Yang, and X. Hu, "Age effects on the agricultural labor force farmers cultivated land use efficiency—evidence from different levels of economic development in the region," *Resources Science*, no. 9, pp. 1691–1698, 2011.

[6] C. Woodsong, "Old farmers, invisible farmers: age and agriculture in Jamaica," *Journal of Cross-Cultural Gerontology*, vol. 9, no. 3, pp. 277–299, 1994.

[7] Y. Zhang, X. Li, and W. Song, "Determinants of cropland abandonment at the parcel, household and village levels in

mountain areas of China: a multi-level analysis," *Land Use Policy*, vol. 41, pp. 186–192, 2014.

[8] X. Hu and F. Zhong, "The impact of population aging on crop production—analysis," *Agricultural Economy Based on Wheat and Cotton Crops Issue*, no. 2, pp. 36–43, 2013.

[9] X. Hu and F. Zhong, "Rural population aging impact on food production—analysis of rural fixed observation point data," *Chinese Rural Economy*, no. 7, pp. 29–39, 2012.

[10] T. Tong, T.-H. E. Yu, S.-H. Cho, K. Jensen, and D. de La Torre Ugarte, "Evaluating the spatial spillover effects of transportation infrastructure on agricultural output across the United States," *Journal of Transport Geography*, vol. 30, pp. 47–55, 2013.

[11] H. Chang, X.-Y. Dong, and F. MacPhail, "Labor migration and time use patterns of the left-behind children and elderly in rural China," *World Development*, vol. 39, no. 12, pp. 2199–2210, 2011.

[12] X. Chen, Y. Chen, and J. Zhang, "Quantitative study of Chinese rural population aging affect agricultural output pairs," *Chinese Population Science*, no. 2, pp. 39–46, 2011.

[13] X. He, "Agricultural labor force aging research—based on surveys in rural Zhejiang Province," *Population and Economy*, no. 2, pp. 69–78, 2013.

[14] L. Li and Y. Li, "Research on aging agricultural labor force—based on the second national agricultural census data analysis," *Agricultural Economy*, no. 6, pp. 61–66, 2009.

[15] K. G. Manton, "Population and labor force aging, effect on socio-economic development in brazil, russia, india and china," *International Encyclopedia of Public Health*, pp. 170–181, 2008.

[16] Q. Zhou, "Factors willingness of farmers to grow grain analysis," *Agricultural Technology Economic*, no. 5, pp. 25–30, 2009.

[17] X. Liu, Z. Weng, and H. Zhu, "Research Jiangxi grain farmer food subsidy policy cognitive behavioral characteristics and production operations analysis," *Business Research*, no. 11, pp. 13–16, 2008.

[18] Y. Ma and Y. Yang, "Impact of policies on food subsidies to grain farmers will, farmers' income and production inputs—an empirical case study of Hebei," *Agricultural Technology Economy*, no. 2, pp. 7–13, 2005.

[19] R. L. Clark, E. A. York, and R. Anker, "Economic development and labor force participation of older persons," *Population Research and Policy Review*, vol. 18, no. 5, pp. 411–432, 1999.

[20] K. D. Roberts, "The determinants of job choice by rural labor migrants in Shanghai," *China Economic Review*, vol. 12, no. 1, pp. 15–39, 2001.

[21] J. M. D'Antoni, A. K. Mishra, and A. P. Barkley, "Feast or flee: government payments and labor migration from U.S. agriculture," *Journal of Policy Modeling*, vol. 34, no. 2, pp. 181–192, 2012.

[22] F. Goletti and P. Chabot, "Food policy research for improving the reform of agricultural input and output markets in Central Asia," *Food Policy*, vol. 25, no. 6, pp. 661–679, 2000.

[23] A. Kontogeorgos, A. Michailidis, F. Chatzitheodoridis, and E. Loizou, "'New Farmers' a crucial parameter for the Greek primary sector: assessments and perceptions," *Procedia Economics and Finance*, vol. 14, pp. 333–341, 2014.

[24] T. Schultz, *Transformation of Traditional Agriculture*, Commercial Press, 1987.

[25] H. Zhang and G. Yang, "Arable land fragmentation and its impact on technical efficiency of food production—beyond the micro data based on stochastic frontier production function and the number of farmers," *Resources Science*, no. 5, pp. 903–910, 2012.

[26] T. Liu, F. Qu, J. Jin, and X. Shi, "Land fragmentation, the impact on farmers land transfer land use efficiency," *Resources Science*, no. 10, pp. 1511–1517, 2008.

[27] R. J. F. Burton, "An alternative to farmer age as an indicator of life-cycle stage: the case for a farm family age index," *Journal of Rural Studies*, vol. 22, no. 4, pp. 485–492, 2006.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 453072, 13 pages
http://dx.doi.org/10.1155/2015/453072

Research Article

Consensus of Noisy Multiagent Systems with Markovian Switching Topologies and Time-Varying Delays

Yilun Shang

Department of Mathematics, Tongji University, Shanghai 200092, China

Correspondence should be addressed to Yilun Shang; shylmath@hotmail.com

Received 25 March 2015; Revised 26 May 2015; Accepted 21 July 2015

Academic Editor: Son Nguyen

Stochastic multiagent systems have attracted much attention during the past few decades. This paper concerns the continuous-time consensus of a network of agents under directed switching communication topologies governed by a time-homogeneous Markovian process. The agent dynamics are described by linear time-invariant systems, with random noises as well as time-varying delays. Two types of network-induced delays are considered, namely, delays affecting only the output of the agents' neighbors and delays affecting both the agents' own output and the output of their neighbors. We present necessary and sufficient consensus conditions for these two classes of multiagent systems, respectively. The design method of consensus gains allows for decoupling the design problem from the graph properties. Numerical simulations are implemented to test the effectiveness of our obtained results as well as the tightness of necessary/sufficient conditions.

1. Introduction

In the past few years, distributed coordination of multiagent systems has been considered by many researchers due to its broad applications in such areas as swarming of animals/robots, cooperative unmanned aerial/underwater vehicles, distributed computation, air traffic control, and distributed sensor networks. One of the important issues in coordinated control is network based consensus protocol design. In this setting, consensus refers to every agent achieving agreement about some common or shared quantity by exchanging information according to a set of rules.

For the purpose of reaching consensus, important interaction details of agents in a system are mostly encoded by the communication graph of the system, which gives a general setting to study consensus and allows for the application of graph-theoretical notations and tools. Distributed computation over networks has been studied in the pioneering work of Tsitsiklis [1] and Chatterjee and Seneta [2] in systems and ergodicity theory. More recently, analytical frameworks for solving consensus problems were introduced by Olfati-Saber and Murray [3] and Jadbabaie et al. [4] based on graph and matrix theory. Since then numerous consensus protocols have been proposed, mostly for simple single- and double-integrator dynamics (see, e.g., [5–9] and references therein). It is pointed out that [10] design of consensus protocols for agent dynamics delineated, more generally, by linear time-invariant systems is more challenging due to the possible existence of strictly unstable eigenvalues (poles) in the open-loop matrix. Necessary and sufficient consensus conditions for linear time-invariant systems were explored in [11–14] recently.

In many cases, the communication between agents is subject to stochastic perturbation—the connections change with time due to packet drops, agent failure, and various external disturbances. Therefore, the communication graphs underpinning the physical systems are better characterized by random switching networks. Stochastic consensus with single- and double-integrator dynamics has been well researched [15–21]. For example, asymptotic agreement of continuous-time single-integrator agent dynamics over Poisson random graphs is considered in [15]. The results are further extended in [16] to solve mean square consensus under directed and weighted independently switching random graphs. When the communication topology is described by a strictly stationary ergodic graph process, a necessary and sufficient condition for almost sure consensus of single-integrator agents is shown

to be the connectivity of the mean topology with respect to a stationary distribution of the process [17]. For both discrete-time and continuous-time multiagent systems with single-integrator dynamics and balanced communication graphs, it has been shown in [21] that the ergodic Markov jump linear system achieves average consensus almost surely if and only if the union of topologies corresponding to the states of the Markov process is strongly connected. Similarly, for second-order discrete systems with (not necessarily ergodic) Markovian switching topologies, the necessary and sufficient condition for mean square consensus becomes that each union of graphs corresponding to the closed sets of positive recurrent states has a spanning tree [20]. Recently, this result is extended to linear time-invariant agent dynamics in [10] for both discrete- and continuous-time consensus.

It is well documented in the literature [22] that unmodeled delay effects in a feedback mechanism may destabilize an otherwise stable system. In multiagent systems, time-varying delays may arise naturally due to the asymmetry of interactions, the congestion of the communication channels, and the finite transmission speed. Moreover, noise/uncertainty frequently occurs to agents through, for example, communication errors and spurious measurements in communication systems. Therefore, it would be desirable to understand consensus problems in the setting of Markovian switching topologies with interactions affected by both time-varying delays and random noises.

In this paper, we investigate consensus problems for continuous-time multiagent systems with linear time-invariant agent dynamics under Markovian switching topologies, time-varying delays, and stochastic noises. In particular, we consider two types of communication delays: delays affecting both the state of the agents and that of their neighbors and delays affecting only the state of the agents' neighbors. A unified framework that considers these delays in continuous-time multiagent systems with fixed topology is first established in [23]. It is worth noting that although other interesting delay-dependent robustness results are reported in, for example, [24–26], the communication topologies considered are either static or switch deterministically.

This work deals with a group of identical agent dynamics, each of which follows a linear time-invariant system with white noises input. The information flow between agents is modeled by a time-homogenous Markov process, whose state space corresponds to all the possible communication patterns (directed graphs). We establish necessary and sufficient conditions to guarantee all agents asymptotically achieve an agreement in the mean square sense and in the almost sure sense for both types of time delays, respectively. When each graph corresponding to a state of the Markov process contains a spanning tree or is d-regular for a fixed $d \geq 1$, we show that the agents can reach consensus for suitable time-varying delays in terms of M-matrices if the agent dynamics is stabilizable. Conversely, if the consensus is achieved, the agent dynamics must be stabilizable and each union of graphs corresponding to the closed sets of positive recurrent states of the Markov process contains a spanning tree. The main mathematical techniques used here are based on the stability analysis of Markovian jump linear systems, stochastic differential delay equations, and graph and matrix theory.

The rest of the paper is organized as follows. Section 2 contains the problem formulation. Section 3 presents the main results. A couple of numerical examples are given in Section 4. The conclusion is drawn in Section 5.

Notation. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ be the n-dimensional column vectors of all ones and all zeros, respectively. I_n represents an $n \times n$ identity matrix. If the dimension is clear from the context, we sometimes suppress the subscript n. The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The closed right half plane is signified by $\mathbb{C}^+$. We say $A > B$ ($A \geq B$) if $A - B$ is positive definite (semidefinite), where A and B are symmetric matrices of the same dimensions. A^T means the transpose of matrix A, while A^H means its conjugate transpose. For a vector x, $\|x\|$ refers to its Euclidean norm; for a matrix A, $\|A\| = \sqrt{\operatorname{trace}(A^T A)}$ represents its trace norm. Let $\|\cdot\|_{\max}$ represent the max norm of a matrix, namely, the maximum of the absolute values of elements. For a matrix $A \in \mathbb{R}^{n\times n}$, its null space is designated by $\operatorname{Null}(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$. By $A \otimes B$ we denote the Kronecker product of matrices A and B, which admits the following properties: $(A \otimes B)(C \otimes D) = AC \otimes BD$, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, and $(A \otimes B)^T = A^T \otimes B^T$. Denote by $\{\lambda_i(A)\}_{i=1}^n$ the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$. Throughout this paper, we will order them in a nondecreasing order according to their modules: $|\lambda_1(A)| \leq |\lambda_2(A)| \leq \cdots \leq |\lambda_n(A)|$.

2. Problem Formulation and Preliminary Results

2.1. Graph and Consensus Properties. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ represent a directed graph of order N, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is the set of nodes (agents) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges. The ordered pair $(v_i, v_j) \in \mathcal{E}$ denotes a directed edge from node v_i to node v_j, indicating that the information can be sent from agent v_i to agent v_j. The weighted adjacency matrix $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is defined by $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Define the in-degree matrix as a diagonal matrix $\mathcal{D} = \operatorname{diag}(d_1^{\text{in}}, \ldots, d_N^{\text{in}})$, with $d_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ being the in-degree of agent v_i. Similarly, the out-degree of agent v_i is defined by $d_i^{\text{out}} = \sum_{j=1}^N a_{ji}$. $\mathcal{G}$ is said to be balanced if $d_i^{\text{in}} = d_i^{\text{out}}$ for all $i = 1, \ldots, N$ [3]. Define the graph Laplacian matrix as $\mathcal{L} = (l_{ij}) = \mathcal{D} - \mathcal{A}$, which has all row sums equal to zero.

A sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{k-1}}, v_{i_k})$, with $(v_{i_{j-1}}, v_{i_j}) \in \mathcal{E}$ for $j = 2, \ldots, k$, is called a directed path from agent v_{i_1} to agent v_{i_k}. We say that $\mathcal{G}$ contains a spanning tree if there is an agent (called root) such that every other agent can be connected by a directed path starting from the root. By Lemma 3.3 of [6], $\mathcal{G}$ contains a spanning tree if and only if $0 = \lambda_1(\mathcal{L}) < |\lambda_2(\mathcal{L})|$. Let s be a positive integer. The union of s graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1, \mathcal{A}_1), \ldots, \mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s, \mathcal{A}_s)$ is denoted by $\bigcup_{i=1}^s \mathcal{G}_i = \{\mathcal{V}, \bigcup_{i=1}^s \mathcal{E}_i, \sum_{i=1}^s \mathcal{A}_i\}$.

For $t \geq 0$, let $\tau(t) \geq 0$ be a differentiable function which will stand for the time-varying communication delay.

For $i = 1, \ldots, N$, the dynamics of each agent v_i in continuous time takes the following two different forms:

(i) with self-delay:

$$\dot{x}_i(t) = Ax_i(t-\tau(t)) + Bu_i^w(t) + \widehat{A}x_i(t-\tau(t))\dot{w}_i(t) + \widehat{B}\widehat{u}_i^w(t), \tag{1}$$

(ii) without self-delay:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i^o(t) + \widehat{A}x_i(t)\dot{w}_i(t) + \widehat{B}\widehat{u}_i^o(t), \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ represents the state of agent v_i at time t, $u_i^w(t), \widehat{u}_i^w(t), u_i^o(t), \widehat{u}_i^o(t) \in \mathbb{R}^m$ are control inputs of agent v_i given by

$$u_i^w(t) = K\sum_{j=1}^{N} a_{ij}(t)\left(x_j(t-\tau(t)) - x_i(t-\tau(t))\right), \tag{3}$$

$$\widehat{u}_i^w(t) = \widehat{K}\sum_{j=1}^{N}\sigma_{ij}(t)\dot{w}_{ij}(t)\left(x_j(t-\tau(t)) - x_i(t-\tau(t))\right), \tag{4}$$

$$u_i^o(t) = K\sum_{j=1}^{N} a_{ij}(t)\left(x_j(t-\tau(t)) - x_i(t)\right), \tag{5}$$

$$\widehat{u}_i^o(t) = \widehat{K}\sum_{j=1}^{N}\sigma_{ij}(t)\dot{w}_{ij}(t)\left(x_j(t-\tau(t)) - x_i(t)\right), \tag{6}$$

respectively, $A, \widehat{A} \in \mathbb{R}^{n\times n}$, $B, \widehat{B} \in \mathbb{R}^{n\times m}$ are system matrices, and $\{w_i(t), w_{ij}(t) : i, j = 1, 2, \ldots, N\}$ are independent standard white noises. Here, $K, \widehat{K} \in \mathbb{R}^{m\times n}$ are common consensus gains to be designed later, and σ_{ij} are referred to as the intensity of noise. To highlight the presence of noise, it is natural to define a noise graph $\widehat{\mathscr{G}} = (\mathscr{V}, \mathscr{E}, \widehat{\mathscr{A}})$ with the adjacency matrix $\widehat{\mathscr{A}} = (\sigma_{ij}) \in \mathbb{R}^{N\times N}$ satisfying $\sigma_{ij} > 0$ if $(v_j, v_i) \in \mathscr{E}$ and $\sigma_{ij} = 0$ otherwise. By definition, if viewing $\mathscr{G}$ and $\widehat{\mathscr{G}}$ as unweighted graphs, that is, the adjacency matrices are taken as binary ones, we have $\mathscr{G} = \widehat{\mathscr{G}}$. Likewise, the corresponding degree and Laplacian matrices are denoted by $\widehat{\mathscr{D}} = \operatorname{diag}(\sigma_1, \ldots, \sigma_N)$ with $\sigma_i = \sum_{j=1}^{N}\sigma_{ij}$ and $\widehat{\mathscr{L}} = (\widehat{l}_{ij}) = \widehat{\mathscr{D}} - \widehat{\mathscr{A}}$, respectively.

Remark 1. The above dynamical models characterize system uncertainties with Gaussian white noise appearing as an exogenous input (similar treatment can be found in, e.g., [27–30]). To see this, take $\widehat{A} = \zeta_1 A$ and $\widehat{B} = \zeta_2 B$ ($\zeta_1, \zeta_2 > 0$). System (1), for example, can be recast as

$$\begin{aligned}\dot{x}_i(t) = {} & A\left(1+\zeta_1\dot{w}_i(t)\right)x_i(t-\tau(t)) \\ & + B\sum_{j=1}^{N}\left(Ka_{ij}(t) + \widehat{K}\zeta_2\sigma_{ij}(t)\dot{w}_{ij}(t)\right) \\ & \cdot\left(x_j(t-\tau(t)) - x_i(t-\tau(t))\right).\end{aligned} \tag{7}$$

The perturbations are represented by a linear combination of gain matrices K and $\widehat{K}$ to be determined. Moreover, if $\zeta_1 = 1$, we take $\widehat{K} = K$. (Although other choices are theoretically allowed as per Theorem 8 below, we make them equal in practice since one usually is not able to separate out the noise from the rest of the state.) Thus, the uncertainty reduces to the conventional form $a_{ij}(t) + \zeta_2\sigma_{ij}(t)\dot{w}_{ij}(t)$. We mention that other commonly studied uncertainties pertaining to the consensus problems include the measurement noises which only affect the received neighbors' states (e.g., [18]) and the additive plant noises (e.g., [31]).

Multiagent system (2) with consensus protocols (5) and (6) considers only propagation delays for information transmitted from agent v_j to agent v_i on the communication network. Propagation delay has been addressed previously, for example, in works [24, 32–35]. Multiagent system (1) with consensus protocols (3) and (4) models both self-delay and neighboring delay. This scheme is relevant for dynamic agents with computation or reaction delays; see, for example, [3, 25, 26, 36, 37]. Although it would be more realistic to explore heterogeneous delays, we consider the uniform delay $\tau(t)$ as a first step, and this simplifies the derivation.

In the current work, we deal with delay robustness in both multiagent systems (1) and (2) over a stochastically time-varying interaction topology $\mathscr{G}(t)$ as well as its associated noise topology $\widehat{\mathscr{G}}(t)$, which is governed by a homogeneous continuous-time Markov process $\theta(t)$, taking value in the finite set $S = \{1, 2, \ldots, s\}$. More precisely, $\mathscr{G}(t) \in \{\mathscr{G}_1, \ldots, \mathscr{G}_s\}$ and $\widehat{\mathscr{G}}(t) \in \{\widehat{\mathscr{G}}_1, \ldots, \widehat{\mathscr{G}}_s\}$; $\mathscr{G}(t) = \mathscr{G}_i$ and $\widehat{\mathscr{G}}(t) = \widehat{\mathscr{G}}_i$ if and only if $\theta(t) = i$. For each i, the adjacency, degree, and Laplacian matrices for $\mathscr{G}_i$ ($\widehat{\mathscr{G}}_i$, resp.) will be denoted by $\mathscr{A}_i$, $\mathscr{D}_i$, and $\mathscr{L}_i$ ($\widehat{\mathscr{A}}_i$, $\widehat{\mathscr{D}}_i$, and $\widehat{\mathscr{L}}_i$, resp.), respectively.

Definition 2. System (1) ((2), resp.) under control protocols (3) and (4) ((5) and (6), resp.) achieves consensus if there exist consensus gains $K, \widehat{K}$ such that, for any $x_i(0) \in \mathbb{R}^n$ and initial distribution of $\theta(0)$,

$$\lim_{t\to\infty}\mathrm{E}\left(\left\|x_i(t) - x_j(t)\right\|^2\right) = 0 \tag{8}$$

for any $i, j \in \{1, 2, \ldots, N\}$.

We say matrix A in (1) and (2) is Hurwitz (or stable) if every eigenvalue of A has strictly negative real part; that is, it belongs to $\mathbb{C}\setminus\mathbb{C}^+$. The pair (A, B) is called stabilizable if there exists $C \in \mathbb{R}^{m\times n}$ such that $A + BC$ is Hurwitz [38].

Assumption 3. The following assumptions are made throughout the paper:

(a) Communication graphs $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_s$ are balanced.

(b) Matrix A is not Hurwitz.

(c) The white noises $w_i(t) \overset{\mathrm{d}}{=} w_{ij}(t)$ for all $i, j = 1, \ldots, N$, where $\overset{\mathrm{d}}{=}$ means equality in distribution.

(d) The Markov process $\theta(t)$ is independent of the Brownian motions $w_i(t)$ $(i = 1, \ldots, N)$.

Item (a) in Assumption 3 is also used in [10, 21]. Item (b) is meant to eliminate the triviality, since consensus can be reached by setting zero consensus gains if A is Hurwitz. In this paper, we assume a special sort of noise—possibly due to the homogeneity of the communication channels between each agent and its neighbors—in which $w_{ij}(t)$ is independent of j as item (c) indicated. The consensus in Definition 2 is defined in the sense of mean square convergence. This implies that the consensus can also be achieved in the almost sure sense in view of item (d) and the homogeneity of the Markov process (see Corollary 3.46 of [39, 40]).

2.2. Exponential Stability for Delay Markovian Jump Systems. Denote by $(\Omega, \mathscr{F}, \mathscr{P})$ the underlying common probability space for the Markov process and Brownian motions discussed above. The homogeneous continuous-time Markov process $\theta(t)$ with generator $Q = (q_{ij}) \in \mathbb{R}^{s\times s}$ is formally given by

$$\begin{aligned} &\mathrm{P}\left(\theta(t+h) = j \mid \theta(t) = i\right) \\ &\quad = \begin{cases} q_{ij}h + o(h), & \text{if } i \neq j, \\ 1 + q_{ii}h + o(h), & \text{if } i = j, \end{cases} \end{aligned} \tag{9}$$

where $h > 0$ and $h \to 0$. Here q_{ij} is the transition rate from i to j if $i \neq j$, while $q_{ii} = -\sum_{j\neq i} q_{ij}$. As is known, the state space $S = \{1, 2, \ldots, s\}$ of $\theta(t)$ can be decomposed uniquely into the form $S = \{J \cup S_1 \cup \cdots \cup S_r\}$, where each S_j $(j = 1, \ldots, r)$ is a closed communication class (i.e., closed set in the Markov process) of positive recurrent states and J is a set of transient states [41].

Let $x(t) \in \mathbb{R}^n$ for $t \geq 0$. Consider a stochastic differential delay equation with Markovian switching of the form

$$\begin{aligned} dx(t) = {} & f(x(t), x(t-\tau(t)), \theta(t))\, dt \\ & + g(x(t), x(t-\tau(t)), \theta(t))\, dw(t), \end{aligned} \tag{10}$$

where $w(t)$ is an m-dimensional standard Brownian motion and $f : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous satisfying the following.

Assumption 4. For each $i \in S$, there exist constants $\alpha_i \in \mathbb{R}$ and $\rho_i, \gamma_i, \beta_i \geq 0$ such that, for all $x, y \in \mathbb{R}^n$,

(a) $2x^T f(x, 0, i) \leq \alpha_i \|x\|^2$;

(b) $\|f(x, 0, i) - f(x, y, i)\| \leq \rho_i \|y\|$;

(c) $\|g(x, y, i)\|^2 \leq \gamma_i \|x\|^2 + \beta_i \|y\|^2$.

Define $F = \mathrm{diag}(-\alpha_1 - \rho_1 - \gamma_1, \ldots, -\alpha_s - \rho_s - \gamma_s) - Q \in \mathbb{R}^{s\times s}$. A square matrix F is called a nonsingular M-matrix if all the off-diagonal entries are nonpositive and F^{-1} is a nonnegative matrix. A list of equivalent conditions for M-matrix can be found in [41]. The following result establishes the exponential stability of system (10).

Lemma 5 (see [42]). *Assume that F is a nonsingular M-matrix and $((\rho_1 + \beta_1)^{-1}, \ldots, (\rho_s + \beta_s)^{-1})^T - F^{-1}\mathbf{1}_s \in \mathbb{R}^s$ is a positive vector. Then the trivial solution $x(t) \equiv 0$ of (10) is exponentially stable in mean square if*

$$\dot{\tau}(t) < 1 - \kappa, \tag{11}$$

for all $t \geq 0$, where $\kappa = \max_{1\leq i\leq s} b_i(\rho_i + \beta_i)$ and $(b_1, \ldots, b_s)^T = F^{-1}\mathbf{1}$.

It is obvious that $\kappa < 1$. Therefore, unbounded time delay is allowed. This is a desirable feature, for example, in delayed cellular neural networks, where delays are variable and in effect unbounded [43–46].

Remark 6. The existence of solutions to dynamical systems (1) and (2), in an even more general nonlinear setting, has been studied in [47]. Let $Z(t) = (x_1^T(t), \ldots, x_N^T(t))^T \in \mathbb{R}^{nN}$. For $\tau > 0$, let $\tau(t) : [0, \infty) \to [0, \tau]$ be a continuous function. Denote by $C([-\tau, 0]; \mathbb{R}^{nN})$ the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^{nN}$. It is shown that [47] for any bounded and $\mathscr{F}$-measurable initial condition $Z_0 \in C([-\tau, 0]; \mathbb{R}^{nN})$ system (1) (as well as (2)) has a unique continuous solution $Z(t; Z_0)$ on $t \geq -\tau$. Our strategy in the sequel is to first transfer the systems to error dynamics (see (13) and (26) below) and then address the stability of zero solution utilizing Lemma 5.

The objective of this paper is to reveal how stability analysis of differential delay equations, together with techniques used in matrix, Markov chain, and graph theory, can be applied to investigate stochastic consensus problems (1) and (2).

3. Main Results

In this section, we derive necessary and sufficient conditions for reaching consensus of noisy linear systems (1) and (2) under Markovian switching topologies and time-varying delays.

3.1. Consensus Conditions for Systems with Self-Delay. We first consider multiagent system (1) with protocols (3) and (4), where both self-delay and neighboring delay are factored in.

Let $\mathscr{L}(t) = (l_{ij}(t))$ and $\widehat{\mathscr{L}}(t) = (\widehat{l}_{ij}(t))$ signify the Laplacian matrices for the switching topologies $\mathscr{G}(t)$ and $\widehat{\mathscr{G}}(t)$ at time t. For $i = 1, \ldots, N$, define the disagreement error for agent v_i as $\delta_i(t) = x_i(t) - \overline{x}(t)$, where $\overline{x}(t) = (1/N)\sum_{i=1}^N x_i(t)$ is the average state vector. Rearranging (1) with (3) and (4) gives

$$\begin{aligned} \dot{\delta}_i(t) = {} & A\delta_i(t-\tau(t)) + BK\sum_{j=1}^N l_{ij}(t)\,\delta_j(t-\tau(t)) \\ & + \left(\widehat{A}\delta_i(t-\tau(t)) + \widehat{B}\widehat{K}\sum_{j=1}^N \widehat{l}_{ij}(t)\,\delta_j(t-\tau(t))\right) \\ & \cdot \dot{w}_i(t), \end{aligned} \tag{12}$$

where we have used Assumption 3(a). Set $\delta(t) = (\delta_1^T(t), \ldots, \delta_N^T(t))^T \in \mathbb{R}^{nN}$ and $w(t) = (w_1(t), \ldots, w_N(t))^T \in \mathbb{R}^N$.

Moreover, let $\delta'(t) = \operatorname{diag}(\delta_1(t), \ldots, \delta_N(t))$ be an $nN \times N$ block diagonal matrix. Consequently, (12) can be recast as

$$\begin{aligned} d\delta(t) &= \left(I_N \otimes A - \mathscr{L}(t) \otimes BK\right) \delta(t - \tau(t))\, dt \\ &\quad + \left(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}(t) \otimes \widehat{B}\widehat{K}\right) \delta'(t - \tau(t))\, dw(t). \end{aligned} \tag{13}$$

Theorem 7 (necessary conditions). *Suppose that Assumption 3 holds and system (1) with protocols (3) and (4) achieves consensus. Furthermore, assume that there exist $\xi_i > 0$ for $i = 1, \ldots, s$, and $\zeta_1, \zeta_2 > 0$ such that $\widehat{\mathscr{A}}_i = \xi_i \mathscr{A}_i$, $\widehat{A} = \zeta_1 A$, and $\widehat{B} = \zeta_2 B$. Then*

(a) *each union of the graphs $\mathscr{G}_i$ $(i = 1, \ldots, s)$ corresponding to all the states in the closed set S_j for $j = 1, \ldots, r$ has a spanning tree;*

(b) *(A, B) is stabilizable.*

Proof. To prove (a), let $\mathscr{G}_{S_1} = \bigcup_{i \in S_1} \mathscr{G}_i$ be the union graph corresponding to the closed set S_1. Without loss of generality, assume that $\mathscr{G}_{S_1}$ does not contain a spanning tree. Denote by $\mathscr{L}_{S_1}$ its Laplacian matrix. By Assumption 3(a), $\mathscr{L}_{S_1}^T$ is also a Laplacian matrix. It follows from Corollary 4.2 of [48] that there is a unit vector $c = (c_1, \ldots, c_N)^T \in \mathbb{R}^N$ such that $c^T \mathbf{1} = 0$ and $\mathscr{L}_{S_1}^T c = \mathbf{0}$. Using Lemma 3.5 of [21], we obtain

$$\operatorname{Null}\left(-\mathscr{L}_{S_1}^T\right) = \bigcap_{i \in S_1} \operatorname{Null}\left(-\mathscr{L}_i^T\right). \tag{14}$$

Hence, $\mathscr{L}_i^T c = \mathbf{0}$ for any $i \in S_1$. There exists some Φ_1 such that $\Phi = (\mathbf{1}/\sqrt{N}, c, \Phi_1) \in \mathbb{R}^{N \times N}$ is an orthogonal matrix.

Define $\widetilde{\delta}(t) = (\Phi^{-1} \otimes I_n)\delta(t)$. By partitioning $\widetilde{\delta}(t)$ in line with $\delta(t)$, that is, $\widetilde{\delta}(t) = (\widetilde{\delta}_1^T(t), \ldots, \widetilde{\delta}_N^T(t))^T$ with each $\widetilde{\delta}_i(t) \in \mathbb{R}^n$, we get $\widetilde{\delta}_1(t) \equiv \mathbf{0}$ and $\widetilde{\delta}_2(t) = \sum_{i=1}^N c_i \delta_i(t)$ by definition. Moreover, by using (13) we have for $t \geq 0$

$$\begin{aligned} \dot{\widetilde{\delta}}(t) &= \left(\Phi^{-1} \otimes A\right) \delta(t - \tau(t)) \\ &\quad - \left(\Phi^{-1} \mathscr{L}(t) \otimes BK\right) \delta(t - \tau(t)) \\ &\quad + \left(\Phi^{-1} \otimes \widehat{A}\right) \delta'(t - \tau(t))\, \dot{w}(t) \\ &\quad - \left(\Phi^{-1} \widehat{\mathscr{L}}(t) \otimes \widehat{B}\widehat{K}\right) \delta'(t - \tau(t))\, \dot{w}(t). \end{aligned} \tag{15}$$

If the initial value $\theta(0) \in S_1$, then $\mathscr{L}(t) \in \{\mathscr{L}_i\}_{i \in S_1}$ and $\widehat{\mathscr{L}}(t) \in \{\widehat{\mathscr{L}}_i\}_{i \in S_1}$. Using the assumption $\widehat{\mathscr{A}}_i = \xi_i \mathscr{A}_i$, we derive

$$\dot{\widetilde{\delta}}_2(t) = A\widetilde{\delta}_2(t - \tau(t)) + \widehat{A} \sum_{i=1}^N c_i \delta_i(t - \tau(t))\, \dot{w}_i(t) \tag{16}$$

for $t \geq 0$.

If $A \neq 0$, take $\widetilde{\delta}_2(-\tau(0)) \notin \operatorname{Null}(A)$. Thus, $\lim_{t \to \infty} \mathrm{E}\|\widetilde{\delta}_2(t)\| \neq 0$ under Assumption 3(b). This contradicts with the consensus definition.

If $A = 0$, we obtain $\mathrm{E}\dot{\widetilde{\delta}}_2(t) = \mathbf{0}$ and recall $\widetilde{\delta}_2(t) = \sum_{i=1}^N c_i \delta_i(t)$. Since $c^T \mathbf{1} = 0$, there exist two components $c_i \neq 0$ and $c_j \neq 0$ satisfying $c_i \neq c_j$. Take $\delta_i(0) = \mathbf{1}$, $\delta_j(0) = -\mathbf{1}$, and $\delta_k(0) = \mathbf{0}$ for all $k \in \{1, \ldots, N\} \setminus \{i, j\}$. It yields that $\lim_{t \to \infty} \mathrm{E}\|\widetilde{\delta}_2(t)\| \neq 0$. Again, it results in a contradiction.

To prove (b), assume that (A, B) is not stabilizable. Then A has an unstable and uncontrollable mode, denoted by $\lambda \in \mathbb{C}^+$. It follows from the Popov-Belevitch-Hautus controllability test [38] that there exists some complex vector $c \neq \mathbf{0}$ satisfying $c^H A = c^H \lambda$ and $c^H B = \mathbf{0}$.

Let $y(t) = (y_1(t), \ldots, y_N(t))^T = (I_N \otimes c)^H \delta(t) \in \mathbb{R}^N$. By means of (13) and the assumptions that $\widehat{A} = \zeta_1 A$ and $\widehat{B} = \zeta_2 B$, we derive

$$\begin{aligned} \dot{y}(t) &= \left(I_N \otimes c^H A\right) \delta(t - \tau(t)) \\ &\quad - \left(\mathscr{L}(t) \otimes c^H BK\right) \delta(t - \tau(t)) \\ &\quad + \left(I_N \otimes c^H \widehat{A}\right) \delta'(t - \tau(t))\, \dot{w}(t) \\ &\quad - \left(\widehat{\mathscr{L}}(t) \otimes c^H \widehat{B}\widehat{K}\right) \delta'(t - \tau(t))\, \dot{w}(t) \\ &= \lambda y(t - \tau(t)) \\ &\quad + \lambda \zeta_1 \operatorname{diag}\left(y_1(t - \tau(t)), \ldots, y_N(t - \tau(t))\right) \dot{w}(t). \end{aligned} \tag{17}$$

Taking $y(-\tau(0)) \neq \mathbf{0}$, we see that $\liminf_{t \to \infty} \mathrm{E}\|y(t)\| > 0$ since λ is unstable. This contradicts with the consensus definition. □

Note that the assumption $\widehat{\mathscr{A}}_i = \xi_i \mathscr{A}_i$ $(i = 1, \ldots, s)$ is only used in the proof of statement (a), while the assumptions $\widehat{A} = \zeta_1 A$ and $\widehat{B} = \zeta_2 B$ are only used in the proof of statement (b).

If (A, B) is stabilizable, there exists an $n \times n$ matrix $P > 0$ such that

$$P > A^T PA - A^T PB\left(B^T PB\right)^{-1} B^T PA \tag{18}$$

by the Riccati inequality [49]. Similarly, if $(\widehat{A}, \widehat{B})$ is stabilizable, there exists an $n \times n$ matrix $\widehat{P} > 0$ such that

$$\widehat{P} > \widehat{A}^T \widehat{P}\widehat{A} - \widehat{A}^T \widehat{P}\widehat{B}\left(\widehat{B}^T \widehat{P}\widehat{B}\right)^{-1} \widehat{B}^T \widehat{P}\widehat{A}. \tag{19}$$

Theorem 8 (sufficient conditions). *Suppose that Assumption 3 holds, both (A, B) and $(\widehat{A}, \widehat{B})$ are stabilizable, and $\mathscr{G}_i$ contains a spanning tree for every $i \in S$. If $F = -\rho I_s - Q$ is a nonsingular M-matrix and $(\rho + \beta)^{-1}\mathbf{1}_s - F^{-1}\mathbf{1}_s$ is a positive vector, then system (1) with protocols (3) and (4) achieves consensus for $\tau(t)$ satisfying*

$$\dot{\tau}(t) < 1 - (\rho + \beta) \cdot \left\|F^{-1}\mathbf{1}_s\right\|_{\max}, \tag{20}$$

where $\rho = \sqrt{\lambda_n(P)/\lambda_1(P)}$, $\beta = \lambda_n(\widehat{P})/\lambda_1(\widehat{P})$, and P and $\widehat{P}$ are given by (18) and (19), respectively.

Moreover, feasible consensus gains K and $\widehat{K}$ are given by $K = \eta(B^T PB)^{-1} B^T PA$ with $\eta \geq 1/\min_{2 \leq j \leq N, 1 \leq i \leq s} |\lambda_j(\mathscr{L}_i)|$ and $\widehat{K} = \widehat{\eta}(\widehat{B}^T \widehat{P}\widehat{B})^{-1} \widehat{B}^T \widehat{P}\widehat{A}$ with $\widehat{\eta} \geq 1/\min_{2 \leq j \leq N, 1 \leq i \leq s} |\lambda_j(\widehat{\mathscr{L}}_i)|$, respectively.

Proof. The idea is to apply Lemma 5 to the error dynamics (13). It suffices to check Assumption 4 holds.

For item (a), $\alpha_i = 0$ for all $i \in S$ clearly. To check item (b), we take a unitary matrix for each $i \in S$: $\Phi_i = (\mathbf{1}_N/\sqrt{N}, \phi_{i,2}, \ldots, \phi_{i,N})$, where $\phi_{i,j} \in \mathbb{C}^N$ satisfies $\mathscr{L}_i\phi_{i,j} = \lambda_j(\mathscr{L}_i)\phi_{i,j}$ for $j = 2, \ldots, N$. Given $i \in S$, by setting $\widetilde{\delta} = (\Phi_i^H \otimes I_n)\delta$ (and partitioning it in conformity with that of δ as in Theorem 7), we derive that

$$\begin{aligned}
&\delta^T\left(I_N \otimes A^T - \mathscr{L}_i^T \otimes K^T B^T\right)(I_N \otimes P) \\
&\quad \cdot (I_N \otimes A - \mathscr{L}_i \otimes BK)\,\delta \\
&\quad = \delta^T\left(I_N \otimes A^T PA - \mathscr{L}_i^T \mathscr{L}_i \otimes K^T B^T PBK\right)\delta \\
&\quad = \widetilde{\delta}^H\left(I_N \otimes A^T PA - \Phi_i^H \mathscr{L}_i^H \mathscr{L}_i \Phi_i \otimes K^T B^T PBK\right)\widetilde{\delta} \\
&\quad = \sum_{j=2}^{N} \widetilde{\delta}_j^H \\
&\quad \cdot \left(A^T PA - \left|\lambda_j(\mathscr{L}_i)\right|^2 \eta^2 A^T PB\left(B^T PB\right)^{-1} B^T PA\right)\widetilde{\delta}_j \\
&\quad \le \sum_{j=1}^{N} \widetilde{\delta}_j^H P \widetilde{\delta}_j = \delta^T (I_N \otimes P)\,\delta,
\end{aligned} \tag{21}$$

where we have taken $K = \eta(B^T PB)^{-1}B^T PA$ with $\eta \ge 1/\min_{2\le j\le N,\, 1\le i\le s}|\lambda_j(\mathscr{L}_i)|$ and used inequality (18). Note that η is well defined, since $\mathscr{G}_i$ contains a spanning tree, and thus there is only one zero eigenvalue of $\mathscr{L}_i$ [48].

Since $P > 0$, using (21) we obtain

$$\begin{aligned}
&\left\|(I_N \otimes A - \mathscr{L}_i \otimes BK)\,\delta\right\|^2 \le \frac{1}{\lambda_1(P)} \\
&\quad \cdot \delta^T\left(I_N \otimes A^T - \mathscr{L}_i^T \otimes K^T B^T\right)(I_N \otimes P) \\
&\quad \cdot (I_N \otimes A - \mathscr{L}_i \otimes BK)\,\delta \le \frac{1}{\lambda_1(P)}\delta^T (I_N \otimes P)\,\delta \\
&\quad \le \frac{\lambda_n(P)}{\lambda_1(P)}\|\delta\|^2 .
\end{aligned} \tag{22}$$

Hence, we take $\rho = \rho_i = \sqrt{\lambda_n(P)/\lambda_1(P)}$ independent of i.

