Abstract and Applied Analysis

Functional Differential and Difference Equations with Applications 2013

Guest Editors: J. Diblík, E. Braverman, I. Györi, Yu. Rogovchenko, M. Růžičková, and A. Zafer

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Editorial Board

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Editorial

Functional Differential and Difference Equations with Applications 2013

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This annual issue comes as a sequel to two special issues, *Recent Progress in Differential and Difference Equations* edited by the four members of the present team and *Functional Differential and Difference Equations with Applications* with the same editorial team, both published by the *Abstract and Applied Analysis* in 2011 and 2012, respectively.

In the call for papers prepared by the Guest Editors and posted on the journal's web page, we encouraged submission of state-of-the-art contributions on a wide spectrum of topics such as asymptotic behavior of solutions, boundedness and periodicity of solutions, nonoscillation and oscillation of solutions, representation of solutions, stability, numerical algorithms, and computational aspects, as well as applications to real-world phenomena. This invitation was warmly welcomed by the mathematical community; more than sixty manuscripts addressing important problems in the above-mentioned fields of qualitative theory of functional differential and difference equations were submitted to the Editorial Office and went through a thorough peer refereeing process. Twenty-seven carefully selected research articles collected in this special issue reflect modern trends and advances in functional differential and difference equations. Sixty-seven authors from fourteen countries (China, Czech Republic, Georgia, Hungary, Israel, Latvia, Norway, Portugal, Saudi Arabia, Serbia, Slovak Republic, Spain, Turkey, and Ukraine) have contributed to the success of this thematic collection of papers.

Questions related to the existence of periodic solutions and their stability properties traditionally attract attention of researchers working in the qualitative theory of differential, functional differential, and difference equations. In this issue, the reader will find results that relate periodicity of linear autonomous nonhomogeneous difference equations to the existence of equilibria. Systems of nonlinear difference equations whose all well-defined solutions are periodic are considered. For some classes of nonlinear systems with delay, it is shown that the presence of the time delay results in the existence of periodic solutions.

Existence of oscillating solutions for half-linear differential equations with delay and for even order damped equations with distributed deviating arguments is studied. Nonlinear oscillations in the context of saddle-center bifurcation in the dynamical system describing a singularly perturbed forced oscillator of Duffing's type with a nonlinear restoring and a nonperiodic external driving force are examined. Boundedness of solutions to a class of second-order periodic systems with singularities is considered as well.

Stability problems always attract interest of researchers, and this special issue is not an exception. The reader will find papers on the stability of impulsive stochastic functional differential equations with delayed impulses and stability of differential systems under permanently arising impulses, an application of moment equations in a model of the stable foreign currency exchange market in conditions of

uncertainty, a study on the global exponential stability of equilibria for impulsive cellular neural network models with piecewise alternately advanced and retarded arguments, and analysis of stability and global Hopf bifurcation phenomena in a ratio-dependent predator-prey model with two time delays.

New results for boundary value problems are also reported, including the existence and uniqueness theorem for solutions of nonhomogeneous impulsive boundary value problem for planar Hamiltonian systems and general results on the solvability of singular initial value problems.

Several articles deal with the behavior of solutions at infinity. Namely, explicit formulas for planar weakly delayed linear discrete systems are derived; results on asymptotic behavior of solutions to generalized Emden-Fowler differential equations with delayed argument, higher-order quasilinear neutral differential equations, vector integral equations with deviating arguments, and linear Volterra integrodifferential systems are reported along with the applications of nonlinear weakly singular inequalities to Volterra-type difference equations.

Other important problems discussed in this special issue are related to the representation of the solutions of linear discrete systems with constant coefficients and two delays, automorphisms of ordinary differential equations, spatial discretization of the Cauchy problem for a multidimensional linear parabolic partial differential equation of the second order with nondivergent operator, and unbounded time- and space-dependent coefficients. Newton-Kantorovich and Smale type convergence theorems are used in a deformed Newton method with the third-order convergence for solving nonlinear equations. Regularity of a mild solution for a stochastic fractional delayed reaction-diffusion equation driven by Lévy space-time white noise and an inverse problem for Dirac differential operators with discontinuity conditions and discontinuous coefficient are studied. Finally, a mathematical model for the incompressible two-dimensional/axisymmetric non-Newtonian fluid flows and heat transfer analysis in the region of stagnation point over a stretching/shrinking sheet and axisymmetric shrinking sheet is presented.

It is not possible to provide in this short editorial note a detailed description for all papers included in this volume. However, it is clear that they reflect contemporary trends in the development of the qualitative theory of functional differential equations and feature important applications. We believe that this special issue challenges researchers with new unsolved problems and introduces many new ideas and useful techniques.

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Research Article

General Explicit Solution of Planar Weakly Delayed Linear Discrete Systems and Pasting Its Solutions

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Planar linear discrete systems with constant coefficients and delays $x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k-m_l)$ are considered where $k \in \mathbb{Z}_0^\infty := \{0, 1, \ldots, \infty\}$, $m_1, m_2, \ldots, m_n$ are constant integer delays, $0 < m_1 < m_2 < \cdots < m_n$, $A, B^1, \ldots, B^n$ are constant 2×2 matrices, and $x : \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$. It is assumed that the considered system is weakly delayed. The characteristic equations of such systems are identical with those for the same systems but without delayed terms. In this case, after several steps, the space of solutions with a given starting dimension $2(m_n+1)$ is pasted into a space with a dimension less than the starting one. In a sense, this situation is analogous to one known in the theory of linear differential systems with constant coefficients and special delays when the initially infinite dimensional space of solutions on the initial interval turns (after several steps) into a finite dimensional set of solutions. For every possible case, explicit general solutions are constructed and, finally, results on the dimensionality of the space of solutions are obtained.

1. Introduction

1.1. Preliminary Notions and Properties. We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \ldots, q\}$, where $s = -\infty$ or $q = \infty$ is admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$, we always assume $s \le q$. In the paper, we deal with the discrete planar system

$$x(k+1) = Ax(k) + \sum_{l=1}^{n} B^l x_l(k-m_l), \tag{1}$$

where $m_1, m_2, \ldots, m_n$ are constant integer delays, $0 < m_1 < m_2 < \cdots < m_n$, $k \in \mathbb{Z}_0^\infty$, $A, B^1, \ldots, B^n$ are constant 2×2 matrices, $A = (a_{ij})$, $B^l = (b_{ij}^l)$, $i, j = 1, 2$, $l = 1, 2, \ldots, n$, and $x : \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$. Throughout the paper, we assume that

$$B^l \neq \Theta, \tag{2}$$

where $l = 1, 2, \ldots, n$ and Θ is 2×2 zero matrix. Together with (1), we consider an initial (Cauchy) problem

$$x(k) = \varphi(k), \tag{3}$$

where $k = -m_n, -m_n+1, \ldots, 0$ with $\varphi : \mathbb{Z}_{-m_n}^0 \to \mathbb{R}^2$. The *existence* and *uniqueness* of the solution of the initial problem (1), (3) on $\mathbb{Z}_{-m_n}^\infty$ are obvious. We recall that the *solution* $x : \mathbb{Z}_{-m_n}^\infty \to \mathbb{R}^2$ of (1), (3) is defined as an *infinite sequence*

$$\begin{aligned} \{&x(-m_n) = \varphi(-m_n), \\ &x(-m_n+1) = \varphi(-m_n+1), \ldots, \\ &x(0) = \varphi(0), \\ &x(1), x(2), \ldots, x(k), \ldots\} \end{aligned} \tag{4}$$

such that, for any $k \in \mathbb{Z}_0^\infty$, equality (1) holds.

The space of all initial data (3) with $\varphi : \mathbb{Z}_{-m_n}^0 \to \mathbb{R}^2$ is obviously $2(m_n+1)$-dimensional. Below, we describe the

fact that, among system (1), there are such systems that their space of solutions, being initially $2(m_n+1)$-dimensional, on a reduced interval turns into a space having a dimension less than $2(m_n+1)$. The problem under consideration (pasting property of solutions) is exactly formulated in Section 1.4.

1.2. Weakly Delayed Systems. We consider system (1) and look for a solution having the form $x(k)=\xi\lambda^k$, where $k\in\mathbb{Z}_{-m_n}^{\infty}$, $\lambda=$ constant with $\lambda\neq 0$, and $\xi=(\xi_1,\xi_2)^T$ is a nonzero constant vector. The usual procedure leads to a characteristic equation

$$D:=\det\left(A+\sum_{l=1}^{n}\lambda^{-m_l}B^l-\lambda I\right)=0, \tag{5}$$

where I is the unit 2×2 matrix. Together with (1), we consider a system with the terms containing delays omitted:

$$x(k+1)=Ax(k) \tag{6}$$

and its characteristic equation

$$\det(A-\lambda I)=0. \tag{7}$$

Definition 1. System (1) is called a weakly delayed system if characteristic equations (5), (7) corresponding to systems (1) and (6) are equal, that is, if, for every $\lambda\in\mathbb{C}\setminus\{0\}$,

$$D=\det\left(A+\sum_{l=1}^{n}\lambda^{-m_l}B^l-\lambda I\right)=\det(A-\lambda I). \tag{8}$$

We consider a linear transformation

$$x(k)=\mathcal{S}y(k), \tag{9}$$

with a nonsingular 2×2 matrix $\mathcal{S}$, then the discrete system for y is

$$y(k+1)=A_{\mathcal{S}}y(k)+\sum_{l=1}^{n}B_{\mathcal{S}}^{l}y(k-m_l), \tag{10}$$

with $A_{\mathcal{S}}=\mathcal{S}^{-1}A\mathcal{S}$, $B_{\mathcal{S}}^{l}=\mathcal{S}^{-1}B^l\mathcal{S}$, where $l=1,2,\ldots,n$. We show that a system's property of being one weakly delayed is preserved by every nonsingular linear transformation.

Lemma 2. *If system* (1) *is a weakly delayed system, then its arbitrary linear nonsingular transformation* (9) *again leads to a weakly delayed system* (10).

Proof. It is easy to show that

$$\det\left(A_{\mathcal{S}}+\sum_{l=1}^{n}\lambda^{-m_l}B_{\mathcal{S}}^{l}-\lambda I\right)=\det\left(A_{\mathcal{S}}-\lambda I\right) \tag{11}$$

holds since

$$\begin{aligned}
&\det\left(A_{\mathcal{S}}+\sum_{l=1}^{n}\lambda^{-m_l}B_{\mathcal{S}}^{l}-\lambda I\right)\\
&\quad=\det\left(A_{\mathcal{S}}+\lambda^{-m_1}B_{\mathcal{S}}^{1}+\lambda^{-m_2}B_{\mathcal{S}}^{2}\right.\\
&\qquad\left.+\cdots+\lambda^{-m_n}B_{\mathcal{S}}^{n}-\lambda I\right)\\
&\quad=\det\left(\mathcal{S}^{-1}A\mathcal{S}+\lambda^{-m_1}\mathcal{S}^{-1}B^1\mathcal{S}\right.\\
&\qquad+\lambda^{-m_2}\mathcal{S}^{-1}B^2\mathcal{S}+\cdots\\
&\qquad\left.+\lambda^{-m_n}\mathcal{S}^{-1}B^n\mathcal{S}-\lambda\mathcal{S}^{-1}I\mathcal{S}\right)\\
&\quad=\det\left[\mathcal{S}^{-1}\left(A+\lambda^{-m_1}B^1+\lambda^{-m_2}B^2\right.\right.\\
&\qquad\left.\left.+\cdots+\lambda^{-m_n}B^n-\lambda I\right)\mathcal{S}\right]\\
&\quad=\det\mathcal{S}^{-1}\det\left(A+\lambda^{-m_1}B^1+\lambda^{-m_2}B^2\right.\\
&\qquad\left.+\cdots+\lambda^{-m_n}B^n-\lambda I\right)\det\mathcal{S}\\
&\quad=\det\left(A+\lambda^{-m_1}B^1+\lambda^{-m_2}B^2\right.\\
&\qquad\left.+\cdots+\lambda^{-m_n}B^n-\lambda I\right)\\
&\quad=\det\left(A+\sum_{l=1}^{n}\lambda^{-m_l}B^l-\lambda I\right),\\
&\det\left(A_{\mathcal{S}}-\lambda I\right)=\det\left(\mathcal{S}^{-1}A\mathcal{S}-\lambda\mathcal{S}^{-1}I\mathcal{S}\right)\\
&\qquad=\det\left[\mathcal{S}^{-1}(A-\lambda I)\mathcal{S}\right]\\
&\qquad=\det\mathcal{S}^{-1}\det(A-\lambda I)\det\mathcal{S}\\
&\qquad=\det(A-\lambda I),\\
&\det\left(A+\sum_{l=1}^{n}\lambda^{-m_l}B^l-\lambda I\right)=\det(A-\lambda I);
\end{aligned} \tag{12}$$

that is, equality (8) is assumed. □

1.3. Necessary and Sufficient Conditions Determining Weakly Delayed Systems. In the next theorem, we give conditions, in terms of determinants, indicating whether a system is weakly delayed.

Theorem 3. *System* (1) *is a weakly delayed system if and only if the following* $3n+n(n-1)/2$ *conditions hold simultaneously:*

$$b_{11}^{l}+b_{22}^{l}=0, \tag{13}$$

$$\begin{vmatrix} b_{11}^{l} & b_{12}^{l}\\ b_{21}^{l} & b_{22}^{l}\end{vmatrix}=0, \tag{14}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ b_{21}^{l} & b_{22}^{l} \end{vmatrix} + \begin{vmatrix} b_{11}^{l} & b_{12}^{l} \\ a_{21} & a_{22} \end{vmatrix} = 0, \tag{15}$$

$$\begin{vmatrix} b_{11}^{l} & b_{12}^{l} \\ b_{21}^{v} & b_{22}^{v} \end{vmatrix} + \begin{vmatrix} b_{11}^{v} & b_{12}^{v} \\ b_{21}^{l} & b_{22}^{l} \end{vmatrix} = 0, \tag{16}$$

where $l, v = 1, 2, \ldots, n$ and $v > l$.

Proof. We start with computing determinant D defined by (5). We get

$$D = \begin{vmatrix} D_1 & D_2 \\ D_3 & D_4 \end{vmatrix}, \tag{17}$$

where

$$\begin{aligned} D_1 &= a_{11} + b_{11}^{1}\lambda^{-m_1} + b_{11}^{2}\lambda^{-m_2} + \cdots + b_{11}^{n}\lambda^{-m_n} - \lambda, \\ D_2 &= a_{12} + b_{12}^{1}\lambda^{-m_1} + b_{12}^{2}\lambda^{-m_2} + \cdots + b_{12}^{n}\lambda^{-m_n}, \\ D_3 &= a_{21} + b_{21}^{1}\lambda^{-m_1} + b_{21}^{2}\lambda^{-m_2} + \cdots + b_{21}^{n}\lambda^{-m_n}, \\ D_4 &= a_{22} + b_{22}^{1}\lambda^{-m_1} + b_{22}^{2}\lambda^{-m_2} + \cdots + b_{22}^{n}\lambda^{-m_n} - \lambda. \end{aligned} \tag{18}$$

Expanding the determinant on the right-hand side along summands of the first column, we get

$$\begin{aligned} D = {} & \begin{vmatrix} a_{11} & a_{12} + b_{12}^{1}\lambda^{-m_1} + b_{12}^{2}\lambda^{-m_2} + \cdots + b_{12}^{n}\lambda^{-m_n} \\ a_{21} & a_{22} + b_{22}^{1}\lambda^{-m_1} + b_{22}^{2}\lambda^{-m_2} + \cdots + b_{22}^{n}\lambda^{-m_n} - \lambda \end{vmatrix} \\ & + \lambda^{-m_1} \begin{vmatrix} b_{11}^{1} & a_{12} + b_{12}^{1}\lambda^{-m_1} + b_{12}^{2}\lambda^{-m_2} + \cdots + b_{12}^{n}\lambda^{-m_n} \\ b_{21}^{1} & a_{22} + b_{22}^{1}\lambda^{-m_1} + b_{22}^{2}\lambda^{-m_2} + \cdots + b_{22}^{n}\lambda^{-m_n} - \lambda \end{vmatrix} \\ & + \lambda^{-m_2} \begin{vmatrix} b_{11}^{2} & a_{12} + b_{12}^{1}\lambda^{-m_1} + b_{12}^{2}\lambda^{-m_2} + \cdots + b_{12}^{n}\lambda^{-m_n} \\ b_{21}^{2} & a_{22} + b_{22}^{1}\lambda^{-m_1} + b_{22}^{2}\lambda^{-m_2} + \cdots + b_{22}^{n}\lambda^{-m_n} - \lambda \end{vmatrix} \\ & + \cdots \\ & + \lambda^{-m_n} \begin{vmatrix} b_{11}^{n} & a_{12} + b_{12}^{1}\lambda^{-m_1} + b_{12}^{2}\lambda^{-m_2} + \cdots + b_{12}^{n}\lambda^{-m_n} \\ b_{21}^{n} & a_{22} + b_{22}^{1}\lambda^{-m_1} + b_{22}^{2}\lambda^{-m_2} + \cdots + b_{22}^{n}\lambda^{-m_n} - \lambda \end{vmatrix} \\ & + \lambda \begin{vmatrix} -1 & a_{12} + b_{12}^{1}\lambda^{-m_1} + b_{12}^{2}\lambda^{-m_2} + \cdots + b_{12}^{n}\lambda^{-m_n} \\ 0 & a_{22} + b_{22}^{1}\lambda^{-m_1} + b_{22}^{2}\lambda^{-m_2} + \cdots + b_{22}^{n}\lambda^{-m_n} - \lambda \end{vmatrix}. \end{aligned} \tag{19}$$

Expanding each of the above determinants along summands of the second column, we have

$$\begin{aligned} D = {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \lambda^{-m_1} \begin{vmatrix} a_{11} & b_{12}^{1} \\ a_{21} & b_{22}^{1} \end{vmatrix} + \lambda^{-m_2} \begin{vmatrix} a_{11} & b_{12}^{2} \\ a_{21} & b_{22}^{2} \end{vmatrix} \\ & + \cdots + \lambda^{-m_n} \begin{vmatrix} a_{11} & b_{12}^{n} \\ a_{21} & b_{22}^{n} \end{vmatrix} + \lambda \begin{vmatrix} a_{11} & 0 \\ a_{21} & -1 \end{vmatrix} \\ & + \lambda^{-m_1} \left[\begin{vmatrix} b_{11}^{1} & a_{12} \\ b_{21}^{1} & a_{22} \end{vmatrix} + \lambda^{-m_1} \begin{vmatrix} b_{11}^{1} & b_{12}^{1} \\ b_{21}^{1} & b_{22}^{1} \end{vmatrix} + \lambda^{-m_2} \begin{vmatrix} b_{11}^{1} & b_{12}^{2} \\ b_{21}^{1} & b_{22}^{2} \end{vmatrix} \right. \\ & \qquad \left. + \cdots + \lambda^{-m_n} \begin{vmatrix} b_{11}^{1} & b_{12}^{n} \\ b_{21}^{1} & b_{22}^{n} \end{vmatrix} + \lambda \begin{vmatrix} b_{11}^{1} & 0 \\ b_{21}^{1} & -1 \end{vmatrix} \right] \\ & + \lambda^{-m_2} \left[\begin{vmatrix} b_{11}^{2} & a_{12} \\ b_{21}^{2} & a_{22} \end{vmatrix} + \lambda^{-m_1} \begin{vmatrix} b_{11}^{2} & b_{12}^{1} \\ b_{21}^{2} & b_{22}^{1} \end{vmatrix} + \lambda^{-m_2} \begin{vmatrix} b_{11}^{2} & b_{12}^{2} \\ b_{21}^{2} & b_{22}^{2} \end{vmatrix} \right. \\ & \qquad \left. + \cdots + \lambda^{-m_n} \begin{vmatrix} b_{11}^{2} & b_{12}^{n} \\ b_{21}^{2} & b_{22}^{n} \end{vmatrix} + \lambda \begin{vmatrix} b_{11}^{2} & 0 \\ b_{21}^{2} & -1 \end{vmatrix} \right] \\ & + \cdots \\ & + \lambda^{-m_n} \left[\begin{vmatrix} b_{11}^{n} & a_{12} \\ b_{21}^{n} & a_{22} \end{vmatrix} + \lambda^{-m_1} \begin{vmatrix} b_{11}^{n} & b_{12}^{1} \\ b_{21}^{n} & b_{22}^{1} \end{vmatrix} + \lambda^{-m_2} \begin{vmatrix} b_{11}^{n} & b_{12}^{2} \\ b_{21}^{n} & b_{22}^{2} \end{vmatrix} \right. \\ & \qquad \left. + \cdots + \lambda^{-m_n} \begin{vmatrix} b_{11}^{n} & b_{12}^{n} \\ b_{21}^{n} & b_{22}^{n} \end{vmatrix} + \lambda \begin{vmatrix} b_{11}^{n} & 0 \\ b_{21}^{n} & -1 \end{vmatrix} \right] \\ & + \lambda \left[\begin{vmatrix} -1 & a_{12} \\ 0 & a_{22} \end{vmatrix} + \lambda^{-m_1} \begin{vmatrix} -1 & b_{12}^{1} \\ 0 & b_{22}^{1} \end{vmatrix} + \lambda^{-m_2} \begin{vmatrix} -1 & b_{12}^{2} \\ 0 & b_{22}^{2} \end{vmatrix} \right. \\ & \qquad \left. + \cdots + \lambda^{-m_n} \begin{vmatrix} -1 & b_{12}^{n} \\ 0 & b_{22}^{n} \end{vmatrix} + \lambda \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \right]. \end{aligned} \tag{20}$$

After simplification, we get

$$\begin{aligned} D = {} & \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} - \lambda^{-m_1+1}\left(b_{11}^{1} + b_{22}^{1}\right) \\ & - \lambda^{-m_2+1}\left(b_{11}^{2} + b_{22}^{2}\right) + \cdots - \lambda^{-m_n+1}\left(b_{11}^{n} + b_{22}^{n}\right) \\ & + \lambda^{-m_1} \left[\begin{vmatrix} a_{11} & a_{12} \\ b_{21}^{1} & b_{22}^{1} \end{vmatrix} + \begin{vmatrix} b_{11}^{1} & b_{12}^{1} \\ a_{21} & a_{22} \end{vmatrix} \right] \\ & + \lambda^{-m_2} \left[\begin{vmatrix} a_{11} & a_{12} \\ b_{21}^{2} & b_{22}^{2} \end{vmatrix} + \begin{vmatrix} b_{11}^{2} & b_{12}^{2} \\ a_{21} & a_{22} \end{vmatrix} \right] \\ & + \cdots + \lambda^{-m_n} \left[\begin{vmatrix} a_{11} & a_{12} \\ b_{21}^{n} & b_{22}^{n} \end{vmatrix} + \begin{vmatrix} b_{11}^{n} & b_{12}^{n} \\ a_{21} & a_{22} \end{vmatrix} \right] \\ & + \lambda^{-m_1-m_2} \left[\begin{vmatrix} b_{11}^{1} & b_{12}^{1} \\ b_{21}^{2} & b_{22}^{2} \end{vmatrix} + \begin{vmatrix} b_{11}^{2} & b_{12}^{2} \\ b_{21}^{1} & b_{22}^{1} \end{vmatrix} \right] \\ & + \cdots + \lambda^{-m_1-m_n} \left[\begin{vmatrix} b_{11}^{1} & b_{12}^{1} \\ b_{21}^{n} & b_{22}^{n} \end{vmatrix} + \begin{vmatrix} b_{11}^{n} & b_{12}^{n} \\ b_{21}^{1} & b_{22}^{1} \end{vmatrix} \right] \\ & + \lambda^{-m_2-m_3} \left[\begin{vmatrix} b_{11}^{2} & b_{12}^{2} \\ b_{21}^{3} & b_{22}^{3} \end{vmatrix} + \begin{vmatrix} b_{11}^{3} & b_{12}^{3} \\ b_{21}^{2} & b_{22}^{2} \end{vmatrix} \right] \\ & + \cdots + \lambda^{-m_2-m_n} \left[\begin{vmatrix} b_{11}^{2} & b_{12}^{2} \\ b_{21}^{n} & b_{22}^{n} \end{vmatrix} + \begin{vmatrix} b_{11}^{n} & b_{12}^{n} \\ b_{21}^{2} & b_{22}^{2} \end{vmatrix} \right] \\ & + \cdots + \lambda^{-m_{n-1}-m_n} \left[\begin{vmatrix} b_{11}^{n-1} & b_{12}^{n-1} \\ b_{21}^{n} & b_{22}^{n} \end{vmatrix} + \begin{vmatrix} b_{11}^{n} & b_{12}^{n} \\ b_{21}^{n-1} & b_{22}^{n-1} \end{vmatrix} \right] \\ & + \lambda^{-2m_1} \begin{vmatrix} b_{11}^{1} & b_{12}^{1} \\ b_{21}^{1} & b_{22}^{1} \end{vmatrix} + \lambda^{-2m_2} \begin{vmatrix} b_{11}^{2} & b_{12}^{2} \\ b_{21}^{2} & b_{22}^{2} \end{vmatrix} \\ & + \cdots + \lambda^{-2m_n} \begin{vmatrix} b_{11}^{n} & b_{12}^{n} \\ b_{21}^{n} & b_{22}^{n} \end{vmatrix}. \end{aligned} \tag{21}$$

Now we see that for (8) to hold; that is,

$$\begin{aligned} D &= \det\left(A + \sum_{l=1}^{n} \lambda^{-m_l} B^l - \lambda I\right) \\ &= \det(A - \lambda I) \\ &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}, \end{aligned} \tag{22}$$

conditions (13)–(16) are both necessary and sufficient. □

Lemma 4. *Conditions* (13)–(16) *are equivalent to*

$$\operatorname{tr} B^l = \det B^l = 0, \tag{23}$$

$$\det\left(A + B^l\right) = \det A, \tag{24}$$

$$\det\left(B^l + B^v\right) = 0, \tag{25}$$

where $l, v = 1, 2, \ldots, n$ *and* $v > l$.

Proof. (I) We show that assumptions (13)–(16) imply (23)–(25). It is obvious that condition (23) is equivalent to (13), (14). Now we consider

$$\det\left(A + B^l\right) = \begin{vmatrix} a_{11} + b_{11}^l & a_{12} + b_{12}^l \\ a_{21} + b_{21}^l & a_{22} + b_{22}^l \end{vmatrix}. \tag{26}$$

Expanding the determinant on the right-hand side along summands of the first column and then expanding each of the determinants along summands of the second column, we have

$$\begin{aligned} \det\left(A + B^l\right) &= \begin{vmatrix} a_{11} & a_{12} + b_{12}^l \\ a_{21} & a_{22} + b_{22}^l \end{vmatrix} + \begin{vmatrix} b_{11}^l & a_{12} + b_{12}^l \\ b_{21}^l & a_{22} + b_{22}^l \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12}^l \\ a_{21} & b_{22}^l \end{vmatrix} \\ &\quad + \begin{vmatrix} b_{11}^l & a_{12} \\ b_{21}^l & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11}^l & b_{12}^l \\ b_{21}^l & b_{22}^l \end{vmatrix} \\ &= [\text{due to (15) and (16)}] = \det A. \end{aligned} \tag{27}$$

Now we consider

$$\det\left(B^l + B^v\right) = \begin{vmatrix} b_{11}^l + b_{11}^v & b_{12}^l + b_{12}^v \\ b_{21}^l + b_{21}^v & b_{22}^l + b_{22}^v \end{vmatrix}. \tag{28}$$

Expanding the determinant on the right-hand side along summands of the first column and then expanding each of the determinants along summands of the second column, we have

$$\begin{aligned} \det\left(B^l + B^v\right) &= \begin{vmatrix} b_{11}^l & b_{12}^l + b_{12}^v \\ b_{21}^l & b_{22}^l + b_{22}^v \end{vmatrix} + \begin{vmatrix} b_{11}^v & b_{12}^l + b_{12}^v \\ b_{21}^v & b_{22}^l + b_{22}^v \end{vmatrix} \\ &= \begin{vmatrix} b_{11}^l & b_{12}^l \\ b_{21}^l & b_{22}^l \end{vmatrix} + \begin{vmatrix} b_{11}^l & b_{12}^v \\ b_{21}^l & b_{22}^v \end{vmatrix} \\ &\quad + \begin{vmatrix} b_{11}^v & b_{12}^l \\ b_{21}^v & b_{22}^l \end{vmatrix} + \begin{vmatrix} b_{11}^v & b_{12}^v \\ b_{21}^v & b_{22}^v \end{vmatrix} \\ &= [\text{due to (15), (17)}] = 0. \end{aligned} \tag{29}$$

(II) Now we prove that assumptions (23)–(25) imply (13) and (16). Due to equivalence of (13) and (14) with (23), it remains to be shown that (23)–(25) imply (15) and (16).

If (24) holds, then, from computations in (27), we see that

$$\begin{vmatrix} a_{11} & b_{12}^l \\ a_{21} & b_{22}^l \end{vmatrix} + \begin{vmatrix} b_{11}^l & a_{12} \\ b_{21}^l & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11}^l & b_{12}^l \\ b_{21}^l & b_{22}^l \end{vmatrix} = 0, \tag{30}$$

and because of (23) we get (15).

Finally, we show that (23) and (25) imply (16). From (29) (using (23)) we get

$$\det\left(B^l + B^v\right) = \begin{vmatrix} b_{11}^l & b_{12}^v \\ b_{21}^l & b_{22}^v \end{vmatrix} + \begin{vmatrix} b_{11}^v & b_{12}^l \\ b_{21}^v & b_{22}^l \end{vmatrix} = 0, \tag{31}$$

that is, (16) holds. □

1.4. Problem under Consideration. The aim of this paper is to give explicit formulas for solutions of weakly delayed systems and to show that, after several steps, the dimension of the space of all solutions, being initially equal to the dimension $2(m_n + 1)$ of the space of initial data (3) generated by discrete functions φ, is reduced to a dimension less than the initial one on an interval of the form $\mathbb{Z}_s^\infty$ with an $s > 0$. In other words, we will show that the $2(m_n + 1)$-dimensional space of all solutions of (1) is pasted to a less-dimensional space of solutions on $\mathbb{Z}_s^\infty$. This problem is solved directly by explicitly computing the corresponding solutions of the Cauchy problems with each of the cases arising being considered. The underlying idea for such investigation is simple. If (1) is a weakly delayed system, then the corresponding characteristic equation has only two eigenvalues instead of $2(m_n + 1)$ eigenvalues in the case of systems with nonweak delays. This explains why the dimension of the space of solutions becomes less than the initial one. The final results (Theorems 10–13) provide the dimension of the space of solutions. Our results generalize the results in [1, 2], where system (1) with $n = 1$ and $n = 2$ was analyzed.

1.5. Auxiliary Formula. For the reader's convenience, we recall one explicit formula (see, e.g., [3]) for the solutions of linear scalar discrete nondelayed equations used in this paper. We consider initial-value problem for the first order linear discrete nonhomogeneous equation

$$w(k+1) = aw(k) + g(k), \quad w(k_0) = w_0, \quad k \in \mathbb{Z}_{k_0}^\infty, \tag{32}$$

with $a \in \mathbb{C}$ and $g : \mathbb{Z}_{k_0}^{\infty} \to \mathbb{C}$. Then, it is easy to verify that unique solution of this problem is

$$w(k) = a^{k-k_0} w_0 + \sum_{r=k_0}^{k-1} a^{k-1-r} g(r), \quad k \in \mathbb{Z}_{k_0+1}^{\infty}. \tag{33}$$

Throughout the paper, we adopt the customary notation for the sum: $\sum_{i=\ell+s}^{\ell} \mathscr{F}(i) = 0$, where ℓ is an integer, s is a positive integer, and "$\mathscr{F}$" denotes the function considered independently of whether it is defined for indicated arguments or not.

Note that the formula (33) is used many times in recent literature to analyze asymptotic properties of solutions of various classes of difference equations, including nonlinear equations. We refer, for example, to [4–8] and to relevant references therein.

2. General Solution of Weakly Delayed System

If (8) holds, then (5) and (7) have only two (and the same) roots simultaneously. In order to prove the properties of the family of solutions of (1) formulated in the introduction, we will discuss each combination of roots, that is, the cases of two real and distinct roots, a pair of complex conjugate roots, and, finally, a double real root.

Although computations in Sections 1.2 and 1.3 were performed under assumption that $\lambda \neq 0$, results of this part remain valid also if one or both roots of characteristic equation (7) are zero.

2.1. Jordan Forms of the Matrix A and Corresponding Solutions of Problem (1) *and* (3). It is known that, for every matrix A, there exists a nonsingular matrix S transforming it to the corresponding Jordan matrix form Λ. This means that

$$\Lambda = S^{-1} A S, \tag{34}$$

where Λ has the following four possible forms (denoted below as $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$), depending on the roots of the characteristic equation (7), that is, on the roots of

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{35}$$

If (35) has two real distinct roots λ_1, λ_2, then

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{36}$$

if the roots are complex conjugate, that is, $\lambda_{1,2} = p \pm iq$ with $q \neq 0$, then

$$\Lambda_2 = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \tag{37}$$

and, finally, in the case of one double real root $\lambda_{1,2} = \lambda$, we have either

$$\Lambda_3 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{38}$$

or

$$\Lambda_4 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{39}$$

The transformation $y(k) = S^{-1}x(k)$ transforms (1) into a system

$$y(k+1) = \Lambda y(k) + \sum_{l=1}^{n} B^{*l} y(k - m_l), \quad k \in \mathbb{Z}_0^{\infty} \tag{40}$$

with $B^{*l} = S^{-1}B^l S$, $B^{*l} = (b_{ij}^{*l})$, $l = 1, \ldots, n$, and $i, j = 1, 2$. Together with (40), we consider an initial problem

$$y(k) = \varphi^*(k), \tag{41}$$

$k \in \mathbb{Z}_{-m_n}^0$ with $\varphi^* : \mathbb{Z}_{-m_n}^0 \to \mathbb{R}^2$ where $\varphi^*(k) = S^{-1}\varphi(k)$ is the initial function corresponding to the initial function φ in (3).

Next, we consider all four possible cases (36)–(39) separately.

We define

$$\Phi_1(k) := (0, \varphi_1^*(k))^T, \qquad \Phi_2(k) := (\varphi_2^*(k), 0)^T, \qquad k \in \mathbb{Z}_{-m_n}^0. \tag{42}$$

Assuming that (1) is a weakly delayed system, by Lemma 2, the system (40) is weakly delayed system again.

2.1.1. Case (36) *of Two Real Distinct Roots.* In this case, we have $\Lambda = \Lambda_1$ and $\Lambda_1^k = \operatorname{diag}(\lambda_1^k, \lambda_2^k)$. The necessary and sufficient conditions (13)–(16) for (40) turn into

$$b_{11}^{*l} + b_{22}^{*l} = 0, \tag{43}$$

$$\begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} = b_{11}^{*l} b_{22}^{*l} - b_{12}^{*l} b_{21}^{*l} = 0, \tag{44}$$

$$\begin{vmatrix} \lambda_1 & 0 \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ 0 & \lambda_2 \end{vmatrix} = \lambda_1 b_{22}^{*l} + \lambda_2 b_{11}^{*l} = 0, \tag{45}$$

$$\begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ b_{21}^{*\nu} & b_{22}^{*\nu} \end{vmatrix} + \begin{vmatrix} b_{11}^{*\nu} & b_{12}^{*\nu} \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} = 0. \tag{46}$$

Since $\lambda_1 \neq \lambda_2$, (43) and (45) yield $b_{11}^{*l} = b_{22}^{*l} = 0$, then, from (44), we get $b_{12}^{*l} b_{21}^{*l} = 0$, so that either $b_{21}^{*l} = 0$ or $b_{12}^{*l} = 0$. In view of assumptions $B^l \neq \Theta$, $l = 1, 2, \ldots, n$, we conclude that only the following cases I, II are possible:

(I) $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$, $l = 1, 2, \ldots, n$,

(II) $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = 0$, $b_{21}^{*l} \neq 0$, $l = 1, 2, \ldots, n$.

In Theorem 5 both cases I, II are analyzed.

Theorem 5. *Let* (1) *be a weakly delayed system and* (35) *has two real distinct roots* λ_1, λ_2. *If case (I) holds, then the solution*

of the initial problem (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\ \Lambda_1^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \left[\sum_{l=1}^{n} b_{12}^{*l} \Phi_2(r - m_l) \right] \\ \qquad \text{if } k \in \mathbb{Z}_1^{m_1+1}, \\ \vdots \\ \Lambda_1^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \left[\sum_{l=s+1}^{n} b_{12}^{*l} \Phi_2(r - m_l) \right] \\ \quad + \sum_{l=1}^{s} b_{12}^{*l} \left[\sum_{r=0}^{m_l} \lambda_1^{k-1-r} \Phi_2(r - m_l) \right. \\ \qquad \left. + \Phi_2(0) \sum_{r=m_l+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_l} \right] \\ \qquad \text{if } k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \\ \qquad s = 1, 2, \ldots, n-1, \\ \vdots \\ \Lambda_1^k \varphi^*(0) + \sum_{l=1}^{n} b_{12}^{*l} \left[\sum_{r=0}^{m_l} \lambda_1^{k-1-r} \Phi_2(r - m_l) \right. \\ \qquad \left. + \Phi_2(0) \sum_{r=m_l+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_l} \right] \\ \qquad \text{if } k \in \mathbb{Z}_{m_n+2}^{\infty}. \end{cases} \tag{47}$$

If case (II) is true, then the solution of initial problem (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \qquad \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\ \Lambda_1^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda_2^{k-1-r} \left[\sum_{l=1}^{n} b_{21}^{*l} \Phi_1(r - m_l) \right] \\ \qquad \text{if } k \in \mathbb{Z}_1^{m_1+1}, \\ \vdots \\ \Lambda_1^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda_2^{k-1-r} \left[\sum_{l=s+1}^{n} b_{21}^{*l} \Phi_1(r - m_l) \right] \\ \quad + \sum_{l=1}^{s} b_{21}^{*l} \left[\sum_{r=0}^{m_l} \lambda_2^{k-1-r} \Phi_1(r - m_l) \right. \\ \qquad \left. + \Phi_1(0) \sum_{r=m_l+1}^{k-1} \lambda_1^{r-m_l} \lambda_2^{k-1-r} \right] \\ \qquad \text{if } k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \\ \qquad s = 1, 2, \ldots, n-1, \\ \vdots \\ \Lambda_1^k \varphi^*(0) + \sum_{l=1}^{n} b_{21}^{*l} \left[\sum_{r=0}^{m_l} \lambda_2^{k-1-r} \Phi_1(r - m_l) \right. \\ \qquad \left. + \Phi_1(0) \sum_{r=m_l+1}^{k-1} \lambda_1^{r-m_l} \lambda_2^{k-1-r} \right] \\ \qquad \text{if } k \in \mathbb{Z}_{m_n+2}^{\infty}. \end{cases} \tag{48}$$

Proof. If case (I) is true, then the transformed system (40) takes the form

$$y_1(k+1) = \lambda_1 y_1(k) + \sum_{l=1}^{n} b_{12}^{*l} y_2(k - m_l), \tag{49}$$

$$y_2(k+1) = \lambda_2 y_2(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{50}$$

and if case (II) holds, then (40) takes the form

$$y_1(k+1) = \lambda_1 y_1(k), \tag{51}$$

$$y_2(k+1) = \lambda_2 y_2(k) + \sum_{l=1}^{n} b_{21}^{*l} y_1(k - m_l), \quad k \in \mathbb{Z}_0^{\infty}. \tag{52}$$

We investigate only the initial problem (49), (50), (41) since the initial problem (51), (52), (41) can be examined in a similar way.

From (50), (41), we get

$$y_2(k) = \begin{cases} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\ \lambda_2^k \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_1^{\infty}, \end{cases} \tag{53}$$

then (49) becomes

$$y_1(k+1) = \begin{cases} \lambda_1 y_1(k) + \sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) & \text{if } k \in \mathbb{Z}_0^{m_1}, \\ \lambda_1 y_1(k) + b_{12}^{*1} \lambda_2^{k-m_1} \varphi_2^*(0) \\ \quad + \sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) & \text{if } k \in \mathbb{Z}_{m_1+1}^{m_2}, \\ \lambda_1 y_1(k) + \sum_{l=1}^{2} b_{12}^{*l} \lambda_2^{k-m_l} \varphi_2^*(0) \\ \quad + \sum_{l=3}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) & \text{if } k \in \mathbb{Z}_{m_2+1}^{m_3}, \\ \vdots \\ \lambda_1 y_1(k) + \sum_{l=1}^{s} b_{12}^{*l} \lambda_2^{k-m_l} \varphi_2^*(0) \\ \quad + \sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) & \text{if } k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}, \\ & s = 3, 4, \ldots, n-1, \\ \vdots \\ \lambda_1 y_1(k) + \sum_{l=1}^{n} b_{12}^{*l} \lambda_2^{k-m_l} \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_{m_n+1}^{\infty}. \end{cases} \tag{54}$$

First, we solve this equation for $k \in \mathbb{Z}_0^{m_1}$. This means that we consider the problem

$$\begin{aligned} y_1(k+1) &= \lambda_1 y_1(k) \\ &\quad + \sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(k-m_l), \quad k \in \mathbb{Z}_0^{m_1}, \\ y_1(0) &= \varphi_1^*(0). \end{aligned} \tag{55}$$

With the aid of formula (33), we get

$$y_1(k) = \lambda_1^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \left[\sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right], \quad k \in \mathbb{Z}_1^{m_1+1}. \tag{56}$$

Now we solve (54) for $k \in \mathbb{Z}_{m_1+1}^{m_2}$ with initial data deduced from (56); that is, we consider the problem

$$\begin{aligned} y_1(k+1) &= \lambda_1 y_1(k) + b_{12}^{*1} \lambda_2^{k-m_1} \varphi_2^*(0) \\ &\quad + \sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(k-m_l), \quad k \in \mathbb{Z}_{m_1+1}^{m_2}, \\ y_1(m_1+1) &= \lambda_1^{m_1+1} \varphi_1^*(0) + \sum_{r=0}^{m_1} \lambda_1^{m_1-r} \left[\sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right]. \end{aligned} \tag{57}$$

Applying formula (33) we get (for $k \in \mathbb{Z}_{m_1+2}^{m_2+1}$)

$$\begin{aligned} y_1(k) &= \lambda_1^{k-(m_1+1)} y_1(m_1+1) \\ &\quad + \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r} \left[b_{12}^{*1} \lambda_2^{r-m_1} \varphi_2^*(0) \right. \\ &\qquad \left. + \sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &= \lambda_1^{k-m_1-1} \left[\lambda_1^{m_1+1} \varphi_1^*(0) \right. \\ &\qquad \left. + \sum_{r=0}^{m_1} \lambda_1^{m_1-r} \left[\sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \right] \\ &\quad + \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r} \left[b_{12}^{*1} \lambda_2^{r-m_1} \varphi_2^*(0) \right. \\ &\qquad \left. + \sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &= \lambda_1^k \varphi_1^*(0) + \sum_{r=0}^{m_1} \lambda_1^{k-1-r} \left[\sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &\quad + \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r} \left[b_{12}^{*1} \lambda_2^{r-m_1} \varphi_2^*(0) \right. \\ &\qquad \left. + \sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &= \lambda_1^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \left[\sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &\quad + b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda_1^{k-1-r} \varphi_2^*(r-m_1) \right. \\ &\qquad \left. + \varphi_2^*(0) \sum_{r=m_1+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m_1} \right]. \end{aligned} \tag{58}$$

Now we solve (54) for $k \in \mathbb{Z}_{m_2+1}^{m_3}$ with initial data deduced from (58); that is, we consider the problem

$$\begin{aligned} y_1(k+1) &= \lambda_1 y_1(k) + \sum_{l=1}^{2} b_{12}^{*l} \lambda_2^{k-m_l} \varphi_2^*(0) \\ &\quad + \sum_{l=3}^{n} b_{12}^{*l} \varphi_2^*(k-m_l), \quad k \in \mathbb{Z}_{m_2+1}^{m_3}, \\ y_1(m_2+1) &= \lambda_1^{m_2+1} \varphi_1^*(0) + \sum_{r=0}^{m_2} \lambda_1^{m_2-r} \left[\sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &\quad + b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda_1^{m_2-r} \varphi_2^*(r-m_1) \right. \\ &\qquad \left. + \varphi_2^*(0) \sum_{r=m_1+1}^{m_2} \lambda_1^{m_2-r} \lambda_2^{r-m_1} \right]. \end{aligned} \tag{59}$$

Applying formula (33) yields (for $k \in \mathbb{Z}_{m_2+2}^{m_3+1}$)

$$\begin{aligned} &y_1(k) \\ &= \lambda_1^{k-(m_2+1)} y_1(m_2+1) \\ &\quad + \sum_{r=m_2+1}^{k-1} \lambda_1^{k-1-r} \left[\sum_{l=1}^{2} b_{12}^{*l} \lambda_2^{r-m_l} \varphi_2^*(0) + \sum_{l=3}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\ &= \lambda_1^{k-m_2-1} \left[\lambda_1^{m_2+1} \varphi_1^*(0) + \sum_{r=0}^{m_2} \lambda_1^{m_2-r} \left[\sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \right. \end{aligned}$$

$$
\begin{aligned}
&+ b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{m_2-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{m_2}\lambda_1^{m_2-r}\lambda_2^{r-m_1}\right]\Bigg]\\
&+\sum_{r=m_2+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{2}b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)+\sum_{l=3}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{m_2}\lambda_1^{k-1-r}\left[\sum_{l=2}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{k-1-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{m_2}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
&+\sum_{r=m_2+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{2}b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)\right.\\
&\qquad\left.+\sum_{l=3}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=3}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{k-1-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
&+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda_1^{k-1-r}\varphi_2^*(r-m_2)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_2+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_2}\right].
\end{aligned}
\tag{60}
$$

From (56), (58), and (60) we deduce that expected form of the solution of the initial problem for $k\in\mathbb{Z}_{m_{s-1}+1}^{m_s}$ with initial data derived from the solution of previous equation for $k\in\mathbb{Z}_{m_{s-2}+1}^{m_{s-1}}$ is

$$
\begin{aligned}
y_1(k)&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=s}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&\quad+\sum_{l=1}^{s-1}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right]\\
&\qquad\text{if } k\in\mathbb{Z}_{m_{s-1}+2}^{m_s+1}.
\end{aligned}
\tag{61}
$$

We solve (54) for $k\in\mathbb{Z}_{m_s+1}^{m_{s+1}}$ with initial data deduced from (61); that is, we consider the problem

$$
\begin{aligned}
y_1(k+1)&=\lambda_1y_1(k)+\sum_{l=1}^{s}b_{12}^{*l}\lambda_2^{k-m_l}\varphi_2^*(0)\\
&\quad+\sum_{l=s+1}^{n}b_{12}^{*l}\varphi_2^*(k-m_l),\quad k\in\mathbb{Z}_{m_s+1}^{m_{s+1}},\\
y_1(m_s+1)&=\lambda_1^{m_s+1}\varphi_1^*(0)+\sum_{r=0}^{m_s}\lambda_1^{m_s-r}\left[\sum_{l=s}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&\quad+\sum_{l=1}^{s-1}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{m_s-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{m_s}\lambda_1^{m_s-r}\lambda_2^{r-m_l}\right].
\end{aligned}
\tag{62}
$$

Applying formula (33) yields (for $k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1}$)

$$
\begin{aligned}
&y_1(k)\\
&=\lambda_1^{k-(m_s+1)}y_1(m_s+1)\\
&\quad+\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{s}b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)+\sum_{l=s+1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&=\lambda_1^{k-(m_s+1)}\left[\lambda_1^{m_s+1}\varphi_1^*(0)+\sum_{r=0}^{m_s}\lambda_1^{m_s-r}\left[\sum_{l=s}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\right.\\
&\qquad+\sum_{l=1}^{s-1}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{m_s-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{m_s}\lambda_1^{m_s-r}\lambda_2^{r-m_l}\right]\right]\\
&\quad+\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{s}b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)\right.\\
&\qquad\left.+\sum_{l=s+1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{m_s}\lambda_1^{k-1-r}\left[\sum_{l=s}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{s-1} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right]\\
&+\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{s} b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)\right.\\
&\qquad\left.+\sum_{l=s+1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{m_s}\lambda_1^{k-1-r}\left[b_{12}^{*s}\varphi_2^*(r-m_s)\right.\\
&\qquad\left.+\sum_{l=s+1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{k-1-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
&+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda_1^{k-1-r}\varphi_2^*(r-m_2)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_2+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_2}\right]\\
&+\cdots\\
&+b_{12}^{*s-1}\left[\sum_{r=0}^{m_{s-1}}\lambda_1^{k-1-r}\varphi_2^*(r-m_{s-1})\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_{s-1}+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_{s-1}}\right]\\
&+b_{12}^{*1}\left[\varphi_2^*(0)\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
&+b_{12}^{*2}\left[\varphi_2^*(0)\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_2}\right]\\
&+\cdots\\
&+b_{12}^{*s}\left[\varphi_2^*(0)\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_s}\right]\\
&+\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=s+1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=s+1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&+\sum_{l=1}^{s} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right].
\end{aligned}
\tag{63}
$$

In the end we solve (54) for $k\in\mathbb{Z}_{m_n+1}^{\infty}$ with initial data deduced from (63); that is, we consider the problem

$$
\begin{aligned}
&y_1(k+1)=\lambda_1 y_1(k)+\sum_{l=1}^{n} b_{12}^{*l}\lambda_2^{k-m_l}\varphi_2^*(0),\quad k\in\mathbb{Z}_{m_n+1}^{\infty},\\
&y_1(m_n+1)=\lambda_1^{m_n+1}\varphi_1^*(0)+\sum_{r=0}^{m_n}\lambda_1^{m_n-r}b_{12}^{*n}\varphi_2^*(r-m_n)\\
&\qquad+\sum_{l=1}^{n-1} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{m_n-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{m_n}\lambda_1^{m_n-r}\lambda_2^{r-m_l}\right].
\end{aligned}
\tag{64}
$$

Applying formula (33) yields (for $k\in\mathbb{Z}_{m_n+2}^{\infty}$)

$$
\begin{aligned}
y_1(k)&=\lambda_1^{k-(m_n+1)}y_1(m_n+1)\\
&\quad+\sum_{r=m_n+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)\right]\\
&=\lambda_1^{k-(m_n+1)}\left[\lambda_1^{m_n+1}\varphi_1^*(0)+\sum_{r=0}^{m_n}\lambda_1^{m_n-r}b_{12}^{*n}\varphi_2^*(r-m_n)\right.\\
&\qquad+\sum_{l=1}^{n-1} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{m_n-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\qquad\left.\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{m_n}\lambda_1^{m_n-r}\lambda_2^{r-m_l}\right]\right]\\
&\quad+\sum_{r=m_n+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{m_n}\lambda_1^{k-1-r}b_{12}^{*n}\varphi_2^*(r-m_n)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{n-1} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{m_n}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right]\\
&+\sum_{r=m_n+1}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\lambda_2^{r-m_l}\varphi_2^*(0)\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{m_n}\lambda_1^{k-1-r}b_{12}^{*n}\varphi_2^*(r-m_n)\\
&+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{k-1-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
&+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda_1^{k-1-r}\varphi_2^*(r-m_2)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_2+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_2}\right]\\
&+\cdots\\
&+b_{12}^{*n-1}\left[\sum_{r=0}^{m_{n-1}}\lambda_1^{k-1-r}\varphi_2^*(r-m_{n-1})\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_{n-1}+1}^{m_s}\lambda_1^{k-1-r}\lambda_2^{r-m_{n-1}}\right]\\
&+b_{12}^{*1}\left[\varphi_2^*(0)\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
&+b_{12}^{*2}\left[\varphi_2^*(0)\sum_{r=m_s+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_2}\right]\\
&+\cdots\\
&+b_{12}^{*n}\left[\varphi_2^*(0)\sum_{r=m_n+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_n}\right]\\
&=\lambda_1^k\varphi_1^*(0)+\sum_{l=1}^{n} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right].
\end{aligned}
\tag{65}
$$

Summing up all particular cases (56)–(65) we have

$$
y_1(k)=\begin{cases}
\varphi_1^*(k) & \text{if } k\in\mathbb{Z}_{-m_n}^{0},\\
\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] & \text{if } k\in\mathbb{Z}_{1}^{m_1+1},\\
\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
\quad+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{k-1-r}\varphi_2^*(r-m_1)\right.\\
\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right] & \text{if } k\in\mathbb{Z}_{m_1+2}^{m_2+1},\\
\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=3}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
\quad+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda_1^{k-1-r}\varphi_2^*(r-m_1)\right.\\
\qquad\left.+\varphi_2^*(0)\sum_{r=m_1+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_1}\right]\\
\quad+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda_1^{k-1-r}\varphi_2^*(r-m_2)\right.\\
\qquad\left.+\varphi_2^*(0)\sum_{r=m_2+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_2}\right] & \text{if } k\in\mathbb{Z}_{m_2+2}^{m_3+1},\\
\vdots\\
\lambda_1^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda_1^{k-1-r}\left[\sum_{l=s+1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
\quad+\sum_{l=1}^{s} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right] & \text{if } k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1},\\
\vdots\\
\lambda_1^k\varphi_1^*(0)\\
\quad+\sum_{l=1}^{n} b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda_1^{k-1-r}\varphi_2^*(r-m_l)\right.\\
\qquad\left.+\varphi_2^*(0)\sum_{r=m_l+1}^{k-1}\lambda_1^{k-1-r}\lambda_2^{r-m_l}\right] & \text{if } k\in\mathbb{Z}_{m_n+2}^{\infty}.
\end{cases}
\tag{66}
$$

Now, taking into account (42), formula (47) is a consequence of (53) and (66). Formula (48) can be proved in a similar way.

Finally, we note that both formulas (47), (48) remain valid for $b_{12}^{*l}=b_{21}^{*l}=0$. In this case, the transformed system

(1) reduces to a system without delays. This possibility is excluded by condition (2). □

2.1.2. Case (37) *of Two Complex Conjugate Roots.* The necessary and sufficient conditions (13)–(16) take the forms (43), (44), (46), and

$$\begin{vmatrix} p & q \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ -q & p \end{vmatrix} = p\left(b_{11}^{*l} + b_{22}^{*l}\right) + q\left(b_{12}^{*l} - b_{21}^{*l}\right) = 0, \tag{67}$$

where $l, v = 1, 2, \ldots, n$ and $v > l$.

The system of conditions (43), (44), and (67) gives $b_{12}^{*l} = b_{21}^{*l}$, $(b_{11}^{*l})^2 = -(b_{12}^{*l})^2$ and admits only one possibility; namely,

$$b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = b_{21}^{*l} = 0. \tag{68}$$

Consequently, $B^{*l} = \Theta$, $B^l = \Theta$.

The initial problem (1), (3) reduces to a problem without delay

$$\begin{gathered} x(k+1) = Ax(k), \\ x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m_n}^0 \end{gathered} \tag{69}$$

and, obviously,

$$x(k) = \begin{cases} \varphi(k) & \text{if } k \in \mathbb{Z}_{-m_n}^0, \\ A^k \varphi(0) & \text{if } k \in \mathbb{Z}_1^\infty. \end{cases} \tag{70}$$

From this discussion, the next theorem follows.

Theorem 6. *There exists no weakly delayed system* (1) *if* Λ *has the form* (37).

Finally, we note that the assumption (2) alone excludes this case.

2.1.3. Case (38) *of Double Real Root.* In this case we have $\Lambda = \Lambda_3$ and $\Lambda_3^k = \operatorname{diag}(\lambda^k, \lambda^k)$. For (40), the necessary and sufficient conditions (13)–(16) are reduced to (43), (44), (46), and

$$\begin{vmatrix} \lambda & 0 \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ 0 & \lambda \end{vmatrix} = \lambda\left(b_{11}^{*l} + b_{22}^{*l}\right) = 0, \tag{71}$$

where $l = 1, 2, \ldots, n$.

From (43), (44), and (71), we get $b_{12}^{*l} b_{21}^{*l} = -(b_{11}^{*l})^2$. From the condition (46) we get

$$b_{11}^{*l} b_{22}^{*v} - b_{12}^{*l} b_{21}^{*v} + b_{22}^{*l} b_{11}^{*v} - b_{21}^{*l} b_{12}^{*v} = 0, \tag{72}$$

where $l, v = 1, 2, \ldots, n$ and $v > l$. Multiplying (72) by $b_{12}^{*l} b_{12}^{*v}$, we have

$$\begin{gathered} b_{11}^{*l} b_{22}^{*v} b_{12}^{*l} b_{12}^{*v} - \left(b_{12}^{*l}\right)^2 b_{21}^{*v} b_{12}^{*v} \\ + b_{22}^{*l} b_{11}^{*v} b_{12}^{*l} b_{12}^{*v} - b_{21}^{*l} b_{12}^{*l} \left(b_{12}^{*v}\right)^2 = 0. \end{gathered} \tag{73}$$

Substituting $b_{12}^{*l} b_{21}^{*l} = -(b_{11}^{*l})^2$, $b_{12}^{*v} b_{21}^{*v} = -(b_{11}^{*v})^2$ into (73) and using (43) we obtain

$$\begin{gathered} -b_{11}^{*l} b_{11}^{*v} b_{12}^{*l} b_{12}^{*v} + \left(b_{12}^{*l}\right)^2 \left(b_{11}^{*v}\right)^2 \\ -b_{11}^{*l} b_{11}^{*v} b_{12}^{*l} b_{12}^{*v} + \left(b_{11}^{*l}\right)^2 \left(b_{12}^{*v}\right)^2 = 0. \end{gathered} \tag{74}$$

The equation (74) can be written as

$$\begin{gathered} \left(b_{12}^{*l} b_{11}^{*v} - b_{12}^{*v} b_{11}^{*l}\right)^2 = 0, \\ b_{12}^{*l} b_{11}^{*v} = b_{12}^{*v} b_{11}^{*l}. \end{gathered} \tag{75}$$

Now we will analyse the two possible cases: $b_{12}^{*l} b_{21}^{*l} = 0$ and $b_{12}^{*l} b_{21}^{*l} \neq 0$.

For the case $b_{12}^{*l} b_{21}^{*l} = 0$, we have from (43), (44) that $b_{11}^{*l} = b_{22}^{*l} = 0$ and $b_{12}^{*l} = 0$ or $b_{21}^{*l} = 0$. For $b_{12}^{*l} = 0$ and $b_{21}^{*l} \neq 0$, condition (46) gives $b_{12}^{*v} = 0$, where $l, v = 1, 2, \ldots, n$ and $v > l$. Then, from (43), (44) for $l = v$, we get $b_{11}^{*v} = b_{22}^{*v} = 0$ and $b_{21}^{*v} \neq 0$.

For $b_{21}^{*l} = 0$ and $b_{12}^{*l} \neq 0$, condition (46) gives $b_{21}^{*v} = 0$, where $l, v = 1, 2, \ldots, n$ and $v > l$, then, from (43), (44) for $l = v$, we get $b_{11}^{*v} = b_{22}^{*v} = 0$ and $b_{12}^{*v} \neq 0$.

Now we discuss the case $b_{12}^{*l} b_{21}^{*l} \neq 0$. From conditions (43), (44), we have $b_{12}^{*l} b_{21}^{*l} = -(b_{11}^{*l})^2$ and $b_{11}^{*l} b_{22}^{*l} \neq 0$. This yields $b_{11}^{*l} \neq 0$, $b_{22}^{*l} \neq 0$ and, from (75), we have $b_{11}^{*v} \neq 0$, $b_{12}^{*v} \neq 0$. By conditions (43), (44) for $v = l$, we get $b_{22}^{*v} \neq 0$, $b_{21}^{*v} \neq 0$.

From the assumptions $B^l \neq \Theta$, we conclude that only the following cases ((I), (II), (III)) are possible:

(I) $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$,

(II) $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = 0$, $b_{21}^{*l} \neq 0$,

(III) $b_{12}^{*l} b_{21}^{*l} \neq 0$,

where $l = 1, 2, \ldots, n$.

2.1.4. Case $b_{12}^{*l} b_{21}^{*l} = 0$

Theorem 7. *Let* (1) *be a weakly delayed system,* (35) *has a twofold root* $\lambda_{1,2} = \lambda$, $b_{12}^{*l} b_{21}^{*l} = 0$ *and the matrix* Λ *has the form* (38). *Then the solution of the initial problem* (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^\infty$, *where in case* $b_{21}^{*l} = 0$, $y(k)$ *has the form*

$$y(k)=\begin{cases}\varphi^*(k) & \text{if } k\in\mathbb{Z}_{-m_n}^{0},\\ \Lambda_3^k\varphi^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=1}^{n}b_{12}^{*l}\Phi_2(r-m_l)\right] & \text{if } k\in\mathbb{Z}_{1}^{m_1+1},\\ \vdots & \\ \Lambda_3^k\varphi^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=s+1}^{n}b_{12}^{*l}\Phi_2(r-m_l)\right] & \\ \quad+\sum_{l=1}^{s}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\Phi_2(r-m_l)\right. & \\ \qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\Phi_2(0)\right] & \\ & \text{if } k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1},\\ & s=1,2,\ldots,n-1,\\ \vdots & \\ \Lambda_3^k\varphi^*(0) & \\ \quad+\sum_{l=1}^{n}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\Phi_2(r-m_l)\right. & \\ \qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\Phi_2(0)\right] & \\ & \text{if } k\in\mathbb{Z}_{m_n+2}^{\infty}.\end{cases}\tag{76}$$

If $b_{12}^{*l}=0$ is true then the solution of initial problem (1), (3) *is $x(k)=Sy(k)$, $k\in\mathbb{Z}_{-m_n}^{\infty}$, where $y(k)$ has the form*

$$y(k)=\begin{cases}\varphi^*(k) & \text{if } k\in\mathbb{Z}_{-m_n}^{0},\\ \Lambda_3^k\varphi^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=1}^{n}b_{21}^{*l}\Phi_1(r-m_l)\right] & \text{if } k\in\mathbb{Z}_{1}^{m_1+1},\\ \vdots & \\ \Lambda_3^k\varphi^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=s+1}^{n}b_{21}^{*l}\Phi_1(r-m_l)\right] & \\ \quad+\sum_{l=1}^{s}b_{21}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\Phi_1(r-m_l)\right. & \\ \qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\Phi_1(0)\right] & \\ & \text{if } k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1},\\ & s=1,2,\ldots,n-1,\\ \vdots & \\ \Lambda_3^k\varphi^*(0) & \\ \quad+\sum_{l=1}^{n}b_{21}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\Phi_1(r-m_l)\right. & \\ \qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\Phi_1(0)\right] & \\ & \text{if } k\in\mathbb{Z}_{m_n+2}^{\infty}.\end{cases}\tag{77}$$

Proof. Case (I) means that $b_{12}^{*l}\neq 0$. Then (40) turns into the system

$$\begin{aligned}y_1(k+1)&=\lambda y_1(k)+\sum_{l=1}^{n}b_{12}^{*l}y_2(k-m_l),\quad k\in\mathbb{Z}_0^{\infty}\\ y_2(k+1)&=\lambda y_2(k),\end{aligned}\tag{78}$$

and, if $b_{21}^{*l}\neq 0$, (40) turns into the system

$$\begin{aligned}y_1(k+1)&=\lambda y_1(k),\\ y_2(k+1)&=\lambda y_2(k)+\sum_{l=1}^{n}b_{21}^{*l}y_1(k-m_l),\quad k\in\mathbb{Z}_0^{\infty}.\end{aligned}\tag{79}$$

System (78) can be solved in much the same way as the systems (49), (50) if we put $\lambda_1=\lambda_2=\lambda$, and the discussion of the system (79) goes along the same lines as that of the systems (51), (52) with $\lambda_1=\lambda_2=\lambda$. Formulas (76) and (77) are consequences of (47), (48). □

2.1.5. Case $b_{12}^{*l}b_{21}^{*l}\neq 0$. For $k\in\mathbb{Z}_{-m_n}^{0}$, we define

$$\Phi_l^*(k):=\left(b_{11}^{*l}\left[\varphi_1^*(k)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k)\right],\right.$$
$$\left.-\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left[\varphi_1^*(k)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k)\right]\right)^T.\tag{80}$$

Theorem 8. *Let system* (1) *be a weakly delayed system,* (35) *admits two repeated roots $\lambda_{1,2}=\lambda$, $b_{12}^{*l}b_{21}^{*l}\neq 0$ and the matrix Λ_3 has the form* (38). *Then the solution of the initial problem*

(1), (3) *is given by* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m_n}^{\infty}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\ \Lambda_3^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum_{l=1}^{n} \Phi_l^*(r-m_l) \right] & \text{if } k \in \mathbb{Z}_1^{m_1+1}, \\ \vdots & \\ \Lambda_3^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum_{l=s+1}^{n} \Phi_l^*(r-m_l) \right] & \\ \quad + \sum_{l=1}^{s} \left[\sum_{r=0}^{m_l} \lambda^{k-1-r} \Phi_l^*(r-m_l) \right. & \\ \quad \left. + (k-1-m_l) \lambda^{k-1-m_l} \Phi_l^*(0) \right] & \text{if } k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \\ & s = 1, 2, \ldots, n-1, \\ \vdots & \\ \Lambda_3^k \varphi^*(0) & \\ \quad + \sum_{l=1}^{n} \left[\sum_{r=0}^{m_l} \lambda^{k-1-r} \Phi_l^*(r-m_l) \right. & \\ \quad \left. + (k-1-m_l) \lambda^{k-1-m_l} \Phi_l^*(0) \right] & \\ & \text{if } k \in \mathbb{Z}_{m_n+2}^{\infty}. \end{cases} \tag{81}$$

Proof. In this case, all the entries of B^{*l} are nonzero and, from (43), (44), and (71), we get

$$B^* = \begin{pmatrix} b_{11}^{*l} & b_{12}^{*l} \\ \dfrac{-\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} & -b_{11}^{*l} \end{pmatrix}, \tag{82}$$

where $l = 1, 2, \ldots, n$, then, the system (40) reduces to

$$y_1(k+1) = \lambda y_1(k) + \sum_{l=1}^{n} \left[b_{11}^{*l} y_1(k-m_l) + b_{12}^{*l} y_2(k-m_l) \right], \tag{83}$$

$$y_2(k+1) = \lambda y_2(k) - \sum_{l=1}^{n} \left[\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} y_1(k-m_l) + b_{11}^{*l} y_2(k-m_l) \right], \tag{84}$$

where $k \in \mathbb{Z}_0^{\infty}$. It is easy to see (multiplying (84) by b_{12}^{*1}/b_{11}^{*1} and summing both equations) that

$$y_1(k+1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} y_2(k+1) = \lambda \left[y_1(k) + \frac{b_{12}^{*1}}{b_{11}^{*1}} y_2(k) \right], \quad k \in \mathbb{Z}_0^{\infty}. \tag{85}$$

Equation (85) is a homogeneous equation with respect to the unknown expression

$$y_1(k) + \left(\frac{b_{12}^{*1}}{b_{11}^{*1}} \right) y_2(k), \tag{86}$$

then, using (33), we obtain

$$y_1(k) + \frac{b_{12}^{*1}}{b_{11}^{*1}} y_2(k) = \begin{cases} \varphi_1^*(k) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\ \lambda^k \left[\varphi_1^*(0) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] & \text{if } k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{87}$$

With the aid of (87), we rewrite the systems (83), (84) as follows:

$$y_1(k+1) = \begin{cases} \lambda y_1(k) + \sum_{l=1}^{n} b_{11}^{*l} \left[\varphi_1^*(k-m_l) \right. & \\ \quad \left. + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k-m_l) \right] & \text{if } k \in \mathbb{Z}_0^{m_1}, \\ \lambda y_1(k) + b_{11}^{*1} \lambda^{k-m_1} \left[\varphi_1^*(0) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] & \\ \quad + \sum_{l=2}^{n} b_{11}^{*l} \left[\varphi_1^*(k-m_l) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k-m_l) \right] & \text{if } k \in \mathbb{Z}_{m_1+1}^{m_2}, \\ \lambda y_1(k) + \sum_{l=1}^{2} b_{11}^{*l} \lambda^{k-m_l} \left[\varphi_1^*(0) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] & \\ \quad + \sum_{l=3}^{n} b_{11}^{*l} \left[\varphi_1^*(k-m_l) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k-m_l) \right] & \text{if } k \in \mathbb{Z}_{m_2+1}^{m_3}, \\ \vdots & \\ \lambda y_1(k) + \sum_{l=1}^{s} b_{11}^{*l} \lambda^{k-m_l} \left[\varphi_1^*(0) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] & \\ \quad + \sum_{l=s+1}^{n} b_{11}^{*l} \left[\varphi_1^*(k-m_l) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k-m_l) \right] & \text{if } k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}, \\ & s = 3, 4, \ldots, n-1, \\ \vdots & \\ \lambda y_1(k) + \sum_{l=1}^{n} b_{11}^{*l} \lambda^{k-m_l} \left[\varphi_1^*(0) + \dfrac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] & \\ & \text{if } k \in \mathbb{Z}_{m_n+1}^{\infty}, \end{cases}$$

$$y_2(k+1)=\begin{cases}\lambda y_2(k)-\sum_{l=1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\right] & \text{if } k\in\mathbb{Z}_0^{m_1},\\ \lambda y_2(k)-\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\lambda^{k-m_1}\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]-\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(k-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_1)\right] & \text{if } k\in\mathbb{Z}_{m_1+1}^{m_2},\\ \lambda y_2(k)-\sum_{l=1}^{2}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]-\sum_{l=3}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\right] & \text{if } k\in\mathbb{Z}_{m_2+1}^{m_3},\\ \vdots & \\ \lambda y_2(k)-\sum_{l=1}^{s}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]-\sum_{l=s+1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\right] & \text{if } k\in\mathbb{Z}_{m_s+1}^{m_{s+1}},\ s=3,4,\ldots,n-1,\\ \vdots & \\ \lambda y_2(k)-\sum_{l=1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\lambda^{k-m_l}\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right] & \text{if } k\in\mathbb{Z}_{m_n+1}^{\infty}.\end{cases}\tag{88}$$

First, we solve this system for $k\in\mathbb{Z}_0^{m_1}$ and consider the problems

$$\begin{aligned}y_1(k+1)&=\lambda y_1(k)+\sum_{l=1}^{n}b_{11}^{*l}\left[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\right]\\&\qquad\text{if } k\in\mathbb{Z}_0^{m_1},\\ y_1(0)&=\varphi_1^*(0),\\ y_2(k+1)&=\lambda y_2(k)-\sum_{l=1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\right]\\&\qquad\text{if } k\in\mathbb{Z}_0^{m_1},\\ y_2(0)&=\varphi_2^*(0).\end{aligned}\tag{89}$$

With the aid of formula (33), we get

$$\begin{aligned}y_1(k)&=\lambda^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\times\left(\sum_{l=1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right),\\&\qquad k\in\mathbb{Z}_1^{m_1+1},\\ y_2(k)&=\lambda^k\varphi_2^*(0)-\sum_{r=0}^{k-1}\lambda^{k-1-r}\times\left(\sum_{l=1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right),\\&\qquad k\in\mathbb{Z}_1^{m_1+1}.\end{aligned}\tag{90, 91}$$

Now we solve system (88) for $k\in\mathbb{Z}_{m_1+1}^{m_2}$; that is, we consider the problem (with initial data deduced from (90), (91))

$$\begin{aligned}y_1(k+1)&=\lambda y_1(k)+b_{11}^{*1}\lambda^{k-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]+\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\right]\\&\qquad\text{if } k\in\mathbb{Z}_{m_1+1}^{m_2},\\ y_1(m_1+1)&=\lambda^{m_1+1}\varphi_1^*(0)+\sum_{r=0}^{m_1}\lambda^{m_1-r}\times\left(\sum_{l=1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right),\end{aligned}$$

$$
\begin{aligned}
&y_2(k+1)\\
&\quad=\lambda y_2(k)-\frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}}\lambda^{k-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\\
&\qquad-\sum_{l=2}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left[\varphi_1^*(k-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_1)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{if } k\in\mathbb{Z}_{m_1+1}^{m_2},\\
&y_2(m_1+1)\\
&\quad=\lambda^{m_1+1}\varphi_2^*(0)-\sum_{r=0}^{m_1}\lambda^{m_1-r}\\
&\qquad\times\left(\sum_{l=1}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right).
\end{aligned}
\tag{92}
$$

Formula (33) yields (for $k\in\mathbb{Z}_{m_1+2}^{m_2+1}$)

$$
\begin{aligned}
&y_1(k)\\
&=\lambda^{k-(m_1+1)}y_1(m_1+1)+\sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\\
&\quad\times\left(b_{11}^{*1}\lambda^{k-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad+\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&=\lambda^{k-m_1-1}\left[\lambda^{m_1+1}\varphi_1^*(0)+\sum_{r=0}^{m_1}\lambda^{m_1-r}\right.\\
&\qquad\times\left(\sum_{l=1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\right]\\
&\quad+\sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^{*1}\lambda^{k-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\qquad+\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\\
&\qquad\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&=\lambda^k\varphi_1^*(0)+\sum_{r=0}^{m_1}\lambda^{k-1-r}\left(\sum_{l=1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+\sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^{*1}\lambda^{k-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\qquad+\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\\
&\qquad\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&=\lambda^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left(\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+\sum_{r=0}^{m_1}\lambda^{k-1-r}b_{11}^{*1}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\\
&\quad+(k-1-m_1)\,b_{11}^{*1}\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\\
&=\lambda^k\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left(\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.\\
&\qquad\qquad+(k-1-m_1)\,\lambda^{k-1-m_1}\left[\varphi_1^*(0)\right.\\
&\qquad\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right),
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
&y_2(k)\\
&=\lambda^{k-(m_1+1)}y_2(m_1+1)-\sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\\
&\quad\times\left(\frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}}\lambda^{r-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad+\sum_{l=2}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)\right.
\end{aligned}
$$

$$+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Big)$$

$$=\lambda^{k-m_1-1}\Bigg[\lambda^{m_1+1}\varphi_2^*(0)-\sum_{r=0}^{m_1}\lambda^{m_1-r}\times\Bigg(\sum_{l=1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Bigg)\Bigg]$$

$$-\sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\Bigg(\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\lambda^{r-m_1}\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]+\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Bigg)$$

$$=\lambda^k\varphi_2^*(0)-\sum_{r=0}^{m_1}\lambda^{k-1-r}\Bigg(\sum_{l=1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Bigg)$$

$$-\sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\Bigg(\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\lambda^{r-m_1}\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]+\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Bigg)$$

$$=\lambda^k\varphi_2^*(0)-\sum_{r=0}^{k-1}\lambda^{k-1-r}\Bigg(\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Bigg)$$

$$-\sum_{r=0}^{m_1}\lambda^{k-1-r}\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\Big[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\Big]$$

$$-(k-1-m_1)\lambda^{k-1-m_1}\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]$$

$$=\lambda^k\varphi_2^*(0)-\sum_{r=0}^{k-1}\lambda^{k-1-r}\Bigg(\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}b_{12}^{*l}\varphi_2^*(r-m_l)\Big]\Bigg)$$

$$-\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\Bigg(\sum_{r=0}^{m_1}\lambda^{k-1-r}\Big[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\Big]+(k-1-m_1)\lambda^{k-1-m_1}\times\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]\Bigg). \tag{94}$$

Now we solve (88) for $k\in\mathbb{Z}_{m_2+1}^{m_3}$; that is, we consider the problem (with initial data deduced from (93), (94))

$$y_1(k+1)=\lambda y_1(k)+\sum_{l=1}^{2}b_{11}^{*l}\lambda^{k-m_l}\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]+\sum_{l=3}^{n}b_{11}^{*l}\Big[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\Big]\quad\text{if } k\in\mathbb{Z}_{m_2+1}^{m_3},$$

$$y_1(m_2+1)=\lambda^{m_2+1}\varphi_1^*(0)+\sum_{r=0}^{m_2}\lambda^{m_2-r}\times\Bigg(\sum_{l=2}^{n}b_{11}^{*l}\Big[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\Big]\Bigg)+b_{11}^{*1}\Bigg(\sum_{r=0}^{m_1}\lambda^{m_2-r}\Big[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\Big]+(m_2-m_1)\lambda^{m_2-m_1}\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]\Bigg),$$

$$y_2(k+1)=\lambda y_2(k)-\sum_{l=1}^{2}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\Big]-\sum_{l=3}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\Big[\varphi_1^*(k-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(k-m_l)\Big]\quad\text{if } k\in\mathbb{Z}_{m_2+1}^{m_3},$$

$$
\begin{aligned}
&y_2(m_2+1)\\
&= \lambda^{m_2+1}\varphi_2^*(0) - \sum_{r=0}^{m_2}\lambda^{m_2-r}\\
&\quad\times\left(\sum_{l=2}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\right)\\
&\quad-\frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}}\left(\sum_{r=0}^{m_1}\lambda^{m_2-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.\\
&\qquad\left.+(m_2-m_1)\lambda^{m_2-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right).
\end{aligned}
\tag{95}
$$

Applying formula (33) yields (for $k\in\mathbb{Z}_{m_2+2}^{m_3+1}$)

$$
\begin{aligned}
&y_1(k)\\
&= \lambda^{k-(m_2+1)}y_1(m_2+1) + \sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\\
&\quad\times\left(\sum_{l=1}^{2}b_{11}^{*l}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\left.+\sum_{l=3}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&= \lambda^{k-m_2-1}\left[\lambda^{m_2+1}\varphi_1^*(0)+\sum_{r=0}^{m_2}\lambda^{m_2-r}\right.\\
&\qquad\times\left(\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\qquad+b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{m_2-r}\left[\varphi_1^*(r-m_1)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.\\
&\qquad\qquad+(m_2-m_1)\lambda^{m_2-m_1}\\
&\qquad\qquad\left.\left.\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\right]\\
&\quad+\sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{2}b_{11}^{*l}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\left.+\sum_{l=3}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&= \lambda^{k}\varphi_1^*(0)+\sum_{r=0}^{m_2}\lambda^{k-1-r}\left(\sum_{l=2}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.\\
&\qquad\left.+(m_2-m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\\
&\quad+\sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{2}b_{11}^{*l}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\left.+\sum_{l=3}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&= \lambda^{k}\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left(\sum_{l=3}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+\sum_{r=0}^{m_2}\lambda^{k-1-r}b_{11}^{*2}\left[\varphi_1^*(r-m_2)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_2)\right]\\
&\quad+b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.\\
&\qquad\left.+(m_2-m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\\
&\quad+(k-1-m_2)\left(b_{11}^{*1}\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\left.+b_{11}^{*2}\lambda^{k-1-m_2}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\\
&= \lambda^{k}\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left(\sum_{l=3}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)
\end{aligned}
$$

$$+ b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.$$
$$\left.+(k-1-m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$
$$+ b_{11}^{*2}\left(\sum_{r=0}^{m_2}\lambda^{k-1-r}\left[\varphi_1^*(r-m_2)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_2)\right]\right.$$
$$\left.+(k-1-m_2)\lambda^{k-1-m_2}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right), \tag{96}$$

$$y_2(k)$$
$$= \lambda^{k-(m_2+1)}y_2(m_2+1)-\sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}$$
$$\times\left(\sum_{l=1}^{2}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.$$
$$\left.+\sum_{l=3}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$
$$= \lambda^{k-m_2-1}\left[\lambda^{m_2+1}\varphi_2^*(0)-\sum_{r=0}^{m_2}\lambda^{m_2-r}\right.$$
$$\times\left(\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)\right.\right.$$
$$\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$
$$-\frac{(b_{11}^{*l})^2}{b_{12}^{*1}}\left(\sum_{r=0}^{m_1}\lambda^{m_2-r}\left[\varphi_1^*(r-m_1)\right.\right.$$
$$\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]$$
$$+(m_2-m_1)\lambda^{m_2-m_1}$$
$$\left.\left.\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\right]$$
$$-\sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{2}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.$$
$$\left.+\sum_{l=3}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)\right.\right.$$
$$\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$
$$= \lambda^k\varphi_2^*(0)-\sum_{r=0}^{m_2}\lambda^{k-1-r}\left(\sum_{l=2}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)\right.\right.$$
$$\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$
$$-\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.$$
$$\left.+(m_2-m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$
$$-\sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{2}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.$$
$$\left.+\sum_{l=3}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)\right.\right.$$
$$\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$
$$= \lambda^k\varphi_2^*(0)-\sum_{r=0}^{k-1}\lambda^{k-1-r}\left(\sum_{l=3}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)\right.\right.$$
$$\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$
$$-\sum_{r=0}^{m_2}\lambda^{k-1-r}\frac{(b_{11}^{*2})^2}{b_{12}^{*2}}\left[\varphi_1^*(r-m_2)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_2)\right]$$
$$-\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)\right.\right.$$
$$\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.$$
$$\left.+(m_2-m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$
$$-(k-1-m_2)$$
$$\times\left(\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.$$
$$\left.+\frac{(b_{11}^{*2})^2}{b_{12}^{*2}}\lambda^{k-1-m_2}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$
\begin{aligned}
&= \lambda^k \varphi_2^*(0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=3}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) \\
&\quad - \frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] + (k-1-m_1) \lambda^{k-1-m_1} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&\quad - \frac{\left(b_{11}^{*2}\right)^2}{b_{12}^{*2}} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_2) \right] + (k-1-m_2) \lambda^{k-1-m_2} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right).
\end{aligned} \tag{97}
$$

From (93)–(97) we deduce that expected form of the solution of the initial problem for $k \in \mathbb{Z}_{m_{s-1}+1}^{m_s}$ with initial data derived from the solution of previous equation for $k \in \mathbb{Z}_{m_{s-2}+1}^{m_{s-1}}$ is

$$
\begin{aligned}
y_1(k) &= \lambda^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \times \left(\sum_{l=s}^{n} b_{11}^{*l} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) \\
&\quad + \sum_{l=1}^{s-1} \left[b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] + (k-1-m_l) \lambda^{k-1-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \right] \\
&\qquad \text{if } k \in \mathbb{Z}_{m_{s-1}+2}^{m_s+1}, \\
y_1(k) &= \lambda^k \varphi_2^*(0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \times \left(\sum_{l=s}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) \\
&\quad - \sum_{l=1}^{s-1} \left[\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] + (k-1-m_l) \lambda^{k-1-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \right] \\
&\qquad \text{if } k \in \mathbb{Z}_{m_{s-1}+2}^{m_s+1}.
\end{aligned} \tag{98}
$$

We solve (88) for $k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}$ with initial data deduced from (98); that is, we consider the problem

$$
\begin{aligned}
y_1(k+1) &= \lambda y_1(k) + \sum_{l=1}^{s} b_{11}^{*l} \lambda^{k-m_l} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] + \sum_{l=s+1}^{n} b_{11}^{*l} \left[\varphi_1^*(k-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k-m_l) \right] \\
&\qquad \text{if } k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}, \\
y_1(m_s+1) &= \lambda^{m_s+1} \varphi_1^*(0) + \sum_{r=0}^{m_s} \lambda^{m_s-r} \left(\sum_{l=s}^{n} b_{11}^{*l} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) \\
&\quad + \sum_{l=1}^{s-1} b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{m_s-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] + (m_s-m_l) \lambda^{m_s-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right), \\
y_2(k+1) &= \lambda y_2(k) - \sum_{l=1}^{s} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] - \sum_{l=s+1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(k-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k-m_l) \right] \\
&\qquad \text{if } k \in \mathbb{Z}_{m_s+1}^{m_{s+1}},
\end{aligned}
$$

$$
\begin{aligned}
&y_2(m_s+1)\\
&=\lambda^{m_s+1}\varphi_2^*(0)\\
&\quad-\sum_{r=0}^{m_s}\lambda^{m_s-r}\left(\sum_{l=s}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad-\sum_{l=1}^{s-1}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left(\sum_{r=0}^{m_l}\lambda^{m_s-r}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right.\\
&\qquad\left.+(m_s-m_l)\lambda^{m_s-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right).
\end{aligned}
\tag{99}
$$

Applying formula (33) yields (for $k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1}$)

$$
\begin{aligned}
&y_1(k)\\
&=\lambda^{k-(m_s+1)}y_1(m_s+1)+\sum_{r=m_s+1}^{k-1}\lambda^{k-1-r}\\
&\quad\times\left(\sum_{l=1}^{s}b_{11}^{*l}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\left.+\sum_{l=s+1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&=\lambda^{k-m_s-1}\left[\lambda^{m_s+1}\varphi_1^*(0)+\sum_{r=0}^{m_s}\lambda^{m_s-r}\right.\\
&\qquad\times\left(\sum_{l=s}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\qquad+\sum_{l=1}^{s-1}b_{11}^{*l}\left(\sum_{r=0}^{m_l}\lambda^{m_s-r}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\\
&\qquad\qquad+(m_s-m_l)\lambda^{m_s-m_l}\\
&\qquad\qquad\left.\left.\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\right]\\
&\quad+\sum_{r=m_s+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{s}b_{11}^{*l}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad\left.+\sum_{l=s+1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&=\lambda^{k}\varphi_1^*(0)+\sum_{r=0}^{m_s}\lambda^{k-1-r}\\
&\quad\times\left(\sum_{l=s}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+\sum_{l=1}^{s-1}b_{11}^{*l}\left(\sum_{r=0}^{m_l}\lambda^{k-1-r}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right.\\
&\qquad+(m_s-m_l)\lambda^{k-1-m_l}\\
&\qquad\left.\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\\
&\quad+\sum_{r=m_s+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{s}b_{11}^{*l}\lambda^{r-m_l}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right.\\
&\qquad+\sum_{l=s+1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)\right.\\
&\qquad\qquad\left.\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&=\lambda^{k}\varphi_1^*(0)+\sum_{r=0}^{k-1}\lambda^{k-1-r}\\
&\quad\times\left(\sum_{l=s+1}^{n}b_{11}^{*l}\left[\varphi_1^*(r-m_l)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)\\
&\quad+b_{11}^{*s}\sum_{r=0}^{m_s}\lambda^{k-1-r}\left[\varphi_1^*(r-m_s)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_s)\right]\\
&\quad+b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right]\right.\\
&\qquad\left.+(m_s-m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\\
&\quad+b_{11}^{*2}\left(\sum_{r=0}^{m_2}\lambda^{k-1-r}\left[\varphi_1^*(r-m_2)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_2)\right]\right.\\
&\qquad+(m_s-m_2)\lambda^{k-1-m_2}\\
&\qquad\left.\times\left[\varphi_1^*(0)+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\\
&\quad+\cdots\\
&\quad+b_{11}^{*s-1}\left(\sum_{r=0}^{m_{s-1}}\lambda^{k-1-r}\left[\varphi_1^*(r-m_{s-1})\right.\right.\\
&\qquad\qquad\left.+\frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_{s-1})\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ (m_s - m_{s-1}) \lambda^{k-1-m_{s-1}} \\
&\times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg) \\
&+ (k-1-m_s) \\
&\times \Bigg(\lambda^{k-1-m_1} b_{11}^{*1} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \\
&+ \lambda^{k-1-m_2} b_{11}^{*2} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] + \cdots \\
&+ \lambda^{k-1-m_{s-1}} b_{11}^{*s-1} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \\
&+ \lambda^{k-1-m_s} b_{11}^{*s} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg) \\
= {} & \lambda^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \\
&\times \left(\sum_{l=s+1}^{n} b_{11}^{*l} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) \\
&+ b_{11}^{*1} \Bigg(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] \\
&+ (k-1-m_1) \lambda^{k-1-m_1} \\
&\times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg) \\
&+ b_{11}^{*2} \Bigg(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_2) \right] \\
&+ (k-1-m_2) \lambda^{k-1-m_2} \\
&\times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg) \\
&+ \cdots \\
&+ b_{11}^{*s-1} \Bigg(\sum_{r=0}^{m_{s-1}} \lambda^{k-1-r} \left[\varphi_1^*(r-m_{s-1}) \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_{s-1}) \right] \\
&+ (k-1-m_{s-1}) \lambda^{k-1-m_{s-1}} \\
&\times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg) \\
&+ b_{11}^{*s} \Bigg(\sum_{r=0}^{m_s} \lambda^{k-1-r} \left[\varphi_1^*(r-m_s) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_s) \right] \\
&+ (k-1-m_s) \lambda^{k-1-m_s} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg) \\
= {} & \lambda^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \\
&\times \left(\sum_{l=s+1}^{n} b_{11}^{*l} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) \\
&+ \sum_{l=1}^{s} b_{11}^{*l} \Bigg(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^*(r-m_l) \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \\
&+ (k-1-m_l) \lambda^{k-1-m_l} \\
&\times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \Bigg),
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
& y_2(k) \\
&= \lambda^{k-(m_s+1)} y_2(m_s+1) - \sum_{r=m_s+1}^{k-1} \lambda^{k-1-r} \\
&\times \Bigg(\sum_{l=1}^{s} \frac{(b_{11}^{*l})^2}{b_{12}^{*l}} \lambda^{r-m_l} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \\
&+ \sum_{l=s+1}^{n} \frac{(b_{11}^{*l})^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \Bigg) \\
&= \lambda^{k-m_s-1} \Bigg[\lambda^{m_s+1} \varphi_2^*(0) - \sum_{r=0}^{m_s} \lambda^{m_s-r} \\
&\times \Bigg(\sum_{l=s}^{n} \frac{(b_{11}^{*l})^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \Bigg) \\
&- \sum_{l=1}^{s-1} \frac{(b_{11}^{*l})^2}{b_{12}^{*l}} \Bigg(\sum_{r=0}^{m_l} \lambda^{m_s-r} \left[\varphi_1^*(r-m_l) \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right]
\end{aligned}
$$

$$+ (m_s - m_l)\lambda^{m_s - m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\Bigg)\Bigg]$$

$$- \sum_{r=m_s+1}^{k-1} \lambda^{k-1-r}\left(\sum_{l=1}^{s}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\lambda^{r-m_l}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right] + \sum_{l=s+1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$

$$= \lambda^k\varphi_2^*(0) - \sum_{r=0}^{m_s}\lambda^{k-1-r} \times \left(\sum_{l=s}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$

$$- \sum_{l=1}^{s-1}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left(\sum_{r=0}^{m_l}\lambda^{k-1-r}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right] + (m_s - m_l)\lambda^{k-1-m_l}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$- \sum_{r=m_s+1}^{k-1} \lambda^{k-1-r}\left(\sum_{l=1}^{s}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\lambda^{r-m_l}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right] + \sum_{l=s+1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$

$$= \lambda^k\varphi_2^*(0) - \sum_{r=0}^{k-1}\lambda^{k-1-r} \times \left(\sum_{l=s+1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$

$$- \frac{(b_{11}^{*s})^2}{b_{12}^{*s}}\sum_{r=0}^{m_s}\lambda^{k-1-r}\left[\varphi_1^*(r-m_s) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_s)\right]$$

$$- \frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right] + (m_s - m_1)\lambda^{k-1-m_1}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$- \frac{(b_{11}^{*2})^2}{b_{12}^{*2}}\left(\sum_{r=0}^{m_2}\lambda^{k-1-r}\left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_2)\right] + (m_s - m_2)\lambda^{k-1-m_2}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$+ \cdots$$

$$- \frac{(b_{11}^{*s-1})^2}{b_{12}^{*s-1}}\left(\sum_{r=0}^{m_{s-1}}\lambda^{k-1-r}\left[\varphi_1^*(r-m_{s-1}) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_{s-1})\right] + (m_s - m_{s-1})\lambda^{k-1-m_{s-1}}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$- (k-1-m_s) \times \left(\lambda^{k-1-m_1}\frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right] + \lambda^{k-1-m_2}\frac{(b_{11}^{*2})^2}{b_{12}^{*2}}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right] + \cdots + \lambda^{k-1-m_{s-1}}\frac{(b_{11}^{*s-1})^2}{b_{12}^{*s-1}}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right] + \lambda^{k-1-m_s}\frac{(b_{11}^{*s})^2}{b_{12}^{*s}}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$= \lambda^k\varphi_2^*(0) - \sum_{r=0}^{k-1}\lambda^{k-1-r} \times \left(\sum_{l=s+1}^{n}\frac{(b_{11}^{*l})^2}{b_{12}^{*l}}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right]\right)$$

$$- \frac{(b_{11}^{*1})^2}{b_{12}^{*1}}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right] + (k-1-m_1)\lambda^{k-1-m_1} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)$$

$$
\begin{aligned}
&- \frac{\left(b_{11}^{*2}\right)^2}{b_{12}^{*2}} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^* (r - m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_2) \right] \right. \\
&\qquad + (k - 1 - m_2) \lambda^{k-1-m_2} \\
&\qquad \left. \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) \\
&+ \cdots \\
&- \frac{\left(b_{11}^{*s-1}\right)^2}{b_{12}^{*s-1}} \left(\sum_{r=0}^{m_{s-1}} \lambda^{k-1-r} \left[\varphi_1^* (r - m_{s-1}) \right. \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_{s-1}) \right] \\
&\qquad + (k - 1 - m_{s-1}) \lambda^{k-1-m_{s-1}} \\
&\qquad \left. \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) \\
&- \frac{\left(b_{11}^{*s}\right)^2}{b_{12}^{*s}} \left(\sum_{r=0}^{m_s} \lambda^{k-1-r} \left[\varphi_1^* (r - m_s) \right. \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_s) \right] \\
&\qquad + (k - 1 - m_s) \lambda^{k-1-m_s} \\
&\qquad \left. \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) \\
&= \lambda^k \varphi_2^* (0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \\
&\quad \times \left(\sum_{l=s+1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right) \\
&\quad - \sum_{l=1}^{s} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^* (r - m_l) \right. \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \\
&\qquad + (k - 1 - m_l) \lambda^{k-1-m_l} \\
&\qquad \left. \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right).
\end{aligned}
\tag{101}
$$

In the end, we solve (88) for $k \in \mathbb{Z}_{m_n+1}^{\infty}$ with initial data deduced from (100) and (101); that is, we consider the problem

$$
\begin{aligned}
&y_1 (k + 1) = \lambda y_1 (k) + \sum_{l=1}^{n} b_{11}^{*l} \lambda^{k-m_l} \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \\
&\qquad \text{if } k \in \mathbb{Z}_{m_n+1}^{\infty}, \\
&y_1 (m_n + 1) \\
&\quad = \lambda^{m_n+1} \varphi_1^* (0) + \sum_{r=0}^{m_n} \lambda^{m_n-r} b_{11}^{*n} \\
&\qquad \times \left[\varphi_1^* (r - m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_n) \right] \\
&\qquad + \sum_{l=1}^{n-1} b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{m_n-r} \left[\varphi_1^* (r - m_l) \right. \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \\
&\qquad + (m_n - m_l) \lambda^{m_n-m_l} \\
&\qquad \left. \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right), \\
&y_2 (k + 1) \\
&\quad = \lambda y_2 (k) - \sum_{l=1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \lambda^{k-m_l} \\
&\qquad \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \quad \text{if } k \in \mathbb{Z}_{m_n+1}^{\infty}, \\
&y_2 (m_n + 1) \\
&\quad = \lambda^{m_n+1} \varphi_2^* (0) - \sum_{r=0}^{m_n} \lambda^{m_n-r} \frac{\left(b_{11}^{*n}\right)^2}{b_{12}^{*n}} \\
&\qquad \times \left[\varphi_1^* (r - m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_n) \right] \\
&\qquad - \sum_{l=1}^{n-1} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left(\sum_{r=0}^{m_l} \lambda^{m_n-r} \left[\varphi_1^* (r - m_l) \right. \right. \\
&\qquad \left. + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \\
&\qquad + (m_n - m_l) \lambda^{m_n-m_l} \\
&\qquad \left. \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right).
\end{aligned}
\tag{102}
$$

Applying formula (33) yields (for $k \in \mathbb{Z}_{m_n+2}^{\infty}$)

$$
\begin{aligned}
&y_1(k) \\
&= \lambda^{k-(m_n+1)} y_1(m_n+1) + \sum_{r=m_n+1}^{k-1} \lambda^{k-1-r} \\
&\quad \times \left(\sum_{l=1}^{n} b_{11}^{*l} \lambda^{r-m_l} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&= \lambda^{k-m_n-1} \Bigg[\lambda^{m_n+1} \varphi_1^*(0) \\
&\qquad + \sum_{r=0}^{m_n} \lambda^{m_n-r} b_{11}^{*n} \left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_n) \right] \\
&\qquad + \sum_{l=1}^{n-1} b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{m_n-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right. \\
&\qquad \left. + (m_n - m_l) \lambda^{m_n-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \Bigg] \\
&\quad + \sum_{r=m_n+1}^{k-1} \lambda^{k-1-r} \left(\sum_{l=1}^{n} b_{11}^{*l} \lambda^{r-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&= \lambda^k \varphi_1^*(0) + \sum_{r=0}^{m_n} \lambda^{k-1-r} b_{11}^{*n} \left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_n) \right] \\
&\quad + \sum_{l=1}^{n-1} b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right. \\
&\qquad \left. + (m_n - m_l) \lambda^{k-1-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&\quad + \sum_{r=m_n+1}^{k-1} \lambda^{k-1-r} \left(\sum_{l=1}^{n} b_{11}^{*l} \lambda^{r-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&= \lambda^k \varphi_1^*(0) + \sum_{r=0}^{m_n} \lambda^{k-1-r} b_{11}^{*n} \left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_n) \right] \\
&\quad + b_{11}^{*1} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] \right. \\
&\qquad \left. + (m_n - m_1) \lambda^{k-1-m_1} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&\quad + b_{11}^{*2} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_2) \right] \right. \\
&\qquad \left. + (m_n - m_2) \lambda^{k-1-m_2} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&\quad + \cdots \\
&\quad + b_{11}^{*n-1} \left(\sum_{r=0}^{m_{n-1}} \lambda^{k-1-r} \left[\varphi_1^*(r-m_{n-1}) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_{n-1}) \right] \right. \\
&\qquad \left. + (m_n - m_{n-1}) \lambda^{k-1-m_{n-1}} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) \\
&\quad + (k-1-m_n) \\
&\quad \times \left(\lambda^{k-1-m_1} b_{11}^{*1} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right. \\
&\qquad + \lambda^{k-1-m_2} b_{11}^{*2} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] + \cdots \\
&\qquad + \lambda^{k-1-m_{n-1}} b_{11}^{*n-1} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \\
&\qquad \left. + \lambda^{k-1-m_n} b_{11}^{*n} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda^k \varphi_1^*(0) + b_{11}^{*1}\left(\sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_1)\right] + (k-1-m_1)\lambda^{k-1-m_1} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right) \\
&\quad + b_{11}^{*2}\left(\sum_{r=0}^{m_2}\lambda^{k-1-r}\left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_2)\right] + (k-1-m_2)\lambda^{k-1-m_2} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right) \\
&\quad + \cdots \\
&\quad + b_{11}^{*n-1}\left(\sum_{r=0}^{m_{n-1}}\lambda^{k-1-r}\left[\varphi_1^*(r-m_{n-1}) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_{n-1})\right] + (k-1-m_{n-1})\lambda^{k-1-m_{n-1}} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right) \\
&\quad + b_{11}^{*n}\left(\sum_{r=0}^{m_n}\lambda^{k-1-r}\left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_n)\right] + (k-1-m_n)\lambda^{k-1-m_n} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right) \\
&= \lambda^k \varphi_1^*(0) + \sum_{l=1}^{n} b_{11}^{*l}\left(\sum_{r=0}^{m_l}\lambda^{k-1-r}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right] + (k-1-m_l)\lambda^{k-1-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right),
\end{aligned}
\tag{103}
$$

$$
\begin{aligned}
&y_2(k) \\
&= \lambda^{k-(m_n+1)} y_2(m_n+1) - \sum_{r=m_n+1}^{k-1}\lambda^{k-1-r} \times \left(\sum_{l=1}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\lambda^{r-m_l}\left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right) \\
&= \lambda^{k-m_n-1} \times \left[\lambda^{m_n+1}\varphi_2^*(0) - \sum_{r=0}^{m_n}\lambda^{m_n-r}\frac{\left(b_{11}^{*n}\right)^2}{b_{12}^{*n}} \times \left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_n)\right]\right. \\
&\quad - \sum_{l=1}^{n-1}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \times \left(\sum_{r=0}^{m_l}\lambda^{m_n-r}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right] + (m_n-m_l)\lambda^{m_n-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)\Bigg] \\
&\quad - \sum_{r=m_n+1}^{k-1}\lambda^{k-1-r}\left(\sum_{l=1}^{n}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\lambda^{r-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right) \\
&= \lambda^k \varphi_2^*(0) - \sum_{r=0}^{m_n}\lambda^{k-1-r}\frac{\left(b_{11}^{*n}\right)^2}{b_{12}^{*n}} \times \left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_n)\right] \\
&\quad - \sum_{l=1}^{n-1}\frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}}\left(\sum_{r=0}^{m_l}\lambda^{k-1-r}\left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(r-m_l)\right] + (m_n-m_l)\lambda^{k-1-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}}\varphi_2^*(0)\right]\right)
\end{aligned}
$$

$$- \sum_{r=m_n+1}^{k-1} \lambda^{k-1-r} \left(\sum_{l=1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \lambda^{r-m_l} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$= \lambda^k \varphi_2^*(0) - \sum_{r=0}^{m_n} \lambda^{k-1-r} \frac{\left(b_{11}^{*n}\right)^2}{b_{12}^{*n}} \left[\varphi_1^*(r-m_n) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_n) \right]$$

$$- \frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] + (m_n - m_1) \lambda^{k-1-m_1} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$- \frac{\left(b_{11}^{*2}\right)^2}{b_{12}^{*2}} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_2) \right] + (m_n - m_2) \lambda^{k-1-m_2} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$+ \cdots$$

$$- \frac{\left(b_{11}^{*n-1}\right)^2}{b_{12}^{*n-1}} \left(\sum_{r=0}^{m_{n-1}} \lambda^{k-1-r} \left[\varphi_1^*(r-m_{n-1}) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_{n-1}) \right] + (m_n - m_{n-1}) \lambda^{k-1-m_{n-1}} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) - (k-1-m_n)$$

$$\times \left(\lambda^{k-1-m_1} \frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] + \lambda^{k-1-m_2} \frac{\left(b_{11}^{*2}\right)^2}{b_{12}^{*2}} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] + \cdots + \lambda^{k-1-m_{n-1}} \frac{\left(b_{11}^{*n-1}\right)^2}{b_{12}^{*n-1}} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] + \lambda^{k-1-m_n} \frac{\left(b_{11}^{*n}\right)^2}{b_{12}^{*n}} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$= \lambda^k \varphi_2^*(0) - \frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}} \times \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] + (k-1-m_1) \lambda^{k-1-m_1} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$- \frac{\left(b_{11}^{*2}\right)^2}{b_{12}^{*2}} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_2) \right] + (k-1-m_2) \lambda^{k-1-m_2} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$+ \cdots$$

$$- \frac{\left(b_{11}^{*n-1}\right)^2}{b_{12}^{*n-1}} \left(\sum_{r=0}^{m_{n-1}} \lambda^{k-1-r} \left[\varphi_1^*(r-m_{n-1}) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_{n-1}) \right] + (k-1-m_{n-1}) \lambda^{k-1-m_{n-1}} \times \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right)$$

$$- \frac{\left(b_{11}^{*n}\right)^2}{b_{12}^{*n}} \left(\sum_{r=0}^{m_n} \lambda^{k-1-r} \left[\varphi_1^*(r-m_n) \right. \right.$$

$$
\begin{aligned}
&\quad + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_n) \Big] \\
&\quad + (k - 1 - m_n) \lambda^{k-1-m_n} \\
&\quad \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \Big) \\
&= \lambda^k \varphi_2^* (0) - \sum_{l=1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \\
&\quad \times \Big(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \\
&\quad + (k - 1 - m_l) \lambda^{k-1-m_l} \\
&\quad \times \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \Big).
\end{aligned} \tag{104}
$$

Summing up all particular cases (90), (93), (96), (100), and (103) we have

$$
y_1 (k) = \begin{cases}
\varphi_1^* (k) & \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\
\lambda^k \varphi_1^* (0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=1}^{n} b_{11}^{*l} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right) & \text{if } k \in \mathbb{Z}_{1}^{m_1+1}, \\
\lambda^k \varphi_1^* (0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=2}^{n} b_{11}^{*l} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right) & \\
\quad + b_{11}^{*1} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^* (r - m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_1) \right] \right. & \\
\qquad \left. + (k - 1 - m_1) \lambda^{k-1-m_1} \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_1+2}^{m_2+1}, \\
\lambda^k \varphi_1^* (0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=3}^{n} b_{11}^{*l} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right) & \\
\quad + b_{11}^{*1} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^* (r - m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_1) \right] \right. & \\
\qquad \left. + (k - 1 - m_1) \lambda^{k-1-m_1} \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) & \\
\quad + b_{11}^{*2} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^* (r - m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_2) \right] \right. & \\
\qquad \left. + (k - 1 - m_2) \lambda^{k-1-m_2} \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_2+2}^{m_3+1}, \\
\vdots & \\
\lambda^k \varphi_1^* (0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=s+1}^{n} b_{11}^{*l} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right) & \\
\quad + \sum_{l=1}^{s} b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right. & \\
\qquad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \\
\vdots & \\
\lambda^k \varphi_1^* (0) + \sum_{l=1}^{n} b_{11}^{*l} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^* (r - m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (r - m_l) \right] \right. & \\
\qquad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \left[\varphi_1^* (0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^* (0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_n+2}^{\infty},
\end{cases} \tag{105}
$$

and from cases (91), (94), (97), (101), and (104) we conclude that

$$
y_2(k) = \begin{cases}
\varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^{0}, \\
\lambda^k \varphi_2^*(0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) & \text{if } k \in \mathbb{Z}_{1}^{m_1+1}, \\
\lambda^k \varphi_2^*(0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=2}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \right) & \\
\quad - \frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] \right. & \\
\qquad \left. + (k-1-m_1) \lambda^{k-1-m_1} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_1+2}^{m_2+1}, \\
\lambda^k \varphi_2^*(0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=3}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) & \\
\quad - \frac{\left(b_{11}^{*1}\right)^2}{b_{12}^{*1}} \left(\sum_{r=0}^{m_1} \lambda^{k-1-r} \left[\varphi_1^*(r-m_1) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_1) \right] \right. & \\
\qquad \left. + (k-1-m_1) \lambda^{k-1-m_1} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) & \\
\quad - \frac{\left(b_{11}^{*2}\right)^2}{b_{12}^{*2}} \left(\sum_{r=0}^{m_2} \lambda^{k-1-r} \left[\varphi_1^*(r-m_2) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_2) \right] \right. & \\
\qquad \left. + (k-1-m_2) \lambda^{k-1-m_2} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_2+2}^{m_3+1}, \\
\vdots & \\
\lambda^k \varphi_2^*(0) - \sum_{r=0}^{k-1} \lambda^{k-1-r} \left(\sum_{l=s+1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right) & \\
\quad - \sum_{l=1}^{s} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right. & \\
\qquad \left. + (k-1-m_l) \lambda^{k-1-m_l} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \\
\vdots & \\
\lambda^k \varphi_2^*(0) - \sum_{l=1}^{n} \frac{\left(b_{11}^{*l}\right)^2}{b_{12}^{*l}} \left(\sum_{r=0}^{m_l} \lambda^{k-1-r} \left[\varphi_1^*(r-m_l) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(r-m_l) \right] \right. & \\
\qquad \left. + (k-1-m_l) \lambda^{k-1-m_l} \left[\varphi_1^*(0) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(0) \right] \right) & \text{if } k \in \mathbb{Z}_{m_n+2}^{\infty}.
\end{cases} \tag{106}
$$

Formula (81) is now a direct consequence of (105), (106), and (80). □

2.1.6. Case (39) *of a Double Real Root.* If the matrix Λ has the form (39), the necessary and sufficient conditions (13)–(16), for (40), are reduced to (43), (44), (46), and

$$
\begin{vmatrix} \lambda & 1 \\ b_{21}^{*l} & b_{22}^{*l} \end{vmatrix} + \begin{vmatrix} b_{11}^{*l} & b_{12}^{*l} \\ 0 & \lambda \end{vmatrix} = \lambda \left(b_{11}^{*l} + b_{22}^{*l} \right) - b_{21}^{*l} = 0. \tag{107}
$$

Then (43), (44), and (107) give $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$.

Theorem 9. *Let* (1) *be a weakly delayed system,* (35) *has a double root* $\lambda_{1,2} = \lambda$ *and the matrix* Λ *has the form* (39). *Then*

$b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$ *and the solution of the initial problem* (1), (3) *is* $x(k) = \mathcal{S}y(k)$, $y(k) = (y_1(k), y_2(k))^T$, *and*

$$y_1(k) = \begin{cases} \varphi_1^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^0, \\ \lambda^k \varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) \\ \quad + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(r - m_l) \right] \\ \qquad \text{if } k \in \mathbb{Z}_1^{m_1+1}, \\ \vdots \\ \lambda^k \varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) \\ \quad + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(r - m_l) \right] \\ \quad + \sum_{l=1}^{s} b_{12}^{*l} \left[\sum_{r=0}^{m_l} \lambda^{k-1-r} \varphi_2^*(r - m_l) \right. \\ \qquad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \varphi_2^*(0) \right] \\ \qquad \text{if } k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}, \\ \qquad s = 1, 2, \ldots, n-1, \\ \vdots \\ \lambda^k \varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) \\ \quad + \sum_{l=1}^{n} b_{12}^{*l} \left[\sum_{r=0}^{m_l} \lambda^{k-1-r} \varphi_2^*(r - m_l) \right. \\ \qquad \left. + (k - 1 - m_l) \lambda^{k-1-m_l} \varphi_2^*(0) \right] \\ \qquad \text{if } k \in \mathbb{Z}_{m_n+2}^{\infty}, \end{cases} \tag{108}$$

$$y_2(k) = \begin{cases} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^0, \\ \lambda^k \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{109}$$

Proof. The system (40) can be written as

$$y_1(k+1) = \lambda y_1(k) + y_2(k) + \sum_{l=1}^{n} b_{12}^{*l} y_2(k - m_l), \tag{110}$$

$$y_2(k+1) = \lambda y_2(k), \quad k \in \mathbb{Z}_0^{\infty}. \tag{111}$$

Solving (111), we get

$$y_2(k) = \begin{cases} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m_n}^0, \\ \lambda^k \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_1^{\infty}, \end{cases} \tag{112}$$

then (110) turns into

$$y_1(k+1) = \begin{cases} \lambda y_1(k) + \lambda^k \varphi_2^*(0) + \sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) \\ \qquad \text{if } k \in \mathbb{Z}_0^{m_1}, \\ \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^{*1} \lambda^{k-m_1} \varphi_2^*(0) \\ \quad + \sum_{l=2}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) \\ \qquad \text{if } k \in \mathbb{Z}_{m_1+1}^{m_2}, \\ \lambda y_1(k) + \lambda^k \varphi_2^*(0) + \sum_{l=1}^{2} b_{12}^{*l} \lambda^{k-m_l} \varphi_2^*(0) \\ \quad + \sum_{l=3}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) \\ \qquad \text{if } k \in \mathbb{Z}_{m_2+1}^{m_3}, \\ \vdots \\ \lambda y_1(k) + \lambda^k \varphi_2^*(0) + \sum_{l=1}^{s} b_{12}^{*l} \lambda^{k-m_l} \varphi_2^*(0) \\ \quad + \sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) \\ \qquad \text{if } k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}, \\ \qquad s = 3, 4, \ldots, n-1, \\ \vdots \\ \lambda y_1(k) + \lambda^k \varphi_2^*(0) + \sum_{l=1}^{n} b_{12}^{*l} \lambda^{k-m_l} \varphi_2^*(0) \\ \qquad \text{if } k \in \mathbb{Z}_{m_n+1}^{\infty}. \end{cases} \tag{113}$$

Equation (113) can be solved in a way similar to that of (54) in the proof of Theorem 5 using (33).

First we solve (113) for $k \in \mathbb{Z}_0^{m_1}$. This means that we consider the problem

$$\begin{aligned} y_1(k+1) &= \lambda y_1(k) + \lambda^k \varphi_2^*(0) \\ &\quad + \sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(k - m_l) \quad \text{if } k \in \mathbb{Z}_0^{m_1}, \\ y_1(0) &= \varphi_1^*(0). \end{aligned} \tag{114}$$

With the aid of formula (33), we get

$$\begin{aligned} y_1(k) &= \lambda^k \varphi_1^*(0) \\ &\quad + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left[\lambda^r \varphi_2^*(0) + \sum_{l=1}^{n} b_{12}^{*l} \varphi_2^*(r - m_l) \right] \end{aligned}$$

$$= \lambda^k \varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right], \quad k \in \mathbb{Z}_1^{m_1+1}. \tag{115}$$

Now we solve (113) for $k \in \mathbb{Z}_{m_1+1}^{m_2}$ with initial data deduced from (115); that is, we consider the problem

$$\begin{aligned} y_1(k+1) &= \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^{*1}\lambda^{k-m_1}\varphi_2^*(0) + \sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(k-m_l), \quad k \in \mathbb{Z}_{m_1+1}^{m_2}, \\ y_1(m_1+1) &= \lambda^{m_1+1}\varphi_1^*(0) + (m_1+1)\lambda^{m_1}\varphi_2^*(0) + \sum_{r=0}^{m_1}\lambda^{m_1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]. \end{aligned} \tag{116}$$

Applying formula (33) we get (for $k \in \mathbb{Z}_{m_1+2}^{m_2+1}$)

$$\begin{aligned} y_1(k) &= \lambda^{k-(m_1+1)} y_1(m_1+1) + \sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\left[\lambda^r\varphi_2^*(0) + b_{12}^{*1}\lambda^{r-m_1}\varphi_2^*(0) + \sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\ &= \lambda^{k-m_1-1}\left[\lambda^{m_1+1}\varphi_1^*(0) + (m_1+1)\lambda^{m_1}\varphi_2^*(0) + \sum_{r=0}^{m_1}\lambda^{m_1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]\right] \\ &\quad + \sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\left[\lambda^r\varphi_2^*(0) + b_{12}^{*1}\lambda^{r-m_1}\varphi_2^*(0) + \sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\ &= \lambda^k\varphi_1^*(0) + (m_1+1)\lambda^{k-1}\varphi_2^*(0) + \sum_{r=0}^{m_1}\lambda^{k-1-r}\left[\sum_{l=1}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\ &\quad + (k-m_1-1)\lambda^{k-1}\varphi_2^*(0) + \sum_{r=m_1+1}^{k-1}\lambda^{k-1-r}\left[b_{12}^{*1}\lambda^{r-m_1}\varphi_2^*(0) + \sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\ &= \lambda^k\varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) + \sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\ &\quad + b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1) + (k-1-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\right]. \end{aligned} \tag{117}$$

Now we solve (113) for $k \in \mathbb{Z}_{m_2+1}^{m_3}$ with initial data deduced from (117); that is, we consider the problem

$$\begin{aligned} y_1(k+1) &= \lambda y_1(k) + \lambda^k\varphi_2^*(0) + \sum_{l=1}^{2} b_{12}^{*l}\lambda^{k-m_l}\varphi_2^*(0) + \sum_{l=3}^{n} b_{12}^{*l}\varphi_2^*(k-m_l) \quad k \in \mathbb{Z}_{m_2+1}^{m_3}, \\ y_1(m_2+1) &= \lambda^{m_2+1}\varphi_1^*(0) + (m_2+1)\lambda^{m_2}\varphi_2^*(0) + \sum_{r=0}^{m_2}\lambda^{m_2-r}\left[\sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\ &\quad + b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda^{m_2-r}\varphi_2^*(r-m_1) + (m_2-m_1)\lambda^{m_2-m_1}\varphi_2^*(0)\right]. \end{aligned} \tag{118}$$

Applying formula (33) yields (for $k \in \mathbb{Z}_{m_2+2}^{m_3+1}$)

$$y_1(k) = \lambda^{k-(m_2+1)} y_1(m_2+1) + \sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\left[\lambda^r\varphi_2^*(0) + \sum_{l=1}^{2} b_{12}^{*l}\lambda^{r-m_l}\varphi_2^*(0) + \sum_{l=3}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right]$$

$$
\begin{aligned}
&= \lambda^{k-m_2-1}\Bigg[\lambda^{m_2+1}\varphi_1^*(0) + (m_2+1)\lambda^{m_2}\varphi_2^*(0) \\
&\quad + \sum_{r=0}^{m_2}\lambda^{m_2-r}\left[\sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\
&\quad + b_{12}^{*1}\Bigg[\sum_{r=0}^{m_1}\lambda^{m_2-r}\varphi_2^*(r-m_1) \\
&\qquad + (m_2-m_1)\lambda^{m_2-m_1}\varphi_2^*(0)\Bigg]\Bigg] \\
&\quad + \sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\Bigg[\lambda^{r}\varphi_2^*(0) + \sum_{l=1}^{2} b_{12}^{*l}\lambda^{r-m_l}\varphi_2^*(0) \\
&\qquad + \sum_{l=3}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\Bigg] \\
&= \lambda^{k}\varphi_1^*(0) + (m_2+1)\lambda^{k-1}\varphi_2^*(0) \\
&\quad + \sum_{r=0}^{m_2}\lambda^{k-1-r}\left[\sum_{l=2}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\
&\quad + b_{12}^{*1}\Bigg[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1) \\
&\qquad + (m_2-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\Bigg] \\
&\quad + (k-1-m_2)\lambda^{k-1}\varphi_2^*(0) \\
&\quad + \sum_{r=m_2+1}^{k-1}\lambda^{k-1-r}\Bigg[\sum_{l=1}^{2} b_{12}^{*l}\lambda^{r-m_l}\varphi_2^*(0) \\
&\qquad + \sum_{l=3}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\Bigg] \\
&= \lambda^{k}\varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) \\
&\quad + \sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=3}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\
&\quad + b_{12}^{*1}\Bigg[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1) \\
&\qquad + (k-1-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\Bigg] \\
&\quad + b_{12}^{*2}\Bigg[\sum_{r=0}^{m_2}\lambda^{k-1-r}\varphi_2^*(r-m_2) \\
&\qquad + (k-1-m_2)\lambda^{k-1-m_2}\varphi_2^*(0)\Bigg].
\end{aligned}
\tag{119}
$$

From (115), (117), and (119) we deduce that expected form of the solution of the initial problem for $k \in \mathbb{Z}_{m_{s-1}+1}^{m_s}$ with initial data derived from the solution of previous equation for $k \in \mathbb{Z}_{m_{s-2}+1}^{m_{s-1}}$ is

$$
\begin{aligned}
y_1(k) &= \lambda^{k}\varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) \\
&\quad + \sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=s}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\
&\quad + \sum_{l=1}^{s-1} b_{12}^{*l}\Bigg[\sum_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l) \\
&\qquad + (k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\Bigg] \\
&\qquad \text{if } k \in \mathbb{Z}_{m_{s-1}+2}^{m_s+1}.
\end{aligned}
\tag{120}
$$

We solve (113) for $k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}$ with initial data deduced from (120); that is, we consider the problem

$$
\begin{aligned}
y_1(k+1) &= \lambda y_1(k) + \lambda^{k}\varphi_2^*(0) + \sum_{l=1}^{s} b_{12}^{*l}\lambda^{k-m_l}\varphi_2^*(0) \\
&\quad + \sum_{l=s+1}^{n} b_{12}^{*l}\varphi_2^*(k-m_l), \quad k \in \mathbb{Z}_{m_s+1}^{m_{s+1}}, \\
y_1(m_s+1) &= \lambda^{m_s+1}\varphi_1^*(0) + (m_s+1)\lambda^{m_s}\varphi_2^*(0) \\
&\quad + \sum_{r=0}^{m_s}\lambda^{m_s-r}\left[\sum_{l=s}^{n} b_{12}^{*l}\varphi_2^*(r-m_l)\right] \\
&\quad + \sum_{l=1}^{s-1} b_{12}^{*l}\Bigg[\sum_{r=0}^{m_l}\lambda^{m_s-r}\varphi_2^*(r-m_l) \\
&\qquad + (m_s-m_l)\lambda^{m_s-m_l}\varphi_2^*(0)\Bigg].
\end{aligned}
\tag{121}
$$

Applying formula (33) yields (for $k \in \mathbb{Z}_{m_s+2}^{m_{s+1}+1}$)

$$
\begin{aligned}
y_1(k) &= \lambda^{k-(m_s+1)} y_1(m_s+1) \\
&\quad + \sum_{r=m_s+1}^{k-1} \lambda^{k-1-r} \left[\lambda^r \varphi_2^*(0) + \sum_{l=1}^{s} b_{12}^{*l} \lambda^{r-m_l} \varphi_2^*(0) \right. \\
&\qquad \left. + \sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&= \lambda^{k-m_s-1} \\
&\quad \times \left[\lambda^{m_s+1} \varphi_1^*(0) + (m_s+1) \lambda^{m_s} \varphi_2^*(0) \right. \\
&\qquad + \sum_{r=0}^{m_s} \lambda^{m_s-r} \left[\sum_{l=s}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&\qquad + \sum_{l=1}^{s-1} b_{12}^{*l} \left[\sum_{r=0}^{m_l} \lambda^{m_s-r} \varphi_2^*(r-m_l) \right. \\
&\qquad \left. \left. + (m_s-m_l) \lambda^{m_s-m_l} \varphi_2^*(0) \right] \right] \\
&\quad + \sum_{r=m_s+1}^{k-1} \lambda^{k-1-r} \left[\lambda^r \varphi_2^*(0) + \sum_{l=1}^{s} b_{12}^{*l} \lambda^{r-m_l} \varphi_2^*(0) \right. \\
&\qquad \left. + \sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&= \lambda^k \varphi_1^*(0) + (m_s+1) \lambda^{k-1} \varphi_2^*(0) \\
&\quad + \sum_{r=0}^{m_s} \lambda^{k-1-r} \left[\sum_{l=s}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&\quad + \sum_{l=1}^{s-1} b_{12}^{*l} \left[\sum_{r=0}^{m_l} \lambda^{k-1-r} \varphi_2^*(r-m_l) \right. \\
&\qquad \left. + (m_s-m_l) \lambda^{k-1-m_l} \varphi_2^*(0) \right] \\
&\quad + (k-1-m_s) \lambda^{k-1} \varphi_2^*(0) \\
&\quad + \sum_{r=m_s+1}^{k-1} \lambda^{k-1-r} \left[\sum_{l=1}^{s} b_{12}^{*l} \lambda^{r-m_l} \varphi_2^*(0) \right. \\
&\qquad \left. + \sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&= \lambda^k \varphi_1^*(0) + k\lambda^{k-1} \varphi_2^*(0) \\
&\quad + \sum_{r=0}^{k-1} \lambda^{k-1-r} \times \left[\sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&\quad + \sum_{r=0}^{m_s} \lambda^{k-1-r} b_{12}^{*s} \varphi_2^*(r-m_s) \\
&\quad + b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda^{k-1-r} \varphi_2^*(r-m_1) \right. \\
&\qquad \left. + (m_s-m_1) \lambda^{k-1-m_1} \varphi_2^*(0) \right] \\
&\quad + b_{12}^{*2} \left[\sum_{r=0}^{m_2} \lambda^{k-1-r} \varphi_2^*(r-m_2) \right. \\
&\qquad \left. + (m_s-m_2) \lambda^{k-1-m_2} \varphi_2^*(0) \right] \\
&\quad + \cdots \\
&\quad + b_{12}^{*s-1} \left[\sum_{r=0}^{m_{s-1}} \lambda^{k-1-r} \varphi_2^*(r-m_{s-1}) \right. \\
&\qquad \left. + (m_s-m_{s-1}) \lambda^{k-1-m_{s-1}} \varphi_2^*(0) \right] \\
&\quad + (k-1-m_s) \left[\lambda^{k-1-m_1} b_{12}^{*1} \varphi_2^*(0) \right. \\
&\qquad + \lambda^{k-1-m_2} b_{12}^{*2} \varphi_2^*(0) \\
&\qquad + \cdots + \lambda^{k-1-m_{s-1}} b_{12}^{*s-1} \varphi_2^*(0) \\
&\qquad \left. + \lambda^{k-1-m_2} b_{12}^{*s} \varphi_2^*(0) \right] \\
&= \lambda^k \varphi_1^*(0) + k\lambda^{k-1} \varphi_2^*(0) \\
&\quad + \sum_{r=0}^{k-1} \lambda^{k-1-r} \left[\sum_{l=s+1}^{n} b_{12}^{*l} \varphi_2^*(r-m_l) \right] \\
&\quad + b_{12}^{*1} \left[\sum_{r=0}^{m_1} \lambda^{k-1-r} \varphi_2^*(r-m_1) \right. \\
&\qquad \left. + (k-1-m_1) \lambda^{k-1-m_1} \varphi_2^*(0) \right] \\
&\quad + b_{12}^{*2} \left[\sum_{r=0}^{m_2} \lambda^{k-1-r} \varphi_2^*(r-m_2) \right. \\
&\qquad \left. + (k-1-m_2) \lambda^{k-1-m_2} \varphi_2^*(0) \right] \\
&\quad + \cdots \\
&\quad + b_{12}^{*s-1} \left[\sum_{r=0}^{m_{s-1}} \lambda^{k-1-r} \varphi_2^*(r-m_{s-1}) \right. \\
&\qquad \left. + (k-1-m_{s-1}) \lambda^{k-1-m_{s-1}} \varphi_2^*(0) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{12}^{*s}\left[\sum_{r=0}^{m_s}\lambda^{k-1-r}\varphi_2^*(r-m_s)\right.\\
&\qquad\left.+(k-1-m_s)\lambda^{k-1-m_s}\varphi_2^*(0)\right]\\
&= \lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
&\quad+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=s+1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
&\quad+\sum_{l=1}^{s}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right].
\end{aligned}
\tag{122}
$$

In the end, we solve (113) for $k\in\mathbb{Z}_{m_n+1}^{\infty}$ with initial data deduced from (122); that is, we consider the problem

$$
\begin{aligned}
y_1(k+1) &= \lambda y_1(k)+\lambda^k\varphi_2^*(0)\\
&\quad+\sum_{l=1}^{n}b_{12}^{*l}\lambda^{k-m_l}\varphi_2^*(0)\quad \text{if } k\in\mathbb{Z}_{m_n+1}^{\infty},\\
y_1(m_n+1) &= \lambda^{m_n+1}\varphi_1^*(0)+(m_n+1)\lambda^{m_n}\varphi_2^*(0)\\
&\quad+\sum_{r=0}^{m_n}\lambda^{m_n-r}b_{12}^{*n}\varphi_2^*(r-m_n)\\
&\quad+\sum_{l=1}^{n-1}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{m_n-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+(m_n-m_l)\lambda^{m_n-m_l}\varphi_2^*(0)\right].
\end{aligned}
\tag{123}
$$

Applying formula (33) yields (for $k\in\mathbb{Z}_{m_n+2}^{\infty}$)

$$
\begin{aligned}
&y_1(k)\\
&= \lambda^{k-(m_n+1)}y_1(m_n+1)\\
&\quad+\sum_{r=m_n+1}^{k-1}\lambda^{k-1-r}\left[\lambda^r\varphi_2^*(0)+\sum_{l=1}^{n}b_{12}^{*l}\lambda^{r-m_l}\varphi_2^*(0)\right]\\
&= \lambda^{k-m_n-1}\left[\lambda^{m_n+1}\varphi_1^*(0)+(m_n+1)\lambda^{m_n}\varphi_2^*(0)\right.\\
&\qquad+\sum_{r=0}^{m_n}\lambda^{m_n-r}b_{12}^{*n}\varphi_2^*(r-m_n)\\
&\qquad+\sum_{l=1}^{n-1}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{m_n-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.\left.+(m_n-m_l)\lambda^{m_n-m_l}\varphi_2^*(0)\right]\right]\\
&\quad+\sum_{r=m_n+1}^{k-1}\lambda^{k-1-r}\left[\lambda^r\varphi_2^*(0)+\sum_{l=1}^{n}b_{12}^{*l}\lambda^{r-m_l}\varphi_2^*(0)\right]\\
&= \lambda^k\varphi_1^*(0)+(m_n+1)\lambda^{k-1}\varphi_2^*(0)\\
&\quad+\sum_{r=0}^{m_n}\lambda^{k-1-r}b_{12}^{*n}\varphi_2^*(r-m_n)\\
&\quad+\sum_{l=1}^{n-1}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+(m_n-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right]\\
&\quad+(k-1-m_n)\lambda^{k-1}\varphi_2^*(0)\\
&\quad+\sum_{r=m_n+1}^{k-1}\lambda^{k-1-r}\left[\sum_{l=1}^{n}b_{12}^{*l}\lambda^{r-m_l}\varphi_2^*(0)\right]\\
&= \lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)+\sum_{r=0}^{m_n}\lambda^{k-1-r}b_{12}^{*n}\varphi_2^*(r-m_n)\\
&\quad+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+(m_n-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\right]\\
&\quad+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda^{k-1-r}\varphi_2^*(r-m_2)\right.\\
&\qquad\left.+(m_n-m_2)\lambda^{k-1-m_2}\varphi_2^*(0)\right]\\
&\quad+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{12}^{*n-1}\left[\sum_{r=0}^{m_{n-1}}\lambda^{k-1-r}\varphi_2^*(r-m_{n-1})\right.\\
&\qquad\left.+(m_n-m_{n-1})\lambda^{k-1-m_{n-1}}\varphi_2^*(0)\right]\\
&+(k-1-m_n)\\
&\times\left[\lambda^{k-1-m_1}b_{12}^{*1}\varphi_2^*(0)+\lambda^{k-1-m_2}b_{12}^{*2}\varphi_2^*(0)\right.\\
&\qquad+\cdots+\lambda^{k-1-m_{n-1}}b_{12}^{*n-1}\varphi_2^*(0)\\
&\qquad\left.+\lambda^{k-1-m_n}b_{12}^{*n}\varphi_2^*(0)\right]\\
&=\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
&+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1)\right.\\
&\qquad\left.+(k-1-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\right]\\
&+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda^{k-1-r}\varphi_2^*(r-m_2)\right.\\
&\qquad\left.+(k-1-m_2)\lambda^{k-1-m_2}\varphi_2^*(0)\right]\\
&+\cdots\\
&+b_{12}^{*n-1}\left[\sum_{r=0}^{m_{n-1}}\lambda^{k-1-r}\varphi_2^*(r-m_{n-1})\right.\\
&\qquad\left.+(k-1-m_{n-1})\lambda^{k-1-m_{n-1}}\varphi_2^*(0)\right]\\
&+b_{12}^{*n}\left[\sum_{r=0}^{m_n}\lambda^{k-1-r}\varphi_2^*(r-m_n)\right.\\
&\qquad\left.+(k-1-m_n)\lambda^{k-1-m_n}\varphi_2^*(0)\right]\\
&=\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
&+\sum_{l=1}^{n}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\
&\qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right].
\end{aligned}
\tag{124}
$$

Summing up all particular cases (115)–(124), we get

$$
y_1(k)=\begin{cases}
\varphi_1^*(k) & \text{if } k\in\mathbb{Z}_{-m_n}^{0},\\
\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
\quad+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right] & \text{if } k\in\mathbb{Z}_{1}^{m_1+1},\\
\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
\quad+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=2}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
\quad+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1)\right.\\
\qquad\left.+(k-1-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\right] & \text{if } k\in\mathbb{Z}_{m_1+2}^{m_2+1},\\
\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
\quad+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=3}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
\quad+b_{12}^{*1}\left[\sum_{r=0}^{m_1}\lambda^{k-1-r}\varphi_2^*(r-m_1)\right.\\
\qquad\left.+(k-1-m_1)\lambda^{k-1-m_1}\varphi_2^*(0)\right]\\
\quad+b_{12}^{*2}\left[\sum_{r=0}^{m_2}\lambda^{k-1-r}\varphi_2^*(r-m_2)\right.\\
\qquad\left.+(k-1-m_2)\lambda^{k-1-m_2}\varphi_2^*(0)\right] & \text{if } k\in\mathbb{Z}_{m_2+2}^{m_3+1},\\
\vdots\\
\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
\quad+\sum_{r=0}^{k-1}\lambda^{k-1-r}\left[\sum_{l=s+1}^{n}b_{12}^{*l}\varphi_2^*(r-m_l)\right]\\
\quad+\sum_{l=1}^{s}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\
\qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right] & \text{if } k\in\mathbb{Z}_{m_s+2}^{m_{s+1}+1},\\
\vdots\\
\lambda^k\varphi_1^*(0)+k\lambda^{k-1}\varphi_2^*(0)\\
\quad+\sum_{l=1}^{n}b_{12}^{*l}\left[\sum_{r=0}^{m_l}\lambda^{k-1-r}\varphi_2^*(r-m_l)\right.\\
\qquad\left.+(k-1-m_l)\lambda^{k-1-m_l}\varphi_2^*(0)\right] & \text{if } k\in\mathbb{Z}_{m_n+2}^{\infty}.
\end{cases}
\tag{125}
$$

Formulas (108) and (109) are consequences of (125), (112). □

3. Dimension of the Set of Solutions

Since all the possible cases of the planar system (1) with weak delay have been analysed, we are ready to formulate results concerning the dimension of the space of solutions of (1) assuming that initial condition (3) is variable. Although case $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = b_{21}^{*l} = 0$ does not lead to a weakly delayed system and is excluded by (2), for completeness of analysis we incorporate such possibility in our analysis as well (such a case can be considered as a degenerated weakly delayed system). Before formulation we remark that if an assumption in the following theorem is assumed to be valid for a fixed index $l \in \{1, 2, \ldots, n\}$, it is easy to see that it must be valid for all indices $l = 1, 2, \ldots, n$.

Theorem 10. *Let* (1) *be a weakly delayed system and let* (35) *having both roots different from zero and* $l \in \{1, 2, \ldots, n\}$ *be fixed. Then the space of solutions, being initially* $2(m_n + 1)$*-dimensional, becomes on* $\mathbb{Z}_{m_n+2}^{\infty}$ *only*

(1) $(m_n + 2)$*-dimensional if* (35) *has*

(a) *two real distinct roots and* $(b_{12}^{*l})^2 + (b_{21}^{*l})^2 > 0$,

(b) *a double real root,* $b_{12}^{*l} b_{21}^{*l} = 0$, *and* $(b_{12}^{*l})^2 + (b_{21}^{*l})^2 > 0$.

(c) *a double real root and* $b_{12}^{*l} b_{21}^{*l} \neq 0$,

(2) 2*-dimensional if* (35) *has*

(a) *two real distinct roots and* $b_{12}^{*l} = b_{21}^{*l} = 0$,

(b) *a pair of complex conjugate roots,*

(c) *a double real root and* $b_{12}^{*l} = b_{21}^{*l} = 0$.

Proof. We will carefully go through all the theorems considered (Theorems 5–9) adding the case of a pair of complex conjugate roots and our conclusion will hold at least on $\mathbb{Z}_{m_n+2}^{\infty}$ (some of the statements hold on a larger interval).

(a) Analysing the statement of Theorem 5 (case (36) of two real distinct roots), we obtain the following subcases.

(a1) If $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals $m_n + 2$ since the last formula in (47) uses only $m_n + 2$ arbitrary parameters:

$$\varphi_1^*(0), \varphi_2^*(-m_n), \varphi_2^*(-m_n + 1), \ldots, \varphi_2^*(0). \tag{126}$$

(a2) If $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = 0$, $b_{21}^{*l} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals $m_n + 2$ since the last formula in (48) uses only $m_n + 2$ arbitrary parameters:

$$\varphi_1^*(-m_n), \varphi_1^*(-m_n + 1), \ldots, \varphi_1^*(0), \varphi_2^*(0). \tag{127}$$

(a3) If $b_{12}^{*l} = b_{21}^{*l} = 0$, then $b_{11}^{*l} = b_{22}^{*l} = 0$ and Theorem 5 is not applicable. The dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals 2 since the solution is determined only by 2 arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(0). \tag{128}$$

This means that all the cases considered are covered by conclusions (1)(a) and (2)(a) of Theorem 10.

(b) In case (37) of two complex conjugate roots, we have $b_{11}^{*l} = b_{22}^{*l} = b_{12}^{*l} = b_{21}^{*l} = 0$ (i.e., we deal not with a weakly delayed system, as noted previosly) and the formula (70) uses only 2 arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(0) \tag{129}$$

for every $k \in \mathbb{Z}_1^{\infty}$. This is covered by case (2)(b) of Theorem 10.

(c) Analysing the statement of Theorems 7 and 8 (case (38) of a double real root), we obtain the following subcases.

(c1) If $b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals m_n+2 since the last formula in (76) uses only $m_n + 2$ arbitrary parameters:

$$\varphi_1^*(0), \varphi_2^*(-m_n), \varphi_2^*(-m_n + 1), \ldots, \varphi_2^*(0). \tag{130}$$

(c2) If $b_{12}^{*l} = 0$, $b_{21}^{*l} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals m_n+2 since the last formula in (77) uses only $m_n + 2$ arbitrary parameters:

$$\varphi_1^*(-m_n), \varphi_1^*(-m_n + 1), \ldots, \varphi_1^*(0), \varphi_2^*(0). \tag{131}$$

(c3) If $b_{12}^{*l} = b_{21}^{*l} = 0$ (degenerated weakly delayed system), then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals 2and solutions are determined only by 2 arbitrary parameters:

$$\varphi_1^*(0), \varphi_2^*(0). \tag{132}$$

(c4) If $b_{12}^{*l} b_{21}^{*l} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals m_n+2 since the last formula in (81) uses only $m_n + 2$ arbitrary parameters:

$$C(-m_n), C(-m_n + 1), \ldots, C(0), \varphi_1^*(0), \tag{133}$$

where

$$C(k) := \left[\varphi_1^*(k) + \frac{b_{12}^{*1}}{b_{11}^{*1}} \varphi_2^*(k)\right], \quad k \in \mathbb{Z}_{-m_n}^{0}. \tag{134}$$

The parameter $\varphi_2^*(0)$ cannot be seen as independent since it depends on the independent parameters $\varphi_1^*(0)$ and $C(0)$.

All the cases considered are covered by conclusions (1)(b), (1)(c), and (2)(c) of Theorem 10.

(d) Analysing the statement of Theorem 9 (case (39) of a double real root), we obtain the following subcases:

(d1) If $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = 0$, $b_{12}^{*l} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals $m_n + 2$ since the last formula in (108) uses only $m_n + 2$ arbitrary parameters:

$$\varphi_1^*(0), \varphi_2^*(-m_n), \varphi_2^*(-m_n + 1), \ldots, \varphi_2^*(0) \tag{135}$$

and the last formula in (109) provides no new information.

(d2) If $b_{11}^{*l} = b_{22}^{*l} = b_{21}^{*l} = b_{12}^{*l} = 0$ (degenerated weakly delayed system), then the dimension of the space of solutions on $\mathbb{Z}_{m_n+2}^{\infty}$ equals 2 since solutions are determined only by 2 arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(0). \tag{136}$$

Both cases are covered by conclusions (1)(b) and (2)(c) of Theorem 10.

Since there are no cases other than cases (a)–(d), the proof is finished. □

Theorem 10 can be formulated simply as follows.

Theorem 11. *Let* (1) *be a weakly delayed system and let* (35) *have both roots different from zero, then the space of solutions, being initially* $2(m_n + 1)$*-dimensional, is on* $\mathbb{Z}_{m_n+2}^{\infty}$ *only*

(1) $(m_n + 2)$*-dimensional if* $(b_{12}^{*l})^2 + (b_{21}^{*l})^2 > 0$,

(2) 2*-dimensional if* $b_{12}^{*l} = b_{21}^{*l} = 0$.

We omit the proofs of the following two theorems since, again, they are much the same as those of Theorems 5–9.

Theorem 12. *Let* (1) *be a weakly delayed system and let* (35) *have a simple root* $\lambda = 0$*, then the space of solutions, being initially* $2(m_n + 1)$*-dimensional, is either* $(m_n + 1)$*-dimensional or* 1*-dimensional on* $\mathbb{Z}_{m_n+2}^{\infty}$.

Theorem 13. *Let* (1) *be a weakly delayed system and let* (35) *have a double root* $\lambda = 0$*, then the space of solutions, being initially* $2(m_n + 1)$*-dimensional, turns into a* 0*-dimensional space on* $\mathbb{Z}_{m_n+2}^{\infty}$*, namely, into the zero solution.*

4. Concluding Remarks

To our best knowledge, weakly delayed systems were firstly defined in [9] for systems of linear delayed differential systems with constant coefficients and in [1] for planar linear discrete systems with a single delay (in these papers such systems are called systems with a weak delay). The weakly delayed systems analyzed in this paper can be simplified and their solutions can be found in explicit analytical forms (results obtained generalize those in [1, 2]). Consequently, analytical forms of solutions can be used directly to solve several problems for weakly delayed systems, for example, problems of asymptotical behavior of their solutions, boundary-value problems, or some problems of control theory (using different methods, such problems have recently been investigated e.g., in [10–18]). For an alternative approach to differential-difference equations using the variational iteration method and new analytical and asymptotic methods see, for example, [19–21].

In the case of discrete systems of two equations investigated in this paper, to obtain the corresponding eigenvalues, it is sufficient to solve only a second-order polynomial equation rather than a polynomial equation of order $2m_n$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Representation of the Solutions of Linear Discrete Systems with Constant Coefficients and Two Delays

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The purpose of this paper is to develop a method for the construction of solutions to initial problems of linear discrete systems with constant coefficients and with two delays $\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k)$, where $m, n \in \mathbb{N}$, $m \neq n$, are fixed, $k = 0, \ldots, \infty$, $B = (b_{ij})$, $C = (c_{ij})$ are constant $r \times r$ matrices, f is a given $r \times 1$ vector, and x is an $r \times 1$ unknown vector. Solutions are expressed with the aid of a special function called the discrete matrix delayed exponential for two delays. Such approach results in a possibility to express an initial Cauchy problem in a closed form. Examples are shown illustrating the results obtained.

1. Introduction

Throughout the paper, we will use the following notation. For integers s, t, $s \leq t$, we define the set $\mathbb{Z}_s^t := \{s, s+1, \ldots, t-1, t\}$. Similarly, we define the sets $\mathbb{Z}_{-\infty}^t := \{\ldots, t-1, t\}$ and $\mathbb{Z}_s^\infty := \{s, s+1, \ldots\}$. The function $\lfloor \cdot \rfloor$ used below is the floor integer function. We will employ the following property of the floor integer function:

$$x - 1 < \lfloor x \rfloor \leq x, \tag{1}$$

where $x \in \mathbb{R}$.

In this paper, we deal with the discrete system

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k), \tag{2}$$

where $m, n \in \mathbb{N}$, $m \neq n$, are fixed, $k \in \mathbb{Z}_0^\infty$, $B = (b_{ij})$, $C = (c_{ij})$ are constant $r \times r$ matrices, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^r$ is a given $r \times 1$ vector, and $x : \mathbb{Z}_0^\infty \to \mathbb{R}^r$ is an $r \times 1$ unknown vector.

Together with (2), we consider an initial (Cauchy) problem

$$x(k) = \varphi(k). \tag{3}$$

Define binomial coefficients as customary; that is, for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$\binom{n}{k} := \begin{cases} \dfrac{n!}{k!\,(n-k)!} & \text{if } n \geq k \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We recall that, for a well-defined discrete function $\omega(k)$, the forward difference operator Δ is defined as $\Delta\omega(k) = \omega(k+1) - \omega(k)$. In the paper, we also adopt the customary notation $\sum_{i=i_1}^{i_2} g_i = 0$ if $i_2 < i_1$. In the case of double sums, we set

$$\sum_{i=i_1, j=j_1}^{i_2, j_2} g_{ij} = 0 \tag{5}$$

if at least one of the inequalities $i_2 < i_1$, $j_2 < j_1$ holds.

In [1, 2], a discrete matrix delayed exponential for a single delay $m \in \mathbb{N}$ was defined.

Definition 1. For an $r \times r$ constant matrix B, $k \in \mathbb{Z}$, and fixed $m \in \mathbb{N}$, one defines the discrete matrix delayed exponential e_m^{Bk} as follows:

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta \\ \quad \text{if } k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I + \sum_{j=1}^{\ell} B^j \cdot \begin{pmatrix} k - m(j-1) \\ j \end{pmatrix} \\ \quad \text{if } \ell = 0, 1, 2, \ldots, k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}, \end{cases} \tag{6}$$

where Θ is an $r \times r$ null matrix and I is an $r \times r$ unit matrix.

Such discrete matrix delayed exponential was used in [1] to construct solutions of the initial problems (2), (3) with $C \equiv \Theta$, where Θ is an $r \times r$ zero matrix. In these constructions, the main property (Theorem 2) of discrete matrix delayed exponential for a single delay $m \in \mathbb{N}$ was utilized in [1].

Theorem 2. *Let B be a constant $r \times r$ matrix. Then, for $k \in \mathbb{Z}_{-m}^{\infty}$,*

$$\Delta\, \mathrm{e}_m^{Bk} = B\, \mathrm{e}_m^{B(k-m)}. \tag{7}$$

The properties of delayed matrix exponential functions for their continuous and discrete variants and their applications are the topic of recent papers [1–18]. We note that the definition of the delayed matrix exponential was first defined for the continuous case in [4] and, for the discrete case, in [1, 2].

The paper is organized as follows. Discrete matrix delayed exponentials for two delays and their main property are considered in Section 2. A representation of the solution to problem (2), (3) is given in Section 3 and examples illustrating the results obtained are shown in Section 4.

2. Discrete Matrix Delayed Exponential for Two Delays and Its Main Property

In order to extend the results proved in [1, 2] to problems (2), (3), a discrete matrix delayed exponential for two delays was proposed in [3]. There is a discrete matrix delayed exponential for two delays $m, n \in \mathbb{N}$, $m \neq n$, defined as follows.

Definition 3. Let B, C be constant $r \times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$, be fixed integers. One defines a discrete $r \times r$ matrix function e_{mn}^{BCk} called the discrete matrix delayed exponential for two delays m, n and for two $r \times r$ constant matrices B, C:

$$\mathrm{e}_{mn}^{BCk} := \begin{cases} \Theta \\ \quad \text{if } k \in \mathbb{Z}_{-\infty}^{-\max\{m,n\}-1}, \\ I \\ \quad \text{if } k \in \mathbb{Z}_{-\max\{m,n\}}^{0}, \\ I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \begin{pmatrix} i+j \\ i \end{pmatrix} \begin{pmatrix} k - mi - nj \\ i+j+1 \end{pmatrix} \\ \quad \text{if } k \in \mathbb{Z}_{1}^{\infty}, \end{cases} \tag{8}$$

where

$$p_{(k)} := \left\lfloor \frac{k+m}{m+1} \right\rfloor, \qquad q_{(k)} := \left\lfloor \frac{k+n}{n+1} \right\rfloor. \tag{9}$$

Let us show an example illustrating this special exponential function.

Example 4. For $k \in \mathbb{Z}_0^{12}$ we will construct the matrix e_{mn}^{BCk} if $m = 2$ and $n = 3$. Computing particular matrices generating $\mathrm{e}_{2,3}^{BCk}$ for $k \in \mathbb{Z}_0^{12}$, we get

$$\begin{aligned}
\mathrm{e}_{2,3}^{BC0} &= I, \qquad \mathrm{e}_{2,3}^{BC1} = I + B + C, \\
\mathrm{e}_{2,3}^{BC2} &= I + (B+C)\,2, \qquad \mathrm{e}_{2,3}^{BC3} = I + (B+C)\,3, \\
\mathrm{e}_{2,3}^{BC4} &= I + (B+C)(4+B), \\
\mathrm{e}_{2,3}^{BC5} &= I + (B+C)(5+3B+C), \\
\mathrm{e}_{2,3}^{BC6} &= I + (B+C)(6+6B+3C), \\
\mathrm{e}_{2,3}^{BC7} &= I + (B+C)\left(7+10B+6C+B^2\right), \\
\mathrm{e}_{2,3}^{BC8} &= I + (B+C)\left(8+15B+10C+4B^2+2BC\right), \\
\mathrm{e}_{2,3}^{BC9} &= I + (B+C)\left(9+21B+15C+10B^2+8BC+C^2\right), \\
\mathrm{e}_{2,3}^{BC10} &= I + (B+C) \\
&\quad \times \left(10+28B+21C+20B^2+20BC+4C^2+B^3\right), \\
\mathrm{e}_{2,3}^{BC11} &= I + (B+C)\left(11+36B+28C+35B^2\right. \\
&\qquad \left. +40BC+10C^2+5B^3+3B^2C\right), \\
\mathrm{e}_{2,3}^{BC12} &= I + (B+C)\left(12+45B+36C+56B^2+70BC\right. \\
&\qquad \left. +20C^2+15B^3+15B^2C+3BC^2\right).
\end{aligned} \tag{10}$$

The main property of e_{mn}^{BCk} was proved in [3].

Theorem 5. *Let B, C be constant $r \times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$, be fixed integers. Then*

$$\Delta\, \mathfrak{e}_{mn}^{BCk} = B\, \mathfrak{e}_{mn}^{BC(k-m)} + C\, \mathfrak{e}_{mn}^{BC(k-n)} \tag{11}$$

for $k \in \mathbb{Z}_0^{\infty}$.

The analysis of $\mathfrak{e}_{mn}^{BCk}$ applicability to a representation of the solution to initial problem (2), (3) unfortunately does not lead to satisfactory results because, as we will see below, an additional condition $\det(B + C) \neq 0$ is necessary. A small difference in the definition results in representations of solutions of initial problems without this assumption. Now we give a second definition of a discrete matrix delayed exponential for two delays $\widetilde{\mathfrak{e}}_{mn}^{BCk}$.

Definition 6. Let B, C be constant $r \times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$, be fixed integers. One defines a discrete $r \times r$ matrix function $\widetilde{\mathfrak{e}}_{mn}^{BCk}$ called the discrete matrix delayed exponential for two delays m, n and for two $r \times r$ constant matrices B, C as follows:

$$\widetilde{\mathfrak{e}}_{mn}^{BCk} := \begin{cases} \Theta \\ \quad \text{if } k \in \mathbb{Z}_{-\infty}^{-1}, \\ I \\ \quad \text{if } k \in \mathbb{Z}_0^{m}, \\ I + B \displaystyle\sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} \\ \quad + C \displaystyle\sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \\ \quad \text{if } k \in \mathbb{Z}_{m+1}^{\infty}, \end{cases} \tag{12}$$

where

$$p_{(k)} := \left\lfloor \frac{k+m}{m+1} \right\rfloor, \qquad q_{(k)} := \left\lfloor \frac{k+n}{n+1} \right\rfloor. \tag{13}$$

Remark 7. For $k \in \mathbb{Z}_0^{n}$, it is easy to deduce that $\widetilde{\mathfrak{e}}_{mn}^{BCk} = \mathfrak{e}_{m}^{B(k-m)}$.

In order to compare both types of discrete delayed matrices for two delays and see the difference between both definitions, we consider the following example where delays are the same as in Example 4.

Example 8. For $k \in \mathbb{Z}_0^{12}$ we will construct the matrix $\widetilde{\mathfrak{e}}_{mn}^{BCk}$ if $m = 2$ and $n = 3$. Computing particular matrices generating $\widetilde{\mathfrak{e}}_{2,3}^{BCk}$ for $k \in \mathbb{Z}_0^{12}$, we get

$$\begin{aligned}
\widetilde{\mathfrak{e}}_{2,3}^{BC0} &= I, \qquad \widetilde{\mathfrak{e}}_{2,3}^{BC1} = I, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC2} &= I, \qquad \widetilde{\mathfrak{e}}_{2,3}^{BC3} = I + B, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC4} &= I + 2B + C, \qquad \widetilde{\mathfrak{e}}_{2,3}^{BC5} = I + 3B + 2C, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC6} &= I + 4B + 3C + B^2, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC7} &= I + 5B + 4C + 3B^2 + 2BC, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC8} &= I + 6B + 5C + 6B^2 + 6BC + C^2, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC9} &= I + 7B + 6C + 10B^2 + 12BC + 3C^2 + B^3, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC10} &= I + 8B + 7C + 15B^2 + 20BC + 6C^2 + 4B^3 + 3B^2C, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC11} &= I + 9B + 8C + 21B^2 + 30BC + 10C^2 \\
&\quad + 10B^3 + 12B^2C + 3BC^2, \\
\widetilde{\mathfrak{e}}_{2,3}^{BC12} &= I + 10B + 9C + 28B^2 + 42BC + 15C^2 \\
&\quad + 20B^3 + 30B^2C + 12BC^2 + C^3 + B^4.
\end{aligned} \tag{14}$$

The main property of $\widetilde{\mathfrak{e}}_{mn}^{BCk}$ is given by the following theorem.

Theorem 9. *Let B, C be constant $r \times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$, be fixed integers. Then*

$$\Delta\, \widetilde{\mathfrak{e}}_{mn}^{BCk} = B\, \widetilde{\mathfrak{e}}_{mn}^{BC(k-m)} + C\, \widetilde{\mathfrak{e}}_{mn}^{BC(k-n)} \tag{15}$$

for $k \in \mathbb{Z}_0^{\infty}$.

Proof. Let $k \geq 1$. From (1) and (13), we can see easily that, for an integer $k \geq 0$ satisfying

$$\begin{aligned}
&(p_{(k)} - 1)(m+1) + 1 \\
&\quad \leq k \leq p_{(k)}(m+1) \wedge (q_{(k)} - 1)(n+1) + 1 \\
&\quad \leq k \leq q_{(k)}(n+1),
\end{aligned} \tag{16}$$

the equation

$$\Delta \widetilde{\mathfrak{e}}_{mn}^{BCk} = \Delta \left[I + B \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \right] \tag{17}$$

holds by Definition 6 of $\widetilde{\mathfrak{e}}_{mn}^{BCk}$. Since $\Delta I = \Theta$, we have

$$\Delta \widetilde{\mathfrak{e}}_{mn}^{BCk} = \Delta \left[B \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \right]. \tag{18}$$

By the definition of the forward difference, that is,

$$\Delta \widetilde{\mathfrak{e}}_{mn}^{BCk} = \widetilde{\mathfrak{e}}_{mn}^{BC(k+1)} - \widetilde{\mathfrak{e}}_{mn}^{BCk}, \tag{19}$$

we conclude that it is reasonable to divide the proof into four parts given by the four values of integer k.

In the first case, k is such that

$$\begin{aligned}&(p_{(k)}-1)(m+1)+1\\&\quad\le k<p_{(k)}(m+1)\wedge(q_{(k)}-1)(n+1)+1\\&\quad\le k<q_{(k)}(n+1),\end{aligned}\tag{20}$$

in the second case

$$k=p_{(k)}(m+1)\wedge(q_{(k)}-1)(n+1)+1\le k<q_{(k)}(n+1),\tag{21}$$

in the third case

$$(p_{(k)}-1)(m+1)+1\le k<p_{(k)}(m+1)\wedge k=q_{(k)}(n+1),\tag{22}$$

and in the fourth case

$$k=p_{(k)}(m+1)\wedge k=q_{(k)}(n+1).\tag{23}$$

We see that the above four cases cover all the possible relations between k, $p_{(k)}$, and $q_{(k)}$.

In the proof, we use obvious identities

$$\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1},\tag{24}$$

where $n,k\in\mathbb{N}$ and

$$\begin{aligned}&\binom{i}{i}=\binom{i-1}{i-1},\qquad\binom{j}{0}=\binom{j-1}{0},\\&\binom{i+j}{i}=\binom{i+j-1}{i-1}+\binom{i+j-1}{i},\end{aligned}\tag{25}$$

where $i,j\in\mathbb{N}$, derived from (4) and (24).

Now we consider (in parts (I)–(IV) below) all four cases and perform auxiliary computations. The proof will be finished in part (V).

(I) $(p_{(k)}-1)(m+1)+1\le k<p_{(k)}(m+1)\wedge(q_{(k)}-1)(n+1)+1\le k<q_{(k)}(n+1)$. From (1) and (13), we get

$$\begin{aligned}p_{(k-m)}&=\left\lfloor\frac{k-m+m}{m+1}\right\rfloor\le\frac{k}{m+1}<p_{(k)},\\p_{(k-m)}&=\left\lfloor\frac{k-m+m}{m+1}\right\rfloor>\frac{k}{m+1}-1=\frac{k-m-1}{m+1}>p_{(k)}-2.\end{aligned}\tag{26}$$

Therefore, $p_{(k-m)}=p_{(k)}-1$ and, by Definition 6,

$$\begin{aligned}&\tilde{\mathfrak{e}}_{mn}^{BC(k-m)}\\&=I+B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1}B^iC^j\binom{i+j}{i}\binom{k-m-m-mi-nj}{i+j+1}\\&\quad+C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1}B^iC^j\binom{i+j}{i}\binom{k-m-n-mi-nj}{i+j+1}.\end{aligned}\tag{27}$$

Similarly, omitting details, we get, using (1) and (13), $q_{(k-n)}=q_{(k)}-1$ and

$$\begin{aligned}&\tilde{\mathfrak{e}}_{mn}^{BC(k-n)}\\&=I+B\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2}B^iC^j\binom{i+j}{i}\binom{k-n-m-mi-nj}{i+j+1}\\&\quad+C\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2}B^iC^j\binom{i+j}{i}\binom{k-n-n-mi-nj}{i+j+1}.\end{aligned}\tag{28}$$

Let $q_{(k-m)}\ge1$. We show that

$$\begin{aligned}&\binom{k-m-m-mi-nj}{i+j+1}=0\quad\text{if } i\ge0,\ j\ge q_{(k-m)},\\&\binom{k-m-n-mi-nj}{i+j+1}=0\quad\text{if } i\ge0,\ j\ge q_{(k-m)}.\end{aligned}\tag{29}$$

By (1),

$$q_{(k-m)}=\left\lfloor\frac{k-m+n}{n+1}\right\rfloor>\frac{k-m+n}{n+1}-1=\frac{k-m-1}{n+1}\tag{30}$$

or

$$\begin{aligned}&k-m-m\\&\quad<(n+1)q_{(k-m)}+1\\&\quad\le(m+1)i+(n+1)j+1\quad\text{if } i\ge0,\ j\ge q_{(k-m)},\\&k-m-n\\&\quad<(n+1)q_{(k-m)}+1\\&\quad\le(m+1)i+(n+1)j+1\quad\text{if } i\ge0,\ j\ge q_{(k-m)}.\end{aligned}\tag{31}$$

From the last inequalities, we get

$$\begin{aligned}&k-m-m-mi-nj<i+j+1\quad\text{if } i\ge0,\ j\ge q_{(k-m)},\\&k-m-n-mi-nj<i+j+1\quad\text{if } i\ge0,\ j\ge q_{(k-m)},\end{aligned}\tag{32}$$

and (29) holds by (4). For that reason and since $q_{(k-m)}\le q_{(k)}$, we can replace $q_{(k-m)}$ by $q_{(k)}$ in (27). Thus, we have

$$\begin{aligned}&\tilde{\mathfrak{e}}_{mn}^{BC(k-m)}\\&=I+B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1}B^iC^j\binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1}\\&\quad+C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1}B^iC^j\binom{i+j}{i}\binom{k-n-m(i+1)-nj}{i+j+1}.\end{aligned}\tag{33}$$

It is easy to see that, due to (5), formula (33) can be used instead of (27) if $q_{(k-m)} < 1$ too. Let $p_{(k-n)} \geq 1$. Similarly, we can show that

$$\begin{aligned} &\binom{k-n-m-mi-nj}{i+j+1} = 0 \quad \text{if } i \geq p_{(k-n)}, j \geq 0, \\ &\binom{k-n-n-mi-nj}{i+j+1} = 0 \quad \text{if } i \geq p_{(k-n)}, j \geq 0, \end{aligned} \tag{34}$$

and, since $p_{(k-n)} \leq p_{(k)}$, we can replace $p_{(k-n)}$ by $p_{(k)}$ in (28). Thus, we have

$$\begin{aligned} &\tilde{e}_{mn}^{BC(k-n)} \\ &= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-m-mi-n(j+1)}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-n(j+1)}{i+j+1}. \end{aligned} \tag{35}$$

It is easy to see that, due to (5), formula (35) can be used instead of (28) if $p_{(k-n)} < 1$ too By Definition 6,

$$\begin{aligned} &\tilde{e}_{mn}^{BC(k+1)} \\ &= I + B \sum_{i=0,j=0}^{p_{(k+1)}-1,q_{(k+1)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k+1)}-1,q_{(k+1)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}. \end{aligned} \tag{36}$$

Due to (1), we also conclude that

$$p_{(k+1)} = p_{(k)}, \qquad q_{(k+1)} = q_{(k)} \tag{37}$$

because

$$\begin{aligned} p_{(k+1)} &= \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \leq \frac{k}{m+1} + 1 < p_{(k)} + 1, \\ p_{(k+1)} &= \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 \\ &= \frac{k}{m+1} \geq p_{(k)} - 1 + \frac{1}{m+1}. \end{aligned} \tag{38}$$

The second formula can be proved similarly.

Then,

$$\begin{aligned} &\tilde{e}_{mn}^{BC(k+1)} \\ &= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}. \end{aligned} \tag{39}$$

Now we are able to prove that

$$\begin{aligned} \Delta\tilde{e}_{mn}^{BCk} &= B\tilde{e}_{mn}^{BC(k-m)} + C\tilde{e}_{mn}^{BC(k-n)} \\ &= B\Bigg[I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m(i+1)-nj}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-m(i+1)-nj}{i+j+1} \Bigg] \\ &\quad + C\Bigg[I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-m-mi-n(j+1)}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-n(j+1)}{i+j+1} \Bigg]. \end{aligned} \tag{40}$$

(II) $k = p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \leq k < q_{(k)}(n+1)$. In this case,

$$\begin{aligned} p_{(k-m)} &= \left\lfloor \frac{k-m+m}{m+1} \right\rfloor = \left\lfloor \frac{k}{m+1} \right\rfloor = p_{(k)}, \\ p_{(k+1)} &= \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \leq \frac{k+1+m}{m+1} = \frac{k}{m+1} + 1 = p_{(k)} + 1, \\ p_{(k+1)} &= \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} = p_{(k)} \end{aligned} \tag{41}$$

and $p_{(k+1)} = p_{(k)} + 1$. In addition to this (see the relevant computations performed in case (I)), we have $q_{(k-n)} = q_{(k)} - 1$ and $q_{(k+1)} = q_{(k)}$.

Then,

$$\begin{aligned} &\tilde{e}_{mn}^{BC(k+1)} \\ &= I + B \sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}, \end{aligned} \tag{42}$$

$$\begin{aligned} &\tilde{e}_{mn}^{BC(k-m)} \\ &= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} \\ &\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1}, \end{aligned} \tag{43}$$

$$\tilde{e}_{mn}^{BC(k-n)} = I + B \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}. \tag{44}$$

For $k = p_{(k)}(m+1)$, $i = p_{(k)}$, and $j \geq 0$, we have

$$\binom{k+1-m-mi-nj}{i+j+1} = \binom{p_{(k)}+1-m-nj}{p_{(k)}+1+j} = 0, \quad \binom{k+1-n-mi-nj}{i+j+1} = \binom{p_{(k)}+1-n-nj}{p_{(k)}+1+j} = 0 \tag{45}$$

and, for $k = p_{(k)}(m+1)$, $i = p_{(k)} - 1$, and $j \geq 0$, we have

$$\binom{k-m-m-mi-nj}{i+j+1} = \binom{p_{(k)}-m-nj}{p_{(k)}+j} = 0, \quad \binom{k-m-n-mi-nj}{i+j+1} = \binom{p_{(k)}-n-nj}{p_{(k)}+j} = 0. \tag{46}$$

Thus, we can substitute $p_{(k)} - 1$ for $p_{(k)}$ in (42) and $p_{(k)} - 2$ for $p_{(k)} - 1$ in (43).

Like with the computations performed in the previous part of the proof, (29), (34) hold. So we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (43) and $p_{(k)}$ for $p_{(k-n)}$ in (44).

Accordingly, we have

$$\tilde{e}_{mn}^{BC(k+1)} = I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}, \tag{47}$$

$$\tilde{e}_{mn}^{BC(k-m)} = I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m(i+1)-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-m(i+1)-nj}{i+j+1}, \tag{48}$$

$$\tilde{e}_{mn}^{BC(k-n)} = I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-m-mi-n(j+1)}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-n(j+1)}{i+j+1}. \tag{49}$$

It is easy to see that, due to (5), formula (48) can also be used instead of (43) if $q_{(k-m)} < 1$ and formula (49) can also be used instead of (44) if $p_{(k-n)} < 1$. Therefore, we see that (like in part (I)) the relation (40) must be proved.