To verify item (c), note that

$$\begin{aligned}
&\left\|\left(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}_i \otimes \widehat{B}\widehat{K}\right)\delta'\right\|^2 \\
&\quad = \operatorname{trace}\left(\delta'^T\left(I_N \otimes \widehat{A}^T - \widehat{\mathscr{L}}_i^T \otimes \widehat{K}^T\widehat{B}^T\right)\right. \\
&\quad \left.\cdot \left(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}_i \otimes \widehat{B}\widehat{K}\right)\delta'\right) = \delta^T\left(I_N \otimes \widehat{A}^T - \widehat{\mathscr{L}}_i^T\right. \\
&\quad \left.\otimes \widehat{K}^T\widehat{B}^T\right)\left(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}_i \otimes \widehat{B}\widehat{K}\right)\delta \\
&\quad = \left\|\left(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}_i \otimes \widehat{B}\widehat{K}\right)\delta\right\|^2 .
\end{aligned} \tag{23}$$

Therefore, similarly as in the proof of (21) and (22) we obtain

$$\left\|\left(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}_i \otimes \widehat{B}\widehat{K}\right)\delta'\right\|^2 \le \frac{\lambda_n\left(\widehat{P}\right)}{\lambda_1\left(\widehat{P}\right)}\|\delta\|^2 , \tag{24}$$

where we have taken $\widehat{K} = \widehat{\eta}(\widehat{B}^T\widehat{P}\widehat{B})^{-1}\widehat{B}^T\widehat{P}\widehat{A}$ with $\widehat{\eta} \ge 1/\min_{2\le j\le N,\, 1\le i\le s}|\lambda_j(\widehat{\mathscr{L}}_i)|$. Since the fact that $\mathscr{G}_i$ contains a spanning tree implies that $\widehat{\mathscr{G}}_i$ also contains a spanning tree, $\widehat{\eta}$ is well defined with the same reason as above. Hence, we take $\gamma_i = 0$ and $\beta = \beta_i = \lambda_n(\widehat{P})/\lambda_1(\widehat{P})$ for all $i \in S$. □

Remark 9. (a) The design of consensus gains K and $\widehat{K}$ splits the design problem from the underlying communication topology. For example, K is constructed based on the system matrices (18) and a multiplicative coefficient η depending only on the graphs. Such a design procedure decouples the effects of agent dynamics and the network topologies, which simplifies the consensus design for the cases where the number of agents is large (see also [10, 13]).

(b) When $n = m$, $A = 0$, and $B = I_n$, we reproduce single-integrator agent dynamics, and (18) and (19) always hold true. This can be viewed as a generalization of results in [21] by introducing random noise and time delay.

(c) The assumption in Theorem 8 about F being a nonsingular M-matrix and $(\rho + \beta)^{-1}\mathbf{1}_s - F^{-1}\mathbf{1}_s$ having positive entries is easy to verify. Indeed, all the off-diagonal entries are nonpositive by the definition of generator Q. Thus, it suffices to show that F^{-1} is nonnegative (this always holds if $n = 1$) and ensure that every row sum of it is less than $(\rho + \beta)^{-1}$.

(d) There is a gap pertaining to graph connectivity between sufficient conditions (Theorem 8) and necessary conditions (Theorem 7). Comparing with the previous work [10] for noise-free and delay-free systems, we understand that the stronger connectivity requirement—each graph $\mathscr{G}_i$ contains a spanning tree—is introduced to accommodate the added noises and time-varying delays. Notice that the results are based on Lemma 5, which is about nonlinear systems. This also suggests the conditions derived here could be conservative. Notwithstanding, the study of weaker sufficient condition (e.g., using some algebraic methods) comparable to that of the necessary condition is an interesting future research.

3.2. Consensus Conditions for Systems without Self-Delay. Next, we study multiagent system (2) with protocols (5) and (6), where only neighboring delay is considered.

Similarly as above, for $i = 1, \ldots, N$, define the disagreement error for agent v_i as $\delta_i(t) = x_i(t) - \overline{x}(t)$, and $\overline{x}(t) = (1/N)\sum_{i=1}^{N} x_i(t)$ is the average state vector. Under Assumption 3(a), (2) together with (5) and (6) yields

$$\begin{aligned}
\dot{\delta}_i(t) &= A\delta_i(t) + BK\sum_{j=1}^{N} a_{ij}(t)\left(\delta_j(t-\tau(t)) - \delta_i(t)\right) \\
&\quad + \left(\widehat{A}\delta_i(t-\tau(t))\right. \\
&\quad \left. + \widehat{B}\widehat{K}\sum_{j=1}^{N}\sigma_{ij}(t)\left(\delta_j(t-\tau(t)) - \delta_i(t)\right)\right)\dot{w}_i(t) .
\end{aligned} \tag{25}$$

Let $\delta(t) = (\delta_1^T(t), \ldots, \delta_N^T(t))^T \in \mathbb{R}^{nN}$, $w(t) = (w_1(t), \ldots, w_N(t))^T \in \mathbb{R}^N$, and $\delta'(t) = \mathrm{diag}(\delta_1(t), \ldots, \delta_N(t))$. We rewrite (25) in a compact form as

$$\begin{aligned} d\delta(t) = {} & \big((I_N \otimes A - \mathscr{D}(t) \otimes BK)\,\delta(t) \\ & + (\mathscr{A}(t) \otimes BK)\,\delta(t-\tau(t))\big)\,dt \\ & + \big(\big(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}(t) \otimes \widehat{B}\widehat{K}\big)\,\delta'(t) \\ & + \big(\widehat{\mathscr{A}}(t) \otimes \widehat{B}\widehat{K}\big)\,\delta'(t-\tau(t))\big)\,dw(t). \end{aligned} \tag{26}$$

Theorem 10 (necessary conditions). *Suppose that Assumption 3 holds and system (2) with protocols (5) and (6) achieves consensus for some constant delay τ. Furthermore, assume that there exist $\xi_i > 0$ for $i = 1, \ldots, s$, and $\zeta_1, \zeta_2 > 0$ such that $\widehat{\mathscr{A}}_i = \xi_i \mathscr{A}_i$, $\widehat{A} = \zeta_1 A$, and $\widehat{B} = \zeta_2 B$. Then*

(a) *each union of the graphs $\mathscr{G}_i$ $(i = 1, \ldots, s)$ corresponding to all the states in the closed set S_j for $j = 1, \ldots, r$ has a spanning tree;*

(b) *(A, B) is stabilizable.*

Proof. Since consensus is achieved, we have $\lim_{t\to\infty}\|\delta(t) - \delta(t-\tau)\| = 0$ and $\lim_{t\to\infty}\|\delta'(t) - \delta'(t-\tau)\| = 0$ for fixed τ. Therefore, system (26) can be recast as

$$\begin{aligned} d\delta(t) = {} & \big((I_N \otimes A - \mathscr{L}(t) \otimes BK)\,\delta(t-\tau) \\ & + O(\|\delta(t) - \delta(t-\tau)\|) \cdot \mathbf{1}_{Nn}\big)\,dt \\ & + \big(\big(I_N \otimes \widehat{A} - \widehat{\mathscr{L}}(t) \otimes \widehat{B}\widehat{K}\big)\,\delta'(t-\tau) \\ & + O\big(\big\|\delta'(t) - \delta'(t-\tau)\big\|\big) \cdot \mathbf{1}_{Nn}\big)\,dw(t). \end{aligned} \tag{27}$$

Hence, Theorem 10 can be proved by the same reasoning as in Theorem 7. □

As is noted below Theorem 7, the assumption $\widehat{\mathscr{A}}_i = \xi_i \mathscr{A}_i$ $(i = 1, \ldots, s)$ is only used in the proof of statement (a), while the assumptions $\widehat{A} = \zeta_1 A$ and $\widehat{B} = \zeta_2 B$ are only used in the proof of statement (b).

Recall that if (A, B) is stabilizable, there exists a $C \in \mathbb{R}^{m\times n}$ such that $A + BC$ is Hurwitz. Therefore, by the Lyapunov stability theorem, there exists an $n \times n$ matrix $P > 0$ such that

$$(A + BC)\,P + P\,(A + BC)^T = -I. \tag{28}$$

Define a symmetric matrix $R \in \mathbb{R}^{n\times n}$ by

$$R = (A + BC)^T P + P\,(A + BC). \tag{29}$$

Theorem 11 (sufficient conditions). *Suppose that Assumption 3 holds, both (A, B) and $(\widehat{A}, \widehat{B})$ are stabilizable, and $\mathscr{G}_i$ is d-regular $(d \geq 1)$ for any $i \in S$. If $F = -(\alpha + \gamma)I_s - \mathrm{diag}(\rho_1, \ldots, \rho_s) - Q$ is a nonsingular M-matrix and $((\rho_1 + \beta_1)^{-1}, \ldots, (\rho_s + \beta_s)^{-1})^T - F^{-1}\mathbf{1}_s$ is a positive vector, then system (2) with protocols (5) and (6) achieves consensus for $\tau(t)$ satisfying*

$$\dot{\tau}(t) < 1 - \max_{1\leq i\leq s} b_i\,(\rho_i + \beta_i), \tag{30}$$

where $(b_1, \ldots, b_s)^T = F^{-1}\mathbf{1}$, $\alpha = (\lambda_n(R) - 1)/(2\lambda_1(P))$, $\gamma = 2\lambda_n(\widehat{P})/\lambda_1(\widehat{P})$, $\rho_i = \sqrt{\lambda_{Nn}(\mathscr{A}_i^T \mathscr{A}_i \otimes K^T B^T BK)}$, and $\beta_i = 2\lambda_{Nn}(\widehat{\mathscr{A}}_i^T \widehat{\mathscr{A}}_i \otimes \widehat{K}^T \widehat{B}^T \widehat{B}\widehat{K})$. Here, P, $\widehat{P}$, and R are given by (28), (19), and (29), respectively.

Moreover, feasible consensus gains K and $\widehat{K}$ are given by $K = -C/d$ and $\widehat{K} = \widehat{\eta}(\widehat{B}^T \widehat{P} \widehat{B})^{-1} \widehat{B}^T \widehat{P} \widehat{A}$ with $\widehat{\eta} \geq 1/\min_{1\leq i\leq s,\, 1\leq j\leq N} \sigma_j^i$, respectively. Here, $\widehat{\mathscr{D}}_i = \mathrm{diag}(\sigma_1^i, \ldots, \sigma_N^i)$ for $i = 1, \ldots, s$.

Proof. As in the proof of Theorem 8, we will apply Lemma 5 to the error dynamics (26).

To check item (a) in Assumption 4, we note that

$$\begin{aligned} 2\delta^T (I_N \otimes A - \mathscr{D}_i \otimes BK)\,\delta = {} & \delta^T \big(I_N \otimes A - \mathscr{D}_i \otimes BK \\ & + I_N \otimes A^T - \mathscr{D}_i \otimes K^T B^T\big)\,\delta \leq \lambda_{Nn}\big(I_N \otimes A - \mathscr{D}_i \\ & \otimes BK + I_N \otimes A^T - \mathscr{D}_i \otimes K^T B^T\big)\,\|\delta\|^2 = \lambda_n\big(A \\ & - dBK + (A - dBK)^T\big)\,\|\delta\|^2, \end{aligned} \tag{31}$$

where we have used the assumption that $\mathscr{G}_i$ is d-regular and the Rayleigh quotient inequality. Taking $K = -C/d$ in (31) and utilizing (28) and (29), we get

$$\begin{aligned} 2\delta^T (I_N \otimes A - \mathscr{D}_i \otimes BK)\,\delta \leq {} & \lambda_n\big(A + BC \\ & + (A + BC)^T\big)\,\|\delta\|^2 = \frac{1}{2\lambda_1(P)} \\ & \cdot \lambda_n\big(\big(A + BC + (A + BC)^T\big)\,\lambda_1(P)\,I \\ & + \lambda_1(P)\,I\,\big(A + BC + (A + BC)^T\big)\big)\,\|\delta\|^2 \leq \frac{1}{2\lambda_1(P)} \\ & \cdot \lambda_n\big(\big(A + BC + (A + BC)^T\big)\,P \\ & + P\,\big(A + BC + (A + BC)^T\big)\big)\,\|\delta\|^2 = \frac{1}{2\lambda_1(P)}\lambda_n(R \\ & - I)\,\|\delta\|^2. \end{aligned} \tag{32}$$

Therefore, we take $\alpha = \alpha_i = (\lambda_n(R) - 1)/(2\lambda_1(P))$ for all $i \in S$.

For item (b), again by applying the Rayleigh quotient inequality we derive

$$\begin{aligned} \|(\mathscr{A}_i \otimes BK)\,\delta\|^2 &= \delta^T \big(\mathscr{A}_i^T \otimes K^T B^T\big)\,(\mathscr{A}_i \otimes BK)\,\delta \\ &\leq \lambda_{Nn}\big(\mathscr{A}_i^T \mathscr{A}_i \otimes K^T B^T BK\big)\,\|\delta\|^2 \\ &= \frac{1}{d^2}\lambda_{Nn}\big(\mathscr{A}_i^T \mathscr{A}_i \otimes C^T B^T BC\big)\,\|\delta\|^2. \end{aligned} \tag{33}$$

Therefore, we take $\rho_i = (1/d)\sqrt{\lambda_{Nn}(\mathscr{A}_i^T \mathscr{A}_i \otimes C^T B^T BC)}$ for $i \in S$.

To show (c), we recall the simple norm inequality $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$. It suffices to find suitable γ_i and β_i so that the following two inequalities hold for $i \in S$:

$$2\,\big\|\big(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\big)\,\delta'\big\|^2 \leq \gamma_i\,\|\delta'\|^2, \tag{34}$$

$$2\,\big\|\big(\widehat{\mathscr{A}}_i \otimes \widehat{B}\widehat{K}\big)\,\delta'\big\|^2 \leq \beta_i\,\|\delta'\|^2. \tag{35}$$

Since $(\widehat{A}, \widehat{B})$ is stabilizable, we get $\widehat{P} > 0$ such that (19) holds. Hence,

$$\begin{aligned}
&\delta^T \left(I_N \otimes \widehat{A}^T - \widehat{\mathscr{D}}_i^T \otimes \widehat{K}^T \widehat{B}^T\right)\left(I_N \otimes \widehat{P}\right) \\
&\quad \cdot \left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta \\
&\quad = \delta^T \left(I_N \otimes \widehat{A}^T \widehat{P}\widehat{A} - \widehat{\mathscr{D}}_i^T \widehat{\mathscr{D}}_i \otimes \widehat{K}^T \widehat{B}^T \widehat{B}\widehat{K}\right)\delta = \sum_{j=1}^{N} \delta_j^T \\
&\quad \cdot \left(\widehat{A}^T \widehat{P}\widehat{A} - \left(\sigma_j^i\right)^2 \widehat{\eta}^2 \widehat{A}^T \widehat{P}\widehat{B}\left(\widehat{B}^T \widehat{P}\widehat{B}\right)^{-1} \widehat{B}^T \widehat{P}\widehat{A}\right)\delta_j \\
&\quad \le \sum_{j=1}^{N} \delta_j^T \widehat{P}\delta_j = \delta^T \left(I_N \otimes \widehat{P}\right)\delta,
\end{aligned} \tag{36}$$

where we have taken $\widehat{K} = \widehat{\eta}(\widehat{B}^T \widehat{P}\widehat{B})^{-1}\widehat{B}^T \widehat{P}\widehat{A}$ with $\widehat{\eta} \ge 1/\min_{1\le i\le s,\, 1\le j\le N}\sigma_j^i$. Here $\{\sigma_j^i\}$ are defined as in the statement of Theorem 11. Since $d \ge 1$, we have $\sigma_j^i > 0$ for all i and j. Therefore, $\widehat{\eta}$ is well defined.

Note that

$$\begin{aligned}
&\left\|\left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta'\right\|^2 \\
&\quad = \operatorname{trace}\left(\delta'^T \left(I_N \otimes \widehat{A}^T - \widehat{\mathscr{D}}_i^T \otimes \widehat{K}^T \widehat{B}^T\right)\right. \\
&\quad \left.\cdot \left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta'\right) = \delta^T \left(I_N \otimes \widehat{A}^T - \widehat{\mathscr{D}}_i^T\right. \\
&\quad \left.\otimes \widehat{K}^T \widehat{B}^T\right)\left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta \\
&\quad = \left\|\left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta\right\|^2.
\end{aligned} \tag{37}$$

Using (36) we obtain

$$\begin{aligned}
&\left\|\left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta'\right\|^2 \le \frac{1}{\lambda_1\left(\widehat{P}\right)} \\
&\quad \cdot \delta^T \left(I_N \otimes \widehat{A}^T - \widehat{\mathscr{D}}_i^T \otimes \widehat{K}^T \widehat{B}^T\right)\left(I_N \otimes \widehat{P}\right) \\
&\quad \cdot \left(I_N \otimes \widehat{A} - \widehat{\mathscr{D}}_i \otimes \widehat{B}\widehat{K}\right)\delta \le \frac{1}{\lambda_1\left(\widehat{P}\right)}\delta^T \left(I_N \otimes \widehat{P}\right)\delta \\
&\quad \le \frac{\lambda_n\left(\widehat{P}\right)}{\lambda_1\left(\widehat{P}\right)}\delta^T\delta \le \frac{\lambda_n\left(\widehat{P}\right)}{\lambda_1\left(\widehat{P}\right)}\left\|\delta'\right\|^2.
\end{aligned} \tag{38}$$

Therefore, (34) holds by taking $\gamma = \gamma_i = 2\lambda_n(\widehat{P})/\lambda_1(\widehat{P})$ for any $i \in S$.

Similarly, (35) is true with $\beta_i = 2\lambda_{Nn}(\widehat{\mathscr{A}}_i^T \widehat{\mathscr{A}}_i \otimes \widehat{K}^T \widehat{B}^T \widehat{B}\widehat{K})$ by using the Rayleigh quotient inequality. □

Remark 12. (a) Similar comments in Remark 9 can be applied here. In addition, we note that the requirement in Theorem 11—$\mathscr{G}_i$ are d-regular graphs for all i—is somewhat restrictive. If this condition is violated, we might use Weyl's inequality (see, e.g., [50]) to bound the maximum eigenvalue in (31), which nonetheless will lead to a more cumbersome expression.

(b) Interestingly, sufficient conditions in Theorem 11 do not explicitly mention connectivity assumptions, whereas necessary conditions in Theorem 10 clearly state certain connectivity assumption. To see how some connectivity is implicitly required in Theorem 11, we consider a special case with $n = m = 1$, $S = \{1, 2\}$, and $\mathscr{G}_i$ $(i \in S)$ being not connected and having two connected components, out of which one is a complete graph with $d + 1$ nodes. We can show that matrix F in Theorem 11 is not a nonsingular M-matrix, violating the assumption of Theorem 11. Indeed, it is straightforward to check that $\gamma = 2$, $\rho_1 = \rho_2 = d|BK|$, and $\alpha = 2(A + BC) < 0$ with $A \ge 0$. Setting $Q = \left(\begin{smallmatrix} -a & a \\ b & -b \end{smallmatrix}\right)$ with $a, b > 0$, we obtain the 2×2 matrix F. F being a nonsingular M-matrix is equivalent to the fact that all its leading principal minors are positive [41], which in turn yields $-2(A + BC) - 2 - d|BK| > 0$. However, this inequality does not hold for any $K \in \mathbb{R}$ when $A + BC \ge -1$.

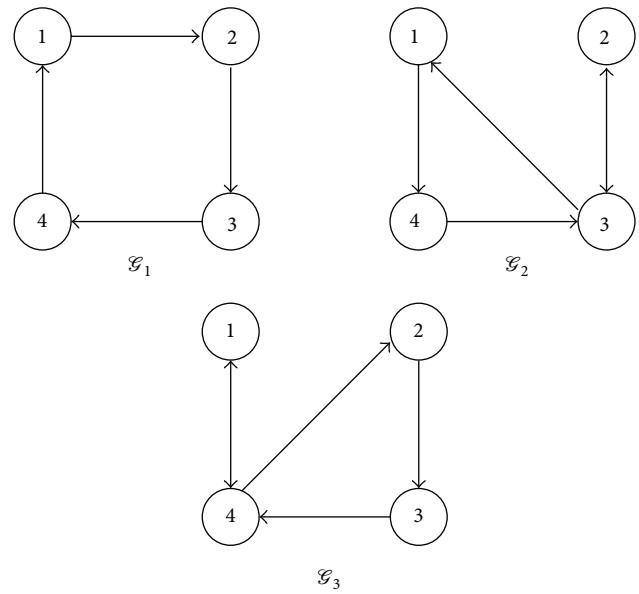


Figure 1: Communication topologies $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$ for system (1).

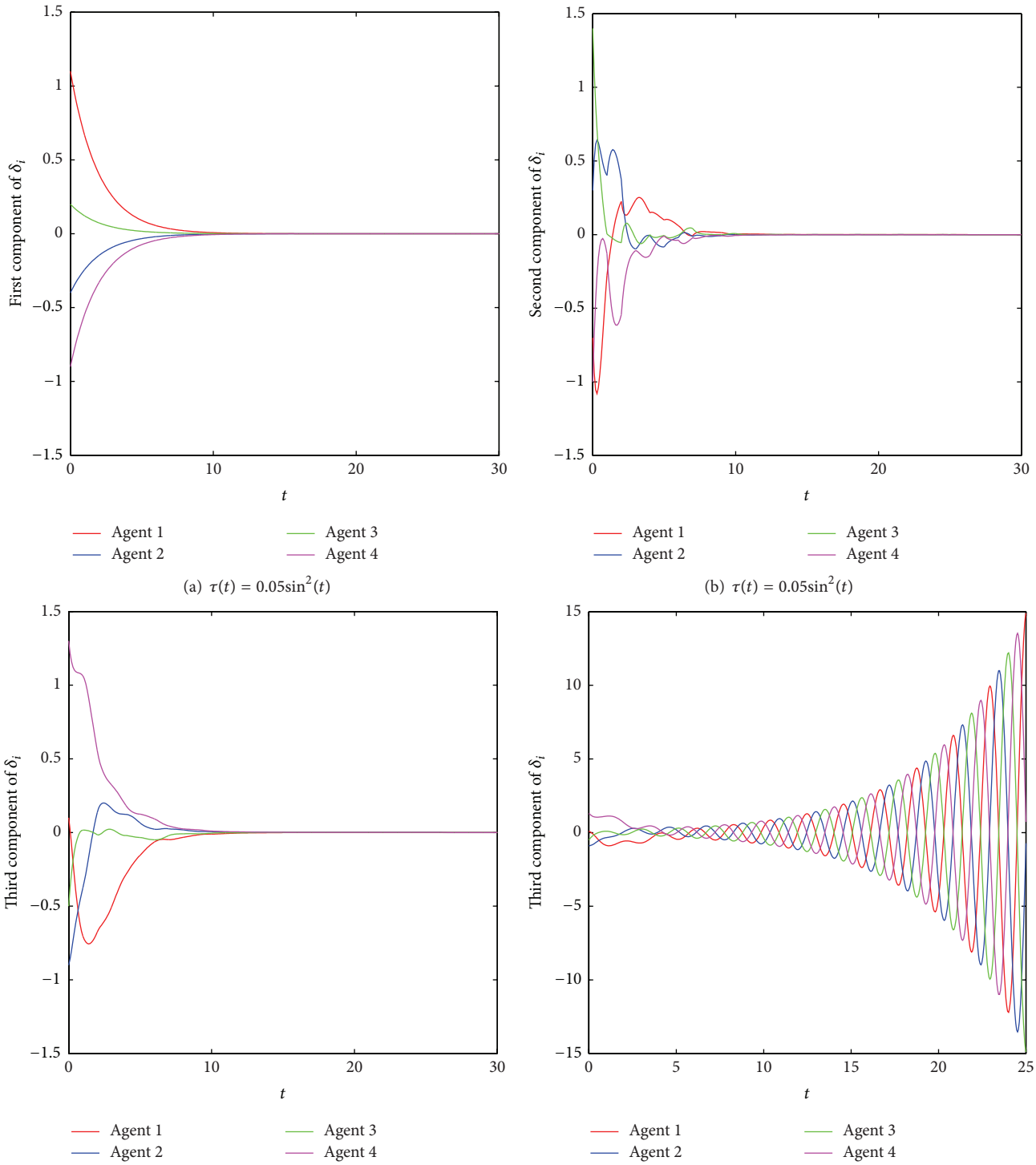
4. Simulation Examples

In this section, we present several examples to illustrate the availability of the proposed results. We consider all the adjacency matrices as binary 0-1 matrices in the following.

Example 1 (system with self-delay). Consider multiagent system (1) with $N = 4$ agents. The communication topologies will randomly switch amongst the triad $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$ of Figure 1 following a homogeneous Markovian process with generator

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 1 & 1 & -2 \end{pmatrix} \tag{39}$$

and state space $S = \{1, 2, 3\}$. Note that each of these three graphs is balanced and contains a spanning tree. For



(a) $\tau(t) = 0.05\sin^2(t)$ (b) $\tau(t) = 0.05\sin^2(t)$

(c) $\tau(t) = 0.05\sin^2(t)$ (d) $\tau(t) = 0.2\sin^2(t)$

FIGURE 2: Consensus errors of multiagent system (1) with respect to different time-delay $\tau(t)$.

$i = 1, 2, 3$, we choose $\widehat{\mathscr{A}}_i = 0.01 \cdot \mathscr{A}_i$. Take $n = 3$, $m = 1$ and let the agent dynamics be specified as

$$A = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \widehat{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{40}$$

and $\widehat{A} = 0.01 \cdot A$. Since $A + BC$ and $\widehat{A} + \widehat{B}\widehat{C}$ are Hurwitz by choosing $C = (-1, -2, -1)$ and $\widehat{C} = 0.01 \cdot C$, both pairs (A, B)

and $(\widehat{A}, \widehat{B})$ are stabilizable. Since Riccati inequalities (18) and (19) can be written as linear matrix inequalities (LMIs) (see, e.g., [51]), we obtain $P = \widehat{P} = \operatorname{diag}(1, 1.1, 1)$ by using LMI Toolbox in MATLAB. We calculate

$$F^{-1} = \begin{pmatrix} 0.1807 & 0.1295 & 0.0904 \\ 0.3168 & 0.0565 & 0.0132 \\ 0.0168 & 0.2832 & 0.0681 \end{pmatrix}, \tag{41}$$

which is a nonnegative matrix. In view of Theorem 8, we solve the consensus gains as $K = \widehat{K} = (2, 2.5, 2)$.

According to Theorem 8, multiagent system (1) with protocols (3) and (4) achieves consensus for $\tau(t)$ satisfying

$$\begin{aligned} \dot{\tau}(t) &< 1 - (\rho + \beta) \cdot \left\| F^{-1} \mathbf{1}_s \right\|_{\max} \\ &= 1 - \left(\sqrt{1.1} + 1.1\right) \times 0.4006 = 0.1392. \end{aligned} \tag{42}$$

With the initial state of each agent being taken randomly from $[-1, 1]$, we show in Figures 2(a)–2(c) a sample path of the consensus errors δ_i $(i = 1, \ldots, 4)$ by choosing $\tau(t) = 0.05\sin^2(t)$. Clearly, (42) is satisfied, and consensus is achieved. In Figure 2(d), we show only the third component of δ_i $(i = 1, \ldots, 4)$ with $\tau(t) = 0.2\sin^2(t)$. This delay violates condition (42), and consensus is not achieved.

Example 2 (system without self-delay). Consider multiagent system (2) with $N = 4$ agents. The communication topologies switch amongst $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ of Figure 3 following the same Markovian process as defined in Example 1. Note that each of these three graphs is balanced and d-regular with $d = 1$. $\mathcal{G}_1$ and $\mathcal{G}_3$ contain spanning trees but $\mathcal{G}_2$ does not. The adjacency matrices $\mathcal{A}_i, \widehat{\mathcal{A}}_i$ $(i = 1, 2, 3)$ and the agent dynamics $A, B, \widehat{A}, \widehat{B}$ are taken as in Example 1. Solving Lyapunov equation (28) by using LYAP function in MATLAB, we obtain

$$P = \begin{pmatrix} 1 & -0.2857 & -0.5714 \\ -0.2857 & 1.2857 & -0.5000 \\ -0.5714 & -0.5000 & 2.3571 \end{pmatrix}, \tag{43}$$

and $\widehat{P} = \operatorname{diag}(1, 1.1, 1)$ as in Example 1. It follows from (29) that

$$R = \begin{pmatrix} -1 & 0.2857 & 0 \\ 0.2857 & -2.5714 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

We calculate $\alpha = -0.9690$, $\gamma = 2.2$, and a nonnegative matrix

$$F^{-1} = \begin{pmatrix} 0.2307 & 0.1453 & 0.0786 \\ 0.1722 & 0.1204 & 0.2548 \\ 0.0470 & 0.3793 & 0.0682 \end{pmatrix}. \tag{45}$$

From Theorem 11, we solve the consensus gains as $K = \widehat{K} = (1, 2, 1)$.

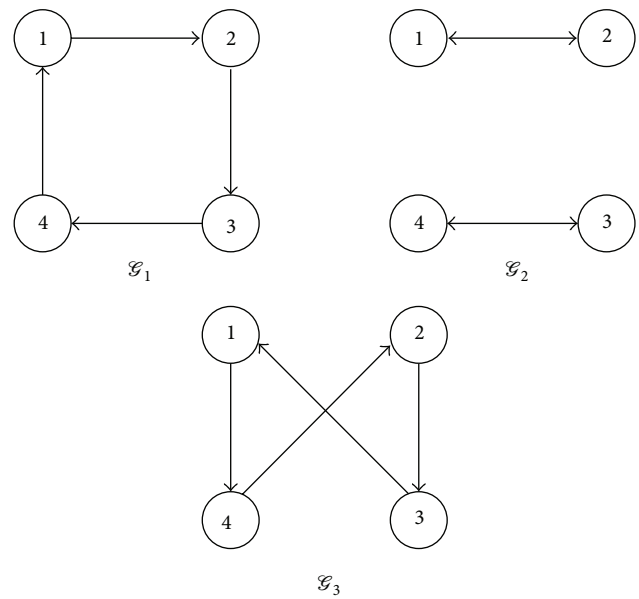


Figure 3: Communication topologies $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ for system (2).

In light of Theorem 11, multiagent system (2) with protocols (5) and (6) achieves consensus for $\tau(t)$ satisfying

$$\begin{aligned} \dot{\tau}(t) &< 1 - \max_{1 \le i \le 3} b_i (\rho_i + \beta_i) \\ &= 1 - \max\{0.2510, 0.7557, 0.0586\} = 0.2443. \end{aligned} \tag{46}$$

With the initial state of each agent being taken randomly from $[-1, 1]$, we show in Figures 4(a)–4(c) a sample path of the consensus errors δ_i $(i = 1, \ldots, 4)$ by choosing $\tau(t) = 0.1\cos^2(t)$. Since (46) is satisfied, consensus is achieved. We plot in Figure 4(d) only the third component of δ_i $(i = 1, \ldots, 4)$ with $\tau(t) = 0.3\cos^2(t)$. Condition (46) does not hold, and consensus is not achieved.

Example 3 (tightness of bounds). In this example, we consider multiagent system (1) with $N = 1000$ agents. The communication topologies will randomly switch amongst 10 random graphs $\mathcal{G}_1, \ldots, \mathcal{G}_{10}$ in the classical Erdős-Rényi $\mathcal{G}(N, p)$ model with p being the connection probability between any two nodes [52]. The generator of the underlying Markovian process $\theta(t)$ with $S = \{1, 2, \ldots, 10\}$ is taken as $Q = -10 I_{10} + \mathbf{1}_{10} \mathbf{1}_{10}^T$. The adjacency matrices $\mathcal{A}_i, \widehat{\mathcal{A}}_i$ $(i = 1, 2, \ldots, 10)$ and the agent dynamics $A, B, \widehat{A}, \widehat{B}$ are taken as in Example 1. Moreover, we fix $\tau(t) \equiv 0.05$.

The program stops if all components of δ_i $(i = 1, 2, \ldots, 10)$ are less than 10^{-5}; if the program does not stop before $t = 10^4$, we regard that the consensus is not achieved. For each given p, we collect 500 samples (1 sample consists of 10 graphs) to check whether the system finally achieves consensus. The fraction of samples that reach consensus is shown in Figure 5 as a function of p. The curve displays a sigmoidal variation with respect to p, saturating at 1 when p is just over 3.1×10^{-3}. It is well known that [52] the random graphs are not connected with high probability if

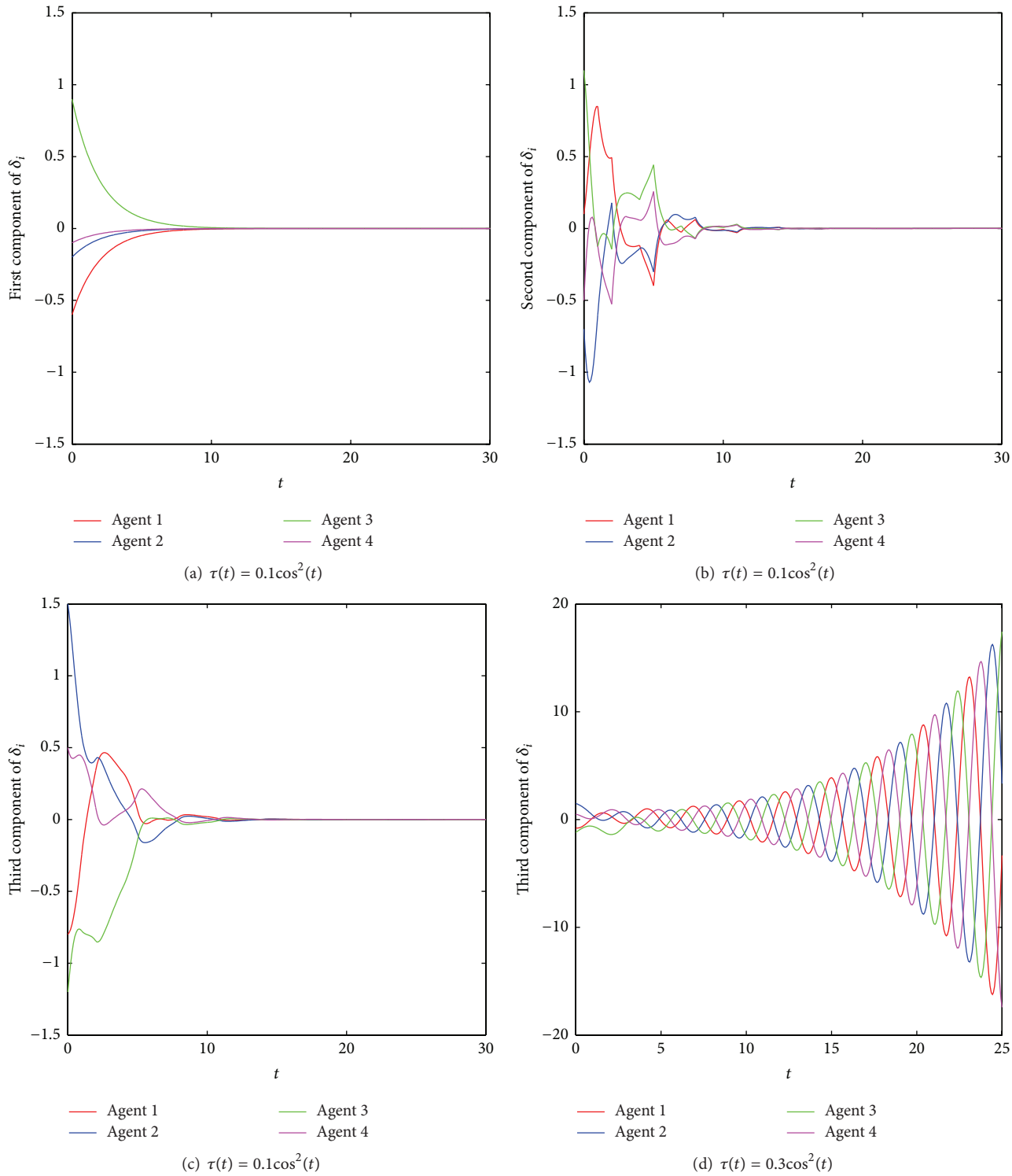


Figure 4: Consensus errors of multiagent system (2) with respect to different time-delay $\tau(t)$.

$p < (\ln N)/N$ (here, about 6.9×10^{-3}) in the large N limit. Figure 5 reveals that consensus can still be achieved with a much smaller p than the connectivity threshold indicating that the sufficient condition regarding connectivity in Theorem 8 can be weakened.

5. Conclusion

This paper has studied the continuous-time consensus problem of linear multiagent systems under Markovian switching interaction topologies, random noises, and time-varying

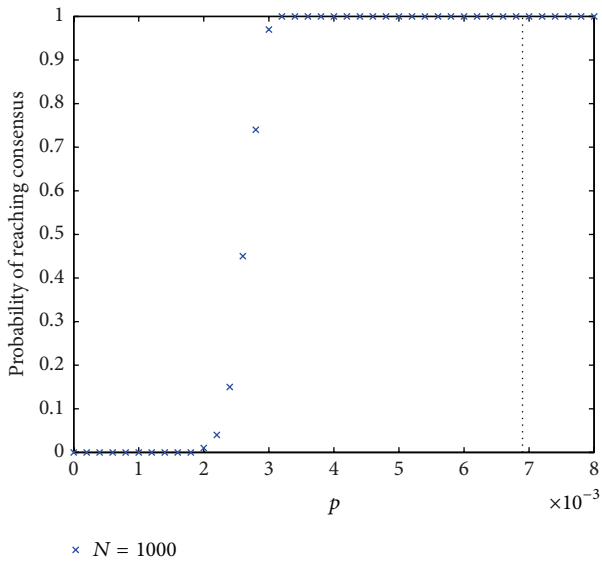


Figure 5: Probability of reaching consensus for multiagent system (1) as a function of p over Markovian switching random graphs $\mathcal{G}(N, p)$.

delays. The agent dynamics are described by linear time-invariant systems, in which two types of network-induced delays are considered, namely, delays affecting only the output of the agents' neighbors and delays affecting both the agents' own output and the output of their neighbors. Necessary and sufficient consensus conditions have been derived, respectively, for these two classes of multiagent systems. The design of consensus gains has a computationally advantageous decoupling feature. Numerical examples are given to demonstrate the effectiveness of the proposed methods. Although there is a gap between necessary and sufficient conditions, the necessary one seems to be tighter in view of the simulations. Future research worth investigation could be the heterogeneous time delays, general uncertainties, and systems with different dynamics (see, e.g., [53]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank Professor Quanxin Zhu for useful suggestions. The author is also grateful to the four anonymous reviewers and the editor for valuable comments that have improved the paper. This work was funded in part by the Program for Young Excellent Talents in Tongji University (2014KJ036) and by Shanghai Pujiang Program (15PJ1408300).

References

[1] J. N. Tsitsiklis, *Problems in decentralized decision making and computation [Ph.D. thesis]*, Massachusetts Institute of Technology, Cambridge, Mass, USA, 1984.

[2] S. Chatterjee and E. Seneta, "Towards consensus: some convergence theorems on repeated averaging," *Journal of Applied Probability*, vol. 14, no. 1, pp. 89–97, 1977.

[3] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[4] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[5] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[6] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[7] Y. Hong, G. Chen, and L. Bushnell, "Distributed observers design for leader-following control of multi-agent networks," *Automatica*, vol. 44, no. 3, pp. 846–850, 2008.

[8] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1503–1509, 2008.

[9] G. Xie and L. Wang, "Consensus control for a class of networks of dynamic agents," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 10-11, pp. 941–959, 2007.

[10] K. You, Z. Li, and L. Xie, "Consensus condition for linear multi-agent systems over randomly switching topologies," *Automatica*, vol. 49, no. 10, pp. 3125–3132, 2013.

[11] C.-Q. Ma and J.-F. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1263–1268, 2010.

[12] K. You and L. Xie, "Network topology and communication data rate for consensusability of discrete-time multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2262–2275, 2011.

[13] Z. Li, Z. Duan, and G. Chen, "Dynamic consensus of linear multi-agent systems," *IET Control Theory & Applications*, vol. 5, no. 1, pp. 19–28, 2011.

[14] Y. Shang, "Group pinning consensus under fixed and randomly switching topologies with acyclic partition," *Networks and Heterogeneous Media*, vol. 9, no. 3, pp. 553–573, 2014.

[15] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1867–1872, 2005.

[16] M. Porfiri and D. J. Stilwell, "Consensus seeking over random weighted directed graphs," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1767–1773, 2007.

[17] A. Tahbaz-Salehi and A. Jadbabaie, "Consensus over ergodic stationary graph processes," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 225–230, 2010.

[18] T. Li and J.-F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2043–2057, 2010.

[19] B. Touri and A. Nedić, "On ergodicity, infinite flow, and consensus in random models," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1593–1605, 2011.

[20] G. Miao, S. Xu, and Y. Zou, "Necessary and sufficient conditions for mean square consensus under Markov switching topologies," *International Journal of Systems Science*, vol. 44, no. 1, pp. 178–186, 2013.

[21] I. Matei, J. S. Baras, and C. Somarakis, "Convergence results for the linear consensus problem under Markovian random graphs," *SIAM Journal on Control and Optimization*, vol. 51, no. 2, pp. 1574–1591, 2013.

[22] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, NY, USA, 1993.

[23] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Delay robustness in consensus problems," *Automatica*, vol. 46, no. 8, pp. 1252–1265, 2010.

[24] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Delay robustness in non-identical multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1597–1603, 2012.

[25] X. Wang, A. Saberi, A. A. Stoorvogel, H. F. Grip, and T. Yang, "Consensus in the network with uniform constant communication delay," *Automatica*, vol. 49, no. 8, pp. 2461–2467, 2013.

[26] J. Liu, X. Liu, W.-C. Xie, and H. Zhang, "Stochastic consensus seeking with communication delays," *Automatica*, vol. 47, no. 12, pp. 2689–2696, 2011.

[27] Y.-Z. Sun and J. Ruan, "Leader-follower consensus problems of multi-agent systems with noise perturbation and time delays," *Chinese Physics Letters*, vol. 25, no. 9, pp. 3493–3495, 2008.

[28] Y. Shang, "Group consensus of multi-agent systems in directed networks with noises and time delays," *International Journal of Systems Science*, vol. 46, no. 14, pp. 2481–2492, 2015.

[29] J. Lai, S. Chen, and X. Lu, "Tracking consensus of nonlinear MASs with asymmetric communication delays in noisy environments," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 7, pp. 2334–2344, 2014.

[30] Y. Shang, "Synchronization in networks of coupled harmonic oscillators with stochastic perturbation and time delays," *Annals of the Academy of Romanian Scientists—Series on Mathematics and its Applications*, vol. 4, no. 1, pp. 44–58, 2012.

[31] Y. Shang and R. Bouffanais, "Influence of the number of topologically interacting neighbors on swarm dynamics," *Scientific Reports*, vol. 4, article 4184, 2014.

[32] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: convergence rates, measurement delays, and asynchronous events," *SIAM Journal on Control and Optimization*, vol. 47, no. 2, pp. 601–623, 2008.

[33] N. Chopra, D. M. Stipanović, and M. W. Spong, "On synchronization and collision avoidance for mechanical systems," in *Proceedings of the American Control Conference (ACC '08)*, pp. 3713–3718, Seattle, Wash, USA, June 2008.

[34] A. Papachristodoulou and A. Jadbabaie, "Synchonization in oscillator networks with heterogeneous delays, switching topologies and nonlinear dynamics," in *Proceedings of the 45th IEEE Conference on Decision and Control*, pp. 4307–4312, December 2006.

[35] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[36] Y. Shang, "Average consensus in multi-agent systems with uncertain topologies and multiple time-varying delays," *Linear Algebra and its Applications*, vol. 459, pp. 411–429, 2014.

[37] Y. G. Sun, L. Wang, and G. Xie, "Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays," *Systems & Control Letters*, vol. 57, no. 2, pp. 175–183, 2008.

[38] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1980.

[39] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*, Springer, London, UK, 2005.

[40] X. Feng and K. A. Loparo, "Stability of linear Markovian jump systems," in *Proceedings of the 29th IEEE Conference on Decision and Control*, pp. 1408–1413, Honolulu, Hawaii, USA, December 1990.

[41] E. Seneta, *Non-Negative Matrices and Markov Chains*, Springer, New York, NY, USA, 2006.

[42] X. Mao, "Robustness of stability of stochastic differential delay equations with Markovian switching," *Stability and Control: Theory and Applications*, vol. 3, no. 1, pp. 48–61, 2000.

[43] Z. Zeng, J. Wang, and X. Liao, "Global asymptotic stability and global exponential stability of neural networks with unbounded time-varying delays," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 52, no. 3, pp. 168–173, 2005.

[44] Q. Zhu, J. Cao, and R. Rakkiyappan, "Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays," *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1085–1098, 2015.

[45] Q. Zhu and J. Cao, "Mean-square exponential input-to-state stability of stochastic delayed neural networks," *Neurocomputing*, vol. 131, pp. 157–163, 2014.

[46] Q. Zhu and J. Cao, "Stability analysis of markovian jump stochastic BAM neural networks with impulse control and mixed time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 3, pp. 467–479, 2012.

[47] X. Mao, A. Matasov, and A. B. Piunovskiy, "Stochastic differential delay equations with Markovian switching," *Bernoulli*, vol. 5, pp. 1–18, 1999.

[48] J. S. Caughman and J. J. P. Veerman, "Kernels of directed graph Laplacians," *Electronic Journal of Combinatorics*, vol. 13, article R39, 2006.

[49] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1994.

[50] H. Weyl, "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)," *Mathematische Annalen*, vol. 71, no. 4, pp. 441–479, 1912.

[51] J. G. VanAntwerp and R. D. Braatz, "Tutorial on linear and bilinear matrix inequalities," *Journal of Process Control*, vol. 10, no. 4, pp. 363–385, 2000.

[52] B. Bollobás, *Random Graphs*, Cambridge University Press, New York, NY, USA, 2001.

[53] Y. Zheng and L. Wang, "Consensus of heterogeneous multi-agent systems without velocity measurements," *International Journal of Control*, vol. 85, no. 7, pp. 906–914, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 348235, 7 pages
http://dx.doi.org/10.1155/2015/348235

Research Article

The Mean Stability Criteria in terms of Two Measures for Stochastic Differential Equations with Coefficient's Uncertainty

Rui Zhang,[1] Yinjing Guo,[1] Xiangrong Wang,[2] and Xueqing Zhang[3]

[1] *College of Electronic Communication and Physics, Shandong University of Science and Technology, Qingdao 266590, China*
[2] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*
[3] *College of Environmental Science and Engineering, Ocean University of China, Qingdao 266100, China*

Correspondence should be addressed to Xueqing Zhang; zxq@ouc.edu.cn

Received 21 April 2015; Revised 23 July 2015; Accepted 9 August 2015

Academic Editor: Son Nguyen

This paper extends the stochastic stability criteria of two measures to the mean stability and proves the stability criteria for a kind of stochastic Itô's systems. Moreover, by applying optimal control approaches, the mean stability criteria in terms of two measures are also obtained for the stochastic systems with coefficient's uncertainty.

1. Introduction

Lyapunov's method, which makes an essential use of auxiliary functions (also called Lyapunov functions), is an important approach to study the stability of differential systems including ordinary differential equations (ODEs) and stochastic differential equations (SDEs). This method started in Lyapunov's original work in 1892 [1] for demonstrating stability of ODEs. In the 1960s, Movchan [2] studied the stability with two measures; such works were also developed and can be seen in [3]. In the past decades, Lyapunov's method is modified to the study of stability of Markovian processes [4], stochastic differential systems based on Brownian motions [5], semimartingales [6], or Lévy processes [7] and is also developed with the form of exponential stability [8] or LaSalle theorem [9], and so forth. Recently, the stability for systems with unknown parameters is also discussed [10], and the theorems of stability are widely applied in aerospace [11], state-feedback control [12], automatic control [13], neural networks [14], and other fields.

In this paper, we will discuss the following stochastic Itô's systems:

$$dx = f(t,x)\,dt + \sigma(t,x)\,dB_t, \qquad x(t_0) = x_0,\ t \geq t_0 \in \mathscr{R}_+, \tag{1}$$

where f, σ satisfy the usual Lipschitzian conditions and B is d-dimensional standard Brownian motion. It is well known that, for a stochastic process $x(t)$ and a given positive function $h(t,x)$, $\lim_{t\to\infty} h(t, x(\omega,t)) = 0$ almost surely discussed in [15] does not imply that $\lim_{t\to\infty}\mathbb{E}h(t,x(t)) \to 0$. So, we extend Lyapunov's methods used by [3] for ODEs to the stochastic cases and study the mean stability criteria in terms of two measures for system (1).

This paper is organized as follows: In Section 2, we first introduce Lyapunov's derivatives for (1) and deduce the basic comparison results in terms of Lyapunov's function. In Section 3, we prove the stochastic two-measure stability criteria for Itô systems, which can be seen as the extension of that of the ODEs. As described in [16], stability, robustness, and optimality can be considered systematically and simultaneously. In Section 4, the optimal control approach is extended to the stochastic systems with coefficient's uncertainty.

2. Basic Comparison Results for Stochastic Differential Equations

Let $(\Omega, \mathscr{F}, P)$ be a given completed probability space, and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion with filtration:

$$\mathscr{F}_t = \sigma(B_s,\ s \leq t). \tag{2}$$

Let $f \in C(\mathscr{R}_+ \times \mathscr{R}^n, \mathscr{R}^n)$, $\sigma \in C(\mathscr{R}_+ \times \mathscr{R}^n, \mathscr{R}^n \times \mathscr{R}^d)$ be deterministic functions and satisfy the following Lipschitz condition and linear growth condtion: there exists $L > 0$, for every $x, y \in \mathscr{R}^n$, such that

$$\begin{aligned} &\|f(t,x) - f(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le L\|x-y\|, \\ &\|f(t,x)\| \le L(1+\|x\|). \end{aligned} \tag{3}$$

For a given function $V \in C(\mathscr{R}_+ \times \mathscr{R}^n, \mathscr{R})$, $(t,x) \in \mathscr{R}_+ \times \mathscr{R}^n$, we denote

$$\begin{aligned} \mathscr{D}^+V(t,x)_{(1)} = &\lim_{h\to 0^+}\sup \frac{1}{h} \\ &\cdot \mathbb{E}\left[V\left(t+h, x+hf(t,x)+\sqrt{h}\sigma(t,x)\xi\right)\right. \\ &\left.-V(t,x)\right], \end{aligned} \tag{4}$$

where ξ represents a d-dimensional random variable with standard normal distribution; that is, $\xi \sim N(0,I)$ and I is a d-order identity matrix.

Remark 1. We use the notation $\mathscr{D}^+V(t,x)_{(1)}$ to emphasize the definition with respect to system (1). For convenience, we use the shortened form $\mathscr{D}^+V$ to substitute $\mathscr{D}^+V(t,x)_{(1)}$.

Remark 2. If $V \in C^{1,2}(\mathscr{R}_+ \times \mathscr{R}^n, \mathscr{R}_+)$, $\mathscr{D}^+V(t,x)$ is Lyapunov's operator $\mathscr{L}$ associated with (1); that is,

$$\mathscr{D}^+V_{(1)} = \mathscr{L}V := V_t + \langle V_x, f\rangle + \frac{1}{2}\left\langle \sigma\sigma^T, V_{xx}\right\rangle, \tag{5}$$

where V_t is the partial derivative for t, V_x is the gradient of V for x, and V_{xx} is the Hessian matrix of V for x.

Since (4) is dependent on expectation calculating, it is not easy to check whether $\mathscr{D}^+V$ exists or not. The following lemma gives a condition for the existence of $\mathscr{D}^+V$.

Lemma 3. *Let $V \in C(\mathscr{R}_+ \times \mathscr{R}^n, \mathscr{R})$ be continuous one-order differentiable for t and also satisfy the following condition:*

$$\begin{aligned} \left|V(t,x+y) - V(t,x) - \langle \alpha(t,x), y\rangle\right| \le K(t)\|y\|^2, \\ \forall x, y \in \mathscr{R}^n, \end{aligned} \tag{6}$$

where $\alpha(t,x)$ is continuous on $\mathscr{R}_+ \times \mathscr{R}^n$ and $K(t) \ge 0$ is locally bounded for t. Then $\mathscr{D}^+V(t,x)$ exists at (t,x).

Proof. Denote $y = hf(t,x) + \sqrt{h}\sigma(t,x)\xi$; then, for fixed t, x,

$$\begin{aligned} &\frac{1}{h}\mathbb{E}\left[V(t+h, x+y) - V(t,x)\right] \\ &\quad = \frac{1}{h}\mathbb{E}\left[V(t+h, x+y) - V(t+h,x)\right] \\ &\qquad + \frac{1}{h}\left[V(t+h,x) - V(t,x)\right]. \end{aligned} \tag{7}$$

By (6) and distribution of ξ, we have the first item of the right side of (7) that is bounded:

$$\begin{aligned} &\langle \alpha(t,x), f(t,x)\rangle \\ &\quad - K(t,x,y)\,\mathbb{E}\left\|\sqrt{h}f(t,x) + \sigma(t,x)\xi\right\|^2 \\ &\quad \le \frac{1}{h}\mathbb{E}\left[V(t+h, x+y) - V(t+h,x)\right] \\ &\quad \le \langle \alpha(t,x), f(t,x)\rangle \\ &\quad - K(t)\,\mathbb{E}\left\|\sqrt{h}f(t,x) + \sigma(t,x)\xi\right\|^2. \end{aligned} \tag{8}$$

Since V is differentiable for t, so the last item of the right side of (7) is also bounded. Therefore, the supremum limit of (7) exists; that is, $\mathscr{D}^+V(t,x)$ exist exactly. □

The following lemmas will be used later.

Lemma 4 (see Theorem 7.1.2 in [6], or Theorems 4.3 and 4.4 in [17]). *Suppose f, σ satisfy (3). Then, stochastic differential equation (1) admits a unique strong solution $\{x(t),\ t \ge 0\}$ such that, for any $h > 0$ ($[t, t+h] \subset [0,T]$, $l \ge 2$), there exists (K_T is a constant dependent only on T, l, and L):*

$$\mathbb{E}\sup_{t\le s\le t+h}\|x(s)\|^l \le K_T\left(1 + E\|x(t)\|^l\right), \tag{9}$$

$$\mathbb{E}\|x(r) - x(s)\|^l \le K_T\left(1 + E\|x(s)\|^l\right)|r-s|^{l/2}. \tag{10}$$

Lemma 5. *Suppose V satisfies (6), and $x(t)$ is the solution of (1); let $m(t) = \mathbb{E}V(t,x(t))$; then*

$$D^+m(t) \le \mathbb{E}\mathscr{D}^+V(t,x(t)), \tag{11}$$

where $\mathscr{D}^+V(t,x(t)) = \mathscr{D}^+V(t,x)|_{x=x(t)}$ and $D^+m(t)$ is the usual right upper Dini derivative defined by

$$D^+m(t) = \lim_{h\to 0^+}\sup \frac{m(t+h) - m(t)}{h}. \tag{12}$$

Proof. For small $h > 0$, we have

$$\begin{aligned} m(t+h) - m(t) = &\mathbb{E}\left[V(t+h, x(t+h)) - V\left(t\right.\right. \\ &\left. + h, x(t) + hf(t,x(t)) + \sigma(t,x(t))\left(B_{t+h} - B_t\right)\right) \\ &+ V\left(t+h, x(t) + hf(t,x(t))\right. \\ &\left.\left. + \sigma(t,x(t))\left(B_{t+h} - B_t\right)\right) - V(t,x(t))\right]. \end{aligned} \tag{13}$$

We now prove that the first two items of the right side are the higher infinitesimal of h. By (1), we know that

$$\begin{aligned} &\hat{x} := x(t+h) \\ &\quad - \left(x(t) + hf(t,x(t)) + \sigma(t,x(t))\left(B_{t+h} - B_t\right)\right) \\ &= \int_t^{t+h}\left[f(s,x(s)) - f(t,x(t))\right]ds \\ &\quad + \int_t^{t+h}\left[\sigma(s,x(s)) - \sigma(t,x(t))\right]dBs. \end{aligned} \tag{14}$$

For convenience, we denote $\widehat{f}_s := f(s, x(s)) - f(t, x(t))$, similar meaning for $\widehat{\sigma}_s$. We have

$$\begin{aligned}\mathbb{E}\|\widehat{x}\|^2 &\le 2\left(\mathbb{E}\left\|\int_t^{t+h}\widehat{f}_s ds\right\|^2 + \mathbb{E}\left\|\int_t^{t+h}\widehat{\sigma}_s dBs\right\|^2\right)\\ &\le 2\left[h\int_t^{t+h}\mathbb{E}\left\|\widehat{f}_s\right\|^2 ds + \int_t^{t+h}\mathbb{E}\left\|\widehat{\sigma}_s\right\|^2 ds\right]\\ &\le 2L\left[h\int_t^{t+h}\mathbb{E}\|x(s)-x(t)\|^2 ds\right.\\ &\quad\left.+\int_t^{t+h}\mathbb{E}\|x(s)-x(t)\|^2 ds\right].\end{aligned} \tag{15}$$

By Lemma 4 and inequality (10), we have

$$\mathbb{E}\|\widehat{x}\|^2 \le 2LK_T\left(1+\mathbb{E}\|x(t)\|^2\right)\left(\frac{1}{2}h^3+h^2\right). \tag{16}$$

Let $\widetilde{x}_t := x(t) + hf(t, x(t)) + \sigma(t, x(t))(B_{t+h} - B_t)$; then ($\alpha(x)$ replaces $\alpha(t, x)$ for shortening)

$$\begin{aligned}&\mathbb{E}\left|V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t) - \langle\alpha(\widetilde{x}_t), \widehat{x}\rangle\right|\\ &\quad\le C(h)h^2.\end{aligned} \tag{17}$$

Now we estimate the order of

$$\begin{aligned}\mathbb{E}\langle\alpha(\widetilde{x}_t), \widehat{x}\rangle &= \mathbb{E}\langle\alpha(x(t)), \widehat{x}\rangle\\ &\quad+\mathbb{E}\langle\alpha(\widetilde{x}_t) - \alpha(x(t)), \widehat{x}\rangle.\end{aligned} \tag{18}$$

Since $\mathbb{E}\langle\alpha(x(t)), \widehat{x}\rangle = \mathbb{E}\langle\alpha(x(t)), \mathbb{E}[\widehat{x} \mid \mathscr{F}_t]\rangle$ and $\mathbb{E}[\widehat{x} \mid \mathscr{F}_t] = \mathbb{E}[\int_t^{t+h}\widehat{f}_s ds \mid \mathscr{F}_t]$, so

$$\begin{aligned}&|\mathbb{E}\langle\alpha(x(t)), \widehat{x}\rangle| = \left|\mathbb{E}\left\langle\alpha(x(t)), \int_t^{t+h}\widehat{f}_s ds\right\rangle\right|\\ &\quad\le C_1(h)h^{3/2},\\ &\left|\mathbb{E}\langle\alpha(\widetilde{x}_t) - \alpha(x(t)), \widehat{x}\rangle\right| \le \left\|\alpha(\widetilde{x}_t) - \alpha(x(t))\right\|\|\widehat{x}\|\\ &\quad\le C_2(h)h^{3/2},\end{aligned} \tag{19}$$

where $C_1(h), C_2(h)$ are continuous positive functions. By (19), we have

$$\left|\mathbb{E}\langle\alpha(\widetilde{x}_t), \widehat{x}\rangle\right| \le C_3(h)h^{3/2}. \tag{20}$$

Since

$$\begin{aligned}&\left|\mathbb{E}\left[V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t)\right]\right|\\ &\quad\le \mathbb{E}\left|V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t) - \langle\alpha(\widetilde{x}_t), \widehat{x}\rangle\right|\\ &\qquad+\left|\mathbb{E}\langle\alpha(\widetilde{x}_t), \widehat{x}\rangle\right|,\end{aligned} \tag{21}$$

by (17) and (20), we see that the first two items of right side are higher infinitesimal of t. So we have

$$\lim_{h\to 0^+}\frac{1}{h}\left|\mathbb{E}\left[V(t+h, x(t+h)) - V(t+h, \widetilde{x}_t)\right]\right| = 0. \tag{22}$$

For the last two items of (13), since $x(t)$ is independent of $B_{t+h} - B_t$ with normal distribution $N(0, h)$, so we have

$$\begin{aligned}&\limsup_{h\to 0}\frac{1}{h}\mathbb{E}\left[V(t+h, x(t)+hf(t, x(t))\right.\\ &\quad\left.+\sigma(t, x(t))(B_{t+h} - B_t)) - V(t, x(t))\right]\\ &\quad\le \mathbb{E}\left\{\limsup_{h\to 0}\frac{1}{h}\right.\\ &\quad\cdot\mathbb{E}\left[V(t+h, y+hf(t, y)+\sigma(t, y)(B_{t+h} - B_t))\right.\\ &\quad\left.\left.-V(t, y)\right]_{y=x(t)}\right\} = \mathbb{E}\left[\mathscr{D}^+V(t, y)\big|_{y=x(t)}\right].\end{aligned} \tag{23}$$

This proves (11). □

The following lemma will be used in the proof of Theorem 7.

Lemma 6 (see Theorem 1.5.2 in [3]). *Let $g \in C(\mathscr{R}_+\times\mathscr{R}_+, \mathscr{R})$ and $r(t)$ be the maximal solution of*

$$\begin{aligned}\dot{u} &= g(t, u),\\ u(t_0) &= u_0,\end{aligned} \tag{24}$$

existing on $[t_0, \infty)$. Suppose $m \in C(\mathscr{R}_+, \mathscr{R}_+)$ and $Dm(t) \le g(t, m(t))$, $t \ge t_0$, where D is any fixed Dini derivative. Then $m(t_0) \le u_0$ implies $m(t) \le r(t)$, $t \ge t_0$.

Now we formulate the basic comparison results in terms of Lyapunov function V.

Theorem 7. *Assume $\mathscr{D}^+V(t, x)$ satisfies*

$$\mathscr{D}^+V(t, x) \le g(t, V(t, x)), \quad (t, x) \in \mathscr{R}_+ \times \mathscr{R}^n, \tag{25}$$

where $g \in C(\mathscr{R}_+^2, R)$ is concave for u. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of the differential equation

$$\begin{aligned}\dot{u} &= g(t, u),\\ u(t_0) &= u_0 \ge 0.\end{aligned} \tag{26}$$

Then, for every solution of (1) $x(t) = x(t, t_0, x_0)$, $V(t_0, x_0) \le u_0$ implies

$$\mathbb{E}V(t, x(t)) \le r(t), \quad t \ge t_0. \tag{27}$$

Proof. Denote $m(t) = \mathbb{E}V(t, x(t))$. By Lemma 5 and the concave of g we have

$$D^+m(t) \le g(t, m(t)), \quad m(t_0) \le u_0. \tag{28}$$

By Lemma 6, we can obtain the result (27). □

Remark 8. If $V \in C^{1,2}(\mathscr{R}_+ \times \mathscr{R}^n, \mathscr{R}_+)$, the inequality (25) became

$$\mathscr{L}V(t, x) \le g(t, V(t, x)), \quad (t, x) \in \mathscr{R}_+ \times \mathscr{R}^n. \tag{29}$$

3. Stability Criteria in terms of Two Measures

Now we discuss the two-measure stability criteria for the stochastic differential system (1). We assume $f(t,0) = \sigma(t,0) = 0$ for all $t \geq t_0$. Firstly, we give some definitions for stochastic stability.

Definition 9. The stochastic differential system (1) is said to be

(S_1) mean (h_0,h)-equistable, if for each $\epsilon > 0$ and $t_0 \in \mathscr{R}_+$, there exists a function $\delta = \delta(t_0,\epsilon) > 0$ which is continuous in t_0 for each ϵ such that

$$\mathbb{E}h_0(t_0,x_0) < \delta \quad \text{implies } \mathbb{E}h(t,x(t)) < \epsilon,\ t \geq t_0, \tag{30}$$

where $x(t) = x(t,t_0,x_0)$ is any solution of (1);

(S_2) mean (h_0,h)-uniformly stable, if (S_1) holds with δ being independent of t_0;

(S_3) mean (h_0,h)-quasiequiasymptotically stable, if for each $\epsilon > 0$ and $t_0 \in \mathscr{R}_+$, there exist positive number $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\epsilon)$ such that

$$\begin{aligned}&\mathbb{E}h_0(t_0,x_0) < \delta_0\\ &\qquad \text{implies } \mathbb{E}h(t,x(t)) < \epsilon,\ t \geq t_0 + T;\end{aligned} \tag{31}$$

(S_4) mean (h_0,h)-quasiuniform asymptotically stable if (S_3) holds with δ_0 and T being independent of t_0;

(S_5) mean (h_0,h)-asymptotically stable if (S_1) holds and, given $t_0 \in \mathscr{R}_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that

$$\mathbb{E}h_0(t_0,x_0) < \delta_0 \text{ implies } \lim_{t\to\infty} \mathbb{E}h(t,x(t)) = 0; \tag{32}$$

(S_6) mean (h_0,h)-uniformly equiasymptotically stable, if (S_1) and (S_3) hold together;

(S_7) mean (h_0,h)-uniformly asymptotically stable if (S_2) and (S_4) hold simultaneously;

(S_8) mean (h_0,h)-unstable if (S_1) fails to hold.

The following classes of functions will be used in this paper:

$\mathscr{K} = \{a \in C(\mathscr{R}_+,\mathscr{R}_+) : a(u)$ is strictly increasing in u and $a(0) = 0\}$,

$\mathscr{CK} = \{a \in C(\mathscr{R}_+^2,\mathscr{R}_+) : a(t,s) \in \mathscr{K}$ for each $t\}$,

$\Gamma = \{h \in C(\mathscr{R}_+ \times \mathscr{R}^n,\mathscr{R}_+) : \inf h(t,x) = 0\}$.

Definition 10. Let $h_0, h \in \Gamma$. Then, we say that h_0 is finer than h if there exists a function $\phi \in \mathscr{CK}$ such that $h(t,x) \leq \phi(t,h_0(t,x))$. Furthermore, if ϕ is independent of t then we call h_0 uniformly finer than h.

Definition 11. Let $V \in C(\mathscr{R}_+ \times \mathscr{R}^n,\mathscr{R}_+)$. If there exists a function convex $b \in \mathscr{K}$ such that $b(h(t,x)) \leq V(t,x)$, then we call V h-positive definite. If there exists a concave function $a \in \mathscr{K}$ such that $V(t,x) \leq a(h(t,x))$, then we call V h-decrescent.

Theorem 12. *Assume that*

(A_0) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than* h,

(A_1) $V \in C(\mathscr{R}_+ \times \mathscr{R}^n,\mathscr{R}_+)$, $V(t,x)$ *satisfies (6), and* V *is* h*-positive definite and* h_0*-decrescent,*

(A_2) $g \in C(\mathscr{R}_+^2,\mathscr{R})$ *and* $g(t,0) \equiv 0$,

(A_3) $\mathscr{D}^+V(t,x) \leq g(t,V(t,x))$ *for* $(t,x) \in \mathscr{R}_+ \times \mathscr{R}^n$.

Then, the stability properties of the trivial solution of (26) imply the corresponding (h_0,h)*-stability properties of (1).*

Proof. Since V is h-positive definite, so there exists a convex $b \in \mathscr{K}$ such that

$$b(h(t,x)) \leq V(t,x), \quad (t,x) \in \mathscr{R}_+ \times \mathscr{R}^n. \tag{33}$$

Suppose the trivial solution of (26) is equistable and $r(t,t_0,r(t_0))$ $(t \geq t_0)$ is its maximal solutions with initial time t_0 and initial value $r(t_0) \geq 0$, then, for every $\epsilon > 0$, there exists $\delta_1 > 0$, when $r(t_0) < \delta_1$,

$$r(t) = r(t,t_0,r(t_0)) < b(\epsilon), \quad t \geq t_0. \tag{34}$$

Let $r(t_0) = \mathbb{E}V(t_0,x(t_0))$; then, by Theorem 7, we have

$$\mathbb{E}V(t,x(t)) \leq r(t), \quad t \geq t_0. \tag{35}$$

Since V is h_0-decrescent, there exists a concave $a \in \mathscr{K}$ such that

$$V(t,x) \leq a(h_0(t,x)), \quad (t,x) \in \mathscr{R}_+ \times \mathscr{R}^n. \tag{36}$$

So

$$r(t_0) \leq \mathbb{E}a(h_0(t_0,x(t_0))) \leq a(\mathbb{E}h_0(t_0,x(t_0))). \tag{37}$$

Since a is continuous and strictly increasing, so let $\delta = a^{-1}(\delta_1)$; then when $\mathbb{E}h_0(t_0,x(t_0)) < \delta$, inequality (34) holds. Combining (33), (34), and (35) and using the strictly increase of b, we can gain

$$\mathbb{E}h(t,t_0,x(t_0)) < \epsilon, \quad t \geq t_0, \tag{38}$$

which implies (1) (h_0,h)-equistability. □

Remark 13. If $V \in C^{1,2}(\mathscr{R}_+ \times \mathscr{R}^n,\mathscr{R}_+)$, then condition ($A_3$) can be replaced by

$$\mathscr{L}V(t,x) \leq g(t,V(t,x)). \tag{39}$$

Remark 14. The stabilities of auxiliary ordinary differential equation (26) are defined by Definition 2.4.1 in [3].

Example 15. Consider the following 2-dimensional Itô's system:

$$\begin{aligned}dx_1(t) &= x_2(t)\,dt,\\ dx_2(t) &= -(bx_1(t) + ax_2(t))\,dt\\ &\quad + (cx_2(t) + ex_1(t))\,dB_t.\end{aligned} \tag{40}$$

Let $h(t,x) = h_0(t,x) = e^{\gamma t}(x_1^2 + x_2^2)$; suppose $V(t,x)$ has the form

$$V(t,x) = e^{\gamma t}\left(\alpha x_1^2 + 2\beta x_1 x_2 + x_2^2\right). \tag{41}$$

Then

$$\begin{aligned}\mathscr{L}V = e^{\gamma s}\Big[&\left(\gamma\alpha - 2\beta b + e^2\right)x_1^2 \\ &+ \left(2\gamma\beta + 2\alpha - 2\beta a - 2b + 2ce\right)x_1(s)x_2 \\ &+ \left(\gamma + 2\beta - 2a + c^2\right)x_2^2\Big].\end{aligned} \tag{42}$$

In order to make $V(t,x)h$-positive, we let

$$\begin{aligned}\alpha &> 0, \\ \alpha - \beta^2 &> 0.\end{aligned} \tag{43}$$

Let $g(t,u) = -\theta u$, $\theta > 0$; in order to find $\theta > 0$ to satisfy

$$\mathscr{L}V \le g(t,V), \tag{44}$$

we set

$$\begin{aligned}2\beta\gamma + 2\alpha - 2\beta a - 2b + 2ce &= 0, \\ \gamma\alpha - 2\beta b + e^2 &< 0, \\ \gamma + 2\beta - 2a + c^2 &< 0.\end{aligned} \tag{45}$$

When $\gamma < a$, combining (43) and (45), we have that, when

$$\begin{aligned}&\max\left\{\frac{1}{2}\left[a - \gamma - \sqrt{(\gamma - a)^2 + 4(b - ce)}\right],\right. \\ &\left.\frac{\gamma b - \gamma ce + e^2}{\gamma^2 + 2b - \gamma a}\right\} < \beta < \min\left\{\frac{b - ce}{\gamma - a}, a\right. \\ &- \frac{1}{2}\left(\gamma + c^2\right), \\ &\left.\frac{1}{2}\left[a - \gamma + \sqrt{(\gamma - a)^2 + 4(b - ce)}\right]\right\},\end{aligned} \tag{46}$$

there exists $\theta > 0$ which satisfies (44). Moreover, the trivial solution of

$$\dot{u} = -\theta u \tag{47}$$

is uniformly asymptotically stable; by Theorem 12, the stochastic differential is mean-(h_0, h)-asymptotically stable.

However, in practice, the coefficients maybe have some uncertainty properties; that is, we only know the range of the parameters a, b. Then how to discuss the stability of such systems with uncertainty is still a very interesting problem. The following section will introduce an optimal control approach to discuss the stability of such systems with uncertainty.

4. The Stability for Systems with Uncertainty

Consider the following stochastic differential equations with uncertainty:

$$dx = \left[f(x) + g(x)k(x)\right]dt + \sigma(x)\,dB_t, \tag{48}$$

where $k(x)$ is an uncertainty function, and $f(0) = k(0) = \sigma(0) = 0$, that is, the trivial solution, is 0 of (48). Furthermore, we also suppose there exists a function $k_{\max}(x) \ge 0$ such that

$$\|k(x)\| \le k_{\max}(x). \tag{49}$$

Now we discuss how to determine the asymptotic stability of system (48) for all uncertainty function k. Similar to the methods applied by [18, 19], we can translate this stability problem into an optimal problem.

For the nominal system

$$dx = \left[f(x) + g(x)u\right]dt + \sigma(x)\,dB_t, \tag{50}$$

suppose we can find a state-feedback control $u = \overline{u}(x)$ that minimizes the cost functional

$$J(x(\cdot), u) = \mathbb{E}^y \int_0^\infty \left[f_{\max}^2(x) + \rho h_0(x) + 3u^T u\right]dt, \tag{51}$$

where $h_0 \in C(\mathscr{R}^n, \mathscr{R}_+)$, y is the initial value of (50), $\rho > 0$, and u is admissible on $[0,T]$ for each $T < \infty$, and satisfies

$$J(x(\cdot), u) < \infty. \tag{52}$$

Let the value function

$$V(y) = \min_u J(x(\cdot), u). \tag{53}$$

Theorem 16. *Suppose $\overline{u}(x)$ is an optimal control of problem (53), and there exists $\overline{\rho} \le \rho$ satisfying*

$$\|\overline{u}\|^2 \le \frac{\overline{\rho}}{2}h_0(x), \tag{54}$$

and the value function $V \in C^2(\mathscr{R}^n, \mathscr{R}_+)$, V is also h positive, and h_0 is decrescent, then system (48) is uniformly mean (h_0, h) uniformly asymptotically stable for all uncertainties $k(x)$.

Proof. The values function V satisfies the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned}\min_{u\in\mathscr{R}^p}\Big(&f_{\max}^2 + x^T x + u^T u + \langle V_x, f + gu\rangle \\ &+ \frac{1}{2}\left\langle \sigma\sigma^T, V_{xx}\right\rangle\Big) = 0.\end{aligned} \tag{55}$$

So, the optimal control $\overline{u}$ satisfies

$$\begin{aligned}&f_{\max}^2 + x^T x + \overline{u}^T\overline{u} + \langle V_x, f + g\overline{u}\rangle + \frac{1}{2}\left\langle \sigma\sigma^T, V_{xx}\right\rangle \\ &\quad = 0, \\ &2\overline{u}^T + V_x^T g = 0.\end{aligned} \tag{56}$$

Then, the Lyapunov generator of (48) for V is given as

$$\begin{aligned}\mathscr{L}V(x) &:= \langle V_x, f+gk\rangle + \frac{1}{2}\left\langle V_{xx}, \sigma\sigma^T\right\rangle \\ &= \langle V_x, gk\rangle - \langle V_x, g\overline{u}\rangle - h_0(x) - k_{\max}^2 - \overline{u}^T\overline{u} \\ &\le \|k\|^2 - k_{\max}^2 - (\rho-\overline{\rho})h_0(x) \\ &\le -(\rho-\overline{\rho})h_0(x).\end{aligned} \tag{57}$$

So, for the solutions $x(t)$ of (48) with uncertainty $k(x)$, applying Itô's formula to $V(x(t))$, we have, when $t \ge t_0$,

$$\begin{aligned}\mathbb{E}V(x(t)) - \mathbb{E}V(x(t_0)) &= \mathbb{E}\int_{t_0}^{t}\mathscr{L}V(x(s))\,ds \\ &\le -(\rho-\overline{\rho})\,\mathbb{E}\int_{t_0}^{t}h_0(x)\,ds.\end{aligned} \tag{58}$$

Hence

$$\mathbb{E}V(x(t)) \le \mathbb{E}V(x(t_0)) - \mathbb{E}\int_{t_0}^{t}h_0(x(s))\,ds, \quad \forall t \ge t_0. \tag{59}$$

So $m(t) = \mathbb{E}V(x(t))$ is decreasing on $[0,\infty]$. Now we show $m(t) \to 0$ when $t \to \infty$. Suppose $m(t) \to m_0 > 0$. Since V is h_0 decrescent, so there exists $c > 0$ and $t_0 > 0$, such that $\mathbb{E}h_0(x(t)) > c > 0$ $(t \ge t_0)$ combining with (59); we have

$$\mathbb{E}V(x(t)) \le \mathbb{E}V(x(t_0)) - c(t-t_0) \longrightarrow -\infty, \quad t \longrightarrow \infty. \tag{60}$$

This contradicts the fact that $V(x) \ge 0$. This implies that system (48) is uniformly mean (h_0, h) uniformly asymptotically stable. □

Corollary 17. *Specially, let*

$$\begin{aligned}f(x) &= Ax, \\ g(x) &= G, \\ \sigma(x) &= Cx, \\ k_{\max}(x) &= M\|x\|.\end{aligned} \tag{61}$$

The value function can be given by $V(x) = x^TPx$ and the optimal control $\overline{u}(x) = -G^TPx$, where $P > 0$ satisfies the following Riccati equation:

$$\left(M^2+1\right)I + PA + A^TP + C^TPC - PGG^TP = 0. \tag{62}$$

Let $\lambda_{\max}$ be the maximal eigenvalue of PGG^TP. If $\lambda_{\max}^2 \le \rho/2$, then the corresponding system with uncertainty is uniformly mean (h_0, h)-equistable for all uncertainties $k(x)$ (which is also mean square asymptotically stable for all uncertainties $k(x)$).

Example 18. Consider (41) with uncertainty coefficients. In (41), replace a, b by $a + \sin x_1$ and $b + \cos x_2$, respectively, and a takes values in $[1.4, 1.6]$, b in $[0.9, 1.1]$, $c = -1$, and $e = 1$. Then the system with uncertainty is obtained:

$$\begin{aligned}dx_1(t) &= x_2(t)\,dt, \\ dx_2(t) &= -\left[\left(b + 0.1\cos B_t\right)x_1(t)\right. \\ &\quad \left. + \left(a + 0.1\sin B_t\right)x_2(t)\right]dt + \left(cx_2(t) + ex_1(t)\right)dB_t.\end{aligned} \tag{63}$$

Let

$$\begin{aligned}A &= \begin{bmatrix}0 & 1\\ -1 & -1.5\end{bmatrix}, \\ B &= \begin{bmatrix}0 & 0\\ 1.5 - b - 0.1\cos B_t & 1 - a - 0.1\sin B_t\end{bmatrix}, \\ C &= \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix},\end{aligned} \tag{64}$$

And let G be the 2-order identical matrix. Let $h(x) = h_0(x) = \|x\|^2$; then

$$\begin{aligned}f(x) &= Ax, \\ g(x) &= G, \\ k(x) &= Bx, \\ \sigma(x) &= Cx.\end{aligned} \tag{65}$$

We can take $k_{\max}(x) = 0.3\sqrt{x_1^2 + x_2^2}$ and the auxiliary optimal problem is

$$dx = (Ax + Gu)\,dt + Cx\,dB_t, \tag{66}$$

with the cost functional

$$J(x(\cdot), u) = \mathbb{E}^y\int_0^\infty \left[3x^Tx + u^Tu\right]dt, \tag{67}$$

$\rho = 2.91$, solving (62) with $M = 0.3$; we have

$$P = \begin{bmatrix}1 & 0\\ 0 & 0.5\end{bmatrix}. \tag{68}$$

By Corollary 17, we can determine that the stochastic system (63) is uniformly mean (h_0, h) uniformly stable for all uncertainties.

5. Conclusion

In this paper, we extend the stability criteria of two measures to the mean stability situations for the stochastic systems with uncertainty. For the usual SDE, we give the results of mean stability criteria which are the basic criteria for such systems. As far as the systems with uncertainty, in order to resolve the difficulties coming from the coefficient uncertainty, we use the optimal control results as an auxiliary method to determine the mean stability. Furthermore, the stability criteria in terms of two measures for other stochastic systems, such as systems with Markovian jumps or Poisson jumps, are worth further studying.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the NSF of China (61471224, 61402265, and 11271007), the NSF of Shandong Province (ZR2011FM018), and the Doctoral Fund of Ministry of Education (20123718110010).