(III) $(p_{(k)} - 1)(m+1) + 1 \leq k < p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)$. In this case, we have (see the relevant computations in cases (I) and (II))

$$p_{(k-m)} = p_{(k)} - 1, \qquad p_{(k+1)} = p_{(k)}, \qquad q_{(k-n)} = q_{(k)}, \qquad q_{(k+1)} = q_{(k)} + 1. \tag{50}$$

Then,

$$\tilde{e}_{mn}^{BC(k+1)} = I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}, \tag{51}$$

$$\tilde{e}_{mn}^{BC(k-m)} = I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1}, \tag{52}$$

$$\tilde{e}_{mn}^{BC(k-n)} = I + B \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}. \tag{53}$$

For $k = q_{(k)}(n+1)$, $j = q_{(k)}$, and $i \geq 0$, we have

$$\binom{k+1-m-mi-nj}{i+j+1} = \binom{q_{(k)}+1-m-mi}{i+q_{(k)}+1} = 0, \quad \binom{k+1-n-mi-nj}{i+j+1} = \binom{q_{(k)}+1-n-mi}{i+q_{(k)}+1} = 0 \tag{54}$$

and, for $k = q_{(k)}(m+1)$, $j = q_{(k)} - 1$, and $i \geq 0$, we get

$$\binom{k-n-m-mi-nj}{i+j+1} = \binom{q_{(k)}-m-mi}{i+q_{(k)}} = 0,$$
$$\binom{k-n-n-mi-nj}{i+j+1} = \binom{q_{(k)}-n-mi}{i+q_{(k)}} = 0. \tag{55}$$

Thus we can replace $q_{(k)}$ by $q_{(k)}-1$ in (51) and $q_{(k)}-1$ by $q_{(k)}-2$ in (53).

Like with the computations performed in cases (I) and (II), formulas (29), (34) hold and we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (52) and $q_{(k)}$ for $q_{(k-n)}$ in (53). This means that

$$\begin{aligned}\tilde{e}_{mn}^{BC(k+1)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-n-mi-nj}{i+j+1},\end{aligned} \tag{56}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-m)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-n-mi-nj}{i+j+1},\end{aligned} \tag{57}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-n)} &= I + B\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-n-mi-nj}{i+j+1}.\end{aligned} \tag{58}$$

It is easy to see that, due to (5), formula (57) can also be used instead of (52) if $q_{(k-m)} < 1$ and formula (58) can also be used instead of (53) if $p_{(k-n)} < 1$. Therefore, we see that (as in parts (I), (II)) (40) must be proved.

(IV) $k = p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)$. In this case, we have (see similar combinations in the cases (II) and (III))

$$p_{(k-m)} = p_{(k)}, \qquad p_{(k+1)} = p_{(k)} + 1,$$
$$q_{(k-n)} = q_{(k)}, \qquad q_{(k+1)} = q_{(k)} + 1. \tag{59}$$

Then,

$$\begin{aligned}\tilde{e}_{mn}^{BC(k+1)} &= I + B\sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^iC^j\binom{i+j}{i}\binom{k+1-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^iC^j\binom{i+j}{i}\binom{k+1-n-mi-nj}{i+j+1},\end{aligned} \tag{60}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-m)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-n-mi-nj}{i+j+1},\end{aligned} \tag{61}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-n)} &= I + B\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-n-mi-nj}{i+j+1}.\end{aligned} \tag{62}$$

As in part (II), for $k = p_{(k)}(m+1)$, $i = p_{(k)}$, and $j \geq 0$, formulas (45) hold and, for $k = p_{(k)}(m+1)$, $i = p_{(k)} - 1$, and $j \geq 0$, formulas (46) hold. Thus we can substitute $p_{(k)} - 1$ for $p_{(k)}$ in (60) and $p_{(k)} - 2$ for $p_{(k)} - 1$ in (61).

As in part (III), for $k = q_{(k)}(n+1)$, $j = q_{(k)}$, and $i \geq 0$, formulas (54) hold and, for $k = q_{(k)}(m+1)$, $j = q_{(k)} - 1$, and $i \geq 0$, formulas (55) hold. Thus we can replace $q_{(k)}$ by $q_{(k)} - 1$ in (60) and $q_{(k)} - 1$ by $q_{(k)} - 2$ in (62).

As before, (29), (34) hold and we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (61) and $p_{(k)}$ for $p_{(k-n)}$ in (62). Thus, we have

$$\begin{aligned}\tilde{e}_{mn}^{BC(k+1)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-m-mi-nj}{i+j+1} \\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-n-mi-nj}{i+j+1},\end{aligned} \tag{63}$$

$$\begin{aligned}
&\tilde{e}_{mn}^{BC(k-m)} \\
&= I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1},
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\tilde{e}_{mn}^{BC(k-n)} \\
&= I + B \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}.
\end{aligned} \tag{65}$$

It is easy to see that, due to (5), formula (64) can also be used instead of (61) if $q_{(k-m)} < 1$ and formula (65) can also be used instead of (62) if $p_{(k-n)} < 1$. Therefore, we see that (as in all the previous parts) (40) must be proved.

(V) The Proof of Formula (40). Now we prove (40). With the aid of (18), (19), (24), and (36), we get

$$\begin{aligned}
&\Delta\tilde{e}_{mn}^{BCk} \\
&= \tilde{e}_{mn}^{BC(k+1)} - \tilde{e}_{mn}^{BCk} \\
&= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1} \\
&\quad - I - B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} \\
&\quad - C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \\
&= B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-m-mi-nj}{i+j+1} - \binom{k-m-mi-nj}{i+j+1} \right] \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-n-mi-nj}{i+j+1} - \binom{k-n-mi-nj}{i+j+1} \right] \\
&= B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j} \\
&= B \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-m-mi}{i} \right. \\
&\quad + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-m-nj}{j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j} \right] \\
&\quad + C \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-n-mi}{i} \right. \\
&\quad + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-n-nj}{j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j} \right].
\end{aligned} \tag{66}$$

By (25), we have

$$\begin{aligned}
&\Delta\tilde{e}_{mn}^{BCk} \\
&= B \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1} \binom{k-m-mi}{i} \right. \\
&\quad + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0} \binom{k-m-nj}{j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-m-mi-nj}{i+j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-m-mi-nj}{i+j} \right] \\
&\quad + C \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1} \binom{k-n-mi}{i} \right. \\
&\quad + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0} \binom{k-n-nj}{j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-n-mi-nj}{i+j}
\end{aligned}$$

$$+\sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-n-mi-nj}{i+j}\Bigg]$$

$$= B\Bigg[I+\sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-m-mi-nj}{i+j}$$

$$+\sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-m-mi-nj}{i+j}\Bigg]$$

$$+C\Bigg[I+\sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-n-mi-nj}{i+j}$$

$$+\sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-n-mi-nj}{i+j}\Bigg]. \tag{67}$$

Now, in the first and third sum, we replace the summation index i by $i+1$ and, in the second and fourth sum, we replace the summation index j by $j+1$. Then,

$$\Delta\tilde{e}_{mn}^{BCk}$$

$$= B\Bigg[I+\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j\binom{i+j}{i}$$

$$\times\binom{k-m-m(i+1)-nj}{i+j+1}$$

$$+\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1}\binom{i+j}{i}\binom{k-m-mi-n(j+1)}{i+j+1}\Bigg]$$

$$+C\Bigg[I+\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j\binom{i+j}{i}$$

$$\times\binom{k-n-m(i+1)-nj}{i+j+1}$$

$$+\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1}\binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}\Bigg]$$

$$= B+B^2\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1}$$

$$+BC\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-m-mi-n(j+1)}{i+j+1}$$

$$+C+BC\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}$$

$$\times\binom{k-n-m(i+1)-nj}{i+j+1}$$

$$+C^2\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}$$

$$= B\Bigg[I+B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1}$$

$$+C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-m(i+1)-nj}{i+j+1}\Bigg]$$

$$+C\Bigg[I+B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}$$

$$\times\binom{k-m-mi-n(j+1)}{i+j+1}$$

$$+C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}\Bigg]$$

$$= B\tilde{e}_{mn}^{BC(k-m)}+C\tilde{e}_{mn}^{BC(k-n)}. \tag{68}$$

Due to (33) and (35), we conclude that formula (40) is valid.

We proved that formula (15) holds in each of the considered cases (I), (II), (III), and (IV) for $k\geq 1$. If $k=0$, the proof can be done directly because $p_{(0)}=q_{(0)}=0$, $p_{(1)}=q_{(1)}=1$,

$$\begin{aligned}
\Delta\tilde{e}_{mn}^{BC0} &= \tilde{e}_{mn}^{BC1}-\tilde{e}_{mn}^{BC0}\\
&= I+B\sum_{i=0,j=0}^{0,0} B^iC^j\binom{i+j}{i}\binom{1-m-mi-nj}{i+j+1}\\
&\quad +C\sum_{i=0,j=0}^{0,0} B^iC^j\binom{i+j}{i}\binom{1-n-mi-nj}{i+j+1}\\
&\quad -I-B\sum_{i=0,j=0}^{-1,-1} B^iC^j\binom{i+j}{i}\binom{-m-mi-nj}{i+j+1}\\
&\quad -C\sum_{i=0,j=0}^{-1,-1} B^iC^j\binom{i+j}{i}\binom{-n-mi-nj}{i+j+1}\\
&= I+B\Theta+C\Theta-I-B\Theta-C\Theta=\Theta,\\
B\tilde{e}_{mn}^{BC(-m)}&+C\tilde{e}_{mn}^{BC(-n)}=B\Theta+C\Theta=\Theta.
\end{aligned} \tag{69}$$

Formula (15) holds again. Theorem 9 is proved. □

3. Representing the Solution of an Initial Problem by Discrete Matrix Delayed Exponential for Two Delays

In this part, we prove the main results of the paper. With the aid of both discrete matrix delayed exponentials we

give formulas for the solution of the homogeneous and nonhomogeneous initial problem (2), (3).

3.1. Representing the Solution of a Homogeneous Initial Problem. Consider the homogeneous initial problem

$$\Delta x(k) = Bx(k-m) + Cx(k-n), \quad k \in \mathbb{Z}_0^{\infty}, \tag{70}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-n}^{0}. \tag{71}$$

First we derive formulas for the solution of (70), (71) with the aid of discrete matrix delayed exponential e_{mn}^{BCk} and then with the aid of discrete matrix delayed exponential $\tilde{e}_{mn}^{BCk}$.

Theorem 10. *Let B, C be constant $r \times r$ matrices such that*

$$BC = CB, \qquad \det(B+C) \neq 0, \tag{72}$$

and let $m, n \in \mathbb{N}$, $m < n$, be fixed integers. Then, the solution of the initial Cauchy problem (70), (71) *can be expressed in the form*

$$x(k) = \sum_{j=0}^{n} e_{mn}^{BC(k+j)} v_j, \tag{73}$$

where $k \in \mathbb{Z}_{-n}^{\infty}$ and

$$\begin{gathered} v_0 = \varphi(-n) - \sum_{s=1}^{n} v_s, \\ v_\ell = (B+C)^{-1}\left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell} \Delta\, e_{mn}^{BCt} v_{t+\ell}\right], \quad \ell \in \mathbb{Z}_1^{n}. \end{gathered} \tag{74}$$

Proof. We are going to find the solution of the problem (70), (71) in the form

$$x(k) = \sum_{j=0}^{n} e_{mn}^{BC(k+j)} v_j, \quad k \in \mathbb{Z}_{-n}^{\infty} \tag{75}$$

with unknown constant vectors v_j. Due to linearity (taking into account that k varies), we have, for $k \geq 0$,

$$\begin{aligned} \Delta x(k) &= \Delta \sum_{j=0}^{n} e_{mn}^{BC(k+j)} v_j = \sum_{j=0}^{n} \Delta\left[e_{mn}^{BC(k+j)} v_j\right] \\ &= \sum_{j=0}^{n} \Delta\left[e_{mn}^{BC(k+j)}\right] v_j. \end{aligned} \tag{76}$$

Using formula (11),

$$\begin{aligned} \Delta x(k) &= \sum_{j=0}^{n} \left(B e_{mn}^{BC(k-m+j)} + C e_{mn}^{BC(k-n+j)}\right) v_j \\ &= B \sum_{j=0}^{n} e_{mn}^{BC(k-m+j)} v_j + C \sum_{j=0}^{n} e_{mn}^{BC(k-n+j)} v_j \\ &= Bx(k-m) + Cx(k-n). \end{aligned} \tag{77}$$

Now we conclude that, for any v_j and $k \in \mathbb{Z}_0^{\infty}$, the equation $\Delta x(k) = Bx(k-m) + Cx(k-n)$ holds. We will try to satisfy initial conditions (71). Due to (75), we have, for $k \in \mathbb{Z}_{-n}^{0}$,

$$\begin{aligned}
&e_{mn}^{BC0} v_0 + e_{mn}^{BC1} v_1 + e_{mn}^{BC2} v_2 + \cdots + e_{mn}^{BC(n-2)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-1)} v_{n-1} + e_{mn}^{BCn} v_n = \varphi(0), \\
&e_{mn}^{BC(-1)} v_0 + e_{mn}^{BC0} v_1 + e_{mn}^{BC1} v_2 + \cdots + e_{mn}^{BC(n-3)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-2)} v_{n-1} + e_{mn}^{BC(n-1)} v_n = \varphi(-1), \\
&e_{mn}^{BC(-2)} v_0 + e_{mn}^{BC(-1)} v_1 + e_{mn}^{BC0} v_2 + \cdots + e_{mn}^{BC(n-4)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-3)} v_{n-1} + e_{mn}^{BC(n-2)} v_n = \varphi(-2), \\
&e_{mn}^{BC(-3)} v_0 + e_{mn}^{BC(-2)} v_1 + e_{mn}^{BC(-1)} v_2 + \cdots + e_{mn}^{BC(n-5)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-4)} v_{n-1} + e_{mn}^{BC(n-3)} v_n = \varphi(-3), \\
&\qquad \vdots \\
&e_{mn}^{BC(-n+3)} v_0 + e_{mn}^{BC(-n+4)} v_1 + e_{mn}^{BC(-n+5)} v_2 + \cdots + e_{mn}^{BC1} v_{n-2} \\
&\quad + e_{mn}^{BC2} v_{n-1} + e_{mn}^{BC3} v_n = \varphi(-n+3), \\
&e_{mn}^{BC(-n+2)} v_0 + e_{mn}^{BC(-n+3)} v_1 + e_{mn}^{BC(-n+4)} v_2 + \cdots + e_{mn}^{BC0} v_{n-2} \\
&\quad + e_{mn}^{BC1} v_{n-1} + e_{mn}^{BC2} v_n = \varphi(-n+2), \\
&e_{mn}^{BC(-n+1)} v_0 + e_{mn}^{BC(-n+2)} v_1 + e_{mn}^{BC(-n+3)} v_2 + \cdots + e_{mn}^{BC(-1)} v_{n-2} \\
&\quad + e_{mn}^{BC0} v_{n-1} + e_{mn}^{BC1} v_n = \varphi(-n+1), \\
&e_{mn}^{BC(-n)} v_0 + e_{mn}^{BC(-n+1)} v_1 + e_{mn}^{BC(-n+2)} v_2 + \cdots + e_{mn}^{BC(-2)} v_{n-2} \\
&\quad + e_{mn}^{BC(-1)} v_{n-1} + e_{mn}^{BC0} v_n = \varphi(-n).
\end{aligned} \tag{78}$$

Due to Definition 3, $e_{mn}^{BCk} = I$ for $k \in \mathbb{Z}_{-n}^{0}$. So we have

$$\begin{aligned}
&v_0 + e_{mn}^{BC1} v_1 + e_{mn}^{BC2} v_2 + \cdots + e_{mn}^{BC(n-2)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-1)} v_{n-1} + e_{mn}^{BCn} v_n = \varphi(0), && (E_0) \\
&v_0 + v_1 + e_{mn}^{BC1} v_2 + \cdots + e_{mn}^{BC(n-3)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-2)} v_{n-1} + e_{mn}^{BC(n-1)} v_n = \varphi(-1), && (E_1) \\
&v_0 + v_1 + v_2 + \cdots + e_{mn}^{BC(n-4)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-3)} v_{n-1} + e_{mn}^{BC(n-2)} v_n = \varphi(-2), && (E_2) \\
&v_0 + v_1 + v_2 + \cdots + e_{mn}^{BC(n-5)} v_{n-2} \\
&\quad + e_{mn}^{BC(n-4)} v_{n-1} + e_{mn}^{BC(n-3)} v_n = \varphi(-3), && (E_3)
\end{aligned}$$

$$\vdots$$

$$v_0 + v_1 + v_2 + \cdots + e_{mn}^{BC1} v_{n-2} + e_{mn}^{BC2} v_{n-1} + e_{mn}^{BC3} v_n = \varphi(-n+3), \quad (E_{n-3})$$

$$v_0 + v_1 + v_2 + \cdots + v_{n-2} + e_{mn}^{BC1} v_{n-1} + e_{mn}^{BC2} v_n = \varphi(-n+2), \quad (E_{n-2})$$

$$v_0 + v_1 + v_2 + \cdots + v_{n-2} + v_{n-1} + e_{mn}^{BC1} v_n = \varphi(-n+1), \quad (E_{n-1})$$

$$v_0 + v_1 + v_2 + \cdots + v_{n-2} + v_{n-1} + v_n = \varphi(-n). \quad (E_n)$$

Subtracting the neighbouring equations $((E_{n-1} - E_n), (E_{n-2} - E_{n-1}), \ldots, (E_0 - E_1))$, we get

$$\left(e_{mn}^{BC1} - I\right) v_n = \varphi(-n+1) - \varphi(-n), \quad (E_{n-1} - E_n)$$

$$\left(e_{mn}^{BC1} - I\right) v_{n-1} + \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_n = \varphi(-n+2) - \varphi(-n+1), \quad (E_{n-2} - E_{n-1})$$

$$\left(e_{mn}^{BC1} - I\right) v_{n-2} + \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_{n-1} + \left(e_{mn}^{BC3} - e_{mn}^{BC2}\right) v_n = \varphi(-n+3) - \varphi(-n+2), \quad (E_{n-3} - E_{n-2})$$

$$\vdots$$

$$\left(e_{mn}^{BC1} - I\right) v_3 + \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_4 + \cdots + \left(e_{mn}^{BC(n-4)} - e_{mn}^{BC(n-5)}\right) v_{n-2} + \left(e_{mn}^{BC(n-3)} - e_{mn}^{BC(n-4)}\right) v_{n-1} + \left(e_{mn}^{BC(n-2)} - e_{mn}^{BC(n-3)}\right) v_n = \varphi(-2) - \varphi(-3), \quad (E_2 - E_3)$$

$$\left(e_{mn}^{BC1} - I\right) v_2 + \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_3 + \cdots + \left(e_{mn}^{BC(n-3)} - e_{mn}^{BC(n-4)}\right) v_{n-2} + \left(e_{mn}^{BC(n-2)} - e_{mn}^{BC(n-3)}\right) v_{n-1} + \left(e_{mn}^{BC(n-1)} - e_{mn}^{BC(n-2)}\right) v_n = \varphi(-1) - \varphi(-2), \quad (E_1 - E_2)$$

$$\left(e_{mn}^{BC1} - I\right) v_1 + \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_2 + \cdots + \left(e_{mn}^{BC(n-2)} - e_{mn}^{BC(n-3)}\right) v_{n-2} + \left(e_{mn}^{BC(n-1)} - e_{mn}^{BC(n-2)}\right) v_{n-1} + \left(e_{mn}^{BCn} - e_{mn}^{BC(n-1)}\right) v_n = \varphi(0) - \varphi(-1). \quad (E_0 - E_1)$$

By Definition 3, we have

$$e_{mn}^{BC1} - I = I + B + C - I = B + C, \quad (79)$$

and, from the foregoing equations, we get

$$\begin{aligned}
v_n &= (B+C)^{-1} \Delta\varphi(-n), \\
v_{n-1} &= (B+C)^{-1} \left[\Delta\varphi(-n+1) - \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_n\right], \\
v_{n-2} &= (B+C)^{-1} \left[\Delta\varphi(-n+2) - \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_{n-1} - \left(e_{mn}^{BC3} - e_{mn}^{BC2}\right) v_n\right], \\
&\vdots \\
v_3 &= (B+C)^{-1} \left[\Delta\varphi(-3) - \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_4 - \cdots - \left(e_{mn}^{BC(n-4)} - e_{mn}^{BC(n-5)}\right) v_{n-2} - \left(e_{mn}^{BC(n-3)} - e_{mn}^{BC(n-4)}\right) v_{n-1} - \left(e_{mn}^{BC(n-2)} - e_{mn}^{BC(n-3)}\right) v_n\right], \\
v_2 &= (B+C)^{-1} \left[\Delta\varphi(-2) - \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_3 - \cdots - \left(e_{mn}^{BC(n-3)} - e_{mn}^{BC(n-4)}\right) v_{n-2} - \left(e_{mn}^{BC(n-2)} - e_{mn}^{BC(n-3)}\right) v_{n-1} - \left(e_{mn}^{BC(n-1)} - e_{mn}^{BC(n-2)}\right) v_n\right], \\
v_1 &= (B+C)^{-1} \left[\Delta\varphi(-1) - \left(e_{mn}^{BC2} - e_{mn}^{BC1}\right) v_2 - \cdots - \left(e_{mn}^{BC(n-2)} - e_{mn}^{BC(n-3)}\right) v_{n-2} - \left(e_{mn}^{BC(n-1)} - e_{mn}^{BC(n-2)}\right) v_{n-1} - \left(e_{mn}^{BCn} - e_{mn}^{BC(n-1)}\right) v_n\right].
\end{aligned} \quad (80)$$

The previous formulas can be shortened as

$$\begin{aligned}
v_\ell &= (B+C)^{-1} \left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell} \left(e_{mn}^{BC(t+1)} - e_{mn}^{BCt}\right) v_{t+\ell}\right] \\
&= (B+C)^{-1} \left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell} \Delta e_{mn}^{BCt} v_{t+\ell}\right],
\end{aligned} \quad (81)$$

where $\ell \in \mathbb{Z}_1^n$. Finally, from (E_n), we get

$$v_0 = \varphi(-n) - \sum_{s=1}^{n} v_s. \quad (82)$$

Theorem 10 is proved. □

Now we express the solution of the homogeneous Cauchy problem by $\tilde{e}_{mn}^{BC(k)}$. In this case, the condition $\det(B+C) \neq 0$ is not necessary.

Theorem 11. *Let B, C be constant $r\times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$, be fixed integers. Then the solution of the initial Cauchy problem* (70), (71) *can be expressed in the form*

$$x(k) = \sum_{j=0}^{n} \widetilde{e}_{mn}^{BC(k+j)} w_j, \tag{83}$$

where $k \in \mathbb{Z}_{-n}^{\infty}$ and

$$\begin{aligned} w_\ell &= \Delta\varphi(-\ell-1) - \Delta\widetilde{e}_{mn}^{BC(-\ell+n-1)}\varphi(-n) \\ &\quad - \sum_{s=-n}^{-\ell-m-2} \Delta\widetilde{e}_{mn}^{BC(-\ell-s-2)}\Delta\varphi(s), \quad \ell \in \mathbb{Z}_0^{n-m-1}, \\ w_\ell &= \Delta\varphi(-\ell-1), \quad \ell \in \mathbb{Z}_{n-m}^{n-1}, \\ w_n &= \varphi(-n). \end{aligned} \tag{84}$$

Proof. We are going to find the solution of the problem (70), (71) in the form

$$x(k) = \sum_{j=0}^{n} \widetilde{e}_{mn}^{BC(k+j)} w_j, \quad k \geq 0 \tag{85}$$

with unknown constant vectors w_j. Due to linearity (taking into account that k varies), we have

$$\begin{aligned} \Delta x(k) &= \Delta\sum_{j=0}^{n} \widetilde{e}_{mn}^{BC(k+j)} w_j = \sum_{j=0}^{n} \Delta\left[\widetilde{e}_{mn}^{BC(k+j)} w_j\right] \\ &= \sum_{j=0}^{n} \Delta\left[\widetilde{e}_{mn}^{BC(k+j)}\right] w_j. \end{aligned} \tag{86}$$

We use formula (15) and we get

$$\begin{aligned} \Delta x(k) &= \sum_{j=0}^{n} \left(B\widetilde{e}_{mn}^{BC(k-m+j)} + C\widetilde{e}_{mn}^{BC(k-n+j)}\right) w_j \\ &= B\sum_{j=0}^{n} \widetilde{e}_{mn}^{BC(k-m+j)} w_j + C\sum_{j=0}^{n} \widetilde{e}_{mn}^{BC(k-n+j)} w_j \\ &= Bx(k-m) + Cx(k-n). \end{aligned} \tag{87}$$

Now we conclude that, for any w_j and $k \in \mathbb{Z}_0^{\infty}$, the equation $\Delta x(k) = Bx(k-m) + Cx(k-n)$ holds. We will try to satisfy initial conditions (71). Due to (83), we have, for $k \in \mathbb{Z}_{-n}^{0}$,

$$\begin{aligned} &\widetilde{e}_{mn}^{BC0} w_0 + \widetilde{e}_{mn}^{BC1} w_1 + \widetilde{e}_{mn}^{BC2} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC(n-2)} w_{n-2} + \widetilde{e}_{mn}^{BC(n-1)} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BCn} w_n = \varphi(0), \\ &\widetilde{e}_{mn}^{BC(-1)} w_0 + \widetilde{e}_{mn}^{BC0} w_1 + \widetilde{e}_{mn}^{BC1} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC(n-3)} w_{n-2} + \widetilde{e}_{mn}^{BC(n-2)} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC(n-1)} w_n = \varphi(-1), \\ &\widetilde{e}_{mn}^{BC(-2)} w_0 + \widetilde{e}_{mn}^{BC(-1)} w_1 + \widetilde{e}_{mn}^{BC0} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC(n-4)} w_{n-2} + \widetilde{e}_{mn}^{BC(n-3)} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC(n-2)} w_n = \varphi(-2), \\ &\widetilde{e}_{mn}^{BC(-3)} w_0 + \widetilde{e}_{mn}^{BC(-2)} w_1 + \widetilde{e}_{mn}^{BC(-1)} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC(n-5)} w_{n-2} + \widetilde{e}_{mn}^{BC(n-4)} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC(n-3)} w_n = \varphi(-3), \\ &\qquad \vdots \\ &\widetilde{e}_{mn}^{BC(-n+3)} w_0 + \widetilde{e}_{mn}^{BC(-n+4)} w_1 + \widetilde{e}_{mn}^{BC(-n+5)} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC1} w_{n-2} + \widetilde{e}_{mn}^{BC2} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC3} w_n = \varphi(-n+3), \\ &\widetilde{e}_{mn}^{BC(-n+2)} w_0 + \widetilde{e}_{mn}^{BC(-n+3)} w_1 + \widetilde{e}_{mn}^{BC(-n+4)} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC0} w_{n-2} + \widetilde{e}_{mn}^{BC1} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC2} w_n = \varphi(-n+2), \\ &\widetilde{e}_{mn}^{BC(-n+1)} w_0 + \widetilde{e}_{mn}^{BC(-n+2)} w_1 + \widetilde{e}_{mn}^{BC(-n+3)} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC(-1)} w_{n-2} + \widetilde{e}_{mn}^{BC0} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC1} w_n = \varphi(-n+1), \\ &\widetilde{e}_{mn}^{BC(-n)} w_0 + \widetilde{e}_{mn}^{BC(-n+1)} w_1 + \widetilde{e}_{mn}^{BC(-n+2)} w_2 + \cdots \\ &\quad + \widetilde{e}_{mn}^{BC(-2)} w_{n-2} + \widetilde{e}_{mn}^{BC(-1)} w_{n-1} \\ &\quad + \widetilde{e}_{mn}^{BC0} w_n = \varphi(-n). \end{aligned} \tag{88}$$

By Definition 6, we have $\widetilde{e}_{mn}^{BCk} = \Theta$ for $k \in \mathbb{Z}_{-\infty}^{-1}$ and $\widetilde{e}_{mn}^{BCk} = I$ for $k \in \mathbb{Z}_0^{m}$. Thus, we have

$$\begin{aligned} &w_0 + w_1 + w_2 + \cdots + w_m + \widetilde{e}_{mn}^{BC(m+1)} w_{m+1} \\ &\quad + \widetilde{e}_{mn}^{BC(m+2)} w_{m+2} + \cdots + \widetilde{e}_{mn}^{BC(n-2)} w_{n-2} \\ &\quad + \widetilde{e}_{mn}^{BC(n-1)} w_{n-1} + \widetilde{e}_{mn}^{BCn} w_n = \varphi(0), \end{aligned} \tag{$\widetilde{E}_0$}$$

$$\begin{aligned} &w_1 + w_2 + w_3 + \cdots + w_{m+1} + \widetilde{e}_{mn}^{BC(m+1)} w_{m+2} \\ &\quad + \widetilde{e}_{mn}^{BC(m+2)} w_{m+3} + \cdots + \widetilde{e}_{mn}^{BC(n-3)} w_{n-2} \\ &\quad + \widetilde{e}_{mn}^{BC(n-2)} w_{n-1} + \widetilde{e}_{mn}^{BC(n-1)} w_n = \varphi(-1), \end{aligned} \tag{$\widetilde{E}_1$}$$

$$\begin{aligned} &w_2 + w_3 + w_4 + \cdots + w_{m+2} + \tilde{e}_{mn}^{BC(m+1)} w_{m+3} \\ &\quad + \tilde{e}_{mn}^{BC(m+2)} w_{m+4} + \cdots + \tilde{e}_{mn}^{BC(n-4)} w_{n-2} \\ &\quad + \tilde{e}_{mn}^{BC(n-3)} w_{n-1} + \tilde{e}_{mn}^{BC(n-2)} w_n = \varphi(-2), \end{aligned} \tag{$\tilde{E}_2$}$$

$$\vdots$$

$$\begin{aligned} &w_{n-m-2} + w_{n-m-1} + w_{n-m} + \cdots + w_{n-2} \\ &\quad + \tilde{e}_{mn}^{BC(m+1)} w_{n-1} + \tilde{e}_{mn}^{BC(m+2)} w_n = \varphi(-n+m+2), \end{aligned} \tag{$\tilde{E}_{n-m-2}$}$$

$$\begin{aligned} &w_{n-m-1} + w_{n-m} + w_{n-m+1} + \cdots + w_{n-2} \\ &\quad + w_{n-1} + \tilde{e}_{mn}^{BC(m+1)} w_n = \varphi(-n+m+1), \end{aligned} \tag{$\tilde{E}_{n-m-1}$}$$

$$\begin{aligned} &w_{n-m} + w_{n-m+1} + w_{n-m+2} + \cdots \\ &\quad + w_{n-2} + w_{n-1} + w_n = \varphi(-n+m), \end{aligned} \tag{$\tilde{E}_{n-m}$}$$

$$\vdots \tag{$\tilde{E}_{n-2}$}$$

$$w_{n-2} + w_{n-1} + w_n = \varphi(-n+2),$$

$$w_{n-1} + w_n = \varphi(-n+1), \tag{$\tilde{E}_{n-1}$}$$

$$w_n = \varphi(-n). \tag{$\tilde{E}_n$}$$

We see directly that $w_n = \varphi(-n)$. Subtracting the neighbouring equations $(\left(\tilde{E}_{n-1} - \tilde{E}_n\right), \left(\tilde{E}_{n-2} - \tilde{E}_{n-1}\right), \ldots, \left(\tilde{E}_{n-m} - \tilde{E}_{n-m+1}\right))$, we immediately get the formulas for $w_{n-1}, w_{n-2}, \ldots, w_{n-m}$ as follows:

$$w_{n-1} = \varphi(-n+1) - \varphi(-n) = \Delta\varphi(-n), \tag{$\tilde{E}_{n-1} - \tilde{E}_n$}$$

$$w_{n-2} = \varphi(-n+2) - \varphi(-n+1) = \Delta\varphi(-n+1), \tag{$\tilde{E}_{n-2} - \tilde{E}_{n-1}$}$$

$$\vdots$$

$$\begin{aligned} w_{n-m+1} &= \varphi(-n+m-1) - \varphi(-n+m-2) \\ &= \Delta\varphi(-n+m-2), \end{aligned} \tag{$\tilde{E}_{n-m+1} - \tilde{E}_{n-m+2}$}$$

$$\begin{aligned} w_{n-m} &= \varphi(-n+m) - \varphi(-n+m-1) \\ &= \Delta\varphi(-n+m-1). \end{aligned} \tag{$\tilde{E}_{n-m} - \tilde{E}_{n-m+1}$}$$

Further, subtracting the neighbouring equations $(\left(\tilde{E}_{n-m-1} - \tilde{E}_{n-m}\right), \left(\tilde{E}_{n-m-2} - \tilde{E}_{n-m-1}\right), \ldots, \left(\tilde{E}_0 - \tilde{E}_1\right))$, we get

$$\begin{aligned} &w_{n-m-1} + \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_n \\ &\quad = \varphi(-n+m+1) - \varphi(-n+m) \\ &\quad \Longrightarrow w_{n-m-1} = \Delta\varphi(-n+m) - \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] \varphi(-n), \end{aligned} \tag{$\tilde{E}_{n-m-1} - \tilde{E}_{n-m}$}$$

$$\begin{aligned} &w_{n-m-2} + \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_{n-1} \\ &\quad + \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] w_n \\ &\quad = \varphi(-n+m+2) - \varphi(-n+m+1) \\ &\quad \Longrightarrow w_{n-m-2} = \Delta\varphi(-n+m+1) \\ &\quad - \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] \varphi(-n) \\ &\quad - \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] \Delta\varphi(-n), \end{aligned}$$

$$\vdots \tag{$\tilde{E}_{n-m-2} - \tilde{E}_{n-m-1}$}$$

$$\begin{aligned} &w_2 + \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_{m+3} \\ &\quad + \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] w_{m+4} + \cdots \\ &\quad + \left[\tilde{e}_{mn}^{BC(n-4)} - \tilde{e}_{mn}^{BC(n-5)}\right] w_{n-2} \\ &\quad + \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right] w_{n-1} \\ &\quad + \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] w_n = \varphi(-2) - \varphi(-3) \\ &\quad \Longrightarrow w_2 = \Delta\varphi(-3) - \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] \varphi(-n) \\ &\quad - \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right] \Delta\varphi(-n) \\ &\quad - \left[\tilde{e}_{mn}^{BC(n-4)} - \tilde{e}_{mn}^{BC(n-5)}\right] \Delta\varphi(-n+1) - \cdots \\ &\quad - \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] \Delta\varphi(-m-5) \\ &\quad - \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] \Delta\varphi(-m-4), \end{aligned} \tag{$\tilde{E}_2 - \tilde{E}_3$}$$

$$\begin{aligned} &w_1 + \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_{m+2} + \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] w_{m+3} \\ &\quad + \cdots + \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right] w_{n-2} \\ &\quad + \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] w_{n-1} \\ &\quad + \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right] w_n \\ &= \varphi(-1) - \varphi(-2) \\ &\Longrightarrow w_1 = \Delta\varphi(-2) \\ &\quad - \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right] \varphi(-n) \end{aligned}$$

$$
\begin{aligned}
&- \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] \Delta\varphi(-n) \\
&- \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right] \Delta\varphi(-n+1) - \cdots \\
&- \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] \Delta\varphi(-m-4) \\
&- \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] \Delta\varphi(-m-3),
\end{aligned}
\qquad \left(\tilde{E}_1 - \tilde{E}_2\right)
$$

$$
\begin{aligned}
&w_0 + \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_{m+1} \\
&\quad + \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] w_{m+2} + \cdots \\
&\quad + \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] w_{n-2} \\
&\quad + \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right] w_{n-1} \\
&\quad + \left[\tilde{e}_{mn}^{BCn} - \tilde{e}_{mn}^{BC(n-1)}\right] w_n \\
&= \varphi(0) - \varphi(-1) \qquad \left(\tilde{E}_0 - \tilde{E}_1\right) \\
&\Longrightarrow w_0 = \Delta\varphi(-1) - \left[\tilde{e}_{mn}^{BCn} - \tilde{e}_{mn}^{BC(n-1)}\right] \varphi(-n) \\
&\quad - \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right] \Delta\varphi(-n) \\
&\quad - \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] \Delta\varphi(-n+1) - \cdots \\
&\quad - \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] \Delta\varphi(-m-3) \\
&\quad - \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] \Delta\varphi(-m-2).
\end{aligned}
$$

The previous formulas can be written as

$$
\begin{aligned}
w_\ell &= \Delta\varphi(-\ell-1) - \left[\tilde{e}_{mn}^{BC(-\ell+n)} - \tilde{e}_{mn}^{BC(-\ell+n-1)}\right] \varphi(-n) \\
&\quad - \sum_{s=-n}^{-\ell-m-2} \left[\tilde{e}_{mn}^{BC(-\ell-s-1)} - \tilde{e}_{mn}^{BC(-\ell-s-2)}\right] \Delta\varphi(s), \\
&= \Delta\varphi(-\ell-1) - \Delta\tilde{e}_{mn}^{BC(-\ell+n-1)} \varphi(-n) \\
&\quad - \sum_{s=-n}^{-\ell-m-2} \Delta\tilde{e}_{mn}^{BC(-\ell-s-2)} \Delta\varphi(s), \quad \ell \in \mathbb{Z}_0^{n-m-1}, \\
w_\ell &= \Delta\varphi(-\ell-1), \quad \ell \in \mathbb{Z}_{n-m}^{n-1}, \\
w_n &= \varphi(-n).
\end{aligned}
\tag{89}
$$

Theorem 11 is proved. □

3.2. Representing the Solution of a Nonhomogeneous Initial Problem. We consider a nonhomogeneous initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{90}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-n}^{0}. \tag{91}$$

By the theory of linear equations, we can obtain its solution as the sum of a solution of adjoint homogeneous problem (70), (71) (satisfying the same initial data) and a particular solution of (90) being zero on an initial interval. Let us, therefore, find such a particular solution.

We need an auxiliary lemma the proof of which is omitted.

Lemma 12. *Let a function $F(k,n)$ of two discrete variables be given. Then,*

$$\Delta_k \left[\sum_{j=1}^{k} F(k,j)\right] = F(k+1,k+1) + \sum_{j=1}^{k} \Delta_k F(k,j). \tag{92}$$

Now we are ready to find a particular solution $x_p(k)$, $k \in \mathbb{Z}_{-n}^{\infty}$ of the initial Cauchy problem:

$$\Delta x(k) = Bx(\mathrm{k}-m) + Cx(k-n) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{93}$$

$$x(k) = 0, \quad k \in \mathbb{Z}_{-n}^{0}. \tag{94}$$

Theorem 13. *The solution $x = x_p(k)$ of the initial Cauchy problem* (93), (94) *can be represented on $\mathbb{Z}_{-n}^{\infty}$ in the form*

$$x_p(k) = \sum_{\ell=1}^{k} \tilde{e}_{mn}^{BC(k-\ell)} f(\ell-1), \ k \in \mathbb{Z}_0^{\infty}. \tag{95}$$

Proof. We are going to find a particular solution $x_p(k)$ of problem (93), (94) in the form (95). We substitute (95) into (93). Then, we get

$$
\begin{aligned}
&\Delta\left[\sum_{\ell=1}^{k} \tilde{e}_{mn}^{BC(k-\ell)} f(\ell-1)\right] \\
&\quad = B\sum_{\ell=1}^{k-m} \tilde{e}_{mn}^{BC(k-m-\ell)} f(\ell-1) \\
&\qquad + C\sum_{\ell=1}^{k-n} \tilde{e}_{mn}^{BC(k-n-\ell)} f(j-1) + f(k).
\end{aligned}
\tag{96}
$$

We modify the left-hand side of (96). With the aid of Lemma 12, we obtain

$$
\begin{aligned}
&\Delta\left[\sum_{\ell=1}^{k} \tilde{e}_{mn}^{BC(k-\ell)} f(j-1)\right] \\
&\quad = \tilde{e}_{mn}^{BC((k+1)-(k+1))} f(k+1-1) \\
&\qquad + \sum_{\ell=1}^{k} \Delta\left[\tilde{e}_{mn}^{BC(k-\ell)} f(j-1)\right],
\end{aligned}
\tag{97}
$$

and, applying Theorem 9, we get

$$\begin{aligned}
&\Delta\left[\sum_{\ell=1}^{k}\tilde{e}_{mn}^{BC(k-\ell)}f(j-1)\right]\\
&=\tilde{e}_{mn}^{BC0}f(k)+\sum_{\ell=1}^{k}\left[B\tilde{e}_{mn}^{BC(k-m-\ell)}+C\tilde{e}_{mn}^{BC(k-n-\ell)}\right]f(j-1)\\
&=\tilde{e}_{mn}^{BC0}f(k)+B\left[\sum_{\ell=1}^{k-m}\tilde{e}_{mn}^{BC(k-m-\ell)}f(j-1)\right.\\
&\qquad\left.+\sum_{\ell=k-m+1}^{k}\tilde{e}_{mn}^{BC(k-m-\ell)}f(j-1)\right]\\
&\quad+C\left[\sum_{\ell=1}^{k-n}\tilde{e}_{mn}^{BC(k-n-\ell)}f(j-1)\right.\\
&\qquad\left.+\sum_{\ell=k-n+1}^{k}\tilde{e}_{mn}^{BC(k-n-\ell)}f(j-1)\right].
\end{aligned}\tag{98}$$

By Definition 6, we have $\tilde{e}_{mn}^{BC0}=I$, $\tilde{e}_{mn}^{BC(k-m-\ell)}=\Theta$ for $\ell\in\mathbb{Z}_{k-m+1}^{k}$ and $\tilde{e}_{mn}^{BC(k-n-\ell)}=\Theta$ for $\ell\in\mathbb{Z}_{k-n+1}^{k}$. Thus, we get

$$\begin{aligned}
&\Delta\left[\sum_{\ell=1}^{k}\tilde{e}_{mn}^{BC(k-\ell)}f(j-1)\right]\\
&\quad=f(k)+B\sum_{\ell=1}^{k-m}\tilde{e}_{mn}^{BC(k-m-\ell)}f(j-1)\\
&\qquad+C\sum_{\ell=1}^{k-n}\tilde{e}_{mn}^{BC(k-n-\ell)}f(j-1)
\end{aligned}\tag{99}$$

and (96) holds. □

Combining the results of Theorems 10, 11, and 13, we get immediately the following two theorems, which describe the solution of (90), (91). The first theorem uses the delayed matrix exponential e_{mn}^{BCk} and the second one uses the delayed matrix exponential $\tilde{e}_{mn}^{BCk}$.

Theorem 14. *Let B, C be constant $r\times r$ matrices with*

$$BC=CB,\qquad \det(B+C)\neq 0,\tag{100}$$

and let $m,n\in\mathbb{N}$, $m<n$, be fixed integers. Then, the solution of the initial Cauchy problem (90), (91) *can be expressed in the form*

$$x(k)=\sum_{j=0}^{n}e_{mn}^{BC(k+j)}v_j+\sum_{\ell=1}^{k}\tilde{e}_{mn}^{BC(k-\ell)}f(\ell-1),\tag{101}$$

where $k\in\mathbb{Z}_{-n}^{\infty}$ and

$$\begin{aligned}
v_0&=\varphi(-n)-\sum_{s=1}^{n}v_s,\\
v_\ell&=(B+C)^{-1}\left[\Delta\varphi(-\ell)-\sum_{t=1}^{n-\ell}\Delta\, e_{mn}^{BCt}v_{t+\ell}\right],\quad \ell\in\mathbb{Z}_1^n.
\end{aligned}\tag{102}$$

Theorem 15. *Let B, C be constant $r\times r$ matrices with $BC=CB$ and let $m,n\in\mathbb{N}$, $m<n$, be fixed integers. Then, the solution of the initial Cauchy problem* (90), (91) *can be expressed in the form*

$$x(k)=\sum_{j=0}^{n}\tilde{e}_{mn}^{BC(k+j)}w_j+\sum_{\ell=1}^{k}\tilde{e}_{mn}^{BC(k-\ell)}f(\ell-1),\tag{103}$$

where $k\in\mathbb{Z}_{-n}^{\infty}$ and

$$\begin{aligned}
w_\ell&=\Delta\varphi(-\ell-1)-\Delta\tilde{e}_{mn}^{BC(-\ell+n-1)}\varphi(-n)\\
&\quad-\sum_{s=-n}^{-\ell-m-2}\Delta\tilde{e}_{mn}^{BC(-\ell-s-2)}\Delta\varphi(s),\quad \ell\in\mathbb{Z}_0^{n-m-1},\\
w_\ell&=\Delta\varphi(-\ell-1),\quad \ell\in\mathbb{Z}_{n-m}^{n-1},\\
w_n&=\varphi(-n).
\end{aligned}\tag{104}$$

4. Examples

Below, we show four examples to demonstrate the results achieved.

Example 16. Let us represent the solution of the scalar ($r=1$) problem (70), (71) where we put $m=2$, $n=3$, $B=b$, $C=c$, $\varphi(-3)=1$, $\varphi(-2)=2$, $\varphi(-1)=3$, and $\varphi(0)=4$, using Theorem 10. We get

$$\Delta x(k)=bx(k-2)+cx(k-3),\quad k\in\mathbb{Z}_0^{\infty},\tag{105}$$

$$\begin{aligned}
x(-3)&=\varphi(-3)=1,\\
x(-2)&=\varphi(-2)=2,\\
x(-1)&=\varphi(-1)=3,\\
x(0)&=\varphi(0)=4.
\end{aligned}\tag{106}$$

By Theorem 10, the solution of problem (105), (106) is

$$x(k)=\sum_{j=0}^{3}e_{2,3}^{bc(k+j)}v_j,\quad k\in\mathbb{Z}_{-3}^{\infty},\tag{107}$$

where

$$\begin{aligned}
v_3 &= (b+c)^{-1}\left[\Delta\varphi(-3) - \sum_{t=1}^{0}\Delta e_{2,3}^{bct} v_{t+3}\right] = (b+c)^{-1},\\
v_2 &= (b+c)^{-1}\left[\Delta\varphi(-2) - \sum_{t=1}^{1}\Delta e_{2,3}^{bct} v_{t+2}\right]\\
&= (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta e_{2,3}^{bc1} v_3\right]\\
&= (b+c)^{-1}\left[1 - \left(e_{2,3}^{bc2} - e_{2,3}^{bc1}\right)(b+c)^{-1}\right]\\
&= (b+c)^{-1}\left[1 - (b+c)(b+c)^{-1}\right] = 0,\\
v_1 &= (b+c)^{-1}\left[\Delta\varphi(-1) - \sum_{t=1}^{2}\Delta e_{2,3}^{bct} v_{t+1}\right]\\
&= (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta e_{2,3}^{bc1} v_2 - \Delta e_{2,3}^{bc2} v_3\right]\\
&= (b+c)^{-1}\left[1 - \left(e_{2,3}^{bc3} - e_{2,3}^{bc2}\right)(b+c)^{-1}\right]\\
&= (b+c)^{-1}\left[1 - (b+c)(b+c)^{-1}\right] = 0,\\
v_0 &= \varphi(-3) - \sum_{s=1}^{3} v_s = 1 - (b+c)^{-1}.
\end{aligned} \tag{108}$$

Thus, we get

$$x(k) = e_{2,3}^{bck}\left[1 - (b+c)^{-1}\right] + e_{2,3}^{bc(k+3)}(b+c)^{-1}. \tag{109}$$

We give values of $x(k)$ for $k \in \mathbb{Z}_1^8$ as follows:

$$\begin{aligned}
x(1) &= 4 + 2b + c,\\
x(2) &= 4 + 5b + 3c,\\
x(3) &= 4 + 9b + 6c,\\
x(4) &= 4 + 13b + 10c + 2b^2 + bc,\\
x(5) &= 4 + 17b + 14c + 7b^2 + 6bc + c^2,\\
x(6) &= 4 + 21b + 18c + 16b^2 + 17bc + 4c^2,\\
x(7) &= 4 + 25b + 22c + 29b^2 + 36bc + 10c^2 + 2b^3 + b^2c,\\
x(8) &= 4 + 29b + 26c + 46b^2 + 63bc + 20c^2\\
&\quad + 9b^3 + 9b^2c + 2bc^2.
\end{aligned} \tag{110}$$

Example 17. Let us represent the solution of the scalar ($r = 1$) problem (90), (91) where we put $m = 2$, $n = 3$, $B = b$, $C = c$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, $\varphi(0) = 4$, and $f(k) = k + 1$, using Theorem 11. Thus, we have

$$\Delta x(k) = bx(k-2) + cx(k-3) + k + 1, \quad k \in \mathbb{Z}_0^{\infty}, \tag{111}$$

$$\begin{aligned}
x(-3) &= \varphi(-3) = 1,\\
x(-2) &= \varphi(-2) = 2,\\
x(-1) &= \varphi(-1) = 3,\\
x(0) &= \varphi(0) = 4.
\end{aligned} \tag{112}$$

By Theorem 11, the solution of problem (111), (112) is

$$x(k) = \sum_{j=0}^{3}\tilde{e}_{2,3}^{bc(k+j)} w_j + \sum_{\ell=1}^{k}\tilde{e}_{2,3}^{bc(k-\ell)}\ell, \quad k \in \mathbb{Z}_{-3}^{\infty}, \tag{113}$$

where

$$\begin{aligned}
w_0 &= \Delta\varphi(-1) - \Delta\tilde{e}_{2,3}^{bc2}\varphi(-3) - \sum_{s=-3}^{-4}\Delta\tilde{e}_{2,3}^{bc(-s-2)}\Delta\varphi(s)\\
&= 1 - \left(\tilde{e}_{2,3}^{bc3} - \tilde{e}_{2,3}^{bc2}\right)1\\
&= 1 - (1 + b - 1) = 1 - b,\\
w_1 &= \Delta\varphi(-2) = 1,\\
w_2 &= \Delta\varphi(-3) = 1,\\
w_3 &= \varphi(-3) = 1.
\end{aligned} \tag{114}$$

Thus, we get

$$\begin{aligned}
x(k) &= \tilde{e}_{2,3}^{bck}(1-b) + \tilde{e}_{2,3}^{bc(k+1)} + \tilde{e}_{2,3}^{bc(k+2)}\\
&\quad + \tilde{e}_{2,3}^{bc(k+3)} + \sum_{\ell=1}^{k}\tilde{e}_{2,3}^{bc(k-\ell)}\ell.
\end{aligned} \tag{115}$$

The first eight values of the homogeneous problem are given in Example 16. Now, we compute the first eight values of a particular solution $x_p(k) = \sum_{\ell=1}^{k}\tilde{e}_{2,3}^{bc(k-\ell)}\ell$ as follows:

$$\begin{aligned}
&x_p(1) = 1, \qquad x_p(2) = 3,\\
&x_p(3) = 6, \qquad x_p(4) = 10 + b,\\
&x_p(5) = 15 + 4b + c, \qquad x_p(6) = 21 + 10b + 4c,\\
&x_p(7) = 28 + 20b + 10c + b^2,\\
&x_p(8) = 36 + 35b + 20c + 5b^2 + 2bc.
\end{aligned} \tag{116}$$

Together, we get

$$\begin{aligned}
&x(1) = 5 + 2b + c, \qquad x(2) = 7 + 5b + 3c,\\
&x(3) = 10 + 9b + 6c,\\
&x(4) = 14 + 14b + 10c + 2b^2 + bc,\\
&x(5) = 19 + 21b + 15c + 7b^2 + 6bc + c^2,\\
&x(6) = 25 + 31b + 22c + 16b^2 + 17bc + 4c^2,\\
&x(7) = 32 + 45b + 32c + 30b^2 + 36bc + 10c^2 + 2b^3 + b^2c,\\
&x(8) = 40 + 64b + 46c + 51b^2 + 65bc + 20c^2\\
&\qquad + 9b^3 + 9b^2c + 2bc^2.
\end{aligned} \tag{117}$$

Example 18. Let us represent the solution of the scalar ($r = 1$) problem (70), (71) where we put $m = 2$, $n = 3$, $B = b = 4$, $C = c = -1$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, and $\varphi(0) = 4$, using Theorem 10. Thus, we have

$$\Delta x(k) = 4x(k-2) - x(k-3), \quad k \in \mathbb{Z}_0^{\infty}, \tag{118}$$

$$\begin{aligned}
x(-3) &= \varphi(-3) = 1,\\
x(-2) &= \varphi(-2) = 2,\\
x(-1) &= \varphi(-1) = 3,\\
x(0) &= \varphi(0) = 4.
\end{aligned} \tag{119}$$

By Theorem 10, the solution of problem (118), (119) is

$$x(k) = \sum_{j=0}^{3} e_{2,3}^{bc(k+j)} v_j, \quad k \in \mathbb{Z}_{-3}^{\infty}, \tag{120}$$

where

$$\begin{aligned}
v_3 &= (b+c)^{-1}\left[\Delta\varphi(-3) - \sum_{t=1}^{0}\Delta e_{2,3}^{bct} v_{t+3}\right] = (b+c)^{-1} = \frac{1}{3},\\
v_2 &= (b+c)^{-1}\left[\Delta\varphi(-2) - \sum_{t=1}^{1}\Delta e_{2,3}^{bct} v_{t+2}\right]\\
&= (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta e_{2,3}^{bc1} v_3\right]\\
&= (b+c)^{-1}\left[1 - \left(e_{2,3}^{bc2} - e_{2,3}^{bc1}\right)(b+c)^{-1}\right]\\
&= (b+c)^{-1}\left[1 - (b+c)(b+c)^{-1}\right] = 0,\\
v_1 &= (b+c)^{-1}\left[\Delta\varphi(-1) - \sum_{t=1}^{2}\Delta e_{2,3}^{bct} v_{t+1}\right]\\
&= (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta e_{2,3}^{bc1} v_2 - \Delta e_{2,3}^{bc2} v_3\right]\\
&= (b+c)^{-1}\left[1 - \left(e_{2,3}^{bc3} - e_{2,3}^{bc2}\right)(b+c)^{-1}\right]\\
&= (b+c)^{-1}\left[1 - (b+c)(b+c)^{-1}\right] = 0,\\
v_0 &= \varphi(-3) - \sum_{s=1}^{3} v_s = 1 - (b+c)^{-1} = \frac{2}{3}.
\end{aligned} \tag{121}$$

Thus, we get

$$\begin{aligned}
x(k) &= e_{2,3}^{bck}\cdot\frac{2}{3} + e_{2,3}^{bc(k+3)}\cdot\frac{1}{3},\\
x(1) &= e_{2,3}^{bc1}\cdot\frac{2}{3} + e_{2,3}^{bc4}\cdot\frac{1}{3} = 4\cdot\frac{2}{3} + 25\cdot\frac{1}{3} = 11,\\
x(2) &= e_{2,3}^{bc2}\cdot\frac{2}{3} + e_{2,3}^{bc5}\cdot\frac{1}{3} = 7\cdot\frac{2}{3} + 49\cdot\frac{1}{3} = 21,\\
x(3) &= e_{2,3}^{bc3}\cdot\frac{2}{3} + e_{2,3}^{bc6}\cdot\frac{1}{3} = 10\cdot\frac{2}{3} + 82\cdot\frac{1}{3} = 34,\\
x(4) &= e_{2,3}^{bc4}\cdot\frac{2}{3} + e_{2,3}^{bc7}\cdot\frac{1}{3} = 25\cdot\frac{2}{3} + 172\cdot\frac{1}{3} = 74,\\
x(5) &= e_{2,3}^{bc5}\cdot\frac{2}{3} + e_{2,3}^{bc8}\cdot\frac{1}{3} = 49\cdot\frac{2}{3} + 343\cdot\frac{1}{3} = 147,\\
x(6) &= e_{2,3}^{bc6}\cdot\frac{2}{3} + e_{2,3}^{bc9}\cdot\frac{1}{3} = 82\cdot\frac{2}{3} + 622\cdot\frac{1}{3} = 262,\\
x(7) &= e_{2,3}^{bc7}\cdot\frac{2}{3} + e_{2,3}^{bc10}\cdot\frac{1}{3} = 172\cdot\frac{2}{3} + 1228\cdot\frac{1}{3} = 524,\\
x(8) &= e_{2,3}^{bc8}\cdot\frac{2}{3} + e_{2,3}^{bc11}\cdot\frac{1}{3} = 343\cdot\frac{2}{3} + 2428\cdot\frac{1}{3} = 1038.
\end{aligned} \tag{122}$$

Example 19. Let us represent the solution of the scalar ($r = 1$) problem (90), (91) where we put $m = 2$, $n = 3$, $B = b = 4$, $C = c = -1$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, $\varphi(0) = 4$, and $f(k) = k + 1$, using Theorem 11. Thus, we have

$$\Delta x(k) = 4x(k-2) - x(k-3) + k + 1, \quad k \in \mathbb{Z}_0^{\infty}, \tag{123}$$

$$\begin{aligned}
x(-3) &= \varphi(-3) = 1,\\
x(-2) &= \varphi(-2) = 2,\\
x(-1) &= \varphi(-1) = 3,\\
x(0) &= \varphi(0) = 4.
\end{aligned} \tag{124}$$

By Theorem 11, the solution of the problem (123), (124) is

$$x(k) = \sum_{j=0}^{3} \tilde{e}_{2,3}^{bc(k+j)} w_j + \sum_{\ell=1}^{k} \tilde{e}_{2,3}^{bc(k-\ell)} \ell, \quad k \in \mathbb{Z}_{-3}^{\infty}, \tag{125}$$

where

$$\begin{aligned} w_0 &= \Delta\varphi(-1) - \Delta\tilde{e}_{2,3}^{bc2}\varphi(-3) - \sum_{s=-3}^{-4}\Delta\tilde{e}_{2,3}^{bc(-s-2)}\Delta\varphi(s) \\ &= 1 - \left(\tilde{e}_{2,3}^{bc3} - \tilde{e}_{2,3}^{bc2}\right)1 \\ &= 1 - (1 + b - 1) = 1 - b = -3, \\ w_1 &= \Delta\varphi(-2) = 1, \\ w_2 &= \Delta\varphi(-3) = 1, \\ w_3 &= \varphi(-3) = 1. \end{aligned} \tag{126}$$

Thus, we get

$$\begin{aligned} x(k) &= \tilde{e}_{2,3}^{bck}(-3) + \tilde{e}_{2,3}^{bc(k+1)} + \tilde{e}_{2,3}^{bc(k+2)} \\ &\quad + \tilde{e}_{2,3}^{bc(k+3)} + \sum_{\ell=1}^{k}\tilde{e}_{2,3}^{bc(k-\ell)}\ell, \\ x(1) &= \tilde{e}_{2,3}^{bc1}(-3) + \tilde{e}_{2,3}^{bc2} + \tilde{e}_{2,3}^{bc3} + \tilde{e}_{2,3}^{bc4} + \sum_{\ell=1}^{1}\tilde{e}_{2,3}^{bc(1-\ell)}\ell \\ &= -3 + 1 + 5 + 8 + 1 = 12, \\ x(2) &= \tilde{e}_{2,3}^{bc2}(-3) + \tilde{e}_{2,3}^{bc3} + \tilde{e}_{2,3}^{bc4} + \tilde{e}_{2,3}^{bc5} + \sum_{\ell=1}^{2}\tilde{e}_{2,3}^{bc(2-\ell)}\ell \\ &= -3 + 5 + 8 + 11 + 3 = 24, \\ x(3) &= \tilde{e}_{2,3}^{bc3}(-3) + \tilde{e}_{2,3}^{bc4} + \tilde{e}_{2,3}^{bc5} + \tilde{e}_{2,3}^{bc6} + \sum_{\ell=1}^{3}\tilde{e}_{2,3}^{bc(3-\ell)}\ell \\ &= -15 + 8 + 11 + 30 + 6 = 40, \\ x(4) &= \tilde{e}_{2,3}^{bc4}(-3) + \tilde{e}_{2,3}^{bc5} + \tilde{e}_{2,3}^{bc6} + \tilde{e}_{2,3}^{bc7} + \sum_{\ell=1}^{4}\tilde{e}_{2,3}^{bc(4-\ell)}\ell \\ &= -24 + 11 + 30 + 57 + 14 = 88, \\ x(5) &= \tilde{e}_{2,3}^{bc5}(-3) + \tilde{e}_{2,3}^{bc6} + \tilde{e}_{2,3}^{bc7} + \tilde{e}_{2,3}^{bc8} + \sum_{\ell=1}^{5}\tilde{e}_{2,3}^{bc(5-\ell)}\ell \\ &= -33 + 30 + 57 + 93 + 30 = 177, \\ x(6) &= \tilde{e}_{2,3}^{bc6}(-3) + \tilde{e}_{2,3}^{bc7} + \tilde{e}_{2,3}^{bc8} + \tilde{e}_{2,3}^{bc9} + \sum_{\ell=1}^{6}\tilde{e}_{2,3}^{bc(6-\ell)}\ell \\ &= -90 + 57 + 93 + 202 + 57 = 319, \\ x(7) &= \tilde{e}_{2,3}^{bc7}(-3) + \tilde{e}_{2,3}^{bc8} + \tilde{e}_{2,3}^{bc9} + \tilde{e}_{2,3}^{bc10} + \sum_{\ell=1}^{7}\tilde{e}_{2,3}^{bc(7-\ell)}\ell \\ &= -171 + 93 + 202 + 400 + 114 = 638, \\ x(8) &= \tilde{e}_{2,3}^{bc8}(-3) + \tilde{e}_{2,3}^{bc9} + \tilde{e}_{2,3}^{bc10} + \tilde{e}_{2,3}^{bc11} + \sum_{\ell=1}^{8}\tilde{e}_{2,3}^{bc(8-\ell)}\ell \\ &= -279 + 202 + 400 + 715 + 228 = 1266. \end{aligned} \tag{127}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Spatial Approximation of Nondivergent Type Parabolic PDEs with Unbounded Coefficients Related to Finance

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We study the spatial discretisation of the Cauchy problem for a multidimensional linear parabolic PDE of second order, with nondivergent operator and unbounded time- and space-dependent coefficients. The equation free term and the initial data are also allowed to grow. Under a nondegeneracy assumption, we consider the PDE solvability in the framework of the variational approach and approximate in space the PDE problem's generalised solution, with the use of finite-difference methods. The rate of convergence is estimated.

1. Introduction

In this paper, we study the discretisation in space of the Cauchy problem

$$\begin{gathered}\frac{\partial u}{\partial t} = Lu + f \quad \text{in } [0,T]\times\mathbb{R}^d,\\ u(0,x) = g(x) \quad \text{on } \mathbb{R}^d,\end{gathered} \tag{1}$$

where L is the second-order partial differential operator in the nondivergence form

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x), \quad i,j = 1,\dots,d, \tag{2}$$

with real coefficients (written with the usual summation convention), f and g are given real-valued functions, and $T \in (0,\infty)$ is a constant. We assume that operator $\partial/\partial t - L$ is uniformly parabolic and allows the growth in the spatial variables of the first- and second-order coefficients in L (linear and quadratic growth, resp.) and of the data f and g (polynomial growth).

Multidimensional partial differential equation (PDE) problems arise in Financial Mathematics and in Mathematical Physics. We are mainly motivated by the application to a large class of stochastic models in Financial Mathematics comprising the non-path-dependent options, with fixed exercise, written on multiple assets (basket options, exchange options, compound options, European options on future contracts and foreign-exchange, and others) and also to a particular type of path-dependent options: the Asian options (see, e.g., [1]).

Let us consider the stochastic modelling of a multiasset option of European type under the framework of a general version of Black-Scholes model, where the vector of asset appreciation rates and the volatility matrix are taken to be time- and space-dependent, and the riskless interest rate is a function of time. Owing to a Feynman-Kač type formula, pricing this option can be reduced to solving the Cauchy problem (with terminal condition) for the degenerate second-order linear parabolic PDE of nondivergent type, with null term and unbounded coefficients (see, e.g., [1]),

$$\begin{gathered}\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^{ij}(t,S)S^iS^j\frac{\partial^2 V}{\partial S^i\partial S^j} + r(t)S^i\frac{\partial V}{\partial S^i} - r(t)V = 0 \quad \text{in } [0,T]\times\mathbb{R}^d_+,\\ V(T,S) = \phi(S) \quad \text{on } \mathbb{R}^d_+,\end{gathered} \tag{3}$$

where $\mathbb{R}^d_+ \equiv \{x \in \mathbb{R}^d : x^i > 0, i = 1, \ldots, d\}$, V is the (unknown) option value, S^i the price of the ith underlying asset, (σ^{ij}) the volatility matrix, r the risk-free interest rate, and ϕ the pay-off function.

Therefore, as an alternative to approximating the option price with probabilistic numerical methods, we can approximate the solution of the corresponding PDE problem (3) with the use of nonprobabilistic techniques.

When problem (1) is considered in connection with option pricing, we see that the growth of the Black-Scholes PDE coefficients is appropriately matched. Also, the general case where the asset appreciation rate vector, the volatility matrix, and the risk-free interest rate are variable is covered. Finally, by imposing weak conditions on the initial data g, we will allow the financial derivative pay-off to be specified in a large class of functions. The free term f is included to further improve generality.

In this paper, we study the approximation in space (for the time approximation, we refer to [2–4], where a general evolution equation problem of parabolic type is discretised) of the second-order parabolic problem (1), in the challenging case where the coefficients are unbounded (as well as the free data f and g). The results are obtained under the strong assumption that the PDE does not degenerate but by imposing weak regularity assumptions. In order to facilitate the approach, we avoid any numerical methods' sophistication and make use of basic one-step finite-difference schemes. Also, an estimate for the rate of convergence of the discretised problem's generalised solution to the exact problem's generalised solution is given.

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDEs. We make use of the L^2 theory of solvability of linear PDEs in weighted Sobolev spaces. In particular, we consider the PDE solvability in a class of weighted Sobolev spaces used by O. G. Purtukhia (the references for Purtukhia's works can be found in [5]) for the treatment of linear stochastic partial differential equations (SPDEs) and further generalised by Gyöngy and Krylov (see [5]), the so-called well-weighted Sobolev spaces. By constructing discrete versions of these spaces, we set a suitable discretised framework and investigate the spatial approximation to the PDE generalised solution with the use of standard variational techniques.

We emphasize some points.

Firstly, we note that many PDE problems related to finance are Cauchy problems: initial-boundary value problems arise mostly after a localisation procedure for the purpose of obtaining implementable numerical schemes. Therefore, we do not find in many of these problems the complex domain geometries which are one important reason to favour other numerical methods (e.g., finite-element methods).

Also, although the finite-difference method for approximating PDEs is a well-developed area, and the theory could be considered reasonably complete since three decades ago, some important research is still currently pursued (see, e.g., the recent works [8–10]). (We refer to [6] for a brief summary of the method's history, and also for the references of the seminal works by R. Courant, K. O. Friedrichs, and H. Lewy, and further major contributions by many others. For the application of the finite-difference method to option pricing, we refer to the review paper [7] for the references of the original publications by M. Brennan and E. S. Schwartz and further major research.)

Secondly, we observe that the usual procedure for obtaining implementable numerical schemes for problem (1) is to localise it to a bounded domain in $[0, T] \times \mathbb{R}^d$ and then to approximate the localised problem (see, e.g., [11–13]; see also [14], where the approximation is pursued for more complex financial models but using the same localisation technique). In this case, there is no need to consider weighted functional spaces for the solvability and approximation study, as the PDE coefficients are bounded in the truncated domain.

An alternative procedure is to (semi)discretise problem (1) in $[0, T] \times Z^d_h$, with Z^d_h being the h-grid on $\mathbb{R}^d$, and then localise the discretised problem to a bounded domain in $[0, T] \times Z^d_h$ by imposing a discrete artificial boundary condition (see, e.g., [15–17], where several types of initial-value problems on unbounded domains are approximated; we refer to [16, 17] for the procedure discussion). Our study is meaningful in this latter case, as the coefficient unboundedness remains a problem that must be dealt with.

Finally, we remark that (i) the partial differential operators arising in finance are of nondivergent type and (ii) we do not assume the operator coefficients to be smooth enough to be possible to obtain an equivalent divergent operator. Therefore, although there are definite advantages in considering the operator in the divergent form when the approach is variational, this is not available for the present work.

We outline the paper. In Section 2, we establish some well-known facts on the solvability of linear PDEs under a general framework and introduce the well-weighted Sobolev spaces. In Section 3, we discretise in space problem (1), with the use of a finite-difference scheme. We set a discrete framework and deduce the existence and uniqueness of the discretised problem's generalised solution. In Section 4, we investigate the approximation properties of the scheme and compute a rate of convergence. In Section 5, we make a few final comments.

2. Preliminaries and Classical Results

We establish some facts on the solvability of PDEs under a general framework.

Let V be a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(,)$. Then H^*, the dual space of H, is also continuously and densely embedded into V^*, the dual of V. Let us use the notation $\langle , \rangle$ for the duality. Let H^* be identified with H in the usual way, by the help of the inner product. Then we have the so-called normal triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$, with continuous and dense embeddings.

Let us consider the Cauchy problem for an evolution equation

$$\frac{du}{dt} = A(t)u + f(t) \quad \text{in } [0,T], \; u(0) = g, \tag{4}$$

with $T \in (0,\infty)$ and where for every $t \in [0,T]$ $A(t)$ is a linear operator from V to V^*, $f(t) \in V^*$, and $g \in H$.

We assume that the operator $A(t)$ is continuous and impose a coercivity condition, as well as some regularity on the free data f and g.

Assumption 1. There exist constants $\lambda > 0$, K, M, and N such that

(1) $\langle A(t)v, v\rangle + \lambda\|v\|_V^2 \le K\|v\|_H^2$, $\forall v \in V$, $\forall t \in [0,T]$;

(2) $\|A(t)v\|_{V^*} \le M\|v\|_V$, $\forall v \in V$, $\forall t \in [0,T]$;

(3) $\int_0^T \|f(t)\|_{V^*}^2 dt \le N$ and $\|g\|_H \le N$.

We define the generalised solution of problem (4).

Definition 2. One says that $u \in C([0,T];H)$ is a generalised solution of (4) on $[0,T]$ if

(1) $u \in L^2([0,T];V)$;

(2) for every $t \in [0,T]$,

$$(u(t), v) = (g, v) + \int_0^t \langle A(s)u(s), v\rangle ds + \int_0^t \langle f(s), v\rangle ds \tag{5}$$

holds for all $v \in V$.

Notation 1. Let W be a Banach space with norm $\| \; \|$. We denote by $C([0,T];W)$ the space of continuous W-valued functions on $[0,T]$ and by $L^2([0,T];W)$ the space of W-valued functions w on $[0,T]$ such that $\|w\|_{L^2([0,T];W)} := \left(\int_0^T \|w(t)\|^2 dt\right)^{1/2} < \infty$.

The following well-known result states that, under Assumption 1, problem (4) has a unique generalised solution (see, e.g., [5], where the result is stated for the more general case of a linear stochastic evolution equation problem and [18]).

Theorem 3. *Under (1)–(3) in Assumption 1, problem* (4) *has a unique generalised solution on* $[0,T]$. *Moreover*

$$\sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 dt \le N\left(\|g\|_H^2 + \int_0^T \|f(t)\|_{V^*}^2 dt\right), \tag{6}$$

where N is a constant.

Let us now consider the particular PDE problem

$$u_t = Lu + f \quad \text{in } Q, \qquad u(0,x) = g(x) \quad \text{on } \mathbb{R}^d, \tag{7}$$

where L is the second-order operator with real coefficients

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x), \tag{8}$$

$Q = [0,T] \times \mathbb{R}^d$, with $T \in (0,\infty)$, and f and g are given functions. We allow the growth, in the spatial variables, of the coefficients $a^{ij}(t,x)$ and $b^i(t,x)$, $i, j = 1,\ldots,d$, and of the free data $f(t,x)$ and $g(x)$.

In order to set the framework for problem (7), we introduce a suitable class of weighted Sobolev spaces, the so-called well-weighted Sobolev spaces (we refer to [5] for a complete description of this class of spaces).

Let U be a domain in $\mathbb{R}^d$, that is, an open subset of $\mathbb{R}^d$. Let $r > 0$, $\rho > 0$ be smooth functions in U and $m \ge 0$ an integer. The weighted Sobolev space $W^{m,2}(U;r,\rho)$ is the Banach space of locally integrable functions $v : U \to \mathbb{R}$ such that for each multi-index α, with $|\alpha| \le m$, $D^\alpha v$ exists in the weak sense, and

$$\|v\|_{W^{m,2}(U;r,\rho)} := \left(\sum_{|\alpha|\le m} \int_U r^2 \left|\rho^{|\alpha|} D^\alpha v\right|^2 dx\right)^{1/2} < \infty. \tag{9}$$

Endowed with the inner product which generates the above norm

$$(v,w)_{W^{m,2}(U;r,\rho)} := \sum_{|\alpha|\le m} \int_U r^2 \rho^{2|\alpha|} D^\alpha v D^\alpha w dx, \tag{10}$$

for all $v, w \in W^{m,2}(U;r,\rho)$, $W^{m,2}(U;r,\rho)$ is a Hilbert space.

Remark 4. Setting the weight functions $r = \rho = 1$, for all $x \in U$, we obtain the particular case of the Sobolev spaces $W^{m,2}(U)$.

Notation 2. In the sequel, when $U = \mathbb{R}^d$ we drop the argument in the function space notation. For instance, we denote $W^{m,2}(\mathbb{R}^d;r,\rho) =: W^{m,2}(r,\rho)$.

We make some assumptions on the behaviour of the weight functions r and ρ (see [5]).

Assumption 5. Let $m \ge 0$ be an integer and $r > 0$, $\rho > 0$ smooth functions on $\mathbb{R}^d$. There exists a constant K such that

(1) $|D^\alpha \rho| \le K\rho^{1-|\alpha|}$, for all multi-indexes α such that $|\alpha| \le m-1$ if $m \ge 2$;

(2) $|D^\alpha r| \le K(r/\rho^{|\alpha|})$, for all multi-indexes α such that $|\alpha| \le m$.

Remark 6. In (1) in Assumption 5, if $m < 2$ nothing is required.

Example 7. The following functions (taken from [5], citing O. G. Purtukhia) satisfy Assumption 5:

(1) $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \le 1/2$;

(2) $r(x) = \exp(\pm(1+|x|^2)^\beta)$, $0 \le \beta \le 1/2$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \le 1/2 - \beta$;

(3) $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \in \mathbb{R}$;

(4) $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq 1/2$;

(5) $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \geq 0$;

(6) $\rho(x) = \exp(-(1+|x|^2)^\gamma)$, $\gamma \geq 0$; each weight function $r(x)$ in examples (1)–(5).

Now, we switch our point of view and consider the functions $w : Q \to \mathbb{R}$ as mappings of t into certain spaces of functions of x we make precise below such that, for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $(w(t))(x) := w(t,x)$.

We impose a coercivity condition and make some assumptions on the growth and regularity of the operator's coefficients and also on the regularity of the free data f and g (see [5], where the assumptions are made for the more general case of an SPDE problem).

Assumption 8. Let $r > 0$, $\rho > 0$ be smooth functions on $\mathbb{R}^d$ and $m \geq 0$ an integer.

(1) There exists a constant $\lambda > 0$ such that $\sum_{i,j=1}^d a^{ij}(t,x)\xi^i\xi^j \geq \lambda\rho^2(x)|\xi|^2$, for all $t \geq 0$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^d$.

(2) The coefficients of L are measurable functions in $[0,T]\times\mathbb{R}^d$. The derivatives in x of the coefficients a^{ij} up to order $m \vee 1$ and of the coefficients b^i and c up to order m exist for any $t \in [0,T]$. Moreover, there exists a constant K such that

$$\begin{gathered}\left|D_x^\alpha a^{ij}\right| \leq K\rho^{2-|\alpha|} \quad \forall\, |\alpha| \leq m \vee 1, \\ \left|D_x^\alpha b^i\right| \leq K\rho^{1-|\alpha|}, \quad \left|D_x^\alpha c\right| \leq K \quad \forall\, |\alpha| \leq m,\end{gathered} \tag{11}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^d$, with D_x^α denoting the $|\alpha|$th partial derivative operator with respect to x.

(3) Consider $f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.

Notation 3. In the above assumption, $p \vee q := \max(p,q)$, with p, q integers. Also, for $m = 0$ we use the notation $W^{-1,2}(r,\rho) := (W^{1,2}(r,\rho))^*$, where $(W^{1,2}(r,\rho))^*$ is the dual of $W^{1,2}(r,\rho)$.

We define the generalised solution of problem (7).

Definition 9. One says that $u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalised solution of (7) on $[0,T]$ if

(1) $u \in L^2([0,T]; W^{1,2}(r,\rho))$;

(2) for every $t \in [0,T]$,

$$\begin{aligned}(u(t),\varphi) &= (g,\varphi) + \int_0^t \Big\{-\left(a^{ij}(s)\, D_{x^i}u(s), D_{x^j}\varphi\right) \\ &\quad + \left(b^i(s)\, D_{x^i}u(s) - D_{x^j}a^{ij}(s)\, D_{x^i}u(s), \varphi\right) \\ &\quad + \left(c(s)\, u(s), \varphi\right) + \langle f(s), \varphi\rangle \Big\}\, ds\end{aligned} \tag{12}$$

holds for all $\varphi \in C_0^\infty$.

Notation 4. The notation $(\,,\,)$ in the above definition stands for the inner product in $W^{0,2}(r,\rho)$. Also, we denote $C_0^\infty := C_0^\infty(\mathbb{R}^d)$.

Remark 10. Note that as an alternative to the infinite differentiability of φ in (2) it can be required that $\varphi \in W^{1,2}(r,\rho)$.

Finally, we state a result on the existence and uniqueness of the solution of problem (7). This result can be obtained from the general result in abstract spaces by using the suitable triple of spaces (see, e.g., [5], where the result is proved for the more general case of an SPDE problem).

Theorem 11. *Under (1)-(2) in Assumption 5, with $m+1$ in place of m, with $m \geq 0$ being an integer, and (1)–(3) in Assumption 8, problem (7) admits a unique generalised solution u on $[0,T]$. Moreover*

$$\begin{gathered}u \in C\left([0,T]; W^{m,2}(r,\rho)\right) \cap L^2\left([0,T]; W^{m+1,2}(r,\rho)\right), \\ \sup_{0\leq t\leq T} \|u(t)\|^2_{W^{m,2}(r,\rho)} + \int_0^T \|u(t)\|^2_{W^{m+1,2}(r,\rho)}\, dt \\ \leq N\left(\|g\|^2_{W^{m,2}(r,\rho)} + \int_0^T \|f(t)\|^2_{W^{m-1,2}(r,\rho)}\, dt\right),\end{gathered} \tag{13}$$

with N being a constant.

3. The Discrete Framework

We now proceed to the discretisation of problem (7) in the spatial variable. We set a suitable discrete framework with the use of a finite-difference scheme and, by showing that discretised problem can be cast into the general problem (4), we prove an existence and uniqueness result for the discretised problem's generalised solution.

We emphasize that this study mirrors the study of problem (7), in the sense that the framework we now set is a discrete version of the framework set for problem (7), and the techniques used for proving the existence and uniqueness results are the same for both problems.

Define the h-grid on $\mathbb{R}^d$, with $h \in (0,1]$,

$$Z_h^d = \left\{x \in \mathbb{R}^d : x = h\sum_{i=1}^d e_i n_i,\ n_i = 0, \pm1, \pm2, \ldots\right\}, \tag{14}$$

where e_i, for $i = 1, 2, \ldots, d$, is the unit vector in $\mathbb{R}^d$ whose ith entry is 1.

For every $x \in Z_h^d$, denote

$$\begin{aligned} \partial_i^+ u &= \partial_i^+ u(t,x) = h^{-1}\left(u(t, x+he_i) - u(t,x)\right), \\ \partial_i^- u &= \partial_i^- u(t,x) = h^{-1}\left(u(t,x) - u(t, x-he_i)\right) \end{aligned} \tag{15}$$

the forward and backward difference quotients in space, respectively. Define the discrete operator

$$L_h(t,x) = a^{ij}(t,x)\,\partial_j^-\partial_i^+ + b^i(t,x)\,\partial_i^+ + c(t,x). \tag{16}$$

We consider the discrete problem

$$u_t = L_h u + f_h \quad \text{in } Q(h), \qquad u(0,x) = g_h(x) \quad \text{on } Z_h^d, \tag{17}$$

where $Q(h) = [0,T] \times Z_h^d$, with $T \in (0,\infty)$, and f_h and g_h are functions such that $f_h : Q(h) \to \mathbb{R}$ and $g_h : Z_h^d \to \mathbb{R}$.

Consider functions $v : Z_h^d \to \mathbb{R}$. We introduce the space $l^{0,2}(r)$, the discrete version of the weighted Sobolev space $W^{0,2}(r,\rho)$,

$$l^{0,2}(r) := \left\{v : \|v\|_{l^{0,2}(r)} < \infty\right\}, \tag{18}$$

where the norm $\|v\|_{l^{0,2}(r)}$ is given by

$$\|v\|_{l^{0,2}(r)} = \left(\sum_{x\in Z_h^d} r^2(x)\,|v(x)|^2 h^d\right)^{1/2}. \tag{19}$$

We define, for any $v, w \in l^{0,2}(r)$, the inner product

$$(v,w)_{l^{0,2}(r)} = \sum_{x\in Z_h^d} r^2(x)\,v(x)\,w(x)\,h^d, \tag{20}$$

which induces the norm.

The inner product space $l^{0,2}(r)$ has a good structure: it can be easily shown that it is complete, therefore a Hilbert space.

For functions $w : Z_h^d \to \mathbb{R}$, we introduce also the discrete version of the weighted Sobolev space $W^{1,2}(r,\rho)$, the space $l^{1,2}(r,\rho)$ defined by

$$l^{1,2}(r,\rho) = \left\{w : \|w\|_{l^{1,2}(r,\rho)} < \infty\right\}, \tag{21}$$

with norm

$$\|w\|^2_{l^{1,2}(r,\rho)} := \|w\|^2_{l^{0,2}(r)} + \sum_{i=1}^d \left\|\rho\partial_i^+ w\right\|^2_{l^{0,2}(r)}. \tag{22}$$

We endow $l^{1,2}(r,\rho)$ with the inner product, inducing the norm,

$$(w,z)_{l^{1,2}(r,\rho)} = (w,z)_{l^{0,2}(r)} + \sum_{i=1}^d \left(\rho\partial_i^+ w, \rho\partial_i^+ z\right)_{l^{0,2}(r)}, \tag{23}$$

for any functions $w, z \in l^{1,2}(r,\rho)$.

We want to show that the discrete framework we have set is a particular case of the general framework considered in Section 2.

It can be easily checked that $l^{1,2}(r,\rho)$ is a reflexive and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}(r)$.

As $l^{1,2}(r,\rho)$, endowed with the inner product $(,)_{l^{1,2}(r,\rho)}$, is clearly a Hilbert space therefore it is reflexive, and the proof for the separability is trivial. The continuity of the embedding follows immediately from $\|v\|_{l^{0,2}(r)} \le \|v\|_{l^{1,2}(r,\rho)}$, for all $v \in l^{1,2}(r,\rho)$. Finally, the denseness can be checked by noticing that, for an arbitrary function $w \in l^{0,2}(r)$, and B a ball in Z_h^d, the function z defined by

$$z(x) = \begin{cases} w(x), & x \in B, \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$

belongs obviously to $l^{1,2}(r,\rho)$ and that, for any given $\varepsilon > 0$, $\|w - z\|_{l^{0,2}(r)} < \varepsilon$ if the diameter of B is chosen sufficiently large.

As in the previous section, we switch our viewpoint and consider the functions $z : Q(h) \to \mathbb{R}$ as mappings of t into certain spaces of functions of x, defined by $(z(t))(x) := z(t,x)$, for all $t \in [0,T]$ and for all $x \in Z_h^d$. For these functions, we consider the space $C([0,T]; l^{0,2}(r))$ of continuous $l^{0,2}(r)$-valued functions on $[0,T]$ and the spaces

$$\begin{aligned} &L^2\left([0,T]; l^{m,2}(r,\rho)\right) \\ &\quad = \left\{z : [0,T] \longrightarrow l^{m,2}(r,\rho) : \int_0^T \|z(t)\|^2_{l^{m,2}(r,\rho)} dt < \infty\right\}, \end{aligned} \tag{25}$$

with $m = 0, 1$.

Notation 5. We identify $l^{0,2}(r,\rho)$ with $l^{0,2}(r)$.

Remark 12. Clearly, if $u \in C([0,T]; l^{0,2}(r))$ then $\sup_{t\in[0,T]} \|u(t)\|_{l^{0,2}(r)} < \infty$.

We make some assumptions on the regularity of the data f_h and g_h in (17).

Assumption 13. Let $r > 0$ be a smooth function on $\mathbb{R}^d$. We assume

(1) $f_h \in L^2([0,T]; l^{0,2}(r))$;

(2) $g_h \in l^{0,2}(r)$.

Remark 14. In Assumption 13, (1) can be replaced with the weaker assumption $f_h \in L^2([0,T]; (l^{1,2}(r,\rho))^*)$, where $(l^{1,2}(r,\rho))^*$ denotes the dual space of $l^{1,2}(r,\rho)$.

Remark 15. We note that $|\partial_i^+ a^{ij}| \le K\rho$ can be obtained from (2) in Assumption 8. In fact, by the mean-value theorem,

$$\begin{aligned} \left|\partial_i^+ a^{ij}(t,x)\right| &= \left|h^{-1}\left(a^{ij}(t, x+he_i) - a^{ij}(t,x)\right)\right| \\ &= \left|\frac{\partial}{\partial x^i} a^{ij}(t, x+\tau e_i)\right|, \end{aligned} \tag{26}$$

for some τ such that $0 < \tau < h$. Thus $|(\partial/\partial x^i)a^{ij}| \le K\rho$ implies $|\partial_i^+ a^{ij}| \le K\rho$.

We define the generalised solution of problem (17).

Definition 16. One says that $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalised solution of (17) if, for every $t \in [0,T]$,

$$\begin{aligned}(u(t),\varphi) = {} & (g_h,\varphi) \\ & + \int_0^t \Big\{ -\left(a^{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi\right) \\ & \quad + \left(b^i(s)\partial_i^+ u(s) - \partial_j^+ a^{ij}(s)\partial_i^+ u(s), \varphi\right) \\ & \quad + \left(c(s)u(s),\varphi\right) + \langle f_h(s),\varphi\rangle \Big\} ds\end{aligned} \tag{27}$$

holds for all $\varphi \in l^{1,2}(r,\rho)$.

Notation 6. In the above definition, $(\ ,\ )$ denotes the inner product in $l^{0,2}(r)$. We keep this convention for the remainder of present section.

Finally, we prove an existence and uniqueness result for the solution of the discrete problem (17). This result shows that the scheme is stable, that is, that, informally, the discrete problem's solution remains bounded independent of the space-step h. The result is obtained as a consequence of Theorem 3, remaining only to prove that, under the discrete framework we constructed, (1)-(2) in Assumption 1 hold.

Theorem 17. *Under (1)-(2) in Assumption 8 and (1)-(2) in Assumption 13, problem* (17) *has a unique generalised solution u in* $[0,T]$. *Moreover*

$$\begin{aligned}\sup_{0\le t\le T} \|u(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t)\|^2_{l^{1,2}(r,\rho)} dt \\ \le N\left(\|g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f_h(t)\|^2_{l^{0,2}(r)} dt\right),\end{aligned} \tag{28}$$

with N a constant independent of h.

Proof. Let $L_h(s) : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$, for every $s \in [0,T]$. We define

$$\begin{aligned}\langle L_h(s)\psi,\varphi\rangle := {} & -\left(a^{ij}(s)\partial_i^+\psi, \partial_j^+\varphi\right) \\ & + \left(b^i(s)\partial_i^+\psi - \partial_j^+ a^{ij}(s)\partial_i^+\psi, \varphi\right) \\ & + \left(c(s)\psi,\varphi\right),\end{aligned} \tag{29}$$

for all $s \in [0,T]$, $\varphi, \psi \in l^{1,2}(r,\rho)$.

It suffices to prove that the following properties hold:

(1) $\exists K, \lambda > 0$ constants: $\langle L_h(s)\psi,\psi\rangle + \lambda\|\psi\|_{l^{1,2}(r,\rho)} \le K\|\psi\|_{l^{0,2}(r)}$,

(2) $\exists K$ constant: $\langle L_h(s)\psi,\varphi\rangle| \le K\|\psi\|_{l^{1,2}(r,\rho)} \cdot \|\varphi\|_{l^{1,2}(r,\rho)}$,

for all $s \in [0,T]$, $\varphi, \psi \in l^{1,2}(r,\rho)$.

For the first property, owing to (1) and (2) in Assumption 8, we have

$$\begin{aligned}\langle L_h(s)\psi,\psi\rangle = {} & -\sum_{i,j}\sum_x r^2 a^{ij}(s)\partial_i^+\psi\partial_j^+\psi h^d \\ & + \sum_i\sum_x r^2\left(b^i(s) - \partial_j^+ a^{ij}(s)\right)\partial_i^+\psi\psi h^d \\ & + \sum_x r^2 c(s)\psi\psi h^d \\ \le {} & -\lambda\sum_i\sum_x r^2|\rho\partial_i^+\psi|^2 h^d \\ & + 2K\sum_i\sum_x r^2\rho|\partial_i^+\psi\psi| h^d + K\sum_x r^2|\psi|^2 h^d \\ = {} & -\lambda\sum_i\|\rho\partial_i^+\psi\|^2_{l^{0,2}(r)} \\ & + 2K\sum_i\sum_x r^2\rho|\partial_i^+\psi\psi| h^d + K\|\psi\|^2_{l^{0,2}(r)},\end{aligned} \tag{30}$$

where the variable $x \in Z_h^d$ is omitted, $\sum_x$ denotes the summation over Z_h^d, and $\sum_i$, $\sum_j$ the summation over $\{1,2,\ldots,d\}$. We use Cauchy's inequality on the second term in estimate (30) and obtain

$$\begin{aligned}\langle L_h(s)\psi,\psi\rangle \le {} & -\lambda\sum_i\|\rho\partial_i^+\psi\|^2_{l^{0,2}(r)} + \varepsilon K\sum_i\sum_x r^2|\rho\partial_i^+\psi|^2 h^d \\ & + \frac{K}{\varepsilon}\sum_i\sum_x r^2|\psi|^2 h^d + K\|\psi\|^2_{l^{0,2}(r)} \\ = {} & -\lambda\sum_i\|\rho\partial_i^+\psi\|^2_{l^{0,2}(r)} - \lambda\|\psi\|^2_{l^{0,2}(r)} \\ & + \varepsilon K\sum_i\|\rho\partial_i^+\psi\|^2_{l^{0,2}(r)} \\ & + \frac{K}{\varepsilon}\|\psi\|^2_{l^{0,2}(r)} + (K+\lambda)\|\psi\|^2_{l^{0,2}(r)} \\ \le {} & -\lambda\|\psi\|^2_{l^{1,2}(r,\rho)} + K\|\psi\|^2_{l^{0,2}(r)},\end{aligned} \tag{31}$$

with $\lambda > 0, K$ constants, by taking ε sufficiently small. The first property is proved.

The second property follows from (2) in Assumption 8 and Cauchy-Schwarz inequality

$$\begin{aligned}|\langle L_h(s)\psi,\varphi\rangle| = \Bigg| & -\sum_{i,j}\sum_x r^2 a^{ij}(s)\partial_i^+\psi\partial_j^+\varphi h^d \\ & + \sum_i\sum_x r^2 b^i(s)\partial_i^+\psi\varphi h^d \\ & - \sum_{i,j}\sum_x r^2\partial_j^+ a^{ij}(s)\partial_i^+\psi\varphi h^d\end{aligned}$$

$$
\begin{aligned}
&\left. + \sum_x r^2 c(s)\psi\varphi h^d \right| \\
&\le K \sum_{i,j}\sum_x r^2 \left|\rho^2 \partial_i^+ \psi \partial_j^+ \varphi\right| h^d \\
&\quad + K\sum_i \sum_x r^2 \left|\rho \partial_i^+ \psi \varphi\right| h^d + K \sum_x r^2 \left|\psi\varphi\right| h^d \\
&\le K \sum_i \left\|\rho\partial_i^+\psi\right\|_{l^{0,2}(r)} \sum_j \left\|\rho\partial_j^+\varphi\right\|_{l^{0,2}(r)} \\
&\quad + K\sum_i \left\|\rho\partial_i^+\psi\right\|_{l^{0,2}(r)} \left\|\varphi\right\|_{l^{0,2}(r)} \\
&\quad + K\left\|\psi\right\|_{l^{0,2}(r)}\left\|\varphi\right\|_{l^{0,2}(r)} \\
&\le K\left\|\psi\right\|_{l^{1,2}(r,\rho)} \cdot \left\|\varphi\right\|_{l^{1,2}(r,\rho)},
\end{aligned}
\tag{32}
$$

where the same writing conventions are kept.

Owing to Theorem 3 the result follows. □

4. Approximation Results

In this section, we study the approximation properties of scheme (17). We begin by investigating the consistency of the scheme and prove that the difference quotients approximate the partial derivatives (with accuracy of order 1). In addition, we estimate the rate of convergence of the difference quotients to the partial derivatives.

The result is obtained by using a Sobolev inequality, under stronger regularity assumptions, and imposing that the weights ρ are bounded from below by a positive constant. In practice, the latter restriction amounts to assuming that the weights ρ are increasing functions of $|x|$, which is precisely the case we are interested in.

Also, we note that the way we set our discrete framework, in strong connection with the framework for problem (7), plays a crucial role in obtaining the convergence rate.

Theorem 18. *Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$ and m an integer strictly greater than $d/2$. Assume that (1)-(2) in Assumption 5 are satisfied and also that $\rho(x) \ge C$ on $\mathbb{R}^d$, with $C > 0$ being a constant. Let $u(t) \in W^{m+2,2}(r,\rho)$, $v(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$. Then there exists a constant N independent of h such that*

(1) $\sum_{x\in Z_h^d} r^2(x)|u_{x^i}(t,x) - \partial_i^+ u(t,x)|^2 \rho^2(x) h^d \le h^2 N \|u(t)\|^2_{W^{m+2,2}(r,\rho)}$,

(2) $\sum_{x\in Z_h^d} r^2(x)|v_{x^i x^j}(t,x) - \partial_j^- \partial_i^+ v(t,x)|^2 \rho^4(x) h^d \le h^2 N \|v(t)\|^2_{W^{m+3,2}(r,\rho)}$,

for all $t \in [0,T]$.

Remark 19. The following remarks will be used in the proof of the theorem.

(1) Under the conditions of the theorem, function $u(t)$ (function $v(t)$) has a modification in x which is continuously differentiable in x up to order 2 (up to order 3), and the derivatives equal the weak derivatives, for every $t \in [0,T]$. This can be proved by Sobolev's embedding of $W^{m,2}(B)$ into $C^n(\overline{B})$, for balls B in $\mathbb{R}^d$, if $m > d/2 + n$ (see, e.g., [18, 19]). We consider these modifications in the theorem's proof.

(2) Note that if U, V are open subsets of $\mathbb{R}^d$ with $V \subset U$ and $w \in W^{m,2}(U)$ then $w \in W^{m,2}(V)$. Also, if $w \in W^{m,2}(U)$ and $\zeta \in C_0^\infty(U)$ then $\zeta \in W^{m,2}(U)$ and $\zeta w \in W^{m,2}(U)$ (see, e.g., [18, 19]).

Proof of Theorem 18. Let us prove (1). We define a suitable geometric setting and then obtain an estimate for

$$
r^2(x)\left|u_{x^i}(t,x) - \partial_i^+ u(t,x)\right|^2 \rho^2(x), \tag{33}
$$

with $x \in Z_h^d$, using Sobolev's inequality on a fixed ball.

Let us consider d-cells

$$
\begin{aligned}
R_h = \Big\{ &\left(x^1, x^2, \ldots, x^d\right) \in \mathbb{R}^d : x_h^i < x^i < x_h^i + h, \\
&i = 1, 2, \ldots, d \Big\},
\end{aligned}
\tag{34}
$$

with $x_h = (x_h^1, x_h^2, \ldots, x_h^d) \in Z_h^d$ fixed. Consider the particular d-cell, where $h = 1$ and $x_1 = (0,0,\ldots,0)$, and denote it by R_1^0. Now, take open balls B_h such that $B_h \supset R_h$, with the vertices $\{x_h^i, x_h^i + h,\ i = 1,2,\ldots,d\}$ lying on the limiting sphere. Denote by B_1^0 the ball containing R_1^0.

For every $x_h \in Z_h^d$, taking in mind (1) in Remark 19, we have, by the mean-value theorem,

$$
\begin{aligned}
\partial_i^+ u(t,x_h) &= h^{-1}\left(u(t,x_h + he_i) - u(t,x_h)\right) \\
&= u_{x^i}(t, x_h + \theta h e_i),
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\left|u_{x^i}(t,x_h) - \partial_i^+ u(t,x_h)\right| &= \left|u_{x^i}(t,x_h) - u_{x^i}(t,x_h + \theta h e_i)\right| \\
&\le h\left|u_{x^i x^i}\left(t, x_h + \theta' h e_i\right)\right|,
\end{aligned}
\tag{36}
$$

for some $0 < \theta' < \theta < 1$. Clearly,

$$
\left|u_{x^i x^i}\left(t, x_h + \theta' h e_i\right)\right| \le \sup_{x\in R_h} \left|u_{x^i x^i}(t,x)\right|, \tag{37}
$$

and then, from (36) and (37),

$$\left|u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)\right|^2\le h^2\sup_{x\in R_h}\left|u_{x^ix^i}(t,x)\right|^2. \tag{38}$$

We change variable in order to have the supremum in (38) calculated over the fixed d-cell R_1^0:

$$\sup_{x\in R_h}\left|u_{x^ix^i}(t,x)\right|=\sup_{x\in R_1^0}\left|u_{x^ix^i}(t,x_h+hx)\right|. \tag{39}$$

As

$$\sup_{x\in R_1^0}\left|u_{x^ix^i}(t,x_h+hx)\right|^2\le\sup_{x\in B_1^0}\left|u_{x^ix^i}(t,x_h+hx)\right|^2, \tag{40}$$

from (38)–(40) we immediately obtain

$$\begin{aligned}&r^2(x_h)\left|u_{x^i}(t,x_h)-\partial_i^+u(t,x_h)\right|^2\rho^2(x_h)\\&\quad\le h^2\sup_{x\in R_1^0}\left(r^2(x_h+hx)\left|u_{x^ix^i}(t,x_h+hx)\right|^2\rho^2(x_h+hx)\right)\\&\quad\le h^2\sup_{x\in B_1^0}\left(r^2(x_h+hx)\left|u_{x^ix^i}(t,x_h+hx)\right|^2\rho^2(x_h+hx)\right).\end{aligned} \tag{41}$$

Now, taking in mind (2) in Remark 19, we have for $m>d/2$ by using Sobolev's inequality

$$\begin{aligned}&\sup_{x\in B_1^0}\left|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)\right|^2\\&\quad\le N\sum_{|\alpha|\le m}\int_{B_1^0}\left|D_x^\alpha\left(r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho\right.\right.\\&\qquad\left.\left.\times(x_h+hx)\right)\right|^2dx,\end{aligned} \tag{42}$$

with N being a constant independent of h. Observe that the Leibniz formula

$$\begin{aligned}\left|D_x^\alpha(ru_{x^ix^i}\rho)\right|&=\left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta(r\rho)D_x^{\alpha-\beta}u_{x^ix^i}\right|\\&=\left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left(\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma rD^{\beta-\gamma}\rho\right)D_x^{\alpha-\beta}u_{x^ix^i}\right|\end{aligned} \tag{43}$$

holds (the arguments of r, ρ and $u_{x^ix^i}$ are omitted). Also, keeping the same convention, owing to Assumption 5

$$\left|D^\gamma r\right|\le Kr\rho^{-|\gamma|},\qquad\left|D^{\beta-\gamma}\rho\right|\le K\rho^{1-(|\beta|-|\gamma|)}, \tag{44}$$

with K a constant, and then

$$\begin{aligned}\left|\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma rD^{\beta-\gamma}\rho\right|&\le N\sum_{\gamma\le\beta}\binom{\beta}{\gamma}r\rho^{-|\gamma|}\rho^{1-(|\beta|-|\gamma|)}\\&\le Nr\rho^{1-|\beta|},\end{aligned} \tag{45}$$

with N a constant. From (42)–(45), we get

$$\begin{aligned}&\sup_{x\in B_1^0}\left|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)\right|^2\\&\quad\le N\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\int_{B_1^0}r^2(x_h+hx)\left|\rho^{1-|\beta|}(x_h+hx)\right|^2\\&\qquad\times\left|D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)\right|^2dx.\end{aligned} \tag{46}$$

Note also that, owing to Hölder inequality and to the hypotheses on function ρ, the integral in (46) can be estimated by

$$\begin{aligned}&\int_{B_1^0}r^2(x_h+hx)\\&\quad\times\left|\rho^{1-|\beta|}(x_h+hx)D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)\right|^2dx\\&\quad\le N\int_{B_1^0}r^2(x_h+hx)\left|\rho^{2+(|\alpha|-|\beta|)}(x_h+hx)\right.\\&\qquad\left.\times D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)\right|^2dx\\&\qquad\cdot\sup_{x\in B_1^0}\left|\rho^{-1-|\alpha|}(x_h+hx)\right|^2\\&\quad\le N\int_{B_1^0}r^2(x_h+hx)\left|\rho^{2+(|\alpha|-|\beta|)}(x_h+hx)\right.\\&\qquad\left.\times D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)\right|^2dx.\end{aligned} \tag{47}$$

Thus, from (46) and (47),

$$\begin{aligned}&\sup_{x\in B_1^0}\left|r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)\right|^2\\&\quad\le N\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\int_{B_1^0}r^2(x_h+hx)\\&\qquad\times\left|\rho^{2+(|\alpha|-|\beta|)}(x_h+hx)\right.\\&\qquad\left.\cdot D_x^{\alpha-\beta}u_{x^ix^i}(t,x_h+hx)\right|^2dx\\&\quad\le N\sum_{|\alpha|\le m}\int_{B_1^0}r^2(x_h+hx)\\&\qquad\times\left|\rho^{2+|\alpha|}(x_h+hx)D_x^\alpha u_{x^ix^i}(t,x_h+hx)\right|^2dx\end{aligned}$$

$$\begin{aligned}
&\le N \sum_{|\alpha|\le m+2} \int_{B_1^0} r^2(x_h+hx) \\
&\qquad \times \left|\rho^{|\alpha|}(x_h+hx) D_x^\alpha u(t,x_h+hx)\right|^2 dx \\
&= N \sum_{|\alpha|\le m+2} \int_{B_h} r^2(x) \left|\rho^{|\alpha|}(x) D_x^\alpha u(t,x)\right|^2 h^{-d} h^{2|\alpha|} dx \\
&\le N \sum_{|\alpha|\le m+2} \int_{B_h} r^2(x) \left|\rho^{|\alpha|}(x) D_x^\alpha u(t,x)\right|^2 h^{-d} dx.
\end{aligned} \tag{48}$$

Finally, owing to the particular geometry of the framework we have set, from (41) and (48) we obtain

$$\begin{aligned}
&\sum_{x\in Z_h^d} r^2(x) \left|u_{x^i}(t,x) - \partial_i^+ u(t,x)\right|^2 \rho^2(x) h^d \\
&\le N h^2 \sum_{|\alpha|\le m+2} \sum_{x_h\in Z_h^d} \int_{B_h(x_h)} r^2(x) \left|\rho^{|\alpha|}(x) D_x^\alpha u(t,x)\right|^2 dx \\
&\le N h^2 \sum_{|\alpha|\le m+2} \sum_{x_h\in Z_h^d} \int_{R_h(x_h)} r^2(x) \left|\rho^{|\alpha|}(x) D_x^\alpha u(t,x)\right|^2 dx \\
&\le h^2 N \|u(t)\|^2_{W^{m+2,2}(r,\rho)},
\end{aligned} \tag{49}$$

where $B_h(x_h) := B_h$, $R_h(x_h) := R_h$, and the proof for (1) is complete.

The proof for (2) follows the same steps. □

Finally, owing to the stability and consistency properties of the scheme (Theorems 17 and 18, resp.), we prove the convergence of the discrete problem's solution to the PDE problem's solution and compute a rate of convergence. The accuracy obtained is of order 1.

Theorem 20. *Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$ and m an integer strictly greater than $d/2$. Assume that the hypotheses of Theorems 11 and 17 are satisfied and that $\rho(x) \ge C$ on $\mathbb{R}^d$, with $C > 0$ being a constant. Denote by u the solution of problem (7) in Theorem 11 and by u_h the solution of problem (17) in Theorem 17. Assume additionally that $u \in L^2([0,T]; W^{m+3,2}(r,\rho))$. Then*

$$\begin{aligned}
&\sup_{0\le t\le T} \|u(t) - u_h(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t) - u_h(t)\|^2_{l^{1,2}(r,\rho)} dt \\
&\le h^2 N \int_0^T \|u(t)\|^2_{W^{m+3,2}(r,\rho)} dt \\
&\quad + N \left(\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}(r)} dt \right),
\end{aligned} \tag{50}$$

with N being a constant independent of h.

Remark 21. Under the conditions of the above theorem, there are modifications in x such that the data $f(t)$ and g are continuous in x, for every $t \in [0,T]$ (see Remark 19). We will consider these modifications in the proof of the theorem.

Proof of Theorem 20. From (7) and (17), we have that $u - u_h$ satisfies the problem

$$\begin{aligned}
&(u-u_h)_t = L_h(u-u_h) + (L-L_h)u + (f-f_h) \quad \text{in } Q(h) \\
&(u-u_h)(0,x) = (g-g_h)(x) \quad \text{on } Z_h^d.
\end{aligned} \tag{51}$$

Taking in mind Remark 21, we see clearly that $f - f_h \in L^2([0,T]; l^{0,2}(r))$ and $g - g_h \in l^{0,2}(r)$.

With respect to the term $(L - L_h)u$, note that if $u(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$,

$$\begin{aligned}
&\sum_{x\in Z_h^d} r^2(x) \left|(L-L_h)(t)u(t)\right|^2 h^d \\
&= \sum_{x\in Z_h^d} r^2(x) \left| a^{ij}(t,x) \left(\frac{\partial^2}{\partial x^i \partial x^j} - \partial_j^- \partial_i^+ \right) u(t,x) \right. \\
&\qquad \left. + b^i(t,x) \left(\frac{\partial}{\partial x^i} - \partial_i^+ \right) u(t,x) \right|^2 h^d \\
&\le h^2 N \|u(t)\|^2_{W^{m+3,2}(r,\rho)} < \infty,
\end{aligned} \tag{52}$$

owing to (2) in Assumption 8 and to Theorem 18. Thus $(L - L_h)(t)u(t) \in l^{0,2}(r)$, for every $t \in [0,T]$. Moreover, as by assumption $u \in L^2([0,T]; W^{m+3,2}(r,\rho))$, we obtain immediately $(L - L_h)u \in L^2([0,T]; l^{0,2}(r))$.

We have shown that problem (51) satisfies the hypotheses of Theorem 17, therefore holding the estimate

$$\begin{aligned}
&\sup_{0\le t\le T} \|u(t) - u_h(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t) - u_h(t)\|^2_{l^{1,2}(r,\rho)} dt \\
&\le N \left(\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}(r)} dt \right. \\
&\qquad \left. + \int_0^T \|(L-L_h)(t)u(t)\|^2_{l^{0,2}(r)} dt \right).
\end{aligned} \tag{53}$$

Owing again to (2) in Assumption 8 and to Theorem 18, the result follows. □

The following result is an immediate consequence of Theorem 20.

Corollary 22. *Let the hypotheses of Theorem 20 be satisfied, and denote by u the solution of (7) in Theorem 11 and by u_h the solution of (17) in Theorem 17. If there is a constant N independent of h such that*

$$\begin{aligned}
&\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}(r)} dt \\
&\le h^2 N \left(\|g\|^2_{W^{m,2}(r,\rho)} + \int_0^T \|f(t)\|^2_{W^{m-1,2}(r,\rho)} dt \right)
\end{aligned} \tag{54}$$

then

$$\sup_{0\le t\le T}\left\|u(t)-u_h(t)\right\|^2_{l^{0,2}(r)}+\int_0^T\left\|u(t)-u_h(t)\right\|^2_{l^{1,2}(r,\rho)}dt$$
$$\le h^2N\left(\int_0^T\left\|u(t)\right\|^2_{W^{m+3,2}(r,\rho)}dt+\left\|g\right\|^2_{W^{m,2}(r,\rho)}\right.$$
$$\left.+\int_0^T\left\|f(t)\right\|^2_{W^{m-1,2}(r,\rho)}dt\right). \tag{55}$$

5. Conclusions

In this paper, we investigated the finite-difference spatial approximation of the Cauchy problem for a second-order linear parabolic PDE, in the framework of the variational approach.

By considering a suitable class of weighted Sobolev spaces, and its zero and first-order discrete versions, we could deal with the growth in space of the PDE coefficients (as well as with the spatial growth of the free data f and g). Moreover, as the framework and techniques used to study the discrete problem mirror the framework and techniques for the corresponding continuous problem we could estimate a rate of convergence.

The approximation was studied under the strong assumption that the PDE does not degenerate. But the framework we used is broadly the appropriate framework for a future investigation of the related degenerate case.

Other possible further research directions include the use of splitting-up methods (see [10]), following Richardson's idea to accelerate numerical schemes, and also the use of techniques reducing the volume of computational work (e.g., sparse grid techniques), in order to deal with the computational challenge posed by the possible high dimensionality of the problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stability Analysis of Impulsive Stochastic Functional Differential Equations with Delayed Impulses via Comparison Principle and Impulsive Delay Differential Inequality

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The problem of stability for nonlinear impulsive stochastic functional differential equations with delayed impulses is addressed in this paper. Based on the comparison principle and an impulsive delay differential inequality, some exponential stability and asymptotical stability criteria are derived, which show that the system will be stable if the impulses' frequency and amplitude are suitably related to the increase or decrease of the continuous stochastic flows. The obtained results complement ones from some recent works. Two examples are discussed to illustrate the effectiveness and advantages of our results.

1. Introduction

Impulsive dynamical equations have received considerable attention during the recent decades since they provide a natural framework for mathematical modeling of many real world evolutionary processes where the states undergo abrupt changes at certain instants (see [1–7]). In particular, more researchers have given special interests to the stability and stabilization analysis of impulsive functional differential equations (IFDEs) and there are extensive literatures in this field (see [8–14] and reference therein).

In the current literature concerning IFDEs, the impulses are assumed to take the form $\Delta x(t_k) = I_k(t_k, x(t_k^-))$, which indicates that the state "jump" at the impulse times t_k is only related to the present state variables. But in most cases, it is more applicable that the state variables on the impulses that we add are also related to the past ones. For example, in the transmission of the impulse information, input delays are often encountered (see, e.g., [15, 16]). So, it is more meaningful if the above impulses are modified as $\Delta x(t_k) = x(t_k) - x(t_k^-) = I_k(t_k, x((t_k - d_k)^-))$. Recently, there have been several attempts in the literature to study the stability and control problems of IFDEs with delayed impulse (IFDEs-DI). For example, by using Lyapunov functions couples with Razumikhin techniques, some Razumikhin-type asymptotic stability and exponential stability criteria for IFDEs-DI were established in [17–19], and some Lyapunov-based sufficient conditions for the exponential stability of the equations were derived in [20].

On the other hand, stochastic perturbations are unavoidable in real equations (see [21, 22] and reference therein). In recent years, the stability analysis of impulsive stochastic functional equations which include delay equations is interesting to many investigators, and many results of stability criteria of these equations have been reported (see, e.g., [23–29]). Very recently, [30, 31] took environment noise into account and generalized delayed impulses to stochastic equations. In particular, applying the Lyapunov functions couples with Razumikhin techniques, [30] investigates both moment and almost sure exponential stability of impulsive stochastic functional differential equations with delayed impulses (ISFDEs-DI), and several Razumikhin-type criteria on the exponential stability and uniform stability in terms of two measures for the equations were established in [31].

But it is worth noting that the stability analysis in [30] and the effects of time delay on the impulses have been ignored. And in [30, 31], the authors only consider the case that the impulsive stabilization. Moreover, it is well known that the Razumikhin techniques are very effective in the study of stability problems for ordinary and functional differential equations. However, when we use the Razumikhin techniques, we need to choose an appropriate minimal class of functionals relative to which the derivative of the Lyapunov function or Lyapunov functional is estimated, which is not entirely convenient.

Motivated by the above discussion, in this paper, we will further investigate the stability of ISFDEs-DI. By using the comparison principle and an impulsive delay differential inequality, some exponential and asymptotical stability criteria are derived, which are more convenient to be applied than those Razumikhin-type conditions. Our results complement ones from some recent works and show that the ISFDE-ID will be stable if the impulses' frequency and amplitude are suitably related to the increase or decrease of the corresponding continuous stochastic flows. The rest of the paper is organized as follows. In Section 2, some relevant notations and definitions are presented. In Section 3, the comparison principle, an impulsive delay differential inequality, and several criteria on the exponential stability and asymptotical stability are established. Section 4 provides two illustrative examples to demonstrate the applications of the obtained results. Finally, conclusions are drawn in Section 5.

2. Preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions; that is, it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t) = (w_1(t), \ldots, w_d(t))^T$ be a d-dimensional Brownian motion defined on the probability space. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}^n$ the n-dimensional real Euclidean space, and $\mathbb{R}^{n\times d}$ the space of $n \times d$ real matrices. I stands for the identity matrix of appropriate dimensions. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. For $A \in \mathbb{R}^{n\times d}$, $\|A\|$ denotes spectral norm of the matrix A. Denote by $\lambda_{\min}(\cdot)$ the minimum eigenvalue of a matrix. If A is a vector or matrix, its transpose is denoted by A^T.

Let $\tau > 0$ and $\mathrm{PC}([-\tau, 0]; \mathbb{R}^n) = \{\varphi : [-\tau, 0] \to \mathbb{R}^n | \varphi(t^+) = \varphi(t)$ for all $t \in [-\tau, 0)$, $\varphi(t^-)$ exist and let $\varphi(t^-) = \varphi(t)$ for all but at most a finite number of points $t \in (-\tau, 0]\}$ be with the norm $\| \varphi \| = \sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$, where $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of function $\varphi(t)$ at t, respectively. Denote $\mathrm{PC}([t_0 - \tau, \infty); \mathbb{R}) = \{\varphi|\varphi|_{[t_0-\tau,b]} \in \mathrm{PC}([t_0 - \tau, b]; \mathbb{R})$ for all $b > t_0 - \tau\}$.

For $p > 0$ and $t \geqslant 0$, let $\mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable $\mathrm{PC}([-\tau, 0]; \mathbb{R}^n)$-valued random variables φ such that $\sup_{-\tau\leqslant\theta\leqslant 0}\mathbb{E}|\varphi(\theta)|^p < \infty$, where $\mathbb{E}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathbb{P}$. And $L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$ measurable $\mathbb{R}^n$-valued random variables X, such that $\mathbb{E}|X|^p < \infty$. Let $\mathrm{PC}^b([-\tau, 0]; \mathbb{R}^n)$ be the family of all bounded $\mathrm{PC}([-\tau, 0]; \mathbb{R}^n)$-valued functions, and let $\mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_{t_0}$ measurable $\mathrm{PC}^b([-\tau, 0]; \mathbb{R}^n)$-valued functions.

Consider the following ISFDE-DI:

$$\begin{aligned} &\mathrm{d}x(t) = f(t, x_t)\,\mathrm{d}t + g(t, x_t)\,\mathrm{d}w(t), \quad t \neq t_k,\ t \geqslant t_0, \\ &x(t_k) = I_k\left(t_k, x(t_k^-), x\left((t_k - d_k)^-\right)\right), \quad k \in \mathbb{N}, \\ &x_{t_0}(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{1}$$

where the initial value $\xi \in \mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$, $x(t) = ((x_1(t), \ldots, x_n(t))^T$, $x_t = x(t + \theta) \in \mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$. Both $f : \mathbb{R}_+ \times \mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times d}$ are Borel measurable. $I_k : \mathbb{R}_+ \times L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \times L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \to \mathbb{R}^n$ represents the impulsive perturbation of x at time t_k. The fixed moments of impulse times $\{t_k, k \in \mathbb{N}\}$ satisfy $0 \leqslant t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty$ (as $k \to \infty$). $\{d_k \geqslant 0, k \in \mathbb{N}\}$ are the impulse input delays satisfying $d = \sup_{k\in\mathbb{N}} d_k < \infty$.

As a standing hypothesis, we assume that for any $\xi \in \mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$ there exists a unique stochastic process satisfying (1) denoted by $x(t; t_0, \xi)$, which is continuous on the right-hand side and limitable on the left-hand side (see [32]). Moreover, we assume that $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, and $I_k(t, 0, 0) \equiv 0$ for all $t \geqslant t_0$, $k \in \mathbb{N}$; then (1) admits a trivial solution $x(t) \equiv 0$.

We introduce the following scalar IFDE-DI as the comparison system:

$$\begin{aligned} &\dot{u}(t) = h(t, u(t), u_t), \quad t \neq t_k,\ t \geqslant t_0, \\ &u(t_k) = \Psi_{1k}(u(t_k^-)) + \Psi_{2k}\left(u(t_k - d_k)^-\right), \quad k \in \mathbb{N}, \\ &u_{t_0}(\theta) = \zeta(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{2}$$

where the initial value $\zeta \in \mathrm{PC}([-\tau, 0]; \mathbb{R}_+)$; $u_t \in \mathrm{PC}([-\tau, 0]; \mathbb{R}_+)$ is defined as $u_t = u(t + \theta)$, $\theta \in [-\tau, 0]$. $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathrm{PC}([-\tau, 0]; \mathbb{R}_+) \to \mathbb{R}_+$ is continuous, Lebesgue measurable, and nondecreasing with respect to the last argument; Ψ_{1k}, $\Psi_{2k} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and nondecreasing. Assume that $h(t, 0, 0) \equiv 0$, $\Psi_{1k}(0) \equiv 0$, and $\Psi_{2k}(0) \equiv 0$; then system (2) admits a trivial solution $u(t) \equiv 0$. We further assume that for any $\zeta \in \mathrm{PC}^b([-\tau, 0]; \mathbb{R}_+)$, there exists a unique solution to system (2) on $[t_0 - \tau, \infty)$ denoted by $u(t; t_0, \zeta)$ (see [5, 6]) which is continuous on the right-handside and limitable on the left-hand side.

For convenience, we introduce the following function classes:

$\mathcal{K} = \{\phi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and strictly increasing, $\phi(0) = 0\}$.

$\mathcal{K}_\infty = \{\phi \in \mathcal{K}, \phi(s) \to \infty$ as $s \to \infty\}$.

$C\mathcal{K} = \{\phi \in \mathcal{K}, \phi$ is concave$\}$.

$V\mathcal{K}_\infty = \{\phi \in \mathcal{K}_\infty,\ \phi$ is convex$\}$.

At the end of this section, let us introduce the following definitions.

Definition 1 (see [23, 26]). The trivial solution of (1) is said to be as follows.

(i) pth moment stable if, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\mathbb{E}\left|x(t;t_0,\xi)\right|^p \leqslant \varepsilon, \quad t \geqslant t_0, \tag{3}$$

whenever $\mathbb{E}\|\xi\|^p < \delta$.

(ii) pth moment asymptotically stable if it is pth moment stable and there exists $\delta_0 = \delta_0(t_0)$ such that

$$\lim_{t\to\infty} \mathbb{E}\left|x(t;t_0,\xi)\right|^p = 0, \quad t \geqslant t_0, \tag{4}$$

whenever $\mathbb{E}\|\xi\|^p < \delta_0$.

(iii) pth moment globally exponentially stable if there is a pair of positive constants λ, C such that

$$\mathbb{E}\left|x(t;t_0,\xi)\right|^p \leqslant C\mathbb{E}\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 \tag{5}$$

for all $\xi \in \mathrm{PC}^b_{\mathscr{F}_{t_0}}([-\tau,0];\mathbb{R}^n)$. When $p = 2$, it is usually said to be globally exponentially stable in mean square.

Definition 2 (see [26]). A function $V : [t_0-\tau,\infty)\times\mathbb{R}^n \to \mathbb{R}_+$ belongs to class v_0 if

(i) V is continuous on each of the sets $[t_{k-1},t_k)\times\mathbb{R}^n$ and for each $x, y \in \mathbb{R}^n$, $t \in [t_{k-1},t_k)$, and $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x)$ exists;

(ii) $V(t,x)$ is continuously once differentiable in t and twice in x in each of the sets $(t_{k-1},t_k)\times\mathbb{R}^n$, $k \in \mathbb{N}$.

If $V \in v_0$, define an operator $\mathscr{L}V$ from $[t_0,\infty)\times \mathrm{PC}([-\tau,0];\mathbb{R}^n)$ to $\mathbb{R}$ by

$$\begin{aligned}\mathscr{L}V(t,\varphi) &= V_t(t,\varphi(0)) + V_x(t,\varphi(0))\, f(t,\varphi)\\ &\quad + \frac{1}{2}\operatorname{trace}\left[g^T(t,\varphi)\, V_{xx}(t,\varphi(0))\, g(t,\varphi)\right],\end{aligned} \tag{6}$$

where

$$\begin{aligned}V_t(t,x) &= \frac{\partial V(t,x)}{\partial t},\\ V_x(t,x) &= \left(\frac{\partial V(t,x)}{\partial x_1},\ldots,\frac{\partial V(t,x)}{\partial x_n}\right),\\ V_{xx}(t,x) &= \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}.\end{aligned} \tag{7}$$

3. Main results

In this section, we will develop an impulsive delay differential inequality and comparison principles and establish some criteria on pth moment exponential stability and asymptotical stability for (1).

Lemma 3 (impulsive delay differential inequality). *Assume that $c \in \mathbb{R}$, $\delta \in \mathbb{R}$, $q \in \mathbb{R}_+$, $a_k > 0$, $b_k \geqslant 0$, $k \in \mathbb{N}$, $\overline{u}(t) := \sup_{\theta\in[-\tau,0]} u(t+\theta)$, and*

(i) $\ln(a_k + b_k e^{cd_k}) \leqslant \delta(t_k - t_{k-1})$ *for each* $k \in \mathbb{N}$;

(ii) $\delta + c + q\gamma < 0$, *where* $\gamma = \sup_{k\in\mathbb{N}}\{e^{\delta(t_k-t_{k-1})}, 1/e^{\delta(t_k-t_{k-1})}\}$.

Then any solution $u \in PC([t_0-\tau,\infty);\mathbb{R}_+)$ of the scalar impulsive delay differential inequality problem

$$\begin{aligned}&D^+u(t) \leqslant cu(t) + q\overline{u}(t), \quad t \neq t_k,\ t \geqslant t_0,\\ &u(t_k) \leqslant a_k u(t_k^-) + b_k u\left((t_k - d_k)^-\right), \quad k \in \mathbb{N}\end{aligned} \tag{8}$$

satisfies

$$u(t) \leqslant \gamma\overline{u}(t_0)\, e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 - \tau, \tag{9}$$

where λ is the unique positive solution of $\lambda + \delta + c + q\gamma e^{\lambda\tau} = 0$.

Proof. Set $v(t) = e^{-c(t-t_0)}u(t)$, $t \in [t_0-\tau,\infty)$. For each $k \in \mathbb{N}$, by the second inequality of (8), we have

$$\begin{aligned}v(t_k) &= e^{-c(t_k-t_0)}u(t_k)\\ &\leqslant e^{-c(t_k-t_0)}\left[a_k u(t_k^-) + b_k u\left((t_k-d_k)^-\right)\right]\\ &= a_k e^{-c(t_k-t_0)}u(t_k^-) + \beta_k b_k u\left((t_k-d_k)^-\right)e^{-c(t_k-d_k-t_0)}\\ &= a_k v(t_k^-) + \beta_k b_k v\left((t_k-d_k)^-\right),\end{aligned} \tag{10}$$

where $\beta_k = e^{cd_k}$.

On the other hand, for any $t \neq t_k$, $k \in \mathbb{N}$,

$$D^+v(t) = e^{-c(t-t_0)}\left[-cu(t) + D^+u(t)\right] \leqslant qe^{-c(t-t_0)}\overline{u}(t). \tag{11}$$

For $t \in [t_0,t_1)$, integrating inequality (11) from t_0 to t, we obtain

$$v(t) \leqslant v(t_0) + \int_{t_0}^{t} qe^{-c(s-t_0)}\overline{u}(s)\,\mathrm{d}s; \tag{12}$$

this implies that

$$v(t_1^-) \leqslant v(t_0) + \int_{t_0}^{t_1} qe^{-c(s-t_0)}\overline{u}(s)\,\mathrm{d}s. \tag{13}$$

For $t \in [t_1, t_2)$, by the same method, together with (10), (11), and (13), we have

$$\begin{aligned} v(t) &\leqslant v(t_1) + \int_{t_1}^{t} qe^{-c(s-t_0)}\overline{u}(s)\,ds \\ &\leqslant a_1 v(t_1^-) + \beta_1 b_1 v\left((t_1 - d_1)^-\right) + \int_{t_1}^{t} qe^{-c(s-t_0)}\overline{u}(s)\,ds \\ &\leqslant a_1\left[v(t_0) + \int_{t_0}^{t_1} qe^{-c(s-t_0)}\overline{u}(s)\,ds\right] \\ &\quad + \beta_1 b_1\left[v(t_0) + \int_{t_0}^{t_1-d_1} qe^{-c(s-t_0)}\overline{u}(s)\,ds\right] \\ &\quad + \int_{t_1}^{t} qe^{-c(s-t_0)}\overline{u}(s)\,ds \\ &\leqslant (a_1 + \beta_1 b_1)\, v(t_0) + (a_1 + \beta_1 b_1) \\ &\quad \times \int_{t_0}^{t_1} qe^{-c(s-t_0)}\overline{u}(s)\,ds + \int_{t_1}^{t} qe^{-c(s-t_0)}\overline{u}(s)\,ds. \end{aligned} \tag{14}$$

By induction, we have, for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$,

$$\begin{aligned} v(t) &\leqslant v(t_0) \prod_{t_0 < t_j \leqslant t} (a_j + \beta_j b_j) \\ &\quad + \int_{t_0}^{t} \prod_{s < t_j \leqslant t} (a_j + \beta_j b_j)\, qe^{-c(s-t_0)}\overline{u}(s)\,ds. \end{aligned} \tag{15}$$

Thus, for $t > t_0$, we get

$$\begin{aligned} u(t) &\leqslant u(t_0)\, e^{c(t-t_0)} \prod_{t_0 < t_j \leqslant t} (a_j + \beta_j b_j) \\ &\quad + \int_{t_0}^{t} \prod_{s < t_j \leqslant t} (a_j + \beta_j b_j)\, qe^{c(t-s)}\overline{u}(s)\,ds. \end{aligned} \tag{16}$$

Let $t_{j_1}, t_{j_2}, \ldots, t_{j_m}$ be impulse points in $(s, t]$, $t > s$. In view of condition (i), we get

$$\begin{aligned} \prod_{s < t_j \leqslant t} (a_j + \beta_j b_j) &= (a_{j_1} + \beta_{j_1} b_{j_1}) \\ &\quad \times (a_{j_2} + \beta_{j_2} b_{j_2}) \cdots (a_{j_m} + \beta_{j_m} b_{j_m}) \\ &\leqslant e^{\delta(t_{j_1} - t_{j_1-1})} e^{\delta(t_{j_2} - t_{j_1})} \cdots e^{\delta(t_{j_m} - t_{j_{m-1}})} \\ &= e^{\delta(t_{j_m} - t_{j_1-1})} = e^{\delta(t-s)} e^{\delta(t_{j_m} - t)} e^{\delta(s - t_{j_l-1})} \\ &\leqslant \gamma e^{\delta(t-s)}, \end{aligned} \tag{17}$$

where t_{j_1-1} is the first impulsive point before t_{j_1} and satisfies $t_{j_1-1} < s$. Submitting this into inequality (16), then, for $t > t_0$,

$$u(t) \leqslant \gamma e^{(c+\delta)(t-t_0)} u(t_0) + \int_{t_0}^{t} \gamma q e^{(c+\delta)(t-s)}\overline{u}(s)\,ds. \tag{18}$$

Let $\Phi(\lambda) = \lambda + c + \delta + \gamma q e^{\lambda\tau}$. Then condition (ii) implies $\Phi(0) < 0$. Moreover, $\Phi(+\infty) = +\infty$ and $\Phi'(\lambda) = 1 + \tau\gamma q e^{\lambda\tau} > 0$. Hence $\Phi(\lambda) = 0$ has a unique positive solution λ. Next, we claim that

$$u(t) \leqslant \gamma\overline{u}(t_0)\, e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 - \tau. \tag{19}$$

Since

$$u(t) \leqslant \overline{u}(t_0) \leqslant \gamma\overline{u}(t_0)\, e^{-\lambda(t-t_0)}, \quad t \in [t_0 - \tau, t_0]. \tag{20}$$

So we only need to prove (19) for $t > t_0$. Suppose not, then there exists a $t^* \in (t_0, +\infty)$ such that

$$u(t^*) > \gamma\overline{u}(t_0)\, e^{-\lambda(t^*-t_0)}, \tag{21}$$

$$u(t) \leqslant \gamma\overline{u}(t_0)\, e^{-\lambda(t-t_0)}, \quad t \in [t_0 - \tau, t^*). \tag{22}$$

Thus from (18), (22), and $\Phi(\lambda) = 0$, we see that

$$\begin{aligned} u(t^*) &\leqslant \gamma\overline{u}(t_0)\, e^{(c+\delta)(t^*-t_0)} + \gamma \int_{t_0}^{t^*} qe^{(c+\delta)(t^*-s)}\overline{u}(s)\,ds \\ &\leqslant \gamma\overline{u}(t_0)\, e^{(c+\delta)(t^*-t_0)} \\ &\quad + \gamma \int_{t_0}^{t^*} \gamma q e^{\lambda\tau} e^{(c+\delta)(t^*-s)} e^{-\lambda(s-t_0)}\overline{u}(t_0)\,ds \\ &= \gamma\overline{u}(t_0)\, e^{-\lambda(t^*-t_0)}, \end{aligned} \tag{23}$$

which is a contradiction. Therefore, (19) holds. This completes the proof. □

Lemma 4 (comparison principle). *Assume that there exists a function $V \in v_0$ such that*

(i) $\mathbb{E}\mathscr{L}V(t, \varphi) \leqslant h(t, \mathbb{E}V(t, \varphi(0)), \mathbb{E}V(t + \theta, \varphi))$ *for any* $(t, \varphi) \in [t_{k-1}, t_k) \times PC^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$, $k \in \mathbb{N}$;

(ii) $\mathbb{E}V(t_k, I_k(t_k, X, Y)) \leqslant \Psi_{1k}(\mathbb{E}V(t_k^-, X)) + \Psi_{2k}(\mathbb{E}V((t_k - d_k)^-, Y))$ *for all* $X, Y \in L^p_{\mathscr{F}_t}(\Omega; \mathbb{R}^n)$, $k \in \mathbb{N}$.

Then,

$$\mathbb{E}V(t, x(t)) \leqslant u(t; t_0, \zeta), \quad t \geqslant t_0 \tag{24}$$

provided $\mathbb{E}V(t_0 + \theta, x(t_0 + \theta)) \leqslant \zeta(\theta)$, $\theta \in [-\tau, 0]$, *where* $x(t) = x(t; t_0, \xi)$ *is the solution process to* (1).

Proof. For any $t \in [t_{k-1}, t_k)$ and $\alpha > 0$ sufficiently small satisfying $t + \alpha < t_k$, by the Itô formula together with condition (i), we have

$$\begin{aligned} &\mathbb{E}V(t + \alpha, x(t + \alpha)) - \mathbb{E}V(t, x(t)) \\ &\quad = \int_t^{t+\alpha} \mathbb{E}\mathscr{L}V(s, x_s)\,ds \\ &\quad \leqslant \int_t^{t+\alpha} h(s, \mathbb{E}V(s, x(s)), \mathbb{E}V(s + \theta, x_s))\,ds; \end{aligned} \tag{25}$$

this implies that

$$\begin{aligned} &D^{+}\mathbb{E}V(t,x(t)) \\ &\quad := \limsup_{\alpha\to 0^{+}} \frac{\mathbb{E}V(t+\alpha, x(t+\alpha)) - \mathbb{E}V(t,x(t))}{\alpha} \\ &\quad \leqslant \limsup_{\alpha\to 0^{+}} \frac{1}{\alpha}\int_{t}^{t+\alpha} h\left(s, \mathbb{E}V(s,x(s)), \mathbb{E}V\left(s+\theta, x_{s}\right)\right) \mathrm{d}s \\ &\quad = h\left(t, \mathbb{E}V(t,x(t)), \mathbb{E}V\left(t+\theta, x_{t}\right)\right). \end{aligned} \tag{26}$$

Write $u(t;t_0,\zeta) = u(t)$ simply. Now supposing that for each $\theta \in [-\tau, 0]$, $\mathbb{E}V(t_0+\theta, x(t_0+\theta)) \leqslant \zeta(\theta)$, we claim that

$$\mathbb{E}V(t,x(t)) \leqslant u(t), \quad t \in \left[t_0-\tau, t_1\right). \tag{27}$$

Consider the system

$$\begin{aligned} &\dot{U}(t) = h\left(t, U(t), U_t\right) + \varepsilon, \quad t \in \left[t_0, t_1\right), \\ &U(\theta) = \zeta(\theta) + \varepsilon, \quad \theta \in \left[t_0-\tau, t_0\right], \end{aligned} \tag{28}$$

where $\varepsilon > 0$ is a constant. We claim that $U(t) \geqslant \mathbb{E}V(t,x(t))$ for $t \in [t_0-\tau, t_1)$.

In fact, if this is not true, then from the continuity of $U(t)$ and $\mathbb{E}V(t,x(t))$ in $t \in [t_0, t_1)$, we know that there exist a $t^* \in (t_0, t_1)$ and a sufficiently small constant $\alpha > 0$ such that $t^* + \alpha < t_1$ and

$$\begin{aligned} &\mathbb{E}V(t,x(t)) \leqslant U(t), \quad t \in \left[t_0-\tau, t^*\right), \\ &\mathbb{E}V\left(t^*, x\left(t^*\right)\right) = U\left(t^*\right), \\ &\mathbb{E}V(t,x(t)) > U(t), \quad t \in \left(t^*, t^*+\alpha\right). \end{aligned} \tag{29}$$

Thus $\dot{U}(t^*) = D^{+}U(t^*) \leqslant D^{+}\mathbb{E}V(t^*, x(t^*))$. On the other hand, by condition (i), we obtain that

$$\begin{aligned} \dot{U}\left(t^*\right) &= h\left(t^*, U\left(t^*\right), U_{t^*}\right) + \varepsilon \\ &\geqslant h\left(t^*, V\left(t^*, x\left(t^*\right)\right), \mathbb{E}V\left(t^*+\theta, x_{t^*}\right)\right) + \varepsilon \\ &> h\left(t^*, V\left(t^*, x\left(t^*\right)\right), \mathbb{E}V\left(t^*+\theta, x_{t^*}\right)\right) \\ &\geqslant D^{+}\mathbb{E}V\left(t^*, x\left(t^*\right)\right). \end{aligned} \tag{30}$$

This is a contradiction. So $U(t) \geqslant \mathbb{E}V(t,x(t))$ holds for all $t \in [t_0-\tau, t_1)$. Let $\varepsilon \to 0$; then $U(t) \to u(t)$, and hence inequality (27) holds.

Noting that $\Psi_{1k}(\cdot)$ and $\Psi_{2k}(\cdot)$ are nondecreasing, by (27) and condition (ii), we get

$$\begin{aligned} &\mathbb{E}V\left(t_1, x\left(t_1\right)\right) \\ &\quad = \mathbb{E}V\left(t_1, I_1\left(t_1, x\left(t_1^-\right), x\left(t_1-d_1\right)^-\right)\right) \\ &\quad \leqslant \Psi_{11}\left(\mathbb{E}V\left(t_1^-, x\left(t_1^-\right)\right)\right) \\ &\qquad + \Psi_{21}\left(\mathbb{E}V\left(\left(t_1-d_1\right)^-, x\left(t_1-d_1\right)^-\right)\right) \\ &\quad \leqslant \Psi_{11}\left(u\left(t_1^-\right)\right) + \Psi_{21}\left(u\left(t_1-d_1\right)^-\right) = u\left(t_1\right). \end{aligned} \tag{31}$$

Thus, it follows from (27) and (31) that

$$\mathbb{E}V\left(t_1+\theta, x\left(t_1+\theta\right)\right) \leqslant u\left(t_1+\theta\right), \quad \theta \in [-\tau, 0]. \tag{32}$$

Similar to the previous process, we have $\mathbb{E}V(t,x(t)) \leqslant u(t)$ when $t \in [t_0-\tau, t_2)$. By induction, it follows that $\mathbb{E}V(t,x(t)) \leqslant u(t)$, $t \in [t_0-\tau, \infty)$. The proof is complete. □

Theorem 5. *Assume that there exist functions $V \in v_0$, $\phi_1 \in V\mathscr{K}_\infty$, and $\phi_2 \in C\mathscr{K}$ such that*

(i) *$\phi_1(|x|^p) \leqslant V(t,x) \leqslant \phi_2(|x|^p)$ for any $(t,x) \in [t_0-\tau, \infty) \times \mathbb{R}^n$;*

(ii) *$\mathbb{E}\mathscr{L}V(t,\varphi) \leqslant h(t, \mathbb{E}V(t,\varphi(0)), \mathbb{E}V(t+\theta,\varphi))$ for any $(t,\varphi) \in [t_{k-1}, t_k) \times PC^{p}_{\mathscr{F}_t}([-\tau,0];\mathbb{R}^n)$, $k \in \mathbb{N}$;*

(iii) *$\mathbb{E}V(t_k, I_k(t_k, X, Y)) \leqslant \Psi_{1k}(\mathbb{E}V(t_k^-, X)) + \Psi_{2k}(\mathbb{E}V((t_k-d_k)^-, Y))$ for all $X, Y \in L^{p}_{\mathscr{F}_t}(\Omega;\mathbb{R}^n)$, $k \in \mathbb{N}$.*

Then the stability properties of the trivial solution of IFDE-DI (2) imply the corresponding stability properties of the trivial solution of ISFDE-DI (1). Moreover, if condition (i) is replaced by

(i*) *there exist positive constants p, c_1, and c_2 such that for all $(t,x) \in [t_0-\tau, \infty) \times \mathbb{R}^n$*

$$c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p \tag{33}$$

then the global exponential stability of the trivial solution of IFDE-DI (2) implies that pth moment global exponential stability of ISFDE-DI (1).

Proof. Firstly, assume that the trivial solution of IFDE-DI (2) is stable. Let $\varepsilon > 0$; then for given $\phi_1(\varepsilon) > 0$, there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that $\delta_1 < \phi_1(\varepsilon)$ and

$$\|\zeta\|^p < \delta_1 \text{ implies } u(t;t_0,\zeta) < \phi_1(\varepsilon), \quad t \geqslant t_0. \tag{34}$$

Let $\zeta(\theta) = \mathbb{E}V(t_0+\theta, x(t_0+\theta))$, $\theta \in [-\tau, 0]$. From conditions (ii) and (iii) and Lemma 4, we get that

$$\mathbb{E}V(t,x(t)) \leqslant u(t;t_0,\zeta), \quad t \geqslant t_0. \tag{35}$$

Let $\delta \leqslant \phi_2^{-1}(\delta_1)$ and $\mathbb{E}\|\xi\|^p < \delta$; then by condition (i) and $\phi_2 \in C\mathscr{K}$, we have $\|\zeta\|^p \leqslant \mathbb{E}\phi_2(\|\xi\|^p) \leqslant \phi_2(\mathbb{E}\|\xi\|^p) < \phi_2(\delta) \leqslant \delta_1$. Hence, by (34) and (35), we have

$$\mathbb{E}V(t,x(t)) < \phi_1(\varepsilon), \quad t \geqslant t_0. \tag{36}$$

If $\mathbb{E}\|\xi\|^p < \delta$, then by conditions (i) and (36), we have

$$\mathbb{E}|x(t)|^p \leqslant \phi_1^{-1}(\mathbb{E}V(t,x(t))) < \varepsilon, \quad t \geqslant t_0; \tag{37}$$

that is, the trivial solution of ISFDE-DI (1) is stable.

Next, let us suppose that the trivial solution of IFDE-DI (2) is asymptotically stable. This implies that the trivial solution of ISFDE-DI (1) is stable. Let $\zeta(\theta) = \mathbb{E}V(t_0+\theta, x(t_0+\theta))$, $\theta \in [-\tau, 0]$. Since $u = 0$ is attractive, for any $\varepsilon > 0$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \delta_0)$ such that

$$\|\zeta\|^p < \delta_0, \text{ implies } u(t;t_0,\zeta) < \phi_1(\varepsilon), \quad t \geqslant t_0 + T. \tag{38}$$

Choose $\mathbb{E}\|\xi\|^p < \delta_0$. Note the fact that $\phi \in V\mathscr{K}$ implies $\phi^{-1} \in C\mathscr{K}$. Then by (35) and (37), we get

$$\mathbb{E}|x(t)|^p \leqslant \phi_1^{-1}(\mathbb{E}V(t, x(t))) < \varepsilon, \quad t \geqslant t_0 + T, \tag{39}$$

which implies that the trivial solution of ISFDE-DI (1) is asymptotically stable.

Thirdly, let us suppose that the trivial solution of IFDE-DI (2) is globally exponentially stable and condition (i*) holds. Then, there exists a couple of positive constants γ and K such that

$$u(t) \leqslant K\|\zeta\| e^{-\gamma(t-t_0)}, \quad t \geqslant t_0. \tag{40}$$

Let $\zeta(\theta) = V(t_0 + \theta, x(t_0 + \theta))$, $\theta \in [-\tau, 0]$. Then by (35) and (40), we get $\mathbb{E}V(t, x(t)) \leqslant u(t) \leqslant K\mathbb{E}\|\xi\|^p e^{-\gamma(t-t_0)}$ for all $t \geqslant t_0$. Thus, by condition (i*), it yields that

$$\mathbb{E}|x(t)|^p \leqslant \frac{Kc_2}{c_1}\mathbb{E}\|\xi\|^p e^{-\gamma(t-t_0)}, \quad t \geqslant t_0. \tag{41}$$

Hence, the trivial solution of ISFDE-DI (1) is pth moment globally exponentially stable. The proof is complete. □

Theorem 6. *Assume that there exist a function $V \in v_0$, positive constants c_1, c_2, q, and a_k, constants c and δ, and $b_k \geqslant 0$ such that*

(i) $c_1|x|^p \leqslant V(t, x) \leqslant c_2|x|^p$ *for any* $(t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$;

(ii) $\mathbb{E}\mathscr{L}V(t, \varphi) \leqslant c\mathbb{E}V(t, \varphi(0)) + q\mathbb{E}V(t + \theta, \varphi)$ *for any* $(t, \varphi) \in [t_{k-1}, t_k) \times PC^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$, $k \in \mathbb{N}$;

(iii) $\mathbb{E}V(t_k, I_k(t_k, X, Y)) \leqslant a_k\mathbb{E}V(t_k^-, X) + b_k\mathbb{E}V((t_k - d_k)^-, Y)$ *for all* $X, Y \in L^p_{\mathscr{F}_t}(\Omega; \mathbb{R}^n)$, $k \in \mathbb{N}$;

(iv) $\ln(a_k + b_k e^{cd_k}) \leqslant \delta(t_k - t_{k-1})$ *for each* $k \in \mathbb{N}$;

(v) $\delta + c + q\gamma < 0$ *where* $\gamma = \sup_{k\in\mathbb{N}}\{e^{\delta(t_k - t_{k-1})}, 1/e^{\delta(t_k - t_{k-1})}\}$.

Then the trivial solution of ISFDE-DI (1) *is pth moment globally exponentially stable.*

Proof. Let $u(t) = \mathbb{E}V(t, \varphi(0))$, $h(t, u(t), u_t) = cu(t) + qu_t$, $\Psi_{1k}(u(t_k^-)) = a_k u(t_k^-)$, and $\Psi_{2k}(u((t_k - d_k)^-)) = b_k u((t_k - d_k)^-)$. We obtain the comparison system (2). It is easy to verify that all conditions of Theorem 5 are satisfied and so the global exponential stability of the trivial solution of IFDE-DI (2) implies that pth moment global exponential stability of ISFDE-DI (1).

Furthermore, let λ be the unique positive solution of $\lambda + \delta + p + q\gamma e^{\lambda\tau} = 0$. Using conditions (ii) and (iii), we find

$$\begin{aligned} &D^+u(t) \leqslant cu(t) + q\overline{u}(t), \quad t \neq t_k,\ t \geqslant t_0, \\ &u(t_k) \leqslant a_k u(t_k^-) + b_k u\left((t_k - d_k)^-\right), \quad k \in \mathbb{N}. \end{aligned} \tag{42}$$

Thus from conditions (iv) and (v) and Lemma 3, we obtain that

$$u(t) \leqslant \gamma\overline{u}(t_0) e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 - \tau, \tag{43}$$

which implies that the trivial solution of IFDE-DI (2) is globally exponentially stable. The proof of Theorem 6 is complete. □

Remark 7. An impulsive stochastic dynamical system can be viewed as a hybrid one comprised of two components: a continuous stochastic dynamic and a discrete dynamic. Theorem 6 can be used to deal will all three cases: the system with stable continuous stochastic dynamic and unstable discrete dynamic, the system with unstable continuous stochastic dynamic and stable discrete dynamic, and the system with stable continuous stochastic dynamic and stable discrete dynamic. When $c < 0$, the continuous stochastic dynamic of (1) may be stable. In this case, in order to ensure the stability of the entire system, the delayed impulses' frequency $\{t_k - t_{k-1}, k \in \mathbb{N}\}$ and amplitude a_k, b_k should be suitably related to the decrease of continuous flows; that is, conditions (iv) and (v) hold. In this sense, Theorem 6 can be used to deal with the robust stabling of continuous stochastic dynamic subject to delayed impulsive perturbations. When $c \geqslant 0$, the continuous stochastic dynamic of (1) may be unstable and the stability of the entire system is determined by the delayed impulse effects. In this case, we need to require that the delayed impulses' frequency and amplitude should be suitablly related to the decrease of of continuous flows.

Remark 8. It is noted that the exponential stability analysis in [30, 31] only considers the case of impulsive stabilization. In this sense, Theorem 6 has a wider adaptive range.

4. Examples

In this section, the effectiveness and advantages of the results derived in the preceding section will be illustrated by two examples.

Example 1. Consider the two-dimensional nonlinear impulsive stochastic delay equation in the form

$$\begin{aligned} \mathrm{d}x_1(t) &= \left[-2x_2(t)\sin\left(x_1(t-\tau)\right) - 5x_1(t) \right. \\ &\qquad \left. + 0.5x_2(t-\tau)\right]\mathrm{d}t + 0.2x_1(t-\tau)\,\mathrm{d}w(t), \\ &\qquad\qquad t \neq t_k, \\ \mathrm{d}x_2(t) &= \left[x_1(t)\sin\left(x_1(t-\tau)\right) - 5x_2(t) \right. \\ &\qquad \left. +0.4x_2(t-\tau)\right]\mathrm{d}t \\ &\qquad + 0.4x_2(t-\tau)\,\mathrm{d}w(t), \quad t \neq t_k, \\ x_1(t_k) &= x_1(t_k^-) + \alpha x_1\left((t_k - d_k)^-\right), \quad k \in \mathbb{N}, \\ x_2(t_k) &= x_2(t_k^-) + \alpha x_2\left((t_k - d_k)^-\right), \quad k \in \mathbb{N}, \end{aligned} \tag{44}$$

where $\tau > 0$, $d_k \in [0, d]$, $\alpha \geqslant 0$. If there exists a positive constant $\varepsilon > 0$ such that

$$\begin{aligned} &\alpha < \sqrt{\frac{9/0.445 - 1 - \varepsilon}{1 + 1/\varepsilon}}, \\ &\varrho = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > \frac{\ln\left[1 + \varepsilon + (1 + 1/\varepsilon)\alpha^2\right]}{9 - 0.445\left[1 + \varepsilon + (1 + 1/\varepsilon)\alpha^2\right]}, \end{aligned} \tag{45}$$

then (44) is globally exponentially stable for any bounded impulsive input delays $\{d_k\}$.

Denote $I_k(t_k, X, Y) = X + \alpha Y$. Choose the Lyapunov function $V(t,x) = (1/4)x_1^2 + (1/2)x_2^2$; then for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{E}V\left(t_k, I_k\left(t_k, X, Y\right)\right) \\
&\quad = \frac{1}{4}\left|X_1 + \alpha Y_1\right|^2 + \frac{1}{2}\left|X_2 + \alpha Y_2\right|^2 \\
&\quad = \mathbb{E}V\left(t_k^-, X\right) + \alpha^2 \mathbb{E}V\left(\left(t_k - d_k\right)^-, Y\right) \\
&\qquad + \frac{\alpha}{2}\mathbb{E}\left(X_1 Y_1\right) + \alpha \mathbb{E}\left(X_2 Y_2\right) \\
&\quad \leqslant (1+\varepsilon)\,\mathbb{E}V\left(t_k^-, X\right) + \left(1 + \frac{1}{\varepsilon}\right)\alpha^2 \mathbb{E}V\left(\left(t_k - d_k\right)^-, Y\right), \\
&\mathbb{E}\mathscr{L}V\left(t, \varphi\right) \\
&\quad = -10\mathbb{E}V\left(t, \varphi(0)\right) \\
&\qquad + \mathbb{E}\left[0.25\varphi_1(0)\varphi_2(-\tau) + 0.4\varphi_2(0)\varphi_2(-\tau)\right. \\
&\qquad\qquad \left. + 0.01\varphi_1^2(-\tau) + 0.08\varphi_2^2(-\tau)\right] \\
&\quad \leqslant -10\mathbb{E}V\left(t, \varphi(0)\right) + \mathbb{E}\left[0.01\varphi_1^2(-\tau) + 0.08\varphi_2^2(-\tau)\right] \\
&\qquad + \mathbb{E}\left[0.25\varphi_1^2(0) + 0.0625\varphi_2^2(-\tau)\right. \\
&\qquad\qquad \left. + 0.5\varphi_2^2(0) + 0.08\varphi_2^2(-\tau)\right] \\
&\quad = -9\mathbb{E}V\left(t, \varphi(0)\right) + \mathbb{E}\left[0.01\varphi_1^2(-\tau) + 0.2225\varphi_2^2(-\tau)\right] \\
&\quad \leqslant -9\mathbb{E}V\left(t, \varphi(0)\right) + 0.445\mathbb{E}V\left(t-\tau, \varphi(-\tau)\right),
\end{aligned} \tag{46}$$

for $t \neq t_k$.

Take $c_1 = 1/4$, $c_2 = 1/2$, $c = -9$, $q = 0.445$, $a_k \equiv 1 + \varepsilon$, $b_k \equiv (1 + 1/\varepsilon)\alpha^2$, $\delta = \ln[1 + \varepsilon + (1 + 1/\varepsilon)\alpha^2]/\varrho$, $\gamma = 1 + \varepsilon + (1+1/\varepsilon)\alpha^2$. It is easy to check that all conditions of Theorem 6 are satisfied under conditions (45), which means that (44) is globally mean square exponentially stable for any bounded impulsive input delays $\{d_k\}$.

Remark. It is noted that (44) without impulses is globally mean square exponentially stable and the impulses are destabilizing since $\alpha \geqslant 0$. Hence, the existing stability theorems in [30, 31] fail to work. This shows that our results have a wider adaptive range.

Example 2. Consider the following impulsive stochastic delayed neural network:

$$\begin{aligned}
\mathrm{d}x(t) &= \left[-x(t) + Af\left(x\left(t - \tau(t)\right)\right)\right]\mathrm{d}t \\
&\qquad + Bx\left(t - \tau(t)\right)\mathrm{d}w(t), \quad t \neq t_k, \\
x\left(t_k\right) &= 0.3x\left(t_k^-\right) \\
&\qquad + 0.2x\left(\left(t_k - d_k\right)^-\right), \quad k \in \mathbb{N},
\end{aligned} \tag{47}$$

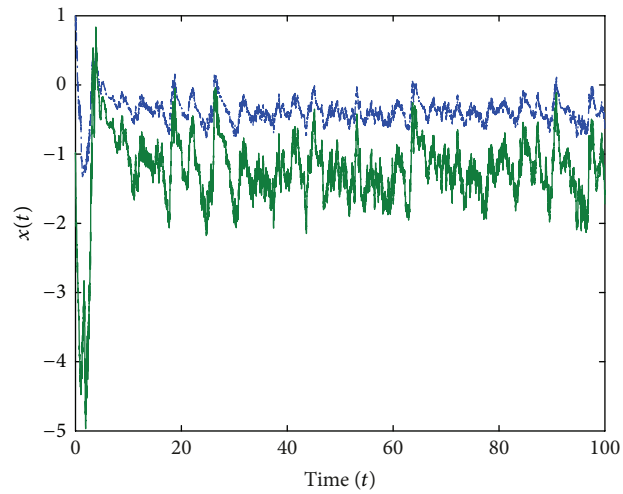

Figure 1: The solution of system (47) without impulses (single sample).

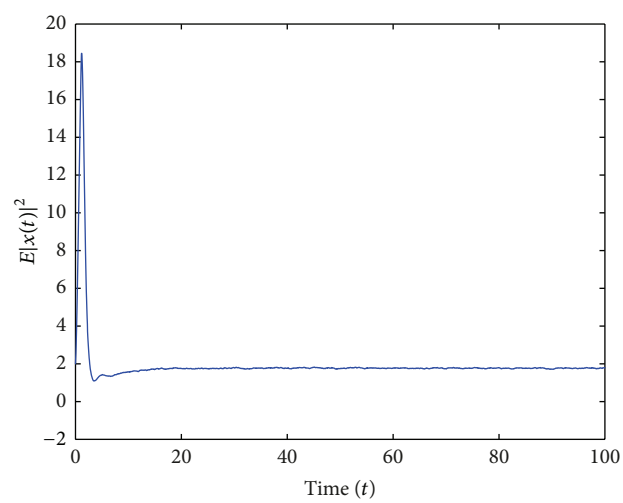

Figure 2: The mean square of the solution of system (47) without impulses (2000 samples).

where

$$A = \begin{bmatrix} -1.5 & 1 \\ -3 & 2.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \tag{48}$$

$f(x) = (f_1(x_1), f_2(x_2))^T$ with $f_1(s) = f_2(s) = (1/2)(|s+1| - |s-1|)$.

It is noted that (47) without impulse is not stable, and its simulation with delay $\tau(t) = 1$ and initial data $\xi(s) = [1, -1]^T$ and $s \in [-1, 0]$ are shown in Figures 1 and 2.

In the following, applying Theorem 5, we will show that under impulsive control law, (47) is mean square exponentially stable if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \leqslant 0.0681$.

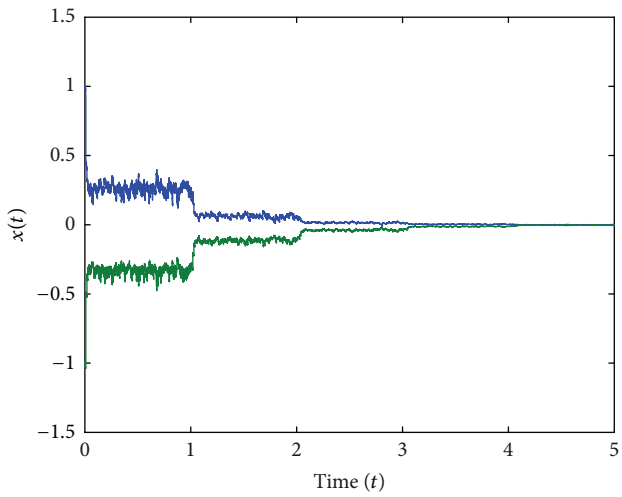

FIGURE 3: The solution of system (47) with impulses (single sample).

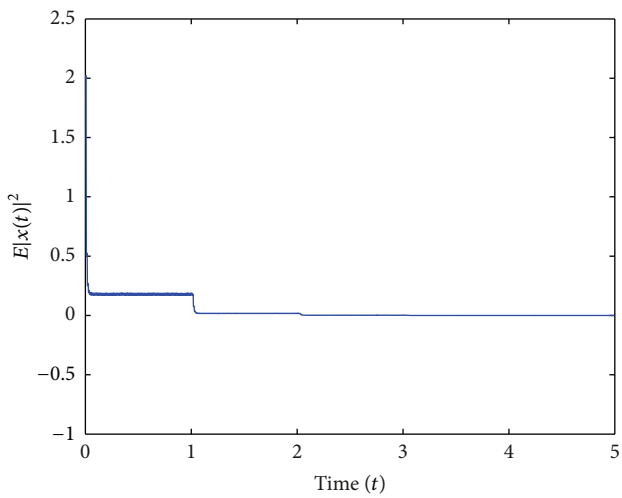

FIGURE 4: The mean square of the solution of system (47) with impulses (2000 samples).

Denote $I_k(t_k, X, Y) = 0.3X + 0.2Y$. Choose $V(t, x) = |x|^2$. Then condition (i) of Theorem 5 holds with $c_1 = c_2 = 1$,

$$\begin{aligned}
&\mathbb{E}V\left(t_k, I_k\left(t_k, X, Y\right)\right) \\
&\quad = [0.3X + 0.2Y]^T [0.3X + 0.2Y] \\
&\quad \leqslant 0.18\mathbb{E}|X|^2 + 0.08\mathbb{E}|Y|^2 \\
&\quad = 0.18\mathbb{E}V\left(t_k^-, X\right) + 0.08\mathbb{E}V\left(\left(t_k - d_k\right)^-, Y\right), \\
&\mathbb{E}\mathscr{L}V\left(t, \varphi\right) \\
&\quad = \mathbb{E}\left[2\varphi^T(0)\left(-\varphi(0) + Af\left(\varphi\left(-\tau(t)\right)\right)\right)\right] \\
&\qquad + \mathbb{E}\left[\varphi^T\left(-\tau(t)\right) B^T B \varphi\left(-\tau(t)\right)\right] \\
&\quad \leqslant \mathbb{E}\left[\left(-2 + \|A\|\right)\left|\varphi(0)\right|^2\right. \\
&\qquad \left. + \left(\|A\| + \|B\|^2\right)\left|\varphi\left(-\tau(t)\right)\right|^2\right] \\
&\quad \leqslant 2.2976\mathbb{E}\left|\varphi(0)\right|^2 + 4.5476\mathbb{E}\left|\varphi\left(-\tau(t)\right)\right|^2 \\
&\quad = 2.2976\mathbb{E}V\left(t, \varphi(0)\right) \\
&\qquad + 4.5476\mathbb{E}V\left(t - \tau(t), \varphi\left(-\tau(t)\right)\right),
\end{aligned} \tag{49}$$

for $t \neq t_k$.

Thus, the comparison system is

$$\begin{aligned}
&\dot{u}(t) = 2.2976u(t) + 4.5476u\left(t - \tau(t)\right), \quad t \neq t_k, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geqslant t_0, \\
&u\left(t_k\right) = 0.18u\left(t_k^-\right) + 0.08u\left(\left(t_k - d_k\right)^-\right), \quad k \in \mathbb{N},
\end{aligned} \tag{50}$$

which according to case (iii) of Corollary 1 in [19] is globally exponentially stable for any bounded impulsive input delays $\{d_k\}$ if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \ln(1/0.26)/(2.2976+4.5476/0.26) = 0.0681$. Hence, we conclude by Theorem 6 that system (47) is mean square exponentially stable if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \leqslant 0.0681$. With the same initial value, the simulations of the impulsive stochastic delay neural network (47) under the delayed impulsive control law $x(t_k) = 0.3x(t_k^-) + 0.2x((t_k - d_k)^-)$, $t_k - t_{k-1} = 0.06$, $d_k = 0.4$ are shown in Figures 3 and 4.

5. Conclusions

This paper has investigated the exponential stability of ISFDEs-DI based on the comparison approach and an impulsive delay differential inequality. Some criteria on the pth moment global exponential stability are established. The obtained results complement some recent works. Two examples have been given to illustrate the effectiveness and the advantages of the results obtained. One of the drawbacks of the proposed method is perhaps that our results require the condition $\delta + c + q\gamma < 0$ and thus cannot deal with the time delay system with $\Delta x(t_k) = B_k x((t_k - d_k)^-)$. There will be future work to establish a criterion for the above system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Automorphisms of Ordinary Differential Equations

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The paper deals with the local theory of internal symmetries of underdetermined systems of ordinary differential equations in full generality. The symmetries need not preserve the choice of the independent variable, the hierarchy of dependent variables, and the order of derivatives. Internal approach to the symmetries of one-dimensional constrained variational integrals is moreover proposed without the use of multipliers.

1. Preface

The theory of symmetries of *determined systems* (the solution depends on constants) of ordinary differential equations was ultimately established in Lie's and Cartan's era in the most possible generality and the technical tools (infinitesimal transformations and moving frames) are well known. Recall that the calculations are performed in finite-dimensional spaces given in advance and the results are expressed in terms of Lie groups or Lie-Cartan pseudogroups.

We deal with *underdetermined systems* (more unknown functions than the number of equations) of ordinary differential equations here. Then the symmetry problem is rather involved. Even the system of three first-order quasilinear equations with four unknown functions (equivalently, three Pfaffian equations with five variables) treated in the famous Cartan's article [1] and repeatedly referred to in actual literature was not yet clearly explained in detail. Paradoxically, the common tools (the calculations in given finite-order jet space) are quite sufficient for this particular example. We will later see that they are insufficient to analyze the seemingly easier symmetry problem of one first-order equation with three unknown functions (alternatively, two Pfaffian equations with five variables) in full generality since the order of derivatives need not be preserved in this case and the finite-order jet spaces may be destroyed. Recall that even the higher-order symmetries (automorphisms) of empty systems of differential equations (i.e., of the infinite order jet spaces without any additional differential constraints) are nontrivial [2–4] and cannot be included into the classical Lie-Cartan theory of transformation groups. Such symmetries need not preserve any finite-dimensional space and therefore the invariant differential forms (the Maurer-Cartan forms, the moving coframes) need not exist.

Let us outline the very core of the subject for better clarity by using the common jet terminology. We start with the *higher-order transformations* of curves $w^i = w^i(x)$ $(i = 1, \dots, m)$ lying in the space $\mathbb{R}^{m+1}$ with coordinates $x, w^1, \dots, w^m$. The transformations are defined by certain formulae

$$\begin{gathered} \overline{x} = W\left(x, \dots, w_s^j, \dots\right), \\ \overline{w}^i = W^i\left(x, \dots, w_s^j, \dots\right) \quad (i = 1, \dots, m), \end{gathered} \tag{1}$$

where the C^∞-smooth real-valued functions W, W^i depend on a finite number of the familiar jet variables

$$w_s^j = \frac{d^s w^j}{dx^s} \quad \left(j = 1, \dots, m; s = 0, 1, \dots\right). \tag{2}$$

The resulting curve $\overline{w}^i = \overline{w}^i(\overline{x})$ $(i = 1, \dots, m)$ again lying in $\mathbb{R}^{m+1}$ appears as follows. We put

$$\overline{x} = W\left(x, \dots, \frac{d^s w^j(x)}{dx^s}, \dots\right) = w(x) \tag{3}$$

and assuming

$$w'(x) = (DW)\left(x, \ldots, \frac{d^s w^j(x)}{dx^s}, \ldots\right) \neq 0$$
$$\left(D = \frac{\partial}{\partial x} + \sum w^j_{s+1} \frac{\partial}{\partial w^j_s}\right), \tag{4}$$

there exists the inverse function $x = w^{-1}(\overline{x})$ which provides the desired result

$$\overline{w}^i(\overline{x}) = W^i\left(w^{-1}(\overline{x}), \ldots, \frac{d^s w^j}{dx^s}\left(w^{-1}(\overline{x})\right), \ldots\right). \tag{5}$$

One can also easily obtain the well-known *prolongation formula*

$$\overline{w}^i_s = W^i_s\left(x, \ldots, w^j_s, \ldots\right)$$
$$\left(W^i_{s+1} = \frac{DW^i_s}{DW}; i = 1, \ldots, m; s = 0, 1, \ldots; W^i_0 = W^i\right) \tag{6}$$

for the derivatives $\overline{w}^i_r = d^r\overline{w}^i/d\overline{x}^r$ by using the Pfaffian equations

$$d\overline{w}^i_r - \overline{w}^i_{r+1} d\overline{x} = 0 \quad (i = 1, \ldots, m; r = 0, 1, \ldots). \tag{7}$$

Functions W satisfying (4) and W^i may be arbitrary here.

At this place, in order to obtain coherent theory, introduction of the familiar *infinite-order jet space of x-parametrized curves* briefly designated as $\mathbf{M}(m)$ with coordinates x, w^i_r $(i = 1, \ldots, m; r = 0, 1, \ldots)$ is necessary. Then formulae ((1), (6)) determine a mapping $\mathbf{m} : \mathbf{M}(m) \to \mathbf{M}(m)$, a *morphism* of the jet space $\mathbf{M}(m)$. If the inverse $\mathbf{m}^{-1}$ given by certain formulae

$$x = \overline{W}\left(\overline{x}, \ldots, \overline{w}^j_s, \ldots\right),$$
$$w^i_r = \overline{W}^i_r\left(\overline{x}, \ldots, \overline{w}^j_s, \ldots\right) \quad (i = 1, \ldots, m; r = 0, 1, \ldots) \tag{8}$$

exists, we speak of an *automorphism* (in alternative common terms, *symmetry*) $\mathbf{m}$ of the jet space $\mathbf{M}(m)$. It should be noted that we tacitly deal with the *local theory* in the sense that all formulae and identities, all mappings, and transformation groups to follow are in fact considered only on certain open subsets of the relevant underlying spaces which is not formally declared by the notation. Expressively saying, in order to avoid the clumsy purism, we follow the reasonable 19th century practice and do not rigorously indicate the true definition domains.

After this preparation, a *system of differential equations* is traditionally identified with the subspace $\mathbf{M} \subset \mathbf{M}(m)$ given by certain equations

$$D^r G^k = 0$$
$$\left(k = 1, \ldots K; r = 0, 1, \ldots; G^k = G^k\left(x, \ldots, w^j_s, \ldots\right)\right). \tag{9}$$

(We tacitly suppose that $\mathbf{M} \subset \mathbf{M}(m)$ is a "reasonable subspace" and omit the technical details.) This is the *infinitely prolonged system*. The total derivative vector field D defined on $\mathbf{M}(m)$ is tangent to the subspace $\mathbf{M} \subset \mathbf{M}(m)$ and may be regarded as a vector field on $\mathbf{M}$, as well. The morphism $\mathbf{m} : \mathbf{M}(m) \to \mathbf{M}(m)$ transforms $\mathbf{M} \subset \mathbf{M}(m)$ into the subspace $\mathbf{mM} \subset \mathbf{mM}(m) \subset \mathbf{M}(m)$ given by the equations

$$\left(D^r G^k\right)\left(W, \ldots, W^j_s, \ldots\right) = 0$$
$$(k = 1, \ldots, K; r = 0, 1, \ldots). \tag{10}$$

This is again a system of differential equations. In our paper, we are interested only in the particular case when $\mathbf{mM} = \mathbf{M}$. Then, if the inverse $\mathbf{m}^{-1}$ locally exists on a neighbourhood of the subspace $\mathbf{M} \subset \mathbf{M}(m)$ in the total jet space, we speak of the *external symmetry* $\mathbf{m}$ of the system of differential equations (9). Let us, however, deal with the natural restriction $\overline{\mathbf{m}} : \mathbf{M} \to \mathbf{M}$ of the mapping $\mathbf{m}$ to the subspace $\mathbf{M}$. If there exists the inverse $\overline{\mathbf{m}}^{-1} : \mathbf{M} \to \mathbf{M}$ of the restriction, we speak of the *internal symmetry*. Internal symmetries do not depend on the localizations of $\mathbf{M}$ in $\mathbf{M}(m)$. More precisely, differential equations can be introduced without any reference to jet spaces and the internal symmetries can be defined without the use of localizations. On this occasion, we are also interested in *groups* of internal symmetries. They are generated by special vector fields, the *infinitesimal symmetries*.

In the actual literature, differential equations are as a rule considered in finite-dimensional jet spaces. Then the internal and external symmetries become rather delicate and differ from our concepts since the higher-order symmetries are not taken into account. We will not discuss such conceptual confusion in this paper with the belief that the following two remarks (and Remark 5) should be quite sufficient in this respect.

Remark 1 (on the symmetries). The true structure of the jet space $\mathbf{M}(m)$ is determined by the *contact module* $\Omega(m)$ which involves all *contact forms*

$$\omega = \sum a^i_r \omega^i_r$$
$$\left(\omega^i_r = dw^i_r - w^i_{r+1} dx, \text{ finite sum, arbitrary coefficients}\right). \tag{11}$$

Then the above *morphisms* $\mathbf{m} : \mathbf{M}(m) \to \mathbf{M}(m)$ given in ((1), (6)) are characterized by the property $\mathbf{m}^*\Omega(m) \subset \Omega(m)$. Recall that invertible morphisms are *automorphisms*. Let us introduce the subspace $\mathbf{i} : \mathbf{M} \subset \mathbf{M}(m)$ of all points (9). This $\mathbf{M}$ is equipped with the restriction $\Omega = \mathbf{i}^*\Omega(m)$ of the contact module. Recall that we are interested only in the case $\mathbf{mM} = \mathbf{M}$ (abbreviation of $\mathbf{miM} = \mathbf{iM}$). Let $\overline{\mathbf{m}} : \mathbf{M} \to \mathbf{M}$ be the restriction of $\mathbf{m}$. If $\mathbf{m}$ is a morphism then $\overline{\mathbf{m}}$ is a morphism in the sense that $\overline{\mathbf{m}}^*\Omega \subset \Omega$. Recall that we have the *internal symmetry*, if $\overline{\mathbf{m}}$ is moreover invertible. If also $\mathbf{m}$ is invertible, we have the *external symmetry* $\overline{\mathbf{m}}$. The internal symmetries can be defined without any reference to $\mathbf{m}$ and $\mathbf{M}(m)$ as follows. Let $\overline{\mathbf{m}} : \mathbf{M} \to \mathbf{M}$ be any invertible mapping such that $\overline{\mathbf{m}}^*\Omega \subset \Omega$. This $\overline{\mathbf{m}}$ can be always extended to a morphism $\mathbf{m} : \mathbf{M}(m) \to \mathbf{M}(m)$ of the ambient jet space.

(Hint, recurrence (6) holds true both in $\mathbf{M}(m)$ and in $\mathbf{M}$.) So we may conclude that such $\overline{\mathbf{m}}$ is just the *internal symmetry*. Moreover, if there exists *invertible* extension $\mathbf{m}$ of $\overline{\mathbf{m}}$, then $\overline{\mathbf{m}}$ is even the *external symmetry* but the latter concept already depends on the localization $\mathbf{i}$ of $\mathbf{M}$ in $\mathbf{M}(m)$.

Remark 2 (on infinitesimal symmetries). Let us consider a vector field

$$Z = z\frac{\partial}{\partial x} + \sum z_r^i \frac{\partial}{\partial w_r^i} \qquad (\text{infinite sum, arbitrary coefficients}) \tag{12}$$

on the jet space $\mathbf{M}(m)$. Let us moreover suppose $\mathscr{L}_Z\Omega(m) \subset \Omega(m)$ from now on (where $\mathscr{L}_Z$ denotes the Lie derivative see also Definition 8). In common terminology, such vector fields Z are called *generalized* (*higher-order*, *Lie-Bäcklund*) *infinitesimal symmetries* of the jet space $\mathbf{M}(m)$. However Z need not in general generate any true group of transformations and we therefore prefer the "unorthodox" term a *variation* Z here. (See Section 7 and especially Remark 35 where the reasons for this term are clarified.) The common term *infinitesimal symmetry* is retained only for the favourable case when Z generates a local one-parameter Lie group [5]. Let us consider the above subspace $\mathbf{i} : \mathbf{M} \subset \mathbf{M}(m)$. If Z is tangent to $\mathbf{M}$, then there exists the natural restriction $\overline{Z}$ of Z to $\mathbf{M}$. Clearly $\mathscr{L}_{\overline{Z}}\Omega \subset \Omega$ and we speak of the (*internal*) *variation* $\overline{Z}$. If $\overline{Z}$ moreover generates a group in $\mathbf{M}$, we have the (*internal*) *infinitesimal symmetry* $\overline{Z}$. The internal concepts on $\mathbf{M}$ can be easily introduced without any reference to the ambient space $\mathbf{M}(m)$. This is not the case for the concept of the *external infinitesimal symmetry* $\overline{Z}$ which supposes that *appropriate extension* Z of $\overline{Z}$ on the ambient space $\mathbf{M}(m)$ *generates a Lie group*.

We deal only with the *internal* symmetries and infinitesimal symmetries in this paper. It is to be noted once more that *infinite-dimensional underlying spaces are necessary* if we wish to obtain a coherent theory. The common technical tools invented in the finite-dimensional spaces will be only slightly adapted; alas, the ingenious methods proposed, for example, in [6–8] seem to be not suitable for this aim and so we undertake the elementary approach [9] here.

2. Technical Tools

We introduce infinite-dimensional manifold $\mathbf{M}$ modelled on the space $\mathbb{R}^\infty$ with local coordinates $h^1, h^2, \ldots$ in full accordance with [9]. The manifold $\mathbf{M}$ is equipped with the *structural algebra* $\mathscr{F}(\mathbf{M})$ of C^∞-smooth functions expressed as $f = f(h^1, \ldots, h^{m(f)})$ in terms of coordinates. Transformations (mappings) $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ are (locally) given by certain formulae

$$\mathbf{m}^* h^i = H^i\left(h^1, \ldots, h^{m(i)}\right) \quad \left(H^i \in \mathscr{F}(\mathbf{M}); i = 1, 2, \ldots\right), \tag{13}$$

and analogous (invertible) formulae describe the change of coordinates at the overlapping coordinate systems.

Let $\Phi(\mathbf{M})$ be the $\mathscr{F}(\mathbf{M})$-module of differential 1-forms

$$\varphi = \sum f^i dg^i \quad \left(f^i, g^i \in \mathscr{F}(\mathbf{M}); \text{finite sum}\right). \tag{14}$$

The familiar rules of exterior calculus can be applied without any change, in particular $\mathbf{m}^*\varphi = \sum \mathbf{m}^* f^i d\mathbf{m}^* g^i$ for the above transformation $\mathbf{m}$.

Let $\mathscr{T}(\mathbf{M})$ be the $\mathscr{F}(\mathbf{M})$-module of vector fields Z. In terms of coordinates we have

$$Z = \sum z^i \frac{\partial}{\partial h^i} \qquad \left(z^i = z^i\left(h^1, \ldots, h^{m(i)}\right) \in \mathscr{F}(\mathbf{M}), \text{infinite sum}\right), \tag{15}$$

where the coefficients z^i may be quite arbitrary. We identify Z with the linear functional on $\Phi(\mathbf{M})$ determined by the familiar duality pairing

$$dh^i(Z) = Z\rfloor dh^i = Zh^i = z^i \quad (i = 1, 2, \ldots). \tag{16}$$

With this principle in mind, if certain forms $\varphi^1, \varphi^2, \ldots \in \Phi(\mathbf{M})$ generate the $\mathscr{F}(\mathbf{M})$-module, then the values

$$\varphi^i(Z) = Z\rfloor \varphi^i = \overline{z}^i \in \mathscr{F}(\mathbf{M}) \quad (i = 1, 2, \ldots) \tag{17}$$

uniquely determine the vector field Z and (17) can be very expressively (and unorthodoxly) recorded by

$$Z = \sum \overline{z}^i \frac{\partial}{\partial \varphi^i} \quad \left(\overline{z}^i = \varphi^i(Z) \in \mathscr{F}(\mathbf{M}), \text{ infinite sum}\right). \tag{18}$$

This is a mere symbolical record, not the true infinite series. However, if $\varphi^1, \varphi^2, \ldots$ is a *basis* of the module $\Phi(\mathbf{M})$ in the sense that every $\varphi \in \Phi(\mathbf{M})$ admits a unique representation $\varphi = \sum f^i\varphi^i$ ($f^i \in \mathscr{F}(\mathbf{M})$, finite sum) then the coefficients $\overline{z}_i$ can be quite arbitrary and (18) may be regarded as a true infinite series. The arising vector fields $\partial/\partial\varphi^1, \partial/\partial\varphi^2, \ldots$ provide a *weak basis* (infinite expansions, see [9]) of $\mathscr{T}(\mathbf{M})$ dual to the basis $\varphi^1, \varphi^1, \ldots$ of $\Phi(\mathbf{M})$. In this transcription, (15) is alternatively expressed as

$$Z = \sum z^i \frac{\partial}{\partial dh^i} \quad \left(z^i \in \mathscr{F}(\mathbf{M}), \text{infinite sum}\right). \tag{19}$$

We recall the *Lie derivative* $\mathscr{L}_Z = Z\rfloor d + dZ\rfloor$ acting on exterior differential forms. The image $\mathbf{m}_* Z$ of a vector field defined by the property

$$\mathbf{m}^*\left(\mathbf{m}_* Z\right) f = Z\mathbf{m}^* f \quad \left(f \in \mathscr{F}(\mathbf{M})\right) \tag{20}$$

need not exist. It is defined if $\mathbf{m}$ is invertible.

We consider various submodules $\Omega \subset \Phi(\mathbf{M})$ of differential forms together with the relevant *orthogonal* submodules $\Omega^\perp \subset \mathscr{T}(\mathbf{M})$ consisting of all vector fields $Z \in \mathscr{T}(\mathbf{M})$ such that $\omega(Z) = 0$ ($\omega \in \Omega$). The existence of (local) $\mathscr{F}(\mathbf{M})$-bases in all submodules of $\Phi(\mathbf{M})$ to appear in our reasonings is tacitly postulated. *Dimension* of an $\mathscr{F}(\mathbf{M})$-module is the number of elements of an $\mathscr{F}(\mathbf{M})$-basis. Omitting some "exceptional points," it may be confused with the dimension

of the corresponding $\mathbb{R}$-module (the *localization*) at a fixed place $\mathbf{P} \in \mathbf{M}$. On this occasion, it should be noted that the image

$$\mathbf{m}_* Z_{\mathbf{P}} \quad \left((\mathbf{m}_* Z_{\mathbf{P}})_{\mathbf{Q}} f = Z_{\mathbf{P}} (\mathbf{m}^* f), \mathbf{mP} = \mathbf{Q}, \mathbf{P} \in \mathbf{M} \right) \quad (21)$$

of a tangent vector $Z_{\mathbf{P}}$ at $\mathbf{P}$ exists as a vector at the place $\mathbf{Q}$.

Let us also remark with regret that any rigorous exposition of classical analysis in the infinite-dimensional space $\mathbb{R}^\infty$ is not yet available; however, certain adjustments of finite-dimensional results are not difficult. For instance, the following invertibility theorem will latently occur in the proof of Theorem 20.

Theorem 3. *A mapping* $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ *is invertible if and only if any of the following equivalent conditions is satisfied: the pull-back* $\mathbf{m}^* : \mathscr{F}(\mathbf{M}) \to \mathscr{F}(\mathbf{M})$ *is invertible, the pull-back* $\mathbf{m}^* : \Phi(\mathbf{M}) \to \Phi(\mathbf{M})$ *is invertible, and if* $\varphi^1, \varphi^2, \ldots$ *is a (fixed, equivalently: arbitrary) basis of module* $\Phi(\mathbf{M})$*, then* $\mathbf{m}^*\varphi^1, \mathbf{m}^*\varphi^2, \ldots$ *again is a basis.*

Hint. A nonlinear version of the familiar Gauss elimination procedure for infinite dimension [9] provides a direct proof with difficulties concerning the definition domain of the resulting inverse mapping. Nevertheless if $\mathbf{m}$ is moreover a morphism of a diffiety (see Definition 8) then the prolongation procedure ensures the local existence of $\mathbf{m}^{-1}$ in the common sense.

3. Fundamental Concepts

We introduce a somewhat unusual intrinsical approach to underdetermined systems of ordinary differential equations in terms of the above underlying space $\mathbf{M}$, a submodule $\Omega \subset \Phi(\mathbf{M})$ of differential 1-forms, and its orthogonal submodule $\mathscr{H} = \Omega^\perp \subset \mathscr{T}(\mathbf{M})$ of vector fields.

Definition 4. A codimension one submodule $\Omega \subset \Phi(\mathbf{M})$ is called a *diffiety* if there exists a *good filtration*

$$\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \cup \Omega_l \quad (22)$$

by finite-dimensional submodules $\Omega_l \subset \Omega$ $(l = 0, 1, \ldots)$ such that

$$\begin{gathered} \mathscr{L}_{\mathscr{H}} \Omega_l \subset \Omega_{l+1} \quad (\text{all } l), \\ \mathscr{L}_{\mathscr{H}} \Omega_l + \Omega_l = \Omega_{l+1} \quad (l \text{ large enough}). \end{gathered} \quad (23)$$

To every subset $\Theta \subset \Phi(\mathbf{M})$, let $\mathscr{L}_{\mathscr{H}}\Theta \subset \Phi(\mathbf{M})$ denote the submodule with generators $\mathscr{L}_Z \vartheta$ $(Z \in \mathscr{H}, \vartheta \in \Theta)$. Since $\Theta \subset \mathscr{L}_{\mathscr{H}}\Theta$ (easy), the second requirement (23) can be a little formally simplified as $\mathscr{L}_{\mathscr{H}}\Omega_l = \Omega_{l+1}$.

Remark 5. This is a global coordinate-free definition; however, we again deal only with the local theory from now on in the sense that the definition domains (of filtrations (22), of independent variable x to follow, and so on) are not specified. It should be noted on this occasion that the common geometrical approach [6–8] to differential equations rests on the use of the rigid structure of finite-order jets. Many classical concepts then become incorrect, if the higher-order mappings are allowed but we cannot adequately discuss this important topic here. Rather subtle difficulties are also passed over already in the common approach to the fundamental jet theory. For instance, smooth curves in the plane $\mathbb{R}^2$ with coordinates x, y are parametrized either by x (i.e., $y = y(x)$) or by coordinate y (i.e., $x = x(y)$) in the common so-called "geometrical" approach [6–8]. However, then already the Lie's classical achievements concerning contact transformations [10, 11] with curves parametrized either by $p = dy/dx$ or by $q = dx/dy$ cannot be involved. Quite analogously, the "higher-order" parameterizations and mappings [2–5] are in fact rejected in the common "rigid" jet theory with a mere point symmetries.

Definition 6. Let a differential dx $(x \in \mathscr{F}(\mathbf{M}))$ generate together with Ω the total module $\Phi(\mathbf{M})$ of all differential 1-forms. Then x is called the *independent variable* to diffiety Ω. The vector field $D = D_x$ (abbreviation) such that

$$\begin{gathered} D \in \mathscr{H}, \quad Dx = dx(D) = 1 \\ \left(\text{symbolically } D = \frac{\partial}{\partial dx} + \sum_{\omega \in \Omega} 0 \cdot \frac{\partial}{\partial \omega} \right) \end{gathered} \quad (24)$$

is called *total* (or *formal*) *derivative* of Ω with respect to the independent variable x. This vector field D is a basis of the one-dimensional module $\mathscr{H} = \Omega^\perp$ for every fixed particular choice of the independent variable x.

Remark 7. Let us state some simple properties of diffieties. The proofs are quite easy and may be omitted. A form $\varphi \in \Phi(\mathbf{M})$ is lying in Ω if and only if $\varphi(D) = 0$. In particular $\mathscr{L}_D \Omega \subset \Omega$ in accordance with the identities

$$\begin{gathered} \mathscr{L}_D \omega = D \rfloor d\omega + d\omega(D) = D \rfloor d\omega, \\ (\mathscr{L}_D \omega)(D) = d\omega(D, D) = 0 \\ (\omega \in \Omega). \end{gathered} \quad (25)$$

(This *trivial* property clarifies the *more restrictive* condition (23).) Moreover clearly

$$\begin{gathered} Dfdg - Dgdf, \quad df - Dfdx \in \Omega \\ (f, g \in \mathscr{F}(\mathbf{M})) \end{gathered} \quad (26)$$

and in particular

$$\begin{gathered} Dh^i dh^j - Dh^j dh^i, \quad dh^i - Dh^i dx \in \Omega \\ (i, j = 1, 2, \ldots) \end{gathered} \quad (27)$$

for all coordinates. We have very useful $\mathscr{F}(\mathbf{M})$-generators of diffiety Ω. The independent variable and the filtrations (22) can be capriciously modified. In particular the *c-lift* [9]

$$\begin{gathered} \Omega_{*+c} = \widetilde{\Omega}_* : \widetilde{\Omega}_0 \subset \widetilde{\Omega}_1 \subset \cdots \subset \Omega = \cup \widetilde{\Omega}_l \\ (\widetilde{\Omega}_l = \Omega_{l+c}, c = 0, 1, \ldots) \end{gathered} \quad (28)$$

with c large enough ensures that $\widetilde{\Omega}_{l+1} = \mathscr{L}_{\mathscr{H}}\widetilde{\Omega}_l + \widetilde{\Omega}_l$ for all $l \geq 0$. We will be, however, interested just in the reverse concept "Ω_{*-c}" latently involved in the "standard adaptation" of filtrations to appear later on.

Definition 8. A transformation $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ is called a *morphism* of the diffiety Ω if $\mathbf{m}^*\Omega \subset \Omega$. Invertible morphisms are *automorphisms* (or *symmetries*) of Ω. A vector field $Z \in \mathscr{T}(\mathbf{M})$ satisfying $\mathscr{L}_Z\Omega \subset \Omega$ is called the *variation* of Ω. If moreover Z (locally) generates a one-parameter group of transformations, we speak of the *infinitesimal symmetry* Z of diffiety Ω.

Remark 9. Let us mention the transformation groups in more detail. A local one-parameter group of transformations $\mathbf{m}(\lambda) : \mathbf{M} \to \mathbf{M}$ is given by certain formulae

$$\mathbf{m}(\lambda)^* h^i = H^i\left(h^1, \ldots, h^{m(i)}; \lambda\right) \quad (i = 1, 2, \ldots; -\varepsilon < \lambda < \varepsilon) \tag{29}$$

in terms of local coordinates, where $\mathbf{m}(\lambda + \mu) = \mathbf{m}(\lambda)\mathbf{m}(\mu)$, $\mathbf{m}(0) = id$ is supposed. Then the special vector field (15) defined by

$$z^i = \frac{\partial}{\partial\lambda}\mathbf{m}(\lambda)^* h^i\Big|_{\lambda=0} = \frac{\partial}{\partial\lambda}H^i\left(h^1, \ldots, h^{m(i)}; 0\right) \quad (i = 1, 2, \ldots) \tag{30}$$

is called the *infinitesimal transformation* of the group (29). In the opposite direction, we recall that a general vector field (15) generates the local group (29) if and only if the *Lie system*

$$\frac{\partial H^i}{\partial\lambda} = z^i\left(H^1, \ldots, H^{n(i)}\right), \qquad H^i\left(h^1, \ldots, h^{m(i)}; 0\right) = h^i \quad (i = 1, 2, \ldots) \tag{31}$$

is satisfied. Alas, a given vector field (19) *need not* in general generate any transformation group since the Lie system need not admit any solution (29).

With all fundamental concepts available, let us eventually recall the familiar and thoroughly discussed in [9] interrelation between the diffieties and the corresponding classical concept of differential equations for the convenience of reader. In brief terms, the idea is quite simple. A given system of differential equations is represented by a system of Pfaffian equations $\omega = 0$ and the module Ω generated by such 1-forms ω is just the diffiety. More precisely, we deal with the infinite prolongations as follows.

In one direction, let a system of underdetermined ordinary differential equations be given. We may deal with the first-order system

$$\frac{dw^j}{dx} = f^j\left(x, w^1, \ldots, w^m, \frac{dw^{J+1}}{dx}, \ldots, \frac{dw^m}{dx}\right) \quad (j = 1, \ldots, J) \tag{32}$$

without any true loss of generality. Then (32) completed with

$$\frac{dw_s^k}{dx} = w_{s+1}^k \quad \left(w_0^k = w^k; k = J+1, \ldots, m; s = 0, 1, \ldots\right) \tag{33}$$

provides the infinite prolongation. The corresponding diffiety Ω is generated by the forms

$$dw^j - f^j dx, \qquad dw_s^k - w_{s+1}^k dx \quad \left(j = 1, \ldots, J; k = J+1, \ldots, m; s = 0, 1, \ldots\right) \tag{34}$$

in the space $\mathbf{M}$ with coordinates

$$w^j \quad (j = 1, \ldots, J), \qquad w_s^k \quad \left(k = J+1, \ldots, m; s = 0, 1, \ldots; w_0^k = w^k\right). \tag{35}$$

Clearly

$$D_x = \frac{\partial}{\partial x} + \sum f^j \frac{\partial}{\partial w^j} + \sum w_{s+1}^k \frac{\partial}{\partial w_s^k} \in \mathscr{H} \tag{36}$$

is the total derivative and the submodules $\Omega_l \subset \Omega$ of all forms (34) with $s \leq l$ determine a quite simple filtration (22) with respect to the order of contact forms. (Hint: use the formulae

$$\mathscr{L}_D\left(dw^j - f^j dx\right) = \sum \frac{\partial f^j}{\partial w^j}\left(dw^j - f^j dx\right) + \sum \frac{\partial f^j}{\partial w_1^j}\left(dw_1^j - w_2^j dx\right) \tag{37}$$

and $\mathscr{L}_D(dw_s^j - w_{s+1}^j dx) = dw_{s+1}^j - w_{s+2}^j dx$.) However, there exist many other and more useful filtrations; see the examples to follow later on.

The particular case $J = 0$ of the empty system (32) can be easily related to the case of the jet space $\mathbf{M}(m)$ of all x-parametrized curves in $\mathbb{R}^{m+1}$ of the Section 1. The relevant diffiety is identified with the module $\Omega(m)$ of all contact forms (11), of course.

In the reverse direction, let a diffiety Ω be given on the space $\mathbf{M}$. In accordance with (27), the forms $dh^i - Dh^i dx$ $(i = 1, 2, \ldots)$ generate Ω. So we have the Pfaffian system $dh^i - Dh^i dx = 0$ $(i = 1, 2, \ldots)$ and therefore the system of differential equations

$$\frac{dh^i}{dx} = g^i\left(x, h^1, \ldots, h^{m(i)}\right) \quad \left(i = 1, 2, \ldots; g^i = Dh^i\right) \tag{38}$$

of rather unpleasant kind. Then, due to the existence of a filtration (22) and (23), one can obtain also the above classical system of differential equations (32) together with the prolongation (33) by means of appropriate change of coordinates [9]. This is, however, a lengthy procedure and a shorter approach can be described as follows. Let the second requirement (23) be satisfied, if $l \geq L$. Suppose that the forms $\omega^j = \sum a_i^j dh^i$ $(j = 1, \ldots, J = \dim \Omega_L)$ generate module Ω_L. Then all forms

$$\mathscr{L}_D^k \omega^j \quad (j = 1, \ldots, J; k = 0, 1, \ldots) \tag{39}$$

generate the diffiety Ω. The corresponding Pfaffian system $\mathscr{L}_D^k\omega^j = 0$ is equivalent to certain infinite prolongation of differential equations, namely,

$$\begin{aligned} \omega^j &= \sum a_i^j dh^i = 0 \text{ is equivalent to } \sum a_i^j \frac{dh^i}{dx} = 0, \\ \mathscr{L}_D\omega^j &= \sum Da_i^j dh^i + \sum a_i^j dDh^i \\ &= 0 \text{ is equivalent to } \frac{d}{dx}\sum a_i^j \frac{dh^i}{dx} = 0 \end{aligned} \tag{40}$$

(direct verification), and in general

$$\begin{aligned} \mathscr{L}_D^k\omega^j &= \sum D^k a_i^j dh^i + \cdots + \sum a_i^j dD^k h^i \\ &= 0 \text{ is equivalent to } \frac{d^k}{dx^k}\sum a_i^j \frac{dh^i}{dx} = 0. \end{aligned} \tag{41}$$

We have the infinite prolongation of the classical system $\sum a_i^j dh^i/dx = 0$ $(j = 1, \ldots, J)$ and this is just the system that corresponds to diffiety Ω.

Altogether taken, differential equations uniquely determine the corresponding diffieties; however, a given diffiety leads to many rather dissimilar but equivalent systems of differential equations with regard to the additional choice of dependent and independent variables.

Remark 10. Definitions 4–8 make good sense even if $\mathbf{M}$ is a finite-dimensional manifold and then provide the well-known intrinsical approach to determined systems of differential equations. They are identified with vector fields (better, fields of directions) in the finite-dimensional space $\mathbf{M}$. Choosing a certain independent variable x, the equations are represented by the vector field D_x or, more visually, by the corresponding D_x-flow. The general theory becomes trivial; we may, for example, choose $\Omega_l = \Omega$ for all l in filtration (22).

4. On the Structure of Diffieties

Definition 11. To every submodule $\Theta \subset \Omega$ of a diffiety $\Omega \subset \Phi(\mathbf{M})$, let $\operatorname{Ker}\Theta \subset \Theta$ be the submodule of all $\vartheta \in \Theta$ such that $\mathscr{L}_{\mathscr{H}}\vartheta \in \Theta$. Filtration (22) and (23) is called a *standard* one, if

$$\begin{aligned} &\operatorname{Ker}\Omega_{l+1} = \Omega_l \quad (l \geq 0), \\ &\operatorname{Ker}^2\Omega_0 = \operatorname{Ker}\Omega_0 \neq \Omega_0. \end{aligned} \tag{42}$$

For every $\omega \in \Omega$, the first condition ensures that the inclusions $\omega \in \Omega_l$, $\mathscr{L}_D\omega \in \Omega_{l+1}$ are equivalent and the second condition ensures that $\mathscr{L}_D\omega \in \Omega_0$ implies $\mathscr{L}_D^2\omega \in \Omega_0$.

Theorem 12. *Appropriate adaptation of some lower-order terms of a given filtration* (22) *and* (23) *provides a standard filtration in a unique manner* [9]. *Equivalently and in more detail, there exists unique standard filtration* $\overline{\Omega}_* : \overline{\Omega}_0 \subset \overline{\Omega}_1 \subset \cdots \subset \Omega = \cup\overline{\Omega}_l$ *such that* $\Omega_l = \overline{\Omega}_{l+c}$ *for appropriate* $c \in \mathbb{N}$ *and all* l *large enough. Equivalently and briefly, there exists unique standard filtration* $\overline{\Omega}_*$ *such that* $\Omega_{*+c'} = \overline{\Omega}_{*+c''}$ *for appropriate* $c', c'' \in \mathbb{N}$.

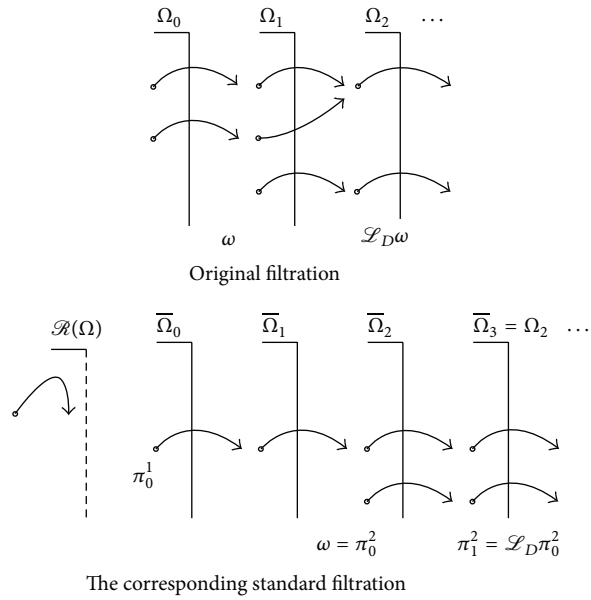


Figure 1

Proof. The mapping $\mathscr{L}_D : \Omega_l \to \Omega_{l+1}$ naturally induces certain $\mathscr{F}(\mathbf{M})$-homomorphism

$$D : \Omega_l/\Omega_{l-1} \longrightarrow \Omega_{l+1}/\Omega_l \quad (l \geq 0, \text{ formally } \Omega_{-1} = 0) \tag{43}$$

of factor modules denoted by D for better clarity. Homomorphisms D are surjective and therefore even bijective for all l large enough, say for $l \geq L$. However, the injectivity of D implies $\operatorname{Ker}\Omega_l = \Omega_{l-1}$ $(l \geq L)$. It follows that we have *strongly decreasing* sequence

$$\begin{aligned} &\cdots \supset \Omega_L\,(= \operatorname{Ker}\Omega_{L+1}) \supset \Omega_{L-1}\,(= \operatorname{Ker}\Omega_L) \\ &\quad \supset \operatorname{Ker}\Omega_{L-1} \supset \operatorname{Ker}^2\Omega_{L-1} \supset \cdots, \end{aligned} \tag{44}$$

which necessarily terminates with the stationarity $\operatorname{Ker}^C\Omega_{L-1} = \operatorname{Ker}^{C+1}\Omega_{L-1}$. Denoting

$$\begin{aligned} &\overline{\Omega}_0 = \operatorname{Ker}^{C-1}\Omega_{L-1}, \ldots, \\ &\overline{\Omega}_{C-1} = \operatorname{Ker}\Omega_{L-1}, \quad \overline{\Omega}_C = \Omega_{L-1}, \quad \overline{\Omega}_{C+1} = \Omega_L, \ldots, \end{aligned} \tag{45}$$

we have the sought *strongly increasing* standard filtration

$$\begin{aligned} \overline{\Omega}_* : \overline{\Omega}_0 &\subset \overline{\Omega}_1 \subset \cdots \subset \overline{\Omega}_C\,(= \Omega_{L-1}) \\ &\subset \overline{\Omega}_{C+1}\,(= \Omega_L) \subset \cdots \subset \Omega = \cup\overline{\Omega}_l \end{aligned} \tag{46}$$

of diffiety Ω. In particular $\operatorname{Ker}^2\overline{\Omega}_0 = \operatorname{Ker}^{C+1}\Omega_{L-1} = \operatorname{Ker}^C\Omega_{L-1} = \operatorname{Ker}\overline{\Omega}_0$. □

Proof of Theorem 12 was of the algorithmical nature and provides a useful *standard basis* of diffiety Ω as follows. Assume that the forms

$$\begin{gathered}\tau^1,\dots,\tau^K \in \overline{\Omega}_0 \\ \text{provide a basis of the submodule } \operatorname{Ker}\overline{\Omega}_0 \subset \overline{\Omega}_0\end{gathered} \tag{47}$$

(recall that $\operatorname{Ker}^2\overline{\Omega}_0 = \operatorname{Ker}\overline{\Omega}_0$ whence $\mathscr{L}_D \operatorname{Ker}\overline{\Omega}_0 \subset \operatorname{Ker}\overline{\Omega}_0$) and moreover the classes of forms

$$\pi^1,\dots,\pi^{j_0} \in \overline{\Omega}_0 \text{ provide a basis of } \overline{\Omega}_0/\operatorname{Ker}\overline{\Omega}_0 \tag{48}$$

(recall that $D : \overline{\Omega}_0/\operatorname{Ker}\overline{\Omega}_0 \to \overline{\Omega}_1/\overline{\Omega}_0$ is injective mapping), the classes of forms

$$\begin{gathered}\pi^{j_0+1},\dots,\pi^{j_1} \in \overline{\Omega}_1 \\ \text{provide a basis of } \overline{\Omega}_1/\left(\overline{\Omega}_0 + \mathscr{L}_{\mathscr{H}}\overline{\Omega}_0\right)\end{gathered} \tag{49}$$

(recall that $D : \overline{\Omega}_1/\overline{\Omega}_0 \to \overline{\Omega}_2/\overline{\Omega}_1$ is injective mapping), and in general the classes of forms

$$\begin{gathered}\pi^{j_{l-1}+1},\dots,\pi^{j_l} \in \overline{\Omega}_l \\ \text{provide a basis of } \overline{\Omega}_l/\left(\overline{\Omega}_{l-1} + \mathscr{L}_{\mathscr{H}}\overline{\Omega}_{l-1}\right).\end{gathered} \tag{50}$$

Alternatively saying, the following forms constitute a basis:

$$\begin{gathered}\tau^1,\dots,\tau^K \text{ of } \operatorname{Ker}\overline{\Omega}_0, \\ \text{together with } \pi^1,\dots,\pi^{j_0} \text{ of } \overline{\Omega}_0, \\ \text{together with } \mathscr{L}_D\pi^1,\dots,\mathscr{L}_D\pi^{j_0},\pi^{j_0+1},\dots,\pi^{j_1} \text{ of } \overline{\Omega}_1, \\ \text{together with } \mathscr{L}_D^2\pi^1,\dots,\mathscr{L}_D^2\pi^{j_0},\mathscr{L}_D\pi^{j_0+1},\dots,\mathscr{L}_D\pi^{j_1}, \\ \pi^{j_1+1},\dots,\pi^{j_2} \text{ of } \overline{\Omega}_2,\end{gathered} \tag{51}$$

and so on. Let us denote

$$\pi_r^j = \mathscr{L}_D^r\pi^j \quad \left(j = j_{l-1}+1,\dots,j_l\right). \tag{52}$$

In terms of this notation

$\tau^1,\dots,\tau^K$ is a basis of $\operatorname{Ker}\overline{\Omega}_0$ and together with the forms

$$\begin{gathered}\tau^1,\dots,\tau^K, \\ \pi_0^1,\dots,\pi_0^{j_0}, \quad \pi_1^1,\dots,\pi_1^{j_0},\dots, \quad \pi_l^1,\dots,\pi_l^{j_0}, \\ \pi_0^{j_0+1},\dots,\pi_0^{j_1},\dots, \quad \pi_{l-1}^{j_0+1},\dots,\pi_{l-1}^{j_1} \\ \dots \\ \pi_0^{j_{l-1}+1},\dots,\pi_0^{j_l},\end{gathered} \tag{53}$$

we have the *standard basis* of $\overline{\Omega}_l$.

Clearly $j_L = j_{L+1} = \cdots$ and it follows that there is only a finite number $\mu(\Omega) = j_L$ of *initial forms*

$$\begin{gathered}\pi^1 = \pi_0^1,\dots,\pi^{j_0} = \pi_0^{j_0},\pi^{j_0+1} = \pi_0^{j_0+1}, \\ \dots,\pi^{j_{L-1}+1} = \pi_0^{j_{L-1}+1},\dots,\pi^{j_L} = \pi_0^{j_L}\end{gathered} \tag{54}$$

with the lower zero indice. The following forms π_r^j $(r > 0)$ satisfy the recurrence and the (equivalent) congruence

$$\mathscr{L}_D\pi_r^j = \pi_{r+1}^j, \quad d\pi_r^j \cong dx \wedge \pi_{r+1}^j \quad (\operatorname{mod}\Omega\wedge\Omega). \tag{55}$$

In this sense, the *linearly independent forms π_r^j are generalizations of the classical contact forms $\omega_r^j = dw_r^j - w_{r+1}^j\,dx$ of the jet theory.*

Theorem 13. *Let $\overline{\Omega}_*$ be a standard filtration of diffiety Ω. Then the submodule $\operatorname{Ker}\overline{\Omega}_0 \subset \Omega$ is generated by all differentials $df \in \Omega$.*

Proof. First assume $df \in \Omega$. Then $Df = df(D) = 0$ whence $\mathscr{L}_D df = dDf = 0$. Clearly $df \in \overline{\Omega}_l$ for appropriate l. This implies $df \in \operatorname{Ker}\overline{\Omega}_{l-1}$, if $l \geq 0$ therefore $df \in \operatorname{Ker}\overline{\Omega}_0$. It follows that $\operatorname{Ker}\overline{\Omega}_0$ contains all differentials $df \in \Omega$.

Conversely let $\tau \in \operatorname{Ker}\overline{\Omega}_0$. Due to the equality $\operatorname{Ker}\overline{\Omega}_0 = \operatorname{Ker}^2\Omega_0$, we have $\mathscr{L}_D\tau \in \operatorname{Ker}\overline{\Omega}_0$ whence $d\tau \cong dx \wedge \mathscr{L}_D\tau$ $(\operatorname{mod}\Omega\wedge\Omega)$, consequently

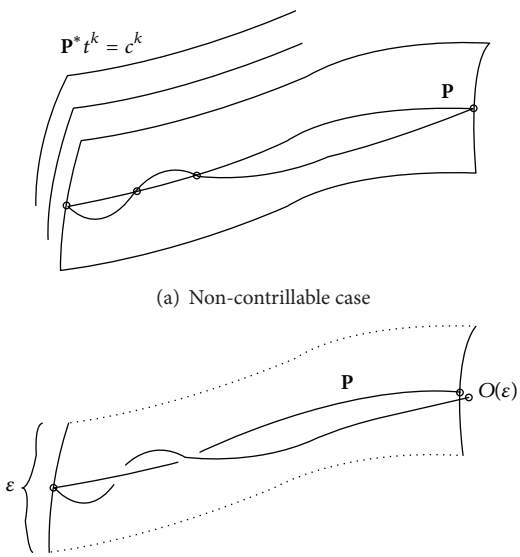
$$d\tau \cong \sum a_{rs}^{ji}\pi_r^j \wedge \pi_s^i \quad \left(\operatorname{mod}\ \operatorname{Ker}\overline{\Omega}_0, \text{ sum over } i \leq j\right). \tag{56}$$

It follows that $a_{rs}^{ji} = 0$ identically by using $d(d\tau) = 0$ and (55). (Hint: look at assumed top order product $\pi_R^j \wedge \pi_s^i$ where $R \geq$ all r. Then $d^2\tau$ involves only one summand with $dx \wedge \pi_{R+1}^j \wedge \pi_s^i$ which is impossible since $d^2 = 0$.) Therefore $d(\operatorname{Ker}\overline{\Omega}_0) \cong 0\,(\operatorname{mod}\operatorname{Ker}\overline{\Omega}_0)$ and the Frobenius theorem can be applied. Module $\operatorname{Ker}\overline{\Omega}_0$ has a basis consisting of total differentials. □

Definition 14. We may denote $\mathscr{R}(\Omega) = \operatorname{Ker}\overline{\Omega}_0$ since this module does not depend on the choice of the filtration (22). Together with the original basis $\tau^1,\dots,\tau^K$ occurring in (53), there exists alternative basis $dt^1,\dots,dt^K$ with differentials. In the particular case $\mathscr{R}(\Omega) = 0$, hence, $K = 0$, we speak of a *controllable* diffiety Ω.

Remark 15. The controllability is a familiar concept of the theory of underdetermined ordinary differential equations or Pfaffian systems in finite-dimensional spaces [12]; however, some aspects due to diffieties are worth mentioning here. If $\mathscr{R}(\Omega) \neq 0$ is a nontrivial module, the underlying space $\mathbf{M}$ is fibered by the leaves $t^k = c^k \in \mathbb{R}$ $(k = 1,\dots,K)$ depending on $K > 0$ parameters. A curve $\mathbf{p} : \mathbf{I} \to \mathbf{M}$ $(\mathbf{I} \subset \mathbb{R})$ is called a *solution* of diffiety Ω, if $\mathbf{p}^*\omega = 0$ $(\omega \in \Omega)$. Since $dt^k \in \mathscr{R}(\Omega) \subset \Omega$, we have

$$\begin{gathered}\mathbf{p}^*dt^k = d\mathbf{p}^*t^k = 0, \qquad \mathbf{p}^*t^k = c^k \in \mathbb{R} \\ (k = 1,\dots,K),\end{gathered} \tag{57}$$



(a) Non-contrillable case

(b) Mayer extermal

Figure 2

therefore every solution of diffiety Ω is contained in a certain leaf (the Figure 2(a)).

In the controllable case, such foliation of the space $\mathbf{M}$ does not exist. However, the construction of the standard filtration need not be of the "universal nature." There may exist some "exceptional points" where the terms π_r^i of the standard basis are not independent. We may even obtain a solution $\mathbf{p}$ consisting of such exceptional points and then there appears the "infinitesimal leaf" of the noncontrollability along $\mathbf{p}$ which means that $\mathbf{p}$ is a *Mayer extremal* (the Figure 2(b)). We refer to article [13] inspired by the beautiful paper [14]. In the present paper, such exceptional points are tacitly excluded. They produce singularities of the symmetry groups and deserve a special, not yet available approach. It should be noted that the noncontrollable case also causes some technical difficulties. We may however suppose $\mathscr{R}(\Omega) = 0$ without much loss of generality since the noncontrollable diffiety can be restricted to a leaf and regarded as a diffiety depending on parameters $c^1, \ldots, c^K$.

Theorem 16. *The total number $\mu(\Omega)$ of initial forms does not depend on the choice of the good filtration* (22).

Proof. Filtration (22) differs from the standard filtration $\overline{\Omega}_*$ only in lower terms whence

$$\dim \Omega_l = \dim \overline{\Omega}_l + \text{const.} = \mu(\Omega)\, l + \text{const.} \qquad (l \text{ large enough}). \tag{58}$$

Let another filtration $\widetilde{\Omega}_* : \widetilde{\Omega}_0 \subset \widetilde{\Omega}_1 \subset \cdots \subset \Omega = \cup \widetilde{\Omega}_l$ of diffiety Ω provide (corresponding standard filtration and therefore) certain number $\widetilde{\mu}(\Omega)$ of (other) initial forms. Then

$$\dim \Omega_l = \dim \widetilde{\Omega}_l + \text{const.} = \widetilde{\mu}(\Omega)\, l + \text{const.} \qquad (l \text{ large enough}). \tag{59}$$

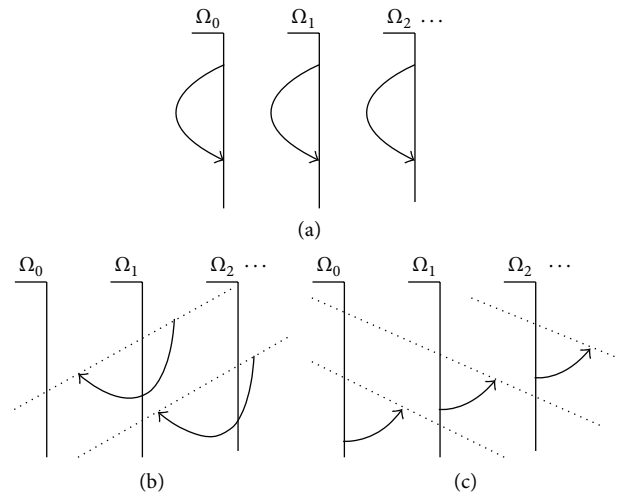


Figure 3

However $\Omega_l \subset \widetilde{\Omega}_{L(l)} \subset \Omega_{M(l)}$ for appropriate $L(l)$ and $M(l)$ whence

$$\Omega_{l+k} \subset \widetilde{\Omega}_{L(l)+k} \subset \Omega_{M(l)+k} \qquad (l, k \text{ large enough}) \tag{60}$$

by using (23) and the equality $\mu(\Omega) = \widetilde{\mu}(\Omega)$ easily follows. □

5. On the Morphisms and Variations

A huge literature on the *point symmetries* (scheme (a) of Figure 3, the order of derivatives is preserved) of differential equations is available. On the contrary, we can mention only a few fundamental principles for the *generalized* (or *higher-order*) *symmetries* (scheme (c) of Figure 3) since the general theory deserves quite another paper. Our modest aim is to clarify a little the mechanisms of the particular examples to follow. We will also deal with *generalized* (or *higher-order*) *groups of symmetries* and the relevant *generalized infinitesimal symmetries* (scheme (b) Figure 3) with ambiguous higher-order invariant subspaces (the dotted lines). Figure 3 should be therefore regarded as a rough description of the topics to follow and we also refer to Section 9 for more transparent details. The main difficulty of the higher-order theory lies in the fact that *the dotted domains are not known in advance*. Modules Ω_l represent the "natural" filtration with respect to the primary order of contact forms in the ambient jet space, see the examples. They depend on the accidental inclusion $\mathbf{M} \subset \mathbf{M}(m)$ mentioned in Section 1 and *do not have any true geometrical sense* in the internal approach. It is to be therefore surprisingly observed that the seemingly "exotic" at the first glance concept of higher-order transformations of Section 1 should be regarded for reasonable and the only possible in the coordinate free theory. On the other hand, an important distinction between the *group-like morphisms* with large number of finite-dimensional invariant subspaces (scheme (a) and (b)) and the genuine *order-destroying morphisms* without such subspaces (scheme (c)) is of the highest importance.

We are passing to rigorous exposition. Let us recall the diffiety $\Omega \subset \Phi(\mathbf{M})$ on the space $\mathbf{M}$, the independent variable $x \in \mathscr{F}(\mathbf{M})$ with the corresponding vector field $D = D_x \in \Omega^{\perp} = \mathscr{H}$, the controllability submodule $\mathscr{R}(\Omega) \subset \Omega$ with the basis $dt^1, \ldots, dt^K$, and a standard basis π_r^j ($j = 1, \ldots, \mu(\Omega); r = 0, 1, \ldots$) of diffiety Ω.

Let us begin with morphisms.

Lemma 17. *If* $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ *is a morphism of* Ω *then* $\mathbf{m}^*\mathscr{R}(\Omega) \subset \mathscr{R}(\Omega)$ *and the recurrence*

$$DW\mathbf{m}^*\pi_{r+1}^j \cong \mathscr{L}_D\mathbf{m}^*\pi_r^j \qquad (W = \mathbf{m}^*x;\ j = 1, \ldots, \mu(\Omega);\ r = 0, 1, \ldots) \tag{61}$$

modulo $\mathscr{R}(\Omega)$ *holds true.*

Proof. If $\mathbf{m}$ is a morphism then $\mathbf{m}^*\Omega \subset \Omega$ therefore $\mathbf{m}^*\mathscr{R}(\Omega) \subset \mathscr{R}(\Omega)$ (use Theorem 13) and $\mathbf{m}^*\pi_r^j \cong \sum a_{rs}^{ji}\pi_s^i \pmod{\mathscr{R}(\Omega)}$. It follows that

$$\begin{aligned} &\mathbf{m}^*d\pi_r^j \cong \mathbf{m}^*\left(dx \wedge \pi_{r+1}^j\right) \cong DWdx \wedge \mathbf{m}^*\pi_{r+1}^j, \\ &d\mathbf{m}^*\pi_r^j \cong d\sum a_{rs}^{ji}\pi_s^i \\ &\quad \cong \sum Da_{rs}^{ji}dx \wedge \pi_s^i \\ &\qquad + \sum a_{rs}^{ji}dx \wedge \pi_{s+1}^i \cong dx \wedge \mathscr{L}_D\mathbf{m}^*\pi_r^j \end{aligned} \tag{62}$$

modulo $\mathscr{R}(\Omega)$ and $\Omega\wedge\Omega$. This implies (61) by comparing both factors of dx. □

Remark 18. On this occasion, the following useful principles of calculation are worth mentioning:

$$\begin{aligned} &\text{if } \alpha, \beta \in \Omega \text{ satisfy } d\alpha \cong dx \wedge \beta \pmod{\Omega \wedge \Omega} \\ &\quad \text{then } DW\mathbf{m}^*\beta = \mathscr{L}_D\mathbf{m}^*\alpha, \end{aligned} \tag{63}$$

$$\text{if } u, v \in \mathscr{F}(\mathbf{M}),\ du - vdx \in \Omega \text{ then } DW\mathbf{m}^*v = D\mathbf{m}^*u, \tag{64}$$

and in general

$$\mathbf{m}^*Df \cdot D\mathbf{m}^*g = \mathbf{m}^*Dg \cdot D\mathbf{m}^*f \quad (f, g \in \mathscr{F}(\mathbf{M})). \tag{65}$$

In terms of notation (21), we conclude that $\mathbf{m}_*D_{\mathbf{P}} = DW(\mathbf{P}) \cdot D_{\mathbf{Q}}$ and therefore

$$\mathbf{m}_*\left(\frac{1}{DW}D\right) = D \quad (W = \mathbf{m}^*x), \tag{66}$$

if the morphism $\mathbf{m}$ of diffiety Ω is invertible.

Let us turn to invertible morphisms.

Lemma 19. *The inverse of a morphism* $\mathbf{m}$ *again is a morphism.*

Proof. Assume $\omega \in \Omega$, $\mathbf{m}^{-1*}\omega \cong fdx \pmod{\Omega}$. Then

$$\omega = \mathbf{m}^*\mathbf{m}^{-1*}\omega \cong \mathbf{m}^*(fdx) = \mathbf{m}^*f \cdot dW \in \Omega, \tag{67}$$

where $dW = d\mathbf{m}^*x = \mathbf{m}^*dx \neq 0$. Hence $\mathbf{m}^*f = 0$, $f = 0$ and therefore $\mathbf{m}^{-1*}\Omega \subset \Omega$. □

We have $\mathbf{m}^*\Omega \subset \Omega$ if $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ is a morphism and moreover $\mathbf{m}^{-1*}\Omega \subset \Omega$ hence $\Omega^* \subset \mathbf{m}^*\Omega$ in the invertible case. The converse and rather useful assertion is as follows.

Theorem 20. *A morphism* $\mathbf{m}$ *of diffiety* Ω *is invertible if and only if* $\mathbf{m}^*\Omega = \Omega$.

This may be obtained easily from the following result.

Lemma 21. *Let* $\mathbf{m}^*\mathscr{R}(\Omega) = \mathscr{R}(\Omega)$ *and* $\pi_0^j \in \mathbf{m}^*\Omega$ ($j = 1, \ldots, \mu(\Omega)$). *Then* $\mathbf{m}$ *is invertible.*

Proof. Proof of the Lemma 21 is analogous as in [2, Theorem 2] and we briefly recall only the main principles here. It is sufficient to prove the invertibility of $\mathbf{m}^* : \Omega \to \Omega$.

Assuming $\pi_r^j \in \mathbf{m}^*\Omega$ then $\pi_{r+1}^j = \mathscr{L}_D\pi_r^j \in \mathbf{m}^*\Omega$ by virtue of recurrence (61). It follows that $\Omega \subset \mathbf{m}^*\Omega$ and $\mathbf{m}^*$ is surjective. We prove that $\mathbf{m}^* : \Omega \to \Omega$ *is even injectivity* by using the well-known algebraical interrelation between filtrations and gradations.

Let us introduce filtrations Ω_* ($\overline{\Omega}_*$, resp.) as follows: the submodule $\Omega_l \subset \Omega$ ($\overline{\Omega}_l \subset \Omega$) is generated by $\mathscr{R}(\Omega)$ and all forms π_r^j ($\mathbf{m}^*\pi_r^j$) where $r \leq l$. We also introduce the gradations

$$\begin{aligned} &\mathscr{M} = \oplus\mathscr{M}_l \quad (\mathscr{M}_l = \Omega_l/\Omega_{l-1}), \\ &\overline{\mathscr{M}} = \oplus\overline{\mathscr{M}}_l \quad (\overline{\mathscr{M}}_l = \overline{\Omega}_l/\overline{\Omega}_{l-1}) \\ &\qquad (l = 0, 1, \ldots) \end{aligned} \tag{68}$$

(formally $\Omega_{-1} = \overline{\Omega}_{-1} = 0$). It follows that the naturally induced mapping $\mathbf{m}^* : \mathscr{M} \to \mathscr{M}$ is surjective and *it is sufficient to prove that this induced* $\mathbf{m}^*$ *is also injective.*

We are passing to the most delicate part of the proof. The surjectivity of $\mathbf{m}^* : \Omega \to \Omega$ implies that $\Omega_0 \subset \overline{\Omega}_L$ for L large enough. Therefore $\Omega_l \subset \overline{\Omega}_{L+l}$ by applying the recursion (61) which implies

$$\begin{aligned} &\dim \Omega_l = \mu(\Omega)\, l + \text{const.} \leq \dim \overline{\Omega}_{L+l} \\ &\quad (\mu(\Omega) = \dim \mathscr{M}_l;\ l = 0, 1, \ldots). \end{aligned} \tag{69}$$

On the other hand, assume the noninjectivity therefore the existence of a nontrivial identity

$$0 = \sum a_r^i\mathbf{m}^*\pi_r^i = \cdots + \sum a_R^i\mathbf{m}^*\pi_R^i \quad (\text{top-order terms}). \tag{70}$$

Then $0 = \cdots + (DW)^{-l}\sum a_R^i\mathbf{m}^*\pi_{R+l}^i$ ($l = 0, 1, \ldots$) by applying operator $\mathscr{L}_D$ and recurrence (61). Due to the existence of such identities, it follows that

$$\begin{aligned} &\dim \overline{\mathscr{M}}_l < \dim \mathscr{M}_l = \mu(\Omega), \\ &\dim \overline{\Omega}_{L+l} \leq (\mu(\Omega) - 1)\, l + \text{const.} \end{aligned} \tag{71}$$

and this is a contradiction. □

Remark 22. Recall that if $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ is a mapping and $\Omega \subset \Phi(\mathbf{M})$ a submodule, then $\mathbf{m}^*\Omega \subset \Phi(\mathbf{M})$ denotes the *submodule* with generators $\mathbf{m}^*\omega$ $(\omega \in \Omega)$ in accordance with the common practice in the algebraical module theory. Let in particular Ω be a diffiety and assume $\mathscr{R}(\Omega) = 0$ for simplicity. Then module $\mathbf{m}^*\Omega$ is generated by all forms $\mathbf{m}^*\pi_r^j$ and therefore by all forms $\mathscr{L}_D^r\mathbf{m}^*\pi_0^j$, see Lemma 17. It follows that the invertibility of the morphism $\mathbf{m}$ depends only on the properties of the forms $\mathbf{m}^*\pi_0^j$, see Lemma 21. In this sense, the invertibility problem is reduced to the finite-dimensional reasonings.

We turn to the variations.

Lemma 23. *A vector field $Z \in \mathscr{T}(\mathbf{M})$ is a variation of diffiety Ω if and only if*

$$\pi_{r+1}^j(Z) = D\pi_r^j(Z) \quad \left(j = 1, \ldots, \mu(\Omega); r = 0, 1, \ldots\right) \tag{72}$$

and all Zt^k $(k = 1, \ldots, K;$ fixed $k)$ are functions only of variables $t^1, \ldots, t^K$.

Proof. We suppose $\mathscr{L}_Z\Omega \subset \Omega$ which is equivalent to the congruences

$$\begin{aligned} \mathscr{L}_Z dt^k &= dZt^k \cong DZt^k dx = 0 \pmod{\Omega}, \\ \mathscr{L}_Z \pi_r^j &= Z\rfloor d\pi_r^j + d\pi_r^j(Z) \\ &\cong \left(-\pi_{r+1}^j(Z) + D\pi_r^j(Z)\right) dx = 0 \pmod{\Omega} \end{aligned} \tag{73}$$

by using ((26) and (55)). So we have obtained (72) and moreover identities $DZt^k = 0$ $(k = 1, \ldots, K)$.

It is sufficient to prove that the latter identities imply $dZt^k = 0 \pmod{dt^1, \ldots, dt^K}$. However, every differential df $(f \in \mathscr{F}(\mathbf{M}))$ can be represented as

$$df = Dfdx + \sum f^{k'} dt^{k'} + \sum f_r^j \pi_r^j \tag{74}$$

in terms of the standard basis. Assuming in particular $f = Zt^k$ (fixed $k = 1, \ldots, K$), we have already obtained the equation $Df = 0$ and then identities $f_r^j = 0$ easily follow by applying the common rule $d(df) = 0$ together with (26). This concludes the proof. □

Theorem 24. *A variation Z of diffiety Ω is infinitesimal symmetry of Ω if and only if all forms $\mathscr{L}_Z^k\pi_0^j$ $(k = 0, 1, \ldots)$ are contained in a finite-dimensional module.*

We omit lengthy proof and refer to more general results [5, Lemma 5.4, Theorem 5.6, and especially Theorem 11.1]. In future examples, we apply other and quite elementary arguments in order to avoid the nontrivial Theorem 24.

Remark 25. It follows from Lemma 23 that variations Z of diffiety Ω can be represented by the universal series

$$Z = \sum c^k \frac{\partial}{\partial t^k} + z\frac{\partial}{\partial x} + \sum D^r p^j \frac{\partial}{\partial \pi_r^j}, \tag{75}$$

where $c^k = c^k(t^1, \ldots, t^K)$ are arbitrary composed functions and $z = Zx$, $p^j = \pi_0^j(Z)$ are arbitrary functions in $\mathscr{F}(\mathbf{M})$. We have *explicit formulae for all variations* (in common terms, *for all Lie-Bäcklund infinitesimal symmetries*) *of a given system of ordinary differential equations*. Recall that these variations Z need not generate any true group, and though the criterion in Theorem 24 is formally simple, it is not easy to be applied. Lemma 17 can be regarded as a counterpart to Lemma 23 since it ensures quite analogous result for the morphism $\mathbf{m}$ or, better saying, for the pullback $\mathbf{m}^* : \Phi(\mathbf{M}) \to \Phi(\mathbf{M})$ of a morphism. In more detail, the quite arbitrary choice of the initial terms $\mathbf{m}^*\pi_0^j$ of recurrence (61) is in principle possible but provides a mere formal result (corresponding to the formal nature of variations Z) and does not ensure the existence of true morphism $\mathbf{m}$. We may refer to articles [2, 3] where the formal part (the *algebra*) is distinguished from the nonformal part (the *analysis*) in the higher-order algorithms.

We conclude this Section with the only gratifying result [9, point (ν) on page 40].

Theorem 26. *The standard filtration is unique in the case $\mu(\Omega) = 1$.*

Proof. Let us take a fixed filtration (22) and the corresponding standard filtration (46). Since $\mu(\Omega) = 1$, we have only one initial form π_0^1 and therefore $\tau^1, \ldots, \tau^K, \pi_0^1, \ldots, \pi_l^1$ is a basis of $\overline{\Omega}_l$; see (53). Let us take another standard filtration $\widetilde{\Omega}_*$. Then the module $\widetilde{\Omega}_0$ has certain basis

$$\begin{aligned} &\tau^1, \ldots, \tau^K \quad (\text{common forms}), \\ &\widetilde{\pi}_0^1 = \sum a_r \pi_r^1 = \cdots + a_R \pi_R^1 \quad \left(a_R \neq 0, \text{ top-order term}\right). \end{aligned} \tag{76}$$

These forms together with all $\widetilde{\pi}_s^1 = \mathscr{L}_D^s\widetilde{\pi}_0^1 = \cdots + a_R\pi_{R+s}^1$ $(s \geq 0)$ generate the module Ω and this is possible only if $R = 0$. We conclude that $\widetilde{\pi}_0^1 = a_R\pi_R^1 = a_0\pi_0^1$ which implies $\widetilde{\Omega}_0 = \overline{\Omega}_0$ hence $\widetilde{\Omega}_l = \overline{\Omega}_l$ for all l. □

Remark 27. It follows that in the particular case $\mu(\Omega) = 1$, every symmetry and infinitesimal symmetry preserves all terms of the (unique) standard filtration. So we have a large family of finite-dimensional subspaces of the underlying space $\mathbf{M}$ which are preserved too. *The classical methods acting in finite-dimensional spaces uniquely determined in advance can be applied and are quite sufficient in this case $\mu(\Omega) = 1$.*

Remark 28. In more generality, one could also consider two diffieties Ω and $\widetilde{\Omega}$ on the underlying spaces $\mathbf{M}$ and $\widetilde{\mathbf{M}}$, respectively. Though we do not deal with the *isomorphism problems* of two diffieties Ω and $\widetilde{\Omega}$ here, let us mention that such *isomorphism* is defined as invertible mapping $\mathbf{m} : \widetilde{\mathbf{M}} \to \mathbf{M}$ of underlying spaces satisfying $\mathbf{m}^*\Omega = \widetilde{\Omega}$. Quite equivalent "absolute equivalence" problem was introduced in [15] and resolved just for the case $\mu(\Omega) = \mu(\widetilde{\Omega}) = 1$ (in our terminology) by using finite-dimensional methods. We have discovered alternative approach here: the isomorphism

m identifies the *unique* standard filtrations of Ω and of $\widetilde{\Omega}$. On this occasion, it is worth mentioning Cartan's pessimistic notice (rather unusual in his work) to the case $\mu(\Omega) > 1$: "Je dois ajourter que la géneralization de la théorie de l'equivalence absolu aux systémes differentiels dont la solution générale dépend de deux functions arbitraires d'un argument n'est pas immédiate et souléve d'asses grosses difficultiés." The same notice can be literally repeated also for the theory of the higher-order symmetries treated in this paper.

6. The Order-Preserving Case of Infinitesimal Symmetries

We are passing to the first example which intentionally concerns the well-known "towering" problem in order to examine our method reliably. Let us deal with infinitesimal symmetries of differential equation

$$\frac{d^2u}{dx^2} = F\left(\frac{dv}{dx}\right) \tag{77}$$

involving two unknown functions $u = u(x)$ and $v = v(x)$. In external theory, (77) is identified with the subspace $\mathbf{i} : \mathbf{M} \subset \mathbf{M}(2)$ defined by the conditions

$$\begin{gathered} D^r\left(u_2 - F\right) = u_{r+2} - D^r F\left(v_1\right) = 0 \\ \left(r = 0, 1, \ldots; D = \frac{\partial}{\partial x} + \sum u_{r+1}\frac{\partial}{\partial u_r} + \sum v_{r+1}\frac{\partial}{\partial v_r}\right) \end{gathered} \tag{78}$$

in the jet space $\mathbf{M}(2)$. We use simplified notation of coordinates and contact forms

$$u_r = w_r^1, \quad v_r = w_r^2, \quad \alpha_r = \omega_r^1, \quad \beta_r = \omega_r^2 \qquad (r = 0, 1, \ldots) \tag{79}$$

here. We are, however, interested in internal theory, that is, in the diffiety Ω corresponding to (77). Diffiety Ω appears if the contact forms

$$\alpha_r = du_r - u_{r+1}dx, \quad \beta_r = dv_r - v_{r+1}dx \qquad (r = 0, 1, \ldots) \tag{80}$$

are restricted to the subspace $\mathbf{i} : \mathbf{M} \subset \mathbf{M}(2)$. In accordance with the common practice, let us again simplify as

$$u_r = \mathbf{i}^* u_r, \quad v_r = \mathbf{i}^* v_r, \quad \alpha_r = \mathbf{i}^* \alpha_r, \quad \beta_r = \mathbf{i}^* \beta_r \tag{81}$$

the notation of the restrictions to **M**, and moreover D will be regarded as a vector field on **M** from now on.

Let us outline the lengthy path of future reasonings for the convenience of reader. We begin with preparatory points (ι)–($\iota\iota\iota$). The underlying space **M** together with the diffiety Ω is introduced and the standard basis $\pi_0, \pi_1, \ldots$ ($\mu(\Omega) = 1$, abbreviation $\pi_r = \pi_r^1$) of diffiety Ω is determined. The standard basis is related to the "common" basis of Ω by means of formulae (93). We obtain explicit representation (99) for the *variations* Z with two *arbitrary functions* $z = Zx$ and $p = \pi_0(Z)$ as the final result. Variations Z generating the true group (i.e., the *infinitesimal symmetries* Z of Ω) satisfy certain strong conditions discovered in points ($\iota\nu$) and (ν). The conditions are expressed by the *resolving system* (107) and (108) or, alternatively, by (112)–(114) only in terms of the functions p, Dp, D^2p, and D^3p. This rather complicated resolving system which does not provide any clear insight is equivalent to much simpler *crucial requirements* (121) or (125) on the actual structure of function p; see the central points ($\nu\iota$)–($\nu\iota\iota$). Then the subsequent points are devoted to the explicit solution of these equations (125). This is a mere technical task of traditional mathematical analysis and we omit comments at this place.

(ι) The diffiety. Let us introduce space **M** equipped with coordinates x, u_0, u_1, v_r ($r = 0, 1, \ldots$). Then

$$\begin{gathered} u_{r+2} = D^r F\left(v_1\right) \\ \left(r = 0, 1, \ldots; D = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_0} + F\frac{\partial}{\partial u_1} + \sum v_{r+1}\frac{\partial}{\partial v_r}\right) \end{gathered} \tag{82}$$

are merely composed functions. The forms

$$\begin{gathered} \alpha_0 = du_0 - u_1 dx, \qquad \alpha_1 = du_1 - F dx, \\ \beta_r = dv_r - v_{r+1}dx \quad (r = 0, 1, \ldots) \end{gathered} \tag{83}$$

provide a basis of the diffiety Ω; however, all forms $\alpha_r = du_r - u_{r+1}dx$ ($r = 2, 3, \ldots$) are also lying in Ω as follows from the obvious rule:

$$\mathscr{L}_D\alpha_r = \alpha_{r+1}, \quad \mathscr{L}_D\beta_r = \beta_{r+1} \quad (r = 0, 1, \ldots) \tag{84}$$

and the inclusion $\mathscr{L}_D\Omega \subset \Omega$.

($\iota\iota$) Standard Filtration. There exists the "natural" filtration Ω_* of diffiety Ω with respect to the order: submodule $\Omega_l \subset \Omega$ involves the forms α_r, β_r with $r \le l$. Alternatively saying, α_0, β_0 is a basis of Ω_0 and

$$\alpha_0, \beta_0, \alpha_1, \beta_1, \beta_2, \ldots, \beta_l \text{ is a basis of } \Omega_l \quad (l \ge 1). \tag{85}$$

Clearly $\operatorname{Ker}\Omega_{l+1} = \Omega_l$ if $l \ge 1$ as follows from (84). However,

$$\begin{aligned} \mathscr{L}_D\alpha_1 &= \mathscr{L}_D\left(du_1 - F dx\right) = dF - DF dx \\ &= F'\left(dv_1 - v_2 dx\right) = F'\beta_1 \in \Omega_1. \end{aligned} \tag{86}$$

(Figure 4(a)) therefore

$$\begin{aligned} \mathscr{L}_D\left(\alpha_1 - F'\beta_0\right) &= F'\beta_1 - DF'\beta_0 - F'\mathscr{L}_D\beta_0 \\ &= -DF'\beta_0 \in \Omega_0. \end{aligned} \tag{87}$$

Then $\alpha_0, \alpha = \alpha_1 - F'\beta_1, \beta_0$ may be taken for a basis of module $\operatorname{Ker}\Omega_1$ (Figure 4(b)).

Moreover

$$\begin{gathered} \mathscr{L}_D\alpha_0 = \alpha_1 = \alpha + F'\beta_0, \\ \mathscr{L}_D\alpha = -DF'\beta_0 \in \operatorname{Ker}\Omega_1, \end{gathered} \tag{88}$$

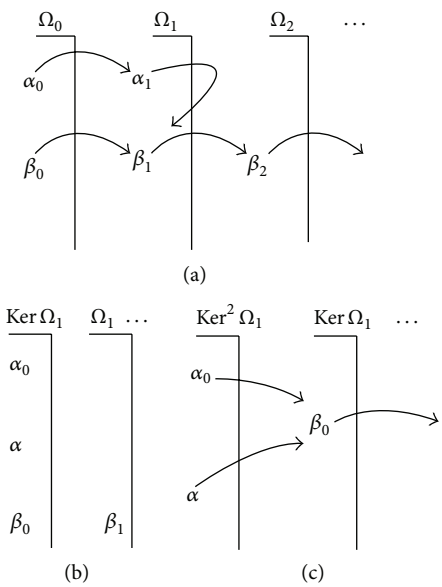


Figure 4

hence α_0, α constitute a basis of module $\mathrm{Ker}^2\Omega_1$ (Figure 4(c)) and finally

$$\begin{aligned}
&\mathscr{L}_D\left(F'\alpha + DF'\alpha\right) \\
&\quad = DF'\alpha + D^2F'\alpha_0 + F'\mathscr{L}_D\alpha + DF'\mathscr{L}_D\alpha_0 \\
&\quad = 2DF'\alpha + D^2F'\alpha_0 \in \mathrm{Ker}^2\Omega_1.
\end{aligned} \tag{89}$$

Therefore assuming

$$DF' = F''v_3 \neq 0 \quad \left(\text{hence } F'' \neq 0\right) \tag{90}$$

from now on, the form $\pi_0 = F'\alpha + DF'\alpha_0$ may be taken for a basis of module $\mathrm{Ker}^3\Omega_0$. We have obtained the standard filtration

$$\begin{aligned}
\overline{\Omega}_* : \overline{\Omega}_0 &= \mathrm{Ker}^3\Omega_1 \subset \overline{\Omega}_1 = \mathrm{Ker}^2\Omega_1 \subset \overline{\Omega}_2 \\
&= \mathrm{Ker}\,\Omega_1 \subset \overline{\Omega}_3 = \Omega_1 \subset \cdots \quad (\mathscr{R}(\Omega) = 0),
\end{aligned} \tag{91}$$

where forms

$$\begin{aligned}
&\pi_r = \mathscr{L}_D^r\pi_0 \\
&\left(r = 0, \ldots, l;\right. \\
&\left.\pi_0 = F'\alpha + DF'\alpha_0 = DF'\alpha_0 + F'\alpha_1 - \left(F'\right)^2\beta_0\right)
\end{aligned} \tag{92}$$

provide a basis of module $\overline{\Omega}_l$.

Abbreviating $f = F'$ from now on, explicit formulae

$$\begin{aligned}
\pi_0 &= f\alpha + Df\alpha_0, \\
\pi_1 &= 2Df\alpha + D^2f\alpha_0, \\
\pi_2 &= 3D^2f\alpha + D^3f\alpha_0 + C\beta_0, \\
C\alpha &= D^2f\pi_0 - Df\pi_1, \\
C\alpha_0 &= -2Df\pi_0 + f\pi_1, \\
C^2\beta_0 &= A\pi_0 + B\pi_1 + C\pi_2,
\end{aligned} \tag{93}$$

where $\alpha = \alpha_1 - f\beta_0$ and

$$\begin{aligned}
A &= 2Df \cdot D^3f - 3\left(D^2f\right)^2, \\
B &= 3Df \cdot D^2f - fD^3f, \\
C &= fD^2f - 2\left(Df\right)^2
\end{aligned} \tag{94}$$

can be easily found. They will be sufficient in calculations to follow. Recall that we suppose that the inequality (90) hold true, hence $C = \cdots + ff'v_3 = \cdots + F'F''v_3 \neq 0$.

(ιι) Variations. We deal with vector fields

$$Z = z\frac{\partial}{\partial x} + z_0^1\frac{\partial}{\partial u_0} + z_1^1\frac{\partial}{\partial u_1} + \sum z_r^2\frac{\partial}{\partial v_r} \tag{95}$$

(the notation (75) with indices is retained) on the space **M**. Recall that Z is a variation if $\mathscr{L}_Z\Omega \subset \Omega$. In terms of coordinates, the conditions are

$$\begin{aligned}
z_1^1 &= Dz_0^1 - u_1Dz, \\
fz_0^1 &= Dz_1^1 - FDz, \\
z_{r+1}^2 &= Dz_r^2 - v_{r+1}Dz \quad (r = 0, 1, \ldots),
\end{aligned} \tag{96}$$

where the first and third equations are merely recurrences while the middle equation causes serious difficulties (a classical result. Hint: use $\mathscr{L}_Z\alpha_0 \in \Omega$, $\mathscr{L}_Z\alpha_1 \in \Omega$, $\mathscr{L}_Z\beta_r \in \Omega$). By using the alternative formula

$$Z = z\frac{\partial}{\partial dx} + a_0\frac{\partial}{\partial \alpha_0} + a_1\frac{\partial}{\partial \alpha_1} + \sum b_r\frac{\partial}{\partial \beta_r}, \tag{97}$$

the conditions slightly simplify

$$\begin{aligned}
&a_1 = Da_0, \qquad Da_1 = fb_1, \\
&b_{r+1} = Db_r \quad (r = 0, 1, \ldots).
\end{aligned} \tag{98}$$

(Hint: apply the rule $\mathscr{L}_Z\varphi = Z\rfloor d\varphi + d\varphi(Z)$ to the forms $\varphi = \alpha_0, \alpha_1, \beta_r$.) However, by virtue of Lemma 23 and standard filtration, we have explicit formula

$$Z = z\frac{\partial}{\partial dx} + \sum D^r p\frac{\partial}{\partial \pi_r} \quad \left(z = Zx, p = \pi_0(Z)\right) \tag{99}$$

for the variations where z and p are arbitrary functions. One can then easily obtain explicit formulae for all coefficients a_0, a_1, b_r in (97) and z_0^1, z^1, z_r^2 in (95) by using the left-hand identities (93). They need not be stated here.

(ιν) Infinitesimal Transformations. We refer to Remark 27: *variation* Z is *infinitesimal symmetry if and only if*

$$\mathscr{L}_Z \pi_0 = Z \rfloor d\pi_0 + dp = \lambda \pi_0 \tag{100}$$

for appropriate multiplier $\lambda \in \mathscr{F}(\mathbf{M})$. In explicit terms, we recall formula

$$\pi_0 = f\alpha + Df\alpha_0 = f\alpha + f' v_2 \alpha_0, \tag{101}$$

where

$$\begin{gathered} d\pi_0 = dx \wedge \pi_1 \quad (\operatorname{mod} \Omega \wedge \Omega), \\ d\alpha = d(\alpha_1 - f\beta_0) \cong -df \wedge \beta_0 = -f'\beta_1 \wedge \beta_0 \\ (\operatorname{mod} dx), \end{gathered} \tag{102}$$

and therefore clearly

$$\begin{aligned} d\pi_0 &= dx \wedge \pi_1 + \left(f'\beta_1 \wedge \alpha - ff'\beta_1 \wedge \beta_0\right) \\ &\quad + \left(f'' v_2 \beta_1 + f'\beta_2\right) \wedge \alpha_0 \\ &= dx \wedge \pi_1 + \beta_1 \wedge \left(f'\alpha - ff'\beta_0 + f'' v_2 \alpha_0\right) \\ &\quad + f'\beta_2 \wedge \alpha_0. \end{aligned} \tag{103}$$

So denoting

$$\begin{gathered} z = Zx = dx(Z), \qquad a_0 = \alpha_0(Z), \qquad a = \alpha(Z), \\ b_r = \beta_r(Z), \qquad p = \pi_0(Z) \quad (r = 0, 1, \ldots), \end{gathered} \tag{104}$$

requirement (100) reads

$$\begin{aligned} &z\pi_1 + b_1\left(f'\alpha - ff'\alpha_0 + f'' v_2 \beta_0\right) + f' b_2 \alpha_0 \\ &\quad - \left(f'a - ff'a_0 + f'' v_2 b_0\right)\beta_1 - f' a_0 \beta_2 + dp \\ &\quad = \lambda\left(f\alpha + Df\alpha_0\right), \end{aligned} \tag{105}$$

where $\pi_1 = 2Df\alpha + D^2 f\alpha_0$ and

$$\begin{aligned} dp &\cong p_{u_0}\alpha_0 + p_{u_1}\alpha_1 + \sum p_{v_r}\beta_r \\ &= p_{u_0}\alpha_0 + p_{u_1}\alpha + \left(fp_{u_1} + p_{v_0}\right)\beta_0 + \sum_{r>0} p_{v_r}\beta_r \end{aligned} \tag{106}$$

$(\operatorname{mod} dx)$ should be moreover inserted. It follows that requirement (100) is equivalent to the so-called *resolving system*

$$\begin{gathered} 2zDf + f' b_1 + p_{u_1} = \lambda f, \\ zD^2 f + f'' v_2 b_1 + p_{u_0} = \lambda Df, \end{gathered} \tag{107}$$

$$\begin{gathered} ff' b_1 = fp_{u_1} + p_{v_0}, \\ f'a - ff' b_0 + f'' v_2 a_0 = p_{v_1}, \\ f' a_0 = p_{v_2}. \end{gathered} \tag{108}$$

Moreover $p_{v_r} = 0$ $(r \geq 2)$ and therefore $p = p(u_0, u_1, v_0, v_1, v_2)$ is of the order 2 at most.

(ν) On the Resolving System. Equations (107) uniquely determine the multiplier λ and the "horizontal" coefficient $z = Zx$ in terms of the "vertical" coefficients a_0, a, b_r, and p. For instance the formula

$$z = \frac{1}{C}\left(\left(f' b_1 + p_{u_1}\right) Df - \left(f'' v_2 b_1 + p_{u_0}\right) f\right) \tag{109}$$

easily follows. So we may focus on (108).

Equations (108) deserve more effort. They depend only on "vertical" components and can be expressed in terms of functions p, Dp, $D^2 p$, and $D^3 p$ if the obvious identities

$$\begin{gathered} p = af + a_0 Df, \\ Dp = 2aDf + a_0 D^2 f, \\ D^2 p = 3aD^2 f + a_0 D^3 f + b_0 C, \\ Ca = D^2 f \cdot p - Df \cdot Dp, \\ Ca_0 = -2Df \cdot p + fDp, \\ C^2 b_0 = Ap + BDp + CD^2 p, \end{gathered} \tag{110}$$

following from (93) together with the prolongation formula

$$C^2 b_1 + 2b_0 CDC = D\left(C^2 b_0\right) = D\left(Ap + BDp + CD^2 p\right) \tag{111}$$

are applied. By using the lucky identity $DC = -B$ (direct verification), one can obtain the *alternative resolving system*

$$\begin{aligned} &ff'\left(C\left(DA \cdot p + (A + DB)Dp + CD^3 p\right)\right. \\ &\quad \left. + 2B\left(Ap + BDp + CD^2 p\right)\right) \\ &\quad = C^3\left(fp_{u_1} + p_{v_0}\right), \end{aligned} \tag{112}$$

$$\begin{aligned} &C\left(\left(f' D^2 f - 2v_2 f'' Df\right)p + \left(v_2 f'' f - f' Df\right)Dp\right) \\ &\quad - ff'\left(Ap + BDp + CD^2 p\right) \\ &\quad = C^2 p_{v_1}, \end{aligned} \tag{113}$$

$$-2f' Df \cdot p + ff' Dp = Cp_{v_2} \tag{114}$$

only in terms of the unknown function p. Recall that *the resolving system is satisfied if and only if the vector field* (99) *is infinitesimal symmetry*.

Our aim is to determine the function p satisfying (112)–(114). Alas, the *resolving system* does not provide any insight into the true structure of function p. It will be therefore

replaced by other conditions of classical nature, the *crucial requirements* and the *simplified requirements* as follows.

(vi) Crucial Requirements. We start with simple formulae

$$\begin{gathered} Df = f'v_2, \qquad D^2 f = f''v_2^2 + f'v_3, \\ D^3 f = f'''v_2^3 + 3f''v_2v_3 + f'v_4, \\ Dp = \cdots + p_{v_2}v_3, \qquad D^2 p = \cdots + p_{v_2v_2}v_3^2 + p_{v_2}v_4 \\ (\text{the top-order terms}). \end{gathered} \tag{115}$$

Using moreover (94), one can see that there is a unique summand in (113) which involves the factor v_3^3, namely the summand

$$-ff' \cdot C \cdot D^2 p \cong -ff' \cdot ff'v_3 \cdot p_{v_2v_2}v_3^2. \tag{116}$$

It follows that $p_{v_2v_2} = 0$ identically and we (temporarily) may denote

$$p = M(x, u_0, u_1, v_0, v_1) + N(x, u_0, u_1, v_0, v_1)\, v_2. \tag{117}$$

The simplest equation (114) of the resolving system then reads

$$\begin{aligned} &-2f' \cdot f'v_2 \cdot (M + Nv_2) + ff'D(M + Nv_2) \\ &\quad = \left(\left(ff'' - 2f'^2\right) {v_2}^2 + ff'v_3\right) N. \end{aligned} \tag{118}$$

Clearly

$$D(M + Nv_2) = \mathscr{D}M + \left(M_{v_1} + \mathscr{D}N\right) v_2 + N_{v_1}v_2^2 + Nv_3, \tag{119}$$

where the reduced operator

$$\mathscr{D} = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_0} + F\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_0} \tag{120}$$

appears and we obtain three so-called *crucial requirements*

$$\mathscr{D}M = 0, \quad 2Mf' = \left(M_{v_1} + \mathscr{D}N\right) f, \quad N_{v_1}f' = Nf'' \tag{121}$$

for the functions M, N by inspection of the variable v_2. Altogether taken, *the last resolving equation* (114) *is equivalent to three requirements* (121). *We will see with great pleasure in (viii)* below that requirements (121) ensure even the remaining equations (112) and (113) of the resolving system.

(vii) The Crucial Requirements Simplified. The right-hand equation (121) reads

$$Q_{v_1} = 0 \quad \left(Q = \frac{N}{f'}\right) \tag{122}$$

and the middle equation (121) reads

$$f^2 P_{v_1} + f'\mathscr{D}Q = 0 \quad \left(P = \frac{M}{f^2}\right), \tag{123}$$

whence altogether

$$p = Pf^2 + Qf'v_2 = Pf^2 + QDf. \tag{124}$$

The left-hand equation (121) does not change much; it may be expressed by $\mathscr{D}P = 0$.

Let us summarize our achievements. *In order to determine function p given by* (124), *we have three simplified requirements*

$$\begin{gathered} \mathscr{D}P = 0, \\ f^2 P_{v_1} + f'\mathscr{D}Q = 0, \qquad Q_{v_1} = 0 \end{gathered} \tag{125}$$

for the coefficients $P = P(x, u_0, u_1, v_0, v_1)$ and $Q = Q(x, u_0, u_1, v_0, v_1)$.

(viii) Resolving System is Deleted. Let us recall the primary transcription (108) of the resolving system. We have already seen that (125) implies (114) and hence the equivalent and simplest right-hand equation (108).

Let us turn to the middle equation (108) equivalent to (113). One can directly find formulae

$$\begin{aligned} Dp &= DP \cdot f^2 + PD\left(f^2\right) + DQ \cdot Df + QD^2 f \\ &= PD\left(f^2\right) + QD^2 f \end{aligned} \tag{126}$$

by using (124) and (125). Moreover

$$\begin{gathered} a = Pf, \qquad a_0 = Q, \qquad a_1 = Da_0 = \mathscr{D}Q, \\ b_0 = \frac{a_1 - a}{f} = \frac{\mathscr{D}Q}{f} - P \end{gathered} \tag{127}$$

by using (124) and right-hand formulae (110). Substitution into middle equation (108) with

$$p_{v_1} = \frac{\partial}{\partial v_1}\left(Pf^2 + Qf'v_2\right) = P_{v_1}f' + 2Pff' + Qf''v_2 \tag{128}$$

gives the identity.

As the right-hand equation (108) equivalent to (112) is concerned, we may use

$$\begin{gathered} b_1 = Db_0 = \frac{\mathscr{D}^2 Q}{f} + \frac{\partial}{\partial v_1}\left(\frac{\mathscr{D}Q}{f}\right) v_2 - P_{v_1}v_2, \\ \mathscr{D}^2 Q = -\mathscr{D}\left(\frac{f^2}{f'}P_{v_1}\right), \end{gathered} \tag{129}$$

where

$$\begin{gathered} (\mathscr{D}P)_{v_1} = \mathscr{D}\left(P_{v_1}\right) + P_{u_1}f + P_{v_0} = 0, \\ (\mathscr{D}Q)_{v_1} = Q_{u_1}f + Q_{v_0}. \end{gathered} \tag{130}$$

Moreover

$$fp_{u_1} + p_{v_0} = f\left(P_{u_1}f^2 + Q_{u_1}f'v_2\right) + P_{v_0}f^2 + Q_{v_0}f'v_2 \tag{131}$$

and (108) again becomes the identity.

(ix) Back to the Crucial Requirements. Passing to the final part of this example, let us eventually solve (125) with the

unknown functions P, Q and given function f. This is already a task of classical mathematical analysis. We abbreviate $v = v_1$ from now on since this variable v frequently occurs in our formulae.

Let us begin with middle equation (125) which reads

$$P_v = \left(\frac{1}{f}\right)' \cdot \left(q + FQ_{u_1} + vQ_{v_0}\right) \quad \left(q = Q_x + u_1 Q_{u_0}\right) \tag{132}$$

whence

$$\begin{aligned} P = {} & \frac{1}{f}q + \int \left(\frac{1}{f}\right)' F dv \cdot Q_{u_1} + \int \left(\frac{1}{f}\right)' v dv \cdot Q_{v_0} \\ & + \overline{P} \quad \left(\overline{P} = \overline{P}\left(x, u_0, u_1, v_0\right)\right) \end{aligned} \tag{133}$$

since Q is independent of variable v due to the right-hand equation (125). We may insert

$$\begin{gathered} \int \left(\frac{1}{f}\right)' F dv = \frac{F}{f} - v, \\ \int \left(\frac{1}{f}\right)' v dv = \frac{v}{f} - \int_{\overline{v}}^{v} \frac{dv}{f} \quad (\text{fixed } \overline{v} \in \mathbb{R}) \end{gathered} \tag{134}$$

and the remaining left-hand equation (125) is expressed by the identity

$$\begin{aligned} & 1 \cdot \left(\overline{P}_x + u_1 \overline{P}_{u_0}\right) + F \cdot \overline{P}_{u_1} + v \cdot \overline{P}_{v_0} \\ & \quad + \frac{1}{f} \cdot \left(q_x + u_1 q_{u_0} + F \cdot q_{u_1} + v \cdot q_{v_0}\right) + \left(\frac{F}{f} - v\right) \\ & \quad \cdot \left(Q_{u_1 x} + u_1 Q_{u_1 u_0} + F \cdot Q_{u_1 u_1} + v \cdot Q_{u_1 v_0}\right) \\ & \quad + \left(\frac{v}{f} - \int \frac{dv}{f}\right) \cdot \left(q_{v_0} + F \cdot Q_{v_0 u_1} + v \cdot Q_{v_0 v_0}\right) \\ & = 0. \end{aligned} \tag{135}$$

Functions $\overline{P}, q, Q$ are independent of v and thereby subjected to very strong conditions by the inspection of the coefficients of functions

$$\begin{gathered} 1, F, v, \frac{1}{f}, \frac{1}{f}F, \frac{1}{f}v, \left(\frac{F}{f} - v\right)F, \left(\frac{F}{f} - v\right)v, \\ \frac{v}{f} - \int \frac{dv}{f}, \left(\frac{v}{f} - \int \frac{dv}{f}\right)F, \left(\frac{v}{f} - \int \frac{dv}{f}\right)v \end{gathered} \tag{136}$$

in identity (135). The final result depends on the properties of function F and we mention only a few instructive subcases here.

(κ) The Generic Subcase. Functions (136) are in general linearly independent over $\mathbb{R}$ and identity (135) implies

$$\overline{P}_x + u_1 \overline{P}_{u_0} = \overline{P}_{u_1} = \overline{P}_{v_0} - Q_{u_1 x} - u_1 Q_{u_1 u_0} = 0, \tag{137}$$

$$\left(q_x + u_1 q_{u_0}\right) = Q_{xx} + 2u_1 Q_{x u_0} + u_1^2 Q_{u_0 u_0} = 0, \tag{138}$$

$$\begin{aligned} & \left(q_{u_1} + Q_{u_1 x} + u_1 Q_{u_1 u_0}\right) = Q_{u_0} \\ & \quad + 2\left(Q_{u_1 x} + u_1 Q_{u_1 u_0}\right) = 0, \end{aligned} \tag{139}$$

$$\begin{aligned} & \left(q_{v_0}\right) = Q_{x v_0} + u_1 Q_{u_0 v_0} = Q_{u_1 u_1} \\ & \quad = Q_{u_1 v_0} = Q_{v_0 v_0} = 0. \end{aligned} \tag{140}$$

The unknown functions $\overline{P}$ and Q can be easily found as follows. We may suppose that

$$\begin{gathered} Q = a\left(x, u_0\right) u_1 + b\left(x, u_0\right) v_0 + c\left(x, u_0\right), \\ b\left(x, u_0\right) = B \in \mathbb{R}, \end{gathered} \tag{141}$$

by using (140). Then $\overline{P}_x = \overline{P}_{u_0} = \overline{P}_{u_1} = 0$; hence $\overline{P} = \overline{P}(v_0)$ due to (139). Moreover $\overline{P}' = a_x + u_1 a_0$ which implies $\overline{P}' = a_x \in \mathbb{R}$, $a_0 = 0$; hence $a = a(x)$ and altogether

$$a = Ax + \overline{A}, \qquad \overline{P} = Av_0 + C \quad \left(A, \overline{A}, C \in \mathbb{R}\right). \tag{142}$$

Then

$$c = C_1 x + C_2 u_0 + C_3 \quad \left(C_1, C_2, C_3 \in \mathbb{R}\right) \tag{143}$$

follows from (138). Hence, $C_2 + 2A = 0$ due to (139) and altogether

$$\begin{gathered} Q = \left(Ax + \overline{A}\right) u_1 + Bv_0 + C_1 x - 2Au_0 + C_3, \\ \overline{P} = Av_0 + C. \end{gathered} \tag{144}$$

Recalling moreover (133), *we have explicit formulae for the solutions P, Q of crucial requirements* (125) *and the symmetry problem is resolved*. While $\overline{P}$ and Q are mere polynomials, the total coefficient P given by (133) depends on the quadrature $\int (dv/f)$ and *this may be globally rather complicated function*. It follows that, in our approach, the elementary and the "transcendental" parts of the solution are in a certain sense separated.

(κι) A Special Case of Function F. Let us choose $F(v) = e^v$. Then series (136) becomes quite explicit; namely,

$$\begin{gathered} 1, e^v, v, e^{-v}, 1, ve^{-v}, (1 - v) e^v, \\ (1 - v) v, v + 1, (v + 1) e^v, (v + 1) ve^v \end{gathered} \tag{145}$$

and these functions are linearly dependent. Identity (135) implies smaller number of requirements; the first term in (137) is combined with (139) into the single equation

$$\overline{P}_x + u_1 \overline{P}_{u_0} + q_{u_1} + Q_{u_1 x} + u_1 Q_{u_1 u_0} = 0 \tag{146}$$

without any other change. We can state the final solution

$$Q = \left(Ax + \overline{A}\right) u_1 + Bv_0 + C_1 x + C_2 u_0 + C_3, \qquad \overline{P} = Av_0 + \left(C_2 + 2A\right) x + C \tag{147}$$

with only one additional parameter $C_2 \in \mathbb{R}$ if compared to the previous formulae (144).

(κιι) Another Special Case. Let us eventually mention the very prominent function $F(v) = v^{1/2}$; see [1, 7, 16]. Then the series

$$1, v^{1/2}, v, 2v^{1/2}, 2v, 2v^{3/2}, v^{3/2}, v^2, \frac{2}{3}v^{3/2}, \frac{2}{3}v^2, \frac{2}{3}v^{5/2} \tag{148}$$

stands for (136) and the relevant identity (135) implies the system of equations

$$\overline{P}_x + u_1 \overline{P}_{u_0} = 0, \tag{149}$$

$$\begin{aligned} &\overline{P}_{u_1} + 2\left(q_x + u_1 q_{u_0}\right) \\ &\quad = \overline{P}_{u_1} + 2\left(Q_{xx} + 2u_1 Q_{xu_0} + u_1^2 Q_{u_0 u_0}\right) = 0, \end{aligned} \tag{150}$$

$$\begin{aligned} &\overline{P}_{v_0} + 2q_{u_1} + Q_{u_1 x} + u_1 Q_{u_1 u_0} \\ &\quad = \overline{P}_{v_0} + 2Q_{u_0} + 3\left(Q_{u_1 x} + u_1 Q_{u_1 u_0}\right) = 0, \end{aligned} \tag{151}$$

$$\begin{aligned} &6q_{v_0} + 3Q_{u_1 u_1} + 2\left(Q_{v_0 x} + u_1 Q_{v_0 u_0}\right) \\ &\quad = 3Q_{u_1 u_1} + 8\left(Q_{v_0 x} + u_1 Q_{v_0 u_0}\right) = 0, \end{aligned} \tag{152}$$

$$Q_{u_1 v_0} = 0, \quad Q_{v_0 v_0} = 0. \tag{153}$$

We are passing to the solution of the system of (149)–(153) with unknown functions $Q = Q(x, u_0, u_1, v_0)$ and $\overline{P} = \overline{P}(x, u_0, u_1, v_0)$. Due to (153), we may put

$$Q = a\left(x, u_0\right) v_0 + b\left(x, u_0, u_1\right) \tag{154}$$

and then (152) is expressed by $3b_{u_1 u_1} + 8(a_x + u_1 a_{u_0}) = 0$, whence easily

$$b = -\frac{4}{9} a_{u_0} u_1^3 - \frac{4}{3} a_x u_1^2 + \overline{b}\left(x, u_0\right) u_1 + \widetilde{b}\left(x, u_0\right). \tag{155}$$

Moreover (151) reads $\overline{P}_{v_0} + 2(a_{u_0} v_0 + b_{u_0}) + 3(b_{u_1 x} + u_1 b_{u_1 u_0}) = 0$, whence

$$\begin{aligned} \overline{P} = &-a_{u_0} v_0^2 - \left(2b_{u_0} + 3\left(b_{u_1 x} + u_1 b_{u_1 u_0}\right)\right) v_0 \\ &+ \overline{p}\left(x, u_0, u_1\right). \end{aligned} \tag{156}$$

Remaining equations (149) and (150) do not admit such simple discussion. Using (154) and (155), identity (149) is equivalent to the system

$$a_{u_0 x} + u_1 a_{u_0 u_0} = 0 \qquad \left(\text{whence } a = Au_0 + \overline{a}\left(x\right), A \in \mathbb{R}\right), \tag{157}$$

$$\begin{aligned} &2b_{u_0 x} + 3\left(b_{u_1 xx} + u_1 b_{u_1 u_0 x}\right) \\ &\quad + u_1\left(2b_{u_0 u_0} + 3\left(b_{u_1 x u_0} + u_1 b_{u_1 u_0 u_0}\right)\right) = 0, \end{aligned} \tag{158}$$

$$\overline{p}_x + u_1 \overline{p}_{u_0} = 0 \tag{159}$$

of three equations and identity (150) is equivalent to the system

$$2b_{u_0 u_1} + 3\left(b_{u_1 x u_1} + b_{u_1 u_0} + u_1 b_{u_1 u_0 u_1}\right) = 2\overline{a}'', \tag{160}$$

$$\overline{p}_{u_1} + 2\left(b_{xx} + 2u_1 b_{x u_0} + u_1^2 b_{u_0 u_0}\right) = 0, \tag{161}$$

if (157) is moreover employed. At the same time, (155) can be improved as

$$b = -\frac{4}{9} A u_1^3 - \frac{4}{3} \overline{a}' u_1^2 + \overline{b}\left(x, u_0\right) u_1 + \widetilde{b}\left(x, u_0\right). \tag{162}$$

With this improvement, (160) reads $2\overline{b}_{u_0} + 3(-(8/4)\overline{a}'' + \overline{b}_{u_0}) = 2\overline{a}''$ and it follows that

$$\overline{b} = 2\overline{a}'' u_0 + \widehat{b}\left(x\right). \tag{163}$$

Analogously (158) reads

$$\begin{aligned} &2\left(\overline{b}_{u_0 x} u_1 + \widetilde{b}_{u_0 x}\right) + 3\left(-\frac{8}{3}\overline{a}''' u_1 + \overline{b}_{xx} + u_1 \overline{b}_{u_0 x}\right) \\ &\quad + u_1\left(2\widetilde{b}_{u_0 u_0} + 3\overline{b}_{u_0 x}\right) = 0, \end{aligned} \tag{164}$$

which is equivalent to the system

$$2\widetilde{b}_{u_0 x} + 6\overline{a}^{(4)} u_0 + 3\widehat{b}'' = 0, \qquad 4\overline{a}''' + \widetilde{b}_{u_0 u_0} = 0, \tag{165}$$

if (163) is inserted. Altogether, it follows that (158) is equivalent to

$$2\widetilde{b} = -3\overline{a}''' u_0^2 - 3\widehat{b}' u_0 + \widetilde{b}_0\left(u_0\right) + \widetilde{b}_1\left(x\right), \qquad \overline{a}''' = -\frac{1}{2}\widetilde{b}_0'' = A_3 \in \mathbb{R}, \tag{166}$$

whence

$$\begin{aligned} &\overline{a} = A_3 \frac{x^3}{6} + A_2 x^2 + A_1 x + A_0, \\ &\widetilde{b}_0 = -A_3 u_0^2 + B_1 u_0 + B_0 \\ &\qquad \left(A_2, A_1, A_0, B_1, B_0 \in \mathbb{R}\right). \end{aligned} \tag{167}$$

At the same time, we have improvements

$$\begin{aligned} &\overline{b} = 2\left(A_3 + 2A_2\right) u_0 + \widehat{b}, \\ &2\widetilde{b} = -3A_3 u_0^2 - 3\widehat{b}' u_0 + \widetilde{b}\left(u_0\right) + \widetilde{b}_1\left(x\right) \end{aligned} \tag{168}$$

of the above formulae. Let us eventually turn to the remaining equations (159) and (161). We begin with (161) which can be simplified to

$$\overline{p}_{u_1} + 2\frac{16}{3} A_3 u_1^2 + 2\widehat{b}'' u_1 - \left(3\widehat{b}''' u_0 + \widetilde{b}_1''\right) = 0, \tag{169}$$

whence

$$\begin{aligned} \overline{p} = &2\frac{16}{3} A_3 \frac{u_1^3}{3} + \widehat{b}'' u_1^2 - \left(3\widehat{b}''' u_0 + \widetilde{b}_1''\right) u_1 \\ &+ \check{p}\left(x, u_0\right). \end{aligned} \tag{170}$$

Then the last requirement (159) is easily simplified as

$$\widehat{b}'''u_1^2 - \left(3\widehat{b}^{(4)}u_0 + \widetilde{b}_1'''\right)u_1 + \check{p}_x + u_1\left(-3\widehat{b}'''u_1 + \check{p}_{u_0}\right) = 0 \tag{171}$$

and it follows that

$$\check{p}_x = 0, \qquad \widetilde{b}_1''' - 3\widehat{b}^{(4)} + \check{p}_{u_0} = 0, \qquad \widehat{b}''' = 0, \tag{172}$$

whence easily

$$\check{p} = \check{p}(u_0), \qquad \widetilde{b}_1''' = -\check{p}' = C_3 \in \mathbb{R}, \qquad \widehat{b} = D_2x^2 + D_1x + D_0, \tag{173}$$

$$\widetilde{b}_1 = C_3\frac{x^3}{6} + C_2x^2 + C_1x + C_0, \qquad \check{p} = -C_3u_0 + C \qquad (C_3, D_2, \ldots, C \in \mathbb{R}). \tag{174}$$

The solution is eventually done. It depends on the parameters

$$A, A_3, A_2, A_1, A_0, B_1, B_0, C_3, C_2, C_1, C_0, C, D_2, D_1, D_0 \in \mathbb{R} \tag{175}$$

in the total number of 15. This is seemingly in contradiction with [1, 7, 16] where 14-dimensional symmetry group (namely, the exceptional simple Lie group $\mathbb{G}_2$) was declared. However, our final symmetry in fact depends on the sum $B_0 + C_0$ as follows from (166), (167), and (174) and therefore no contradiction appears. We will not explicitly state the resulting symmetries Z for obvious reason here. Recall that they are given by (99) where z, p are clarified in (109) and (124). Coefficients appearing in (124) are clarified in (133), (156), and (170) and in (154), (157), (162), (168), (173), and (174).

It should be moreover noted that our approach is of the universal nature while the method of explicit calculations which provides the infinitesimal transformations in [7] *rests on a lucky accident*; see [7, Theorem 3.2, and the subsequent discussion].

Remark 29. Variations Z were easily found in (μ). Due to Theorem 26 and Remark 27, infinitesimal symmetries satisfy moreover $\mathscr{L}_Z\pi_0 = \lambda\pi_0$ or, alternatively saying, *they preserve the Pfaffian equation* $\pi_0 = 0$, and this property was just employed. We will now prove the converse without use of Theorem 24. The reasoning is as follows. *Let a variation Z preserve Pfaffian equation* $\pi_0 = 0$. Then Z preserves the space of *adjoint variables* x, u_0, u_1, v_0, v_1 of this Pfaffian equation. In this *finite-dimensional space*, the variation Z generates a group which can be prolonged to the higher-order jet variables. It follows that Z is indeed an *infinitesimal transformation*.

Remark 30. Let us briefly mention the case $F(v_1) = Av_1 + B$ $(A, B \in \mathbb{R}; A \neq 0)$ as yet excluded by condition (90). In this linear case, clearly

$$\begin{aligned} \mathscr{L}_D\alpha_1 &= \mathscr{L}_D\left(du_1 - (Av_1 + B)\,dx\right) \\ &= d(Av_1 + B) - Av_2dx = A\beta_1, \\ \mathscr{L}_D(\alpha_1 - A\beta_0) &= 0, \\ \tau = \alpha_1 - A\beta_0 &= d(u_1 - Av_0 - Bx) \in \mathscr{R}(\Omega) \end{aligned} \tag{176}$$

and we may introduce standard filtration

$$\begin{aligned} \mathscr{R}(\Omega) \subset \overline{\Omega}_0 &= \operatorname{Ker}^2\Omega_1 \subset \overline{\Omega}_1 \\ &= \operatorname{Ker}\Omega_1 \subset \overline{\Omega}_0 = \Omega_1 \subset \overline{\Omega}_1 = \Omega_2 \subset \cdots, \end{aligned} \tag{177}$$

where τ is a basis of $\mathscr{R}(\Omega)$ and the forms

$$\begin{aligned} \pi_0 = \alpha_0, \qquad \pi_1 = \alpha_1, \\ \pi_2 = \mathscr{L}_D\alpha_1 = A\beta_1, \ldots, \pi_l = A\beta_{l-1} \end{aligned} \tag{178}$$

provide a basis of module $\overline{\Omega}_l$ $(l \geq 1)$. The symmetries can be easily found. They are the prolonged contact transformations $\mathbf{m}$ defined by $\mathbf{m}^*\alpha_0 = \lambda\alpha_0$ depending moreover on the parameter $t = u_1 - Av_0 - Bx$. Roughly saying, the geometry of the linear second-order equation $u_2 = Av_1 + B$ is identical with the contact geometry of curves in $\mathbb{R}^2$. Quite analogous result can be obtained also for the Monge equation $F(x, u_0, u_1, v_0, v_1) = 0$ and, in much greater generality, for the system of two Pfaffian equations in four-dimensional space [17].

Remark 31. Let us once more return to the crucial requirement (125) where operators $\mathscr{D}$ and $\partial/\partial v_1$ are applied to unknown functions P and Q. We have employed the simplicity of the second operator $\partial/\partial v_1$ in the above solution; see formula (133). However, analogous "complementary" method can be applied to the first operator $\mathscr{D}$ as follows. Let us introduce new variables

$$\begin{aligned} \overline{x} = x, \qquad \overline{u}_0 = u_0 - u_1x + \frac{Fx^2}{2}, \\ \overline{u}_1 = u_1 - Fx, \qquad \overline{v}_0 = v_0 - v_1x, \qquad \overline{v} = v_1 \end{aligned} \tag{179}$$

with the obvious inverse transformation (not stated here). Then

$$\begin{aligned} \mathscr{D} &= \frac{\partial}{\partial\overline{x}}, \\ \frac{\partial}{\partial v_1} &= \frac{f}{2}\overline{x}^2\frac{\partial}{\partial\overline{u}_0} - \overline{x}\left(f\frac{\partial}{\partial\overline{u}_1} + \frac{\partial}{\partial\overline{v}_0}\right) + \frac{\partial}{\partial\overline{v}} \\ &\qquad (f = f(v_1) = f(\overline{v})) \end{aligned} \tag{180}$$

in terms of new variables. We again abbreviate $v = \overline{v} = v_1$. Passing to new coordinates, the left-hand requirement (125)

is simplified as $P = \overline{P}(\overline{u}_0, \overline{u}_1, \overline{v}_0, v)$. The middle requirement (125) reads

$$f^2\left(\frac{f}{2}\overline{x}^2\overline{P}_{\overline{u}_0} - \overline{x}\left(f\overline{P}_{\overline{u}_1} + \overline{P}_{\overline{v}_0}\right) + \overline{P}_v\right) + f'\overline{Q}_{\overline{x}} = 0 \qquad \left(Q = \overline{Q}\left(\overline{x}, \overline{u}_0, \overline{u}_1, \overline{v}_0, v\right)\right) \tag{181}$$

and determines the function $\overline{Q}$ in terms of new variables as

$$\overline{Q} = -\frac{f^2}{f'}\left(\frac{f}{6}\overline{x}^3\overline{P}_{\overline{u}_0} - \frac{\overline{x}^2}{2}\left(f\overline{P}_{\overline{u}_1} + \overline{P}_{\overline{v}_0}\right) + \overline{x}\overline{P}_v\right) + \overline{q}, \qquad \overline{q} = \overline{q}\left(\overline{u}_0, \overline{u}_1, \overline{v}_0, v\right), \tag{182}$$

where $\overline{q}$ is constant of integration. This is a *polynomial in variable* $\overline{x}$ and it follows easily that the remaining right-hand requirement (130) applied to function $\overline{Q}$ is equivalent to the system

$$\begin{gathered}\overline{P}_{\overline{u}_0\overline{u}_0} = 0, \qquad \mathscr{P}_{\overline{u}_0} = 0, \qquad \overline{q}_v = 0,\\ f\overline{q}_{\overline{u}_1} + \overline{q}_{\overline{v}_0} + \frac{\partial}{\partial v}\left(\frac{f^2}{f'}\overline{P}_v\right) = 0,\\ \frac{f^3}{f'}\overline{P}_{\overline{u}_0 v} + \frac{f^2}{f'}\left(f\mathscr{P}_{\overline{u}_1} + \mathscr{P}_{\overline{v}_0}\right) + \frac{1}{3}\frac{\partial}{\partial v}\left(\frac{f^3}{f'}\overline{P}_{\overline{u}_0}\right) = 0,\\ f\overline{q}_{\overline{u}_0} + 2\frac{f^2}{f'}\mathscr{P}_v + \frac{\partial}{\partial v}\left(\frac{f^2}{f'}\mathscr{P}\right) = 0 \qquad \left(\mathscr{P} = f\overline{P}_{\overline{u}_1} + \overline{P}_{\overline{v}_0}\right).\end{gathered} \tag{183}$$

We will not discuss this alternative approach here in more detail.

Remark 32. Though the symmetries of (77) can be *completely determined* by applying the *common methods*, several formally quite different ways of the calculation are possible. It would certainly be of practical interest which of them is the "most economical" one. Let us mention such an alternative way for better clarity. We start with the "opposite" transcription

$$\frac{dv}{dx} = G\left(\frac{d^2u}{dx^2}\right) \quad \left(G = F^{-1}, \text{the inverse function}\right) \tag{184}$$

of (77). The primary concepts are retained, the same underlying space $\mathbf{M}$, diffiety Ω, and contact forms α_r, β_r ($r = 0, 1, \ldots$). However, we choose $x, u_0, u_1, \ldots, v_0$ for *new coordinates* on $\mathbf{M}$ from now on and the forms

$$\alpha_r = du_r - u_{r+1}dx \quad (r = 0, 1, \ldots), \qquad \beta_0 = dv_0 - G\left(u_2\right)dx \tag{185}$$

for *new basis* of Ω. We have moreover

$$D = \frac{\partial}{\partial x} + \sum u_{r+1}\frac{\partial}{\partial u_r} + G\frac{\partial}{\partial v_0} \tag{186}$$

in terms of new coordinates. The standard filtration is formally simplified. The forms

$$\pi_r = \mathscr{L}_D^r\pi_0 \qquad \left(r = 0, 1, \ldots; \pi_0 = \beta_0 - G'\alpha_1 + DG'\alpha_0\right) \tag{187}$$

may be taken for *new standard basis* if the inequality $D^2G' \neq 0$ is supposed. This follows from the obvious formulae

$$\pi_1 = \mathscr{L}_D\pi_0 = D^2G'\alpha_0, \qquad \pi_2 = \mathscr{L}_D\pi_1 = D^3G'\alpha_0 + D^2G'\alpha_1, \ldots, \tag{188}$$

simplifying the analogous left-hand side (93). Then, analogously to (95) and (99), we introduce the *variations*

$$Z = z\frac{\partial}{\partial x} + \sum z_r^1\frac{\partial}{\partial u_r} + z_0^2\frac{\partial}{\partial v_0} = z\frac{\partial}{\partial dx} + \sum D^r p\frac{\partial}{\partial \pi_r} \tag{189}$$

of diffiety Ω where $z = Zx$ and $p = \pi_0(Z)$ may be arbitrary functions. Recall that we have even *infinitesimal symmetry* of Ω if and only if the requirement (100) is satisfied. However clearly

$$d\pi_0 = dx \wedge \pi_1 + G''\alpha_1 \wedge \alpha_2 + \left(G'''u_3\alpha_2 + G''\alpha_3\right) \wedge \alpha_0 \tag{190}$$

and one can obtain the *resolving equations* as follows. First of all, we obtain equations

$$zD^2G' + G''u_3a_2 + G''a_3 + p_{u_0} = \lambda DG', \qquad p_{v_0} = \lambda \qquad \left(a_r = \alpha_r\left(Z\right)\right) \tag{191}$$

which determine coefficients z and λ analogously to (107). Moreover

$$\begin{aligned}&G''a_2 - p_{u_1} - G'p_{v_0}\\ &\quad = G''a_1 - G'''u_3a_0 + p_{u_2}\\ &\quad = G''a_0 - p_{u_3} = p_{u_r} = 0 \quad (r > 3)\end{aligned} \tag{192}$$

are conditions for the unknown function $p = p(x, u_0, \ldots, u_3, v_0)$ analogous to (108). The "vertical" coefficients a_r can be expressed in terms of functions p, Dp, D^2p and D^3p, by using the equation

$$D^2G' \cdot a_0 = D^2G' \cdot \alpha_0\left(Z\right) = \pi_1\left(Z\right) = D\pi_0\left(Z\right) = Dp \tag{193}$$

and the recurrence $a_{r+1} = Da_r$. As yet the calculations are much easier then for the above case of formulae (110); however, the resulting *resolving system* of three equations analogous to (112)–(114) is again complicated and will not

be explicitly stated here. Remarkable task appears when we investigate the corresponding *crucial requirements* and try to determine the structure of function p in terms of new coordinates. For instance, the "very prominent" and seemingly rather artificial case (κu) turns into the "simplest possible" and quite natural equation $dv/dx = (d^2u/dx^2)^2$ in new coordinates.

7. Brief Digression to the Calculus of Variations

The classical *Lagrange problem* of the calculus of variations deals with an underdetermined system of differential equations (better with a diffiety) together with a variational integral. We are interested in internal symmetries of this variational problem.

Let us start with a diffiety $\Omega \subset \Phi(\mathbf{M})$. We choose a standard filtration $\overline{\Omega}_*$ and the corresponding standard basis π_r^j ($j = 1, \ldots, \mu(\Omega); r = 0, 1, \ldots$). For better clarity, we suppose the controllable case $\mathscr{R}(\Omega) = 0$. Let $x \in \mathscr{F}(\mathbf{M})$ be an independent variable. Let us consider x-parametrized solutions $\mathbf{p}$ of diffiety Ω in the sense

$$\begin{gathered} \mathbf{p} : \mathbf{I} \longrightarrow \mathbf{M} \quad (\mathbf{I} \subset \mathbb{R}), \\ \mathbf{p}^* \omega = 0 \quad (\omega \in \Omega), \\ \mathbf{p}^* x = x \in \mathbf{I} \subset \mathbb{R}. \end{gathered} \tag{194}$$

Here $\mathbf{I} \subset \mathbb{R}$ is a closed interval $a \le x \le b$ with a little confusion: letter x denotes both a function on $\mathbf{M}$ and the common coordinate (that is, a point) in $\mathbb{R}$.

Definition 33. A vector field $V \in \mathscr{T}(\mathbf{M})$ is called a *variation* of solution $\mathbf{p}$ of diffiety Ω if $\mathbf{p}^* \mathscr{L}_V \omega = 0$ ($\omega \in \Omega$). This is a mere slight adaptation of the familiar classical concept.

Lemma 34. *A vector field $V \in \mathscr{T}(\mathbf{M})$ is a variation of $\mathbf{p}$ if and only if*

$$\begin{gathered} \mathbf{p}^* \pi_{r+1}^j (V) = \mathbf{p}^* D\pi_r^j (V) = \frac{d}{dx} \mathbf{p}^* \pi_r^j (V) \\ \left(j = 1, \ldots, \mu(\Omega); r = 0, 1, \ldots\right). \end{gathered} \tag{195}$$

Proof. A variation V satisfies $\mathbf{p}^* \mathscr{L}_V \pi_r^j = 0$, where

$$\begin{aligned} \mathbf{p}^* \mathscr{L}_V \pi_r^j &= \mathbf{p}^* \left(V \rfloor d\pi_r^j + d\pi_r^j (V) \right) \\ &= \mathbf{p}^* \left(-\pi_{r+1}^j (V) + D\pi_r^j (V) \right) dx \end{aligned} \tag{196}$$

by virtue of (55). □

Remark 35. It follows easily that a vector field $Z \in \mathscr{T}(\mathbf{M})$ is a variation of diffiety Ω in the sense of Definition 8 if and only if Z is a variation of every solution $\mathbf{p}$ of Ω; see Lemma 23. Conversely, if V is a variation of a solution $\mathbf{p}$ then there exist many variations Z of Ω such that $Z = V$ at every point of $\mathbf{p}$, and they are characterized by the identities $\mathbf{p}^* \pi_0^j(Z) = \mathbf{p}^* \pi_0^j(V)$ ($j = 1, \ldots, \mu(\Omega)$) along the curve $\mathbf{p}$; see formula (72). We conclude that the concepts "variation Z of Ω" and "variations V of $\mathbf{p}$" are closely related. Roughly saying, variations V of $\mathbf{p}$ are "restrictions" of variations Z of Ω to the curve $\mathbf{p}$.

Definition 36. A couple $\{\Omega, \varphi\}$ where $\Omega \subset \Phi(\mathbf{M})$ is a diffiety and $\varphi \in \Phi(\mathbf{M})$ is a differential form will be identified with a *variational problem* in the (common) sense that diffiety Ω represents the *differential constraints* to the *variational integral* $\int \varphi$. A solution $\mathbf{p}$ of Ω is called an *extremal* of this variational problem, if

$$\int \mathbf{p}^* \mathscr{L}_V \varphi = \int \mathbf{p}^* \; V \rfloor d\varphi = 0 \quad (\text{special variations } V) \tag{197}$$

for every variation V of $\mathbf{p}$ which is vanishing at the endpoints $\mathbf{p}(a), \mathbf{p}(b) \in \mathbf{M}$. This definition provides the common classical extremals; see Remark 43.

Remark 37. The phrase "variation V of $\mathbf{p}$" can be replaced with "variation V of Ω". The form φ can be replaced with arbitrary form $\varphi + \omega$ ($\omega \in \Omega$). The extremals do not change.

Theorem 38. *To every standard basis of Ω and given $\varphi \in \Phi(\mathbf{M})$ there exists unique form $\breve{\varphi} \in \Phi(\mathbf{M})$ such that*

$$\begin{gathered} \breve{\varphi} \cong \varphi \quad (\operatorname{mod} \Omega), \\ d\breve{\varphi} \cong 0 \quad \left(\operatorname{mod} \Omega \wedge \Omega \text{ and all initial forms } \pi_0^j\right). \end{gathered} \tag{198}$$

In accordance with (198) *we assume that*

$$d\breve{\varphi} \cong \sum e^j \pi_0^j \wedge dx \quad (\operatorname{mod} \Omega \wedge \Omega). \tag{199}$$

Then a solution $\mathbf{p}$ of Ω is extremal if and only if $\mathbf{p}^* e^j = 0$ ($j = 1, \ldots, \mu(\Omega)$) and therefore if and only if

$$\mathbf{p}^* \; Z \rfloor d\breve{\varphi} = 0 \quad (Z \in \mathscr{T}(\mathbf{M})) \tag{200}$$

for all vector fields $Z \in \mathscr{T}(\mathbf{M})$.

Proof (see [9]). For a given $\varphi \in \Phi(\mathbf{M})$, let us look at a top-order summand

$$\begin{gathered} d\varphi \cong \sum a_r^j \pi_r^j \wedge dx = \cdots + a_R^I \pi_R^I \wedge dx \\ (\operatorname{mod} \Omega \wedge \Omega). \end{gathered} \tag{201}$$

If $R > 0$, the summand can be deleted if the primary differential form φ is replaced with the new form $\varphi + a_R^I \pi_{R-1}^I$. The extremals do not change. The procedure is unique and terminates in form $\breve{\varphi}$ satisfying (198). Then (200) follows from the identity

$$\mathbf{p}^* \; Z \rfloor d\breve{\varphi} = \mathbf{p}^* \; Z \rfloor \sum e^j \pi_0^j \wedge dx = \mathbf{p}^* \sum e^j \pi_0^j (Z) \, dx, \tag{202}$$

where the functions $\pi_0^j(Z)$ may be quite arbitrary if Z is a variation, see Lemma 34. □

Definition 39. The differential form $\breve{\varphi}$ can be regarded for the internal *Poincaré-Cartan form* of our variational problem and equations $e^j = 0$ $(j = 1, \ldots, \mu(\Omega))$ for the *Euler-Lagrange system*.

We turn to the symmetries.

Definition 40. A symmetry $\mathbf{m}$ of diffiety Ω is called a *symmetry of variational problem* $\{\Omega, \varphi\}$, if $\mathbf{m}^*\varphi \cong \varphi \,(\mathrm{mod}\,\Omega)$. A variation (infinitesimal symmetry) Z of Ω is called a *variation* (*infinitesimal symmetry*, resp.) of *variational problem* $\{\Omega, \varphi\}$, if $\mathscr{L}_Z\varphi \in \Omega$. Let $V \in \mathscr{T}(\mathbf{M})$ be a variation of a solution $\mathbf{p}$ of diffiety Ω. Then V is called a *Jacobi vector field* of $\mathbf{p}$, if moreover $\mathbf{p}^*\mathscr{L}_V\varphi = 0$. Roughly saying, variations Z of variational problem $\{\Omega, \varphi\}$ are "universal" Jacobi vector fields for all solutions $\mathbf{p}$ of Ω. In classical theory, Jacobi vector fields are introduced only for the particular case when $\mathbf{p}$ is an extremal.

We will see in the following example that Poincaré-Cartan forms $\breve{\varphi}$ simplify the calculation of symmetries and variations. On this occasion, we also recall the following admirable result.

Theorem 41 (E. Noether). *If Z is a variation of variational problem $\{\Omega, \varphi\}$ and $\breve{\varphi}$ is a Poincaré-Cartan form then $\mathbf{p}^*\breve{\varphi}(Z) = const.$ for every extremal $\mathbf{p}$.*

Proof. We have $\mathscr{L}_Z\breve{\varphi} \in \Omega$, $\mathbf{p}^*\omega = 0$ $(\omega \in \Omega)$, and therefore

$$\begin{aligned} 0 &= \mathbf{p}^*\mathscr{L}_Z\breve{\varphi} = \mathbf{p}^*\left(Z\rfloor d\breve{\varphi} + d\breve{\varphi}(Z)\right) \\ &= \mathbf{p}^* d\breve{\varphi}(Z) = d\mathbf{p}^*\breve{\varphi}(Z) \end{aligned} \tag{203}$$

by virtue of (200). □

Remark 42. Many concepts of the classical calculus of variations lose the geometrical meaning if the higher-order symmetries are accepted; for example, this concerns the common concept of a nondegenerate variational problem and even the order of a variational integral. On the other hand, the most important concepts can be appropriately modified; for example, the Hilbert-Weierstrass extremality theory together with the Hamilton-Jacobi equations [18–21] since the Poincaré-Cartan forms $\breve{\varphi}$ make "absolute sense" along the extremals.

Remark 43. In the common classical calculus of variations, extremals $\mathbf{p}$ are defined by the property $\int \mathbf{p}^*\mathscr{L}_V\varphi = 0$, where variations V satisfy certain weak *boundary conditions* at the endpoints ("fixed ends" or transversality) in order to delete some "boundary effects" of the variational integral. Much stronger conditions appear in Definition 36. Therefore

$$\text{classical extremals} \subset \text{our extremals}. \tag{204}$$

However, φ can be replaced by the form $\breve{\varphi}$. Then

$$\int \mathbf{p}^*\mathscr{L}_V\breve{\varphi} = \int \mathbf{p}^*\, V\rfloor d\breve{\varphi} + \text{boundary term}. \tag{205}$$

For the above special variations V, the boundary term vanishes. If $\mathbf{p}$ is extremal in the sense of Definition 36, then (200) and Remark 15 may be applied and it follows that

$$\text{classical extremals} \supset \text{our extremals}. \tag{206}$$

In topical Griffiths' theory [22], extremals are defined by the property

$$\begin{aligned} &\mathbf{p}^*\, Z\rfloor d(\varphi + \omega) = 0 \\ &\left(\text{all } Z \in \mathscr{T}(\mathbf{M}), \text{appropriate } \omega \in \Omega \text{ depending on } \mathbf{p}\right) \end{aligned} \tag{207}$$

which is clearly equivalent to the condition

$$\begin{aligned} &\int \mathbf{p}^*\, Z\rfloor d(\varphi + \omega) = \int \mathbf{p}^*\mathscr{L}_Z(\varphi + \omega) = 0 \\ &\left(\text{special vector fields } Z, \text{appropriate } \omega \in \Omega\right), \end{aligned} \tag{208}$$

where Z are *vector fields vanishing at the endpoints*. This condition trivially implies

$$\begin{aligned} &\int \mathbf{p}^*\mathscr{L}_V(\varphi + \omega) = 0 \text{ hence } \int \mathbf{p}^*\mathscr{L}_V\varphi = 0 \\ &\left(\text{special variations } V\right) \end{aligned} \tag{209}$$

with *variations* V vanishing at the endpoints; see Remark 35. Therefore

$$\text{Griffiths extremals} \subset \text{our extremals}. \tag{210}$$

The converse inclusion

$$\text{Griffiths extremals} \supset \text{our extremals} \tag{211}$$

is, however, trivial since the *universal form* $\breve{\varphi} = \varphi + \breve{\omega}$ $(\breve{\omega} \in \Omega)$ satisfies $\mathbf{p}^*Z\rfloor d\breve{\varphi} = 0$ even for every extremal in the sense of Definition 36. We conclude that all the mentioned concepts of extremals are identical. (We apologize for this hasty exposition. Roughly saying, the Griffiths' theory and our approach are almost identical. The Griffiths' *correction* $\omega \in \Omega$ *depending on* $\mathbf{p}$ is made *universal* here. The classical approach rests on a *special choice* of boundary conditions for the variations V. However, such a special choice is misleading since it does not affect the resulting family of extremals and we prefer a *universal choice* here as well.)

8. Particular Example of a Variational Integral

A simple illustrative example is necessary at this place. Let us again deal with diffiety Ω of Section 6. So we recall coordinates $x, u_0, u_1, v_0, v_1, \ldots$ of the underlying space $\mathbf{M}$, the contact forms α_r, β_r $(r = 0, 1, \ldots)$ generating Ω, the vector field

$$\begin{aligned} D &= \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_0} + F\frac{\partial}{\partial u_1} + \sum v_{r+1}\frac{\partial}{\partial v_r} \\ &= \frac{\partial}{\partial x} + \sum_{\omega\in\Omega} 0 \cdot \frac{\partial}{\partial \omega} \in \mathscr{H}, \end{aligned} \tag{212}$$

and the standard basis $\pi_0, \pi_1, \ldots$ of Ω. We moreover introduce variational integrals

$$\int \varphi \quad (\varphi = g dx, g \in \mathscr{F}(\mathbf{M})). \tag{213}$$

Assuming $\partial g/\partial \pi_r = 0$ $(r > R)$ and therefore

$$\begin{aligned} dg &= Dg dx + \frac{\partial g}{\partial u_0}\alpha_0 + \frac{\partial g}{\partial u_1}\alpha_1 + \sum \frac{\partial g}{\partial v_r}\beta_r \\ &= Dg dx + \frac{\partial g}{\partial \pi_0}\pi_0 + \cdots + \frac{\partial g}{\partial \pi_R}\pi_R, \end{aligned} \tag{214}$$

we introduce the functions

$$\begin{aligned} g_R &= \frac{\partial g}{\partial \pi_R}, \\ g_{r-1} &= \frac{\partial g}{\partial \pi_{r-1}} - Dg_r \quad (r = R, \ldots, 1). \end{aligned} \tag{215}$$

Then

$$\breve{\varphi} = g dx + g_1\pi_0 + \cdots + g_R\pi_{R-1} \tag{216}$$

is the Poincaré-Cartan form since the identity

$$d\breve{\varphi} = g_0 \cdot \pi_0 \wedge dx \quad (\mathrm{mod}\ \Omega \wedge \Omega) \tag{217}$$

can be directly verified. In accordance with formula (199) where $e = e^1$, $\pi_0 = \pi_0^1$ is abbreviated, we have $e = g_0$. Let us denote

$$\begin{aligned} e = e[g] &= g_0 \\ &= \frac{\partial g}{\partial \pi_0} - D\frac{\partial g}{\partial \pi_1} + \cdots + (-1)^R D^R \frac{\partial g}{\partial \pi_R}, \end{aligned} \tag{218}$$

for better clarity. The following simple result will be needed.

Lemma 44. *Identity $e[g] = 0$ is equivalent to the equation $g = DG$ with appropriate $G \in \mathscr{F}(\mathbf{M})$.*

Proof. By virtue of (200), the identity is equivalent to the congruence $d\breve{\varphi} \cong 0$ (mod$\Omega \wedge \Omega$). However, if the rule $d(d\breve{\varphi}) = 0$ is applied to the congruence, it follows easily that $d\breve{\varphi} = 0$ identically. Therefore $\breve{\varphi} = dG \cong DG dx$ (mod Ω) by using the Poincaré lemma. □

Let us mention symmetries $\mathbf{m}$ and variations Z of our variational problem in more detail. In the favourable case $\mu(\Omega) = 1$, the task is not difficult.

The symmetry $\mathbf{m}$ of our variational problem $\{\Omega, \varphi\}$ clearly preserves the *unique* Poincaré-Cartan form $\breve{\varphi}$ and therefore also the vector field $\mathscr{D} = D/g \in \mathscr{H}$ determined by the condition $\breve{\varphi}(\mathscr{D}) = 1$. We suppose $g \neq 0$ here. It follows that all differential forms

$$\begin{aligned} \breve{\pi}_0 &= \mathscr{L}_{\mathscr{D}}\breve{\varphi} = \mathscr{D}\rfloor d\breve{\varphi} + d\breve{\varphi}(\mathscr{D}) = e\pi_0 \cdot \mathscr{D}x = \frac{e}{g}\pi_0, \\ \breve{\pi}_{r+1} &= \mathscr{L}_{\mathscr{D}}\breve{\pi}_r \quad (r = 0, 1, \ldots) \end{aligned} \tag{219}$$

are preserved, too. Let us moreover suppose $e = e[g] \neq 0$. Clearly

$$\begin{aligned} \breve{\pi}_1 &= \frac{1}{g}\mathscr{L}_D\pi_0 = \frac{1}{g}\left(D\frac{e}{g}\pi_0 + \frac{e}{g}\pi_1\right), \\ \breve{\pi}_{r+1} &= \frac{1}{g}\mathscr{L}_{\mathscr{D}}\pi_r \cong \frac{e}{g^{r+1}}\pi_r \\ &\quad (\mathrm{mod}\ dx, \pi_0, \ldots, \pi_r). \end{aligned} \tag{220}$$

Therefore $\breve{\varphi}, \breve{\pi}_0, \breve{\pi}_1, \ldots$ is *invariant basis* of module $\Phi(\mathbf{M})$ in the sense

$$\begin{aligned} \mathbf{m}^*\breve{\varphi} &= \breve{\varphi}, \\ \mathbf{m}^*\breve{\pi}_r &= \breve{\pi}_r \quad (r = 0, 1, \ldots). \end{aligned} \tag{221}$$

It follows that the *symmetries* $\mathbf{m}$ *of our variational problem* $\{\Omega, \varphi\}$ *can be comfortably determined*. Quite analogous conclusion can be made for the infinitesimal symmetries, of course.

Passing to the variations Z of the variational problem, we have explicit formula (99) for the variations of Ω and moreover condition $\mathscr{L}_Z\varphi \in \Omega$ equivalent to $\mathscr{L}_Z\breve{\varphi} \in \Omega$. However,

$$\begin{aligned} \mathscr{L}_Z\breve{\varphi} &= Z\rfloor d\breve{\varphi} + d\breve{\varphi}(Z) \\ &\cong Z\rfloor(e\pi_0 \wedge dx) + D\breve{\varphi}(Z)\,dx \\ &= (ep + D\breve{\varphi}(Z))\,dx \quad (\mathrm{mod}\ \Omega) \end{aligned} \tag{222}$$

and therefore

$$\begin{aligned} 0 &= ep + D\breve{\varphi}(Z) = ep + DG \\ &(G = gz + g_0p + g_1Dp + \cdots + g_RD^Rp). \end{aligned} \tag{223}$$

Assume $g \neq 0$. We obtain condition $e[ep] = 0$ for the unknown function p. In more precise notation and in full detail

$$\begin{aligned} &e[e[g]p] \\ &= \left(\frac{\partial}{\partial \pi_0} - D\frac{\partial}{\partial \pi_1} + \cdots\right)\left(\frac{\partial g}{\partial \pi_0} - D\frac{\partial g}{\partial \pi_1} + \cdots\right)p \\ &= 0. \end{aligned} \tag{224}$$

This is formally a very simple condition concerning the unknown function p; alas, it is not easy to be resolved. Paradoxically, *variations Z cause serious difficulties.*

For better clarity, we continue this example with particular choice of the variational integral. Let us consider variational integral $\int g(x, u_0, v_0)dx$. Equation (214) reads

$$\begin{aligned} dg &= Dg dx + \frac{\partial g}{\partial u_0}\alpha_0 + \frac{\partial g}{\partial v_0}\beta_0 \\ &= Dg dx + \frac{\partial g}{\partial \pi_0}\pi_0 + \frac{\partial g}{\partial \pi_1}\pi_1 + \frac{\partial g}{\partial \pi_2}\pi_2 \end{aligned} \tag{225}$$

and it follows that

$$\frac{\partial g}{\partial \pi_0} = -2g_{u_0}\frac{Df}{C} + g_{v_0}\frac{A}{C^2},$$
$$\frac{\partial g}{\partial \pi_1} = g_{u_0}\frac{f}{C} + g_{v_0}\frac{B}{C^2}, \qquad \frac{\partial g}{\partial \pi_2} = \frac{g_{v_0}}{C} \tag{226}$$

by using (93). We have $R = 2$ and therefore

$$\breve{\varphi} = g dx + \left(\frac{\partial g}{\partial \pi_1} - D\frac{\partial g}{\partial \pi_2}\right)\pi_0 + \frac{\partial g}{\partial \pi_2}\pi_1,$$
$$e = \frac{\partial g}{\partial \pi_0} - D\frac{\partial g}{\partial \pi_1} + D^2\frac{\partial g}{\partial \pi_2} \tag{227}$$

by virtue of (215)–(218). Both the Poincaré-Cartan form $\breve{\varphi}$ and the Euler-Lagrange equation $e = 0$ can be expressed in terms of common coordinates, if derivatives (226) are inserted. We omit the final formulae here. Passing to the symmetries $\mathbf{m}$, we may simulate the moving frames method and express the differential

$$d\breve{\varphi} = \sum C_r \breve{\pi}_r \wedge \breve{\varphi} + \sum_{r<s} C_{rs}\breve{\pi}_r \wedge \breve{\pi}_s$$
$$= \breve{\pi}_0 \wedge \breve{\varphi} + \sum_{r<s} C_{rs}\breve{\pi}_r \wedge \breve{\pi}_s \tag{228}$$

in terms of the invariant basis (221). Then *all coefficients C_{rs} are invariants of symmetry* $\mathbf{m}$; that is,

$$\mathbf{m}^* C_{rs} = C_{rs} \text{ hence } \mathbf{m}^* \mathscr{D}^k C_{rs} = \mathscr{D}^k C_{rs} \quad \left(\mathscr{D} = \frac{D}{g}\right). \tag{229}$$

In fact *we have obtained all invariants*. (Hint: for instance, differential

$$d\breve{\pi}_0 = d\mathscr{L}_{\mathscr{D}}\breve{\varphi} = \mathscr{L}_{\mathscr{D}} d\breve{\varphi}$$
$$= \sum \mathscr{D} C_{rs}\breve{\pi}_r \wedge \breve{\pi}_s + \mathscr{L}_{\mathscr{D}}\left(\breve{\pi}_0 \wedge \breve{\varphi}\right)$$
$$+ \sum C_{rs}\mathscr{L}_{\mathscr{D}}\left(\breve{\pi}_r \wedge \breve{\pi}_s\right) \tag{230}$$

does not provide any novelty.) It follows that the symmetry problem is resolved. *Compatibility of the system of* (229) *ensures the existence of symmetries* $\mathbf{m}$ *of the variational problem* $\{\Omega, \varphi\}$ *since the Frobenius theorem can be applied to the Pfaffian system* (221). In the most favourable case, C_{rs} are even constants. Explicit calculation of invariants C_{rs} is a lengthy but routine procedure. First of all

$$d\breve{\varphi} \cong dg_1 \wedge \pi_0 + dg_2 \wedge \pi_1 + g_1 d\pi_0$$
$$+ g_2 d\pi_1 \pmod{dx} \tag{231}$$

by using the primary formula (216). Then

$$dg_1 \cong \sum \frac{\partial g_1}{\partial \pi_r}\pi_r, \qquad dg_2 \cong \sum \frac{\partial g_2}{\partial \pi_r}\pi_r \pmod{dx} \tag{232}$$

may be substituted where the coefficient can be determined analogously as in (226). As the differential

$$d\pi_0 \cong \beta_1 \wedge \left(f'\alpha - ff'\beta_0 + f''\nu_2\alpha_0\right)$$
$$+ f'\beta_2 \wedge \alpha_0 \pmod{dx} \tag{233}$$

is concerned, we refer to formula in Section 6. The contact forms must be replaced with the standard basis by using the right-hand formulae (93). Then we may use the lucky identity

$$d\pi_1 = d\mathscr{L}_D\pi_0 = \mathscr{L}_D d\pi_0 \tag{234}$$

in order to determine the last summand in (231). In the end, the standard basis π_r in (231) can be easily replaced by the invariant forms $\breve{\pi}_r$ $(r = 0, 1, \ldots)$ and we are done.

9. The Order-Increasing Case

Let us eventually return to the main topic, the differential equations. We will finish this paper with decisive examples of higher-order symmetries, namely, with symmetries of the Monge equation

$$\frac{dw}{dx} = F\left(x, u, v, w, \frac{du}{dx}, \frac{dv}{dx}\right) \tag{235}$$

involving three unknown functions $u = u(x)$, $v = v(x)$, and $w = w(x)$. Let us directly turn to the internal theory carried out by using the underlying space $\mathbf{M}$ with coordinates

$$x, u_r, v_r, w_0 \quad (r = 0, 1, \ldots), \tag{236}$$

diffiety $\Omega \subset \Phi(\mathbf{M})$ with the basis

$$\alpha_r = du_r - u_{r+1}dx,$$
$$\beta_r = dv_r - v_{r+1}dx \quad (r = 0, 1, \ldots),$$
$$\gamma_0 = dw_0 - F\left(x, u_0, v_0, w_0, u_1, v_1\right)dx \tag{237}$$

and the total derivative

$$D = \frac{\partial}{\partial x} + \sum u_{r+1}\frac{\partial}{\partial u_r} + \sum v_{r+1}\frac{\partial}{\partial v_r} + F\frac{\partial}{\partial w_0} \in \mathscr{H}. \tag{238}$$

We also introduce functions and differential forms

$$w_r = D^r w_0 \in \mathscr{F}(\mathbf{M}),$$
$$\gamma_r = \mathscr{L}_D^r \gamma_0 = dw_r - w_{r+1}dx \in \Omega$$
$$(r = 0, 1, \ldots) \tag{239}$$

for the formal reasons. The natural filtration Ω_* in accordance with the order is such that the forms $\alpha_0, \ldots, \alpha_l, \beta_0, \ldots, \beta_l, \gamma_0$ are taken for the basis of submodule $\Omega_l \subset \Omega$ $(l = 0, 1, \ldots)$. Let us determine the corresponding standard filtration $\overline{\Omega}_*$. Clearly

$$\mathscr{L}_D\gamma_0 = \gamma_1 = F_{u_0}\alpha_0 + F_{v_0}\beta_0 + F_{w_0}\gamma_0$$
$$+ F_{u_1}\alpha_1 + F_{v_1}\beta_1 \tag{240}$$

and therefore

$$\begin{aligned}&\mathscr{L}_D\left(\gamma_0 - F_{u_1}\alpha_0 - F_{v_1}\beta_0\right)\\&\quad = \left(F_{u_0} - DF_{u_1}\right)\alpha_0 + \left(F_{v_0} - DF_{v_1}\right)\beta_0 + F_{w_0}\gamma_0 \in \Omega_0.\end{aligned} \tag{241}$$

Denoting $\pi = \gamma_0 - F_{u_1}\alpha_0 - F_{v_1}\beta_0$, we obtain

$$\begin{aligned}\mathscr{L}_D\pi &= \left(F_{u_0} - DF_{u_1}\right)\alpha_0 + \left(F_{v_0} - DF_{v_1}\right)\beta_0 + F_{w_0}\gamma_0\\&= A\alpha_0 + B\beta_0 + F_{w_0}\pi \in \Omega_0\\&\left(A = F_{u_0} - DF_{u_1} + F_{w_0}F_{u_1}, B = F_{v_0} - DF_{v_1} + F_{w_0}F_{v_1}\right).\end{aligned} \tag{242}$$

We will not deal with the case when $A = B = 0$ identically. Let us instead suppose that $A \neq 0$ from now on. Then $\mathscr{R}(\Omega) = 0$ and we may introduce standard filtration $\overline{\Omega}_*$ of diffiety Ω where the form

$$\pi_0^1 = \pi = \gamma_0 - F_{u_1}\alpha_0 - F_{v_1}\beta_0 \tag{243}$$

generates $\overline{\Omega}_0$ and in general the forms

$$\begin{aligned}\pi_r^1 &= \mathscr{L}_D^r\pi \quad (r = 0, \ldots, l),\\ \pi_r^2 &= \beta_r \quad (r = 0, \ldots, l-1)\end{aligned} \tag{244}$$

generate module $\overline{\Omega}_l$ ($l \geq 1$). Notation (53) with indices is retained here. With this preparation, we are passing to the symmetries of diffiety Ω. Theorem 26 and Remark 27 fail since $\mu(\Omega) = 2$ in our case. There exist many standard filtrations of Ω and we may also expect the existence of the order-destroying symmetries.

The preparation is done; however, before passing to quite explicit examples, certain general aspects are worth mentioning. We recall Figure 3 which can be transparently illustrated just at this place for the first time.

First of all, every *order-preserving symmetry* $\mathbf{m}$ on scheme (a) of Figure 3 obviously satisfies certain formulae

$$\begin{aligned}\mathbf{m}^*\alpha_0 &= a^1\alpha_0 + a^2\beta_0 + a^3\gamma_0,\\ \mathbf{m}^*\beta_0 &= b^1\alpha_0 + b^2\beta_0 + b^3\gamma_0,\\ \mathbf{m}^*\gamma_0 &= c^1\alpha_0 + c^2\beta_0 + c^3\gamma_0,\\ &\det\begin{pmatrix} a^1 & a^2 & a^3\\ b^1 & b^2 & b^3\\ c^1 & c^2 & c^3\end{pmatrix} \neq 0,\end{aligned} \tag{245}$$

where the coefficients cannot be in fact arbitrary since they are subjected to identity (240). In more detail, we have

$$\begin{aligned}&DW\cdot\mathbf{m}^*\gamma_1\\&\quad = \mathscr{L}_D\mathbf{m}^*\gamma_0\\&\quad = Dc^1\alpha_0 + Dc^2\beta_0 + Dc^3\gamma_0 + c^1\alpha_1 + c^2\beta_1 + c^3\gamma_1\\&\qquad\qquad (W = \mathbf{m}^*x)\end{aligned} \tag{246}$$

in accordance with (63). Alternatively (240) implies

$$\mathbf{m}^*\gamma_1 = \mathbf{m}^*F_{u_0}\cdot\mathbf{m}^*\alpha_0 + \cdots + \mathbf{m}^*F_{v_1}\cdot\mathbf{m}^*\beta_1, \tag{247}$$

where the forms $\mathbf{m}^*\alpha_0, \ldots, \mathbf{m}^*\beta_1$ can be expressed in terms of forms $\alpha_0, \ldots, \gamma_1$. The comparison provides many unpleasant interrelations among coefficients $a^1, \ldots, c^3$.

However, by using the standard basis, the same symmetry satisfies shorter formulae

$$\begin{aligned}&\mathbf{m}^*\pi_0^1 = a\pi_0^1\\&\left(\text{hence, automatically } DW\mathbf{m}^*\pi_1^1 = Da\pi_0^1 + a\pi_1^1\right),\\&\mathbf{m}^*\pi_0^2 = b_0^1\pi_0^1 + b_0^2\pi_0^2 + b_1^1\pi_1^1\end{aligned} \tag{248}$$

with coefficients subjected only to the inequalities $a \neq 0$ and $b_0^2 \neq 0$ at this place. We employ the fact that both triples $\alpha_0, \beta_0, \gamma_0$ and $\pi_0^1, \pi_0^2, \pi_1^1$ are bases of module Ω_0. Moreover $\mathbf{m}^*$ preserves the natural filtration Ω_* and therefore also the corresponding standard filtration $\overline{\Omega}_*$. Especially, the initial term $\overline{\Omega}_0$ is preserved and $\mathbf{m}^*\pi_0^1$ is a mere multiple of π_0^1.

The *order-preserving infinitesimal symmetry* Z corresponding to scheme (a) satisfies either the system

$$\begin{aligned}\mathscr{L}_Z\alpha_0 &= \lambda_1\alpha_0 + \lambda_2\beta_0 + \lambda_3\gamma_0,\\ \mathscr{L}_Z\beta_0 &= \mu_1\alpha_0 + \mu_2\beta_0 + \mu_3\gamma_0,\\ \mathscr{L}_Z\gamma_0 &= \nu_1\alpha_0 + \nu_2\beta_0 + \nu_3\gamma_0\end{aligned} \tag{249}$$

with coefficients subjected to many identities analogous as above or, alternatively, the equivalent and shorter system

$$\begin{aligned}&\mathscr{L}_Z\pi_0^1 = \mu\pi_0^1\\&\left(\text{hence } \mathscr{L}_Z\pi_1^1 = D\mu\pi_0^1 + (\mu - DZx)\pi_1^1\right),\\&\mathscr{L}_Z\pi_0^2 = \lambda_0^1\pi_0^1 + \lambda_0^2\pi_0^2 + \lambda_1^1\pi_1^1\end{aligned} \tag{250}$$

with arbitrary coefficients in terms of the standard basis. For the middle equation use the identity

$$\mathscr{L}_Z\pi_1^1 = \mathscr{L}_Z\mathscr{L}_D\pi_0^1 = \mathscr{L}_D\mathscr{L}_Z\pi_0^1 - Dz\pi_1^1 \quad (z = Zx). \tag{251}$$

This follows from the Lie bracket formula $[D, Z] = Dz\cdot D$ which is true if and only if Z is a variation of diffiety Ω. The *Cartan's general equivalence method* [23] *can be applied* to this order-preserving symmetry problem; however, we will mention the *Lie approach* later on.

With this result, the simplest possible *order-increasing symmetry* $\mathbf{m}$ on scheme (c) of Figure 3 can be introduced by the equations

$$\begin{aligned}\mathbf{m}^*\pi_0^1 &= a^1\pi_0^1 + a^2\pi_0^2,\\ \mathbf{m}^*\pi_0^2 &= b^1\pi_0^1 + b^2\pi_0^2 + b\left(a^1\pi_1^1 + a^2\pi_1^2\right),\\ &\det\begin{pmatrix} a^1 & b^1 - bDa^1\\ a^2 & b^2 - bDa^2\end{pmatrix} \neq 0.\end{aligned} \tag{252}$$

Let us prove the invertibility of $\mathbf{m}$. Clearly

$$\begin{aligned} DW\mathbf{m}^*\pi_1^1 &= \mathscr{L}_D\mathbf{m}^*\pi_0^1 \\ &= Da^1\pi_0^1 + Da^2\pi_0^2 + a^1\pi_1^1 + a^2\pi_1^2 \end{aligned} \tag{253}$$

and it follows that

$$\begin{aligned} &\mathbf{m}^*\pi_0^2 - bDW\mathbf{m}^*\pi_1^1 \\ &\quad = \left(b^1 - bDa^1\right)\pi_0^1 + \left(b^2 - bDa^2\right)\pi_0^2 \in \mathbf{m}^*\Omega. \end{aligned} \tag{254}$$

Inclusions $\pi_0^1, \pi_0^2 \in \mathbf{m}^*\Omega$ therefore hold true and Lemma 21 can be applied.

In order to state another example to scheme (c), let us consider the equations

$$\begin{aligned} \mathbf{m}^*\pi_0^1 &= a_0^1\pi_0^1 + a_0^2\pi_0^2 + a_1^1\pi_1^1 + a_1^2\pi_1^2, \\ \mathbf{m}^*\pi_0^2 &= b_0^1\pi_0^1 + b_0^2\pi_0^2 + b_1^1\pi_1^1 + b_1^2\pi_1^2. \end{aligned} \tag{255}$$

Invertibility of such morphism $\mathbf{m}$ is ensured if $b_1^i = ba_1^i$, $b_0^i - ba_0^i = aa_1^i$ $(i = 1, 2)$ for appropriate factors $a \neq 0, b$ and if moreover

$$\det\begin{pmatrix} a_1^1 & a_0^1 - Da_1^1 \\ a_1^2 & a_0^2 - Da_1^2 \end{pmatrix} \neq 0. \tag{256}$$

For the proof of invertibility, apply $\mathscr{L}_D$ to the inclusion

$$\frac{1}{a}\left(\mathbf{m}^*\pi_0^2 - b\mathbf{m}^*\pi_0^1\right) = a_1^1\pi_0^1 + a_1^2\pi_0^2 \in \mathbf{m}^*\Omega \tag{257}$$

and verify that $\pi_0^1, \pi_0^2 \in \mathbf{m}^*\Omega$.

In both examples, *the common general equivalence method* [23] *fails*. The corresponding *variations* Z can be introduced and are rather interesting though *they do not generate* any symmetry groups. See Remark 46 below.

It is also easy to illustrate scheme (b) of Figure 3 by using the symmetries $\mathbf{m}$ and infinitesimal symmetries Z such that

$$\begin{aligned} &\mathbf{m}^*\pi_0^1 = a^1\pi_0^1 + a^2\pi_0^2, \\ &\mathbf{m}^*\pi_0^2 = b^1\pi_0^1 + b^2\pi_0^2, \\ &\det\begin{pmatrix} a^1 & a^2 \\ b^1 & b^2 \end{pmatrix} \neq 0, \\ &\mathscr{L}_Z\pi_0^1 = \lambda^1\pi_0^1 + \lambda^2\pi_0^2, \\ &\mathscr{L}_Z\pi_0^2 = \mu^1\pi_0^1 + \mu^2\pi_0^2. \end{aligned} \tag{258}$$

(Hint: Theorem 24 can be trivially applied and the natural filtration is not preserved, if $a^2 \neq 0$ and $\lambda^2 \neq 0$.) Another example is provided by the equations

$$\begin{aligned} &\mathbf{m}^*\pi_0^1 = a_0^1\pi_0^1 + a_0^2\pi_0^2 + a_1^2\pi_1^2, \\ &\mathbf{m}^*\pi_0^2 = b\pi_0^2, \\ &\mathscr{L}_Z\pi_0^1 = \mu_0^1\pi_0^1 + \mu_0^2\pi_0^2 + \mu_1^2\pi_1^2, \\ &\mathscr{L}_Z\pi_0^2 = \mu\pi_0^2 \end{aligned} \tag{259}$$

"symmetrical" to the order-preserving case. The classical Lie's infinitesimal symmetries and the Cartan's equivalence method *can be both applied without any change*.

We have briefly indicated only the simplest devices here and refer to [2, Section 4] for the universal construction. A complete overview of all possible higher-order symmetries of (235) is lying beyond any actual imagination. For instance, the composition $\mathbf{m}_1 \circ \mathbf{m}_2$ of symmetries and the conjugate groups $\mathbf{m} \circ \mathbf{m}(\lambda) \circ \mathbf{m}^{-1}$ to a given group provide much more complicated examples than the original components $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}$, and $\mathbf{m}(\lambda)$. The definition equations for such *composition of symmetries* can be directly found and they look rather depressively for the time being.

10. Concluding Examples on Infinitesimal Symmetries

We deal only with a simplified equation (235), namely, with the equation

$$\frac{dw}{dx} = F\left(\frac{du}{dx}, \frac{dv}{dx}\right) \tag{260}$$

for good reasons to be clarified in the Appendix. Let us abbreviate

$$\begin{aligned} &F_1 = F_{u_1}, \qquad F^1 = F_{v_1}, \qquad F_{11} = F_{u_1u_1}, \\ &F_1^1 = F_{u_1v_1}, \qquad F^{11} = F_{v_1v_1} \end{aligned} \tag{261}$$

from now on. The crucial identity (240) then reads

$$\mathscr{L}_D\gamma_0 = \gamma_1 = F_1\alpha_1 + F^1\beta_1 \tag{262}$$

and we recall the standard basis $\pi_r^1 = \mathscr{L}_D^r\pi_0^1$, $\pi_r^2 = \mathscr{L}_D^r\pi_0^2 = \beta_r$ $(r = 0, 1, \ldots)$, where

$$\begin{aligned} \pi_0^1 &= \gamma_0 - F_1\alpha_0 - F^1\beta_0, \\ \pi_1^1 &= -DF_1\alpha_0 - DF^1\beta_0 \end{aligned} \tag{263}$$

in terms of the simplified notation. The formulae

$$\begin{aligned} d\pi_0^1 &= dx \wedge \pi_1^1 + \left(F_{11}\alpha_0 + F_1^1\beta_0\right) \wedge \alpha_1 \\ &\quad + \left(F_1^1\alpha_0 + F^{11}\beta_0\right) \wedge \beta_1, \\ d\pi_0^2 &= dx \wedge \beta_1 \end{aligned} \tag{264}$$

easily follow. On this occasion, we also recall more general adjustments

$$\begin{aligned} &\lambda\mathbf{m}^*\mathscr{L}_D\omega = \mathscr{L}_D\mathbf{m}^*\omega \\ &\left(\omega \in \Omega, \lambda = DW, W = \mathbf{m}^*x\right), \end{aligned} \tag{265}$$

$$\left(\mathscr{L}_D\omega\right)(Z) = D\left(\omega(Z)\right) \quad (\omega \in \Omega) \tag{266}$$

of Lemmas 17 and 23. The factor $\lambda \neq 0$ appearing here can be defined by the congruence $\mathbf{m}^*dx \cong \lambda dx \pmod{\Omega}$ as well.

Several symmetry problems for (260) will be mentioned. We start with examples on *infinitesimal symmetries* Z and demonstrate our approach both using the traditional order-preserving case and then employing two technically quite analogous order-increasing symmetry problems. The calculations are elementary but not of a mere mechanical nature and the concise form of the final results is worth attention. That is, by using the series (268) with the standard basis, the unknown functions z, p, and q satisfy quite reasonable and explicitly solvable conditions. Denoting

$$\begin{gathered} a_0 = \alpha_0(Z), \qquad a_1 = \alpha_1(Z) = Da_0, \\ p = \pi_0^1(Z), \qquad q = \pi_0^2(Z) = \beta_0(Z), \\ z = Zx, \end{gathered} \tag{267}$$

we simulate the procedure of Section 6 and our method again rests on the explicit formula

$$Z = z\frac{\partial}{\partial x} + \sum D^r p \frac{\partial}{\partial \pi_r^1} + \sum D^r q \frac{\partial}{\partial \pi_r^2} \tag{268}$$

for all *variations* Z. We recall that *infinitesimal symmetries* Z moreover satisfy certain additional requirements in order to ensure the conditions of Theorem 24. The choice of such requirements which is arbitrary to a large extent (dotted lines in Figure 3(b)) strongly affects the final result, the resulting symmetries Z. Altogether taken, reasonings of this Section 10 belong to the *Lie's theory* appropriately adapted to the infinite-dimensional spaces. On the contrary, we will conclude this paper with only few remarks on the true (not group-like) higher-order *symmetries* **m** in subsequent Section 11. The reasonings can be related to the E. Cartan's *general equivalence method* [16, 23] and they would deserve more space than it is possible here.

Let us turn to proper examples.

(ι) The Order-Preserving Symmetry Problem. We again intentionally start with a mere "traditional" case. Let us deal with infinitesimal symmetries Z satisfying

$$\begin{aligned} \mathscr{L}_Z \pi_0^1 &= \mu\pi_0^1 = \mu\left(\gamma_0 - F_1\alpha_0 - F^1\beta_0\right), \\ \mathscr{L}_Z \pi_0^2 &= \lambda_0^1\pi_0^1 + \lambda_0^2\pi_0^2 + \lambda_1^1\pi_1^1 \\ &= \mu^1\alpha_0 + \mu^2\beta_0 + \mu^3\gamma_0. \end{aligned} \tag{269}$$

We use the "hybrid" equations involving both the standard basis and the contact forms. Let us recall the explicit formula (268) for all *variations*. We have moreover the above equations (269) in order to obtain the true *infinitesimal symmetries*. In more detail

$$\begin{aligned} Z\rfloor d\pi_0^1 + dp &= \mu\left(\gamma_0 - F_1\alpha_0 - F^1\beta_0\right), \\ Z\rfloor d\pi_0^2 + dq &= \mu^1\alpha_0 + \mu^2\beta_0 + \mu^3\gamma_0 \end{aligned} \tag{270}$$

should be satisfied. Analogously as in Section 6, this is expressed by the *resolving system*

$$\begin{gathered} zDF_1 + F_{11}a_1 + F_1^1 Dq = p_{u_0} + \mu F_1, \\ zDF^1 + F_1^1 a_1 + F^{11} Dq = p_{v_0} + \mu F^1, \\ F_{11}a_0 + F_1^1 q + p_{u_1} = 0, \\ F_1^1 a_0 + F^{11} q + p_{v_1} = 0, \end{gathered} \tag{271}$$

$$\begin{gathered} p_{w_0} = \mu, \\ p_{u_r} = p_{v_r} = 0 \quad (r \geq 2), \\ q_{u_0} = \mu^1, \qquad q_{v_0} = \mu^2, \qquad q_{w_0} = \mu^3, \\ q_{u_r} = 0 \quad (r \geq 1), \\ z + q_{v_1} = 0, \\ q_{v_r} = 0 \quad (r \geq 2) \end{gathered} \tag{272}$$

by using (264) and $\beta_1(Z) = D\beta_0(Z) = Dq$. It follows that only (271) with $\mu = p_{w_0}$, $z = -q_{v_1}$ inserted and coefficients a_0, a_1 given by

$$a_0 DF_1 + qDF^1 + Dp = 0, \qquad a_1 = Da_0 \tag{273}$$

are the most important.

Let us denote $\Delta = (F_1^1)^2 - F_{11}F^{11}$ and *assume* $\Delta \neq 0$ *from now on.* Equations (271) are equivalent to

$$\begin{gathered} \Delta Dq = \det\begin{pmatrix} p_{u_0} + p_{w_0}F_1 + q_{v_1}DF_1 & F_{11} \\ p_{v_0} + p_{w_0}F^1 + q_{v_1}DF^1 & F_1^1 \end{pmatrix}, \\ \Delta a_1 = \det\begin{pmatrix} F_1^1 & p_{u_0} + p_{w_0}F_1 + q_{v_1}DF_1 \\ F^{11} & p_{v_0} + p_{w_0}F^1 + q_{v_1}DF^1, \end{pmatrix}, \end{gathered} \tag{274}$$

$$\begin{gathered} \Delta q = \det\begin{pmatrix} F_{11} & p_{u_1} \\ F_1^1 & p_{v_1} \end{pmatrix}, \\ \Delta a_0 = \det\begin{pmatrix} p_{u_1} & F_1^1 \\ p_{v_1} & F^{11} \end{pmatrix}. \end{gathered} \tag{275}$$

We have unknown functions

$$\begin{gathered} p = p(x, u_0, v_0, w_0, u_1, v_1), \\ q = q(x, u_0, v_0, w_0, v_1) \end{gathered} \tag{276}$$

and let us pass to the solution of (273), (274), and (275).

The first equation (273) multiplied by function Δ reads

$$\begin{aligned} &\left(F_{11}u_2 + F_1^1 v_2\right)\Delta a_0 + \left(F_1^1 u_2 + F^{11} v_2\right)\Delta q \\ &\quad + \left(\mathscr{D}p + p_{u_1}u_2 + p_{v_1}v_2\right)\Delta = 0 \end{aligned} \tag{277}$$

and therefore implies only the identity

$$\mathscr{D}p = 0 \quad \left(\mathscr{D} = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_0} + v_1\frac{\partial}{\partial v_0} + F\frac{\partial}{\partial w_0}\right), \tag{278}$$

if both equations (275) are accepted (direct verification). Alternatively saying, second equation (275) can be regarded for a *definition of function* a_0 if (278) is taken into account. Let us denote $G = \Delta a_0$ for a moment. Then

$$DG = D\Delta \cdot a_0 + \Delta \cdot Da_0 = D\Delta \cdot \frac{G}{\Delta} + \Delta \cdot a_1, \qquad \Delta \cdot DG - G \cdot D\Delta = a_1 \cdot (\Delta)^2 \tag{279}$$

and the second equation (274) reads

$$\Delta \cdot DG - G \cdot D\Delta = \Delta \cdot \det(\cdots) \tag{280}$$

with the same determinant. Lower-order terms clearly provide the equation

$$\mathscr{D}G = \det\begin{pmatrix} F_1^1 & p_{u_0} + p_{w_0}F_1 \\ F^{11} & p_{v_0} + p_{w_0}F^1 \end{pmatrix} \tag{281}$$

and coefficients of v_2 give

$$\Delta \cdot G_{v_1} - G \cdot \Delta_{v_1} = 0, \text{hence } G = g(x, u_0, v_0, w_0)\,\Delta \tag{282}$$

for appropriate function g; however trivially $g = a_0$. With this result, we obtain

$$\Delta\left(g_{u_1}\Delta + g\Delta_{u_1}\right) - g\Delta \cdot \Delta_{u_1} = \Delta \det\begin{pmatrix} F_1^1 & q_{v_1}F_{11} \\ F^{11} & q_{v_1}F_1^1 \end{pmatrix} = (\Delta)^2 q_{v_1} \tag{283}$$

by inspection of coefficients of v_2. It follows that $g_{u_1} = q_{v_1}$, whence

$$g = R(x, u_0, v_0, w_0)\,u_1 + S(x, u_0, v_0, w_0), \qquad q = R(x, u_0, v_0, w_0)\,v_1 + T(x, u_0, v_0, w_0). \tag{284}$$

Analogously the lower-order terms of the first equation (274) give

$$\Delta \cdot \mathscr{D}q = \det\begin{pmatrix} p_{u_0} + p_{w_0}F_1 & F_{11} \\ p_{v_0} + p_{w_0}F^1 & F_1^1 \end{pmatrix}, \tag{285}$$

while the second-order terms do not provide any new requirements.