References

[1] A. M. Lyapunov, *The General Problem of the Stability of Motion*, translated by A. T. Fuller, Taylor & Francis, 1992 (Russian).

[2] A. A. Movchan, "Stability of processes with respect to two measures," *Prikladnaia Matematika i Mekhanika*, vol. 24, pp. 988–1001, 1960.

[3] V. Lakshmikantham, S. Leela, and A. A. Martynyuk, *Stability Analysis of Nonlinear Systems*, Marcel Dekker, 1989.

[4] H. J. Kushner, *Stochastic Stability and Control*, Academic Press, 1967.

[5] R. Z. Has'minskii, *Stochastic Stability of Differential Equations*, translated by D. Louvish, Sijthoff and Noordhoff, 1981.

[6] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, John Wiley & Sons, 1974.

[7] Q. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise ," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 126–142, 2014.

[8] X. Mao, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, 1994.

[9] X. Mao, "Stochastic versions of the LaSalle theorem," *Journal of Differential Equations*, vol. 153, no. 1, pp. 175–195, 1999.

[10] Q. Zhu, "Stabilization of stochastically singular nonlinear jump systems with unknown parameters and continuously distributed delays," *International Journal of Control, Automation and Systems*, vol. 11, no. 4, pp. 683–691, 2013.

[11] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method*, Springer, 1977.

[12] H. Wang and Q. Zhu, "Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form," *Automatica*, vol. 54, pp. 284–291, 2015.

[13] Y. Xu and X. Chen, "Rules mining of power system transient stability assessment based on clustering analysis and rough set theory," *Applied Mechanics and Materials*, vol. 521, pp. 418–422, 2014.

[14] Q. Zhu, J. Cao, and R. Rakkiyappan, "Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays," *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1085–1098, 2015.

[15] C. Yuan, "Stability in terms of two measures for stochastic differential equations," *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, vol. 10, no. 6, pp. 895–910, 2003.

[16] W. Grote, D. Kastsian, and M. Mönnigmann, "Guaranteed stability in optimal power generation dispatching under uncertainty," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 1103–1112, 2013.

[17] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Publishing, 2007.

[18] F. Lin, R. D. Brandt, and J. Sun, "Robust control of nonlinear systems: compensating for uncertainty," *International Journal of Control*, vol. 56, no. 6, pp. 1453–1459, 1992.

[19] F. Lin, "An optimal control approach to robust control design," *International Journal of Control*, vol. 73, no. 3, pp. 177–186, 2000.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 269695, 8 pages
http://dx.doi.org/10.1155/2015/269695

Research Article

Dynamic Inventory and Pricing Policy in a Periodic-Review Inventory System with Finite Ordering Capacity and Price Adjustment Cost

Baimei Yang,[1] Chunyan Gao,[2] Na Liu,[3] and Liang Xu[4]

[1] *School of Business, Shanghai Dianji University, Shanghai 201306, China*
[2] *Department of Management Science, Southwestern University of Finance and Economics, Chengdu 611130, China*
[3] *Business School, Beijing Institute of Fashion Technology, Beijing 100029, China*
[4] *Department of Logistics Management, Southwestern University of Finance and Economics, Chengdu 611130, China*

Correspondence should be addressed to Na Liu; nanalaufashion@gmail.com

Received 23 April 2015; Accepted 21 September 2015

Academic Editor: Ruihua Liu

We consider a dynamic inventory control and pricing optimization problem in a periodic-review inventory system with price adjustment cost. Each order occurs with a fixed ordering cost; the ordering quantity is capacitated. We consider a sequential decision problem, where the firm first chooses the ordering quantity and then the sale price to maximize the expected total discounted profit over the sale horizon. We show that the optimal inventory control is partially characterized by a (s, s', p) policy in four regions, and the optimal pricing policy is dependent on the inventory level after the replenishment decision. We present some numerical examples to explore the effects of various parameters on the optimal pricing and replenishment policy.

1. Introduction

Traditional literature on the multistage inventory system mainly focuses on replenishment decision with or without setup cost. The well-known result is that the order-up-to policy is optimal for the systems without setup cost and the (s, S) policy is optimal for the systems with setup cost. Increasing researchers are devoted to the study of joint price and inventory control in the multistage inventory system. Our paper belongs to this stream, but our paper considers a sequential decision problem in a periodic-review inventory system with fixed ordering cost and price adjustment cost. The ordering quantity is capacitated; this may be limited by the storage capacity or the supply capability. The firm first decides its inventory level and then chooses a sale price to maximize its long-run profit. Our result shows that the optimal inventory and pricing decision still preservers a threshold-type structure.

Our paper is related to literature on the optimal control of a single product system with finite capacity and setup cost. Several studies have been devoted to this area. Shaoxiang and Lambrecht [1] obtain the generally known result; that is, the optimal policy can only be partially characterized in the form of X-Y bands. In particular, when the inventory level is below the first band X, then produce/order the capacity, and when the inventory level is over the second band Y, produce/order nothing. If the inventory level is between the two bands, the ordering policy is complicated and depends on the instance. Gallego and Scheller-Wolf [2] extend their work. They derive the structure of the policy between the bands. The optimal policy is characterized by two numbers s and s' which divide the state space into four possible regions. However, none of them have studied the pricing problem in the inventory control problem. Zhang et al. [3] consider a single-item, finite-horizon, periodic-review coordinated decision model on pricing and inventory control with capacity constraints and fixed ordering cost. They show that the profit-to-go function is strongly CK-concave, and the optimal policy has an (s, S, P)-like structure. However, the price adjustment cost has not been addressed. Chao et al. [4] recently consider

the joint pricing and inventory decisions. They study a periodic-review inventory system with setup cost and finite ordering capacity in each period. They show that the optimal inventory control is characterized by an (s, s', p) policy in four regions of the starting inventory level. However, in their paper, the selling price can be adjusted without any cost.

In reality, changing price is costly and incurs a price adjustment cost. In the economics literature, there are two major types of price adjustment costs: the managerial costs and the physical costs. Rotemberg [5], Levy et al. [6], Slade and Groupe de Recherche en Economie Quantitative d'Aix-Marseille [7], Aguirregabiria [8], Bergen et al. [9], and Zbaracki et al. [10] have stated that both types of costs are significant in retailing and other industries. According to these empirical studies, Chen et al. [11] consider a periodic-review inventory model with price adjustment cost. The price adjustment cost consists of both fixed and variable components. They develop the general model and characterize the optimal policies for two special scenarios, a model with inventory carryover and no fixed price-change costs and a model with fixed price-change costs and no inventory carryover. Although there is price adjustment cost, they do not consider the finite ordering capacity.

Under the assumption of random additive demand model, our paper tries to investigate the structure of the optimal inventory control and pricing policy in each period. We show that the optimal inventory policy is partially characterized by an (s, s', p) policy on four regions; in two of these regions the optimal policy is completely specified while, in the other two, it is partially specified. More specifically, the optimal ordering quantity in the first region is the full capacity, while in the last region it is optimal to order nothing; in the two middle regions, the optimal decision is either to order to the maximum capacity, to order to at least a prespecified level s', or to order nothing. The optimal pricing policy $p(y)$ in each period is dependent on the inventory level after the replenishment decision, y, which is in general not a monotone function. The key concept utilized is strong CK-concavity, which is an extension of K-concavity, and was first introduced by Gallego and Scheller-Wolf [2].

The rest of this paper is organized as follows. In Section 2, we induce the model description. The structural properties of the optimal inventory and pricing policy are characterized in Section 3. We present some numerical examples to show the effects of various parameters on the optimal control policy in Section 4. Finally, we conclude with some future research direction in Section 5.

2. The Model

Consider a periodic-review inventory system with finite ordering capacity and price adjustment cost. There are N periods, with the first period being 1 and last period being N. In each period, the sequence of events is given as follows: (1) inventory level is reviewed and replenishment order is placed; (2) replenishment order arrives; (3) a selling price is set; (4) random demand is realized; and (5) all costs are computed.

In period n, the selling price is p_n, which is taken in interval $[p_l, p_h]$, and the demand is D_n. We assume that the demand D_n is sensitive to the selling price p_n. Moreover, we consider an additive demand function. The demand function is $D_n(p_n) = d(p_n) + \epsilon_n$, $n = 1, \ldots, N$, where ϵ_n is a random variable with mean zero and $d(p_n)$ is the average demand. Furthermore, $d(p_n)$ is a decreasing linear function of p_n. When the selling price p_n increases from p_l to p_h, the average demand decreases from d_h to d_l; that is, $d_h = d(p_l)$ and $d_l = d(p_h)$. Each demand arrives requiring only one unit of product and is satisfied from inventory if any. If the demand cannot be satisfied from the on-hand inventory immediately, then it is backlogged and incurs a backorder cost. The structure of demand function indicates that determining the selling price p_n is equivalent to setting the average demand d_n.

Each replenishment incurs a fixed ordering cost K and the variable unit ordering cost c. There is a finite ordering capacity C for each period, which means the ordering quantity in each period cannot exceed C, where $C > 0$. If C is sufficiently large, it generalizes to the incapacitated case. Let x_n be the inventory level at the beginning of period n before placing an order and let y_n be the inventory level after the order delivered. At the end of each period, the demand is realized and a revenue is received. The expected revenue is given by $r_n(d) = d_n \cdot p_n(d_n)$, which is assumed to be a concave function. Meanwhile, an inventory holding and shortage cost occurs denoted by $h(y_n - d_n)$. If $x \geq 0$, $h(x)$ represents the holding cost; if $x < 0$, $h(x)$ represents the shortage cost. For ease of presentation, we let $G(y) = E[h(y - \epsilon_n)]$. Therefore, given that the inventory level after replenishment is y_n and the expected demand for period n is d_n, the expected holding and shortage cost is $G(y_n - d_n)$.

We assume that there is a fixed guide price p_0 for deciding the selling price p_n. Price changing from the guide price to the actual price is costly. The cost of a price adjustment from guide price to the actual selling price in period n is denoted by $U_n(p_n - p_0)$. Zbaracki et al. [10] and Chen et al. [11] pointed out that as the price adjustment cost becomes larger, it would cost more on decision and internal communication. Here, we assume that the variable cost $U_n(\cdot)$ is convex and increases with $|p_n - p_0|$. The forms of $U_n(\cdot)$ could be either piecewise linear functions or quadratic functions. The ordering quantity in period n is $y_n - x_n$; therefore, we have $x_n \leq y_n \leq x_n + C$ due to the capacitated ordering quantity C. Therefore, the expected total cost incurs in period n including setup cost, ordering cost, holding and shortage cost and price adjustment price is given by

$$\begin{aligned} &K1\left[y_n > x_n\right] + c\left(y_n - x_n\right) + G\left(y_n - d_n\right) \\ &\quad + U_n\left(p_n - p_0\right), \end{aligned} \tag{1}$$

where $1[A]$ is the indicating function taking value 1 if statement A is true and zero otherwise.

We aim to obtain the optimal pricing and inventory decisions in each period to maximize the expected total discounted profit over the n periods. Let $V_n(x)$ denote the maximum expected total discounted profit from period n to

the end of the planning horizon with the starting inventory level (before ordering decision) x. The optimality equation is

$$\begin{aligned} V_n(x) = \max_{x\le y\le x+C} \max_{p\in[p_l,p_h]} \{&-K1[y>x] + r(p) \\ &- c(y-x) - G(y-d) - U(p-p_0) \\ &+ \alpha E[V_{n+1}(y-d(p)-\epsilon_n)]\}, \end{aligned} \tag{2}$$

where α is the one-period discount factor, $\alpha \in [0,1]$. The terminal condition is $V_{N+1}(x) \equiv 0$. Note that the price p_n can be indicated in the form of demand d_n by the inverse demand function; that is, $p_n = p_n(d_n)$, and the price adjustment cost can be written in the form of d_n instead of p_n, that is, $U_n(d_n - p_0)$, such that optimizing over the selling price p_n is equivalent to optimizing over the average demand d_n. Therefore, the optimality equation is rewritten as follows:

$$\begin{aligned} V_n(x) = \max_{x\le y\le x+C} \max_{d\in[d_l,d_h]} \{&-K1[y>x] + r(d) \\ &- c(y-x) - G(y-d) - U(d-p_0) \\ &+ \alpha E[V_{n+1}(y-d-\epsilon_n)]\}. \end{aligned} \tag{3}$$

For notation convenience, we define another function

$$W_n(y) = -G(y) + \alpha E[V_{n+1}(y-\epsilon_n)]. \tag{4}$$

Then the optimality equation is further simplified to

$$\begin{aligned} V_n(x) = cx + \max_{x\le y\le x+C} \max_{d\in[d_l,d_h]} \{&-K1[y>x] + r_n(d) - cy - U_n(d-p_0) + W_n(y-d)\} = cx \\ &+ \max_{x\le y\le x+C} \left\{-K1[y>x] - cy + \max_{d\in[d_l,d_h]} \{r_n(d) - U_n(d-p_0) + W_n(y-d)\}\right\}. \end{aligned} \tag{5}$$

3. The Optimal Policy

In order to characterize the structural properties of the optimal replenishment and pricing policy, we first introduce the definition of strongly CK-concave and properties of CK-concave functions as well, which is defined in Chao et al. [4]. This definition and the properties are very important in studying inventory models with finite capacity and setup cost.

Definition 1. A function $g(\cdot) : R \to R$ is strongly CK-concave if, for all $a \ge 0$, $b > 0$, and $z \in [0, C]$, we have

$$\frac{z}{b} g(y-a) + g(y) \ge \frac{z}{b} g(y-a-b) + g(y+z) - K. \tag{6}$$

The structure of strong CK-concave function is shown in Figure 1. If $G(x)$ is strong CK-concave, it implies that the slope of the line made of points $(x, G(x))$ and $(x+z, G(x+z)-K)$ is smaller than the slope of the line made of points $(x-a-b, G(x-a-b))$ and $(x-a, G(x-a))$.

Chao et al. [4] also pointed out that the strongly CK-concave function possesses some additional properties as follows:

(1) If G is strongly CK-concave, then it is also strongly DL-concave for $0 \le D \le C$ and $L \ge K$.

(2) If G is concave, it is also strongly CK-concave for any nonnegative C and K.

(3) If G_1 is strongly CK_1-concave and G_2 is strongly CK_2-concave, then for $\alpha, \beta \ge 0$, $\alpha G_1 + \beta G_2$ is strongly $C(\alpha K_1 + \beta K_2)$-concave.

(4) If G is strongly CK-concave and X is a random variable such that $E[|G(y-X)|] < \infty$, then $E[G(y-X)]$ is also strongly CK-concave.

In the following, we aim to show that $V_n(x)$ preservers the property of strong CK-concavity. Before going further, we first show that each term on the right hand side of (3) possesses some certain properties, which will facilitate our analysis of objective function $V_n(x)$.

Lemma 2. *$U_n(d-p_0)$ is convex in d.*

Proof. Considering that $U_n(d-p_0)$ is continuous and second-order derivable, the convexity of $U_n(d-p_0)$ can be proved by its second derivative. We have

$$\begin{aligned} \frac{dU_n(d-p_0)}{dd} &= \frac{dU_n(p-p_0)}{dp}\frac{dp(d)}{dd}, \\ \frac{d^2U_n(d-p_0)}{dd^2} &= \frac{d^2U_n(p-p_0)}{dp^2}\left(\frac{dp(d)}{dd}\right)^2 \\ &\quad + \frac{dU_n(p-p_0)}{dp}\frac{d^2p(d)}{dd^2}. \end{aligned} \tag{7}$$

Since $d(p_n)$ is linear and decreasing on p_n, which means $p(d_n)$ is also linear and decreasing on d_n, then $d^2p(d)/dd^2 = 0$. At the same time, due to the convexity of $U_n(\cdot)$, $d^2U_n(p-p_0)/dp^2 \ge 0$. Therefore,

$$\frac{d^2U_n(d-p_0)}{dd^2} = \frac{d^2U_n(p-p_0)}{dp^2}\left(\frac{dp(d)}{dd}\right)^2 \ge 0, \tag{8}$$

which indicates that $U_n(d-p_0)$ is convex in d. Lemma 2 is proved. □

Lemma 3. *Let $d_n(y)$ be the maximizer of $r_n(d) - U_n(d-p_0) + W_n(y-d)$; then $y - d_n(y)$ is increasing in y.*

Proof. Due to the concavity of $r(\cdot)$ and Lemma 2, Lemma 3 can be conducted directly by using the properties of supermodularity. □

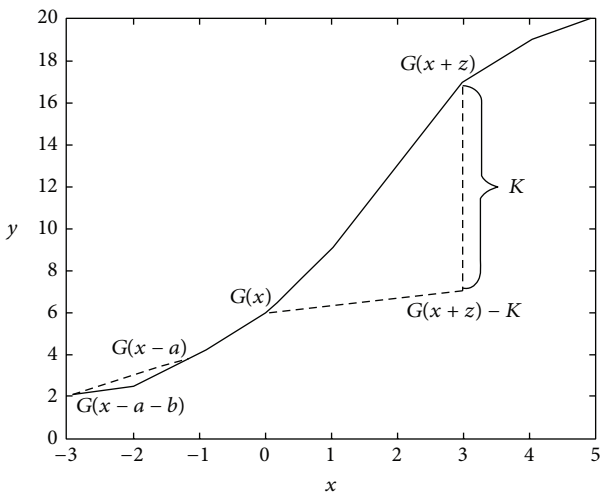


FIGURE 1: CK-concave function.

Lemma 4. *If $W_n(y)$ is strongly CK-concave, then*

$$g(y) = \max_{d\in[d_l,d_h]} \{r_n(d) - U_n(d - p_0) + W_n(y - d)\} \tag{9}$$

is also strongly CK-concave.

The proof of Lemma 4 is similar to that in Chao et al. [4]. We omit it for simplicity.

Lemma 5. *$V_n(x)$ is strongly CK-concave.*

Proof. Lemma 5 can be proved by induction. When $n = N+1$, we have $V_{N+1} \equiv 0$, such that V_{N+1} is strongly CK-concave. Now suppose that $V_{n+1}(x)$ is strongly CK-concave; then we proceed to prove that $V_n(x)$ is also strongly CK-concave.

Due to the property of strong CK-concavity, we obtain that $\alpha E[V_{n+1}(y-\epsilon_n)$ is strongly $C(\alpha K)$-concave. Consider that $-G_n(y)$ is concave; then $W_n(y)$ is strongly $C(\alpha K)$-concave, which is also strongly CK-concave. Lemma 4 shows that $g(y)$ is also strongly CK-concave. Therefore,

$$V_n(x) = cx + \max_{x\le y\le x+C} \{-K1[y > x] - cy + g(y)\} \tag{10}$$

is also strongly CK-concave. Readers are referred to Gallego and Scheller-Wolf [2] for more details. Lemma 5 is concluded. □

The strong CK-concavity of $V_n(x)$ characterizes the structural properties of the optimal inventory and pricing policy for each period as given in the following theorem.

Theorem 6. *Suppose that x is the starting inventory level at the beginning of period n. The optimal inventory policy is threshold-type policy which is characterized by two numbers s_n and s_n', where $s_n \le s_n'$. Furthermore, the optimal inventory policy possesses the following additional properties:*

(a) *If $s_n' - C \le s_n$, then the optimal ordering policy is*

(i) *order capacity C if $x < s_n' - C$;*

(ii) *order at least up to s_n' if $s_n' - C \le x < s_n$;*

(iii) *either order nothing or order at least up to s_n' if $s_n \le x < s_n'$;*

(iv) *order nothing if $x \ge s_n'$.*

(b) *If $s_n' - C > s_n$, then the optimal ordering policy is*

(i) *order capacity C if $x < s_n$;*

(ii) *either order nothing or order C if $s_n \le x < s_n' - C$;*

(iii) *either order nothing or order at least up to s_n' if $s_n' - C \le x < s_n'$;*

(iv) *order nothing if $x \ge s_n'$.*

The optimal pricing decision is characterized by $p_n^*(y)$, which depends on the postorder inventory position y. Furthermore, the optimal pricing decision $p_n^*(y)$, as well as the optimal average demand $d_n^*(y)$, is in general not monotone in y.

Proof. Suppose

$$\begin{aligned} H_n(y) &= -cy \\ &\quad + \max_{d\in[d_l,d_h]} \{r_n(d) - U_n(d - p_0) + W_n(y - d)\}. \end{aligned} \tag{11}$$

Define s_n, S_n, and s_n' by

$$\begin{aligned} S_n &= \inf\left\{y \in R \mid G(y) = \sup_{y\in R} H_n(y)\right\}, \\ s_n &= \inf\left\{x \mid -K + \sup_{x\le y\le x+C} H_n(y) \le H_n(x)\right\}, \\ s_n' &= \max\left\{x \le S_n \mid -K + \sup_{x\le y\le x+C} H_n(y) \ge H_n(x)\right\}. \end{aligned} \tag{12}$$

Obviously, $s_n \le s_n'$.

The optimal pricing decision is determined by the maximizer in Lemma 4. Let

$$\begin{aligned} d_n^*(y) &= \arg\max_{d\in[d_l,d_h]} \{r_n(d) - U_n(d - p_0) + W_n(y - d)\}, \end{aligned} \tag{13}$$

which means that the optimal average demand in period n depends on the replenished inventory level y. Since $p = p(d)$ is the inverse function of $d = d_n(p)$, then we will obtain that the optimal pricing decision is

$$p_n^*(y) = p(d_n^*(y)), \tag{14}$$

when the replenished inventory level is y. Therefore, the optimal selling price in period n also depends on the replenished inventory level y. However, $p_n^*(y)$ is not monotone in y. We will give one example in Section 4. The proof of Theorem 6 is concluded. □

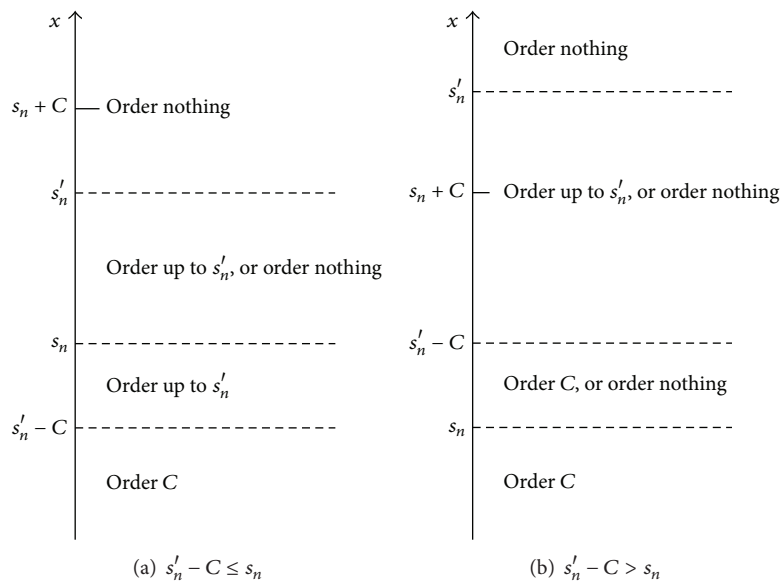


Figure 2: The structure of the optimal replenishment policy.

The structure of the optimal inventory policy is presented in Figure 2.

Our results are similar to Gallego and Scheller-Wolf [2] in that the optimal inventory policy can only be partially characterized. When the inventory level before replenishment x is less than $\min\{s'_n - C, s_n\}$, the optimal ordering policy is to order the full capacity. When x is larger than s'_n, the optimal ordering policy is no order. When $\min\{s'_n - C, s_n\} \le x < s'_n$, the optimal strategy is complicated. When $\max\{s'_n - C, s_n\} \le x < s'_n$, the optimal strategy may be to either order nothing or order at least up to s'_n. When $\min\{s'_n - C, s_n\} \le x < \max\{s'_n - C, s_n\}$, there would be two possibilities. If $s'_n - C \le s_n$, the optimal policy is to order at least up to s'_n. If $s'_n - C > s_n$, the optimal policy is no order or ordering full capacity. Moreover, the optimal pricing decision depends on the inventory level after replenishment.

4. Numerical Tests

In order to explore the effects of the setup cost, the ordering capacity, the guide price, and the adjustment cost function on the optimal control policy, we conduct several numerical experiments for a simple inventory problem with $N = 4$ periods. In the subsequent numerical experiments, we use the following basic settings: the discount factor is $\alpha = 0.9$, purchasing unit cost $c = 3$, guide price $p_0 = 3$, ordering capacity $C = 10$, and setup cost $K = 10$. We adopt $h(x) = h\max\{0, x\} + b\max\{0, -x\}$ as the holding shortage cost rate function, where $h = 2$ and $b = 4$. Suppose that $U_n(\cdot)$ is piecewise linear, which means that $U_n(p - p_0) = a_1 \max\{p_0 - p, 0\} + a_2 \max\{p - p_0, 0\}$, where $a_1 = a_2 = 0.5$. The demand in period n is $D_n(p_n) = d(p_n) + \epsilon_n$, where $d(p_n) = 10 - p_n$ and random error $\epsilon_n \in \{-1, 1\}$, with probability mass function $P\{\epsilon_n = 1\} = P\{\epsilon_n = -1\} = 0.5$. Here, p_n takes values in $[1, 9]$. Thus, as p_n increases from 1 to 9, the average demand decreases from 9 to 1.

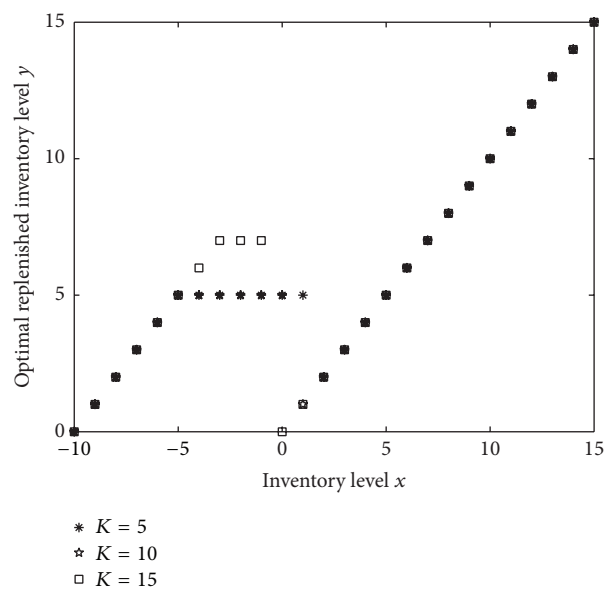


Figure 3: Optimal replenished inventory level y for different K.

4.1. Effect of Setup Cost. We study the effect of the setup cost K on the optimal inventory and pricing policy. The results are shown in Figures 3 and 4. In Figure 3, the x-axis represents the inventory level before ordering x and y-axis represents the inventory level after ordering y. The value of x goes from -10 to 15, with the increment of 1. In Figure 4, the x-axis represents the inventory level before ordering x and y-axis represents the optimal selling price p. The value of x also goes

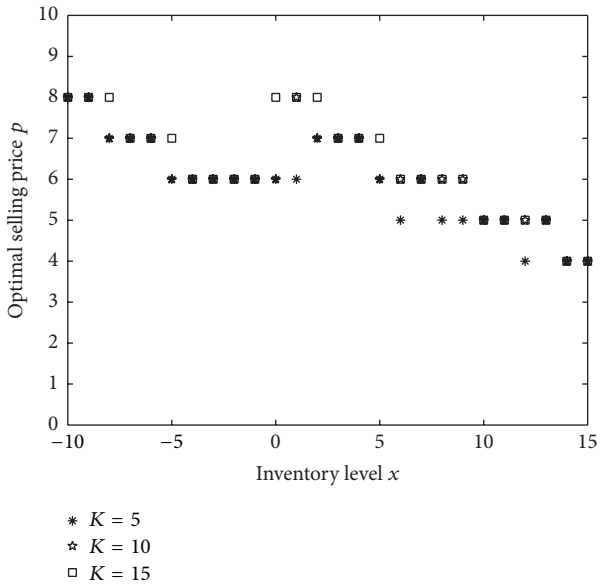


Figure 4: Optimal selling price p for different K.

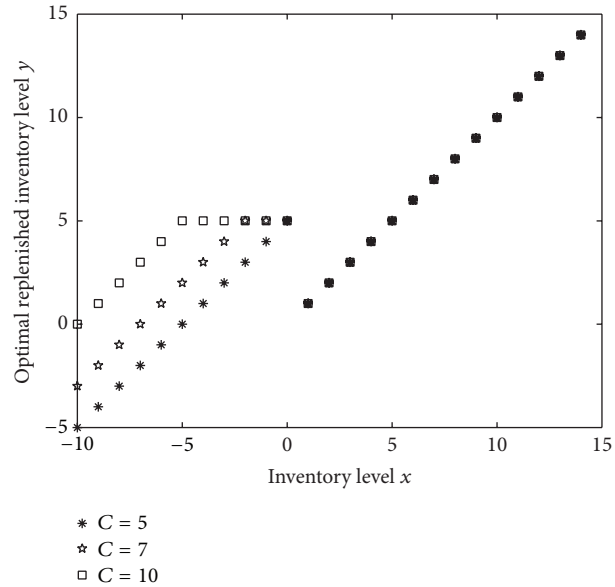


Figure 5: Optimal replenished inventory level y for different C.

from −10 to 15, with the increment of 1. Here, we consider $K = 5$, $K = 10$, and $K = 15$ separately.

Figure 3 shows that the higher setup cost K implies lower inventory level at which the optimal ordering policy changes from ordering to not ordering, while Figure 4 shows that the higher setup cost indicates higher optimal selling price. The results are intuitive. The trade-off is the setup cost, holding cost, and sale revenue. When the setup cost is high, we will decrease the replenishment frequency in order to reduce the ordering cost. Hence, we would place no order at low inventory level and order up to a higher inventory level if we place an order. The other alternative way is to increase the selling price to reduce the demand, in the purpose of saving setup cost.

Observation 1. The optimal selling price p is not always monotonic in y.

When $K = 5$ and the inventory level before ordering x is no less than 2, the optimal replenished inventory level y is equal to x. Furthermore, in Figure 4, when $K = 5$ and $x \geq 2$, the optimal selling price p is not monotonic in x; in other words, p is not monotonic in y.

4.2. Effect of Ordering Capacity. The effects of ordering capacity K on the optimal inventory and pricing policy are shown in Figures 5 and 6. Higher ordering capacity means that we may order more every time without increasing cost. Particularly, when the inventory level is high enough, the optimal policy is not to order. Then the ordering capacity has no effect on the optimal ordering policy and selling price. When the inventory level is small, higher ordering capacity indicates higher optimal replenished inventory level and lower optimal selling price, which induces higher demand.

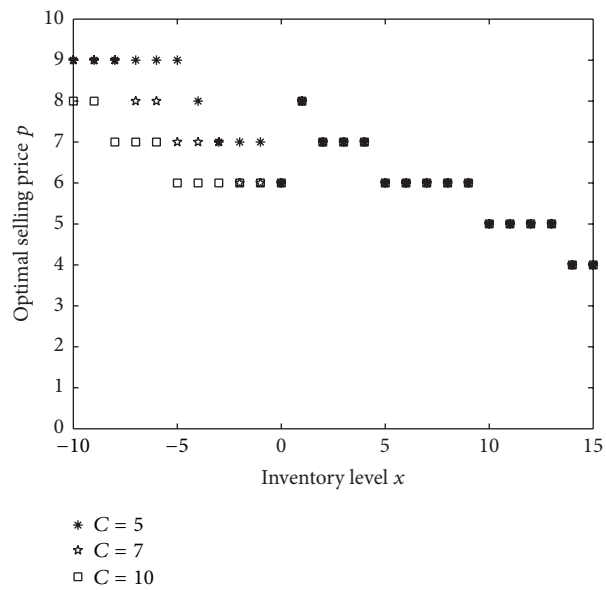


Figure 6: Optimal selling price p for different C.

4.3. Effect of Guide Price. The effects of guide price p_0 on the optimal inventory and pricing policy are shown in Figures 7 and 8. Compared with no guide price, the existence of the guide price indicates higher inventory level at which the optimal ordering policy changes from ordering to not ordering. The guide price has no obvious effect on the optimal replenishment inventory level, but it influences the optimal selling price. The optimal selling price would be closer to the guide price compared with the initial optimal selling price without guide price. For instance, when the guide price is 5, the optimal selling price would be lower than the initial one under small inventory level, while the optimal selling price

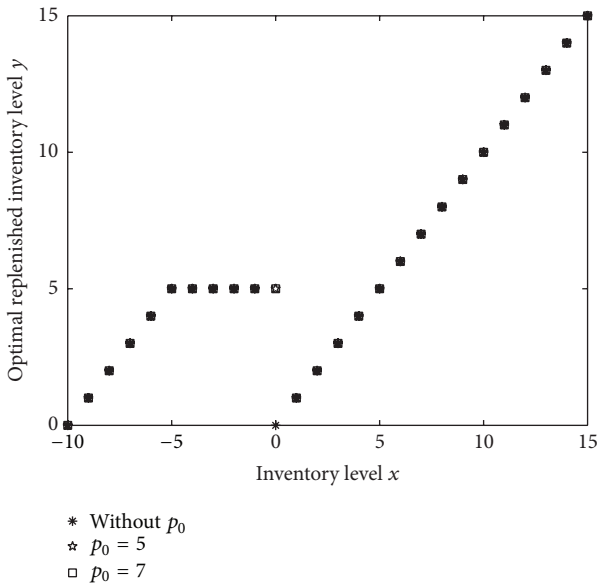


Figure 7: Optimal replenished inventory level y for different p_0.

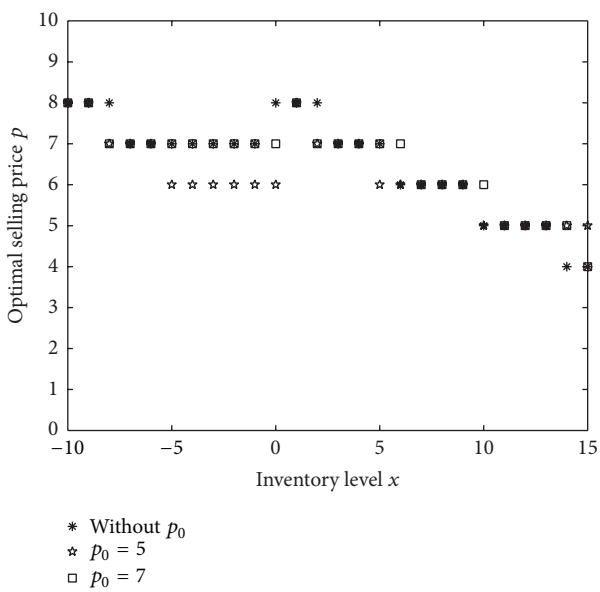


Figure 8: Optimal selling price p for different p_0.

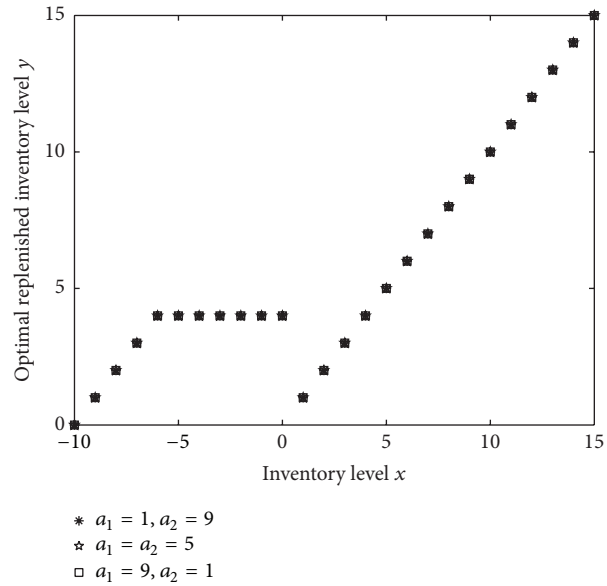


Figure 9: Optimal replenished inventory level y for different a_1 and a_2.

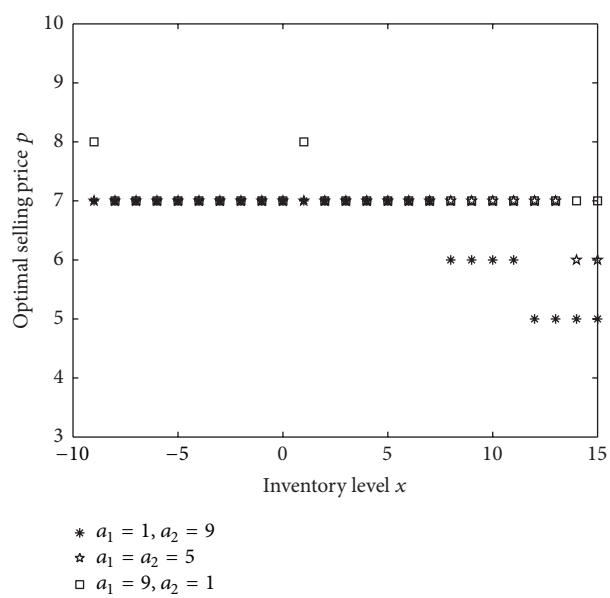


Figure 10: Optimal selling price p for different a_1 and a_2.

would be higher than the initial one under high inventory level.

In Figure 8, when the inventory level is 15, the optimal selling price is 5 when the guide price is 5, while the optimal selling price is 4 when the guide price is 7. It leads to the following observation.

Observation 2. Higher guide price does not indicate higher optimal selling price.

4.4. Effect of Price Adjustment Cost Function. The effects of price adjustment cost on the optimal ordering and pricing policy are shown in Figures 9 and 10. We consider three cases: $a_1 = 1, a_2 = 9, a_1 = a_2 = 5$, and $a_1 = 9, a_2 = 1$. From Figure 9, we find that the price adjustment cost function has no obvious effect on the optimal replenishment inventory level; however, it influences the optimal selling price obviously. $a_1 < a_2$ implies that it would be more costly when the selling price is higher than the guide price. $a_1 > a_2$ implies that it would be more costly when the selling price is smaller than the guide price. $a_1 = a_2$ implies that it would be more costly when the selling price is not equal to the guide price. Hence, in Figure 10, under the same inventory level, the optimal selling price is the highest when $a_1 > a_2$, while the optimal selling price is the smallest when $a_1 < a_2$.

5. Conclusions

In this paper, we consider a dynamic inventory control and pricing optimization problem in a periodic-review inventory system with fixed ordering cost and price adjustment cost. At the same time, the ordering quantity is limited. Here, we assume that the price adjustment cost functions are piecewise linear. We show that the optimal inventory control, similar to Chao et al. [4], is also partially characterized by (s, s', p) policy in four regions, and the optimal pricing policy is dependent on the inventory level after the replenishment decision. From the numerical tests, we present some statistical analysis to study the effects of various parameters on the optimal control policy. For example, the higher setup cost K implies lower inventory level at which the optimal ordering policy changes from ordering to not ordering and higher optimal selling price. When the inventory level is small, higher ordering capacity indicates high optimal replenished inventory level and lower optimal selling price. When the inventory level is high enough, the optimal ordering policy and selling price are the same under different inventory level. Optimal selling price would be closer to the guide price compared with the initial optimal selling price without guide price, while higher guide price does not indicate higher optimal selling price. Under the same inventory level, the optimal selling price is the highest when it would be more costly when the selling price is smaller than the guide price, while the optimal selling price is the smallest when it would be more costly when the selling price is larger than the guide price.

There are still many interesting issues worth studying in the future research. Our paper studied increasing convex price adjustment cost; exploring price adjustment cost function with more complicated form may be one of potential research directions. In our paper, the decision sequence is first inventory decision and then price decision, but in reality the firm may first set price to serve the target market and then build up the inventory. In this case, what is the optimal pricing and replenishment policy? Does the optimal control policy still possess the similar structure?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grants nos. 71201128 and 71201127, Project for Training and Supporting Young Teachers in Shanghai Universities (no. ZZSDJ13007), and Leading Academic Discipline Project of Shanghai Dianji University (no. 10XKJ01).

References

[1] C. Shaoxiang and M. Lambrecht, "X-Y band and modified (s, S) policy," *Operations Research*, vol. 44, no. 6, pp. 1013–1019, 1996.

[2] G. Gallego and A. Scheller-Wolf, "Capacitated inventory problems with fixed order costs: some optimal policy structure," *European Journal of Operational Research*, vol. 126, no. 3, pp. 603–613, 2000.

[3] J.-L. Zhang, J. Chen, and C.-Y. Lee, "Coordinated pricing and inventory control problems with capacity constraints and fixed ordering cost," *Naval Research Logistics*, vol. 59, no. 5, pp. 376–383, 2012.