Let us finally recall (275) with $a_0 = g$ and q given by (284) inserted. These equations turn into the compatible system

$$p_{u_1} + (Ru_1 + S)\,F_{11} + (Rv_1 + T)\,F_1^1 = 0, \qquad p_{v_1} + (Ru_1 + S)\,F_1^1 + (Rv_1 + T)\,F^{11} = 0 \tag{286}$$

for the function p with the solution

$$p = \left(F - F_1u_1 - F^1v_1\right)R - F_1S - F^1T + C \qquad (C = C(x, u_0, v_0, w_0)). \tag{287}$$

Then (278) is expressed by the *crucial requirement*

$$\left(F - F_1u_1 - F^1v_1\right) \cdot \mathscr{D}R - F_1 \cdot \mathscr{D}S - F^1 \cdot \mathscr{D}T + \mathscr{D}C = 0 \tag{288}$$

for the functions R, S, T and C. One can moreover verify with the help of

$$0 = (\mathscr{D}p)_{u_1} = \mathscr{D}\left(p_{u_1}\right) + p_{u_0} + p_{w_0}F_1, \qquad 0 = (\mathscr{D}p)_{v_1} = \mathscr{D}\left(p_{v_1}\right) + p_{v_0} + p_{w_0}F^1 \tag{289}$$

that the remaining equations (281) and (285) become identities.

Let us summarize our achievements. *Assuming* $(F_1^1)^2 \neq F_{11}F^{11}$, *all infinitesimal symmetries* (268) *are determined by formula* (287), the second equation (284) and $z = -q_{v_1} = -R$ *with functions* R, S, T, C *of variables* x_0, u_0, v_0, w_0 *satisfying* (288).

Traditional methods are sufficient to analyze thoroughly (288). Passing to more details, we have

$$\begin{aligned} &\left(F - F_1u_1 - F^1v_1\right)\left(u_1R_{u_0} + v_1R_{v_0} + FR_{w_0}\right) \\ &\quad - F_1\left(S_x + u_1\left(S_{u_0} + R_x\right) + v_1S_{v_0} + FS_{w_0}\right) \\ &\quad - F^1\left(T_x + u_1T_{u_0} + v_1\left(T_{v_0} + R_x\right) + FT_{w_0}\right) \\ &\quad + C_x + u_1C_{u_0} + v_1C_{v_0} + F\left(C_{w_0} + R_x\right) = 0. \end{aligned} \tag{290}$$

Analogously as in Section 6, the large series of coefficients

$$Fu_1, F_1(u_1)^2, F^1v_1u_1, Fv_1, F_1u_1v_1, \ldots, 1, u_1, v_1, F \tag{291}$$

appears. If these functions are $\mathbb{R}$-linearly independent, only the solution R, S, T, C such that

$$R_{u_0} = R_{v_0} = R_{w_0} = S_x = S_{u_0} + R_x = \cdots = C_{v_0} = C_{w_0} + R_x = 0 \tag{292}$$

is possible. It follows that

$$R = a_1x + a_2, \qquad S = -a_1u_0 + a_3, \qquad T = -a_1v_0 + a_4, \qquad C = -a_1w_0 + a_5, \tag{293}$$

where $a_1, \ldots, a_5 \in \mathbb{R}$ are arbitrary constants. This result provides the obvious symmetries which are self-evident at a first glance, the coordinate shifts and the similarity.

For a special choice of function F, the symmetry group may be very large and less trivial. We can mention the case $F = u_1v_1$. Then the arising system of five equations

$$\begin{aligned} &R_x + S_{u_0} + T_{v_0} - C_{w_0} \\ &\quad = R_{u_0} + T_{w_0} \\ &\quad = R_{v_0} + S_{w_0} = T_x - C_{u_0} = S_x - C_{v_0} = 0 \end{aligned} \tag{294}$$

for the unknown functions

$$R = R(x, u_0, v_0), \qquad S = S(x, u_0, w_0), \qquad T = T(x, v_0, w_0) = C(u_0, v_0, w_0), \tag{295}$$

can be resolved by

$$\begin{aligned} R &= a_1 x + a_2 u_0 + a_3 v_0 + a_4, \\ S &= a_5 x + a_6 u_0 - a_3 w_0 + a_7, \\ T &= a_8 x + a_9 v_0 - a_2 w_0 + a_{10}, \\ C &= a_8 u_0 + a_6 v_0 + (a_1 + a_6 + a_9) w_0 + a_{11}, \end{aligned} \tag{296}$$

where $a_1, \ldots, a_{11} \in \mathbb{R}$ are arbitrary constants.

We omit more examples, in particular the interesting cases (with $\mathbb{R}$-linear dependence of functions $F - F_1 u_1 - F^1 v_1, F_1, F^1, 1$) where the infinitesimal symmetries depend on arbitrary functions and the "degenerate" cases when either $\Delta = 0$ or $DF_1 = DF^1 = 0$ identically.

(ii) The Order-Increasing Infinitesimal Symmetry. Let us mention variations (268) satisfying moreover the equations

$$\mathscr{L}_Z \pi_0^1 = \mu_0^1 \pi_0^1 + \mu_0^2 \pi_0^2 + \mu_1^2 \pi_1^2, \qquad \mathscr{L}_Z \pi_0^2 = \mu \pi_0^2 \tag{297}$$

which provide the order-increasing case, if $\mu_1^2 \neq 0$. One can then obtain the *resolving system*

$$\begin{aligned} &zDF_1 + F_{11} a_1 + F_1^1 Dq - p_{u_0} - p_{w_0} F_1 \\ &\quad = F_{11} a_0 + F_1^1 q + p_{u_1} = z + q_{v_1} = 0 \end{aligned} \tag{298}$$

for the unknown functions

$$\begin{aligned} &p = p(x, u_0, v_0, w_0, u_1, v_1), \\ &q = q(x, v_0, v_1), \qquad z = z(x, v_0, v_1) \end{aligned} \tag{299}$$

and moreover formula

$$\mu_1^2 = F_1^1 a_0 + F^{11} q + p_{v_1} = -F_1^1 \frac{qDF^1 + Dp}{DF_1} + F^{11} q + p_{v_1} \tag{300}$$

for the coefficient μ_1^2. We mention only the particular case $F = u_1 v_1$. Then the resolving system reads $-q_{v_1} v_2 + Dq - p_{u_0} - p_{w_0} v_1 = q + p_{u_1} = 0$ and admits the solution

$$p = -qu_1 + \left(q_x + q_{v_0} v_1\right) u_0 + P(x, v_1 u_0 - w_0, v_0, v_1), \tag{301}$$

where the functions $q = q(x, v_0, v_1)$ and $P = P(x, v_1 u_0 - w_0, v_0, v_1)$ may be arbitrarily chosen. Since the above coefficient

$$\mu_1^2 = -\frac{1}{v_2}(qu_2 + Dp) + p_{v_1} \tag{302}$$

does not in general vanish, we have a large family of order-increasing infinitesimal symmetries.

(iii) Another Order-Increasing Case. Let us mention variations (268) satisfying the equations

$$\mathscr{L}_Z \pi_0^1 = \lambda^1 \pi_0^1 + \lambda^2 \pi_0^2, \qquad \mathscr{L}_Z \pi_0^2 = \mu^1 \pi_0^1 + \mu^2 \pi_0^2 \tag{303}$$

which provide an order-increasing case if $\lambda^2 \neq 0$. The *resolving system*

$$\begin{aligned} &zDF_1 + F_{11} a_1 + F_1^1 Dq - p_{u_0} - p_{w_0} F_1 = 0, \\ &F_{11} a_0 + F_1^1 q + p_{u_1} = F_1^1 a_0 + F^{11} q + p_{v_1} = 0, \\ &q_{u_0} + q_{w_0} F_1 = q_{v_1} + z = 0 \end{aligned} \tag{304}$$

for the unknown functions

$$\begin{aligned} p &= p(x, u_0, v_0, w_0, u_1, v_1), \\ q &= q(x, u_0, v_0, w_0, v_1), \\ z &= z(x, u_0, v_0, w_0, v_1) \end{aligned} \tag{305}$$

looks more complicated. One can also obtain the formula

$$\lambda^2 = q_{v_1} DF^1 - F_1^1 a_1 - F^{11} Dq + p_{v_0} + p_{w_0} F^1 \tag{306}$$

for the important coefficient λ^2. Let us again mention only the particular case $F = u_1 v_1$. Then the resolving system is simplified as

$$\begin{aligned} &q_{v_1} v_2 - Dq + p_{u_0} + p_{w_0} v_1 = q + p_{u_1} \\ &\quad = qu_2 + Dp - p_{v_1} v_2 = q_{u_0} + q_{w_0} v_1 = 0. \end{aligned} \tag{307}$$

It follows immediately that $p = -qu_1 + P$ and the resolving system is reduced to the equations

$$\begin{aligned} &\mathscr{D} q = P_{u_0} + P_{w_0} v_1, \qquad u_1 \mathscr{D} q = \mathscr{D} P, \\ &q_{u_0} + q_{w_0} v_1 = 0 \\ &\left(\mathscr{D} = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u_0} + v_1 \frac{\partial}{\partial v_0} + F \frac{\partial}{\partial w_0}\right) \end{aligned} \tag{308}$$

for the unknown functions P and q of variables x, u_0, v_0, w_0, v_1. This implies that $q = Q(x, w, v_0, v_1)$, where $w = w_0 - v_1 u_0$ and we obtain two equations

$$Q_x + Q_{v_0} v_1 = P_{u_0} + P_{w_0} v_1, \qquad P_x + P_{v_0} v_1 = 0 \tag{309}$$

with the solution $P = \overline{P}(v_0 - xv_1, w_0 - u_0 v_1, v_1) + \overline{Q}$, where $\overline{P}$ may be arbitrary function while $\overline{Q} = \overline{Q}(x, u_0, v_0, w_0, v_1)$ is a fixed particular solution of differential equation

$$Q_x + Q_{v_0} v_1 = \overline{Q}_{u_0} + \overline{Q}_{w_0} v_1 \tag{310}$$

satisfying moreover the identity $\overline{Q}_x + \overline{Q}_{v_0} v_1 = 0$. We may choose the particular solution $\overline{Q} = (Q_x + Q_{v_0} v_1) u_0$. Then the identity turns into the requirement $(\partial/\partial x + v_1 \partial/\partial v_0)^2 Q = 0$ which is satisfied if

$$Q = Q_1(w, v_1) x + Q_0(w, v_1) \quad (w = w_0 - v_1 u_0). \tag{311}$$

Altogether taken, we have obtained the final solution

$$\begin{aligned} p &= -qu_1 + \overline{P} + \left(Q_x + Q_{v_0} v_1\right) u_0, \\ q &= Q = Q_1 x + Q_0, \end{aligned} \tag{312}$$

where

$$\overline{P} = \overline{P}\left(v_0 - xv_1, w, v_1\right), \qquad Q_1 = Q_1\left(w, v_1\right),$$
$$Q_0 = Q_0\left(w, v_1\right), \qquad w = w_0 - v_1 u_0 \tag{313}$$

and $\overline{P}$, Q_1, Q_0 are quite arbitrary functions. The above-mentioned coefficient λ^2 does not in general vanish. (Indeed, look at the top-order summands

$$\lambda^2 = \cdots - F_1^1 a_1 + \cdots = \cdots - a_1 + \cdots, \tag{314}$$

where $v_2 a_0 = \cdots + Dp$ by virtue of (263); hence, $v_2 a_1 = \cdots + D^2 p = \cdots - D^2(qu_1) = \cdots - qu_3$ may be substituted.) We again have an order-increasing infinitesimal symmetry.

Remark 45. Variations Z satisfying (269) preserve the Pfaffian system $\pi_0^1 = \pi_0^2 = \pi_1^1 = 0$ and therefore generate a group for analogous reasons as in Remark 29. Variations Z satisfying (297) preserve the Pfaffian system $\pi_0^1 = \pi_0^2 = \pi_1^2 = 0$ and the case of requirements (303) is quite trivial in this respect. It follows that we have indeed obtained the *infinitesimal symmetries* Z.

11. Concluding Example on Order-Increasing Symmetries

Passing from infinitesimal symmetries Z to the true symmetries $\mathbf{m}$, the linear theory is replaced with highly nonlinear area of Pfaffian equations and the prolongation into involutiveness. In accordance with E. Cartan's notice, nobody should expect such easily available results as in the Lie's infinitesimal theory. Our modest aim is twofold: to perform an economical reduction of the symmetry problem to finite dimension and to point out a useful interrelation between appropriate *variations* Z and *one-parameter families* $\mathbf{m}(t)$ of higher-order symmetries. We again deal only with (260).

(ι) Setting the Problem. Let us deal with symmetries $\mathbf{m}$ such that

$$\begin{aligned}
&\mathbf{m}^* \pi_0^i \\
&= a^i \alpha_0 + b^i \beta_0 + c^i \gamma_0 \\
&= \left(a^i + F_1 c^i\right)\alpha_0 + \left(b^i + F^1 c^i\right)\beta_0 + c^i \pi_0^1 \quad (i = 1, 2).
\end{aligned} \tag{315}$$

Invertibility of $\mathbf{m}$ is obviously ensured if

$$\det\begin{pmatrix} a^1 + F_1 c^1 & a^2 + F_1 c^2 \\ b^1 + F^1 c^1 & b^2 + F^1 c^2 \end{pmatrix} = 0,$$
$$\det\begin{pmatrix} a^1 & a^2 & Da^1 - uDa^2 \\ b^1 & b^2 & Db^1 - uDb^2 \\ c^1 & c^2 & Dc^1 - uDc^2 \end{pmatrix} \neq 0, \tag{316}$$

where

$$u = \frac{a^1 + F_1 c^1}{a^2 + F_1 c^2} = \frac{b^1 + F_1 c^1}{b^2 + F_1 c^2}. \tag{317}$$

We tacitly suppose $a^2 + F_1 c^2 \neq 0$, $b^2 + F^1 c^2 \neq 0$ and one can observe that the particular case $u = 0$ provides the traditional order-preserving symmetries. Equations (315) can be simplified to the equivalent system of equations

$$\mathbf{m}^* \pi_0^1 - u\mathbf{m}^* \pi_0^2 = v\pi_0^1,$$
$$\mathbf{m}^* \pi_0^2 = a\alpha_0 + b\beta_0 + c\gamma_0, \tag{318}$$

where $v = c^1 - uc^2$ and $a = a^2$, $b = b^2$, $c = c^2$. The invertibility is ensured by the inequalities

$$v \neq 0, \qquad \det\begin{pmatrix} a & F_1 & DF_1 \\ b & F^1 & DF^1 \\ c & -1 & 0 \end{pmatrix} \neq 0. \tag{319}$$

Equations (318) will be represented by a Pfaffian system in a certain finite-dimensional space; however, let us again simplify the notation by bars; for example,

$$\overline{x} = \mathbf{m}^* x, \qquad \overline{\pi}_r^i = \mathbf{m}^* \pi_r^i, \qquad \overline{\alpha}_r = \mathbf{m}^* \alpha_r,$$
$$\overline{\beta}_r = \overline{\pi}_r^2 = \mathbf{m}^* \beta_r = \mathbf{m}^* \pi_r^2, \tag{320}$$

and so like. Then we have the system

$$\overline{\pi}_0^1 - u\overline{\pi}_0^2 = v\pi_0^1, \qquad \overline{\pi}_0^2 = a\alpha_0 + b\beta_0 + c\gamma_0 \tag{321}$$

which should be completed by the exterior derivatives

$$\begin{aligned}
&d\overline{\pi}_0^1 - ud\overline{\pi}_0^2 - du \wedge \overline{\pi}_0^2 = dv \wedge \pi_0^1 + vd\pi_0^1, \\
&d\overline{\pi}_0^2 = da \wedge \alpha_0 + db \wedge \beta_0 + dc \wedge \gamma_0 \\
&\qquad + dx \wedge \left(\left(a + F_1 c\right)\alpha_1 + \left(b + F^1 c\right)\beta_1\right).
\end{aligned} \tag{322}$$

We refer to (264) for terms $d\pi_0^1$, $d\overline{\pi}_0^1$ appearing here. We have obtained the *compatibility problem* of (322). The familiar *prolongation criterion* can be shortly expressed as follows. All coefficients and variables with bars are functions of the primary jet variables. So we may suppose, for example,

$$du = Udx + \sum u_r^1 \alpha_r + \sum u_r^2 \beta_r + u_0^3 \gamma_0$$
$$(U = Du) \tag{323}$$

(with summands of uncertain lengths) and analogously for dv, da, db, dc with a little adjustment for the differential

$$d\overline{x} = \lambda\left(dx + \sum \lambda_r^1 \alpha_r + \sum \lambda_r^2 \beta_r + \lambda_0^3 \gamma_0\right)$$
$$\left(\lambda = D\overline{x} = D\mathbf{m}^* x = DW\right). \tag{324}$$

Such substitutions into (322) should give identities. However, a short inspection of the summand $\overline{F}_{11}\overline{\alpha}_0 \wedge \overline{\alpha}_1$ in differential $d\overline{\pi}_0^1$ implies that then necessarily either $u = 0$ (the group case) or $F_{11} = 0$ identically.

(ιι) A Particular Case. It follows that the assumption $F = f(v_1)u_1 + g(v_1)$ is necessary; however, let us again suppose

$F = u_1 v_1$ from now on. Then (321) may be retained and (322) become more explicit

$$d\overline{x} \wedge \left(\overline{\pi}_1^1 - u\overline{\pi}_1^2\right) + \overline{\beta}_0 \wedge \overline{\alpha}_1 + \overline{\alpha}_0 \wedge \overline{\beta}_1 - du \wedge \overline{\pi}_0^2 = dv \wedge \pi_0^1 + v\left(dx \wedge \pi_1^1 + \beta_0 \wedge \alpha_1 + \alpha_0 \wedge \beta_1\right), \tag{325}$$

$$d\overline{x} \wedge \overline{\pi}_1^2 = da \wedge \alpha_0 + db \wedge \beta_0 + dc \wedge \gamma_0 + dx \wedge \left((a + v_1 c)\alpha_1 + (b + u_1 c)\beta_1\right). \tag{326}$$

We turn to the prolongation procedure in more detail.

(ιι) On the Equation (326). The prolongation should satisfy the identity

$$\begin{aligned} &\lambda\left(dx + \sum \lambda_r^1 \alpha_r + \sum \lambda_r^2 \beta_r + \lambda_0^3 \gamma_0\right) \\ &\quad \wedge \frac{1}{\lambda}\left(A\alpha_0 + B\beta_0 + C\gamma_0 + (a + v_1 c)\alpha_1 + (b + u_1 c)\beta_1\right) \\ &= \left(Adx + \sum a_r^1 \alpha_r + \sum a_r^2 \beta_r + a_0^3 \gamma_0\right) \wedge \alpha_0 \\ &\quad + \cdots + \left(Cdx + \sum c_r^1 \alpha_r + \sum c_r^2 \beta_r + c_0^3 \gamma_0\right) \wedge \gamma_0 \\ &\quad + dx \wedge \left((a + v_1 c)\alpha_1 + (b + u_1 c)\beta_1\right), \end{aligned} \tag{327}$$

where $A = Da$, $B = Db$, $C = Dc$. All summands with factor $dx\wedge$ mutually cancel. Then we conclude that necessarily $\lambda_r^1 = \lambda_r^2 = 0$ $(r > 1)$ and also $a_r^1 = \cdots = c_r^2 = 0$ $(r > 1)$. It follows that *the problem is reduced to finite dimension:* $\overline{x}, a, b, c$ are functions only of coordinates x, u_0, v_0, u_1, v_1. Even the explicit formulae can be easily obtained as follows

$$\begin{gathered} a_1^1 = \lambda_1^1 A - \lambda_0^1 (a + v_1 c), \ldots, b_1^2 = \lambda_1^2 B - \lambda_0^2 (b + u_1 c), \\ b_0^1 - a_0^2 = \lambda_0^1 B - \lambda_0^2 A, \\ c_0^1 - a_0^3 = \lambda_0^1 C - \lambda_0^3 A, \\ c_0^2 - b_0^3 = \lambda_0^2 C - \lambda_0^3 B \end{gathered} \tag{328}$$

for the prolongation where moreover $\lambda_1^1(b+u_1 c) = \lambda_1^2(a+v_1 c)$ is supposed.

(ιν) On the Equation (325). Calculations modulo dx are also sufficient here. The prolongation should satisfy

$$\begin{aligned} &\left(\lambda_0^1 \alpha_0 + \lambda_0^2 \beta_0 + \lambda_0^3 \gamma_0 + \lambda_1^1 \alpha_1 + \lambda_1^2 \beta_1\right) \\ &\quad \wedge \left(V\pi_0^1 + v\pi_1^1 + U\overline{\pi}_0^2\right) \\ &\quad - \left(\sum u_r^1 \alpha_r + \sum u_r^2 \beta_r + u_0^3 \gamma_0\right) \wedge \overline{\pi}_0^2 \\ &\quad - \left(\sum v_r^1 \alpha_r + \sum v_r^2 \beta_r + v_0^3 \gamma_0\right) \wedge \pi_0^1 \\ &= -\overline{\beta}_0 \wedge \overline{\alpha}_1 - \overline{\alpha}_0 \wedge \overline{\beta}_1 + v\left(\beta_0 \wedge \alpha_1 + \alpha_0 \wedge \beta_1\right), \end{aligned} \tag{329}$$

where $U = Du$, $V = Dv$. We have used the identity

$$\overline{\pi}_1^1 - u\overline{\pi}_1^2 = \frac{1}{\lambda}\mathscr{L}_D\left(\overline{\pi}_0^1 - u\overline{\pi}_0^2\right) + \frac{U}{\lambda}\overline{\pi}_0^2, \tag{330}$$

where $\overline{\pi}_0^1 - u\overline{\pi}_0^2 = v\pi_0^1$ is moreover substituted. In order to prove the existence of prolongation, the right-hand side terms should be made more explicit. We recall the identity (263) which gives

$$\begin{gathered} \pi_1^1 + v_2 \alpha_0 + u_2 \pi_0^2 = 0, \\ \pi_2^1 + v_2 \alpha_0 + v_1 \alpha_1 + u_3 \pi_0^2 + u_2 \pi_1^2 = 0 \end{gathered} \tag{331}$$

and uniquely determines the forms $\overline{\alpha}_0$ and $\overline{\alpha}_1$ in terms of forms $\overline{\pi}_1^1, \overline{\pi}_0^2, \overline{\pi}_2^1, \overline{\pi}_1^2$ and therefore in terms of contact forms α_r, β_r, and γ_0, if the rule (265) is applied to the primary equations (321). The result is that

$$\begin{aligned} \overline{\beta}_0 \wedge \overline{\alpha}_1 &= \overline{\pi}_0^2 \wedge (\text{a certain sum of forms } \alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \alpha_2, \beta_2) \\ \overline{\alpha}_0 \wedge \overline{\beta}_1 &= -\frac{1}{\overline{v}_2}\left(\overline{u}_2 \overline{\pi}_0^2 + \overline{\pi}_1^1\right) \wedge (\text{a certain sum of forms } \alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1). \end{aligned} \tag{332}$$

On the other hand, inequalities (266) imply that the factors

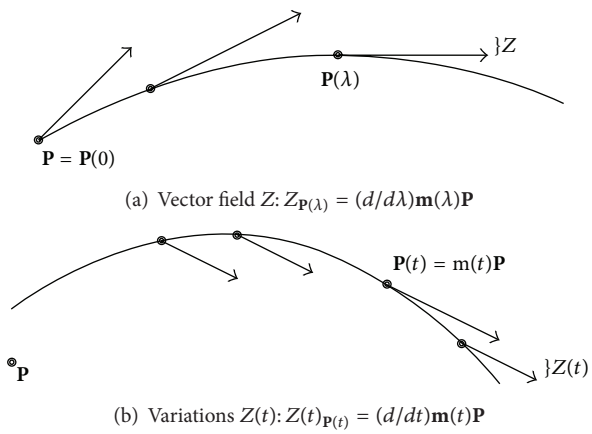
$$V\pi_0^1 + v\pi_1^1 + U\overline{\pi}_0^2, \overline{\pi}_0^2, \pi_1^1 \tag{333}$$

on the left-hand side of (329) are linearly independent and we conclude that the prolongation can be realized. Moreover u becomes a function of second-order coordinates while x and v are functions of first-order coordinates x, u_0, v_0, w_0, u_1, v as before. *The problem is again reduced to finite-dimension*; however, we do not state explicit formula for the prolongation here.

Remark 46. In accordance with Lie's classical theory, the existence of *infinitesimal symmetries* Z (Figure 5(a)) is equivalent to the existence of a one-parameter *group* $\mathbf{m}(\lambda)$ of symmetries (Figures 3(a) and 3(b)) due to the solvability of the Lie system ensured by Theorem 24. Alas, the "genuine" higher-order symmetries (Figure 3(c)) cannot be obtained in this way and they rest on the toilsome mechanisms of Pfaffian systems. We nevertheless propose a hopeful conjecture as follows. Every one-parameter *family* $\mathbf{m}(t)$ of symmetries ensures the existence of many *variations* $Z(t)$ depending on parameter t (Figure 5(b)). We believe that the converse can be proved as well: one-parameter *families* of symmetries can be reconstructed from a "sufficiently large" supply of *variations*. Indeed, if t is regarded as additional variable of the underlying space, then the family $Z(t)$ turns into a single vector field.

In any case, the existence of many variations is a *necessary condition* for the existence of "genuine" higher-order symmetries and the following point (ν) will be instructive in this respect.

(ν) On the Variations. If a one-parameter family $\mathbf{m}(t) = \mathbf{m}$ (abbreviation) satisfies (318) then the corresponding family



(a) Vector field Z: $Z_{\mathbf{P}(\lambda)} = (d/d\lambda)\mathbf{m}(\lambda)\mathbf{P}$

(b) Variations $Z(t)$: $Z(t)_{\mathbf{P}(t)} = (d/dt)\mathbf{m}(t)\mathbf{P}$

Figure 5

$Z(t) = Z$ (abbreviation) of variations clearly satisfies the system

$$\begin{aligned} &\mathscr{L}_Z\pi_0^1 - u\mathscr{L}_Z\pi_0^2 - u'\pi_0^2 = v'\pi_0^1, \\ &\mathscr{L}_Z\pi_0^2 = a'\alpha_0 + b'\beta_0 + c'\gamma_0, \end{aligned} \tag{334}$$

where $u' = Zu, \ldots, c' = Zc$ may be regarded as new parameters. Assuming formula (268), one can obtain the resolving system

$$\begin{aligned} &zv_2 + Dq - p_{u_0} - a'v_1 + uq_{u_0} \\ &\quad = q_{u_0} + a'v_1 + c'v_2 = 0, \end{aligned} \tag{335}$$

$$\begin{aligned} &zu_2 + a_1 - p_{v_0} - a'u_1 + u' + uq_{v_0} \\ &\quad = q_{v_0} + a'u_1 - b' + c'u_2 = 0, \end{aligned} \tag{336}$$

$$\begin{aligned} &q + p_{u_1} \\ &\quad = q_{u_1} = a_0 + p_{v_1} = z + q_{v_1} \\ &\quad = p_{w_0} - v' - uq_{w_0} = q_{w_0} - a' = 0. \end{aligned} \tag{337}$$

It follows from right-hand equations (337) that $p = -qu_1 + \overline{q}$, where $q, \overline{q}$ do not depend on u_1. Recalling the identity

$$a_0v_2 + qu_2 + Dp = 0, \tag{338}$$

then the middle equations (337) yield the conditions

$$\begin{aligned} q_{u_0} + q_{w_0}v_1 &= \overline{q}_x + \overline{q}_{v_0}v_1 \\ &= q_x + q_{v_0}v_1 - \overline{q}_{u_0} - \overline{q}_{w_0}v_1 = 0 \end{aligned} \tag{339}$$

and $q = q(x, u_0, v_0, v_1, w_0)$, $\overline{q} = \overline{q}(x, u_0, v_0, v_1, w_0)$. With this result, (335) turns into identity and (336) reduces to the equation

$$u' = u\left(q_{w_0}u_1 - q_{v_0}\right) + u_2q_{v_1} + D\left(q_{v_1}u_1 - \overline{q}_{v_1}\right) + \overline{q}_{v_0} \tag{340}$$

for the parameter u'. Equations (339) are trivially satisfied if

$$\begin{aligned} q &= Q\left(xv_1 - v_0, u_0v_1 - w_0, v_1\right), \\ \overline{q} &= \overline{Q}\left(xv_1 - v_0, u_0v_1 - w_0, v_1\right). \end{aligned} \tag{341}$$

There exist many variations corresponding to (318). The *necessary condition* for the existence of higher-order symmetries is satisfied.

Remark 47. Let us briefly sketch the connection to the general equivalence method [23] by using slightly adapted Cartan's notation. We consider space $\mathbb{R}^n$ (and its counterpart $\overline{\mathbb{R}}^n$) with coordinates $(x) = (x_1, \ldots, x_n)$ (or $(\overline{x}) = (\overline{x}_1, \ldots, \overline{x}_n)$, resp.) and linearly independent 1-forms $\omega_1, \ldots, \omega_n$ (and $\overline{\omega}_1, \ldots, \overline{\omega}_n$). In the classical equivalence problem, a mapping $\mathbf{m}$ should be determined such that

$$\begin{aligned} &\mathbf{m}^*\overline{\omega}_i = \sum a_{ij}(u)\,\omega_j \\ &\left(i, j = 1, \ldots, n; (x) = \mathbf{m}^*(\overline{x}), u = u(x)\right), \end{aligned} \tag{342}$$

where $(a_{ij}(u))$ is a matrix of a linear group with parameters $(u) = (u_1, \ldots, u_r)$. In Cartan's approach, this requirement is made symmetrical:

$$\begin{aligned} &\mathbf{m}^* \sum a_{ij}(\overline{u})\,\overline{\omega}_i = \sum a_{ij}(u)\,\omega_j \\ &\left((x) = \mathbf{m}^*(\overline{x}), \overline{u} = \overline{u}(x, u)\right). \end{aligned} \tag{343}$$

This provides the invariant differential forms by appropriate simultaneous adjustments of both sides (343). Such procedure fails, if $(a_{ij}(x))$ is not a matrix of a linear group which happens just in the case of higher-order symmetries on Figure 3(c). Then the corresponding total system (342) with $i, j = 1, 2, \ldots$ is invertible only in the infinite-dimensional underlying space $\mathbf{M}$ and $(a_{ij}(u))$ need not be even a square matrix in any finite portion of the system (342). On the other hand, such a finite portion is quite sufficient since Lemma 17 ensures the extension on the total space $\mathbf{M}$. The "symmetrization" procedure cannot be applied, invariant differential forms need not exist, and only the common prolongation procedure is available, if the problem is reduced to a finite-dimensional subspace of $\mathbf{M}$.

12. Concluding Survey

Our approach to differential equations and our methods differ from the common traditional use. For better clarity, let us briefly report the main novelties as follows: clear interrelation between the external and internal concepts in Remarks 1 and 2; introduction and frequent use of "nonholonomic" series (18); the "absolute" and coordinate-free Definition 4 of ordinary differential equations; the distinction between variations and infinitesimal symmetries in Definition 8; the main tool, the standard bases generalizing the common contact forms in jet spaces; the invariance of constants $K = K(\Omega)$ and $\mu = \mu(\Omega)$, the controllability concept related to the Mayer problem; the distinction between order-preserving,

group-like, and true higher-order symmetries in Figure 1; technical Lemmas 17, 19, and 23 and Theorem 24 which provide new universal method of solution of the higher-order symmetry problem; new explicit formula as (136) for the famous and "well-known" symmetry problem of a Monge equation with two unknown functions; the Lagrange variational problem without Lagrange multipliers and with easy proofs; see Theorem 41; particular results of new kind for the Monge equation with three unknown functions; a note on the insufficience of G-structures in Remark 47.

All these achievements can be carried over the partial differential equations.

On this occasion, the actual extensive theory of the *control systems*

$$\frac{dx}{dt} = f(x,u) \quad \left(t \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathbb{R}^m\right) \tag{344}$$

is worth mentioning. It may be regarded as a mere formally adapted individual subcase of the theory of underdetermined systems of ordinary differential equations. However, the exceptional role of the *independent variable* t (the change of notation), the *state variables* x, and the *control* u is emphasized in applications; see [24–26] and references therein. In particular, only the t-preserving and moreover t-independent symmetries of the system (344) are accepted. So in our notation (1), such restriction means that we suppose $\overline{x} = W = x$ and functions W^i are independent of x. This is a fatal restriction of the impact of the theory of control systems. It follows that the results of this theory *do not imply* the classical results by Lie and Cartan; they are of rather special nature. The lack of new effective methods adapted to the control systems theory should be moreover noted. The absence of explicit solutions of particular examples is also symptomatic. Last but not least, unlike our diffieties, the control systems *cannot* be reasonably generalized for the partial differential equations.

We believe that the internal and higher-order approach to some nonholonomic theories are possible, for instance, in the case of the higher-order subriemannian geometry [12]. It seems that the advanced results [27] in the theory of geodesics can be appropriately adapted and rephrased in terms of invariants (as in [28]) instead of adjoint tensor fields.

Appendix

A nontrivial automorphism $\mathbf{m}$ of the jet space $\mathbf{M}(3)$ related to the theory of differential equation (260) is worth mentioning [9, pp. 44–46] without additional comments. In terms of usual jet coordinates x, w_r^i $(i = 1,2,3; r = 0,1,\ldots)$ on the space $\mathbf{M}(3)$, we put

$$\mathbf{m}^* w_0^1 = w_1^1, \qquad \mathbf{m}^* w_0^2 = w_1^2, \qquad \mathbf{m}^* x = w_1^3, \tag{A.1}$$

and moreover

$$\mathbf{m}^* w_0^3 = \det\begin{pmatrix} x w_1^1 - w_0^1 & \mathbf{m}^* w_1^1 & \mathbf{m}^* w_2^1 \\ x w_1^2 - w_0^2 & \mathbf{m}^* w_1^2 & \mathbf{m}^* w_2^2 \\ x w_1^3 - w_0^3 & 1 & 0 \end{pmatrix}. \tag{A.2}$$

The morphism $\mathbf{m}$ is rigorously defined since the transforms

$$\mathbf{m}^* w_1^j = \frac{\mathbf{m}^* w_0^j}{\mathbf{m}^* x}, \qquad \mathbf{m}^* w_2^j = \frac{\mathbf{m}^* w_1^j}{\mathbf{m}^* x} \quad (j = 1,2) \tag{A.3}$$

are well-known due to the prolongation (6). The point of construction is as follows. We have

$$\begin{aligned} \mathbf{m}^*\left(x - F\left(w_0^1, w_0^2\right)\right) &= \mathbf{m}^* x - F\left(\mathbf{m}^* w_0^1, \mathbf{m}^* w_0^2\right) \\ &= w_1^3 - F\left(w_1^1, w_1^2\right). \end{aligned} \tag{A.4}$$

So, assuming the invertibility of $\mathbf{m}$, *differential equations* (260) *are identified with subspaces* $\mathbf{N} \subset \mathbf{M}(3)$ *given by equations* $x = F(w_0^1, w_0^2)$. Every such a subspace $\mathbf{N}$ with given $F \neq$ const. is clearly isomorphic to the jet space $\mathbf{M}(2)$. We conclude that *the diffiety* Ω *corresponding to given equation* (260) *is isomorphic to the diffiety* $\Omega(2)$ *of all curves in three-dimensional space* $\mathbb{R}^3$ and therefore admits huge supply of higher-order symmetries; see [4, Section 7] for quite simple examples.

Let us turn to the invertibility problem. We introduce a morphism $\mathbf{n}$ which will be identified with the sought inverse $\mathbf{m}^{-1}$. The definition is as follows. Let us introduce functions a, b, c determined by three linear equations

$$\begin{gathered} \det\begin{pmatrix} a & w_1^1 & w_2^1 \\ b & w_1^2 & w_2^2 \\ c & 1 & 0 \end{pmatrix} = w_0^3, \\ \det\begin{pmatrix} a & w_1^1 & w_3^1 \\ b & w_1^2 & w_3^2 \\ c & 1 & 0 \end{pmatrix} = w_1^3, \\ \det\begin{pmatrix} a & w_2^1 & w_3^1 \\ b & w_2^2 & w_3^2 \\ c & 0 & 0 \end{pmatrix} + \det\begin{pmatrix} a & w_1^1 & w_4^1 \\ b & w_1^2 & w_4^2 \\ c & 1 & 0 \end{pmatrix} = w_2^3. \end{gathered} \tag{A.5}$$

It follows that functions a, b, c moreover satisfy

$$\det\begin{pmatrix} Da & w_1^1 & w_2^1 \\ Db & w_1^2 & w_2^2 \\ Dc & 1 & 0 \end{pmatrix} = \det\begin{pmatrix} Da & w_1^1 & w_3^1 \\ Db & w_1^2 & w_3^2 \\ Dc & 1 & 0 \end{pmatrix} = 0 \tag{A.6}$$

whence the equations

$$\begin{gathered} Da = w_1^1 \mathbf{n}^* x, \qquad Db = w_1^2 \mathbf{n}^* x, \\ Dc = 1 \cdot \mathbf{n}^* x = \mathbf{n}^* x \end{gathered} \tag{A.7}$$

uniquely define function $\mathbf{n}^* x$. We finally put

$$\begin{gathered} \mathbf{n}^* w_0^1 = w_0^1 \mathbf{n}^* x - a, \qquad \mathbf{n}^* w_0^2 = w_0^2 \mathbf{n}^* x - b, \\ \mathbf{n}^* w_0^3 = x \mathbf{n}^* x - c \end{gathered} \tag{A.8}$$

and then

$$\begin{gathered} D\mathbf{n}^* w_0^1 = w_1^1 \mathbf{n}^* x + w_0^1 D\mathbf{n}^* x - Da = w_0^1 D\mathbf{n}^* x, \\ D\mathbf{n}^* w_0^2 = \cdots = w_0^2 D\mathbf{n}^* x, \\ D\mathbf{n}^* w_0^3 = \cdots = x. \end{gathered} \tag{A.9}$$

It follows that functions w_0^1, w_0^2, x and hence a, b, c and even the coordinate w_0^3 can be expressed in terms of certain pullbacks $\mathbf{n}^*$. Therefore $\mathbf{n}$ is invertible and moreover $\mathbf{n} = \mathbf{m}^{-1}$. Indeed, the last three equations read

$$w_0^1 = \frac{1}{D\mathbf{n}^* x} D\mathbf{n}^* w_0^1 = \mathbf{n}^* D w_0^1 = \mathbf{n}^* w_1^1,$$
$$w_0^2 = \cdots = \mathbf{n}^* w_1^2, \qquad x = \cdots = \mathbf{n}^* w_1^3 \tag{A.10}$$

in full accordance with the initial equations (A.1) and formula (A.2) follows from (A.8).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On the Periodicity of Some Classes of Systems of Nonlinear Difference Equations

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Some classes of systems of difference equations whose all well-defined solutions are periodic are presented in this note.

1. Introduction

There has been a great recent interest in studying difference equations and systems of difference equations which do not stem from differential ones (see, e.g., [1–19] and the references therein). For some results on concrete systems of nonlinear difference equations, see, for example, [1, 3–5, 9–12, 18, 19]. Some classical results in the topic can be found, for example, in book [20].

Solution $(x_n^{(1)},\dots,x_n^{(l)})_{n\geq -k}$, of the system of difference equations

$$\begin{aligned} x_n^{(1)} &= f_1\left(x_{n-1}^{(1)},\dots,x_{n-k_1}^{(1)},\dots,x_{n-1}^{(l)},\dots,x_{n-k_l}^{(l)}\right),\\ x_n^{(2)} &= f_2\left(x_{n-1}^{(1)},\dots,x_{n-k_1}^{(1)},\dots,x_{n-1}^{(l)},\dots,x_{n-k_l}^{(l)}\right),\\ &\vdots\\ x_n^{(l)} &= f_l\left(x_{n-1}^{(1)},\dots,x_{n-k_1}^{(1)},\dots,x_{n-1}^{(l)},\dots,x_{n-k_l}^{(l)}\right), \end{aligned} \tag{1}$$

where $n\in\mathbb{N}_0$ and $k=\max\{k_1,\dots,k_l\}$, is called eventually periodic with period p, if there is an $n_1\geq -k$ such that

$$x_{n+p}^{(j)} = x_n^{(j)}, \tag{2}$$

for every $j=\overline{1,l}$, and $n\geq n_1$. It is periodic with period p if $n_1=-k$. Period p is prime if there is no $\widehat{p}\in\mathbb{N}$, $\widehat{p}<p$, which is a period. If all well-defined solutions of an equation or a system of difference equations are eventually periodic with the same period, then such an equation or system is called periodic. For some results on the periodicity, asymptotic periodicity and periodic equations or systems of difference equations see, for example, [1–10, 12–14, 16–19] and the related references therein.

In recent paper [19], the authors formulated four results which claim that the following systems of difference equations are periodic with period ten:

$$x_{n+1}=\frac{y_n}{x_{n-1}(1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(1+x_n)},\quad n\in\mathbb{N}_0; \tag{3}$$

$$x_{n+1}=\frac{y_n}{x_{n-1}(-1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(-1+x_n)},\quad n\in\mathbb{N}_0; \tag{4}$$

$$x_{n+1}=\frac{y_n}{x_{n-1}(1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(-1+x_n)},\quad n\in\mathbb{N}_0; \tag{5}$$

$$x_{n+1}=\frac{y_n}{x_{n-1}(-1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(1+x_n)},\quad n\in\mathbb{N}_0. \tag{6}$$

First, we show that all the results in [19] follow from known ones in the literature and also present some extensions

of these results in the spirit of systems (3)–(6). To do this, we will use a system of difference equations related to the following, so called, Lyness difference equation:

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{7}$$

It is easy to see that every well-defined solution of (7) is periodic with period five. The equation arises in frieze patterns (for the original sources, see [21–23]).

Studying max-type equations and systems of difference equations is another topic of a recent interest (see, e.g, [2, 3, 5–7, 10, 11, 15–19]).

Some special cases of the following max-type difference equation:

$$x_n = \max\left\{\frac{A_n}{x_{n-s}}, x_{n-k}\right\}, \quad n \in \mathbb{N}_0, \tag{8}$$

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$, have been studied, for example, in [2, 16]. Positive solutions of (8) are periodic in many cases. However, if $(A_n)_{n\in\mathbb{N}_0}$ is not a positive sequence, it was shown in [2] that (8) can have unbounded solutions.

In [5], it was shown that all solutions of the following max-type system of difference equations:

$$x_{n+1} = \max\left\{\frac{A_n}{y_n}, x_{n-1}\right\}, \qquad cy_{n+1} = \max\left\{\frac{B_n}{x_n}, y_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{9}$$

where $x_0, x_{-1}, y_0, y_{-1} \in (0, +\infty)$ and $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ are positive two-periodic sequences, are eventually periodic with, not necessarily prime, period two. This was done by direct calculation.

By using direct calculation, it can be easily shown that positive solutions of the following max-type system of difference equations:

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, y_{n-1}\right\}, \qquad y_{n+1} = \max\left\{\frac{B_n}{y_n}, x_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{10}$$

where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are positive two-periodic sequences, are also periodic.

Here, we give a noncalculatory explanation of the fact by proving that positive solutions of the following max-type system of difference equations:

$$x_n = \max\left\{\frac{A_n}{x_{n-s}}, y_{n-k}\right\}, \qquad y_n = \max\left\{\frac{B_n}{y_{n-s}}, x_{n-k}\right\}, \quad n \in \mathbb{N}_0, \tag{11}$$

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ are positive periodic sequences of a certain period, are also periodic. We also present another extension of the result.

2. Some Extensions of Systems (3)–(6)

In this section, we present some periodic systems of difference equations in the spirit of systems (3)–(6).

Theorem 1. *Consider the following system of difference equations*

$$\begin{aligned} x_{n+1}^{(1)} &= f_1^{-1}\left(\frac{1 + f_2\left(x_n^{(2)}\right)}{f_3\left(x_{n-1}^{(3)}\right)}\right), \\ &\vdots \\ x_{n+1}^{(k-1)} &= f_{k-1}^{-1}\left(\frac{1 + f_k\left(x_n^{(k)}\right)}{f_1\left(x_{n-1}^{(1)}\right)}\right), \\ x_{n+1}^{(k)} &= f_k^{-1}\left(\frac{1 + f_1\left(x_n^{(1)}\right)}{f_2\left(x_{n-1}^{(2)}\right)}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{12}$$

where $k \in \mathbb{N}\setminus\{1\}$, and functions f_j, $j = \overline{1,k}$, are continuous on their domains; map the set $\mathbb{R}\setminus\{0\}$ onto itself and, for each fixed $j \in \{1,\ldots,k\}$, f_j is simultaneously increasing or decreasing on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

Then the following statements hold.

(a) *If $k \not\equiv 0 \pmod 5$, then every well-defined solution of system (12) is periodic with period $5k$.*

(b) *If $k \equiv 0 \pmod 5$, then every well-defined solution of system (12) is periodic with period k.*

Proof. From the conditions of the theorem, it follows that for each $j \in \{1,\ldots,k\}$, there is f_j^{-1} which continuously map the set $\mathbb{R}\setminus\{0\}$ onto itself. Using the change of variables

$$y_n^{(j)} = f_j\left(x_n^{(j)}\right), \quad j = \overline{1,k}, \tag{13}$$

system (12) is easily transformed into the next one

$$y_{n+1}^{(1)} = \frac{1 + y_n^{(2)}}{y_{n-1}^{(1)}}, \qquad y_{n+1}^{(2)} = \frac{1 + y_n^{(1)}}{y_{n-1}^{(2)}}, \tag{14}$$

for $k = 2$,

$$y_{n+1}^{(1)} = \frac{1 + y_n^{(2)}}{y_{n-1}^{(3)}}, \qquad y_{n+1}^{(2)} = \frac{1 + y_n^{(3)}}{y_{n-1}^{(1)}}, \qquad y_{n+1}^{(3)} = \frac{1 + y_n^{(1)}}{y_{n-1}^{(2)}}, \tag{15}$$

for $k = 3$, and

$$y_{n+1}^{(1)} = \frac{1 + y_n^{(2)}}{y_{n-1}^{(3)}}, \qquad y_{n+1}^{(2)} = \frac{1 + y_n^{(3)}}{y_{n-1}^{(4)}}, \ldots, y_{n+1}^{(k)} = \frac{1 + y_n^{(1)}}{y_{n-1}^{(2)}}, \tag{16}$$

for $k \geq 4$. In [4], it was proved that, if $k \not\equiv 0 \pmod 5$, then every well-defined solution of systems (14)–(16) is periodic

with period $5k$, and, if $k \equiv 0 \pmod 5$, then every well-defined solution of systems (14)–(16) is periodic with period k. Using this along with the fact

$$x_n^{(j)} = f_j^{-1}\left(y_n^{(j)}\right), \quad j = \overline{1,k}, \tag{17}$$

following from (13), the results in (a) and (b) follow. □

The following theorem is proved in a similar way. Therefore, the proof will be omitted.

Theorem 2. *Consider the following system of difference equations*

$$\begin{aligned} x_{n+1}^{(1)} &= f_1^{-1}\left(\frac{1+f_k\left(x_n^{(k)}\right)}{f_{k-1}\left(x_{n-1}^{(k-1)}\right)}\right), \\ x_{n+1}^{(2)} &= f_2^{-1}\left(\frac{1+f_1\left(x_n^{(1)}\right)}{f_k\left(x_{n-1}^{(k)}\right)}\right), \\ &\vdots \\ x_{n+1}^{(k)} &= f_k^{-1}\left(\frac{1+f_{k-1}\left(x_n^{(k-1)}\right)}{f_{k-2}\left(x_{n-1}^{(k-2)}\right)}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{18}$$

where $k \in \mathbb{N} \setminus \{1\}$, and functions f_j, $j = \overline{1,k}$, are continuous on their domains; map the set $\mathbb{R} \setminus \{0\}$ onto itself and, for each fixed $j \in \{1,\ldots,k\}$, f_j is simultaneously increasing or decreasing on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

Then the following statements hold.

(a) *If $k \not\equiv 0 \pmod 5$, then every well-defined solution of system (18) is periodic with period $5k$.*

(b) *If $k \equiv 0 \pmod 5$, then every well-defined solution of system (18) is periodic with period k.*

Now, we show that all the results on the periodicity of the solutions of systems (3)–(6) in [19] follow from Theorems 1 and 2.

Corollary 3. *Systems of difference equations (3)–(6) are all periodic with period ten.*

Proof. For the systems of difference equations (3)–(6), we use the following changes of variables, respectively:

$$\begin{aligned} &x_n = \frac{1}{u_n}, \quad y_n = \frac{1}{v_n}, \quad n \in \mathbb{N}_0; \\ &x_n = -\frac{1}{u_n}, \quad y_n = -\frac{1}{v_n}, \quad n \in \mathbb{N}_0; \\ &x_n = -\frac{1}{u_n}, \quad y_n = \frac{1}{v_n}, \quad n \in \mathbb{N}_0; \\ &x_n = \frac{1}{u_n}, \quad y_n = -\frac{1}{v_n}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{19}$$

By using them, systems (3)–(6) are transformed into system (14). By applying Theorem 1(a), ten-periodicity of all well-defined solutions of system (14) follows, from which ten-periodicity of all well-defined solutions of systems (3)–(6) follows. □

The following four examples are natural extensions of systems (3)–(6).

Example 4. If we use the following functions:

$$f_1(x) = \frac{1}{x^l}, \qquad f_2(x) = \frac{1}{x^m}, \tag{20}$$

where l and m are odd integers, we see that all the conditions in Theorem 1 are applied with $k = 2$, so by using the theorem we obtain that the system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-1}}\left(\frac{y_n^m}{1+y_n^m}\right)^{1/l}, \\ y_{n+1} &= \frac{1}{y_{n-1}}\left(\frac{x_n^l}{1+x_n^l}\right)^{1/m}, \end{aligned} \tag{21}$$

$n \in \mathbb{N}_0$, is also periodic with period ten.

Example 5. For

$$f_1(x) = -\frac{1}{x^l}, \qquad f_2(x) = -\frac{1}{x^m}, \tag{22}$$

where l and m are also odd integers, all the conditions of Theorem 1 are again satisfied with $k = 2$. Using the theorem, it follows that the system

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-1}}\left(\frac{y_n^m}{-1+y_n^m}\right)^{1/l}, \\ y_{n+1} &= \frac{1}{y_{n-1}}\left(\frac{x_n^l}{-1+x_n^l}\right)^{1/m}, \end{aligned} \tag{23}$$

$n \in \mathbb{N}_0$, is ten-periodic.

Example 6. For

$$f_1(x) = \frac{1}{x^l}, \qquad f_2(x) = -\frac{1}{x^m}, \tag{24}$$

where l and m are odd integers, all the conditions of Theorem 1 are also satisfied with $k = 2$. Applying the theorem, it follows that the system

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-1}}\left(\frac{y_n^m}{-1+y_n^m}\right)^{1/l}, \\ y_{n+1} &= \frac{1}{y_{n-1}}\left(\frac{x_n^l}{1+x_n^l}\right)^{1/m}, \end{aligned} \tag{25}$$

$n \in \mathbb{N}_0$, is ten-periodic.

Example 7. Finally, for

$$f_1(x) = -\frac{1}{x^l}, \qquad f_2(x) = \frac{1}{x^m}, \tag{26}$$

where l and m are odd integers and applying Theorem 1 with $k = 2$, we get that the system

$$x_{n+1} = \frac{1}{x_{n-1}}\left(\frac{y_n^m}{1+y_n^m}\right)^{1/l}, \qquad y_{n+1} = \frac{1}{y_{n-1}}\left(\frac{x_n^l}{-1+x_n^l}\right)^{1/m}, \tag{27}$$

$n \in \mathbb{N}_0$, is ten-periodic.

The main results in [4] can be relatively easily extended to a very general situation, which have been noticed by Iričanin and Stević soon after publishing [4], and later also proved by several other authors. Namely, the following result holds (see, e.g., [1]).

Theorem 8. *Assume that the following difference equation*

$$x_n = f\left(x_{n-1}, \ldots, x_{n-k}\right), \quad n \in \mathbb{N}_0, \tag{28}$$

is periodic with period p.

Then the following system of difference equations

$$x_n^{(i)} = f\left(x_{n-1}^{(\sigma(i))}, x_{n-2}^{(\sigma^{[2]}(i))}, \ldots, x_{n-k}^{(\sigma^{[k]}(i))}\right), \quad i = \overline{1,l}, \quad n \in \mathbb{N}_0, \tag{29}$$

where $\sigma(i) = i+1$, for $1 \le i \le l-1$, $\sigma(l) = 1$ and $\sigma^{[j]}(i) = \sigma(\sigma^{[j-1]}(i))$, $j = \overline{1,k}$, and $\sigma^{[0]}(i) = i$, $i = \overline{1,l}$, is periodic with period $\operatorname{lcm}(p,l)$ (the least common multiple of numbers p and l).

Theorem 8 can be used in constructing numerous periodic cyclic systems of difference equations based on scalar periodic difference equations, which, with some changes of variables, give some other periodic cyclic systems of difference equations.

3. Periodicity of Positive Solutions of System (11)

In this section, we study positive solutions of system (11). By $\gcd(s,k)$, we denote the greatest common divisor of natural numbers s and k.

Theorem 9. *Consider system* (11). *Assume that $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are positive $k\gcd(s,k)$-periodic sequences. Then every positive solution of system* (11) *is periodic with, not necessarily prime, period*

$$p = 2k\gcd(s,k). \tag{30}$$

Proof. Let $r = \gcd(s,k)$. Then we have that $s = rs_1$ and $k = rk_1$ for some $s_1, k_1 \in \mathbb{N}$ such that

$$\gcd\left(s_1, k_1\right) = 1. \tag{31}$$

Since every $n \in \mathbb{N}_0$ can be written as $n = mr + i$, for some $m \in \mathbb{N}_0$ and $i = \overline{0, r-1}$, system (11) becomes

$$x_{mr+i} = \max\left\{\frac{A_{mr+i}}{x_{r(m-s_1)+i}}, y_{r(m-k_1)+i}\right\}, \qquad y_{mr+i} = \max\left\{\frac{B_{mr+i}}{y_{r(m-s_1)+i}}, x_{r(m-k_1)+i}\right\}, \tag{32}$$

for every $m \in \mathbb{N}_0$ and $i = \overline{0, r-1}$.

Using the next change of variables

$$x_t^{(i)} = x_{tr+i}, \qquad y_t^{(i)} = y_{tr+i}, \tag{33}$$

where $t \ge -\max\{s_1, k_1\}$, $i = \overline{0, r-1}$, in (32), we have that $(x_t^{(i)})_{t\ge-\max\{s_1,k_1\}}$, $(y_t^{(i)})_{t\ge-\max\{s_1,k_1\}}$, $i = \overline{0, r-1}$, are r independent solutions of the next systems

$$x_t = \max\left\{\frac{A_{tr+i}}{x_{t-s_1}}, y_{t-k_1}\right\}, \qquad y_t = \max\left\{\frac{B_{tr+i}}{y_{t-s_1}}, x_{t-k_1}\right\}, \tag{34}$$

which are systems of the form in (11) with s_1 and k_1 instead of s and k, and where the sequences $(A_{tr+i})_{t\in\mathbb{N}_0}$ and $(B_{tr+i})_{t\in\mathbb{N}_0}$, $i = \overline{1, r}$, are k-periodic.

Hence, it is enough to prove the theorem when $\gcd(s,k) = 1$ and the sequences $(A_n)_{n\in\mathbb{N}_0}$, and $(B_n)_{n\in\mathbb{N}_0}$ are positive k-periodic.

Now note that from the equations in (11), we have that

$$x_n \ge y_{n-k}, \quad y_n \ge x_{n-k}, \quad \text{for } n \in \mathbb{N}_0. \tag{35}$$

Further, by using the equations in (11), we also get

$$x_n = \max\left\{\frac{A_n}{x_{n-s}}, y_{n-k}\right\} = \max\left\{\frac{A_n}{x_{n-s}}, \frac{B_{n-k}}{y_{n-k-s}}, x_{n-2k}\right\}, \qquad y_n = \max\left\{\frac{B_n}{y_{n-s}}, x_{n-k}\right\} = \max\left\{\frac{B_n}{y_{n-s}}, \frac{A_{n-k}}{x_{n-k-s}}, y_{n-2k}\right\}, \tag{36}$$

for $n \ge k$.

Using relations (36), we get

$$\begin{aligned} x_n &= \max\left\{\frac{A_n}{x_{n-s}}, \frac{B_{n-k}}{y_{n-k-s}}, x_{n-2k}\right\} \\ &= \max\left\{\frac{A_n}{x_{n-s}}, \frac{B_{n-k}}{y_{n-k-s}}, \frac{A_{n-2k}}{x_{n-2k-s}}, \frac{B_{n-3k}}{y_{n-3k-s}}, x_{n-4k}\right\}, \\ y_n &= \max\left\{\frac{B_n}{y_{n-s}}, \frac{A_{n-k}}{x_{n-k-s}}, y_{n-2k}\right\} \\ &= \max\left\{\frac{B_n}{y_{n-s}}, \frac{A_{n-k}}{x_{n-k-s}}, \frac{B_{n-2k}}{y_{n-2k-s}}, \frac{A_{n-3k}}{x_{n-3k-s}}, y_{n-4k}\right\}, \end{aligned} \tag{37}$$

for $n \ge 3k$.

Now, note that, from the inequalities in (35), we have that

$$x_n \ge x_{n-2k}, \quad y_n \ge y_{n-2k}, \quad \text{for } n \ge k. \tag{38}$$

Using (38) and k-periodicity of the sequences A_n and B_n, we obtain

$$\begin{aligned}
\frac{A_n}{x_{n-s}} &= \frac{A_{n-2k}}{x_{n-s}} \le \frac{A_{n-2k}}{x_{n-2k-s}}, \\
\frac{B_{n-k}}{y_{n-k-s}} &= \frac{B_{n-3k}}{y_{n-k-s}} \le \frac{B_{n-3k}}{y_{n-3k-s}}, \\
\frac{B_n}{y_{n-s}} &= \frac{B_{n-2k}}{y_{n-s}} \le \frac{B_{n-2k}}{y_{n-2k-s}}, \\
\frac{A_{n-k}}{x_{n-k-s}} &= \frac{A_{n-3k}}{x_{n-k-s}} \le \frac{A_{n-3k}}{x_{n-3k-s}}.
\end{aligned} \tag{39}$$

Employing (39) into (37), we get

$$\begin{aligned}
x_n &= \max\left\{\frac{A_{n-2k}}{x_{n-2k-s}}, \frac{B_{n-3k}}{y_{n-3k-s}}, x_{n-4k}\right\} = x_{n-2k}, \\
y_n &= \max\left\{\frac{B_{n-2k}}{y_{n-2k-s}}, \frac{A_{n-3k}}{x_{n-3k-s}}, y_{n-4k}\right\} = y_{n-2k},
\end{aligned} \tag{40}$$

from which it follows that in this case the solutions of system (11) are $2k$-periodic. From all the above, the theorem follows. □

By a slight modification of the proof of Theorem 9, the next result can be proved. We omit the proof.

Theorem 10. *Consider the following system of difference equations*

$$\begin{gathered}
x_n = \max\left\{\frac{A_n}{x_{n-s}}, y_{n-k}\right\}, \qquad y_n = \max\left\{\frac{B_n}{y_{n-s}}, z_{n-k}\right\}, \\
z_n = \max\left\{\frac{C_n}{z_{n-s}}, x_{n-k}\right\}, \quad n \in \mathbb{N}_0,
\end{gathered} \tag{41}$$

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$, and $(C_n)_{n\in\mathbb{N}_0}$ are positive $k\ \gcd(s,k)$-periodic sequences. Then, every positive solution of system (41) is periodic with, not necessarily prime, period

$$p = 3k\gcd(s,k). \tag{42}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Asymptotic Behavior of Solutions of Generalized Emden-Fowler Differential Equations with Delay Argument

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The following differential equation $u^{(n)}(t) + p(t)|u(\sigma(t))|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0$ is considered. Here $p \in L_{\mathrm{loc}}(R_+; R_+)$, $\mu \in C(R_+; (0, +\infty))$, $\sigma \in C(R_+; R_+)$, $\sigma(t) \le t$, and $\lim_{t\to+\infty}\sigma(t) = +\infty$. We say that the equation is almost linear if the condition $\lim_{t\to+\infty}\mu(t) = 1$ is fulfilled, while if $\limsup_{t\to+\infty}\mu(t) \ne 1$ or $\liminf_{t\to+\infty}\mu(t) \ne 1$, then the equation is an essentially nonlinear differential equation. In the case of almost linear and essentially nonlinear differential equations with advanced argument, oscillatory properties have been extensively studied, but there are no results on delay equations of this sort. In this paper, new sufficient conditions implying Property **A** for delay Emden-Fowler equations are obtained.

1. Introduction

This work deals with oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + p(t)|u(\sigma(t))|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0, \tag{1}$$

where

$$\begin{gathered} p \in L_{\mathrm{loc}}(R_+; R), \qquad \mu \in C(R_+; (0; +\infty)), \\ \sigma \in C(R_+; R_+), \qquad \sigma(t) \le t \quad \text{for } t \in R_+, \\ \lim_{t\to+\infty} \sigma(t) = +\infty. \end{gathered} \tag{2}$$

It will always be assumed that the condition

$$p(t) \ge 0 \quad \text{for } t \in R_+ \tag{3}$$

is fulfilled.

Let $t_0 \in R_+$. A function $u : [t_0; +\infty) \to R$ is said to be a proper solution of (1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive,

$$\sup\{|u(s)| : s \in [t; +\infty)\} > 0 \quad \text{for } t \ge t_0, \tag{4}$$

and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0; +\infty)$ and the equality $\overline{u}^{(n)}(t) + p(t)|\overline{u}(\sigma(t))|^{\mu(t)} \operatorname{sign} \overline{u}(\sigma(t)) = 0$ holds for $t \in [t_0 : +\infty)$. A proper solution $u : [t_0 : +\infty) \to R$ of (1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1. We say that (1) has Property **A** if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{for } t \uparrow +\infty \ \ (i = 0, \ldots, n-1), \tag{5}$$

when n is odd.

Definition 2. We say that (1) is almost linear if the condition $\lim_{t\to+\infty}\mu(t) = 1$ holds, while if $\limsup_{t\to+\infty}\mu(t) \ne 1$ or $\liminf_{t\to+\infty}\mu(t) \ne 1$, then we say that the equation is an essentially nonlinear differential equation.

The Emden-Fowler equation originated from theories concerning gaseous dynamics in astrophysics in the middle of the nineteenth century. In the study of stellar structure at

that time it was important to investigate the equilibrium configuration of the mass of spherical clouds of gas. Lord Kelvin in 1862 assumed that the gaseous cloud is under convective equilibrium and then Lane [1] studied the equation

$$\frac{1}{t^2}\frac{d}{dt}\left(t^2\frac{du}{dt}\right)+u^{\gamma}=0. \tag{6}$$

The Emden-Fowler equations were first considered only for second-order equations and written in the form

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right)+q(t)u^{\gamma}=0, \quad t\geq 0, \tag{7}$$

which could be reduced in the case of positive and continuous coefficients to the equation

$$x''+a(t)x^{\gamma}=0, \quad t\geq 0. \tag{8}$$

To avoid difficulties of defining x^{γ} when $x(t)$ is negative and γ is not an integer, the equation

$$x''(t)+a(t)|x(t)|^{\gamma}\operatorname{sign} x(t)=0, \quad t\geq 0, \tag{9}$$

was usually considered. The mathematical foundation of the theory of such equations was built by Fowler [2] and the description of the results can be found in Chapter 7 of [3].

We see also the Emden-Fowler equation in gas dynamics and fluid mechanics (see Sansone [4], page 431 and the paper [5]). Nonoscillation of these equations is important in various applications. Note that the zero of such solutions corresponds to an equilibrium state in a fluid with spherical distribution of density and under mutual attraction of its particles. The Emden-Fowler equations can be either oscillatory (i.e., all proper solutions have a sequence of zeros tending to zero) or nonoscillatory, if solutions are eventually positive or negative, or, in contrast with the case of linear differential equations of second order, may possess both oscillating and nonoscillating solutions. For example, for the equation

$$x''(t)+t^{\mu}|x(t)|^{\gamma}\operatorname{sign} x(t)=0, \quad t\geq 0, \tag{10}$$

it was proven in [2] that for $\mu\geq -2>-(\gamma+3)/2$ all solutions oscillate, for $\mu<-(\gamma+3)/2$—all solutions nonoscillate, and for $-(\gamma+3)/2\leq\mu<-2$ there are both oscillating and nonoscillating solutions.

The Emden-Fowler equation presents one of the classical objects in the theory of differential equations. Tests for oscillation and nonoscillation of all solutions and existence of oscillating solutions were obtained in the works [6–8]. In [9] for the case $0<\gamma<1$, it was obtained that all solutions of the equation

$$x''(t)+a(t)\left|x^{\gamma}(t)\right|\operatorname{sign} x(t)=0, \quad t\geq 0, \tag{11}$$

oscillate if and only if

$$\int_0^{\infty} t^{\gamma}a(t)\,dt=\infty. \tag{12}$$

The latest research results in this area are presented in the book [8]. Behavior of solutions to nth order Emden-Fowler equations can be essentially more complicated. Properties A and B defined by Kiguradze are studied in the abovementioned book.

There are essentially less results on oscillation of delay Emden-Fowler equations. Oscillation properties of nonlinear delay differential equations, where Emden-Fowler equations were also included as a particular case, were studied in [10–20]. Results of these papers are discussed in [13, 15], where various examples demonstrating essentialities of conditions are also presented. Note that for delay differential equations there are no results on nonoscillation of all solutions and only existence of nonoscillating solutions is studied. Actually, the results on oscillation of delayed equations are based on the approaches existing for ordinary differential equations with development in the direction of preventing the obstructive influence of delay. In the paper [15] the following equation

$$x^{(n)}(t)+a(t)f[x(\sigma(t))]=0, \quad t\geq 0,\ n \text{ even}, \tag{13}$$

and its particular case

$$x^{(n)}(t)+a(t)|x(\sigma(t))|^{\gamma}\operatorname{sign} x(\sigma(t))=0, \quad t\geq 0,\ n \text{ even}, \tag{14}$$

are considered. It was obtained for the last equation under some standard assumptions on the coefficients [15] that in the case $0<\gamma<1$,

$$\int_0^{\infty}\sigma^{(n-1)\gamma}(t)a(t)\,dt=\infty, \tag{15}$$

all solutions oscillate. We see that the integral depends on deviation of argument $\sigma(t)$ and the power of the equation n. For the equation

$$x^{(n)}(t)+a(t)\prod_{i=1}^{m}\left|x(\sigma_i(t))\right|^{\gamma_i}\operatorname{sign} x(\sigma_i(t))=0, \quad t\geq 0, \quad n \text{ even}, \tag{16}$$

where γ_i is the ratio of two positive odd integers, $\sigma(t)\leq\sigma_i(t)\leq t$ for $i=1,\ldots,m$, and $\sigma(t)\to\infty$ as $t\to\infty$, each of the following conditions (a), (b), and (c) ensures oscillation of all solutions:

(a)

$$\int_0^{\infty}\sigma^{(n-1)\gamma}(t)a(t)\,dt=\infty \quad \text{for } \gamma=\sum_{i=1}^{m}\gamma_i<1; \tag{17}$$

(b)

$$\int_0^{\infty}\sigma^{(n-1)\alpha}(t)a(t)\,dt=\infty \quad \text{for } \gamma=1, 0<\alpha<1; \tag{18}$$

(c)

$$\int_0^{\infty}\sigma^{n-1}(t)a(t)\,dt=\infty, \quad \sigma'(t)\geq 0 \quad \text{for } \gamma>1. \tag{19}$$

Most proofs of results on oscillation of all solutions to second order equations utilize the fact that if a nonoscillating solution exists, the signs of the solution $x(t)$ and its second derivative $x''(t)$ are opposite to each other for sufficiently large t. Then a growth of nonoscillating solution is estimated and the authors come to contradiction with conditions that proves oscillation of all solutions. Note that delays disturb oscillation. Instead of $t^{\gamma}, \sigma^{\gamma}(t)$ appears. The principle is clear: for oscillation of all solutions we have to achieve a corresponding smallness of the delay $t-\sigma(t)$. All this is more complicated if we study nth order equations. In this case also the fact that $x(t)$ and its nth derivative $x^{(n)}(t)$ have different signs for sufficiently large t is used, but the technique is more complicated.

In the papers [21–28] a generalization of Emden-Fowler equations was considered. The powers in these papers can be functions and not constants. In many cases, it leads to essentially new oscillation properties of such equations. Surprisingly, oscillation behavior of equations, with the power λ and with functional power $\mu(t)$ such that $\lim_{t\to\infty}\mu(t) = \lambda$, can be quite different. The main purpose of our paper is to study conditions under which the generalized (in this sense) equations preserve the known oscillation properties of Emden-Fowler equations and conditions under which these properties are not preserved. Oscillatory properties of almost linear and essentially nonlinear differential equation with advanced argument have already been studied in [21–28]. In this paper we study oscillation properties of nth order delay Emden-Fowler equations.

2. Some Auxiliary Lemmas

In the sequel, $\widetilde{C}_{\mathrm{loc}}([t_0;+\infty))$ will denote the set of all functions $u:[t_0;+\infty)\to R$ absolutely continuous on any finite subinternal of $[t_0;+\infty)$ along with their derivatives of order up to including $n-1$.

Lemma 3 (see [28]). *Let $u\in\widetilde{C}^{n-1}_{\mathrm{loc}}([t_0;+\infty))$, $u(t)>0$, $u^{(n)}(t)\le 0$ for $t\ge t_0$, and $u^{(n)}(t)\not\equiv 0$ in any neighborhood of $+\infty$. Then there exist $t_1\ge t_0$ and $\ell\in\{0,\dots,n-1\}$ such that $\ell+n$ is odd and*

$$\begin{gathered} u^{(i)}(t)>0 \quad \text{for } t\ge t_1 \ \ (i=0,\dots,\ell-1), \\ (-1)^{i+\ell}u^{(i)}(t)>0 \quad \text{for } t\ge t_1 \ \ (i=\ell,\dots,n-1). \end{gathered} \tag{20_ℓ}$$

Remark 4. If n is odd and $\ell=0$, then it means that in (20_0) only the second inequalities are fulfilled.

Lemma 5 (see [29]). *Let $u\in\widetilde{C}^{n-1}_{\mathrm{loc}}([t_0;+\infty))$ and let (20_ℓ) be fulfilled for some $\ell\in\{0,\dots,n-1\}$ with $\ell+n$ odd. Then*

$$\int_{t_0}^{+\infty} t^{n-\ell-1}\left|u^{(n)}(t)\right|dt<+\infty. \tag{21}$$

If, moreover,

$$\int_{t_0}^{+\infty} t^{n-\ell}\left|u^{(n)}(t)\right|dt=+\infty, \tag{22}$$

then there exists $t_*>t_0$ such that

$$\frac{u^{(i)}(t)}{t^{\ell-i}}\downarrow,\quad \frac{u^{(i)}(t)}{t^{\ell-i-1}}\uparrow \quad (i=0,\dots,\ell-1), \tag{23_i}$$

$$u(t)\ge\frac{t^{\ell-1}}{\ell!}u^{(\ell-1)}(t) \quad \text{for } t\ge t_*, \tag{24}$$

$$\begin{aligned} u^{(\ell-1)}(t)\ge{}& \frac{t}{(n-\ell)!}\int_t^{+\infty} s^{n-\ell-1}\left|u^{(n)}(s)\right|ds \\ &+\frac{1}{(n-\ell)!}\int_{t_*}^{t} s^{n-\ell}\left|u^{(n)}(s)\right|ds \quad \text{for } t\ge t_*. \end{aligned} \tag{25}$$

3. Necessary Conditions for the Existence of a Solution of Type (20_ℓ)

The following notation will be used throughout the work:

$$\alpha=\inf\{\mu(t):t\in R_+\},\qquad \beta=\sup\{\mu(t):t\in R_+\}, \tag{26}$$

$$\begin{gathered} \sigma_{(-1)}(t)=\sup\{s\ge 0,\ \sigma(s)\le t\}, \\ \sigma_{(-k)}=\sigma_{(-1)}\circ\sigma_{(-(k-1))},\quad k=2,3,\dots. \end{gathered} \tag{27}$$

Clearly $\sigma_{(-1)}(t)\ge t$, and $\sigma_{(-1)}$ is nondecreasing and coincides with the inverse of σ when the latter exists.

Definition 6. Let $t_0\in R_+$. By $\mathbf{U}_{\ell,t_0}$ one denotes the set of all proper solutions $u:[t_0,+\infty)\to R$ of (1) satisfying the condition (20_ℓ) with some $t_1\ge t_0$.

Lemma 7. *Let the conditions* (2), (3) *be fulfilled, let $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd, and let $u\in\mathbf{U}_{\ell,t_0}$ be a positive proper solution of* (1). *If, moreover, $\alpha\ge 1$, $\beta<+\infty$,*

$$\int_0^{+\infty} t^{n-\ell}(\sigma(t))^{(\ell-1)\mu(t)}p(t)\,dt=+\infty, \tag{28_ℓ}$$

then for any $M\in(1;+\infty)$ there exists $t_*>t_0$ such that for any $k\in N$

$$u^{(\ell-1)}(t)\ge\rho^{(\alpha)}_{k,\ell,t_*}(t) \quad \text{for } t\ge\sigma_{(-k)}(t_*), \tag{29}$$

where α is given by the first equality of (26) *and*

$$\rho_{1,\ell,t_*}^{(\alpha)}(t) = \ell!\exp\left\{M_\ell(\alpha)\int_{\sigma_{(-1)}(t_*)}^{t}\int_s^{+\infty}\xi^{n-\ell-2}\times(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}\times p(\xi)\,d\xi\,ds\right\}, \tag{30}$$

$$\rho_{i,\ell,t_*}^{(\alpha)}(t) = \ell! + \frac{1}{(n-\ell)!}\times\int_{\sigma_{(-i)}(t_*)}^{t}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}\times p(\xi)\times\left(\frac{1}{\ell!}\rho_{i-1,\ell,t_*}(\sigma(\xi))\right)^{\mu(\xi)}d\xi\,ds \quad (i=2,\dots,k), \tag{31}$$

$$M_\ell(\alpha) = \begin{cases}\dfrac{1}{\ell!(n-\ell)!} & \text{if } \alpha = 1,\\ M & \text{if } \alpha > 1.\end{cases} \tag{32}$$

Proof. Let $t_0 \in R_+$, $\ell \in \{1,\dots,n-1\}$ with $\ell + n$ odd and $u \in \mathbf{U}_{\ell,t_0}$ (see Definition 6) is solution of (1). Since $\beta < +\infty$, according to (1), (20_ℓ), and (28_ℓ), it is clear that condition (22) holds. Thus, by Lemma 5 there exists $t_2 > t_1$ such that the conditions (23_i)–(25) with $t_* = t_2$ are fulfilled and

$$u^{(\ell-1)}(t) \geq \frac{t}{(n-\ell)!}\int_t^{+\infty}s^{n-\ell-1}p(s)(u(\sigma(s)))^{\mu(s)}ds + \frac{1}{(n-\ell)!}\int_{t_2}^{t}s^{n-\ell}p(s)(u(\sigma(s)))^{\mu(s)}ds \quad \text{for } t \geq t_2. \tag{33}$$

Observe that there exists $t_3 > t_2$ such that $\sigma(t) \geq t_2$ for $t \geq t_3$. Thus, by (24), for any $t \geq t_3$ we get

$$\begin{aligned} u^{(\ell-1)}(t) &\geq \frac{t}{(n-\ell)!}\int_t^{+\infty}s^{n-\ell-1}p(s)(u(\sigma(s)))^{\mu(s)}ds \\ &\quad - \frac{1}{(n-\ell)!}\int_{t_2}^{t}sd\int_s^{+\infty}\xi^{n-\ell-1}p(\xi)\times(u(\sigma(\xi)))^{\mu(\xi)}d\xi\,ds \\ &\geq \frac{1}{(n-\ell)!}\int_{t_3}^{t}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}\times\left(\frac{1}{\ell!}u^{(\ell-1)}(\sigma(\xi))\right)^{\mu(\xi)}p(\xi)\,d\xi\,ds. \end{aligned} \tag{34}$$

According to (28_ℓ) and $(23_{\ell-1})$, choose $t_* > t_3$ such that

$$\frac{1}{(n-\ell)!}\int_{t_3}^{t_*}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}\times\left(\frac{u^{(\ell-1)}(\sigma(\xi))}{\ell!}\right)^{\mu(\xi)}p(\xi)\,d\xi\,ds > \ell!. \tag{35}$$

By (34) and (35) we have

$$u^{(\ell-1)}(t) \geq \ell! + \frac{1}{(n-\ell)!}\int_{t_*}^{t}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}p(\xi)\times\left(\frac{u^{(\ell-1)}(\sigma(\xi))}{\ell!}\right)^{\mu(\xi)}d\xi\,ds \quad \text{for } t \geq t_*. \tag{36}$$

Let $\alpha = 1$. Since $u^{(\ell-1)}(t) \to +\infty$ as $t \to +\infty$, without loss of generally we can assume that $u^{(\ell-1)}(\sigma(\xi)) \geq \ell!$ for $\xi \geq t_3$. Then by (23_ℓ) from (36) we get

$$u^{(\ell-1)}(t) \geq \ell! + \frac{1}{\ell!(n-\ell)!}\int_{t_*}^{t}\int_s^{+\infty}\xi^{n-\ell-2}\times(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}u^{(\ell-1)}(\xi)\,p(\xi)\,d\xi\,ds \quad \text{for } t \geq t_*. \tag{37}$$

It is obvious that

$$x'(t) \geq \frac{u^{(\ell-1)}(t)}{\ell!(n-\ell)!}\int_t^{+\infty}\xi^{n-\ell-2}(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}p(\xi)\,d\xi, \tag{38}$$

where

$$x(t) = \ell! + \frac{1}{\ell!(n-\ell)!}\int_{t_*}^{t}\int_s^{+\infty}\xi^{n-\ell-2}\times(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}p(\xi)\,u^{(\ell-1)}(\xi)\,d\xi\,ds. \tag{39}$$

Thus, according to $(23_{\ell-1})$ (37), and (39) from (38) we get

$$x'(t) \geq \frac{x(t)}{\ell!\,(n-\ell)!}\int_t^{+\infty}\xi^{n-\ell-2}(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}p(\xi)\,d\xi \quad \text{for } t\geq t_*. \tag{40}$$

Therefore

$$x(t) \geq \ell!\exp\left\{\frac{1}{\ell!\,(n-\ell)!}\times\int_{t_*}^{t}\int_s^{+\infty}\xi^{n-\ell-2}(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}p(\xi)\,d\xi\,ds\right\} \quad \text{for } t\geq t_*. \tag{41}$$

Hence, according to (37) and (39)

$$u^{(\ell-1)}(t)\geq \rho^{(1)}_{1,\ell,t_*}(t) \quad \text{for } t\geq \sigma_{(-1)}(t_*), \tag{42}$$

where

$$\rho^{(1)}_{1,\ell,t_*}(t) = \ell!\exp\left\{\frac{1}{\ell!\,(n-\ell)!}\times\int_{\sigma_{(-1)}(t_*)}^{t}\int_s^{+\infty}\xi^{n-\ell-2}\times(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}p(\xi)\,d\xi\,ds\right\}. \tag{43}$$

Thus, according to (36) and (42)

$$u^{(\ell-1)}(t)\geq \rho^{(1)}_{i,\ell,t_*}(t) \quad \text{for } t\geq \sigma_{(-i)}(t_*)\ \ (i=1,\dots,k), \tag{44}$$

where

$$\rho^{(1)}_{i,\ell,t_*}(t) = \ell! + \frac{1}{(n-\ell)!}\times\int_{\sigma_{(-i)}(t_*)}^{t}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}p(\xi)\times\left(\frac{1}{\ell!}\,\rho^{(1)}_{i-1,\ell,t_*}(\sigma(\xi))\right)^{\mu(\xi)}d\xi\,ds \quad (i=2,\dots,k). \tag{45}$$

Now assume that $\alpha>1$ and $M\in(1,+\infty)$. Since $u^{(\ell-1)}(t)\uparrow+\infty$ for $t\uparrow+\infty$, without loss of generality we can assume that $(u^{(\ell-1)}(\sigma(\xi))/\ell!)^{\alpha-1}\geq \ell!(n-\ell)!\ M$ for $\xi\geq t_*$. Therefore, from (36) we have

$$u^{(\ell-1)}(t) \geq \ell! + M\int_{t_*}^{t}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}\times p(\xi)\,u^{(\ell-1)}(\sigma(\xi))\,d\xi\,ds \quad \text{for } t\geq t_*. \tag{46}$$

Taking into account (46), as above we can find that if $\alpha>1$, then

$$u^{(\ell-1)}(t)\geq \rho^{(\alpha)}_{k,\ell,t_*}(t) \quad \text{for } t\geq \sigma_{(-k)}(t_*), \tag{47}$$

where

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell!\exp\left\{M\int_{\sigma_{(-1)}(t_*)}^{t}\int_s^{+\infty}\xi^{n-\ell-2}(\sigma(\xi))^{1+(\ell-1)\mu(\xi)}\times p(\xi)\,d\xi\,ds\right\} \quad \text{for } t\geq \sigma_{(-1)}(t_*), \tag{48}$$

$$\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell! + \frac{1}{(n-\ell)!}\times\int_{\sigma_{(-i)}(t_*)}^{t}\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}p(\xi)\times\left(\frac{1}{\ell!}\,\rho^{(\alpha)}_{i-1,\ell,t_*}(\sigma(\xi))\right)^{\mu(\xi)}d\xi\,ds \quad \text{for } t\geq \sigma_{(-i)}(t_*)\ \ (i=2,\dots,k). \tag{49}$$

According to (43)–(45) and (47)–(49), it is obvious that for any $\alpha\geq 1$, $k\in N$, and $M>1$ there exists $t_*\in R_+$ such that (29)–(31) hold, where $M_\ell(\alpha)$ is defined by (32). This proves the validity of the lemma. □

Analogously we can prove.

Lemma 8. *Let conditions* (2), (3), (28_ℓ) *be fulfilled, let* $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *odd,* $1\leq\beta<+\infty$, *and let* $u\in\mathbf{U}_{\ell,t_0}$ *be a positive proper solution of* (1). *Then for any* $M\in(1;+\infty)$ *there exists* $t_*>t_0$ *such that for any* $k\in N$

$$u^{(\ell-1)}(t)\geq \tilde{\rho}^{(\beta)}_{k,\ell,t_*}(t) \quad \textit{for } t\geq \sigma_{(-k)}(t_*), \tag{50}$$

where β is defined by the second equality of (26) *and*

$$\widetilde{\rho}_{1,\ell,t_*}^{(\beta)}(t) = \ell! \exp \times \left\{ M(\beta) \times \int_{\sigma_{(-1)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-2} \times (\sigma(\xi))^{1+\ell\mu(\xi)-\beta} \times p(\xi)\, d\xi\, ds \right\}, \tag{51}$$

$$\widetilde{\rho}_{i,\ell,t_*}^{(\beta)}(t) = \ell! + \frac{1}{(n-\ell)!} \times \int_{\sigma_{(-i)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} \times \left(\frac{1}{\ell!}\, \widetilde{\rho}_{i-1,\ell,t_*}(\sigma(\xi)) \right)^{\mu(\xi)} p(\xi)\, d\xi\, ds \quad (i = 2, \ldots, k), \tag{52}$$

$$M(\beta) = \begin{cases} \dfrac{1}{\ell!\,(n-\ell)!} & \text{if } \beta = 1, \\ M & \text{if } \beta > 1. \end{cases} \tag{53}$$

Remark 9. In Lemma 7, the condition $\beta < +\infty$ cannot be replaced by the condition $\beta = +\infty$. Indeed, let $c \in (0,1)$. Consider (1), where n is even and

$$\sigma(t) \equiv t, \qquad p(t) = \frac{n!\, t^{\log_{1/c} t}}{t^{n+1}(ct-1)^{\log_{1/c} t}}, \qquad \beta(t) = \log_{1/c} t, \quad t \geq \frac{2}{c}. \tag{54}$$

It is obvious that the function $u(t) = c - (1/t)$ is the solution of (1) and it satisfies the condition (20_1) for $t \geq (2/c)$. On the other hand, the condition (28_1) holds, but the condition (22) is not fulfilled.

Theorem 10. *Let $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ be odd, let $\beta < +\infty$ and the conditions* (2), (3), (28_ℓ), *and let*

$$\int_0^{+\infty} t^{n-\ell-1} (\sigma(t))^{\ell\mu(t)} p(t)\, dt = +\infty \tag{55_ℓ}$$

be fulfilled, and for some $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} \neq \emptyset$. Then for any $M > 1$ there exists $t_* > t_0$ such that if $\alpha = 1$,

$$\lim_{t \to +\infty} \frac{1}{t} \int_{\sigma_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} \times \left(\frac{1}{\ell!}\, \rho_{k,\ell,t_*}^{(1)}(\sigma(\xi)) \right)^{\mu(\xi)} p(\xi)\, d\xi\, ds = 0 \tag{56}$$

and if $\alpha > 1$, then for any $k \in N$ and $\delta \in (1;\alpha]$,

$$\int_{\sigma_{(-k)}(t_*)}^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\sigma(\xi))^{(\ell-1)\mu(\xi)+\delta} \times \left(\frac{1}{\ell!}\, \rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi)) \right)^{\mu(\xi)-\delta} p(\xi)\, d\xi\, ds < +\infty, \tag{57}$$

where α is defined by first equality of (26) *and $\rho_{k,\ell,t_*}^{(\alpha)}$ is given by* (30)–(32).

Proof. Let $M > 1$ and $t_0 \in R_+$ such that $\mathbf{U}_{\ell,t_0} \neq \emptyset$. By definition, (1) has a proper solution $u \in \mathbf{U}_{\ell,t_0}$ satisfying the condition (20_ℓ) with some $t_\ell \geq t_0$. Due to (1), (20_ℓ), and (28_ℓ), it is obvious that condition (22) holds. Thus by Lemma 5 there exists $t_2 > t_1$ such that conditions (23_i)-(24) with $t_* = t_2$ are fulfilled. On the other hand, according to Lemma 7 (and its proof), we see that

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int_{t_2}^{t} \int_s^{+\infty} \xi^{n-\ell-1} p(\xi) (u(\sigma(\xi)))^{\mu(\xi)} d\xi\, ds \quad \text{for } t \geq t_2, \tag{58}$$

and there exists $t_* > t_2$ such that relation (30) is fulfilled. Without loss of generality we can assume that $\sigma(t) \geq t_2$ for $t \geq t_*$. Therefore, by (24), from (58) we have

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int_{\sigma_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi) \times \left(\frac{1}{\ell!}\, u^{(\ell-1)}(\sigma(\xi)) \right)^{\mu(\xi)} d\xi\, ds. \tag{59}$$

Assume that $\alpha = 1$. Then by (44) and (59), we have

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int_{\sigma_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi) \times \left(\frac{1}{\ell!}\, \rho_{k-1,\ell,t_*}^{(1)}(\sigma(\xi)) \right)^{\mu(\xi)} d\xi\, ds \quad \text{for } t \geq \sigma_{(-k)}(t_*). \tag{60}$$

On the other hand, according to $(23_{\ell-1})$ and (55_ℓ) it is obvious that

$$\frac{u^{(\ell-1)}(t)}{t} \downarrow 0 \quad \text{for } t \uparrow +\infty. \tag{61}$$

Therefore, from (60) we get

$$\lim_{t \to +\infty} \frac{1}{t} \int_{\sigma_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi) \times \left(\frac{1}{\ell!}\, \rho_{k,\ell,t_*}^{(1)}(\sigma(\xi)) \right)^{\mu(\xi)} d\xi\, ds = 0. \tag{62}$$

Now assume that $\alpha > 1$ and $\delta \in (1, \alpha]$. Then according to (47), ($23_{\ell-1}$), and (61), from (59) we have

$$\begin{aligned} u^{(\ell-1)}(t) &\geq \frac{1}{(n-\ell)!}\int_{\sigma_{(-k)}(t_*)}^{t}\int_{s}^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}p(\xi) \\ &\quad\times\left(\frac{1}{\ell!}\,u^{(\ell-1)}(\sigma(\xi))\right)^{\mu(\xi)-\delta} \\ &\quad\times\left(\frac{1}{\ell!}\,u^{(\ell-1)}(\sigma(\xi))\right)^{\delta}d\xi\,ds \\ &\geq \frac{1}{(n-\ell)!}\int_{\sigma_{(-k)}(t_*)}^{t}\int_{s}^{+\infty}\xi^{n-\ell-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)}p(\xi) \\ &\quad\times\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta} \\ &\quad\times\left(\frac{1}{\ell!}\,u^{(\ell-1)}(\xi)\right)^{\delta}d\xi\,ds \\ &\geq \frac{1}{(n-\ell)!}\int_{\sigma_{(-k)}(t_*)}^{t}\left(\frac{u^{\ell-1}(s)}{\ell!}\right)^{\delta} \\ &\quad\times\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)}p(\xi) \\ &\quad\times\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds. \end{aligned} \tag{63}$$

Thus, we obtain

$$\begin{aligned} (v(t))^{\delta} &\geq \frac{1}{(\ell!\,(n-\ell)!)^{\delta}} \\ &\quad\times\left(\int_{\sigma_{(-k)}(t_*)}^{t}v^{\delta}(s)\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)}\right. \\ &\quad\left.\times\,p(\xi)\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds\right), \end{aligned} \tag{64}$$

where $v(t) = (1/\ell!)u^{(\ell-1)}(t)$.

It is obvious that there exist $t_1 > \sigma_{(-k)}(t_*)$ such that

$$\begin{aligned} &\int_{\sigma_{(-k)}(t_*)}^{t}v^{\delta}(s)\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)}p(\xi) \\ &\qquad\times\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds > 0 \\ &\qquad\text{for } t \geq t_1. \end{aligned} \tag{65}$$

Therefore, from (64)

$$\begin{aligned} &\int_{t_1}^{t}\frac{\varphi'(s)}{(\varphi(s))^{\delta}}\,ds \\ &\quad\geq \frac{1}{(\ell!\,(n-\ell)!)^{\delta}} \\ &\quad\times\int_{t_1}^{t}\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)} \\ &\qquad\times p(\xi)\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds, \end{aligned} \tag{66}$$

where

$$\begin{aligned} &\varphi(t) \\ &\quad= \int_{\sigma_{(-k)}(t_*)}^{t}(v(s))^{\delta}\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)} \\ &\qquad\times p(\xi)\left(\frac{1}{\ell!}\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds. \end{aligned} \tag{67}$$

From the last inequality we get

$$\begin{aligned} &\int_{t_1}^{t}\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)}p(\xi) \\ &\qquad\times\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds \\ &\quad\leq \frac{(\ell!\,(n-\ell)!)^{\delta}}{\delta-1}\left[\varphi^{1-\delta}(t_1)-\varphi^{1-\delta}(t)\right] \\ &\quad\leq \frac{(\ell!\,(n-\ell)!)^{\delta}}{\delta-1}\,\varphi^{1-\delta}(t_1)\quad\text{for } t \geq t_1. \end{aligned} \tag{68}$$

Passing to the limit in the latter inequality, we get

$$\begin{aligned} &\int_{t_1}^{+\infty}\int_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)}p(\xi) \\ &\qquad\times\left(\frac{1}{\ell!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta}d\xi\,ds < +\infty; \end{aligned} \tag{69}$$

that is, according to (62) and (69), (56) and (57) hold, which proves the validity of the theorem. □

Using Lemma 8 in a similar manner one can prove the following.

Theorem 11. *Let $\ell \in \{1, \ldots, n-1\}$ with $\ell+n$ be odd, let* (2), (3), (28_ℓ), *and* (55_ℓ) *be fulfilled, and for some $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \emptyset$. Then there exists $t_* > t_0$ such that if $\beta = 1$, for any $k \in N$*

$$\begin{aligned} &\limsup_{t\to+\infty}\frac{1}{t} \\ &\quad\times\int_{\sigma_{(-k)}(t_*)}^{t}\int_{s}^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)} \\ &\qquad\times\left(\widetilde{\rho}_{k,\ell,t_*}^{(1)}(\sigma(\xi))\right)^{\mu(\xi)}p(\xi)\,d\xi\,ds = 0, \end{aligned} \tag{70}$$

and if $1 < \beta < +\infty$, *then for any* $k \in N$ *and* $\delta \in (1, \beta]$

$$\int_{\sigma_{(-k)}(t_*)}^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\sigma(\xi))^{\ell\mu(\xi)+\delta-\beta} \times \left(\widetilde{\rho}_{k,\ell,t_*}^{(\beta)}(\sigma(\xi))\right)^{\beta-\delta} p(\xi)\, d\xi\, ds < +\infty, \tag{71}$$

where β *is defined by the second equality of* (26) *and* $\widetilde{\rho}_{k,\ell,t_*}^{(\beta)}$ *is given by* (51)–(53).

4. Sufficient Conditions for Nonexistence of Solutions of the Type (20_ℓ)

Theorem 12. *Let* $\beta < +\infty$, $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, let the conditions* (2), (3), (28_ℓ), *and* (55_ℓ) *be fulfilled, and if* $\alpha = 1$, *for any large* $t_* \in R_+$ *and for some* $k \in N$

$$\limsup_{t \to +\infty} \frac{1}{t} \int_{\sigma_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} \times \left(\frac{1}{\ell!}\, \rho_{k,\ell,t_*}^{(1)}(\sigma(\xi))\right)^{\mu(\xi)} p(\xi)\, d\xi\, ds > 0 \tag{72_ℓ}$$

or if $\alpha > 1$, *for same* $k \in N$ *and* $\delta \in (1, \alpha]$

$$\int_{\sigma_{(-k)}(t_*)}^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)} \times \left(\frac{1}{\ell!}\, \rho_{k,\ell,t_*}^{(\alpha)}(\sigma(\xi))\right)^{\mu(\xi)-\delta} p(\xi)\, d\xi\, ds = +\infty. \tag{73_ℓ}$$

Then for any $t_0 \in R_+$ *one has* $\mathbf{U}_{\ell,t_0} = \emptyset$, *where* α *and* β *are defined by* (26) *and* $\rho_{k,\ell,t_*}^{(\alpha)}$ *is given by* (30)–(32).

Proof. Assume the contrary. Let there exist $t_0 \in R_+$ such that $\mathbf{U}_{\ell,t_0} \neq \emptyset$ (see Definition 6). Then (1) has a proper solution $u : [t_0, +\infty) \to R$ satisfying the condition (20_ℓ). Since the condition of Theorem 10 is fulfilled, there exists $t_* > t_0$ such that if $\alpha = 1$ (if $\alpha > 1$), the condition (56) (the condition (57)) holds, which contradicts (72_ℓ) and (73_ℓ). The obtained contradiction proves the validity of the theorem. □

Using Theorem 11 analogously we can prove the following.

Theorem 13. *Let* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, let the conditions* (2), (3), (28_ℓ), *and* (55_ℓ) *be fulfilled, and if* $\beta = 1$, *for any large* $t_* \in R_+$ *and for some* $k \in N$

$$\limsup_{t \to +\infty} \frac{1}{t} \int_{\sigma_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} \times \left(\frac{1}{\ell!}\, \widetilde{\rho}_{k,\ell,t_*}^{(1)}(\sigma(\xi))\right)^{\mu(\xi)} p(\xi)\, d\xi\, ds > 0 \tag{74_ℓ}$$

or if $1 < \beta < +\infty$ *for same* $k \in N$ *and* $\delta \in (1, \beta]$

$$\int_{\sigma_{(-k)}(t_*)}^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\sigma(\xi))^{\ell\mu(\xi)+\delta-\beta} \times \left(\widetilde{\rho}_{k,\ell,t_*}^{(\beta)}(\sigma(\xi))\right)^{\beta-\delta} p(\xi)\, d\xi\, ds = +\infty. \tag{75_ℓ}$$

Then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \emptyset$, *where* β *is defined by the second equality of* (26) *and* $\widetilde{\rho}_{k,\ell,t_*}$ *is given by* (51)–(53).

Corollary 14. *Let* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, let the conditions* (2), (3), *and* (55_ℓ) *be fulfilled,* $\alpha = 1$, $\beta < +\infty$, *and*

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds > 0. \tag{76_ℓ}$$

Then $\mathbf{U}_{\ell,t_0} = \emptyset$ *for any* $t_0 \in R_+$.

Proof. Since

$$\rho_{1,\ell,t}^{(\alpha)}(\sigma(t_*)) \geq \ell! \quad \text{for } t \geq \sigma_{(-1)}(t_*), \tag{77}$$

it suffices to note that by (76_ℓ) the conditions (72_ℓ) and (28_ℓ) are fulfilled for $k = 1$. □

Corollary 15. *Let* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, let the conditions* (2) *and* (3) *be fulfilled,* $\alpha = 1$, $\beta < +\infty$, *and*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-\ell-2} (\sigma(s))^{1+(\ell-1)\mu(s)} p(s)\, ds = \gamma > 0. \tag{78_ℓ}$$

If, moreover, for some $\varepsilon \in (0, \gamma)$

$$\limsup_{t \to +\infty} \frac{1}{t} \times \int_0^t \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{\mu(\xi)(\ell-1+((\gamma-\varepsilon)/(\ell!(n-\ell)!)))} p(\xi)\, d\xi\, ds > 0, \tag{79_ℓ}$$

then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \emptyset$.

Proof. Clearly by virtue of (78_ℓ) conditions (28_ℓ) and (55_ℓ) are fulfilled. Let $\varepsilon \in (0, \gamma)$. According to (78_ℓ) and (79_ℓ) it is obvious that, for large t, $\rho_{1,\ell,t_*}^{(1)}(t) \geq \ell! t^{(\gamma-\varepsilon)/(\ell!(n-\ell)!)}$. Therefore, by (79_ℓ), for $k = 1$, (72_ℓ) holds, which proves the validity of the corollary. □

Corollary 16. *Let* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, let the conditions* (2), (3), (28_ℓ), *and* (55_ℓ) *be fulfilled,* $\alpha > 1$, $\beta < +\infty$, *and for some* $\delta \in (1, \alpha]$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds = +\infty. \tag{80_ℓ}$$

Then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \emptyset$, *where* α *is defined by the first condition of* (3).

Proof. By (80_ℓ) and (77), for $k = 1$, the condition (73_ℓ) holds, which proves the validity of the corollary. □

Corollary 17. *Let $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd and let the conditions* (2), (3), *and* (78_ℓ) *be fulfilled. Moreover, if $\alpha > 1$, $\beta < +\infty$, and there exists $m \in N$ such that*

$$\liminf_{t \to +\infty} \frac{\sigma^m(t)}{t} > 0, \tag{81}$$

then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \emptyset$, where α is defined by the first condition of (3).

Proof. By (78_ℓ) there exist $c > 0$ and $t_1 \in R_+$ such that

$$t \int_t^{+\infty} \xi^{n-\ell-2} (\sigma(\xi))^{1+(\ell-1)\mu(\xi)} p(\xi)\, d\xi \geq c \quad \text{for } t \geq t_1. \tag{82_ℓ}$$

Let $\delta = (1+\alpha)/2$ and $M = m(1+\alpha)/c(\alpha - \delta)$. Then by (82_ℓ) and (30), for large $t_* > t_1$,

$$\rho_{1,\ell,t_*}^{(\alpha)}(t) \geq t^{Mc} \quad t \geq t_*. \tag{83}$$

Therefore

$$\begin{aligned}
&\left(\frac{\sigma(t)}{t}\right)^{\delta} \left(\frac{1}{\ell!}\, \rho_{1,\ell,t_*}^{(\alpha)}(\sigma(t))\right)^{\mu(t)-\delta} \\
&\quad \geq \left(\frac{\sigma(t)}{t}\right)^{\delta} \left(\frac{1}{\ell!}\, \sigma^{Mc}(t)\right)^{\alpha-\delta} \\
&\quad = \frac{1}{(\ell!)^{\alpha-\delta}} \left(\frac{(\sigma(t))^{1+(Mc(\alpha-\delta)/\delta)}}{t}\right)^{\delta} \\
&\quad > \frac{1}{(\ell!)^{\alpha-\delta}} \left(\frac{(\sigma(t))^{m}}{t}\right)^{\delta}.
\end{aligned} \tag{84}$$

Thus, by (81) and (78_ℓ), it is obvious that (73_ℓ) holds, which proves the corollary. □

Quite similarly one can prove the following.

Corollary 18. *Let $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, let the conditions* (2), (3), (28_ℓ), *and* (55_ℓ) *be fulfilled, $\alpha > 1$, and $\beta < +\infty$. Moreover, if*

$$\liminf_{t \to +\infty} t \ln t \int_t^{+\infty} \xi^{n-\ell-2} (\sigma(\xi))^{1+(\ell-1)\mu(\xi)} p(\xi)\, d\xi > 0 \tag{85_ℓ}$$

and for some $\delta \in (1, \alpha]$ and $m \in N$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\sigma(\xi))^{\delta+(\ell-1)\mu(\xi)} (\ln \sigma(\xi))^m d\xi\, ds = +\infty, \tag{86_ℓ}$$

then for any $t_0 \in R_+$ one has $\mathbf{U}_{\ell,t_0} = \emptyset$, where α and β are defined by the condition of (26).

Corollary 19. *Let $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ be odd, let the conditions* (2), (3), *and* (28_ℓ) *be fulfilled, $\alpha > 1$, and $\beta < +\infty$. Moreover, let there exist $\gamma \in (0,1)$ and $r \in (0,1]$ such that*

$$\liminf_{t \to +\infty} t^{\gamma} \int_t^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi > 0, \tag{87_ℓ}$$

$$\liminf_{t \to +\infty} \frac{\sigma(t)}{t^r} > 0, \tag{88}$$

and let at least one of the conditions

$$r\alpha \geq 1 \tag{89}$$

or $r\alpha < 1$ and for some $\varepsilon > 0$ and $\delta \in (1, \alpha]$

$$\begin{aligned}
&\int_0^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta+(\alpha r(1-\gamma)/(1-\alpha r))-\varepsilon} \\
&\qquad \times (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds = +\infty
\end{aligned} \tag{90_ℓ}$$

be fulfilled. Then for any $t_0 \in R_+$ one has $\mathbf{U}_{\ell,t_0} = \emptyset$, where α is defined by (26).

Proof. It suffices to show that the condition (73_ℓ) is satisfied for some $k \in N$ and $\delta = (1+\alpha)/2$. Indeed, according to (87_ℓ) and (88), there exist $\gamma \in (0,1)$, $r \in (0,1]$, $c > 0$, and $t_1 \in R_+$ such that

$$t^{\gamma} \int_t^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi > c \quad \text{for } t \geq t_1, \tag{91}$$

$$\sigma(t) \geq c t^r \quad \text{for } t \geq t_1. \tag{92}$$

By (77), (31), and (91), from (31) we get

$$\begin{aligned}
&\rho_{2,\ell,t_*}^{(\alpha)}(t) \\
&\quad \geq \frac{c}{(n-\ell)!} \int_{\sigma_{(-1)}(t_*)}^{t} s^{-\gamma} ds \\
&\qquad = \frac{c\left(t^{1-\gamma} - \sigma_{(-1)}^{1-\gamma}(t_*)\right)}{(n-\ell)!\ (1-\gamma)} \quad \text{for } t \geq \sigma_{(-1)}(t_*).
\end{aligned} \tag{93}$$

Let $\gamma_1 \in (\gamma, 1)$. Choose $t_2 > \sigma_{(-1)}(t_*)$ such that

$$\rho_{2,\ell,t_*}^{(\alpha)}(t) \geq t^{1-\gamma_1} \quad \text{for } t \geq t_2. \tag{94}$$

Therefore, by (91) from (31) we can find $t_3 > t_2$ such that

$$\rho_{3,\ell,t_*}^{(\alpha)}(t) \geq t^{(1-\gamma_1)(1+\alpha r)} \quad \text{for } t \geq t_3. \tag{95}$$

Hence for any $k_0 \in N$ there exists t_{k_0} such that

$$\rho_{k_0,\ell,t_*}^{(\alpha)}(t) \geq t^{(1-\gamma_1)(1+\alpha r+\cdots+(\alpha r)^{k_\alpha-2})} \quad \text{for } t \geq t_{k_0}. \tag{96}$$

Assume that (89) is fulfilled. Choose $k_0 \in N$ such that $k_0 - 1 \geq ((1-r)(1+\alpha))/((1-\gamma_1)(\alpha-1))$. Then according

to (92), (96), and (28_ℓ), the condition (73_ℓ) holds for $k = k_0$ and $\delta = (1+\alpha)/2$. In this case, the validity of the corollary has already been proven.

Assume now that (90_ℓ) is fulfilled. Let $\varepsilon > 0$ and choose $k_0 \in N$ and $\gamma_1 \in (\gamma, 1)$ such that

$$(1-\gamma_1)\left(1+\alpha r+\cdots+(\alpha r)^{k_0-2}\right) > \frac{(1-\gamma)\alpha r}{1-\alpha r} - \varepsilon. \quad (97)$$

Then according to (96), (92), and (90_ℓ), it is obvious that (73_ℓ) holds for $k = k_0$. The proof of the corollary is complete. □

Using Theorem 13, in a manner similar to above we can prove the following.

Corollary 20. *Let $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, let the conditions* (2), (3), *and (28_ℓ) be fulfilled, $\beta = 1$, and*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}p(\xi)\,d\xi\,ds > 0. \quad (98_\ell)$$

Then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \emptyset$, where β is given by (26).

Corollary 21. *Let $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, let the conditions* (2), (3), *and (28_ℓ) be fulfilled, $\beta = 1$, and*

$$\liminf_{t\to+\infty} t\int_t^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}p(\xi)\,d\xi = \gamma > 0. \quad (99_\ell)$$

Moreover, let for some $\varepsilon \in (0,\gamma)$

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\int_s^{+\infty}\xi^{n-\ell-1}(\sigma(\xi))^{\mu(\xi)(\ell-1+((\gamma-\varepsilon)/\ell!(n-\ell)!))} \times p(\xi)\,d\xi\,ds > 0. \quad (100_\ell)$$

Then for any $t_0 \in R_+$ one has $\mathbf{U}_{\ell,t_0} = \emptyset$, where β is given by (26).

Corollary 22. *Let $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, let the conditions* (2), (3), *(28_ℓ), and (55_ℓ) be fulfilled, $1 < \beta < +\infty$, and for some $\delta \in (1,\beta]$*

$$\int_0^{+\infty}\int_s^{+\infty}\xi^{n-\ell-1-\delta}(\sigma(\xi))^{\ell\mu(\xi)+\delta-\beta}p(\xi)\,d\xi\,ds = +\infty. \quad (101_\ell)$$

Then for any $t_0 \in R_+$, $\mathbf{U}_{\ell,t_0} = \emptyset$.

5. Differential Equation with Property A

Theorem 23. *Let the conditions* (2), (3) *be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, the conditions (74_ℓ) and (75_ℓ) hold. Moreover for any large $t_* \in R_+$, if $\alpha = 1$ and $\beta < +\infty$ for some $k \in N$ let (72_ℓ) be fulfilled or if $\alpha > 1$ and $\beta < +\infty$, for some $k \in N$, $M \in (1,+\infty)$, and $\delta \in (1,\alpha]$, let (72_ℓ) be fulfilled. Then, if for odd n*

$$\int_0^{+\infty} t^{n-1}p(t)\,dt = +\infty, \quad (102)$$

then (1) *has Property* **A**, *where α and β are defined by* (26) *and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by* (30)–(32).

Proof. Let (1) have a proper nonoscillatory solution $u : [t_0,+\infty) \to (0,+\infty)$ (the case $u(t) < 0$ is similar). Then by (2), (3), and Lemma 3 there exists $\ell \in \{0,\dots,n-1\}$ such that $\ell+n$ is odd and conditions (20_ℓ) hold. Since, for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, the conditions of Theorem 12 are fulfilled we have $\ell \notin \{1,\dots,n-1\}$. Now assume that $\ell = 0$, n is odd, and there exists $c \in (0,1)$ such that $u(t) \geq c$ for sufficiently large t. According to (20_0) since $\beta < +\infty$, from (1) we have

$$\begin{aligned}\sum_{i=0}^{n-1}(n-i-1)!t_1^i\left|u^{(i)}(t_1)\right| &\geq \int_{t_1}^t s^{n-1}p(s)\,c^{\mu(s)}ds \\ &\geq c^\beta\int_{t_1}^t s^{n-1}p(s)\,ds \quad \text{for } t \geq t_1,\end{aligned} \quad (103)$$

where t_1 is a sufficiently large number. The last inequality contradicts the condition (102). The obtained contradiction proves that (1) has Property **A**. □

Theorem 24. *Let the conditions* (2), (3) *be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd the conditions (28_ℓ) and (55_ℓ) hold. Let moreover, if $\beta = 1$, for some $k \in N$ (74_ℓ) hold or if $1 < \beta < +\infty$ and for some $k \in N$, $M \in (1,+\infty)$, and $\delta \in (1,\beta]$, (75_ℓ) hold. Then, if for odd n* (102) *is fulfilled, then* (1) *has Property* **A**, *where β is defined by the second equality of* (26) *and $\tilde{\rho}^{(\beta)}_{k,\ell,t_*}$ is given by* (51)–(53).

Proof. The proof of the theorem is analogous to that of Theorem 23. We simply use Theorem 13 instead of Theorem 12. □

Theorem 25. *Let $\alpha > 1$, $\beta < +\infty$, let the conditions* (2), (3), (28_1), *and* (55_1) *be fulfilled, and*

$$\liminf_{t\to+\infty}\frac{(\sigma(t))^{\mu(t)}}{t} > 0. \quad (104)$$

If, moreover, for some $k \in N$, $M \in (1,+\infty)$, and $\delta \in (1,\alpha]$, (73_1) *holds and for odd n* (102) *is fulfilled, then* (1) *has Property* **A**, *where α and β are defined by* (26) *and $\rho^{(\alpha)}_{k,1,t_*}$ is given by* (30)–(32).

Proof. It suffices to note that by (104) and (72_1), for any $\ell \in \{2,\dots,n-1\}$ there exist $M > 1$, $k \in N$, and $\delta \in (1,\alpha)$ such that condition (72_ℓ) is fulfilled. □

Theorem 26. *Let $1 < \beta < +\infty$, and conditions* (2), (3), (28_1), (29), *and* (104) *be fulfilled. If, moreover, for some $k \in N$, $M \in (1,+\infty)$, and $\delta \in (1,\beta]$,* (75_1) *holds and for odd n* (102) *is fulfilled, then* (1) *has Property* **A**, *where β is defined by the second condition of* (26) *and $\tilde{\rho}^{(\beta)}_{k,1,t_*}$ is given by* (51)–(55).

Proof. The theorem is proved similarly to Theorem 25 if we replace the condition (73_1) by the condition (75_1). □

Corollary 27. *Let $\alpha = 1$, $\beta < +\infty$, and conditions* (2), (3), (28_1), (76_1), *and* (104) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Proof. By (28_1), (76_1), and (104), condition (102), and for any $\ell \in \{1, \ldots, n-1\}$ (76_ℓ) holds. Now assume that (1) has a proper nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$. Then, by (2), (3), and Lemma 3, there exists $\ell \in \{0, \ldots, n-1\}$ such that $\ell + n$ is odd and the condition (20_ℓ) holds. Since for any $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd the conditions of Corollary 14 are fulfilled, we have $\ell \notin \{1, \ldots, n-1\}$. Therefore n is odd and $\ell = 0$. According to (102) and (20_0) it is obvious that the condition (5) holds. Therefore, (1) has Property **A**. □

Using Corollaries 15–19, the validity of Corollaries 28–32 can be proven similarly to Corollary 27.

Corollary 28. *Let $\alpha = 1$, $\beta < +\infty$, and conditions* (2), (3), (78_1), (79_1), *and* (104) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Corollary 29. *Let $\alpha > 1$, $\beta < +\infty$, conditions* (2), (3), (104) *be fulfilled, and* (80_1) *for some $\delta \in (1, \alpha]$ hold and if n is odd, condition* (102) *holds. Then* (1) *has Property* **A**.

Corollary 30. *Let $\alpha > 1$, $\beta < +\infty$, the conditions* (2), (3), (28_1), (78_1), (104) *be fulfilled, and* (81) *for some $m \in N$ holds. Then* (1) *has Property* **A**.

Corollary 31. *Let $\alpha > 1$, $\beta < +\infty$, the conditions* (2), (3), (28_1), (104), *and* (85_1) *and for some $\delta \in (1, \alpha]$ and $m \in N$* (86_1) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Corollary 32. *Let $\alpha > 1$, $\beta < +\infty$, and the conditions* (2), (3), (28_1), *and* (104) *be fulfilled. Let moreover, there exist $\gamma \in (0, 1)$ and $r \in (0, 1]$ such that* (87_1) *and* (88) *hold. Then either condition* (89) *or condition* (90_1) *is sufficient for* (1) *to have Property* **A**.

Corollary 33. *Let $\beta = 1$ and the conditions* (2), (3), (28_1), (104), *and* (98_1) *be fulfilled. Then* (1) *has Property* **A**, *where β is defined by the second condition of* (26).

Proof. By (28_1), (104), and (98_1), the condition (102), and for any $\ell \in \{1, \ldots, n-1\}$ (98_ℓ) holds. Therefore, by Corollary 20, for any $t_0 \in R_+$ and $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ is odd $\mathbf{U}_{\ell, t_0} = \emptyset$. On the other hand, if n is odd and $\ell = 0$, according to (102) it is obvious that the condition (5) holds, which proves that (1) has Property **A**. □

Using Corollaries 21 and 22, we can analogously prove the following corollaries.

Corollary 34. *Let $\beta = 1$ and the conditions* (2), (3), (104), (99_1), *and* (100_1) *be fulfilled. Then* (1) *has Property* **A**, *where β is given by* (26).

Corollary 35. *Let $1 < \beta < +\infty$, the conditions* (2), (3), (104), (28_1), *and* (29_1) *be fulfilled, and if n is odd* (102) *holds. Moreover, if for some $\delta \in (1, \beta)$* (101_1) *holds, then* (1) *has Property* **A**.

Theorem 36. *Let $\alpha > 1$, $\beta < +\infty$, the conditions* (2), (3), (28_{n-1}), *and* (29_{n-1}) *be fulfilled, and*

$$\limsup_{t \to +\infty} \frac{(\sigma(t))^{\mu(t)}}{t} < +\infty. \tag{105}$$

Moreover, for some $k \in N$, $M \in (1, +\infty)$, and $\delta \in (1, \alpha]$, let (73_{n-1}) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are defined by* (26).

Proof. It suffices to note that by (28_{n-1}), (29_{n-1}), (105), and (73_{n-1}) there exist $M > 1$, $k \in N$, and $\delta \in (1, \alpha]$ such that (28_ℓ), (29_ℓ), and (73_ℓ) hold for any $\ell \in \{1, \ldots, n-2\}$. □

Theorem 37. *Let $1 < \beta < +\infty$ and the conditions* (2), (3), (28_{n-1}), (29_{n-1}), *and* (105) *be fulfilled, and for some $k \in N$, $M \in (1, +\infty)$, and $\delta \in (1, \beta]$,* (75_{n-1}) *holds. Then* (1) *has Property* **A**, *where β is given by the second condition of* (26).

Proof. The proof is similar to that of Theorem 36 we replace the condition (73_{n-1}) by the condition (75_{n-1}). □

Corollary 38. *Let $\alpha = 1$, $\beta < +\infty$, and the conditions* (2), (3), (28_{n-1}), (76_{n-1}), *and* (105) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Proof. By (28_{n-1}), (76_{n-1}), and (105), the condition (102), and for any $\ell \in \{1, \ldots, n-1\}$ the condition (76_ℓ) holds; it is obvious that (1) has Property **A**. □

Using Corollaries 15–19, the validity of Corollaries 39–43 below can be proven similarly to Corollary 38.

Corollary 39. *Let $\alpha = 1$, $\beta < +\infty$, and the conditions* (2), (3), (78_{n-1}), (79_{n-1}), *and* (105) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Corollary 40. *Let $\alpha > 1$, $\beta < +\infty$, and the conditions* (2), (3), (105), *and, for some $\delta \in (1, \alpha]$,* (80_{n-1}) *be fulfilled. Then* (1) *has Property* **A**, *where α is given by* (26).

Corollary 41. *Let $\alpha > 1$, $\beta < +\infty$, and the conditions* (2), (3), (28_{n-1}), (78_{n-1}), (105), *and for some $m \in N$* (81) *be fulfilled. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Corollary 42. *Let $\alpha > 1$, $\beta < +\infty$, and the conditions* (2), (3), (28_{n-1}), (85_{n-1}) *and* (105) *be fulfilled and for some $\delta \in (1, \alpha]$ and $m \in N$* (86_{n-1}) *holds. Then* (1) *has Property* **A**, *where α and β are given by* (26).

Corollary 43. *Let $\alpha > 1$, $\beta < +\infty$, and the conditions* (2), (3), (28_{n-1}), *and* (105) *be fulfilled. Let, moreover, there exist $\gamma \in (0, 1)$ and $r \in (0, 1)$ such that* (87_{n-1}) *and* (88) *hold. Then either condition* (89) *or condition* (90_{n-1}) *is sufficient for* (1) *to have Property* **A**.

Corollary 44. *Let $\beta = 1$ and the conditions* (2), (3), (28_{n-1}), (105), *and* (98_{n-1}) *be fulfilled. Then* (1) *has Property* **A**.

Proof. By (28_{n-1}), (105), and (98_{n-1}), the conditions (102), and for any $\ell \in \{1,\dots,n-1\}$ (98_ℓ) holds. Therefore, by Corollary 20, it is clear that (1) has Property **A**. □

Using Corollaries 21 and 22, analogously we can prove Corollaries 45 and 5.18.

Corollary 45. *Let $\beta = 1$ and the conditions* (2), (3), (105), (99_{n-1}), *and* (100_{n-1}) *be fulfilled. Then* (1) *has Property* **A**, *where β is given by* (26).

Corollary 46. *Let $1 < \beta < +\infty$ and the conditions* (2), (3), (105), *and* (29_{n-1}) *be fulfilled. If for some $\delta \in (1,\beta]$,* (101_{n-1}) *holds, then* (1) *has Property* **A**.

6. Necessary and Sufficient Conditions

Theorem 47. *Let $\alpha > 1$ and $\beta < +\infty$, let the conditions* (2) *and* (3) *be fulfilled and*

$$\liminf_{t\to+\infty}\frac{\sigma(t)}{t} > 0. \tag{106}$$

Then the condition (102) *is necessary and sufficient for* (1) *to have Property* **A**, *where α and β are given by* (26).

Proof. Necessity. Assume that (1) has Property **A** and

$$\int_0^{+\infty} t^{n-1}p(t)\,dt < +\infty. \tag{107}$$

Therefore, by Lemma 4.1 from [28], there exists $c \neq 0$ such that (1) has a proper solution $u : [0,+\infty) \to R$ satisfying the condition $\lim_{t\to+\infty}u(t) = c$. But this contradicts the fact that (1) has Property **A**.

Sufficiency. By (106) and (102) it is obvious that the condition (80_1) holds. Therefore the sufficiency follows from Corollary 29. □

Remark 48. In Theorem 47 the condition $\beta < +\infty$ cannot be replaced by the condition $\beta = +\infty$. Indeed, let $c \in (0,1/2)$, $\alpha = 1/2c$, and

$$p(t) = \frac{n!t^{\lg_\alpha t}}{t^{1+n}(1+ct)^{\lg_\alpha t}} \quad t \geq 1. \tag{108}$$

It is obvious that the condition (102) is fulfilled, but equation

$$u^{(n)}(t) + p(t)\,|u(t)|^{\lg_\alpha t}\,\mathrm{sign}\,u(t) = 0 \tag{109}$$

has solution $u(t) = (1/t + c)$. Therefore, (109) does not have Property **A**.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Asymptotic Behavior of Higher-Order Quasilinear Neutral Differential Equations

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We study asymptotic behavior of solutions to a class of higher-order quasilinear neutral differential equations under the assumptions that allow applications to even- and odd-order differential equations with delayed and advanced arguments, as well as to functional differential equations with more complex arguments that may, for instance, alternate indefinitely between delayed and advanced types. New theorems extend a number of results reported in the literature. Illustrative examples are presented.

1. Introduction

In this paper, we study asymptotic behavior of solutions to a class of higher-order quasilinear neutral functional differential equations

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\,x^{\beta}(\sigma(t)) = 0, \tag{1}$$

where $t \in \mathbb{I} := [t_0, \infty)$, $t_0 \in \mathbb{R}$, $z(t) := x(t) + p(t)x(\tau(t))$, $r, \tau, \sigma \in C^1(\mathbb{I}, \mathbb{R})$, $r'(t) \geq 0$, $r(t) > 0$, $\lim_{t\to\infty}\sigma(t) = \infty$, $p, q \in C(\mathbb{I}, \mathbb{R})$, $q(t) \geq 0$, and $q(t)$ does not vanish eventually. We also assume that $\alpha, \beta \in \mathfrak{R}$, where $\mathfrak{R}$ stands for the set containing all quotients of odd positive integers. Analysis of qualitative properties of (1) is important not only for the sake of further development of the oscillation theory, but for practical reasons too. In fact, a particular case of (1), an Emden-Fowler type equation

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\,x^{\beta}(\sigma(t)) = 0, \tag{2}$$

has numerous applications in physics and engineering; see, for instance, the papers by Ou and Wong [1] or Wong [2].

As customary, by a solution of (1) we understand a function $x \in C([T_x, \infty), \mathbb{R})$, $T_x \geq t_0$, which has the property $r(z^{(n-1)})^{\alpha} \in C^1([T_x, \infty), \mathbb{R})$ and turns (1) into identity for all $t \in [T_x, \infty)$. We deal only with proper solutions x of (1) that satisfy the condition $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and tacitly assume that (1) possesses such solutions. A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros on the ray $[T_x, \infty)$; otherwise, it is termed *nonoscillatory*. Equation (1) is called oscillatory if all its proper solutions are oscillatory.

For several decades, an increasing interest in obtaining sufficient conditions for oscillatory and nonoscillatory behavior of different classes of differential equations has been observed; see, for instance, the monographs [3–6], the papers [1, 2, 7–25], and the references cited therein. Let us briefly comment on a number of related results which motivated our study. Questions regarding the oscillation and asymptotic behavior of solutions to (2) have been studied by Džurina and Baculíková [12] and Zhang et al. [23, 25]. In particular, Zhang et al. [23, 25] derived some results on the oscillation and asymptotic behavior of solutions to (2) in the case where $\alpha \geq \beta$, $\sigma(t) < t$, and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t < \infty. \tag{3}$$

Oscillation criteria for (1) for $n = 2$, $\alpha = \beta = 1$, and $0 \leq p(t) \leq p_0 < \infty$ can be found in the papers by Baculíková and Džurina [8] and Li et al. [18]. A number of oscillation results for (1) have been established by Baculíková et al. [11]

and Xing et al. [22] under the assumptions that $\alpha = \beta$, $0 \le p(t) \le p_0 < \infty$, and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t = \infty. \tag{4}$$

We conclude by mentioning that Baculíková and Džurina [10] studied another particular case of (1) assuming that $\alpha = \beta = 1$, $0 \le p(t) \le p_0 < \infty$, and

$$\int_{t_0}^{\infty} r^{-1}(t)\,\mathrm{d}t = \infty. \tag{5}$$

It should be noted that research in this paper was strongly motivated by the recent contributions of Baculíková and Džurina [10], Li et al. [18], and Zhang et al. [23, 25]. Our principal goal is to analyze the asymptotic behavior of solutions to (1) in the case where condition (3) holds. We provide sufficient conditions which ensure that solutions to (1) are either oscillatory or approach zero at infinity. In some cases, we reveal oscillatory nature of (1). However, we do not discuss in this paper nonoscillation results referring to the recent monograph by Agarwal et al. [3] for an excellent analysis of recent advances in this direction.

As usual, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1) since, under our assumptions, if $x(t)$ is a solution, then $-x(t)$ is a solution of this equation too.

In the sequel, we denote by τ^{-1} the function which is inverse to τ. We also adopt the following notation for a compact presentation of our results:

$$\begin{gathered}
A(t) := \int_t^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s,\\
Q(t) := \min\{q(t), q(\tau(t))\},\\
R(t) := \max\{r(t), r(\tau(t))\},\\
Q_\gamma(t) := Q(t)\left(\frac{(\eta_1(t))^{n-1}}{r^{1/\alpha}(\eta_1(t))}\right)^{\gamma},\\
Q_\beta(t) := Q(t)\left(\sigma^{n-2}(t)\right)^{\beta},\\
\widetilde{Q}_\beta(t) := Q(t)\left(\frac{(\eta_1(t))^{n-1}}{r^{1/\beta}(\eta_1(t))}\right)^{\beta},\\
Q_\theta(t) := Q(t)\left(\int_{\eta_3(t)}^{\infty}(\eta-\eta_3(t))^{n-3}A(\eta)\,\mathrm{d}\eta\right)^{\theta},\\
\overline{Q}_\beta(t) := Q(t)\left(\int_{\eta_3(t)}^{\infty}(\eta-\eta_3(t))^{n-3}A(\eta)\,\mathrm{d}\eta\right)^{\beta},\\
\widehat{Q}_\beta(t) := Q(t)\left(\frac{\sigma^{n-1}(t)}{r^{1/\beta}(\sigma(t))}\right)^{\beta},\\
\widetilde{Q}_\gamma(t) := Q(t)\left(\frac{\sigma^{n-1}(t)}{r^{1/\alpha}(\sigma(t))}\right)^{\gamma},
\end{gathered} \tag{6}$$

where the meaning of γ, θ, η_1, and η_3 will be explained later.

2. Asymptotic Behavior of Solutions to Even-Order Equations

In what follows, $\tau(t)$ can be both a delayed or an advanced argument. Throughout this section, in addition to the basic assumptions listed in the introduction, it is also supposed that (3) holds along with

(H_1) $0 \le p(t) \le p_0 < \infty$, for some constant p_0;

(H_2) $\tau'(t) \ge \tau_* > 0$, $\tau \circ \sigma = \sigma \circ \tau$.

We need the following auxiliary results.

Lemma 1 (see [20]). *Let $f \in C^n([t_0,\infty),\mathbb{R}^+)$. If the nth derivative $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist a $t_1 \ge t_0$ and an integer l, $0 \le l \le n$ with $n+l$ even for $f^{(n)}(t) \ge 0$, or $n+l$ odd for $f^{(n)}(t) \le 0$ such that*

$$\begin{gathered}
l > 0 \text{ yields } f^{(k)}(t) > 0 \quad \text{for } t \ge t_1,\ k = 0,1,\ldots,l-1,\\
l \le n-1 \text{ yields } (-1)^{l+k} f^{(k)}(t) > 0 \quad \text{for } t \ge t_1,\\
k = l,\ l+1,\ldots,n-1.
\end{gathered} \tag{7}$$

Lemma 2 (see [5, Lemma 2.2.3]). *Let f be as in Lemma 1,*

$$f^{(n)}(t)\, f^{(n-1)}(t) \le 0 \tag{8}$$

for $t \ge t_1$, and assume also that

$$\lim_{t\to\infty} f(t) \ne 0. \tag{9}$$

Then, for every constant $\lambda \in (0,1)$, there exists a $t_\lambda \in [t_1,\infty)$ such that

$$f(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} \left|f^{(n-1)}(t)\right|, \tag{10}$$

for all $t \in [t_\lambda,\infty)$.

We are in a position now to state and prove principal results of this paper for even-order equations.

Theorem 3. *Let $n \ge 2$ be even and let $0 < \beta \le 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist two numbers $\gamma, \lambda \in \mathfrak{R}$ such that $\gamma \le \beta \le \lambda$ and $\gamma < \alpha < \lambda$. Suppose further that there exist two functions $\eta_1, \eta_2 \in C(\mathbb{I},\mathbb{R})$ such that*

$$\begin{gathered}
\eta_1(t) \le \sigma(t) \le \eta_2(t), \qquad \eta_1(t) < t \le \tau(t) < \eta_2(t),\\
\lim_{t\to\infty} \eta_1(t) = \infty.
\end{gathered} \tag{11}$$

If

$$\int^{\infty} Q_\gamma(t)\,\mathrm{d}t = \infty, \tag{12}$$

$$\int^{\infty} Q_\beta(t) A^{\lambda}(\eta_2(t))\,\mathrm{d}t = \infty, \tag{13}$$

every solution $x(t)$ of (1) is either oscillatory or satisfies

$$\lim_{t\to\infty} x(t) = 0. \tag{14}$$

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ which is eventually positive and such that

$$\lim_{t\to\infty} x(t) \neq 0. \tag{15}$$

Then z satisfies

$$\begin{aligned} z(\sigma(t)) &= x(\sigma(t)) + p(\sigma(t))\, x(\tau(\sigma(t))) \\ &\le x(\sigma(t)) + p_0 x(\tau(\sigma(t))). \end{aligned} \tag{16}$$

In view of (1), we have

$$\begin{aligned} 0 &= \frac{{p_0}^{\beta}}{\tau'(t)}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + {p_0}^{\beta} q(\tau(t))\, x^{\beta}(\sigma(\tau(t))) \\ &\ge \frac{{p_0}^{\beta}}{\tau_*}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + {p_0}^{\beta} q(\tau(t))\, x^{\beta}(\sigma(\tau(t))). \end{aligned} \tag{17}$$

Using (16) and [9, Lemma 2], we obtain

$$\begin{aligned} &q(t)\, x^{\beta}(\sigma(t)) + {p_0}^{\beta} q(\tau(t))\, x^{\beta}(\sigma(\tau(t))) \\ &\quad = q(t)\, x^{\beta}(\sigma(t)) + {p_0}^{\beta} q(\tau(t))\, x^{\beta}(\tau(\sigma(t))) \\ &\quad \ge Q(t)\, z^{\beta}(\sigma(t)). \end{aligned} \tag{18}$$

It follows from (1), (17), and (18) that

$$\begin{aligned} &\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + Q(t)\, z^{\beta}(\sigma(t)) \le 0. \end{aligned} \tag{19}$$

As in the proof of [25, Theorem 2.1], we conclude that, by virtue of (1) and Lemma 1, there are two possibilities, either

$$\begin{aligned} &z(t) > 0, \qquad z'(t) > 0, \qquad z^{(n-1)}(t) > 0, \\ &z^{(n)}(t) \le 0, \qquad \left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' \le 0, \end{aligned} \tag{20}$$

or

$$\begin{aligned} &z(t) > 0, \qquad z^{(n-2)}(t) > 0, \\ &z^{(n-1)}(t) < 0, \qquad \left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' \le 0, \end{aligned} \tag{21}$$

for all $t \ge t_1$, where $t_1 \ge t_0$ is large enough.

Case I. Suppose first that conditions (20) hold. Using inequality (19) and assumption $\eta_1(t) \le \sigma(t)$, we conclude that

$$\begin{aligned} &\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + Q(t)\, z^{\beta}(\eta_1(t)) \le 0. \end{aligned} \tag{22}$$

Furthermore, by the monotonicity of $z(t)$, there exists a constant $M > 0$ such that

$$z^{\beta}(\eta_1(t)) = z^{\beta-\gamma}(\eta_1(t))\, z^{\gamma}(\eta_1(t)) \ge M^{\beta-\gamma} z^{\gamma}(\eta_1(t)). \tag{23}$$

Combining (22) and (23), we have

$$\begin{aligned} &\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + M_1 Q(t)\, z^{\gamma}(\eta_1(t)) \le 0, \end{aligned} \tag{24}$$

where $M_1 = M^{\beta-\gamma}$. An application of conditions (20) allows us to deduce that the function

$$w(t) := r(t)\left(z^{(n-1)}(t)\right)^{\alpha} \tag{25}$$

is positive and nonincreasing. By Lemma 2, we have

$$\begin{aligned} z(t) &\ge \frac{\lambda t^{n-1}}{(n-1)!\, r^{1/\alpha}(t)} r^{1/\alpha}(t)\, z^{(n-1)}(t) \\ &= \frac{\lambda t^{n-1}}{(n-1)!\, r^{1/\alpha}(t)} w^{1/\alpha}(t), \end{aligned} \tag{26}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Using (26) in (24), we conclude that $w(t)$ is a positive solution of a delay differential inequality

$$\begin{aligned} &\left(w(t) + \frac{{p_0}^{\beta}}{\tau_*} w(\tau(t))\right)' \\ &\quad + M_1\left(\frac{\lambda}{(n-1)!}\right)^{\gamma} Q_{\gamma}(t)\, w^{\gamma/\alpha}(\eta_1(t)) \le 0. \end{aligned} \tag{27}$$

Define now a function $y(t)$ by

$$y(t) := w(t) + \frac{{p_0}^{\beta}}{\tau_*} w(\tau(t)). \tag{28}$$

Then, by the monotonicity of $w(t)$,

$$y(t) \le w(t)\left(1 + \frac{{p_0}^{\beta}}{\tau_*}\right). \tag{29}$$

Substituting (29) into (27), we observe that $y(t)$ is a positive solution of a delay differential inequality

$$y'(t) + M_1\left(\frac{\lambda}{(n-1)!}\right)^{\gamma}\left(\frac{\tau_*}{\tau_* + {p_0}^{\beta}}\right)^{\gamma/\alpha} Q_{\gamma}(t)\, y^{\gamma/\alpha}(\eta_1(t)) \le 0. \tag{30}$$

Then, by virtue of [21, Theorem 1], the associated delay differential equation

$$y'(t) + M_1\left(\frac{\lambda}{(n-1)!}\right)^{\gamma}\left(\frac{\tau_*}{\tau_* + {p_0}^{\beta}}\right)^{\gamma/\alpha} Q_{\gamma}(t)\, y^{\gamma/\alpha}(\eta_1(t)) = 0 \tag{31}$$

also has a positive solution. However, the result by Kitamura and Kusano [15, Theorem 2] implies that, under assumption (12), (31) is oscillatory. Therefore, (1) cannot have positive solutions.

Case II. Assume now that conditions (21) hold. By virtue of (15), we have that

$$\lim_{t\to\infty} z(t) \neq 0. \tag{32}$$

An application of Lemma 2 yields

$$z(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} z^{(n-2)}(t), \tag{33}$$

for any $\lambda \in (0,1)$ and for all sufficiently large t. Hence, by (19) and (33), we obtain

$$\begin{aligned}&\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + \left(\frac{\lambda}{(n-2)!}\right)^{\beta} Q_{\beta}(t)\left(z^{(n-2)}(\sigma(t))\right)^{\beta} \leq 0.\end{aligned} \tag{34}$$

Using conditions $z^{(n-1)}(t) < 0$, $\sigma(t) \leq \eta_2(t)$, and inequality (34), we have

$$\begin{aligned}&\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + \left(\frac{\lambda}{(n-2)!}\right)^{\beta} Q_{\beta}(t)\left(z^{(n-2)}(\eta_2(t))\right)^{\beta} \leq 0.\end{aligned} \tag{35}$$

Furthermore, by the monotonicity of $z^{(n-2)}(t)$, there exists a constant $N > 0$ such that

$$\begin{aligned}\left(z^{(n-2)}(\eta_1(t))\right)^{\beta} &= \left(z^{(n-2)}(\eta_1(t))\right)^{\beta-\lambda}\left(z^{(n-2)}(\eta_1(t))\right)^{\lambda} \\ &\geq N^{\beta-\lambda}\left(z^{(n-2)}(\eta_2(t))\right)^{\lambda}.\end{aligned} \tag{36}$$

Combining (35) and (36), we arrive at

$$\begin{aligned}&\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad + N_1\left(\frac{\lambda}{(n-2)!}\right)^{\beta} Q_{\beta}(t)\left(z^{(n-2)}(\eta_2(t))\right)^{\lambda} \leq 0,\end{aligned} \tag{37}$$

where $N_1 = N^{\beta-\lambda}$. Using the monotonicity of $w(t)$, for $s \geq t \geq t_1$, we conclude that

$$r^{1/\alpha}(s) z^{(n-1)}(s) \leq r^{1/\alpha}(t) z^{(n-1)}(t). \tag{38}$$

Dividing (38) by $r^{1/\alpha}(s)$ and integrating the resulting inequality from t to l, we obtain

$$\begin{aligned}&z^{(n-2)}(l) \\ &\quad \leq z^{(n-2)}(t) + r^{1/\alpha}(t) z^{(n-1)}(t) \int_t^l r^{-1/\alpha}(s)\,ds.\end{aligned} \tag{39}$$

Passing to the limit as $l \to \infty$, we deduce that

$$0 \leq z^{(n-2)}(t) + r^{1/\alpha}(t) z^{(n-1)}(t) A(t), \tag{40}$$

which yields

$$z^{(n-2)}(t) \geq -A(t) r^{1/\alpha}(t) z^{(n-1)}(t) = -A(t) w^{1/\alpha}(t). \tag{41}$$

Combining (37) and (41), we have

$$\begin{aligned}&\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{{p_0}^{\beta}}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\ &\quad - N_1\left(\frac{\lambda}{(n-2)!}\right)^{\beta} Q_{\beta}(t) A^{\lambda}(\eta_2(t)) w^{\lambda/\alpha}(\eta_2(t)) \leq 0.\end{aligned} \tag{42}$$

Using again monotonicity of $w(t)$, we conclude that

$$y(t) \geq w(\tau(t))\left(1 + \frac{{p_0}^{\beta}}{\tau_*}\right). \tag{43}$$

Substituting (43) into (42), we observe that $y(t)$ is a negative solution of an advanced differential inequality

$$\begin{aligned}&y'(t) - N_1\left(\frac{\lambda}{(n-2)!}\right)^{\beta}\left(\frac{\tau_*}{\tau_* + {p_0}^{\beta}}\right)^{\lambda/\alpha} \\ &\quad \times Q_{\beta}(t) A^{\lambda}(\eta_2(t)) y^{\lambda/\alpha}\left(\tau^{-1}(\eta_2(t))\right) \leq 0,\end{aligned} \tag{44}$$

which implies that $u(t) := -y(t)$ is a positive solution of an advanced differential inequality

$$\begin{aligned}&u'(t) - N_1\left(\frac{\lambda}{(n-2)!}\right)^{\beta}\left(\frac{\tau_*}{\tau_* + {p_0}^{\beta}}\right)^{\lambda/\alpha} \\ &\quad \times Q_{\beta}(t) A^{\lambda}(\eta_2(t)) u^{\lambda/\alpha}\left(\tau^{-1}(\eta_2(t))\right) \geq 0.\end{aligned} \tag{45}$$

Consequently, by [7, Lemma 2.3], the associated advanced differential equation

$$\begin{aligned}&u'(t) - N_1\left(\frac{\lambda}{(n-2)!}\right)^{\beta}\left(\frac{\tau_*}{\tau_* + {p_0}^{\beta}}\right)^{\lambda/\alpha} \\ &\quad \times Q_{\beta}(t) A^{\lambda}(\eta_2(t)) u^{\lambda/\alpha}\left(\tau^{-1}(\eta_2(t))\right) = 0\end{aligned} \tag{46}$$

also has a positive solution. However, it follows from [15, Theorem 1] that if condition (13) holds, (46) is oscillatory. Therefore, (1) cannot have positive solutions. This contradiction with our initial assumption completes the proof. □

Theorem 4. *Let $n \geq 2$ be even, and let $0 < \alpha = \beta \leq 1$. Assume that conditions (H_1) and (H_2) hold, and there exist*

two functions $\eta_1, \eta_2 \in C(\mathbb{I}, \mathbb{R})$ *satisfying* (11). *Suppose also that conditions*

$$\frac{\tau_*}{((n-1)!)^\beta (\tau_* + p_0{}^\beta)} \liminf_{t\to\infty} \int_{\eta_1(t)}^{t} \widetilde{Q}_\beta(s)\,\mathrm{d}s > \frac{1}{e}, \tag{47}$$

$$\frac{\tau_*}{((n-2)!)^\beta (\tau_* + p_0{}^\beta)} \liminf_{t\to\infty} \int_{t}^{\tau^{-1}(\eta_2(t))} Q_\beta(s) A^\beta(\eta_2(s))\,\mathrm{d}s > \frac{1}{e} \tag{48}$$

are satisfied. Then conclusion of Theorem 3 remains intact.

Proof. Assume that $x(t)$ is an eventually positive solution of (1) that satisfies (15). Proceeding as in the proof of Theorem 3, one comes to the conclusion that, for every $\lambda \in (0,1)$, a delay differential equation

$$y'(t) + \left(\frac{\lambda}{(n-1)!}\right)^\beta \frac{\tau_*}{\tau_* + p_0{}^\beta} \widetilde{Q}_\beta(t)\, y(\eta_1(t)) = 0 \tag{49}$$

and an advanced differential equation

$$u'(t) - \left(\frac{\lambda}{(n-2)!}\right)^\beta \frac{\tau_*}{\tau_* + p_0{}^\beta} Q_\beta(t) A^\beta(\eta_2(t)) \times u\left(\tau^{-1}(\eta_2(t))\right) = 0 \tag{50}$$

both have positive solutions. On the other hand, condition (47) and [9, Lemma 4] imply that (49) is oscillatory, a contradiction. Likewise, by virtue of [6, Theorem 2.4.1], condition (48) yields that (50) has no positive solutions. This contradiction completes the proof. □

Theorem 5. *Let* $n \geq 2$ *be even and* $0 < \beta \leq 1$. *Assume that conditions* (H_1) *and* (H_2) *are satisfied, and there exist two numbers* $\gamma, \lambda \in \mathfrak{R}$ *as in Theorem 3 and two functions* $\eta_1, \eta_2 \in C(\mathbb{I}, \mathbb{R})$ *such that*

$$\eta_1(t) \leq \sigma(t) \leq \eta_2(t), \qquad \eta_1(t) < \tau(t) \leq t < \eta_2(t), \qquad \lim_{t\to\infty} \eta_1(t) = \infty. \tag{51}$$

If conditions (12) *and* (13) *hold, the conclusion of Theorem 3 remains intact.*

Proof. As above, let $x(t)$ be an eventually positive solution of (1) that satisfies (15). As in the proof of Theorem 3, we split the argument into two parts.

Case I. Assume first that (20) is satisfied. It has been established in the proof of Theorem 3 that the function $w(t)$ defined by (25) is positive, nonincreasing, and satisfies inequality (27). Introducing again $y(t)$ by (28) and using the monotonicity of $w(t)$, we conclude that

$$y(t) \leq w(\tau(t)) \left(1 + \frac{p_0{}^\beta}{\tau_*}\right). \tag{52}$$

Substitution of (52) into (27) implies that, for sufficiently large t, $y(t)$ is a positive solution of a delay differential inequality

$$y'(t) + M_1 \left(\frac{\lambda}{(n-1)!}\right)^\gamma \left(\frac{\tau_*}{\tau_* + p_0{}^\beta}\right)^{\gamma/\alpha} Q_\gamma(t) \times y^{\gamma/\alpha}\left(\tau^{-1}(\eta_1(t))\right) \leq 0. \tag{53}$$

Then, by virtue of [21, Theorem 1], the associated delay differential equation

$$y'(t) + M_1 \left(\frac{\lambda}{(n-1)!}\right)^\gamma \left(\frac{\tau_*}{\tau_* + p_0{}^\beta}\right)^{\gamma/\alpha} Q_\gamma(t) \times y^{\gamma/\alpha}\left(\tau^{-1}(\eta_1(t))\right) = 0 \tag{54}$$

also has a positive solution. However, [15, Theorem 2] implies that if (12) holds, (54) is oscillatory. Therefore, (1) cannot have positive solutions.

Case II. Assume now that (21) is satisfied. It has been established in the proof of Theorem 3 that the function $w(t)$ defined by (25) is negative, nonincreasing, and satisfies the inequality (42). Introducing again $y(t)$ by (28) and using the monotonicity of $w(t)$, we conclude that

$$y(t) \geq w(t) \left(1 + \frac{p_0{}^\beta}{\tau_*}\right). \tag{55}$$

Substituting (55) into (42), we observe that $y(t)$ is a negative solution of an advanced differential inequality

$$y'(t) - N_1 \left(\frac{\lambda}{(n-2)!}\right)^\beta \left(\frac{\tau_*}{\tau_* + p_0{}^\beta}\right)^{\lambda/\alpha} Q_\beta(t) \times A^\lambda(\eta_2(t))\, y^{\lambda/\alpha}(\eta_2(t)) \leq 0. \tag{56}$$

That is, $u(t) := -y(t)$ is a positive solution of an advanced differential inequality

$$u'(t) - N_1 \left(\frac{\lambda}{(n-2)!}\right)^\beta \left(\frac{\tau_*}{\tau_* + p_0{}^\beta}\right)^{\lambda/\alpha} Q_\beta(t) \times A^\lambda(\eta_2(t))\, u^{\lambda/\alpha}(\eta_2(t)) \geq 0. \tag{57}$$

Then, by [7, Lemma 2.3], the associated advanced differential equation

$$u'(t) - N_1 \left(\frac{\lambda}{(n-2)!}\right)^\beta \left(\frac{\tau_*}{\tau_* + p_0{}^\beta}\right)^{\lambda/\alpha} Q_\beta(t) \times A^\lambda(\eta_2(t))\, u^{\lambda/\alpha}(\eta_2(t)) = 0 \tag{58}$$

also has a positive solution. However, [15, Theorem 1] implies that, under assumption (13), (58) is oscillatory. Therefore, (1) cannot have positive solutions. This contradiction with our initial assumption completes the proof. □

Theorem 6. *Let $n \geq 2$ be even and $0 < \alpha = \beta \leq 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist two functions $\eta_1, \eta_2 \in C(\mathbb{I}, \mathbb{R})$ satisfying* (51). *Suppose also that*

$$\frac{\tau_*}{((n-1)!)^\beta (\tau_* + p_0{}^\beta)} \liminf_{t\to\infty} \int_{\tau^{-1}(\eta_1(t))}^{t} \widetilde{Q}_\beta(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \quad (59)$$

$$\frac{\tau_*}{((n-2)!)^\beta (\tau_* + p_0{}^\beta)} \liminf_{t\to\infty} \int_{t}^{\eta_2(t)} Q_\beta(s) A^\beta(\eta_2(s))\,\mathrm{d}s > \frac{1}{\mathrm{e}}. \quad (60)$$

Then the conclusion of Theorem 3 remains intact.

Proof. Assuming that $x(t)$ is an eventually positive solution of (1) that satisfies (15) and proceeding as in the proof of Theorem 5, one concludes that, for every $\lambda \in (0,1)$, a delay differential equation

$$y'(t) + \left(\frac{\lambda}{(n-1)!}\right)^\beta \frac{\tau_*}{\tau_* + p_0{}^\beta} \widetilde{Q}_\beta(t)\, y\left(\tau^{-1}(\eta_1(t))\right) = 0 \quad (61)$$

and an advanced differential equation

$$u'(t) - \frac{\tau_*}{\tau_* + p_0{}^\beta}\left(\frac{\lambda}{(n-2)!}\right)^\beta Q_\beta(t) A^\beta(\eta_2(t))\, u(\eta_2(t)) = 0 \quad (62)$$

have positive solutions. On the other hand, application of condition (59) along with [9, Lemma 4] implies that (61) is oscillatory, a contradiction. Likewise, by virtue of [6, Theorem 2.4.1], condition (60) yields that (62) has no positive solutions. This contradiction completes the proof. □

Note that Theorems 3–6 ensure that every solution $x(t)$ of (1) is either oscillatory or tends to zero as $t \to \infty$ and, unfortunately, cannot distinguish solutions with different behaviors. In the remaining part of this section, we establish several results which guarantee that all solutions of (1) are oscillatory.

Theorem 7. *Let $n \geq 4$ be even and $0 < \beta \leq 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist three numbers $\gamma, \lambda, \theta \in \mathfrak{R}$ such that $\gamma \leq \beta \leq \lambda$, $\gamma < \alpha < \lambda$, $\theta \geq \beta$, and $\theta > \alpha$. Suppose further that there exist three functions $\eta_1, \eta_2, \eta_3 \in C(\mathbb{I}, \mathbb{R})$*

$$\eta_3(t) \geq \sigma(t), \qquad \eta_3(t) > \tau(t), \quad (63)$$

and such that (11) *holds. Assume that conditions* (12), (13), *and*

$$\int^\infty Q_\theta(t)\,\mathrm{d}t = \infty \quad (64)$$

hold. Then (1) *is oscillatory.*

Proof. Without loss of generality, suppose that $x(t)$ is a nonoscillatory solution of (1) which is eventually positive. As in the proof of Theorem 3, we obtain (19). Applying the same argument as in the paper by Zhang et al. [23, Theorem 2.1], we conclude that, by virtue of (1) and Lemma 1, in addition to the case (20), there are two more possible types of behavior of solutions for $t \geq t_1$, where $t_1 \geq t_0$ is large enough in the proof of Theorem 3. Namely, one can also have

$$\begin{gathered} z(t) > 0, \qquad z'(t) > 0, \qquad z^{(n-2)}(t) > 0, \\ z^{(n-1)}(t) < 0, \qquad \left(r(t)\left(z^{(n-1)}(t)\right)^\alpha\right)' \leq 0, \end{gathered} \quad (65)$$

or

$$\begin{gathered} z(t) > 0, \qquad z^{(j)}(t) < 0, \qquad z^{(j+1)}(t) > 0, \\ z^{(n-1)}(t) < 0, \qquad \left(r(t)\left(z^{(n-1)}(t)\right)^\alpha\right)' \leq 0, \end{gathered} \quad (66)$$

for all odd integers $j \in \{1, 2, \ldots, n-3\}$. However, conditions (12) and (13) yield that neither (20) nor (65) is possible.

Therefore, we have to analyze the only remaining case, and we assume now that all the conditions in (66) are satisfied. Then, inequality (41) holds. Integrating (41) from t to ∞ $n-2$ times, we obtain

$$\begin{aligned} z(t) &\geq -\frac{\int_t^\infty (\eta - t)^{n-3} A(\eta)\,\mathrm{d}\eta}{(n-3)!} r^{1/\alpha}(t)\, z^{(n-1)}(t) \\ &= -\frac{\int_t^\infty (\eta - t)^{n-3} A(\eta)\,\mathrm{d}\eta}{(n-3)!} w^{1/\alpha}(t), \end{aligned} \quad (67)$$

where $w(t)$ is defined by (25). Taking into account that $z'(t) < 0$, $\sigma(t) \leq \eta_3(t)$, and using (19), we have

$$\begin{aligned} &\left(r(t)\left(z^{(n-1)}(t)\right)^\alpha + \frac{p_0{}^\beta}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^\alpha\right)' \\ &\quad + Q(t) z^\beta(\eta_3(t)) \leq 0. \end{aligned} \quad (68)$$

By virtue of monotonicity of $z(t)$, there exists a constant $M_2 > 0$ such that

$$z^\beta(\eta_3(t)) = z^{\beta-\theta}(\eta_3(t)) z^\theta(\eta_3(t)) \geq M_2 z^\theta(\eta_3(t)). \quad (69)$$

Combining (68) and (69), we obtain

$$\begin{aligned} &\left(r(t)\left(z^{(n-1)}(t)\right)^\alpha + \frac{p_0{}^\beta}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^\alpha\right)' \\ &\quad + M_2 Q(t) z^\theta(\eta_3(t)) \leq 0. \end{aligned} \quad (70)$$

Using (67) in (70), we conclude that in this case, the function $w(t)$ defined by (25) is negative, nonincreasing, and satisfies the inequality

$$\begin{aligned} &\left(r(t)\left(z^{(n-1)}(t)\right)^\alpha + \frac{p_0{}^\beta}{\tau_*} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^\alpha\right)' \\ &\quad - \frac{M_2}{((n-3)!)^\theta} Q_\theta(t) w^{\theta/\alpha}(\eta_3(t)) \leq 0. \end{aligned} \quad (71)$$

Introducing again $y(t)$ by (28) and using the monotonicity of $w(t)$, we arrive at (43). Substitution of (43) into (71) leads to the conclusion that $y(t)$ is a negative solution of an advanced differential inequality

$$y'(t) - \frac{M_2}{((n-3)!)^{\theta}}\left(\frac{\tau_*}{\tau_* + p_0{}^{\beta}}\right)^{\theta/\alpha} Q_{\theta}(t)\, y^{\theta/\alpha}\left(\tau^{-1}(\eta_3(t))\right) \leq 0, \tag{72}$$

in which case the function $u(t) := -y(t)$ is a positive solution of an advanced differential inequality

$$u'(t) - \frac{M_2}{((n-3)!)^{\theta}}\left(\frac{\tau_*}{\tau_* + p_0{}^{\beta}}\right)^{\theta/\alpha} Q_{\theta}(t)\, u^{\theta/\alpha}\left(\tau^{-1}(\eta_3(t))\right) \geq 0. \tag{73}$$

Then, by [7, Lemma 2.3], the associated advanced differential equation

$$u'(t) - \frac{M_2}{((n-3)!)^{\theta}}\left(\frac{\tau_*}{\tau_* + p_0{}^{\beta}}\right)^{\theta/\alpha} Q_{\theta}(t)\, u^{\theta/\alpha}\left(\tau^{-1}(\eta_3(t))\right) = 0 \tag{74}$$

also has a positive solution. However, [15, Theorem 1] implies that (74) is oscillatory under assumption (64). Therefore, (1) cannot have positive solutions. This contradiction with our initial assumption completes the proof. □

Theorem 8. *Let $n \geq 4$ be even and $0 < \alpha = \beta \leq 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist three functions $\eta_1, \eta_2, \eta_3 \in C(\mathbb{I}, \mathbb{R})$ as in Theorem 7. Suppose also that conditions* (47) *and* (48) *hold. If*

$$\frac{\tau_*}{((n-3)!)^{\beta}(\tau_* + p_0{}^{\beta})} \liminf_{t\to\infty} \int_t^{\tau^{-1}(\eta_3(t))} \overline{Q}_{\beta}(s)\, ds > \frac{1}{e}, \tag{75}$$

(1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) which is eventually positive. As in the proof of Theorem 7, one can have either (20) or (65), or (66). However, conditions (47) and (48) exclude cases (20) and (65). Thus, all the inequalities in (66) should be satisfied. Along the same lines as in the proof of Theorem 7, one comes to the conclusion that an advanced differential equation

$$u'(t) - \frac{\tau_*}{((n-3)!)^{\beta}(\tau_* + p_0{}^{\beta})} \overline{Q}_{\beta}(t)\, u\left(\tau^{-1}(\eta_3(t))\right) = 0 \tag{76}$$

has positive solutions. On the other hand, if condition (75) holds, a well-known result [6, Theorem 2.4.1] implies that (76) has no positive solutions. This contradiction completes the proof. □

Theorem 9. *Let $n \geq 4$ be even, $0 < \beta \leq 1$, and assume that conditions (H_1) and (H_2) are satisfied. Suppose further that there exist three numbers $\gamma, \lambda, \theta \in \mathfrak{R}$ as in Theorem 7 and three functions $\eta_1, \eta_2, \eta_3 \in C(\mathbb{I}, \mathbb{R})$ such that* (51) *is satisfied, $\eta_3(t) \geq \sigma(t)$, and $\eta_3(t) > t$. If* (12), (13), *and* (64) *hold,* (1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive nonoscillatory solution of (1). The same argument as in the proof of Theorem 7 yields that (66) holds. Define the function $w(t)$ by (25). From the proof of Theorem 7, we already know that $w(t)$ is negative, nonincreasing, and satisfies the inequality (71). Introducing then the function $y(t)$ by (28) and using the monotonicity of $w(t)$, we arrive at (55). Substituting (55) into (71), we observe that $y(t)$ is a negative solution of an advanced differential inequality

$$y'(t) - \frac{M_2}{((n-3)!)^{\theta}}\left(\frac{\tau_*}{\tau_* + p_0{}^{\beta}}\right)^{\theta/\alpha} Q_{\theta}(t)\, y^{\theta/\alpha}(\eta_3(t)) \leq 0, \tag{77}$$

while $u(t) := -y(t)$ is a positive solution of an advanced differential inequality

$$u'(t) - \frac{M_2}{((n-3)!)^{\theta}}\left(\frac{\tau_*}{\tau_* + p_0{}^{\beta}}\right)^{\theta/\alpha} Q_{\theta}(t)\, u^{\theta/\alpha}(\eta_3(t)) \geq 0. \tag{78}$$

In this case, the result due to Baculíková [7, Lemma 2.3] allows one to deduce that the associated advanced differential equation

$$u'(t) - \frac{M_2}{((n-3)!)^{\theta}}\left(\frac{\tau_*}{\tau_* + p_0{}^{\beta}}\right)^{\theta/\alpha} Q_{\theta}(t)\, u^{\theta/\alpha}(\eta_3(t)) = 0 \tag{79}$$

also has a positive solution. However, it has been established by Kitamura and Kusano [15, Theorem 1] that if condition (64) is satisfied, (79) is oscillatory. Therefore, (1) cannot have positive solutions, and this contradiction with the assumptions of the theorem completes the proof. □

Theorem 10. *Let $n \geq 4$ be even and $0 < \alpha = \beta \leq 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist three functions $\eta_1, \eta_2, \eta_3 \in C(\mathbb{I}, \mathbb{R})$ as in Theorem 9. Suppose further that* (59), (60) *hold, and*

$$\frac{\tau_*}{((n-3)!)^{\beta}(\tau_* + p_0{}^{\beta})} \liminf_{t\to\infty} \int_t^{\eta_3(t)} \overline{Q}_{\beta}(s)\, ds > \frac{1}{e}. \tag{80}$$

Then (1) *is oscillatory.*

Proof. Assuming that $x(t)$ is an eventually positive nonoscillatory solution of (1) and reasoning as in the proof of Theorem 7, one concludes that (66) holds. As in the proof of Theorem 9, we observe that an advanced differential equation

$$u'(t) - \frac{\tau_*}{((n-3)!)^{\beta}(\tau_* + p_0{}^{\beta})} \overline{Q}_{\beta}(t)\, u(\eta_3(t)) = 0 \tag{81}$$

has positive solutions. On the other hand, if condition (80) is satisfied, a result reported by Ladde et al. [6, Theorem 2.4.1] yields that (81) has no positive solutions. This contradiction completes the proof. □

3. Asymptotic Behavior of Solutions to Odd-Order Equations

In this section, in addition to conditions (H_1), (H_2), and (3), we also assume that

(H_3) $\sigma(t) < t$.

The validity of the following four propositions can be established in the same manner as it has been done for Theorems 3–6. Therefore, to avoid unnecessary repetition, we only formulate counterparts of Theorems 3–6 for the case of odd-order equations.

Theorem 11. *Let $n \geq 3$ be odd and let $0 < \beta \leq 1$. Assume that conditions (H_1)–(H_3) are satisfied, and there exist two numbers $\gamma, \lambda \in \mathfrak{R}$ as in Theorem 3 and a function $\eta_4 \in C(\mathbb{I}, \mathbb{R})$ such that $t \leq \tau(t) < \eta_4(t)$. Suppose further that*

$$\int^{\infty} \widetilde{Q}_{\gamma}(t)\,\mathrm{d}t = \infty, \qquad \int^{\infty} Q_{\beta}(t) A^{\lambda}(\eta_4(t))\,\mathrm{d}t = \infty. \tag{82}$$

Then the conclusion of Theorem 3 remains intact.

Theorem 12. *Let $n \geq 3$ be odd, and let $0 < \alpha = \beta \leq 1$. Assume that conditions (H_1)–(H_3) are satisfied, and there exists a function $\eta_4 \in C(\mathbb{I}, \mathbb{R})$ as in Theorem 11. Suppose also that*

$$\frac{\tau_*}{((n-1)!)^{\beta}(\tau_* + p_0{}^{\beta})} \liminf_{t\to\infty} \int_{\sigma(t)}^{t} \widehat{Q}_{\beta}(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}},$$
$$\frac{\tau_*}{((n-2)!)^{\beta}(\tau_* + p_0{}^{\beta})} \liminf_{t\to\infty} \int_{t}^{\tau^{-1}(\eta_4(t))} Q_{\beta}(s) \times A^{\beta}(\eta_4(s))\,ds > \frac{1}{\mathrm{e}}. \tag{83}$$

Then the conclusion of Theorem 3 remains intact.

Theorem 13. *Let $n \geq 3$ be odd and let $0 < \beta \leq 1$. Assume that conditions (H_1)–(H_3) are satisfied, and there exist two numbers $\gamma, \lambda \in \mathfrak{R}$ as in Theorem 3 and a function $\eta_4 \in C(\mathbb{I}, \mathbb{R})$ such that $\sigma(t) < \tau(t) \leq t < \eta_4(t)$. Suppose further that conditions (82) are satisfied. Then the conclusion of Theorem 3 remains intact.*

Theorem 14. *Let $n \geq 3$ be odd, and let $0 < \alpha = \beta \leq 1$. Assume that conditions (H_1)–(H_3) are satisfied, and there exists a function $\eta_4 \in C(\mathbb{I}, \mathbb{R})$ as in Theorem 13. If*

$$\frac{\tau_*}{((n-1)!)^{\beta}(\tau_* + p_0{}^{\beta})} \liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \widehat{Q}_{\beta}(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}},$$
$$\frac{\tau_*}{((n-2)!)^{\beta}(\tau_* + p_0{}^{\beta})} \liminf_{t\to\infty} \int_{t}^{\eta_4(t)} Q_{\beta}(s) A^{\beta}(\eta_4(s))\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{84}$$

the conclusion of Theorem 3 remains intact.

Note that Theorems 11–14 apply only if σ is a delayed argument, $\sigma(t) < t$. Hence, it is important to complement such results with the following theorems that can be applied in the case where σ is an advanced argument, $\sigma(t) \geq t$.

Theorem 15. *Let $n \geq 3$ be odd and let $0 < \beta \leq 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist two numbers $\gamma, \lambda \in \mathfrak{R}$ as in Theorem 3 and two functions $\eta_1, \eta_2 \in C(\mathbb{I}, \mathbb{R})$ satisfying (11). Suppose also that*

$$\int_{t_0}^{\infty} \xi^{n-2} \left[\frac{1}{R(\xi)} \int_{\xi}^{\infty} Q(s)\,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}\xi = \infty. \tag{85}$$

If (12) and (13) are satisfied, the conclusion of Theorem 3 remains intact.

Proof. Assume that (1) has an eventually positive solution $x(t)$ satisfying (15). Proceeding as in the proof of Theorem 3, we arrive at (19) and observe that (1) yields that either (20) or (21) holds.

Indeed, it follows from the condition $(r(t)(z^{(n-1)}(t))^{\alpha})' \leq 0$ that either $z^{(n-1)}(t) > 0$ or $z^{(n-1)}(t) < 0$. Assume first that $z^{(n-1)}(t) < 0$; this immediately leads us to conditions (21). On the other hand, if $z^{(n-1)}(t) > 0$, then $z^{(n)}(t) \leq 0$ due to the fact that $r'(t) \geq 0$. We claim that $z'(t) > 0$ eventually. In fact, if this is not the case, then $z'(t) < 0$ eventually. Since $z(t) > 0$, $z'(t) < 0$, and (15) holds, there should exist a positive constant a such that

$$\lim_{t\to\infty} z(t) = a. \tag{86}$$

On the other hand, if $z^{(n-1)}(t) > 0$ and $z^{(n)}(t) \leq 0$, there exists a constant $b \geq 0$ such that

$$\lim_{t\to\infty} z^{(n-1)}(t) = b \geq 0. \tag{87}$$

Hence,

$$\lim_{t\to\infty} z^{(i)}(t) = 0, \tag{88}$$

for $i = 1, 2, \ldots, n-1$. Integrating (19) from t to ∞ and using the fact that the limit

$$\lim_{t\to\infty} r(t)\left(z^{(n-1)}(t)\right)^{\alpha} \geq 0 \tag{89}$$

is finite, we have

$$-r(t)\left(z^{(n-1)}(t)\right)^{\alpha}-\frac{{p_0}^{\beta}}{\tau_*}r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}+\int_t^{\infty}Q(s)\,z^{\beta}(\sigma(s))\,\mathrm{d}s\le 0. \tag{90}$$

Consequently,

$$-R(t)\left[\left(z^{(n-1)}(t)\right)^{\alpha}+\left(\left(\frac{{p_0}^{\beta}}{\tau_*}\right)^{1/\alpha}\right)^{\alpha}\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right]+\int_t^{\infty}Q(s)\,z^{\beta}(\sigma(s))\,\mathrm{d}s\le 0. \tag{91}$$

Assume first that $\alpha\le 1$. Using the result due to Baculíková [7, Lemma 2.2], we obtain

$$\left(z^{(n-1)}(t)\right)^{\alpha}+\left(\left(\frac{{p_0}^{\beta}}{\tau_*}\right)^{1/\alpha}\right)^{\alpha}\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\le 2^{1-\alpha}\left[z^{(n-1)}(t)+\left(\frac{{p_0}^{\beta}}{\tau_*}\right)^{1/\alpha}z^{(n-1)}(\tau(t))\right]^{\alpha}. \tag{92}$$

Substituting (92) into (91), we have

$$-2^{1-\alpha}R(t)\left[z^{(n-1)}(t)+\left(\frac{{p_0}^{\beta}}{\tau_*}\right)^{1/\alpha}z^{(n-1)}(\tau(t))\right]^{\alpha}+\int_t^{\infty}Q(s)\,z^{\beta}(\sigma(s))\,\mathrm{d}s\le 0, \tag{93}$$

which yields

$$-\left[z^{(n-1)}(t)+\left(\frac{{p_0}^{\beta}}{\tau_*}\right)^{1/\alpha}z^{(n-1)}(\tau(t))\right]^{\alpha}\le-\frac{1}{2^{1-\alpha}R(t)}\int_t^{\infty}Q(s)\,z^{\beta}(\sigma(s))\,\mathrm{d}s. \tag{94}$$

Therefore,

$$-\left[z^{(n-1)}(t)+\left(\frac{{p_0}^{\beta}}{\tau_*}\right)^{1/\alpha}z^{(n-1)}(\tau(t))\right]+\left[\frac{1}{2^{1-\alpha}R(t)}\int_t^{\infty}Q(s)\,z^{\beta}(\sigma(s))\,\mathrm{d}s\right]^{1/\alpha}\le 0. \tag{95}$$

Integrate (95) $n-2$ times from t to ∞ and then one more time from t_1 to ∞. Using (88) and changing the order of integration, we obtain

$$\int_{t_1}^{\infty}\frac{(\xi-t_1)^{n-2}}{(n-2)!}\left[\frac{1}{2^{1-\alpha}R(\xi)}\int_{\xi}^{\infty}Q(s)\,z^{\beta}(\sigma(s))\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}\xi<\infty. \tag{96}$$

Inequality (96) yields

$$\int_{t_1}^{\infty}\xi^{n-2}\left[\frac{1}{R(\xi)}\int_{\xi}^{\infty}Q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}\xi<\infty, \tag{97}$$

which contradicts (85).

For the case $\alpha>1$, one arrives at the contradiction with the assumptions of the theorem by using another auxiliary result obtained by Baculíková [7, Lemma 2.1]. Thus, we conclude that $z'(t)>0$ eventually. The rest of the proof follows the same lines as in Theorem 3 and is omitted. □

Combining the ideas exploited in the proofs of Theorems 4–6 and 15, one can derive the following results.

Theorem 16. *Let $n\ge 3$ be odd, and let $0<\alpha=\beta\le 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist two functions $\eta_1,\eta_2\in C(\mathbb{I},\mathbb{R})$ satisfying* (11). *If* (47), (48), *and* (85) *hold, the conclusion of Theorem 3 remains intact.*

Theorem 17. *Let $n\ge 3$ be odd, and let $0<\beta\le 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist two numbers $\gamma,\lambda\in\mathfrak{R}$ as in Theorem 3 and two functions $\eta_1,\eta_2\in C(\mathbb{I},\mathbb{R})$ satisfying* (51). *If conditions* (12), (13), *and* (85) *are satisfied, the conclusion of Theorem 3 remains intact.*

Theorem 18. *Let $n\ge 3$ be odd, and let $0<\alpha=\beta\le 1$. Assume that conditions (H_1) and (H_2) are satisfied, and there exist two functions $\eta_1,\eta_2\in C(\mathbb{I},\mathbb{R})$ satisfying* (51). *If conditions* (59), (60), *and* (85) *are satisfied, the conclusion of Theorem 3 remains intact.*

4. Examples and Discussion

The following examples illustrate applications of some of theoretical results presented in the previous sections. In all the examples, p_0 is a constant such that $0\le p_0<\infty$.

Example 1. For $t\ge 1$, consider a fourth-order neutral differential equation

$$\left(\mathrm{e}^{t}\left(x(t)+p_0x(t-2)\right)'''\right)'+\frac{1}{16}\left(1+p_0\mathrm{e}\right)\mathrm{e}^{t-1/2}x(t-1)=0. \tag{98}$$

Let $\eta_1(t)=t-3$ and $\eta_2(t)=t+1$. An application of Theorem 6 yields that every solution $x(t)$ of (98) is either oscillatory or satisfies (14). As a matter of fact, $x(t)=\mathrm{e}^{-t/2}$ is an exact solution to (98) satisfying (14).

Example 2. For $t\ge 1$, consider a fourth-order neutral differential equation

$$\left(\mathrm{e}^{t}\left(x(t)+p_0x(t+2\pi)\right)'''\right)'+2\sqrt{10}\left(1+p_0\mathrm{e}^{2\pi}\right)\times\mathrm{e}^{t+\arcsin\sqrt{10}/10}x\left(t-\arcsin\frac{\sqrt{10}}{10}\right)=0. \tag{99}$$

Let $\eta_1(t)=t-3$ and $\eta_2(t)=\eta_3(t)=t+3\pi$. Using Theorem 8, we deduce that (99) is oscillatory. It is not hard to verify that one oscillatory solution of this equation is $x(t)=\mathrm{e}^{t}\sin t$.

Example 3. For $t \geq 1$, consider a third-order neutral differential equation

$$\left(e^{t}\left(x(t)+p_0 x(t-2)\right)''\right)' + \frac{9}{8}\left(1+p_0 e^{3}\right)e^{t-3/2}x(t-1) = 0. \tag{100}$$

Let $\eta(t) = t+1$. It follows from Theorem 14 that every solution $x(t)$ of (100) is either oscillatory or satisfies (14). In fact, one solution of this equation satisfying (14) is $x(t) = e^{-t/2}$.

Remark 4. In the case of (2), oscillation criteria established in this paper complement theorems reported by Zhang et al. [23, 25] because our criteria apply also in the case where $\sigma(t) \geq t$ and $\beta > \alpha$. On the other hand, our results for (1) supplement those reported by Baculíková and Džurina [10], Baculíková et al. [11], and Xing et al. [22] since our theorems can be applied if $\alpha \neq \beta$ and (3) holds.

Remark 5. By using inequality

$$x_1^{\beta} + x_2^{\beta} \geq 2^{1-\beta}\left(x_1+x_2\right)^{\beta} \tag{101}$$

which holds for any $\beta \geq 1$ and for all $x_1, x_2 \in [0,\infty)$, results reported in this paper can be extended to (1) for all $\beta \in \mathfrak{R}$ which satisfy $\beta > 1$. In this case, one has to replace $Q(t) := \min\{q(t), q(\tau(t))\}$ with a function $Q(t) := 2^{1-\beta}\min\{q(t), q(\tau(t))\}$ and proceed as above.

Remark 6. Our main assumptions on functional arguments do not specify whether $\tau(t)$ is a delayed or an advanced argument. Remarkably, $\sigma(t)$ can even switch its nature between an advanced and delayed argument. However, as in the paper by Baculíková and Džurina [10, condition (H_3)], such flexibility is achieved at the cost of requiring that the function τ is monotonic and satisfies $\tau \circ \sigma = \sigma \circ \tau$. The question regarding the analysis of the asymptotic behavior of solutions to (1) with other methods that do not require these assumptions remains open at the moment.

Conflict of Interests

The authors declare that they have no competing interests and no financial issues to disclose.

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Research Article

Existence and Global Exponential Stability of Equilibrium for Impulsive Cellular Neural Network Models with Piecewise Alternately Advanced and Retarded Argument

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We introduce impulsive cellular neural network models with piecewise alternately advanced and retarded argument (in short IDEPCA). The model with the advanced argument is system with strong anticipation. Some sufficient conditions are established for the existence and global exponential stability of a unique equilibrium. The approaches are based on employing Banach's fixed point theorem and a new IDEPCA integral inequality of Gronwall type. The criteria given are easily verifiable, possess many adjustable parameters, and depend on impulses and piecewise constant argument deviations, which provides exibility for the design and analysis of cellular neural network models. Several numerical examples and simulations are also given to show the feasibility and effectiveness of our results.

1. Introduction

Chua and Yang [1] proposed a novel class of information-processing systems called cellular neural networks (CNNs) in 1988. Like neural networks, it is a large-scale nonlinear analog circuit which processes signals in real time. Like cellular automata [2] it is made of a massive aggregate of regularly spaced circuit clones, called cells, which communicate with each other directly only through its nearest neighbors. Each cell is made of a linear capacitor, a nonlinear voltage-controlled current source, and a few resistive linear circuit elements. The key features of neural networks are asynchronous parallel processing and global interaction of network elements. Impressive applications of neural networks have been proposed for various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision. For the circuit diagram and connection pattern implementing the CNN, one can refer to [1]. The CNN can be applied in signal processing and can also be used to solve some image processing and pattern recognition problems [3]. However, it is necessary to solve some dynamic image processing and pattern recognition problems by using delayed cellular neural networks (DCNN) [4–6]. The study of the stability of CNN and DCNN is known to be an important problem in theory and applications.

On the other hand, in real world, many evolutionary processes are characterized by abrupt changes at certain time. These changes are known to be impulsive phenomena, which are included in many fields such as physics, chemistry, population dynamics, and optimal control. Fundamental theory of impulsive differential equations has been developed in [7]. Furthermore, researches of impulsive differential equations have been received much interesting in recent years [8–18]. Meanwhile, several kinds of neural networks with impulse have been investigated. In particular, Xu and Yang established the delay differential inequalities with impulsive initial conditions; some new sufficient conditions for global exponential stability of impulsive delay model were obtained [15, 16].

Most neural networks can be classified into two types, continuous or discrete. However, many real-world systems and natural processes cannot be categorized into one of them. They display characteristics both continuous and discrete

styles. For instance, some biological neural networks in biology, bursting rhythm models in pathology, and optimal control models in economics are characterized by abrupt changes of state. These are the familiar impulsive phenomena.

It is well known that applications of CNN depend crucially on the dynamical behavior of the networks. In these applications, stability and convergence of neural networks are prerequisites. However, in the design of neural networks one is interested not only in the uniform asymptotic stability but also in the global exponential stability, which guarantees a neural network to converge fast enough in order to achieve fast response. In addition, in the analysis of dynamical neural networks for parallel computation and optimization, to increase the rate of convergence to the equilibrium point of the networks and reduce the neural computing time, it is necessary to ensure a desired exponential convergence rate of the networks' trajectories, starting from arbitrary initial states to the equilibrium point which corresponds to the optimal solution. Thus, from the mathematical and engineering points of view, it is required that the neural networks have a unique equilibrium point which is globally exponentially stable. Therefore, the problem of stability analysis has received great attention and many results on this topic have been reported in the literature. See, for instance, [4, 9, 13, 19–27] and references cited therein.

1.1. Piecewise Constant Impulsive Systems. Differential equations with piecewise constant argument (in short DEPCA) are first considered by Shah and Wiener [28] and Cooke and Wiener [29] in the 80s and have been developed by many authors. Applications of DEPCAs are discussed in [30]. Theory and practice of DEPCA of general type, have been discussed extensively in [31–37]. Piecewise constant systems exist in widely expanded areas such as biomedicine, chemistry, mechanical engineering, and physics. The systematical studies with mathematical models involving piecewise constant arguments were initiated for solving some biomedical problems. These kinds of equations are similar in structure to those found in certain sequential-continuous models of disease dynamics. In [38], the following system of equations describing the dynamics of the disease for generation $n = 1, 2, \ldots$ is investigated:

$$\begin{aligned} \frac{dI^{(n)}}{dt}(t) &= -c(t)\, I^{(n)}(t) + k(t)\, S^{(n)}(t)\, I^{(n)}(t), \\ &\qquad n < t \le n+1, \\ \frac{dS^{(n)}}{dt}(t) &= -c(t)\, S^{(n)}(t) - k(t)\, S^{(n)}(t)\, I^{(n)}(t), \\ &\qquad n < t \le n+1, \end{aligned} \tag{1}$$

while

$$I^{(1)}(1) = I_0, \qquad S^{(1)}(1) = S_0, \tag{2}$$

where c is the death rate and k is the horizontal transmission factor. These types of models are special cases of the general form

$$\begin{aligned} \frac{dx(t)}{dt} &= F(t, x_t), \quad [t] < t \le [t]+1, \ x_{[t]} = \phi_{[t]}, \\ \phi_{[t]} &= G([t], x_{[t]}), \quad [t] \ge 2, \ \phi_1 = H, \end{aligned} \tag{3}$$

which arise naturally in a number of models of epidemic. DEPCAs usually describe hybrid dynamical systems (a combination of continuous and discrete) and so combine properties of both differential and difference equations.

Impulsive differential equations with discontinuous argument are proposed as an open problem by Wiener [30] in 1994, namely, the impulsive differential equations with piecewise constant argument: IDEPCA. As we know, impulsive differential equations with piecewise constant arguments (in short IDEPCA) are studied in a few papers [8, 39, 40].

1.2. Model Description. First, let us give a general description of the mathematical model of ICNNs with piecewise alternately advanced and retarded argument:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) \\ &\quad + \sum_{j=1}^{n} \left\{ b_{ij} f_j(x_j(t)) + c_{ij} g_j\left(x_j\left(m\left[\frac{t+l}{m}\right]\right)\right)\right\} \\ &\quad + d_i, \quad t \ne mk - l, \end{aligned} \tag{4a}$$

$$\Delta x_i\big|_{t=mk-l} = J_{ik}(x_i(mk - l^-)), \quad i = 1, 2, \ldots, n, \ k \in \mathbb{N}, \tag{4b}$$

where $[\cdot]$ signifies the greatest integer function, l and m are positive real numbers such that $l < m$, t, $x_i \in \mathbb{R}^+$, $i = 1, 2, \ldots, n$, $\Delta x_i(mk - l) = x_i(mk - l) - x_i(mk - l^-)$, and $x_i(mk - l^-) = \lim_{h \to 0^-} x_i(mk - l + h)$. Moreover, n denotes the number of neurons in the network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(m[(t + l)/m]))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and discrete-time $m[(t + l)/m]$, a_i denotes the rate with which the unit i resets its potential to the resting state when isolated from other units and inputs, b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t, c_{ij} denotes the synaptic connection weight of the unit j on the unit i at discrete-time $m[(t + l)/m]$, and d_i is the input from outside the network to the unit i. The numbers $x_i(mk - l^-)$ and $x_i(mk - l)$ are, respectively, the states of the ith unit before and after impulse perturbation at the moment $mk - l$, $k \in \mathbb{N}$, and represent the abrupt change of the state $J_{ik}(x_i(mk - l^-))$ at the impulsive moment $mk - l$.

Let us clarify why the IDEPCA (4a)-(4b) is of alternately advanced and retarded type; that is, the argument can change its deviation character during the motion. The argument is deviated if it is advanced or retarded. Fix $k \in \mathbb{N}$, and consider the IDEPCA on the interval $I_k = [mk -$

$l, m(k+1)-l)$. Then, the identification function $m[(t+l)/m]$ is equal to mk. If $t \in I_k^+ = [mk-l, mk)$, then $m[(t+l)/m] \geq t$ and IDEPCA (4a)-(4b) is an equation with advanced argument. Similarly, if $t \in I_k^- = (mk, m(k+1)-l)$ then $m[(t+l)/m] < t$ and IDEPCA (4a)-(4b) is an equation with retarded argument. Consequently, IDEPCA (4a)-(4b) changes the type of deviation of the argument during the process. In other words, the IDEPCA (4a)-(4b) is of alternately advanced and retarded type.

For any solution $x(t) = (x_1(t), \ldots, x_n(t))^T$ of IDEPCA (4a)-(4b), the model can be summarized as follows:

$$\frac{dx(t)}{dt} = -Ax(t) + Bf(x(t)) + Cg\left(x\left(m\left[\frac{t+l}{m}\right]\right)\right) + D, \quad t \neq mk-l, \tag{5a}$$

$$\Delta x|_{t=mk-l} = J_k\left(x\left(mk-l^-\right)\right), \quad k \in \mathbb{N}, \tag{5b}$$

where $A = \operatorname{diag}(a_1, \ldots, a_n)$, $B = (b_{ij})_{n\times n}$, and $C = (c_{ij})_{n\times n}$ are constant matrices and $D = (d_1, \ldots, d_n)$ is a constant vector. Moreover, the functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfy $(\partial f_i/\partial x_j) = (\partial g_i/\partial x_j) = 0$ when $i \neq j$.

To the best of our knowledge, cellular neural network with piecewise constant argument has been developed by few authors, for example, Huang et al. Reference [41] considered first the following cellular neural network with piecewise constant delay:

$$\frac{dx_i(t)}{dt} = -a_i([t])\, x_i(t) + \sum_{j=1}^{n}\left\{c_{ij}([t])\, g_j\left(x_j([t])\right)\right\} + d_i([t]), \tag{6}$$

where $[\cdot]$ signifies the greatest integer function. Some sufficient conditions of existence and attractivity of almost periodic sequence solution were given for the corresponding discrete-time analogue:

$$x_i(k+1) = x_i(k)\, e^{-a_i(k)} + \frac{1-e^{-a_i(k)}}{a_i(k)} \times \left\{\sum_{j=1}^{n} c_{ij}(k)\, g_j\left(x_j(k)\right) + d_i(k)\right\}. \tag{7}$$

In 2010, Akhmet and Yılmaz [8] considered first the following impulsive neural network with only piecewise constant retarded argument:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{m}\left\{b_{ij} f_j\left(x_j(t)\right) + c_{ij} g_j\left(x_j(\beta(t))\right)\right\} \\ &\quad + d_i, \quad t \neq \theta_k, \\ \Delta x_i|_{t=\theta_k} &= I_k\left(x_i\left(\theta_k^-\right)\right), \quad i = 1, 2, \ldots, m,\ k \in \mathbb{N}, \end{aligned} \tag{8}$$

where $\beta(t) = \theta_k$ if $\theta_k < t < \theta_{k+1}$, $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, is an identification function and $\theta_k > 0$, $k \in \mathbb{N}$, is a sequence of real numbers. Several sufficient conditions are obtained for the existence and stability of a unique ω-periodic solution.

In this paper, we for the first time study the dynamic behavior of impulsive cellular neural network models that combine the properties of impulsive differential equations and discrete-time difference equations, that is, the following impulsive cellular neural network with piecewise alternately advanced and retarded argument:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) \\ &\quad + \sum_{j=1}^{n}\left\{b_{ij} f_j\left(x_j(t)\right) + c_{ij} g_j\left(x_j\left(m\left[\frac{t+l}{m}\right]\right)\right)\right\} \\ &\quad + d_i, \quad t \neq mk-l, \\ \Delta x_i|_{t=mk-l} &= J_{ik}\left(x_i\left(mk-l^-\right)\right), \quad i = 1, 2, \ldots, n,\ k \in \mathbb{N}. \end{aligned} \tag{9}$$

The purpose of this paper is to derive some new and simple sufficient conditions for the existence and uniqueness of solutions of the ICNNs with IDEPCA system (5a)-(5b), which is globally exponentially stable. This paper is organized as follows. In Section 2, we establish several criteria for the existence and uniqueness of a unique equilibrium of the ICNNs with IDEPCA system and the equivalence lemma for (5a)-(5b). Here, a new IDEPCA Gronwall-type inequality is very useful. In Section 3, we derive some sufficient conditions which ensure that a unique equilibrium of the ICNNs with IDEPCA system (5a)-(5b) is globally exponentially stable. In Section 4, two illustrative examples and the numerical simulations are given to demonstrate the effectiveness of our results. The conclusions are drawn in Section 5.

2. Existence and Uniqueness Theorems

In this section, sufficient conditions that govern the network parameters and the activation functions are established for the existence of a unique equilibrium state of the impulsive cellular neural network models (5a)-(5b).

2.1. Preliminaries and Definition. In this section, we will focus our attention on some preliminary results which will be used in the existence and uniqueness of solutions of the ICNNs with IDEPCA system (5a)-(5b).

For every $t \in \mathbb{R}$, let $i = i(t) \in \mathbb{N}$ be the unique integer such that $t \in I_i = [mi-l, m(i+1)-l)$.

For the sake of convenience, two of the standing assumptions are formulated below.

Lipschitz Condition

(L) The activation functions f_j and g_j with $f_j(0) = 0$, $g_j(0) = 0$ $(j = 1, 2, \ldots, n)$ satisfy the Lipschitz

condition; that is, there are constants $\mathscr{L}_j^f, \mathscr{L}_j^g > 0$ such that

$$\begin{aligned} \left|f_j(u) - f_j(v)\right| &\le \mathscr{L}_j^f |u - v|, \\ \left|g_j(u) - g_j(v)\right| &\le \mathscr{L}_j^g |u - v| \end{aligned} \tag{10}$$

for all $u, v \in \mathbb{R}^+$.

The impulsive operator J_k satisfies

$$\left|J_{jk}(u) - J_{jk}(v)\right| \le \mathscr{L}_k^J |u - v|, \tag{11}$$

for all $u, v \in \mathbb{R}^+$, $j = 1, \ldots, n$, $k \in \mathbb{N}$, where $\mathscr{L}_k^J$ is a positive Lipschitz constant.

Existence condition

(E) Consider

$$\max_{i\in[1,\ldots,n]} \left\{ \frac{1 - e^{-l\cdot a_*}}{a_*} \left(\sum_{j=1}^{n} \left[\mathscr{L}_j^f \left|b_{ij}\right| + \mathscr{L}_j^g \left|c_{ij}\right| \right] \right) \right\} < 1, \tag{12}$$

where $\min_{i\in[1,\ldots,n]} a_i = a_*$.

First, we prove the existence and uniqueness of solutions of IDEPCA system (5a)-(5b). A natural extension of the original definition of a solution of DEPCA [28–30, 42] allows us to define a solution of IDEPCA system.

Definition 1. A function x is a solution of IDEPCA system (5a)-(5b) in $\mathbb{R}^+ = [0, \infty)$ if

(i) $x(t)$ is continuous for $t \in \mathbb{R}^+$ with the possible exception of the points $t = mk - l$, $k \in \mathbb{N}$,

(ii) $x(t)$ is right continuous and has left-hand limits at the points $t = mk - l$, $k \in \mathbb{N}$,

(iii) $x(t)$ is differentiable and satisfies (5a) for any $t \in \mathbb{R}^+$, with the possible exception of the points $t = mk - l$, $k \in \mathbb{N}$, where one-sided derivatives exist,

(iv) $x(n)$ satisfies (5b) for $n = km - l$, $k \in \mathbb{N}$.

To study nonlinear IDEPCA system, we will use the approach based on the construction of an equivalent integral equation. Let us give the following proposition.

Proposition 2. *Let $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$. The function $x(t) = x(t, \tau, x_0)$ is a solution on $\mathbb{R}^+$ of the IDEPCA system* (5a)-(5b) *in the sense of Definition 1 if and only if it is a solution of the integral equation*

$$\begin{aligned} x(t) &= e^{-A(t-\tau)} x_0 \\ &\quad + \int_\tau^t e^{-A(t-s)} \left[Bf(x(s)) + Cg\left(x\left(m\left[\frac{s+l}{m}\right]\right)\right) + D \right] ds \\ &\quad + \sum_{k=i(\tau)+1}^{i(t)} e^{-A(t-(mk-l))} J_k\left(x\left(mk - l^-\right)\right), \quad t \in \mathbb{R}^+. \end{aligned} \tag{13}$$

In particular, one has the following integral equations: for $i = 1, \ldots, n$, $t \in \mathbb{R}^+$,

$$\begin{aligned} x_i(t) &= e^{-a_i(t-\tau)} x_i(\tau) \\ &\quad + \int_\tau^t e^{-a_i(t-s)} \left[\sum_{j=1}^{n} b_{ij} f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(m\left[\frac{s+l}{m}\right]\right)\right) + d_i \right] ds \\ &\quad + \sum_{k=i(\tau)+1}^{i(t)} e^{-a(t-(mk-l))} J_{ik}\left(x_i\left(mk - 1^-\right)\right). \end{aligned} \tag{14}$$

The proof of Proposition 2 is almost identical to the verification in [7] with slight changes which are caused by the piecewise constant argument.

In the next, we give the following lemma about IDEPCA integral inequality of Gronwall type, which is one of the most important auxiliary results of the present paper.

Lemma 3. *Let $u : \mathbb{R} \to [0, \infty)$ be a function such that u is continuous with possible points of discontinuity of the first kind at $t = mk - l$, $k \in \mathbb{N}$, and η_1, η_2 are nonnegative real constants satisfying*

$$\upsilon := (\eta_1 + \eta_2) l < 1. \tag{15}$$

Suppose that for $t \ge \tau$ the inequality

$$\begin{aligned} u(t) &\le u(\tau) + \int_\tau^t \left(\eta_1 u(s) + \eta_2 u\left(m\left[\frac{s+l}{m}\right]\right)\right) ds \\ &\quad + \sum_{k=i(\tau)+1}^{i(t)} \beta_k u\left(mk - l^-\right) \end{aligned} \tag{16}$$

holds. Then for $t \ge \tau$,

$$u(t) \le u(\tau) \prod_{k=i(\tau)+1}^{i(t)} (1 + \beta_k) \exp\left\{ \left(\eta_1 + \frac{\eta_2}{1-\upsilon} \right)(t - \tau) \right\}, \tag{17}$$

$$\begin{aligned} u\left(m\left[\frac{t+l}{m}\right]\right) &\le \frac{u(\tau)}{1-\upsilon} \prod_{k=i(\tau)+1}^{i(t)} (1 + \beta_k) \\ &\quad \times \exp\left\{ \left(\eta_1 + \frac{\eta_2}{1-\upsilon} \right)(t - \tau) \right\}, \end{aligned} \tag{18}$$

$$u(mi) \le (1 - \upsilon)^{-1} u(mi - l), \quad i \in \mathbb{N}. \tag{19}$$

Proof. Call $v(t)$ the right member of (16). So $v(\tau) = u(\tau)$, $u \le v$, and v is a piecewise differentiable and nondecreasing function and, by (16), it satisfies

$$\begin{aligned} v'(t) &\le \eta_1 v(t) + \eta_2 v\left(m\left[\frac{t+l}{m}\right]\right), \\ v(mk - l) &\le (1 + \beta_k) v\left(mk - l^-\right), \end{aligned} \tag{20}$$

$k \in \mathbb{N}$ and for any $t \geq r$ with $t,\ r \in I_i$

$$v(t) - v(r) \leq \int_r^t \left(\eta_1 v(s) + \eta_2 v\left(m\left[\frac{s+l}{m}\right]\right)\right) ds. \quad (21)$$

With $t = mi$ and $r = mi - l$ in (21) for $t \in I_i$, since v is a nondecreasing function, we get

$$\begin{aligned} v(mi) &\leq v(mi-l) + \int_{mi-l}^{mi} \left(\eta_1 v(s) + \eta_2 v(mi)\right) ds \\ &\leq v(mi-l) + (\eta_1 + \eta_2)\, l \cdot v(mi). \end{aligned} \quad (22)$$

Considering the particular case $\tau = t_i$ and taking $v(t_i) = u(t_i)$ and $u \leq v$, by (15) and (22), estimate (19) follows. Take now in (21) $t \in I_i$ and $r = mi - l$ to obtain

$$\begin{aligned} v(t) &\leq v(mi-l) + \int_{mi-l}^{t} \left(\eta_1 v(s) + \eta_2 v(mi)\right) ds \\ &\leq v(mi-l) + \int_{mi-l}^{t} \left(\eta_1 v(s) + \frac{\eta_2}{1-\upsilon} v(mi-l)\right) ds \\ &\leq v(mi-l) + \int_{mi-l}^{t} \left(\eta_1 + \frac{\eta_2}{1-\upsilon}\right) v(s)\, ds \end{aligned} \quad (23)$$

because v is a nondecreasing function. Now, we can apply the classical Gronwall's Lemma to get

$$v(t) \leq v(mi-l) \exp\left\{\left(\eta_1 + \frac{\eta_2}{1-\upsilon}\right)(t - (mi-l))\right\} \quad \text{for } t \in I_i. \quad (24)$$

By the impulsive effect (20), we have

$$\begin{aligned} v(m(i+1)-l) &\leq (1+\beta_{i+1})\, v(mi-l) \\ &\quad \times \exp\left\{m \cdot \left(\eta_1 + \frac{\eta_2}{1-\upsilon}\right)\right\}. \end{aligned} \quad (25)$$

From (25), recursively we obtain

$$\begin{aligned} u(t) \leq v(t) &\leq v(\tau) \prod_{k=i(\tau)+1}^{i(t)} (1+\beta_k) \\ &\quad \times \exp\left\{\left(\eta_1 + \frac{\eta_2}{1-\upsilon}\right)(t-\tau)\right\}, \end{aligned} \quad (26)$$

using $v(\tau) = u(\tau)$ and (19); then we give (17) and (18). The proof is complete. This IDEPCA inequality of Gronwall type seems to be new. □

We need to have the global unique existence of solutions $x(t) = x(t, \tau, x_0)$ on $\mathbb{R}^+$ of the nonlinear IDEPCA system (5a)-(5b).

One can easily see that IDEPCA system (5a)-(5b) has the form of DEPCA system without impulsive effect within the intervals $[mi - l, m(i+1) - l)$, $i \in \mathbb{N}$; then using the same technique of [34, 35, 37] we have the following results.

Proposition 4. *Suppose that conditions (L) and (E) hold. For any $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ there exists a unique solution $x(t) = x(t, \tau, x_0)$ of the IDEPCA system* (5a)-(5b) *on $[mi(\tau) - l, m(i(\tau)+1) - l)$.*

Theorem 5. *Under conditions (L) and (E), for every $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(t) = x(t, \tau, x_0)$ of the IDEPCA system* (5a)-(5b) *with $x(\tau) = x_0$ for $t \in [\tau, \infty)$ in the sense of Definition 1.*

Proof. Fix $\tau \in \mathbb{R}^+$; then $\tau \in I_{i(\tau)} = [mi(\tau) - l, mi(\tau) + m - l)$. Use Proposition 4 with $x(\tau) = x_0$ to obtain the unique solution $x(t) = x(t, \tau, x_0)$ on $I_{i(\tau)}$. Then apply the impulse condition to evaluate uniquely

$$\begin{aligned} &x(mi(\tau) + m - l, \tau, x_0) \\ &\quad = x(mi(\tau) + m - l^-, \tau, x_0) \\ &\qquad + J_{i(\tau)+1}\left(x(mi(\tau) + m - l^-, \tau, x_0)\right). \end{aligned} \quad (27)$$

Next, on the interval $I_{i(\tau)+1} = [mi(\tau) + m - l, mi(\tau) + 2m - l)$ the solution satisfies the DEPCA:

$$\begin{aligned} \frac{dy_i(t)}{dt} &= -a_i y_i(t) \\ &\quad + \sum_{j=1}^{n} \left\{b_{ij} f_j(y_j(t)) + c_{ij} g_j\left(y_j\left(m\left[\frac{t+l}{m}\right]\right)\right)\right\} \\ &\quad + d_i, \quad i = 1, 2, \ldots, n. \end{aligned} \quad (28)$$

The IDEPCA system has a unique solution $y(t, mi(\tau) + m - l, x(mi(\tau) + m - l, \tau, x_0))$. By definition of the solution of IDEPCA system (5a)-(5b), $x(t, \tau, x_0) = y(t, mi(\tau) + m - l, x(mi(\tau) + m - l, \tau, x_0))$ on $I_{i(\tau)+1} = [mi(\tau) + m - l, mi(\tau) + 2m - l)$. The mathematical induction completes the proof. □

2.2. Existence and Uniqueness of Equilibrium. When impulsive cellular neural network models are used for the solution of optimization problems, one of the fundamental issues in the design of a network is concerned with the existence of a unique globally exponentially stable equilibrium state of network (5a)-(5b). Without requiring the boundedness, differentiability, or monotonicity, we establish easily verifiable sufficient conditions for the existence of a unique equilibrium state in this section.

Let us denote an equilibrium state of the impulsive cellular neural network models (5a)-(5b) by the constant vector $x^* = (x_1^*, x_2^* \ldots, x_n^*)^T \in \mathbb{R}^n$, where each x_i^* is governed by the algebraic system

$$\begin{aligned} 0 &= -a_i x_i^* + \sum_{j=1}^{n} \left\{b_{ij} f_j(x_j^*) + c_{ij} g_j(x_j^*)\right\} \\ &\quad + d_i, \quad i = 1, \ldots, n. \end{aligned} \quad (29)$$

Here, it is assumed that the impulse functions $J_{ik}(\cdot)$ satisfy $J_{ik}(x_i^*) = 0$ for all $i = 1, \ldots, n$, $k \in \mathbb{N}$.

In the following theorem, we obtain sufficient conditions for the existence of a unique equilibrium, $x^* = (x_1^*, x_2^* \ldots, x_n^*)^T \in \mathbb{R}^n$, of the impulsive cellular neural network models (5a)-(5b).

Theorem 6. *Suppose that conditions (L) and (E) hold and the neural parameters a_i, b_{ij}, and c_{ij} and Lipschitz constants $\mathscr{L}_i^f$, $\mathscr{L}_i^g$ satisfy*

$$a_i > \mathscr{L}_i^f \sum_{j=1}^{n} \left|b_{ij}\right| + \mathscr{L}_i^g \sum_{j=1}^{n} \left|c_{ij}\right|, \quad i, j = 1, \ldots, n. \tag{30}$$

Then there exists a unique equilibrium state x^* of the ICNNs with IDEPCA system (5a)-(5b).

Proof. Let us consider a mapping $G(u) = (G_1(u), G_2(u), \ldots, G_n(u))^T \in \mathbb{R}^n$, where $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ and

$$G_i(u) = \frac{1}{a_i}\left[\sum_{j=1}^{n}\left\{b_{ij} f_j\left(u_j\right) + c_{ij} g_j\left(u_j\right)\right\} + d_i\right], \quad i = 1, \ldots, n. \tag{31}$$

By applying the hypotheses,

$$\begin{aligned}
&\max_{1\le i\le n}\left|G_i(u) - G_i(v)\right| \\
&= \max_{1\le i\le n}\left|\frac{1}{a_i}\left[\sum_{j=1}^{n}\left\{b_{ij} f_j\left(u_j\right) + c_{ij} g_j\left(u_j\right)\right\} + d_i\right]\right. \\
&\qquad \left. - \frac{1}{a_i}\left[\sum_{j=1}^{n}\left\{b_{ij} f_j\left(v_j\right) + c_{ij} g_j\left(v_j\right)\right\} + d_i\right]\right| \\
&\le \max_{1\le i\le n}\left\{\frac{1}{a_i}\sum_{j=1}^{n}\left\{\left|b_{ij}\right|\left|f_j\left(u_j\right) - f_j\left(v_j\right)\right|\right\}\right. \\
&\qquad \left. + \frac{1}{a_i}\sum_{j=1}^{n}\left\{\left|c_{ij}\right|\left|g_j\left(u_j\right) - g_j\left(v_j\right)\right|\right\}\right\} \\
&\le \max_{1\le i\le n}\left\{\frac{1}{a_i}\sum_{j=1}^{n}\left\{\mathscr{L}_j^f\left|b_{ij}\right|\left|u_j - v_j\right|\right\}\right. \\
&\qquad \left. + \frac{1}{a_i}\sum_{j=1}^{n}\left\{\mathscr{L}_j^g\left|c_{ij}\right|\left|u_j - v_j\right|\right\}\right\} \\
&\le \max_{1\le i\le n}\left\{\frac{1}{a_i}\sum_{j=1}^{n}\left\{\mathscr{L}_j^f\left|b_{ij}\right| + \mathscr{L}_j^g\left|c_{ij}\right|\right\}\right\}\cdot \max_{1\le j\le n}\left|u_j - v_j\right|,
\end{aligned} \tag{32}$$

where the number

$$\rho = \frac{1}{a_i}\sum_{j=1}^{n}\left\{\mathscr{L}_j^f\left|b_{ij}\right| + \mathscr{L}_j^g\left|c_{ij}\right|\right\} \tag{33}$$

satisfies $0 < \rho < 1$ by virtue of condition (30). Thus,

$$\max_{1\le i\le n}\left|G_i(u) - G_i(v)\right| \le \rho \max_{1\le j\le n}\left|u_j - v_j\right| \tag{34}$$

for any two vectors $u, v \in \mathbb{R}^n$ implying that the mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is a global contraction on $\mathbb{R}^n$ endowed with the supremum norm. Hence, there is a unique fixed point $x^* \in \mathbb{R}^n$ that satisfies $G(x^*) = x^*$ (i.e., $G_i(x^*) = x_i^*$ for $i = 1, \ldots, n$). This point defines the unique equilibrium state of the impulsive cellular neural network models (5a)-(5b). The proof is now complete. □

3. Global Exponential Stability of Equilibrium

The existence and stability of a unique equilibrium state are usually a requirement in the design of cellular neural network models for various applications, particularly when there are destabilizing agents such as retarded arguments and impulses. However, even if the unique stable state exists, these agents may affect the convergence speed of the network, which in turn can downgrade the performance of the network in applications that demand fast computation in real-time mode. Thus, exponential stability is usually desirable for an impulsive network, and sufficient conditions for the global exponential stability of the unique equilibrium state x^* of the ICNNs with IDEPCA system (5a)-(5b) are obtained in this section.

For analytical convenience, the ICNNs with IDEPCA system (5a)-(5b) can be simplified as follows. Let

$$\begin{gathered}
z_i(t) = x_i(t) - x_i^*, \qquad \widehat{f}\left(z_i(t)\right) = f\left(x_i(t) + x_i^*\right) - f\left(x_i^*\right), \\
\widehat{g}\left(z_i\left(m\left[\frac{t+l}{m}\right]\right)\right) = g\left(x_i\left(m\left[\frac{t+l}{m}\right]\right) + x_i^*\right) - g\left(x_i^*\right), \\
\widehat{J}_{ik}\left(z_i\left(mk - l^-\right)\right) = J_{ik}\left(x_i\left(mk - l^-\right) + x_i^*\right),
\end{gathered} \tag{35}$$

so that the ICNNs with IDEPCA system (5a)-(5b) can be written as

$$\begin{aligned}
\frac{dz(t)}{dt} &= -Az(t) + B\widehat{f}\left(z(t)\right) \\
&\quad + C\widehat{g}\left(z\left(m\left[\frac{t+l}{m}\right]\right)\right), \quad t \ne mk - l, \\
\Delta x\big|_{t=mk-l} &= \widehat{J}_k\left(z\left(mk - l^-\right)\right), \quad k \in \mathbb{N},
\end{aligned} \tag{36}$$

where $\widehat{f}(z(t)) = [\widehat{f}_1(z_1(t)), \ldots, \widehat{f}_n(z_n(t))]^T$, $\widehat{g}(z(m[(t+l)/m])) = [\widehat{g}_1(z_1(m[(t+l)/m])), \ldots, \widehat{g}_n(z_n(m[(t+l)/m]))]^T$ and $\widehat{J}_k(z(mk - l^-)) = [J_{1k}(x_1(mk - l^-) + x_1^*), \ldots, J_{nk}(x_n(mk - l^-) + x_n^*)]^T$.

The activation functions $\widehat{f}_i(\cdot)$, inheriting the properties of $f_i(\cdot)$, satisfy

$$\widehat{f}_i(0) = 0, \qquad \left|\widehat{f}_i(u) - \widehat{f}_i(v)\right| \le \mathscr{L}_i^f\left|u - v\right|, \tag{37}$$

the functions $\widehat{g}_i(\cdot)$ inherit the properties of $g_i(\cdot)$, namely,

$$\widehat{g}_i(0) = 0, \qquad \left|\widehat{g}_i(u) - \widehat{g}_i(v)\right| \le \mathscr{L}_i^g\left|u - v\right|, \tag{38}$$

and the impulsive operator $\widehat{J}_{ik}$ satisfies

$$\widehat{J}_{ik}(0) = 0, \qquad \left|\widehat{J}_{ik}(u) - \widehat{J}_{ik}(v)\right| \le \mathscr{L}_k^J \left|u_i - v_i\right|, \tag{39}$$

for all $u, v \in \mathbb{R}^+$, $i = 1, \ldots, n$, $k \in \mathbb{N}$.

It is clear that the stability of the zero solution of (36) is equivalent to that of the equilibrium x^* of the ICNNs with IDEPCA system (4a)-(4b). Therefore, we restrict our discussion to the stability of the zero solution of (36).

First of all, we give the following definition and lemma, which will be used in the proof of the stability of the zero solution for the ICNNs with IDEPCA system.

Definition 7. The equilibrium x^* of the ICNNs with IDEPCA system (5a)-(5b) is said to be globally exponentially stable if there exist positive constants α and λ such that the estimation

$$\left|x(t) - x^*\right| \le \alpha \left|x(\tau) - x^*\right| e^{-\lambda(t-\tau)} \tag{40}$$

is valid for all $t \ge \tau$.

Lemma 8. *If (L) and (E) are satisfied, then the solutions φ and ψ of the IDEPCA system* (5a)-(5b) *satisfy for all $t \ge \tau$ the inequality*

$$\left|\varphi(t) - \psi(t)\right| \le \left|\varphi(\tau) - \psi(\tau)\right| \exp\left(-\lambda_{i(t)} \cdot (t-\tau)\right), \tag{41}$$

where $\lambda_{i(t)} = a_* - \beta^* - \mathscr{L}_{i(t)}$, $a_* = \min_{i\in[1,\ldots,n]} a_i$, $\mathscr{L}_{i(t)} = \max_{i(\tau)+1\le k\le i(t)} (\ln(1+\mathscr{L}_k^J)/m)$,

$$\begin{aligned}
\beta^* &= \max_{i\in[1,\ldots,n]} \beta_i \\
&= \max_{i\in[1,\ldots,n]} \sum_{j=1}^{n} \left(\mathscr{L}_j^f \left|b_{ij}\right| + \mathscr{L}_j^g \left|c_{ij}\right| (1-\widetilde{v})^{-1} \right. \\
&\qquad \left. \times \exp\left(a_* \cdot (m-l)\right)\right), \\
\widetilde{v} &= \max_{i\in[1,\ldots,n]} \left(\sum_{j=1}^{n} \mathscr{L}_j^f \left|b_{ij}\right| + \mathscr{L}_j^g \left|c_{ij}\right| \exp\left(a_* \cdot (m-l)\right)\right) \\
&\quad \cdot l < 1.
\end{aligned} \tag{42}$$

Proof. Suppose that $\varphi(t) = (\varphi_1, \ldots, \varphi_n)^T$ and $\psi(t) = (\psi_1, \ldots, \psi_n)^T$ are arbitrary solutions of the IDEPCA system (5a)-(5b). Let $y(t) = \varphi(t) - \psi(t)$ and by (5a) and (5b) it follows that $y(\cdot)$ satisfies

$$\begin{aligned}
\dot{y}(t) &= -Ay(t) + B\left(f\left(y(t)+\psi(t)\right) - f\left(\psi(t)\right)\right) \\
&\quad + C\left\{g\left(y\left(m\left[\frac{t+l}{m}\right]\right) + \psi\left(m\left[\frac{t+l}{m}\right]\right)\right) \right. \\
&\quad \left. - g\left(\psi\left(m\left[\frac{t+l}{m}\right]\right)\right)\right\}, \\
\Delta y\big|_{t=mk-l} &= J_k\left(y\left(mk-l^-\right) + \psi\left(mk-l^-\right)\right) \\
&\quad - J_k\left(\psi\left(mk-l^-\right)\right), \quad k \in \mathbb{N}.
\end{aligned} \tag{43}$$

By the variation of parameters formula, it can be proved that

$$\begin{aligned}
y(t) &= e^{-A(t-\tau)} y(\tau) + \int_\tau^t e^{-A(t-s)} \mathscr{R}(s, y(s))\, ds \\
&\quad + \sum_{k=i(\tau)+1}^{i(t)} e^{-A(t-(mk-l))} \mathscr{J}_k\left(y\left(mk-l^-\right)\right),
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
\mathscr{R}(s, y(s)) &:= B \\
&\quad \cdot \left\{f\left(y(s)+\psi(s)\right) - f\left(\psi(s)\right)\right\} \\
&\quad + C \cdot \left\{g\left(y\left(m\left[\frac{s+l}{m}\right]\right)\right.\right. \\
&\quad \left. + \psi\left(m\left[\frac{s+l}{m}\right]\right)\right) \\
&\quad \left. - g\left(\psi\left(m\left[\frac{s+l}{m}\right]\right)\right)\right\}, \\
\mathscr{J}_k\left(y\left(mk-l^-\right)\right) &:= J_k\left(y\left(mk-l^-\right) + \psi\left(mk-l^-\right)\right) \\
&\quad - J_k\left(\psi\left(mk-l^-\right)\right).
\end{aligned} \tag{45}$$

Notice that (L) implies that

$$\begin{aligned}
\left|\mathscr{R}_i(s, y(s))\right| &\le \left(\sum_{j=1}^{n} \mathscr{L}_j^f \left|b_{ij}\right| \left|yi(s)\right| \right. \\
&\quad \left. + \sum_{j=1}^{n} \mathscr{L}_j^g \left|c_{ij}\right| \left|yi\left(m\left[\frac{s+l}{m}\right]\right)\right|\right), \\
\left|\mathscr{R}_i(s, y(s))\right| &\le \max_{i\in[1,\ldots,n]} \left(\sum_{j=1}^{n} \mathscr{L}_j^f \left|b_{ij}\right| \left|y(s)\right| \right. \\
&\quad \left. + \sum_{j=1}^{n} \mathscr{L}_j^g \left|c_{ij}\right| \left|y\left(m\left[\frac{s+l}{m}\right]\right)\right|\right), \\
\left|\mathscr{J}_k\left(y\left(mk-l^-\right)\right)\right| &\le \mathscr{L}_k^J \left|y\left(mk-l^-\right)\right|.
\end{aligned} \tag{46}$$

By (44), we can deduce that $v_i(t) = \exp(a_* \cdot (t-\tau))|y_i(t)|$ satisfies

$$\begin{aligned}
\left|v_i(t)\right| &\le \left|\varphi_i(\tau) - \psi_i(\tau)\right| \\
&\quad + \int_\tau^t \left[\sum_{j=1}^{n} \mathscr{L}_j^f \left|b_{ij}\right| \left|v_j(s)\right| \right. \\
&\quad + \sum_{j=1}^{n} \mathscr{L}_j^g \left|c_{ij}\right| \left|v_j\left(m\left[\frac{s+l}{m}\right]\right)\right|
\end{aligned}$$

$$\times \exp\left(a_* \cdot \left(s - m\left[\frac{s+l}{m}\right]\right)\right)\Bigg] ds + \sum_{k=i(\tau)+1}^{i(t)} \mathscr{L}_k^J \left|v_i\left(mk - l^-\right)\right|, \tag{47}$$

or

$$\begin{aligned}
|v(t)| &\le \left|\varphi(\tau) - \psi(\tau)\right| \\
&\quad + \max_{i\in[1,\dots,n]} \int_\tau^t \Bigg[\sum_{j=1}^n \mathscr{L}_j^f \left|b_{ij}\right| |v(s)| \\
&\qquad + \sum_{j=1}^n \mathscr{L}_j^g \left|c_{ij}\right| \left|v\left(m\left[\frac{s+l}{m}\right]\right)\right| \\
&\qquad \times \exp\left(a_* \cdot \left(s - m\left[\frac{s+l}{m}\right]\right)\right)\Bigg] ds \\
&\quad + \sum_{k=i(\tau)+1}^{i(t)} \mathscr{L}_k^J \left|v\left(mk - l^-\right)\right| \\
&\le \left|\varphi(\tau) - \psi(\tau)\right| + \max_{i\in[1,\dots,n]} \\
&\quad \times \int_\tau^t \Bigg[\sum_{j=1}^n \mathscr{L}_j^f \left|b_{ij}\right| |v(s)| + \sum_{j=1}^n \mathscr{L}_j^g \left|c_{ij}\right| \\
&\qquad \times \exp\left(a_* \cdot (m-l)\right) \left|v\left(m\left[\frac{s+l}{m}\right]\right)\right|\Bigg] ds \\
&\quad + \sum_{k=i(\tau)+1}^{i(t)} \mathscr{L}_k^J \left|v\left(mk - l^-\right)\right|,
\end{aligned} \tag{48}$$

for any finite $t \in [\tau, \infty)$.

Hence, by Lemma 3 of IDEPCA Gronwall's inequality implies

$$\begin{aligned}
|v(t)| &\le \left|\varphi(\tau) - \psi(\tau)\right| \prod_{k=i(\tau)+1}^{i(t)} \left(1 + \mathscr{L}_k^J\right) \\
&\quad \times \exp\Bigg(\max_{i\in[1,\dots,n]} \Bigg\{\sum_{j=1}^n \mathscr{L}_j^f \left|b_{ij}\right| + \frac{1}{1-\tilde{\nu}} \sum_{j=1}^n \mathscr{L}_j^g \left|c_{ij}\right| \\
&\qquad \times \exp\left(a_* \cdot (m-l)\right)\Bigg\} (t-\tau)\Bigg).
\end{aligned} \tag{49}$$

Then, we have

$$\begin{aligned}
\left|\varphi(t) - \psi(t)\right| &\le \left|\varphi(\tau) - \psi(\tau)\right| \prod_{k=i(\tau)+1}^{i(t)} \left(1 + \mathscr{L}_k^J\right) \\
&\quad \times \exp\left\{-\left(a_* - \max_{i\in[1,\dots,n]} \beta_i\right)(t-\tau)\right\},
\end{aligned} \tag{50}$$

or

$$\begin{aligned}
&\left|\varphi(t) - \psi(t)\right| \\
&\quad \le \left|\varphi(\tau) - \psi(\tau)\right| \\
&\qquad \times \exp\Bigg\{-\Bigg(a_* - \max_{i\in[1,\dots,n]} \beta_i \\
&\qquad\qquad - \max_{i(\tau)+1\le k\le i(t)} \frac{\ln\left(1 + \mathscr{L}_k^J\right)}{m}\Bigg)(t-\tau)\Bigg\},
\end{aligned} \tag{51}$$

and the statement (41) follows. □

The following result will show sufficient conditions for the global exponential stability of the unique equilibrium of the ICNNs with IDEPCA system (5a)-(5b).

Theorem 9. *If the assumptions of Theorem 6,* (42) *and*

$$a_* - \beta^* - \mathscr{L}_{i(t)} > 0, \quad t \in \mathbb{R}^+, \tag{52}$$

are satisfied, then the unique equilibrium x^* of the ICNNs with IDEPCA system (5a)-(5b) *is globally exponentially stable.*

Proof. By Theorem 6, we know that the IDEPCA system (5a)-(5b) has a unique equilibrium x^*. Let $x(t, x_0)$ be an arbitrary solution of (5a)-(5b) with initial condition x_0 and define $z(t) = x(t, x_0) - x^*$. By Lemma 8 and (42), we obtain

$$|z(t)| \le |z(\tau)| \exp\left(-\lambda_{i(t)} \cdot (t-\tau)\right), \tag{53}$$

where $\lambda_{i(t)} = a_* - \beta^* - \mathscr{L}_{i(t)}$. So, using (52), we see that $|z(t)| \to 0$ as $t \to \infty$. That is, the zero solution of ICNNs with IDEPCA system (36) is globally exponentially stable. Therefore, the unique equilibrium x^* of the ICNNs with IDEPCA system (5a)-(5b) is globally exponentially stable. □

Remark 10. To the best of the author's knowledge, this is the first time we investigate impulsive cellular neural network models with piecewise alternately advanced and retarded argument in equilibrium case. Sufficient conditions are gained for the existence and exponential stability of a unique equilibrium of the ICNNs with IDEPCA system. And our results can be extended to a unique equilibrium of the CNNs with DEPCA system. See Corollaries 11–12. Our results about exponential stability of a unique equilibrium of the ICNNs with IDEPCA system may give some insight into the application of neural networks.

As immediate corollaries of Lemma 8 and Theorem 9, the following results without impulsive effects are true.

Corollary 11. *If (L) and (E) are satisfied, then the solutions φ and ψ of the DEPCA system* (5a) *satisfy for all $t \geq \tau$ the inequality*

$$\left|\varphi(t)-\psi(t)\right| \leq \left|\varphi(\tau)-\psi(\tau)\right| \exp\left(-\lambda\cdot(t-\tau)\right), \tag{54}$$

where $\lambda = a_* - \beta^*$, $a_* = \min_{i\in[1,\dots,n]} a_i$,

$$\beta^* = \max_{i\in[1,\dots,n]} \beta_i = \max_{i\in[1,\dots,n]} \sum_{j=1}^{n} \left(\mathscr{L}_j^f \left|b_{ij}\right| + \mathscr{L}_j^g \left|c_{ij}\right| (1-\widetilde{v})^{-1} \times \exp\left(a_*\cdot(m-l)\right)\right),$$

$$\widetilde{v} = \max_{i\in[1,\dots,n]} \left(\sum_{j=1}^{n} \mathscr{L}_j^f \left|b_{ij}\right| + \mathscr{L}_j^g \left|c_{ij}\right| \exp\left(a_*\cdot(m-l)\right)\right)\cdot l < 1. \tag{55}$$

Corollary 12. *If the assumptions of Corollary 11 and*

$$a_* - \beta^* > 0 \tag{56}$$

are satisfied, then the unique equilibrium x^* of the CNNs with DEPCA system (5a) *is globally exponentially stable.*

Remark 13. In [41], authors investigated discrete-time cellular neural network without impulsive effects in almost periodic case. Simple sufficient conditions are gained for a unique almost periodic sequence solution which is globally attractive. When $m = 1$, $l = 0$, this conclusion of Corollary 12 cannot be derived by applying the corresponding stability result for cellular neural networks given in the literature [41] with a_i, b_{ij}, c_{ij}, and d_i being constant coefficients.

4. Examples and Simulations

In this section, we give two examples with numerical simulations to illustrate the effectiveness of the proposed method and results.

Example 1. Consider the following impulsive cellular neural networks with piecewise alternately advanced and retarded argument:

$$\begin{aligned}
\frac{dx(t)}{dt} &= -\begin{pmatrix} 1.2 & 0 \\ 0 & 0.9 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.15 & 0.25 \\ 0.25 & 0.15 \end{pmatrix}\begin{pmatrix} \tanh\left(\dfrac{x_1(t)}{2}\right) \\ \tanh\left(\dfrac{x_2(t)}{8}\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0.15 & 0.25 \\ 0.25 & 0.15 \end{pmatrix}\begin{pmatrix} \tanh\left(\dfrac{x_1(3[(t+1)/3])}{8}\right) \\ \tanh\left(\dfrac{x_2(3[(t+1)/3])}{2}\right) \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}, \\
\Delta x|_{t=3k-1} &= \begin{pmatrix} J_{1k}\left(x_1(3k-1^-)\right) \\ J_{2k}\left(x_2(3k-1^-)\right) \end{pmatrix} = \begin{pmatrix} \dfrac{x_1(3k-1^-)-x_1^*}{2} \\ \dfrac{x_2(3k-1^-)-x_2^*}{3} \end{pmatrix}, \quad k\in\mathbb{N},
\end{aligned} \tag{57}$$

where $x_1^* = 1.943$, $x_2^* = 1.57$. One can check that the point $x^* = (x_1^*, x_2^*)^T$ satisfies the algebraic system

$$-a_i x_i^* + \sum_{j=1}^{2} \left\{b_{ij} f_j\left(x_j^*\right) + c_{ij} g_j\left(x_j^*\right)\right\} + d_i = 0, \quad i = 1, 2, \tag{58}$$

approximately. And it is clear that $J_{ik}(x_i^*) = 0$ for $i = 1, 2$. By simple calculation, we can see that $a_* = 0.9$, $\mathscr{L}_1^f = \mathscr{L}_2^g = \mathscr{L}_1^J = 1/2$, $\mathscr{L}_2^f = \mathscr{L}_1^g = 1/8$, $\mathscr{L}_2^J = 1/3$, and $\sup_{t\in\mathbb{R}^+} \mathscr{L}_{i(t)} = \ln(1+\mathscr{L}_1^J)/3 \approx 0.1351$. Then

$$\begin{aligned}
&\frac{1-e^{-l\cdot a_*}}{a_*}\left(\sum_{j=1}^{n}\left[\mathscr{L}_j^f\left|b_{1j}\right| + \mathscr{L}_j^g\left|c_{1j}\right|\right]\right) \approx 0.1813 < 1, \\
&\frac{1-e^{-l\cdot a_*}}{a_*}\left(\sum_{j=1}^{n}\left[\mathscr{L}_j^f\left|b_{2j}\right| + \mathscr{L}_j^g\left|c_{2j}\right|\right]\right) \approx 0.1648 < 1, \\
&a_1 = 1.2 > 0.275 = \mathscr{L}_1^f \sum_{j=1}^{2}\left|b_{1j}\right| + \mathscr{L}_1^g \sum_{j=1}^{2}\left|c_{1j}\right|, \\
&a_2 = 0.9 > 0.25 = \mathscr{L}_2^f \sum_{j=1}^{2}\left|b_{2j}\right| + \mathscr{L}_2^g \sum_{j=1}^{2}\left|c_{2j}\right|.
\end{aligned} \tag{59}$$

By Theorem 6, we know that the ICNNs with IDEPCA system (57) have a unique equilibrium state x^*, approximately with the error, which is less than 10^{-11} (evaluated by MATLAB).

Moreover, we have

$$\widetilde{v} = \max_{i\in[1,\dots,n]}\left(\sum_{j=1}^{n}\mathscr{L}_j^f\left|b_{ij}\right| + \mathscr{L}_j^g\left|c_{ij}\right|\exp\left(a_*\cdot(m-l)\right)\right)\cdot l \approx 0.7522 < 1,$$

$$\sum_{j=1}^{n}\left(\mathscr{L}_j^f\left|b_{1j}\right| + \mathscr{L}_j^g\left|c_{1j}\right|(1-\widetilde{v})^{-1}\exp\left(a_*\cdot(m-l)\right)\right)$$

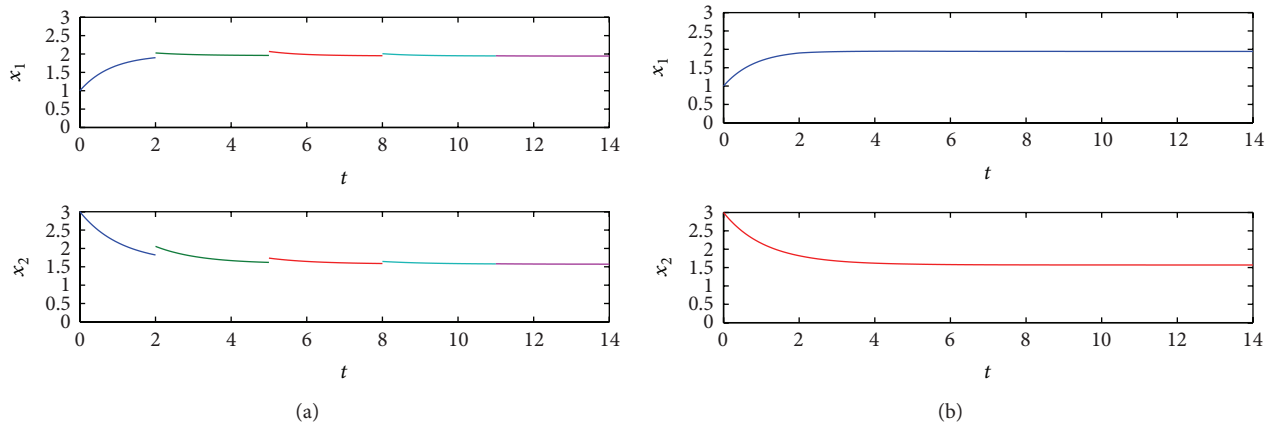

FIGURE 1: (a) The simulation, where the initial value is chosen as $(1,3)^T$, illustrates that all trajectories uniformly converge to the unique equilibrium $x^* = (1.943, 1.57)^T$ for the ICNNs (57) with impulsive effects. (b) The simulation, where the initial value is chosen as $(1,3)^T$, illustrates that all trajectories uniformly converge to the unique equilibrium $x^* = (1.943, 1.57)^T$ for the ICNNs (57) without impulsive effects.

$$\begin{aligned}
&\approx 0.7607 < 0.7648 \approx a_* - \sup_{t\in\mathbb{R}^+}\mathscr{L}_{i(t)},\\
&\sum_{j=1}^{n}\left(\mathscr{L}_j^f\left|b_{2j}\right| + \mathscr{L}_j^g\left|c_{2j}\right|(1-\tilde{v})^{-1}\exp\left(a_*\cdot(m-l)\right)\right)\\
&\approx 0.58302 < 0.7648 \approx a_* - \sup_{t\in\mathbb{R}^+}\mathscr{L}_{i(t)}.
\end{aligned} \tag{60}$$

Thus, according to Theorem 9, the ICNNs with IDEPCA system (57) have a unique globally exponentially stable equilibrium. The numerical simulations, showing the convergence of the unique equilibrium x^* of the ICNNs with and without impulses (57), are given in Figures 1(a) and 1(b).

Example 2. Consider the following impulsive cellular neural networks model with piecewise constant argument:

$$\begin{aligned}
&\frac{dx(t)}{dt}\\
&= -\begin{pmatrix}1.3 & 0\\ 0 & 0.8\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0.1 & 0.25\\ 0.25 & 0.35\end{pmatrix}\\
&\quad\times\begin{pmatrix}\tanh\left(\dfrac{x_1(t)}{3}\right)\\ \tanh\left(\dfrac{x_2(t)}{4}\right)\end{pmatrix} + \begin{pmatrix}0.16 & 0.26\\ 0.25 & 0.15\end{pmatrix}\\
&\quad\times\begin{pmatrix}\dfrac{\left|x_1\left(4\left[\frac{t+1}{4}\right]\right)+1\right| - \left|x_1\left(4\left[\frac{t+1}{4}\right]\right)-1\right|}{10}\\ \dfrac{\left|x_2\left(4\left[\frac{t+1}{4}\right]\right)+1\right| - \left|x_2\left(4\left[\frac{t+1}{4}\right]\right)-1\right|}{8}\end{pmatrix}\\
&\quad+\begin{pmatrix}3\\ 4\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\Delta x|_{t=4k-1} &= \begin{pmatrix}J_{1k}\left(x_1\left(4k-1^-\right)\right)\\ J_{2k}\left(x_2\left(4k-1^-\right)\right)\end{pmatrix}\\
&= \begin{pmatrix}\dfrac{x_1\left(4k-1^-\right)-x_1^*}{4}\\ \dfrac{x_2\left(4k-1^-\right)-x_2^*}{10}\end{pmatrix}, \quad k\in\mathbb{N},
\end{aligned} \tag{61}$$

where $x_1^* = 2.608$, $x_2^* = 5.719$. One can check that the point $x^* = (x_1^*, x_2^*)^T$ satisfies the algebraic system (29) approximately and it is clear that $J_{ik}(x_i^*) = 0$ for $i = 1,2$. The output functions are $f_1(x_1) = \tanh(x_1/3)$, $f_2(x_2) = \tanh(x_2/4)$, $g_1(x_1) = (|x_1+1| - |x_1-1|)/10$, and $g_2(x_2) = (|x_2+1| - |x_2-1|)/8$.

We can easily obtain that $a_* = 0.8$, $\mathscr{L}_1^f = 1/3$, $\mathscr{L}_2^f = \mathscr{L}_2^g = \mathscr{L}_1^J = 0.25$, $\mathscr{L}_1^g = 0.2$, $\mathscr{L}_2^J = 0.1$, and $\sup_{t\in\mathbb{R}^+}\mathscr{L}_{i(t)} = \ln(1+\mathscr{L}_1^J)/4 \approx 0.0557$.

Then we give

$$\begin{aligned}
&\frac{1-e^{-l\cdot a_*}}{a_*}\left(\sum_{j=1}^{n}\left[\mathscr{L}_j^f\left|b_{1j}\right| + \mathscr{L}_j^g\left|c_{1j}\right|\right]\right) \approx 0.1904 < 1,\\
&\frac{1-e^{-l\cdot a_*}}{a_*}\left(\sum_{j=1}^{n}\left[\mathscr{L}_j^f\left|b_{2j}\right| + \mathscr{L}_j^g\left|c_{2j}\right|\right]\right) \approx 0.1996 < 1,\\
&a_1 = 1.3 > 0.2766 \approx \mathscr{L}_1^f\sum_{j=1}^{2}\left|b_{1j}\right| + \mathscr{L}_1^g\sum_{j=1}^{2}\left|c_{1j}\right|,\\
&a_2 = 0.8 > 0.29 = \mathscr{L}_2^f\sum_{j=1}^{2}\left|b_{2j}\right| + \mathscr{L}_2^g\sum_{j=1}^{2}\left|c_{2j}\right|.
\end{aligned} \tag{62}$$

By Theorem 6, we know that the ICNNs with IDEPCA system (61) have a unique equilibrium.

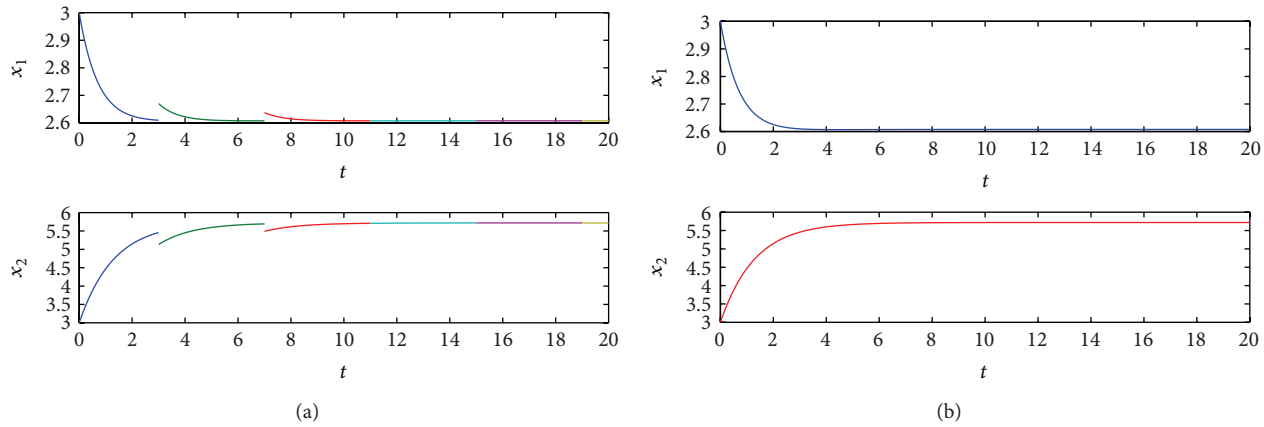

FIGURE 2: (a) The simulation, where the initial value is chosen as $(3,3)^T$, illustrates that all trajectories uniformly converge to the unique equilibrium $x^* = (2.608, 5.719)^T$ for the ICNNs (61) with impulsive effects. (b) The simulation, where the initial value is chosen as $(3,3)^T$, illustrates that all trajectories uniformly converge to the unique equilibrium $x^* = (2.608, 5.719)^T$ for the ICNNs (61) without impulsive effects.

In addition, we have

$$\begin{aligned}
\widetilde{\nu} &= \max_{i\in[1,\dots,n]}\left(\sum_{j=1}^{n}\mathscr{L}_j^f\left|b_{ij}\right| + \mathscr{L}_j^g\left|c_{ij}\right|\exp\left(a_*\cdot(m-l)\right)\right)\\
&\quad\cdot l \approx 0.4615 < 1,\\
&\sum_{j=1}^{n}\left(\mathscr{L}_j^f\left|b_{1j}\right| + \mathscr{L}_j^g\left|c_{1j}\right|(1-\widetilde{\nu})^{-1}\exp\left(a_*\cdot(m-l)\right)\right)\\
&\qquad\approx 0.5826 < 0.7442 \approx a_* - \sup_{t\in\mathbb{R}^+}\mathscr{L}_{i(t)},\\
&\sum_{j=1}^{n}\left(\mathscr{L}_j^f\left|b_{2j}\right| + \mathscr{L}_j^g\left|c_{2j}\right|(1-\widetilde{\nu})^{-1}\exp\left(a_*\cdot(m-l)\right)\right)\\
&\qquad\approx 0.7286 < 0.7442 \approx a_* - \sup_{t\in\mathbb{R}^+}\mathscr{L}_{i(t)}.
\end{aligned} \tag{63}$$

From Theorem 9, the ICNNs with IDEPCA system (61) have the unique equilibrium x^* which is globally asymptotically stable and all other solutions of the IDEPCA system (61) converge exponentially to it as $t \to \infty$. The numerical simulations, showing the convergence of the unique equilibrium x^* of the ICNNs with and without impulses (61), are given in Figures 2(a) and 2(b).

5. Conclusions

This is the first time that impulsive differential equations with alternately advanced and retarded argument have been applied to the model of cellular neural network models, and this paper has provided sufficient conditions guaranteeing the existence, uniqueness, and global exponential stability of the unique equilibrium of the impulsive cellular neural network models for the considered system based on a new IDEPCA integral inequality of Gronwall type and fixed point theorem. In addition, our method gives new ideas not only from the modeling point of view but also from that of theoretical opportunities since the impulsive cellular neural network model equation involves piecewise constant arguments of both advanced and delayed types. The obtained results could be useful in the design and applications of impulsive cellular neural network models. Furthermore, the examples with numerical simulations are given to show the effectiveness of the proposed method and results.

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Research Article

Newton-Kantorovich and Smale Uniform Type Convergence Theorem for a Deformed Newton Method in Banach Spaces

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Newton-Kantorovich and Smale uniform type of convergence theorem of a deformed Newton method having the third-order convergence is established in a Banach space for solving nonlinear equations. The error estimate is determined to demonstrate the efficiency of our approach. The obtained results are illustrated with three examples.

1. Introduction

In this paper, we study the problem of approximating a unique solution x^* of a nonlinear operator equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on an open convex Ω of a Banach space X with values in a Banach space Y.

There are many iterative methods (see [1–3]), which have been used for finding a solution of (1). For example, the well-known iterative method for solving (1) is Newton's method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad (n \geq 0) \ (x_0 \in \Omega). \tag{2}$$

Under the appropriate assumptions, Newton's method is the second-order convergence. Kantorovich (see [4]) presented the famous convergence result regarding a solution of (1). Many Newton-Kantorovich type of convergence theorems were given in papers [5–11]. Frontini and Sormani (see [12]) presented a new deformed Newton method with

$$\int_{x_n}^{x} f'(t)\,dt \simeq (x - x_n) f'\left(\frac{x_n + x}{2}\right). \tag{3}$$

The deformed Newton method can be written as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/2f'(x_n))}, \tag{4}$$

where f is a real or a complex function. In papers [13–17], the local convergence theorem has been established and the deformed method in a real or a complex space was discussed.

In the paper, we generalize the deformed Newton method [18] in a Banach space. The deformed Newton method [18] is shown as follows:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= x_n - F'\left(\frac{x_n + y_n}{2}\right)^{-1} F(x_n), \end{aligned} \tag{5}$$

where F is defined on an open convex subset Ω of a Banach space X with values in a Banach space Y, $F(x)$ has Fréchet derivatives in Ω, and $F'(x)^{-1}$ exists.

We establish Newton-Kantorovich and Smale uniform type convergence theorem (see [18]) for the deformed Newton method with the third-order in a Banach space with new sufficient conditions for the existence of a well-defined sequence which converges to a unique solution x^* of (1).

2. Main Results

Denote $g(t) = \int_0^t (t-u)L(u)du - t + \eta$, $u \in (0,R)$, $\eta > 0$, and suppose $L(u)$, $L'(u)$ are the positive and nondecreasing continuous functions, $\lim_{t\to R^+} g(t) = g(R^+) > 0$, $\int_0^R L(u)du > 1$, $\int_0^\alpha L(u)du = 1$ for $\alpha \in (0,R)$, $\beta = \alpha - \int_0^\alpha (\alpha - u)L(u)du = \int_0^\alpha uL(u)du$.

Assume that sequences $\{t_n\}$, $\{s_n\}$ are generated by the following formulae [18]:

$$s_n = t_n - g'(t_n)^{-1} g(t_n),$$
$$t_{n+1} = t_n - g'\left(\frac{t_n + s_n}{2}\right)^{-1} g(t_n), \quad t_0 = 0. \tag{6}$$

Firstly, we give some lemmas.

Lemma 1. *If $\eta \le \beta$, then the function $g(t)$ has two positive real roots r_1, r_2 $(0 < r_1 \le \alpha \le r_2 < R)$.*

Proof. Because $g(0) = \eta > 0$, $g(R^+) > 0$, and $g''(t) = L(t) > 0$, we know that $g(t)$ is the convex function for $t \in (0,R)$. Hence, α is a unique positive root of $g'(t) = \int_0^t L(u)du - 1$. So, the necessary and sufficient condition that $g(t)$ has two positive roots for $t \in (0,R)$ is that the minimum of $g(t)$ satisfies the condition $g(\alpha) \le 0$, which holds for $\eta \le \beta$. This completes the proof of Lemma 1. □

Lemma 2. *Suppose the sequences $\{t_n\}$, $\{s_n\}$ are generated by* (6). *Then, for $\eta \le \beta$, the sequences $\{t_n\}$, $\{s_n\}$ are increasing and converge to the minimum positive root of $g(t)$, and*

$$0 \le t_n \le s_n \le t_{n+1} < r_1. \tag{7}$$

Proof. Denote

$$U(x) = x - \frac{g(x)}{g'(x)}, \qquad V(x) = x - \frac{g(x)}{g'((x+U(x))/2)}. \tag{8}$$

On $[0,r_1)$, we know $g(t) > 0$, $g'(t) < 0$, $g''(t) > 0$, and $g''(t)$ is increasing. Denoting $y = (x+U(x))/2 = x - g(x)/2g'(x)$, then

$$\begin{aligned}
U'(x) &= \frac{g(x)g''(x)}{g'(x)^2} > 0,\\
\left[g'(y) - g'(x)\right] &= g''(\xi)(y-x) = -g''(\xi)\frac{g(x)}{2g'(x)},\\
&\qquad \xi \in (x,y),\\
V'(x) &= 1 - \left(g'(x)g'(y) - \frac{1}{2}g(x)g''(y)\right.\\
&\quad \left.\times 1 + \left(\frac{g(x)g''(x)}{g'(x)^2}\right)\right)\\
&\quad \times \left(g'(y)^2\right)^{-1}\\
&= \frac{1}{g'(y)}\left[g'(y) - g'(x)\right] + \frac{g(x)g''(y)}{2g'(y)^2}\\
&\quad + \frac{g(x)^2 g''(x)g''(y)}{2g'(x)^2 g'(y)^2} \ge -\frac{g''(\xi)}{g'(y)}\cdot\frac{g(x)}{2g'(x)}\\
&\quad + \frac{g(x)g''(y)}{2g'(y)^2} = -\frac{g(x)g''(\xi)}{2g'(y)^2 g'(x)}\left[g'(y) - g'(x)\right]\\
&\quad + \frac{g(x)g''(y) - g(x)g''(\xi)}{2g'(y)^2} = \frac{g(x)g''(\xi)}{2g'(y)^2 g'(x)}\\
&\quad \cdot \frac{g(x)g''(\xi)}{2g'(x)} + \frac{g(x)g''(y) - g(x)g''(\xi)}{2g'(y)^2} > 0.
\end{aligned} \tag{9}$$

Therefore, $U(x)$, $V(x)$ are increasing on $[0,r_1]$. Thus, for $x \in [0,r_1)$, $U(x) < U(r_1) = r_1$, $V(x) < V(r_1) = r_1$. Moreover,

$$s_n = U(t_n), \qquad t_{n+1} = V(t_n), \qquad t_0 = 0 < r_1; \tag{10}$$

hence we can easily prove Lemma 2 by the induction.

Suppose X and Y are the Banach spaces, $\Omega \subset X$ is an open convex subset, $F : \Omega \subset X \to Y$ has the second-order Fréchet derivative, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$, and the following conditions hold:

$$\begin{aligned}
&\left\|F'(x_0)^{-1}F(x_0)\right\| \le \eta, \qquad \left\|F'(x_0)^{-1}F''(x_0)\right\| \le L(0),\\
&\left\|F'(x_0)^{-1}\left(F''(y) - F''(x)\right)\right\|\\
&\quad \le \int_{\rho(x)}^{\rho(\overline{x,y})} L'(u)\,du, \quad x,y \in \Omega,\ \rho(\overline{x,y}) < \alpha,
\end{aligned} \tag{11}$$

where $\rho(x) = \|x - x_0\|$ and $\rho(\overline{x,y}) = \|y - x\| + \|x - x_0\|$. □

Lemma 3. *Suppose F satisfies* (11) *and $\|x - x_0\| < \alpha$. Then $F'(x)^{-1}$ exists, and*

$$\begin{aligned}
&\left\|F'(x_0)^{-1}F''(x)\right\| \le g''(\|x - x_0\|),\\
&\left\|F'(x)^{-1}F'(x_0)\right\| \le -\frac{1}{g'(\|x - x_0\|)}.
\end{aligned} \tag{12}$$

Proof. Firstly, by the conditions (11), we know that

$$\begin{aligned}
\left\|F'(x_0)^{-1}F''(x)\right\| &\le \left\|F'(x_0)^{-1}F''(x_0)\right\|\\
&\quad + \left\|F'(x_0)^{-1}F''(x) - F'(x_0)^{-1}F''(x_0)\right\|\\
&\le L(0) + \int_0^{\|x-x_0\|} L'(u)\,du\\
&= L(\|x - x_0\|) = g''(\|x - x_0\|).
\end{aligned} \tag{13}$$

Secondly, we know $g'(t) < 0$ for $t < \alpha$. Hence

$$\begin{aligned}\left\|F'(x_0)^{-1}F'(x) - I\right\| &= \left\|F'(x_0)^{-1}\left[F'(x) - F'(x_0)\right]\right\| \\ &= \left\|F'(x_0)^{-1}\int_0^1 F''(x_0 + t(x - x_0)) \times (x - x_0)\,dt\right\| \\ &\le \int_0^1 g''(t\|x - x_0\|)\|x - x_0\|\,dt \\ &= g'(\|x - x_0\|) - g'(0) \\ &= g'(\|x - x_0\|) + 1 < 1.\end{aligned} \tag{14}$$

By Banach Theorem, we know $F'(x)^{-1}$ exists, and

$$\begin{aligned}\left\|F'(x)^{-1}F'(x_0)\right\| &\le \frac{1}{1 - \left\|F'(x_0)^{-1}F'(x) - I\right\|} \\ &= -\frac{1}{g'(\|x - x_0\|)}.\end{aligned} \tag{15}$$

This completes the proof of Lemma 3. □

Lemma 4. *Suppose X and Y are Banach spaces, Ω is an open convex of the Banach space X, $F : \Omega \subset X \to Y$ has the second-order Fréchet derivative, and the sequences $\{x_n\}$, $\{y_n\}$ are generated by* (5). *Then, for any natural number n, the following formula holds:*

$$\begin{aligned}F(x_{n+1}) &= \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times (x_{n+1} - y_n)^2 \\ &\quad + \frac{1}{2}\int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times (x_{n+1} - y_n)(y_n - x_n) \\ &\quad + \frac{1}{2}\int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times (y_n - x_n)(x_{n+1} - y_n) \\ &\quad + \frac{1}{4}\int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times (y_n - x_n)^2 \\ &\quad - \frac{1}{4}\int_0^1 F''\left(\frac{x_n + y_n}{2} - t\left(\frac{y_n - x_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times (y_n - x_n)^2.\end{aligned} \tag{16}$$

Proof. By (5), we have

$$\begin{aligned}F(x_{n+1}) &= F(x_{n+1}) - F\left(\frac{x_n + y_n}{2}\right) \\ &\quad - F'\left(\frac{x_n + y_n}{2}\right)\left(x_{n+1} - \frac{x_n + y_n}{2}\right) \\ &\quad + F\left(\frac{x_n + y_n}{2}\right) + F'\left(\frac{x_n + y_n}{2}\right) \\ &\quad \times \left(x_{n+1} - \frac{x_n + y_n}{2}\right) \\ &= \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times \left(x_{n+1} - \frac{x_n + y_n}{2}\right)^2 + F\left(\frac{x_n + y_n}{2}\right) \\ &\quad + F'\left(\frac{x_n + y_n}{2}\right)\left(x_{n+1} - \frac{x_n + y_n}{2}\right), \\ &F\left(\frac{x_n + y_n}{2}\right) + F'\left(\frac{x_n + y_n}{2}\right)\left(x_{n+1} - \frac{x_n + y_n}{2}\right) \\ &= F\left(\frac{x_n + y_n}{2}\right) + F'\left(\frac{x_n + y_n}{2}\right) \\ &\quad \times \left(x_{n+1} - x_n - \frac{y_n - x_n}{2}\right) \\ &= F\left(\frac{x_n + y_n}{2}\right) - F(x_n) - F'\left(\frac{x_n + y_n}{2}\right)\frac{y_n - x_n}{2} \\ &= -\frac{1}{4}\int_0^1 F''\left(\frac{x_n + y_n}{2} - t\left(\frac{y_n - x_n}{2}\right)\right)(1 - t)\,dt \\ &\quad \times (y_n - x_n)^2.\end{aligned} \tag{17}$$

Hence

$$\begin{aligned}F(x_{n+1}) &= \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\ &\quad \times (1 - t)\,dt(x_{n+1} - y_n)^2 \\ &\quad + \frac{1}{2}\int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\ &\quad \times (1 - t)\,dt\,(x_{n+1} - y_n)(y_n - x_n) \\ &\quad + \frac{1}{2}\int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\ &\quad \times (1 - t)\,dt\,(y_n - x_n)(x_{n+1} - y_n) \\ &\quad + \frac{1}{4}\int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\ &\quad \times (1 - t)\,dt(y_n - x_n)^2\end{aligned}$$

$$-\frac{1}{4}\int_0^1 F''\left(\frac{x_n+y_n}{2}-t\left(\frac{y_n-x_n}{2}\right)\right)(1-t)\,dt \times (y_n-x_n)^2. \tag{18}$$

This completes the proof of Lemma 4. □

Theorem 5. *Suppose X and Y are Banach spaces, $\Omega \subset X$ is an open convex subset, $F : \Omega \subset X \to Y$ satisfies condition* (11), *$\eta \le \beta$, and*

$$\overline{S(x_0, r_1)} = \{x \mid \|x - x_0\| \le r_1, x \in X\} \subset \Omega. \tag{19}$$

Then the sequence $\{x_n\}_{n\ge 0}$ generated by (5) *is well defined, $x_n \in \overline{S(x_0, r_1)}$, and converges to the unique solution x^* in $S(x_0, \alpha)$ and*

$$\|x_n - x^*\| \le r_1 - t_n. \tag{20}$$

Proof. By induction, we can prove that the following formulae hold:

$$\begin{gathered}
\|x_n - x_0\| \le t_n;\\
\left\|F'(x_n)^{-1}F'(x_0)\right\| \le -g'(t_n)^{-1};\\
\|y_n - x_n\| \le s_n - t_n;\\
\|y_n - x_0\| \le s_n;\\
\left\|F'\left(\frac{x_n+y_n}{2}\right)^{-1}F'(x_0)\right\| \le -g'\left(\frac{t_n+s_n}{2}\right)^{-1};\\
\|x_{n+1} - y_n\| \le t_{n+1} - s_n;\\
\|x_{n+1} - x_n\| \le t_{n+1} - t_n.
\end{gathered} \tag{21}$$

In fact, by Lemma 2 we know $t_n < r_1$ for any natural number n. It is easy to prove that for $n = 0$ the above formulae hold. Suppose the above formulae also hold for $n > 0$. Then

$$\begin{aligned}
\|x_{n+1} - x_0\| &\le \|x_{n+1} - x_n\| \\
&\quad + \|x_n - x_0\| \le t_{n+1} - t_n + t_n = t_{n+1}.
\end{aligned} \tag{22}$$

By Lemma 3, we get

$$\begin{aligned}
\left\|F'(x_{n+1})^{-1}F'(x_0)\right\| &\le -g'(\|x_{n+1} - x_0\|)^{-1} \\
&\le -g'(t_{n+1})^{-1}.
\end{aligned} \tag{23}$$

By Lemmas 3 and 4 and the fact that $-g'(t)^{-1}$, $g''(t)$ are positive and increasing on $[0, \alpha)$, we have

$$\begin{aligned}
&\left\|F'(x_0)^{-1}\left[F''\left(\frac{x_n+y_n}{2}+t\left(x_{n+1}-\frac{x_n+y_n}{2}\right)\right)\right.\right.\\
&\qquad\left.\left.-F''\left(\frac{x_n+y_n}{2}-t\left(\frac{y_n-x_n}{2}\right)\right)\right]\right\|\\
&\le \int_0^{t\|x_{n+1}-x_n\|} L'\left(u+\left\|\frac{x_n+y_n}{2}-t\left(\frac{y_n-x_n}{2}\right)-x_0\right\|\right)du\\
&\le \int_0^{t(t_{n+1}-t_n)} L'\left(u+\frac{t_n+s_n}{2}-t\frac{s_n-t_n}{2}\right)du\\
&= L\left(\frac{t_n+s_n}{2}+t\left(t_{n+1}-\frac{t_n+s_n}{2}\right)\right)\\
&\quad - L\left(\frac{t_n+s_n}{2}-t\left(\frac{s_n-t_n}{2}\right)\right)\\
&= g''\left(\frac{t_n+s_n}{2}+t\left(t_{n+1}-\frac{t_n+s_n}{2}\right)\right)\\
&\quad - g''\left(\frac{t_n+s_n}{2}-t\left(\frac{s_n-t_n}{2}\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\left\|F'(x_0)^{-1}F(x_{n+1})\right\|\\
&\le \int_0^1 g''\left(\frac{t_n+s_n}{2}+t\left(t_{n+1}-\frac{t_n+s_n}{2}\right)\right)\\
&\quad \times (1-t)\,dt(t_{n+1}-s_n)^2\\
&\quad + \frac{1}{2}\int_0^1 g''\left(\frac{t_n+s_n}{2}+t\left(t_{n+1}-\frac{t_n+s_n}{2}\right)\right)(1-t)\,dt\\
&\quad \times (t_{n+1}-s_n)(s_n-t_n)\\
&\quad + \frac{1}{2}\int_0^1 g''\left(\frac{t_n+s_n}{2}+t\left(t_{n+1}-\frac{t_n+s_n}{2}\right)\right)(1-t)\,dt\\
&\quad \times (s_n-t_n)(t_{n+1}-s_n)\\
&\quad + \frac{1}{4}\int_0^1 g''\left(\frac{t_n+s_n}{2}+t\left(t_{n+1}-\frac{t_n+s_n}{2}\right)\right)(1-t)\,dt\\
&\quad \times (s_n-t_n)^2\\
&\quad - \frac{1}{4}\int_0^1 g''\left(\frac{t_n+s_n}{2}-t\left(\frac{s_n-t_n}{2}\right)\right)(1-t)\,dt\\
&\quad \times (s_n-t_n)^2 = g(t_{n+1}).
\end{aligned} \tag{24}$$

Hence we get

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &= \left\|-F'(x_{n+1})^{-1}F(x_{n+1})\right\| \\
&\le \left\|-F'(x_{n+1})^{-1}F'(x_0)\right\| \\
&\quad \times \left\|F'(x_0)^{-1}F(x_{n+1})\right\|
\end{aligned}$$

$$\begin{aligned}
&\le -g'(t_{n+1})^{-1} g(t_{n+1}) \\
&= s_{n+1} - t_{n+1}, \\
\|y_{n+1} - x_0\| &\le \|y_{n+1} - x_{n+1}\| \\
&\quad + \|x_{n+1} - x_0\| \le s_{n+1}.
\end{aligned} \tag{25}$$

By Lemma 3, we get

$$\left\| F'\left(\frac{x_{n+1}+y_{n+1}}{2}\right)^{-1} F'(x_0) \right\| \le -g'\left(\frac{t_{n+1}+s_{n+1}}{2}\right)^{-1}. \tag{26}$$

Moreover, we have

$$\begin{aligned}
&\|x_{n+2} - y_{n+1}\| \\
&= \Bigg\| F'(x_{n+1})^{-1} F(x_{n+1}) \\
&\quad - F'\left(\frac{x_{n+1}+y_{n+1}}{2}\right)^{-1} F(x_{n+1}) \Bigg\| \\
&= \Bigg\| F'\left(\frac{x_{n+1}+y_{n+1}}{2}\right)^{-1} \left[F'\left(\frac{x_{n+1}+y_{n+1}}{2}\right) \right. \\
&\qquad \left. - F'(x_{n+1}) \right] \\
&\quad \times F'(x_{n+1})^{-1} F(x_{n+1}) \Bigg\| \\
&= \Bigg\| F'\left(\frac{x_{n+1}+y_{n+1}}{2}\right)^{-1} F'(x_0) F'(x_0)^{-1} \\
&\quad \times \int_0^1 F''\left(x_{n+1} + \frac{t}{2}(y_{n+1} - x_{n+1})\right) dt \\
&\quad \times \frac{y_{n+1} - x_{n+1}}{2} F'(x_{n+1})^{-1} \\
&\quad \times F'(x_0) F'(x_0)^{-1} F(x_{n+1}) \Bigg\| \\
&\le g'\left(\frac{t_{n+1}+s_{n+1}}{2}\right)^{-1} \\
&\quad \times \int_0^1 g''\left(t_{n+1} + \frac{t}{2}(s_{n+1} - t_{n+1})\right) dt \\
&\quad \times \frac{(s_{n+1} - t_{n+1})}{2} g'(t_{n+1})^{-1} g(t_{n+1}) \\
&\le g'\left(\frac{t_{n+1}+s_{n+1}}{2}\right)^{-1} \\
&\quad \times \left[g'\left(\frac{t_{n+1}+s_{n+1}}{2}\right) - g'(t_{n+1}) \right] \\
&\quad \times g'(t_{n+1})^{-1} g(t_{n+1}) \\
&= g'(t_{n+1})^{-1} g(t_{n+1}) \\
&\quad - g'\left(\frac{t_{n+1}+s_{n+1}}{2}\right)^{-1} \\
&\quad \times g(t_{n+1}) = t_{n+2} - s_{n+1}, \\
\|x_{n+2} - x_{n+1}\| &\le \|x_{n+2} - y_{n+1}\| \\
&\quad + \|y_{n+1} - x_{n+1}\| \le t_{n+2} - t_{n+1}.
\end{aligned} \tag{27}$$

Hence, the sequence $\{x_n\}_{n\ge 0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and $\{x_n\}$ converges to the solution $x^* \in \overline{S(x_0, r_1)}$ of (1).

Now we prove the uniqueness. Suppose y^* is also a solution of (1) on $S(x_0, \alpha)$. We know that $g'(t) < 0$ for $t \in [0, \alpha)$. Then

$$\begin{aligned}
&\left\| F'(x_0)^{-1} \int_0^1 F'(x^* + t(y^* - x^*))\, dt - I \right\| \\
&\le \left\| F'(x_0)^{-1} \int_0^1 \left\{ F'[x^* + t(y^* - x^*)] - F'(x_0) \right\} dt \right\| \\
&\le \Bigg\| F'(x_0)^{-1} \int_0^1 \int_0^1 F''(x_0 + s(x^* - x_0 + t(y^* - x^*)))\, ds dt \\
&\quad \times (x^* - x_0 + t(y^* - x^*)) \Bigg\| \\
&\le \int_0^1 \int_0^1 g''(s\|x^* - x_0 + t(y^* - x^*)\|)\, ds dt \\
&\quad \times \|x^* - x_0 + t(y^* - x^*)\| \\
&= \int_0^1 g'(\|x^* - x_0 + t(y^* - x^*)\|)\, dt - g'(0) \\
&= \int_0^1 g'(\|(1-t)(x^* - x_0) + t(y^* - x_0)\|)\, dt + 1 < 1.
\end{aligned} \tag{28}$$

By Banach Theorem, we know the inverse of $\int_0^1 F'[x^* + t(y^* - x^*)]dt$ exists and

$$\begin{aligned}
0 &= F(y^*) - F(x^*) \\
&= \int_0^1 F'[x^* + t(y^* - x^*)]\, dt\, (y^* - x^*);
\end{aligned} \tag{29}$$

hence we get $y^* = x^*$. This completes the proof of the uniqueness of the solution of (1).

For $m > n$, we know that

$$\begin{aligned}
\|x_m - x_n\| &\le \|x_m - x_{m-1}\| \\
&\quad + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \le t_m - t_n.
\end{aligned} \tag{30}$$

When $m \to \infty$, we get

$$\|x_n - x^*\| \le r_1 - t_n. \tag{31}$$

This completes the proof of Theorem 5. □

Suppose that $L(u) = \gamma + Ku$, $u \in (0, +\infty)$, $\gamma, K > 0$. Then $\int_{\rho(x)}^{\rho(\overline{x,y})} L'(u)du = K\|x - y\|$, $g(t) = (1/6)Kt^3 + (1/2)\gamma t^2 - t + \eta\alpha = 2/(\gamma + \sqrt{\gamma^2 + 2K})$, and $\beta = \alpha - (1/6)K\alpha^3 - (1/2)\gamma\alpha^2 = 2(\gamma + 2\sqrt{\gamma^2 + 2K})/3(\gamma + \sqrt{\gamma^2 + 2K})^2$.

Corollary 6. *Suppose X and Y are the Banach spaces, Ω is an open convex subset of the Banach space X, $F : \Omega \subset X \to Y$ has the second-order Fréchet derivative, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$, and the following conditions hold:*

$$\begin{gathered}\left\|F'(x_0)^{-1}F(x_0)\right\| \le \eta, \qquad \left\|F'(x_0)^{-1}F''(x_0)\right\| \le \gamma, \\ \left\|F'(x_0)^{-1}\left(F''(x) - F''(y)\right)\right\| \le K\|x - y\| \quad x, y \in \Omega, \\ \eta \le \frac{2\left(\gamma + 2\sqrt{\gamma^2 + 2K}\right)}{3\left(\gamma + \sqrt{\gamma^2 + 2K}\right)^2}, \qquad \overline{S(x_0, r_1)} \subset \Omega.\end{gathered} \tag{32}$$

Then the sequence $\{x_n\}_{n\ge0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and $\{x_n\}$ converges to the unique solution x^* on $S(x_0, \alpha)$ of (1), where $r_1 \le r_2$ are two positive roots of $g(t) = (1/6)Kt^3 + (1/2)\gamma t^2 - t + \eta$.

Suppose $L(u) = 2\gamma(1 - \gamma u)^{-3}$, $u \in (0, 1/\gamma)$, $g(t) = \eta - t + \gamma t^2/(1 - \gamma t)$, $\alpha = (1 - \sqrt{2}/2)(1/\gamma)$ and $\beta = (3 - 2\sqrt{2})/\gamma$ and for $\|x - x_0\| < \alpha$, $\|F'(x_0)^{-1}F'''(x)\| \le 6\gamma^2/(1 - \gamma\|x - x_0\|)^4$. Hence, for $\|x - x_0\| + \|y - x\| < \alpha$, we get

$$\begin{aligned}&\left\|F'(x_0)^{-1}\left[F''(y) - F''(x)\right]\right\| \\ &\quad = \left\|\int_0^1 F'(x_0)^{-1}F'''(x + t(y - x))\,dt\,(y - x)\right\| \\ &\quad \le \int_0^1 \frac{6\gamma^2}{\left[1 - \gamma\|x - x_0 + t(y - x)\|\right]^4}dt\,\|y - x\| \\ &\quad \le \int_0^1 \frac{6\gamma^2}{\left[1 - \gamma\left(\|x - x_0\| + t\|y - x\|\right)\right]^4}dt\,\|y - x\| \\ &\quad = \int_{\|x - x_0\|}^{\|x - x_0\| + \|y - x\|} \frac{6\gamma^2}{(1 - \gamma u)^4}u \\ &\quad = \int_{\|x - x_0\|}^{\|x - x_0\| + \|y - x\|} L'(u)\,du.\end{aligned} \tag{33}$$

Corollary 7 (see [10]). *Suppose X and Y are Banach spaces, Ω is an open convex subset of the Banach space X, $F : \Omega \subset X \to Y$ has the third-order Fréchet derivative, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$, and the following conditions hold:*

$$\begin{gathered}\left\|F'(x_0)^{-1}F(x_0)\right\| \le \eta, \qquad \left\|F'(x_0)^{-1}F''(x_0)\right\| \le 2\gamma, \\ \left\|F'(x_0)^{-1}F'''(x)\right\| \le \frac{6\gamma^2}{(1 - \gamma\|x - x_0\|)^4} \\ = g'''(\|x - x_0\|), \quad x \in \Omega, \\ \|x - x_0\| \le \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}, \qquad \eta\gamma \le 3 - 2\sqrt{2}, \\ \overline{S(x_0, r_1)} \subset \Omega.\end{gathered} \tag{34}$$

Then the sequence $\{x_n\}_{n\ge0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and $\{x_n\}$ converges to the unique solution x^* of (1) on $S(x_0, (1 - 1/\sqrt{2})(1/\gamma))$, where

$$\begin{gathered}r_1 = \frac{1 + \eta\gamma - \sqrt{(1 + \eta\gamma)^2 - 8\eta\gamma}}{4\gamma}, \\ r_2 = \frac{1 + \eta\gamma + \sqrt{(1 + \eta\gamma)^2 - 8\eta\gamma}}{4\gamma}\end{gathered} \tag{35}$$

are two positive roots of the equation $g(t) = \eta - t + \gamma t^2/(1 - \gamma t)$.

3. Numerical Examples

In this section, we apply the convergence theorem and show three numerical examples.

Example 1. Consider the equation

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3} = 0. \tag{36}$$

We choose the initial point $x_0 = 0$, $\Omega = [-1, 1]$; then

$$\begin{gathered}\eta = \left|F'(0)^{-1}F(0)\right| = \frac{2}{5}, \qquad \gamma = \left|F'(0)^{-1}F''(0)\right| = \frac{2}{5}, \\ K = \frac{6}{5}, \\ \frac{2\left(\gamma + 2\sqrt{\gamma^2 + 2K}\right)}{3\left(\gamma + \sqrt{\gamma^2 + 2K}\right)^2} = \frac{3}{5} > \eta.\end{gathered} \tag{37}$$

Hence, by Corollary 6, the sequence $\{x_n\}_{n\ge0}$ generated by (5) is well defined, and $\{x_n\}$ converges to the solution x^* of (36).

Now, we will analyze errors $\|x_n - x^*\|$ by Corollary 6 (see Table 1). In this case, we take $x_0 = 0$; then $r_1 = 0.462598422\cdots$.

Example 2. Consider the system of equation [18] $F(u, v) = 0$, where

$$F(u, v) = (uv - 1, uv + u - 2v)^T. \tag{38}$$

Table 1: Error results for Corollary 6 ($\|x_n - x^*\| \le r_1 - t_n$).

Step	$r_1 - t_n$	Step	$r_1 - t_n$
$k = 1$	1.616985×10^{-2}	$k = 2$	2.236349×10^{-6}
$k = 3$	6.225929×10^{-18}	$k = 4$	1.343387×10^{-52}
$k = 5$	$1.349560 \times 10^{-156}$	$k = 6$	$1.368249 \times 10^{-468}$

Then, we have

$$F'(u,v) = \begin{pmatrix} v & u \\ v+1 & u-2 \end{pmatrix},$$
$$F'(u,v)^{-1} = -\frac{1}{u+2v}\begin{pmatrix} u-2 & -u \\ -v-1 & v \end{pmatrix}, \tag{39}$$
$$F''(u,v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \hline 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We choose $x_0 = (u_0, v_0) = (1.75, 1.75)$ and $\Omega = \{x \mid \|x - x_0\| \le 1.75\}$. We take the max-norm in R^2 and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$. Define the norm of a bilinear operator B on R^2 by

$$\|B\| = \sup_{\|u\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} u_k \right|, \tag{40}$$

where $u = (u_1, u_2)^T$ and

$$B = \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ \hline b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{pmatrix}. \tag{41}$$

Then we get the following results:

$$\eta = \left\|F'(x_0)^{-1} F(x_0)\right\| = \frac{9}{14},$$
$$\gamma = \left\|F'(x_0)^{-1} F''(x_0)\right\| = \frac{16}{21}, \tag{42}$$
$$K = 0, \qquad \frac{2\left(\gamma + 2\sqrt{\gamma^2 + 2K}\right)}{3\left(\gamma + \sqrt{\gamma^2 + 2K}\right)^2} > \eta.$$

This means that the hypotheses of Corollary 6 are satisfied.

Now, we will analyze errors $\|x_n - x^*\|$ by Corollary 6 (see Table 2). In this case, we take $x_0 = (u_0, v_0) = (1.75, 1.75)$; then $r_1 = 1.125$.

Example 3. Consider the following integral equations:

$$x(s) = 1 + \frac{1}{4} x(s) \int_0^1 \frac{s}{s+t} x(t)\, dt \tag{43}$$

Table 2: Error results for Corollary 6 ($\|x_n - x^*\| \le r_1 - t_n$).

Step	$r_1 - t_n$	Step	$r_1 - t_n$
$k = 1$	2.736486×10^{-1}	$k = 2$	3.044252×10^{-2}
$k = 3$	1.588069×10^{-4}	$k = 4$	2.844419×10^{-11}
$k = 5$	1.636509×10^{-30}	$k = 6$	3.116680×10^{-92}

Table 3: Error results for Corollary 7 ($\|x_n - x^*\| \le r_1 - t_n$).

Step	$r_1 - t_n$	Step	$r_1 - t_n$
$k = 1$	2.764303×10^{-3}	$k = 2$	4.099223×10^{-9}
$k = 3$	1.344301×10^{-26}	$k = 4$	4.741124×10^{-79}
$k = 5$	$2.079868 \times 10^{-236}$	$k = 6$	$<1.0 \times 10^{-500}$

and the space $X = C[0,1]$ with the norm

$$\|x\| = \max_{0 \le s \le 1} |x(s)|. \tag{44}$$

This equation arises in the theory of radiative transfer and neutron transport and in the kinetic theory of gases. Define the operator F on X by

$$F(x) = \frac{1}{4} x(s) \int_0^1 \frac{s}{s+t} x(t)\, dt - x(s) + 1. \tag{45}$$

Then, for $x_0 = 1$, we obtain

$$\eta = \left\|F'(x_0)^{-1} F(x_0)\right\| = 0.2652,$$
$$2\gamma = \left\|F'(x_0)^{-1} F''(x_0)\right\| = 1.5304 \times 2$$
$$\cdot \frac{1}{4} \max_{0 \le s \le 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = 1.5304 \times \frac{\ln 2}{2} = 0.5303, \tag{46}$$
$$\eta\gamma = 0.07032 < 3 - 2\sqrt{2},$$
$$\left\|F'(x_0)^{-1} F'''(x)\right\| = 0 < \frac{6\gamma^2}{\left(1 - \gamma\|x - x_0\|\right)^4}.$$

This means that the hypotheses of Corollary 7 are satisfied.

Now, we will analyze errors $\|x_n - x^*\|$ by Corollary 7 (see Table 3). In this case, we take $x_0 = 1$; then $r_1 = 0.289222\cdots$.

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Research Article

Asymptotic Behavior of Solutions to a Vector Integral Equation with Deviating Arguments

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In this paper, we propose the study of an integral equation, with deviating arguments, of the type $y(t) = \omega(t) - \int_0^\infty f(t,s,y(\gamma_1(s)),\dots,y(\gamma_N(s)))ds$, $t \geq 0$, in the context of Banach spaces, with the intention of giving sufficient conditions that ensure the existence of solutions with the same asymptotic behavior at ∞ as $\omega(t)$. A similar equation, but requiring a little less restrictive hypotheses, is $y(t) = \omega(t) - \int_0^\infty q(t,s)F(s,y(\gamma_1(s)),\dots,y(\gamma_N(s)))ds$, $t \geq 0$. In the case of $q(t,s) = (t-s)_+$, its solutions with asymptotic behavior given by $\omega(t)$ yield solutions of the second order nonlinear abstract differential equation $y''(t) - \omega''(t) + F(t,y(\gamma_1(t)),\dots,y(\gamma_N(t))) = 0$, with the same asymptotic behavior at ∞ as $\omega(t)$.

1. Introduction

From the pioneering work of Atkinson [1], and subsequent works found in the literature (see, e.g., [2–10] for recent papers on the subject), we consider the following differential problem, with deviating arguments:

$$y''(t) - \omega''(t) + F(t, y(\gamma_1(t)), \dots, y(\gamma_N(t))) = 0, \quad t \geq 0, \tag{1}$$

with the task of finding solutions y with the same behavior at ∞ as ω. Solutions with this prescription are given by the solutions of the following integral equation:

$$y(t) = \omega(t) - \int_t^\infty (s-t) F(s, y(\gamma_1(s)), \dots, y(\gamma_N(s)))\, ds, \quad t \geq 0, \tag{2}$$

which, by writing $f(t,s,y(\gamma_1(s)),\dots,y(\gamma_N(s))) = (s-t)_+F(s,y(\gamma_1(s)),\dots,y(\gamma_N(s)))$, is of the type

$$y(t) = \omega(t) - \int_0^\infty f(t, s, y(\gamma_1(s)), \dots, y(\gamma_N(s)))\, ds, \quad t \geq 0. \tag{3}$$

The purpose of this note is to provide conditions that ensure the existence of solutions to the above integral equation, whose asymptotic behavior at ∞ is the same as that of ω, thus giving a procedure to show existence of solutions with prescribed asymptotic behavior of differential equation of the type (1). Our wish is to also work out this integral equation in the setting of Banach spaces.

Denote by $\mathbb{R}^+$ the set $[0,\infty)$ of nonnegative real numbers. Assume that $\{\gamma_j\}_{j=1}^N$ is a finite set of continuous mappings from $\mathbb{R}^+$ to $\mathbb{R}^+$, that $(X, \|\cdot\|_X)$ is a Banach space (with norm $\|\cdot\|_X$), and also that ω is a continuous mapping from $\mathbb{R}^+$ to X. Finally, assume that $f : \mathbb{R}^+ \times \mathbb{R}^+ \times X^N \to X$ is a given continuous mapping with certain regularity and integrability conditions, to be specified later. In order to give a better aspect to our equation, define, for each continuous $y : \mathbb{R}^+ \to X$, the mapping $\Gamma(y) : \mathbb{R}^+ \to X^N$ given by

$$\Gamma(y)(t) = (y(\gamma_1(t)), \dots, y(\gamma_N(t))), \quad t \geq 0. \tag{4}$$

Then, our equation becomes

$$y(t) = \omega(t) - \int_0^\infty f(t, s, \Gamma(y)(s))\, ds, \quad t \geq 0, \tag{E}$$

which, by writing $x(t) = y(t) - \omega(t)$, $t \in \mathbb{R}^+$, is transformed into

$$x(t) = -\int_0^\infty f(t, s, \Gamma(\omega + x)(s))\, ds, \quad t \geq 0. \qquad (\tilde{E})$$

A bit more of notation and preliminary results are needed. As customary, $B_X(x, r)$ denotes the open ball in X centered at x with radius r. The closure in X of any set $A \subseteq X$ is written $\overline{A}$, and its closed convex hull, $\overline{\text{co}}(A)$. The space of continuous X-valued functions defined on $\mathbb{R}^+$ is denoted by $\mathscr{C}(\mathbb{R}^+, X)$, while the space of bounded ones is $\mathscr{C}_b(\mathbb{R}^+, X)$. The latter is a Banach space when endowed with the sup norm $\|\cdot\|_\infty$, (i.e., for $x \in \mathscr{C}_b(\mathbb{R}^+, X)$, $\|x\|_\infty = \sup_{t\in\mathbb{R}^+}\|x(t)\|_X$).

The Schauder fixed point theorem states that any continuous operator T defined on a nonempty, bounded, closed and convex subset C of a Banach space has necessarily a fixed point, provided that $T(C)$ is a relatively compact subset of C. We will also be needing a well-known version of the Arzelà-Ascoli theorem which, in the case that occupies us, is as follows: if a family $\mathscr{F} \subseteq \mathscr{C}(\mathbb{R}^+, X)$ is equicontinuous at each $t \in \mathbb{R}^+$, and each section $\mathscr{F}(t) := \{u(t) : u \in \mathscr{F}\}$ is relatively compact in X, then each sequence $\{u_n\} \subseteq \mathscr{F}$ contains a subsequence that converges uniformly on compact subsets of $\mathbb{R}^+$ to a given X-valued function u.

Let us also say a word about vector integrals. For a brief introduction see, for example, [11] or [12]. If $f : [a, b] \to X$ is a bounded vector function, the Riemann sum of f associated to a finite partition of $[a, b]$, $\wp = \{a = t_0 < t_1 < \cdots < t_n = b\}$ (with norm $\|\wp\| = \max_j |t_j - t_{j-1}|$), and to a selection $\{t_j^*\}_{j=1}^n$, $t_j^* \in [t_{j-1}, t_j]$, is $\sum_{j=1}^n f(t_j^*)(t_j - t_{j-1})$. Without specifying the type of limit alluded, we say that f is Riemann integrable on $[a, b]$ if there exists the limit of its Riemann sums as the norm of the partitions of $[a, b]$ tends to 0, in which case, this limit (which is unique) is called the integral of f over $[a, b]$ and is denoted by $\int_a^b f(t)dt$. That is, being a little bit sloppy with the notation, we have

$$\int_a^b f(t)\, dt = \lim_{\|\wp\| \to 0} \sum_j f\left(t_j^*\right)\left(t_j - t_{j-1}\right). \qquad (5)$$

Observe that whenever f is integrable on $[a, b]$, then each Riemann sum associated to f is $(b - a)$ times a linear convex combination of elements of $f([a, b])$. Therefore, the integral of f over $[a, b]$, being a limit of Riemann sums, is a multiple of an element of the closed convex hull of $f([a, b])$, that is,

$$\int_a^b f(t)\, dt = \lim_{\|\wp\| \to 0} (b - a) \times \sum_j f\left(t_j^*\right) \frac{t_j - t_{j-1}}{b - a} \in (b - a)\ \overline{\text{co}} f([a, b]). \qquad (6)$$

2. Existence of Solutions

We begin this section enumerating the conditions that will be basic for our results on existence of solutions of (E). Take f, γ_j, and ω as above and, in analogy with what would happen with continuous mappings on finite dimensional spaces, assume that

f is uniformly continuous on bounded sets of $\mathbb{R}^+ \times \mathbb{R}^+ \times X^N$, and maps bounded sets into relatively compact sets of X. (H0)

The following conditions have already been motivated in previous work [7]. Recall that $\omega : \mathbb{R}^+ \to X$ is continuous, but not necessarily bounded.

$$\text{There exists } g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ bounded, with } g(t) \xrightarrow{t\to\infty} 0,$$
$$\text{such that for } C = \{y \in \mathscr{C}(\mathbb{R}^+, X) : \|y(t) - \omega(t)\|_X \leq g(t), t \in \mathbb{R}^+\},$$
$$\int_0^\infty \|f(t, s, \Gamma(y)(s))\|_X ds \leq g(t), \quad \forall t \in \mathbb{R}^+,\ \forall y \in C. \qquad (H1)$$

$$\text{There exists } h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ with } h(\tau) \xrightarrow{\tau\to\infty} 0, \text{ such that}$$
$$\int_\tau^\infty \|f(t, s, \Gamma(y)(s))\|_X ds \leq h(\tau), \quad \forall t, \tau \in \mathbb{R}^+,\ \forall y \in C. \qquad (H2)$$

The result on existence of solutions to the integral equation is the following.

Theorem 1. *Under hypotheses* $(H0)$, $(H1)$, *and* $(H2)$, *the integral equation* (E) *has a solution* $y(t)$ *asymptotically equal to* $\omega(t)$ *as* $t \to \infty$.

Remark 2. This theorem represents a generalization of the one presented in the work [7] in two aspects. First, we have made the jump to deal with integral equations in the setting of infinite dimensional spaces. And second, we have included deviating arguments in the equation.

Proof of Theorem 1. First observe that it suffices to find a solution to the integral equation $(\tilde{E})$ in $\widetilde{C} = \{x \in \mathscr{C}_b(\mathbb{R}^+, X) : \|x(t)\|_X \leq g(t), t \in \mathbb{R}^+\}$, and this will be achieved by proving the existence of a fixed point in $\widetilde{C}$ of the operator:

$$T : x \in \widetilde{C} \longmapsto Tx,$$
$$Tx(t) = -\int_0^\infty f(t, s, \Gamma(\omega + x)(s))\, ds, \quad t \geq 0. \qquad (7)$$

We proceed to check that the conditions of the Schauder fixed point Theorem are fulfilled. First observe that $\widetilde{C}$ is a nonempty, bounded, closed, and convex subset of the Banach space $\mathscr{C}_b(\mathbb{R}^+, X)$. Also, by (H1), $\|Tx(t)\|_X \leq g(t)$ for all $t \in \mathbb{R}^+$ and all $x \in \widetilde{C}$. So $T\widetilde{C} \subseteq \widetilde{C}$ provided $Tx \in \mathscr{C}(\mathbb{R}^+, X)$ for all $x \in \widetilde{C}$.

In fact, we will prove four assertions. (a) $T\widetilde{C}$ is uniformly equicontinuous on $\mathbb{R}^+$. This will give the desired continuity

of Tx for each $x \in \widetilde{C}$. (b) T is uniformly continuous on $\widetilde{C}$, hence it will be continuous on $\widetilde{C}$. (c) $T\widetilde{C}(\mathbb{R}^+)$ is relatively compact in X. Thus, (a) and (c) will imply, by the Arzelà-Ascoli theorem, that each sequence in $T\widetilde{C}$ has a subsequence which converges uniformly on each compact subset of $\mathbb{R}^+$ to a given function in $\mathscr{C}(\mathbb{R}^+, X)$ (actually in $\widetilde{C}$). Finally, in order to obtain that $T\widetilde{C}$ is relatively compact in $\mathscr{C}_b(\mathbb{R}^+, X)$, we will use the "funnel" structure of $\widetilde{C}$ to prove (d), that any sequence in $\widetilde{C}$ which converges uniformly on each compact subset of $\mathbb{R}^+$ to a given function in $\widetilde{C}$ must indeed converge uniformly to that function in all of $\mathbb{R}^+$. With all these assertions, the Schauder fixed point theorem can be applied to conclude the existence of a fixed point of T, as we want.

Start fixing an arbitrary $\varepsilon > 0$ once for all. In what follows, we will build up different objects indexed by this ε, $(t_\varepsilon, \tau_\varepsilon, D_\varepsilon, B_\varepsilon)$, knowing that even if for each assertion we have to start taking an arbitrary $\varepsilon > 0$, the objects will vary accordingly but not the way to obtain them.

Since $g(t) \to 0$ as $t \to \infty$, there exists $t_\varepsilon > 0$ such that

$$g(t) < \frac{\varepsilon}{2}, \quad \forall t \geq t_\varepsilon. \tag{8}$$

We start proving (d), as it is quite independent of the rest of assertions. Assume that a sequence $\{u_n\} \subseteq \widetilde{C}$ converges uniformly on each compact subset of $\mathbb{R}^+$ to a function $u \in \widetilde{C}$, and let us show that indeed converges uniformly to u in all of $\mathbb{R}^+$. For the $\varepsilon > 0$ above (so for any $\varepsilon > 0$), find the corresponding $t_\varepsilon > 0$ to satisfy (8). Since u_n, $n \in \mathbb{N}$, and u all belong to $\widetilde{C}$, then

$$\|u_n(t) - u(t)\|_X \leq 2g(t), \quad \forall n \in \mathbb{N}, \ \forall t \in \mathbb{R}^+. \tag{9}$$

In particular,

$$\|u_n(t) - u(t)\|_X \leq 2g(t) < \varepsilon, \quad \forall n \in \mathbb{N}, \ \forall t \geq t_\varepsilon. \tag{10}$$

Now, since $\{u_n\}$ converges uniformly to u in $[0, t_\varepsilon]$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|u_n(t) - u(t)\|_X < \varepsilon, \quad \forall n \geq n_\varepsilon, \forall t \in [0, t_\varepsilon]. \tag{11}$$

This tells us that if $n \geq n_\varepsilon$, then $\|u_n - u\|_\infty \leq \varepsilon$, proving that the convergence of $\{u_n\}$ to u is uniform on $\mathbb{R}^+$, and thus (d) is proven.

Next, continue building up other objects associated with the arbitrary ε fixed above. Observe that, by (8) and (H1),

$$\begin{aligned} \|Tx(t)\|_X &\leq \int_0^\infty \|f(t, s, \Gamma(\omega + x)(s))\|_X ds \\ &\leq g(t) < \frac{\varepsilon}{2}, \quad \forall t \geq t_\varepsilon, \ \forall x \in \widetilde{C}. \end{aligned} \tag{12}$$

Also, since $h(\tau) \to 0$ as $\tau \to \infty$, there exists $\tau_\varepsilon > 0$ such that

$$h(\tau) < \frac{\varepsilon}{4}, \quad \forall \tau \geq \tau_\varepsilon, \tag{13}$$

so, by (H2),

$$\begin{aligned} &\int_{\tau_\varepsilon}^\infty \|f(t, s, \Gamma(\omega + x)(s))\|_X ds \\ &\quad \leq h(\tau_\varepsilon) < \frac{\varepsilon}{4}, \quad \forall t \in \mathbb{R}^+, \ \forall x \in \widetilde{C}. \end{aligned} \tag{14}$$

The continuity of ω and the γ_j's, $j = 1, \ldots, N$, and the uniform bound for functions in $\widetilde{C}$ (given by the bound of g) imply that there exists a bounded set $D_\varepsilon \subseteq X^N$, depending on τ_ε, ω, the γ_j's, and g, but not on $x \in \widetilde{C}$, such that

$$\begin{aligned} &\Gamma(\omega + x)(s) \\ &= (\omega(\gamma_1(s)) + x(\gamma_1(s)), \ldots, \omega(\gamma_N(s)) + x(\gamma_N(s))) \in D_\varepsilon, \\ &\qquad \forall x \in \widetilde{C}, \ \forall s \in [0, \tau_\varepsilon]. \end{aligned} \tag{15}$$

Now, observe that $B_\varepsilon := [0, t_\varepsilon + 1] \times [0, \tau_\varepsilon] \times D_\varepsilon$ is bounded in $\mathbb{R}^+ \times \mathbb{R}^+ \times X^N$, so by (H0),

$$\begin{aligned} &f \text{ is uniformly continuous on } B_\varepsilon, \\ &\text{and } f(B_\varepsilon) \text{ is relatively compact in } X. \end{aligned} \tag{16}$$

By the uniform continuity of f on B_ε, there exists $\delta_\varepsilon \in (0, 1)$ such that, whenever $(t_1, s_1, y_{1,1}, \ldots, y_{N,1}) \in B_\varepsilon$ and $(t_2, s_2, y_{1,2}, \ldots, y_{N,2}) \in B_\varepsilon$ with $|t_1 - t_2| < \delta_\varepsilon$, $|s_1 - s_2| < \delta_\varepsilon$, and $\|y_{j,1} - y_{j,2}\|_X < \delta_\varepsilon$, $j = 1, \ldots, N$, then

$$\|f(t_1, s_1, y_{1,1}, \ldots, y_{N,1}) - f(t_2, s_2, y_{1,2}, \ldots, y_{N,2})\|_X < \frac{\varepsilon}{4\tau_\varepsilon}. \tag{17}$$

Notice that if $t_1, t_2 \in \mathbb{R}^+$ with $|t_1 - t_2| < \delta_\varepsilon$ and, without loss of generality, $t_1 \leq t_2$, then several cases are possible. If $t_2 \geq t_\varepsilon + 1$, then, as $\delta_\varepsilon < 1$, $t_2 \geq t_1 > t_2 - \delta_\varepsilon > t_\varepsilon + 1 - 1 = t_\varepsilon$, and so, by (12),

$$\begin{aligned} &\|Tx(t_1) - Tx(t_2)\|_X \leq \|Tx(t_1)\|_X + \|Tx(t_2)\|_X < \varepsilon, \\ &\qquad \forall x \in \widetilde{C}. \end{aligned} \tag{18}$$

If, on the other hand, $t_2 \leq t_\varepsilon + 1$, then, by (15), $\Gamma(\omega + x)([0, \tau_\varepsilon]) \subseteq D_\varepsilon$ for all $x \in \widetilde{C}$, and, by (17) and (14), we have, for any $x \in \widetilde{C}$,

$$\begin{aligned} &\|Tx(t_1) - Tx(t_2)\|_X \\ &\quad \leq \int_0^{\tau_\varepsilon} \|f(t_2, s, \Gamma(\omega + x)(s)) \\ &\qquad - f(t_1, s, \Gamma(\omega + x)(s))\|_X ds \\ &\qquad + \int_{\tau_\varepsilon}^\infty \|f(t_1, s, \Gamma(\omega + x)(s))\|_X ds \\ &\qquad + \int_{\tau_\varepsilon}^\infty \|f(t_2, s, \Gamma(\omega + x)(s))\|_X ds \\ &\quad < \tau_\varepsilon \frac{\varepsilon}{4\tau_\varepsilon} + 2\frac{\varepsilon}{4} < \varepsilon. \end{aligned} \tag{19}$$

This proves (a), the uniform equicontinuity of $T\widetilde{C}$ over $\mathbb{R}^+$.

Now notice that if $x_1, x_2 \in \widetilde{C}$ with $\|x_1 - x_2\|_\infty < \delta_\varepsilon$ and $t \in \mathbb{R}^+$, then, again, two cases are possible. If $t \geq t_\varepsilon$, then, by (12),

$$\|Tx_1(t) - Tx_2(t)\|_X \leq \|Tx_1(t)\|_X + \|Tx_2(t)\|_X \leq 2g(t) < \varepsilon, \tag{20}$$

while for $t \leq t_\varepsilon$, using (15), (17), (14), and (13),

$$\begin{aligned}
&\|Tx_1(t) - Tx_2(t)\|_X \\
&\quad \leq \int_0^{\tau_\varepsilon} \|f(t, s, \Gamma(\omega + x_2)(s)) \\
&\qquad\qquad - f(t, s, \Gamma(\omega + x_1)(s))\|_X ds \\
&\qquad + \int_{\tau_\varepsilon}^{\infty} \|f(t, s, \Gamma(\omega + x_1)(s))\|_X ds \\
&\qquad + \int_{\tau_\varepsilon}^{\infty} \|f(t, s, \Gamma(\omega + x_2)(s))\|_X ds \\
&\quad < \tau_\varepsilon \frac{\varepsilon}{4\tau_\varepsilon} + 2h(\tau_\varepsilon) < \varepsilon,
\end{aligned} \tag{21}$$

This proves that $\|Tx_1 - Tx_2\|_\infty < \varepsilon$, showing (b), that T is uniformly continuous on $\widetilde{C}$.

For the compactness of $\overline{T\widetilde{C}(\mathbb{R}^+)}$ in X, it suffices to show that $T\widetilde{C}(\mathbb{R}^+)$ is totally bounded [13, page 298]; that is, for the given $\varepsilon > 0$ (so for any $\varepsilon > 0$) there exists a finite covering of $T\widetilde{C}(\mathbb{R}^+)$ with balls of radii not bigger than ε. Observe first that, by (12), $\|Tx(t)\|_X < \varepsilon/2$ for all $x \in \widetilde{C}$ and all $t \geq t_\varepsilon$, that is,

$$T\widetilde{C}([t_\varepsilon, \infty)) \subseteq B_X\left(0, \frac{\varepsilon}{2}\right). \tag{22}$$

Now, in order to control the elements of $T\widetilde{C}([0, t_\varepsilon])$, observe that each of these can be decomposed as the sum of a "head" and a "tail,"

$$\begin{aligned}
Tx(t) = &-\int_0^{\tau_\varepsilon} f(t, s, \Gamma(\omega + x)(s))\, ds \\
&- \int_{\tau_\varepsilon}^{\infty} f(t, s, \Gamma(\omega + x)(s))\, ds, \\
&\forall x \in \widetilde{C} \text{ and all } t \in [0, t_\varepsilon].
\end{aligned} \tag{23}$$

The "head" can be approximated by Riemann sums, which, in turn, are nothing else but τ_ε times a convex linear combination of elements of $-f(B_\varepsilon)$, that is, the "head" is an element of $\tau_\varepsilon\, \overline{\text{co}}(-f(B_\varepsilon))$. By (16), $f(B_\varepsilon)$ has compact closure in X, so by Mazur's theorem [14], $\overline{\text{co}}(-f(B_\varepsilon))$ is compact, and therefore it can be covered with a finite number of balls, say $B_1, \ldots, B_\ell$, of radii not bigger than $\varepsilon/(2\tau_\varepsilon)$. This yields a finite covering of $\tau_\varepsilon \overline{\text{co}}(-f(B_\varepsilon))$ with balls of radii not bigger than $\varepsilon/2$, precisely the collection $\{\tau_\varepsilon B_j\}_{j=1}^{\ell}$. On the other hand, by (14), the "tail" of each of the above integrals is bounded by $h(\tau_\varepsilon) < \varepsilon/4$, so they are elements of $B_X(0, \varepsilon/4)$. All this can be summarized as follows:

$$\begin{aligned}
T\widetilde{C}([0, t_\varepsilon]) &\subseteq \tau_\varepsilon \overline{\text{co}}(-f(B_\varepsilon)) + B_X\left(0, \frac{\varepsilon}{4}\right) \\
&\subseteq \left(\bigcup_{j=1}^{\ell} \tau_\varepsilon B_j\right) + B_X\left(0, \frac{\varepsilon}{4}\right) \\
&= \bigcup_{j=1}^{\ell} \left(\tau_\varepsilon B_j + B_X\left(0, \frac{\varepsilon}{4}\right)\right);
\end{aligned} \tag{24}$$

that is, $T\widetilde{C}([0, t_\varepsilon])$ can be covered with a finite collection of balls of radii smaller than ε, because each $\tau_\varepsilon B_j + B_X(0, \varepsilon/4)$ is readily seen to be a ball of radius not bigger than $3\varepsilon/4$.

At the end, by (22) and (24), we have

$$\begin{aligned}
T\widetilde{C}(\mathbb{R}^+) &\subseteq T\widetilde{C}([0, t_\varepsilon]) \cup T\widetilde{C}([t_\varepsilon, \infty)) \\
&\subseteq \bigcup_{j=1}^{\ell} \left(\tau_\varepsilon B_j + B_X\left(0, \frac{\varepsilon}{4}\right)\right) \cup B_X\left(0, \frac{\varepsilon}{2}\right);
\end{aligned} \tag{25}$$

that is, we have given a finite covering of $T\widetilde{C}(\mathbb{R}^+)$ with balls of radii not bigger than ε. With this we conclude (c) and, with all four assertions proved, the theorem too. □

Coming back to the integral equation (2) underlying the differential equation (1), we just need to adapt the hypotheses presented above to the function $f(t, s, y_1, \ldots, y_N) = (s - t)_+ F(s, y_1, \ldots, y_N)$, to obtain a corresponding result on existence of solutions to (2) with asymptotic behavior given by $\omega(t)$. Notice that these hypotheses, (H0), (H1), and (H2), are natural generalizations from the 1-dimensional case. However, the proof in the abstract setting has shown many more properties for the operator T that needed in order to show the existence of a fixed point. This, somehow, is telling us that the hypotheses could be weakened. For the moment, we content ourselves noticing that, in (2), the corresponding hypothesis of uniform continuity on bounded sets is "not needed," because it will be "consequence" of the other hypotheses, and a little trick of changing the domain of definition of the operator. Thus, for our next result, we will be using the following hypothesis:

$$\begin{aligned}
&F : \mathbb{R}^+ \times X^N \longrightarrow X \text{ is continuous and} \\
&\text{maps bounded sets into relatively compact ones.}
\end{aligned} \tag{H0$'$}$$

Also, instead of considering just the integral equation (2), we generalize a little bit to a convolution type integral equation,

$$y(t) = \omega(t) - \int_0^{\infty} q(t, s) \quad F(s, \Gamma(y)(s))\, ds, \quad t \geq 0, \tag{E_1}$$

with a kernel $q(t,s)$ satisfying the right properties for us. Observe that, when $q(t,s) = (s-t)_+$, and $0 \le t_1 \le t_2$,

$$\begin{aligned} q(t_1,s) - q(t_2,s) &= (s-t_1)_+ - (s-t_2)_+ \\ &= \begin{cases} 0 & \text{if } 0 \le s \le t_1 \\ s - t_1 & \text{if } t_1 \le s \le t_2 \\ t_2 - t_1 & \text{if } t_2 \le s \end{cases} \\ &\le t_2 - t_1. \end{aligned} \tag{26}$$

That is, $q(t,s)$ satisfies a Lipschitz condition on the first variable, independent of the second. Actually, much less is needed, just continuity of $q(t,s)$ suffices.

Theorem 3. *Let ω, γ_j, $j = 1,\dots,N$, Γ be as for Theorem 1, and let $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be continuous. Under hypotheses $(H0')$, $(H1)$, and $(H2)$ (where $f(t,s,y_1,\dots,y_N) = q(t,s)\, F(s,y_1,\dots,y_N)$), the integral equation (E_1) has a solution $y(t)$ asymptotically equal to $\omega(t)$ as $t \to \infty$.*

Remark 4. Observe, as in [7], that if the kernel q is the one we started with, $q(t,s) = (s-t)_+$, then hypothesis (H2) is redundant, because in that case, if $s \ge \tau$, we always have, for $\tau \le t$, $(s-t)_+ \le (s-\tau)_+ \le 2(s-\tau/2)_+$, and for $\tau > t$,

$$(s-t)_+ = s - t \le s \le 2\left(s - \frac{\tau}{2}\right) = 2\left(s - \frac{\tau}{2}\right)_+. \tag{27}$$

Consequently, from hypothesis (H1), taking $h(\tau) = 3g(\tau/2)$, we have $h(\tau) \to 0$ as $\tau \to \infty$, and

$$\begin{aligned} &\int_\tau^\infty (s-t)_+ \left\| F(s,\Gamma(y)(s)) \right\|_X ds \\ &\qquad \le 3g\left(\frac{\tau}{2}\right) = h(\tau), \quad \forall \tau, t \in \mathbb{R}^+, \forall y \in C. \end{aligned} \tag{28}$$

In general, hypothesis (H2) is superfluous whenever there exist constants, $a \in (0,1)$ and $A > 1$, such that

$$\left| q(t,s) \right| \le A \left| q(a\,\tau, s) \right|, \quad \forall t, s, \tau \in \mathbb{R}^+, \text{with } s \ge \tau. \tag{29}$$

Proof of Theorem 3. We proceed as before. Consider initially the same nonempty, bounded, closed, and convex subset $\widetilde{C}$ as before, as well as the same operator T. The idea is to find a fixed point for T using the Schauder fixed point theorem. For that, we start repeating the same scheme of four assertions established in the previous theorem. The two assertions that do not depend on the uniform continuity of f on bounded sets are done the same way as before; hence we omit their proofs. These are (c), that $T\widetilde{C}(\mathbb{R}^+)$ is compact in X, and (d), that uniform convergence on compact subsets of $\mathbb{R}^+$ of a sequence in $\widetilde{C}$ turns into actual uniform convergence on $\mathbb{R}^+$.

Let us now prove (a), that $T\widetilde{C}$ is uniformly equicontinuous on $\mathbb{R}^+$. This will finish showing that $T\widetilde{C} \subseteq \widetilde{C}$, and, by (c), (d), and the Arzelà-Ascoli theorem, that $T\widetilde{C}$ has compact closure in $\mathscr{C}_b(R^+, X)$.

Let $\varepsilon > 0$ be fixed. Find, as before, $t_\varepsilon > 0$, $\tau_\varepsilon > 0$, and D_ε bounded subset of X^N, so as to satisfy (8), (12), (13), (14), and (15). Now, observe that $[0,\tau_\varepsilon] \times D_\varepsilon$ is bounded in $\mathbb{R}^+ \times X^N$, so by $(\mathrm{H}0')$, $F([0,\tau_\varepsilon] \times D_\varepsilon)$ is relatively compact in X; hence there exists $M_\varepsilon > 0$ such that

$$\left\| F(s,\Gamma(\omega + x)(s)) \right\|_X \le M_\varepsilon, \quad \forall s \in [0,\tau_\varepsilon], \forall x \in \widetilde{C}. \tag{30}$$

Notice now that q is uniformly continuous on $R_\varepsilon = [0, t_\varepsilon + 1] \times [0,\tau_\varepsilon]$. So there is $\delta_\varepsilon \in (0,1)$ such that, whenever (t_1,s_1) and (t_2,s_2) are in R_ε with $|t_1 - t_2| < \delta$ and $|s_1 - s_2| < \delta$, then

$$\left| q(t_1,s_1) - q(t_2,s_2) \right| < \frac{\varepsilon}{4\tau_\varepsilon M_\varepsilon}. \tag{31}$$

Now take $t_1, t_2 \in \mathbb{R}^+$ with $|t_1 - t_2| < \delta_\varepsilon$ and, without loss of generality, assume that $t_1 \le t_2$. Then again, several cases are possible. If $t_2 \ge t_\varepsilon + 1$, then, as $\delta_\varepsilon < 1$, $t_2 \ge t_1 > t_2 - \delta_\varepsilon > t_\varepsilon + 1 - 1 = t_\varepsilon$, and so, by (12),

$$\begin{aligned} \left\| Tx(t_1) - Tx(t_2) \right\|_X \le \left\| Tx(t_1) \right\|_X + \left\| Tx(t_2) \right\|_X < \varepsilon, \\ \forall x \in \widetilde{C}. \end{aligned} \tag{32}$$

If, on the other hand, $t_2 \le t_\varepsilon + 1$, then, by (31), (30), and (14), we have, for any $x \in \widetilde{C}$,

$$\begin{aligned} &\left\| Tx(t_1) - Tx(t_2) \right\|_X \\ &\quad \le \int_0^{\tau_\varepsilon} \left| q(t_2,s) - q(t_1,s) \right| \\ &\qquad \times \left\| F(s,\Gamma(\omega + x)(s)) \right\|_X ds \\ &\qquad + \int_{\tau_\varepsilon}^\infty \left| q(t_1,s) \right| \left\| F(s,\Gamma(\omega + x)(s)) \right\|_X ds \\ &\qquad + \int_{\tau_\varepsilon}^\infty \left| q(t_2,s) \right| \left\| F(s,\Gamma(\omega + x)(s)) \right\|_X ds \\ &\quad < \frac{\varepsilon}{4\tau_\varepsilon M_\varepsilon} M_\varepsilon \tau_\varepsilon + 2\,\frac{\varepsilon}{4} < \varepsilon. \end{aligned} \tag{33}$$

This proves the uniform equicontinuity of $T\widetilde{C}$ over $\mathbb{R}^+$.

To finish the proof, we have to prove (b), that T is uniformly continuous. It is here that we restrict the domain of definition of T. Observe that $\overline{\mathrm{co}}(T\widetilde{C})$ is nonempty, closed and convex. Also, $\overline{\mathrm{co}}(T\widetilde{C}) \subseteq \widetilde{C}$ because $T\widetilde{C} \subseteq \widetilde{C}$ and $\widetilde{C}$ is closed and convex. More is true, since $T\widetilde{C}$ has compact closure, then, by Mazur's Theorem, $\overline{\mathrm{co}}(T\widetilde{C})$ is compact too. One more thing, T leaves invariant $\overline{\mathrm{co}}(T\widetilde{C})$:

$$\overline{\mathrm{co}}\left(T\widetilde{C}\right) \subseteq \widetilde{C} \Longrightarrow T\left(\overline{\mathrm{co}}\left(T\widetilde{C}\right)\right) \subseteq T\widetilde{C} \subseteq \overline{\mathrm{co}}\left(T\widetilde{C}\right). \tag{34}$$

With all of this, it suffices to prove that T is continuous on this new T-invariant set. Let us prove that, indeed, T is uniformly continuous on $\overline{\mathrm{co}}(T\widetilde{C})$. Let $\varepsilon > 0$ be given. Find that $t_\varepsilon > 0$ and $\tau_\varepsilon > 0$ as before, so as to satisfy (8), (12), (13), and (14). The continuity of ω and the γ_j's, $j = 1,\dots,N$, and the compactness of $\overline{\mathrm{co}}(T\widetilde{C})$ tells us that $D_\varepsilon = \{\Gamma(\omega + x)(s) : s \in [0,\tau_\varepsilon],\ x \in \overline{\mathrm{co}}(T\widetilde{C})\}$ is a compact subset of X^N, giving us the opportunity to say that

F is uniformly continuous on $[0,\tau_\varepsilon]\times D_\varepsilon$ and, consequently, that $f(t,s,y_1,\dots,y_N)=q(t,s)\,F(s,y_1,\dots,y_N)$ is uniformly continuous on $B_\varepsilon:=[0,t_\varepsilon+1]\times[0,\tau_\varepsilon]\times D_\varepsilon$. So there exists $\delta_\varepsilon\in(0,1)$ such that, for $(t_1,s_1,y_{1,1},\dots,y_{N,1})\in B_\varepsilon$ and $(t_2,s_2,y_{1,2},\dots,y_{N,2})\in B_\varepsilon$ with $|t_1-t_2|<\delta_\varepsilon$, $|s_1-s_2|<\delta_\varepsilon$, and $\|y_{j,1}-y_{j,2}\|_X<\delta_\varepsilon$, $j=1,\dots,N$, then

$$\left\|f\left(t_1,s_1,y_{1,1},\dots,y_{N,1}\right)-f\left(t_2,s_2,y_{1,2},\dots,y_{N,2}\right)\right\|_X<\frac{\varepsilon}{4\tau_\varepsilon}. \tag{35}$$

Now, it is a matter of repeating the same steps as was done in Theorem 1 to prove the uniform continuity of T.

This concludes the proof of the theorem. □

Remark 5. Instead of working with functions defined on $\mathbb{R}^+$, we could have worked with functions defined on any interval of the type $[t_0,\infty)$, obtaining a result completely similar. Also, many times, one is just interested in giving partial solutions; that is, solutions defined not on the whole interval $[t_0,\infty)$, but on some other interval $[t_1,\infty)$ with $t_1\geq t_0$.

Next, we just mention an easy result on the existence of solutions of the underlying differential equation, just to illustrate the type of functions F that could generate a condition like (H1). This result is inspired from [7, Thm. 2] and [8, Thm. 1].

Theorem 6. *Let $F:\mathbb{R}^+\times X^N\to X$ be a continuous function mapping bounded sets into relatively compact ones, such that,*

$$\begin{gathered}\left\|F\left(t,y_1,\dots,y_N\right)\right\|_X\leq\psi\left(t\right),\\ \forall t\in\mathbb{R}^+,\ \forall\left(y_1,\dots,y_N\right)\in X^N,\end{gathered} \tag{36}$$

where

$$\psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+,\ \textit{satisfies}\int_0^\infty s\psi\left(s\right)ds<\infty. \tag{37}$$

Let also $\{\gamma_j\}_{j=1}^N$ be a set of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Then, for any $\omega\in\mathscr{C}^2(\mathbb{R}^+,X)$, (1) has a solution $y\in\mathscr{C}^2(\mathbb{R}^+,X)$ with $y(t)-\omega(t)\to 0$ as $t\to\infty$.

Proof. Take $\omega\in\mathscr{C}^2(\mathbb{R}^+,X)$, consider the corresponding integral equation (2), and define

$$g\left(t\right)=\int_t^\infty s\psi\left(s\right)ds,\quad t\geq 0. \tag{38}$$

Observe that, by (37), $g(t)\to 0$ as $t\to\infty$. This allows us to consider the set

$$C=\left\{y\in\mathscr{C}\left(\mathbb{R}^+,X\right):\left\|y\left(t\right)-\omega\left(t\right)\right\|_X\leq g\left(t\right),\ t\geq 0\right\}, \tag{39}$$

which, adopting the notation used throughout the paper, gives by (36), for $y\in C$ and $t\geq 0$,

$$\int_t^\infty\left\|\left(s-t\right)F\left(s,\Gamma\left(y\right)\left(s\right)\right)\right\|_X ds\leq\int_t^\infty\left(s-t\right)\psi\left(s\right)ds\leq g\left(t\right). \tag{40}$$

With this, Theorem 3 applies (hypothesis (H2) need not be verified by Remark 4) and then there exists $y\in\mathscr{C}(\mathbb{R}^+,X)$, solution of (2) with $y(t)-\omega(t)\to 0$ as $t\to\infty$. Finally, it is just an exercise to check that y is twice continuously differentiable and that satisfies the differential equation (1). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stability and Global Hopf Bifurcation Analysis on a Ratio-Dependent Predator-Prey Model with Two Time Delays

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A ratio-dependent predator-prey model with two time delays is studied. By means of an iteration technique, sufficient conditions are obtained for the global attractiveness of the positive equilibrium. By comparison arguments, the global stability of the semitrivial equilibrium is addressed. By using the theory of functional equation and Hopf bifurcation, the conditions on which positive equilibrium exists and the quality of Hopf bifurcation are given. Using a global Hopf bifurcation result of Wu (1998) for functional differential equations, the global existence of the periodic solutions is obtained. Finally, an example for numerical simulations is also included.

1. Introduction

The main purpose of this paper is to investigate the bifurcation phenomena from the delays for the following predator-prey system:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}x(t)\,y(t-\tau_2)}{my^2(t-\tau_2)+x^2(t)}\right], \\ \dot{y}(t) &= \frac{a_{21}x^2(t-\tau_1)\,y(t)}{my^2(t)+x^2(t-\tau_1)} - r_2 y(t), \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ stand for the population (or density) of the prey and the predator at time t, respectively. From the biological sense, we assume that $x^2 + y^2 \neq 0$. r_1, r_2, a_{11}, a_{12}, a_{21}, and m are positive constants, in which r_1 denotes the intrinsic growth rate of the prey, a_{11} is the intraspecific competition rate of the prey, a_{12} is the capturing rate of the predator, a_{21}/a_{12} describes the efficiency of the predator in converting consumed prey into predator offspring, m is the interference coefficient of the predators, and r_2 is the predator mortality rate. The delay $\tau_1 \geq 0$ denotes the gestation period of the predator; $\tau_2 \geq 0$ is the hunting delay of the predator to prey.

This model is labeled "ratio-dependent," which means that the functional and numerical responses depend on the densities of both prey and predators, especially when predator has to search for food. Such a functional response is called a ratio-dependent response function (see [1] for more details). In system (1), the ratio-dependent response function is of the form $g(x/y) = c(x/y)^2/(m + (x/y)^2) = cx^2/(my^2 + x^2)$.

The ratio-dependent predator-prey model has been studied by several researchers recently and very rich dynamics have been observed [2–5]. For example, Xu et al. [4] studied a delayed ratio-dependent predator-prey model with the same ratio-dependent response function of system (1). By means of an iteration technique, they obtained the sufficient conditions for the global attractiveness of the positive equilibrium. By comparison arguments, they proved the global stability of the semitrivial equilibrium. Finally using the theory of functional equation and Hopf bifurcation, they gave the condition on which positive equilibrium exists and the formulae to determine the quality of Hopf bifurcation. But in their work, the global continuation of local Hopf bifurcation was not mentioned.

In general, periodic solutions through the Hopf bifurcation in delay differential equations are local for the values

of parameters which are only in a small neighborhood of the critical values (see, e.g., [6, 7]). Therefore we would like to know if these nonconstant periodic solutions obtained through local bifurcation can continue for a large range of parameter values. Recently, a great deal of research has been devoted to the topics [8–12]. One of the methods used in them is the global Hopf bifurcation theorem by Wu [13]. For example, Song et al. [12] studied a predator-prey system with two delays, and using the methods in [13], they get the global existence of periodic solutions.

Motivated by [12], we will study the system (1); special attention is paid to the global continuation of local Hopf bifurcation. We suppose that the initial condition for system (1) takes the form

$$x(\theta) = \phi(\theta), \quad y(\theta) = \psi(\theta), \quad \phi(\theta) \geq 0, \quad \psi(\theta) \geq 0,$$
$$\theta \in [-\tau, 0] \left(\tau = \tau_1 + \tau_2\right), \quad \phi(0) > 0, \quad \psi(0) > 0, \tag{2}$$

where $(\phi(\theta), \psi(\theta)) \in \mathscr{C}([-\tau, 0], \mathbf{R}^2_{+0})$, which is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^2_{+0}$, where $\mathbf{R}^2_{+0} = \{(x, y) \mid x \geq 0, y \geq 0\}$.

By the fundamental theory of functional differential equations [14], system (1) has a unique solution $(x(t), y(t))$ satisfying initial condition (2).

The rest of the paper is organized as follows. In Section 2, we show the positivity and the boundedness of solutions of system (1) with initial condition (2). In Section 3, we study the existence of Hopf bifurcation for system (1) at the positive equilibrium. In Section 4, using the normal form theory and the center manifold reduction, explicit formulae are derived to determine the direction of bifurcation and the stability and other properties of bifurcating periodic solutions. In Section 5, by means of an iteration technique, sufficient conditions are obtained for the global attractiveness of the positive equilibrium. In Section 6, we consider the global existence of bifurcating periodic solutions and give some numerical simulations. In Section 7, a brief discussion is given.

2. Positivity and Boundedness

In this section, we study the positivity and boundedness of solutions of system (1) with initial conditions (2).

Theorem 1. *Solutions of system* (1) *with initial condition* (2) *are positive for all* $t \geq 0$.

Proof. Assume $(x(t), y(t))$ to be a solution of system (1) with initial condition (2). Let us consider $y(t)$ for $t \geq 0$. It follows from the second equation of system (1) that

$$y(t) = y(0)\, e^{\int_0^t ((a_{21}x^2(s-\tau_1)/my^2(s)+x^2(s-\tau_1))-r_2)ds}; \tag{3}$$

then, from initial condition (2), we have $y(t) > 0$, for $t \geq 0$. We derive from the first equation of system (1) that

$$x(t) = x(0)\, e^{\int_0^t (r_1 - a_{11}x(s) - (a_{12}x(s)y(s-\tau_2)/my^2(s-\tau_2)+x^2(s)))ds}; \tag{4}$$

that is, $x(t) > 0$ for $t \geq 0$. This ends the proof. □

For the following discussion of boundedness, we first consider the following ordinary differential equation:

$$\dot{u} = \frac{a_{21}A_1^2 u(t)}{mu^2(t) + A_1^2} - r_2 u(t), \quad u(0) > 0, \tag{5}$$

where a_{21}, r_2, A_1, and m are positive constants. From Lemma 2.1 in [5], it is easy to verify the following result.

Lemma 2. *If* $a_{21} < r_2$, *the trivial equilibrium* $u^0 = 0$ *of* (5) *is globally stable. If* $a_{21} > r_2$, *then* (5) *admits a unique positive equilibrium* $u^* = \sqrt{(a_{21} - r_2)/mr_2}A_1$ *which is globally asymptotically stable in* $\Lambda = \{u \mid u \geq 0\}$.

Theorem 3. *Positive solutions of system* (1) *with initial condition* (2) *are ultimately bounded.*

Proof. Let $(x(t), y(t))$ be a positive solution of system (1) with initial condition (2). From the first equation of system (1), we have

$$\dot{x}(t) \leq x(t)\left[r_1 - a_{11}x(t)\right], \tag{6}$$

which yields

$$\limsup_{t \to +\infty} x(t) \leq \frac{r_1}{a_{11}}; \tag{7}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, $x(t) < (r_1/a_{11}) + \epsilon$.

We now consider the boundedness of $y(t)$. If $a_{21} \leq r_2$, we derive from the second equation of system (1) that

$$\dot{y}(t) \leq \left(a_{21} - r_2\right) y(t) \leq 0; \tag{8}$$

from monotone bounded theorem, it is easy to show that $\lim_{t \to +\infty} y(t) \leq y(0)$.

Therefore, we assume below that $a_{21} > r_2$. We derive from the second equation of system (1) that, for $t > T_1 + \tau$,

$$\dot{y}(t) \leq \frac{a_{21}\left(r_1/a_{11} + \epsilon\right)^2 y(t)}{my^2(t) + \left(r_1/a_{11} + \epsilon\right)^2} - r_2 y(t); \tag{9}$$

noting that $a_{21} > r_2$, by Lemma 2, a comparison argument shows that

$$\limsup_{t \to +\infty} y(t) \leq \sqrt{\frac{a_{21} - r_2}{mr_2}}\left(\frac{r_1}{a_{11}} + \epsilon\right). \tag{10}$$

This completes the proof. □

3. Local Stability and Hopf Bifurcation

In this section, we discuss the local stability of the positive equilibrium and the semitrivial equilibrium of system (1) and establish the existence of Hopf bifurcation at the positive equilibrium.

It is easy to show that system (1) always has a semitrivial equilibrium $E_1(r_1/a_{11}, 0)$. Further, if the following condition holds:

$$\text{(H1) } r_1^2 a_{21}^2 m > a_{12}^2 r_2 (a_{21} - r_2) > 0,$$

then system (1) has a unique positive equilibrium $E^*(x^*, y^*)$, where

$$x^* = \frac{r_1 a_{21} - r_2 a_{12} h}{a_{11} a_{21}}, \quad y^* = h x^*, \tag{11}$$

where

$$h = \sqrt{\frac{a_{21} - r_2}{m r_2}}. \tag{12}$$

For convenience, let us introduce new variables $X(t) = x(t - \tau_1), Y(t) = y(t), \tau = \tau_1 + \tau_2$, rewriting $X(t), Y(t)$ as $x(t), y(t)$, so that system (1) can be written as the following system with a single delay:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r_1 - a_{11} x(t) - \frac{a_{12} x(t) y(t-\tau)}{m y^2(t-\tau) + x^2(t)}\right], \\ \dot{y}(t) &= \frac{a_{21} x^2(t) y(t)}{m y^2(t) + x^2(t)} - r_2 y(t). \end{aligned} \tag{13}$$

Clearly, system (13) has the same equilibrium as system (1).

The characteristic equation of system (13) at the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ is of the form

$$(\lambda + r_1)(\lambda + r_2 - a_{21}) = 0. \tag{14}$$

Clearly, (14) always has a root $\lambda = -r_1$, and if $a_{21} < r_2$, the other root of (14) is negative; if $a_{21} > r_2$, the other root of (14) is positive. Hence the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ is locally asymptotically stable (unstable) if $a_{21} < r_2$ ($a_{21} > r_2$).

The characteristic equation of system (13) at the positive equilibrium $E^*(x^*, y^*)$ is of the form

$$\lambda^2 + p_0 \lambda + p_1 + p_2 e^{-\lambda\tau} = 0, \tag{15}$$

where

$$\begin{aligned} p_0 &= r_1 - \frac{2a_{12} r_2^2 h}{a_{21}^2} + \frac{2r_2(a_{21} - r_2)}{a_{21}}, \\ p_1 &= \left(r_1 - \frac{2a_{12} r_2^2 h}{a_{21}^2}\right)\frac{2r_2(a_{21} - r_2)}{a_{21}}, \\ p_2 &= \frac{2a_{12} r_2^2 h (2r_2 - a_{21})(a_{21} - r_2)}{a_{21}^3}, \end{aligned} \tag{16}$$

where h is defined as (12).

When $\tau = 0$, (15) becomes

$$\lambda^2 + p_0 \lambda + p_1 + p_2 = 0. \tag{17}$$

It is easy to show that

$$p_1 + p_2 = \frac{2r_2(a_{21} - r_2)(r_1 a_{21} - a_{12} r_2 h)}{a_{21}^2}. \tag{18}$$

Obviously, if (H1) holds, then $p_1 + p_2 > 0$. Hence, the positive equilibrium $E^*(x^*, y^*)$ of system (13) is locally stable when $\tau = 0$ if

$$r_1 > \frac{2a_{12} r_2^2 h}{a_{21}^2} - \frac{2r_2(a_{21} - r_2)}{a_{21}}, \tag{19}$$

and it is unstable when $\tau = 0$ if

$$r_1 < \frac{2a_{12} r_2^2 h}{a_{21}^2} - \frac{2r_2(a_{21} - r_2)}{a_{21}}. \tag{20}$$

We assume that $\lambda = i\omega (\omega > 0)$ is a root of (15); this is the case if and only if ω satisfies the following equation:

$$-\omega^2 + p_0 \omega i + p_1 + p_2 e^{-i\omega\tau} = 0. \tag{21}$$

Separating the real and imaginary parts, we obtain the following system for ω:

$$\begin{aligned} p_2 \cos\omega\tau &= \omega^2 - p_1, \\ p_2 \sin\omega\tau &= p_0 \omega. \end{aligned} \tag{22}$$

It follows that

$$\omega^4 + (p_0^2 - 2p_1)\omega^2 + p_1^2 - p_2^2 = 0. \tag{23}$$

Letting $z = \omega^2$, (42) becomes

$$z^2 + (p_0^2 - 2p_1) z + p_1^2 - p_2^2 = 0. \tag{24}$$

By a direct calculation, it follows that

$$\begin{aligned} p_0^2 - 2p_1 &= \left(r_1 - \frac{2a_{12} r_2^2 h}{a_{21}^2}\right)^2 + \left(\frac{2r_2(a_{21} - r_2)}{a_{21}}\right)^2 > 0, \\ p_1 - p_2 &= \frac{2r_2(a_{21} - r_2)}{a_{21}}\left(r_1 - \frac{4a_{12} r_2^2 h + a_{12} a_{21} r_2 h}{a_{21}^2}\right). \end{aligned} \tag{25}$$

Note that if (H1) holds, then $p_1 + p_2 > 0$. Hence if (H1) and $p_1 - p_2 > 0$ hold, (24) has no positive roots. Accordingly, if (H1) and $p_1 - p_2 > 0$ hold, the positive equilibrium E^* of system (13) exists and is locally asymptotically stable for all $\tau \geq 0$. If (H1) and $p_1 - p_2 < 0$ hold, then (24) has a unique positive root ω_0, where

$$\omega_0^2 = \frac{1}{2}\left(2p_1 - p_0^2 + \sqrt{p_0^4 - 4p_0^2 p_1 + 4p_2^2}\right). \tag{26}$$

Then, we can get

$$\tau_n = \frac{1}{\omega_0}\arccos\frac{\omega_0^2 - p_1}{p_2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots, \tag{27}$$

at which (15) admits a pair of purely imaginary roots of the form $\pm\omega_0$.

Let $p_1 - p_2 < 0$ and τ_0 be defined above. Denote

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \tag{28}$$

the root of (15) satisfying

$$\alpha(\tau_n) = 0, \qquad \omega(\tau_n) = \omega_0. \tag{29}$$

It is not difficult to verify that the following result holds.

Lemma 4. *If (H1) and $p_1 - p_2 < 0$ hold, the transversal condition $(d(\mathrm{Re}\,\lambda)/d\tau)|_{\tau=\tau_n} > 0$ holds.*

Proof. Differentiating (15) with respect τ, we obtain that

$$2\lambda\frac{d\lambda}{d\tau} + p_0\frac{d\lambda}{d\tau} - p_2\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau} = p_2\lambda e^{-\lambda\tau}; \tag{30}$$

it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p_0}{-\lambda p_2 e^{-\lambda\tau}} - \frac{\tau}{\lambda}; \tag{31}$$

from (15) and (31), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p_0}{-\lambda\left(\lambda^2 + p_0\lambda + p_1\right)} - \frac{\tau}{\lambda}. \tag{32}$$

We therefore derive that

$$\begin{aligned}
&\mathrm{sign}\left\{\left.\frac{d\,(\mathrm{Re}\,\lambda)}{d\tau}\right|_{\tau=\tau_n}\right\} \\
&\quad = \mathrm{sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_n}\right\} \\
&\quad = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{2\lambda + p_0}{-\lambda\left(\lambda^2 + p_0\lambda + p_1\right)}\right]_{\tau=\tau_n}\right\} \\
&\quad = \mathrm{sign}\left\{\frac{\omega_0^2\left(p_0^2 - 2p_1 + \omega_0^2\right)}{\omega_0^4 p_0^2 + \left(\omega_0 p_1 - \omega_0^3\right)^2}\right\}.
\end{aligned} \tag{33}$$

Noting that $p_0^2 - 2p_1 > 0$, hence, if (H1) and $p_1 - p_2 < 0$ hold, we have $(d(\mathrm{Re}\,\lambda)/d\tau)|_{\tau=\tau_n} > 0$. Accordingly, the transversal condition holds and a Hopf bifurcation occurs at $\tau = \tau_n$. □

By Lemma B in [5], we have the following results.

Theorem 5. *Suppose (H1) holds and let h be defined in* (12), *for system* (13), *one has the following.*

(i) *If $r_1 > (2a_{12}r_2^2h/a_{21}^2) - (2r_2(a_{21} - r_2)/a_{21})$ and $r_1 > (4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2$, then the positive equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.*

(ii) *If $r_1 > (2a_{12}r_2^2h/a_{21}^2) - (2r_2(a_{21} - r_2)/a_{21})$ and $r_1 < (4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2$, then there exists a positive number τ_0 such that the positive equilibrium E^* is locally asymptotically stable if $\tau \in [0, \tau_0)$ and is unstable if $\tau > \tau_0$. Further, system* (13) *undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.*

4. Direction and Stability of Hopf Bifurcations

In Section 3, we have shown that system (13) admits a periodic solution bifurcated from the positive equilibrium E^* at the critical value τ_0. In this section, we derive explicit formulae to determine the direction of Hopf bifurcations and stability of periodic solutions bifurcated from the positive equilibrium E^* at critical value τ_0 by using the normal form theory and the center manifold reduction (see, e.g., [15, 16]).

Set $\tau = \tau_0 + \mu$; then $\mu = 0$ is a Hopf bifurcation value of system (13). Thus we can consider the problem above in the phase space $\mathscr{C} = \mathscr{C}([-\tau, 0], \mathbf{R}^2)$.

Let $u_1(t) = x(t) - x^*$, $u_2(t) = y(t) - y^*$. System (13) is transformed into

$$\begin{aligned}
\dot{u}_1(t) &= c_1u_1(t) + c_4u_2(t-\tau) \\
&\quad + \sum_{i+j\geq 2}\frac{1}{i!\,j!}f_{ij}^{(1)}u_1^i(t)\,u_2^j(t-\tau), \\
\dot{u}_2(t) &= c_2u_1(t) + c_3u_2(t) + \sum_{i+j\geq 2}\frac{1}{i!\,j!}f_{ij}^{(2)}u_1^i(t)\,u_2^j(t),
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
c_1 &= -r_1 + \frac{2a_{12}r_2^2h}{a_{21}^2}, \qquad c_2 = \frac{2r_2h\left(a_{21} - r_2\right)}{a_{21}}, \\
c_3 &= -r_2 + \frac{r_2\left(2r_2 - a_{21}\right)}{a_{21}}, \qquad c_4 = -\frac{a_{12}r_2\left(2r_2 - a_{21}\right)}{a_{21}^2}, \\
f^{(1)} &= x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}x(t)\,y(t-\tau)}{my^2(t-\tau) + x^2(t)}\right], \\
f^{(2)} &= \frac{a_{21}x^2(t)\,y(t)}{my^2(t) + x^2(t)} - r_2y(t), \\
f_{ij}^{(1)} &= \left.\frac{\partial^{i+j}f^{(1)}}{\partial x^i\partial y(t-\tau)^j}\right|_{(x^*,y^*)}, \\
f_{ij}^{(2)} &= \left.\frac{\partial^{i+j}f^{(2)}}{\partial x^i\partial y^j}\right|_{(x^*,y^*)}, \quad i, j \geq 0.
\end{aligned} \tag{35}$$

For the simplicity of notations, we rewrite (34) as

$$\dot{u}(t) = L_\mu u_t + f\left(\mu, u_t\right), \tag{36}$$

where $u(t) = (u_1(t), u_1(t))^T \in \mathbf{R}^2$, $u_t(\theta) \in \mathscr{C}$ is defined by $u_t(\theta) = u(t+\theta)$, and $L_\mu : \mathscr{C} \to \mathbf{R}, f : \mathbf{R}\times\mathscr{C} \to \mathbf{R}$ are given, respectively, by

$$L_\mu\phi = \begin{bmatrix} c_1 & 0 \\ c_2 & c_3 \end{bmatrix}\phi(0) + \begin{bmatrix} 0 & c_4 \\ 0 & 0 \end{bmatrix}\phi(-\tau), \tag{37}$$

$$f\left(\mu, \phi\right) = \begin{bmatrix} \sum_{i+j\geq 2}\frac{1}{i!\,j!}f_{ij}^{(1)}\phi_1^i(t)\,\phi_2^j(t-\tau) \\ \sum_{i+j\geq 2}\frac{1}{i!\,j!}f_{ij}^{(2)}\phi_1^i(t)\,\phi_2^j(t) \end{bmatrix}. \tag{38}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-\tau, 0]$ such that

$$L_\mu\phi = \int_{-\tau}^{0} d\eta\left(\theta, \mu\right)\phi(\theta), \quad \text{for } \phi \in \mathscr{C}. \tag{39}$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{bmatrix} c_1 & 0\\ c_2 & c_3\end{bmatrix}\delta(\theta)+\begin{bmatrix} 0 & c_4\\ 0 & 0\end{bmatrix}\delta(\theta+\tau), \tag{40}$$

where δ is the Dirac delta function. For $\phi\in\mathscr{C}^1([-\tau,0],\mathbf{R}^2)$, define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & \theta\in[-\tau,0),\\ \displaystyle\int_{-\tau}^{0}d\eta(s,\mu)\phi(s), & \theta=0,\end{cases} \tag{41}$$

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-\tau,0),\\ f(\mu,\phi), & \theta=0.\end{cases}$$

Then when $\theta=0$, system (36) is equivalent to

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t, \tag{42}$$

where $u_t(\theta)=u(t+\theta)$ for $\theta\in[-\tau,0]$.

For $\psi\in\mathscr{C}^1([0,\tau],(\mathbf{R}^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,\tau],\\ \displaystyle\int_{-\tau}^{0}d\eta^T(t,0)\psi(-t), & s=0,\end{cases} \tag{43}$$

and a bilinear inner product,

$$\begin{aligned}\langle\psi(s),\phi(\theta)\rangle&=\overline{\psi}(0)\phi(0)\\&\quad-\int_{-\tau}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi,\end{aligned} \tag{44}$$

where $\eta(\theta)=\eta(\theta,0)$ and $\overline{(\cdot)}$ denotes the conjugate complex of $(\cdot)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 3, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively.

Suppose that $q(\theta)=(1,\rho)^Te^{i\omega_0\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0$. Then $A(0)q(\theta)=i\omega_0q(\theta)$. From the definition of $A(0)$, it is easy to get $\rho=(i\omega_0-c_3)/c_2$.

Similarly, let $q^*(s)=D(1,\rho^*)e^{-i\omega_0s}$ be the eigenvector of A^* corresponding to $-i\omega_0$. By the definition of A^*, we can compute $\rho^*=(-i\omega_0-c_1)/c_2$.

In order to assure $\langle q^*(s),q(\theta)\rangle=1$, we need to determine the value of D. From (44) and the definitions of q and q^*, we have $D=1/(1+\overline{\rho}^*\rho+c_4\rho\tau_0e^{i\tau_0\omega_0})$ such that $\langle q^*(s),q(\theta)\rangle=1$ and $\langle q^*(s),\overline{q}(\theta)\rangle=0$.