[4] X. Chao, B. Yang, and Y. Xu, "Dynamic inventory and pricing policy in a capacitated stochastic inventory system with fixed ordering cost," *Operations Research Letters*, vol. 40, no. 2, pp. 99–107, 2012.

[5] J. J. Rotemberg, "Sticky prices in the United States," *Journal of Political Economy*, vol. 90, no. 6, pp. 1187–1211, 1982.

[6] D. Levy, M. Bergen, S. Dutta, and R. Venable, "The magnitude of menu costs: direct evidence from large U. S. supermarket chains," *The Quarterly Journal of Economics*, vol. 112, no. 3, pp. 790–825, 1997.

[7] M. E. Slade and Groupe de Recherche en Economie Quantitative d'Aix-Marseille, "Optimal pricing with costly adjustment: evidence from retail-grocery prices," *The Review of Economic Studies*, vol. 65, no. 1, pp. 87–107, 1998.

[8] V. Aguirregabiria, "The dynamics of markups and inventories in retailing firms," *The Review of Economic Studies*, vol. 66, no. 2, pp. 275–308, 1999.

[9] M. Bergen, M. Ritson, S. Dutta, D. Levy, and M. Zbarachi, "Shattering the myth of costless price changes," *European Management Journal*, vol. 21, no. 6, pp. 663–669, 2003.

[10] M. J. Zbaracki, M. Ritson, D. Levy, S. Dutta, and M. Bergen, "Managerial and customer costs of price adjustment: direct evidence from industrial markets," *The Review of Economics and Statistics*, vol. 86, no. 2, pp. 514–533, 2004.

[11] X. Chen, S. X. Zhou, and Y. Chen, "Integration of inventory and pricing decisions with costly price adjustments," *Operations Research*, vol. 59, no. 5, pp. 1144–1158, 2011.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 687428, 16 pages
http://dx.doi.org/10.1155/2015/687428

Research Article

A Random Parameter Model for Continuous-Time Mean-Variance Asset-Liability Management

Hui-qiang Ma,[1,2] Meng Wu,[3] and Nan-jing Huang[4]

[1] *School of Economics, Southwest University for Nationalities, Chengdu, Sichuan 610041, China*
[2] *School of Finance and Statistics, Hunan University, Changsha, Hunan 410079, China*
[3] *Business School, Sichuan University, Chengdu, Sichuan 610064, China*
[4] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*

Correspondence should be addressed to Meng Wu; shancherish@hotmail.com

Received 15 May 2015; Accepted 6 September 2015

Academic Editor: Leonid Shaikhet

We consider a continuous-time mean-variance asset-liability management problem in a market with random market parameters; that is, interest rate, appreciation rates, and volatility rates are considered to be stochastic processes. By using the theories of stochastic linear-quadratic (LQ) optimal control and backward stochastic differential equations (BSDEs), we tackle this problem and derive optimal investment strategies as well as the mean-variance efficient frontier analytically in terms of the solution of BSDEs. We find that the efficient frontier is still a parabola in a market with random parameters. Comparing with the existing results, we also find that the liability does not affect the feasibility of the mean-variance portfolio selection problem. However, in an incomplete market with random parameters, the liability can not be fully hedged.

1. Introduction

Mean-variance portfolio selection model was pioneered by Markowitz [1] in the single-period setting. In his seminal paper, Markowitz proposed the variance as the measure of the risk. The advantage of using variance for measuring the risk of a portfolio is due to the simplicity of computation. Thus, the mean-variance approach has inspired literally hundreds of extensions and applications and also has been commonly used in practical financial decisions. For example, Wang and Xia [2] gave an excellent review on portfolio selection problem. Li and Ng [3] employed the framework of multiobjective optimization and an embedding technique to obtain the exact mean-variance efficient frontier for multiperiod investment. Wu and Li [4] investigated a multiperiod mean-variance portfolio selection with regime switching and uncertain exit time. Zhou and Li [5] studied a continuous-time mean-variance portfolio selection problem under a stochastic LQ framework. Furthermore, Li et al. [6] considered a continuous-time mean-variance portfolio selection problem with no-shorting constraints. Under partial information, Xiong and Zhou [7] and Wang and Wu [8] considered a continuous-time mean-variance portfolio selection problem and a problem of hedging contingent claims by portfolios, respectively.

Among several extensions of the classic mean-variance portfolio selection model, asset and liability management problem is an important subject in both academic literatures and the real world situations. In the real world, liability is so important that almost all financial institutions and individual investors should manage their debt. Thus, incorporating liability into the portfolio selection model can make investment strategies more practical. The research on mean-variance asset-liability management also evokes recent concern. Sharpe and Tint [9] first investigated mean-variance asset-liability management in a single-period setting. Leippold et al. [10] considered a multiperiod asset-liability management problem and derived both the analytical optimal policy and the efficient frontier. Chiu and Li [11] studied a mean-variance asset-liability management problem in the continuous-time case where the liability was governed by a geometric Brownian motion (GBM). Xie et al. [12] also considered a continuous-time asset-liability management problem under the mean-variance criterion where the

dynamic of liability is a Brownian motion with drift. Further, Xie [13] studied a mean-variance portfolio selection model with stochastic liability in a Markovian regime switching financial market. Zeng and Li [14] investigated an asset-liability management problem in a jump diffusion market. Yao et al. [15] studied continuous-time mean-variance asset-liability management with endogenous liabilities. By using the time-consistent approach, Wei et al. [16] considered a mean-variance asset-liability management problem with regime switching.

Among these studies, we note that all market parameters are assumed to be deterministic. However, in the real world, market parameters observed in many situations are always uncertain (see, e.g., [17–20]). In order to capture the features of optimal investment strategies with random parameters, random parameter models have drawn more attention over last few years. For example, Lim and Zhou [21] investigated a mean-variance portfolio selection problem with random parameters in a complete market and derived efficient investment strategies as well as the efficient frontier analytically in terms of the solution of BSDEs. Further, Lim [22] extended Lim and Zhou's [21] results to the case where the market is incomplete.

Up to now, the studies on the asset-liability management problem are under a common assumption that all parameters are assumed to be known with certainty. An interesting and unexplored question is what happens in a more realistic situation with random parameters. This is the main focus of our research. In view of this, we study a mean-variance asset-liability management problem with random parameters and derive both the mean-variance optimal portfolio strategies and the efficient frontier. Referring to Lim [22], we consider a market where the related market parameters are random, such as interest rate, the appreciation rates, and the volatility rates of stocks' price. Further, we routinely assume that the liability is dynamically exogenous and evolves according to a Brownian motion with drift. Note that this description of liability has been widely used (see, e.g., [12, 23, 24]). Under the above assumptions, we introduce an unconstrained stochastic control problem with random parameters and derive the optimal control strategies in terms of the solutions of BSDEs. Then, by using the Lagrange multiplier technique, we derive both the mean-variance optimal investment strategies and the efficient frontier.

Our model is most closely related to the model of Lim [22]. The main differences between our model and Lim's model are in two dimensions. Firstly, we consider a portfolio selection problem with liability. Since the liability is dynamically exogenous, the driving factors of the wealth in our model include that of stocks' price and liability, which is an essential difficulty in our model but not encountered in [22]. Secondly, due to the introduction of random liability, the wealth process derived from our model is no longer homogenous with respect to the control variables, whereas the wealth process in the model without liability (see, e.g., [21, 22]) is homogenous.

This paper proceeds as follows. In Section 2, we give some preliminaries and formulate a continuous-time mean-variance portfolio selection model with liability and random parameters. In Section 3, we introduce an unconstrained stochastic LQ control problem and derive the optimal policies and value function in closed forms in terms of the solution of BSDEs. Further, Section 4 presents the optimal investment strategies and the efficient frontier for the mean-variance asset-liability management problem with random parameters. Section 5 concludes the paper.

2. Model Formulation

In this section, we describe the financial market, the liability, and the mean-variance asset-liability management problem, respectively. Throughout this paper, let T be a fixed terminal time, $(\Omega, \mathcal{F}, P)$ a complete probability space, and M' the transpose of the vector or matrix M.

2.1. The Financial Market. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered complete probability space on which a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $m+d$-dimensional Brownian motion $\overline{W}(t) := (W(t)', B(t)')' := (W^1(t), \ldots, W^m(t), B^1(t), \ldots, B^d(t))'$ for $m \geq 1$ and $d \geq 0$ is defined. It is assumed that $\mathcal{F}_t = \sigma\{\overline{W}(s) : s \leq t\}$. In this paper, we use $B(t)$ to model the financial market incompleteness as Lim [22] did. When $d = 0$, the financial market corresponds to a complete market.

Consider a financial market with $m+1$ securities which consists of a bond and m stocks. The price of bond $A^0(t)$ satisfies the following differential equation:

$$\begin{aligned} dA^0(t) &= r(t) A^0(t)\, dt, \quad t \in [0, T], \\ A^0(0) &= A_0^0 > 0, \end{aligned} \tag{1}$$

where the interest rate is as follows: $r(t) > 0$. The price of the ith stock, $A^i(t)$, is described by the following stochastic differential equation (SDE):

$$\begin{aligned} dA^i(t) &= A^i(t) \left[\mu^i(t)\, dt + \sum_{j=1}^{m} \sigma^{ij}(t)\, dW^j(t) \right], \\ A^i(0) &= A_0^i > 0, \end{aligned} \tag{2}$$

where $\mu^i(t) > 0$ and $\sigma^i(t) := (\sigma^{i1}(t), \ldots, \sigma^{im}(t))$ are appreciation rate and volatility rate of the ith stock, respectively. The $\mathbb{R}^{m\times m}$-valued process of volatility coefficients

$$\sigma(t) := \left(\sigma^1(t)', \ldots, \sigma^m(t)'\right)' \tag{3}$$

is known as the volatility. In addition, we assume that the market parameters $r(\cdot)$, $\mu^i(\cdot)$, and $\sigma^{ij}(\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes.

2.2. Liability. We assume that an exogenous accumulative liability $L(t)$ is governed by

$$\begin{aligned} dL(t) &= u(t)\, dt + v(t)\, dB_L(t), \\ L(0) &= L_0 > 0, \end{aligned} \tag{4}$$

where $B_L(t)$ is a one-dimensional standard Brownian motion. We assume that the diffusion term of the liability, B_L, is correlated with $\overline{W}(t)$, and $\rho = (\rho_1, \ldots, \rho_{m+d})'$ is the correlation coefficient. Then, $B_L(t)$ can be further expressed as follows (see (2.6) of [25] for more details):

$$B_L(t) = \rho' \overline{W}(t) + \sqrt{1 - \rho'\rho} W^0(t), \tag{5}$$

where $W^0(t)$ is a standard Brownian motion which is independent of $\overline{W}(t)$. It follows from Itô's formula that

$$dB_L(t) = \rho' d\overline{W}(t) + \sqrt{1 - \rho'\rho} dW^0(t). \tag{6}$$

Thus, the liability $L(t)$ can be rewritten as

$$\begin{aligned} dL(t) = {} & u(t)\,dt - \delta_1(t)'\,dW(t) - \delta_2(t)'\,dB(t) \\ & - \delta_0(t)\,dW^0(t), \end{aligned} \tag{7}$$

where $\delta_1(t) := -v(t)(\rho_1, \ldots, \rho_m)'$, $\delta_2(t) := -v(t)(\rho_{m+1}, \ldots, \rho_{m+d})'$, and $\delta_0(t) := -v(t)\sqrt{1 - \rho'\rho}$. Further, we assume that $u(\cdot)$ and $\delta(\cdot) := (\delta_1(\cdot)', \delta_2(\cdot)', \delta_0(\cdot))'$ are $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$-adapted stochastic processes, where $\widehat{\mathcal{F}}_t := \sigma\{(\overline{W}(s)', W^0(s))' : s \leq t\}$.

Remark 1. When $B_L(t)$ is independent of $\overline{W}(t)$, that is, $\rho = 0$, $B_L(t)$ is equal to $W^0(t)$. When $\rho'\rho = 1$, $B_L(t)$ can be expressed as a linear combination of $W^1(t), \ldots, W^m(t), B^1(t), \ldots, B^d(t)$.

Remark 2. Since $r(\cdot)$, $\mu^i(\cdot)$, and $\sigma^{ij}(\cdot)$ are the parameters for describing the financial market and $u(\cdot)$ and $v(\cdot)$ are used to describe the exogenous liability, it is reasonable to assume that $r(\cdot)$, $\mu^i(\cdot)$, and $\sigma^{ij}(\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted for $i, j = 1, \ldots, m$, and $u(\cdot)$, $v(\cdot)$, and $\delta(\cdot)$ are $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$-adapted.

2.3. The Mean-Variance Asset-Liability Management Model. Suppose that the trading of shares takes place continuously in a self-financing fashion and there are no transaction costs. We assume that an investor has an initial endowment w and a liability $L(t)$. We denote by $X(t)$ the net total wealth of the investor at time $t \in [0, T]$ and by $\pi^i(t)$, $i = 1, \ldots, m$, the market value of the investor's wealth in the ith stock. Then, $\pi(t) := (\pi^1(t), \ldots, \pi^m(t))'$ is a portfolio. The net total wealth satisfies the following equation:

$$\begin{aligned} dX(t) = {} & (r(t)X(t) + b(t)\pi(t) - u(t))\,dt \\ & + \left(\pi(t)'\sigma(t) + \delta_1(t)'\right) dW(t) \\ & + \delta_2(t)'\,dB(t) + \delta_0(t)\,dW^0(t), \\ X(0) = {} & X_0 = w - L_0, \end{aligned} \tag{8}$$

where $b(t) = (\mu^1(t) - r(t), \ldots, \mu^m(t) - r(t))$.

Next, we introduce the following notations.

One has $|x| := \sqrt{\sum_{i=1}^n x_i^2}$, where $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$.

$\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ is the set of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, $\mathbb{R}^n$-valued stochastic processes on $[0, T]$ such that

$$E\int_0^T |f(t)|^2\,dt < \infty. \tag{9}$$

$\mathcal{L}^\infty_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}))$ is the set of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted essentially bounded stochastic processes on $[0, T]$ with continuous sample paths.

$\mathcal{L}^2_{\widehat{\mathcal{F}}}(0, T; \mathbb{R}^n)$ is the set of $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$-adapted, $\mathbb{R}^n$-valued stochastic processes on $[0, T]$ such that

$$E\int_0^T |f(t)|^2\,dt < \infty. \tag{10}$$

$\mathcal{L}^2_{\widehat{\mathcal{F}}}(\Omega; C(0, T; \mathbb{R}))$ is the set of $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$-adapted, $\mathbb{R}$-valued stochastic processes on $[0, T]$ with P-a.s. continuous sample paths such that $E\sup_{t\in[0,T]}|f(t)|^2 < \infty$.

$\mathcal{L}^{2,\mathrm{loc}}_{\widehat{\mathcal{F}}}(0, T; \mathbb{R}^n)$ is the set of $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$-adapted, $\mathbb{R}^n$-valued stochastic processes on $[0, T]$ such that

$$\int_0^T |f(t)|^2\,dt < \infty, \quad P\text{-a.s.} \tag{11}$$

$\mathcal{L}^2_{\widehat{\mathcal{F}}_T}(\Omega; \mathbb{R})$ is the set of $\widehat{\mathcal{F}}_T$-measurable, square-integrable random variables.

$\mathcal{L}^\infty_{\widehat{\mathcal{F}}}(0, T; \mathbb{R}^m)$ is the set of $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$-adapted essentially bounded stochastic processes on $[0, T]$.

Definition 3. A portfolio policy $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in \mathcal{L}^2_{\widehat{\mathcal{F}}}(0, T; \mathbb{R}^m)$ and there exists a unique solution of (8). In this case, we refer to $(X(\cdot), \pi(\cdot))$ as an admissible pair.

In this paper, we study the classical mean-variance asset-liability management problem where the liability is an exogenous liability $L(t)$. The objective of the investor is to find a portfolio $\pi(\cdot)$ to minimize his/her risk which is measured by the variance of the net terminal wealth subject to archiving a prescribed expected terminal wealth. Then, the mean-variance asset-liability management problem can be formulated as follows:

$$\begin{aligned} J^* := \min \quad & \mathrm{Var}(X(T)) = E[X(T) - c]^2, \\ \text{subject to:} \quad & EX(T) \\ & = c, \quad (X(\cdot), \pi(\cdot)) \text{ is admissible for } (8), \end{aligned} \tag{12}$$

where $c \in \mathbb{R}$ is the prescribed expected terminal wealth. It is clear that (12) is a linearly constrained convex program problem. Thus, it can be reduced to an unconstrained problem by introducing a Lagrange multiplier. Therefore, in Section 3, we first consider the following unconstrained problem parameterized by $l \in \mathbb{R}$,

$$\begin{aligned} \min \quad & E[X(T) - l]^2, \\ \text{subject to:} \quad & (X(\cdot), \pi(\cdot)) \text{ is admissible for } (8), \end{aligned} \tag{13}$$

and approach it from the perspective of stochastic LQ optimal control and BSDEs. Further, in Section 4, based on the results in Section 3, we employ the Lagrange multiplier method to derive the mean-variance efficient portfolio and the efficient frontier.

In addition, we assume that the following assumptions are satisfied throughout this paper.

Assumption 4. Consider the following:

$$\begin{gathered} r(\cdot), \mu^i(\cdot), \sigma^{ij}(\cdot) \in \mathscr{L}^{\infty}_{\mathscr{F}}(0,T;\mathbb{R}), \quad i,j=1,\ldots,m; \\ u(\cdot), v(\cdot) \in \mathscr{L}^{\infty}_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}); \\ \delta(\cdot) \in \mathscr{L}^{\infty}_{\widetilde{\mathscr{F}}}\left(0,T;\mathbb{R}^{m+d+1}\right); \\ \sigma(t)\sigma(t)' \geq \epsilon I_m, \\ \forall t \in [0,T], \text{ for some } \epsilon > 0, \end{gathered} \tag{14}$$

where I_m is the $m \times m$ identity matrix. Note that $\sigma(t)\sigma(t)' \geq \epsilon I_m$ is the so-called nondegeneracy condition and implies that $\sigma(t)$ is invertible.

3. The Unconstrained Asset-Liability Management Problem

The aim of this section is to derive the optimal solution for the unconstrained problem (13).

Consider the following BSDEs:

$$\begin{aligned} dp(t) &= \Big[\left(-2r(t) + |\theta(t)|^2\right)p(t) + 2\theta(t)'\Lambda_1(t) \\ &\quad + \frac{1}{p(t)}|\Lambda_1(t)|^2\Big]dt + \Lambda_1(t)'dW(t) + \Lambda_2(t)'dB(t), \\ p(T) &= 1, \quad p(t) > 0, \ \forall t \in [0,T], \end{aligned} \tag{15}$$

$$\begin{aligned} dh(t) &= \Big(r(t)h(t) + \theta(t)'\eta_1(t) - \frac{\Lambda_2(t)'}{p(t)}\eta_2(t) \\ &\quad - u(t) - \theta(t)'\delta_1(t) + \delta_2(t)'\frac{\Lambda_2(t)}{p(t)}\Big)dt + \eta_1(t)'dW(t) \\ &\quad + \eta_2(t)'dB(t) + \eta_0(t)dW^0(t), \\ h(T) &= l, \end{aligned} \tag{16}$$

where $\theta(t) = \sigma(t)^{-1}b(t)'$. Throughout this paper, a pair of processes $(p(\cdot), \Lambda(\cdot))$ is called a solution of BSDE (15) if it satisfies BSDE (15) and

$$\begin{gathered} \Lambda(\cdot) = \left(\Lambda_1(\cdot)', \Lambda_2(\cdot)'\right)', \\ \left(p(\cdot), \Lambda(\cdot)\right) \in \mathscr{L}^{\infty}_{\mathscr{F}}(\Omega; C(0,T;\mathbb{R})) \\ \cdot \mathscr{L}^{2}_{\mathscr{F}}\left(0,T;\mathbb{R}^{m+d}\right), \\ \frac{1}{p(\cdot)} \in \mathscr{L}^{\infty}_{\mathscr{F}}(\Omega; C(0,T;\mathbb{R})). \end{gathered} \tag{17}$$

On the other hand, a pair $(h(\cdot), \eta(\cdot))$ is called a solution of BSDE (16) if $(h(\cdot), \eta(\cdot))$ satisfies BSDE (16) and

$$\begin{gathered} \eta(\cdot) = \left(\eta_1(\cdot)', \eta_2(\cdot)', \eta_0(\cdot)\right)', \\ \left(h(\cdot), \eta(\cdot)\right) \in \mathscr{L}^{2}_{\widetilde{\mathscr{F}}}(\Omega; C(0,T;\mathbb{R})) \\ \cdot \mathscr{L}^{2}_{\widetilde{\mathscr{F}}}\left(0,T;\mathbb{R}^{m+d+1}\right). \end{gathered} \tag{18}$$

Before deriving the optimal solution for problem (13), we will prove the existence and uniqueness of solutions of BSDEs (15) and (16), respectively. The following result can be found in [22] (see Theorem 6.1 of [22]).

Lemma 5. *If Assumption 4 holds, then the following BSDE,*

$$\begin{aligned} dp(t) &= \Big[\left(-2r(t) + |\theta(t)|^2\right)p(t) + 2\theta(t)'\Lambda_1(t) \\ &\quad + \frac{|\Lambda_1(t)|^2}{p(t)}\Big]dt + \Lambda_1(t)'dW(t) + \widetilde{\Lambda}_2(t)'d\widetilde{B}(t), \\ p(T) &= 1, \quad p(t) > 0, \ \forall t \in [0,T], \end{aligned} \tag{19}$$

has a solution $(p(\cdot), \widetilde{\Lambda}(\cdot))$ where $\widetilde{B}(\cdot) := (B(\cdot)', W^0(\cdot))'$. Moreover, if $(\overline{p}(\cdot), \overline{\Lambda}(\cdot))$ and $(p(\cdot), \widetilde{\Lambda}(\cdot))$ are solutions of (19), then $\overline{p}(\cdot) \equiv p(\cdot)$.

Here, we claim that $\overline{\Lambda}(\cdot) \equiv \widetilde{\Lambda}(\cdot)$ holds. In fact, by applying Itô's formula to $\Delta p(t) := p(t) - \overline{p}(t)$, we have

$$\begin{aligned} d\Delta p(t) &= \Delta F(t)dt + \Delta\Lambda_1(t)'dW(t) \\ &\quad + \Delta\Lambda_2(t)'d\widetilde{B}(t), \\ \Delta p(T) &= 0, \end{aligned} \tag{20}$$

where $\Delta\Lambda_1(\cdot) := \widetilde{\Lambda}_1(\cdot) - \overline{\Lambda}_1(\cdot)$, $\Delta\Lambda_2(\cdot) := \widetilde{\Lambda}_2(\cdot) - \overline{\Lambda}_2(\cdot)$, and

$$\begin{aligned} \Delta F(\cdot) &:= \left(-2r(\cdot) + |\theta(\cdot)|^2\right)\Delta p(\cdot) + 2\theta(\cdot)'\Delta\Lambda_1(\cdot) \\ &\quad + \frac{1}{p(\cdot)}\left(\left|\widetilde{\Lambda}_1(\cdot)\right|^2 - \left|\overline{\Lambda}_1(\cdot)\right|^2\right). \end{aligned} \tag{21}$$

Once again, it follows from Itô's formula that

$$\begin{aligned} &d\left(\Delta p(t)\right)^2 \\ &\quad = \left(2\Delta p(t)\Delta F(t) + |\Delta\Lambda_1(t)|^2 + |\Delta\Lambda_2(t)|^2\right)dt \\ &\qquad + 2\Delta p(t)\Delta\Lambda_1(t)'dW(t) \\ &\qquad + 2\Delta p(t)\Delta\Lambda_2(t)'d\widetilde{B}(t), \\ &\left(\Delta p(T)\right)^2 = 0. \end{aligned} \tag{22}$$

Thus we have

$$-E(\Delta p(t))^2 = E\int_t^T \left(2\Delta p(s)\Delta F(s) + |\Delta\Lambda_1(s)|^2 + |\Delta\Lambda_2(s)|^2\right) ds. \tag{23}$$

From Lemma 5, we know that $\Delta p(\cdot) \equiv 0$ and so

$$0 = E\int_t^T \left(|\Delta\Lambda_1(s)|^2 + |\Delta\Lambda_2(s)|^2\right) ds. \tag{24}$$

This implies that $\overline{\Lambda}(\cdot) \equiv \widetilde{\Lambda}(\cdot)$ holds.

Since $r(\cdot)$ and $\theta(\cdot)$ are $\{\mathscr{F}_t\}_{t\geq 0}$-adapted, BSDE (19) can reduce to (15). This implies that BSDE (15) has a unique solution $(p(\cdot), \Lambda(\cdot))$ under Assumption 4. Moreover, $(p(\cdot), \widetilde{\Lambda}(\cdot))$ is the unique solution of BSDE (19), where $\widetilde{\Lambda}(\cdot) = (\Lambda(\cdot)', 0)'$.

From the discussion of Section 4 in [22], we have the following lemma.

Lemma 6. *If Assumption 4 holds, then*

$$\begin{pmatrix} \widetilde{X}(t) \\ \widetilde{Y}(t) \end{pmatrix} = \begin{pmatrix} W(t) \\ \widetilde{B}(t) \end{pmatrix} + \int_0^t \begin{pmatrix} \theta(s) \\ -\dfrac{\widetilde{\Lambda}_2(s)}{p(s)} \end{pmatrix} ds \tag{25}$$

is a standard Brownian motion under $\overline{P}$ where

$$\frac{d\overline{P}}{dP} := \exp\left\{-\int_0^T \theta(s)' dW(s) + \int_0^T \frac{\widetilde{\Lambda}_2(s)'}{p(s)} d\widetilde{B}(s) - \frac{1}{2}\int_0^T \left(|\theta(s)|^2 + \left|\frac{\widetilde{\Lambda}_2(s)}{p(s)}\right|^2\right) ds\right\}. \tag{26}$$

For the existence and uniqueness of solution of BSDE (16), we have the following result.

Proposition 7. *If Assumption 4 holds, then BSDE (16) has a unique solution.*

Proof. The assumption guarantees that there is a unique optimal control $\widehat{\pi}(\cdot) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(0, T; \mathbb{R}^m)$ for (13). Denote by $\widehat{X}(\cdot)$ the net wealth process associated with the optimal control $\widehat{\pi}(\cdot)$. The optimal condition (see [26]) implies that

$$b(t)' y(t) + \sigma(t) z_1(t) = 0, \tag{27}$$

where $(y(\cdot), z(\cdot))$ is the unique solution of the following linear BSDE (called adjoint equation):

$$\begin{aligned} dy(t) &= -r(t) y(t) dt + z_1(t)' dW(t) + z_2(t)' dB(t) \\ &\quad + z_0(t) dW^0(t), \\ y(T) &= \widehat{X}(T) - l. \end{aligned} \tag{28}$$

By using Itô's formula, we have

$$\begin{aligned} d\left(\frac{1}{p(t)}\right) &= \left(\frac{2r(t) - |\theta(t)|^2}{p(t)} - \frac{2\theta(t)'\Lambda_1(t)}{p^2(t)} + \frac{|\Lambda_2(t)|^2}{p^3(t)}\right) dt - \frac{\Lambda_1(t)'}{p^2(t)} dW(t) - \frac{\Lambda_2(t)'}{p^2(t)} dB(t), \\ d\left(\frac{y(t)}{p(t)}\right) &= \left[\frac{\left(r(t) - |\theta(t)|^2\right) y(t)}{p(t)} - \frac{2\theta(t)'\Lambda_1(t) y(t) + \Lambda_1(t)' z_1(t) + \Lambda_2(t)' z_2(t)}{p^2(t)} + \frac{y(t)|\Lambda_2(t)|^2}{p^3(t)}\right] dt \\ &\quad + \left(\frac{z_1(t)'}{p(t)} - \frac{y(t)\Lambda_1(t)'}{p^2(t)}\right) dW(t) + \left(\frac{z_2(t)'}{p(t)} - \frac{y(t)\Lambda_2(t)'}{p^2(t)}\right) dB(t) + \frac{z_0(t)}{p(t)} dW^0(t), \\ d\left(\widehat{X}(t) - \frac{y(t)}{p(t)}\right) &= \left[r(t)\left(\widehat{X}(t) - \frac{y(t)}{p(t)}\right) + \theta(t)' \cdot \left(\sigma(t)'\widehat{\pi}(t) + \delta_1(t) - \frac{z_1(t)}{p(t)} + \frac{y(t)\Lambda_1(t)}{p^2(t)}\right)\right] dt \\ &\quad - \left[\frac{\Lambda_2(t)'}{p(t)}\left(\delta_2(t) - \frac{z_2(t)}{p(t)} + \frac{y(t)\Lambda_2(t)}{p^2(t)}\right) + u(t) + \theta'(t)\delta_1(t) - \frac{\delta_2(t)'\Lambda_2(t)}{p(t)}\right] dt + \left[\frac{y(t)|\theta(t)|^2}{p(t)} \right. \\ &\quad \left. + \frac{\theta(t)' z_1(t)}{p(t)} + \frac{y(t)\theta(t)'\Lambda_1(t) + \Lambda_1(t)' z_1(t)}{p^2(t)}\right] dt \\ &\quad + \left(\sigma(t)'\widehat{\pi}(t) + \delta_1(t) - \frac{z_1(t)}{p(t)} + \frac{y(t)\Lambda_1(t)}{p^2(t)}\right)' dW(t) \\ &\quad + \left(\delta_2(t) - \frac{z_2(t)}{p(t)} + \frac{y(t)\Lambda_2(t)}{p^2(t)}\right)' dB(t) + \left(\delta_0(t) - \frac{z_0(t)}{p(t)}\right) dW^0(t). \end{aligned} \tag{29}$$

It follows from (27) that $\theta(t)y(t) + z_1(t) = 0$. Further,

$$\begin{aligned} d\left(\widehat{X}(t) - \frac{y(t)}{p(t)}\right) &= \left[r(t)\left(\widehat{X}(t) - \frac{y(t)}{p(t)}\right) + \theta(t)' \cdot \left(\sigma(t)'\widehat{\pi}(t) + \delta_1(t) - \frac{z_1(t)}{p(t)} + \frac{y(t)\Lambda_1(t)}{p^2(t)}\right)\right] dt \\ &\quad - \left[\frac{\Lambda_2(t)'}{p(t)}\left(\delta_2(t) - \frac{z_2(t)}{p(t)} + \frac{y(t)\Lambda_2(t)}{p^2(t)}\right) + u(t) + \theta(t)'\delta_1(t) - \frac{\delta_2(t)'\Lambda_2(t)}{p(t)}\right] dt + \left(\sigma(t)'\widehat{\pi}(t) \right. \\ &\quad \left. + \delta_1(t) - \frac{z_1(t)}{p(t)} + \frac{y(t)\Lambda_1(t)}{p^2(t)}\right)' dW(t) + \left(\delta_2(t) - \frac{z_2(t)}{p(t)} + \frac{y(t)\Lambda_2(t)}{p^2(t)}\right)' dB(t) + \left(\delta_0(t) - \frac{z_0(t)}{p(t)}\right) dW^0(t). \end{aligned} \tag{30}$$

Comparing with BSDE (16), we conclude that $(\widetilde{h}(\cdot),\widetilde{\eta}(\cdot))$ is a solution of (16), where

$$\begin{aligned}
\widetilde{h}(t) &:= \widehat{X}(t) - \frac{y(t)}{p(t)},\\
\widetilde{\eta}_1(t) &:= \sigma(t)'\widehat{\pi}(t) + \delta_1(t) - \frac{z_1(t)}{p(t)} + \frac{y(t)\Lambda_1(t)}{p^2(t)},\\
\widetilde{\eta}_2(t) &:= \delta_2(t) - \frac{z_2(t)}{p(t)} + \frac{y(t)\Lambda_2(t)}{p^2(t)},\\
\widetilde{\eta}_0(t) &:= \delta_0(t) - \frac{z_0(t)}{p(t)}.
\end{aligned} \tag{31}$$

Now we show the uniqueness of the solution for (16). Assume that $(h(\cdot),\eta(\cdot))$ and $(\widetilde{h}(\cdot),\widetilde{\eta}(\cdot))$ are two solutions of BSDE (16). It follows from Itô's formula that

$$\begin{aligned}
&d\Delta h(t)\\
&\quad= \left(r(t)\Delta h(t) + \theta(t)'\Delta\eta_1(t) - \frac{\Lambda_2(t)'}{p(t)}\Delta\eta_2(t)\right)dt\\
&\qquad+ \Delta\eta_1(t)'dW(t) + \Delta\eta_2(t)'dB(t)\\
&\qquad+ \Delta\eta_0(t)dW^0(t),\\
&\Delta h(T) = 0,
\end{aligned} \tag{32}$$

where $\Delta h(\cdot) := h(\cdot) - \widetilde{h}(\cdot)$, $\Delta\eta_1(\cdot) := \eta_1(\cdot) - \widetilde{\eta}_1(\cdot)$, $\Delta\eta_2(\cdot) := \eta_2(\cdot) - \widetilde{\eta}_2(\cdot)$, and $\Delta\eta_0(\cdot) := \eta_0(\cdot) - \widetilde{\eta}_0(\cdot)$.

By using the transformation defined by (25) to (32), we have

$$\begin{aligned}
d\Delta h(t) &= r(t)\Delta h(t)dt + \Delta\eta_1(t)'d\widetilde{X}(t)\\
&\quad+ \left(\Delta\eta_2(t)', \Delta\eta_0(t)\right)d\widetilde{Y}(t),\\
\Delta h(T) &= 0
\end{aligned} \tag{33}$$

which is a linear BSDE and has a unique solution $(0,0)$ under Assumption 4. In consequence, we have $h(\cdot) = \widetilde{h}(\cdot)$ and $\eta(\cdot) = \widetilde{\eta}(\cdot)$.

This completes the proof. □

The following lemma is a generalization of Lemma 3.1 in [22].

Lemma 8. *Suppose that Assumption 4 holds. Let $\overline{\pi}(\cdot) \in \mathscr{L}^{2,\mathrm{loc}}_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^m)$ be given and fixed. If net wealth equation (8) corresponding to $\overline{\pi}(\cdot)$ has a unique solution $X(\cdot)$ such that $X(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R})$ and $X(T) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}_T}(\Omega;\mathbb{R})$, then $\overline{\pi}(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^m)$ is admissible.*

Proof. Assume that $\overline{\pi}(\cdot) \in \mathscr{L}^{2,\mathrm{loc}}_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^m)$ is given and fixed and SDE (8) corresponding to $\overline{\pi}(\cdot)$ has a unique solution $X(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R})$. It follows from Itô's formula that

$$\begin{aligned}
X(t)^2 &= X(0)^2 + \int_0^t 2X(s)(r(s)X(s) + b(s)\overline{\pi}(s)\\
&\quad- u(s))ds + \int_0^t \left(\left|\sigma(s)'\overline{\pi}(s) + \delta_1(s)\right|^2 + \left|\delta_2(s)\right|^2\right.\\
&\quad\left.+ \left|\delta_0(s)\right|^2\right)ds + \int_0^t 2X(s)\delta_1(s)'dW(s)\\
&\quad+ \int_0^t 2X(s)\delta_2(s)'dB(s) + \int_0^t 2X(s)\\
&\quad\cdot \delta_0(s)dW^0(s) + \int_0^t 2X(s)\\
&\quad\cdot \left(\sigma(s)'\overline{\pi}(s)\right)'dW(s).
\end{aligned} \tag{34}$$

Under Assumption 4, we have $2X(\cdot)\delta_1(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^m)$, $2X(\cdot)\delta_2(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^d)$, and $2X(\cdot)\delta_0(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R})$, which imply that

$$\begin{aligned}
&\int_0^t 2X(s)\delta_1(s)'dW(s) + \int_0^t 2X(s)\delta_2(s)'dB(s)\\
&\quad+ \int_0^t 2X(s)\delta_0(s)dW^0(s)
\end{aligned} \tag{35}$$

is a martingale and so

$$\begin{aligned}
&E\left[\int_0^t 2X(s)\delta_1(s)'dW(s) + \int_0^t 2X(s)\delta_2(s)'dB(s)\right.\\
&\quad\left.+ \int_0^t 2X(s)\delta_0(s)dW^0(s)\right] = 0.
\end{aligned} \tag{36}$$

Because $X(\cdot)$ is continuous (and bounded on $[0,T]$, a.s.), we have $2X(\cdot)\sigma(\cdot)'\overline{\pi}(\cdot) \in \mathscr{L}^{2,\mathrm{loc}}_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^m)$ and $\int_0^t 2X(s)(\sigma(s)'\overline{\pi}(s))'dW(s)$ is a local martingale. Therefore, there exists a localizing sequence $\{\tau_i\}$ for the local martingale such that

$$\int_0^{t\wedge\tau_i} 2X(s)\left(\sigma(s)'\overline{\pi}(s)\right)'dW(s) \tag{37}$$

is a martingale.

Putting $t = T\wedge\tau_i$ and taking expectations on both sides of (34), we have

$$\begin{aligned}
EX(T\wedge\tau_i)^2 &= X(0)^2 + E\int_0^{T\wedge\tau_i} 2X(s)(r(s)X(s)\\
&\quad+ b(s)\overline{\pi}(s) - u(s))ds\\
&\quad+ E\int_0^{T\wedge\tau_i} \left(\left|\sigma(s)'\overline{\pi}(s) + \delta_1(s)\right|^2 + \left|\delta_2(s)\right|^2\right.\\
&\quad\left.+ \left|\delta_0(s)\right|^2\right)ds.
\end{aligned} \tag{38}$$

Then it can be rewritten as

$$\begin{aligned} &X(0)^2 + E\int_0^{T\wedge\tau_i} \overline{\pi}(s)'\sigma(s)\sigma(s)'\overline{\pi}(s)\,ds \\ &\quad + E\int_0^{T\wedge\tau_i} \left(|\delta_1(s)|^2 + |\delta_2(s)|^2 + |\delta_0(s)|^2\right) ds \\ &\quad = EX(T\wedge\tau_i)^2 - E\int_0^{T\wedge\tau_i} 2r(s)X(s)^2\,ds \\ &\quad + E\int_0^{T\wedge\tau_i} 2X(s)u(s)\,ds \\ &\quad - E\int_0^{T\wedge\tau_i} 2\delta_1(s)'\sigma(s)'\overline{\pi}(s)\,ds \\ &\quad - E\int_0^{T\wedge\tau_i} 2X(s)b(s)\overline{\pi}(s)\,ds. \end{aligned} \tag{39}$$

Since

$$\begin{aligned} &-2X(s)b(s)\overline{\pi}(s) = -2\left(\sqrt{\frac{4}{\epsilon}}X(s)b(s)\right)\left(\sqrt{\frac{\epsilon}{4}}\overline{\pi}(s)\right) \\ &\quad \le \frac{4}{\epsilon}X(s)^2|b(s)|^2 + \frac{\epsilon}{4}|\overline{\pi}(s)|^2, \\ &-2\delta_1(s)'\sigma(s)'\overline{\pi}(s) \\ &\quad = -2\left(\sqrt{\frac{4}{\epsilon}}\delta_1(s)'\sigma(s)'\right)\left(\sqrt{\frac{\epsilon}{4}}\overline{\pi}(s)\right) \\ &\quad \le \frac{4}{\epsilon}|\sigma(s)\delta_1(s)|^2 + \frac{\epsilon}{4}|\overline{\pi}(s)|^2, \end{aligned} \tag{40}$$

we have

$$\begin{aligned} &X(0)^2 + E\int_0^{T\wedge\tau_i} \overline{\pi}(s)'\sigma(s)\sigma(s)'\overline{\pi}(s)\,ds \\ &\quad + E\int_0^{T\wedge\tau_i} \left(|\delta_1(s)|^2 + |\delta_2(s)|^2 + |\delta_0(s)|^2\right) ds \\ &\quad \le EX(T\wedge\tau_i)^2 - E\int_0^{T\wedge\tau_i} 2r(s)X(s)^2\,ds \\ &\quad + E\int_0^{T\wedge\tau_i} 2X(s)u(s)\,ds \\ &\quad + E\int_0^{T\wedge\tau_i} \frac{4}{\epsilon}|\sigma(s)\delta_1(s)|^2 + \frac{\epsilon}{4}|\overline{\pi}(s)|^2\,ds \\ &\quad + E\int_0^{T\wedge\tau_i} \frac{4}{\epsilon}X(s)^2|b(s)|^2 + \frac{\epsilon}{4}|\overline{\pi}(s)|^2\,ds. \end{aligned} \tag{41}$$

Rewriting the inequality above, we have

$$\begin{aligned} &X(0)^2 + E\int_0^{T\wedge\tau_i} \left(|\delta_1(s)|^2 + |\delta_2(s)|^2 + |\delta_0(s)|^2\right) ds \\ &\quad + E\int_0^{T\wedge\tau_i} \overline{\pi}(s)'\sigma(s)\sigma(s)'\overline{\pi}(s)\,ds \\ &\quad - \frac{\epsilon}{2}E\int_0^{T\wedge\tau_i} |\overline{\pi}(s)|^2\,ds \\ &\quad \le EX(T\wedge\tau_i)^2 \\ &\quad + E\int_0^{T\wedge\tau_i} X(s)^2\left(-2r(s) + \frac{4}{\epsilon}|b(s)|^2\right) ds \\ &\quad + E\int_0^{T\wedge\tau_i} 2X(s)u(s)\,ds \\ &\quad + \frac{4}{\epsilon}E\int_0^{T\wedge\tau_i} |\sigma(s)\delta_1(s)|^2\,ds. \end{aligned} \tag{42}$$

From Fatou's lemma, we obtain

$$\begin{aligned} &X(0)^2 + E\int_0^{T} \left(|\delta_1(s)|^2 + |\delta_2(s)|^2 + |\delta_0(s)|^2\right) ds \\ &\quad + E\int_0^{T} \overline{\pi}(s)'\sigma(s)\sigma(s)'\overline{\pi}(s)\,ds \\ &\quad - \frac{\epsilon}{2}E\int_0^{T} |\overline{\pi}(s)|^2\,ds \le EX(T)^2 \\ &\quad + E\int_0^{T} X(s)^2\left(-2r(s) + \frac{4}{\epsilon}|b(s)|^2\right) ds \\ &\quad + E\int_0^{T} 2X(s)u(s)\,ds + \frac{4}{\epsilon}E\int_0^{T} |\sigma(s)\delta_1(s)|^2\,ds \\ &\quad < +\infty, \end{aligned} \tag{43}$$

where the last inequality comes from Assumption 4 and $X(\cdot) \in \mathscr{L}^2_{\widehat{\mathcal{F}}}(0,T;\mathbb{R})$.