In the following, we first compute the coordinates to describe the center manifold C_0 at $\mu=0$. Define

$$z(t)=\langle q^*,u_t\rangle,\quad W(t,\theta)=u_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\}. \tag{45}$$

On the center manifold C_0, we have

$$\begin{aligned}W(t,\theta)&=W(z(t),\overline{z}(t),\theta)\\&=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}\\&\quad+W_{30}(\theta)\frac{z^3}{6}+\cdots,\end{aligned} \tag{46}$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the directions of q and $\overline{q}$. Note that W is real if u_t is real. We consider only real solutions. For the solution $u_t\in C_0$, since $\mu=0$, we have

$$\begin{aligned}\dot{z}&=\omega_0z+i\langle q^*(\theta),f(0,W(z(t),\overline{z}(t),\theta)\\&\qquad+2\operatorname{Re}\{z(t)q(\theta)\})\rangle\\&=i\omega_0z+\overline{q}^*(0)f(0,W(z(t),\overline{z}(t),0)\\&\qquad+2\operatorname{Re}\{z(t)q(0)\})\\&\triangleq i\omega_0z+\overline{q}^*(0)f_0(z,\overline{z})=i\omega_0z+g(z,\overline{z}),\end{aligned} \tag{47}$$

where

$$\begin{aligned}g(z,\overline{z})&=\overline{q}^*(0)f_0(z,\overline{z})\\&=g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+g_{21}\frac{z^2\overline{z}}{2}+\cdots.\end{aligned} \tag{48}$$

By (45), we have

$$u_t(\theta)=(u_{1t}(\theta),u_{2t}(\theta))^T=W(t,\theta)+zq(\theta)+\overline{zq}(\theta). \tag{49}$$

It follows from (38) and (48) that

$$\begin{aligned}g_{20}&=2\overline{D}\Big[\frac{1}{2}f_{20}^{(1)}\rho^2+f_{11}^{(1)}\rho e^{-i\tau_0\omega_0}+\frac{1}{2}f_{02}^{(1)}e^{-2i\tau_0\omega_0}\\&\qquad+\overline{\rho}^*\Big(\frac{1}{2}f_{20}^{(2)}\rho^2+f_{11}^{(2)}\rho+\frac{1}{2}f_{02}^{(2)}\Big)\Big],\\g_{11}&=\overline{D}\Big[f_{20}^{(1)}\rho\overline{\rho}+f_{11}^{(1)}\Big(\rho e^{i\tau_0\omega_0}+\overline{\rho}e^{-i\tau_0\omega_0}\Big)\\&\qquad+f_{02}^{(1)}+\overline{\rho}^*\Big(f_{20}^{(2)}\rho\overline{\rho}+f_{11}^{(2)}(\rho+\overline{\rho})+f_{02}^{(2)}\Big)\Big],\\g_{02}&=2\overline{D}\Big[\frac{1}{2}f_{20}^{(1)}\overline{\rho}^2+f_{11}^{(1)}\rho e^{i\tau_0\omega_0}\\&\qquad+\frac{1}{2}f_{02}^{(1)}e^{2i\tau_0\omega_0}+\overline{\rho}^*\Big(\frac{1}{2}f_{20}^{(2)}\overline{\rho}^2\\&\qquad\qquad+f_{11}^{(2)}\overline{\rho}+\frac{1}{2}f_{02}^{(2)}\Big)\Big],\\g_{21}&=2\overline{D}\Big[\frac{1}{2}f_{20}^{(1)}\Big(2\rho W_{11}^{(1)}(0)+\overline{\rho}W_{20}^{(1)}(0)\Big)\\&\qquad+f_{11}^{(1)}\Big(\rho W_{11}^{(2)}(-\tau_0)+\frac{1}{2}\overline{\rho}W_{20}^{(2)}(-\tau_0)\\&\qquad\qquad+\frac{1}{2}W_{20}^{(1)}(0)e^{i\tau_0\omega_0}+W_{11}^{(1)}(0)e^{-i\tau_0\omega_0}\Big)\\&\qquad+\frac{1}{2}f_{02}^{(1)}\Big(2W_{11}^{(2)}(-\tau_0)e^{-i\tau_0\omega_0}+W_{20}^{(2)}(-\tau_0)e^{i\tau_0\omega_0}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}f_{21}^{(1)}\left(\rho^2 e^{i\tau_0\omega_0}+2\rho\overline{\rho}e^{-i\tau_0\omega_0}\right)\\
&+\frac{1}{2}f_{12}^{(1)}\left(\overline{\rho}e^{-2i\tau_0\omega_0}+2\rho\right)\\
&+\frac{1}{2}f_{30}^{(1)}\rho^2\overline{\rho}+\frac{1}{2}f_{03}^{(1)}e^{-i\tau_0\omega_0}\Big]\\
&+2\overline{D}\overline{\rho}^*\Big[\frac{1}{2}f_{20}^{(2)}\left(2\rho W_{11}^{(1)}(0)+\overline{\rho}W_{20}^{(1)}(0)\right)\\
&\quad+f_{11}^{(2)}\Big(\rho W_{11}^{(2)}(0)+\frac{1}{2}\overline{\rho}W_{20}^{(2)}(0)\\
&\qquad+\frac{1}{2}W_{20}^{(1)}(0)+W_{11}^{(1)}(0)\Big)\\
&\quad+\frac{1}{2}f_{02}^{(2)}\left(2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)\right)\\
&\quad+\frac{1}{2}f_{21}^{(2)}\left(\rho^2+2\rho\overline{\rho}\right)\\
&\quad+\frac{1}{2}f_{12}^{(2)}\left(\overline{\rho}+2\rho\right)+\frac{1}{2}f_{30}^{(2)}\rho^2\overline{\rho}+\frac{1}{2}f_{03}^{(2)}\Big].
\end{aligned}
\tag{50}
$$

In order to assure the value of g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. By (42) and (45), we have

$$
\begin{aligned}
\dot{W}&=\dot{u}_t-\dot{z}q-\dot{\overline{z}}\overline{q}\\
&=\begin{cases}AW-2\operatorname{Re}\{\overline{q}^*(0)f_0q(\theta)\}, & \theta\in[-\tau_0,0),\\ AW-2\operatorname{Re}\{\overline{q}^*(0)f_0q(\theta)\}+f_0, & \theta=0,\end{cases}\\
&\triangleq AW+H(z,\overline{z},\theta),
\end{aligned}
\tag{51}
$$

where

$$
H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots.
\tag{52}
$$

Notice that near the origin on the center manifold C_0, we have

$$
\dot{W}=W_z\dot{z}+W_{\overline{z}}\dot{\overline{z}};
\tag{53}
$$

thus, we have

$$
\begin{aligned}
(A-2i\omega_k\tau_kI)W_{20}(\theta)&=-H_{20}(\theta),\\
AW_{11}(\theta)&=-H_{11}(\theta).
\end{aligned}
\tag{54}
$$

By (51), for $\theta\in[-\tau_0,0)$, we have

$$
\begin{aligned}
H(z,\overline{z},\theta)&=-\overline{q}^*(0)f_0q(\theta)-q^*(0)\overline{f}_0\overline{q}(\theta)\\
&=-gq(\theta)-\overline{gq}(\theta).
\end{aligned}
\tag{55}
$$

Comparing the coefficients with (51) gives that

$$
\begin{aligned}
H_{20}(\theta)&=-g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta),\\
H_{11}(\theta)&=-g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta).
\end{aligned}
\tag{56}
$$

From (56), (54), and the definition of $A(0)$, we can get

$$
\dot{W}_{20}(\theta)=2i\omega_0W_{20}(\theta)+g_{20}q(\theta)+\overline{g}_{02}\overline{q}(\theta).
\tag{57}
$$

Notice that $q(\theta)=q(0)e^{i\omega_0\theta}$; we have

$$
W_{20}(\theta)=\frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta}+\frac{i\overline{g}_{02}}{3\omega_0}\overline{q}(0)e^{-i\omega_0\theta}+E_1e^{2i\omega_0\theta},
\tag{58}
$$

where $E_1=(E_1^{(1)},E_1^{(2)})\in\mathbf{R}^2$ is a constant vector. In the same way, we can also obtain

$$
W_{11}(\theta)=-\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta}+\frac{i\overline{g}_{11}}{\omega_0}\overline{q}(0)e^{-i\omega_0\theta}+E_2,
\tag{59}
$$

where $E_2=(E_2^{(1)},E_2^{(2)})\in\mathbf{R}^2$ is also a constant vector. In what follows, we will compute E_1 and E_2. From the definition of $A(0)$ and (54), we have

$$
\int_{-\tau_0}^{0}d\eta(\theta)W_{20}(\theta)=2i\omega_0W_{20}(0)-H_{20}(0),
\tag{60}
$$

$$
\int_{-\tau_0}^{0}d\eta(\theta)W_{11}(\theta)=-H_{11}(0),
\tag{61}
$$

where $\eta(\theta)=\eta(0,\theta)$.

From (51), (58), and (60) and noting that

$$
\left[i\omega_0I-\int_{-\tau_0}^{0}e^{i\omega_0\theta}d\eta(\theta)\right]q(0)=0,
\tag{62}
$$

we have

$$
E_1^{(1)}=\frac{1}{A_1}\begin{vmatrix}e_1 & -c_4e^{-2i\omega_0\tau_0}\\ e_2 & 2i\omega_0-c_3\end{vmatrix},\qquad E_1^{(2)}=\frac{1}{A_1}\begin{vmatrix}2i\omega_0-c_1 & e_1\\ -c_2 & e_2\end{vmatrix},
\tag{63}
$$

where

$$
\begin{aligned}
A_1&=(2i\omega_0-c_1)(2i\omega_0-c_3)-c_2c_4e^{-2i\omega_0\tau_0},\\
e_1&=f_{20}^{(1)}\rho^2+2f_{11}^{(1)}\rho e^{-i\tau_0\omega_0}+f_{02}^{(1)}e^{-2i\tau_0\omega_0},\\
e_2&=f_{20}^{(2)}\rho^2+2f_{11}^{(2)}\rho+f_{02}^{(2)}.
\end{aligned}
\tag{64}
$$

From (52), (59), and (61) and noting that

$$
\left[-i\omega_0I-\int_{-\tau_0}^{0}e^{-i\omega_0\theta}d\eta(\theta)\right]\overline{q}(0)=0,
\tag{65}
$$

we have

$$
E_2^{(1)}=\frac{1}{A_2}\begin{vmatrix}e_3 & -c_4\\ e_4 & -c_3\end{vmatrix},\qquad E_2^{(2)}=\frac{1}{A_2}\begin{vmatrix}-c_1 & e_3\\ -c_2 & e_4\end{vmatrix},
\tag{66}
$$

where

$$
\begin{aligned}
A_2&=c_1c_3-c_2c_4,\\
e_3&=f_{20}^{(1)}\rho\overline{\rho}+f_{11}^{(1)}\left(\rho e^{i\tau_0\omega_0}+\overline{\rho}e^{-i\tau_0\omega_0}\right)+f_{02}^{(1)},\\
e_4&=f_{20}^{(2)}\rho\overline{\rho}+f_{11}^{(2)}(\rho+\overline{\rho})+f_{02}^{(2)}.
\end{aligned}
\tag{67}
$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (58) and (59). Furthermore, we can determine each g_{ij}. Therefore, each g_{ij} is determined by the parameters and delay in (13). Thus, we can compute the following values [15]:

$$\begin{gathered} c_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\ T_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}, \\ \beta_2 = 2\operatorname{Re}\{c_1(0)\}, \end{gathered} \tag{68}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_k; that is, μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0\,(< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcation exists for $\tau > \tau_0\,(< \tau_0)$; β_2 determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0\,(> 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0\,(< 0)$.

5. Global Attractiveness

In this section, following Chaplygin [17], taking into account the upper and lower solution technique and using monotone iterative methods [18, 19], we discuss the global attractiveness of the positive equilibrium $E^*(x^*, y^*)$ and the global stability of the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ of system (1), respectively.

Theorem 6. *Suppose (H1) holds and let h be defined above, then the positive equilibrium $E^*(x^*, y^*)$ of system* (1) *is globally attractive provided that the following holds:*

$$\text{(H2) } r_1 > \max\{a_{12}/2\sqrt{m}, (3a_{12}/m) + (2a_{12}r_2/a_{21})h\},$$

Proof. Let $(x(t), y(t))$ be any positive solution of system (1) with initial conditions (2).

Let

$$\begin{aligned} U_1 &= \limsup_{t\to+\infty} x(t), & V_1 &= \liminf_{t\to+\infty} x(t), \\ U_2 &= \limsup_{t\to+\infty} y(t), & V_2 &= \liminf_{t\to+\infty} y(t). \end{aligned} \tag{69}$$

Using iteration method, we will proof that $U_1 = V_1 = x^*$, $U_2 = V_2 = y^*$.

From the first equation of system (1), we have

$$\dot{x}(t) \le x(t)\left[r_1 - a_{11}x(t)\right]; \tag{70}$$

by comparison, it follows that

$$U_1 = \limsup_{t\to+\infty} x(t) \le \frac{r_1}{a_{11}} := M_1^x; \tag{71}$$

hence, for $\epsilon > 0$ sufficiently small, there exists a $T_1 > 0$ such that if $t > T_1$, $x(t) \le M_1^x + \epsilon$.

From the second equation of system (1), we have, for $t > T_1 + \tau$,

$$\dot{y}(t) \le \frac{a_{21}(M_1^x + \epsilon)^2 y(t)}{my^2(t) + (M_1^x + \epsilon)^2} - r_2 y(t). \tag{72}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \frac{a_{21}(M_1^x + \epsilon)^2 u(t)}{mu^2(t) + (M_1^x + \epsilon)^2} - r_2 u(t). \tag{73}$$

Since (H1) holds, by Lemma 2, it follows from (73) that

$$\lim_{t\to+\infty} u(t) = (M_1^x + \epsilon)h, \tag{74}$$

where h is defined in (12). By comparison, we obtain that

$$U_2 = \limsup_{t\to+\infty} y(t) \le (M_1^x + \epsilon)h; \tag{75}$$

since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, it follows that $U_2 \le M_1^y$, where

$$M_1^y = M_1^x h. \tag{76}$$

Hence, for $\epsilon > 0$ sufficiently small, there is a $T_2 > T_1 + \tau$ such that if $t > T_2$, $y(t) \le M_1^y + \epsilon$.

For $\epsilon > 0$ sufficiently small, noting that $my^2(t-\tau_2) + x^2 \ge 2\sqrt{m}xy(t-\tau_2)$, we derive from the first equation of system (1) that, for $t > T_2$,

$$\dot{x}(t) \ge x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}}{2\sqrt{m}}\right]; \tag{77}$$

by comparison, it follows that

$$V_1 = \liminf_{t\to+\infty} x(t) \ge \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}}{2\sqrt{m}}\right) := N_1^x; \tag{78}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_3 > T_2 + \tau$, such that if $t > T_3$, $x(t) \ge N_1^x - \varepsilon$.

For $\epsilon > 0$ sufficiently small, we derive from the second equation of system (1) that, for $t > T_3 + \tau$,

$$\dot{y}(t) \ge \frac{a_{21}(N_1^x - \epsilon)^2 y(t)}{my^2(t) + (N_1^x - \epsilon)^2} - r_2 y(t). \tag{79}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \frac{a_{21}(N_1^x - \epsilon)^2 u(t)}{mu^2(t) + (N_1^x - \epsilon)^2} - r_2 u(t). \tag{80}$$

Since (H1) holds, by Lemma (5), it follows from (80) that

$$\lim_{t\to+\infty} u(t) = (N_1^x - \epsilon)h; \tag{81}$$

by comparison we derive that

$$V_2 = \liminf_{t\to+\infty} y(t) \ge (N_1^x - \epsilon)h. \tag{82}$$

Since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_1^y$, where

$$N_1^y = N_1^x h. \tag{83}$$

Therefore, for $\epsilon > 0$ sufficiently small, there is a $T_4 > T_3 + \tau$ such that if $t > T_4$, $y(t) \geq N_1^y - \epsilon$.

Again, for $\epsilon > 0$ sufficiently small, it follows from the first equation of system (1) that, for $t > T_4$,

$$\dot{x}(t) \leq x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}(N_1^x - \epsilon)(N_1^y - \epsilon)}{m(M_1^y + \epsilon)^2 + (M_1^x + \epsilon)^2}\right]; \tag{84}$$

by comparison we derive that

$$U_1 = \limsup_{t\to+\infty} x(t) \leq \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}(N_1^x - \epsilon)(N_1^y - \epsilon)}{m(M_1^y + \epsilon)^2 + (M_1^x + \epsilon)^2}\right). \tag{85}$$

Since the above inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, it follows that $U \leq M_2^x$, where

$$M_2^x = \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}N_1^x N_1^y}{m(M_1^y)^2 + (M_1^x)^2}\right); \tag{86}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_5 > T_4 + \tau$ such that if $t > T_5$, $x(t) \leq M_2^x + \epsilon$.

It follows from the second equation of system (1) that, for $t > T_5$,

$$\dot{y}(t) \leq \frac{a_{21}(M_2^x + \epsilon)^2 y(t)}{my^2(t) + (M_2^x + \epsilon)^2} - r_2 y(t). \tag{87}$$

By Lemma 2 and a comparison argument we derive from (87) that

$$U_2 = \limsup_{t\to+\infty} y(t) \leq (M_2^x + \epsilon)h; \tag{88}$$

since this inequality holds true for $\epsilon > 0$ sufficiently small, we get $U_2 \leq M_2^y$, where

$$M_2^y = M_2^x h; \tag{89}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_6 > T_5 + \tau$ such that if $t > T_6$, $y(t) \leq M_2^y + \epsilon$.

For $\epsilon > 0$ sufficiently small, it follows from the first equation of system (1) that, for $t > T_6$,

$$\dot{x}(t) \geq x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}(M_2^x + \epsilon)(M_2^y + \epsilon)}{m(N_1^y - \epsilon)^2 + (N_1^x - \epsilon)^2}\right]; \tag{90}$$

by comparison, we can obtain that

$$V_1 = \liminf_{t\to+\infty} x(t) \geq \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}(M_2^x + \epsilon)(M_2^y + \epsilon)}{m(N_1^y - \epsilon)^2 + (N_1^x - \epsilon)^2}\right). \tag{91}$$

Since the above inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, it follows that $V \geq N_2^x$, where

$$N_2^x = \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}M_2^x M_2^y}{m(N_1^y)^2 + (N_1^x)^2}\right); \tag{92}$$

therefore, for $\epsilon > 0$ sufficiently small, there is a $T_7 > T_6 + \tau$ such that if $t > T_7$, $x(t) \geq N_2^x - \epsilon$.

For $\epsilon > 0$ sufficiently small, we derive from the second equation of system (1) that, for $t > T_7 + \tau$,

$$\dot{y}(t) \geq \frac{a_{21}(N_2^x - \epsilon)^2 y(t)}{my^2(t) + (N_2^x - \epsilon)^2} - r_2 y(t). \tag{93}$$

Since (H1) holds, by Lemma 2 and a comparison argument, it follows (93) that

$$V_2 = \liminf_{t\to+\infty} y(t) \geq (N_2^x - \epsilon)h; \tag{94}$$

since, for arbitrary $\epsilon > 0$ sufficiently small, this inequality holds true, we conclude that $V_2 \geq N_2^y$, where

$$N_2^y = N_2^x h. \tag{95}$$

Continuing this process, we obtain four sequences M_n^x, M_n^y, V_n^x, and V_n^y $(n = 1, 2, \ldots)$ such that, for $n \geq 2$,

$$\begin{aligned} M_n^x &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}N_{n-1}^x N_{n-1}^y}{m(M_{n-1}^y)^2 + (M_{n-1}^x)^2}\right), \\ N_n^x &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}M_n^x M_n^y}{m(N_{n-1}^y)^2 + (N_{n-1}^x)^2}\right), \\ M_n^y &= M_n^x h, \qquad N_n^y = N_n^x h, \end{aligned} \tag{96}$$

where h is defined in (12). It is readily seen that

$$N_n^x \leq V_1 \leq U_1 \leq M_n^x, \qquad N_n^y \leq V_2 \leq U_2 \leq M_n^y. \tag{97}$$

It is easy to know that the sequences M_n^x, M_n^y are not increasing and the sequences N_n^x, N_n^y are not decreasing; from accumulation point theorem, the limit of each sequence in M_n^x, M_n^y, N_n^x, and N_n^y exists, Denote

$$\begin{aligned} \overline{x} &= \lim_{t\to+\infty} M_n^x, \qquad \underline{x} = \lim_{t\to+\infty} N_n^x, \\ \overline{y} &= \lim_{t\to+\infty} M_n^y, \qquad \underline{y} = \lim_{t\to+\infty} N_n^y. \end{aligned} \tag{98}$$

We therefore obtain from (96) and (98) that

$$\begin{aligned} \overline{x} &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}\underline{x}\,\underline{y}}{m\overline{y}^2 + \overline{x}^2}\right), \\ \underline{x} &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}\overline{x}\,\overline{y}}{m\underline{y}^2 + \underline{x}^2}\right), \\ \overline{y} &= \overline{x}h, \qquad \underline{y} = \underline{x}h. \end{aligned} \tag{99}$$

To complete the proof, it is sufficient to prove that $\overline{x} = \underline{x}, \overline{y} = \underline{y}$. It follows from (99) that

$$a_{11}\left(1+mh^2\right)\overline{x}^3 = r_1\left(1+mh^2\right)\overline{x}^2 - a_{12}h\underline{x}^2, \tag{100}$$

$$a_{11}\left(1+mh^2\right)\underline{x}^3 = r_1\left(1+mh^2\right)\underline{x}^2 - a_{12}h\overline{x}^2. \tag{101}$$

Letting (100) minus (101), we have

$$\begin{aligned} &a_{11}\left(1+mh^2\right)(\overline{x}-\underline{x})\left(\overline{x}^2+\overline{x}\underline{x}+\underline{x}^2\right) \\ &\quad = \left[r_1\left(1+mh^2\right)+a_{12}h\right](\overline{x}-\underline{x})(\overline{x}+\underline{x}). \end{aligned} \tag{102}$$

If $\overline{x} \neq \underline{x}$, we derive from (102) that

$$\begin{aligned} &a_{11}\left(1+mh^2\right)\left(\overline{x}^2+\overline{x}\underline{x}+\underline{x}^2\right) \\ &\quad = \left[r_1\left(1+mh^2\right)+a_{12}h\right](\overline{x}+\underline{x}). \end{aligned} \tag{103}$$

Letting $A = a_{11}(1+mh^2), B = r_1(1+mh^2)+a_{12}h$, we derive from (103) that

$$\overline{x}\underline{x} = (\overline{x}+\underline{x})^2 - \frac{B}{A}(\overline{x}+\underline{x}). \tag{104}$$

It follows from (104) that

$$\begin{aligned} (\overline{x}+\underline{x})^2 - 4\overline{x}\underline{x} &= (\overline{x}+\underline{x})^2 - 4\left[(\overline{x}+\underline{x})^2 - \frac{B}{A}(\overline{x}+\underline{x})\right] \\ &= (\overline{x}+\underline{x})\left[\frac{4B}{A} - 3(\overline{x}+\underline{x})\right]; \end{aligned} \tag{105}$$

noting that $\overline{x} \geq N_1^x$, $\underline{x} \geq N_1^x$, we derive from (105) that

$$(\overline{x}+\underline{x})^2 - 4\overline{x}\underline{x} \leq 2(\overline{x}+\underline{x})\left[\frac{2B}{A} - 3N_1^x\right]. \tag{106}$$

Substituting (78) into (106), it follows that

$$(\overline{x}+\underline{x})^2 - 4\overline{x}\underline{x} \leq -\frac{2(\overline{x}+\underline{x})}{a_{11}}\left[r_1 - \frac{3a_{12}}{m} - \frac{2a_{12}h}{1+mh^2}\right]. \tag{107}$$

Hence, if (H2) holds, we have $(\overline{x}+\underline{x})^2 - 4\overline{x}\underline{x} < 0$; this is a contradiction. Accordingly, we have $\overline{x} = \underline{x}$. Therefore, from (99), we have $\overline{y} = \underline{y}$. Hence, the positive equilibrium E^* is globally attractive. The proof is complete. □

Using the same methods in [4, 20], we can also get a similar result.

Theorem 7. *If $r_1 > a_{12}/2\sqrt{m}$ and $a_{21} < r_2$, the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ of system (1) is globally asymptotically stable.*

6. Global Continuation of Local Hopf Bifurcations

In this section, we study the global continuation of periodic solutions bifurcating from the positive equilibrium E^* of system (13). Throughout this section, we follow closely the notations in [13]. For simplification of notations, setting $z(t) = (z_1(t), z_2(t))^T = (x(t), y(t))^T$, we may rewrite system (13) as the following functional differential equation:

$$\dot{z}(t) = \mathscr{F}(z_t, \tau, p), \tag{108}$$

where $z_t(\theta) = (z_{1t}(\theta), z_{2t}(\theta))^T = (z_1(t+\theta)$, and $z_2(t+\theta))^T \in \mathscr{C}([-\tau, 0], \mathbf{R}^2)$. It is obvious that if (H1) holds, then system (13) has a semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ and a positive equilibrium $E^*(x^*, y^*)$. Following the work of [13], we need to define

$$\begin{aligned} &\mathbf{X} = \mathscr{C}\left([-\tau, 0], \mathbf{R}^2\right), \\ &\quad \Gamma = \mathrm{Cl}\left\{(z, \tau, p) \in \mathbf{X} \times \mathbf{R} \times \mathbf{R}^+;\ z \text{ is a nonconstant} \right. \\ &\qquad \left. \text{periodic solution of } (108)\right\}, \\ &\mathscr{N} = \left\{(\overline{z}, \overline{\tau}, \overline{p}); \mathscr{F}(\overline{z}, \overline{\tau}, \overline{p}) = 0\right\}. \end{aligned} \tag{109}$$

Let $\ell_{(E^*, \tau_j, 2\pi/\omega_0)}$ denote the connected component passing through $(E^*, \tau_j, 2\pi/\omega_0)$ in Γ, where τ_j is defined by (26). From Theorem 5, we know that $\ell_{(E^*, \tau_j, 2\pi/\omega_0)}$ is nonempty.

We first state the global Hopf bifurcation theory due to Wu [13] for functional differential equations.

Lemma 8. *Assume that (z_*, τ, p) is an isolated center satisfying the hypotheses (A_1)–(A_4) in [13]. Denote by $\ell_{(z_*, \tau, p)}$ the connected component of (z_*, τ, p) in Γ. Then either*

(i) *$\ell_{(z_*, \tau, p)}$ is unbounded or*

(ii) *$\ell_{(z_*, \tau, p)}$ is bounded; $\ell_{(z_*, \tau, p)} \cap \Gamma$ is finite and*

$$\sum_{(z, \tau, p) \in \ell_{(z_*, \tau, p)} \cap \mathscr{N}} \gamma_m(z_*, \tau, p) = 0, \tag{110}$$

for all $m = 1, 2, \ldots$, where $\gamma_m(z_*, \tau, p)$ is the mth crossing number of (z_*, τ, p) if $m \in J(z_*, \tau, p)$ or it is zero if otherwise.

Clearly, if (ii) in Lemma 8 is not true, then $\ell_{(z_*, \tau, p)}$ is unbounded. Thus, if the projections of $\ell_{(z_*, \tau, p)}$ onto z-space and onto p-space are bounded, then the projection onto τ-space is unbounded. Further, if we can show that the projection of $\ell_{(z_*, \tau, p)}$ onto τ-space is away from zero, then the projection of $\ell_{(z_*, \tau, p)}$ onto τ-space must include interval $[\tau, +\infty)$. Following this ideal, we can prove our results on the global continuation of local Hopf bifurcation.

Lemma 9. *If condition (H1) holds, then all nonconstant periodic solutions of (13) with initial conditions,*

$$\begin{aligned} &x(\theta) = \phi(\theta), \quad y(\theta) = \psi(\theta), \quad \phi(\theta) \geq 0, \quad \psi(\theta) \geq 0, \\ &\theta \in [-\tau, 0]\left(\tau = \tau_1 + \tau_2\right), \quad \phi(0) > 0, \quad \psi(0) > 0, \end{aligned} \tag{111}$$

are uniformly bounded.

Proof. Suppose that $x = x(t), y = y(t)$ are nonconstant periodic solutions of system (13) and define

$$x(\xi_1) = \min\{x(t)\}, \quad x(\eta_1) = \max\{x(t)\},$$
$$y(\xi_2) = \min\{y(t)\}, \quad y(\eta_2) = \max\{y(t)\}. \tag{112}$$

It follows from system (13) that

$$x(t) = x(0)\exp\left\{\int_0^t \left(r_1 - a_{11}x(s) - \frac{a_{12}x(s)y(s-\tau)}{my^2(s-\tau)+x^2(s)}\right)ds\right\},$$
$$y(t) = y(0)\exp\left\{\int_0^t \left(-r_2 + \frac{a_{21}x^2(s)}{my^2(s)+x^2(s)}\right)ds\right\}, \tag{113}$$

which implies that the solutions of system (13) cannot cross the x-axis and y-axis. Thus the nonconstant periodic orbits must be located in the interior of each quadrant. It follows from initial conditions of system (13) that $(t) > 0, y(t) > 0$. From system (13), we can get

$$0 = r_1 - a_{11}x(\eta_1) - \frac{a_{12}x(\eta_1)y(\eta_1-\tau)}{my^2(\eta_1-\tau)+x^2(\eta_1)},$$
$$0 = -r_2 + \frac{a_{21}x^2(\eta_2)}{my^2(\eta_2)+x^2(\eta_2)}. \tag{114}$$

Since $x(t) > 0, y(t) > 0$, it follows from the first equation of (114) that

$$0 < x(\eta_1) \le \frac{r_1}{a_{11}}; \tag{115}$$

on the other hand, by the second equation of (114) and (115), we have

$$0 < y(\eta_2) \le h\frac{r_1}{a_{11}}, \tag{116}$$

where h is defined in (12). From the discussion above, the lemma follows immediately. □

Lemma 10. *If conditions (H1) and (H2) hold, then system* (13) *has no nonconstant periodic solution with period* τ.

Proof. Suppose for a contradiction that system (13) has nonconstant periodic solution with period τ. Then the following system (117) of ordinary differential equations has nonconstant periodic solution:

$$\dot{x}(t) = x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}x(t)y(t)}{my^2(t)+x^2(t)}\right],$$
$$\dot{y}(t) = \frac{a_{21}x^2(t)y(t)}{my^2(t)+x^2(t)} - r_2y(t), \tag{117}$$

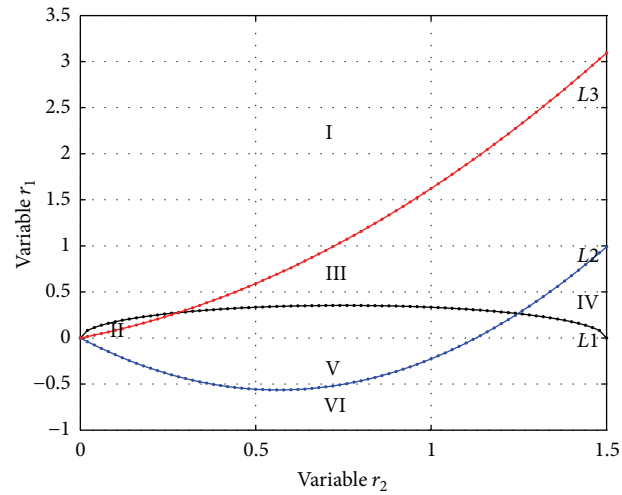

FIGURE 1: The bifurcation diagram of system (1) with $a_{11} = 0.1$, $a_{12} = 1$, $a_{21} = 3/2$, and $m = 2$, where $L1 : r_1 = -(2/9)r_2^2 + r_2/3$, $L2 : r_1 = (8/9)r_2^2\sqrt{((3/2)-r_2)/(2r_2)} - (4/3)r_2((3/2)-r_2)$, and $L3 : r_1 = ((16/9)r_2 + (1/6))\sqrt{((3/2)-r_2)/(2r_2)}$.

which has the same equilibria as system (13), that is, $E_1(r_1/a_{11}, 0)$ and a positive equilibrium $E^*(x^*, y^*)$. Note that x-axis and y-axis are the invariable manifold of system (13) and the orbits of system (13) do not intersect each other. Thus, there is no solution crossing the coordinate axis. On the other hand, note the fact that if system (117) has a periodic solution, then there must be the equilibrium in its interior and E_1 are located on the coordinate axis. Thus, we conclude that the periodic orbit of system (117) must lie in the first quadrant. From the proof of Theorem 6, we known that if (H1) and (H2) hold, the positive equilibrium is asymptotically stable and globally attractive; thus, there is no periodic orbit in the first quadrant. This ends the proof. □

Theorem 11. *Suppose the conditions (H1) and (H2) hold; let* ω_0 *and* $\tau_j (j = 0, 1, \ldots)$ *be defined in* (26). *If* $(2a_{12}r_2^2h/a_{21}^2) - (2r_2(a_{21}-r_2)/a_{21}) < r_1 < ((4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2)$, *then system* (13) *has at least* $j-1$ *periodic solutions for every* $\tau > \tau_j, (j = 1, 2, \ldots)$.

Proof. It is sufficient to prove that the projection of $\ell_{(E^*,\tau_j,2\pi/\omega_0)}$ onto τ-space is $[\overline{\tau}, +\infty)$ for each $j > 0$, where $\overline{\tau} \le \tau_j$.

The characteristic matrix of (108) at an equilibrium $\overline{z} = (\overline{z}^{(1)}, \overline{z}^{(2)}) \in \mathbf{R}^2$ takes the following form:

$$\Delta(\overline{z}, \tau, p)(\lambda) = \lambda \mathrm{Id} - D\mathcal{F}(\overline{z}, \overline{\tau}, \overline{p})\left(e^{\lambda}\mathrm{Id}\right). \tag{118}$$

$(\overline{z}, \overline{\tau}, \overline{p})$ is called a center if $\mathcal{F}(\overline{z}, \overline{\tau}, \overline{p}) = 0$ and $\det(\Delta(\overline{z}, \overline{\tau}, \overline{p})((2\pi/p)i)) = 0$. A center is said to be isolated if it is the only center in some neighborhood of $(\overline{z}, \overline{\tau}, \overline{p})$. It follows from (118) that

$$\det(\Delta(E_1, \tau, p)(\lambda)) = (\lambda + r_1)(\lambda + r_2 - a_{21}) = 0, \tag{119}$$

$$\det(\Delta(E^*, \tau, p)(\lambda)) = \lambda^2 + p_0\lambda + p_1 + p_2e^{-\lambda\tau} = 0, \tag{120}$$

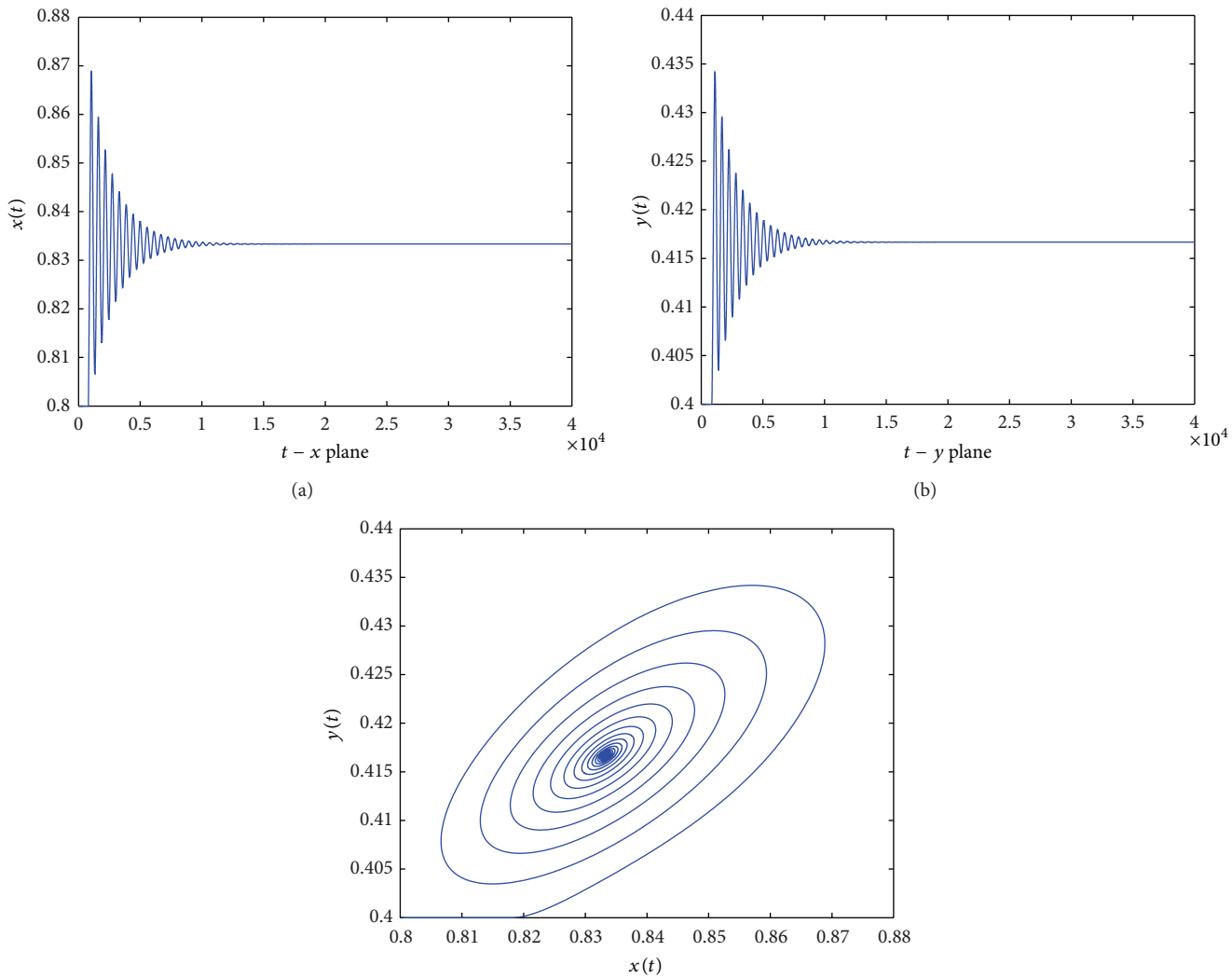

Figure 2: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 6 + 4 = 10$.

where p_0, p_1, and p_2 are defined as in Section 3. From the discussion in Section 3, each of (119) and (120) has no purely imaginary root provided that $r_1 > (4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2$. Thus, we conclude that (108) has no the center of the form as (E_1, τ, p) and (E^*, τ, p). On the other hand, from the discussion in Section 3 about the local Hopf bifurcation, it is easy to verify that $(E^*, \tau_j, 2\pi/\omega_0)$ is an isolated center, and there exist $\epsilon > 0$, $\delta > 0$, and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to \mathscr{C}$ such that $\det(\Delta(\lambda(\tau))) = 0$, $|\lambda(\tau) - \omega_0| < \epsilon$ for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$ and

$$\lambda(\tau_j) = \omega_0 i, \quad \left.\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\right|_{\tau=\tau_j} > 0. \tag{121}$$

Let

$$\Omega_{\epsilon,(2\pi/\omega_0)} = \left\{(\eta, p); 0 < \eta < \epsilon, \left|p - \frac{2\pi}{\omega_0}\right| < \epsilon\right\}. \tag{122}$$

It is easy to verify that, on $[\tau_j - \delta, \tau_j + \delta] \times \partial\Omega_{\epsilon,2\pi/\omega_0}$,

$$\begin{gathered}\det\left(\Delta(E^*, \tau, p)\left(\eta + \frac{2\pi}{p}i\right)\right) = 0 \\ \text{if and only if } \eta = 0,\ \tau = \tau_j,\ p = \frac{2\pi}{\omega_0}.\end{gathered} \tag{123}$$

Therefore, the hypotheses (A_1)–(A_4) in [13] are satisfied. Moreover, if we define

$$\begin{aligned} &H^{\pm}\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right)(\eta, p) \\ &\quad = \det\left(\Delta\left(E^*, \tau_j \pm \delta, p\right)\left(\eta + \frac{2\pi}{p}i\right)\right), \end{aligned} \tag{124}$$

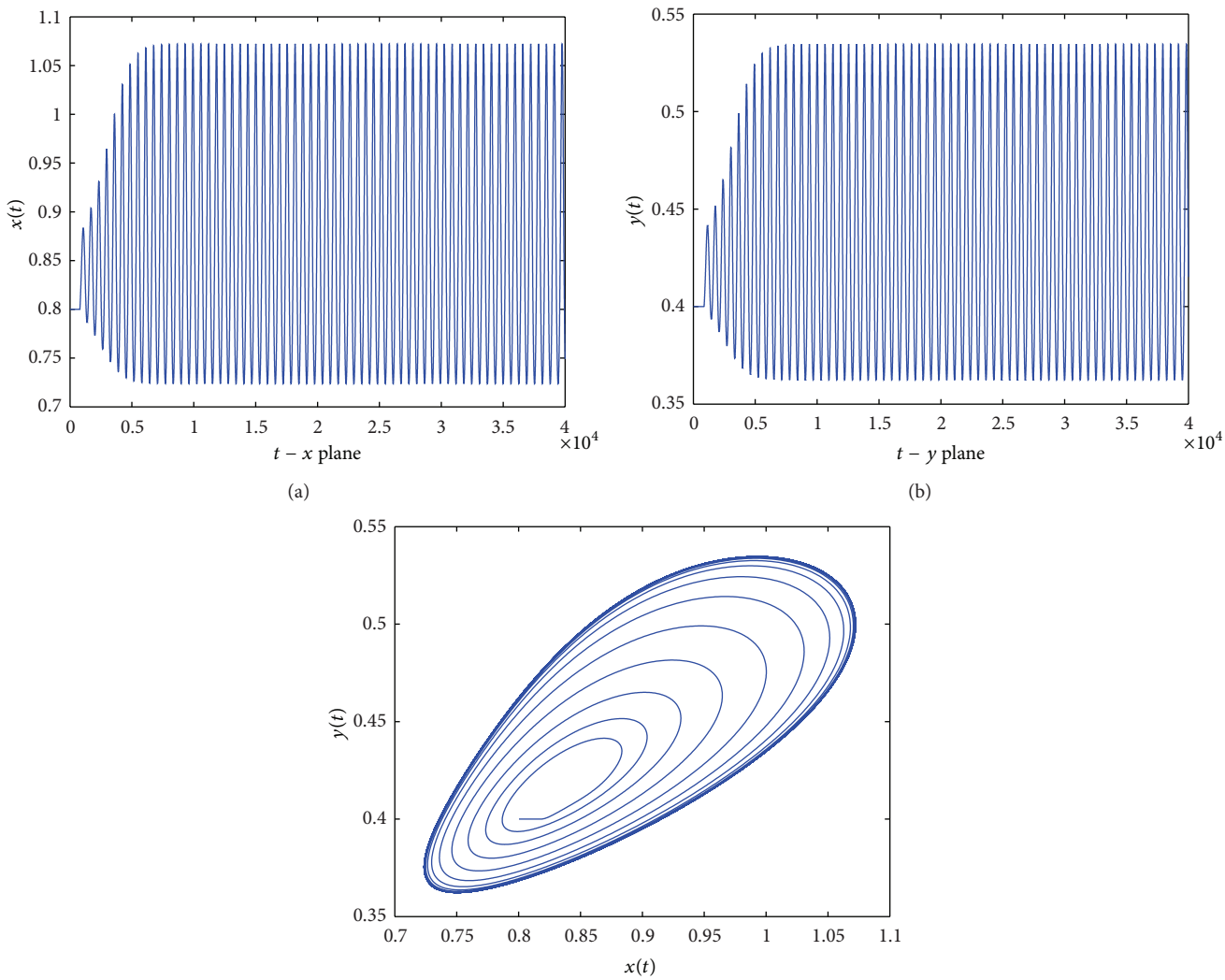

FIGURE 3: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 6 + 6 = 12$.

then we have the crossing number of isolated center $(E^*, \tau_j, (2\pi/\omega_0))$ as follows:

$$\gamma\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right) = \deg_B\left(H^-\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right), \Omega_{\epsilon,2\pi/\omega_0}\right) - \deg_B\left(H^+\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right), \Omega_{\epsilon,2\pi/\omega_0}\right) = -1. \tag{125}$$

Thus, we have

$$\sum_{(\overline{z},\overline{\tau},\overline{p})\in\mathscr{C}_{(E^*,\tau_j,2\pi/\omega_0)}} \gamma(\overline{z},\overline{\tau},\overline{p}) < 0, \tag{126}$$

where $(\overline{z},\overline{\tau},\overline{p})$ has all or parts of the form $(E^*, \tau_k, 2\pi/\omega_0)(k = 0, 1, \ldots)$. It follows from Lemma 8 that the connected component $\ell_{(E^*,\tau_j,2\pi/\omega_0)}$ through $(E^*, \tau_j, 2\pi/\omega_0)$ in Γ is unbounded. From (26), we can know that if (H1) holds, for $j \geq 1$,

$$\tau_j = \frac{1}{\omega_0}\arccos\frac{\omega_0^2 - p_1}{p_2} + \frac{2j\pi}{\omega_0} > \frac{2\pi}{\omega_0}. \tag{127}$$

Now we prove that the projection of $\ell_{(E^*,\tau_j,2\pi/\omega_0)}$ onto τ-space is $[\overline{\tau}, +\infty)$, where $\overline{\tau} \leq \tau_j$. Clearly, it follows from the proof of Lemma 10 that system (13) with $\tau = 0$ has no nontrivial periodic solution. Hence, the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto τ-space is away from zero.

For a contradiction, we suppose that the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto τ-space is bounded; this means that the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto τ-space is included in an interval $(0, \tau^*)$. Noticing $(2\pi/\omega_0) < \tau_j$ and applying Lemma 10 we have $0 < p < \tau^*$ for $(z(t), \tau, p)$ belonging

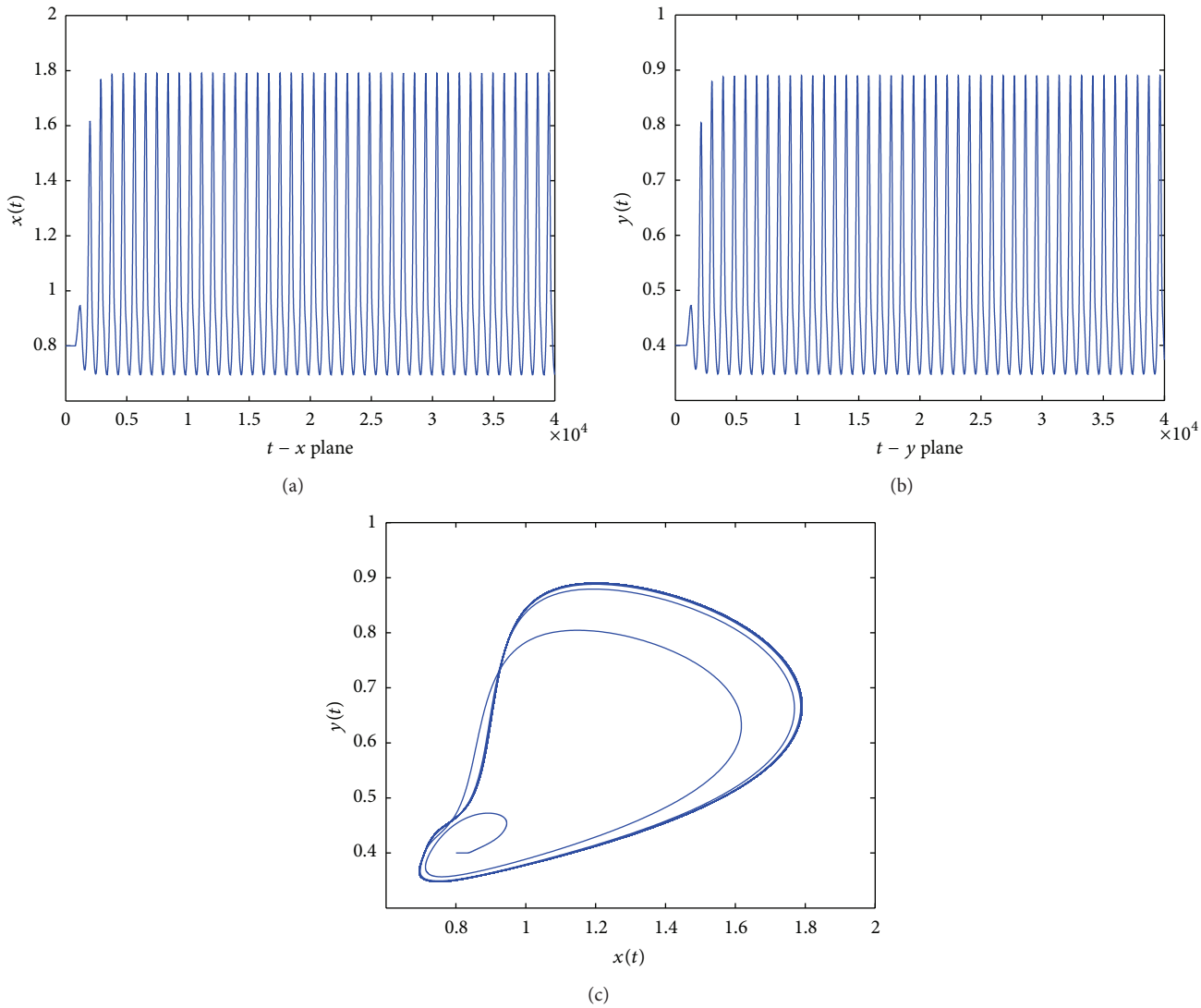

FIGURE 4: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 10 + 8 = 18$.

to $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$. Applying Lemma 9, we know that the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto z-space is bounded. So the component of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ is bounded. This contradicts our conclusion that $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ is unbounded. The contradiction implies that the projection of $\ell_{(E^*\,\tau_j,(2\pi/\omega_0))}$ onto τ-space is unbounded above.

Hence, system (13) has at least $j-1$ periodic solution for every $\tau > \tau_j$, $(j = 1, 2, \ldots)$. This completes the proof. □

Example 12. In system (1), we first choose $a_{11} = 0.1$, $a_{12} = 1$, $a_{21} = 3/2$, and $m = 2$. As depicted in Figure 1, a bifurcation diagram is given for system (1) with respect to the parameters r_1 and r_2. By the discussion in Section 3, system (1) always has a semitrivial equilibrium E_1, and if $r_2 > a_{21}$, E_1 is asymptotically stable; otherwise, E_1 is unstable. So if we choose $0 < r_2 < a_{21} = 3/2$, as depicted in Figure 1, E_1 is always unstable. In domains II, V, and VI, the positive equilibrium is not feasible. In domains I, III, and IV, system (1) has a unique positive equilibrium; it is locally asymptotically stable in domain I and is unstable in domain IV. In domain III, system (1) undergoes a Hopf bifurcation at the positive equilibrium at some τ_0. Further, we choose $r_1 = 5/12$, $r_2 = 1$, $a_{11} = 0.1$,$a_{12} = 1$, $a_{21} = 3/2$, and $m = 2$. In this case, system (1) has a positive equilibrium $E^* = (5/6, 5/12)$. By computation, we have $\omega_0 \approx 0.1063$, $\tau_0 \approx 10.8795$, and $\tau_1 \approx 69.9876$. From Theorem 5, E^* is stable when $\tau < \tau_0$ as illustrated by numerical simulations (see Figure 2). When τ passes through the critical value τ_0, the equilibrium E^* loses its stability and a Hopf bifurcation occurs; that is, a family of periodic solution bifurcates from E^*. By the algorithm derived in Section 3 and Section 4, we have $\lambda'(\tau_0) = 0.0053 - 0.0058i$, $c_1(0) = -0.4357 + 0.0265i$, which implies that $\mu_2 > 0$, $\beta_2 < 0$, and $T_2 > 0$. Thus, by the discussion in Section 4, the Hopf bifurcation is supercritical for $\tau > \tau_0$, the bifurcating periodic solutions from E^* at τ_0 are asymptotically stable, and the period of these periodic solutions is increasing with the increasing of τ, which are depicted in Figures 3, 4, and 5.

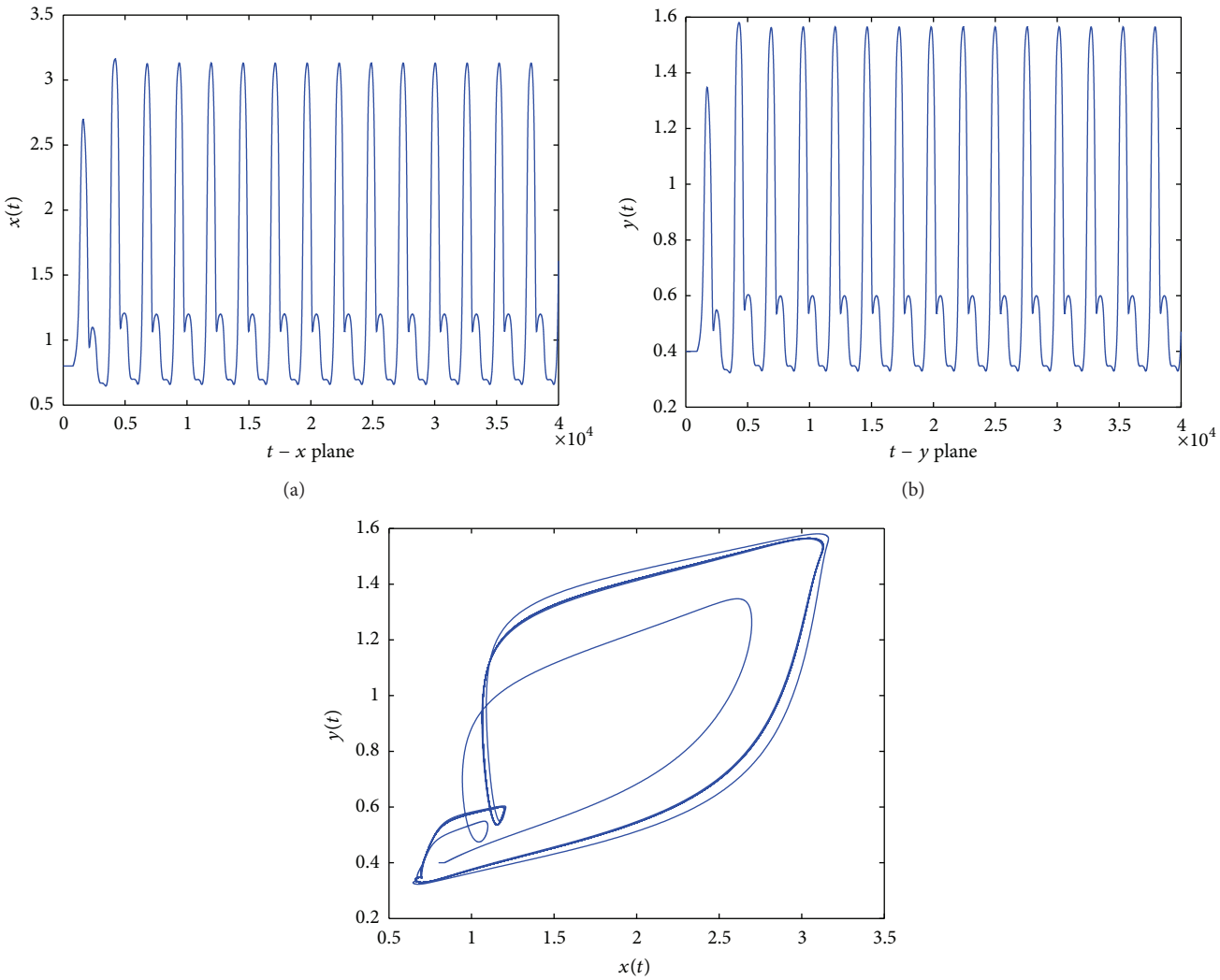

FIGURE 5: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 10 + 60 = 70$.

Furthermore, Figure 5 shows that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of $\tau_1 = 69.9876$.

7. Discussion

In this paper, we have studied a ratio-dependent predator-prey model with two time delays. By analyzing the corresponding characteristic equation, the local stability of the positive equilibrium and the semitrivial equilibrium of system (1) was discussed. We have obtained the estimated length of gestation delay which would not affect the stable coexistence of both prey and predator species at their equilibrium values. The existence of Hopf bifurcation for system (1) at the positive equilibrium was also established. From theoretical analysis it was shown that the larger values of gestation time delay cause fluctuation in individual population density and hence the system becomes unstable. As the estimated length of delay to preserve stability and the critical length of time delay for Hopf bifurcation are dependent upon the parameters of system, it is possible to impose some control, which will prevent the possible abnormal oscillation in population density. The global attractiveness result in Theorem 6 implied that system (1) is permanent if the intrinsic growth rate of the prey and the conversion rate and the interference rate of the predator are high, and the death rate of the predator is low. From Theorem 7 we see that if the death rate of the predator is greater than the conversion rate of the predator, the predator population become extinct for any gestation delay. In particular, the results about boundedness and attractiveness are similar to the results of [4]. From the discussion in Sections 3 and 4, we see that if the values of r_1, r_2, a_{11}, a_{12}, a_{21}, and m are given, we can get the Hopf bifurcation value of τ, and further we may determine the direction of Hopf bifurcation and the stability of periodic solutions bifurcating from the positive equilibrium E^* at the critical point τ_0.

Furthermore, we show that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of delay.

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Research Article

Boundedness of Solutions for a Class of Second-Order Periodic Systems

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In this paper we study the following second-order periodic system: $x'' + V'(x) + p(x,t) = 0$, where $V(x)$ has a singularity. Under some assumptions on the $V(x)$ and $p(x,t)$ by Ortega' small twist theorem, we obtain the existence of quasi-periodic solutions and boundedness of all the solutions.

1. Introduction and Main Result

In 1991, Levi [1] considered the following equation:

$$x'' + V'(x,t) = 0, \tag{1}$$

where $V(x,t)$ satisfies some growth conditions and $V(x,t) = V(x,t+1)$. The author reduced the system to a normal form and then applied Moser twist theorem to prove the existence of quasi-periodic solution and the boundedness of all solutions. This result relies on the fact that the nonlinearity $V(x,t)$ can guarantee the twist condition of KAM theorem. Later, several authors improved Levi's result; we refer to [2–4] and the references therein.

Recently, Capietto et al. [5] studied the following equation:

$$x'' + V'(x) = F(x,t), \tag{2}$$

wher $F(x,t) = p(t)$ is a π-periodic function and $V(x) = (1/2)x_+^2 + (1/(1-x_-^2)^\nu) - 1$, where $x_+ = \max\{x,0\}$, $x_- = \max\{-x,0\}$, and ν is a positive integer. Under the Lazer-Leach assumption that

$$1 + \frac{1}{2}\int_0^\pi p(t_0+\theta)\sin\theta d\theta > 0, \quad \forall t_0 \in R, \tag{3}$$

they prove the boundedness of solutions and the existence of quasi-periodic solution by Moser twist theorem. It is the first time that the equation of the boundedness of all solutions is treated in case of a singular potential.

We observe that $F(x,t) = p(t)$ in (2) is smooth and bounded, so a natural question is to find sufficient conditions on $F(x,t)$ such that all solutions of (2) are bounded when $F(x,t)$ is unbounded. The purpose of this paper is to deal with this problem.

Motivated by the papers [1, 5, 6], we consider the following equation:

$$x'' + V'(x) + p(x,t) = 0, \tag{4}$$

where

$$V = \frac{1}{2}x_+^2 + \frac{1}{1-x_-^2} - 1, \quad x > -1. \tag{5}$$

In order to state our main results, we give some notation and assumptions. Let $\alpha \in (0,1)$ be some fixed constant. Let

$$\overline{p} = \frac{p(x,t)}{|x|^\alpha}, \qquad P(x,t) = \int_0^x p(s,t)\,ds. \tag{6}$$

(A1) Assume $p(x,t) \in C^{7,6}(S^1 \times R)$ and $\lim_{x\to+\infty}\overline{p}(x,t) = \overline{p}_+(t)$ uniformly in t.

(A2) $\lim_{x\to+\infty} x^m(\partial^{m+n}\overline{p}(x,t)/\partial x^m \partial t^n) = \overline{p}_{+,m,n}(t)$ uniformly in t for $(m,n) = (6,0),(6,7),(0,7)$, where $\overline{p}_{+,m,0}(t) \equiv 0$ and $\overline{p}_{+,6,7}(t) \equiv 0$.

(A3) We suppose Lazer-Leach assumption holds:

$$\int_0^{\pi} \overline{p}_+ (t_0 + \theta) (\sin\theta)^{1+\alpha} d\theta > 0, \quad \forall t_0 \in R. \tag{7}$$

Our main result is the following theorem.

Theorem 1. *Under the assumptions (A1)–(A3), all the solutions of* (4) *are defined for all* $t \in (-\infty, +\infty)$, *and for each solution* $x(t)$, *one has* $\sup_{t\in R}(|x(t)| + |x'(t)|) < +\infty$.

The main idea of our proof is acquired from [6]. The proof of Theorem 1 is based on a small twist theorem due to Ortega [7]. The hypotheses (A1)–(A3) of our theorem are used to prove that the Poincaré mapping of (4) satisfies the assumptions of Ortega's theorem.

Moreover, we have the following theorem on solutions of Mather type.

Theorem 2. *Assume that* $p(t) \in C$ *satisfies* (7); *then, there is an* $\epsilon_0 > 0$ *such that, for any* $\omega \in (1/\pi, 1/(\pi + \epsilon_0))$, (4) *has a solution* $(x_\omega(t), x'_\omega(t))$ *of Mather type with rotation number* ω. *More precisely,*

Case 1 *(*$\omega = p/q$ *is rational).* The solutions $(x_\omega(t+2i\pi), x'_\omega(t+2i\pi))$, $1 \le i \le q-1$, are mutually unlinked periodic solution of periodic $q\pi$; moreover, in this case,

$$\lim_{q\to\infty} \min_{t\in R} |x_\omega(t)| + |x'_\omega(t)| = +\infty. \tag{8}$$

Case 2 *(*ω *is irrational).* The solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one.

2. Proof of Theorem

2.1. Action-Angle Variables and Some Estimates. Observe that (4) is equivalent to the following Hamiltonian system:

$$x' = \frac{\partial H}{\partial y}, \qquad y' = -\frac{\partial H}{\partial x} \tag{9}$$

with the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}y^2 + V(x) + P(x, t). \tag{10}$$

In order to introduce action and angle variables, we first consider the auxiliary autonomous equation:

$$x' = y, \qquad y' = -V'(x), \tag{11}$$

which is an integrable Hamiltonian system with Hamiltonian function

$$H_1(x, y, t) = \frac{1}{2}y^2 + V(x). \tag{12}$$

The closed curves $H_1(x, y, t) = h > 0$ are just the integral curves of (11).

Denote by $T_0(h)$ the time period of the integral curve Γ_h of (11) defined by $H_1(x, y, t) = h$ and by I the area enclosed by the closed curve Γ_h for every $h > 0$. Let $-1 < -\alpha_h < 0 < \beta_h$ be such that $V(-\alpha_h) = V(\beta_h) = h$. It is easy to see that

$$\begin{aligned} I_0(h) &= 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h - V(s))}ds, \quad \forall h > 0, \\ T_0(h) &= I'_0(h) = 2\int_{\alpha_h}^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}}ds, \quad \forall h > 0. \end{aligned} \tag{13}$$

By direct computation we get

$$\begin{aligned} I_0(h) &= 2\int_0^{\beta_h} \sqrt{2(h - V(s))}ds + 2\int_{-\alpha_h}^{0} \sqrt{2(h - V(s))}ds \\ &= \pi h + 2\int_0^{\alpha_h} \sqrt{2(h - V(-s))}ds, \end{aligned} \tag{14}$$

so

$$T_0(h) = \pi + \int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}ds. \tag{15}$$

We then have

$$I_0(h) = I_-(h) + I_+(h), \qquad T_0(h) = T_-(h) + T_+(h), \tag{16}$$

where

$$\begin{aligned} I_-(h) &= 2\int_0^{-\alpha_h} \sqrt{2(h - V(s))}ds, \qquad I_+(h) = \pi h, \\ T_-(h) &= 2\int_0^{-\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}ds, \qquad T_+(h) = \pi. \end{aligned} \tag{17}$$

We now give the estimates on the functions I_- and T_-.

Lemma 3. *One has*

$$\begin{aligned} h^n \left| \frac{d^n T_-(h)}{dh^n} \right| &\le Ch^{-1/2}, \\ h^n \left| \frac{d^n I_-(h)}{dh^n} \right| &\le Ch^{1/2}, \end{aligned} \tag{18}$$

where $n = 0, 1, \ldots, 6$, $h \to \infty$. *Note that here and below one always uses* C, C_0, *or* C'_0 *to indicate some constants.*

Proof. Now we estimate the first inequality. We chose $V(s)/h = \eta$ as the new variable of integration; then we have

$$\begin{aligned} T_-(h) &= \int_{-\alpha_h}^{0} \frac{1}{\sqrt{2(h - V(s))}}ds \\ &= \int_0^1 \frac{\sqrt{h}}{V'(s(\eta, h))} \frac{1}{\sqrt{2(1-\eta)}}d\eta. \end{aligned} \tag{19}$$

Since $V(s) = (1/(1-s^2)) - 1$ and $V(s)/h = \eta$, we have $s = \sqrt{\eta h/(1+\eta h)}$. By direct computation, we have

$$V'(s) = \frac{2s}{(1-s^2)^2} = \frac{2\sqrt{\eta h}(1+\eta h)^2}{\sqrt{1+\eta h}}, \tag{20}$$

and then we get

$$T_{-}^{(n)}(h) = \frac{(-3/2)!}{((-3/2)-n)!}\int_0^1 \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta, \quad n = 0, 1, \ldots, 6. \tag{21}$$

When $0 \le \eta \le h^{-1}$ and h is sufficiently large, there exits C_0 such that $1-\eta > C_0$, so we have

$$\begin{aligned}\int_0^{h^{-1}} &\frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta \\ &\le C\int_0^{h^{-1}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}}d\eta \\ &\le \frac{C}{C_0}\int_0^{h^{-1}} \eta^{n-(1/2)}d\eta \le Ch^{-(1/2)-n}.\end{aligned} \tag{22}$$

Since $h^{-2/3} \le \eta \le 1$, we have

$$h^{1/3} < 1 + h^{1/3} \le 1 + \eta h \le 1 + h, \tag{23}$$

and then

$$\begin{aligned}\int_{h^{-2/3}}^1 &\frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta \\ &\le C\int_{h^{-2/3}}^1 \frac{\eta^n h^n}{\sqrt{2\eta(1-\eta)}h^n(1+\eta h)^n(1+\eta h)^{3/2}}d\eta \\ &\le C\int_{h^{-2/3}}^1 \frac{1}{\sqrt{2\eta(1-\eta)}h^n(1+\eta h)^{3/2}}d\eta \\ &\le C\int_{h^{-2/3}}^1 \frac{1}{\sqrt{2\eta(1-\eta)}h^n h^{1/2}}d\eta \\ &\le Ch^{(-1/2)-n}\int_0^1 \frac{1}{\sqrt{2\eta(1-\eta)}}d\eta \le Ch^{(-1/2)-n}.\end{aligned} \tag{24}$$

Observing that there is $C_0 > 0$ such that $\sqrt{1-\eta} \ge C_0$ when $h^{-1} \le \eta \le h^{-2/3}$ and $h \to +\infty$, we have

$$\begin{aligned}\int_{h^{-1}}^{h^{-2/3}} &\frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta \\ &\le C_1 h^{(-3/2)-n}\int_{h^{-1}}^{h^{-2/3}} \frac{1}{\sqrt{2\eta(1-\eta)}\eta^{3/2}}d\eta \\ &\le \frac{C_1}{C_0}h^{(-3/2)-n}\int_{h^{-1}}^{h^{-2/3}} \frac{1}{\eta^2}d\eta = \frac{C_1}{C_0}h^{(-3/2)-n}\left.\frac{1}{\eta}\right|_{h_{-1}}^{h^{-2/3}} \\ &= \frac{C_1}{C_0}h^{(-3/2)-n}\left(h - h^{2/3}\right) \le Ch^{-(1/2)-n}.\end{aligned} \tag{25}$$

By (22)–(25) we have $T_{-}^{(n)}(h) \le Ch^{(-1/2)-n}$, $n = 0, 1, \ldots, 6$.

The proof of the second inequality is similar to the first one, so we only give a brief proof.

We choose $V(s)/h = \eta$ as the new variable of integration, so we have

$$\begin{aligned}\frac{\partial s}{\partial h} &= \frac{\eta}{V'}, \qquad s = \sqrt{\frac{\eta h}{1+\eta h}}, \\ V'(s) &= \frac{2s}{(1-s^2)^2} = \frac{2\sqrt{\eta h}(1+\eta h)^2}{\sqrt{1+\eta h}}.\end{aligned} \tag{26}$$

By direct computation, we have

$$I_{-}(h) = 2\int_{-\alpha_h}^0 \sqrt{2(h - V(s))}ds = h\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}(1+\eta h)^{3/2}}d\eta. \tag{27}$$

By (27), we can easily get

$$\begin{aligned}I_{-}^{(n)}(h) &= I_{-1}^{(n)}(h) + I_{-2}^{(n)}(h) \\ &= n\frac{(-3/2)!}{((-3/2)-n+1)!} \\ &\quad\times \int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}}\frac{\eta^{n-1}}{(1+\eta h)^{(3/2)+n-1}}d\eta \\ &\quad + \frac{(-3/2)!}{((-3/2)-n)!}h\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}}\frac{\eta^n}{(1+\eta h)^{(3/2)+n}}d\eta,\end{aligned} \tag{28}$$

where $n = 0, 1, \ldots, 6$.

By a similar way to that used in estimating $T_{-}^{(n)}(h)$, we get

$$I_{-1}^{(n)}(h) \le Ch^{(1/2)-n}, \qquad I_{-2}^{(n)}(h) \le Ch^{(1/2)-n}, \tag{29}$$

which means that

$$I_{-}^{(n)}(h) \le Ch^{(1/2)-n}, \quad n = 0, 1, \ldots, 6. \tag{30}$$

Thus Lemma 3 is proved. □

Remark 4. It follows from the definitions of $T_{+}(h)$, $T_{-}(h)$ and Lemma 3 that

$$\lim_{h\to+\infty} T_{-}(h) = 0, \qquad \lim_{h\to+\infty} T_{+}(h) = \pi. \tag{31}$$

Thus the time period $T_0(h)$ is dominated by $T_{+}(h)$ when h is sufficiently large. From the relation between $T_{-}(h)$ and $I_{-}(h)$, we know $I_0(h)$ is dominated by $I_{+}(h)$ when h is sufficiently large.

Remark 5. It also follows from the definition of $I(h)$, $I_{-}(h)$, $I_{+}(h)$ and Remark 4 that

$$\left|h^n\frac{d^n I_0(h)}{dh^n}\right| \le C_0 I_0(h), \quad \text{for } n \ge 1. \tag{32}$$

Remark 6. Note that $h = h_0(I_0)$ is the inverse function of I_0. By Remark 5, we have

$$\left| I^n \frac{d^n h(I)}{dI^n} \right| \le C_0 h(I) \quad \text{for } n \ge 1. \tag{33}$$

We now carry out the standard reduction to the action-angle variables. For this purpose, we define the generating function $S(x, I) = \int_C \sqrt{2(h - V(s))}ds$, where C is the part of the closed curve Γ_h connecting the point on the y-axis and point (x, y).

We define the well-known map $(\theta, I) \to (x, y)$ by

$$y = \frac{\partial S}{\partial x}(x, I), \qquad \theta = \frac{\partial S}{\partial I}(x, I), \tag{34}$$

which is symplectic since

$$\begin{aligned} dx \wedge dy &= dx \wedge (S_{xx}dx + S_{xI}dI) = S_{xI}dx \wedge dI, \\ d\theta \wedge dI &= (S_{Ix}dx + S_{II}dI) \wedge dI = S_{Ix}d \wedge dI. \end{aligned} \tag{35}$$

From the above discussion, we can easily get

$$\theta = \begin{cases} \dfrac{\pi}{T_0(h(x,y))} \times \left(\dfrac{T_-(h(x,y))}{2} + \arcsin \dfrac{x}{\sqrt{2(h(x,y))}} \right), & \text{if } x > 0,\ y > 0, \\ \dfrac{\pi}{T_0(h(x,y))} \times \left(\dfrac{T_-(h(x,y))}{2} + \pi + \arcsin \dfrac{x}{\sqrt{2(h(x,y))}} \right), & \text{if } x > 0,\ y < 0, \\ \dfrac{\pi}{T_0(h(x,y))} \times \left(\displaystyle\int_{-\alpha_h}^{x} \dfrac{1}{\sqrt{2\left(h(x,y) + 1 - (1-s^2)^{-1}\right)}} ds \right), & \text{if } x < 0,\ y > 0, \\ \dfrac{\pi}{T_0(h(x,y))} \times \left(T_0(h(x,y)) - \displaystyle\int_{-\alpha_h}^{x} \dfrac{1}{\sqrt{2\left(h(x,y) + 1 - (1-s^2)^{-1}\right)}} ds \right), & \text{if } x < 0,\ y < 0, \end{cases}$$

$$I(x, y) = I_0(h(x, y)) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h(x, y) - V(s))}ds. \tag{36}$$

In the new variables (θ, I), the system (9) becomes

$$\theta' = \frac{\partial H}{\partial I}, \qquad I' = -\frac{\partial H}{\partial \theta}, \tag{37}$$

where

$$H(\theta, I, t) = \pi h_0(I) + \pi P(x(I, \theta), t). \tag{38}$$

In order to estimate $\pi P(I, \theta)$, we need the estimate on the function $x(I, \theta)$.

Lemma 7. *For I sufficiently large and $-\alpha_h \le x < 0$, the following estimates hold:*

$$\left| I^n \frac{\partial^n x(I, \theta)}{\partial I^n} \right| \le c\sqrt{I}, \quad \text{for } 0 \le n \le 6. \tag{39}$$

The Lemma was first proved in [1], and later [5] gives a different proof; [8] using the method of induction hypothesis also gives another proof. So, for concision, we omit the proof.

2.2. New Action and Angle Variables. Now we are concerned with the Hamiltonian system (37) with Hamiltonian function $H(\theta, I, t)$ given by (38). Note that

$$Id\theta - Hdt = -(Hdt - Id\theta). \tag{40}$$

This means that if one can solve I from (38) as a function of H (using θ and t as parameters), then

$$\frac{dH}{d\theta} = -\frac{\partial I}{\partial t}(t, H, \theta), \qquad \frac{dt}{d\theta} = \frac{\partial I}{\partial H}(t, H, \theta) \tag{41}$$

is also a Hamiltonian system with Hamiltonian function I and now the action, angle, and time variables are H, t, and θ.

From (38) and Lemma 3, we have

$$\frac{\partial H}{\partial I} \longrightarrow 1, \quad \text{as } I \longrightarrow +\infty. \tag{42}$$

So we assume that I can be written as

$$I = I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right), \tag{43}$$

where R satisfies $|R| < H/\pi$. Recalling that h_0 is the inverse function of I_0, we have

$$\frac{H}{\pi} + R(H, t, \theta) = h_0(I), \tag{44}$$

which implies that

$$R(H, t, \theta) = P(x(I, \theta), t). \tag{45}$$

As a consequence, R is implicitly defined by

$$R(H, t, \theta) = P\left(x\left(I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right), \theta\right), t\right). \tag{46}$$

For the estimates of R, we need the following lemmas.

Lemma 8. *Let $f(x, t)$ and $g(x, t)$ be continuously differentiable for $(x, t) \in [0, +\infty) \times I$, where I is an interval of R. If*

(1) *$g(x, t) \to \infty$ as $x \to +\infty$, uniformly with respect to $t \in I$,*

(2) $f'_x(x,t)/g'_x(x,t) \to h(t)$ *as* $x \to +\infty$, *uniformly with respect to* $t \in I$,

then one has $f(x,t)/g(x,t) \to h(t)$ *as* $x \to +\infty$, *uniformly with respect to* $t \in I$.

Proof. For any $0 < \epsilon < 1$, there exits X_1, such that if $x > X_1$, we have

$$\left|\frac{f'_x(x,t)}{g'_x(x,t)} - h(t)\right| \le \frac{\epsilon}{2}, \quad \forall t \in I. \tag{47}$$

Let $x_0 = X_1 + 1$. Then by Lagrangian differential mean value theorem, it follows that, for all $x > X_1$, we have

$$\left|\frac{f(x,t) - f(x_0,t)}{g(x,t) - g(x_0,t)} - h(t)\right| \le \frac{\epsilon}{2}, \quad \forall t \in I. \tag{48}$$

Moreover, there exists a constant $M > 0$ such that

$$\left|g(x_0,t)\frac{f(x,t) - f(x_0,t)}{g(x,t) - g(x_0,t)} - f(x_0,t)\right| \le M, \quad \forall t \in I. \tag{49}$$

By condition (A1), there exists $X > X_1$; we have $|g(x,t)| > 2M/\epsilon$.

Thus

$$\begin{aligned}\left|\frac{f(x,t)}{g(x,t)} - h(t)\right| &\le \left|\frac{f(x,t) - f(x_0,t)}{g(x,t) - g(x_0,t)} - h(t)\right| \\ &+ \left|\frac{1}{g(x,t)}\left(g(x_0,t)\frac{f(x,t) - f(x_0,t)}{g(x,t) - g(x_0,t)} - f(x_0,t)\right)\right| < \epsilon.\end{aligned} \tag{50}$$

□

Lemma 9. *Under the assumptions (A1) and (A2), the following results hold:*

(1) $\lim_{x\to+\infty}(x^m(\partial^{m+n}\overline{p}(x,t)/\partial x^m\partial t^n)) = \overline{p}_{+,m,n}(t)$ *uniformly in* t, *for* $0 \le m \le 6$ *and* $0 \le n \le 7$, *where* $\overline{p}_{+,m,n}(t) = 0$ *for* $1 \le m \le 6$, $0 \le n \le 7$ *and* $\overline{p}_{+,0,n}(t) = \overline{p}^{(n)}(t)$ *for* $0 \le n \le 7$.

(2) $\lim_{x\to+\infty}(x^m(\partial^{m+n}/\partial x^m\partial t^n)(P(x,t)/|x|^\sigma x)) = (1/(\sigma+1))\overline{p}_{+,m,n}(t)$, *for* $0 \le m \le 6$ *and* $0 \le n \le 7$, *where* $\overline{p}_{+,7,n}(t) = 0$ *for* $0 \le n \le 7$.

Proof. Result (1) is similar to Lemma 2.1 in [9], so we omit the proof.

For result (2), we first prove that, for $1 \le m \le 6$ and $0 \le n \le 7$,

$$\begin{aligned}\lim_{x\to+\infty} x^m \frac{(\partial^{m+n}/\partial x^m\partial t^n)\, p(t,x)}{|x|^\sigma} \\ = \sigma\cdots(\sigma - m + 1)\,\overline{p}_+^{(n)}(t).\end{aligned} \tag{51}$$

For $m = 1$ and $0 \le n \le 7$, by result (1), we have that

$$\lim_{x\to+\infty} x\left\{-\sigma\cdot\frac{(\partial^n/\partial t^n)\, p(t,x)}{|x|^\sigma\cdot x} + \frac{(\partial^{1+n}/\partial x\partial t^n)\, p(t,x)}{|x|^\sigma}\right\} = 0, \tag{52}$$

uniformly in t. It follows that

$$\lim_{x\to+\infty} x\cdot\frac{(\partial^{1+n}/\partial x\partial t^n)\, p(t,x)}{|x|^\sigma} = \sigma\cdot\overline{p}_+^{(n)}(t). \tag{53}$$

This means that (51) holds for $m = 1$. For $m = 2$, combining this with result (1) and the result of (51) for $m = 1$, we have that (51) still holds. Inductively we can prove that (51) holds for all $1 \le m \le 6$.

Obviously, it follows that

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\frac{P(t,x)}{|x|^\sigma\cdot x} - \frac{1}{1+\alpha}\cdot\overline{p}_+^{(n)}(t) \\ &\quad = \frac{\int_0^x\left[(\partial^n/\partial t^n)\, p(t,x) - \overline{p}_+^{(n)}(t)\,|s|^\sigma\right]ds}{|x|^\sigma\cdot x}.\end{aligned} \tag{54}$$

Using Lemma 8 and the first result (1) for $m = 0$, it follows that

$$\lim_{x\to+\infty}\frac{\partial^n}{\partial t^n}\frac{P(t,x)}{|x|^\sigma\cdot x} = \frac{1}{\sigma+1}\overline{p}_+^{(n)}(t), \tag{55}$$

for $0 \le n \le 7$. In a similar way to the proof of (51), we have

$$\begin{aligned}&\lim_{x\to+\infty} x^m\frac{\partial^{m+n}}{\partial x^m\partial t^n}\frac{P(x,t)}{|x|^\sigma x} \\ &\quad = \frac{1}{\sigma+1}\overline{p}_{+,m,n}(t), \quad \text{for } 0 \le m \le 6,\ 0 \le n \le 7,\end{aligned} \tag{56}$$

where $\overline{p}_{+,7,n}(t) = 0$. Thus we proved Lemma 9. □

Now we give the estimates of R. By Lemma 9, in a similar way to that for Lemma 2.3 in [5], we have the following lemma.

Lemma 10. *The function* $R(H,t,\theta)$ *satisfies the following estimates:*

$$\left|\frac{\partial^{m+l}R(H,t,\theta)}{\partial H^m\partial t^l}\right| \le H^{(\alpha+1)/2}, \quad \text{for } m + l \le 6. \tag{57}$$

Moreover, by the implicit function theorem, there exists a function $R_1 = R_1(t,H,\theta)$ such that

$$R(H,t,\theta) = P(x(H,\theta),t) + R_1(H,t,\theta). \tag{58}$$

Since

$$\begin{aligned}R_1(H,t,\theta) &= R(H,t,\theta) - P(x(H,\theta),t) \\ &= P\left\{x\left[I_0\left(\frac{H}{\pi} + R(H,t,\theta)\right),\theta\right],t\right\} \\ &\quad - P(x(H,\theta),t) \\ &= \int_0^1 p\{x[H + s(\pi R + I_-),\theta],t\} \\ &\quad\cdot\frac{\partial x}{\partial I}(H + s(\pi R + I_-),\theta)\cdot(\pi R + I_-)\,ds,\end{aligned} \tag{59}$$

by Lemmas 3 and 10, we have the estimates on $R_1(H,t,\theta)$.

Lemma 11. *Consider the following:*

$$\left|\frac{\partial^{k+l}R_1(H,t,\theta)}{\partial^k H\partial^l t}\right| < H^{\alpha/2} \quad \text{for } k+l\le 6. \tag{60}$$

For the estimate of $I((H/\pi)+R)$, we need the estimate on $I_-((H/\pi)+R)$. By Lemma 3 and noticing that $|R|<H/\pi$, we have the following lemma.

Lemma 12. *Consider the following:*

$$\left|\frac{\partial^{k+l}I_-((H/\pi)+R)}{\partial^k H\partial^l t}\right| < H^{1/2} \quad \text{for } k+l\le 6. \tag{61}$$

Now the new Hamiltonian function $I = I(t,H,\theta)$ is written in the form

$$\begin{aligned} I &= I_0\left(\frac{H}{\pi}+R\right) = I_+\left(\frac{H}{\pi}+R\right) + I_-\left(\frac{H}{\pi}+R\right) \\ &= H+\pi R(H,t,\theta) + I_-\left(\frac{H}{\pi}+R\right) \\ &= H+\pi P(x(H,\theta),t) + R_1(H,t,\theta) + I_-\left(\frac{H}{\pi}+R\right). \end{aligned} \tag{62}$$

The system (41) is of the form

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{\partial I}{\partial H} = 1+\pi\frac{\partial x}{\partial H}(H,\theta)\,p(x(H,\theta),t) \\ &\quad + \frac{\partial R_1}{\partial H}(H,t,\theta) + \frac{\partial I_-}{\partial H}(H,t,\theta), \\ \frac{dH}{d\theta} &= -\frac{\partial I}{\partial t} = -\pi\frac{\partial P}{\partial t}(x(\theta,H),t) \\ &\quad - \frac{\partial R_1}{\partial t}(t,H,\theta) - \frac{\partial I_-}{\partial t}(H,t,\theta). \end{aligned} \tag{63}$$

Introduce a new action variable $\rho\in[1,2]$ and a parameter $\epsilon>0$ by $H=\epsilon^{-2}\rho$. Then, $H\gg 1 \Leftrightarrow 0<\epsilon\ll 1$. Under this transformation, the system (63) is changed into the form

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{\partial I}{\partial H} = 1+\pi\frac{\partial x}{\partial H}(H,\theta)\,p(x(H,\theta),t) \\ &\quad + \frac{\partial R_1}{\partial H}(H,t,\theta) + \frac{\partial I_-}{\partial H}(H,t,\theta), \\ \frac{d\rho}{d\theta} &= -\frac{\partial I}{\partial t} = -\epsilon^2\left[\pi\frac{\partial P}{\partial t}(x(\theta,H),t)\right. \\ &\quad \left. + \frac{\partial R_1}{\partial t}(H,t,\theta) + \frac{\partial I_-}{\partial t}(H,t,\theta)\right], \end{aligned} \tag{64}$$

which is also Hamiltonian system with the new Hamiltonian function

$$\begin{aligned} \Gamma(t,\rho,\theta;\epsilon) &= \rho+\pi\epsilon^{-2}P\left(x\left(\theta,\epsilon^{-2}\rho\right),t\right) \\ &\quad + \epsilon^{-2}R_1\left(\epsilon^{-2}\rho,\theta,t\right) + \epsilon^{-2}I_-\left(\epsilon^{-2}\rho,\theta,t\right). \end{aligned} \tag{65}$$

Obviously, if $\epsilon\ll 1$, the solution $(t(\theta,t_0,\rho_0),\rho(\theta,t_0,\rho_0))$ of (64) with the initial date $(t_0,\rho_0)\in R\times[1,2]$ is defined in the interval $\theta\in[0,2\pi]$ and $\rho(\theta,t_0,\rho_0)\in[(1/2),3]$. So the Poincaré map of (64) is well defined in the domain $R\times[1,2]$.

Lemma 13 (see [6] Lemma 5.1). *The Poincaré map of* (64) *has intersection property.*

The proof is similar to the corresponding one in [6].

For convenience we introduce the notation $O_k(1)$ and $o_k(1)$. We say a function $f(t,\rho,\theta,\epsilon)\in O_k(1)$ if f is smooth in (t,ρ) and, for $k_1+k_2\le k$,

$$\left|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial\rho^{k_2}}f(t,\rho,\theta,\epsilon)\right| \le C, \tag{66}$$

for some constant $C>0$ which is independent of the arguments t, ρ, θ, and ϵ.

Similarly, we say $f(t,\rho,\theta,\epsilon)\in o_k(1)$ if f is smooth in (t,ρ) and, for $k_1+k_2\le k$,

$$\lim_{\epsilon\to 0}\left|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial\rho^{k_2}}f(t,\rho,\theta,\epsilon)\right| = 0, \tag{67}$$

uniformly in (t,ρ,θ).

2.3. Poincaré Map and Twist Theorems. We will use Ortega' small twist theorem to prove that the Pioncaré map P has an invariant closed curve, if ϵ is sufficiently small. Let us first recall the theorem in [7].

Lemma 14 (Ortega's Theorem). *Let $A=\mathbb{S}^1\times[a,b]$ be a finite cylinder with universal cover $\mathbb{A}=\mathbb{R}\times[a,b]$. The coordinate in $\mathbb{A}$ is denoted by (τ,ν). Consider a map*

$$\overline{f}: A\longrightarrow \mathbb{S}\times\mathbb{R}. \tag{68}$$

One assumes that the map has the intersection property. Suppose that $f: A\to\mathbb{R}\times\mathbb{R}$, $(\tau_0,\nu_0)\to(\tau_1,\nu_1)$ is a lift of $\overline{f}$ and it has the form

$$\begin{aligned} \tau_1 &= \tau_0+2N\pi+\delta l_1(\tau_0,\nu_0)+\delta\widetilde{g}_1(\tau_0,\nu_0), \\ \nu_1 &= \nu_0+\delta l_2(\tau_0,\nu_0)+\delta\widetilde{g}_2(\tau_0,\nu_0), \end{aligned} \tag{69}$$

where N is an integer and $\delta\in(0,1)$ is a parameter. The functions l_1, l_2, $\widetilde{g}_1$, and $\widetilde{g}_2$ satisfy

$$\begin{aligned} &l_1\in C^6(A), \qquad l_1(\tau_0,\nu_0)>0, \qquad \frac{\partial l_1}{\partial\nu_0}(\tau_0,\nu_0)>0, \\ &\qquad\qquad \forall(\tau_0,\nu_0)\in A, \\ &l_2(\cdot,\cdot),\widetilde{g}_1(\cdot,\cdot,\epsilon),\widetilde{g}_2(\cdot,\cdot,\epsilon)\in C^5(A). \end{aligned} \tag{70}$$

In addition, one assumes that there is a function $I: A\to R$ satisfying

$$\begin{aligned} &I\in C^6(A), \qquad \frac{\partial I}{\partial\nu_0}(\tau_0,\nu_0)>0, \quad \forall(\tau_0,\nu_0)\in A, \\ &l_1(\tau_0,\nu_0)\cdot\frac{\partial I}{\partial\tau_o}(\tau_0,\nu_0)+l_2(\tau_0,\nu_0)\cdot\frac{\partial I}{\partial\nu_0}(\tau_0,\nu_0)=0, \\ &\qquad\qquad \forall(\tau_0,\nu_0)\in A. \end{aligned} \tag{71}$$

Moreover, suppose that there are two numbers $\tilde{a}$ and $\tilde{b}$ such that $a < \tilde{a} < \tilde{b} < b$ and

$$I_M(a) < I_m(\tilde{a}) \le I_M(\tilde{a}) < I_m(\tilde{b}) \le I_M(\tilde{b}) < I_m(b), \quad (72)$$

where

$$I_M(r) = \max_{\rho\in S^1} I(\rho_o, \tau_o), \qquad I_m(r) = \min_{\rho\in S^1} I(\rho_o, \tau_o). \quad (73)$$

Then there exist $\epsilon > 0$ and $\Delta > 0$ such that, if $\delta < \Delta$ and

$$\|\tilde{g}_1(\cdot,\cdot,\epsilon)\|_{C^5(A)} + \|\tilde{g}_2(\cdot,\cdot,\epsilon)\|_{C^5(A)} < \epsilon, \quad (74)$$

the mapping $\overline{f}$ has an invariant curve in Γ_A. The constant ϵ is independent of δ.

We make the ansatz that the solution of (64) with the initial condition $(t(0), \rho(0)) = (t_0, \rho_0)$ is of the form

$$\begin{aligned} t &= t_0 + \theta + \epsilon^{1-\alpha}\Sigma_1(t_0, \rho_0, \theta; \epsilon), \\ \rho &= \rho_0 + \epsilon^{1-\alpha}\Sigma_2(t_0, \rho_0, \theta; \epsilon). \end{aligned} \quad (75)$$

Then, the Poincaré map of (64) is

$$\begin{aligned} P: t_1 &= t_0 + 2\pi + \epsilon^{1-\alpha}\Sigma_1(t_0, \rho_0, 2\pi; \epsilon), \\ \rho_1 &= \rho_0 + \epsilon^{1-\alpha}\Sigma_2(t_0, \rho_0, 2\pi; \epsilon). \end{aligned} \quad (76)$$

The functions Σ_1 and Σ_2 satisfy

$$\begin{aligned} \Sigma_1 &= \pi\epsilon^{\alpha-1}\int_0^\theta \frac{\partial x}{\partial H}(\epsilon^{-2}\rho, \theta)\, p(x(\epsilon^{-2}\rho, \theta), t)\, d\theta \\ &\quad + \epsilon^{\alpha-1}\int_0^\theta \left(\frac{\partial R_1}{\partial H}(\epsilon^{-2}\rho, t, \theta) + \frac{\partial I_-}{\partial H}(\epsilon^{-2}\rho, t, \theta)\right) d\theta, \\ \Sigma_2 &= -\pi\epsilon^{\alpha+1}\int_0^\theta \frac{\partial P}{\partial t}(x(\epsilon^{-2}\rho, \theta), t)\, d\theta \\ &\quad - \epsilon^{\alpha+1}\int_0^\theta \left(\frac{\partial R_1}{\partial t}(\epsilon^{-2}\rho, t, \theta) - \frac{\partial I_-}{\partial t}(\epsilon^{-2}\rho, t, \theta)\right) d\theta, \end{aligned} \quad (77)$$

where $t = t_0 + \theta + \epsilon^{1-\alpha}\Sigma_1$ and $\rho = \rho_0 + \epsilon^{1-\alpha}\Sigma_2$. By Lemmas 9, 11, and 12, we know that

$$|\Sigma_1| + |\Sigma_2| \le C \quad \text{for } \theta \in [0, 2\pi]. \quad (78)$$

Hence, for $\rho_0 \in [1, 2]$, we may choose ϵ sufficiently small such that

$$\rho_0 + \epsilon\Sigma_2 \ge \frac{\rho_0}{2} \ge \frac{1}{2}. \quad (79)$$

Moreover we can prove that

$$\Sigma_1, \Sigma_2 \in O_6(1). \quad (80)$$

In a similar way to that used for estimating R_1, by direct calculation we have the following lemma.

Lemma 15. *The following estimates hold:*

$$\begin{aligned} &P(x(\epsilon^{-2}\rho, \theta), t) - P(x(\epsilon^{-2}\rho_0, \theta), t_0) \in \epsilon^{-\alpha}O_6(1), \\ &\frac{\partial P}{\partial t}(x(\epsilon^{-2}\rho, \theta), t) - \frac{\partial P}{\partial t}(x(\epsilon^{-2}\rho_0, \theta), t_0) \in \epsilon^{2-\alpha}O_6(1). \end{aligned} \quad (81)$$

Now we give an asymptotic expression of Poincaré map of (63); that is, we study the behavior of the functions Σ_1 and Σ_2 at $\theta = \pi$ as $\epsilon \to 0$. In order to estimate Σ_1 and Σ_2, we need to introduce the following definition and lemma. Let

$$\begin{aligned} \Theta_+(I) &= \text{meas}\{\theta \in [0, \pi], x(H_0, \theta) > 0\}, \\ \Theta_-(I) &= T_0 - \Theta_+(I), \end{aligned} \quad (82)$$

where $H_0 = \epsilon^{-2}\rho_0$.

Lemma 16. *Consider the following:*

$$\Theta_+(I) = \pi + \epsilon O_6(1), \qquad \Theta_-(I) = \epsilon O_6(1). \quad (83)$$

Proof. This Lemma was proved in [5], so we omit the details.

For estimate Σ_1 and Σ_2, we need the estimates of x and x_H.

We recall that, when $x < 0$, we have

$$|x(H_0, \theta)| = O_6(1), \qquad |x_H(H_0, \theta)| = \epsilon^2 O_5(1). \quad (84)$$

When $x > 0$, by the definition of θ, we have

$$\arcsin\frac{x(H_0, \theta)}{\sqrt{2h}} = \frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2} = \theta + \epsilon^2 O_5(1), \quad (85)$$

which yields that

$$\begin{aligned} x(H_0, \theta) &= \sqrt{\frac{2H_0}{\pi}}\sin\theta + O_5(1), \\ x_H(H_0, \theta) &= \sqrt{\frac{1}{2H_0\pi}}\sin\theta + \epsilon^2 O_5(1). \end{aligned} \quad (86)$$

Now we can give the estimates of Σ_1 and Σ_2. □

Lemma 17. *The following estimates hold true:*

$$\Sigma_1(t_0,\rho_0,2\pi;\epsilon)=\left(\frac{\pi}{2\rho_0}\right)^{(\alpha-1)/2}\times\int_0^{\pi}(\sin\theta)^{1+\alpha}\overline{p}_+(t_0+\theta)\,d\theta+o_6(1),\tag{87}$$

$$\Sigma_2(t_0,\rho_0,2\pi;\epsilon)=-\pi^{(1-\alpha)/2}(2\rho_0)^{(\alpha+1)/2}\times\int_0^{\pi}(\sin\theta)^{1+\alpha}\overline{p}_+'(t_0+\theta)\,d\theta+o_6(1),\tag{88}$$

for $\epsilon\to 0$.

Proof. Firstly we consider Σ_1. By Lemmas 11, 12, and 15 and (77), we have

$$\begin{aligned}\Sigma_1(t_0,\rho_0,2\pi;\epsilon)&=\pi\epsilon^{\alpha-1}\int_0^{\pi}\frac{\partial x}{\partial H}\left(\epsilon^{-2}\rho,\theta\right)p\left(x\left(\epsilon^{-2}\rho,\theta\right),t\right)d\theta\\&\quad+\epsilon^{\alpha-1}\int_0^{\pi}\frac{\partial R_1}{\partial H}\left(x\left(\epsilon^{-2}\rho,\theta\right),t\right)\\&\quad+\frac{\partial I_-}{\partial H}\left(x\left(\epsilon^{-2}\rho,\theta\right),t\right)d\theta\\&=\pi\epsilon^{\alpha-1}\int_0^{\pi}\frac{\partial x}{\partial H}\left(\epsilon^{-2}\rho_0,\theta\right)p\left(x\left(\epsilon^{-2}\rho_0,\theta\right),t_0+\theta\right)d\theta\\&\quad+\epsilon^{\alpha}O_6(1)\\&=\pi\epsilon^{\alpha-1}\int_{\Theta_+}\frac{\partial x}{\partial H}\left(\epsilon^{-2}\rho_0,\theta\right)p\left(x\left(\epsilon^{-2}\rho_0,\theta\right),t_0+\theta\right)d\theta\\&\quad+\pi\epsilon^{\alpha-1}\int_{\Theta_-}\frac{\partial x}{\partial H}\left(\epsilon^{-2}\rho_0,\theta\right)p\left(x\left(\epsilon^{-2}\rho_0,\theta\right),t_0+\theta\right)d\theta\\&\quad+\epsilon^{\alpha}O_6(1).\end{aligned}\tag{89}$$

By result (2) of Lemma 9, as $\epsilon\to 0$ which means $x\to\infty$,we have

$$\begin{aligned}&\pi\epsilon^{\alpha-1}\int_{\Theta_+}\frac{\partial x}{\partial H}\left(\epsilon^{-2}\rho_0,\theta\right)p\left(x\left(\epsilon^{-2}\rho_0,\theta\right),t_0+\theta\right)d\theta\\&\quad=\pi\epsilon^{\alpha-1}\int_{\Theta_+}\frac{\partial x}{\partial H}\left(\theta,\epsilon^{-2}\rho\right)|x|^{\alpha}\overline{p}_+(t_0+\theta)\,d\theta\\&\qquad+\epsilon^{\alpha}O_6(1).\end{aligned}\tag{90}$$

By the measure of Θ_-, we have

$$\pi\epsilon^{\alpha-1}\int_{\Theta_-}\frac{\partial x}{\partial H}\left(\epsilon^{-2}\rho_0,\theta\right)\times p\left(x\left(\epsilon^{-2}\rho_0,\theta\right),t_0+\theta\right)d\theta=\epsilon^{\alpha}O_6(1).\tag{91}$$

By (90) and (91), we have

$$\begin{aligned}\Sigma_1(t_0,\rho_0,2\pi;\epsilon)&=\pi\epsilon^{\alpha-1}\int_{\Theta_+}\frac{\partial x}{\partial H}\left(\theta,\epsilon^{-2}\rho\right)|x|^{\alpha}\overline{p}_+(t_0+\theta)\,d\theta+\epsilon^{\alpha}O_6(1)\\&=\pi\epsilon^{\alpha-1}\int_0^{\pi}\frac{\partial x}{\partial H}\left(\theta,\epsilon^{-2}\rho\right)|x|^{\alpha}\overline{p}_+(t_0+\theta)\,d\theta+\epsilon^{\alpha}O_6(1)\\&=\left(\frac{\pi}{2\rho_0}\right)^{(1-\alpha)/2}\int_0^{\pi}(\sin\theta)^{\alpha+1}\overline{p}_+(t_0+\theta)\,d\theta+o_6(1).\end{aligned}\tag{92}$$

Now we consider Σ_2. By Lemmas 11, 12, and 15 and (77), we have

$$\begin{aligned}\Sigma_2(t_0,\rho_0,2\pi;\epsilon)&=-\pi\epsilon^{\alpha+1}\int_0^{\pi}\frac{\partial P}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho\right),t\right)d\theta\\&\quad-\epsilon^{\alpha+1}\int_0^{\pi}\left[\frac{\partial R_1}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho\right),t\right)\right.\\&\qquad\left.+\frac{\partial I_-}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho\right),t\right)\right]d\theta\\&=-\pi\epsilon^{\alpha+1}\int_0^{\pi}\frac{\partial P}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho_0\right),t_0+\theta\right)d\theta+\epsilon^{\alpha}O_6(1)\\&=-\pi\epsilon^{\alpha+1}\int_{\Theta_+}\frac{\partial P}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho_0\right),t_0+\theta\right)d\theta\\&\quad-\pi\epsilon^{\alpha+1}\int_{\Theta_-}\frac{\partial P}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho_0\right),t_0+\theta\right)d\theta+\epsilon^{\alpha}O_6(1).\end{aligned}\tag{93}$$

By result (2) of Lemma 9, as $\epsilon\to 0$, we have

$$\begin{aligned}&-\pi\epsilon^{\alpha+1}\int_{\Theta_+}\frac{\partial P}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho_0\right),t_0+\theta\right)d\theta\\&\quad=-\frac{\pi\epsilon^{\alpha+1}}{\alpha+1}\int_{\Theta_+}\left|x\left(\theta,\epsilon^{-2}\rho_0\right)\right|^{\alpha}x\left(\theta,\epsilon^{-2}\rho_0\right)\overline{p}_+'(t_0+\theta)\,d\theta\\&\qquad+\epsilon^{\alpha}O_6(1).\end{aligned}\tag{94}$$

By the measure of Θ_-, we have

$$-\pi\epsilon^{\alpha+1}\int_{\Theta_-}\frac{\partial P}{\partial t}\left(x\left(\theta,\epsilon^{-2}\rho_0\right),t_0+\theta\right)d\theta=\epsilon^{\alpha}O_6(1).\tag{95}$$

By (94) and (95), we have

$$\begin{aligned}\Sigma_2 &= -\frac{\pi\epsilon^{\alpha+1}}{\alpha+1}\int_{\Theta_+}\left|x\left(\theta,\epsilon^{-2}\rho_0\right)\right|^{\alpha}x\left(\theta,\epsilon^{-2}\rho_0\right)\overline{p}'_+ \\ &\qquad\times\left(t_0+\theta\right)d\theta+\epsilon^{\alpha}O_6(1) \\ &= -\frac{\pi\epsilon^{\alpha+1}}{\alpha+1}\int_0^{\pi}\left|x\left(\theta,\epsilon^{-2}\rho_0\right)\right|^{\alpha}x\left(\theta,\epsilon^{-2}\rho_0\right)\overline{p}'_+ \\ &\qquad\times\left(t_0+\theta\right)d\theta+\epsilon^{\alpha}O_6(1) \\ &= -\frac{1}{\alpha+1}\pi^{(1-\alpha)/2}\left(2\rho_0\right)^{(\alpha+1)/2}\int_0^{\pi}(\sin\theta)^{1+\alpha}\overline{p}'_+ \\ &\qquad\times\left(t_0+\theta\right)d\theta+o_6(1).\end{aligned} \tag{96}$$

Thus Lemma 17 is proved. □

2.4. Proof of Theorem 1. Let

$$\begin{aligned}\Psi_1\left(t_0,\rho_0\right) &= \left(\frac{\pi}{2\rho_0}\right)^{(1-\alpha)/2} \\ &\qquad\times\int_0^{\pi}(\sin\theta)^{1+\alpha}\overline{p}_+\left(t_0+\theta\right)d\theta, \\ \Psi_2\left(t_0,\rho_0\right) &= -\frac{1}{\alpha+1}\pi^{(1-\alpha)/2}\left(2\rho_0\right)^{(\alpha+1)/2} \\ &\qquad\times\int_0^{\pi}(\sin\theta)^{1+\alpha}\overline{p}'_+\left(t_0+\theta\right)d\theta.\end{aligned} \tag{97}$$

Then there are two functions ϕ_1 and ϕ_2 such that the Poincaré map of (64), given by (76) of the form

$$\begin{aligned}P: t_1 &= t_0+2\pi+\epsilon^{1-\alpha}\Psi_1\left(t_0,\rho_0\right)+\epsilon^{1-\alpha}\phi_1, \\ \rho_1 &= \rho_0+\epsilon^{1-\alpha}\Psi_2\left(t_0,\rho_0\right)+\epsilon^{1-\alpha}\phi_2,\end{aligned} \tag{98}$$

where $\phi_1,\phi_2\in O_6(1)$.

Since $\int_0^{\pi}\overline{p}_+(t_0+\theta)\sin\theta d\theta>0$, $\forall t_0\in R$,we have

$$\Psi_1>0,\qquad \frac{\partial\Psi_1}{\partial\rho_0}\neq 0. \tag{99}$$

Let

$$L=\frac{\rho_0^{-(1+\alpha)/2}}{\int_0^{\pi}(\sin\theta)^{1+\alpha}\overline{p}_+\left(t_0+\theta\right)d\theta}. \tag{100}$$

Then

$$\frac{\partial L}{\partial t_0}\Psi_1\left(t_0,\rho_0\right)+\frac{\partial L}{\partial\rho_0}\Psi_2\left(t_0,\rho_0\right)=0. \tag{101}$$

The other assumptions of Ortega's theorem are easily verified. Hence, there is an invariant curve of P in the annulus $(t_0,\rho_0)\in S^1\times[1.2]$ which implies the boundedness of our original equation (4). Then Theorem 1 is proved.

2.5. Proof of Theorem 2. We apply Aubry-Mather theory. By Theorem B in [10] and the monotone twist property of the Pioncaré map P guaranteed by $\partial\Psi_1/\partial\rho_0<0$, it is straightforward to check that Theorem 2 is correct.

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Research Article

Moment Equations in Modeling a Stable Foreign Currency Exchange Market in Conditions of Uncertainty

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The paper develops a mathematical model of foreign currency exchange market in the form of a stochastic linear differential equation with coefficients depending on a semi-Markov process. The boundaries of the domain of its instability is determined by using moment equations.

1. Introduction

The economic growth of a given country is based on the government policy that includes numerous control moments. An important part of this policy is the correct financial policy, which defines the priorities in the development of financial relations and its function is to ensure the financial stability of the state. Finance and energy markets have been an active scientific field for some time, even though the development and applications of sophisticated quantitative methods in these areas are relatively new and referred to in a broader context as energy finance. Energy finance is often viewed as a branch of mathematical finance, yet this area continues to provide a rich source of issues that are fuelling new and exciting research developments [1].

The foreign currency exchange market is one of the most liquid financial markets with banks as major participants. Income from the foreign currency exchange transactions makes up a significant proportion of the banks income. The currency exchange risks associated with open positions are especially imminent in periods of significant fluctuations in exchange rates. The main feature of risky cases related to the market risk is that such cases occur as a result of adverse changes in the general market situation. Whenever such cases occur, the value of the assets has a tendency to decrease for a short-term period, causing liquidity gap.

In view of the disbalance of the foreign currency exchange market, the negative trade balance, the high inflation, an effective foreign currency exchange rate policy determining the optimal level of foreign currency exchange rates is an important problem.

Under such conditions, it is especially important to perceive the "bank" as a comprehensive dynamic system that works in the conditions of unstable economy under high foreign currency exchange risks. Thus, a more widespread use of economic-mathematical methods and models is necessary to study the processes taking place in the "bank", evaluating the effectiveness of its work and identifying the trends and ways to improve the management of the banking activities.

Significant scientific achievements in the field of banking and construction of some models can be found in [2, 3], and some economical models are studied in [4–6]. However, many other issues of bank practices require further research and elaboration of approaches to their solution. One aspect of the model is to build stable functioning of the foreign currency exchange transactions of the "bank" as a factor of effective functioning of the banking system in general [7, 8].

Most scientists understand under the category "financial stability of the banking system" the establishment of an effective mechanism preventing the emergence of banking crises and facilitating further development of economy. Depending

on the tasks, the stability of the banks may be defined as in the model presented in this paper.

The paper develops a stability model of foreign currency bank transactions with semi-Markov fluctuations. An example illustrates the theory in the special case when the semi-Markov process can take three possible states. This means that a commercial bank operates in a foreign currency exchange market that can be in three states: stable foreign currency exchange market, market in the crisis, and market with currency restrictions. In the example, we assume that the bank remains in each state for the same period of time.

In addition, the present paper contains the necessary and sufficient conditions for the mean square stability and conditions for the L_2-stability of systems with semi-Markov coefficients and random transformations of solutions. There are constructed moment equations as a tool for studying the stochastic system stability which is working in uncertainty conditions.

2. Statement of the Problem

Let $(\Omega, \mathscr{F}, F, \mathbb{P})$ be a filtered probability space (or stochastic basis) consisting of a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration

$$F = \{\mathscr{F}_t, \forall t \geq 0\} \subset \mathscr{F}. \tag{1}$$

The space Ω is called the sample space, $\mathscr{F}$ is the set of all possible events (the σ-algebra), and $\mathbb{P}$ is some probability measure on Ω. A family $\xi = \{\xi(t) : t \geq 0\}$ of random variables $\xi(t) : \Omega \to \mathbb{L}_2$ is called a continuous-time stochastic process on the state space $\mathbb{L}_2$. In our considerations, $\xi(t)$ is a random semi-Markov process and the state space $\mathbb{L}_2$ is the space of all random variables for which there exists squared mathematical expectation. On such a probability space, we consider initial problem formulated for the stochastic system of linear differential equations in the form

$$\frac{dx(t)}{dt} = A(\xi(t))\, x(t), \tag{2}$$

$$x(0) = \varphi(\omega), \tag{3}$$

where $A(\xi(t))$ is an $m \times m$ matrix whose elements depend on the semi-Markov process $\xi(t)$. The state function $x(t)$ is an m-dimensional column vector-function with the initial state $\varphi(\omega)$ at $t = 0$. For simplicity, we denote

$$A(\xi(t)) = A_k, \quad \text{if } \xi(t) = \theta_k,\ k = 1, 2, \ldots, n, \tag{4}$$

where A_k are constant $m \times m$ matrices and θ_k are real numbers.

An m-dimensional vector-function $x(t)$ is called a solution of the initial value problem (2) and (3) if $x(t)$ satisfies (2) and initial condition (3) within the meaning of a strong solution of the initial Cauchy problem.

Our considerations are subject to the following assumptions.

Assumption 1. The random semi-Markov process $\xi(t)$ can take n possible states

$$\theta_1, \theta_2, \ldots, \theta_n \tag{5}$$

with transition probabilities

$$\pi_{sk}(t) = P\left\{\xi\left(t_{j+1}\right) = \theta_s \mid \xi\left(t_j\right) = \theta_k, j = 1, 2, \ldots\right\}, \quad s, k = 1, 2, \ldots, n, \tag{6}$$

where $t_0 = 0 < t_1 < t_2 < \cdots$ are the moments of time at which the jump from one state to another is realized.

If we fix any moment $t > 0$, then the semi-Markov process takes some of states, $\xi(t) = \theta_k$, $k = 1, 2, \ldots, n$, and the state function $x(t) \equiv x(t, \omega)$ changes in accordance with the deterministic system of differential equations

$$\frac{dx(t)}{dt} = A_k x(t), \quad k = 1, 2, \ldots, n. \tag{7}$$

So the solution of such a system is in the form

$$x(t) = e^{A_k t} \varphi(\omega), \quad k = 1, 2, \ldots, n. \tag{8}$$

Assumption 2. The jumping time during which the process is in state θ_s before it jumps to state θ_k, $s, k = 1, 2, \ldots, n$ is given by a discrete integer-valued random variable T_{sk} whose probability density function is a known function $d_{sk}(t)$. Then, the intensity $q_{sk}(t)$ of the jump from state θ_s to state θ_k is given by the formula

$$q_{sk}(t) = \pi_{sk}(t)\, d_{sk}(t), \quad s, k = 1, 2, \ldots, n, \tag{9}$$

and the semi-Markov process $\xi(t)$ is defined by the transition intensity matrix

$$Q(t) = \left(q_{sk}(t)\right)_{s,k=1}^{n}, \tag{10}$$

whose elements satisfy the relationships

$$q_{sk}(t) \geq 0, \qquad \int_0^\infty \sum_{s=1}^{n} q_{sk}(t)\, dt = \int_0^\infty q_k(t)\, dt = 1, \tag{11}$$

where $q_k(t)$ is the probability density of the elapsed time T_k in state θ_k if the process jumps to it at time t_j. If $\psi_k(t)$ denotes the probability of the event that no jump takes place during the interval (t_j, t_{j+1}), provided that the process jumps to the state θ_k at time t_j, then

$$\psi_k(t) = \int_t^\infty q_k(\tau)\, d\tau, \quad k = 1, 2, \ldots, n. \tag{12}$$

In our considerations, it will be convenient to denote the block-diagonal matrix,

$$\Psi(t) = \operatorname{diag}\left(\psi_1(t), \psi_2(t), \ldots, \psi_n(t)\right). \tag{13}$$

Assumption 3. At the moments of jumps t_j, $j = 1, 2, \ldots$ that are caused by some perturbations, solutions of (2) submit to the random transformations

$$x\left(t_j + 0\right) = C_{sk} x\left(t_j - 0\right), \quad s, k = 1, \ldots, n, \tag{14}$$

where C_{sk} are $m \times m$ constant matrices and $\det C_{sk} \neq 0$.

Our aim is to transform the stochastic system with random coefficients to a deterministic system with solutions whose stability can be considered by using classical methods. To complete this task below, we present a method of moment equations. We will show that the method is effective and useful for solving an economical model problem.

3. Construction of the Moment Equations

We define the moments of the first or second order of a random variable x before we derive the moments equations. We use some notation. In the sequel, $\mathbb{E}_m$ denotes an m-dimensional Euclidean space, functions $f_k(t,x)$, $k = 1,2,\ldots,n$ are the particular density functions of the random variable x, and the vector-function

$$f(t,x) = \left(f_1(t,x), f_2(t,x), \ldots, f_n(t,x)\right)^T, \tag{15}$$

where the operation T denotes transposition, is called the vector of particular density functions. Moreover, we define

$$\begin{gathered} S(t) := \left(q_{sk}S_{sk}\right)_{s,k=1}^{n}, \quad s,k = 1,\ldots,n, \\ R(t) := \operatorname{diag}\left(R_1(t),\ldots,R_n(t)\right), \end{gathered} \tag{16}$$

where S_{sk} are operators defined, for a given function f, as

$$S_{sk}f(t,x) \equiv f_k\left(t, C_{sk}^{-1}x\right)\det C_{sk}^{-1} \tag{17}$$

and R_k, $k = 1,\ldots,n$ are operators defined, for a given function f, as

$$R_k(t)f(t,x) \equiv f_k\left(t, e^{-A_k t}x\right)\det\left(e^{-A_k t}\right). \tag{18}$$

Definition 4. Let $x \in \mathbb{E}_m$ be a continuous random variable depending on a random semi-Markov process $\xi(t)$ with n possible states θ_k, $k = 1,2,\ldots,n$. The n-dimensional column vectors $E^{(1)}\{x(t)\}$ and $n\times n$ matrices $E^{(2)}\{x(t)\}$ of the form

$$E^{(1)}\{x(t)\} = \sum_{k=1}^{n} E_k^{(1)}\{x(t)\}, \qquad E^{(2)}\{x(t)\} = \sum_{k=1}^{n} E_k^{(2)}\{x(t)\}, \tag{19}$$

where

$$\begin{gathered} E_k^{(1)}\{x(t)\} = \int_{\mathbb{E}_m} x f_k(t,x)\,dx, \\ E_k^{(2)}\{x(t)\} = \int_{\mathbb{E}_m} x x^T f_k(t,x)\,dx, \end{gathered} \tag{20}$$

are called moments of the first or second order of the random variable x, respectively. The values $E_k^{(1)}\{x(t)\}$ and $E_k^{(2)}\{x(t)\}$, $k = 1,\ldots,n$ are called particular moments of the first or second order, respectively.

Theorem 5. *Let the coefficients of the linear differential system* (2) *depend on a random semi-Markov process* $\xi(t)$ *with transition intensity matrix* (10) *and, for solutions of system* (2), *there occur jumps* (14) *simultaneously with jumps of the process* $\xi(t)$. *Then, the following three statements are true.*

(1) *The stochastic process* $(x(t),\xi(t))$ *is defined by the operator equation*

$$f(t,x) = L(t)f(0,x), \tag{21}$$

where the matrix operator $L(t) \equiv (L_{ij}(t))_{i,j=1}^{n}$ *satisfies*

$$L(t) = \psi(t)R(t) + \int_0^t L(t-\tau)S(\tau)R(\tau)\,d\tau. \tag{22}$$

(2) *The vectors of particular moments of first order satisfy*

$$\begin{aligned} E_k^{(1)}\{x(t)\} &= \psi_k(t)e^{A_k t}E_k^{(1)}\{x(0)\} \\ &\quad + \int_0^t \psi(t-\tau)e^{A_k(t-\tau)}Z_k(\tau)\,d\tau, \end{aligned} \tag{23}$$

where

$$\begin{aligned} Z_k(t) &= \sum_{s=1}^{n} q_{ks}(t)C_{ks}e^{A_s t}E_s^{(1)}\{x(0)\} \\ &\quad + \int_0^t \sum_{s=1}^{n} q_{ks}(t-\tau)C_{ks}e^{A_s(t-\tau)}Z_s(\tau)\,d\tau \end{aligned} \tag{24}$$

and $k = 1,\ldots,n$.

(3) *The matrix of particular moments of second order satisfies*

$$\begin{aligned} E_k^{(2)}\{x(t)\} &= \psi_k(t)e^{A_k t}E_k^{(2)}\{x(0)\}e^{A_k^T t} \\ &\quad + \int_0^t \psi_k(t-\tau)e^{A_k(t-\tau)}W_k(\tau)e^{A_k^T(t-\tau)}d\tau, \end{aligned} \tag{25}$$

where

$$\begin{aligned} W_k(t) &= \sum_{s=1}^{n} q_{ks}(t)C_{ks}e^{A_s t}E_s^{(2)}\{x(0)\}e^{A_s^T t}C_{ks}^T \\ &\quad + \int_0^t \sum_{s=1}^{n} q_{ks}(t-\tau)C_{ks}e^{A_s(t-\tau)}W_s(\tau)e^{A_s^T(t-\tau)}C_{ks}^T d\tau, \end{aligned} \tag{26}$$

and $k = 1,\ldots,n$.

Proof. (1) The stochastic process $(x(t),\xi(t))$ is also semi-Markov because all probabilistic properties of the process for $t > t_j$ are defined by particular probability density functions at the moment t_j of jump. Thus, there exists a linear operator $L(\tau)$ such that

$$f\left(t_j+\tau, x\right) = L(\tau)f\left(t_j, x\right), \quad j = 0,1,2,\ldots. \tag{27}$$

Let the stochastic process $\xi(t)$ move to state θ_k at the moment $t = 0$. Then,

$$p_k(0) = 1, \quad p_s(0) = 0, \quad \text{for } s \neq k,\ s = 1,\ldots,n, \tag{28}$$

$$f_k(0,x) \geq 0, \qquad \int_{E_m} f_k(0,x)\,dx = 1, \qquad f_s(0,x) \equiv 0, \quad s \neq k. \tag{29}$$

For particular density functions, when $t \geq 0$, we obtain

$$f_s(t,x) = L_{sk}(t) f_k(0,x), \quad k,s = 1,\dots,n. \tag{30}$$

Also, with probability $\psi_k(t)$, in view of $x(0) = e^{-A_k t}x$, we obtain the equality

$$f_k(t,x) = f_k\left(0, e^{-A_k t}x\right) \det\left(e^{-A_k t}\right), \qquad f_s(t,x) \equiv 0, \quad k \neq s. \tag{31}$$

On the interval $(\tau, \tau + d\tau)$, there could be a jump of the stochastic process $\xi(\tau)$ from state θ_k to state θ_s with probability $q_{ks}(\tau)d\tau$. Taking into account that functions $q_{ks}(t)$ are continuous, we obtain the probability

$$P\{\tau < t_1 < \tau + d\tau\} = \int_\tau^{\tau+d\tau} q_k(s)\,ds \approx q_k(\tau)\,d\tau. \tag{32}$$

After a jump at the moment lying between moments τ and $\tau + d\tau$, we can use (27). At the moment t_1 of the first jump of the stochastic process $\xi(t)$, in accordance with (14), we have

$$f_s(t_1 + 0, x) = f_k\left(t_1 - 0, C_{sk}^{-1}x\right) \det C_{sk}^{-1}. \tag{33}$$

Therefore, by using operators R, S defined in (16), to remain in state θ_k at the moment t, we get

$$\begin{aligned} L_{kk}(t) f_k(0,x) &= \psi_k(t) R_k(t) f_k(0,x) \\ &\quad + \int_0^t \sum_{s=1}^n L_{ks}(t-\tau) q_{sk}(\tau) S_{sk} R_k(\tau) f_k(0,x)\,d\tau, \\ &\qquad k = 1,\dots,n. \end{aligned} \tag{34}$$

For transition from state θ_k to state θ_τ at the moment t, we obtain

$$\begin{aligned} L_{\tau k}(t) f_k(0,x) &= \int_0^t \sum_{s=1}^n L_{\tau s}(t-\tau) q_{sk}(\tau) S_{sk} R_k(\tau) f_k(0,x)\,d\tau, \end{aligned} \tag{35}$$

where $\tau \neq k, \tau, k = 1,\dots,n$.

These two systems can be written in the form (22).

(2) Before we derive the system of moment equations in (23), we establish an auxiliary operator equation. Let us find the solution of (22) in the form

$$L(t) = \psi(t) R(t) + \int_0^t \psi(t-\tau) R(t-\tau) U(\tau)\,d\tau, \tag{36}$$

where U is an unknown matrix. If we put $L(t)$ expressed in the form (36) into (22), we obtain

$$\begin{aligned} &\int_0^t \psi(t-\tau) R(t-\tau) U(\tau)\,d\tau \\ &\quad = \int_0^t \psi(t-\tau) T(t-\tau) S(\tau) R(\tau)\,d\tau \\ &\qquad + \int_0^t d\tau \int_0^{t-\tau} \psi(t-\tau-s) R(t-\tau-s) U(s) S(\tau) R(\tau)\,ds. \end{aligned} \tag{37}$$

After substituting $r = \tau + s$, $\tau = \tau$ in the double integral and after changing the order of integration, we obtain

$$\begin{aligned} &\int_0^t d\tau \int_0^{t-\tau} \psi(t-\tau-s) R(t-\tau-s) U(s) S(\tau) R(\tau)\,ds \\ &\quad = \int_0^t \psi(t-r) \left(\int_0^r U(r-\tau) S(\tau) R(\tau)\,d\tau \right) dr. \end{aligned} \tag{38}$$

Therefore, a suitable matrix U in (22) is the matrix

$$U(t) = S(t) R(t) + \int_0^t U(t-\tau) S(\tau) R(\tau)\,d\tau. \tag{39}$$

In the other way, we can find a matrix $U(t)$ as a solution of (36) in the form

$$U(t) = S(t) R(t) + \int_0^t S(t-\tau) R(t-\tau) V(\tau)\,d\tau, \tag{40}$$

where V is an unknown matrix. From this, we get

$$V(t) = S(t) R(t) + \int_0^t V(t-\tau) S(\tau) R(\tau)\,d\tau. \tag{41}$$

A comparison of (39) and (41) implies that we can set $V(t) \equiv U(t)$. Then, (40) can be written as

$$U(t) = S(t) R(t) + \int_0^t S(t-\tau) R(t-\tau) U(\tau)\,d\tau, \tag{42}$$

or

$$U(t) = S(t) R(t) + \int_0^t S(\tau) R(\tau) U(t-\tau)\,d\tau. \tag{43}$$

Multiplying (36) and (38) on the right by the vector $f(0,x)$, we obtain

$$\begin{aligned} f(t,x) &= \psi(t) R(t) f(0,x) \\ &\quad + \int_0^t \psi(t-\tau) R(t-\tau) U(\tau) f(0,x)\,d\tau, \\ U(t) f(0,x) &= S(t) R(t) f(0,x) \\ &\quad + \int_0^t S(t-\tau) R(t-\tau) U(\tau) f(0,x)\,d\tau. \end{aligned} \tag{44}$$

Denote

$$\begin{aligned} h(t,x) &= U(t) f(0,x), \\ h(t,x) &= \left(h_1(t,x), h_2(t,x), \dots, h_n(t,x)\right)^T, \end{aligned} \tag{45}$$

and

$$\begin{aligned} z_k(t) &= \int_{\mathbb{E}_m} x h_k(t,x)\,dx, \\ W_k(t) &= \int_{\mathbb{E}_m} x x^T h_k(t,x)\,dx, \end{aligned} \tag{46}$$

where $k = 1, \ldots, n$. The system (44) can be rewritten into the scalar form

$$f_k(t,x) = \psi_k(t) R_k(t) f_k(0,x) + \int_0^t \psi_k(t-\tau) R_k(t-\tau) h_k(\tau,x)\, d\tau, \tag{47}$$

$$h_k(t,x) = \sum_{s=1}^{n} q_{ks}(t) S_{ks} R_s(t) f_s(0,x) + \int_0^t \sum_{s=1}^{n} q_{ks}(t-\tau) S_{ks} R_s(t-\tau) h_s(\tau,x)\, d\tau, \quad k = 1, \ldots, n. \tag{48}$$

The system of (23) can be obtained by multiplying each equation of (48) by vector x and integrating it over the space $\mathbb{E}_m$. In doing so, it is necessary to use

$$\begin{aligned}
&\int_{\mathbb{E}_m} x f_k(t,x)\, dx = E_k^{(1)}\{x(t)\},\\
&\int_{\mathbb{E}_m} x R_k(t) f_k(0,x) = \int_{\mathbb{E}_m} x f_k\left(0, e^{-A_k t} x\right) \det e^{-A_k t} dx = \int_{\mathbb{E}_m} e^{A_k t} y f_k(0,y)\, dy = e^{A_k t} E_k^{(1)}\{x(t)\},\\
&\int_{\mathbb{E}_m} x R_k(t-\tau) h_k(\tau,x)\, dx = \int_{\mathbb{E}_m} x h_k\left(\tau, e^{-A_k(t-\tau)}\right) \det e^{-A_k(t-\tau)} dx = e^{A_k(t-\tau)} \int_{\mathbb{E}_m} y h_k(\tau,y)\, dy = e^{A_k(t-\tau)} z_k(\tau),\\
&\int_{\mathbb{E}_m} x S_{ks} R_k(t) f_s(0,x)\, dx = \int_{\mathbb{E}_m} x S_{ks} f_s\left(0, e^{-A_s t} x\right) \det e^{-A_s t} dx = \int_{\mathbb{E}_m} x f_s\left(0, e^{-A_s t} C_{ks}^{-1} x\right) \det e^{-A_s t} C_{ks}^{-1} dx = C_{ks} e^{A_s t}\\
&\qquad \int_{\mathbb{E}_m} x f_s(0,x)\, dx = C_{ks} e^{A_s t} E_s^{(1)}\{x(0)\}.
\end{aligned} \tag{49}$$

(3) The system of (25) can be obtained by multiplying each equation in (48) by matrix xx^T and integrating it over the space $\mathbb{E}_m$ by using matrix equalities

$$\begin{aligned}
&\int_{\mathbb{E}_m} x x^T R_k(t) f_k(0,x)\, dx = \int_{\mathbb{E}_m} x x^T f_k\left(0, e^{-A_k t} x\right) \det e^{-A_k t} dx = \int_{\mathbb{E}_m} e^{A_k t} y y^T e^{A_k^T t} f_k(0,y)\, dy = e^{A_k t} E_k^{(2)}\{x(0)\} e^{A_k^T t},\\
&\int_{\mathbb{E}_m} x x^T S_{ks} R_s(t) f_s(0,x)\, dx = \int_{\mathbb{E}_m} x x^T f_s\left(0, e^{-A_k t} C_{ks}^{-1} x\right) \det e^{-A_k t} \det C_{ks}^{-1} dx = C_{ks} e^{A_s t} E_s^{(2)}\{x(0)\} e^{A_s^T t} C_{ks}^T.
\end{aligned} \tag{50}$$

□

4. Necessary and Sufficient Conditions of L_2-Stability

Several different stability definitions are useful. Here, we recall the mean stability and the mean square stability definitions, the L_2-stability, and the classical definition of asymptotic stability.

Definition 6. The trivial solution of system (2) is said to be mean square stable on the interval $[0,\infty)$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that any solution $x(t)$ corresponding to the initial data $x(0)$ exists for all $t \geq 0$ and the mathematical expectation

$$E^{(1)}\left\{\|x(t)\|^2\right\} < \varepsilon, \text{ whenever } t \geq 0,\ \|x(0)\| < \delta. \tag{51}$$

The mean stability of the zero solution of system (2) is much defined in the same way with only $\|x(t)\|^2$ being replaced by $\|x(t)\|$.

Definition 7. The trivial solution of system (2) is said to be asymptotically mean square stable on the interval $[0,\infty)$ if it is stable and, moreover,

$$\lim_{t\to\infty} E^{(2)}\{x(t)\} = 0. \tag{52}$$

Remark 8. It is obvious that the mean stability of the zero solution of system (2) is equivalent to the asymptotic stability of the solutions of system (23) and (24) and the mean square stability of the solutions of system (2) is equivalent to the asymptotic stability of the solutions of system (25) and (26).

Definition 9. The trivial solution of the differential systems (2) is said to be L_2-stable if the integral

$$\int_0^\infty E^{(1)}\left\{\|x(t)\|^2\right\} dt \tag{53}$$

converges.

Remark 10. It is easy to see that the integral (53) converges if and only if the matrix integral

$$\int_0^\infty E^{(2)}\{x(t)\}\, dt \tag{54}$$

is convergent.

Lemma 11. *The following three inequalities hold:*

(1) $E_k^{(2)}\{x(t)\} \geq 0,\ k = 1, \ldots, n.$

(2) $e^{A_k t} E_k^{(2)}\{x(0)\} e^{A_k^T t} x \geq 0,\ k = 1, \ldots, n.$

(3) $C_{ks} e^{A_s t} E_s^{(2)}\{x(0)\} e^{A_s^T t} C_{ks}^T \geq 0,\ k, s = 1, \ldots, n.$

Proof. All inequalities follow from property $f_k(t, x) \geq 0$, $k = 1, \ldots, n$ in accordance with (23) and (24). □

It is convenient to derive some sufficient and necessary conditions of L_2-stability by using a matrix operator with suitable properties. Such an operator is defined by the following lemma.

Lemma 12. *The linear matrix operators $N_{ks}(t)$, $k, s = 1, \ldots, n$, defined as*

$$N_{ks}(t) \circ W_s(t) := \int_0^t q_{ks}(t-\tau) C_{ks} e^{A_s(t-\tau)} W_s(\tau) e^{A_s^T(t-\tau)} C_{ks}^T d\tau, \tag{55}$$

are monotonous.

Proof. Because $q_{ks}(t-\tau) \geq 0$, in accordance with the third statement of Lemma 11, we have

$$x^T N_{ks}(t) \circ W_s(t) x \geq 0, \text{ for } x \geq 0. \tag{56}$$

So $W_s(\tau) \geq 0$ implies $N_{ks}(t) \circ W_s(t) \geq 0$ and the operator is monotone. □

Remark 13. The linear monotonous operator $N_{ks}(t)$ transforms any symmetric matrix D into a symmetric matrix $N_{ks}(t) \circ D$. Its monotonicity guarantees that inequality $D_1 \geq D_2$ implies inequality

$$N_{ks}(t) \circ D_1 \geq N_{ks}(t) \circ D_2. \tag{57}$$

Lemma 14. *The symmetric matrices $W_k(t)$, $k = 1, \ldots, n$ defined by* (26) *satisfy the inequalities $W_k(t) \geq 0$, $k = 1, \ldots, n$.*

Proof. System (26) can be rewritten in the form

$$W_k(t) = \sum_{s=1}^n q_{ks}(t) C_{ks} e^{A_s t} E_s^{(2)}\{x(0)\} e^{A_s^T t} C_{ks}^T + \sum_{s=1}^n N_{ks}(t) \circ W_s(t), \tag{58}$$

$k = 1, \ldots, n$, by using the matrix operators (55). System (26) as well as system (58) can be solved by the method of successive approximations

$$W_k^{(0)}(t) = \sum_{s=1}^n q_{ks}(t) C_{ks} e^{A_s t} E_s^{(2)}\{x(0)\} e^{A_s^T t} C_{ks}^T,$$
$$W_k^{(l+1)}(t) = W_k^{(0)}(t) + \sum_{s=1}^n N_{ks}(t) \circ W_s^{(l)}(t), \quad l = 1, 2, \ldots. \tag{59}$$

Hence, $W_k^{(0)}(t) \geq 0$ and $W_k^{(l)}(t) \geq 0$ implies $W_k^{(l+1)} \geq 0$, $k = 1, \ldots, n$. So the solution of system (58),

$$W_k(t) = \lim_{l \to \infty} W_k^{(l)}(t), \tag{60}$$

satisfies $W_k(t) \geq 0$, $k = 1, \ldots, n$. □

Now, we rewrite the moment equations (25) and (26) into a more compact form by using the denotations

$$D_k = \int_0^\infty E_k^{(2)}\{x(t)\} dt, \qquad W_k = \int_0^\infty W_k(\tau) d\tau, \quad k = 1, \ldots, n. \tag{61}$$

Then, integrating systems (25) and (26) from 0 to ∞ with respect to t, we obtain

$$D_k = \int_0^\infty \psi_k(t) e^{A_k t} \left(E_k^{(2)}\{x(0)\} + W_k\right) e^{A_k^T t} dt, \quad k = 1, \ldots, n, \tag{62}$$

$$W_k = \sum_{s=1}^n \int_0^\infty q_{ks}(t) C_{ks} e^{A_s t} \left(E_s^{(2)}\{x(0)\} + W_s\right) e^{A_s^T t} C_{ks}^T dt, \quad k = 1, \ldots, n. \tag{63}$$

Corollary 15. *Let the zero solution of the system* (2) *with jumps* (14) *at random time moments t_j, $j = 0, 1, 2, \ldots$ determined by jumps of stochastic process $\xi(t)$ be L_2-stable. Then, the integrals*

$$I_k = \int_0^\infty \psi_k(t) e^{A_k t} E_k^{(2)}\{x(0)\} e^{A_k^T t} dt, \quad k = 1, \ldots, n \tag{64}$$

are convergent.

Proof. This immediately follows from Lemma 14. In fact, since $W_k \geq 0$, then $D_k \geq I_k$, $k = 1, \ldots, n$. □

Theorem 16. *Let the integrals* (64) *be convergent. Then, the zero solution of system* (2) *is L_2-stable if and only if the solutions $W_k \geq 0$, $k = 1, \ldots, n$ of system* (63) *are bounded.*

Proof. (1) *Sufficiency*. Integrating system (59) from 0 to ∞ with respect to t, using notation $W_k^{(l)} = \int_0^\infty W_k^{(l)}(t) dt$, $k = 1, \ldots, n$, we obtain the system of matrix successive approximations

$$W_k^{(l+1)} = W_k^{(0)} + \sum_{s=1}^n N_{ks} \circ W_s^{(l)}, \quad k = 1, \ldots, n, \quad l = 0, 1, 2, \ldots. \tag{65}$$

As the linear operators N_{ks}, $k, s = 1, \ldots, n$ defined by (55) are monotonous, the zero solution of system (2) is L_2-stable if the sequence of matrices $W_k^{(l)}$, $l = 0, 1, 2, \ldots$ is convergent.

(2) *Necessity*. Let us assume that the solution $W_k = Z_k \geq 0$, $k = 1, \ldots, n$ of the system

$$W_k = W_k^{(0)} + \sum_{s=1}^{n} N_{ks} \circ W_s, \quad k = 1, \ldots, n \tag{66}$$

is bounded. Obviously, $Z_k \geq W_k^{(0)}$, $k = 1, \ldots, n$ and, in view of successive approximations (65), we get

$$\begin{aligned} W_k^{(l+1)} &= W_k^{(0)} + N_{ks} \circ W_s^{(l)} \\ &\leq W_k^{(0)} + N_{ks} \circ Z_s = Z_k, \quad k = 1, \ldots, n. \end{aligned} \tag{67}$$

So, for all $l = 0, 1, 2, \ldots$, we have $W_k^{(l)} \leq Z_k$, $k = 1, \ldots, n$.

Next, from (65), we obtain

$$\begin{aligned} W_k^{(l+1)} &= W_k^{(0)} + N_{ks} \circ W_s^{(l)} \geq W_k^{(0)} \\ &\quad + N_{ks} \circ W_s^{(l-1)} = W_k^{(l)}, \quad k = 1, \ldots, n. \end{aligned} \tag{68}$$

Moreover, because $W_k^{(0)} \geq 0$, $W_k^{(l+1)} \geq W_k^{(l)}$, $k = 1, \ldots, n$ is satisfied for all $l = 0, 1, \ldots$.

Finally, the boundedness and monotonicity of the matrix sequences $W_k^{(l)}$, $l = 0, 1, \ldots$ imply the existence of limits

$$W_k = \lim_{l \to \infty} W_k^{(l)}, \tag{69}$$

$k = 1, \ldots, n$. Consequently, independently of the initial value $W_k^{(0)}$, in view of $0 \leq W_k^{(l)} \leq Z_k$, the linear operator $N = (N_{ks})_{k,s=1}^{n}$ has the spectral radius less than 1. This means that system (65) has a unique solution $W_k = Z_k$, $k = 1, \ldots, n$. □

5. Model Problem

The functioning of the foreign currency exchange market in conditions of uncertainty can be modelled by using stochastic differential equations. Such convenient mathematical model is the scalar case of the initial problem (2), (3), that is, the initial problem formulated for the stochastic linear differential equation

$$\frac{dx(t)}{dt} = a(\xi(t))\, x(t), \tag{70}$$

$$x(0) = \varphi(\omega), \tag{71}$$

where coefficient a depends on a semi-Markov process $\xi(t)$. The possible states $\theta_1, \ldots, \theta_n$ of the stochastic process $\xi(t)$ express the conditions in which the bank works, for example, in a currency crisis, in a stable foreign currency exchange market, and so forth. Let the stochastic process $\xi(t)$ take the states θ_k, $k = 1, 2, \ldots, n$. If $\xi(t) = \theta_k$, we denote $a(\xi(t)) = a_k$. Further assume that the intensities $q_{sk}(t)$ are determined by the formulas

$$q_{ss}(t) = 0, \tag{72}$$

$$q_{sk}(t) = \begin{cases} \dfrac{1}{T_{sk}} & \text{for } 0 \leq t < T_{sk}, \\ 0 & \text{for } t \geq T_{sk}, \end{cases} \tag{73}$$

where $s, k = 1, 2, \ldots, n$.

Perturbations in the foreign currency exchange market cause the changes of the stochastic process $\xi(t)$, and consequently, solutions of (2) in this scalar case are subject to the random transformations

$$x(t_j + 0) = p_k x(t_j - 0), \quad p_k \neq 0, \ k = 1, \ldots, n \tag{74}$$

at the moments of jumps t_j, $j = 1, 2, \ldots$.

We derive the domain of stability of the foreign currency exchange market using the results of the previous section. The moment equations in (25) for $E_k^{(2)}\{x(t)\}$, $k = 1, \ldots, n$ take, in the scalar case, the form

$$\begin{aligned} E_k^{(2)}\{x(t)\} &= \psi_k e^{2a_k t} E_k^{(2)}\{x(0)\} \\ &\quad + \int_0^t \psi_k(t-\tau)\, e^{2a_k(t-\tau)} W_k(\tau)\, d\tau, \end{aligned} \tag{75}$$

$$\begin{aligned} W_k(t) &= \sum_{s=1}^{n} q_{ks}(t)\, C_{ks}^2 e^{2a_s t} E_s^{(2)}\{x(0)\} \\ &\quad + \int_0^t \sum_{s=1}^{n} q_{ks}(t-\tau)\, C_{ks}^2 e^{2a_s(t-\tau)} W_s(\tau)\, d\tau, \\ &\qquad k = 1, \ldots, n. \end{aligned} \tag{76}$$

By definition, the zero solution of (75) is asymptotically stable if $E^{(2)}\{x(t)\} \to 0$ for $t \to \infty$.

Lemma 17. *If*

$$\lim_{t \to \infty} E_k^{(2)}\{x(t)\} = 0, \quad k = 1, \ldots, n, \tag{77}$$

then

$$\lim_{t \to \infty} \psi_k(t)\, e^{2a_k t} = 0, \quad k = 1, \ldots, n. \tag{78}$$

Proof. Formula (20) implies $E_k^{(2)}\{x(t)\} \geq 0$, $k = 1, \ldots, n$ if $t \geq 0$. Therefore, $W_k(t) \geq 0$, $k = 1, \ldots, n$, $t \geq 0$. From (75), it follows that inequalities

$$E_k^{(2)}\{x(t)\} \geq \psi_k e^{2a_k t} E_k^{(2)}\{x(0)\}, \quad k = 1, \ldots, n \tag{79}$$

are always satisfied. Then, for any constant $E_k^{(2)}\{x(0)\}$, property (77) implies (78). □

Note that, in general, condition (78) does not imply the property (77). In the following theorem, it is shown what additional assumptions are needed.

Theorem 18. *Let* (78) *hold and let*

$$\int_0^\infty \psi_k(t)\, e^{2a_k t} dt < \infty, \quad k = 1, \ldots, n. \tag{80}$$

Then, if there exist limits

$$\lim_{t \to \infty} W_k(t) = 0, \quad k = 1, \ldots, n, \tag{81}$$

limits (77) *exist too.*

Proof. We denote

$$I_k = \int_0^\infty \psi_k(t) e^{2a_s t} dt, \quad I_k \geq 0, \ k = 1, \ldots, n. \tag{82}$$

As the integrals I_k, $k = 1, \ldots, n$ are convergent, for all $\varepsilon > 0$, there exists $T_1 > 0$ such that, for all $t > T_1$, the inequalities

$$\int_t^\infty \psi_k(t) e^{2a_k t} < \varepsilon, \quad k = 1, \ldots, n \tag{83}$$

hold. Similarly, assumption (81) means that $\forall \varepsilon > 0 \ \exists \ T_2 > 0$ such that $\forall t > T_2$ the inequalities

$$0 \leq W_k(t) < \varepsilon, \quad k = 1, \ldots, n \tag{84}$$

hold. Moreover, there exists constant W_0 such that

$$|W_k(t)| \leq W_0, \quad k = 1, \ldots, n, \ t \geq T_2. \tag{85}$$

The integral part of (75) can be now estimated if $t > T_1 + T_2$. We have

$$\begin{aligned} &\int_0^t \psi_k(t-\tau) e^{2a_k(t-\tau)} W_k(\tau) d\tau \\ &\quad = \int_0^{T_1+T_2} \psi_k(t) e^{2a_k \tau} W_k(t-\tau) d\tau \\ &\qquad + \int_{T_1+T_2}^t \psi_k(t) e^{2a_k \tau} W_k(t-\tau) d\tau \\ &\quad \leq I_k \varepsilon + W_0 \varepsilon. \end{aligned} \tag{86}$$

Thus, there exist limits

$$\lim_{t\to\infty} \int_0^t \psi_k(t-\tau) e^{2a_k(t-\tau)} W_k(\tau) d\tau = 0, \quad k = 1, \ldots, n. \tag{87}$$

□

Corollary 19. *Let the assumptions of Theorem 18 hold. Then, the asymptotical stability of solutions of system* (75) *implies the asymptotical mean square stability of the zero solution of* (70).

Proof. Under the given assumptions, this follows from the existence of limits (77). □

The results obtained make it possible to examine the stability of the stochastic equation (70) by using the deterministic system of (76). Here, we can use the known methods such as the Laplace transformation.

If we denote

$$\Upsilon_k(p) = \int_0^\infty e^{-pt} W_k(t) dt, \qquad \Theta_{ks}(p) = \int_0^\infty e^{-pt} q_{ks}(t) dt, \qquad k, s = 1, \ldots, n, \tag{88}$$

then, multiplying (76) by e^{-pt} and integrating it from 0 to ∞ with respect to t, (76) can be transformed into the system of linear algebraic equations with respect to the functions $\Upsilon_k(p)$. We get the system

$$\begin{aligned} \Upsilon_k(p) &= \sum_{s=1}^n \Theta_{ks}(p - 2a_s) C_{ks}^2 E_s^{(2)} \{x(0)\} \\ &\quad + \sum_{s=1}^n \Theta_{ks}(p - 2a_s) C_{ks}^2 \Upsilon_s(p), \end{aligned} \tag{89}$$

where $k = 1, \ldots, n$.

The determinant $\Delta(p)$ of the system of linear equations (89) is in the form

$$\Delta(p) = \begin{vmatrix} 1 - \Theta_{11}(p-2a_1) C_{11}^2 & -\Theta_{12}(p-2a_2) C_{12}^2 & \cdots & -\Theta_{1n}(p-2a_n) C_{1n}^2 \\ -\Theta_{21}(p-2a_1) C_{21}^2 & 1 - \Theta_{22}(p-2a_2) C_{22}^2 & \cdots & -\Theta_{2n}(p-2a_n) C_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\Theta_{n1}(p-2a_1) C_{n1}^2 & -\Theta_{n2}(p-2a_2) C_{n2}^2 & \cdots & 1 - \Theta_{nn}(p-2a_n) C_{nn}^2 \end{vmatrix}. \tag{90}$$

The singular points of mappings $\Theta_{ks}(p)$, $k, s = 1, \ldots, n$ are determined by the roots of $\Delta(p) = 0$. If all the functions $\Theta_{ks}(p)$, $k, s = 1, \ldots, n$ are entire, then there are no singular points, except for the point $p = \infty$.

Remark 20. In a particular case, solutions of (89) are located on the boundary of the stability domain. If $p = 0$, then $\Delta(0) = 0$ is the equation determining the boundary of the stability domain.

To solve the model problem formulated at the beginning of this section, we use the stochastic operators $S_{sk} f(x)$, $s, k = 1, 2, \ldots, n$ in the form

$$S_{sk} f(t, x) = \begin{cases} f(t, x), & s = k, \\ \displaystyle\sum_{k=1}^n \frac{p_k}{p_s} f\left(t, \frac{x}{p_s}\right), & s \neq k, \end{cases} \tag{91}$$

associated to the intensities $q_{sk}(t)$ determined by (72). The domain of stability of banking operations in the foreign currency exchange market can be derived from the behavior of the solutions of the moment equations (76). Before we use the moment equations, we express the probabilities C_{ks}, $k, s = 1, 2, \ldots, n$ from formula (14) by using the probabilities p_k, $k = 1, 2, \ldots, n$ from formula (74) in the form

$$C_{ks} = \begin{cases} 1, & s = k, \\ \sum_{k=1}^{n} p_k^2, & s \neq k. \end{cases} \tag{92}$$

Then, the moment equations (76) can be rewritten into the form

$$W_k(t) = \rho \left[\sum_{\substack{s=1 \\ s \neq k}}^{n} q_{ks}(t) e^{2a_s t} E_s^{(2)} \{x(0)\} + \int_0^t \sum_{s=1}^{n} q_{ks}(t-\tau) e^{2a_s(t-\tau)} W_s(\tau)\, d\tau \right], \quad k = 1, \ldots, n, \tag{93}$$

where $\rho = \sum_{k=1}^{n} p_k^2$. The system (93) can be solved in the same way as above, that is, by Laplace transformation. Then, multiplying (93) by e^{-pt} and integrating it from 0 to ∞ with respect to t, we get a preliminary form of the system

$$\Upsilon_k(p) = \rho \left[\int_0^\infty \sum_{\substack{s=1 \\ k \neq s}}^{n} q_{ks} e^{2a_s t} e^{-pt} E_s^{(2)} \{x(0)\}\, dt + \int_0^\infty \int_0^t \sum_{\substack{s=1 \\ k \neq s}}^{n} q_{ks}(t-\tau) e^{2a_s(t-\tau)} e^{-pt} W_s(\tau)\, d\tau dt \right], \quad k = 1, \ldots, n, \tag{94}$$

which we will still have to modify using some properties of the Laplace transformation. In accordance with the property of the delay and with regard to the equality

$$\int_0^\infty q_{ks} e^{-pt}\, dt = \frac{1 - e^{-pT_{ks}}}{T_{ks} p}, \quad k, s = 1, \ldots, n, \ k \neq s, \tag{95}$$

we obtain

$$\int_0^\infty q_{ks} e^{2a_s t} e^{-pt} dt = \frac{1 - e^{-T_{ks}(p-2a_s)}}{T_{ks}(p - 2a_s)}, \quad k, s = 1, \ldots, n, \ k \neq s. \tag{96}$$

In accordance with the property of convolution, we get

$$\int_0^\infty \int_0^t q_{ks}(t-\tau) e^{2a_s(t-\tau)} e^{-pt} W_s(\tau)\, d\tau dt = \frac{1 - e^{-T_{ks}(p-2a_s)}}{T_{ks}(p - 2a_s)}. \tag{97}$$

Therefore, the system of (94) can be written in the form

$$\Upsilon_k(p) = \rho \left[\sum_{\substack{s=1 \\ k \neq s}}^{n} E_s^{(2)} \{x(0)\} \frac{1 - e^{-T_{ks}(p-2a_s)}}{T_{ks}(p - 2a_s)} + \sum_{\substack{s=1 \\ k \neq s}}^{n} f_s(p) \frac{1 - e^{-T_{ks}(p-2a_s)}}{T_{ks}(p - 2a_s)} \right], \tag{98}$$

where $k = 1, \ldots, n$, or, using the notations

$$b_k(p) \equiv \sum_{\substack{s=1 \\ k \neq s}}^{n} E_s^{(2)} \{x(0)\} \frac{1 - e^{-T_{ks}(p-2a_s)}}{T_{ks}(p - 2a_s)}, \qquad a_{ks} \equiv \frac{1 - e^{-T_{ks}(p-2a_s)}}{T_{ks}(p - 2a_s)}, \tag{99}$$

$k, s = 1, \ldots, n$, $k \neq s$, in a simpler form

$$\Upsilon_k(p) = \rho \left(b_k(p) + \sum_{\substack{s=1 \\ k \neq s}}^{n} a_{ks} \Upsilon_s(p) \right), \quad k = 1, \ldots, n. \tag{100}$$

By the Cramer theorem, we can solve the system (100). The singular points are determined by the roots of

$$\det(E - \rho a) = 0, \tag{101}$$

where $a \equiv (a_{ks}(p, T_{ks}))_{k,s=1}^{n}$, while $\rho a_{kk}(p, T_{kk}) \equiv -1$.

The character of the roots of (101) determines the stability of the solutions of the system of integral equations in (93). If the real parts of all the roots of (101) are negative, then the solutions of (93) are asymptotically stable. If there is at least one root of (101) with a positive real part, then the solutions of integral equations (93) are unstable.

The character of the dependence between parameters p and T_{ks} can be determined by solving the system of algebraic equations in (101) by numerical methods.

Example 21. The real boundaries of the instability domain of foreign currency exchange market can be determined in a particular case. Suppose that the random semi-Markov process can take three states:

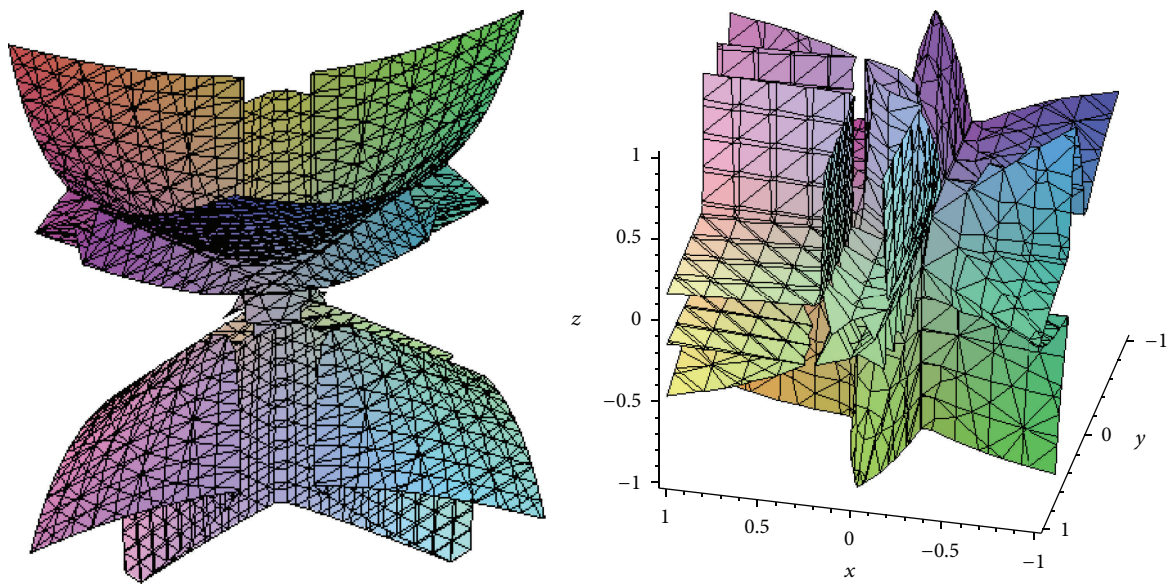
θ_1—if the bank operates in a currency crisis, then $a(\xi(t)) = a_1$;

θ_2—if the bank operates in a stable foreign currency exchange market, then $a(\xi(t)) = a_2$;

θ_3—if the bank operates in a market with currency restrictions, then $a(\xi(t)) = a_3$,

with intensities

$$q_{11}(t) = q_{22}(t) = q_{33}(t) \equiv 0,$$
$$q_{12}(t) = q_{13}(t) = q_{21}(t) = q_{23}(t) = q_{31}(t) = q_{32}(t) \equiv \begin{cases} \frac{1}{T} & \text{for } 0 \le t < T, \\ 0 & \text{for } t > T, \end{cases} \tag{102}$$



(a) (b)

Figure 1: The boundary of instability of solutions of (70) constructed in the plane of parameters a_1, a_2, and a_3 for $p = 0$ and for different values of ρ.

which means that the bank remains in each state for the same period of time $1/T$. In the above case, the system (100) takes the form

$$\begin{aligned} \Upsilon_1(p) &= \rho b_1(p) + \rho a_{12}\Upsilon_2(p) + \rho a_{13}\Upsilon_3(p), \\ \Upsilon_2(p) &= \rho b_2(p) + \rho a_{21}\Upsilon_1(p) + \rho a_{23}\Upsilon_3(p), \\ \Upsilon_3(p) &= \rho b_3(p) + \rho a_{31}\Upsilon_1(p) + \rho a_{32}\Upsilon_2(p), \end{aligned} \tag{103}$$

where

$$\begin{aligned} a_{12} &= \frac{1-e^{-T(p-2a_2)}}{T(p-2a_2)}, \qquad a_{13} = \frac{1-e^{-T(p-2a_3)}}{T(p-2a_3)}, \\ a_{21} &= \frac{1-e^{-T(p-2a_1)}}{T(p-2a_1)}, \qquad a_{23} = \frac{1-e^{-T(p-2a_3)}}{T(p-2a_3)}, \\ a_{31} &= \frac{1-e^{-T(p-2a_1)}}{T(p-2a_1)}, \qquad a_{32} = \frac{1-e^{-T(p-2a_2)}}{T(p-2a_2)}. \end{aligned} \tag{104}$$

The value ρ expresses the mean value of the bank's income from foreign currency transactions during time T. The singular point is $p = 0$, when the solution is situated on the boundary of the domain of instability on the plane of coefficients a_1, a_2, and a_3.

Equation (101) is in the form

$$\begin{vmatrix} 1 & -\rho a_{12} & -\rho a_{13} \\ -\rho a_{21} & 1 & -\rho a_{23} \\ -\rho a_{31} & -\rho a_{32} & 1 \end{vmatrix} = 0, \tag{105}$$

or

$$\begin{aligned} 1 - \rho^3 a_{21}a_{32}a_{13} - \rho^3 a_{12}a_{23}a_{31} - \rho^2 a_{13}a_{31} \\ - \rho^2 a_{23}a_{32} - \rho^2 a_{12}a_{21} = 0. \end{aligned} \tag{106}$$

If $p = 0$, the boundaries of instability of solutions of (70) are constructed in the plane of parameters a_1, a_2, and a_3 for different values ρ (see Figure 1 where some admissible boundaries are constructed).

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Research Article

Singular Initial Value Problem for Certain Classes of Systems of Ordinary Differential Equations

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The paper is devoted to the study of the solvability of a singular initial value problem for systems of ordinary differential equations. The main results give sufficient conditions for the existence of solutions in the right-hand neighbourhood of a singular point. In addition, the dimension of the set of initial data generating such solutions is estimated. An asymptotic behavior of solutions is determined as well and relevant asymptotic formulas are derived. The method of functions defined implicitly and the topological method (Ważewski's method) are used in the proofs. The results generalize some previous ones on singular initial value problems for differential equations.

1. Introduction

Let $x_0 > 0$ and $y_0 > 0$ be given constants. Define auxiliary set of points

$$D_n := (0, x_0] \times \prod_{i=1}^{n} (0, y_{0i}], \tag{1}$$

where $y_{0i} = y_0$.

In the paper, we consider a system of ordinary differential equations in the form

$$g_i(x, y)\, y_i' = \alpha_i(x, y), \tag{2}$$

where $i = 1, \ldots, n$ and functions $g_i, \alpha_i : D_n \to (0, \infty)$ can satisfy

$$g_r(0^+, \theta) = \alpha_r(0^+, \theta) = 0 \tag{3}$$

for some indices $r \in \{1, 2, \ldots, n\}$ or

$$\frac{1}{g_s(0^+, \theta)} = \frac{1}{\alpha_s(0^+, \theta)} = 0 \tag{4}$$

for some indices $s \in \{1, 2, \ldots, n\}$, where $\theta = (0, 0, \ldots, 0)$ is the n-dimensional zero vector. Together with system (2) we consider the initial problem

$$y_i(0^+) = 0, \quad i = 1, \ldots, n. \tag{5}$$

Because of the above properties, the initial problem (2), (5) is called a singular initial problem.

Definition 1. Denote by $M(x_0, y_0)$ a class of vector-functions $\varphi : (0, x_0] \to \mathbb{R}^n$ having the following properties:

(1) φ is continuously differentiable on $(0, x_0]$;

(2) $\varphi_i(0^+) = 0$, $i = 1, \ldots, n$;

(3) $0 < \varphi_i^{(j)}(x)$ for $x \in (0, x_0]$, $j = 0, 1$, $i = 1, \ldots, n$;

(4) $\varphi_i(x_0) < y_0$, $i = 1, \ldots, n$.

For $\varphi \in M(x_0, y_0)$, define an auxiliary vector-function

$$G(x, \varphi(x)) = (G_1(x, \varphi(x)), \ldots, G_n(x, \varphi(x))), \tag{6}$$

where

$$G_i(x,\varphi(x)) := -\varphi_i'(x)\, g_i(x,\varphi(x)) + \alpha_i(x,\varphi(x)), \quad i = 1,\dots,n. \tag{7}$$

In the paper, sufficient conditions which guarantee the existence of a parametric class of solutions of initial value problem (2), (5) are given and asymptotic formulas

$$y_i(x) = \varphi_i(x)(1+o(1)), \qquad x \longrightarrow 0^+,\ i = 1,\dots,n \tag{8}$$

are derived, where φ_i are the coordinates of a function $\varphi \in M(x_0, y_0)$ and the symbol $o(1)$ is the well-known Landau order symbol.

There are numerous papers and books dealing with singular initial value problems (see, e.g., [1–16] and the references therein). Among others, we should mention pioneering results on the solvability of singular problems for ordinary differential equations achieved by Chechyk [15] and Kiguradze [13]. The results of the paper generalize previous investigation of the first author on the solvability of singular problems [5–7]. The main differences are as follows. In [5], a scalar singular differential equation was studied for the case that a function similar to the function G above does not change the sign for $x \to 0^+$. In [6], system (2) is investigated under the assumption that the ith right-hand side of the system is bounded by the product of two functions, with the first depending only on the variable x while the second one only depends on the variable y_i, $i = 1,\dots,n$. In comparison with the results of [7], we cannot expect that a first approximation of system (2) consists of equations with separable variables.

The structure of the paper is the following. In Section 2, auxiliary results on implicit functions are given. We refer to Corollary 4 where formula (27) is crucial for the proofs of the asymptotic behavior of solutions. The main results of the paper are formulated in Section 3. New results are proved and a progress is achieved by implicit construction of funnels, where solutions of the singular problem are expected. To prove the existence of such solutions, the topological method of Ważewski (see, e.g., [17–19]) is used. A simple illustrative example is shown here as well. A generalization of the results derived is discussed in Section 4.

2. Auxiliary Results on Implicit Functions

First, we give some properties of implicit functions used in the following proofs.

Lemma 2. *Assume that a function $\omega : D_1 \to \mathbb{R}$ satisfies the following conditions:*

(1) *$\omega(x,y)$ is continuously differentiable with respect to x and y;*

(2) *$\omega(0^+,0^+) = 0$;*

(3) *for every $y \in (0,y_0]$, there exists a finite limit $\omega(0^+,y)$, for every $x \in (0,x_0]$, there exists a finite limit $\omega(x,0^+)$, and $\omega(0^+,y)\omega(x,0^+) < 0$;*

(4) *$\omega_x'(x,y)\omega_y'(x,y) < 0$, where $(x,y) \in D_1$.*

Then,

$$\omega(x,y) = 0 \tag{9}$$

defines a unique implicit function $y = y(x)$ on some interval $(0,x_{00}]$, $0 < x_{00} \le x_0$ such that $y(x) \in M(x_{00},y_0)$.

Proof. Analysing assumptions (1)–(4), we deduce that only the following two cases can occur: either

$$\begin{aligned} &\omega(0^+,y) < 0, \qquad \omega_y'(x,y) < 0,\\ &\omega(x,0^+) > 0, \qquad \omega_x'(x,y) > 0 \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\omega(0^+,y) > 0, \qquad \omega_y'(x,y) > 0,\\ &\omega(x,0^+) < 0, \qquad \omega_x'(x,y) < 0 \end{aligned} \tag{11}$$

while the remaining two cases

$$\begin{aligned} &\omega(0^+,y) < 0, \qquad \omega_y'(x,y) > 0, \qquad \omega(x,0^+) > 0,\\ &\omega_x'(x,y) < 0, \qquad \omega(0^+,y) > 0, \qquad \omega_y'(x,y) < 0,\\ &\omega(x,0^+) < 0, \qquad \omega_x'(x,y) > 0 \end{aligned} \tag{12}$$

are in contradiction with assumptions (1) and (2). The rest of the proof is analogous to the proofs of the well-known implicit-function theorems and, therefore, we leave it out. □

To formulate the second lemma, we need some auxiliary notions. Define, for a given $y_{00} > 0$ and $\varepsilon^* > 0$ satisfying the inequality $0 < y_{00}(1+\varepsilon^*) < y_0$, the set

$$D^*(y_{00},\varepsilon^*) := \{(y,\varepsilon) : y \in [0,y_{00}], \varepsilon \in (-\varepsilon^*,\varepsilon^*)\}. \tag{13}$$

Moreover, for a given continuously differentiable function $R : (0,y_0) \to (0,\infty)$, let

$$F(y,\varepsilon) := \frac{R(y(1+\varepsilon))}{R(y)}, \tag{14}$$

where argument $y(1+\varepsilon)$ is assumed to be positive. In addition, we define

$$\begin{aligned} F(0,\varepsilon) &:= \lim_{y\to 0^+} F(y,\varepsilon),\\ F_\varepsilon'(0,\varepsilon) &:= \lim_{y\to 0^+} F_\varepsilon'(y,\varepsilon) \end{aligned} \tag{15}$$

provided that the limits exist and are finite.

Lemma 3. *Assume that functions*

$$R : (0,y_0) \longrightarrow (0,\infty), \qquad F : D^*(y_{00},\varepsilon^*) \longrightarrow [0,\infty) \tag{16}$$

satisfy the following conditions:

(1) *$R(y)$ is continuously differentiable and $R'(y) < 0$;*

(2) *$F(y, \varepsilon)$ is continuous with respect to y and continuously differentiable with respect to ε;*

(3) $F'_{\varepsilon}(y, 0) \neq 0$.

Then, for an arbitrary $y \in [0, y_{00}]$ and $\varepsilon_1 \in (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$, where $\tilde{\varepsilon}_1$ is a positive and sufficiently small constant, there exists a unique continuous solution

$$\varepsilon_1^0 = \varepsilon_1^0(y, \varepsilon_1) \tag{17}$$

of the equation

$$1 + \varepsilon_1 - F\left(y, \varepsilon_1^0\right) = 0, \tag{18}$$

where

$$\begin{aligned} &\varepsilon_1^0(y, \varepsilon_1) : [0, y_{00}] \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) \\ &\longrightarrow \begin{cases} \left(-\tilde{\varepsilon}_1^0, 0\right) & \text{if } \varepsilon_1 \in (0, \tilde{\varepsilon}_1), \\ 0 & \text{if } \varepsilon_1 = 0, \\ \left(0, \tilde{\varepsilon}_1^0\right) & \text{if } \varepsilon_1 \in (-\tilde{\varepsilon}_1, 0), \end{cases} \end{aligned} \tag{19}$$

and $\tilde{\varepsilon}_1^0$ is a sufficiently small positive constant, $\tilde{\varepsilon}_1^0 \leq \varepsilon^*$.