Since $\sigma(t)\sigma(t)' \ge \epsilon I_m$, we have

$$\begin{aligned} &\frac{\epsilon}{2}E\int_0^{T} |\overline{\pi}(s)|^2\,ds \\ &\quad \le X(0)^2 \\ &\quad + E\int_0^{T} \left(|\delta_1(s)|^2 + |\delta_2(s)|^2 + |\delta_0(s)|^2\right) ds \\ &\quad + E\int_0^{T} \overline{\pi}(s)'\sigma(s)\sigma(s)'\overline{\pi}(s)\,ds \\ &\quad - \frac{\epsilon}{2}E\int_0^{T} |\overline{\pi}(s)|^2\,ds < +\infty \end{aligned} \tag{44}$$

which implies that $\overline{\pi}(\cdot) \in \mathscr{L}^2_{\widehat{\mathcal{F}}}(0,T;\mathbb{R}^m)$ is admissible.

This completes the proof. □

The following result concerns the admissibility of (46).

Proposition 9. *If Assumption 4 holds, then*

$$\begin{aligned} dX(t) &= \Big\{ r(t)X(t) - \theta(t)' \Big[\delta_1(t) - \eta_1(t) \\ &\quad + (X(t) - h(t)) \Big(\theta(t) + \frac{\Lambda_1(t)}{p(t)} \Big) \Big] - u(t) \Big\} dt \\ &\quad + \Big[\eta_1(t) - (X(t) - h(t)) \Big(\theta(t) + \frac{\Lambda_1(t)}{p(t)} \Big) \Big]' dW(t) \\ &\quad + \delta_2(t)' dB(t) + \delta_0(t) dW^0(t), \\ X(0) &= X_0 \end{aligned} \tag{45}$$

(replace (8) by (46)) has a unique solution with $X(\cdot) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(0,T;\mathbb{R})$ *and* $X(T) \in \mathscr{L}^2_{\widehat{\mathscr{F}}_T}(\Omega;\mathbb{R})$. *Moreover,*

$$\begin{aligned} \overline{\pi}(t) := -\left(\sigma(t)^{-1}\right)' \Big[\delta_1(t) - \eta_1(t) \\ + (X(t) - h(t)) \Big(\theta(t) + \frac{\Lambda_1(t)}{p(t)} \Big) \Big] \end{aligned} \tag{46}$$

is admissible.

Proof. Consider the following SDE:

$$\begin{aligned} dY(t) &= -r(t)Y(t)dt - Y(t)\theta(t)' dW(t) \\ &\quad + [Y(t)\alpha_1(t) + \alpha_2(t)]' dB(t) \\ &\quad + \beta_2(t) dW^0(t), \\ Y(0) &= p(0)(X(0) - h(0)), \end{aligned} \tag{47}$$

where

$$\begin{aligned} \alpha_1(t) &= \frac{\Lambda_2(t)}{p(t)}, \\ \alpha_2(t) &= p(t)(\delta_2(t) - \eta_2(t)), \\ \beta_2(t) &= p(t)(\delta_0(t) - \eta_0(t)). \end{aligned} \tag{48}$$

It can be shown that

$$\begin{aligned} Y(t) &= \Phi(t) \Big[p(0)(X(0) - h(0)) \\ &\quad - \int_0^t \alpha_1(s)' \alpha_2(s) \Phi^{-1}(s) ds \Big] + \Phi(t) \\ &\quad \cdot \Big(\int_0^t \alpha_2(s)' \Phi^{-1}(s) dB(s) \\ &\quad + \int_0^t \beta_2(s) \Phi^{-1}(s) dW^0(s) \Big) \end{aligned} \tag{49}$$

is the unique solution of (47), where

$$\begin{aligned} \Phi(t) &= \exp \Big\{ -\frac{1}{2} \int_0^t \left(|\theta(s)|^2 + |\alpha_1(s)|^2 + 2r(s) \right) ds \\ &\quad - \int_0^t \theta(s)' dW(s) + \int_0^t \alpha_1(s)' dB(s) \Big\}. \end{aligned} \tag{50}$$

By using Itô's formula, we have that

$$X(t) = h(t) + \frac{Y(t)}{p(t)} \tag{51}$$

is the unique solution of SDE (45).

It follows from Itô's formula that

$$\begin{aligned} d(X(t) - h(t)) &= \Big[r(t)(X(t) - h(t)) \\ &\quad - (X(t) - h(t))\theta(t)' \Big(\theta(t) + \frac{\Lambda_1(t)}{p(t)} \Big) \\ &\quad + \frac{\Lambda_2(t)'}{p(t)} (\eta_2(t) - \delta_2(t)) \Big] dt - (X(t) - h(t)) \\ &\quad \cdot \Big(\theta(t) + \frac{\Lambda_1(t)}{p(t)} \Big)' dW(t) \\ &\quad + (\delta_2(t) - \eta_2(t))' dB(t) + (\delta_0(t) - \eta_0(t)) dW^0(t), \\ d(X(t) - h(t))^2 &= \Big[(X(t) - h(t))^2 \Big(2r(t) - |\theta(t)|^2 + \Big| \frac{\Lambda_1(t)}{p(t)} \Big|^2 \Big) \\ &\quad + 2(X(t) - h(t))(\eta_2(t) - \delta_2(t))' \frac{\Lambda_2(t)}{p(t)} \Big] dt \\ &\quad + \left(|\delta_2(t) - \eta_2(t)|^2 + |\delta_0(t) - \eta_0(t)|^2 \right) dt \\ &\quad - 2(X(t) - h(t))^2 \Big(\theta(t) + \frac{\Lambda_1(t)}{p(t)} \Big)' dW(t) \\ &\quad + 2(X(t) - h(t))(\delta_2(t) - \eta_2(t))' dB(t) + 2(X(t) \\ &\quad - h(t))(\delta_0(t) - \eta_0(t)) dW^0(t), \\ dp(t)(X(t) - h(t))^2 &= p(t) \Big(|\delta_2(t) - \eta_2(t)|^2 \\ &\quad + |\delta_0(t) - \eta_0(t)|^2 \Big) dt - p(t)(X(t) - h(t))^2 \Big(2\theta(t) \\ &\quad + \frac{\Lambda_1(t)}{p(t)} \Big)' dW(t) \\ &\quad + \Big[2p(t)(X(t) - h(t))(\delta_2(t) - \eta_2(t))' \\ &\quad + (X(t) - h(t))^2 \Lambda_2(t)' \Big] dB(t) + 2p(t)(X(t) \\ &\quad - h(t))(\delta_0(t) - \eta_0(t)) dW^0(t). \end{aligned} \tag{52}$$

Then, we have

$$\begin{aligned} p(t)(X(t)-h(t))^2 &= p(0)(X(0)-h(0))^2 \\ &+ \int_0^t p(s)\left(|\delta_2(s)-\eta_2(s)|^2 + |\delta_0(s)-\eta_0(s)|^2\right) ds \\ &- \int_0^t p(s)(X(s)-h(s))^2 \left(2\theta(s) \right. \\ &\left. + \frac{\Lambda_1(s)}{p(s)}\right)' dW(s) \\ &+ \int_0^t \left[2p(s)(X(s)-h(s))(\delta_2(s)-\eta_2(s))' \right. \\ &\left. + (X(s)-h(s))^2 \Lambda_2(s)'\right] dB(s) + \int_0^t 2p(s) \\ &\cdot (X(s)-h(s))(\delta_0(s)-\eta_0(s)) dW^0(s). \end{aligned} \tag{53}$$

Taking $X(t) = h(t) + Y(t)/p(t)$ into account, we conclude that

$$\begin{aligned} &- \int_0^t p(s)(X(s)-h(s))^2 \left(2\theta(s) + \frac{\Lambda_1(s)}{p(s)}\right)' dW(s) \\ &+ \int_0^t 2p(s)(X(s)-h(s))(\delta_2(s)-\eta_2(s))' dB(s) \\ &+ \int_0^t (X(s)-h(s))^2 \Lambda_2(s)' dB(s) \\ &+ \int_0^t 2p(s)(X(s)-h(s))(\delta_0(s)-\eta_0(s)) dW^0(s) \\ &= - \int_0^t \frac{Y(s)^2}{p(s)} \left(2\theta(s) + \frac{\Lambda_1(s)}{p(s)}\right)' dW(s) \\ &+ \int_0^t 2Y(s)(\delta_2(s)-\eta_2(s))' dB(s) \\ &+ \int_0^t \frac{Y(s)^2}{p(s)^2} \Lambda_2(s)' dB(s) \\ &+ \int_0^t 2Y(s)(\delta_0(s)-\eta_0(s)) dW^0(s) \end{aligned} \tag{54}$$

is a local martingale under Assumption 4. Let $\{\tau_i\}$ be a localizing sequence for the local martingale above. Then, for any $t \in [0,T]$,

$$\begin{aligned} E\left[p(t\wedge\tau_i)(X(t\wedge\tau_i)-h(t\wedge\tau_i))^2\right] &= p(0)(X(0) \\ &- h(0))^2 + E\int_0^{t\wedge\tau_i} p(s) \\ &\cdot \left(|\delta_2(s)-\eta_2(s)|^2 + |\delta_0(s)-\eta_0(s)|^2\right) ds. \end{aligned} \tag{55}$$

It follows from Fatou's lemma and Assumption 4 that

$$\begin{aligned} E\left[p(t)(X(t)-h(t))^2\right] &\le p(0)(X(0)-h(0))^2 \\ &+ E\int_0^t p(s) \\ &\cdot \left(|\delta_2(s)-\eta_2(s)|^2 + |\delta_0(s)-\eta_0(s)|^2\right) ds \\ &\le p(0)(X(0)-h(0))^2 + E\int_0^T p(s) \\ &\cdot \left(|\delta_2(s)-\eta_2(s)|^2 + |\delta_0(s)-\eta_0(s)|^2\right) ds. \end{aligned} \tag{56}$$

Since $p(\cdot) > 0$ and $1/p(\cdot) \in \mathscr{L}^\infty_{\mathscr{F}}(\Omega; C(0,T;\mathbb{R}))$, there exists a constant $\varepsilon > 0$ such that, for any $t \in [0,T]$, $p(t) \ge \varepsilon$. Thus, we have

$$\varepsilon E\left[(X(t)-h(t))^2\right] \le E\left[p(t)(X(t)-h(t))^2\right] \le H, \tag{57}$$

where $H := p(0)(X(0)-h(0))^2 + E\int_0^T p(s)(|\delta_2(s)-\eta_2(s)|^2 + |\delta_0(s)-\eta_0(s)|^2)ds < +\infty$.

Further, we have

$$\begin{aligned} E\left[(X(T)-h(T))^2\right] &\le \frac{1}{\varepsilon} E\left[p(T)(X(T)-h(T))^2\right] \\ &\le \frac{H}{\varepsilon} < +\infty, \\ E\left[\int_0^T (X(t)-h(t))^2 dt\right] &= \int_0^T E(X(t)-h(t))^2 dt \\ &\le \frac{H}{\varepsilon} T < +\infty \end{aligned} \tag{58}$$

which means that $(X(T)-h(T)) \in \mathscr{L}^2_{\widehat{\mathscr{F}}_T}(\Omega;\mathbb{R})$ and $(X(\cdot)-h(\cdot)) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(0,T;\mathbb{R})$.

Because $h(T) \in \mathscr{L}^2_{\widehat{\mathscr{F}}_T}(\Omega;\mathbb{R})$ and $h(\cdot) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(0,T;\mathbb{R})$, we have

$$\begin{aligned} X(\cdot) &= (X(\cdot)-h(\cdot)) + h(\cdot) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(0,T;\mathbb{R}), \\ X(T) &= (X(T)-h(T)) + h(T) \in \mathscr{L}^2_{\widehat{\mathscr{F}}_T}(\Omega;\mathbb{R}). \end{aligned} \tag{59}$$

Since $\Lambda_1(\cdot) \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$ and $h(\cdot) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(\Omega; C(0,T;\mathbb{R}))$, it follows from (46) that $\overline{\pi}(\cdot) \in \mathscr{L}^{2,\mathrm{loc}}_{\widehat{\mathscr{F}}}(0,T;\mathbb{R}^m)$. Further, we conclude from Lemma 8 that $\overline{\pi}(\cdot) \in \mathscr{L}^2_{\widehat{\mathscr{F}}}(0,T;\mathbb{R}^m)$ and $\overline{\pi}(\cdot)$ is admissible.

This completes the proof. □

Next, we formulate the optimal control policy and the cost for the unconstrained control problem (13).

Theorem 10. *Let $\overline{\pi}(\cdot)$ be given by (46). If Assumption 4 holds, then $\overline{\pi}(\cdot)$ is the unique optimal control policy for problem (13) and*

$$\begin{aligned} J^* &= p(0)(X(0)-h(0))^2 + E\int_0^T p(t) \\ &\cdot \left(|\delta_2(t)-\eta_2(t)|^2 + |\delta_0(t)-\eta_0(t)|^2\right) dt \end{aligned} \tag{60}$$

is the optimal cost.

Proof. From Itô's formula, (16) and (8) give

$$\begin{aligned}
d(X(t)-h(t)) &= \Big[r(t)(X(t)-h(t))+b(t)\pi(t)\\
&\quad+\theta(t)'(\delta_1(t)-\eta_1(t))+\frac{\Lambda_2(t)'}{p(t)}(\eta_2(t)-\delta_2(t))\Big]dt\\
&\quad+\left(\sigma(t)'\pi(t)+\delta_1(t)-\eta_1(t)\right)'dW(t)+(\delta_2(t)\\
&\quad-\eta_2(t))'dB(t)+\left(\delta^0(t)-\eta_0(t)\right)dW^0(t),\\
d(X(t)-h(t))^2 &= 2(X(t)-h(t))\\
&\quad\cdot\Big[r(t)(X(t)-h(t))\\
&\quad+b(t)\pi(t)+\theta(t)'(\delta_1(t)-\eta_1(t))\\
&\quad+\frac{\Lambda_2(t)'}{p(t)}(\eta_2(t)-\delta_2(t))\Big]dt+2(X(t)-h(t))\\
&\quad\cdot\left(\sigma(t)'\pi(t)+\delta_1(t)\right.\\
&\quad\left.-\eta_1(t)\right)'dW(t)+2(X(t)-h(t))(\delta_2(t)-\eta_2(t))'dB(t)\\
&\quad+2(X(t)-h(t))(\delta_0(t)-\eta_0(t))dW^0(t)\\
&\quad+\left(\left|\sigma(t)'\pi(t)+\delta_1(t)-\eta_1(t)\right|^2+|\delta_2(t)-\eta_2(t)|^2\right.\\
&\quad\left.+(\delta_0(t)-\eta_0(t))^2\right)dt.
\end{aligned}\tag{61}$$

By using Itô's formula again, we have

$$\begin{aligned}
dp(t)(X(t)-h(t))^2 &= p(t)\Big\{\pi(t)'\sigma(t)\sigma(t)'\pi(t)\\
&\quad+2\pi(t)'\Big[\sigma(t)(\delta_1(t)-\eta_1(t))\\
&\quad+(X(t)-h(t))\left(b(t)'+\frac{\sigma(t)\Lambda_1(t)}{p(t)}\right)\Big]\Big\}dt\\
&\quad+p(t)\Big[|\delta_0(t)-\eta_0(t)|^2+|\delta_2(t)-\eta_2(t)|^2\\
&\quad+\Big|\delta_1(t)-\eta_1(t)\\
&\quad+(X(t)-h(t))\left(\theta(t)+\frac{\Lambda_1(t)}{p(t)}\right)\Big|^2\Big]dt+(X(t)\\
&\quad-h(t))\left[(X(t)-h(t))\Lambda_1(t)+2p(t)\left(\sigma(t)'\pi(t)\right.\right.\\
&\quad\left.\left.+\delta_1(t)-\eta_1(t)\right)\right]'dW(t)+(X(t)-h(t))\left[(X(t)\right.\\
&\quad\left.-h(t))\Lambda_2(t)+2p(t)(\delta_2(t)-\eta_2(t))\right]'dB(t)\\
&\quad+2p(t)(X(t)-h(t))(\delta_0(t)-\eta_0(t))dW^0(t),
\end{aligned}\tag{62}$$

or

$$\begin{aligned}
dp(t)(X(t)-h(t))^2 &= p(t)\\
&\quad\cdot\left[(\pi(t)-\overline{\pi}(t))'\sigma(t)\sigma(t)'(\pi(t)-\overline{\pi}(t))\right]dt\\
&\quad+p(t)\left[|\delta_2(t)-\eta_2(t)|^2+|\delta_0(t)-\eta_0(t)|^2\right]dt\\
&\quad+(X(t)-h(t))\left[(X(t)-h(t))\Lambda_1(t)\right.\\
&\quad\left.+2p(t)\left(\sigma(t)'\pi(t)+\delta_1(t)-\eta_1(t)\right)\right]'dW(t)\\
&\quad+(X(t)-h(t))\left[(X(t)-h(t))\Lambda_2(t)\right.\\
&\quad\left.+2p(t)(\delta_2(t)-\eta_2(t))\right]'dB(t)+2p(t)(X(t)\\
&\quad-h(t))(\delta_0(t)-\eta_0(t))dW^0(t),
\end{aligned}\tag{63}$$

where $\overline{\pi}(t)$ is given by (46).

Then, by integrating from $[0,T]$ and taking expectations, we have

$$\begin{aligned}
Ep(T)(X(T)-h(T))^2 &= p(0)(X(0)-h(0))^2\\
&\quad+E\int_0^T p(t)\left[(\pi(t)-\overline{\pi}(t))'\sigma(t)\sigma(t)'\right.\\
&\quad\left.\cdot(\pi(t)-\overline{\pi}(t))\right]dt+E\int_0^T p(t)\left(|\delta_2(t)-\eta_2(t)|^2\right.\\
&\quad\left.+|\delta_0(t)-\eta_0(t)|^2\right)dt+E\int_0^T(X(t)-h(t))\\
&\quad\cdot\left[(X(t)-h(t))\Lambda_1(t)+2p(t)\right.\\
&\quad\left.\cdot\left(\sigma(t)'\pi(t)+\delta_1(t)-\eta_1(t)\right)\right]'dW(t)\\
&\quad+E\int_0^T(X(t)-h(t))\left[(X(t)-h(t))\Lambda_2(t)\right.\\
&\quad\left.+2p(t)(\delta_2(t)-\eta_2(t))\right]'dB(t)\\
&\quad+E\int_0^T 2p(t)(X(t)-h(t))(\delta_0(t)\\
&\quad-\eta_0(t))dW^0(t)=p(0)(X(0)-h(0))^2\\
&\quad+E\int_0^T p(t)\left[(\pi(t)-\overline{\pi}(t))'\sigma(t)\sigma(t)'\right.\\
&\quad\left.\cdot(\pi(t)-\overline{\pi}(t))\right]dt+E\int_0^T p(t)\left(|\delta_2(t)-\eta_2(t)|^2\right.\\
&\quad\left.+|\delta_0(t)-\eta_0(t)|^2\right)dt.
\end{aligned}\tag{64}$$

Due to the fact that $p(t) > 0$ and $\sigma(t)\sigma(t)' > \epsilon I_m$, we have

$$\begin{aligned} E[X(T)-l]^2 &= EP(T)(X(T)-h(T))^2 \geq p(0) \\ &\cdot (X(0)-h(0))^2 + E\int_0^T p(t) \\ &\cdot \left(\left|\delta_2(t)-\eta_2(t)\right|^2 + \left|\delta_0(t)-\eta_0(t)\right|^2\right) dt, \end{aligned} \tag{65}$$

where the equality holds only when $\pi(\cdot) = \overline{\pi}(\cdot)$. This completes the proof. □

Remark 11. If there is no liability, that is, $u(\cdot) = v(\cdot) = 0$ and $\mathscr{F}_t = \widetilde{\mathscr{F}}_t$, then $\delta(\cdot) = 0$, and (16) boils down to

$$\begin{aligned} dh(t) &= \left(r(t)h(t) + \theta(t)'\eta_1(t) - \frac{\Lambda_2(t)'}{p(t)}\eta_2(t)\right)dt \\ &\quad + \eta_1(t)'dW(t) + \eta_2(t)'dB(t) \\ &\quad + \eta_0(t)dW^0(t), \\ h(T) &= l, \end{aligned} \tag{66}$$

and the unique optimal control policy for problem (13) is

$$\begin{aligned} \overline{\pi}(t) &= -\left(\sigma(t)^{-1}\right)' \\ &\cdot \left[-\eta_1(t) + (X(t)-h(t))\left(\theta(t) + \frac{\Lambda_1(t)}{p(t)}\right)\right], \end{aligned} \tag{67}$$

which is the same as (12) in Lim [22].

4. The Mean-Variance Asset-Liability Management Problem

An admissible portfolio π is said to be a feasible portfolio for (12) if it satisfies the constraint in (12). Then, problem (12) is said to be feasible if it has a feasible portfolio. Following the methodology of Lim [22], we get a necessary and sufficient condition for feasibility of problem (12) as follows.

Proposition 12. *Let $(\Psi(\cdot), \xi(\cdot))$ be a unique solution of the following BSDE:*

$$\begin{aligned} d\Psi(t) &= -r(t)\Psi(t)dt + \xi_1(t)'dW(t) \\ &\quad + \xi_2(t)'dB(t), \\ \Psi(T) &= 1. \end{aligned} \tag{68}$$

If Assumption 4 holds, then mean-variance problem (12) is feasible for any $c \in \mathbb{R}$ if and only if

$$E\int_0^T \left|\Psi(t)b(t)' + \sigma(t)\xi_1(t)\right|^2 dt > 0. \tag{69}$$

Proof. Let $\pi(\cdot)$ be admissible and $\widetilde{\pi}(\cdot) = \lambda\pi(\cdot)$ for some $\lambda \in \mathbb{R}$. Assume that $\widetilde{X}(\cdot)$ is the solution of (8) corresponding to $\widetilde{\pi}(\cdot)$. It follows from Itô's formula that $\widetilde{X}(t) = Z_1(t) + \lambda Z_2(t)$, where

$$\begin{aligned} dZ_1(t) &= \left(r(t)Z_1(t) - u(t)\right)dt + \delta_1(t)'dW(t) \\ &\quad + \delta_2(t)'dB(t) + \delta^0(t)dW^0(t), \\ Z_1(0) &= X_0, \\ dZ_2(t) &= \left(r(t)Z_2(t) + b(t)\pi(t)\right)dt \\ &\quad + \pi(t)'\sigma(t)dW(t), \\ Z_2(0) &= 0. \end{aligned} \tag{70}$$

Then we have $E\widetilde{X}(T) = EZ_1(T) + \lambda EZ_2(T)$, where

$$EZ_2(T) = E\int_0^T \left(\Psi(t)b(t)' + \sigma(t)\xi_1(t)\right)'\pi(t)dt, \tag{71}$$

which has been shown in Yong and Zhou [27] (see pp. 353 of [27]). If (69) holds, then we can choose $\pi(t) = \Psi(t)b(t)' + \sigma(t)\xi_1(t)$ such that

$$EZ_2(T) = E\int_0^T \left|\Psi(t)b(t)' + \sigma(t)\xi_1(t)\right|^2 dt > 0. \tag{72}$$

Hence, for any $c \in \mathbb{R}$, $\lambda_c = (EZ_2(T))^{-1}(c - EZ_1(T))$ is well defined and

$$E\widetilde{X}(T) = EZ_1(T) + \lambda_c EZ_2(T) = c. \tag{73}$$

This implies that (12) is feasible for any $c \in \mathbb{R}$.

Conversely, if (12) is feasible for any $c \in \mathbb{R}$, then, for any $c \in \mathbb{R}$, there exists an admissible portfolio $\pi(\cdot)$ such that $EX(T) = EZ_1(T) + EZ_2(T) = c$. Since $EZ_1(T)$ is independent of $\pi(\cdot)$, we conclude that $EZ_2(T) \neq 0$ for some $\pi(\cdot)$. From (71), we know that (69) is true.

This completes the proof. □

Remark 13. The necessary and sufficient condition (69) is the same as that in [22] in which Lim studied the mean-variance portfolio problem without liability. This implies that the liability does not affect the feasibility of mean-variance problem.

Remark 14. As claimed in [22], necessary and sufficient condition (69) is very mild.

In the case of mean-variance asset-liability management problem, we can replace the unique solution $(h(\cdot), \eta(\cdot))$ of BSDE (16) by

$$\begin{aligned} h(t) &= h(T)\, g_1(t) + g_2(t), \\ \eta_1(t) &= h(T)\, q_1(t) + \overline{q}_1(t), \\ \eta_2(t) &= h(T)\, q_2(t) + \overline{q}_2(t), \\ \eta_0(t) &= \overline{q}_0(t), \end{aligned} \tag{74}$$

where $(g_1(\cdot), q(\cdot))$ and $(g_2(\cdot), \overline{q}(\cdot))$ are the unique solutions of the following BSDEs:

$$\begin{aligned} dg_1(t) &= \left(r(t)\, g_1(t) + \theta(t)'\, q_1(t) - \frac{\Lambda_2(t)'}{p(t)} q_2(t)\right) dt \\ &\quad + q_1(t)'\, dW(t) + q_2(t)'\, dB(t), \\ g_1(T) &= 1, \end{aligned} \tag{75}$$

$$\begin{aligned} dg_2(t) &= \left(r(t)\, g_2(t) + \theta(t)'\, \overline{q}_1(t) - \frac{\Lambda_2(t)'}{p(t)} \overline{q}_2(t) \right. \\ &\quad \left. - u(t) - \theta(t)'\, \delta_1(t) + \delta_2(t)'\, \frac{\Lambda_2(t)}{p(t)}\right) dt + \overline{q}_1(t)'\, dW(t) \\ &\quad + \overline{q}_2(t)'\, dB(t) + \overline{q}_0(t)\, dW^0(t), \\ g_2(T) &= 0, \end{aligned} \tag{76}$$

respectively.

By employing the results in Section 3 and Lagrange multiplier technique (or duality theory), we give our main result as follows.

Theorem 15. *If Assumption 4 holds and (69) is satisfied, then mean-variance asset-liability management problem (12) is feasible for every $c \in \mathbb{R}$, and the inequality*

$$1 - M - p(0)\, g_1^2(0) > 0 \tag{77}$$

holds and the following constants,

$$\begin{aligned} M &:= E \int_0^T p(t) \left|q_2(t)\right|^2 dt, \\ k_1 &:= E \int_0^T p(t)\, q_2(t)' \left(\delta_2(t) - \overline{q}_2(t)\right) dt, \\ k &:= \frac{-c + p(0)\, g_1(0) \left(X_0 - g_2(0)\right) + k_1}{1 - M - p(0)\, g_1^2(0)}, \\ D &:= E \int_0^T p(t) \\ &\quad \cdot \left[\left|\delta_2(t) - \overline{q}_2(t)\right|^2 + \left(\delta_0(t) - \overline{q}_0(t)\right)^2\right] dt, \end{aligned} \tag{78}$$

are well defined. The efficient frontier of problem (12) is given by

$$\begin{aligned} \operatorname{Var} X^*(T) &= \frac{M + p(0)\, g_1^2(0)}{1 - M - p(0)\, g_1^2(0)} \left[c \right. \\ &\quad \left. - \frac{k_1 + p(0)\, g_1(0) \left(X_0 - g_2(0)\right)}{M + p(0)\, g_1^2(0)}\right]^2 + D \\ &\quad + p(0) \left(X_0 - g_2(0)\right)^2 \\ &\quad - \frac{\left[k_1 + p(0)\, g_1(0) \left(X_0 - g_2(0)\right)\right]^2}{M + p(0)\, g_1^2(0)}, \end{aligned} \tag{79}$$

where $c = EX^*(T)$ and the optimal portfolio associated with the expected net terminal wealth c is given as follows:

$$\begin{aligned} \pi^*(t) &= -\left(\sigma(t)^{-1}\right)' \left[\delta_1(t) - \overline{q}_1(t) + k q_1(t) \right. \\ &\quad \left. + \left(X(t) - g_2(t) + k g_1(t)\right) \left(\theta(t) + \frac{\Lambda_1(t)}{p(t)}\right)\right]. \end{aligned} \tag{80}$$

Proof. It is easy to verify that problem (12) has a convex constrained set and a convex cost which is bounded below. These imply that (12) is a linearly constrained convex problem. Because problem (12) is feasible, it follows from Lagrange multiplier technique (see [28] for more details) that

$$J^* = \max_{\lambda \in \mathbb{R}} \inf_{(X(\cdot), \pi(\cdot)) \text{ is admissible}} J(\pi(\cdot), \lambda) < +\infty, \tag{81}$$

where

$$\begin{aligned} J(\pi(\cdot), \lambda) &:= E\left(X(T) - c\right)^2 + 2\lambda \left[EX(T) - c\right] \\ &= E\left(X(T) - c + \lambda\right)^2 - \lambda^2. \end{aligned} \tag{82}$$

For each fixed λ, the unconstrained problem

$$J(\lambda) := \inf_{(X(\cdot), \pi(\cdot)) \text{ is admissible}} J(\pi(\cdot), \lambda) \tag{83}$$

has the same form as (13). Then, it follows from Theorem 10 that

$$\begin{aligned} J(\lambda) &= -\lambda^2 + p(0) \left(X_0 - h(0)\right)^2 + E \int_0^T p(t) \\ &\quad \cdot \left[\left|\delta_2(t) - \eta_2(t)\right|^2 + \left(\delta_0(t) - \eta_0(t)\right)^2\right] dt \\ &= E \int_0^T p(t) \left[\left|\delta_2(t) - (c - \lambda)\, q_2(t) - \overline{q}_2(t)\right|^2 \right. \\ &\quad \left. + \left(\delta_0(t) - \overline{q}_0(t)\right)^2\right] dt - \lambda^2 + p(0) \left[X_0 - (c - \lambda) \right. \\ &\quad \left. \cdot g_1(0) - g_2(0)\right]^2, \end{aligned} \tag{84}$$

and the optimal investment strategy is

$$\begin{aligned}\pi(t) = &-\left(\sigma(t)^{-1}\right)'\Big[\delta_1(t) - \eta_1(t) \\ &+ (X(t) - h(t))\left(\theta(t) + \frac{\Lambda_1(t)}{p(t)}\right)\Big] = -\left(\sigma(t)^{-1}\right)' \\ &\cdot [\delta_1(t) - (c-\lambda) q_1(t) \\ &- \bar{q}_1(t)] - \left(\sigma(t)^{-1}\right)'[X(t) - (c-\lambda) g_1(t) \\ &- g_2(t)]\left(\theta(t) + \frac{\Lambda_1(t)}{p(t)}\right).\end{aligned} \tag{85}$$

Rewriting $J(\lambda)$, we have

$$\begin{aligned}J(\lambda) = &(\lambda - c)^2\Big(-1 + p(0) g_1^2(0) \\ &+ E\int_0^T p(t)|q_2(t)|^2 dt\Big) + 2(\lambda - c)\Big[-c \\ &+ g_1(0) p(0)(X_0 - g_2(0)) \\ &+ E\int_0^T p(t) q_2(t)'(\delta_2(t) - \bar{q}_2(t)) dt\Big] - c^2 \\ &+ p(0)(X_0 - g_2(0))^2 + D = (\lambda - c)^2\Big(-1 \\ &+ p(0) g_1^2(0) + M\Big) + 2(\lambda - c)[-c \\ &+ p(0) g_1(0)(X_0 - g_2(0)) + k_1] - c^2 + p(0)(X_0 \\ &- g_2(0))^2 + D.\end{aligned} \tag{86}$$

Since $J(\lambda)$ is quadratic in λ and J^* is finite, we have

$$1 - M - p(0) g_1^2(0) > 0. \tag{87}$$

In fact, if $1 - M - p(0)g_1^2(0) = 0$, then J^* can only be finite when $-c + g_1(0)p(0)(X_0 - g_2(0)) + k_1 = 0$ for any c, which is a contradiction. So it must be the case that $1 - M - p(0)g_1^2(0) > 0$.

Rewriting (86), we have

$$\begin{aligned}J(\lambda) = &-\left(1 - M - p(0) g_1^2(0)\right)(\lambda - c - k)^2 \\ &+ \left(1 - M - p(0) g_1^2(0)\right)k^2 - c^2 \\ &+ p(0)(X_0 - g_2(0))^2 + D.\end{aligned} \tag{88}$$

Then we have the optimal $\lambda^* = c + k$ for (81). Taking λ^* in (88) and (85),

$$\begin{aligned}J^* = &\left(1 - M - p(0) g_1^2(0)\right)k^2 - c^2 + p(0)(X_0 \\ &- g_2(0))^2 + D \\ = &\frac{[-c + k_1 + p(0) g_1(0)(X_0 - g_2(0))]^2}{1 - M - p(0) g_1^2(0)} - c^2 \\ &+ p(0)(X_0 - g_2(0))^2 + D \\ = &\frac{M + p(0) g_1^2(0)}{1 - M - p(0) g_1^2(0)}\Big[c \\ &- \frac{k_1 + p(0) g_1(0)(X_0 - g_2(0))}{M + p(0) g_1^2(0)}\Big]^2 + D + p(0) \\ &\cdot (X_0 - g_2(0))^2 \\ &- \frac{[k_1 + p(0) g_1(0)(X_0 - g_2(0))]^2}{M + p(0) g_1^2(0)}, \\ \pi^*(t) = &-\left(\sigma(t)^{-1}\right)'\Big[\delta_1(t) + k q_1(t) - \bar{q}_1(t) \\ &+ (X(t) + k g_1(t) - g_2(t))\left(\theta(t) + \frac{\Lambda_1(t)}{p(t)}\right)\Big].\end{aligned} \tag{89}$$

This completes the proof. □

We claim that $D + p(0)(X_0 - g_2(0))^2 - [k_1 + p(0)g_1(0)(X_0 - g_2(0))]^2/(M + p(0)g_1^2(0)) \geq 0$. In fact, since

$$\begin{aligned}&p(\cdot) \in \mathscr{L}^\infty_{\mathscr{F}}(\Omega; C(0,T;\mathbb{R})), \\ &q_2(\cdot), \bar{q}_2(\cdot) \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^d), \\ &\delta_2(\cdot) \in \mathscr{L}^\infty_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R}^d),\end{aligned} \tag{90}$$

we have $\sqrt{p(\cdot)}|\delta_2(\cdot) - \bar{q}_2(\cdot)|$, $\sqrt{p(\cdot)}|q_2(\cdot)| \in \mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R})$. Further, because $\mathscr{L}^2_{\widetilde{\mathscr{F}}}(0,T;\mathbb{R})$ is a Hilbert space, it follows from Cauchy-Schwarz's inequality that

$$\begin{aligned}&[k_1 + p(0) g_1(0)(X_0 - g_2(0))]^2 = \Big[E\int_0^T p(t) q_2(t)' \\ &\cdot (\delta_2(t) - \bar{q}_2(t)) dt\Big]^2 + [p(0) g_1(0)(X_0 \\ &- g_2(0))]^2 + 2p(0) g_1(0)(X_0 - g_2(0)) E\int_0^T p(t) \\ &\cdot q_2(t)'(\delta_2(t) - \bar{q}_2(t)) dt \\ &\leq \Big[E\int_0^T \left(\sqrt{p(t)}|\delta_2(t) - \bar{q}_2(t)|\right) \\ &\cdot \left(\sqrt{p(t)}|q_2(t)|\right) dt\Big]^2 + [p(0) g_1(0)(X_0\end{aligned}$$

$$
\begin{aligned}
&- g_2(0))]^2 + E\int_0^T 2\left(\sqrt{p(0)p(t)}\,|g_1(0)|\right.\\
&\cdot |\delta_2(t) - \overline{q}_2(t)|\Big)\left(\sqrt{p(0)p(t)}\,|X_0 - g_2(0)|\right.\\
&\cdot |q_2(t)|\Big)dt \le E\int_0^T p(t)|\delta_2(t) - \overline{q}_2(t)|^2 dt\\
&\cdot E\int_0^T p(t)|q_2(t)|^2 dt + [p(0)g_1(0)(X_0\\
&- g_2(0))]^2 + E\int_0^T p(0)p(t)|g_1(0)|^2|\delta_2(t)\\
&- \overline{q}_2(t)|^2 dt + E\int_0^T p(0)p(t)|X_0 - g_2(0)|^2\\
&\cdot |q_2(t)|^2 dt,
\end{aligned}
\tag{91}
$$

and so

$$
\begin{aligned}
&\left(M + p(0)g_1^2(0)\right)\left\{D + p(0)(X_0 - g_2(0))^2\right.\\
&\left.- \frac{[k_1 + p(0)g_1(0)(X_0 - g_2(0))]^2}{M + p(0)g_1^2(0)}\right\} = p(0)\\
&\cdot g_1^2(0)E\int_0^T p(t)\\
&\cdot \left[|\delta_2(t) - \overline{q}_2(t)|^2 + (\delta_0(t) - \overline{q}_0(t))^2\right]dt\\
&+ [p(0)g_1(0)(X_0 - g_2(0))]^2 + p(0)(X_0\\
&- g_2(0))^2 E\int_0^T p(t)|q_2(t)|^2 dt + E\int_0^T p(t)\\
&\cdot |\delta_2(t) - \overline{q}_2(t)|^2 dtE\int_0^T p(t)|q_2(t)|^2 dt\\
&+ E\int_0^T p(t)(\delta_0(t) - \overline{q}_0(t))^2 dtE\int_0^T p(t)\\
&\cdot |q_2(t)|^2 dt - [k_1 + p(0)g_1(0)(X_0 - g_2(0))]^2\\
&\ge p(0)g_1^2(0)E\int_0^T p(t)(\delta_0(t) - \overline{q}_0(t))^2 dt\\
&+ E\int_0^T p(t)(\delta_0(t) - \overline{q}_0(t))^2 dtE\int_0^T p(t)\\
&\cdot |q_2(t)|^2 dt \ge 0.
\end{aligned}
\tag{92}
$$

Since $M + p(0)g_1^2(0) > 0$, we have

$$
\begin{aligned}
&D + p(0)(X_0 - g_2(0))^2\\
&\quad - \frac{[k_1 + p(0)g_1(0)(X_0 - g_2(0))]^2}{M + p(0)g_1^2(0)} \ge 0.
\end{aligned}
\tag{93}
$$

Remark 16. Theorem 15 shows that efficient frontier (79) is a parabola. Further, for a given mean target, the risk that the investor has to bear is given by (79). In particular, if the investor wants to take the global minimal risk, he/she can obtain the expected terminal wealth $(k_1 + p(0)g_1(0)(X_0 - g_2(0)))/(M + p(0)g_1^2(0))$ by choosing the optimal strategy.