Proof. Define an auxiliary function

$$\omega : D^*(y_{00}, \varepsilon^*) \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) \longrightarrow \mathbb{R} \tag{20}$$

as

$$\omega\left(y, \varepsilon_1^0, \varepsilon_1\right) := 1 + \varepsilon_1 - F\left(y, \varepsilon_1^0\right) \tag{21}$$

and consider implicit equation (18) in the form

$$\omega\left(y, \varepsilon_1^0, \varepsilon_1\right) = 0 \tag{22}$$

with respect to ε_1^0. In what follows, we will assume that $y \in [0, y_{00}]$ is a parameter. Since

$$\omega(y, 0, 0) = 0, \tag{23}$$

$\omega(y, \varepsilon_1^0, \varepsilon_1)$ is continuous with respect to all y, ε_1^0, and ε_1 and continuously differentiable with respect to ε_1^0, and

$$\omega'_{\varepsilon_1^0}(y, 0, 0) = -F'_{\varepsilon_1^0}(y, 0) > 0 \tag{24}$$

for arbitrary $y \in [0, y_{00}]$. Hence we can apply the classical implicit-function theorem. As a result, we state that (18) is uniquely solvable with respect to ε_1^0. Thus

$$\varepsilon_1^0 = \varepsilon_1^0(y, \varepsilon_1), \tag{25}$$

where $\varepsilon_1^0 : [0, y_{00}] \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) \to \mathbb{R}$ is a continuous function with respect to both y and ε_1 and $\tilde{\varepsilon}_1$ is a sufficiently small positive number. The sign of the function $\varepsilon_1^0(y, \varepsilon_1)$ can be specified. In particular, since R is a decreasing function, the function F is decreasing with respect to ε,

$$\begin{aligned} &\varepsilon_1^0 = \varepsilon_1^0(y, \varepsilon_1) : [0, y_{00}] \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) \\ &\longrightarrow \begin{cases} \left(-\tilde{\varepsilon}_1^0, 0\right) & \text{if } \varepsilon_1 \in (0, \tilde{\varepsilon}_1), \\ 0 & \text{if } \varepsilon_1 = 0, \\ \left(0, \tilde{\varepsilon}_1^0\right) & \text{if } \varepsilon_1 \in (-\tilde{\varepsilon}_1, 0), \end{cases} \end{aligned} \tag{26}$$

and $\tilde{\varepsilon}_1^0$ is a sufficiently small positive constant satisfying $\tilde{\varepsilon}_1^0 \leq \varepsilon^*$. □

Corollary 4. *It is possible to reformulate the statement of Lemma 3 as follows. Since (18) is uniquely solvable, one can use the definition of $F(y, \varepsilon)$ given by (14) to get*

$$R(y)(1 + \varepsilon_1) \equiv R\left(y\left(1 - \varepsilon_1^{00}(y, \varepsilon_1)\right)\right), \tag{27}$$

where $\varepsilon_1^{00}(y, \varepsilon_1) := -\varepsilon_1^0(y, \varepsilon_1)$.

3. Main Results

In this part, the main results related to the solvability of problem (2), (5) are proved. We will discuss the dimension of the set of initial conditions generating solutions of this problem as well.

Using $\varphi \in M(x_0, y_0)$, define the sets

$$\begin{aligned} N^{\varphi_i} &:= \{(x, y) : (x, y) \in D_n, \varphi_i(x) < y_i < \varphi_i(x_0)\}, \\ N_{\varphi_i} &:= \{(x, y) : (x, y) \in D_n, y_i < \varphi_i(x)\}, \end{aligned} \tag{28}$$

where $i = 1, 2, \ldots, n$. To formulate the results we need auxiliary functions

$$W_i : N^{\varphi_i} \cup N_{\varphi_i} \longrightarrow \mathbb{R}^n \tag{29}$$

defined as follows:

$$\begin{aligned} W_i(x, y) := {} & g_i\left(\varphi_i^{-1}(y_i), \varphi\left(\varphi_i^{-1}(y_i)\right)\right) \frac{\alpha_i(x, y)}{g_i(x, y)} \\ & - \alpha_i(x, \varphi(x)) \end{aligned} \tag{30}$$

for $i = 1, 2, \ldots, n$.

Theorem 5. *Let $g_i : D_n \to \mathbb{R}^n$ and $\alpha_i : D_n \to \mathbb{R}^n$, $i = 1, \ldots, n$, be continuous functions. Let, moreover, for a function $\varphi \in M(x_0, y_0)$, the following conditions be true:*

(1)

$$0 < g_i(x, \varphi(x)), \quad 0 < \alpha_i(x, \varphi(x)), \ x \in (0, x_0], \quad i = 1, \ldots, n; \tag{31}$$

(2)

$$\int_{0^+} g_i(x, \varphi(x)) \varphi_i'(x)\, dx = \int_{0^+} \alpha_i(x, \varphi(x))\, dx = \infty, \quad i = 1, \ldots, n; \tag{32}$$

(3)

$$\left| \int_x^{x_{00}} G_i(t, \varphi(t))\, dt \right| < \left| \int_{x_{ki}}^{x_{00}} g_i(t, \varphi(t)) \varphi_i'(t)\, dt \right|, \tag{33}$$

where $x_{00} \in (0, x_0)$ is a sufficiently small constant, $x \in (0, x_{00}]$, $x_{ki} = \varphi_i^{-1}(\psi_{ki})$, and ψ_{ki}, $k = 1, 2$, $i = 1, \ldots, n$, are constants such that

$$\psi_{1i} < \varphi_i(x_{00}) < \psi_{2i} < y_0, \quad x_{2i} < x_0; \tag{34}$$

(4) there is an integer $n_1 \in \{0, 1, \ldots, n\}$ such that

(a) $W_i(x, y) > 0$ if $(x, y) \in N^{\varphi_i}$ and $i = 1, \ldots, n_1$;

(b) $W_i(x, y) < 0$ if $(x, y) \in N_{\varphi_i}$ and $i = 1, \ldots, n_1$;

(c) $W_i(x, y) < 0$ if $(x, y) \in N^{\varphi_i}$ and $i = n_1 + 1, \ldots, n$;

(d) $W_i(x, y) > 0$ if $(x, y) \in N_{\varphi_i}$ and $i = n_1 + 1, \ldots, n$.

Here, if $n_1 = 0$, conditions (a), (b) are omitted and, if $n_1 = n$, conditions (c), (d) are omitted.

Then, problem (2), (5) has at least n_1-parametric class of solutions $y(x) = (y_1(x), \ldots, y_n(x))$ such that $(x, y(x)) \in D_n$ for $x \to 0^+$.

Proof. The proof is divided into two parts. First, implicit curves are constructed and their properties are derived. Then, Ważewski's method is applied to special domains having the shape of funnels with sides constructed using implicitly defined hypersurfaces. In this construction, we use implicit curves from the first part of the proof.

Implicitly Defined Curves and Their Properties. Let $\varphi \in M(x_0, y_0)$ be fixed. Define auxiliary functions

$$z_{ki}, \tilde{z}_{ki} : (0, x_{00}] \times (0, y_0] \longrightarrow \mathbb{R}, \quad k = 1, 2, \ i = 1, \ldots, n \tag{35}$$

as

$$\begin{aligned} z_{ki}(x, y_{ki}) &:= \int_{\psi_{ki}}^{y_{ki}} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \\ &\quad - \int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt, \\ \tilde{z}_{ki}(x, y_{ki}) &:= \left[\int_{\psi_{ki}}^{y_{ki}} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt\right]^{-1} \\ &\quad - \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt\right]^{-1}. \end{aligned} \tag{36}$$

We prove that

$$z_{ki}(x, y_{ki}) = 0, \quad k = 1, 2, \ i = 1, \ldots, n, \tag{37}$$

define unique implicit functions

$$y_{ki} = y_{ki}(x) \equiv \psi_{ki}(x) \in M(x_{00}, y_0) \tag{38}$$

on the interval $(0, x_{00}]$. Observe that the function $\psi_{ki}(x)$ is a solution of

$$\tilde{z}_{ki}(x, y_{ki}) = 0, \quad k = 1, 2, \ i = 1, \ldots, n, \tag{39}$$

on the interval $(0, x_{00})$ as well. Therefore, we consider the latter equation and investigate its solvability using Lemma 2. Set

$$\omega(x, y) := \tilde{z}_{ki}(x, y), \tag{40}$$

where $k \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$. We show that the function $\omega(x, y)$ satisfies all assumptions (1)–(4) of Lemma 2, where, instead of the region D_1, we assume the region $(0, x_{00}^*] \times (0, y_0^*]$ with sufficiently small x_{00}^*, y_0^* such that $0 < x_{00}^* < x_{00}$, $0 < y_0^* < y_0$.

(a) It is easy to see (in view of the above assumptions) that the function

$$\omega : (0, x_{00}^*] \times (0, y_0^*] \longrightarrow \mathbb{R} \tag{41}$$

defined by (39) is continuously differentiable with respect to x and y and assumption (1) of Lemma 2, holds.

(b) Compute the limit $\omega(0^+, 0^+)$. We get

$$\begin{aligned} &\omega(0^+, 0^+) \\ &\quad \equiv \tilde{z}_{ki}(0^+, 0^+) \\ &\quad = \lim_{x \to 0^+, y \to 0^+} \left[\left[\int_{\psi_{ki}}^{y} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt\right]^{-1} \right. \\ &\qquad\qquad \left. - \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt\right]^{-1}\right] \\ &\quad = \lim_{y \to 0^+} \left[\int_{\psi_{ki}}^{y} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt\right]^{-1} \\ &\qquad - \lim_{x \to 0^+} \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t)) dt\right]^{-1} \end{aligned} \tag{42}$$

if the last two limits exist and are finite. Substituting $t = \varphi_i(x)$ into the first integral of the last expression and using condition (2), we get

$$\begin{aligned} &\omega(0^+, 0^+) \equiv \tilde{z}_{ki}(0^+, 0^+) \\ &\quad = \lim_{y \to 0^+} \left[\int_{\varphi_i^{-1}(\psi_{ki})}^{\varphi_i^{-1}(y)} g_i(x, \varphi(x)) \varphi_i'(x) dx\right]^{-1} \\ &\qquad - \lim_{x \to +0} \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t)) dt\right]^{-1} \\ &\quad = \left[\int_{x_{ki}}^{0^+} g_i(x, \varphi(x)) \varphi_i'(x)\, dx\right]^{-1} \\ &\qquad - \left[\int_{x_{00}}^{0^+} \alpha_i(x, \varphi(x)) dx\right]^{-1} = 0 \end{aligned} \tag{43}$$

and assumption (2) of Lemma 2 holds.

(c) Now we consider the existence of the product $\omega(0^+, y)\omega(x, 0^+)$ for $y \in (0, y_0^*]$, $x \in (0, x_{00}^*]$ and determine its sign. We get

$$\begin{aligned}
&\omega(0^+, y)\,\omega(x, 0^+) \\
&\quad \equiv \tilde{z}_{ki}(0^+, y)\,\tilde{z}_{ki}(x, 0^+) \\
&\quad = \lim_{x \to 0^+} \left[\left[\int_{\psi_{ki}}^{y} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \right]^{-1} \right. \\
&\qquad \left. - \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt \right]^{-1} \right] \\
&\quad \times \lim_{y \to 0^+} \left[\left[\int_{\psi_{ki}}^{y} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \right]^{-1} \right. \\
&\qquad \left. - \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt \right]^{-1} \right].
\end{aligned} \tag{44}$$

Substituting $t = \varphi_i(x)$ into the first integrals in the square brackets, using conditions (1) and (2) of Theorem 5 and the property $x_{00}^* < x_{00}$, and assuming $y_0^* < \min_{k=1,2;\, i=1,\dots,n}\{\psi_{ki}\}$, we have

$$\begin{aligned}
&\omega(0^+, y)\,\omega(x, 0^+) \\
&\quad \equiv \tilde{z}_{ki}(0^+, y)\,\tilde{z}_{ki}(x, 0^+) \\
&\quad = \left[\left[\int_{\varphi_i^{-1}(\psi_{ki})}^{\varphi_i^{-1}(y)} g_i(x, \varphi(x))\, \varphi_i'(x)\, dx \right]^{-1} \right. \\
&\qquad \left. - \left[\int_{x_{00}}^{0^+} \alpha_i(x, \varphi(x))\, dx \right]^{-1} \right] \\
&\quad \times \left[\left[\int_{\varphi_i^{-1}(\psi_{ki})}^{0^+} g_i(x, \varphi(x))\, \varphi_i'(x)\, dx \right]^{-1} \right. \\
&\qquad \left. - \left[\int_{x_{00}}^{x} \alpha_i(x, \varphi(x))\, dx \right]^{-1} \right] \\
&\quad = \left[\int_{x_{ki}}^{\varphi_i^{-1}(y)} g_i(x, \varphi(x))\, \varphi_i'(x)\, dx \right]^{-1} \\
&\qquad \times \left[\int_{x_{00}}^{x} \alpha_i(x, \varphi(x)) dx \right]^{-1} \cdot (-1) < 0.
\end{aligned} \tag{45}$$

(d) Determine the sign of $\omega_x'(x, y)\omega_y'(x, y)$. We get

$$\begin{aligned}
&\omega_x'(x, y) \cdot \omega_y'(x, y) \\
&\quad \equiv \left(\tilde{z}_{ki}(x, y)\right)_x' \cdot \left(\tilde{z}_{ki}(x, y)\right)_y' \\
&\quad = (-1) \cdot \alpha_i(x, \varphi(x)) \left[\int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt \right]^{-2} \\
&\qquad \times \left[\int_{\psi_{ki}}^{y} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \right]^{-2} \\
&\qquad \times g_i\left(\varphi_i^{-1}(y), \varphi\left(\varphi_i^{-1}(y)\right)\right).
\end{aligned} \tag{46}$$

From condition (1), it follows that

$$\alpha_i(x, \varphi(x)) > 0, \qquad g_i\left(\varphi_i^{-1}(y), \varphi\left(\varphi_i^{-1}(y)\right)\right) > 0. \tag{47}$$

Hence

$$\omega_x'(x, y) \cdot \omega_y'(x, y) \equiv \left(\tilde{z}_{ki}(x, y)\right)_x' \cdot \left(\tilde{z}_{ki}(x, y)\right)_y' < 0. \tag{48}$$

Because of (a)–(d) all assumptions (1)–(4) of Lemma 2 on $(0, x_{00}^*] \times (0, y_0^*]$ hold and (39) defines an implicit function

$$y(x) = y_{ki}(x) = \psi_{ki}(x) \tag{49}$$

on some interval $(0, x_{00}^{**}]$, where $x_{00}^{**} \le x_{00}^*$.

Now we turn to (37) and show that its solution given by formula (49) can be extended beyond x_{00}^{**}.

We show that $x_{00}^{**} = x_{00}$. On the contrary, assume $x_{00}^{**} < x_{00}$. Then, (after a proper transformation of variables) we can apply Lemma 2 to the point $(x_{00}^{**}, \psi_{ki}(x_{00}^{**}))$ again and, by well-known procedure, implicit function can be continued up to the boundary of the region $(0, x_{00}] \times (0, y_0]$.

If $\psi_{ki}(x_{00}^{**}) = y_0$, then $\tilde{z}_{ki}(x_{00}^{**}, y_0) = z_{ki}(x_{00}^{**}, y_0) = 0$. Moreover, we have

$$\begin{aligned}
z_{ki}(x_{00}^{**}, y_0) &= \int_{\psi_{ki}}^{y_0} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \\
&\quad - \int_{x_{00}}^{x_{00}^{**}} \alpha_i(t, \varphi(t))\, dt > 0
\end{aligned} \tag{50}$$

since $x_{00}^{**} < x_{00}$ and $\psi_{ki} < y_0$ by condition (3). This is a contradiction and $x_{00}^{**} = x_{00}$. Therefore, the implicit function $y_{ki} = \psi_{ki}(x)$ can be continued on the whole interval $(0, x_{00}]$. Similarly we will show that the inequality

$$\psi_{ki}(x_{00}) < y_0 \tag{51}$$

holds. We have

$$\begin{aligned}
&z_{ki}(x_{00}, y_0) \\
&\quad = \int_{\psi_{ki}}^{y_0} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \\
&\qquad - \int_{x_{00}}^{x_{00}} \alpha_i(t, \varphi(t))\, dt \\
&\quad = g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt > 0,
\end{aligned} \tag{52}$$

because, by condition (3), $\psi_{ki} < y_0$ holds. It is obvious that

$$\begin{aligned}
&z_{ki}(x_{00}, \psi_{ki}(x_{00})) = 0, \\
&\left(z_{ki}(x_{00}, y)\right)_y' = g_i\left(\varphi_i^{-1}(y), \varphi\left(\varphi_i^{-1}(y)\right)\right) > 0.
\end{aligned} \tag{53}$$

Because the function $z_{ki}(x_{00}, y)$ is monotonously increasing and

$$0 = z_{ki}\left(x_{00}, \psi_{ki}\left(x_{00}\right)\right) < z_{ki}\left(x_{00}, y_0\right), \tag{54}$$

we get $\psi_{ki}(x_{00}) < y_0$. Hence inequality (51) is proved.

Now we will investigate the behavior of implicit curves in a neighborhood of the function φ. Since

$$\psi_{1i} < \varphi_i\left(x_{00}\right) < \psi_{2i}, \tag{55}$$

we have (by condition (8))

$$\begin{gathered}\varphi_i\left(x_{1i}\right) < \varphi_i\left(x_{00}\right) < \varphi_i\left(x_{2i}\right), \\ x_{1i} < x_{00} < x_{2i}.\end{gathered} \tag{56}$$

Thus, (in the first integral we substitute $t = \varphi_i(s)$)

$$\begin{aligned} & z_{ki}\left(x, \varphi_i(x)\right) \\ &= \int_{\psi_{ki}}^{\varphi_i(x)} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt \\ &\quad - \int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt \\ &= \int_{\varphi_i^{-1}(\psi_{ki})}^{x} g_i(s, \varphi(s))\, \varphi_i'(s)\, ds \\ &\quad - \int_{x_{00}}^{x} \alpha_i(t, \varphi(t))\, dt \\ &= \int_{x_{00}}^{x} \left[g_i(t, \varphi(t))\, \varphi_i'(t) - \alpha_i(t, \varphi(t))\right] dt \\ &\quad + \int_{x_{ki}}^{x_{00}} g_i(t, \varphi(t))\, \varphi_i'(t)\, dt \\ &= -\int_{x_{00}}^{x} G_i(t, \varphi(t))\, dt \\ &\quad - \int_{x_{00}}^{x_{ki}} g_i(t, \varphi(t))\, \varphi_i'(t)\, dt, \end{aligned} \tag{57}$$

where $k = 1, 2$ and $i = 1, \ldots, n$. Consequently, we deduce that

$$z_{1i}\left(x, \varphi_i(x)\right) > 0, \quad z_{2i}\left(x, \varphi_i(x)\right) < 0, \quad x \in \left(0, x_{00}\right]. \tag{58}$$

Since functions z_{1i}, z_{2i} increase with respect to their second co-ordinates and

$$z_{1i}\left(x, \psi_{1i}(x)\right) \equiv 0, \qquad z_{2i}\left(x, \psi_{2i}(x)\right) \equiv 0 \tag{59}$$

on $(0, x_{00}]$, we get

$$\psi_{1i}(x) < \varphi_i(x) < \psi_{2i}(x), \quad x \in \left(0, x_{00}\right] \tag{60}$$

for each $i \in \{1, \ldots, n\}$. Finally, we recall that

$$\psi_{ki}\left(0^+\right) = \varphi_i\left(0^+\right) = 0, \quad k = 1, 2, \quad i = 1, \ldots, n. \tag{61}$$

Application of Ważewski's Method to an Implicitly Defined Domain. In the next part of the proof we will apply the topological method of Ważewski. We use the above mentioned functions given implicitly to define an open set

$$\begin{aligned} \Omega^0 := \Big\{(x, y) : (x, y) \in D_n, v_k(x, y) < 0, \\ k = 0, \ldots, n_1, u_j(x, y) < 0, j = n_1 + 1, \ldots, n\Big\}, \end{aligned} \tag{62}$$

where

$$\begin{aligned} & v_0(x, y) \equiv v_0(x) := x - \hat{x}, \quad 0 < \hat{x} < x_{00}, \ \hat{x} \text{ is a constant}, \\ & v_k(x, y) \equiv v_k(x, y_k) := \left(y_k - \psi_{1k}(x)\right)\left(y_k - \psi_{2k}(x)\right), \\ & \qquad k = 1, \ldots, n_1, \\ & u_j(x, y) \equiv u_j\left(x, y_j\right) := \left(y_j - \psi_{1j}(x)\right)\left(y_j - \psi_{2j}(x)\right), \\ & \qquad j = n_1 + 1, \ldots, n. \end{aligned} \tag{63}$$

Now we start to investigate the behavior of the integral curves of system (2) with respect to the boundary of the set Ω^0, that is, on the sets

$$\begin{aligned} V_\beta = \Big\{(x, y) : (x, y) \in D_n, v_\beta(x, y) = 0, v_k(x, y) \le 0, \\ k = 0, \ldots, n_1, \ k \ne \beta, u_j(x, y) \le 0, j = n_1 + 1, \ldots, n\Big\}, \\ \beta = 0, \ldots, n_1, \\ U_\alpha = \Big\{(x, y) : (x, y) \in D_n, u_\alpha(x, y) = 0, v_k(x, y) \le 0, \\ k = 0, \ldots, n_1, u_j(x, y) \le 0, j = n_1 + 1, \ldots, n, j \ne \alpha\Big\}, \\ \alpha = n_1 + 1, \ldots, n. \end{aligned} \tag{64}$$

First, we calculate the full derivative $\dot{v}_\beta(x, y)$ of the function $v_\beta(x, y)$ along trajectories of system (2) on the set V_β, $\beta = 0, \ldots, n_1$. It is clear that $\dot{v}_0(x) = 1 > 0$. Further, for $\beta = 1, \ldots, n_1$, we have

$$\begin{aligned} \dot{v}_\beta(x, y) &= \frac{d}{dx}\left[\left(y_\beta - \psi_{1\beta}(x)\right)\left(y_\beta - \psi_{2\beta}(x)\right)\right] \\ &= \left[\frac{d}{dx}\left(y_\beta - \psi_{1\beta}(x)\right)\right]\left(y_\beta - \psi_{2\beta}(x)\right) \\ &\quad + \left(y_\beta - \psi_{1\beta}(x)\right)\left[\frac{d}{dx}\left(y_\beta - \psi_{2\beta}(x)\right)\right] \\ &= \left[\frac{\alpha_\beta(x, y)}{g_\beta(x, y)} - \psi_{1\beta}'(x)\right]\left(y_\beta - \psi_{2\beta}(x)\right) \\ &\quad + \left[\frac{\alpha_\beta(x, y)}{g_\beta(x, y)} - \psi_{2\beta}'(x)\right]\left(y_\beta - \psi_{1\beta}(x)\right). \end{aligned} \tag{65}$$

On the set V_β, as it follows from the condition $v_\beta(x, y) = 0$, we have either $y_\beta = \psi_{1\beta}(x)$ or $y_\beta = \psi_{2\beta}(x)$.

(1) Let $y_\beta = \psi_{1\beta}(x)$. Then, we can see that

$$\begin{aligned}\dot{v}_\beta\left(x,\psi_{1\beta}(x)\right) &= \left[\frac{\alpha_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}{g_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}-\psi'_{1\beta}(x)\right]\\ &\quad\times\left(\psi_{1\beta}(x)-\psi_{2\beta}(x)\right).\end{aligned} \tag{66}$$

The derivative $\psi'_{1\beta}(x)$ of the function $\psi_{1\beta}(x)$ can be calculated using the well-known rules for differentiation of implicit functions given by identities

$$z_{1\beta}\left(x,\psi_{1\beta}(x)\right)\equiv 0. \tag{67}$$

We get

$$\psi'_{1\beta}(x)\equiv\frac{\alpha_\beta\left(x,\varphi(x)\right)}{g_\beta\left[\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right),\varphi\left(\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right)\right)\right]}. \tag{68}$$

Using that relation, we have

$$\begin{aligned}\dot{v}_\beta\left(x,\psi_{1\beta}(x)\right) &= \left[\frac{\alpha_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}{g_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}\right.\\ &\quad\left.+(-1)\frac{\alpha_\beta\left(x,\varphi(x)\right)}{g_\beta\left[\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right),\varphi\left(\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right)\right)\right]}\right]\\ &\quad\times\left(\psi_{1\beta}(x)-\psi_{2\beta}(x)\right)\\ &= \frac{\psi_{1\beta}(x)-\psi_{2\beta}(x)}{g_\beta\left[\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right),\varphi\left(\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right)\right)\right]}\\ &\quad\times\left[g_\beta\left[\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right),\varphi\left(\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right)\right)\right]\right.\\ &\quad\times\frac{\alpha_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}{g_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}\\ &\quad\left.-\alpha_\beta\left(x,\varphi(x)\right)\right].\end{aligned} \tag{69}$$

Since, by (60), $\psi_{1\beta}(x)<\psi_{2\beta}(x)$ and $\psi_{1\beta}(x)<\varphi_\beta(x)<\psi_{2\beta}(x)$, assumption (1) of the theorem yields

$$g_\beta\left[\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right),\varphi\left(\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right)\right)\right]>0. \tag{70}$$

In view of assumption (4b) ($W_\beta(x,y)<0$ when $(x,y)\in N_{\varphi_\beta}$), we get

$$\begin{aligned}&g_\beta\left[\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right),\varphi\left(\varphi_\beta^{-1}\left(\psi_{1\beta}(x)\right)\right)\right]\\ &\quad\times\frac{\alpha_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}{g_\beta\left(x,y_1,\ldots,y_{\beta-1},\psi_{1\beta}(x),y_{\beta+1},\ldots,y_n\right)}\\ &\quad-\alpha_\beta\left(x,\varphi(x)\right)<0\end{aligned} \tag{71}$$

and, consequently,

$$\dot{v}_\beta\left(x,\psi_{1\beta}(x)\right)>0. \tag{72}$$

(2) Let $y_\beta=\psi_{2\beta}(x)$. Then, by similar calculations and using assumption (4a) ($W_\beta(x,y)>0$ when $(x,y)\in N^{\varphi_\beta}$), we obtain

$$\dot{v}_\beta\left(x,\psi_{2\beta}(x)\right)>0. \tag{73}$$

Hence $\dot{v}_\beta(x,y)>0$ for all $\beta=1,\ldots,n_1$.

Now we will calculate the full derivative $\dot{u}_\alpha(x,y)$ of the function $u_\alpha(x,y)$ along trajectories of system (2) on the set U_α, where $\alpha=n_1+1,\ldots,n$. As above, we get

$$\begin{aligned}\dot{u}_\alpha(x,y) &= \frac{d}{dx}\left[\left(y_\alpha-\psi_{1\alpha}(x)\right)\left(y_\alpha-\psi_{2\alpha}(x)\right)\right]\\ &= \left[\frac{d}{dx}\left(y_\alpha-\psi_{1\alpha}(x)\right)\right]\left(y_\alpha-\psi_{2\alpha}(x)\right)\\ &\quad+\left(y_\alpha-\psi_{1\alpha}(x)\right)\left[\frac{d}{dx}\left(y_\alpha-\psi_{2\alpha}(x)\right)\right]\\ &= \left[\frac{\alpha_\alpha(x,y)}{g_\alpha(x,y)}-\psi'_{1\alpha}(x)\right]\left(y_\alpha-\psi_{2\alpha}(x)\right)\\ &\quad+\left[\frac{\alpha_\alpha(x,y)}{g_\alpha(x,y)}-\psi'_{2\alpha}(x)\right]\left(y_\alpha-\psi_{1\alpha}(x)\right).\end{aligned} \tag{74}$$

On the set U_α, as it follows from the condition $u_\alpha(x,y)=0$, we have either $y_\alpha=\psi_{1\alpha}(x)$ or $y_\alpha=\psi_{2\alpha}(x)$.

(1) Let $y_\alpha=\psi_{1\alpha}(x)$. Then, we get (proceeding like in the previous part of the proof)

$$\begin{aligned}\dot{u}_\alpha\left(x,\psi_{1\alpha}(x)\right) &= \left[\frac{\alpha_\alpha\left(x,y_1,\ldots,y_{\alpha-1},\psi_{1\alpha}(x),y_{\alpha+1},\ldots,y_n\right)}{g_\alpha\left(x,y_1,\ldots,y_{\alpha-1},\psi_{1\alpha}(x),y_{\alpha+1},\ldots,y_n\right)}\right.\\ &\quad\left.+(-1)\frac{\alpha_\alpha\left(x,\varphi(x)\right)}{g_\alpha\left[\varphi_\alpha^{-1}\left(\psi_{1\alpha}(x)\right),\varphi\left(\varphi_\alpha^{-1}\left(\psi_{1\alpha}(x)\right)\right)\right]}\right]\\ &\quad\times\left(\psi_{1\alpha}(x)-\psi_{2\alpha}(x)\right)\end{aligned}$$

$$= \frac{\psi_{1\alpha}(x) - \psi_{2\alpha}(x)}{g_\alpha\left[\varphi_\alpha^{-1}(\psi_{1\alpha}(x)), \varphi\left(\varphi_\alpha^{-1}(\psi_{1\alpha}(x))\right)\right]} \times \left[g_\alpha\left[\varphi_\alpha^{-1}(\psi_{1\alpha}(x)), \varphi\left(\varphi_\alpha^{-1}(\psi_{1\alpha}(x))\right)\right] \times \frac{\alpha_\alpha(x, y_1, \ldots, y_{\alpha-1}, \psi_{1\alpha}(x), y_{\alpha+1}, \ldots, y_n)}{g_\alpha(x, y_1, \ldots, y_{\alpha-1}, \psi_{1\alpha}(x), y_{\alpha+1}, \ldots, y_n)} - \alpha_\alpha(x, \varphi(x)) \right]. \tag{75}$$

Since, by (60), $\psi_{1\alpha}(x) < \psi_{1\beta}(x)$ and $\psi_{1\alpha}(x) < \varphi_\alpha(x) < \psi_{2\alpha}(x)$, applying assumption (1) of the theorem, we have

$$g_\alpha\left[\varphi_\alpha^{-1}(\psi_{1\alpha}(x)), \varphi\left(\varphi_\alpha^{-1}(\psi_{1\alpha}(x))\right)\right] > 0. \tag{76}$$

In view of assumption (4d) ($W_\alpha(x, y) > 0$ when $(x, y) \in N_{\varphi_\alpha}$), we have

$$g_\alpha\left[\varphi_\alpha^{-1}(\psi_{1\alpha}(x)), \varphi\left(\varphi_\alpha^{-1}(\psi_{1\alpha}(x))\right)\right] \times \frac{\alpha_\alpha(x, y_1, \ldots, y_{\alpha-1}, \psi_{1\alpha}(x), y_{\alpha+1}, \ldots, y_n)}{g_\alpha(x, y_1, \ldots, y_{\alpha-1}, \psi_{1\alpha}(x), y_{\alpha+1}, \ldots, y_n)} - \alpha_\alpha(x, \varphi(x)) > 0 \tag{77}$$

and, consequently

$$\dot{u}_\alpha(x, \psi_{1\alpha}(x)) < 0. \tag{78}$$

(2) Let $y_\alpha = \psi_{2\alpha}(x)$. Then, by similar calculations and using assumption (4)-(c) ($W_\alpha(x, y) < 0$ when $(x, y) \in N^{\varphi_\alpha}$), we obtain

$$\dot{u}_\alpha(x, \psi_{2\alpha}(x)) < 0. \tag{79}$$

Thus, $\dot{u}_\alpha(x, y) < 0$ for all $\alpha = n_1 + 1, \ldots, n$.

By [18, Lemma 3.1, page 281], for decreasing values of the variable x, the set of all egress points Ω_e^0, $\Omega_e^0 \subset \Omega^0$, from the set Ω^0 equals the set of all strict egress points Ω_{se}^0, $\Omega_{se}^0 \subset \Omega^0$ from the set Ω^0 (for definitions of Ω_e^0 and Ω_{se}^0 see, for example, [18, page 37 and page 278]); that is,

$$\Omega_e^0 = \Omega_{se}^0 = \bigcup_{\alpha=n_1+1}^{n} U_\alpha \setminus \bigcup_{\beta=0}^{n_1} V_\beta. \tag{80}$$

Let S be a subset of $\Omega^0 \cup \Omega_e^0$ defined as

$$S = \left\{ \left(\tilde{x}, y_1^0, \ldots, y_{n_1}^0, y_{n_1+1}, \ldots, y_n\right) : u_j\left(\tilde{x}, y_j\right) \le 0, \; j = n_1 + 1, \ldots, n, v_0(\tilde{x}) < 0, v_k\left(\tilde{x}, y_k^0\right) < 0, \; k = 1, \ldots, n_1 \right\}, \tag{81}$$

where y_k^0, $k = 1, \ldots, n_1$, are fixed. Then,

$$S \cap \Omega_e^0 = \left\{ \left(\tilde{x}, y_1^0, \ldots, y_{n_1}^0, y_{n_1+1}, \ldots, y_n\right) : u_\alpha(\tilde{x}, y_\alpha) = 0, u_j\left(\tilde{x}, y_j\right) \le 0, \alpha, j = n_1 + 1, \ldots, n, \; \alpha \ne j, v_k\left(\tilde{x}, y_k^0\right) < 0, k = 1, \ldots, n_1 \right\}. \tag{82}$$

We can see that the set $S \cap \Omega_e^0$ is a subset of the boundary ∂S of the set S, but it is not a retract of S. The explanation is simple and is based on the well-known fact that the boundary of an $(n - n_1)$-dimensional ball is not its retract [20], and the set S is topologically equivalent to an $(n - n_1)$-dimensional ball.

We show that $S \cap \Omega_e^0$ is a retract of Ω_e^0. Define a mapping

$$\pi : \Omega_e^0 \ni (x, y) \longmapsto \left(\tilde{x}, y_1^0, \ldots, y_{n_1}^0, \tilde{y}_{n_1+1}, \ldots, \tilde{y}_n\right) \in S \cap \Omega_e^0, \tag{83}$$

where

$$\tilde{y}_j = \psi_{1j}(\tilde{x}) + \left(y_j - \psi_{1j}(x)\right) \frac{\psi_{2j}(\tilde{x}) - \psi_{1j}(\tilde{x})}{\psi_{2j}(x) - \psi_{1j}(x)}, \quad j = n_1 + 1, \ldots, n. \tag{84}$$

With respect to the behavior of functions $\psi_{kj}(x)$, $k = 1, 2$, $j = n_1+1, \ldots, n$, the mapping π is continuous. From the definition of the mapping π, we get that $S \cap \Omega_e^0$ is a retract of Ω_e^0 and, furthermore, S is a compact set.

By seeing Corollary 3.1, [18, page 282] of Ważewski's theorem [18, Theorem 3.1, page 282] there exists a solution of system (2) with the initial conditions in the set $S \cap \Omega_e^0$ and it is contained in Ω^0 for $x \in (0, \tilde{x}]$. This solution satisfies (5) since the set Ω_e^0 is contracted to the initial point for $x \to 0^+$.

As we can change the constants y_k^0, $k = 1, \ldots, n_1$ within the inequality $v_k(\tilde{x}, y_k^0) < 0$, we can repeat the above-mentioned construction for every admissible fixed set $(y_1^0, \ldots, y_{n_1}^0)$. Then, there exists a class of solutions depending on n_1 parameters and lying in Ω^0 for $x \in (0, \tilde{x}]$. If $n_1 = n$, the assertion of theorem remains true as well. The proof is complete. □

Remark 6. If $\varphi \in M(x_0, y_0)$ is a solution of initial problem (2), (5), then

$$G_i(x, \varphi(x)) = -\varphi_i'(x) g_i(x, \varphi(x)) + \alpha_i(x, \varphi(x)) \equiv 0, \tag{85}$$

where $x \in (0, x_0]$ and $i = 1, \ldots, n$, and

$$G(x, \varphi(x)) \equiv \theta, \quad x \in (0, x_0]. \tag{86}$$

Then, condition (3) of Theorem 5 is satisfied and, in this case, we obtain the result on the dimension of the set of initial data generating solutions of initial problem (2), (5).

Denote

$$\begin{aligned} R_i(y) &:= \int_y^{y_0} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt, \\ F_i(y, \varepsilon) &:= \frac{R_i(y(1+\varepsilon))}{R_i(y)}, \\ F_i(0, \varepsilon) &:= \lim_{y \to 0^+} F_i(y, \varepsilon), \\ (F_i)_\varepsilon'(0, \varepsilon) &:= \lim_{y \to 0^+} (F_i)_\varepsilon'(y, \varepsilon), \end{aligned} \tag{87}$$

$i = 1, \ldots, n$, provided that the limits exist and are finite.

Theorem 7. *Let all assumptions of Theorem 5 hold and, moreover, the functions*

$$F_i : D^*(y_{00}, \varepsilon^*) \longrightarrow [0, \infty), \quad i = 1, \ldots, n, \tag{88}$$

satisfy the following:

(1) $F_i(y, \varepsilon)$ *is continuous with respect to* y *and continuously differentiable with respect to* ε*;*

(2) $(F_i)'_\varepsilon(y, 0) \neq 0$.

Then arbitrary solution $y(x) = (y_1(x), \ldots, y_n(x))$ *of initial problem* (2), (5) *mentioned in Theorem 5 has the asymptotic form*

$$y_i(x) = \varphi_i(x)(1 + o(1)), \quad x \longrightarrow 0^+, i = 1, \ldots, n. \tag{89}$$

Proof. From the proof of Theorem 5, it follows that, for coordinates $y_i(x)$, $i = 1, \ldots, n$, of the solution $y(x)$ of initial problem (2), (5), the inequalities

$$\psi_{1i}(x) < y_i(x) < \psi_{2i}(x), \quad i = 1, \ldots, n \tag{90}$$

are valid on an interval $(0, \tilde{x}]$ and we can assume that the inequalities

$$\psi_{1i}(x) < \varphi_i(x) < \psi_{2i}(x), \quad i = 1, \ldots, n \tag{91}$$

are valid on $(0, \tilde{x}]$ as well. Thus, to prove (89), it is sufficient to prove that

$$\lim_{x \to 0^+} \frac{\psi_{2i}(x)}{\psi_{1i}(x)} = 1, \quad i = 1, \ldots, n. \tag{92}$$

Applying L'Hospital's rule to the limits (92), we do not obtain the desired result. Therefore we will apply L'Hospital's rule to the limits

$$\lim_{x \to 0^+} \frac{R_i(\psi_{2i}(x))}{R_i(\psi_{1i}(x))}, \quad i = 1, \ldots, n. \tag{93}$$

This is possible because, in view of condition (2) of the theorem, we obviously have

$$\lim_{x \to 0^+} R_i(\psi_{ki}(x)) = \infty, \quad i = 1, \ldots, n, \ k = 1, 2. \tag{94}$$

Then, we use the auxiliary results for implicit functions. Applying the above-mentioned procedure, we get (for $i \in \{1, \ldots, n\}$)

$$\begin{aligned}
&\lim_{x \to 0^+} \frac{R_i(\psi_{2i}(x))}{R_i(\psi_{1i}(x))} \\
&\quad = \lim_{x \to 0^+} \frac{[R_i(\psi_{2i}(x))]'}{[R_i(\psi_{1i}(x))]'} \\
&\quad = \lim_{x \to 0^+} \frac{-g_i\left[\varphi_i^{-1}(\psi_{2i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{2i}(x))\right)\right]\psi'_{2i}(x)}{-g_i\left[\varphi_i^{-1}(\psi_{1i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{1i}(x))\right)\right]\psi'_{1i}(x)} \\
&\quad = \lim_{x \to 0^+} \frac{g_i\left[\varphi_i^{-1}(\psi_{2i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{2i}(x))\right)\right]}{g_i\left[\varphi_i^{-1}(\psi_{1i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{1i}(x))\right)\right]} \\
&\qquad \times \frac{\alpha_i(x, \varphi(x)) / g_i\left[\varphi_i^{-1}(\psi_{2i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{2i}(x))\right)\right]}{\alpha_i(x, \varphi(x)) / g_i\left[\varphi_i^{-1}(\psi_{1i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{1i}(x))\right)\right]} \\
&\quad = \lim_{x \to 0^+} \frac{g_i\left[\varphi_i^{-1}(\psi_{2i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{2i}(x))\right)\right]}{g_i\left[\varphi_i^{-1}(\psi_{2i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{2i}(x))\right)\right]} \\
&\qquad \times \frac{g_i\left[\varphi_i^{-1}(\psi_{1i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{1i}(x))\right)\right]}{g_i\left[\varphi_i^{-1}(\psi_{1i}(x)), \varphi\left(\varphi_i^{-1}(\psi_{1i}(x))\right)\right]} \\
&\qquad \times \frac{\alpha_i(x, \varphi(x))}{\alpha_i(x, \varphi(x))} = 1.
\end{aligned} \tag{95}$$

Hence it follows that there exists a sufficiently small constant $\varepsilon_i > 0$ such that the inequalities

$$R_i(\psi_{1i}(x))(1 - \varepsilon_i) < R_i(\psi_{2i}(x)) < R_i(\psi_{1i}(x))(1 + \varepsilon_i), \quad i = 1, \ldots, n, \tag{96}$$

hold on some interval $(0, x^*]$, $0 < x^* \leq \tilde{x}$, where x^* is a sufficiently small constant. Applying Lemma 3 and Corollary 4 (formula (27)) we show that inequalities (96) can be written as

$$\begin{aligned}
&R_i\left[\psi_{1i}(x)\left(1 + \varepsilon_i^2(\psi_{1i}(x), \varepsilon_i)\right)\right] \\
&\quad < R_i(\psi_{2i}(x)) < R_i\left[\psi_{1i}(x)\left(1 - \varepsilon_i^1(\psi_{1i}(x), \varepsilon_i)\right)\right], \\
&\qquad i = 1, \ldots, n,
\end{aligned} \tag{97}$$

where $\varepsilon_i^j(\psi_{1i}(x), \varepsilon_i)$, $j = 1, 2$, are constants depending on $\psi_{1i}(x)$ and ε_i such that

$$0 < \varepsilon_i^j(\psi_{1i}(x), \varepsilon_i), \quad \lim_{\varepsilon_i \to 0^+} \varepsilon_i^j(\psi_{1i}(x), \varepsilon_i) = 0, \quad j = 1, 2. \tag{98}$$

In Lemma 3 we set, for every fixed index $i = 1, \ldots, n$,

$$R(y) := R_i(y) = \int_y^{y_0} g_i\left(\varphi_i^{-1}(t), \varphi\left(\varphi_i^{-1}(t)\right)\right) dt,$$
$$F(y,\varepsilon) := F_i(y,\varepsilon) = \frac{R_i(y(1+\varepsilon))}{R_i(y)}. \tag{99}$$

Then, we have $R : (0, y_0) \to (0, \infty)$, $F : D^*(y_{00}, \varepsilon^*) \to [0, \infty)$ and we have the following:

(a) $R(y)$ is continuously differentiable, $R'(y) = -g_i(\varphi_i^{-1}(y), \varphi(\varphi_i^{-1}(y))) < 0$;

(b) $F(y,\varepsilon)$ is continuous with respect to y and continuously differentiable with respect to ε;

(c) $F'_\varepsilon(y, 0) \neq 0$ by condition (2) of the theorem.

Hence, Lemma 3 holds. By Corollary 4, we can write inequalities (96) in the form of (97). From (97), we get

$$\psi_{1i}(x)\left(1 - \varepsilon_i^1\left(\psi_{1i}(x), \varepsilon_i\right)\right) < \psi_{2i}(x) < \psi_{1i}(x)\left(1 + \varepsilon_i^2\left(\psi_{1i}(x), \varepsilon_i\right)\right) \tag{100}$$

for $(0, x^*]$. The last inequalities are equivalent to

$$\lim_{x \to +0} \frac{\psi_{2i}(x)}{\psi_{1i}(x)} = 1, \quad i = 1, \ldots, n. \tag{101}$$

The proof is complete. □

Example 8. Consider the following simple particular case of initial problem (2), (5):

$$x^2\left(1 + x^2 y^6\right) y' = y^2\left(1 + xy^7\right), \tag{102}$$

$$y(0^+) = 0. \tag{103}$$

This problem is a singular one. To apply Theorem 5, we rewrite (102), (103) as

$$\frac{1}{y^2} y' = \frac{1 + xy^7}{x^2\left(1 + x^2 y^6\right)}, \qquad y(0^+) = 0. \tag{104}$$

Set $n = 1$ (below we omit this index since we deal with a scalar problem), $\varphi(x) = x$, and

$$g(x,y) = \frac{1}{y^2}, \qquad \alpha(x,y) = \frac{1 + xy^7}{x^2\left(1 + x^2 y^6\right)}. \tag{105}$$

Without loss of generality, we assume that x_0 and y_0 in definition of D_1 are positive and sufficiently small (from Definition (1), property (4), it follows $x_0 < y_0$). Condition (1), obviously holds. Condition (2), is valid as well because

$$\int_{0^+} g(x, \varphi(x))\, \varphi'(x)\, dx = \int_{0^+} \frac{1}{x^2}\, dx = \infty,$$
$$\int_{0^+} \alpha(x, \varphi(x))\, dx = \int_{0^+} \frac{1 + x^8}{x^2\left(1 + x^8\right)}\, dx = \int_{0^+} \frac{1}{x^2} dx = \infty. \tag{106}$$

Since φ is a solution of (104), we have $G(x, \varphi(x)) \equiv 0$ and Condition (3) holds (see Remark 6 as well). Next,

$$\begin{aligned} W(x,y) &= g\left(\varphi^{-1}(y), \varphi\left(\varphi^{-1}(y)\right)\right) \frac{\alpha(x,y)}{g(x,y)} - \alpha(x, \varphi(x)) \\ &= g(y,y) \frac{\alpha(x,y)}{g(x,y)} - \alpha(x,x) \\ &= \frac{1}{y^2} \frac{\left(1 + xy^7\right) / \left(x^2\left(1 + x^2 y^6\right)\right)}{1/y^2} - \frac{1}{x^2} \\ &= \frac{y^6(y - x)}{x\left(1 + x^2 y^6\right)}. \end{aligned} \tag{107}$$

Set $n_1 = 1$ (i.e., conditions (4c) and (4d) are omitted). For x and y sufficiently small and positive, it is easy to see that $W(x,y) > 0$ if $y > x$ and $W(x,y) < 0$ if $y < x$. Hence, conditions (4a) and (4b) hold. According to Theorem 5, problem (102), (103) has at least one-parametric class of solutions.

Now we apply Theorem 7. We get

$$\begin{aligned} R(y) &= \int_y^{y_0} g\left(\varphi^{-1}(t), \varphi\left(\varphi^{-1}(t)\right)\right) dt \\ &= \int_y^{y_0} g(t,t)\, dt = \int_y^{y_0} \frac{1}{t^2} dt = \frac{1}{y} - \frac{1}{y_0}, \\ F(y,\varepsilon) &= \frac{R(y(1+\varepsilon))}{R(y)} = \frac{1/y(1+\varepsilon) - 1/y_0}{1/y - 1/y_0}. \end{aligned} \tag{108}$$

Now we see that condition (1) holds since F is continuous with respect to y and continuously differentiable with respect to ε for $(y,\varepsilon) \in D^*(y_{00}, \varepsilon^*)$. The values $F(0,\varepsilon)$ and $F'_\varepsilon(0,\varepsilon)$ are computed by above given formulas. Moreover, as

$$\frac{\partial F}{\partial \varepsilon}(y, 0) = \left.\frac{-1/y(1+\varepsilon)^2}{1/y - 1/y_0}\right|_{\varepsilon=0} = \frac{y_0}{y - y_0} \neq 0 \tag{109}$$

for $y \in (0, y_{00}]$ with $y_{00} < y_0$ and

$$\begin{aligned} \frac{\partial F}{\partial \varepsilon}(0,0) &= \lim_{y \to 0^+} \frac{\partial F}{\partial \varepsilon}(y,0) = -\lim_{y \to 0^+} \frac{1/y}{1/y - 1/y_0} \\ &= \lim_{y \to 0^+} \frac{y_0}{y - y_0} = -1 \neq 0. \end{aligned} \tag{110}$$

condition (2) holds too. Theorem 7 is valid and every solution of problem (102), (103) mentioned in Theorem 5 satisfies (89); that is,

$$y(x) = x(1 + o(1)) \tag{111}$$

for $x \to 0^+$.

Finally, we remark that, instead of (102), the same investigation can be performed, for example, for

$$x^2\left(1 + x^2 y^6\right) y' = y^2\left(1 + xy^7 + y^{20}\right), \tag{112}$$

where the function φ is no longer a solution and condition (3) of Theorem 5 holds (the moment x_{00} can be sufficiently small).

4. Generalization

The following theorem improves Theorems 5 and 7. In particular, comparing assumptions of Theorems 5 and 7 with Theorem 9, we see that the corresponding assumptions in Theorem 9 are assumed to be valid on some sets that are reductions of sets used in Theorems 5 and 7.

For given constants $C_i^* > 1$ and $0 < C_i^{**} < 1$, $i = 1, \ldots, n$, and for $\varphi \in M(x_0, y_0)$, define sets

$$\begin{gathered} N^{\varphi_i}(C_i^*) := \{(x,y) : (x,y) \in D_n, \varphi_i(x) < y_i < C_i^* \varphi_i(x)\}, \\ N_{\varphi_i}(C_i^{**}) := \{(x,y) : (x,y) \in D_n, C_i^{**} \varphi_i(x) < y_i < \varphi_i(x)\}, \\ N := \{(x,y) : (x,y) \in D_n, \\ C_i^{**} \varphi_i(x) < y_i < C_i^* \varphi_i(x), \ i = 1, \ldots, n\}. \end{gathered} \tag{113}$$

Theorem 9. *Let, for some constants C_i^*, C_i^{**}, $C_i^* > 1$, $0 < C_i^{**} < 1$, $i = 1, \ldots, n$, and some function $\varphi \in M(x_0, y_0)$, the following conditions be fulfilled.*

(a) *Consider $g_i(x,y) \in C(N)$, $\alpha_i(x,y) \in C(N)$, $i = 1, \ldots, n$.*

(b) *Assumptions (1)–(3) of Theorem 5 hold.*

(c) *For $i = 1, \ldots, n_1$,*

$$\begin{aligned} W_i(x,y) > 0 \quad &\text{if} \quad (x,y) \in N^{\varphi_i}(C_i^*) \cap N, \\ W_i(x,y) < 0 \quad &\text{if} \quad (x,y) \in N_{\varphi_i}(C_i^{**}) \cap N. \end{aligned} \tag{114}$$

We omit this assumption if $n_1 = 0$.

(d) *For $i = n_1 + 1, \ldots, n$,*

$$\begin{aligned} W_i(x,y) < 0 \quad &\text{if} \quad (x,y) \in N^{\varphi_i}(C_i^*) \cap N, \\ W_i(x,y) > 0 \quad &\text{if} \quad (x,y) \in N_{\varphi_i}(C_i^{**}) \cap N. \end{aligned} \tag{115}$$

We omit this assumption if $n_1 = n$.

(e) *Functions $F_i(y, \varepsilon)$, $i = 1, \ldots, n$, satisfy the assumptions of Theorem 7.*

Then, there exists at least an n_1-parametric family of solutions

$$y(x) = (y_1(x), \ldots, y_n(x)) \tag{116}$$

such that

$$y_i(x) = \varphi_i(x)(1 + o(1)), \quad i = 1, \ldots, n, \tag{117}$$

holds for $x \to 0^+$ and $(x, y(x)) \in D_n$.

Proof. The proof is omitted since it is similar to the proofs of Theorems 5 and 7. □

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Research Article

Stability of Impulsive Differential Systems

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The asymptotic phase property and reduction principle for stability of a trivial solution is generalized to the case of the noninvertible impulsive differential equations in Banach spaces whose linear parts split into two parts and satisfy the condition of separation.

1. Introduction

The reduction principle in the theory of stability for systems of autonomous differential equations for the first time was proved by Pliss [1]. For systems of nonautonomous differential equations it was extended by Aulbach [2]; see also Pötzsche [3]. The analogy of the reduction principle for differential equations in Banach spaces was proved by Lykova [4] and for nonautonomous difference equations in Banach spaces by Reinfelds and Janglajew [5]. Several works [6, 7] are devoted to different modifications and applications of the reduction principle. In this paper, we generalize the reduction principle to the case of the noninvertible impulsive differential equations in Banach spaces whose linear part split into two parts and satisfy the condition of separation.

2. The Statement of the Problem

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. By $\mathfrak{L}(\mathbf{X})$ and $\mathfrak{L}(\mathbf{Y})$ we mean the Banach spaces of bounded linear operators. Consider the following system of impulsive differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= A(t)x + f(t,x,y),\\
\frac{dy}{dt} &= B(t)y + g(t,x,y),\\
\Delta x|_{t=\tau_i} &= x(\tau_i+0) - x(\tau_i-0)\\
&= C_i x(\tau_i-0) + p_i(x(\tau_i-0), y(\tau_i-0)),\\
\Delta y|_{t=\tau_i} &= y(\tau_i+0) - y(\tau_i-0)\\
&= D_i y(\tau_i-0) + q_i(x(\tau_i-0), y(\tau_i-0)),
\end{aligned} \tag{1}$$

where

(i) the mappings $A:\mathbb{R}\to\mathfrak{L}(\mathbf{X})$ and $B:\mathbb{R}\to\mathfrak{L}(\mathbf{Y})$ are locally integrable in the Bochner sense;

(ii) the mappings $f:\mathbb{R}\times\mathbf{X}\times\mathbf{Y}\to\mathbf{X}$ and $g:\mathbb{R}\times\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ are locally integrable in the Bochner sense with respect to t for fixed x and y, and in addition they satisfy the uniform Lipschitz conditions

$$\begin{aligned}
\left|f(t,x,y) - f(t,x',y')\right| &\le \varepsilon\left(\left|x-x'\right| + \left|y-y'\right|\right),\\
\left|g(t,x,y) - g(t,x',y')\right| &\le \varepsilon\left(\left|x-x'\right| + \left|y-y'\right|\right);
\end{aligned} \tag{2}$$

(iii) for $i\in\mathbb{Z}$, $C_i\in\mathfrak{L}(\mathbf{X})$, and $D_i\in\mathfrak{L}(\mathbf{Y})$, the mappings $p_i:\mathbf{X}\times\mathbf{Y}\to\mathbf{X}$ and $q_i:\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ satisfy the uniform Lipschitz conditions

$$\begin{aligned}
\left|p_i(x,y) - p_i(x',y')\right| &\le \varepsilon\left(\left|x-x'\right| + \left|y-y'\right|\right),\\
\left|q_i(x,y) - q_i(x',y')\right| &\le \varepsilon\left(\left|x-x'\right| + \left|y-y'\right|\right);
\end{aligned} \tag{3}$$

(iv) the mappings $x\mapsto x + C_i x$ are homeomorphisms;

(v) the moments τ_i of impulse form a strictly increasing sequence

$$\cdots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots, \tag{4}$$

where the limit point may be only ∞.

Without loss of generality we assume that the system (1) has the equilibrium points $x = 0$, $y = 0$,

$$\begin{aligned} &f(t,0,0) = 0, \quad g(t,0,0) = 0, \\ &p_i(t,0,0) = 0, \quad q_i(t,0,0) = 0. \end{aligned} \tag{5}$$

Using the suitable bump function it is possible for the analysis of local stability of the trivial solution to reduce to investigation of the global stability of the trivial solution if the nonlinear terms of (1) are uniform Lipschitz with respect to time and with a sufficient small constant in a fixed radius tubular neighbourhood of the trivial solution.

For simplicity, we assume that the linear part of (1) is decoupled in two separate parts. In many cases, this can be reached via the so-called kinematic similarity transformation [8, 9]. More generally via kinematic similarity transformation, the linear system can be reduced to the same almost reducible system [10], a system with a diagonal part and a small nondiagonal part. However, the kinematic transformation can grow unboundedly as the nondiagonal part tends to zero.

Definition 1 (see [11, 12]). By the solution to an impulsive system one means a piecewise absolutely continuous mapping with discontinuities of the first kind at the points $t = \tau_i$ which for almost all t satisfies system (1) and for $t = \tau_i$ satisfies the conditions of a "jump."

Note that condition (v) together with the Lipschitz property with respect to x and y of the right-hand side ensures that there is a unique solution.

Let $\Phi(\cdot, s, x, y) = (x(\cdot, s, x, y), y(\cdot, s, x, y)) : [s, +\infty) \to \mathbf{X} \times \mathbf{Y}$ be the solution of system (1), where $\Phi(s+0, s, x, y) = (x(s+0, s, x, y), y(s+0, s, x, y)) = (x, y)$. At the break points τ_i the values for all solutions are taken at $\tau_i + 0$ unless otherwise indicated. For short, we will use the notation $\Phi(t) = (x(t), y(t))$.

Let $X(t, s)$ and $Y(t, s)$ be the evolutionary operators of the impulsive linear differential equations

$$\begin{aligned} &\frac{dx}{dt} = A(t)\,x, \\ &\Delta x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) = C_i x(\tau_i - 0), \end{aligned} \tag{6}$$

and, respectively,

$$\begin{aligned} &\frac{dy}{dt} = B(t)\,y, \\ &\Delta y|_{t=\tau_i} = y(\tau_i + 0) - y(\tau_i - 0) = D_i y(\tau_i - 0). \end{aligned} \tag{7}$$

We assume that the operators $X(t, s)$ and $Y(t, s)$ satisfy the *condition of separation* [7]:

$$\begin{aligned} \nu = \max\Bigg(&\sup_s \int_{-\infty}^{s} |Y(s,t)|\,|X(t,s)|\,dt \\ &+ \sup_s \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|X(\tau_i - 0, s)|, \\ &\sup_s \int_{s}^{+\infty} |X(s,t)|\,|Y(t,s)|\,dt \\ &+ \sup_s \sum_{s<\tau_i} |X(s,\tau_i)|\,|Y(\tau_i - 0, s)| \Bigg) < +\infty, \end{aligned} \tag{8}$$

where ν *is the constant of separation*.

To prove the theorems and lemmas, we use integrals which include evolutionary operators in their integrands. That is why it is more useful to estimate not the evolutionary operators but the corresponding integrals. Doing so, on the one hand, the conditions of theorems and lemmas are released from unnecessary technical limitations and, on the other hand, we obtain the conditions that are close to the necessary conditions.

If $A(t) = A$, $B(t) = B$, $C_i = 0$, and $D_i = 0$, then $\nu = \int_0^{+\infty} |e^{-At}||e^{Bt}|dt$. Consequently, the integral converges if the spectrum of the mapping B is located to the left of the spectrum of the mapping A and the spectra are separated by a vertical line in the complex plane.

Let $\mathbf{PC}(\mathbb{R} \times \mathbf{X}, \mathbf{Y})$ be a set of mappings $u : \mathbb{R} \times \mathbf{X} \to \mathbf{Y}$ that are continuous for $(t, x) \in [\tau_i, \tau_{i+1}) \times \mathbf{X}$ and have discontinuities of the first kind for $t = \tau_i$.

The set

$$\begin{aligned} \mathfrak{M} = \Bigg\{ &u \in \mathbf{PC}(\mathbb{R} \times \mathbf{X}, \mathbf{Y}) \mid u(s,0) = 0, \\ &\sup_{s, x \ne 0} \frac{|u(s,x)|}{|x|} < +\infty \Bigg\} \end{aligned} \tag{9}$$

is a Banach space with the norm

$$\begin{aligned} \|u\| &= \sup_{s, x \ne 0} \frac{|u(s,x)|}{|x|}, \\ \mathfrak{M}(k) &= \Big\{ u \in \mathfrak{M} \mid \|u\| \le k, \\ &\qquad \left|u(s,x) - u(s,x')\right| \le k\left|x - x'\right| \Big\} \end{aligned} \tag{10}$$

are a closed subsets of $\mathbf{M}$.

3. Auxiliary Lemma

Lemma 2. *Let $u, u' \in \mathfrak{M}(k)$ and $\varepsilon\nu(k+1) < 1$. Then the following estimations are valid:*

$$\begin{aligned}
&\int_{-\infty}^{s} |Y(s,t)|\,|z(t)|\,dt + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \\
&\quad \le \frac{\nu|x|}{1-\varepsilon\nu(k+1)}, \\
&\int_{-\infty}^{s} |Y(s,t)|\,|z(t) - z'(t)|\,dt \\
&\quad + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0) - z'(\tau_i - 0)| \\
&\quad \le \frac{\nu}{1-\varepsilon\nu(k+1)} \\
&\quad \times \left(|x - x'| + \frac{\varepsilon\nu}{1-\varepsilon\nu(k+1)} |x| \, \|u - u'\| \right),
\end{aligned} \tag{11}$$

where $z : \mathbb{R} \to \mathbf{X}$ is the solution of the impulsive differential equations

$$\begin{aligned}
&\frac{dz}{dt} = A(t)z + f(t, z, u(t,z)), \\
&\Delta z|_{t=\tau_i} = C_i z(\tau_i - 0) + p_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0)))
\end{aligned} \tag{12}$$

satisfying the initial condition $z(s) = x$.

We remark that $X(\tau_i - 0, \tau_i) = (id_x + C_i)^{-1}$ and $|X(\tau_i - 0, \tau_i)| \le \nu$. It follows that (12) has a unique backward solution if $\varepsilon(k+1)\nu < 1$.

Proof. The solution of the impulsive system (12) for $t \le s$ is

$$\begin{aligned}
z(t) = {} & X(t,s)x + \int_s^t X(t,\tau) f(\tau, z(\tau), u(\tau, z(\tau)))\,d\tau \\
& - \sum_{t < \tau_i \le s} X(t,\tau_i) p_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))).
\end{aligned} \tag{13}$$

Taking into account that f and p_i satisfy the uniform Lipschitz conditions and u properties, the solution $z(t)$ can be estimated by

$$\begin{aligned}
|z(t)| \le {} & |X(t,s)|\,|x| \\
& + \varepsilon(k+1) \left(\int_t^s |X(t,\tau)|\,|z(\tau)|\,d\tau \right. \\
& \left. + \sum_{t < \tau_i \le s} |X(t,\tau_i)|\,|z(\tau_i - 0)| \right).
\end{aligned} \tag{14}$$

Multiplying the solution $z(t)$ by $|Y(s,t)|$ and integrating from $-\infty$ to s, we obtain

$$\begin{aligned}
&\int_{-\infty}^{s} |Y(s,t)|\,|z(t)|\,dt \le |x| \int_{-\infty}^{s} |Y(s,t)|\,|X(t,s)|\,dt \\
&\quad + \varepsilon(k+1) \sup_{\tau} \int_{-\infty}^{\tau} |Y(\tau,t)|\,|X(t,\tau)|\,dt \\
&\quad \times \left(\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\,d\tau + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \right).
\end{aligned} \tag{15}$$

Multiplying $z(\tau_i - 0)$ by $|Y(s,\tau_i)|$ and summing for all i with respect to $\tau_i \le s$, we obtain

$$\begin{aligned}
&\sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \\
&\quad \le |x| \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|X(\tau_i - 0, s)| \\
&\quad + \varepsilon(k+1) \sup_{\tau} \sum_{\tau_i \le \tau} |Y(\tau,\tau_i)|\,|X(\tau_i - 0, \tau)| \\
&\quad \times \left(\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\,d\tau + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \right).
\end{aligned} \tag{16}$$

Summing up we get that

$$\begin{aligned}
&\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\,d\tau + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \\
&\quad \le \nu|x| + \varepsilon\nu(k+1) \\
&\quad \times \left(\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\,d\tau + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \right).
\end{aligned} \tag{17}$$

From the last inequality, we get that

$$\begin{aligned}
&\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\,d\tau + \sum_{\tau_i \le s} |Y(s,\tau_i)|\,|z(\tau_i - 0)| \\
&\quad \le \frac{\nu|x|}{1 - \nu\varepsilon(k+1)}.
\end{aligned} \tag{18}$$

Now we estimate the difference $|z(t) - z'(t)|$ taking into consideration the properties of f, p_i, and u:

$$\begin{aligned}
&\left|z(t) - z'(t)\right| \\
&\quad \le |X(t,s)| \left|x - x'\right| + \varepsilon(k+1) \\
&\qquad \times \left(\int_t^s |X(t,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. + \sum_{t<\tau_i\le s} \left|X(t,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \right) \\
&\quad + \varepsilon \left\|u - u'\right\| \left(\int_t^s |X(t,\tau)| \left|z(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. + \sum_{t<\tau_i\le s} \left|X(t,\tau_i)\right| \left|z(\tau_i - 0)\right| \right).
\end{aligned} \tag{19}$$

Multiplying the difference $|z(t) - z'(t)|$ by $|Y(s,t)|$ and integrating from $-\infty$ to s, we obtain

$$\begin{aligned}
&\int_{-\infty}^s |Y(s,t)| \left|z(t) - z'(t)\right| dt \\
&\quad \le \left|x - x'\right| \int_{-\infty}^s |Y(s,t)| \left|X(t,s)\right| dt \\
&\qquad + \varepsilon(k+1) \sup_\tau \int_{-\infty}^\tau |Y(\tau,t)| \left|X(t,\tau)\right| dt \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \right) \\
&\qquad + \varepsilon \left\|u - u'\right\| \sup_\tau \int_{-\infty}^\tau |Y(\tau,t)| \left|X(t,\tau)\right| dt \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau)\right| d\tau + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0)\right| \right).
\end{aligned} \tag{20}$$

Multiplying the difference $|z(\tau_i - 0) - z'(\tau_i - 0)|$ by $|Y(s,\tau_i)|$ and summing for all i with respect to $\tau_i \le s$, we obtain

$$\begin{aligned}
&\sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \\
&\quad \le \left|x - x'\right| \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|X(\tau_i - 0, s)\right| \\
&\qquad + \varepsilon(k+1) \sup_\tau \sum_{\tau_i\le\tau} \left|Y(\tau,\tau_i)\right| \left|X(\tau_i - 0,\tau)\right| \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \right) \\
&\qquad + \varepsilon \left\|u - u'\right\| \sup_\tau \sum_{\tau_i\le\tau} \left|Y(\tau,\tau_i)\right| \left|X(\tau_i - 0,\tau)\right| \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0)\right| \right).
\end{aligned} \tag{21}$$

Summing up, we get that

$$\begin{aligned}
&\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \\
&\qquad + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \\
&\quad \le \nu \left(\left|x - x'\right| + \varepsilon(k+1) \right. \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. \left. + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \right) \right) \\
&\qquad + \varepsilon\nu \left\|u - u'\right\| \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0)\right| \right).
\end{aligned} \tag{22}$$

Applying the first result of Lemma 2, we get

$$\begin{aligned}
&\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \\
&\qquad + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \\
&\quad \le \nu \left(\left|x - x'\right| + \varepsilon(k+1) \right. \\
&\qquad \times \left(\int_{-\infty}^s |Y(s,\tau)| \left|z(\tau) - z'(\tau)\right| d\tau \right. \\
&\qquad\qquad \left. \left. + \sum_{\tau_i\le s} \left|Y(s,\tau_i)\right| \left|z(\tau_i - 0) - z'(\tau_i - 0)\right| \right) \right) \\
&\qquad + \varepsilon\nu \left\|u - u'\right\| \frac{\nu |x|}{1 - \varepsilon\nu(k+1)}.
\end{aligned} \tag{23}$$

From the last inequality we easily obtain (11). □

4. Existence of a Lipschitz Invariant Manifold

Theorem 3. *If $4\varepsilon\nu < 1$, then there exists a unique piecewise continuous mapping $u \in \mathfrak{M}(k)$ satisfying the following properties:*

(i) $u(t, x(t, s, x, u(s, x))) = y(t, s, x, u(s, x))$ *for* $t \geq s$;

(ii) $|u(s, x) - u(s, x')| \leq k|x - x'|$;

(iii) $u(t, 0) = 0$.

Proof. Consider in $\mathfrak{M}(k)$ the functional equation

$$\begin{aligned} u(s,x) &= \int_{-\infty}^{s} Y(s,\tau)\, g(\tau, z(\tau), u(\tau, z(\tau)))\, d\tau \\ &\quad + \sum_{\tau_i \leq s} Y(s,\tau_i)\, q_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))), \end{aligned} \tag{24}$$

where $z : \mathbb{R} \to \mathbf{X}$ is the solution of the impulsive differential equation system (12) satisfying the initial condition $z(s) = x$.

Consider the operator $\mathscr{L} : \mathfrak{M}(k) \to \mathfrak{M}(k)$ defined by the formula

$$\begin{aligned} \mathscr{L}u(s,x) &= \int_{-\infty}^{s} Y(s,\tau)\, g(\tau, z(\tau), u(\tau, z(\tau)))\, d\tau \\ &\quad + \sum_{\tau_i \leq s} Y(s,\tau_i)\, q_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))). \end{aligned} \tag{25}$$

If $4\varepsilon\nu < 1$, then

$$k = (2\varepsilon\nu)^{-1}\left(1 - 2\varepsilon\nu - \sqrt{1 - 4\varepsilon\nu}\right) = \frac{1 - \sqrt{1 - 4\varepsilon\nu}}{1 + \sqrt{1 - 4\varepsilon\nu}} < 1 \tag{26}$$

satisfies the equality

$$k = \varepsilon\nu(k+1)(1 - \varepsilon\nu(k+1))^{-1}. \tag{27}$$

Then

$$\begin{aligned} &|\mathscr{L}u(s,x)| \\ &\quad \leq \varepsilon(k+1) \\ &\qquad \left(\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\, d\tau + \sum_{\tau_i \leq s} |Y(s,\tau_i)|\,|z(\tau_i - 0)|\right) \\ &\quad \leq \frac{\varepsilon\nu(k+1)|x|}{1 - \varepsilon\nu(k+1)} = k|x|. \end{aligned} \tag{28}$$

It follows that $\|\mathscr{L}u\| \leq k$.

Taking into account that g and q_i satisfy the uniform Lipschitz conditions, we get that

$$\begin{aligned} &\left|\mathscr{L}u(s,x) - \mathscr{L}u'(s,x')\right| \\ &\quad \leq \varepsilon(k+1) \\ &\qquad \times \left(\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau) - z'(\tau)|\, d\tau \right. \\ &\qquad\quad \left. + \sum_{\tau_i \leq s} |Y(s,\tau_i)|\,|z(\tau_i - 0) - z'(\tau_i - 0)|\right) \\ &\qquad + \varepsilon\|u - u'\| \left(\int_{-\infty}^{s} |Y(s,\tau)|\,|z(\tau)|\, d\tau \right. \\ &\qquad\quad \left. + \sum_{\tau_i \leq s} |Y(s,\tau_i)|\,|z(\tau_i - 0)|\right) \\ &\quad \leq k\left(|x - x'| + \frac{\varepsilon\nu}{1 - \varepsilon\nu(k+1)}|x|\,\|u - u'\|\right) \\ &\qquad + \frac{\varepsilon\nu|x|}{1 - \varepsilon\nu(k+1)}\|u - u'\| \\ &\quad = k|x - x'| + k|x|\,\|u - u'\|. \end{aligned} \tag{29}$$

We have that $\mathscr{L}u \in \mathfrak{M}(k)$ and $\mathscr{L}$ is a contraction in $\mathfrak{M}(k)$, and therefore there is only one solution satisfying the functional equation $\mathscr{L}u = u$.

In addition for $t \geq s$

$$\begin{aligned} u(t, z(t)) &= \int_{-\infty}^{t} Y(t,\tau)\, g(\tau, z(\tau), u(\tau, z(\tau)))\, d\tau \\ &\quad + \sum_{\tau_i \leq t} Y(t,\tau_i)\, q_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))) \\ &= Y(t,s)\, u(s,x) + \int_{s}^{t} Y(t,\tau)\, g(\tau, z(\tau), u(\tau, z(\tau)))\, d\tau \\ &\quad + \sum_{s < \tau_i \leq t} Y(t,\tau_i)\, q_i(z(\tau_i - 0), u(\tau_i - 0, z(\tau_i - 0))). \end{aligned} \tag{30}$$

Therefore for uniqueness of solutions we get for $t \geq s$

$$\begin{aligned} z(t) &= x(t, s, x, u(s,x)), \\ u(t, z(t)) &= y(t, s, x, u(s,x)). \end{aligned} \tag{31}$$

The theorem is proven. □

5. Behaviour of Solutions in the Neighbourhood of an Invariant Manifold

Theorem 4. *Let $4\varepsilon\nu < 1$. Then the following estimation is valid:*

$$\begin{aligned}\int_s^{+\infty} &|X(s,t)|\,|y(t)-u(t,x(t))|\,dt \\ &+\sum_{s<\tau_i}|X(s,\tau_i)|\,|y(\tau_i-0)-u(\tau_i-0,x(\tau_i-0))| \\ &\le \frac{\nu|y-u(s,x)|}{1-\varepsilon\nu(k+1)}.\end{aligned} \tag{32}$$

The inequality characterizes the integral distance between an arbitrary solution and an invariant manifold.

Proof. For an arbitrary map $\xi : \mathbb{R} \to \mathbf{Y}$, piecewise continuous from the right with points of discontinuity $t = \tau_i$ of the first type, we have the following relation:

$$\xi(t) = \xi(s) + \lim_{\delta\to+0}\frac{1}{\delta}\int_s^t (\xi(r+\delta)-\xi(r))\,dr. \tag{33}$$

Set $\xi(r) = Y(t,r)u(r,x(r))$. Then for $t \ge s$ we obtain

$$\begin{aligned}u(t,x(t)) &= Y(t,s)\,u(s,x) \\ &\quad+\lim_{\delta\to+0}\frac{1}{\delta}\int_s^t (Y(t,r+\delta)\,u(r+\delta,x(r+\delta)) \\ &\qquad -Y(t,r)\,u(r,x(r)))\,dr \\ &= Y(t,s)\,u(s,x) \\ &\quad+\lim_{\delta\to+0}\frac{1}{\delta}\int_s^t Y(t,r+\delta) \\ &\qquad\times(u(r+\delta,x(r+\delta)) \\ &\qquad -y(r+\delta,r,x(r),u(r,x(r))))\,dr \\ &\quad+\lim_{\delta\to+0}\frac{1}{\delta}\int_s^t (Y(t,r+\delta)\,y \\ &\qquad\times(r+\delta,r,x(r),u(r,x(r))) \\ &\qquad -Y(t,r)\,u(r,x(r)))\,dr.\end{aligned} \tag{34}$$

Let us note that

$$\begin{aligned}y(r+\delta,r,x_1,y_1) &= Y(r+\delta,r)\,y_1 \\ &\quad+\int_r^{r+\delta} Y(r+\delta,\tau)\,g(\tau,\Phi(\tau,r,x_1,y_1))\,d\tau \\ &\quad+\sum_{r<\tau_i\le r+\delta} Y(r+\delta,\tau_i)\,q_i(\Phi(\tau_i-0,r,x_1,y_1)).\end{aligned} \tag{35}$$

The third countable can be simplified:

$$\begin{aligned}\lim_{\delta\to+0}\frac{1}{\delta}\int_s^t &\left(\int_r^{r+\delta} Y(t,\tau)\,g(\tau,\Phi(\tau,r,x(r),u(r,x(r))))\,d\tau\right. \\ &\quad+\sum_{r<\tau_i\le r+\delta} Y(t,\tau_i)\,q_i \\ &\qquad\left.\times(\Phi(\tau_i-0,r,x(r),u(r,x(r))))\right)dr \\ &= \int_s^t Y(t,r)\,g(r,x(r),u(r,x(r)))\,dr \\ &\quad+\sum_{s<\tau_i\le t} Y(t,\tau_i)\,q_i(x(\tau_i-0),u(\tau_i-0,x(\tau_i-0))).\end{aligned} \tag{36}$$

Next we obtain

$$\begin{aligned}y(t)-u(t,x(t)) &= Y(t,s)(y-u(s,x)) \\ &\quad+\int_s^t Y(t,r)(g(r,x(r),y(r)) \\ &\qquad -g(r,x(r),u(r,x(r))))\,dr \\ &\quad+\lim_{\delta\to+0}\frac{1}{\delta}\int_s^t Y(t,r+\delta) \\ &\qquad\times(y(r+\delta,r,x(r),u(r,x(r))) \\ &\qquad -u(r+\delta,x(r+\delta)))\,dr \\ &\quad+\sum_{s<\tau_i\le t} Y(t,\tau_i) \\ &\qquad\times(q_i(x(\tau_i-0),y(\tau_i-0)) \\ &\qquad -q_i(x(\tau_i-0),u(\tau_i-0,x(\tau_i-0)))).\end{aligned} \tag{37}$$

Now we consider

$$\begin{aligned}x(r+\delta,r,x_1,y_1) &- x(r+\delta,r,x_1,u(r,x_1)) \\ &= \int_r^{r+\delta} X(r+\delta,\tau)(f(\tau,\Phi(\tau,r,x_1,y_1)) \\ &\qquad -f(\tau,\Phi(\tau,r,x_1,u(r,x_1))))\,d\tau \\ &\quad+\sum_{r<\tau_i\le r+\delta} X(r+\delta,\tau_i)(p_i(\Phi(\tau_i-0,r,x_1,y_1)) \\ &\qquad -p_i(\Phi(\tau_i-0,r,x_1,u(r,x_1))))).\end{aligned} \tag{38}$$

Thus,

$$\begin{aligned}
&\lim_{\delta\to+0}\frac{1}{\delta}\left|\int_s^t Y(t,r+\delta)\left(y(r+\delta,r,x(r),u(r,x(r)))\right.\right.\\
&\qquad\left.\left.-u(r+\delta,x(r+\delta)))\,dr\right|\right.\\
&\le k\lim_{\delta\to+0}\frac{1}{\delta}\\
&\quad\times\int_s^t|Y(t,r+\delta)|\\
&\qquad\times|x(r+\delta,r,x(r),u(r,x(r)))\\
&\qquad\quad-x(r+\delta,r,x(r),y(r))|\,dr\\
&\le k\lim_{\delta\to+0}\frac{1}{\delta}\\
&\quad\times\int_s^t|Y(t,r+\delta)|\\
&\qquad\times\left(\int_r^{r+\delta}|X(r+\delta,\tau)|\right.\\
&\qquad\quad\times|f(\tau,\Phi(\tau,r,x(r),y(r)))\\
&\qquad\quad-f(\tau,\Phi(\tau,r,x(r)u(r,x(r))))|\,d\tau\\
&\qquad+\sum_{r<\tau_i\le r+\delta}|X(r+\delta,\tau_i)|\\
&\qquad\quad\times|p_i(\Phi(\tau_i-0,r,x(r),y(r)))\\
&\qquad\quad-p_i(\Phi(\tau_i-0,r,x(r),u(r,x(r))))|)\,dr\\
&=k\left(\int_s^t|Y(t,r)|\right.\\
&\quad\times|f(r,x(r),y(r))-f(r,x(r),u(r,x(r)))|\,dr\\
&\quad+\sum_{s<\tau_i\le t}|Y(t,\tau_i)|\\
&\qquad\times\left|p_i(x(\tau_i-0),y(\tau_i-0))\right.\\
&\qquad\left.\left.-p_i(x(\tau_i-0),u(\tau_i-0,x(\tau_i-0)))\right|\right).
\end{aligned}\tag{39}$$

We introduce the expression $\eta(t)=|y(t)-u(t,x(t))|$. For $t\ge s$, we obtain the estimation

$$\begin{aligned}
\eta(t)\le\ &|Y(t,s)|\,\eta(s)\\
&+\varepsilon(k+1)\\
&\times\left(\int_s^t|Y(t,r)|\,\eta(r)\,dr+\sum_{s<\tau_i\le t}|Y(t,\tau_i)|\,\eta(\tau_i-0)\right).
\end{aligned}\tag{40}$$

Multiplying by $X(s,t)$, integrating, and summing analogously as in auxiliary Lemma 2, we obtain the inequality

$$\begin{aligned}
&\int_s^{+\infty}|X(s,t)|\,|y(t)-u(t,x(t))|\,dt\\
&\quad+\sum_{s<\tau_i}|X(s,\tau_i)|\,|y(\tau_i-0)-u(\tau_i-0,x(\tau_i-0))|\\
&\le\frac{\nu|y-u(s,x)|}{1-\varepsilon\nu(k+1)}.
\end{aligned}\tag{41}$$

□

6. Asymptotic Phase Type Property

Theorem 5. *Let $4\varepsilon\nu<1$. Then for every solution $(x(\cdot),y(\cdot)):[s,+\infty)\to\mathbf{X}\times\mathbf{Y}$ of the impulsive system (1) there is a such solution $\zeta(\cdot):[s,+\infty)\to\mathbf{X}$ of the impulsive system (12) that for all $t\ge s$ the following estimation is fulfilled:*

$$|\zeta(t)-x(t)|\le k_1|y(t)-u(t,x(t))|,\tag{42}$$

where

$$k_1=\frac{\varepsilon\nu}{\sqrt{1-4\varepsilon\nu}}.\tag{43}$$

Proof. The set of mappings

$$\mathfrak{M}_1=\left\{\kappa\in\mathbf{PC}(\mathbb{R}\times\mathbf{X}\times\mathbf{Y},\mathbf{X})\mid\sup_{s,x,y}\frac{|\kappa(s,x,y)|}{|y-u(s,x)|}<+\infty\right\}\tag{44}$$

is a Banach space with the norm

$$\|\kappa\|=\sup_{s,x,y}\frac{|\kappa(s,x,y)|}{|y-u(s,x)|},\tag{45}$$

respectively.

Consider the functional equation in $\mathfrak{M}_1$

$$\begin{aligned}
&\kappa(s,x,y)\\
&\quad=\int_s^{+\infty}X(s,\tau)\left(f(\tau,\Phi(\tau))\right.\\
&\qquad\qquad-f(\tau,x(\tau)+\kappa(\tau,\Phi(\tau)),\\
&\qquad\qquad\quad\left.u(\tau,x(\tau)+\kappa(\tau,\Phi(\tau)))\right))\,d\tau\\
&\qquad+\sum_{s<\tau_i}X(s,\tau_i)\\
&\qquad\quad\times(p_i(\Phi(\tau_i-0))\\
&\qquad\quad-p_i(x(\tau_i-0)+\kappa(\tau_i-0,\Phi(\tau_i-0)),\\
&\qquad\qquad u(\tau_i-0,x(\tau_i-0)\\
&\qquad\qquad+\kappa(\tau_i-0,\Phi(\tau_i-0)))))).
\end{aligned}\tag{46}$$

Consider the operator $\mathscr{L} : \mathfrak{M}_1 \to \mathfrak{M}_1$ defined by formula

$$\begin{aligned}
&\mathscr{L}\kappa(s,x,y)\\
&\quad= \int_s^{+\infty} X(s,\tau)\big(f(\tau,\Phi(\tau))\\
&\qquad\qquad - f(\tau, x(\tau)+\kappa(\tau,\Phi(\tau)),\\
&\qquad\qquad\quad u(\tau, x(\tau)+\kappa(\tau,\Phi(\tau))))\big)\,d\tau\\
&\quad+\sum_{s<\tau_i} X(s,\tau_i)\\
&\qquad\times\big(p_i(\Phi(\tau_i-0))\\
&\qquad\quad - p_i(x(\tau_i-0)+\kappa(\tau_i-0,\Phi(\tau_i-0)),\\
&\qquad\qquad u(\tau_i-0, x(\tau_i-0)+\kappa\\
&\qquad\qquad\quad((\tau_i-0,\Phi(\tau_i-0)))))\big).
\end{aligned} \tag{47}$$

We have the following estimation:

$$\begin{aligned}
&\left|\mathscr{L}\kappa(s,x,y)-\mathscr{L}\kappa'(s,x,y)\right|\\
&\quad\le \varepsilon(k+1)\\
&\qquad\times\int_s^{+\infty}|X(s,\tau)|\\
&\qquad\quad\times\left|\kappa(\tau,\Phi(\tau))-\kappa'(\tau,\Phi(\tau))\right|d\tau\\
&\qquad+\varepsilon(k+1)\sum_{s<\tau_i}\left|X(s,\tau_i)\right|\\
&\qquad\qquad\times\left|\kappa(\tau_i-0,\Phi(\tau_i-0))\right.\\
&\qquad\qquad\quad\left.-\kappa'(\tau_i-0,\Phi(\tau_i-0))\right|\\
&\quad\le \varepsilon(k+1)\left\|\kappa-\kappa'\right\|\\
&\qquad\times\int_s^{+\infty}|X(s,\tau)|\left|y(\tau)-u(\tau,x(\tau))\right|d\tau\\
&\qquad+\varepsilon(k+1)\left\|\kappa-\kappa'\right\|\sum_{s<\tau_i}\left|X(s,\tau_i)\right|\\
&\qquad\times\left|y(\tau_i-0)-u(\tau_i-0,x(\tau_i-0))\right|\\
&\quad\le \varepsilon(k+1)\left\|\kappa-\kappa'\right\|\frac{\nu\left|y-u(s,x)\right|}{1-\varepsilon\nu(k+1)}\\
&\quad= k\left\|\kappa-\kappa'\right\|\left|y-u(s,x)\right|.
\end{aligned} \tag{48}$$

Besides

$$\begin{aligned}
&\left|\mathscr{L}\kappa(s,x,y)\right|+\varepsilon((k+1)\|\kappa\|+1)\\
&\qquad\times\int_s^{+\infty}|X(s,\tau)|\left|y(\tau)-u(\tau,x(\tau))\right|d\tau\\
&\qquad+\varepsilon((k+1)\|\kappa\|+1)\\
&\qquad\times\sum_{s<\tau_i}\left|X(s,\tau_i)\right|\left|y(\tau_i-0)-u(\tau_i-0,x(\tau_i-0))\right|\\
&\quad\le\frac{\varepsilon\nu((k+1)\|\kappa\|+1)}{1-\varepsilon\nu(k+1)}\left|y-u(s,x)\right|\\
&\quad=\left(k\|\kappa\|+\frac{k}{k+1}\right)\left|y-u(s,x)\right|.
\end{aligned} \tag{49}$$

If

$$\|\kappa\|\le k_1=\frac{k}{1-k^2}=\frac{\varepsilon\nu}{\sqrt{1-4\varepsilon\nu}}, \tag{50}$$

then $\|\mathscr{L}\kappa\|\le k_1$. We have that $\mathscr{L}$ is a contraction and $\mathscr{L}\kappa\in\mathfrak{M}_1$. It follows that there is only one solution satisfying the functional equation $\mathscr{L}\kappa=\kappa$. In addition for $t\ge s$

$$\begin{aligned}
&\kappa(t,\Phi(t))\\
&\quad= \int_t^{+\infty} X(t,\tau)\big(f(\tau,\Phi(\tau))\\
&\qquad\qquad - f(\tau, x(\tau)+\kappa(\tau,\Phi(\tau)),\\
&\qquad\qquad\quad u(\tau, x(\tau)+\kappa(\tau,\Phi(\tau)))))\,d\tau\\
&\quad+\sum_{t<\tau_i} X(t,\tau_i)\\
&\qquad\times\big(p_i(\Phi(\tau_i-0))\\
&\qquad\quad - p_i(x(\tau_i-0)+\kappa(\tau_i-0,\Phi(\tau_i-0)),\\
&\qquad\qquad u(\tau_i-0, x(\tau_i-0)+\kappa(\tau_i-0,\Phi(\tau_i-0)))))\\
&\quad= X(t,s)\kappa(s,x,y)\\
&\qquad-\int_s^t X(t,\tau)\\
&\qquad\times\big(f(\tau,\Phi(\tau))\\
&\qquad\quad - f(\tau, x(\tau)+\kappa(\tau,\Phi(\tau)),\\
&\qquad\qquad\quad u(\tau, x(\tau)+\kappa(\tau,\Phi(\tau)))))\,d\tau\\
&\qquad-\sum_{s<\tau_i\le t} X(t,\tau_i)\big(p_i(\Phi(\tau_i-0))\\
&\qquad\qquad - p_i(x(\tau_i-0)+\kappa(\tau_i-0,\Phi(\tau_i-0)),\\
&\qquad\qquad\quad u(\tau_i-0, x(\tau_i-0)\\
&\qquad\qquad\quad +\kappa(\tau_i-0,\Phi(\tau_i-0)))))
\end{aligned}$$

$$
\begin{aligned}
&= -x(t) + X(t,s)(x + \kappa(s,x,y)) \\
&\quad + \int_s^t X(t,\tau) \\
&\qquad \times f(\tau, x(\tau) + \kappa(\tau, \Phi(\tau)), \\
&\qquad\quad u(\tau, x(\tau) + \kappa(\tau, \Phi(\tau)))) \, d\tau \\
&\quad + \sum_{s<\tau_i\le t} X(t,\tau_i) p_i \\
&\qquad \times (x(\tau_i - 0) + \kappa(\tau_i - 0, \Phi(\tau_i - 0)), \\
&\qquad\quad u(\tau_i - 0, x(\tau_i - 0) + \kappa(\tau_i - 0, \Phi(\tau_i - 0)))).
\end{aligned} \tag{51}
$$

Let

$$\zeta(t) = x(t) + \kappa(t, \Phi(t)), \tag{52}$$

where $\zeta(s) = x + \kappa(s,x,y)$. It follows that $\zeta(t)$ is a solution of (12) and

$$
\begin{aligned}
&|\zeta(t) - x(t)| \\
&\quad = |\kappa(t, \Phi(t))| \le \frac{\varepsilon\nu}{\sqrt{1-4\varepsilon\nu}} |y(t) - u(t, x(t))|.
\end{aligned} \tag{53}
$$

This completes the proof of the theorem. □

7. Stability of the Impulsive Equations

We assume in addition that

$$\mu = \sup_s \int_s^{+\infty} |Y(t,s)| \, dt + \sup_s \sum_{\tau_i > s} |Y(\tau_i - 0, s)| < +\infty. \tag{54}$$

Note that in case $B(t) = B$ and $D_i = 0$ we have $\mu = \int_0^{+\infty} |e^{Bt}| dt$.

Theorem 6. *Let $4\varepsilon\nu < 1$ and $2\varepsilon\mu < 1 + \sqrt{1-4\varepsilon\nu}$. Then the following estimation is valid:*

$$
\begin{aligned}
&\int_s^{+\infty} |y(t) - u(t, x(t))| \, dt \\
&\qquad + \sum_{s<\tau_i} |y(\tau_i - 0) - u(\tau_i - 0, x(\tau_i - 0))| \\
&\quad \le \frac{\mu |y - u(s,x)|}{1 - \varepsilon\mu(k+1)}.
\end{aligned} \tag{55}
$$

Proof. Since

$$
\begin{aligned}
k + 1 &= \frac{1 - \sqrt{1-4\varepsilon\nu}}{2\varepsilon\nu} = \frac{2}{1 + \sqrt{1-4\varepsilon\nu}}, \\
&2\varepsilon\mu < 1 + \sqrt{1-4\varepsilon\nu},
\end{aligned} \tag{56}
$$

we get

$$1 - \varepsilon\mu(k+1) = 1 - \frac{2\varepsilon\mu}{1 + \sqrt{1-4\varepsilon\mu}} > 0. \tag{57}$$

From Theorem 4 of behaviour of solutions, we get inequality (40). Then doing the integration and summing up, inequality (55) is obtained. □

Definition 7. A trivial solution of impulsive equation (1) is *integral stable* if for all $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that for all $|x| < \delta$ and $|y| < \delta$ and $t \ge s$ one has

$$\int_t^{t+1} |x(\tau)| \, d\tau < \varepsilon_1, \qquad \int_t^{t+1} |y(\tau)| \, d\tau < \varepsilon_1. \tag{58}$$

Definition 8. A trivial solution of impulsive equation (1) is asymptotically integral stable if it is integral stable and if there exists a $\delta > 0$ such that for all $|x| < \delta$ and $|y| < \delta$ one has

$$\lim_{t\to+\infty} \int_t^{t+1} |x(\tau)| \, d\tau = 0, \qquad \lim_{t\to+\infty} \int_t^{t+1} |y(\tau)| \, d\tau = 0. \tag{59}$$

Theorem 9. *Let $4\nu\varepsilon < 1$ and $2\varepsilon\mu < 1 + \sqrt{1-4\varepsilon\nu}$. The trivial solution of impulsive equation* (1) *is integral stable, asymptotically integral stable, or integral unstable if and only if the trivial solution of impulsive equation* (12) *is integral stable, asymptotically integral stable, or integral unstable.*

Proof. Suppose that the trivial solution of the system (12) is integral stable. Then for every $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that for all $|\zeta(s)| < \delta_1$ and $t \ge s$ we have

$$\int_t^{t+1} |\zeta(\tau)| \, d\tau < \frac{\varepsilon_1}{2}. \tag{60}$$

Let $|x| < \delta$ and $|y| < \delta$ where

$$\delta < \min\left\{ \frac{1 - \varepsilon\mu(k+1)}{2\mu(k_1+1)} \varepsilon_1, \delta_1 \right\}. \tag{61}$$

Then for $t \ge s$ we get

$$
\begin{aligned}
&\int_t^{t+1} |\zeta(\tau) - x(\tau)| \, d\tau \\
&\quad \le k_1 \int_t^{t+1} |y(\tau) - u(\tau, x(\tau))| \, d\tau \\
&\quad \le \frac{\mu k_1 (k+1)}{1 - \varepsilon\mu(k+1)} \delta < \frac{\varepsilon_1}{2}, \\
&\int_t^{t+1} |y(\tau) - u(\tau, \zeta(\tau))| \, d\tau \\
&\quad \le \int_t^{t+1} |y(\tau) - u(\tau, x(\tau))| \, d\tau \\
&\qquad + \int_t^{t+1} |u(\tau, x(\tau)) - u(\tau, \zeta(\tau))| \, d\tau \\
&\quad \le (1 + kk_1) \int_t^{t+1} |y(\tau) - u(\tau, x(\tau))| \, d\tau \\
&\quad \le \frac{\mu(1 + kk_1)(k+1)}{1 - \varepsilon\mu(k+1)} \delta < \frac{\varepsilon_1}{2}.
\end{aligned} \tag{62}
$$

Therefore

$$\begin{aligned}
&\int_t^{t+1} |x(\tau)|\, d\tau \\
&\quad \le \int_t^{t+1} \left|x(\tau) - \zeta(\tau)\right| d\tau \\
&\qquad + \int_t^{t+1} \left|\zeta(\tau)\right| d\tau < \varepsilon_1, \\
&\int_t^{t+1} \left|y(\tau)\right| d\tau \\
&\quad \le \int_t^{t+1} \left|y(\tau) - u(\tau, \zeta(\tau))\right| d\tau \\
&\qquad + \int_t^{t+1} \left|u(\tau, \zeta(\tau))\right| \\
&\quad < \frac{(k+1)\varepsilon_1}{2} < \varepsilon_1.
\end{aligned} \tag{63}$$

Suppose that the trivial solution of the system (12) is asymptotically integral stable. Then

$$\lim_{t \to +\infty} \int_t^{t+1} \left|\zeta(\tau)\right| d\tau = 0. \tag{64}$$

It follows that

$$\begin{aligned}
&\lim_{t \to +\infty} \int_t^{t+1} |x(\tau)|\, d\tau \\
&\quad \le \lim_{t \to +\infty} \int_t^{t+1} \left|x(\tau) - \zeta(\tau)\right| d\tau \\
&\qquad + \lim_{t \to +\infty} \int_t^{t+1} \left|\zeta(\tau)\right| d\tau = 0,
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\lim_{t \to +\infty} \int_t^{t+1} \left|y(\tau)\right| d\tau \\
&\quad \le \lim_{t \to +\infty} \int_t^{t+1} \left|y(\tau) - u(\tau, x(\tau))\right| d\tau \\
&\qquad + \lim_{t \to +\infty} \int_t^{t+1} |u(\tau, x(\tau))|\, d\tau = 0,
\end{aligned} \tag{66}$$

taking into account that

$$|u(t, x(t))| \le k\,|x(t)|. \tag{67}$$

If the trivial solution of (12) is integral unstable, then the trivial solution of (1) is integral unstable.

If the trivial solution of (1) is integral stable or asymptotically integral stable, then the trivial solution of (12) is also integral stable or asymptotically integral stable.

Let the trivial solution of (1) be integral unstable; then the trivial solution of (12) is integral unstable. Otherwise as before it follows that the trivial solution of (1) is integral stable. We get a contraction. The theorem is proven. □

Theorem 10. *Assume that the estimates*

$$\begin{aligned}
\mu(\beta) &= \sup_s \int_s^{+\infty} |Y(t,s)|\, e^{\beta(t-s)} dt \\
&\quad + \sup_s \sum_{\tau_i > s} \left|Y(\tau_i - 0, s)\right| e^{\beta(\tau_i - s)} < +\infty,
\end{aligned} \tag{68}$$

$$|Y(t,s)|\, e^{\beta(t-s)} \le M(\beta) \quad \forall t \ge s$$

are satisfied for some $\beta \ge 0$. If $4\varepsilon\nu < 1$ and $2\varepsilon\mu(\beta) < 1 + \sqrt{1 - 4\varepsilon\nu}$, then

$$\begin{aligned}
&\left|y(t) - u(t, x(t))\right| \\
&\quad \le \alpha(\beta)\, e^{-\beta(t-s)} \left|y - u(s,x)\right| \quad \text{for } t \ge s.
\end{aligned} \tag{69}$$

Proof. From Theorem 4 of behaviour of solutions, we get inequality (40). Multiplying by $e^{\beta(t-s)}$ and doing the integration and summing up, the inequality

$$\begin{aligned}
&\int_s^{+\infty} e^{\beta(t-s)} \left|y(t) - u(t, x(t))\right| dt \\
&\qquad + \sum_{s<\tau_i} e^{\beta(\tau_i - s)} \left|y(\tau_i - 0) - u(\tau_i - 0, x(\tau_i - 0))\right| \\
&\quad \le \frac{\mu(\beta)\left|y - u(s,x)\right|}{1 - \varepsilon\mu(\beta)(k+1)}
\end{aligned} \tag{70}$$

is obtained.

Then from inequality (40) for $t \ge s$ we get the estimation

$$\begin{aligned}
&\left|y(t) - u(t, x(t))\right| \\
&\quad \le M(\beta)\, e^{-\beta(t-s)} \left|y - u(s,x)\right| \\
&\qquad + \varepsilon(k+1) M(\beta)\, e^{-\beta(t-s)} \\
&\qquad \times \left(\int_s^{+\infty} e^{\beta(r-s)} \left|y(r) - u(r, x(r))\right| dr \right. \\
&\qquad\quad \left. + \sum_{s<\tau_i} \left|y(\tau_i - 0) - u(\tau_i - 0, x(\tau_i - 0))\right| \right) \\
&\quad \le M(\beta)\, e^{-\beta(t-s)} \left(1 + \frac{\varepsilon\mu(\beta)(k+1)}{1 - \varepsilon\mu(\beta)(k+1)}\right) \left|y - u(s,x)\right| \\
&\quad = \alpha(\beta)\, e^{-\beta(t-s)} \left|y - u(s,x)\right|.
\end{aligned} \tag{71}$$

□

Theorem 11. *Let $4\nu\varepsilon < 1$, $2\varepsilon\mu < 1 + \sqrt{1 - 4\varepsilon\nu}$,*

$$\left|y(t) - u(t, x(t))\right| \le \alpha \left|y - u(s,x)\right| \quad \text{if } t \ge s, \tag{72}$$

$$\lim_{t \to +\infty} \left(y(t) - u(t, x(t))\right) = 0. \tag{73}$$

The trivial solution of impulsive equation (1) *is stable, asymptotically stable, or unstable if and only if the trivial solution of impulsive equation* (12) *is stable, asymptotically stable, or unstable.*

Proof. Suppose that the trivial solution of the system (12) is stable. Then for every $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that for all $|\zeta(s)| < \delta_1$ and $t \geq s$ we have $|\zeta(t)| < \varepsilon_1/2$.

Let $|x| < \delta$ and $|y| < \delta$ where

$$\delta < \min\left\{\frac{\varepsilon_1}{2\alpha(k_1+1)(k+1)}, \delta_1\right\}. \tag{74}$$

Then for $t \geq s$ we get

$$\begin{aligned}
&|\zeta(t) - x(t)| \\
&\quad \leq k_1 |y(t) - u(t, x(t))| \\
&\quad \leq \alpha k_1 (k+1)\delta < \frac{\varepsilon_1}{2}, \\
&|y(t) - u(t, \zeta(t))| \\
&\quad \leq |y(t) - u(t, x(t))| \\
&\qquad + |u(t, x(t)) - u(t, \zeta(t))| \\
&\quad \leq (1 + kk_1)|y(t) - u(t, x(t))| \\
&\quad \leq \alpha(1 + kk_1)(k+1)\delta < \frac{\varepsilon_1}{2}.
\end{aligned} \tag{75}$$

Therefore

$$\begin{aligned}
&|x(t)| \leq |x(t) - \zeta(t)| + |\zeta(t)| < \varepsilon_1, \\
&|y(t)| \leq |y(t) - u(t, \zeta(t))| + |u(t, \zeta(t))| \\
&\qquad < \frac{(k+1)\varepsilon_1}{2} < \varepsilon_1.
\end{aligned} \tag{76}$$

Suppose that the trivial solution of the system (12) is asymptotically stable. Then

$$\lim_{t \to +\infty} \zeta(t) = 0. \tag{77}$$

It follows that

$$\begin{aligned}
&\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} (x(t) - \zeta(t)) + \lim_{t \to +\infty} \zeta(t) = 0, \\
&\lim_{t \to +\infty} y(t) = \lim_{t \to +\infty} (y(t) - u(t, x(t))) \\
&\qquad + \lim_{t \to +\infty} u(t, x(t)) = 0.
\end{aligned} \tag{78}$$

If the trivial solution of (12) is unstable, then the trivial solution of (1) is unstable.

If the trivial solution of (1) is stable or asymptotically stable, then the trivial solution of (12) is also stable or asymptotically stable.

Let the trivial solution of (1) be unstable; then the trivial solution of (12) is unstable. Otherwise as before it follows that the trivial solution of (1) is stable. We get a contraction. The theorem is proven. □

Remark 12. Let $\eta(t) = |y(t) - u(t, x(t))|$ be uniformly continuous on $t \in [s, +\infty)$ and let improper integral $\int_s^{+\infty} \eta(t)dt$ converge. Then $\lim_{t \to +\infty} \eta(t) = 0$ [13, page 32].

Remark 13. If we replace assumption (54) by the stronger one

$$\begin{aligned}
\mu_1 = &\int_0^{+\infty} \sup_s |Y(t+s, s)|\, dt \\
&+ \sum_{\tau_i > 0} \sup_s |Y(s + \tau_i - 0, s)| < +\infty,
\end{aligned} \tag{79}$$

then for $t \geq s$ it is possible to prove that $|Y(t,s)| \leq K\exp(-\lambda(t-s))$, where $K \geq 1$ and $\lambda > 0$. Further, if $\varepsilon > 0$ is sufficiently small, then using Gronwall's lemma for all $t \geq s$ the following estimation is valid:

$$|y(t) - u(t, x(t))| \leq Ke^{-\lambda_1(t-s)} |y - u(s, x)|, \tag{80}$$

where $0 < \lambda_1 < \lambda$.

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Research Article

Infinitely Many Periodic Solutions to Delay Differential Equations via Critical Point Theory

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By the critical point theory, infinitely many 4σ-periodic solutions are obtained for the system of delay differential equations $\dot{x}(t) = -f(x(t-\sigma))$, where $\sigma \in (0, +\infty)$ and $f \in C(\mathbb{R}^n, \mathbb{R}^n)$. It is shown that all the periodic solutions derived here are brought about by the time delay.

1. Introduction

This paper is concerned with the existence of periodic solutions to the system of delay differential equations

$$\dot{x}(t) = -f(x(t-\sigma)), \tag{1}$$

where $\sigma \in (0, +\infty)$ and $f \in C(\mathbb{R}^n, \mathbb{R}^n)$.

Delay differential equations have widely been applied to describe the dynamics phenomena in both natural and manmade processes such as chemistry, physics, engineering, and economics. The existence of the periodic solutions for delay differential equations has been extensively investigated by using various methods, including fixed point theorems [1–5], Hopf bifurcation theorems [6–8], variational methods [9–14], the methods of differential inequalities [15–21], and other effective approaches (e.g., see [22–24]). In [25–31], the minimal periods of the periodic solutions to Lipschitzian differential equations are estimated through the Lipschitz constants (see Remark 4).

The use of variational methods in the study of 4σ-periodic solutions of system (1) having a variational structure was introduced in 2005 by Guo and Yu [9]. Assume that

(F_1) f is odd in x; that is, $f(-x) = -f(x)$, for all $x \in \mathbb{R}^n$;

(F_2) there exists $F \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $F_x(x) = f(x)$, for all $x \in \mathbb{R}^n$, where F_x denotes the gradient of F.

In [9], the authors obtained the multiplicity results for periodic solutions to (1) in the case that f is asymptotically linear. Later, the existence of the periodic solution of (1) was investigated by using Morse theory and Galerkin methods [10]. For the other relative investigations, we refer the reader to [11–14].

Many practical problems, such as nonlinear population growth models and control systems working with potentially explosive chemical reactions, can be transformed into the form of (1). For example, by the change of variables $y = a\tanh(ax)$, the following generalized food-limited population model

$$\dot{y}(t) = -\theta \operatorname{sign}(y(t-1)) |y(t-1)|^{\gamma} \left(a^2 - y^2(t)\right) \tag{2}$$

is transformed equivalently into (1) with $n = 1$, $f(x) = \theta a^{\gamma} \operatorname{sign}(x) |\tanh(ax)|^{\gamma}$, and $\sigma = 1$, where θ and a are positive numbers. When $\gamma = 1$, $f'(0) = \theta a^2$. It is known from [24] that, with the slope $f'(0)$ increasing and tending to infinity, the number of the periodic solutions of (2) increases and tends to infinity. Naturally, one would conjecture that when $0 < \gamma < 1$, (2) possesses infinitely many periodic solutions, since in this case $\lim_{x \to 0} f'(x) = +\infty$.

Motivated by the above observation, in this paper, we study the existence of infinitely many periodic solutions to the system (1) under the assumptions (F_1), (F_2), and

(F_3) there are $1 < \alpha, \beta < 2$ and $d_1, r_0 > 0$ such that

(i) $0 < (f(x), x) \leqslant \alpha F(x)$, for all $x \in B_{r_0} \setminus \{\mathbf{0}\}$;

(ii) $d_1 |f(x)|^{\beta'} \leqslant F(x)$, for all $x \in B_{r_0}$,

where $1/\beta + 1/\beta' = 1$, $B_{r_0} = \{x \in \mathbb{R}^n : |x| \leqslant r_0\}$.

Here and subsequently, $(\cdot, \cdot)$, $|\cdot|$ denote the inner product and the standard norm in $\mathbb{R}^n$, respectively, and the bold face $\mathbf{0}$ represents the coordinate origin of $\mathbf{R}^n$. The main result of this paper is stated as follows.

Theorem 1. *Assume that* (F_1)–(F_3) *hold. Then* (1) *possesses a sequence of nonconstant* 4σ*-periodic solutions* $\{x_m\}$ *satisfying* $\|x_m\|_\infty \to 0$ *as* $m \to \infty$.

Example 2. When $0 < \gamma < 1$, it is easy to check that $f(x) = \theta a \operatorname{sign}(x) |\tanh(ax)|^\gamma$ satisfies (F_1)–(F_3) with $\alpha = \beta = 1 + \gamma$; then (2) has a sequence of nonconstant 4-periodic solutions $\{x_m\}$ satisfying $\|x_m\|_\infty \to 0$ as $m \to \infty$.

Remark 3. Let us compare the result here with that in the case of ordinary differential equations (ODE). Without the time delay, (1) reduces to the following system of ODE

$$x'(t) = f(x(t)). \tag{3}$$

Let $x(t) = x(t; x_0)$ be the solution of (3) satisfying the initial condition $x(0) = x_0 \neq \mathbf{0}$. Then the derivative of the Lyapunov function $V(x) = |x|^2$ along $x(t)$ reads

$$\left.\frac{dV}{dt}\right|_{(3)} = (-f(x(t)), x(t)). \tag{4}$$

From (F_3)-(i), we see that $dV/dt|_{(3)} < 0$ for $0 < |x| < r_0$, which implies that there is no any periodic orbit of (3) across $B_{r_0} \setminus \{\mathbf{0}\}$; that is, the trivial solution is an isolated periodic solution. However, by the above theorem, with the time delay, the system (1) possesses infinitely many periodic solutions in any neighborhood of the origin.

Remark 4. Consider the following system of mth order functional differential equations:

$$x^{(m)}(t) = f(x(\tau(t))), \quad t \in \mathbb{R}, \tag{5}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies the Lipschitz condition and $\tau : \mathbb{R}^1 \mapsto \mathbb{R}^1$ is a measurable function. The lower bounds for the periods of the periodic solutions to (5) and their special forms are estimated in [25–31]. From this perspective, Theorem 1 complements the information in the case of non-Lipschitzian differential equations. For the unique solvability of the periodic problems on functional differential equations, we refer the reader to [1, 15–21].

The remainder of this paper is divided into two parts. In the next section, we state the preliminaries on the variational structure for (1). In the final section, the proof of Theorem 1 will be given via the $\mathbb{Z}_2$-genus theory, together with an approximating argument.

2. Preliminaries

Let $L^2(S^1, \mathbb{R}^n)$ denote the set of n-tuples of 2π-periodic functions which are square integrable. If $x \in L^2(S^1, \mathbb{R}^n)$, it has a Fourier expansion

$$x(t) = a_0 + \sum_{j \in \mathbb{N}} \left(a_j \cos jt + b_j \sin jt\right), \tag{6}$$

where $a_i, b_j \in \mathbb{R}^n$ and the series converges in the space $L^2(S^1, \mathbb{R}^n)$. For $x \in L^2(S^1, \mathbb{R}^n)$ with its expansion (6), set $H := \{x \in L^2(S^1, \mathbb{R}^n) \mid \|x\|_H < \infty\}$, where

$$\|x\|_H := |a_0|^2 + \sum_{j \in \mathbb{N}} (1 + j) \left(|a_j|^2 + |b_j|^2\right). \tag{7}$$

Then H, equipped with the norm $\|\cdot\|_H$, is a Sobolev space.

On the other hand, for $x \in H$ with its expansion (6), set

$$\|x\| := |a_0|^2 + \sum_{j \in \mathbb{N}} j \left(|a_j|^2 + |b_j|^2\right). \tag{8}$$

Then H possesses another norm $\|\cdot\|$ which is equivalent to $\|\cdot\|_H$. In the following, we always employ $\|\cdot\|$ as the norm of H. The associated inner product with $\|\cdot\|$ is denoted by $\langle \cdot, \cdot \rangle$.

Now set

$$E := \{x \in H \mid x(\cdot + \pi) = -x(\cdot)\}. \tag{9}$$

Then E is a closed subspace of H and the Fourier expansion of $x \in E$ reduces to

$$x(t) = \sum_{j=1}^{\infty} \left[a_{2j-1} \cos(2j-1)t + b_{2j-1} \sin(2j-1)t\right]. \tag{10}$$

Thus with $x_1, x_2 \in E$ being expanded as

$$x_i(t) = \sum_{j=1}^{\infty} \left[a_{2j-1}^{(i)} \cos(2j-1)t + b_{2j-1}^{(i)} \sin(2j-1)t\right], \quad i = 1, 2, \tag{11}$$

we have

$$\langle x_1, x_2 \rangle = \sum_{j=1}^{\infty} (2j-1) \left\{\left(a_{2j-1}^{(1)}, a_{2j-1}^{(2)}\right) + \left(b_{2j-1}^{(1)}, b_{2j-1}^{(2)}\right)\right\}. \tag{12}$$

For $x, y \in E$, we call y a weak derivative of x and denote it by $\dot{x} = y$ if

$$\int_0^{2\pi} \left(x(t), z'(t)\right) dt = -\int_0^{2\pi} (y(t), z(t))\, dt, \quad \forall z \in C^\infty\left(S^1, \mathbb{R}^n\right). \tag{13}$$

Further, for $x \in C^\infty(S^1, \mathbb{R}^n) \cap E$ with its expansion (10), define

$$\begin{aligned} A(x) &:= \frac{1}{2} \int_0^{2\pi} \left(\dot{x}\left(t + \frac{\pi}{2}\right), x(t)\right) dt \\ &= \frac{1}{2} \sum_{j=1}^{\infty} (-1)^j (2j-1) \left(\left|a_{2j-1}\right|^2 + \left|b_{2j-1}\right|^2\right). \end{aligned} \tag{14}$$

Then it is easy to check that $|A(x)| \leqslant \|x\|^2$ for $x \in C^\infty(S^1, \mathbb{R}^n) \cap E$. Therefore A extends to all of E as a continuous quadratic form. This extension will still be denoted by A.

Let $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfy

$$\widetilde{F}(-x) = \widetilde{F}(x), \quad \left|\widetilde{F}(x)\right| \leqslant C_1 + C_2|x|^s, \quad x \in \mathbb{R}^n \tag{15}$$

for some $s \in [1, \infty)$. Define

$$\Phi(x) := \int_0^{2\pi} \widetilde{F}(x(t))\, dt, \quad x \in E \tag{16}$$

and $I(x) = A(x) + \Phi(x)$, $x \in E$. The following lemma is derived from [9, Lemma 2.2].

Lemma 5 (see [9]). *Let $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfy* (15). *Then $I \in C^1(E, \mathbb{R})$ and*

$$I'(x)\, y = A'(x)\, y + \int_0^{2\pi} \left(\widetilde{F}_x(x(t)), y(t)\right), \quad y \in E, \tag{17}$$

where

$$A'(x)\, y = \int_0^{2\pi} \left(\dot{x}\left(t + \frac{\pi}{2}\right), y(t)\right) dt, \quad y \in E. \tag{18}$$

Moreover, the existence of 2π-periodic solutions $x(t)$ for

$$x'(t) = -\widetilde{F}_x\left(x\left(t - \frac{\pi}{2}\right)\right) \tag{19}$$

satisfying $x \in E$ is equivalent to the existence of critical points of functional I.

Let $\{e_1, \ldots, e_n\}$ be the orthonormal basis of $\mathbb{R}^n$. For $k \in \mathbb{N}$, set

$$\begin{aligned} E_+(k) &:= \operatorname{span}\{\cos[(4k-1)t]\, e_i, \sin[(4k-1)t]\, e_i : \\ &\qquad i = 1, 2, \ldots, n\}, \\ E_-(k) &:= \operatorname{span}\{\cos[(4k-3)t]\, e_i, \sin[(4k-3)t]\, e_i : \\ &\qquad i = 1, 2, \ldots, n\}. \end{aligned} \tag{20}$$

For $l, m \in \mathbb{N} \cup \{+\infty\}$, define

$$V_l^\pm = \overline{\oplus_{k=1}^l E_\pm(k)}, \qquad V_l^m = V_l^- \oplus V_m^+, \tag{21}$$

where the closure is of E sense. Set $V^\pm := V_{+\infty}^\pm$; then $E = V^+ \oplus V^-$. In the rest of this paper, this decomposition will always be referred to when a point $x \in E$ is written as $x = x^+ + x^-$, where $x^\pm \in V^\pm$.

Remark 6. In view of (12), (14), and (18), we see that

$$A(x) = \frac{1}{2}\left(\|x^+\|^2 - \|x^-\|^2\right) \tag{22}$$

and that

$$A'(x)\, y = \langle x^+, y^+\rangle - \langle x^-, y^-\rangle, \quad x, y \in E. \tag{23}$$

The following lemma is derived from [32, Lemma 2.1].

Lemma 7 (see [32]). *For each $s \in [1, \infty)$ there is $\gamma_s > 0$ such that*

$$\|x\|_s \leqslant \gamma_s m^{-1/s} \|x\| \tag{24}$$

for all $x \in (V_{m-1}^{m-1})^\perp$ with $m \geqslant 2$, the orthogonal complement in E, where (and below) $\|\cdot\|_s$ denotes the usual L^s-norm.

3. Proof of Theorem 1

Without loss of generality we assume that $\sigma = \pi/2$ since, under the change of variables $y(t) = x(2\sigma t/\pi)$, (1) can be transformed into the system

$$\dot{y}(t) = -\widetilde{f}\left(y\left(t - \frac{\pi}{2}\right)\right), \tag{1$'$}$$

where $\widetilde{f}(y) = (2\sigma/\pi) f(y)$ still satisfies (F_1–F_3) with f being replaced by $\widetilde{f}$.

Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be such that $\chi(s) = 0$ for $s \leqslant r_0/2$, $\chi(s) = 1$ for $s \geqslant r_0$, and $\chi'(s) > 0$ for $s \in (r_0/2, r_0)$. Define $\widetilde{F}: \mathbb{R}^n \mapsto \mathbb{R}$ by

$$\widetilde{F}(x) := (1 - \chi(|x|))\, F(x) + \chi(|x|)\, M_0|x|^\alpha, \tag{25}$$

where $M_0 = \inf\{F(x)/r_0^\alpha : r_0/2 \leqslant |x| \leqslant r_0\}$.

Let $\alpha' > 0$ be such that $1/\alpha + 1/\alpha' = 1$. By (F_3) we get

$$0 < \left(\widetilde{F}_x(x), x\right) \leqslant \alpha \widetilde{F}(x), \quad \forall x \in \mathbb{R}^n, \tag{26}$$

$$\widetilde{F}(x) \geqslant \begin{cases} C_1\left|\widetilde{F}_x(x)\right|^{\beta'}, & |x| \leqslant 1, \\ C_1\left|\widetilde{F}_x(x)\right|^{\alpha'}, & |x| > 1, \end{cases} \tag{27}$$

where (and below) C_j's stand for positive constants.

Lemma 8. *Let $\widetilde{F}: \mathbb{R}^n \mapsto \mathbb{R}$ be defined by* (25); *then $1 < \beta < \alpha < 2$, $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$, and*

$$C_2|x|^\alpha \leqslant \widetilde{F}(x) \leqslant \begin{cases} C_3|x|^\beta, & |x| \leqslant 1, \\ C_3|x|^\alpha, & |x| > 1. \end{cases} \tag{28}$$

Proof. From (25), it is easy to see that $\widetilde{F} \in C^1(\mathbb{R}^n, \mathbb{R})$. Now we start to prove (28). Let M be such a constant that $|\ln \widetilde{F}(x) - \alpha \ln|x|| \leqslant M$ for $x \in S_1 \equiv \partial B_1$. For $x \in \mathbb{R}^n$, $|x| \leqslant 1$, set $x_0 = x/|x|$; then $x_0 \in S_1$. Define $g(t) = \ln \widetilde{F}(tx_0) - \alpha \ln|tx_0|$, $t \in (0, 1]$. Then, by (26),

$$g'(t) = \left(\frac{\widetilde{F}_x(tx_0)}{\widetilde{F}(tx_0)}, x_0\right) - \alpha \leqslant 0, \tag{29}$$

which implies that $g(|x|) \geqslant g(1)$; that is,

$$\ln \widetilde{F}(x) - \alpha \ln|x| \geqslant \ln \widetilde{F}(x_0) - \alpha \ln|x_0| \geqslant -M. \tag{30}$$

It follows that $\widetilde{F}(x) \geqslant e^{-M}|x|^\alpha$ for $|x| \leqslant 1$, which, combining with (25), leads to the inequality on the left hand of (28) with C_2 being chosen adequately.

Again, for $x \in \mathbb{R}^n$, $|x| \leqslant 1$, set $x_0 = x/|x|$ and define $h(t) = (\widetilde{F}(tx_0))^{1/\beta} - t|x_0|/(\beta C_1^{1/\beta'})$, $t \in [0,1]$. Then by the first inequality in (27),

$$\begin{aligned} h'(t) &= \frac{1}{\beta}\left(\frac{\left(\widetilde{F}_x(tx_0), x_0\right)}{\left(\widetilde{F}(tx_0)\right)^{1/\beta'}} - \frac{1}{C_1^{1/\beta'}}|x_0|\right) \\ &\leqslant \frac{|x_0|}{\beta}\left(\frac{\left|\widetilde{F}_x(tx_0)\right|}{\left(\widetilde{F}(tx_0)\right)^{1/\beta'}} - \frac{1}{C_1^{1/\beta'}}\right) < 0. \end{aligned} \tag{31}$$

Thus $h(|x|) \leqslant h(0) = 0$, which leads to $\widetilde{F}(x) \leqslant C_3'|x|^\beta$, where $C_3' = 1/(\beta C_1^{1/\beta'})^\beta$. In the same way, from the second inequality in (27), we can arrive at $\widetilde{F}(x) \leqslant C_3''|x|^\alpha$ for $|x| > 1$, where the constant C_3'' only depends on α and C_1. With $C_3 = \max\{C_3', C_3''\}$, the inequalities on the right hand of (28) hold. Thus we get (28), which implies that $C_2|x|^\alpha \leqslant C_3|x|^\beta$ for $|x| \leqslant 1$ and that $1 < \beta < \alpha < 2$. The proof is complete. □

Now we consider the functional

$$I(x) = A(x) + \int_0^{2\pi} \widetilde{F}(x)\,dt, \quad x \in E. \tag{32}$$

Lemma 9. *I satisfies (PS) condition; that is, every sequence $\{x_k\} \subset E$ such that $\{I(x_k)\}$ is bounded and $I'(x_k) \to 0$ as $k \to \infty$ has a convergent subsequence.*

Proof. By Lemma 5, for $x \in E$, $I'(x)$ is defined by

$$I'(x)\,y = A'(x)\,y + \int_0^{2\pi}\left(\widetilde{F}_x(x), y\right)dt, \quad \forall y \in E. \tag{33}$$

To verify that I satisfies (PS) condition, we suppose $|I(x_k)| \leqslant C_4$ and $I'(x_k) \to 0$ as $k \to \infty$. Note that, for large k, $|I'(x_k)x| \leqslant \|x\|$. Thus for large k and $x = x_k$, from (32) and (33),

$$\begin{aligned} C_4 + \|x\| &\geqslant I(x) - \frac{1}{2}I'(x)\,x \\ &= \int_0^{2\pi}\left[\widetilde{F}(x) - \frac{1}{2}\left(\widetilde{F}_x(x), x\right)\right]dt. \end{aligned} \tag{34}$$

Noticing that $1 < \beta < \alpha < 2$, we see from (25) that, for all $x \in \mathbb{R}^n$,

$$\left(\widetilde{F}_x(x), x\right) \geqslant C_5 \max\left\{|x|^\alpha, |x|^\beta\right\} - C_6, \tag{35}$$

which, combining with (26) and (34), implies

$$\begin{aligned} C_4 + \|x\| &\geqslant \left(\alpha^{-1} - 2^{-1}\right)\int_0^{2\pi}\left(\widetilde{F}_x(x), x\right)dt \\ &\geqslant C_7 \max\left\{\|x\|_\alpha^\alpha, \|x\|_\beta^\beta\right\} - C_8, \end{aligned} \tag{36}$$

$$\max\left\{\|x\|_\alpha^\alpha, \|x\|_\beta^\beta\right\} \leqslant C_9(\|x\| + 1). \tag{37}$$

Next for large k, taking $x = x_k$ and $\varsigma = x_k^+$ in

$$\left|\int_0^{2\pi}\left(\widetilde{F}_x(x), \varsigma\right)dt + A'(x)\,\varsigma\right| = \left|I'(x)\,\varsigma\right| \leqslant \|\varsigma\| \tag{38}$$

and using (23), (27), and (28) and the Hölder inequality ($1/\alpha + 1/\alpha' = 1$, $1/\beta + 1/\beta' = 1$), we get

$$\begin{aligned} \|x^+\|^2 &\leqslant \left|\int_0^{2\pi}\left(\widetilde{F}_x(x), x^+\right)dt\right| + \|x^+\| \\ &\leqslant C_{10}\left(\int_{|x(t)|>1}|x|^{\alpha/\alpha'}|x^+|\,dt + \int_{|x(t)|\leqslant 1}|x|^{\beta/\beta'}|x^+|\,dt\right) \\ &\quad + \|x^+\| \\ &\leqslant C_{10}\left(\|x\|_\alpha^{\alpha/\alpha'}\|x^+\|_\alpha + \|x\|_\beta^{\beta/\beta'}\|x^+\|_\beta\right) + \|x^+\| \\ &\leqslant C_{11}\left(\|x\|_\alpha^{\alpha/\alpha'} + \|x\|_\beta^{\beta/\beta'} + 1\right)\|x^+\|, \end{aligned} \tag{39}$$

where the last inequality holds since E is compactly embedded in $L^s(S^1, \mathbb{R}^n)$ for $s \geqslant 1$. It follows from (36) that

$$\|x^+\| \leqslant C_{12}\left(\|x\|^{1/\alpha'} + \|x\|^{1/\beta'} + 1\right). \tag{40}$$

Similarly, (40) works with x^+ being replaced by x^-. Combining these inequalities shows

$$\|x\| \leqslant C_{13}\left(\|x\|^{1/\alpha'} + \|x\|^{1/\beta'} + 1\right), \tag{41}$$

which implies that $\{x_k\}$ is bounded in E.

Let Φ be defined by (16). By [33, Proposition B.37], $\{\Phi'(x_k)\}$ is precompact in E. Moreover, from (23) and (33),

$$I'(x_k) = x_k^+ - x_k^- + \Phi'(x_k). \tag{42}$$

It follows that $\{x_k\}$ has a convergent subsequence. The proof is complete. □

Lemma 10. *For each $l \in \mathbb{N}$, there are $\rho_l > 0$, $a_l > 0$, and $0 < b_l \to 0$ such that*

(a) *$I(x) \geqslant 0$, for all $x \in B_{\rho_l} \cap V_l^{+\infty}$ and $\inf I(\partial S_{\rho_l} \cap V_l^{+\infty}) \geqslant a_l$, where $B_{\rho_l} = \{x \in E : \|x\| \leqslant \rho_l\}$;*

(b) *$\sup I((V_{l-1}^{+\infty})^\perp) \leqslant b_l$.*

Proof. Noticing that $V_l^{+\infty} = V_l^- \oplus V_{+\infty}^+$ and that $\dim(V_l^-) < \infty$, we have, for $x \in B_{\rho_l} \cap V_l^{+\infty}$,

$$\begin{aligned} \|x\|_\alpha &= \sup\left\{\int_{S^1}(x(t), y(t))\,dt \mid y \in L^{\alpha'}\left(S^1, \mathbb{R}^n\right), \|y\|_{\alpha'} = 1\right\} \\ &\geqslant \sup\left\{\int_{S^1}(x(t), y(t))\,dt \mid y \in V_l^-, \|y\|_{\alpha'} = 1\right\} \\ &= \sup\left\{\int_{S^1}(x^-(t), y(t))\,dt \mid y \in V_l^-, \|y\|_{\alpha'} = 1\right\} \\ &= \|x^-\|_\alpha \geqslant \eta_l\|x^-\|, \end{aligned} \tag{43}$$

where η_l is a positive constant depending on l. It follows by (28) that, for $x \in B_{\rho_l} \cap V_l^{+\infty}$,

$$\begin{aligned} I(x) &\geqslant \frac{1}{2}\|x^+\|^2 + C_2\eta_l\|x^-\|^\alpha - \frac{1}{2}\|x^-\|^2 \\ &= \frac{1}{2}\|x^+\|^2 + \left(C_2\eta_l - \frac{1}{2}\|x^-\|^{2-\alpha}\right)\|x^-\|^\alpha, \end{aligned} \tag{44}$$

which implies (a) by setting $\rho_l := \min\{1, (C_2\eta_l)^{1/(2-\alpha)}\}$ and $a_l = \rho_l^2(1 + C_2\eta_l)/2$.

Let $x \in (V_{l-1}^{+\infty})^\perp$. By (28) and Lemma 7,

$$\begin{aligned} I(x) &\leqslant C_3\left(\|x\|_\alpha^\alpha + \|x\|_\beta^\beta\right) - \frac{1}{2}\|x\|^2 \\ &\leqslant C_{16}l^{-1}\left(\|x\|^\alpha + \|x\|^\beta\right) - \frac{1}{2}\|x\|^2 \\ &\leqslant b_l := \sup_{s\geq 0} g(s), \end{aligned} \tag{45}$$

where $g(s) := C_{16}l^{-1}(s^\alpha + s^\beta) - s^2/2$. Noticing $1 < \beta \leqslant \alpha < 2$, one can see that $b_l \to 0$ as $l \to \infty$ and (b) follows. The proof is complete. □

In the following, let Σ denote the family of closed (in E) subsets of $E \setminus \{\mathbf{0}\}$ symmetric with respect to the origin, and

$$\gamma : \Sigma \longmapsto \mathbb{N} \cup \{0, \infty\}, \tag{46}$$

the $\mathbb{Z}_2$-genus map (see [33]). For $l, m \in \mathbb{N}$, set

$$\Sigma_l^m = \{A \in \Sigma : A \subset V_{+\infty}^m, \gamma(A) \geqslant n(l+m)\}, \tag{47}$$

and define

$$c_{l,m} = \sup_{A\in\Sigma_l^m} \inf_{x\in A} I(u). \tag{48}$$

Lemma 11. *For all $l, m \in \mathbb{N}$, $c_{l,m}$ is a critical value of I and*

$$a_l \leqslant c_{l,m} \leqslant b_l. \tag{49}$$

Proof. We first prove that (49) holds. For each $m \in \mathbb{N}$, let ρ_l be chosen as that in Lemma 10; then it follows by Lemma 10(a) that $\inf I(\partial S_{\rho_l} \cap V_l^{+\infty}) \geqslant a_l$. Denote $\widetilde{A} = \partial S_{\rho_l} \cap V_l^m$; then $\gamma(\widetilde{A}) = n(m+l)$ and $\widetilde{A} \in \Sigma_l^m$. Since $\widetilde{A} \subset \partial S_{\rho_l} \cap V_l^{+\infty}$, we have

$$c_{l,m} \geqslant \inf_{x\in\widetilde{A}} I(x) \geqslant \inf I\left(\partial S_{\rho_l} \cap V_l^{+\infty}\right) \geqslant a_l. \tag{50}$$

On the other hand, for every $A \in \Sigma_l^m$, by the property of genus, $A \cap (V_{l-1}^{+\infty})^\perp \neq \varnothing$, which, from Lemma 10(b), leads to $\inf_{x\in A} I(x) \leqslant b_l$ for every $A \in \Sigma_l^m$. Thus $c_{l,m} \leqslant b_l$ and (49) holds.

By (F_1) and (25), $\widetilde{F}(x)$ is even with respect to x, which implies that I is even. We claim that $c = c_{l,m}$ is a critical point of I. Otherwise, there exists $\epsilon > 0$, such that there is no any critical point in the interval $(c-\epsilon, c+\epsilon)$. By the definition of $c_{l,m}$, there exists $A \in \Sigma_l^m$, such that

$$\inf_{x\in A} I(x) > c - \epsilon. \tag{51}$$

For $a \in \mathbb{R}$, denote $I^a = \{x \in E : I(x) \geqslant a\}$. Use a positive rather than a negative gradient flow [33, Remark A.17], we get $\eta \in C([0,1] \times E, E)$ such that $\eta(1, \cdot)$ is odd and

$$\eta\left(1, I^{c-\epsilon}\right) \subset I^{c+\epsilon}. \tag{52}$$

Since $A \subset I^{c-\epsilon}$, we have $\eta(1, A) \subset I^{c+\epsilon}$; that is,

$$\inf_{x\in\eta(1,A)} I(x) \geqslant c + \epsilon. \tag{53}$$

On the other hand, by the property of genus, we know that $\gamma(\eta(1, A)) \in \Sigma_l^m$, which, by the definition of c, leads to

$$c \geqslant \inf_{x\in\eta(1,A)} I(x) \geqslant c + \epsilon. \tag{54}$$

This contradiction implies that $c_{l,m}$ is a critical value of I. The proof is complete. □

Now we are in a position to give the following proof.

Proof of Theorem 1. In view of Lemma 11, let $x_{l,m} \in V_{+\infty}^m$ be such that

$$I(x_{l,m}) = c_{l,m}, \qquad I'(x_{l,m}) = 0. \tag{55}$$

Then by (PS) condition, along a subsequence as $m \to \infty$, $x_{l,m} \to x_l \in E$ such that

$$a_l \leqslant I(x_l) \leqslant b_l, \qquad I'(x_l) = 0, \tag{56}$$

which implies that x_l is nonzero. Moreover, by Lemma 5,

$$x_l'(t) = -\widetilde{F}_x\left(x_l\left(t - \frac{\pi}{2}\right)\right). \tag{57}$$

We claim that, for sufficiently large l, x_l solves (1). In fact, from (26) and (56)

$$\begin{aligned} b_l \geqslant I(x_l) &= I(x_l) - \frac{1}{2}I'(x_l)x_l \\ &\geqslant \left(1 - \frac{\alpha}{2}\right)\int_0^{2\pi} \widetilde{F}(x_l)\,dt. \end{aligned} \tag{58}$$

By (27), (58), and Hölder inequality

$$\begin{aligned} \|x_l^+\|^2 &= \int_0^{2\pi} \left(\widetilde{F}_x(x_l), x_l^+\right) dt \\ &\leqslant \|x_l^+\|_\alpha \left(\int_0^{2\pi} \left|\widetilde{F}_x(x_l)\right|^{\alpha'} dt\right)^{1/\alpha'} \\ &\quad + \|x_l^+\|_\beta \left(\int_0^{2\pi} \left|\widetilde{F}_x(x_l)\right|^{\beta'} dt\right)^{1/\beta'} \\ &\leqslant C_{17}\|x_l^+\| \left(\int_0^{2\pi} \widetilde{F}(x_l)\,dt\right)^{1/\alpha'} \\ &\quad + C_{18}\|x_l^+\| \left(\int_0^{2\pi} \widetilde{F}(x_l)\,dt\right)^{1/\beta'} \\ &\leqslant \left(C_{17}b_l^{1/\alpha'} + C_{18}b_l^{1/\beta'}\right)\|x^+\|. \end{aligned} \tag{59}$$

Similarly, the above inequality works with x_l^+ replaced by x_l^-. These inequalities yield

$$\|x_l\| \le C_{19} b_l^{1/\alpha'} + C_{20} b_l^{1/\beta'}. \tag{60}$$

Since $b_l \to 0$ as $l \to \infty$, it follows that

$$\|x_l\| \longrightarrow 0 \quad \text{as } l \longrightarrow \infty. \tag{61}$$

Furthermore, from (27) and (57), we have

$$\begin{aligned}\int_0^{2\pi} |\dot{x}_l(t)|^2 dt &= \int_0^{2\pi} \left|\widetilde{F}_x(x_l)\right|^2 dt \\ &\leqslant C_{21}\left(\int_0^{2\pi}\left[\widetilde{F}(x_l)\right]^{2/\beta'} dt + \int_0^{2\pi}\left[\widetilde{F}(x_l)\right]^{2/\alpha'} dt\right).\end{aligned} \tag{62}$$

It follows from (58) that $\|\dot{x}_l\|_2 \to 0$ as $l \to \infty$. Recalling (61), we get

$$\|x_l\|_{W^{1,2}} \longrightarrow 0 \quad \text{as } l \longrightarrow \infty, \tag{63}$$

which implies that $\|x_l\|_\infty \to 0$ as $l \to 0$. Thus for m sufficiently large, $\|x_l\|_\infty < r_0/2$ and therefor $\widetilde{F}_x(x_l) = F_x(x_l)$. It follow from (57) that, for l sufficiently large, x_l solves (1). In addition, by (1) and (F_3)(i), the only constant solution of (1) is the trivial solution. Then (56) yields that x_l is nonconstant and the proof of Theorem 1 is complete. □

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Research Article

On Linear Difference Equations for Which the Global Periodicity Implies the Existence of an Equilibrium

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It is proved that any first-order globally periodic linear inhomogeneous autonomous difference equation defined by a linear operator with closed range in a Banach space has an equilibrium. This result is extended for higher order linear inhomogeneous system in a real or complex Euclidean space. The work was highly motivated by the early works of Smith (1934, 1941) and the papers of Kister (1961) and Bas (2011).

1. Introduction

Let X be a set and let p be a positive integer. It is said that the transformation $T : X \to X$ is p-periodic if

$$T^p := T \circ \cdots \circ T = id_X, \tag{1}$$

where id_X is the identical function on X and p is the least positive integer with this property. It follows from (1) that T is a bijection. If there is a topology on X and T is continuous, then (1) implies that T is a homeomorphism.

The following question was posed by Smith (see [1]): does any continuous periodic transformation of a Euclidean n-space always admit a fixed point? Smith knew that the answer is true if the period p of the transformation is a prime number (see [2]) or a power of a prime number (see [1]). Moreover, Smith was able to answer the question affirmatively when $n \leq 3$ and for suitably regular transformations, when $n = 4$. But it was shown by Kister (see [3]) that there exist periodic transformations of a Euclidean space without fixed points. Kister's example is based on the results in the paper [4].

Special periodic transformations can be derived from difference equations.

Consider the sth order difference equation:

$$x(n) = h(x(n-1), \ldots, x(n-s)) \quad n \geq 0, \tag{2}$$

where,

(G) s is a positive integer, X is a set, and $h : X^s \to X$.

It is clear that the solutions of (2) are uniquely determined by their initial values:

$$x(n) = \varphi(n), \quad -s \leq n \leq -1, \tag{3}$$

where $\varphi(n) \in X$. The unique solution of (2) and (3) is denoted by $x(\boldsymbol{\varphi}) = (x(\boldsymbol{\varphi})(n))_{n \geq -s}$, where $\boldsymbol{\varphi} := (\varphi(-s), \ldots, \varphi(-1))^T \in X^s$.

We give some basic definitions about the periodicity of (2).

Definition 1. Assume (G).

(a) A sequence $v = (v(n))_{n \geq -s}$ in X is called periodic if there is a positive integer p such that v is p-periodic, which means that $v(n+p) = v(n)$ for all $n \geq -s$.

(b) We say that (2) is globally periodic if there is a positive integer $p \geq s$ for which the equation is globally p-periodic; that is, every solution of it is p-periodic.

(c) We say that (2) is globally p-periodic with prime period p if it is globally p-periodic and p is the least positive integer with this property.

It is easy to see that (2) is globally p-periodic with prime period p if and only if the transformation $T : X^s \to X^s$ defined by

$$T(x_{-s}, \ldots, x_{-1}) := (x_{-s+1}, \ldots, x_{-2}, h(x_{-s}, \ldots, x_{-1})) \tag{4}$$

is p-periodic.

About periodicity of general difference equations, see [5, 6]. Periodicity of linear difference equations is considered in [7].

We recall that the solution $(x(n))_{n\geq -s}$ of (2) is a steady state solution if $x(n) = v(n \geq -s)$, where $v \in X$ is an equilibrium of (2); that is, v obeys

$$v = h(v, \ldots, v). \tag{5}$$

It is obvious that $v \in X$ is an equilibrium of (2) exactly if $(v, \ldots, v)$ is a fixed point of the transformation T given in (4).

Even if there is a metric on X and h is continuous, it is still an open problem to determine whether (2) has or not an equilibrium point, or equivalently, the transformation (4) has a fixed point, if (2) is globally periodic.

In this paper we solve this problem for some linear equations.

Let $\mathbb{K}$ stand for either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Throughout this paper, the term vector space in which the scalar field is not explicitly mentioned will refer to a vector space over $\mathbb{R}$ or over $\mathbb{C}$.

Consider the sth order inhomogeneous linear difference equation:

$$x(n) = \sum_{i=1}^{s} L_i(x(n-i)) + b, \quad n \geq 0, \tag{6}$$

where,

(A) s is a positive integer, X is a vector space, $L_i : X \to X$ is a linear transformation $(1 \leq i \leq s)$, and $b \in X$ is a vector.

The sth order homogeneous linear difference equation associated (6) is

$$x(n) = \sum_{i=1}^{s} L_i(x(n-i)), \quad n \geq 0. \tag{7}$$

Clearly, if that (6) is globally p-periodic, the difference of any two solutions of it is also p-periodic. On the other hand, the general solution of the inhomogeneous equation (6) can be written as the sum of the general solution of the homogeneous equation (7) and an arbitrarily fixed particular solution of the inhomogeneous equation. Thus the global p-periodicity of the inhomogeneous equation implies the global p-periodicity of the related homogeneous equation. One can easily see that the opposite statement is also true if the inhomogeneous equation has a steady state solution which is obviously p-periodic for any $p \geq 1$.

From this we conclude the following.

Conclusion. If (6) has an equilibrium, then (6) and (7) both behave in the same way regarding the global periodicity; that is, they both are globally periodic or both are not globally periodic.

The crux in the application of the above self-evident statement is that not all autonomous inhomogeneous linear difference equations have an equilibrium. But this crux is eliminated by the main theorems of this work in two special cases of (6).

In the first result X is finite dimensional.

Theorem 2. *Consider the system of the sth order inhomogeneous linear difference equations:*

$$x(n) = \sum_{i=1}^{s} A_i x(n-i) + b, \quad n \geq 0, \tag{8}$$

where,

(B) *s is a positive integer, $A_i \in \mathbb{K}^{d\times d}$ $(1 \leq i \leq s)$ are matrices, and $b \in \mathbb{K}^d$ is vector.*

If (8) *is globally periodic, then it has an equilibrium.*

Let X be a vector space. I and O mean the identity and the zero operator on X, respectively. If $L : X \to X$ is a linear transformation, we define the kernel and the image of L in the usual way:

$$\ker(L) := \{x \in X \mid L(x) = 0\}, \qquad \operatorname{im}(L) := \{L(x) \mid x \in X\}. \tag{9}$$

In the next result first-order equations are investigated.

Theorem 3. *Consider the first order inhomogeneous linear difference equation:*

$$x(n) = L(x(n-1)) + b, \quad n \geq 0, \tag{10}$$

where,

(C) *L is a bounded linear operator of the Banach space X into itself such that* $\operatorname{im}(I - L)$ *is closed and $b \in X$ is a vector.*

If (10) *is globally periodic, then it has an equilibrium.*

2. Existence of an Equilibrium in an Abstract First-Order Inhomogeneous Linear Equation

In this section we prove Theorem 3.

First, we need the following lemma about global periodicity.

Lemma 4. *Consider the first order inhomogeneous linear difference equation:*

$$x(n) = L(x(n-1)) + b, \quad n \geq 0, \tag{11}$$

where,

(D) *X is a vector space, $L : X \to X$ is a linear transformation, and $b \in X$ is a vector.*

Let p be a positive integer. Equation (11) is globally p-periodic if and only if

$$L^p = I, \quad \sum_{i=0}^{p-1} L^i(b) = 0. \tag{12}$$

Proof. It is easy to check that (11) is globally p-periodic if and only if

$$(L^p - I)\varphi + \sum_{i=0}^{p-1} L^i(b) = 0, \tag{13}$$

for every $\varphi \in X$, but this condition and (12) are equivalent. □

Remark 5. (a) Condition (12) is equivalent to

$$\left(\sum_{i=0}^{p-1} L^i\right)(I - L) = O, \quad b \in \ker\left(\sum_{i=0}^{p-1} L^i\right). \tag{14}$$

The first part of (14) implies that

$$\operatorname{im}(I - L) \subset \ker\left(\sum_{i=0}^{p-1} L^i\right). \tag{15}$$

Since (11) has an equilibrium point exactly if the linear equation

$$(I - L)x = b \tag{16}$$

has a solution, it follows from the previous establishments that the following two assertions are equivalent. Let p be a positive integer.

(i) If (11) is globally p-periodic, then it has an equilibrium.

(ii) If $L^p = I$, then

$$\operatorname{im}(I - L) = \ker\left(\sum_{i=0}^{p-1} L^i\right). \tag{17}$$

(b) $L^p = I$ implies that L is invertible. If $I - L$ is also invertible, then (16) obviously has a solution (or (17) holds), and therefore the only interesting case is when $I - L$ is not invertible.

We can see that if (11) is globally periodic, then the problem of the existence or nonexistence of an equilibrium leads to a pure linear algebraic problem.

Problem. Let X be a vector space and let $L : X \to X$ be a linear transformation such that $L^p = I$ for some integer $p \geq 2$. Either prove that

$$\operatorname{im}(I - L) = \ker\left(\sum_{i=0}^{p-1} L^i\right), \tag{18}$$

or give an example when $\operatorname{im}(I - L)$ is a proper subset of

$$\ker\left(\sum_{i=0}^{p-1} L^i\right). \tag{19}$$

If L is a linear operator of the Banach space X into itself such that $\operatorname{im}(I - L)$ is closed, then Theorem 3 shows that (18) holds.

Henceforth we need some notations (see [8]).

Definition 6. Let X be a Banach space.

(a) X^* means its dual space, and let (w, u) denote the value of the functional $w \in X^*$ at $u \in X$. For a bounded linear operator L of X into itself, $L^* : X^* \to X^*$ denotes its adjoint operator.

(b) Suppose that M is a subspace of X and N is a subspace of X^*. Their annihilators are defined as follows:

$$\begin{aligned} M^\perp &:= \{w \in X^* \mid (w, u) = 0, u \in M\}, \\ {}^\perp N &:= \{u \in X \mid (w, u) = 0, w \in N\}. \end{aligned} \tag{20}$$

In the proof of Theorem 3 the following result will be used, which is related to the Fredholm alternative (see [9]).

Lemma 7. *Let X be a Banach space and let L be a bounded linear operator of X into itself such that* $\operatorname{im}(I - L)$ is closed. The equation $(I - L)x = b$ is solvable for given $b \in X$ if and only if $b \in {}^\perp(\ker(I - L^*))$.

Proof. It is well known (see [8]) that

$${}^\perp(\ker(I - L^*)) = {}^\perp(\operatorname{im}(I - L)^\perp), \tag{21}$$

and ${}^\perp(\operatorname{im}(I-L)^\perp)$ is the norm closure of $\operatorname{im}(I-L)$ in X. Since $\operatorname{im}(I - L)$ is closed,

$${}^\perp(\ker(I - L^*)) = \operatorname{im}(I - L), \tag{22}$$

which gives the result. □

Remark 8. If X is finite dimensional, then $\operatorname{im}(I - L)$ is closed, since every subspace of X is closed. In this case Lemma 7 is exactly the Fredholm alternative.

Proof of Theorem 3. We can obviously suppose that $p \geq 2$.

Equation (10) has an equilibrium point exactly if the linear equation

$$(I - L)x = b \tag{23}$$

has a solution. By Lemma 7, it is enough to show that

$$b \in {}^\perp(\ker(I - L^*)). \tag{24}$$

To prove (24), assume that

$$w \in \ker(I - L^*). \tag{25}$$

Recalling Lemma 4, we have

$$\begin{aligned}(w,b) &= \left(L^*(w), -\sum_{i=1}^{p-1} L^i(b)\right)\\ &= -\left(w, \sum_{i=1}^{p-1} L^{i+1}(b)\right)\\ &= -\left(w, b+\sum_{i=2}^{p-1} L^i(b)\right)\\ &= -\langle w,b\rangle - \left(\sum_{i=2}^{p-1}(L^*)^i w, b\right).\end{aligned} \tag{26}$$

$w = L^*(w)$ gives $w = (L^*)^i(w)$. Consequently,

$$(w,b) = -(p-1)(w,b), \tag{27}$$

which means that $(w,b) = 0$.

The proof is complete. □

By Remark 8, we have the following.

Corollary 9. *Consider the first order inhomogeneous linear difference equation:*

$$x(n) = L(x(n-1)) + b, \quad n \geq 0, \tag{28}$$

where L is a linear operator of the finite dimensional space X into itself and $b \in X$ is a vector. If (28) *is globally periodic, then it has an equilibrium.*

We illustrate by an example that the conditions involved in Theorem 3 can be satisfied and not only the finite dimensional case.

Example 10. Let $B([0,1])$ be the Banach space of bounded scalar-valued functions on $[0,1]$, with the supremum norm

$$\|f\|_\infty := \sup\{|f(t)| \mid t \in [0,1]\}. \tag{29}$$

Define the function $\alpha \in B([0,1])$ by

$$\alpha(t) := \begin{cases} 1, & \text{if } t \text{ is rational}\\ -1, & \text{if } t \text{ is irrational,}\end{cases} \tag{30}$$

and introduce the following bounded linear operator L on $B([0,1])$:

$$L(f) := \alpha f, \quad f \in B([0,1]). \tag{31}$$

Then $L^2 = I$, $I - L$ is not invertible (by Remark 5 (b), this is an interesting case), and

$$\operatorname{im}(I-L) = \{g \in B([0,1]) \mid g(t) = 0 \text{ if } t \text{ is rational}\} \tag{32}$$

is a closed subspace of $B([0,1])$.

It is easy to see that equation

$$\begin{aligned} x(n) &= L(x(n-1)) + b,\\ x, b &\in B([0,1]),\ n \geq 0,\end{aligned} \tag{33}$$

or equivalently, for every $t \in [0,1]$

$$\begin{aligned} x(n)(t) &= \alpha(t)\, x(n-1)(t) + b(t),\\ x, b &\in B([0,1]),\ n \geq 0,\end{aligned} \tag{34}$$

is globally 2-periodic if and only if $b \in \operatorname{im}(I-L)$, and in this case it has the equlibrium point $(1/2)b$.

The previous example can be extended if the scalars are the complex numbers. Let $p \geq 3$ be an integer, and define the function $\alpha \in B([0,1])$ by

$$\alpha(t) := \begin{cases} 1, & \text{if } t \text{ is rational}\\ \varepsilon_p, & \text{if } t \text{ is irrational,}\end{cases} \tag{35}$$

where

$$\varepsilon_p := e^{(2\pi/p)i} \tag{36}$$

is a primitive pth root of unity. Then $L^p = I$; equation

$$\begin{aligned} x(n) &= L(x(n-1)),\\ x &\in B([0,1]),\ n \geq 0,\end{aligned} \tag{37}$$

is globally p-periodic, and it has solutions with prime period p.

3. The Proof of Theorem 2

We will use the following notations.

Definition 11. Let $m \geq 1$ be an integer.

(a) $BV^{m,d}$ will mean the md-dimensional real vector space of block vectors with entries in $\mathbb{K}^d$.

(b) The real vector space of $m \times m$ block matrices with entries in $\mathbb{K}^{d\times d}$ will be denoted by $BM^{m,d}$ ($BM^{m,d}$ and $\mathbb{K}^{md\times md}$ can be treated as being identical).

(c) The zero matrix and the identity matrix in $\mathbb{K}^{d\times d}$ are denoted by O_d and I_d, respectively.

Let $(x(n))_{n\geq -s}$ be a given sequence in $\mathbb{K}^d$. Then for any fixed $n \geq 0$ we introduce an sd-dimensional state vector:

$$\mathbf{x}_n = (\mathbf{x}_n(-s), \ldots, \mathbf{x}_n(-1))^T \in BV^{s,d}, \tag{38}$$

defined by $\mathbf{x}_n(i) := x(n+i)$ $(-s \leq i \leq -1)$.

As it is well known (see [10]), by using the state vector notation, (8) may be written as an sd-dimensional system of first order difference equations.

Lemma 12. *For any $\boldsymbol{\varphi} = (\varphi(-s), \ldots, \varphi(-1))^T \in BV^{s,d}$, $x(\boldsymbol{\varphi}) = (x(\boldsymbol{\varphi})(n))_{n\geq -s}$ is the solution of* (8) *and* (3) *exactly if*

$$(\mathbf{x}_k(\boldsymbol{\varphi}))_{k\geq 1} = \left((\mathbf{x}_k(\boldsymbol{\varphi})(-s), \ldots, \mathbf{x}_k(\boldsymbol{\varphi})(-1))^T\right)_{k\geq 1} \tag{39}$$

is the solution of

$$\begin{aligned} \mathbf{x}_k &= \mathscr{C}\mathbf{x}_{k-1} + \mathscr{B}, \quad k \geq 1,\\ \mathbf{x}_0 &= \boldsymbol{\varphi},\end{aligned} \tag{40}$$

where the companion matrix $\mathscr{C} \in BM_{\mathbb{K}}^{s,d}$ *and the block vector* $\mathscr{B}$ *can be written in the forms*

$$\mathscr{C} = \begin{pmatrix} O_d & I_d & O_d & \dots & O_d \\ O_d & O_d & I_d & \dots & O_d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_d & O_d & O_d & \dots & I_d \\ A_s & A_{s-1} & A_{s-2} & \dots & A_1 \end{pmatrix}, \tag{41}$$

$$\mathscr{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}. \tag{42}$$

Another companion matrix is developed in [11].

There is a one-to-one correspondence between the global periodicity of (8) and that of (40) and also between equilibrium of (8) and that of (40).

Lemma 13. *(a) Let* $p \geq s$ *be an integer. Equation* (8) *is globally* p *-periodic if and only if* (40) *is also globally* p*-periodic.*

(b) $c \in \mathbb{K}^d$ *is an equilibrium of* (8) *exactly if* $\mathbf{c} = (c, \dots, c)^T \in BV^{s,d}$ *is an equilibrium of* (40).

Now we prove the first main result.

Proof of Theorem 2. We can apply Theorem 3 and Lemma 13. □

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Research Article

Oscillation Theorems for Even Order Damped Equations with Distributed Deviating Arguments

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A class of even order damped differential equations with distributed deviating arguments are investigated. Several new criteria that ensure the oscillation of solutions are obtained. To demonstrate the validity of the results obtained, two examples are given.

1. Introduction and Lemmas

Oscillatory behavior of solutions for different types of second-order differential equations with damping has been widely discussed by using different techniques. Here, we particularly refer the reader to the papers [1–9] and the references quoted therein. However, very little is known for the case of higher order damped functional differential equations with deviating arguments, especially the case with distributed deviating arguments. In this paper, we deal with the following class of even order functional differential equations with damping:

$$x^{(n)}(t) + p(t)x^{(n-1)}(t) + \int_{\alpha}^{\beta} q(t,\xi) f\left(x\left[g_1(t,\xi)\right],\ldots,x\left[g_m(t,\xi)\right]\right) d\mu(\xi) = 0, \quad t \geq t_0 > 0. \tag{1}$$

Our aim is to get the criteria for the oscillatory solutions of (1).

Throughout this paper, we assume that the following conditions hold:

(H_1) n is an even positive integer;

(H_2) $p(t) \in C([t_0,\infty),R_+)$, $q(t,\xi) \in C([t_0,\infty)\times[\alpha,\beta],R_+)$ is not identically zero on any $[T,\infty)\times[\alpha,\beta]$ for $T \geq t_0$, and

$$\lim_{t\to\infty}\int_{t_1}^{t}\exp\left(-\int_{t_1}^{s} p(\tau)\,d\tau\right)ds = \infty, \quad t_1 \geq t_0; \tag{2}$$

(H_3) $f(u_1,u_2,\ldots,u_m) \in C(R^m,R)$ has the same sign as $u_1,u_2,\ldots,u_m$ when $u_1,u_2,\ldots,u_m$ have the same sign, $g_i(t,\xi) \in C([t_0,\infty)\times[\alpha,\beta],R_+)$, $\mu(\xi) \in ([\alpha,\beta],R)$ is nondecreasing, and the integral of (1) is a Stieltjes one.

In the sequel, it will be always assumed that solutions of (1) exist for any $t_0 \geq 0$. A solution $x(t)$ of (1) is called eventually positive solution (or negative solution) if there exists a sufficiently large positive number $t_1 \geq t_0$, such that $x(t) > 0$ (or $x(t) < 0$) for all $t \geq t_1$. A nontrivial solution $x(t)$ of (1) is called oscillatory if it has arbitrary large zeros; otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

Remark 1. Since the integral of (1) is a Stieltjes one, it includes the following equations:

$$x^{(n)}(t) + p(t)x^{(n-1)}(t) + \sum_{i=1}^{m} q_i(t) f\left(x\left[g_1(t)\right],\ldots,x\left[g_m(t)\right]\right) = 0, \quad t \geq t_0 > 0. \tag{1$'$}$$

The following lemmas will be useful to the proof of the main results to be presented in this paper.

Lemma 2 (see [10]). *Let $u(t)$ be a positive and n times differentiable function on R_+. If $u^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_1, +\infty)$ for $t_1 > 0$, then there exists a $t_u \geq t_1$ and an integer l $(0 \leq l \leq n)$, with $n+l$ even for $u(t)u^{(n)}(t) \geq 0$ or $n+l$ odd for $u(t)u^{(n)}(t) \leq 0$; and for $t \geq t_u$,*

$$\begin{gathered} u(t)\,u^{(k)}(t) > 0, \quad 0 \leq k \leq l; \\ (-1)^{k-l} u(t)\,u^{(k)}(t) > 0, \quad l \leq k \leq n. \end{gathered} \tag{3}$$

Lemma 3 (see [11]). *Suppose that the conditions of Lemma 2 are satisfied, and*

$$u^{(n-1)}(t)\,u^{(n)}(t) \leq 0, \quad t \geq t_u, \tag{4}$$

then there exists a constant $\theta \in (0,1)$ such that for sufficiently large t, there exists a constant $M_\theta > 0$ satisfying

$$\left| u'\left(\frac{t}{2}\right) \right| \geq M_\theta t^{n-2} \left| u^{(n-1)}(t) \right|. \tag{5}$$

We say that a function $H = H(t,s)$ belongs to a function class Φ, denoted by $H \in \Phi$, if $H \in C(D, R_+)$, where $D = \{(t,s) : -\infty < s \leq t < \infty\}$, satisfies

(i) $H(t,t) = 0$, for $t \geq t_0$ and $H(t,s) > 0$, for $t > s \geq t_0$;

(ii) partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ exist, and

$$\begin{gathered} \frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \\ \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}, \end{gathered} \tag{6}$$

where $h_1, h_2 \in L_{\mathrm{loc}}(D, R)$.

2. Oscillation Results for $f(u_1, \ldots, u_m)$ with Monotonicity

Throughout this section, we assume that the following conditions hold.

(A_1) There exist functions $\sigma_i(t) \in C'([t_0, \infty), (0, \infty))$, such that $\sigma_i(t) = \min\{t, \inf_{\xi \in [\alpha,\beta]} g_i(t,\xi)\}$, $\lim_{t\to\infty} \sigma_i(t) = \infty$, $\sigma_i'(t) > 0$, and $i = 1, 2, \ldots, m$.

(A_2) $(\partial/\partial u_i) f(u_1, \ldots, u_m) \equiv f_i'(u_1, \ldots, u_m)$ exists, and $f_i'(u_1, \ldots, u_m) \geq \lambda_i > 0$ for $u_i \neq 0$, $i = 1, 2, \ldots, m$, where $\lambda_i > 0$ are some constants, and $i = 1, 2, \ldots, m$.

Lemma 4. *Let $x(t)$ be an eventually positive solution of* (1). *Then, there exists a sufficiently large $T_0 \geq t_0$, such that for all $t \geq T_0$*

$$x'(t) > 0, \qquad x^{(n-1)}(t) > 0, \qquad x^{(n)}(t) \leq 0. \tag{7}$$

Proof. From the assumption, there exists a sufficiently large $t_1 \geq t_0$, such that $x(t) > 0$ for $t \geq t_1$. Further from (A_1), there exists $t_2 \geq t_1$ such that for all $t \geq t_2$

$$\begin{gathered} \sigma_i(t) \geq t_1, \qquad g_i(t,\xi) \geq \sigma_i(t) \geq t_1, \\ i = 1, 2, \ldots, m; \ \xi \in [\alpha, \beta]. \end{gathered} \tag{8}$$

Hence, for all $t \geq t_2$

$$\begin{gathered} x[\sigma_i(t)] > 0, \qquad x[g_i(t,\xi)] > 0, \\ i = 1, 2, \ldots, m; \ \xi \in [\alpha, \beta], \end{gathered} \tag{9}$$

and from (H_3), we have for all $t \geq t_2$ and $\xi \in [\alpha, \beta]$

$$\begin{gathered} f(x[\sigma_1(t)], \ldots, x[\sigma_m(t)]) > 0, \\ f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)]) > 0. \end{gathered} \tag{10}$$

Let

$$v(t) = \exp\int_{t_2}^{t} p(s)\,ds, \quad w(t) = x^{(n-1)}(t)\,v(t), \quad t \geq t_2, \tag{11}$$

then it is easy to know that

$$\begin{aligned} w'(t) &= \left(x^{(n)}(t) + p(t)\,x^{(n-1)}(t)\right) v(t) \\ &= -\int_\alpha^\beta q(t,\xi)\, f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)]) \\ &\quad \times d\mu(\xi)\, v(t) \leq 0, \end{aligned} \tag{12}$$

which implies that $w(t)$ is nonincreasing on $[t_2, +\infty)$.

Now, we claim that $x^{(n-1)}(t) \geq 0$, $t \geq t_2$. Otherwise, there exists $t_3 \geq t_2$ such that $x^{(n-1)}(t_3) < 0$. Therefore,

$$\begin{gathered} x^{(n-1)}(t)\,v(t) \leq x^{(n-1)}(t_3)\,v(t_3), \quad t \geq t_3, \\ \int_{t_3}^{t} x^{(n-1)}(\tau)\,d\tau \leq x^{(n-1)}(t_3)\,v(t_3) \int_{t_3}^{t} \frac{1}{v(\tau)} d\tau, \quad t \geq t_3, \\ x^{(n-2)}(t) \leq x^{(n-2)}(t_3) + x^{(n-1)}(t_3)\,v(t_3) \int_{t_3}^{t} \frac{1}{v(\tau)} d\tau, \\ t \geq t_3. \end{gathered} \tag{13}$$

Using (H_2), we see that $\lim_{t\to+\infty} x^{(n-2)}(t) = -\infty$. Ulteriorly, we can prove $\lim_{t\to+\infty} x(t) = -\infty$, which contradicts $x(t) > 0$, $t \geq t_1$.