Remark 17. Theorem 15 also shows that the global minimal risk is

$$
\begin{aligned}
&D + p(0)(X_0 - g_2(0))^2\\
&\quad - \frac{[k_1 + p(0)g_1(0)(X_0 - g_2(0))]^2}{M + p(0)g_1^2(0)}
\end{aligned}
\tag{94}
$$

which is nonnegative. This implies that when the market parameters are random and the financial market is incomplete, the liability can not be completely hedged.

Remark 18. Now we consider a financial market without liability; that is, $u(\cdot) = v(\cdot) = 0$. Then, we have that $\delta(\cdot) = 0$, $(0, 0)$ is the unique solution of BSDE (76) and the constants in Theorem 15 are given by

$$
\begin{aligned}
M &= E\int_0^T p(t)|q_2(t)|^2 dt,\\
k_1 &= 0,\\
k &= \frac{-c + p(0)g_1(0)X_0}{1 - M - p(0)g_1^2(0)},\\
D &= 0.
\end{aligned}
\tag{95}
$$

It follows from Theorem 15 that the efficient frontier in this case is given by

$$
\begin{aligned}
&\operatorname{Var} X^*(T)\\
&\quad = \frac{M + p(0)g_1^2(0)}{1 - M - p(0)g_1^2(0)}\left(c - \frac{p(0)g_1(0)X_0}{M + p(0)g_1^2(0)}\right)^2\\
&\qquad + \frac{Mp(0)X_0^2}{M + p(0)g_1^2(0)},
\end{aligned}
\tag{96}
$$

which is the same as that of Lim [22]. This implies that Lim's result is a special case of our results. Therefore, our results generalize and improve Lim's results.

5. Conclusions

This paper studies the mean-variance asset-liability management problem with random market parameters. Since market parameters observed in the real world are always uncertain, it is more realistic to consider how to manage both assets and liabilities in a market with random market parameters. By using the theories of stochastic LQ control and BSDE, we derive both optimal investment strategies and the mean-variance efficient frontier. Compared with the existing results, the efficient frontier is still a parabola and liability does not

affect the feasibility of the mean-variance portfolio selection problem in a complete market with random parameters. However, the liability can not be fully hedged in an incomplete market with random parameters.

Future studies can go one step further by considering this problem in a more complex market, whose prices are governed by SDEs with Lévy noise or Markovian switching. By using the methods and techniques proposed by Zhu [29, 30], it would be more interesting to discuss the optimal investment strategies and the efficient frontiers in the market mentioned above.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are very grateful to the referee and the editor for patiently giving critical and insightful comments and suggestions, which helped to significantly improve the models, results, and presentation. The research was supported in part by National Natural Sciences Foundation of China under Grant 71571125, the Fundamental Research Funds for the Central Universities under Grant 2014SZYTD01, and Sichuan University under Grant 2014SCU04A06.

References

[1] H. Markowitz, "Portfolio selection," *Journal of Finance*, vol. 7, no. 1, pp. 77–91, 1952.

[2] S. Y. Wang and Y. S. Xia, *Portfolio Selection and Asset Pricing*, Springer, Berlin, Germany, 2002.

[3] D. Li and W.-L. Ng, "Optimal dynamic portfolio selection: multi-period mean-variance formulation," *Mathematical Finance*, vol. 10, no. 3, pp. 387–406, 2000.

[4] H. L. Wu and Z. F. Li, "Multi-period mean-variance portfolio selection with Markov regime switching and uncertain time-horizon," *Journal of Systems Science and Complexity*, vol. 24, no. 1, pp. 140–155, 2011.

[5] X. Y. Zhou and D. Li, "Continuous-time mean-variance portfolio selection: a stochastic LQ framework," *Applied Mathematics and Optimization*, vol. 42, no. 1, pp. 19–33, 2000.

[6] X. Li, X. Y. Zhou, and A. E. Lim, "Dynamic mean-variance portfolio selection with no-shorting constraints," *SIAM Journal on Control and Optimization*, vol. 40, no. 5, pp. 1540–1555, 2002.

[7] J. Xiong and X. Y. Zhou, "Mean-variance portfolio selection under partial information," *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 156–175, 2007.

[8] G. C. Wang and Z. Wu, "Mean-variance hedging and forward-backward stochastic differential filtering equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 310910, 20 pages, 2011.

[9] W. F. Sharpe and L. G. Tint, "Liabilities—a new approach," *The Journal of Portfolio Management*, vol. 16, no. 2, pp. 5–10, 1990.

[10] M. Leippold, F. Trojani, and P. Vanini, "A geometric approach to multiperiod mean variance optimization of assets and liabilities," *Journal of Economic Dynamics & Control*, vol. 28, no. 6, pp. 1079–1113, 2004.

[11] M. C. Chiu and D. Li, "Asset and liability management under a continuous-time mean-variance optimization framework," *Insurance: Mathematics & Economics*, vol. 39, no. 3, pp. 330–355, 2006.

[12] S. X. Xie, Z. F. Li, and S. Y. Wang, "Continuous-time portfolio selection with liability: meanvariance model and stochastic LQ approach," *Insurance: Mathematics and Economics*, vol. 42, no. 3, pp. 943–953, 2008.

[13] S. Xie, "Continuous-time mean-variance portfolio selection with liability and regime switching," *Insurance: Mathematics & Economics*, vol. 45, no. 1, pp. 148–155, 2009.

[14] Y. Zeng and Z. F. Li, "Asset-liability management under benchmark and mean-variance criteria in a jump diffusion market," *Journal of Systems Science and Complexity*, vol. 24, no. 2, pp. 317–327, 2011.

[15] H. Yao, Y. Lai, and Y. Li, "Continuous-time mean-variance asset-liability management with endogenous liabilities," *Insurance: Mathematics & Economics*, vol. 52, no. 1, pp. 6–17, 2013.

[16] J. Wei, K. C. Wong, S. C. Yam, and S. P. Yung, "Markowitz's mean-variance asset-liability management with regime switching: a time-consistent approach," *Insurance: Mathematics & Economics*, vol. 53, no. 1, pp. 281–291, 2013.

[17] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The Review of Financial Studies*, vol. 6, no. 2, pp. 327–343, 1993.

[18] J. Hull and A. White, "The pricing of options on assets with stochastic volatilities," *Journal of Finance*, vol. 42, no. 2, pp. 281–300, 1987.

[19] J. C. Cox and S. A. Ross, "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 145–166, 1976.

[20] S. Beckers, "The constant elasticity of variance model and its implications for option pricing," *Journal of Finance*, vol. 35, no. 3, pp. 661–673, 1980.

[21] A. E. Lim and X. Y. Zhou, "Mean-variance portfolio selection with random parameters in a complete market," *Mathematics of Operations Research*, vol. 27, no. 1, pp. 101–120, 2002.

[22] A. E. B. Lim, "Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market," *Mathematics of Operations Research*, vol. 29, no. 1, pp. 132–161, 2004.

[23] R. Norberg, "Ruin problems with assets and liabilities of diffusion type," *Stochastic Processes and their Applications*, vol. 81, no. 2, pp. 255–269, 1999.

[24] M. Decamps, A. de Schepper, and M. Goovaerts, "A path integral approach to asset-liability management," *Physica A: Statistical Mechanics and its Applications*, vol. 363, no. 2, pp. 404–416, 2006.

[25] H. K. Koo, "Consumption and portfolio selection with labor income: a continuous time approach," *Mathematical Finance*, vol. 8, no. 1, pp. 49–65, 1998.

[26] S. G. Peng, "A general stochastic maximum principle for optimal control problems," *SIAM Journal on Control and Optimization*, vol. 28, no. 4, pp. 966–979, 1990.

[27] J. M. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Syetems and HJB Equations*, Springer, New York, NY, USA, 1999.

[28] D. G. Luenberger, *Optimization by Vector Space Methods*, John Wiley & Sons, New York, NY, USA, 1968.

[29] Q. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 126–142, 2014.

[30] Q. X. Zhu, "Pth Moment exponential stability of impulsive stochastic functional differential equations with Markovian switching," *Journal of the Franklin Institute*, vol. 351, no. 7, pp. 3965–3986, 2014.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 989542, 14 pages
http://dx.doi.org/10.1155/2015/989542

Research Article

Optimal Design of Stochastic Distributed Order Linear SISO Systems Using Hybrid Spectral Method

Pham Luu Trung Duong,[1] Ezra Kwok,[2] and Moonyong Lee[1]

[1] *School of Chemical Engineering, Yeungnam University, Gyeongsan 712-749, Republic of Korea*
[2] *Chemical & Biological Engineering, University of British Columbia, Vancouver, Canada*

Correspondence should be addressed to Moonyong Lee; mynlee@yu.ac.kr

Received 20 May 2015; Revised 18 August 2015; Accepted 2 September 2015

Academic Editor: Son Nguyen

The distributed order concept, which is a parallel connection of fractional order integrals and derivatives taken to the infinitesimal limit in delta order, has been the main focus in many engineering areas recently. On the other hand, there are few numerical methods available for analyzing distributed order systems, particularly under stochastic forcing. This paper proposes a novel numerical scheme for analyzing the behavior of a distributed order linear single input single output control system under random forcing. The method is based on the operational matrix technique to handle stochastic distributed order systems. The existing Monte Carlo, polynomial chaos, and frequency methods were first adapted to the stochastic distributed order system for comparison. Numerical examples were used to illustrate the accuracy and computational efficiency of the proposed method for the analysis of stochastic distributed order systems. The stability of the systems under stochastic perturbations can also be inferred easily from the moment of random output obtained using the proposed method. Based on the hybrid spectral framework, the optimal design was elaborated on by minimizing the suitably defined constrained-optimization problem.

1. Introduction

Fractional/distributed order calculus is applied widely across a range of disciplines, such as physics, biology, chemistry, finance, physiology, and control engineering [1–6]. The memory property of fractional order calculus provides a novel tool to model real-world plants better than integer order ones such as diffusion plants [5]. Fractional calculus has been used for modeling of turbulence in [2]. In [3], the concept of fractional calculus is used for interpreting the underlying mechanism of dielectric relaxation. A method for design fractional order $PI^{\lambda}D^{\mu}$ controllers for deterministic systems is proposed in [6].

The distributed order (DO) equation, which is a generalized concept fractional order, was first proposed by Caputo in 1969 [7] and solved by him in 1995 [8]. The general solution of linear DO was then discussed systematically [9]. Later, the DO concept was used to examine the diffusion equation [10], the rheological properties of composite materials, and other real complex physical phenomena [11–14]. Several different methods for the time domain analysis of DO systems have been reported [15–18]. On the other hand, a numerical method for the analysis of a DO operator is still immature and requires further development. In particular, there are few methods to analyze DO systems under the excitation of random processes. This motivated the theme of this study: the development of a computational scheme for the analysis basic of a DO system with stochastic settings. The operational matrix (OP) has attracted considerable attention for the analysis of a range of dynamic systems [19–21]. The main characteristic of this technique is that different analysis problems can be reduced to a system of algebraic equations using different types of orthogonal functions, which greatly simplifies the problem [19]. On the other hand, to the best of the author's knowledge, there are no reports on the analysis of stochastic DO systems using an OP. Many natural systems often suffer from stochastic noise that causes fluctuations in their behavior, making them deviate from deterministic models. Therefore, it is important to examine the statistical characteristics of states (mean, variance) for those stochastic

systems. This problem is often called statistical analysis (or uncertainty quantification) of a system [22–24]. This paper proposes a numerical scheme based on the OP technique for the statistical analysis of DO systems.

The Monte Carlo (MC) method is commonly used to simulate a stochastic model [25, 26]. The method relies on the sampling of independent realizations of random inputs according to their prescribed probability distribution. The data is fixed for each realization and the problem becomes deterministic. Solving the multiple deterministic realizations builds an ensemble of solutions, that is, the realization of random solutions, from which statistical information can be extracted, for example, the mean and variance. Nevertheless, this method typically reveals slow convergence and has a large computational demand. For example, the mean values typically converge as $1/\sqrt{M}$, where M is the number of samples.

Generalized polynomial chaos (gPC) [27–32] represents a more recent tool for quantifying the uncertainty within system models. The approach involves expressing stochastic quantities as the orthogonal polynomials of random input parameters. This method is actually a spectral representation in random space and converges rapidly when the expanded function depends smoothly on random parameters. On the other hand, the stochastic inputs of many systems involve random processes parameterized by truncated Karhunen-Loeve (KL) expansions, and the dimensionality of the KL expansions depends on the correlation lengths of these processes. For input processes with low correlation lengths, the number of dimensions required for an accurate representation can be extremely large.

The OP method [29], where a system is described by a stochastic operator (operational matrix), is an alternative approach for the simulation of stochastic integer order systems. This method involves the inverse of the stochastic operators as Neumann series and is most effective for systems with inputs with low correlation lengths. On the other hand, it is restricted to small random parametric uncertainty.

In a recent study [33], the authors introduced a hybrid spectral method, which combines the advantages of both the OP and polynomial chaos (PC), to simulate single input single output (SISO) stochastic fractional order systems. In the present study, the method reported in [33] was extended to the statistical analysis of DO systems affected by stochastic fluctuations. Here, the stochastic operator was approximated using PC instead of a Neumann series. This method provides the algebraic relationships between the first- and second-order stochastic moments of the input and output of a system, hence bypassing the KL expansions that can require large dimensions for accurate results. In contrast to the traditional OP method, the proposed method is not limited by the magnitude of the uncertainty.

Section 2 briefly introduces a DO system and the OP technique for uncertainty quantification in this system, leading to computation of the moments of random matrices. Section 3 summarizes the process of calculating the moments of the random matrices using a stochastic collocation. Section 4 defines the suitable performance objectives coupled with the spectral method for the design of a stochastic linear DO system. Section 5 provides examples to demonstrate the use of the proposed method. The results of the proposed deterministic system with a DO were compared with those of other existing numerical and analytical methods. To assess a stochastic DO system, the MC, gPC, and frequency methods were first adopted to the stochastic DO system for comparison because the analytical results were unavailable. The results from the proposed method were then compared with the numerical results from the MC, gPC, and frequency methods.

2. Preliminary of Fractional and Distributed Order System

In this section, we give some necessary definitions and preliminaries of the fractional calculus theory which will be used in this paper.

2.1. Governing Equation for System Dynamics with Fractional Order Dynamics. Fractional calculus considers the generalization of the integration and differentiation operator to a noninteger order [34, 35]:

$$D_0{}^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}} & \alpha > 0 \\ 1 & \alpha = 0 \\ \displaystyle\int_0^t (d\tau)^{-\alpha} & \alpha < 0, \end{cases} \tag{1}$$

where $\alpha \in R$ is the order of the operator.

Among many formulations of the generalized derivative, the Riemann-Liouville (RL) definition is used most often:

$${}_{\mathrm{RL}}D_0{}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau, \tag{2}$$

where $\Gamma(x)$ denotes the gamma function and m is an integer satisfying $m-1 < \alpha < m$.

The RL fractional integral of a function $f(t)$ is defined as follows:

$${}_{\mathrm{RL}}I_0{}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{3}$$

Another popular definition of a fractional order derivative is the Caputo (C) definition [36],

$${}_{C}D_t{}^{\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau. \tag{4}$$

The Laplace transform for a fractional order derivative under zero initial conditions can be defined as $L\{D_0{}^{\alpha} f(t)\} = s^{\alpha} F(s)$.

Note that, under a zero initial condition, the two Riemann-Liouville and Caputo definitions are equivalent.

Therefore, a fractional order single input single output (SISO) system can be described by a fractional order differential equation as $a_0 D_0{}^{\alpha_0} y(t) + a_1 D_0{}^{\alpha_1} y(t) + \cdots + a_l D_0{}^{\alpha_l} y(t) = b_0 D_0{}^{\beta_0} u(t) + \cdots + b_m D_0{}^{\beta_m} u(t)$ or by a transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + \cdots + b_0 s^{\beta_0}}{a_l s^{\alpha_l} + \cdots + a_0 s^{\alpha_0}}, \tag{5}$$

where α_i and β_i are the arbitrary real positive numbers and $u(t)$ and $y(t)$ are the input and output of the system, respectively.

2.2. Distributed Order Systems. The DO differential operation is defined as follows [17]:

$$D_t^{\rho(\alpha)} f(t) = \int_{\gamma_1}^{\gamma_2} \rho(\alpha) D_t^{\alpha} f(t)\, d\alpha, \tag{6}$$

where $\rho(\alpha)$ denotes the distribution function of order α.

Therefore, the general form of the DO differential equation is

$$\sum_{i=1}^{n} a_i D_t^{\rho_i(\alpha)} y(t) = \sum_{j=1}^{m} b_j D_t^{\rho_j(\alpha)} u(t). \tag{7}$$

For time domain analysis of the DO system, the integral in (7) is discretized using the quadrature formula as follows [16, 17]:

$$\int_{\gamma_1}^{\gamma_2} \rho(\alpha) D_t^{\alpha} f(t)\, d\alpha \approx \sum_{l=1}^{Q} \rho(\alpha_l) \left(D_t^{\alpha_l} f(t)\right) \upsilon_l, \tag{8}$$

where α_l, υ_l are the node and weight from the quadrature formula, respectively. In other words, the DO equation is approximated as a multiterm fractional order equation and can be rearranged as (5).

2.3. Operational Matrices of Block Pulse Function for the Analysis of Distributed Order Systems. Block pulse functions (BPFs) are a complete set of orthogonal functions that are defined over the time interval, $[0, \tau]$,

$$\psi_i = \begin{cases} 1 & \dfrac{i-1}{N}\tau \le t \le \dfrac{i}{N}\tau \\ 0 & \text{elsewhere,} \end{cases} \tag{9}$$

where N is the number of block pulse functions.

Therefore, any function that can be absolutely integrated on the time interval $[0, \tau]$ can be expanded to a series from the block pulse basis as follows:

$$f(t) = \boldsymbol{\psi}_N^{\,T}(t) C_f = \sum_{i=1}^{N} c_{f_i} \psi_i(t), \tag{10}$$

where $\boldsymbol{\psi}_N^{\,T}(t) = [\psi_1(t), \ldots, \psi_N(t)]$ constitutes the block pulse basis. From here, the subscript N of $\boldsymbol{\psi}_N^{\,T}(t)$ is dropped out for the convenience of notation.

The expansion coefficients (or spectral characteristics) can be evaluated as follows:

$$c_{f_i} = \frac{N}{\tau} \int_{[(i-1)/N]\tau}^{(i/N)\tau} f(t) \psi_i(t)\, dt. \tag{11}$$

Furthermore, any function $g(t_1, t_2)$ that is absolutely integrable on the time interval $[0, \tau] \times [0, \tau]$ can be expanded as follows:

$$g(t_1, t_2) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} \psi_i(t_1) \psi_j(t_2) = \boldsymbol{\psi}^T(t_1) C_g \boldsymbol{\psi}(t_2), \tag{12}$$

with expansion coefficients (or spectral characteristics) of

$$c_{ij} = \left(\frac{N}{\tau}\right)^2 \int_{[(i-1)/N]\tau}^{(i/N)\tau} \int_{[(i-1)/N]\tau}^{(i/N)\tau} g(t_1, t_2) \psi_i(t_1) \cdot \psi_j(t_2)\, dt_1 dt_2. \tag{13}$$

Equation (3) can be expressed in terms of the OP [19],

$$I_0^{\alpha} f(t) = \boldsymbol{\psi}(t)^T A_\alpha C_f, \tag{14}$$

where the generalized OP integration of the block pulse function, A_α, is

$$A_\alpha = P_\alpha^{\,T} = \left(\frac{\tau}{N}\right)^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_N \\ 0 & f_1 & f_2 & \cdots & f_{N-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & f_1 \end{pmatrix}^T. \tag{15}$$

The elements of the generalized OP integration can be given by

$$f_1 = 1; \quad f_p = p^{\alpha+1} - 2(p-1)^{\alpha+1} + (p-2)^{\alpha+1} \quad \text{for } p = 2, 3, \ldots, N. \tag{16}$$

The generalized OP of a derivative of order α is

$$B_\alpha A_\alpha = I, \tag{17}$$

where I is the identity matrix.

The generalized OP of derivative can be used to approximate (2) as follows:

$$D_0^{\alpha} f(t) = \boldsymbol{\psi}(t)^T B_\alpha C_f. \tag{18}$$

Therefore, using the OP, the discretization of DO can be expressed as

$$\begin{aligned} D_t^{\rho(\alpha)} f(t) &= \int_{\gamma_1}^{\gamma_2} \rho(\alpha) D_t^{\alpha} f(t)\, d\alpha \\ &\approx \sum_{l=1}^{Q} \rho(\alpha_l) \left(D_t^{\alpha_l} f(t)\right) \upsilon_l \\ &= \sum_{l=1}^{Q} \upsilon_l \rho(\alpha_l) \left(\boldsymbol{\psi}(t)^T B_{\alpha_l} C_f\right). \end{aligned} \tag{19}$$

The DO system in (7) can be rewritten in terms of the OP, A_G, as follows:

$$A_G = \left[\sum_{i=1}^{n} a_i \sum_{l=1}^{Q} \upsilon_l \rho_i(\alpha_l) B_{\alpha_l}\right]^{-1} \left[\sum_{j=1}^{m} b_j \sum_{l=1}^{Q} \upsilon_l \rho_j(\alpha_l) B_{\alpha_l}\right]. \tag{20}$$

The input and output are related by the following equation:

$$C_Y = A_G C_U;$$
$$Y(t) = (C_Y)^T \boldsymbol{\psi}(t); \tag{21}$$
$$U(t) = (C_U)^T \boldsymbol{\psi}(t).$$

2.4. Stochastic Analysis of Distributed Order Systems. Consider the system described by (7), which has the spectral characteristics of input and output linked by (21). Assume that the system is excited by random forcing with a given mean and covariance function as follows:

$$\begin{aligned} M_U(t) &= \mathbb{E}[U(t)] = \left(C_{m_U}\right)^T \boldsymbol{\psi}(t), \\ \kappa_{UU} &= \mathbb{E}\{[U(t_1) - M_U(t_1)][U(t_2) - M_U(t_2)]\} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \psi_i(t_1)\psi_j(t_2) c_{ij} \\ &= \boldsymbol{\psi}(t_1)^T C_{K_{UU}} \boldsymbol{\psi}(t_2), \end{aligned} \tag{22}$$

where $\mathbb{E}[]$ denotes the expectation operator and the spectral characteristics of the mean and covariance function of the input are calculated in (11) and (13).

Using the OP, the spectral characteristics of the mean and covariance of the output are given by the following [33] (the details are available in Appendices):

$$\begin{aligned} C_{m_Y} &= \mathbb{E}[A_G] C_{m_U}, \\ C_{\kappa_{YY}} &= \mathbb{E}\left[A_G\left\{C_{\kappa_{UU}} + \left(C_{m_U}\right)\left(C_{m_U}\right)^T\right\} A_G\right]^T \\ &\quad - C_{m_Y}\left(C_{m_Y}\right)^T. \end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned} m_Y(t) &= \boldsymbol{\psi}(t_1)^T C_{m_Y} = \boldsymbol{\psi}(t_1)^T \mathbb{E}[A_G] C_{m_U}, \\ \kappa_{YY}(t_1, t_2) &= \boldsymbol{\psi}(t_1)^T C_{\kappa_{YY}} \boldsymbol{\psi}(t_2) = \boldsymbol{\psi}(t_1)^T \\ &\quad \cdot \mathbb{E}\left[A_G\left\{C_{\kappa_{UU}} + \left(C_{m_U}\right)\left(C_{m_U}\right)^T\right\} A_G{}^T\right] \boldsymbol{\psi}(t_2) \\ &\quad - \boldsymbol{\psi}(t_1)^T C_{m_Y}\left(C_{m_Y}\right)^T \boldsymbol{\psi}(t_2). \end{aligned} \tag{24}$$

The random parameters a_i, b_j result in the random OP A_G in (23) and (24), and its (OP A_G) moment can be estimated using a stochastic collocation method, which is described in the next section.

When the parameters, a_i, b_j, are deterministic, (23) and (24) become

$$\begin{aligned} m_Y(t) &= \boldsymbol{\psi}(t_1)^T A_G C_{m_U}, \\ \kappa_{YY}(t_1, t_2) &\\ &= \boldsymbol{\psi}(t_1)^T A_G\left\{C_{\kappa_{UU}} + \left(C_{m_U}\right)\left(C_{m_U}\right)^T\right\} A_G{}^T \boldsymbol{\psi}(t_2) \\ &\quad - \boldsymbol{\psi}(t_1)^T C_{m_Y}\left(C_{m_Y}\right)^T \boldsymbol{\psi}(t_2). \end{aligned} \tag{25}$$

Remarks. The relationship in (25) is invariant with respect to the orthogonal polynomial used to construct the OP of the fractional order integral and derivative. The relationships in (24) and (25) are only available for a linear system.

3. Stochastic Collocation for the Operational Matrix

A stochastic collocation method, which is described briefly below, is based on the gPC and can easily estimate the means and variances of complex dynamics. Therefore, it has been used to estimate the moment of the random matrix in (24).

(i) Assume that a random OP has the form, $A = A(\boldsymbol{\xi})$, where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ is a vector of independent random parameters with the probability density functions $\rho_i(\xi_i) : \Gamma_i \to R^+$. Vector $\boldsymbol{\xi}$ has the joint probability density function, $\boldsymbol{\rho} = \prod_{i=1}^{n}\rho_i$, with the support, $\boldsymbol{\Gamma} \equiv \prod_{i=1}^{n}\Gamma_i \in R^{+n}$.

(ii) Choose a suitable quadrature set $\{\xi_i^{(m)}, w^{(m)}\}_{m=1}^{q_i}$ for each random parameter according to the probability density so that a one-dimensional integration can be approximated as accurately as possible by $\int_{\Gamma_i} A(\xi_i)\rho_i(\xi_i)d\xi_i = \sum_{i=1}^{q_i} A(\xi_i^{(m)})w_i^{(m)}$, where $\xi_i^{(m)}$ is the mth node and $w^{(m)}$ is the corresponding weight.

(iii) Construct a multidimensional cubature set by tensorization of the one-dimensional quadrature set over all the combined multi-index $(j_1, \ldots, j_n)$. Because manipulation of the multi-index $(j_1, \ldots, j_n)$ is cumbersome in practice, a single index is preferable for manipulating these equations. The multi-index is often replaced by a graded lexicographic order index, $\mathbf{j}$ [27]. Because the weighting functions of the cubature are the same as the probability density functions, the moment of the random matrix can be approximated by

$$\begin{aligned} \mathbb{E}[A] &= \int_{\boldsymbol{\Gamma}} A(\boldsymbol{\xi})\boldsymbol{\rho}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \sum_{\mathbf{j}=1}^{Q} A\left(\boldsymbol{\xi}^{(\mathbf{j})}\right)\mathbf{w}^{(\mathbf{j})} \\ &= \sum_{j_1=1}^{q_1}\cdots\sum_{j_n=1}^{q_n} A\left(\xi_1{}^{(j_1)}, \ldots, \xi_n{}^{(j_n)}\right)\left(w_1{}^{(j_1)}, \ldots, w_n{}^{(j_n)}\right). \end{aligned} \tag{26}$$

The Matlab suite, OPQ, can be used to obtain the one-dimensional quadrature sets and their corresponding orthogonal polynomials (polynomial chaos) with respect to the different density weights [36].

The algorithm of the proposed method for the analysis of a stochastic system can be summarized as follows:

(a) Calculate the coefficients C_{m_R}, $C_{\kappa_{RR}}$ of the expansions of the mean and covariance of the input as shown in (11) and (13).

(b) Rewrite the DO differential equation in (7) in terms of OP as (20).

(c) The coefficients of expansions of the mean and covariance function of the output are obtained from (23). In (23), the moments of several random matrices need to be calculated. The moment of a random matrix is calculated by the stochastic collocation method as (26).

(d) Finally, the mean and covariance of the output are obtained as (24).

For a clearer understanding of the algorithm, a similar algorithm is depicted graphically in [33] for the analysis of stochastic linear fractional order systems.

4. Optimal $PI^{\lambda}D^{\mu}$ Controller Design

Assume that the system is described by (7), where coefficients a_i, b_j are independent random variables with given distributions. The set point input is a random process with a given mean and covariance function as follows:

$$\begin{aligned} M_R(t) &= \mathbb{E}[R(t)] = \left(C_{m_R}\right)^T \boldsymbol{\psi}(t), \\ \kappa_{RR} &= \mathbb{E}\{[R(t_1) - M_R(t_1)][R(t_2) - M_R(t_2)]\} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \psi_i(t_1)\psi_j(t_2) c_{ij}{}^{R} \\ &= \boldsymbol{\psi}(t_1)^T C_{K_{RR}} \boldsymbol{\psi}(t_2). \end{aligned} \tag{27}$$

The system is in the closed loop configuration, as shown in Figure 1, with a fractional order $PI^{\lambda}D^{\mu}$ controller [35] $C(s) = K_p + (K_i/s^{\lambda}) + K_d s^{\mu}$.

The OP for this controller is

$$A_C = K_p I + K_i A_{\lambda} + K_d B_{\mu}, \tag{28}$$

where I, A_{λ}, and B_{μ} are the identity matrix, the integration OP of fractional order, λ, and the OP of the fractional order derivative, μ, respectively.

Denote the OP for the system as A_G. Using block algebra for OP operator, the OP for a closed loop system can be obtained as follows:

$$A = \left(I + A_G A_C\right)^{-1} A_G A_C. \tag{29}$$

This closed loop OP can be used to obtain the first- and second-order moment of the random output from (24).

The parameters of the $PI^{\lambda}D^{\mu}$ controller can be obtained by optimizing the cost function defined as [37]

$$\begin{aligned} \min_{Kp,Ki,Kd,\lambda,\mu} J &= \min_{Kp,Ki,Kd,\lambda,\mu} \mathbb{E}\int_0^T \left(y_d(t) - y(t)\right)^2 dt \\ &= \min_{Kp,Ki,Kd,\lambda,\mu} \int_0^T \left\{\mathbb{E}\left[y^2{}_d(t)\right] + \mathbb{E}\left[y(t)^2\right]\right. \\ &\quad \left. - 2\mathbb{E}\left[y_d(t)\right]\mathbb{E}\left[y(t)\right]\right\} dt, \end{aligned} \tag{30}$$

where $\mathbb{E}[y(t)^2]$ and $\mathbb{E}[y(t)]$ are obtained from (24).

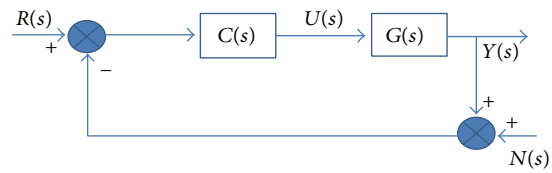


Figure 1: Closed loop control system.

5. Examples

Before going to detailed examples, let us give some information about the existing methods. Ito calculus can be used for the statistical analysis of integer order linear/nonlinear systems only with ideal white noise (noise with direct delta covariance function and infinite bandwidth). On the other hand, the PC method can be used for a system with low bandwidth noise. However, in the PC method the computational load increases significantly as the bandwidth of noise increases. The MC and Quasi-MC methods can be used for arbitrary cases (i.e., with arbitrary type of noise). However, they require a large computational effort for obtaining accurate results. To overcome these limitations in each existing method, a hybrid spectral method [33] was proposed for the statistical analysis of fractional order linear SISO systems with arbitrary type of random input. In this paper, the methodology in [33] was extended to the DO case. Several different case studies were considered to show the efficiency of the proposed method handling different kind of random inputs in a unified frame work: band-limited white noise (noise with low bandwidth), ideal white noise, and fractional Brownian motion with Hurst parameter H. It should be noted that when $H = 1/2$, fractional Brownian motion becomes Brownian motion, whose derivative is ideal white noise.

5.1. Examples 1(a) and 1(b): Band-Limited White Noise Input. Because an integer order system can be considered as a special case of DO systems, this example considers a simple linear integer order system, $G(s) = 1/(1+Ts)$ from [38] with a band-limited white noise as the input.

Let the input have a zero mean and covariance function of $\kappa_{UU}(t_1, t_2) = (W_B/\pi)\,\mathrm{sinc}((t_1 - t_2)W_B/\pi)$, where the sinc function is defined as

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{elsewhere} \\ 1 & \text{for } x = 0. \end{cases} \tag{31}$$

The power spectral density of the input is

$$S_U(\omega) = \begin{cases} \dfrac{1}{(2\pi)} & |\omega| \le W_B \\ 0 & |\omega| > W_B, \end{cases} \tag{32}$$

where W_B is known as bandwidth of the noise. As W_B approaches to infinity, the process will become ideal white noise.

Therefore, the power spectral density function of stochastic output is given by

$$S_Y(\omega) = \begin{cases} \dfrac{1}{2\pi}\dfrac{1}{1+(T\omega)^2} & |\omega| \le W_B \\ 0 & |\omega| > W_B. \end{cases} \tag{33}$$

This is a linear system; the means of the input and output are zero.

Using the frequency method, the exact steady state variance of output can be expressed as [38]

$$D_{y_{ss}} = \frac{1}{2\pi}\int_{-W_B}^{W_B}\frac{1}{1+(T\omega)^2}d\omega = \frac{1}{\pi T}\arctan\left(W_B T\right). \tag{34}$$

The OP for this linear system is

$$A_G = \left(TI + A_1\right)^{-1} A_1, \tag{35}$$

where I is the identity matrix; A_1 is the OP of integration (of order one), which was calculated using (15) and (16).

This system lacks random parameters. The covariance function of the system output is approximated by (25). From (25), it can be seen that the mean of the output by the proposed method is zero.

Two numerical cases are considered.

(a) Consider $W_B = \pi$; $T = 2$. Figure 2 shows the output variance obtained by the proposed method for $W_B = \pi$. For comparison, the results by the gPC, MC, and frequency methods are also given. Note that the frequency method in (34) can provide the exact (analytical) steady state variance. The random process input, $U(t)$, was parameterized using a noncanonical decomposition [33, 39] (the details are available in Appendices). The results from the proposed method were quite satisfactory. Table 1 lists the simulation parameters and computational times required for each method.

(b) Consider $W_B = 4\pi$; $T = 2$. Figure 3 presents the variance obtained by the proposed method for $W_B = 4\pi$. If the number of cubature nodes is kept as in case (a), the gPC method cannot obtain an accurate result in the steady state. The result indicates that in the gPC method the number of cubature nodes needs to increase with increasing bandwidth of the noise, and the computational load increases for obtaining the same accurate result accordingly.

Table 1 presents the computational times and simulation parameters for all methods. From this table and Figures 2 and 3, the proposed method provides better performance in terms of accuracy and computational load.

Remark. The gPC approaches can be divided into two subcategories: intrusive Galerkin [27, 31] approaches and nonintrusive projection approaches [27–30, 32]. The advantage of nonintrusive approach is ease of implementation. For this reason, nonintrusive (collocation) methods have become very popular. The intrusive Galerkin method offers the most accurate solutions involving the least number of equations in multidimensional random spaces, but it is more cumbersome to implement. Thus, in this paper, the nonintrusive method is referred to as the gPC method.

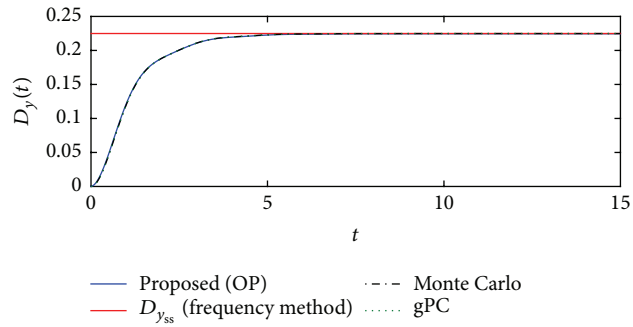


Figure 2: Variances of the output in Example 1(a).

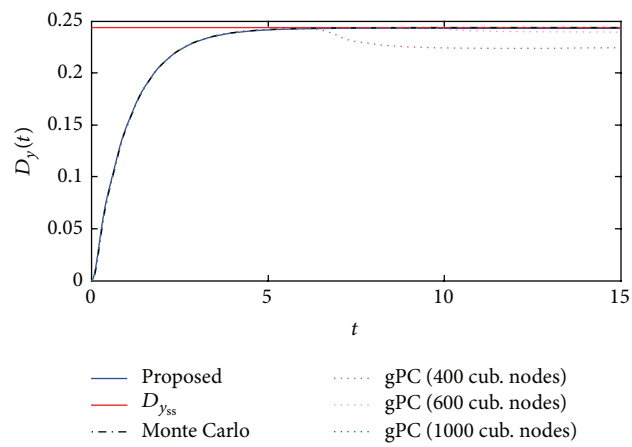


Figure 3: Variances of the output in Example 1(b).

5.2. Examples 2(a), 2(b), 2(c), and 2(d): Ideal White Noise

(a) Example 2(a): Double Delta Function Distributed Order System. This example considers the statistical analysis of a special case of DO integrator taken from the literature [40] as follows:

$$D_t^{\rho(\alpha)} y(t) = u, \tag{36}$$

where $\rho(\alpha) = a_1\delta(\alpha-\alpha_1) + a_2\delta(\alpha-\alpha_2)$ and $\delta()$ is the Dirac delta function. Therefore, (36) is actually a double fractional integrator,

$$a_1 D_0^{\alpha_1} y(t) + a_2 D_0^{\alpha_2} y(t) = u(t). \tag{37}$$

The case where the input $u(t)$ is an ideal white noise with a zero mean and covariance function was considered:

$$\kappa_{UU}(\tau) = \delta(\tau) = \delta\left(t_1 - t_2\right). \tag{38}$$

The exact variance of the output is given by [40]

$$\begin{aligned} D_{y(t)} &= \sigma^2(t) \\ &= \frac{1}{a_2^2}\int_0^t u^{2(\alpha_2-1)}\left[\mathscr{E}_{\alpha_2-\alpha_1,\alpha_2}\left(-\frac{a_1 u^{(\alpha_2-\alpha_1)}}{a_2}\right)\right]^2 du, \end{aligned} \tag{39}$$

where $\mathscr{E}_{\alpha_2-\alpha_1,\alpha_2}()$ is the Mittag-Leffler function, which can be calculated using the Matlab mlf.m function [41].

Table 1: Simulation parameters and time profiles for obtaining the statistical characteristics by the MC, gPC, and proposed methods in Examples 1(a), 1(b), 2(a), 2(b), 3, and 4(a).

Example	Simulation parameters			Computational time (sec.)		
	MC (Halton sampling)	gPC	Proposed	MC	gPC	Proposed
Example 1(a)	10000 samples	400 cubature nodes	512 basis functions	205.30	1.10	0.01
Example 1(b)	10000 samples	1000 cubature nodes	512 basis functions	197.18	2.67	0.01
Example 2(a)	10000 samples	N/A	512 basis functions	199.64	N/A	0.01
Example 2(b)	10000 samples	N/A	2048 basis functions	827.76	N/A	3.22
Example 3	10000 samples	N/A	512 basis functions	210.66	N/A	0.01
Example 4(a)	10000 samples	625 cubature nodes	512 basis functions	4503.90	609.51	15.32

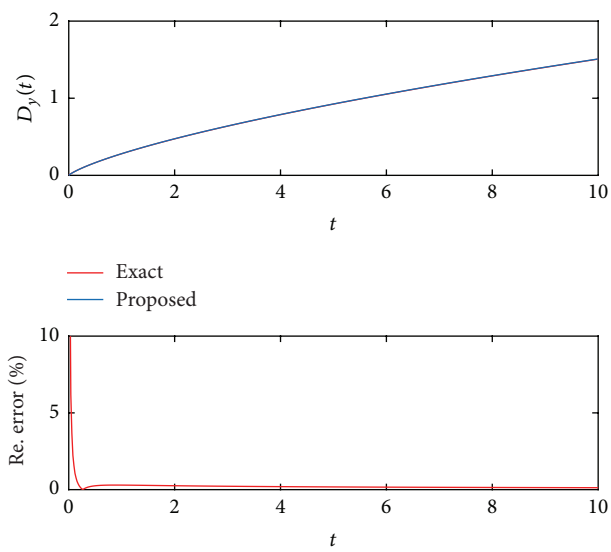


Figure 4: Variances of the output in Example 2(a) with $a_1 = a_2 = 1$, $\alpha_1 = 3/4$, and $\alpha_2 = 1$.

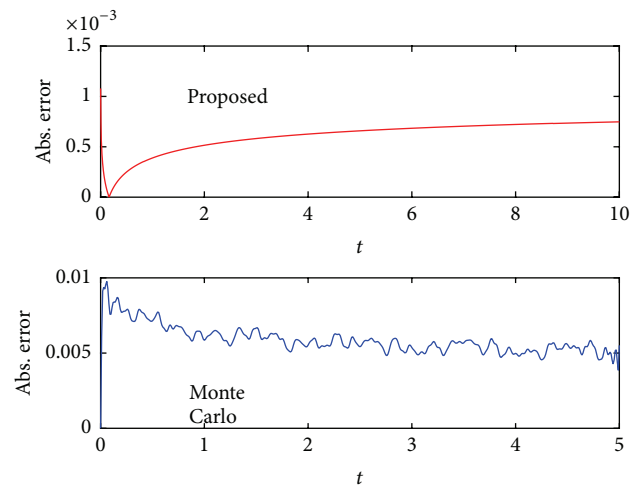


Figure 5: Absolute errors by the proposed and MC methods in Example 2(a).

The OP for system (37) is given by

$$A_G = \left[\sum_{i=1}^{2} B_{\alpha_i} a_i\right]^{-1}, \tag{40}$$

where B_{α_i} is the OP of derivative of order α_i.

This system does not have random parameters. Therefore, the covariance function of system output can be approximated by (25). The regularization technique [33] is used to approximate the Dirac delta covariance function. Figure 4 presents the variance obtained using the proposed method for $a_1 = a_2 = 1$, $\alpha_1 = 3/4$, and $\alpha_2 = 1$. The relative error of the proposed method with respect to (39) is also shown in Figure 4. The result by the proposed method is quite satisfactory. Figure 5 compares the absolute error for the output variance by the proposed and MC methods (with respect to the exact variance in (39)). The simulation times are listed in Table 1 for both methods. For MC simulations, the Matlab code, fode_sol.m, from [42] was used. Again, Table 1 and Figure 5 show that the proposed method has better accuracy with less computational burden than the MC method.

Remarks. The fact that the gPC method becomes computationally intractable for ideal white noise input makes the proposed method more attractive.

(b) Example 2(b): Uniform Distributed Order Integrator. This example considers the statistical analysis of a DO integrator taken from [40]:

$$D_t^{\rho(\alpha)} y(t) = u, \quad \rho(\alpha) = 1, \ 0 \le \alpha \le 1. \tag{41}$$

Again, this study considered the case where the input $u(t)$ is an ideal white noise with a zero mean and covariance function as shown in (38).

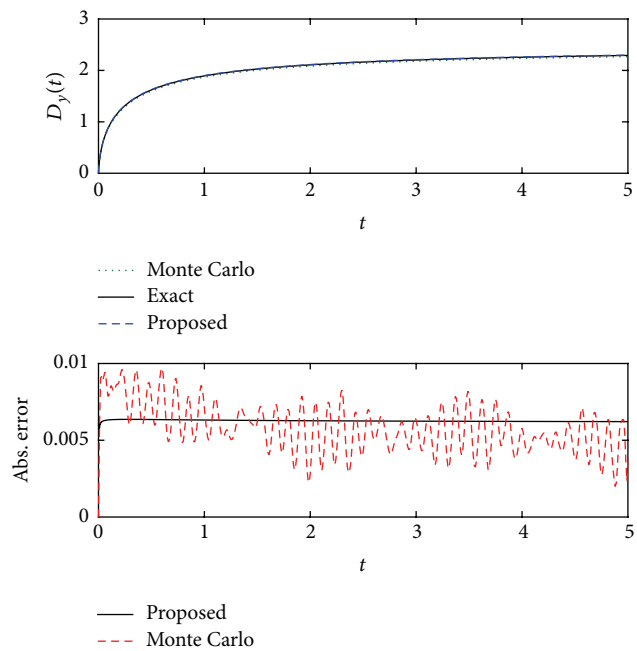


FIGURE 6: Output variance and absolute errors by the proposed and MC methods in Example 2(b).

The variance of the output is given by the following [40]:

$$D_{y(t)} = \sigma^2(t) = \int_0^t \left[e^u E_1(u)\right]^2 du,$$
$$E_1(u) = \int_u^\infty \frac{e^y}{y} dy. \tag{42}$$

The OP for system (41) is given by

$$A_G = \left[\sum_{i=1}^{5} B_{\alpha_i} w_i\right]^{-1}, \tag{43}$$

where B_{α_i} is the OP of derivative of order α_i and $\{\alpha_i, w_i\}_{i=1}^5$ are the nodes and weights from the Legendre quadrature.

Figure 6 shows the variance of the output by the proposed method. The absolute error with respect to the exact variance (42) is also shown.

(c) Example 2(c): $PI^\lambda D^\mu$ Controller Design for Uniform Distributed Order Integrator with Stochastic Input. From the examples above, it can be seen that the proposed method for predicting the mean and variance of the system output provides better accuracy and lower computational load than the other methods such as the MC and gPC. Therefore, it is more suitable for direct optimal design by minimization of the objective function in (30).

Consider a $PI^\lambda D^\mu$ controller as a closed loop configuration with the uniform DO integrator above. The set point $r(t)$ is a random process with a unit mean and covariance function $\kappa_{RR}(\tau) = 0.01\delta(\tau)$. This input $r(t)$ can be viewed as a combination of the deterministic set point and zero mean measurement noise [37].

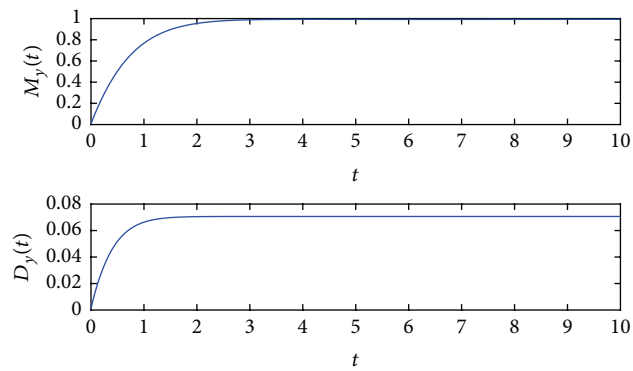


FIGURE 7: Mean and variances of the output of the system in Example 2(c) by the proposed controller.

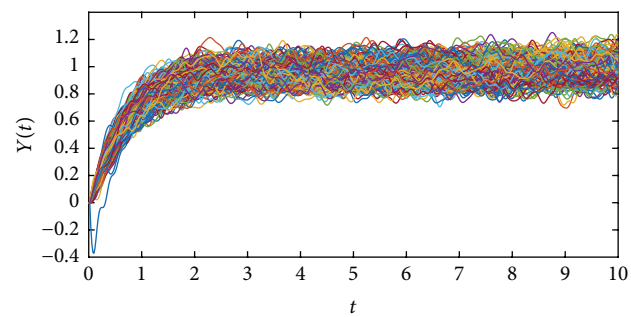


FIGURE 8: 500 responses of stochastic system in Example 2(c) by the proposed controller.

The control objective is to track the deterministic unit step input. This can be achieved by minimizing objective function defined in Section 4. The search space for the optimal parameters of the controller was limited to $0 \le K_p \le 5$; $0 \le K_i \le 5$; $0 \le K_d \le 5$; $0.1 \le \lambda \le 1.9$; $0.1 \le \mu \le 1.9$ for simplicity, as in other studies on the probabilistic approach [29, 37]. The resulting controller can be expressed as $C_1(s) = 0.2805 + 1.1458/s^{0.6422}$.

Figure 7 shows mean and variance of system output with this controller. Figure 8 shows 500 possible responses of the uncertain system with the proposed controller. From the finiteness of the output variance, the stability of system can be determined.

(d) Example 2(d): Improved Mean Tracking Control with Iterative Learning Control. Since the proposed method allows lower computational time for prediction of the mean of system output under random forcing, it can be used with iterative learning control in which input sequence is refined from one trial to next trial [43].

Consider a problem where the mean of closed loop system in Example 2(c) needs to track desired mean $m_{Y_{\text{des}}} = 0.1t(10 - t)$. The following iterative learning control scheme can be used for refining the mean of set point input:

$$m_{Y_k}(t) = \boldsymbol{\psi}(t)^T C_{m_Y} = \boldsymbol{\psi}(t)^T \mathbb{E}\left[A_{\text{cl}}\right] C_{m_{R_k}},$$

$$e_k(t) = m_{Y_{\text{des}}}(t) - m_{Y_k}(t) = \boldsymbol{\psi}(t)^T C_{m_{e_k}},$$

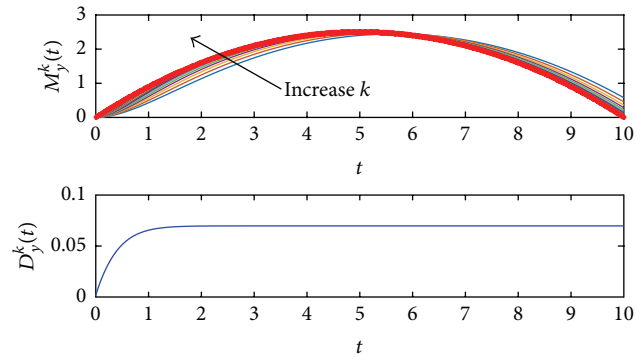


FIGURE 9: Mean and variances of the output of the system in Example 2(d). Line with red o: desired mean.

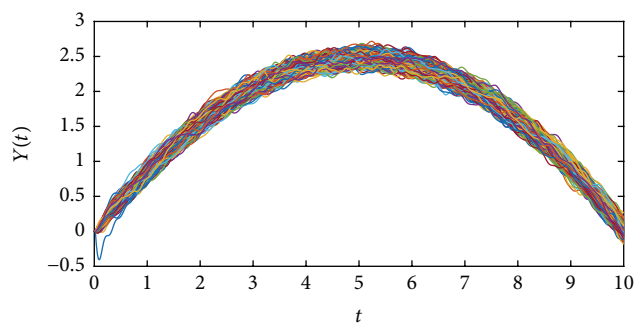


FIGURE 10: 512 responses of stochastic system in Example 2(d) with iterative learning control algorithm.

$$\begin{aligned} m_{R_{k+1}}(t) &= m_{R_k}(t) + e_k(t) \\ &= \boldsymbol{\psi}(t)^T C_{m_{R_k}} + 0.5\boldsymbol{\psi}(t)^T C_{m_{e_k}}, \end{aligned} \tag{44}$$

where $k \in \{0, 1, 2, \ldots\}$ and A_G is the closed loop OP.

Figure 9 shows the simulation results by the proposed method. It can be seen from the figure that the iterative learning algorithm in (44) improves the tracking error in the mean as k increases. The MC simulations with 512 sample responses are shown in Figure 10. It can be seen that the designed control algorithm can track the desired mean despite the random forcing.

Remarks. Note that the low computational cost of the proposed method enables using the iterative learning control algorithm.

5.3. Example 3: Linear Integer Order with Fractional Brownian Motion Input. For each $H \in (0, 1)$, there is a real-value Gaussian process $(\mathscr{B}_H(t), t \geq 0)$ such that $M_{\mathscr{B}_H(t)} = \mathbb{E}(\mathscr{B}_H(t)) = 0$ and the covariance function of $\mathbb{E}(\mathscr{B}_H(t)\mathscr{B}_H(s)) = (1/2)[|t|^{2H} + |s|^{2H} - |t - s|^{2H}]$, $s, t \in \mathbb{R}_+$. This process is called standard fractional Brownian motion with the Hurst parameter H. If $H = 1/2$, then the corresponding standard fractional Brownian motion is the well-known standard Brownian motion.

Recently, there has been growing interest in linear systems with fractional Brownian motion input [44–46]. On the other hand, in contrast to standard Brownian motion, where the moment of output can be obtained by Ito calculus, there are very few methods available for obtaining the moment of the stochastic output of a general linear system with a fractional Brownian motion input.

Consider a random process $X(t)$ that satisfies the following:

$$\begin{aligned} \dot{X}(t) &= -X(t) + \mathscr{B}_H(t), \\ X(0) &= 0, \end{aligned} \tag{45}$$

where $\mathscr{B}_H(t)$ is a fractional Brownian motion with the Hurst parameter H. The mean of $X(t)$ is $M_X(t) = 0$. The variance of $X(t)$ satisfies a differential equation [44]:

$$\dot{D}_X(t) = -2D_X(t) + 2d(t, t), \tag{46}$$

where $d(t, t)$ is given by

$$\begin{aligned} d(s, t) &= \frac{1}{2}\left(t^{2H}\left(1 - e^{-s}\right) + j_1(s) + j_2(s, t)\right), \\ j_1(s) &= e^{-s}\int_0^s u^{2H} e^u du, \\ j_2(s, t) &= e^{-s}\int_0^s (t - u)^{2H} e^u du. \end{aligned} \tag{47}$$

Figure 11 shows the evolution of variance of $X(t)$ for $H = 0.6$ obtained from (46) and (47).

Equation (45) can be rewritten in terms of OP as follows:

$$C_X = (I + B)^{-1} C_{\mathscr{B}} = A_G C_{\mathscr{B}}, \tag{48}$$

where B is the OP of the derivative of order one. Therefore, one can easily obtain the covariance of the random process, $X(t)$, by utilizing the OP for this system, $A_G = (I + B)^{-1}$, and covariance function of fractional Brownian motion and (25). Figure 11 shows the variance of $X(t)$ obtained using the proposed method. Finally, the variance of $X(t)$ by a MC estimation is also given in the same figure.

5.4. Example 4: Linear Distributed Order with Stochastic Parametric and Additive Uncertainties

(a) Example 4(a). This case considers a DO system with both random parameters and random forcing. The system can be expressed as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{\tau \int_0^1 s^\alpha d\alpha + 1}, \tag{49}$$

where k and τ are uniform random variables in [0.5, 1.5]. The system is in a closed loop configuration with a fractional PI controller, $C_1(s) = 0.187 + 36.35/s^{1.216}$ from [46]. Note that the controller, $C_1(s)$, was designed for a nominal system, that is, for $G = 1/(s^{0.5} + 1)$. The input $R(t)$ is a band-limited white

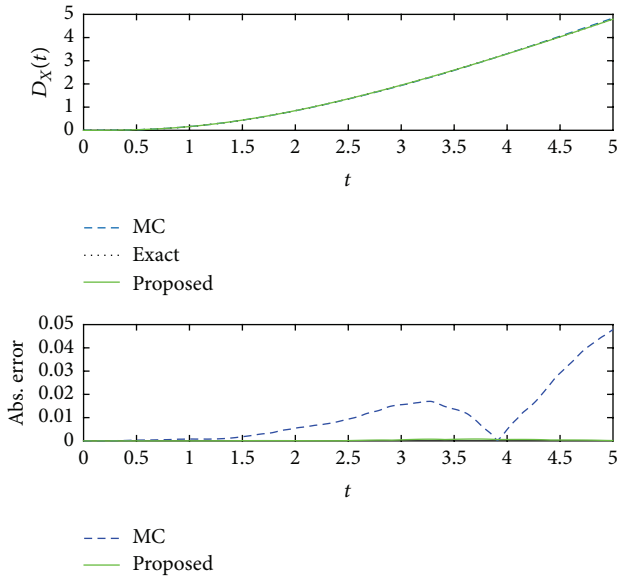


Figure 11: Variances of $X(t)$ in Example 3.

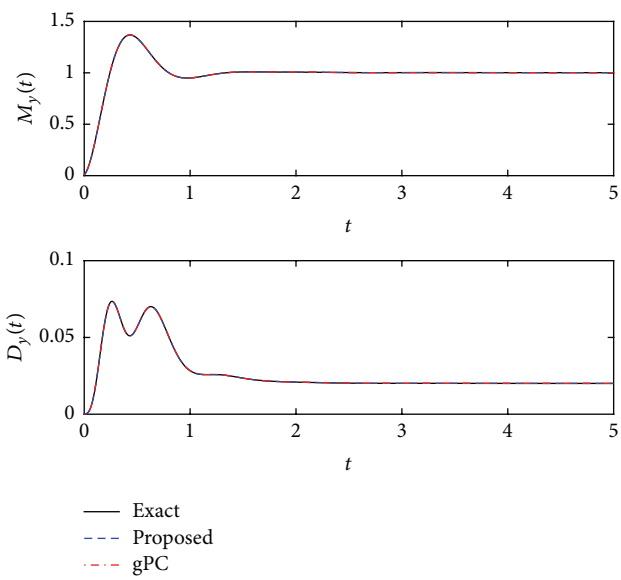


Figure 12: Mean and variances of the output of the system in Example 4(a).

noise process with a unit mean and covariance function of the following: $\kappa_{RR}(t_1, t_2) = 0.02\,\mathrm{sinc}((t_1 - t_2)W_B/\pi)$, where $W_B = 0.02\pi$.

Figure 12 compares the mean and variance of the output calculated by the proposed method using (24) with the results from the gPC and MC methods. In this study, the gPC and MC methods were first applied to the DO systems under stochastic forcing for comparison. In the MC and gPC methods, the DO term was discretized first and the routine fode_sol.m [42] was then used to integrate the multiterm fractional order version of the DO system. Again, the random process input, $U(t)$, was parameterized using a noncanonical decomposition. Table 1 lists the simulation parameters and

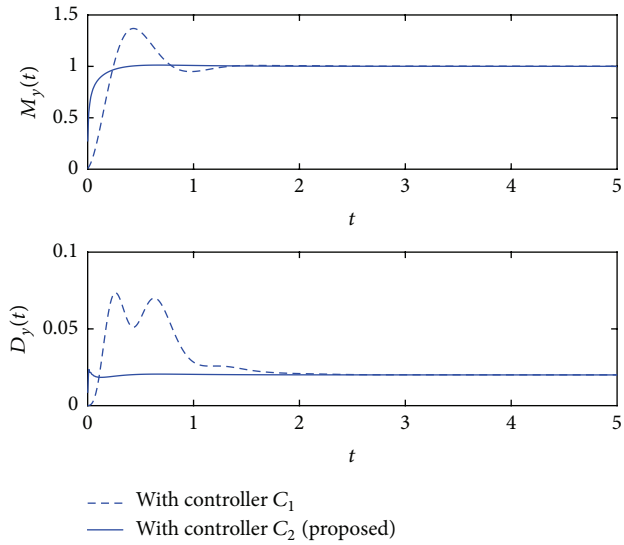


Figure 13: Mean and variance of system output for Example 4(b) by the proposed and initial controllers.

computational times needed for each method. Figure 12 and Table 1 show that the proposed method can provide similar accuracy with much less computational effort than the other methods. The advantage of the proposed method lies in its use of operational matrices: the mean and covariance of the output can be obtained directly from those of the input without parameterization of the input.

(b) Example 4(b). The proposed method was applied to the design of fractional order PID controller for this system. The covariance of the output can be obtained directly from those of the input without parameterization of the input. The control objective is to track the deterministic unit step input. The search space for the parameters of the controller was limited to $0 \le K_p \le 5; 0 \le K_i \le 50; 0 \le K_d \le 5; 0.5 \le \lambda \le 1.5;$ $0.1 \le \mu \le 1.5$. The result is as follows: $C_2(s) = 5+50/(s)+5s^{0.1}$.

Figure 13 shows the mean and variance of the system output by the proposed controller and the controller, $C_1(s) = 0.187 + 36.35/s^{1.216}$ from [46]. Figure 14 shows a bounded region for 1000 possible responses of the uncertain system with the proposed and initial controllers. The proposed controller outperformed the initial controller.

6. Conclusions

A hybrid spectral method was proposed to analyze DO systems in a stochastic setting with arbitrary random forcing and parametric uncertainties. To analyze the system with stochastic parameter perturbation, the stochastic collocation was used to estimate the random operator. This combines the advantages of both the OP technique and PC method. The use of operational matrices explicitly provides the relationship between the first- and second-order moment for the input and output of a system, bypassing parameterization of the random input when predicting the statistical characteristics and reducing the dimensions of the random space.

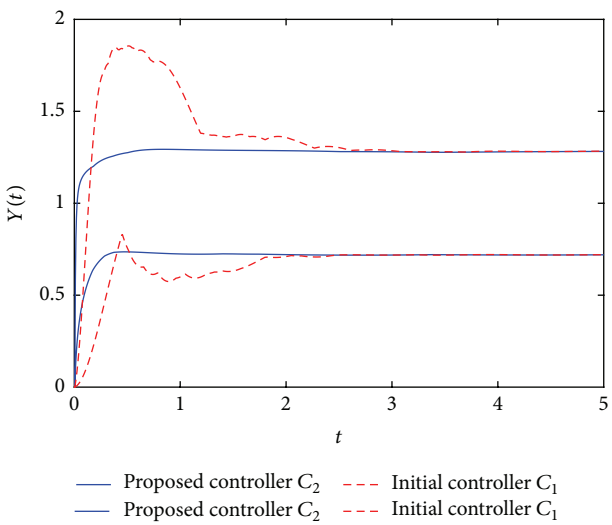


FIGURE 14: Bounded regions for 1000 MC simulations of the stochastic system by the proposed and initial controllers in Example 4(b).

This can also effectively handle a system with a low correlation length input (i.e., ideal white noise) by regularization. The numerical examples show that the proposed method provides superior accuracy and computational efficiency for analyzing stochastic DO systems over other existing methods, such as the gPC, MC, and frequency methods: the frequency method can give the only result at the steady state; the accuracy and efficiency of the gPC method are degraded for a wideband process. Although the MC method is straightforward, its accuracy and computational burden are problematic. On the other hand, the explicit relationship in (24) is only available for a linear system; the applicability of the proposed method is restricted to linear systems.

Appendices

A. Derivation of (24)

Consider a system with its input and output linked by

$$\begin{aligned} C_Y &= A_G C_U, \\ U(t) &= \boldsymbol{\psi}(t)^T C_U, \\ Y(t) &= \boldsymbol{\psi}(t)^T C_U = \boldsymbol{\psi}(t)^T A_G C_U, \end{aligned} \tag{A.1}$$

where A_G is the OP of the system. The input and parameters a_i, b_j are random.

Therefore, the mean of the input and the output in (A.1) was calculated as

$$\begin{aligned} m_U(t) &= \mathbb{E}[U(t)] = \boldsymbol{\psi}(t)^T \mathbb{E}[C_U] = \boldsymbol{\psi}(t)^T C_{m_U}, \\ m_Y(t) &= \mathbb{E}[Y(t)] = \boldsymbol{\psi}(t)^T \mathbb{E}[C_Y] = \boldsymbol{\psi}(t)^T C_{m_Y} \\ &= \boldsymbol{\psi}(t)^T \mathbb{E}[A_G C_U], \end{aligned} \tag{A.2}$$

where $\mathbb{E}[]$ denotes the expectation operator; $C_{m_U} = \mathbb{E}[C_U]$; $C_{m_Y} = \mathbb{E}[C_Y]$.

The statistical independence of A_G and C_U leads to

$$m_Y(t) = \boldsymbol{\psi}(t)^T C_{m_Y} = \boldsymbol{\psi}(t)^T \mathbb{E}[A_G] C_{m_U}. \tag{A.3}$$

Therefore, the spectral characteristics (or expansion coefficients) of the mathematical expectations of input and output are related by

$$C_{m_Y} = \mathbb{E}[A_G] C_{m_U}. \tag{A.4}$$

Introducing the system's signal in the spectral form leads to an equation defining the correlation function of the output to be written as follows:

$$\begin{aligned} \theta_{YY}(t_1, t_2) &= \mathbb{E}[Y(t_1) Y(t_2)] \\ &= \mathbb{E}\left[\boldsymbol{\psi}(t_1)^T C_Y C_Y{}^T \boldsymbol{\psi}(t_2)\right] \\ &= \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[C_Y C_Y{}^T\right] \boldsymbol{\psi}(t_2) \\ &= \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[A_G C_U (C_U)^T A_G{}^T\right] \boldsymbol{\psi}(t_2). \end{aligned} \tag{A.5}$$

Therefore, (A.5) becomes

$$\theta_{YY}(t_1, t_2) = \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[A_G C_{\theta_{UU}} A_G{}^T\right] \boldsymbol{\psi}(t_2), \tag{A.6}$$

where $C_{\theta_{UU}}$ is the square matrix of expansion coefficients (spectral characteristics) of the input's correlation function, which is given by

$$\begin{aligned} \theta_{UU}(t_1, t_2) &= \mathbb{E}[U(t_1) U(t_2)] \\ &= \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[C_U (C_U)^T\right] \boldsymbol{\psi}(t_2) \\ &= \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[C_{\theta_{UU}}\right] \boldsymbol{\psi}(t_2) \\ &= \boldsymbol{\psi}(t_1)^T C_{\theta_{UU}} \boldsymbol{\psi}(t_2). \end{aligned} \tag{A.7}$$

The covariance function of the system's input is defined as

$$\begin{aligned} &\kappa_{UU}(t_1, t_2) \\ &\quad = \mathbb{E}\{[U(t_1) - m_U(t_1)][U(t_2) - m_U(t_2)]\} \\ &\quad = \mathbb{E}[U(t_1) U(t_2)] - m_U(t_1) m_U(t_2) \\ &\quad = \theta_{UU}(t_1, t_2) - m_U(t_1) m_U(t_2). \end{aligned} \tag{A.8}$$

Expanding (A.8) in terms of the orthogonal functions gives the following:

$$\begin{aligned} \kappa_{UU}(t_1, t_2) &= \boldsymbol{\psi}(t_1)^T C_{\kappa_{UU}} \boldsymbol{\psi}(t_2) \\ &= \boldsymbol{\psi}(t_1)^T C_{\theta_{UU}} \boldsymbol{\psi}(t_2) \\ &\quad - \boldsymbol{\psi}(t_1)^T C_{m_U} \left(C_{m_U}\right)^T \boldsymbol{\psi}(t_2). \end{aligned} \tag{A.9}$$

The spectral characteristics of the input signal's moments are given by

$$C_{\kappa_{UU}} = C_{\theta_{UU}} - C_{m_U} \left(C_{m_U}\right)^T. \tag{A.10}$$

Substituting (A.10) into (A.6) gives

$$\begin{aligned}\theta_{YY}(t_1,t_2) &= \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[A_G C_{\theta_{UU}} A_G{}^T\right]\boldsymbol{\psi}(t_2)\\ &= \boldsymbol{\psi}(t_1)^T \mathbb{E}\left[A_G\left\{C_{\kappa_{UU}} + \left(C_{m_U}\right)\left(C_{m_U}\right)^T\right\}A_G{}^T\right]\\ &\quad\cdot \boldsymbol{\psi}(t_2).\end{aligned} \tag{A.11}$$

The covariance function of the system's output is then given by

$$\begin{aligned}\kappa_{YY}(t_1,t_2) &= \boldsymbol{\psi}(t_1)^T C_{\kappa_{YY}}\boldsymbol{\psi}(t_2) = \theta_{YY}(t_1,t_2)\\ &\quad - m_Y(t_1)\,m_Y(t_2) = \boldsymbol{\psi}(t_1)^T\\ &\quad\cdot \mathbb{E}\left[A_G\left\{C_{\kappa_{UU}} + \left(C_{m_U}\right)\left(C_{m_U}\right)^T\right\}A_G{}^T\right]\boldsymbol{\psi}(t_2)\\ &\quad - \boldsymbol{\psi}(t_1)^T C_{m_Y}\left(C_{m_Y}\right)^T\boldsymbol{\psi}(t_2),\end{aligned} \tag{A.12}$$

or in spectral form,

$$\begin{aligned}C_{\kappa_{YY}} &= \mathbb{E}\left[A_G\left\{C_{\kappa_{UU}} + \left(C_{m_U}\right)\left(C_{m_U}\right)^T\right\}A_G{}^T\right]\\ &\quad - C_{m_Y}\left(C_{m_Y}\right)^T.\end{aligned} \tag{A.13}$$

Equation (A.13) gives the relationship between the spectral characteristics of the moments of the system's output and input. Equations (A.4) and (A.13) are then combined to form (24) in the paper

B. Noncanonical Decomposition of Stationary Random Processes [39]

Consider a stationary random process with mean m_X, covariance $\kappa_{XX}(t_1 - t_2) = \kappa_{XX}(\tau)$, and variance $\sigma_X{}^2 = \kappa_{XX}(0)$. This process can be represented as $Z(t,\xi_1,\xi_2) = \sigma_X(\sin(\xi_2 t) + \xi_1\cos(\xi_2 t)) + m_X$ with

$$\begin{aligned}\mathbb{E}[\xi_1] &= 0;\\ \mathbb{E}\left[\xi_1{}^2\right] &= 1;\\ f(\xi_2) &= \frac{S_{XX}(\xi_2)}{\sigma_X{}^2},\end{aligned} \tag{B.1}$$

where ξ_1 and ξ_2 are independent, ξ_1 is Gaussian, and ξ_2 is a random variable with a probability density function (pdf) given in (B.1).

Proof. $Z(t)$ has a mean of

$$\begin{aligned}m_Z &= \mathbb{E}[Z(t,\xi_1,\xi_2)]\\ &= \sigma_X\int_{-\infty}^{\infty}\sin(\xi_2 t)\,f(\xi_2)\,d\xi_2 + m_X = m_X\end{aligned} \tag{B.2}$$

and a covariance function of

$$\begin{aligned}\mathbb{E}\left[\overset{o}{Z}(t_1)\,\overset{o}{Z}(t_2)\right] &= \mathbb{E}\left[\sigma_X{}^2\left\{\sin(\xi_2 t_1)\sin(\xi_2 t_2)\right.\right.\\ &\quad + \xi_1{}^2\cos(\xi_2 t_1)\cos(\xi_2 t_2)\\ &\quad + \xi_1\cos(\xi_2 t_1)\sin(\xi_2 t_2)\\ &\quad \left.\left. + \xi_1\sin(\xi_2 t_1)\cos(\xi_2 t_2)\right\}\right] = \sigma_X{}^2\mathbb{E}\left[\sin(\xi_2 t_1)\right.\\ &\quad \left.\cdot\sin(\xi_2 t_2) + \cos(\xi_2 t_1)\cos(\xi_2 t_2)\right]\\ &= \sigma_X{}^2\int_{-\infty}^{\infty}\cos(\xi_2\tau)\,f(\xi_2)\,d\xi_2,\end{aligned} \tag{B.3}$$

where $\overset{o}{Z}(t) = Z(t) - m_Z = Z(t) - m_X$ is the central component of the random process $Z(t)$. In (B.3), the properties of $\mathbb{E}[\xi_1] = 0$ and $\mathbb{E}[\xi_1{}^2] = 1$ and the independence of ξ_1, ξ_2 are used to simplify the equation.

The covariance function also can be calculated as the inverse Fourier transform of the power spectral density

$$\begin{aligned}\kappa_{ZZ}(\tau) = \kappa_{XX}(\tau) &= \int_{-\infty}^{\infty} S_{XX}(\omega)\,e^{j\omega\tau}d\omega\\ &= \int_{-\infty}^{\infty} S_{XX}(\omega)\cos(\omega\tau)\,d\omega.\end{aligned} \tag{B.4}$$

Comparing (B.3) and (B.4) gives the pdf of ξ_2 in (B.1). Because $\int_{-\infty}^{\infty} S_{XX}(\xi_2)/\sigma_X{}^2 d\xi_2 = 1$, $f(\xi_2)$ is a proper pdf.

(A) A first-order Markov process with a mean m_R and exponential covariance $\kappa_{RR}(\tau) = \sigma_X{}^2 e^{-\alpha|\tau|}$ can be parameterized as $R = \sigma_R(\sin(\xi_2 t) + \xi_1\cos(\xi_2 t)) + m_R(t)$, where ξ_1 is Gaussian, as in (B.1), and $f(\xi_2) = \alpha/\pi(\alpha^2 + \xi_2{}^2)$, $\xi_2 \in (-\infty,\infty)$.

(B) Band-limited white noise with a mean m_R and covariance $\kappa_{RR}(\tau) = c(W_B/\pi)\,\mathrm{sinc}((W_B/\pi)\tau)$ can be parameterized as $R = \sqrt{cW_B/\pi}(\sin(\xi_2 t) + \xi_1\cos(\xi_2 t)) + m_R(t)$, where ξ_1 is Gaussian, as in (B.1), and $f(\xi_2) = 1/2W_B$, $\xi_2 \in [-W_B, W_B]$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2015R1D1A3A01015621). This work was also supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014R1A6A1031189).

References

[1] T. T. Hartley and C. F. Lorenzo, "Order-distributions and the Laplace-domain logarithmic operator," *Advances in Difference Equations*, vol. 2011, article 59, 19 pages, 2011.

[2] W. Chen, "A speculative study of 2/3-order fractional Laplacian modeling of turbulence; some though and conjectures," *Chaos*, vol. 16, Article ID 023126, 7 pages, 2006.

[3] K. Weron and M. Kotulski, "On the Cole-Cole relaxation function and related Mittag-Leffler distribution," *Physica A*, vol. 232, no. 1-2, pp. 180–188, 1996.

[4] D. Baleanu, Z. B. Guvenc, and T. J. A. Machado, Eds., *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, Dordrecht, The Netherlands, 2010.

[5] R. L. Magin, B. S. Akpa, T. Neuberger, and A. G. Webb, "Fractional order analysis of Sephadex gel structures: NMR measurements reflecting anomalous diffusion," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 12, pp. 4581–4587, 2011.

[6] M. K. Bouafoura and N. B. Braiek, "$PI^{\lambda}D^{\mu}$ controller design for integer and fractional plants using piecewise orthogonal functions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 5, pp. 1267–1278, 2010.

[7] M. Caputo, *Elasticida e Dissipazione*, Zanichelli, Bologna, Italy, 1969.

[8] M. Caputo, "Mean-fractional-order-derivative differential equation and filters," *Annali dell' Universita di Ferrara*, vol. 41, no. 1, pp. 73–84, 1995.

[9] R. L. Bagley and P. J. Torvik, "On the existence of the order domain and the solution of distributed order equation (part I, II)," *International Journal of Applied Mechanics*, vol. 2, no. 1, pp. 865–987, 2000.

[10] T. M. Atanackovic, S. Pilipovic, and D. Zorica, "Time distributed-order diffusion-wave equation. I. Volterra-type equation," *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, vol. 465, no. 2106, pp. 1869–1891, 2009.

[11] T. M. Atanackovic, S. Pilipovic, and D. Zorica, "Time distributed-order diffusion-wave equation. II. Applications of Laplace and Fourier transformations," *Proceedings of the Royal Society A*, vol. 465, no. 2106, pp. 1893–1917, 2009.

[12] M. Caputo, "Distributed order differential equations modeling dielectric induction and diffusion," *Fractional Calculus and Applied Analysis*, vol. 4, no. 4, pp. 421–442, 2001.

[13] F. Mainardi and G. Pagnini, "The role of the Fox-Wright functions in fractional sub-diffusion of distributed order," *Journal of Computational and Applied Mathematics*, vol. 207, no. 2, pp. 245–257, 2007.

[14] C. F. Lorenzo and T. T. Hartley, "Variable order and distributed order fractional operators," *Nonlinear Dynamics*, vol. 29, no. 1–4, pp. 57–98, 2002.

[15] H. Sheng, Y. Q. Li, and Y. Chen, "Application of numerical inverse Laplace transform algorithms in fractional calculus," *Journal of the Franklin Institute*, vol. 348, no. 2, pp. 315–330, 2011.

[16] K. Diethelm and N. J. Ford, "Numerical analysis for distributed-order differential equations," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 96–104, 2009.

[17] Z. Jiao, Y. Q. Chen, and I. Podlubny, *Distributed Order Dynamic System Stability, Simulation and Perspective*, Springer, Berlin, Germany, 2012, http://www.mathworks.com/matlabcentral/fileexchange/36570.

[18] C. H. Wang, "On the generalization of block pulse operational matrices for fractional and operational calculus," *Journal of the Franklin Institute*, vol. 315, no. 2, pp. 91–102, 1983.

[19] A. H. Bhrawy, D. Baleanu, L. M. Assas, and J. A. T. Machado, "On a generalized laguerre operational matrix of fractional integration," *Mathematical Problems in Engineering*, vol. 2013, Article ID 569286, 7 pages, 2013.

[20] S. V. Lapin and N. D. Egupov, *Theory of Matrix Operator and Its Application in Automatic Control*, BMSTU Press, Moscow, Russia, 1997, (Russian).

[21] K. Maleknejad and K. Mahdiani, "Solving nonlinear mixed Volterra-Fredholm integral equations with two dimensional block-pulse functions using direct method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 9, pp. 3512–3519, 2011.

[22] Z. L. Huang, X. L. Jin, C. W. Lim, and Y. Wang, "Statistical analysis for stochastic systems including fractional derivatives," *Nonlinear Dynamics*, vol. 59, no. 1-2, pp. 339–349, 2010.

[23] P. L. T. Duong and M. Lee, "Statistical analysis of dead-time system using a deterministic equivalent modeling method," *Asia-Pacific Journal of Chemical Engineering*, vol. 6, no. 3, pp. 369–378, 2011.

[24] M. D. Paola, G. Failla, and A. Pirrotta, "Stationary and non-stationary stochastic response of linear fractional viscoelastic systems," *Probabilistic Engineering Mechanics*, vol. 28, no. 4, pp. 85–90, 2012.

[25] L. Dunn and J. K. Shultis, *Exploring Monte Carlo Methods*, Elsevier, New York, NY, USA, 2011.

[26] D. P. Kroese, T. Taimre, and Z. I. Botev, *Handbook of Monte Carlo Method*, John Wiley & Son, Hoboken, NJ, USA, 2011.

[27] D. Xiu, *Numerical Method for Stochastic Computation: Spectral Approach*, Princeton University Press, Princeton, NJ, USA, 2011.

[28] D. Lucor, C.-H. Su, and G. E. Karniadakis, "Generalized polynomial chaos and random oscillators," *International Journal for Numerical Methods in Engineering*, vol. 60, no. 3, pp. 571–596, 2004.

[29] K. A. Pupkov, N. D. Egupov, A. M. Makarenkov, and A. I. Trofimov, *Theory and Numerical Methods for Studying Stochastic Systems*, Fizmatlits, Moscow, Russia, 2003 (Russian).

[30] D. Xiu and G. E. Karniadakis, "Modeling uncertainty in flow simulations via generalized polynomial chaos," *Journal of Computational Physics*, vol. 187, no. 1, pp. 137–167, 2003.

[31] D. Xiu and J. Shen, "Efficient stochastic Galerkin methods for random diffusion equations," *Journal of Computational Physics*, vol. 228, no. 2, pp. 266–281, 2009.

[32] O. P. Le Maître, O. M. Knio, H. N. Najm, and R. G. Ghanem, "A stochastic projection method for fluid flow. I. Basic formulation," *Journal of Computational Physics*, vol. 173, no. 2, pp. 481–511, 2001.

[33] P. L. T. Duong and M. Lee, "Uncertainty propagation in stochastic fractional order processes using spectral methods: a hybrid approach," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 11, pp. 4262–4273, 2012.

[34] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[35] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[36] W. Gaustchi, *Orthogonal Polynomials: Computation and Approximations*, Oxford University Press, Oxford, UK, 2003.

[37] P. L. T. Duong and M. Lee, "Optimal design of fractional order linear system with stochastic/input/parametric uncertainties by hybrid spectral method," *Journal of Process Control*, vol. 24, no. 10, pp. 1639–1645, 2014.

[38] R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signal and Applied Kalman Filtering with MATLAB Exercises*, Wiley, New York, NY, USA, 4th edition, 2012.

[39] V. I. Chernhecki, *Analysis Accuracy of Nonlinear Control Systems*, Masintroenhie, 1968 (Russian).

[40] C. H. Eab and S. C. Lim, "Fractional Langevin equations of distributed order," *Physical Review E*, vol. 83, no. 3, Article ID 031136, 10 pages, 2011.

[41] I. Podlubny and M. Kacenak, http://www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-function.

[42] A. C. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-Order Systems and Controls*, Springer, London, UK, 2010.

[43] Y. Li, Y. Chen, and H.-S. Ahn, "Fractional-order iterative learning control for fractional-order linear systems," *Asian Journal of Control*, vol. 13, no. 1, pp. 54–63, 2011.

[44] M. Grigoriu, "Linear systems with fractional Brownian motion and Gaussian noise," *Probabilistic Engineering Mechanics*, vol. 22, no. 3, pp. 276–284, 2007.

[45] T. E. Duncan, "Some application of fractional Brownian motion to linear system," in *System Theory: Modeling, Analysis and Control*, T. E. Djaferis, Ed., Kluwer Academic Publishers, 2000.

[46] T. E. Duncan and B. Pasik-Duncan, "Control of some linear with a fractional Brownian motion," in *Proceedings of the 48th IEEE Conference on Decision and Control (CDC '09)*, pp. 8118–8522, Shanghai, China, 2009.