Furthermore, from (1), for all $t \geq t_2$, we have

$$\begin{aligned} x^{(n)}(t) &= -p(t)\,x^{(n-1)}(t) \\ &\quad - \int_\alpha^\beta q(t,\xi)\, f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)]) \\ &\quad \times d\mu(\xi) \leq 0. \end{aligned} \tag{14}$$

Thus, from Lemma 2, there exist $T_0 \geq t_2$ and an odd number l $(0 < l < n)$, such that for $t \geq T_0$, we have

$$\begin{gathered} x^{(k)}(t) > 0, \quad 0 \leq k \leq l; \\ (-1)^{k-l} x^{(k)}(t) > 0, \quad l \leq k \leq n. \end{gathered} \tag{15}$$

By choosing $k = 1$ and $n-1$, we have $x'(t) > 0$ and $x^{(n-1)}(t) > 0$ for $t \geq T_0$. The proof is completed. □

Lemma 5. *Let $x(t)$ be an eventually positive solution of* (1). *Then, there exists a sufficiently large $T_0 \geq t_0$, such that for any interval $[c,b) \subset [T_0,\infty)$, if let*

$$y(t) = \frac{\rho(t)\, x^{(n-1)}(t)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)}, \quad t \in [c,b), \tag{16}$$

where $\rho(t) \in C'([t_0,\infty),(0,\infty))$, then for any $H \in \Phi$,

$$\begin{aligned}\int_c^b H(b,s)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds &\leq H(b,c)\,y(c) + \frac{1}{2}\int_c^b \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \\ &\quad\times\left[h_2(b,s) - \sqrt{H(b,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds.\end{aligned} \tag{17}$$

Proof. From (1) and (16), we have that for $t \in [c,b)$,

$$\begin{aligned}y'(t) &= \frac{\rho(t)\,x^{(n)}(t) + \rho'(t)\,x^{(n-1)}(t)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)} \\ &\quad - \frac{y(t)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)} \\ &\quad\times\left(\frac{1}{2}\sum_{i=1}^m f_i'\left(x\left[\frac{\sigma_1(t)}{2}\right],\ldots,x\left[\frac{\sigma_m(t)}{2}\right]\right)x'\left[\frac{\sigma_i(t)}{2}\right]\sigma_i'(t)\right) \\ &= -\rho(t)\,\frac{\int_\alpha^\beta q(t,\xi)\,f\left(x\left[g_1(t,\xi)\right],\ldots,x\left[g_m(t,\xi)\right]\right)d\mu(\xi)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)} \\ &\quad + \left(\frac{\rho'(t)}{\rho(t)} - p(t)\right)y(t) \\ &\quad - \frac{y(t)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)} \\ &\quad\times\left(\frac{1}{2}\sum_{i=1}^m f_i'\left(x\left[\frac{\sigma_1(t)}{2}\right],\ldots,x\left[\frac{\sigma_m(t)}{2}\right]\right)\times x'\left[\frac{\sigma_i(t)}{2}\right]\sigma_i'(t)\right).\end{aligned} \tag{18}$$

From Lemma 4, there exists a sufficiently large $T_0 \geq t_0$ such that $x'(t) > 0$ and $x^{(n)}(t) \leq 0$ for $t \geq T_0$. Further from (A_1), for all $t \geq T_0$

$$\sigma_i(t) \leq t, \qquad g_i(t,\xi) \geq \sigma_i(t) \geq \frac{\sigma_i(t)}{2}, \qquad i = 1,2,\ldots,m;\ \xi \in [\alpha,\beta]. \tag{19}$$

Hence, for all $t \geq T_0$, we have

$$x\left[g_i(t,\xi)\right] \geq x\left[\frac{\sigma_i(t)}{2}\right], \qquad x^{(n-1)}\left[\sigma_i(t)\right] \geq x^{(n-1)}(t), \qquad i = 1,2,\ldots,m;\ \xi \in [\alpha,\beta]. \tag{20}$$

In view of (20) and (A_2), for all $t \geq T_0$

$$\begin{aligned}&f\left(x\left[g_1(t,\xi)\right],\ldots,x\left[g_m(t,\xi)\right]\right) \\ &\quad\geq f\left(x\left[\frac{\sigma_1(t)}{2}\right],\ldots,x\left[\frac{\sigma_m(t)}{2}\right]\right), \quad \xi \in [\alpha,\beta].\end{aligned} \tag{21}$$

Thus, for all $t \geq T_0$

$$\begin{aligned}&\frac{\int_\alpha^\beta q(t,\xi)\,f\left(x\left[g_1(t,\xi)\right],\ldots,x\left[g_m(t,\xi)\right]\right)d\mu(\xi)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)} \\ &\quad\geq \int_\alpha^\beta q(t,\xi)\,d\mu(\xi).\end{aligned} \tag{22}$$

Therefore, from (18)–(22) and Lemma 3, we obtain

$$\begin{aligned}y'(t) &\leq -\rho(t)\int_\alpha^\beta q(t,\xi)\,d\mu(\xi) + \left(\frac{\rho'(t)}{\rho(t)} - p(t)\right)y(t) \\ &\quad - \frac{x^{(n-1)}(t)}{f\left(x\left[\sigma_1(t)/2\right],\ldots,x\left[\sigma_m(t)/2\right]\right)} \\ &\quad\times\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(t)\,\sigma_i'(t)\right)y(t) \\ &= -\rho(t)\int_\alpha^\beta q(t,\xi)\,d\mu(\xi) + \left(\frac{\rho'(t)}{\rho(t)} - p(t)\right)y(t) \\ &\quad - \rho^{-1}(t)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(t)\,\sigma_i'(t)\right)y^2(t)\end{aligned} \tag{23}$$

for all $t \geq T_0$.

Multiplying (23) by $H(t,s)$, then integrating it with respect to s from c to t for $t \in [c,b)$ and using (i) and (ii), we get that

$$\begin{aligned}&\int_c^t H(t,s)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\ &\leq -\int_c^t H(t,s)\,y'(s)\,ds \\ &\quad + \int_c^t H(t,s)\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)y(s)\,ds \\ &\quad - \int_c^t H(t,s)\,\rho^{-1}(s)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right)y^2(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&= H(t,c)\,y(c) + \int_c^t \frac{\partial H(t,s)}{\partial s} y(s)\,ds \\
&\quad + \int_c^t H(t,s)\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right) y(s)\,ds \\
&\quad - \int_c^t H(t,s)\,\rho^{-1}(s)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right) y^2(s)\,ds \\
&= H(t,c)\,y(c) - \int_c^t \sqrt{H(t,s)} \\
&\quad \times \left[h_2(t,s) - \sqrt{H(t,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right] y(s)\,ds \\
&\quad - \int_c^t H(t,s)\,\rho^{-1}(s)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right) y^2(s)\,ds \\
&= H(t,c)\,y(c) \\
&\quad + \frac{1}{2}\int_c^t \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \\
&\qquad \times \left[h_2(t,s) - \sqrt{H(t,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds \\
&\quad - \int_c^t \left[\sqrt{\frac{H(t,s)}{\rho(s)}\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right)} y(s)\right. \\
&\qquad \left. + \frac{h_2(t,s) - \sqrt{H(t,s)}\left(\left(\rho'(s)/\rho(s)\right) - p(s)\right)}{\sqrt{2\rho^{-1}(s)\left(\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right)}}\right]^2 ds \\
&\le H(t,c)\,y(c) \\
&\quad + \frac{1}{2}\int_c^t \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \\
&\qquad \times \left[h_2(t,s) - \sqrt{H(t,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds.
\end{aligned} \tag{24}
$$

Letting $t \to b^-$ in the above, we obtain (17). The proof is completed. □

Lemma 6. *Let $x(t)$ be an eventually positive solution of* (1). *Then, there exists a sufficiently large $T_0 \ge t_0$ such that for any interval $(a,c] \subset [T_0,\infty)$, if let $y(t)$ be defined by* (16) *on $(a,c]$, then for any $H \in \Phi$,*

$$
\begin{aligned}
&\int_a^c H(s,a)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right) ds \\
&\quad \le -H(c,a)\,y(c) \\
&\quad + \frac{1}{2}\int_a^c \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma^{n-2}(s)\,\sigma_i'(s)} \\
&\qquad \times \left[h_1(s,a) + \sqrt{H(s,a)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds.
\end{aligned} \tag{25}
$$

Proof. Similar to the proof of Lemma 5, by multiplying (23) by $H(s,t)$, then integrating it with respect to s from t to c for $t \in (a,c]$, and then using (i) and (ii), we get that

$$
\begin{aligned}
&\int_t^c H(s,t)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right) ds \\
&\le -\int_t^c H(s,t)\,y'(s)\,ds \\
&\quad + \int_t^c H(s,t)\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right) y(s)\,ds \\
&\quad - \int_t^c H(s,t)\,\rho^{-1}(s)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right) y^2(s)\,ds \\
&= -H(c,t)\,y(c) + \int_t^c \frac{\partial H(s,t)}{\partial s} y(s)\,ds \\
&\quad + \int_t^c H(s,t)\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right) y(s)\,ds \\
&\quad - \int_t^c H(s,t)\,\rho^{-1}(s)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right) y^2(s)\,ds \\
&= -H(c,t)\,y(c) \\
&\quad + \int_t^c \sqrt{H(s,t)}\left[h_1(s,t)\right. \\
&\qquad \left. + \sqrt{H(s,t)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right] y(s)\,ds \\
&\quad - \int_t^c H(s,t)\,\rho^{-1}(s)\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right) y^2(s)\,ds \\
&= -H(c,t)\,y(c) + \frac{1}{2}\int_t^c \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \\
&\quad \times \left[h_1(s,t) + \sqrt{H(s,t)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds \\
&\quad - \int_t^c \left[\sqrt{\frac{H(s,t)}{\rho(s)}\left(\frac{1}{2}\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right)} y(s)\right. \\
&\qquad \left. - \frac{h_1(s,t) + \sqrt{H(s,t)}\left(\left(\rho'(s)/\rho(s)\right) - p(s)\right)}{\sqrt{2\rho^{-1}(s)\left(\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)\right)}}\right]^2 ds
\end{aligned}
$$

$$\leq -H(c,t)\,y(c) + \frac{1}{2}\int_t^c \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left[h_1(s,t) + \sqrt{H(s,t)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds. \tag{26}$$

Letting $t \to a^+$ in the above, we obtain (25). The proof is completed. □

The following theorem is an immediate result from Lemmas 5 and 6.

Theorem 7. *Assume that for each $T \geq t_0$ there exist $H \in \Phi$, $\rho \in C'([t_0,\infty),(0,\infty))$ and $a,b,c \in R$, such that $T \leq a < c < b$ and*

$$\begin{aligned}
&\frac{1}{H(c,a)}\int_a^c H(s,a)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\
&\quad + \frac{1}{H(b,c)}\int_c^b H(b,s)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\
&> \frac{1}{2}\left\{\frac{1}{H(c,a)}\int_a^c \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left[h_1(s,a) + \sqrt{H(s,a)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds \right. \\
&\quad + \frac{1}{H(b,c)} \times \int_c^b \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left.\left[h_2(b,s) - \sqrt{H(b,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds\right\}.
\end{aligned} \tag{27}$$

Then (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution of (1). Then from Lemmas 5 and 6, there exists a sufficiently large $T_0 \geq t_0$, such that for any $(a,b) \subset [T_0,\infty)$, and for any $c \in (a,b)$, $H \in \Phi$ and $\rho \in C'([t_0,\infty),(0,\infty))$, (17) and (25) hold. By dividing (17) and (25) by $H(b,c)$ and $H(c,a)$, respectively, and then adding them, we have

$$\begin{aligned}
&\frac{1}{H(c,a)}\int_a^c H(s,a)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\
&\quad + \frac{1}{H(b,c)}\int_c^b H(b,s)\,\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\
&\leq \frac{1}{2}\left\{\frac{1}{H(c,a)}\int_a^c \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left[h_1(s,a) + \sqrt{H(s,a)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds \right. \\
&\quad + \frac{1}{H(b,c)} \times \int_c^b \frac{\rho(s)}{\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left.\left[h_2(b,s) - \sqrt{H(b,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2 ds\right\},
\end{aligned} \tag{28}$$

which contradicts the assumption (27) and completes the proof. □

Theorem 8. *Assume that for some $H \in \Phi$, $\rho \in C'([t_0,\infty),(0,\infty))$ and for each $r \geq t_0$,*

$$\limsup_{t\to\infty}\int_r^t \left\{H(s,r)\,\rho(s)\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) - \frac{\rho(s)}{2\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left[h_1(s,r) + \sqrt{H(s,r)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2\right\} ds > 0, \tag{29}$$

$$\limsup_{t\to\infty}\int_r^t \left\{H(t,s)\,\rho(s)\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) - \frac{\rho(s)}{2\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \times \left[h_2(t,s) - \sqrt{H(t,s)}\left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2\right\} ds > 0. \tag{30}$$

Then (1) *is oscillatory.*

Proof. For any $T \geq t_0$, let $a = T$. In (29), we choose $r = a$. Then there exists $c > a$ such that

$$\int_a^c \left\{H(s,a)\,\rho(s)\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right.$$

$$-\frac{\rho(s)}{2\sum_{i=1}^{m}\lambda_i M_\theta \sigma_i^{n-2}(s)\sigma_i'(s)} \times\left[h_1(s,a)+\sqrt{H(s,a)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\Bigg\} ds>0. \tag{31}$$

In (30), we choose $r=c$, then there exists $b>c$ such that

$$\int_c^b \Bigg\{H(b,s)\rho(s)\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) -\frac{\rho(s)}{2\sum_{i=1}^{m}\lambda_i M_\theta \sigma_i^{n-2}(s)\sigma_i'(s)} \times\left[h_2(b,s)-\sqrt{H(b,s)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\Bigg\} ds>0. \tag{32}$$

By dividing (31) and (32) by $H(c,a)$ and $H(b,c)$, respectively, and then adding them, we obtain (27). The conclusion thus comes from Theorem 7. The proof is completed. □

For the case of $H := H(t-s) \in \Phi$, we have that $h_1(t-s)=h_2(t-s)$ and thus denote them by $h(t-s)$. The subclass of Φ containing such $H(t-s)$ is denoted by Φ_0. Applying Theorem 7 to Φ_0, and choosing $\rho=1$, we obtain the following.

Theorem 9. *Assume that for each $T\ge t_0$ there exist $H\in\Phi_0$ and $a,c\in R$ such that $T\le a<c$ and*

$$\int_a^c H(s-a)\left(\int_\alpha^\beta [q(s,\xi)+q(2c-s,\xi)]\,d\mu(\xi)\right)ds > \frac{1}{2}\int_a^c\left\{\frac{\left[h(s-a)-p(s)\sqrt{H(s-a)}\right]^2}{\sum_{i=1}^{m}\lambda_i M_\theta\sigma_i^{n-2}(s)\sigma_i'(s)} + \frac{\left[h(s-a)+p(2c-s)\sqrt{H(s-a)}\right]^2}{\sum_{i=1}^{m}\lambda_i M_\theta\sigma_i^{n-2}(2c-s)\sigma_i'(2c-s)}\right\}ds. \tag{33}$$

Then (1) *is oscillatory.*

Proof. Let $b=2c-a$. Then $H(b-c)=H(c-a)=H((b-a)/2)$, and for any $\varphi\in L[a,b]$, we have

$$\int_c^b \varphi(s)\,ds=\int_a^c \varphi(2c-s)\,ds. \tag{34}$$

Hence

$$\int_c^b H(b-s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds = \int_a^c H(s-a)\left(\int_\alpha^\beta q(2c-s,\xi)\,d\mu(\xi)\right)ds,$$

$$\int_c^b \frac{\left[h(b-s)+p(s)\sqrt{H(b-s)}\right]^2}{\sum_{i=1}^{m}\lambda_i M_\theta\sigma_i^{n-2}(s)\sigma_i'(s)}ds = \int_a^c \frac{\left[h(s-a)+p(2c-s)\sqrt{H(s-a)}\right]^2}{\sum_{i=1}^{m}\lambda_i M_\theta\sigma_i^{n-2}(2c-s)\sigma_i'(2c-s)}ds. \tag{35}$$

Thus (33) holds and implies that (27) holds for $H\in\Phi_0$, $\rho=1$ and therefore (1) is oscillatory by Theorem 7. The proof is completed. □

From the above oscillation criteria, we can obtain different sufficient conditions for oscillation of (1) by different choices of $H(t,s)$ and $\rho(s)$. For example, let

$$H(t,s)=(t-s)^\lambda,\quad t\ge s\ge t_0, \tag{36}$$

where $\lambda>1$ is a constant. Then, $H\in\Phi_0$ and $h(t-s)=\lambda(t-s)^{(\lambda/2)-1}$. From Theorem 8, we have the following result.

Corollary 10. *If there exists a function $\rho\in C'([t_0,\infty),(0,\infty))$ and a constant $\lambda>1$ such that for each $r\ge t_0$,*

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}} \times\int_r^t (s-r)^\lambda\rho(s) \times\Bigg\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) -\frac{1}{2\sum_{i=1}^{m}\lambda_i M_\theta\sigma_i^{n-2}(s)\sigma_i'(s)} \times\left[\frac{\lambda}{s-r}+\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\Bigg\}ds>0,$$

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}} \times\int_r^t (t-s)^\lambda\rho(s) \times\Bigg\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) -\frac{1}{2\sum_{i=1}^{m}\lambda_i M_\theta\sigma_i^{n-2}(s)\sigma_i'(s)} \times\left[\frac{\lambda}{t-s}-\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\Bigg\}ds>0. \tag{37}$$

Then (1) *is oscillatory.*

3. Oscillation Results for $f(u_1,\dots,u_m)$ without Monotonicity

Throughout this section we assume that the following conditions hold:

(A_1') there exists a function $\sigma(t) \in C'([t_0,\infty),(0,\infty))$ such that $\sigma(t) = \min\{t, \min_{1\le i\le m}\{\inf_{\xi\in[\alpha,\beta]} g_i(t,\xi)\}\}$, $\lim_{t\to\infty}\sigma(t) = \infty$, $\sigma'(t) > 0$.

(A_2') there exists a constant $\gamma > 0$ and $i_0 \in \{1,2,\dots,m\}$ such that for sufficiently large $|u_i|$ $(i \ne i_0)$

$$\liminf_{|u_{i_0}|\to\infty}\left|\frac{f(u_1,\dots,u_m)}{u_{i_0}}\right| \ge \gamma > 0. \tag{38}$$

Lemma 11. *Let $x(t)$ be an eventually positive solution of* (1). *Then, there exists a sufficiently large $T_0 \ge t_0$ such that for $t \ge T_0$, we have*

$$x'(t) > 0, \qquad x^{(n-1)}(t) > 0, \qquad x^{(n)}(t) \le 0. \tag{39}$$

The proof is similar to that of Lemma 4, thus we omit the details here.

Lemma 12. *Let $x(t)$ be an eventually positive solution of* (1). *Then, there exists a sufficiently large $T_0 \ge t_0$ such that for any interval $[c,b) \subset [T_0,\infty)$, if let*

$$u(t) = \frac{\rho(t)\,x^{(n-1)}(t)}{x[\sigma(t)/2]}, \quad t\in[c,b), \tag{40}$$

where $\rho(t) \in C'([t_0,\infty),(0,\infty))$, then for any $H \in \Phi$,

$$\begin{aligned}\int_c^b \gamma H(b,s)\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds &\le H(b,c)\,u(c)\\ &\quad+\frac{1}{2}\int_c^b \frac{\rho(s)}{M_\theta\sigma^{n-2}(s)\sigma'(s)}\\ &\quad\times\left[h_2(b,s)-\sqrt{H(b,s)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2 ds.\end{aligned} \tag{41}$$

Proof. From (1) and (40) we have that for $t\in[c,b)$

$$\begin{aligned}u'(t) &= \frac{\rho(t)\,x^{(n)}(t)+\rho'(t)\,x^{(n-1)}(t)}{x[\sigma(t)/2]}\\ &\quad-\frac{u(t)}{2x[\sigma(t)/2]}x'\left[\frac{\sigma(t)}{2}\right]\sigma'(t)\\ &= -\rho(t)\frac{\int_\alpha^\beta q(t,\xi)\,f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)])\,d\mu(\xi)}{x[\sigma(t)/2]}\\ &\quad+\left(\frac{\rho'(t)}{\rho(t)}-p(t)\right)u(t)-\frac{x'[\sigma(t)/2]}{2x[\sigma(t)/2]}\sigma'(t)\,u(t).\end{aligned} \tag{42}$$

From Lemma 11, there exists a sufficiently large $T_0 \ge t_0$ such that for all $t \ge T_0$ (39) hold and further from (A_1')

$$\frac{\sigma(t)}{2} \le \sigma(t) \le t, \qquad g_i(t,\xi) \ge \sigma(t) \ge \frac{\sigma(t)}{2}, \qquad i = 1,2,\dots,m;\ \xi\in[\alpha,\beta]. \tag{43}$$

Hence, we have for all $t \ge T_0$,

$$x^{(n-1)}\left[\frac{\sigma(t)}{2}\right] \ge x^{(n-1)}(t), \qquad x[g_i(t,\xi)] \ge x[\sigma(t)/2], \qquad i = 1,2,\dots,m;\ \xi\in[\alpha,\beta]. \tag{44}$$

From (44) and (A_2'), for all $t \ge T_0$

$$\begin{aligned}&f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)])\\ &\quad\ge \gamma x[g_{i_0}(t,\xi)] \ge \gamma x\left[\frac{\sigma(t)}{2}\right], \quad \xi\in[\alpha,\beta].\end{aligned} \tag{45}$$

Thus, for all $t \ge T_0$

$$\begin{aligned}&\frac{\int_\alpha^\beta q(t,\xi)\,f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)])\,d\mu(\xi)}{x[\sigma(t)/2]}\\ &\quad\ge \gamma\int_\alpha^\beta q(t,\xi)\,d\mu(\xi).\end{aligned} \tag{46}$$

Therefore, from (42)–(46) and Lemma 3, we obtain

$$\begin{aligned}u'(t) &\le -\gamma\rho(t)\int_\alpha^\beta q(t,\xi)\,d\mu(\xi)\\ &\quad+\left(\frac{\rho'(t)}{\rho(t)}-p(t)\right)u(t)\\ &\quad-\frac{1}{2}\rho^{-1}(t)\,M_\theta\sigma^{n-2}(t)\,\sigma'(t)\,u^2(t).\end{aligned} \tag{47}$$

The rest of the proof is similar to that of Lemma 5 and thus we omit the details here. □

Similar to the proof in Section 2, we have the following results.

Lemma 13. *Let $x(t)$ be an eventually positive solution of* (1). *Then, there exists a sufficiently large $T_0 \ge t_0$ such that, for any interval $(a,c] \subset [T_0,\infty)$, if let $u(t)$ be defined by* (40) *on $(a,c]$, then for any $H \in \Phi$,*

$$\begin{aligned}\int_a^c \gamma H(s,a)\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds &\le -H(c,a)\,u(c)\\ &\quad+\frac{1}{2}\int_a^c \frac{\rho(s)}{M_\theta\sigma^{n-2}(s)\sigma'(s)}\\ &\quad\times\left[h_1(s,a)+\sqrt{H(s,a)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2 ds.\end{aligned} \tag{48}$$

The following theorem is an immediate result from Lemmas 12 and 13.

Theorem 14. *Assume that for each $T \geq t_0$ there exist $H \in \Phi$, $\rho \in C'([t_0,\infty),(0,\infty))$ and $a,b,c \in R$, such that $T \leq a < c < b$ and*

$$\begin{aligned}
&\frac{1}{H(c,a)}\int_a^c \gamma H(s,a)\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\
&\quad+\frac{1}{H(b,c)}\int_c^b \gamma H(b,s)\rho(s)\left(\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right)ds \\
&>\frac{1}{2M_\theta}\left\{\frac{1}{H(c,a)}\int_a^c \frac{\rho(s)}{\sigma^{n-2}(s)\sigma'(s)}\right. \\
&\quad\times\left[h_1(s,a)+\sqrt{H(s,a)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2 ds \\
&\quad+\frac{1}{H(b,c)}\int_c^b \frac{\rho(s)}{\sigma^{n-2}(s)\sigma'(s)} \\
&\quad\left.\times\left[h_2(b,s)-\sqrt{H(b,s)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2 ds\right\}.
\end{aligned} \tag{49}$$

Then (1) *is oscillatory.*

Theorem 15. *Assume that for some $H \in \Phi$ and $\rho \in C'([t_0,\infty),(0,\infty))$, and for each $r \geq t_0$,*

$$\begin{aligned}
&\limsup_{t\to\infty}\int_r^t\left\{\gamma H(s,r)\rho(s)\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right. \\
&\quad-\frac{\rho(s)}{2M_\theta\sigma^{n-2}(s)\sigma'(s)} \\
&\quad\times\left[h_1(s,r)\right. \\
&\quad\left.\left.+\sqrt{H(s,r)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\right\}ds>0,
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\limsup_{t\to\infty}\int_r^t\left\{\gamma H(t,s)\rho(s)\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right. \\
&\quad-\frac{\rho(s)}{2M_\theta\sigma^{n-2}(s)\sigma'(s)} \\
&\quad\times\left[h_2(t,s)\right. \\
&\quad\left.\left.-\sqrt{H(t,s)}\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\right\}ds>0.
\end{aligned} \tag{51}$$

Then (1) *is oscillatory.*

Theorem 16. *Assume that for each $T \geq t_0$, there exist $H \in \Phi_0$ and $a,c \in R$ such that $T \leq a < c$ and*

$$\begin{aligned}
&\int_a^c \gamma H(s-a)\left(\int_\alpha^\beta [q(s,\xi)+q(2c-s,\xi)]\,d\mu(\xi)\right)ds \\
&\quad>\frac{1}{2M_\theta}\int_a^c\left\{\frac{\left[h(s-a)-p(s)\sqrt{H(s-a)}\right]^2}{\sigma^{n-2}(s)\sigma'(s)}\right. \\
&\quad\left.+\frac{\left[h(s-a)+p(2c-s)\sqrt{H(s-a)}\right]^2}{\sigma^{n-2}(2c-s)\sigma'(2c-s)}\right\}ds.
\end{aligned} \tag{52}$$

Then (1) *is oscillatory.*

Corollary 17. *If there exists a function $\rho \in C'([t_0,\infty),(0,\infty))$ and a constant $\lambda > 1$ such that for each $r \geq t_0$, the following two inequalities hold*

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_r^t \gamma(s-r)^\lambda\rho(s) \\
&\quad\times\left\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)-\frac{1}{2M_\theta\sigma^{n-2}(s)\sigma'(s)}\right. \\
&\quad\left.\times\left[\frac{\lambda}{s-r}+\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\right\}ds>0,
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_r^t \gamma(t-s)^\lambda\rho(s) \\
&\quad\times\left\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)-\frac{1}{2M_\theta\sigma^{n-2}(s)\sigma'(s)}\right. \\
&\quad\left.\times\left[\frac{\lambda}{t-s}-\left(\frac{\rho'(s)}{\rho(s)}-p(s)\right)\right]^2\right\}ds>0.
\end{aligned} \tag{54}$$

Then (1) *is oscillatory.*

4. Examples

In this section we demonstrate the applications of our oscillation criteria through two examples. We will see that the equations in the examples are oscillatory based on the results in Sections 2 and 3.

Example 1. Consider the following nonlinear damped differential equation:

$$\begin{aligned}
&x^{(4)}(t)+\frac{2t}{\exp(t^2)}x^{(3)}(t) \\
&\quad+\int_0^1 e^{2t+\xi}\left[x(t+\xi)+x\left(3t+\xi^2\right)\right. \\
&\quad\left.+x^3(t+\xi)+x^5\left(3t+\xi^2\right)\right]d\xi=0,
\end{aligned} \tag{55}$$

where $t \geq 1$, $p(t) = (2t/\exp(t^2))$, $q(t,\xi) = e^{2t+\xi}$, $f(u_1,u_2) = u_1 + u_2 + u_1^3 + u_2^5$, $g_1(t,\xi) = t + \xi$, $g_2(t,\xi) = 3t + \xi^2$, $\mu(\xi) = \xi$. It is clear that for $t_1 \geq 1$

$$\begin{aligned}
&\lim_{t\to\infty}\int_{t_1}^{t}\exp\left(-\int_{t_1}^{s}p(\tau)\,d\tau\right)ds \\
&\quad = \lim_{t\to\infty}\int_{t_1}^{t}\exp\left(-\int_{t_1}^{s}\frac{2\tau}{\exp(\tau^2)}d\tau\right)ds = \infty, \\
&\sigma_1(t) = t, \qquad \sigma_2(t) = t, \\
&\frac{\partial f}{\partial u_1} = 1 + 3u_1^2 \geq 1 = \lambda_1, \\
&\frac{\partial f}{\partial u_2} = 1 + 5u_2^4 \geq 1 = \lambda_2.
\end{aligned} \tag{56}$$

Applying Corollary 10 with $\lambda = 2$ and $\rho(s) = s^3$, we have through a straightforward computation that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_r^t (s-r)^{\lambda}\rho(s) \\
&\quad\times\left\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) - \frac{1}{2\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)}\right. \\
&\qquad\left.\times\left[\frac{\lambda}{s-r} + \left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2\right\}ds \\
&= \limsup_{t\to\infty}\frac{1}{t}\int_r^t (s-r)^2 s^3 \\
&\quad\times\left\{\int_0^1 e^{2s+\xi}d\xi - \frac{1}{4M_\theta s^2}\left[\frac{5s-3r}{s(s-r)}\right.\right. \\
&\qquad\left.\left. - \frac{2s}{\exp(s^2)}\right]^2\right\}ds = \infty,
\end{aligned}$$

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}} \\
&\quad\times\int_r^t (t-s)^{\lambda}\rho(s) \\
&\quad\times\left\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi)\right. \\
&\qquad - \frac{1}{2\sum_{i=1}^m \lambda_i M_\theta \sigma_i^{n-2}(s)\,\sigma_i'(s)} \\
&\qquad\left.\times\left[\frac{\lambda}{t-s} + \left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2\right\}ds \\
&\quad = \limsup_{t\to\infty}\frac{1}{t} \\
&\quad\times\int_r^t (t-s)^2 s^3 \\
&\quad\times\left\{\int_0^1 e^{2s+\xi}d\xi\right. \\
&\qquad\left. - \frac{1}{4M_\theta s^2}\left[\frac{5s-3t}{s(t-s)} + \frac{2s}{\exp(s^2)}\right]^2\right\} \\
&\quad\times ds = \infty.
\end{aligned} \tag{57}$$

Therefore (37) hold and we conclude by Corollary 10 that (55) is oscillatory.

Example 2. Consider the following nonlinear damped differential equation:

$$\begin{aligned}
&x^{(4)}(t) + \exp(-t)\,x^{(3)}(t) \\
&\quad + \int_0^{\pi/2}\frac{t^2\sin 2\xi}{1+\sin^2\xi}\frac{x(t+\sin\xi)}{2-\exp\left(-x^2(t+\cos\xi)\right)}d\xi = 0, \\
&\qquad t \geq 1,
\end{aligned} \tag{58}$$

where $p(t) = 1/e^t$, $q(t,\xi) = t^2\sin 2\xi/(1+\sin^2\xi)$, $f(u_1,u_2) = u_2/(2-\exp(-u_1^2))$, $g_1(t,\xi) = t + \cos\xi$, $g_2(t,\xi) = t + \sin\xi$, $\mu(\xi) = \xi$. In this example,

$$\frac{\partial f}{\partial u_1} = -\frac{2u_1u_2\exp\left(-u_1^2\right)}{\left(2-\exp\left(-u_1^2\right)\right)^2}. \tag{59}$$

Clearly, Corollary 10 does not apply to (58). However, with $\lambda = 2$ and $\rho(t) = 1$, we can prove the oscillatory character of (58) by Corollary 17. Noting that

$$\begin{aligned}
&\frac{f(u_1,u_2)}{u_2} = \frac{1}{2-\exp\left(-u_1^2\right)} \geq \frac{1}{2} = \gamma, \quad \forall\, u_2 \neq 0, \\
&\lim_{t\to\infty}\int_{t_1}^{t}\exp\left(-\int_{t_1}^{s}p(\tau)\,d\tau\right)ds \\
&\quad = \lim_{t\to\infty}\int_{t_1}^{t}\exp\left(-\int_{t_1}^{s}\frac{1}{e^\tau}d\tau\right)ds = \infty,
\end{aligned} \tag{60}$$

for $t_1 \geq 1$ and $\sigma(t) = t$, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}} \\
&\quad\times\int_r^t \gamma(s-r)^{\lambda}\rho(s) \\
&\quad\times\left\{\int_\alpha^\beta q(s,\xi)\,d\mu(\xi) - \frac{1}{2M_\theta\sigma^{n-2}(s)\,\sigma'(s)}\right. \\
&\qquad\left.\times\left[\frac{\lambda}{s-r} + \left(\frac{\rho'(s)}{\rho(s)} - p(s)\right)\right]^2\right\}ds
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\to\infty} \frac{1}{t}\int_r^t \frac{1}{2}(s-r)^2 \\
&\quad \times \left\{ \int_0^{\pi/2} \frac{s^2 \sin 2\xi}{1+\sin^2\xi} d\xi \right. \\
&\qquad \left. - \frac{1}{2M_\theta s^2}\left[\frac{2}{s-r} - \frac{1}{e^s}\right]^2 \right\} ds \\
&= \limsup_{t\to\infty} \frac{1}{t}\int_r^t \left\{ \frac{1}{2}\ln 2s^2 (s-r)^2 \right. \\
&\qquad \left. - \frac{\left(2e^s - s + r\right)^2}{4M_\theta s^2 e^{2s}} \right\} ds = \infty, \\
&\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_r^t \gamma (t-s)^\lambda \rho(s) \\
&\quad \times \left\{ \int_\alpha^\beta q(s,\xi)\, d\mu(\xi) \right. \\
&\qquad - \frac{1}{2M_\theta \sigma^{n-2}(s)\,\sigma'(s)} \\
&\qquad \left. \times \left[\frac{\lambda}{t-s} - \left(\frac{\rho'(s)}{\rho(s)} - p(s) \right) \right]^2 \right\} ds = \infty,
\end{aligned}
\tag{61}
$$

therefore (53) and (54) hold and we conclude by Corollary 10 that (58) is oscillatory.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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Research Article

Impulsive Boundary Value Problems for Planar Hamiltonian Systems

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We give an existence and uniqueness theorem for solutions of inhomogeneous impulsive boundary value problem (BVP) for planar Hamiltonian systems. Green's function that is needed for representing the solutions is obtained and its properties are listed. The uniqueness of solutions is connected to a Lyapunov type inequality for the corresponding homogeneous BVP.

1. Introduction

The planar Hamiltonian system of 2-linear first-order equations has the form

$$y' = JH(t)\,y, \quad t \in \mathbb{R}, \tag{1}$$

where

$$H(t) = \begin{bmatrix} c(t) & a(t) \\ a(t) & b(t) \end{bmatrix} \tag{2}$$

is a symmetric matrix with piecewise continuous real-valued entries, and

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3}$$

is the so called symplectic identity. Setting

$$y_1(t) = x(t), \qquad y_2(t) = u(t), \tag{4}$$

the Hamiltonian system can be rewritten in a more convenient way as

$$\begin{aligned} x' &= a(t)\,x + b(t)\,u, \\ u' &= -c(t)\,x - a(t)\,u. \end{aligned} \tag{5}$$

Our aim in this work is to prove an existence and uniqueness theorem for solutions of the related BVP for inhomogeneous Hamiltonian system under impulse effect of the form

$$\begin{aligned} x' &= a(t)\,x + b(t)\,u + f_1(t), \\ u' &= -c(t)\,x - a(t)\,u + f_2(t), \\ &\qquad t \in (t_1, t_2) \setminus \{\tau_i\}, \end{aligned} \tag{6a}$$

$$\begin{aligned} x(\tau_i^+) &= \alpha_i x(\tau_i^-) + a_{i1}, \\ u(\tau_i^+) &= -\beta_i x(\tau_i^-) + \alpha_i u(\tau_i^-) + a_{i2}, \\ &\qquad i = 1, 2, \ldots, p, \end{aligned} \tag{6b}$$

$$x(t_1) = A, \quad x(t_2) = B, \tag{6c}$$

where

(i) $\{\tau_i\}$, $\{\alpha_i\}$, $\{\beta_i\}$, $\{a_{i1}\}$, and $\{a_{i2}\}$ are real sequences for $i = 1, 2, \ldots, p$ with

$$t_1 < \tau_1 < \tau_2 < \cdots < \tau_p < t_2; \tag{7}$$

(ii) $a, b, c, f_1, f_2 \in \mathrm{PLC}(J_0)$, where $J_0 = [t_1, t_2]$ and $\mathrm{PLC}(J_0) = \{\omega : J_0 \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $\omega(\tau_i^{\pm})$ exist and $\omega(\tau_i^-) = \omega(\tau_i)$ for $i = 1, 2, \ldots, p\}$;

(iii) $b(t) > 0$ for $t \in (t_1, t_2)$ and $\alpha_i \neq 0$ for $i = 1, 2, \ldots, p$; A and B are given real numbers.

We also set $\tau_0 = t_1$ and $\tau_{p+1} = t_2$ for convenience.

By a solution of the impulsive BVP (6a)–(6c), we mean nontrivial functions $x, u \in \mathrm{PLC}(J_0)$ such that (x, u) satisfies system (6a)–(6c) for all $t \in J_0$.

The corresponding homogeneous BVP takes the form

$$x' = a(t)x + b(t)u, \qquad u' = -c(t)x - a(t)u, \qquad t \in (t_1, t_2) \setminus \{\tau_i\}, \tag{8a}$$

$$x(\tau_i^+) = \alpha_i x(\tau_i^-), \qquad u(\tau_i^+) = -\beta_i x(\tau_i^-) + \alpha_i u(\tau_i^-), \qquad i = 1, 2, \ldots, p, \tag{8b}$$

$$x(t_1) = 0, \qquad x(t_2) = 0. \tag{8c}$$

Note that if we take

$$a(t) \equiv 0, \qquad b(t) = \frac{1}{p(t)}, \qquad c(t) = q(t), \qquad f_1(t) \equiv 0, \qquad f_2(t) = f(t), \tag{9}$$

then we obtain as a special case of (6a), (6b), and (6c) the impulsive BVP for second-order differential equations of the form

$$\left(p(t)x'\right)' + q(t)x = f(t), \quad t \in (t_1, t_2) \setminus \{\tau_i\}, \tag{10a}$$

$$x(\tau_i^+) = \alpha_i x(\tau_i^-) + a_{i1}, \tag{10b}$$

$$\left(px'\right)(\tau_i^+) = -\beta_i x(\tau_i^-) + \alpha_i \left(px'\right)(\tau_i^-) + a_{i2}, \quad i = 1, 2, \ldots, p, \tag{10c}$$

$$x(t_1) = A, \qquad x(t_2) = B. \tag{10d}$$

To the best of our knowledge although many results have been obtained for linear impulsive boundary value problems by using different techniques, there is little known for the linear 2×2 Hamiltonian systems under impulse effect.

The existence and uniqueness of linear impulsive boundary value problem for the first-order equations are considered in [1–4]. For the second-order case we refer to [5, 6] in which the integral representation of the solution of second order linear impulsive boundary value problems is given by using Green's function and the existence and uniqueness of the solutions are obtained. Variational technique approach for the existence of the solutions of linear and nonlinear impulsive boundary value problems can be found in [7–10]. In [11], the method of upper and lower solutions is employed for the existence of solutions of nonlinear impulsive boundary value problems. For a detailed discussion on boundary value problems for higher-order linear impulsive equations we refer to [12]. Basic theory of impulsive differential equations is contained in the seminal book [13].

Our method of proof is based on Green's function formulation and Lyapunov type inequalities for linear Hamiltonian system under impulse effect. There are many studies on Lyapunov type inequalities and their applications for linear ordinary differential equations [14] and for systems [15–17] as well as for linear impulsive differential equations and systems [18, 19]. However, the use of a Lyapunov type inequality in connection with BVPs seems to be limited.

2. Preliminaries

2.1. Lyapunov Type Inequality for Homogeneous Problem. In this section we provide a Lyapunov type inequality to be used for the uniqueness of the inhomogeneous BVP. The obtained inequality is sharper than the one given by the present authors in [20] in the sense that $2|a(t)|$ is replaced by $|a(t)|$.

Theorem 1. *If the homogeneous BVP* (8a), (8b), *and* (8c) *has a real solution* $(x(t), u(t))$ *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2), *then one has the Lyapunov type inequality:*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|\, dt\right)\left[\int_{t_1}^{t_2} b(t)\, dt\right] \times \left[\int_{t_1}^{t_2} c^+(t)\, dt + \sum_{t_1 \le \tau_i < t_2} \left(\frac{\beta_i}{\alpha_i}\right)^+\right] \ge 4, \tag{11}$$

where $c^+(t) = \max\{c(t), 0\}$ *and* $(\beta_i/\alpha_i)^+ = \max\{\beta_i/\alpha_i, 0\}$.

Proof. Define

$$z(t) = \frac{1}{\alpha_1\alpha_2\cdots\alpha_i} x(t), \qquad v(t) = \frac{1}{\alpha_1\alpha_2\cdots\alpha_i} u(t) \tag{12}$$

for $t \in (\tau_i, \tau_{i+1})$ and $i = 0, 1, \ldots, p$, where we put again $\tau_0 = t_1$, $\tau_{p+1} = t_2$ and make a convention that $\alpha_1\alpha_2\cdots\alpha_i = 1$ if $i = 0$.

It is not difficult to see from (8a), (8b), (8c), and (12) that

$$z' = a(t)z + b(t)v, \qquad v' = -c(t)z - a(t)v, \qquad t \in (t_1, t_2) \setminus \{\tau_i\}, \tag{13}$$

$$z(\tau_i^+) = z(\tau_i^-), \qquad v(\tau_i^+) = -\frac{\beta_i}{\alpha_i} z(\tau_i^-) + v(\tau_i^-), \qquad i = 1, 2, \ldots, p, \tag{14}$$

$$z(t_1) = 0, \qquad z(t_2) = 0. \tag{15}$$

Since we assumed that $z(\tau_i) = z(\tau_i^-)$, $z(t)$ is continuous on $[t_1, t_2]$. Moreover, $z' \in \mathrm{PLC}(J_0)$, $z(t_1) = z(t_2) = 0$, and $z(t) \not\equiv 0$ for all $t \in (t_1, t_2)$. We may assume without loss of generality that $z(t) \ge 0$ on (t_1, t_2).

Using (13) and (14) we obtain

$$(vz)' = -c(t)z^2 + b(t)v^2, \quad t \ne \tau_i, \tag{16}$$

$$(vz)(\tau_i^+) - (vz)(\tau_i^-) = -\frac{\beta_i}{\alpha_i} z^2(\tau_i). \tag{17}$$

Integrating (16) from t_1 to t_2 and using (15) and (17) lead to

$$\sum_{t_1 \le \tau_i < t_2} \frac{\beta_i}{\alpha_i} z^2(\tau_i) = \int_{t_1}^{t_2} \left[b(t)v^2(t) - c(t)z^2(t)\right] dt, \tag{18}$$

from which we have

$$\int_{t_1}^{t_2} b(t) v^2(t)\,\mathrm{d}t \le \int_{t_1}^{t_2} c^+(t) z^2(t)\,\mathrm{d}t + \sum_{t_1\le\tau_i<t_2}\left(\frac{\beta_i}{\alpha_i}\right)^+ z^2(\tau_i). \tag{19}$$

On the other hand, from the first equation in (13), we have

$$\left[z(t)\exp\left(-\int_{t_1}^{t} a(u)\mathrm{d}u\right)\right]' = b(t) v(t)\exp\left(-\int_{t_1}^{t} a(u)\,\mathrm{d}u\right), \tag{20}$$

$$\left[z(t)\exp\left(\int_{t}^{t_2} a(u)\mathrm{d}u\right)\right]' = b(t) v(t)\exp\left(\int_{t}^{t_2} a(u)\,\mathrm{d}u\right). \tag{21}$$

Let

$$\max\{|z(t)| : t\in(t_1,t_2)\} = z(t_0) > 0. \tag{22}$$

If we integrate (20) from t_1 to t_0, we see that

$$z(t_0) = \int_{t_1}^{t_0} b(t) v(t)\exp\left(\int_{t}^{t_0} a(u)\,\mathrm{d}u\right)\mathrm{d}t \tag{23}$$

and so

$$z(t_0) \le \int_{t_1}^{t_0} b(t)|v(t)|\exp\left(\int_{t}^{t_0} |a(u)|\,\mathrm{d}u\right)\mathrm{d}t. \tag{24}$$

Using the obvious estimate

$$\int_{t}^{t_0} |a(u)|\,\mathrm{d}u \le \int_{t_1}^{t_0} |a(u)|\,\mathrm{d}u \tag{25}$$

and then applying Cauchy-Schwarz inequality, we have

$$z^2(t_0) \le \exp\left(2\int_{t_1}^{t_0} |a(u)|\,\mathrm{d}u\right)\left[\int_{t_1}^{t_0} b(t)\,\mathrm{d}t\right] \times\left[\int_{t_1}^{t_0} b(t) v^2(t)\,\mathrm{d}t\right]. \tag{26}$$

Similarly, by integrating (21) from t_0 to t_2 and following the above procedure, we get

$$z^2(t_0) \le \exp\left(2\int_{t_0}^{t_2} |a(u)|\,\mathrm{d}u\right)\left[\int_{t_0}^{t_2} b(t)\,\mathrm{d}t\right] \times\left[\int_{t_0}^{t_2} b(t) v^2(t)\,\mathrm{d}t\right]. \tag{27}$$

Now we recall the elementary inequality:

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} \ge 4xy, \quad \alpha,\beta>0,\ \alpha+\beta=1 \tag{28}$$

for $x\ge 0$ and $y\ge 0$. In view of (26) and (27) setting

$$\alpha = \frac{\int_{t_1}^{t_0} b(t)\,\mathrm{d}t}{\int_{t_1}^{t_2} b(t)\,\mathrm{d}t}, \qquad \beta = \frac{\int_{t_0}^{t_2} b(t)\,\mathrm{d}t}{\int_{t_1}^{t_2} b(t)\,\mathrm{d}t},$$
$$x = z(t_0)\exp\left(-\int_{t_1}^{t_0} |a(u)|\,\mathrm{d}u\right), \tag{29}$$
$$y = z(t_0)\exp\left(-\int_{t_0}^{t_2} |a(u)|\,\mathrm{d}u\right)$$

we have

$$\frac{4z^2(t_0)}{\exp\left(\int_{t_1}^{t_2}|a(t)|\,\mathrm{d}t\right)} \le \left[\int_{t_1}^{t_2} b(t)\,\mathrm{d}t\right]\left[\int_{t_1}^{t_2} b(t) v^2(t)\,\mathrm{d}t\right]. \tag{30}$$

Combining (19) and (30) results in

$$\frac{4z^2(t_0)}{\exp\left(\int_{t_1}^{t_2}|a(t)|\,\mathrm{d}t\right)} \le \left[\int_{t_1}^{t_2} b(t)\,\mathrm{d}t\right] \times\left[\int_{t_1}^{t_2} c(t) z^2(t)\,\mathrm{d}t + \sum_{t_1\le\tau_i<t_2}\left(\frac{\beta_i}{\alpha_i}\right) z^2(\tau_i)\right]. \tag{31}$$

Finally, since $z(t_0)\ge z(t)$ for $t\in[t_1,t_2]$, from (31) we obtain the desired inequality:

$$\exp\left(\int_{t_1}^{t_2}|a(t)|\,\mathrm{d}t\right)\left[\int_{t_1}^{t_2} b(t)\,\mathrm{d}t\right] \times\left[\int_{t_1}^{t_2} c^+(t)\,\mathrm{d}t + \sum_{t_1\le\tau_i<t_2}\left(\frac{\beta_i}{\alpha_i}\right)^+\right] \ge 4. \tag{32}$$

□

2.2. Green's Function. Here we derive Green's function to be used for the representation of the solutions of the inhomogeneous BVP.

Let

$$\Phi(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ u_1(t) & u_2(t)\end{bmatrix}, \qquad \Phi(0) = I \tag{33}$$

be a fundamental matrix for (8a), (8b) and set

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}. \tag{34}$$

Define the rectangles

$$\begin{aligned} R_{11} &= [t_1,\tau_1]\times[t_1,\tau_1], \\ R_{i1} &= (\tau_{i-1},\tau_i]\times[t_1,\tau_1], \quad i=2,3,\ldots,p+1, \\ R_{1j} &= [t_1,\tau_1]\times(\tau_{j-1},\tau_j], \quad j=2,3,\ldots,p+1, \\ R_{ij} &= (\tau_{i-1},\tau_i]\times(\tau_{j-1},\tau_j], \quad i,j=2,3,\ldots,p+1, \end{aligned} \tag{35}$$

and the triangles

$$\begin{aligned} T^u &= \{(t,s) \in [t_1,t_2] \times [t_1,t_2] : s > t\}, \\ T^l &= \{(t,s) \in [t_1,t_2] \times [t_1,t_2] : s < t\}, \\ T^u_{ii} &= \{(t,s) \in R_{ii} : s > t\}, \quad i = 1,2,3,\ldots,p+1, \\ T^l_{ii} &= \{(t,s) \in R_{ii} : s < t\}, \quad i = 1,2,3,\ldots,p+1. \end{aligned} \tag{36}$$

Green's function (pair) and its properties are given in the next theorem.

Theorem 2. *Suppose that the homogeneous BVP* (8a)–(8c) *has only the trivial solution. Let*

$$K = -[M\Phi(t_1) + N\Phi(t_2)]^{-1} N\Phi(t_2). \tag{37}$$

Note that the inverse of matrix $M\Phi(t_1)+N\Phi(t_2)$ *exists in view of the assumption (see also the proof of Theorem 4).*

Then the pair of functions

$$\begin{aligned} G(t,s) &= \begin{cases} \Phi(t)(I+K)\Phi^{-1}(s), & s < t, \\ \Phi(t)K\Phi^{-1}(s), & s \geq t, \end{cases} \\ \widetilde{G}(t,\tau_i^+) &= \begin{cases} \Phi(t)(I+K)\Phi^{-1}(\tau_i^+), & \tau_i < t, \\ \Phi(t)K\Phi^{-1}(\tau_i^+), & \tau_i \geq t, \end{cases} \end{aligned} \tag{38}$$

constitutes Green's function for (6a), (6b), *and* (6c). *Moreover, we have the following properties:*

(G1) $G(t,s)$ *is continuous and bounded on* R_{ij},

(G2) $(\partial G(t,s))/\partial t$ *is continuous and bounded on the rectangles* R_{ij} *with* $i \neq j$ *and on the triangles* T^u_{ii} *and* T^l_{ii},

(G3) $G(t,s)$ *satisfies the following jump conditions:*

(a) $G(\tau_i^+,\tau_i) - G(\tau_i^-,\tau_i) = B_i + (B_i - I)G(\tau_i^-,\tau_i)$ *where* $B_i = \begin{bmatrix} \alpha_i & 0 \\ -\beta_i & \alpha_i \end{bmatrix}$,

(b) $G(s^+,s) - G(s^-,s) = I$, $s \neq \tau_i$,

(c) $(\partial G(s^+,s)/\partial t) - (\partial G(s^-,s)/\partial t) = JH(s)$, $s \neq \tau_i$,

(G4) $G(t,s)$, *considered as a function of* t, *is left continuous and satisfies*

$$\begin{aligned} y' &= JH(t)y, \quad t \in J_s \setminus \{\tau_i\}, \\ y(\tau_i^+) &= B_i y(\tau_i^-), \quad i \in \{i : \tau_i \in J_s\}, \\ My(t_1) &+ Ny(t_2) = 0, \end{aligned} \tag{39}$$

where J_s *is any of the intervals* $[t_1,s)$ *or* $(s,t_2]$,

(G5) $\Delta|_{t=\tau_i}\widetilde{G}(t,\tau_i^+) = \widetilde{G}(\tau_i^+,\tau_i^+) - \widetilde{G}(\tau_i^-,\tau_i^+) = (B_i - I)\widetilde{G}(\tau_i^-,\tau_i^+)$,

(G6) $\widetilde{G}(t,s)$, *considered as a function of* t, *is left continuous and satisfies* (39).

Proof. (G1) and (G2) are trivial. Let us consider (G3)(a) follows from

$$\begin{aligned} &G(\tau_i^+,\tau_i) - G(\tau_i^-,\tau_i) \\ &\quad = \Phi(\tau_i^+)(I+K)\Phi^{-1}(\tau_i) - \Phi(\tau_i^-)K\Phi^{-1}(\tau_i) \\ &\quad = B_i + (B_i - I)G(\tau_i^-,\tau_i). \end{aligned} \tag{40}$$

To see (b), we write for $s \neq \tau_i$,

$$\begin{aligned} G(s^+,s) - G(s^-,s) &= \Phi(s^+)(I+K)\Phi^{-1}(s) \\ &\quad - \Phi(s^-)K\Phi^{-1}(s) = I. \end{aligned} \tag{41}$$

For (c), let $t \neq \tau_i$; then

$$\begin{aligned} \frac{\partial G(t,s)}{\partial t} &= \begin{cases} \Phi'(t)(I+K)\Phi^{-1}(s) \\ \quad = JH(t)\Phi(t)(I+K)\Phi^{-1}(s), & s < t, \\ \Phi'(t)K\Phi^{-1}(s) \\ \quad = JH(t)\Phi(t)K\Phi^{-1}(s), & s \geq t, \end{cases} \\ \frac{\partial G(s^+,s)}{\partial t} - \frac{\partial G(s^-,s)}{\partial t} &= JH(s)\Phi(s)(I+K)\Phi^{-1}(s) \\ &\quad - JH(s)\Phi(s)K\Phi^{-1}(s) \\ &= JH(s). \end{aligned} \tag{42}$$

Next, we consider (G4). By definition, it is easy to see that $G(t,s)$ is left continuous function at $t = \tau_i$. Let us consider the interval $[t_1,s)$. The later is similar. The first equation in (39) is direct consequences of (c) and the definition of $G(t,s)$. Clearly,

$$\begin{aligned} G(\tau_i^+,s) &= \Phi(\tau_i^+)K\Phi^{-1}(s) \\ &= B_i\Phi(\tau_i^-)K\Phi^{-1}(s) = B_i G(\tau_i^-,s), \\ MG(t_1,s) &+ NG(t_2,s) \\ &= M\Phi(t_1)K\Phi^{-1}(s) \\ &\quad + N\Phi(t_2)(I+K)\Phi^{-1}(s) \\ &= [M\Phi(t_1) + N\Phi(t_2)]K\Phi^{-1}(s) \\ &\quad + N\Phi(t_2)\Phi^{-1}(s) = 0. \end{aligned} \tag{43}$$

The proofs of (G5) and (G6) are similar to (a) and (G4), respectively. □

Remark 3. One can easily rewrite Green's function (pair) in terms of the solutions of system (8a), (8b). Indeed,

$$\begin{aligned} K &= -[M\Phi(t_1) + N\Phi(t_2)]^{-1} N\Phi(t_2) \\ &= \frac{1}{x_1(t_1)x_2(t_2) - x_1(t_2)x_2(t_1)} \\ &\quad \times \begin{bmatrix} x_1(t_2)x_2(t_1) & x_2(t_1)x_2(t_2) \\ -x_1(t_1)x_1(t_2) & -x_1(t_1)x_2(t_2) \end{bmatrix}, \end{aligned} \tag{44}$$

and since

$$\det \Phi(t) = \det \Phi(0) \exp\left(\int_0^t \operatorname{trace}(JH(s))\,ds\right) \times \prod_{i=1}^{p} \det B_i = \prod_{i=1}^{p} \alpha_i^2, \tag{45}$$

we may write

$$\Phi^{-1}(t) = \frac{1}{\det \Phi(t)} \begin{bmatrix} u_2(t) & -x_2(t) \\ -u_1(t) & x_1(t) \end{bmatrix} = \left(\prod_{i=1}^{p} \alpha_i^{-2}\right) \begin{bmatrix} u_2(t) & -x_2(t) \\ -u_1(t) & x_1(t) \end{bmatrix}. \tag{46}$$

3. The Main Result

Our main result is the following theorem.

Theorem 4. *Let (i)–(iii) hold. If*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|\,dt\right)\left[\int_{t_1}^{t_2} b(t)\,dt\right] \times \left[\int_{t_1}^{t_2} c^+(t)\,dt + \sum_{t_1 \le \tau_i < t_2} \left(\frac{\beta_i}{\alpha_i}\right)^+\right] < 4, \tag{47}$$

then BVP (6a), (6b), *and* (6c) *has a unique solution* $(x(t), u(t))$. *Moreover,* $y = (x(t), u(t))$ *is expressible as*

$$y(t) = w(t) + \int_{t_1}^{t_2} G(t,s) f(s)\,ds + \sum_{t_1 \le \tau_i < t_2} \widetilde{G}(t, \tau_i^+) a_i, \tag{48}$$

where

$$w(t) = \Phi(t)\left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1}\eta, \quad \eta = (A, B)^T, \tag{49}$$

and Green's function pair $(G, \widetilde{G})$ *is given by* (38).

Proof. We first prove the uniqueness. It suffices to show that the homogeneous BVP (8a)–(8c) has only the trivial solution. Let $x(t) \not\equiv 0$ on (t_1, t_2). By Theorem 1, we see that Lyapunov type inequality (11) holds contradicting the inequality (47). Thus $x(t) = 0$ for all $t \in [t_1, t_2]$. Moreover, by (6a), (6b), and (6c) we have

$$b(t)\,u = 0, \quad t \ne \tau_i, \tag{50}$$

which results in $u(t) = 0$ for $t \ne \tau_i$. Taking limit we see that $u(\tau_i^{\pm}) = 0$. As a result we obtain $(x(t), u(t)) = (0, 0)$ for all $t \in [t_1, t_2]$. This completes the uniqueness of the solutions.

For the existence, we start with the variation of parameters formula and write the general solution of system (6a), (6b) as

$$y(t) = \Phi(t)\,c + \int_{t_1}^{t} \Phi(t)\,\Phi^{-1}(s) f(s)\,ds + \sum_{t_1 \le \tau_i < t} \Phi(t)\,\Phi^{-1}(\tau_i^+)\,a_i. \tag{51}$$

Clearly, the boundary condition is satisfied if

$$\left[M\Phi(t_1) + N\Phi(t_2)\right] c = \eta - N\Phi(t_2) \times \left[\int_{t_1}^{t_2} \Phi^{-1}(s) f(s)\,ds + \sum_{t_1 \le \tau_i < t_2} \Phi^{-1}(\tau_i^+)\,a_i\right], \tag{52}$$

where $\eta = (A, B)^T$.

Since we have the uniqueness of solutions, the matrix $M\Phi(t_1) + N\Phi(t_2)$ must have an inverse. Setting

$$K = -\left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1} N\Phi(t_2), \tag{53}$$

we may solve c from (52) uniquely:

$$c = \left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1}\eta + K\left[\int_{t_1}^{t_2} \Phi^{-1}(s) f(s)\,ds + \sum_{t_1 \le \tau_i < t_2} \Phi^{-1}(\tau_i^+)\,a_i\right]. \tag{54}$$

Hence,

$$y(t) = \Phi(t)\left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1}\eta + \Phi(t)(I + K) \times \left[\int_{t_1}^{t} \Phi^{-1}(s) f(s)\,ds + \sum_{t_1 \le \tau_i < t} \Phi^{-1}(\tau_i^+)\,a_i\right] + \Phi(t)\,K\left[\int_{t}^{t_2} \Phi^{-1}(s) f(s)\,ds + \sum_{t \le \tau_i < t_2} \Phi^{-1}(\tau_i^+)\,a_i\right]. \tag{55}$$

Therefore the unique solution of the BVP (6a)–(6c) can be expressed as

$$y(t) = w(t) + \int_{t_1}^{t_2} G(t,s) f(s)\,ds + \sum_{t_1 \le \tau_i < t_2} \widetilde{G}(t, \tau_i^+)\,a_i. \tag{56}$$

□

Let us now consider the BVP (10a), (10b), (10c), and (10d). In this case it is not difficult to see that the corresponding Green's function (pair) becomes

$$G(t,s)=\begin{cases}\psi(t)(I+K)\Psi^{-1}(s)\dfrac{1}{p(s)}e_2, & s<t,\\ \psi(t)K\Psi^{-1}(s)\dfrac{1}{p(s)}e_2, & s\geq t,\end{cases}$$
$$\widetilde{G}(t,\tau_i^+)=\begin{cases}\psi(t)(I+K)\Psi^{-1}(\tau_i^+), & \tau_i<t,\\ \psi(t)K\Psi^{-1}(\tau_i^+), & \tau_i\geq t,\end{cases} \tag{57}$$

where $\psi(t)=[\psi_1,\psi_2]$ is the first row of the (Wronskian) matrix:

$$\Psi(t)=\begin{bmatrix}\psi_1(t) & \psi_2(t)\\ \psi_1'(t) & \psi_2'(t)\end{bmatrix},$$
$$K=-\left[M\Psi(t_1)+N\Psi(t_2)\right]^{-1}N\Psi(t_2),$$
$$e_2=[0,1]^T. \tag{58}$$

Corollary 5. *Suppose that p and c are piece-wise continuous on $[t_1,t_2]$, $p(t)>0$, and $\alpha_i\neq 0$ for $i=1,2,\ldots,p$. If*

$$\left[\int_{t_1}^{t_2}\frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2}c^+(t)\,dt+\sum_{t_1\leq\tau_i<t_2}\left(\frac{\beta_i}{\alpha_i}\right)^+\right]<4, \tag{59}$$

then the BVP (10a), (10b), (10c), *and* (10d) *has a unique solution $x(t)$ which is expressible as*

$$x(t)=w(t)+\int_{t_1}^{t_2}G(t,s)f(s)\,ds+\sum_{t_1\leq\tau_i<t_2}\widetilde{G}(t,\tau_i^+)a_i, \tag{60}$$

where

$$w(t)=\psi(t)\left[M\Psi(t_1)+N\Psi(t_2)\right]^{-1}\eta, \tag{61}$$

and Green's function pair $(G,\widetilde{G})$ is given by (57).

Remark 6. The results in this work are new even if the impulses are absent. The statements of the corresponding theorems are left to the reader.

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Research Article

Oscillation of Half-Linear Differential Equations with Delay

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We study the half-linear delay differential equation $(r(t)\Phi(x'(t)))' + c(t)\Phi(x(\tau(t))) = 0$, $\Phi(x) := |x|^{p-2}x$, $p > 1$. We establish a new a priori bound for the nonoscillatory solution of this equation and utilize this bound to derive new oscillation criteria for this equation in terms of oscillation criteria for an ordinary half-linear differential equation. The presented results extend and improve previous results of other authors. An extension to neutral equations is also provided.

1. Introduction

In this paper we study oscillatory properties of the delay second-order half-linear differential equation

$$\begin{gathered}\left(r(t)\Phi\left(x'(t)\right)\right)' + c(t)\Phi(x(\tau(t))) = 0,\\ \Phi(x) := |x|^{p-2}x, \quad p > 1.\end{gathered} \tag{1}$$

We suppose that r, c, τ are continuous functions defined on $[t_0, \infty)$ such that $r(t) > 0$, $c(t) > 0$ for large t, $\tau(t) \le t$ for all t, and $\lim_{t\to\infty}\tau(t) = \infty$. By q we denote the conjugate number to the number p, that is, $q = p/(p-1)$.

Under the solution of (1) we understand any differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi(x'(t))$ is differentiable and (1) holds for large t.

The solution of (1) is said to be oscillatory if it has infinitely many zeros tending to infinity. Equation (1) is said to be oscillatory if all its solutions are oscillatory. In the opposite case, that is, if there exists an eventually positive solution of (1), (1) is said to be nonoscillatory.

It is well known that the behavior of delay differential equations is very different from the behavior of ordinary differential equations. Among others, the Sturm theory fails and oscillatory solutions may coexist with nonoscillatory solutions.

In certain special cases, it is possible to compare asymptotics of (1) with some other simpler equation. One of the typical objects for this comparison is the first order delay differential equation; see, for example, [1–3] for results on comparing (1) or its extension in the form of neutral differential equation with the first order delay differential inequality. Another simpler object than (1) suitable for comparison with (1) is the half-linear second-order ordinary differential equation

$$\left(r(t)\Phi\left(x'(t)\right)\right)' + c(t)\Phi(x(t)) = 0; \tag{2}$$

see, for example, [4–7]. Note that some of these papers deal with a slightly more general equation

$$\left(r(t)\Phi\left(x'(t)\right)\right)' + f(t, x(t), x(\tau(t))) = 0. \tag{3}$$

However, if this more general equation is considered, conditions imposed on the nonlinearity f usually state that (3) is a kind of majorant of (1) (in the sense used in the Sturmian theory of ordinary differential equations) and allow to extend the results readily from (1) to (3). An example of such conditions is

$$\frac{f(t,u,v)}{\Phi(v)} \ge c(t) \quad \text{or} \quad \frac{f(t,u,v)}{\Phi(u)} \ge c(t) \tag{4}$$

for some (positive) function $c(t)$ and all positive numbers u, v. Note also that some of the above cited papers deal more generally with neutral differential equations and (or) dynamic equations on time scales.

In this paper we compare (1) with the ordinary half-linear equation of the form (2). To make our paper more readable we restrict our attention to differential equations rather than equations on time scales. An extension of our results to neutral differential equations is provided at the end of this paper.

Let us recall the Riccati technique, which is one of the methods frequently used in oscillation theory of both (1) and (2) (it is easy to see that if $\tau(t) = t$, then (1) reduces to (2)). Suppose that (1) is nonoscillatory and let x be its eventually positive solution. Then the function $w(t) = r(t)\Phi(x'(t))/\Phi(x(t))$ satisfies the Riccati type equation

$$w'(t) + c(t)\Phi\left(\frac{x(\tau(t))}{x(t)}\right) + (p-1)r^{1-q}(t)|w(t)|^q = 0. \tag{5}$$

The following lemma plays a crucial role in the qualitative theory of half-linear second order ordinary differential equations.

Lemma 1 (see [8, Theorem 2.2.1]). *Denote* $\mathscr{L}[x] = (r(t)\Phi(x'(t)))' + c(t)\Phi(x(t))$ *and* $\mathscr{R}[w] = w' + c(t) + (p-1)r^{1-q}(t)|w|^q$. *The following statements are equivalent:*

(i) (2) *is nonoscillatory,*

(ii) *there is* $a \in \mathbb{R}$ *and a continuously differentiable function* $w : [a,\infty) \to \mathbb{R}$ *such that*

$$R[w](t) = 0 \quad \text{for } t \in [a,\infty), \tag{6}$$

(iii) *there is* $a \in \mathbb{R}$ *and a continuously differentiable function* $w : [a,\infty) \to \mathbb{R}$ *such that*

$$R[w](t) \le 0 \quad \text{for } t \in [a,\infty), \tag{7}$$

(iv) *there is* $a \in \mathbb{R}$ *and a positive function* $x : [a,\infty) \to \mathbb{R}$ *with* $r\Phi(x')$ *continuously differentiable such that*

$$\mathscr{L}[x](t) \le 0 \quad \text{for } t \in [a,\infty). \tag{8}$$

As we show below, the assumptions used in the paper ensure that the positive solutions are eventually increasing and concave down. The main step when we compare the ordinary half-linear differential equation and its delay counterpart (1) is to reduce (5) to the Riccati inequality of the form (7). The usual approach on how to remove the term $\Phi(x(\tau(t))/x(t))$ from (5) is the following lemma, originally proved in [9] and then used in many subsequent papers.

Lemma 2. *Suppose that x is a function defined for some $T > 0$ such that $x(t) \in C^2[T,\infty)$, $x(t) > 0$, $x'(t) > 0$, and $x''(t) \le 0$ for $t \ge T$. Then, for each $k \in (0,1)$ there exists $T_k \ge T$ such that*

$$\frac{x(\tau(t))}{x(t)} \ge k\frac{\tau(t)}{t} \quad \text{for } t \ge T_k. \tag{9}$$

Note that the proof of Lemma 2 does not exploit the fact that x is a solution of (1) and the lemma holds for any positive increasing concave down function. The proof of (9) can be based on the fact that if $x''(t) \le 0$ on $[T,\infty)$ and $x(T) \ge 0$, then $x(t)/(t-T)$ is decreasing with respect to t on $[T,\infty)$ (see [10, Theorem 128]). Thus

$$\frac{x(\tau(t))}{x(t)} \ge \frac{\tau(t)-T}{t-T} = \frac{\tau(t)}{t}\frac{1-(T/\tau(t))}{1-(T/t)}, \tag{10}$$

where $T \le \tau(t) \le t$. Removing the dependence on T may be implemented by using of a constant $k \in (0,1)$. The presence of one of the constants T or k in the estimates (9) and (10) is an important attribute of these estimates. As a consequence, the resulting integral oscillation citeria have to be formulated either with the constant $k \in (0,1)$, or as interval-type or Kamenev-type criteria, where the dependence on T is usually not disturbing. A typical result looks like the following Theorem A.

Theorem A (see [11, Theorem 2.6]). *Equation* (1) *with* $r \equiv 1$ *is oscillatory if the differential equation*

$$\left(\Phi\left(x'(t)\right)\right)' + \lambda c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi(x(t)) = 0 \tag{11}$$

is oscillatory for some $\lambda \in (0,1)$.

As another particular example of a criterion which suffers from the presence of the constants $m_i \in (0,1)$ see [12, Theorem 2.1].

The above mentioned disadvantage has been removed for the linear delay equation

$$x''(t) + c(t)x(\tau(t)) = 0 \tag{12}$$

under the condition

$$\int_0^\infty sc(s)\,ds = \infty. \tag{13}$$

Opluštil and Šremr utilized in recent papers [13, 14] (12) to derive a sharper estimate than the estimate from Lemma 2. Note that imposing (13) on c does not yield any restriction in oscillation criteria for (12) since (12) is already known to be nonoscillatory if (13) fails. The same approach has been used for linear dynamic equations on time scales by Erbe, Peterson and Saker in [15].

The aim of this paper is to derive a result analogical to the estimate from [13, 14] and make it available also for delay half-linear differential equation. The nonlinearity of the equation causes, that the method from [13, 14] does not extend to (1) directly and we have to use an indirect approach which originates in the fact that the half-linear extension does not yield (13) as its special case, but includes the term $\tau(s)$ instead of s. This estimate suggests a new tool which can be used to improve some oscillation criteria for (1).

2. Preliminaries

The proof of the following statement can be found in [16].

Lemma 3. *Let x be an eventually positive solution of* (1). *If* $\int^{\infty} r^{1-q}(t)\,dt = \infty$, *then* $x'(t) > 0$ *for large* t. *Moreover, if* $r'(t) \geq 0$, *then* $x''(t) \leq 0$ *for large* t.

The following lemma shows that under certain additional conditions we can utilize (1) to derive a sharper version of the estimate from Lemma 2.

Lemma 4. *Suppose that* (1) *is nonoscillatory, and let* $x(t) > 0$ *be a solution of* (1). *If the conditions*

$$\int^{\infty} r^{1-q}(t)\,dt = \infty, \quad r'(t) \geq 0 \text{ for large } t, \tag{14}$$

$$\int^{\infty} c(t)\,\tau^{p-1}(t)\,dt = \infty \tag{15}$$

hold, then there exists $T \in \mathbb{R}$ *such that*

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t)}{t}, \quad t \geq T. \tag{16}$$

Proof. Conditions (14) and Lemma 3 imply that there exists T_0 such that $x(t) > 0$, $x'(t) > 0$, $x''(t) \leq 0$ for $t \geq T_0$.

We show that

$$tx'(t) - x(t) \leq 0 \tag{17}$$

for large t. Since $(tx'(t) - x(t))' = tx''(t) \leq 0$, it is sufficient to show that (17) holds for some $T_1 \geq T_0$. Suppose, by contradiction, that $tx'(t) - x(t) > 0$ for all $t \geq T_0$. Solving this inequality we get $x(t) > Kt$ for $t \geq T_0$, where $K = x(T_0)/T_0 > 0$. Hence, there exists $T_2 \geq T_0$ such that

$$\Phi(x(\tau(t))) \geq K^{p-1}(\tau(t))^{p-1}, \quad t \geq T_2. \tag{18}$$

Since x is a solution of (1), we have

$$\begin{aligned}\left(r(t)\,\Phi\left(x'(t)\right)\right)' &= -c(t)\,\Phi(x(\tau(t))) \\ &\leq -K^{p-1}c(t)\,(\tau(t))^{p-1}, \\ &\qquad t \geq T_2.\end{aligned} \tag{19}$$

Integrating the last inequality from T_2 to t we obtain

$$\begin{aligned}&r(t)\,\Phi\left(x'(t)\right) - r(T_2)\,\Phi\left(x'(T_2)\right) \\ &\quad \leq -K^{p-1}\int_{T_2}^{t} c(s)\,(\tau(s))^{p-1}ds,\end{aligned} \tag{20}$$

and from the fact that $r(t)\Phi(x'(t))$ is positive we get the following finite upper bound for the integral of $c(s)(\tau(s))^{p-1}$:

$$\begin{aligned}&K^{p-1}\int_{T_2}^{t} c(s)\,(\tau(s))^{p-1}ds \\ &\quad \leq r(T_2)\,\Phi\left(x'(T_2)\right) - r(t)\,\Phi\left(x'(t)\right) < r(T_2)\,\Phi\left(x'(T_2)\right)\end{aligned} \tag{21}$$

for $t \geq T_2$. However the condition (15) ensures that the left hand side of this inequality is unbounded. This contradiction proves (17) for large t.

Hence there exists $T_1 \geq T_0$ such that (17) holds for $t \geq T_1$. This inequality together with the computation

$$\left(\frac{x(t)}{t}\right)' = \frac{tx'(t) - x(t)}{t^2} \leq 0, \quad t \geq T_1, \tag{22}$$

shows that the function $x(t)/t$ is decreasing on (T_1, ∞). This fact and the fact that $\tau(t) \leq t$ reveal that there exists $T \geq T_1$ such that

$$\frac{x(t)}{t} \leq \frac{x(\tau(t))}{\tau(t)}, \quad t \geq T, \tag{23}$$

which is equivalent to (16). □

3. Oscillation of Delay Differential Equation

Theorem 5. *Suppose that conditions* (14) *and* (15) *hold. If the ordinary differential equation*

$$\left(r(t)\,\Phi\left(x'(t)\right)\right)' + c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi(x(t)) = 0 \tag{24}$$

is oscillatory, then (1) *is also oscillatory.*

Proof. Suppose, by contradiction, that (1) is nonoscillatory and (24) is oscillatory. Let x be an eventually positive solution of (1). Using Lemma 4 we see that x satisfies the inequality

$$\left(r(t)\,\Phi\left(x'(t)\right)\right)' + c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi(x(t)) \leq 0 \tag{25}$$

and hence, using equivalence between parts (i) and (iv) of Lemma 1, we see that (24) is nonoscillatory which contradicts our assumptions. □

Remark 6. The oscillation criterion from Theorem 5 is general in the sense that the oscillation is given in terms of oscillation of a certain half-linear differential equation rather than in terms of explicit conditions on the coefficients of the equation. Most of the related papers continue the proofs by utilizing techniques used in the theory of half-linear ordinary differential equations (often simply copy of the proofs of known oscillation citeria) to reach effective conditions for oscillation. However, we feel our approach as an advantage, since it allows to utilize arbitrary from large family of oscillation criteria for half-linear oscillation equations to detect oscillation of delay equation. See also [8] for a comprehensive survey on oscillation criteria known up to 2005.

Remark 7. Note that a similar result like Theorem 5 can be proved also without Lemma 4 and using Lemma 2 instead. This results in a comparison of (1) with the equation

$$\left(r(t)\,\Phi\left(x'(t)\right)\right)' + c(t)\,\lambda\left(\frac{\tau(t)}{t}\right)^{p-1}\Phi(x(t)) = 0, \tag{26}$$

where λ is a real parameter which satisfies $\lambda \in (0, 1)$. (Note that for $r \equiv 1$ we get Theorem A.) Equation (24) can be viewed

in a certain sense as a continuation of (26) with respect to λ to the border value $\lambda = 1$. Note that the problems related to oscillation of equation of the type (26) and dependence of oscillatory properties on the parameter λ are referred to as conditional oscillation. In general, oscillation of (26) implies oscillation of (24), but the opposite implication need not be true in general, see the paper [17] which (based on the results from [18]) suggests a method on how to construct a pair of equations of the type (24) and (26) with (24) oscillatory and (26) nonoscillatory.

Remark 8. Theorem 5 extends Theorem A, where oscillation of (1) is deduced from oscillation of (26). The following example shows that this extension is nonempty.

Example 9. Consider the perturbed Euler type half-linear delay differential equation

$$\left(\Phi\left(x'\right)\right)' + \left(\frac{p-1}{p}\right)^p \left(\frac{1}{t^p} + \frac{\mu}{t^p \ln t}\right) \times \left(\frac{t}{\tau(t)}\right)^{p-1} \Phi\left(x\left(\tau(t)\right)\right) = 0, \tag{27}$$

where $\mu > 0$ is real constant. According to Theorem 5, (27) is oscillatory if

$$\left(\Phi\left(x'\right)\right)' + \left(\frac{p-1}{p}\right)^p \left(\frac{1}{t^p} + \frac{\mu}{t^p \ln t}\right) \Phi\left(x(t)\right) = 0 \tag{28}$$

is oscillatory. Following [8, Theorem 5.2.2] (see also [19]) we treat (28) as a perturbation of the nonoscillatory equation

$$\left(\Phi\left(x'\right)\right)' + \left(\frac{p-1}{p}\right)^p \frac{1}{t^p} \Phi(x) = 0 \tag{29}$$

with principal solution $h(t) = t^{(p-1)/p}$. A simple computation shows

$$\int^{\infty} \left(\frac{p-1}{p}\right)^p \frac{\mu}{t^p \ln t} t^{p-1}\, dt = \infty \tag{30}$$

hence (28) is oscillatory by [8, Theorem 5.2.2]. Consequently, (27) is oscillatory for every μ.

We claim that the oscillation of (27) cannot be proved with Theorem A. Really, in our example (11) becomes

$$\left(\Phi\left(x'\right)\right)' + \lambda\left(\frac{p-1}{p}\right)^p \left(\frac{1}{t^p} + \frac{\mu}{t^p \ln t}\right) \Phi\left(x(t)\right) = 0, \tag{31}$$

where $\lambda \in (0,1)$. This equation is nonoscillatory for every $\mu > 0$ by Kneser type nonoscillation criterion [8, Theorem 1.4.5], and thus Theorem A fails to apply.

4. Oscillation of Neutral Differential Equation

In this section we use a slight modification of the estimates from the first part of the paper to derive similar results for the second order neutral differential equation

$$\left(r(t)\,\Phi\left(z'(t)\right)\right)' + c(t)\,\Phi\left(x\left(\tau(t)\right)\right) = 0, \tag{32}$$

where

$$z(t) = x(t) + a(t)\,x(\theta(t)), \tag{33}$$

$0 \le a(t) < 1$, $r(t) > 0$, $c(t) \ge 0$, $\tau(t) \le t$, $\theta(t) \le t$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\theta(t) = \infty$.

Similarly as for (1), if x is a solution of (32) on $[t_0,\infty)$ such that $z(t)$ is positive on $[t_0,\infty)$, then the function $w(t) = r(t)(\Phi(z'(t))/\Phi(z(t)))$ satisfies the Riccati type equation

$$w' + c(t)\,\Phi\left(\frac{x\left(\tau(t)\right)}{z(t)}\right) + (p-1)\,r^{1-q}(t)\,|w|^q = 0 \tag{34}$$

on $[t_0,\infty)$.

Similarly like for the delay equation, the positive solution is increasing and concave down. More precisely, the following lemma holds. For linear version of this lemma see [2, Lemma 1] and for $p \ge 2$ see [1, Lemma 2.1].

Lemma 10. *Let $x(t)$ be an eventually nonoscillatory solution of (32). If $\int^{\infty} r^{1-q}(t)dt = \infty$, then the corresponding function $z(t) = x(t) + a(t)x(\theta(t))$ satisfies*

$$z(t) > 0, \qquad z'(t) > 0 \tag{35}$$

eventually. Moreover, if $r'(t) \ge 0$, then $z''(t) < 0$ for large t.

Proof. The proof is essentially the same as the proof of [1, Lemma 2.1]. We just relax the restriction on p.

Without loss of generality we can suppose that x is eventually positive solution of (32). There exists $T \in \mathbb{R}$ such that $x(t)$, $x(\tau(t))$ and $x(\theta(t))$ are positive on (T,∞) and

$$\left(r(t)\,\Phi\left(z'(t)\right)\right)' = -c(t)\,\Phi\left(x\left(\tau(t)\right)\right) < 0 \tag{36}$$

for $t \in (T,\infty)$. Hence, $r(t)\Phi(z'(t))$ is decreasing and either

$$\Phi\left(z'(t)\right) > 0 \quad \text{or} \quad \Phi\left(z'(t)\right) < 0 \tag{37}$$

for large t.

Suppose that there exists $T_1 > T$ such that $\Phi(z'(t)) < 0$ for $t \ge T_1$. There exists a positive constant M such that

$$r(t)\,\Phi\left(z'(t)\right) < -M < 0, \qquad z'(t) < -\Phi^{-1}(M)\,r^{1-q}(t) \tag{38}$$

for $t \ge T_1$. Integrating this inequality over the interval (T_1,t) we get

$$z(t) \le z(T_1) - \Phi^{-1}(M)\int_{T_1}^{t} r^{1-q}(s)\,ds. \tag{39}$$

Letting $t \to \infty$ we have a negative upper bound for the function z and large t. However, the positivity of both $x(t)$ and $x(\theta(t))$ implies positivity of z. This contradiction proves that $\Phi(z'(t)) > 0$ and $z'(t) > 0$ eventually.

If $r'(t) \ge 0$, then

$$\begin{aligned} 0 &> \left(r(t)\,\Phi\left(z'(t)\right)\right)' \\ &= r'(t)\,\Phi\left(z'(t)\right) + r(t)(p-1)\left|z'(t)\right|^{p-2} z''(t) \\ &\ge r(t)(p-1)\left|z'(t)\right|^{p-2} z''(t), \end{aligned} \tag{40}$$

and hence $z''(t) < 0$. □

Lemma 11. *Suppose that x is an eventually positive nonoscillatory solution of* (32) *and z is the corresponding function defined by* (33). *If $\int^{\infty} r^{1-q}(t)dt = \infty$, then*

$$x(\tau(t)) \geq [1 - a(\tau(t))]\, z(\tau(t)) \tag{41}$$

eventually.

Proof. According to Lemma 10 there exists T such that

$$x(\theta(\theta(t))) > 0, \qquad z(t) > 0, \qquad z'(t) > 0 \tag{42}$$

holds for $t \geq \tau(T)$. From here and from the fact that z is increasing and θ is delay we have

$$\begin{aligned} z(t) &= x(t) + a(t)\, x(\theta(t)) \leq x(t) + a(t)\, z(\theta(t)) \\ &\leq x(t) + a(t)\, z(t). \end{aligned} \tag{43}$$

From here we conclude

$$z(t)(1 - a(t)) \leq x(t) \tag{44}$$

and hence (41) holds for $t > T$. □

The following lemma is an alternative to Lemma 4 for neutral differential equations.

Lemma 12. *Suppose that* (32) *is nonoscillatory and $x(t)$ is an eventually positive solution of* (32). *If*

$$\int^{\infty} c(s)(1 - a(\tau(s)))^{p-1}(\tau(s))^{p-1} ds = \infty \tag{45}$$

and (14) *holds, then the function $z(t)/t$ is decreasing eventually.*

Proof. Similarly like in Lemma 4 we find the derivative

$$\left(\frac{z(t)}{t}\right)' = \frac{z'(t)\, t - z(t)}{t^2}. \tag{46}$$

It is sufficient to show that $z'(t)t - z(t) < 0$ eventually. Lemma 10 implies that there exists t_0 such that $z''(t) < 0$ on (t_0, ∞). This shows that $z'(t)t - z(t)$ is decreasing on (t_0, ∞). As a consequence, if $z'(t_1)t_1 - z(t_1) < 0$ for some $t_1 > t_0$, then $z'(t)t - z(t) < 0$ on (t_1, ∞).

Suppose by contradiction that there exists t_2 such that $z'(t)t - z(t) > 0$ on (t_2, ∞). Solving this inequality we get

$$z(t) \geq \frac{z(t_2)}{t_2} t. \tag{47}$$

Now integrating (32) from t_2 to t and using (41) we get

$$\begin{aligned} r(t)\,\Phi\left(z'(t)\right) &= r(t_2)\,\Phi\left(z'(t_2)\right) - \int_{t_2}^{t} c(s)\,(x(\tau(s)))^{p-1} ds \\ &\leq r(t_2)\,\Phi\left(z'(t_2)\right) \\ &\quad - \int_{t_2}^{t} c(s)\,[1 - a(\tau(s))]^{p-1}(z(\tau(s)))^{p-1} ds \\ &\leq r(t_2)\,\Phi\left(z'(t_2)\right) - \frac{z(\tau(t_2))}{\tau(t_2)} \\ &\quad \times \int_{t_2}^{t} c(s)\,[1 - a(\tau(s))]^{p-1}(\tau(s))^{p-1} ds. \end{aligned} \tag{48}$$

Taking t sufficiently large and using (45) we obtain a negative upper bound for a positive function $r(t)\Phi(z'(t))$. This contradiction proves the lemma. □

Now we can formulate the comparison theorem which relates neutral differential equations to ordinary second-order half-linear differential equations.

Theorem 13. *Suppose that* (45) *and* (14) *hold. If the ordinary half-linear differential equation*

$$\begin{aligned} &\left(r(t)\,\Phi\left(x'(t)\right)\right)' + c(t)\,[1 - a(\tau(t))]^{p-1} \\ &\quad \times \left(\frac{\tau(t)}{t}\right)^{p-1} \Phi(x(t)) = 0 \end{aligned} \tag{49}$$

is oscillatory, then (32) *is also oscillatory.*

Proof. Having proved important estimates in the preceding two lemmas, the proof of the theorem is a modification of the proof of Theorem 5. If $x(t)$ is an eventually positive solution of (32), then the function w defined by $w(t) = r(t)\Phi(z'(t))/\Phi(z(t))$ satisfies (34). Using Lemmas 11 and 12 we see that

$$\begin{aligned} 0 &= w' + c(t)\,\Phi\left(\frac{x(\tau(t))}{z(t)}\right) + (p-1)\, r^{1-q}(t)\,|w|^q \\ &\geq w' + c(t)\,[1 - a(\tau(t))]^{p-1}\Phi\left(\frac{z(\tau(t))}{z(t)}\right) \\ &\quad + (p-1)\, r^{1-q}(t)\,|w|^q \\ &\geq w' + c(t)\,[1 - a(\tau(t))]^{p-1}\left(\frac{\tau(t)}{t}\right)^{p-1} \\ &\quad + (p-1)\, r^{1-q}(t)\,|w|^q. \end{aligned} \tag{50}$$

Hence (49) is nonoscillatory by Lemma 1. □

Remark 14. A version of Theorem 13 has been used implicitly in the proof of [20, Theorem 2.2] for dynamic equations. A closer estimation of the proof shows that one of the important steps is an application of inequality which in the continuous case reads as (10). However, Lemma 12 allows the estimate

$$\frac{z(\tau(t))}{z(t)} \geq \frac{\tau(t)}{t}, \tag{51}$$

which appears to be sharper, since

$$\frac{1 - (T/\tau(t))}{1 - (T/t)} \leq 1 \tag{52}$$

and the annoying dependence of the left-hand side on T usually necessitates to replace it by a constant $k < 1$ which may appear in the resulting oscillation criterion.

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Research Article

Some New Nonlinear Weakly Singular Inequalities and Applications to Volterra-Type Difference Equation

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Some new nonlinear weakly singular difference inequalities are discussed, which generalize some known weakly singular inequalities and can be used in the analysis of nonlinear Volterra-type difference equations with weakly singular kernel. An application to the upper bound of solutions of a nonlinear difference equation is also presented.

1. Introduction

The discrete version of the well-known Gronwall-Bellman inequality is an important tool in the development of the theory of difference equations as well as the analysis of the numerical schemes of differential equations. A great deal of interest has been given to these inequalities, and many results on their generalizations have been found; for example, see [1–4]. Among them, one of the fundamental cases is Pachpatte's result [3] for the difference inequality:

$$u(n) \le a(n) + \sum_{s=0}^{n-1} f(s)\, u(s). \tag{1}$$

In particular, due to the study of the behavior and numerical solutions for the singular integral equations, some discrete weakly singular integral inequalities also have drawn more and more attention [5–7]. Dixon and McKee [8] investigated the convergence of discretization methods for the Volterra integral and integrodifferential equations, by using the following inequality:

$$x_i \le \psi_i + Mh^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^{\alpha}}, \quad i = 1, 2, \ldots, N, \qquad n > 0,\ Nh = T. \tag{2}$$

Henry [9] presented a linear integral inequality with weakly kernel:

$$x(t) \le a(t) + \int_0^t (t-s)^{\beta-1} b(s)\, x(s)\, ds \tag{3}$$

to investigate some qualitative properties for a parabolic equation. The corresponding discrete version was discussed by Slodička [10]. But he studied the case $\tau_k = \tau$, that is, the case of constant differences. Furthermore, the first formulation of the inequality with a nonlinearity and τ_k nonconstant was studied in [6], in which the general nonlinear discrete case as follows:

$$x_n \le a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k) \tag{4}$$

was considered. However, his results are based on the so-called "(q) condition": (1) ω satisfies $e^{-qt}[\omega(u)]^q \le R(t)\omega(e^{-qt})u^q$; (2) there exists $c > 0$ such that $a_n e^{-\tau t_n} \le c$. Recently, a new nonlinear difference inequality:

$$x_n^{\alpha} \le a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k^{\lambda} \tag{5}$$

was discussed by Yang et al. [11]. For other new weakly singular inequalities, lots of work can be found, for example, in [12–22] and references therein.

In this paper, we investigate the new nonlinear weakly singular inequality:

$$x_n \le a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k), \tag{6}$$

where $0 < \beta \le 1$, $t_0 = 0$, $\tau_k = t_{k+1} - t_k$, $\sup_{k\in\mathbb{N}} \tau_k = \tau$, and $\lim_{t\to\infty} t_k = \infty$. Compared to the existing result, our result does not need the so-called "(q) condition" proposed in [6] and can be used to obtain pointwise explicit bounds on solutions for a class of more general weakly singular inequalities of Volterra type. Finally, we also present an application to Volterra-type difference equation with weakly singular kernel.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ = (0,\infty)$, and $\mathbb{N} = \{0,1,2,\ldots\}$. $C(X,Y)$ denotes the collection of continuous functions from the set X to the set Y. As usual, the empty sum is taken to be 0.

Lemma 1 (Discrete Jensen inequality, [11]). *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers, and let $r > 1$ be a real number. Then,*

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1} (A_1^r + A_2^r + \cdots + A_n^r). \tag{7}$$

Lemma 2 (Discrete Hölder inequality, [11]). *Let a_i, b_i $(i = 1,2,\ldots,n)$ be nonnegative real numbers, and let p, q be positive numbers such that $(1/p)+(1/q) = 1$ (or $p = 1, q = \infty$). Then,*

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}. \tag{8}$$

Furthermore, take $p = q = 2$; then, one gets the discrete Cauchy-Schwarz inequality.

Lemma 3. *Suppose that $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing. Let a_n, c_n be nonnegative and nondecreasing in n. If y_n is nonnegative such that*

$$y_n \le a_n + c_n \sum_{k=0}^{n-1} b_k \omega(y_k), \quad n \in \mathbb{N}. \tag{9}$$

Then,

$$y_n \le \Omega^{-1}\left[\Omega(a_n) + c_n \sum_{k=0}^{n-1} b_k\right], \quad 0 \le n \le M, \tag{10}$$

where $\Omega(v) = \int_{v_0}^{v} (1/\omega(s))ds$, $v \ge v_0$, Ω^{-1} is the inverse function of Ω, and M is defined by

$$M = \sup\left\{i : \Omega(a_i) + c_i \sum_{k=0}^{i-1} b_k \in \operatorname{Dom}\left(\Omega^{-1}\right)\right\}. \tag{11}$$

3. Main Results

Assume that

(A_1) a_n, b_n are nonnegative functions for $n \in \mathbb{N}$, respectively;

(A_2) $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing and $\omega(0) = 0$.

Define $\tilde{a}_n = \max_{0\le k\le n, k\in\mathbb{N}} a_k$ and $\tau = \max_{0\le k\le n-1, k\in\mathbb{N}} \tau_k$, where τ_k is the variable time step.

Theorem 4. *Under assumptions (A_1) and (A_2), if x_n is nonnegative such that (6), then*

(1) *for $0 < \beta \le 1/2$, letting $p = 1 + \beta$ and $q = (1+\beta)/\beta$, one has*

$$x_n \le \left[\Omega^{-1}\left(\Omega\left(2^{q-1}\tilde{a}_n^q\right) + 2^{q-1}\tau^{1-(q/p)\beta^2} \times K^{q/p}(\beta) e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q\right)\right]^{1/q}, \tag{12}$$

for $0 \le n \le N_1$, where $\Omega(u) = \int_{u_0}^{u} (1/\omega^q(s^{1/q}))\, ds$, $u \ge u_0 \ge 0$, Ω^{-1} is the inverse function of Ω,

$$K(\beta) = (1+\beta)^{-\beta^2} \Gamma\left(\beta^2\right), \tag{13}$$

and N_1 is the largest integer number such that

$$\Omega\left(2^{q-1}\tilde{a}_n^q\right) + 2^{q-1}\tau^{1-(q/p)\beta^2} K^{q/p}(\beta) e^{q\tau t_n} \times \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \in \operatorname{Dom}\left(\Omega^{-1}\right); \tag{14}$$

(2) *for $1/2 < \beta \le 1$, letting $p = 2$ and $q = 2$, one has*

$$x_n \le \left[\Omega^{-1}\left(\Omega\left(2\tilde{a}_n^2\right) + B(\beta)\tau^{2-2\beta} e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2\right)\right]^{1/2}, \tag{15}$$

for $0 \le n \le N_2$, where $\Omega(u) = \int_{u_0}^{u} (1/\omega^2(s^{1/2}))ds$, $u \ge u_0 \ge 0$,

$$B(\beta) = 4^{1-\beta}\Gamma(2\beta - 1), \quad \beta > \frac{1}{2}, \tag{16}$$

and N_2 is the largest integer number such that

$$\Omega\left(2\tilde{a}_n^2\right) + B(\beta)\tau^{2-2\beta} e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \in \operatorname{Dom}\left(\Omega^{-1}\right). \tag{17}$$

Proof. By definition of $\tilde{a}_n$ and assumption (A_1), $\tilde{a}_n$ is nonnegative and nondecreasing and $\tilde{a}_n \ge a_n$. It follows from (6) that

$$x_n \le \tilde{a}_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k). \tag{18}$$

(1) If $0 < \beta \le 1/2$, using Lemma 2 with the indices $p = 1+\beta$, $q = (1+\beta)/\beta$ for (18), we get

$$\begin{aligned} x_n &\le \tilde{a}_n + \sum_{k=0}^{n-1}(t_n - t_k)^{\beta-1}\tau_k^{1/p}\tau_k^{1/q}e^{\tau t_k}e^{-\tau t_k}b_k\omega(x_k) \\ &\le \tilde{a}_n + \tau^{1/q}\sum_{k=0}^{n-1}(t_n - t_k)^{\beta-1}\tau_k^{1/p}e^{\tau t_k}e^{-\tau t_k}b_k\omega(x_k) \\ &\le \tilde{a}_n + \tau^{1/q}\left[\sum_{k=0}^{n-1}(t_n - t_k)^{p(\beta-1)}\tau_k e^{p\tau t_k}\right]^{1/p} \\ &\quad\times\left[\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\omega^q(x_k)\right]^{1/q}. \end{aligned} \tag{19}$$

By Lemma 1, the inequality above yields

$$\begin{aligned} x_n^q &\le 2^{q-1}\tilde{a}_n^q + 2^{q-1}\tau\left[\sum_{k=0}^{n-1}(t_n - t_k)^{p(\beta-1)}\tau_k e^{p\tau t_k}\right]^{q/p} \\ &\quad\times\left[\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\omega^q(x_k)\right]. \end{aligned} \tag{20}$$

Consider that

$$\begin{aligned} &\sum_{k=0}^{n-1}(t_n - t_k)^{p(\beta-1)}\tau_k e^{p\tau t_k} \\ &\le \int_0^{t_n}(t_n - s)^{p(\beta-1)}e^{p\tau s}ds \\ &= e^{p\tau t_n}\int_0^{t_n}\eta^{p(\beta-1)}e^{-p\tau\eta}d\eta, \\ &= \frac{e^{p\tau t_n}}{(p\tau)^{1+p(\beta-1)}}\int_0^{p\tau t_n}\sigma^{p(\beta-1)e^{-\sigma}}d\sigma \le K(\beta)\tau^{-\beta^2}e^{p\tau t_n}, \end{aligned} \tag{21}$$

where $K(\beta) = (1+\beta)^{-\beta^2}\Gamma(\beta^2)$ and $\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du$, $(Rez > 0)$ is the well-known G-function. Thus, we have

$$\begin{aligned} x_n^q &\le 2^{q-1}\tilde{a}_n^q + 2^{q-1}\tau^{1-(q/p)\beta^2} \\ &\quad\times K^{q/p}(\beta)e^{q\tau t_n}\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\omega^q(x_k). \end{aligned} \tag{22}$$

Let $v_n = x_n^q$, $A_n = 2^{q-1}\tilde{a}_n^q$, and $C_n = 2^{q-1}\tau^{1-(q/p)\beta^2}K^{q/p}(\beta)e^{q\tau t_n}$. Obviously, A_n, C_n are nondecreasing for $n \in \mathbb{N}$ and $\omega^q(v_k^{1/q})$ satisfies the assumption (A_2). Equation (22) can be rewritten as

$$v_n \le A_n + C_n\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\omega^q\left(v_k^{1/q}\right), \tag{23}$$

which is similar to inequality (9). Using Lemma 3, from (23), we have

$$v_n \le \Omega^{-1}\left[\left(\Omega(A_n) + C_n\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\right)\right], \tag{24}$$

for $0 \le n \le N_1$, where N_1 is the largest integer number such that

$$\Omega(A_n) + C_n\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q \in \mathrm{Dom}\left(\Omega^{-1}\right). \tag{25}$$

Therefore, by $x_n = v_n^{1/q}$, (12) holds for $0 \le n \le N_1$.

(2) If $1/2 < \beta \le 1$, applying Cauchy-Schwarz inequality for (18), that is, $p = q = 2$, we get

$$\begin{aligned} x_n &\le \tilde{a}_n + \sum_{k=0}^{n-1}(t_n - t_k)^{\beta-1}\tau_k^{1/2}\tau_k^{1/2}e^{\tau t_k}e^{-\tau t_k}b_k\omega(x_k) \\ &\le \tilde{a}_n + \tau^{1/2}\sum_{k=0}^{n-1}(t_n - t_k)^{\beta-1}\tau_k^{1/2}e^{\tau t_k}e^{-\tau t_k}b_k\omega(x_k) \\ &\le \tilde{a}_n + \tau^{1/2}\left[\sum_{k=0}^{n-1}(t_n - t_k)^{2(\beta-1)}\tau_k e^{2\tau t_k}\right]^{1/2} \\ &\quad\times\left[\sum_{k=0}^{n-1}e^{-2\tau t_k}b_k^2\omega^2(x_k)\right]^{1/2}. \end{aligned} \tag{26}$$

By Lemma 1, the inequality above yields

$$\begin{aligned} x_n^2 &\le 2\tilde{a}_n^2 + 2\tau\left[\sum_{k=0}^{n-1}(t_n - t_k)^{2(\beta-1)}\tau_k e^{2\tau t_k}\right] \\ &\quad\times\left[\sum_{k=0}^{n-1}e^{-2\tau t_k}b_k^2\omega^2(x_k)\right]. \end{aligned} \tag{27}$$

Because

$$\begin{aligned} &\sum_{k=0}^{n-1}(t_n - t_k)^{2(\beta-1)}\tau_k e^{2\tau t_k} \\ &\le \int_0^{t_n}(t_n - s)^{2(\beta-1)}e^{2\tau s}ds \\ &= \frac{e^{2\tau t_n}}{(2\tau)^{2\beta-1}}\int_0^{2\tau t_n}\sigma^{2(\beta-1)e^{-\sigma}}d\sigma \\ &\le \frac{1}{2}B(\beta)\tau^{1-2\beta}e^{2\tau t_n}, \end{aligned} \tag{28}$$

where $B(\beta) = 4^{1-\beta}\Gamma(2\beta-1)$, $\beta > 1/2$, it follows from (27) that

$$x_n^2 \le 2\tilde{a}_n^2 + B(\beta)\tau^{2-2\beta}e^{2\tau t_n}\left[\sum_{k=0}^{n-1}e^{-2\tau t_k}b_k^2\omega^2(x_k)\right]. \tag{29}$$

Let $v_n = x_n^2$, $A_n = 2\tilde{a}_n^2$, and $C_n = B(\beta)\tau^{2-2\beta}e^{2\tau t_n}$. Similarly, A_n, C_n also are nondecreasing for $n \in \mathbb{N}$ and $\omega^2(v_k^{1/2})$ also satisfies the assumption (A_2). Equation (29) can be rewritten as

$$v_n \le A_n + C_n\left(\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \omega^2\left(v_k^{1/2}\right)\right), \tag{30}$$

which also is similar to inequality (9). Using Lemma 3, from (30), we have

$$v_n \le \left[\Omega^{-1}\left(\Omega\left(A_n\right) + C_n\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2\right)\right], \tag{31}$$

for $0 \le n \le N_2$, and N_2 is the largest integer number such that

$$\Omega\left(A_n\right) + C_n\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \in \operatorname{Dom}\left(\Omega^{-1}\right). \tag{32}$$

Clearly, by $x_n = v_n^{1/2}$, (15) also holds for $0 \le n \le N_2$.

□

Remark 5. Here, we note that the most significant work in the study of weakly singular inequalities is Medveď's method, originally presented in the paper [6] and also applied in the paper [18]. But his result holds under the assumption "$\omega(u)$ satisfies the condition (q)," that is, "$e^{-qt}[\omega(u)]^q \le R(t)\omega(e^{-qt}u^q)$, where $R(t)$ is a continuous, nonnegative function." In our result, the condition (q) is eliminated.

Corollary 6. *Under assumptions (A_1) and (A_2), let $\nu > 0$, $\mu > 0 (\nu > \mu)$. If x_n is nonnegative such that*

$$x_n^\nu \le a_n + \sum_{k=0}^{n-1}\left(t_n - t_k\right)^{\beta-1}\tau_k b_k x_k^\mu, \tag{33}$$

then

(1) *if $0 < \beta \le 1/2$, let $p = 1+\beta$ and $q = (1+\beta)/\beta$, and one gets*

$$x_n \le \left[\left(2^{q-1}\tilde{a}_n^q\right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu}2^{q-1}\tau^{1-(q/p)\beta^2} \times K^{q/p}(\beta)\, e^{q\tau t_n}\sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q\right]^{1/(\nu-\mu)q} \tag{34}$$

for $n \ge 0$, where $K(\beta)$ is defined as in Theorem 4;

(2) *if $1/2 < \beta \le 1$, let $p = q = 2$, and one gets*

$$x_n \le \left[\left(2\tilde{a}_n^2\right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu}B(\beta)\,\tau^{2-2\beta} \times e^{2\tau t_n}\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2\right]^{1/2(\nu-\mu)}, \tag{35}$$

for $n \ge 0$, where $B(\beta)$ is defined as in Theorem 4

Proof. Let $z_n = x_n^\nu$, then $x_n = z_n^{1/\nu}$ and $x_n^\mu = z_n^{\mu/\nu}$. From (33), we have

$$z_n \le a_n + \sum_{k=0}^{n-1}\left(t_n - t_k\right)^{\beta-1}\tau_k b_k z_k^{\mu/\nu}. \tag{36}$$

Clearly, $\omega(z_k) = z_k^{\mu/\nu}$ satisfies the assumption (A_2). According to the definition of Ω in Theorem 4, for $0 < \beta \le 1/2$, letting $u_0 = 0$, we have

$$\Omega(u) = \int_{u_0}^{u}\frac{1}{\omega^q\left(s^{1/q}\right)}ds = \int_0^u \frac{ds}{s^{\mu/\nu}} = \frac{\nu}{\nu-\mu}u^{(\nu-\mu)/\nu}, \tag{37}$$

$$\Omega^{-1}(u) = \left(\frac{\nu-\mu}{\nu}u\right)^{\nu/(\nu-\mu)}, \qquad \operatorname{Dom}\left(\Omega^{-1}\right) = [0,\infty). \tag{38}$$

It can be seen easily from (38) that $N_1 = \infty$. Substituting (37) and (38) into (12), we get

$$z_n \le \left[\left(2^{q-1}\tilde{a}_n^q\right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu}2^{q-1}\tau^{1-(q/p)\beta^2} \times K^{q/p}(\beta)\, e^{q\tau t_n}\sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q\right]^{\nu/(\nu-\mu)q}. \tag{39}$$

In view of $x_n = z_n^{1/\nu}$, we can obtain (34). For the case that $1/2 < \beta \le 1$, in fact, Ω and Ω^{-1} are the same as (37) and (38), respectively. So, it follows from (37), (38), and (15) that

$$x_n \le \left[\left(2\tilde{a}_n^2\right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu}B(\beta)\,\tau^{2-2\beta} \times e^{2\tau t_n}\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2\right]^{1/2(\nu-\mu)}, \tag{40}$$

for $n > 0$. □

Remark 7. In [11], Yang et al. investigated inequality (33), under the assumption that a_n is nondecreasing. Clearly, our result does not need such condition, and we get a more concise formula.

Remark 8. Letting $\nu = 2$ and $\mu = 1$, we can get the interesting Henry version of the Ou-Iang-Pachpatte-type difference inequality [3]. Thus, our result is a more general discrete analogue for such inequality.

Corollary 9. *Under assumptions (A_1) and (A_2), if x_n is nonnegative such that*

$$x_n \le a_n + \sum_{k=0}^{n-1}\left(t_n - t_k\right)^{\beta-1}\tau_k b_k x_k, \tag{41}$$

then

(1) *if* $0 < \beta \le 1/2$, *let* $p = 1 + \beta$ *and* $q = (1+\beta)/\beta$, *and one gets*

$$x_n \le 2^{(q-1)/q}\widetilde{a}_n \exp\left(2^{(q-1)/q}\tau^{1-(q/p)\beta^2}K^{q/p}(\beta) \times e^{q\tau t_n}\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\right), \tag{42}$$

for $n \ge 0$, *where* $K(\beta)$ *is defined as in Theorem 4;*

(2) *if* $1/2 < \beta \le 1$, *let* $p = q = 2$, *and one gets*

$$x_n \le \sqrt{2}\widetilde{a}_n \exp\left(\frac{1}{2}B(\beta)\tau^{2-2\beta}e^{2\tau t_n}\sum_{k=0}^{n-1}e^{-2\tau t_k}b_k^2\right), \tag{43}$$

for $n \ge 0$, *where* $B(\beta)$ *is defined as in Theorem 4.*

Proof. In (41), $\omega(u) = u$ also satisfies the assumption (A_2). Thus, we have

$$\Omega(u) = \int_{u_0}^{u}\frac{ds}{s} = \ln\frac{u}{u_0}, \qquad \Omega^{-1}(u) = u_0\exp(u), \qquad \mathrm{Dom}\left(\Omega^{-1}\right) = [0,\infty). \tag{44}$$

Similarly to the computation in Corollary 6, the estimates (42) and (43) hold, respectively. □

4. Application

In this section, we apply our results to discuss the upper bound of solution of a Volterra type difference equation with weakly singular kernel.

Consider the following the inequality:

$$x_n \le 1 + \sum_{k=0}^{n-1}(t_n - t_k)^{-1/2}\tau_k\sqrt{x_k}. \tag{45}$$

Obviously, (45) is the special case of inequality (6), then we get

$$a_n = 1, \qquad \beta = \frac{1}{2}, \qquad \omega = \sqrt{u}. \tag{46}$$

Thus, we can take $p = 1 + \beta = 3/2$ and $q = (1+\beta)/\beta = 3$; then, $q/p = 2$. Moreover,

$$\widetilde{a}_n = 1,$$
$$K(\beta) = (1+\beta)^{-\beta^2}\Gamma\left(\beta^2\right) = \left(\frac{3}{2}\right)^{-1/4}\Gamma\left(\frac{1}{4}\right), \tag{47}$$
$$\Omega(u) = \int_0^u\frac{ds}{\sqrt{s}} = 2\sqrt{u}, \qquad \Omega^{-1}(u) = \frac{u^2}{4}.$$

According to Theorem 4, we obtain

$$\begin{aligned} x_n &\le \left[\Omega^{-1}\left(\Omega\left(2^{q-1}\widetilde{a}_n^q\right) + 2^{q-1}\tau^{1-(q/p)\beta^2}\right) \times K^{q/p}(\beta)e^{q\tau t_n}\sum_{k=0}^{n-1}e^{-q\tau t_k}b_k^q\right]^{1/q} \\ &= \left[\Omega^{-1}\left(\Omega(4) + 4\tau^{1/2}\left(\frac{3}{2}\right)^{-1/2}\right) \times\Gamma^2\left(\frac{1}{4}\right)e^{3\tau t_n}\sum_{k=0}^{n-1}e^{-3\tau t_k}b_k^3\right]^{1/3} \\ &= \left[\Omega^{-1}\left(4 + \frac{4}{3}\sqrt{6}\tau^{1/2}\Gamma^2\left(\frac{1}{4}\right)e^{3\tau t_n}\sum_{k=0}^{n-1}e^{-3\tau t_k}b_k^3\right)\right]^{1/3} \\ &= 4^{-1/3}\left(4 + \frac{4}{3}\sqrt{6}\tau^{1/2}\Gamma^2\left(\frac{1}{4}\right)e^{3\tau t_n}\sum_{k=0}^{n-1}e^{-3\tau t_k}b_k^3\right)^{2/3} \end{aligned} \tag{48}$$

for $n > 0$, which indicates that we get the upper bound of x_n.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Heat Transfer Analysis on the Hiemenz Flow of a Non-Newtonian Fluid: A Homotopy Method Solution

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The mathematical model for the incompressible two-dimensional/axisymmetric non-Newtonian fluid flows and heat transfer analysis in the region of stagnation point over a stretching/shrinking sheet and axisymmetric shrinking sheet is presented. The governing equations are transformed into dimensionless nonlinear ordinary differential equations by similarity transformation. Analytical technique, namely, the homotopy perturbation method (HPM) with general form of linear operator is used to solve dimensionless nonlinear ordinary differential equations. The series solution is obtained without using the diagonal Padé approximants to handle the boundary condition at infinity which can be considered as a clear advantage of homotopy perturbation technique over the decomposition method. The effects of the pertinent parameters on the velocity and temperature field are discussed through graphs. To the best of authors' knowledge, HPM solution with general form of linear operator for two-dimensional/axisymmetric non-Newtonian fluid flows and heat transfer analysis in the region of stagnation point is presented for the first time in the literature.

1. Introduction

Stagnation point flow is of great importance in the prediction of skin friction as well as heat/mass transfer near stagnation regions of bodies in high speed flows and also in the design of thrust bearings and radial diffusers, drag reduction, transpiration cooling, and thermal oil. In 1911, Hiemenz [1] revealed that stagnation point flow can be examined by the Navier-Stokes (NS) equations. He used the similarity of the solution to reduce number of variables by means of a coordinate transformation. Later Howann [2] discovered the stagnation point flow in case of axisymmetric situation. Recently, a number of researchers studied the stagnation point flow considering different fluids models, geometries, and assumptions that were proposed in the literature. The literature on the topic is quite extensive and hence cannot be described here in detail. However some most recent works of eminent researchers regarding the analytical/numerical solution of stagnation point for different geometries may be mentioned in [3–5]. Attia [6], Massoudi and Ramezan [7], and Garg [8] extended the stagnation point flow for heat transfer.

The main aim of this paper is to extend the HPM [9–17] for solving non-Newtonian fluid flow and heat transfer analysis in the region of stagnation point flow towards a stretching/shrinking and axisymmetric shrinking sheet. Also the main motivation is to perform such analysis [3] (shrinking/axisymmetric shrinking sheet) for a non-Newtonian fluid in the presence of heat transfer. Heat transfer plays very important role in nuclear energy because nuclear chain reaction creates heat, and it is used to boil water, produce steam, and drive a steam turbine. The steady Navier-Stokes equations are reduced to the nonlinear ordinary differential equations by using similarity solutions. Graphical results explicitly reveal the complete reliability and efficiency of the suggested algorithm.

2. Governing Equations

The flow and heat characteristics are governed by the following equations [3]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\begin{aligned} u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial z^2} \\ &\quad+\frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x\partial z^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z}+w\frac{\partial^3 u}{\partial z^3}\right), \end{aligned} \tag{2}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}\right)=k\nabla^2 T. \tag{3}$$

The similarity transformations for two-dimensional stagnation flow case are as follows [3]:

$$\begin{gathered} \eta=\sqrt{\frac{a}{\nu}}z,\quad u=axf'(\eta)+bch(\eta),\quad v=0, \\ w=-\sqrt{a\nu}f(\eta),\quad \theta=\frac{T-T_\infty}{T_0-T_\infty}. \end{gathered} \tag{4}$$

The steady Navier-Stokes equations yield a system of nonlinear ordinary differential equations in the form

$$\begin{gathered} f'''+ff''-f'^2+1+\beta\left(2f'f'''+f''^2-ff^{(IV)}\right)=0, \\ h''+fh'-f'h+\beta\left(hf'''+f'h''+f''h'-fh'''\right)=0, \\ \theta''+\Pr f\theta'=0, \end{gathered} \tag{5}$$

and corresponding boundary conditions take the form

$$\begin{gathered} f(0)=0,\qquad f'(0)=\frac{b}{a}=\alpha,\qquad \theta(0)=1, \\ f'(\infty)=1,\quad h(0)=1,\quad h(\infty)=0,\quad \theta(\infty)=0. \end{gathered} \tag{6}$$

The similarity transformations for axisymmetric stagnation flow towards an axisymmetric shrinking surface are as follows [3]:

$$\begin{gathered} \eta(x,y)=\sqrt{\frac{a}{\nu}}z,\qquad u=axg'(\eta)+bcl(\eta), \\ v=ayg'(\eta),\qquad w=-2\sqrt{a\nu}g(\eta),\qquad \theta=\frac{T-T_\infty}{T_0-T_\infty}. \end{gathered} \tag{7}$$

Upon making use of the above substitutions in (2) and (3), the resulting nonlinear system has the following form:

$$\begin{gathered} g'''+2gg''-g'^2+1+\beta\left(2g'g'''+g''^2-2gg^{(IV)}\right)=0, \\ l''+2gl'-g'l+\beta\left(\lg'''+g'l''+g''l'-2gl'''\right)=0, \\ \theta''+2\Pr g\theta'=0, \\ g(0)=0,\quad g'(0)=\frac{b}{a}=\alpha,\qquad \theta(0)=1, \\ g'(\infty)=1,\quad l(0)=1,\quad l(\infty)=0,\qquad \theta(\infty)=0. \end{gathered} \tag{8}$$

3. Analytical Solution

For the HPM [9] solution, we select

$$f_0(\eta)=(1-\alpha)\left(e^{-\eta}-1\right)+\eta,\qquad h_0(\eta)=e^{-\eta}, \tag{9}$$

$$\theta_0(\eta)=e^{-\eta}, \tag{10}$$

as initial approximations of f, h, and θ. We further choose the following auxiliary linear operators:

$$L_1=\frac{\partial^3}{\partial^3\eta}+\frac{\partial^2}{\partial^2\eta},\quad L_2=\frac{\partial^2}{\partial^2\eta}+\frac{\partial}{\partial\eta},\quad L_3=\frac{\partial^2}{\partial^2\eta}+\frac{\partial}{\partial\eta}. \tag{11}$$

In view of the basic idea of the HPM [9], (5) is expressed as

$$\begin{aligned} &(1-p)L_1(f-f_0) \\ &+p\left(f'''+ff''-f'^2+1+\beta\left(2f'f'''+f''^2-ff^{(IV)}\right)\right) \\ &=0, \\ &(1-p)L_2(h-h_0) \\ &+p\left(h''+fh'-f'h+\beta\left(hf'''+f'h''+f''h'-fh'''\right)\right) \\ &=0, \\ &(1-p)L_3(\theta-\theta_0)+p\left(\theta''+\Pr f\theta'\right)=0, \end{aligned} \tag{12}$$

$$\begin{aligned} f&=f_0+pf_1+p^2f_2+\cdots, \\ h&=h_0+ph_1+p^2h_2+\cdots, \\ \theta&=\theta_0+p\theta_1+p^2\theta_2+\cdots. \end{aligned} \tag{13}$$

Assuming $L_1f=0$, $L_2h=0$, and $L_3\theta=0$ and substituting f, h, and θ from (13) into (12) and some simplification and rearrangement based on powers of p-terms, we have

$$\begin{aligned} p^1:{}&L_1f_1+f_0'''+f_0f_0''-f_0'^2+1 \\ &+\beta\left(2f_0'f_0'''+f_0''^2-f_0f_0^{(IV)}\right)=0, \\ &f_1(0)=f_1'(0)=f_1'(\infty)=0, \\ &\vdots \\ p^j:{}&L_1f_j-L_1f_{j-1}+f_{j-1}'''+\sum_{k=0}^{j-1}f_kf_{j-1-k}'' \\ &-\sum_{k=0}^{j-1}f_k'f_{j-1-k}'+1 \\ &+\beta\left(2\sum_{k=0}^{j-1}f_k'f_{j-1-k}'''+\sum_{k=0}^{j-1}f_k''f_{j-1-k}''-\sum_{k=0}^{j-1}f_kf_{j-1-k}^{(IV)}\right) \\ &=0, \\ &f_j(0)=f_j'(0)=f_j'(\infty)=0,\quad j\geq 2; \end{aligned}$$

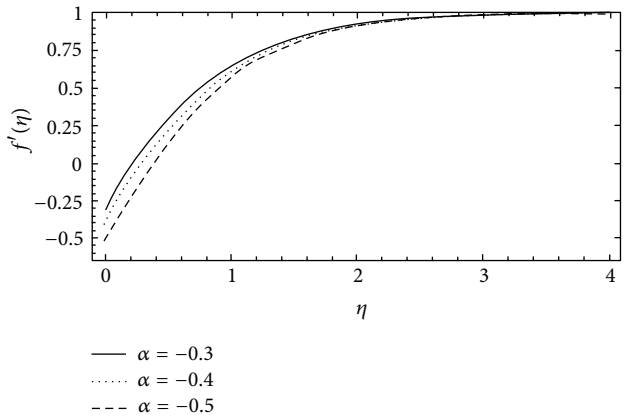

Figure 1: Effect of α on f' for two-dimensional case.

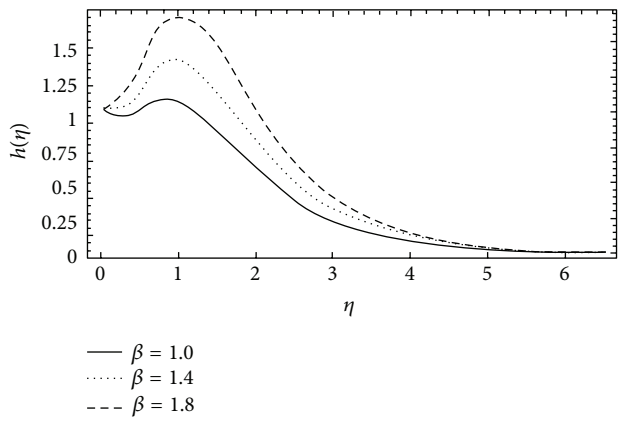

Figure 2: Effect of β on h for two-dimensional case.

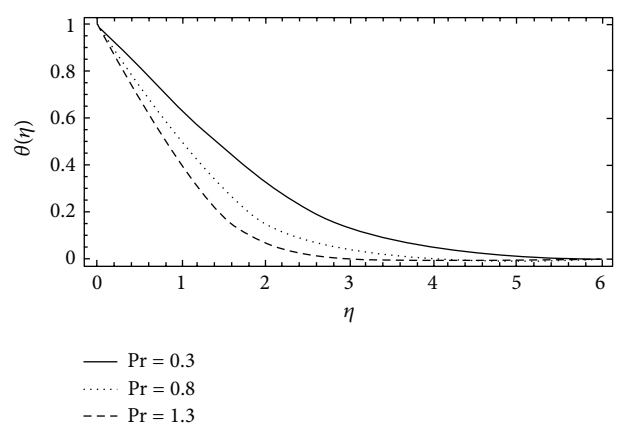

Figure 3: Effect of Pr on θ for two-dimensional case.

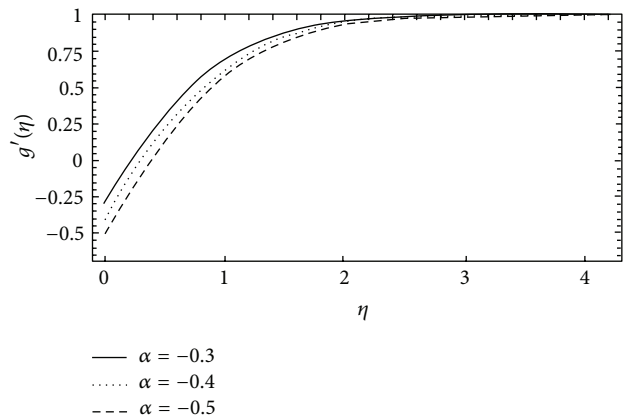

Figure 4: Effect of α on g' for axisymmetric case.

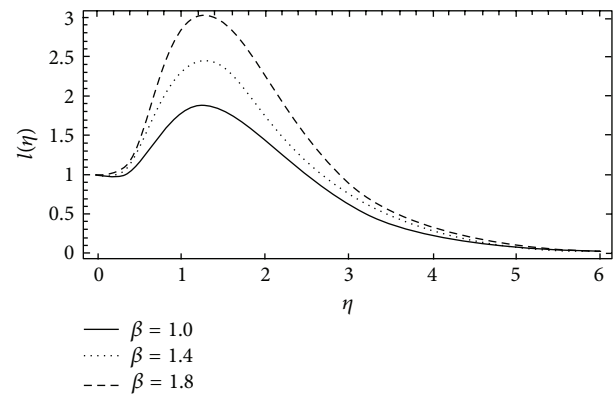

Figure 5: Effect of β on l for axisymmetric case.

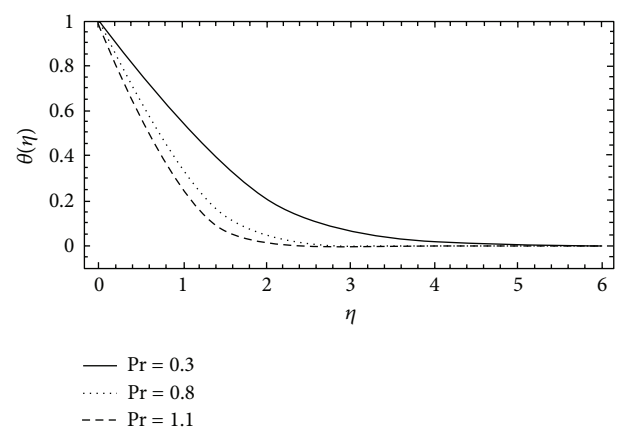

Figure 6: Effect of Pr on θ for axisymmetric case.

$$
\begin{aligned}
&p^1 : L_2 h_1'' + h_0''' + f_0 h_0' - f_0' h_0 \\
&\quad + \beta\left(h_0 f_0''' + f_0' h_0'' + f_0'' h_0' - f_0 h_0'''\right) = 0, \\
&h_1(0) = h_1(\infty) = 0, \\
&\qquad \vdots \\
&p^j : L_2 h_j - L_2 h_{j-1} + h_{j-1}'' + \sum_{k=0}^{j-1} f_k h_{j-1-k}' - \sum_{k=0}^{j-1} f_k' h_{j-1-k} \\
&+ \beta\left(\sum_{k=0}^{j-1} h_k f_{j-1-k}''' + \sum_{k=0}^{j-1} f_k' h_{j-1-k}''\right. \\
&\qquad \left. + \sum_{k=0}^{j-1} f_k'' h_{j-1-k}' - \sum_{k=0}^{j-1} f_k h_{j-1-k}'''\right) = 0, \\
&h_j(0) = h_j(\infty) = 0, \quad j \geq 2; \\
&\qquad p^1 : L_3 \theta_1'' + \theta_0'' + \Pr f_0 \theta_0' = 0, \\
&\qquad \theta_1(0) = \theta_1(\infty) = 0, \\
&\qquad \vdots \\
&p^j : L_3 \theta_j - L_3 \theta_{j-1} + \theta_{j-1}'' + \Pr \sum_{k=0}^{j-1} f_k \theta_{j-1-k}' = 0, \\
&\theta_j(0) = \theta_j(\infty) = 0, \quad j \geq 2.
\end{aligned} \tag{14}
$$

On solving (14) in any software like Mathematica, Maple or MATLAB we can get any order of approximation.

Adopting the same procedure for axisymmetric stagnation flow towards an axisymmetric shrinking surface (8), we can get the required solution for (8)

$$
\begin{aligned}
g &= g_0 + g_1 + g_2 + \cdots, \\
l &= l_0 + l_1 + l_2 + \cdots, \\
\theta &= \theta_0 + \theta_1 + \theta_2 + \cdots.
\end{aligned} \tag{15}
$$

4. Concluding Remarks

In this paper, we have studied non-Newtonian Stagnation point flow in the presence of heat transfer by using HPM. The HPM is used in a direct way without using linearization, discretization, or restrictive assumption. The variations of various emerging parameters on the velocities (f', h, g', l) and temperature field (θ) are discussed through Figures 1, 2, 3, 4, 5, and 6. The main results of the present analysis are as follows:

(i) for two-dimensional case, the velocity f' decreases for shrinking parameter α while for axisymmetric shrinking surface, the velocity g' shows opposite behavior for α;

(ii) for two dimensional case and axisymmetric shrinking surface, the velocity profiles h and l increase with increasing value of β;

(iii) the effects of Prandtl number Pr are same on the temperature field for both cases.

Notations

ρ:	Density of fluid
υ:	Kinematic viscosity
α_1:	Second grade parameter
T:	Temperature
α:	Stretching and shrinking parameter
k:	Thermal conductivity
c_p:	Specific heat
T_0 and T_∞:	The temperatures at and far away from the plate
Pr:	Prandtl number
β:	Dimensionless second grade parameter
f, g, h, l:	Dimensionless velocity profiles
θ:	Dimensionless temperature profile
u:	Velocity component in x direction
v:	Velocity component in y direction
w:	Velocity component in z direction
η:	Independent dimensionless parameter.

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Research Article

Asymptotic Behavior of Solutions to a Linear Volterra Integrodifferential System

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We investigate the asymptotic behavior of solutions to a linear Volterra integrodifferential system $x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^{n}\int_0^t K_{ij}(t,s)x_j(s)ds, t \in \mathbb{R}^+, i = 1,2,\ldots,n$. We show that under some suitable conditions, there exists a solution for the above integrodifferential system, which is asymptotically equivalent to some given functions. Two examples are given to illustrate our theorem.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all nonnegative real numbers, and by $\mathbb{R}^n$ the set of all n-dimensional real vectors. Moreover, $BC(\mathbb{R}^+,\mathbb{R}^n)$ denotes the Banach space of all bounded and continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^n$ with the norm

$$\|f\| = \sup_{t\in\mathbb{R}^+}\max_{1\le j\le n}\left|f_j(t)\right|, \tag{1}$$

where $f(t) = (f_1(t),\ldots,f_n(t))^T$ for $t \in \mathbb{R}^+$.

The aim of this paper is to study some asymptotic behavior of solutions to the following linear Volterra integrodifferential system:

$$x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^{n}\int_0^t K_{ij}(t,s)x_j(s)\,ds, \quad t \in \mathbb{R}^+,\ i = 1,2,\ldots,n, \tag{2}$$

where $a_i, b_i : \mathbb{R}^+ \to \mathbb{R}$ and $K_{ij} : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$, $i, j = 1,2,\ldots,n$ are all continuous functions.

Definition 1. We call $x = (x_1,x_2,\ldots,x_n)^T : \mathbb{R}^+ \to \mathbb{R}^n$ a solution of system (2) if x is continuously differentiable and satisfies (2).

The asymptotic behavior of solutions has been an important and interesting topic in the qualitative theory of differential and difference equations. Especially, recently, many authors have made interesting and important contributions on the asymptotic behavior of solutions for Volterra type difference equations (e.g., we refer the reader to [1–10] and references therein).

Very recently, Diblík and Schmeidel [6] obtained a very interesting result concerning the asymptotic behavior of solutions for the following linear Volterra difference equation:

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n}K(n,i)x(i). \tag{3}$$

More specifically, they proved that for every admissible constant $c \in \mathbb{R}$, there exists a solution $x = x(n)$ of (3) such that

$$x(n) \sim \left(c + \sum_{i=0}^{n-1}\frac{a(i)}{\beta(i+1)}\right)\beta(n), \quad n \longrightarrow \infty, \tag{4}$$

where $\beta(n) = \prod_{i=0}^{n-1}b(i)$. However, to the best of our knowledge, it seems that there is no literature concerning such asymptotic behavior of solutions for Volterra type differential equations. That is the main motivation of this paper. In this paper, we will adopt the idea in the proof of [6] to investigate

some asymptotic behaviors of solutions for Volterra differential system (2).

2. Main Result

Before establishing our main result, we first give an "Arzela-Ascoli" type theorem for the subsets of $BC(\mathbb{R}^+, \mathbb{R}^n)$.

Lemma 2. *Let $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{R}^n)$, satisfying (i) $\mathcal{F}$ is uniformly bounded; (ii) $\mathcal{F}$ is equiuniformly continuous on every compact subset of $\mathbb{R}^+$; (iii) for every $\varepsilon > 0$, there exist $f_\varepsilon \in BC(\mathbb{R}^+, \mathbb{R}^n)$ and $T_\varepsilon > 0$ such that $\|f(t) - f_\varepsilon(t)\| < \varepsilon$ for all $f \in \mathcal{F}$ and $t \geq T_\varepsilon$. Then $\mathcal{F}$ is precompact in $BC(\mathbb{R}^+, \mathbb{R}^n)$.*

Proof. By the condition (iii), for every $k \in \mathbb{N}$, there exist $T_k > 0$ such that

$$\|F_1(t) - F_2(t)\| < \frac{1}{k} \tag{5}$$

for all $F_1, F_2 \in \mathcal{F}$ and $t \geq T_k$.

Let $\{f_n\}$ be a sequence in $\mathcal{F}$. By (i) and (ii), it follows from Arzela-Ascoli theorem that for every $k \in \mathbb{N}$, there exists a subsequence $\{f_n^k\} \subset \{f_n\}$ such that $\{f_n^k\}$ is uniformly convergent on $[0, T_k]$. Then, by choosing the diagonal sequence, we can get a subsequence $\{f_m\} \subset \{f_n\}$ such that, for every $k \in \mathbb{N}$, $\{f_m\}$ is uniformly convergent on $[0, T_k]$.

It remains to show that $\{f_m\}$ is uniformly convergent on $\mathbb{R}^+$. For every $\varepsilon > 0$, choose $k_0 \in \mathbb{N}$ with $1/k_0 < \varepsilon$. Since $\{f_m\}$ is uniformly convergent on $[0, T_{k_0}]$, for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{t \in [0, T_{k_0}]} \|f_{m_1}(t) - f_{m_2}(t)\| \leq \varepsilon \tag{6}$$

for all $m_1, m_2 \geq N$; Combining this with (5), we conclude that

$$\sup_{t \in \mathbb{R}^+} \|f_{m_1}(t) - f_{m_2}(t)\| \leq \varepsilon \tag{7}$$

for all $m_1, m_2 \geq N$, that is, $\{f_m\}$ is uniformly convergent on $\mathbb{R}^+$. This completes the proof. □

Throughout the rest of this paper, for every $i \in \{1, 2, \ldots, n\}$, we assume that

$$A_i = \sup_{t \in \mathbb{R}^+} |A_i(t)| < +\infty, \tag{8}$$

where

$$\begin{gathered} A_i(t) = \int_0^t \frac{a_i(s)}{\beta_i(s)} ds, \\ \beta_i(t) = \exp\left(\int_0^t b_i(s)\, ds\right), \quad t \in \mathbb{R}^+, \\ 0 \leq M_i := \sum_{j=1}^n \int_0^{+\infty} \left(\int_0^t \left|K_{ij}(t,s) \frac{\beta_j(s)}{\beta_i(t)}\right| ds\right) dt < 1. \end{gathered} \tag{9}$$

Theorem 3. *Assume that*

$$\begin{gathered} 0 < \liminf_{t \to +\infty} \beta_i(t) \leq \limsup_{t \to +\infty} \beta_i(t) < +\infty, \\ i = 1, 2, \ldots, n. \end{gathered} \tag{10}$$

Let $c = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$ with $c_i + A_i(t) \geq 0$ for all $i \in \{1, 2, \ldots, n\}$ and $t \in \mathbb{R}^+$. Then, there exists a solution $x = (x_1, x_2, \ldots, x_n)^T : \mathbb{R}^+ \to \mathbb{R}^n$ of system (2) such that

$$x_i(t) \sim (c_i + A_i(t)) \beta_i(t), \quad t \longrightarrow +\infty, \ i = 1, 2, \ldots, n, \tag{11}$$

provided that $\liminf_{t \to +\infty}(c_i + A_i(t)) > 0$.

Proof. We define that

$$\begin{gathered} \alpha_i(0) = \frac{(c_i + A_i) M_i}{1 - M_i}, \\ \alpha_i(t) = (c_i + A_i + \alpha_i(0)) \\ \times \sum_{j=1}^n \int_t^{+\infty} \left(\int_0^s \left|K_{ij}(s,p) \frac{\beta_j(p)}{\beta_i(s)}\right| dp\right) ds, \end{gathered} \tag{12}$$

for all $i \in \{1, 2, \ldots, n\}$ and $t \in \mathbb{R}^+$. Moreover, we define an operator $\rho = (\rho_1, \rho_2, \ldots, \rho_n)^T$ on

$$\begin{aligned} S = \{ & z \in BC(\mathbb{R}^+, \mathbb{R}^n) : c_i + A_i(t) - \alpha_i(0) \\ & \leq z_i(t) \leq c_i + A_i(t) + \alpha_i(0), \\ & i = 1, 2, \ldots, n, \ t \in \mathbb{R}^+ \} \end{aligned} \tag{13}$$

by

$$\begin{gathered} (\rho_i z)(t) = c_i + A_i(t) \\ - \sum_{j=1}^n \int_t^{+\infty} \left(\int_0^s K_{ij}(s,p) z_j(p) \frac{\beta_j(p)}{\beta_i(s)} dp\right) ds, \\ i = 1, 2 \ldots, n, \end{gathered} \tag{14}$$

for $t \in \mathbb{R}^+$ and $z \in S$. It is easy to see that S is a nonempty, closed, and convex set in $BC(\mathbb{R}^+, \mathbb{R}^n)$. Next, we divide the remaining proof into two steps.

Step 1. $\rho(S) \subset S$, ρ is continuous, and $\overline{\rho(S)}$ is compact.

Let $z \in S$. We have

$$\begin{gathered} |(\rho_i z)(t)| \leq |c_i| + A_i + M_i \cdot \|z\| < +\infty, \\ t \in \mathbb{R}^+, \quad i = 1, 2, \ldots, n. \end{gathered} \tag{15}$$

In addition, since $z \in S$, we have

$$|z_i(t)| \leq c_i + A_i + \alpha_i(0), \quad t \in \mathbb{R}^+, \ i = 1, 2, \ldots, n. \tag{16}$$

Then, it follows that

$$\begin{aligned}&\left|(\rho_i z)(t)-(c_i+A_i(t))\right|\\&\quad\le\sum_{j=1}^{n}\int_t^{+\infty}\left(\int_0^s\left|K_{ij}(s,p)\,z_j(p)\,\frac{\beta_j(p)}{\beta_i(s)}\right|dp\right)ds\\&\quad\le\left[c_i+A_i+\alpha_i(0)\right]\cdot M_i=\alpha_i(0)\end{aligned}\tag{17}$$

for all $i\in\{1,2,\dots,n\}$ and $t\in\mathbb{R}^+$. Thus, we conclude that $\rho(S)\subset S$.

For every $\varepsilon>0$, there exists a constant $\delta=\varepsilon/\max_{1\le i\le n}M_i$ such that for all $z,y\in S$ with $\|z-y\|<\delta$, we have

$$\begin{aligned}&\left|(\rho_i z)(t)-(\rho_i y)(t)\right|\\&\quad=\left|\sum_{j=1}^{n}\int_t^{+\infty}\left(\int_0^s K_{ij}(s,p)\,z_j(p)\,\frac{\beta_j(p)}{\beta_i(s)}dp\right)ds\right.\\&\qquad\left.-\sum_{j=1}^{n}\int_t^{+\infty}\left(\int_0^s K_{ij}(s,p)\,y_j(p)\,\frac{\beta_j(p)}{\beta_i(s)}dp\right)ds\right|\\&\quad\le\delta\cdot M_i\le\varepsilon,\quad i\in\{1,2,\dots,n\},\ t\in\mathbb{R}^+,\end{aligned}\tag{18}$$

which means that ρ is continuous.

Next, we show that $\rho(S)$ is precompact. Firstly, for every $x\in S$, we have

$$\begin{aligned}\|\rho x\|&=\sup_{t\in\mathbb{R}^+}\max_{1\le i\le n}\left|(\rho_i x)(t)\right|\\&\le\max_{1\le i\le n}\left[c_i+A_i+\alpha_i(0)\right],\end{aligned}\tag{19}$$

which means that $\rho(S)$ is uniformly bounded. Secondly, for every $z\in S$, $t_1,t_2\in\mathbb{R}^+$ and $i=1,2,\dots,n$, we have

$$\begin{aligned}&\left|(\rho_i z)(t_1)-(\rho_i z)(t_2)\right|\\&\quad=\left|\sum_{j=1}^{n}\int_{t_1}^{+\infty}\left(\int_0^s K_{ij}(s,p)\,z_j(p)\,\frac{\beta_j(p)}{\beta_i(s)}dp\right)ds\right.\\&\qquad\left.-\sum_{j=1}^{n}\int_{t_2}^{+\infty}\left(\int_0^s K_{ij}(s,p)\,z_j(p)\,\frac{\beta_j(p)}{\beta_i(s)}dp\right)ds\right|\\&\quad\le\max_{1\le i\le n}\left[c_i+A_i+\alpha_i(0)\right]\\&\qquad\cdot\left|\sum_{j=1}^{n}\int_{t_1}^{t_2}\left(\int_0^s\left|K_{ij}(s,p)\,\frac{\beta_j(p)}{\beta_i(s)}\right|dp\right)ds\right|,\end{aligned}\tag{20}$$

which yields that $\rho(S)$ is equiuniformly continuous on every compact subsets of $\mathbb{R}^+$. Thirdly, by the definition of M_i, for every $\varepsilon>0$, there exists $T>0$ such that for all $t\ge T$ and $z\in S$, we have

$$\begin{aligned}&\sum_{j=1}^{n}\int_t^{+\infty}\left(\int_0^s\left|K_{ij}(s,p)\,\frac{\beta_j(p)}{\beta_i(s)}\right|dp\right)ds\\&\qquad<\frac{\varepsilon}{\max_{1\le i\le n}\left[c_i+A_i+\alpha_i(0)\right]},\quad i=1,2,\dots,n,\end{aligned}\tag{21}$$

which yields that

$$\left\|\rho_i z-\rho_i 0\right\|<\varepsilon,\quad i=1,2,\dots,n,\tag{22}$$

and thus $\|\rho z-\rho 0\|<\varepsilon$. Then, by Lemma 2, we know that $\rho(S)$ is precompact.

Step 2. By Step 1 and Schauder's fixed-point theorem, ρ has a fixed point in S; that is, there exists $z^0=(z_1^0,z_2^0,\dots,z_n^0)^T\in S$ such that

$$\begin{aligned}z_i^0(t)&=c_i+A_i(t)\\&\quad-\sum_{j=1}^{n}\int_t^{+\infty}\left(\int_0^s K_{ij}(s,p)\,z_j^0(p)\,\frac{\beta_j(p)}{\beta_i(s)}dp\right)ds,\end{aligned}\tag{23}$$

for all $i\in\{1,2,\dots,n\}$ and $t\in\mathbb{R}^+$. Noting that

$$\sup_{z\in S}\|z\|\le\max_{1\le i\le n}\left[c_i+A_i+\alpha_i(0)\right],\tag{24}$$

we have

$$\begin{aligned}&\left|z_i^0(t)-(c_i+A_i(t))\right|\\&\quad\le\max_{1\le i\le n}\left[c_i+A_i+\alpha_i(0)\right]\\&\qquad\cdot\sum_{j=1}^{n}\int_t^{+\infty}\left(\int_0^s\left|K_{ij}(s,p)\,\frac{\beta_j(p)}{\beta_i(s)}\right|dp\right)ds,\end{aligned}\tag{25}$$

for all $i\in\{1,2,\dots,n\}$ and $t\in\mathbb{R}^+$. Then, it is easy to see that

$$\lim_{t\to+\infty}\left|z_i^0(t)-(c_i+A_i(t))\right|=0,\quad i=1,2,\dots,n,\tag{26}$$

Combining this with

$$\liminf_{t\to+\infty}(c_i+A_i(t))>0,\tag{27}$$

we have

$$\lim_{t\to+\infty}\frac{z_i^0(t)}{c_i+A_i(t)}=1,\quad i=1,2,\dots,n,\tag{28}$$

that is,

$$z_i^0(t)\sim c_i+A_i(t),\quad t\longrightarrow+\infty,\ i=1,2,\dots,n.\tag{29}$$

Now, define a function $x=(x_1,x_2,\dots,x_n)^T:\mathbb{R}^+\to\mathbb{R}^n$ by

$$x_i(t)=z_i^0(t)\,\beta_i(t),\quad i=1,2,\dots,n,\ t\in\mathbb{R}^+.\tag{30}$$

It follows from (23) that

$$\frac{d}{dt}z_i^0(t) = \frac{a_i(t)}{\beta_i(t)} + \sum_{j=1}^{n}\int_0^t K_{ij}(t,p)\, z_j^0(p)\,\frac{\beta_j(p)}{\beta_i(t)}dp, \quad i=1,2,\ldots,n, \quad t\in\mathbb{R}^+, \tag{31}$$

which yields that

$$\frac{x_i'(t)\beta_i(t) - x_i(t)\beta_i'(t)}{\beta_i^2(t)} = \frac{a_i(t)}{\beta_i(t)} + \sum_{j=1}^{n}\int_0^t K_{ij}(t,s)\,\frac{x_j(s)}{\beta_i(t)}ds, \quad i=1,2,\ldots,n, \quad t\in\mathbb{R}^+. \tag{32}$$

Then, we get

$$x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^{n}\int_0^t K_{ij}(t,s)\,x_j(s)\,ds, \quad i=1,2,\ldots,n, \quad t\in\mathbb{R}^+, \tag{33}$$

which means that x is a solution to system (2). In addition, combining (28) with the assumption

$$0 < \liminf_{t\to+\infty}\beta_i(t) \le \limsup_{t\to+\infty}\beta_i(t) < +\infty, \quad i=1,2,\ldots,n, \tag{34}$$

we get

$$\lim_{t\to+\infty}\frac{x_i(t)}{(c_i + A_i(t))\,\beta_i(t)} = 1, \quad i=1,2,\ldots,n, \tag{35}$$

which yields (11). □

Example 4. Let $n=1$, and for all $t,s\in\mathbb{R}^+$,

$$a_1(t) = \exp(\sin\pi t)\cos t, \quad b_1(t) = \pi\cos\pi t, \quad K_{11}(t,s) = \frac{\exp(\sin\pi t)}{(1+t+s)^3\exp(\sin\pi s)}. \tag{36}$$

Then, for all $t\in\mathbb{R}^+$, we have $\beta_1(t) = \exp(\sin\pi t)$,

$$A_1(t) = \int_0^t\frac{a_1(s)}{\beta_1(s)}ds = \sin t,$$

$$A_1 = \sup_{t\in\mathbb{R}^+}\{|A_1(t)|\} = 1\in(0,+\infty),$$

$$\begin{aligned} M_1 &= \int_0^{+\infty}\left(\int_0^t\left|K_{11}(t,s)\frac{\beta(s)}{\beta(t)}\right|ds\right)dt \\ &= \int_0^{+\infty}\left(\int_0^t\frac{1}{(1+t+s)^3}ds\right)dt \\ &= \int_0^{+\infty}\frac{1}{2}\left[\frac{1}{(1+t)^2} - \frac{1}{(1+2t)^2}\right]dt \\ &= \frac{1}{2}\int_0^{+\infty}\frac{1}{(1+t)^2}dt - \frac{1}{2}\int_0^{+\infty}\frac{1}{(1+2t)^2}dt \\ &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}\in(0,1). \end{aligned} \tag{37}$$

In addition, it is easy to see that

$$0 < e^{-1} = \liminf_{t\to+\infty}\beta_1(t) \le \limsup_{t\to+\infty}\beta_1(t) = e < +\infty. \tag{38}$$

Thus, by Theorem 3, we conclude that for every $c>1$, there exists a solution $x:\mathbb{R}^+\to\mathbb{R}$ for (2) such that

$$x(t)\sim(c+\sin t)\exp(\sin\pi t), \quad t\longrightarrow+\infty. \tag{39}$$

Remark 5. It is needed to note that in the above example, $(c+\sin t)\exp(\sin\pi t)$ is not a solution to (2).

Example 6. Consider the following system:

$$x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^{2}\int_0^t K_{ij}(t,s)\,x_j(s)\,ds, \quad i=1,2, \tag{40}$$

where

$$\begin{gathered} a_1(t) = \exp(\sin\pi t)\cos t, \\ a_2(t) = -\exp(\cos\pi t)\sin t, \\ b_1(t) = \pi\cos\pi t, \\ b_2(t) = -\pi\sin\pi t, \\ K_{ij}(t,s) = \frac{(-1)^{i+j}\exp(\sin\pi t)}{16\,(1+t+s)^3\exp(\sin\pi s)} \end{gathered} \tag{41}$$

for all $i,j=1,2$, and $t,s\in\mathbb{R}^+$. By a direct calculation, we get

$$\beta_1(t) = \exp(\sin\pi t), \quad \beta_2(t) = \exp(-1+\cos\pi t),$$

$$A_1(t) = \int_0^t\frac{a_1(s)}{\beta_1(s)}ds = \sin t, \quad A_1 = 1,$$

$$A_2(t) = \int_0^t\frac{a_2(s)}{\beta_2(s)}ds = (-1+\cos t)\,e, \quad t\in\mathbb{R}^+, \quad A_2 = 2e,$$

$$\begin{aligned} M_1 &= \sum_{j=1}^{2} \int_0^{+\infty} \left(\int_0^t \left| K_{1j}(t,s) \frac{\beta_j(s)}{\beta_1(t)} \right| ds \right) dt \\ &\le \frac{1+e}{64} < 1, \\ M_2 &= \sum_{j=1}^{2} \int_0^{+\infty} \left(\int_0^t \left| K_{2j}(t,s) \frac{\beta_j(s)}{\beta_2(t)} \right| ds \right) dt \\ &\le \frac{e+2e^3}{64} < 1. \end{aligned} \tag{42}$$

Moreover, we have

$$\begin{aligned} 0 < e^{-1} &= \liminf_{t\to+\infty} \beta_1(t) \\ &\le \limsup_{t\to+\infty} \beta_1(t) = e < +\infty, \\ 0 < e^{-2} &= \liminf_{t\to+\infty} \beta_2(t) \\ &\le \limsup_{t\to+\infty} \beta_2(t) = 1 < +\infty. \end{aligned} \tag{43}$$

Then, by Theorem 3, for every $c = (c_1, c_2)^T \in \mathbb{R}^2$ with $c_1 > 1$ and $c_2 > 2e$, there exists a solution $x = (x_1, x_2) : \mathbb{R}^+ \to \mathbb{R}^2$ of system (40) such that

$$\begin{aligned} x_1(t) &\sim (c_1 + \sin t) \exp(\sin \pi t), \quad t \longrightarrow +\infty, \\ x_2(t) &\sim [c_2 + (-1 + \cos t) e] \exp(-1 + \cos \pi t), \quad t \longrightarrow +\infty. \end{aligned} \tag{44}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Regularity of a Stochastic Fractional Delayed Reaction-Diffusion Equation Driven by Lévy Noise

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The current paper is devoted to the regularity of the mild solution for a stochastic fractional delayed reaction-diffusion equation driven by Lévy space-time white noise. By the Banach fixed point theorem, the existence and uniqueness of the mild solution are proved in the proper working function space which is affected by the delays. Furthermore, the time regularity and space regularity of the mild solution are established respectively. The main results show that both time regularity and space regularity of the mild solution depend on the regularity of initial value and the order of fractional operator. In particular, the time regularity is affected by the regularity of initial value with delays.

1. Introduction

Recently, fractional partial differential equations attract more and more attention. They appear more and more frequently in different research areas and engineering applications. They have been applied to model various phenomena in image analysis, risk management, and statistical mechanics (see, e.g., [1, 2]). There are many papers concerning the existence and regularity of the solution for fractional Navier-Stokes, fractional Ginzburg-Landau equation, fractional Burgers equation, fractional Langevin equation, and so on (see [3, 4] and references therein).

Stochastic partial differential equations driven by Gaussian noise and non-Gaussian noise such as Lévy noise have also attracted a lot of attention. It seems more significant to investigate fractional partial differential equations with some random force, and some authors have investigated the existence and regularity of the solutions for stochastic fractional partial differential equations ([2, 5–7] and the references therein). The authors in [6, 7] proved the existence and uniqueness of the solution for a stochastic fractional partial differential equation driven by a space-time white noise in one dimension. Truman and Wu in [8] applied the Banach fixed point theorem to show the existence and uniqueness of the mild solution for fractal Burgers equations driven by Lévy noise on real line. Brzeźniak and Debbi in papers [9, 10] proved the existence and ergodicity of the solution for fractal Burgers equation driven by Gaussian space-time white noise, and we refer to [9, 10] for more details. In mathematical biology and other fields, delays are often considered in the model such as maturation time for population dynamics. Some efforts have been devoted to the development of the theory of PDEs with delay. Such equations are naturally more difficult since they are infinite dimensional both in time and space variables. We refer to the monographs [11, 12] for more details. To our knowledge, there is no paper to study the stochastic fractional reaction-diffusion equation with delays.

It is worth to point out that the authors in [8] study the existence of the mild solution for stochastic fractional Burgers equation driven by Lévy noise, but they could not provide the regularity of the mild solution. The authors in [7, 13] study the regularity of the mild solution for stochastic fractional partial differential equations driven by Gaussian white noise, but not Lévy noise. There is a natural question, how about the regularity of the mild solution for the stochastic fraction delayed reaction-diffusion equation driven by Lévy noise?

Motivated by [8], in the present paper, we will study the stochastic fractional reaction-diffusion equation with delays driven by Lévy process followed as:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \lambda \Delta_\alpha u(t,x) + f(t,x,u_t) \\ &\quad + g(t,x,u_t) Z_{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) &= u_0(x), \qquad u(\eta,x) = \phi(\eta,x), \quad \eta \in [-r,0], \end{aligned} \tag{1}$$

where $\Delta_\alpha := -(-d^2/dx^2)^{\alpha/2}$ is the fractional Laplacian operator with $\alpha \in (0,2]$, the constants $\lambda \in \mathbb{R}$, $f, g : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are measurable, the function $u_t = u(t+\eta)$, and $Z_{t,x}$ is the one-dimensional Lévy process (see Section 2 for the definition). Recall that D_α reduces to be the Laplacian operator when $\alpha = 2$.

In this paper, the existence, uniqueness, time regularity, and space regularity of the mild solution for (1) are shown for $\alpha \in (1,2]$ in the proper working function space which is affected by the delays. The main results show that both time regularity and space regularity of the mild solution for (1) depend on the regularity of initial value and the order of fractional operator. In particular, the time regularity is affected by the the regularity of initial value with delays.

The rest of this paper is organized as follows. In Section 2, we introduce the definition of the Lévy space-time white noise. Then, some useful properties for the fractional Green kernel are presented. In Section 3, the proper working function space is constructed. Then the existence and uniqueness of the mild solution for (1) are proved by the Banach fixed point theorem in the proper working function space. Finally, the time regularity and space regularity of the mild solution are provided, respectively, in Section 4.

2. Preliminaries

In this section, we first introduce the Lévy space-time white noise. Then, some useful properties for the fractional Green kernel are presented.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual condition. For one-dimensional Lévy process $Z_{t,x}$, it follows from Lévy-Itô decomposition that there exist a constant β_1 and a nonnegative constant β_2, and a one-dimensional space-time white noise $W_{t,x} = (\partial^2 W/\partial t \partial x)(t,x)$ ($W(t,x)$ is a Brownian sheet on $[0,\infty) \times \mathbb{R}$) such that

$$\begin{aligned} Z_{t,x} = \beta_1 t + \beta_2 W_{t,x} &+ \int_{|z|<1} z\widetilde{N}(t,x,dz) \\ &+ \int_{|z|\geq 1} zN(t,x,dz), \end{aligned} \tag{2}$$

where

$$\widetilde{N}(t,x,A) := N(t,x,A) - t\nu(A), \tag{3}$$

where $\nu(A) := E[N(1,A)]$ is the Lévy measure of $Z_{t,x}$.

Similar to [14], for any p, we denote

$$\widehat{c}_p := \left(\int_{\mathbb{R}} |z|^p \nu(dz)\right)^{1/p}. \tag{4}$$

In what follows, we assume that

$$\widehat{c} := \sup_{p\geq 1} \widehat{c}_p < \infty. \tag{5}$$

Recalling that

$$\int_{|z|<1} zN(t,x,dz) = \int_{|z|<1} z\widetilde{N}(t,x,dz) + t\int_{|z|<1} z\nu(dz). \tag{6}$$

By absorbing $\widetilde{\beta} := -\int_{|z|<1} z\nu(dz)$ into β_1, we can rewrite (2) into the following equation:

$$Z_{t,x} = \beta_1 t + \beta_2 W_{t,x} + \int_{\mathbb{R}} zN(t,x,dz). \tag{7}$$

Let $\beta_1 = 0$; then (1) can be written as

$$\begin{aligned} du(t,x) = {} & [\lambda\Delta_\alpha u(t,x) + f(t,x,u_t)]\,dt \\ & + h(t,x,u_t)\,dW_{t,x} \\ & + g(t,x,u_t)\,dY_{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & \quad u(\eta,x) = \phi(\eta,x), \quad \eta \in (r,0], \end{aligned} \tag{8}$$

where $W_{t,x}$ is a one-dimensional space-time white noise and $Y_{t,x} := \int_{\mathbb{R}} zN(t,x,dz)$ is a one-dimensional pure jump Lévy process with Lévy measure of ν. We suppose that W generates a $\{\mathcal{F}\}_{t\geq 0}$-martingale measure in the sense of Walsh [15].

The following assumptions are imposed to the initial data u_0 and $\phi(\eta,x)$, $f(u_t)$, $h(u_t)$, and $g(u_t)$ to show the existence and uniqueness of the mild solution.

(H1) The initial data u_0 which is $\mathcal{F}_0$-measurable and ϕ satisfy

$$\sup_{x\in\mathbb{R}} |u(0,x)|^2 < \infty, \qquad \sup_{x\in\mathbb{R}} \int_{-r}^{0} |\phi(\eta,x)|^2 d\eta < \infty. \tag{9}$$

(H2) There exists a constant K such that for all $t \geq 0$,

$$\begin{aligned} & |f(u_t)|^2 + |h(u_t)|^2 + \left(\int_{\mathbb{R}} |g(u_t)|\,|z|\,\nu(dz)\right)^2 \\ & \quad < K\left(|u|^2 + \int_{-r}^{0} |u(t+\eta)|^2 d\eta\right), \\ & |f(u_t) - f(v_t)|^2 + |h(u_t) - h(v_t)|^2 \\ & \quad + \left(\int_{\mathbb{R}} |g(u_t) - g(v_t)|\,|z|\,\nu(dz)\right)^2 \\ & \quad < K\left(|u-v|^2 + \int_{-r}^{0} |u(t+\eta) - v(t+\eta)|^2 d\eta\right). \end{aligned} \tag{10}$$

Let the Green kennel $G_\alpha(t,x)$ be the fundamental solution of the Cauchy problem:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \lambda\Delta_\alpha v, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ v(0,x) &= \delta_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{11}$$

where $\delta_0(x)$ denotes the Dirac function. By Fourier transform,

$$G_\alpha(t,x) = \left[\mathcal{F}^{-1}\left(e^{\lambda t|\cdot|^\alpha}\right)\right](x). \tag{12}$$

A higher order fractional Green kennel is introduced in [16].

The following lemma gives some useful properties about $G_\alpha(t,x)$, which are key technique tools to get the estimation for the existence and uniqueness of the mild solution.

Lemma 1 (see [7]). *The Green kernel function $G_\alpha(t,x)$ satisfies the following properties.*

(1) *For any $t \geq 0$ $G_\alpha(t,x) = t^{-1/\alpha} G_\alpha(1, t^{-1/\alpha} x)$.*

(2) *For $n \in (1/(\alpha+1), \alpha+1)$, $\int_0^T \int_{\mathbb{R}} |G_\alpha(t,x)|^n dt\, dx < \infty$.*

(3) *For any $x \in \mathbb{R}$, $\int_{\mathbb{R}} G_\alpha(t,x) dx = 1$.*

(4) *For any $t, s \in \mathbb{R}$, $G_\alpha(t,x) * G_\alpha(s,x) = G_\alpha(t+s,x)$.*

(5) *For any $x \in \mathbb{R}$, there exists a constant C such that*

$$G_\alpha(1,x) \leq \frac{C}{1+|x|^{1+\alpha}},$$
$$\partial_x^m G_\alpha(1,x) \leq C \frac{|x|^{\alpha+m-1}}{1+|x|^{1+\alpha}}. \tag{13}$$

3. Existence of the Mild Solution

In this section, we will first construct the proper working function space.

Let T be a fixed positive time and $\mathbb{B}$ the class of all $\mathcal{F}_t$-adapted càdlàg process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\left[|u(t,x)|^2\right] < \infty. \tag{14}$$

Let $\lambda > 0$ be arbitrarily fixed; we define

$$|u|_\lambda^2 = \left\{\int_0^T e^{-\lambda t} \sup_{x\in\mathbb{R}} E|u(t,x)|^2 dt\right\}_{t\geq 0} + \left\{\int_{-r}^0 e^{-\lambda t} E|u(t,x)|^2 dt\right\}_{t\in(-r,0)}. \tag{15}$$

For any $u \in \mathbb{B}$, $|u|_\lambda^2 < \infty$. It is easy to verify that $|\cdot|_\lambda$ is a norm and $(\mathbb{B}, |\cdot|_\lambda)$ is a Banach space.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and $G_\alpha(t,x)$ be given as in the previous section. Following the idea in [17], we represent a mild solution of (8) for $t \geq 0$.

Definition 2. An $\mathcal{F}_t$-adapted random field $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ is said to be a mild solution of (8) with initial value u_0 satisfying (H1) if the following integral equation is fulfilled:

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} G_\alpha(t, x-y) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) f(u_s) dy\, ds \\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) h(u_s) W(dy, ds) \\ &+ \int_0^t \iint_{\mathbb{R}} G_\alpha(t-s, x-y) g(u_s) zN(ds, dy, dz), \end{aligned} \tag{16}$$

where the stochastic integral with respect to $W(t,x)$ is understood in the sense of that introduced by Walsh [15].

Theorem 3. *For $t \geq 0$ and $\alpha \in (1,2]$, assume that (H1) and (H2) hold, then there exists a unique mild solution $u \in \mathbb{B}$ for* (8).

Remark 4. In the following proof, C is a local constant which may change from line to line.

Proof. We will prove the theorem by the following two steps.

Step 1. Suppose that $u \in \mathbb{B}$ and denote

$$\begin{aligned} \mathcal{T}u(t,x) = &\int_{\mathbb{R}} G_\alpha(t, x-y) u_0(y) dy + \mathcal{T}_1 u(t,x) \\ &+ \mathcal{T}_2 u(t,x) + \mathcal{T}_3 u(t,x), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \mathcal{T}_1 u(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) f(u_s) dy\, ds, \\ \mathcal{T}_2 u(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) h(u_s) W(dy, ds), \\ \mathcal{T}_3 u(t,x) &= \int_0^t \iint_{\mathbb{R}} G_\alpha(t-s, x-y) \\ &\qquad \times g(u_s) zN(ds, dy, dz). \end{aligned} \tag{18}$$

It follows from Hölder's inequality, Lemma 1, (H1), and (H2) that

$$\begin{aligned} &E|\mathcal{T}_1 u(t,x)|^2 \\ &= E\left|\int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) f(u_s) dy\, ds\right|^2 \\ &\leq C \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) dy\, ds \\ &\quad \times \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) E|f(u_s)|^2 dy\, ds \\ &\leq CK \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) \\ &\qquad \times E\left(|u(s,y)|^2 + \int_{-r}^0 |u(s+\eta, y)|^2 d\eta\right) dy\, ds \\ &\leq C \int_0^t \sup_{y\in\mathbb{R}} E\left(|u(s,y)|^2 + \int_{-r}^0 |u(s+\eta, y)|^2 d\eta\right) ds \\ &\leq Ct + C \int_0^t \int_{-r}^0 \sup_{y\in\mathbb{R}} E|u(s+\eta, y)|^2 d\eta\, ds \\ &\leq Ct + C \int_{-r}^0 \int_{-r}^t \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds\, d\eta \end{aligned}$$

$$\begin{aligned}
&\le Ct + Cr\int_{-r}^{0}\sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds \\
&\quad + Cr\int_{0}^{t}\sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds \\
&\le Ct(r+1) + Cr < \infty.
\end{aligned} \tag{19}$$

Applying Burkholder-Davis-Gundy inequality, Lemma 1, (H1), and (H2), we have

$$\begin{aligned}
&E|\mathcal{T}_2 u(t,x)|^2 \\
&= E\left|\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\,h(u_s)\,W(ds\,dy)\right|^2 \\
&\le C\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y)\,E|h(u_s)|^2 ds\,dy \\
&\le CK\int_0^t (t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}} E\left(|u(s,y)|^2 + \int_{-r}^{0}|u(s+\eta,y)|^2 d\eta\right) ds \\
&\le Ct^{1-(1/\alpha)} + C\int_{-r}^{0}\int_{-r}^{t}|(t+\eta-s)|^{-1/\alpha} \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds\,d\eta \\
&= Ct^{1-(1/\alpha)} + C\left(\int_{-r}^{t+\eta}\int_{-r}^{0}(t+\eta-s)^{-1/\alpha} d\eta \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds \right. \\
&\qquad \left. + \int_{t+\eta}^{t}\int_{-r}^{0}(s-\eta-t)^{-1/\alpha} d\eta \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds\right) \\
&\le Ct^{1-(1/\alpha)} + C\left(\int_{-r}^{t+\eta}(t-s)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds \right. \\
&\qquad \left. + \int_{t+\eta}^{t}(s+r-t)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds\right) \\
&= Ct^{1-(1/\alpha)} + C\left(\int_{-r}^{0}(t-s)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds \right. \\
&\qquad + \int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds \\
&\qquad \left. + \int_{t+\eta}^{t}(s+r-t)^{1-(1/\alpha)} \times \sup_{y\in\mathbb{R}} E|u(s,y)|^2 ds\right) \\
&\le C\left(t^{1-(1/\alpha)} + (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} + r^{2-(1/\alpha)}\right) < \infty,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&E|\mathcal{T}_3 u(t,x)|^2 \\
&= E\left|\int_0^t\iint_{\mathbb{R}} G_\alpha(t-s,x-y)\,g(u_s)\,zN(ds,dy,dz)\right|^2 \\
&\le C\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y) \times E\left(\int_{\mathbb{R}}|g(u_s)|\,|z|\,\nu(dz)\right)^2 dy\,ds \\
&\le CK\int_0^t (t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}} E\left(|u(s,y)|^2 + \int_{-r}^{0}|u(s+\eta,y)|^2 d\eta\right) ds \\
&\le C\left(t^{1-(1/\alpha)} + (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} + r^{2-(1/\alpha)}\right) < \infty.
\end{aligned} \tag{21}$$

Thus, combining (19) and (20) with (21), we derive

$$\begin{aligned}
E|u(t,x)|^2 \le C\Big[&t(r+1) + r + t^{1-(1/\alpha)} \\
&+ (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} \\
&+ r^{2-(1/\alpha)}\Big].
\end{aligned} \tag{22}$$

Taking Laplace transform formula and (22), we deduce that

$$\begin{aligned}
&|\mathcal{T}u(t,x)|_\lambda^2 \\
&= \int_0^T e^{-\lambda t}\sup_{x\in\mathbb{R}}|\mathcal{T}u(t,x)|^2 dt \\
&\le C\int_0^\infty e^{-\lambda t}\Big[t(r+1) + r + t^{1-(1/\alpha)} \\
&\qquad + (t+r)^{1-(1/\alpha)} + t^{2-(1/\alpha)} \\
&\qquad + r^{2-(1/\alpha)}\Big] dt \\
&\le C\Big[(r+1)\Gamma(2)\lambda^{-2} + r\lambda^{-1} \\
&\qquad + (e^r+1)\Gamma\left(2-\frac{1}{\alpha}\right)\lambda^{-(2-(1/\alpha))}
\end{aligned}$$

$$
\begin{aligned}
&+\Gamma\left(3-\frac{1}{\alpha}\right)\lambda^{-(3-(1/\alpha))}\\
&+r^{2-(1/\alpha)}\lambda^{-1}\Big]\\
\le{}& C\Big[\lambda^{-2}+\lambda^{-1}+\lambda^{-(2-(1/\alpha))}\\
&+\lambda^{-(3-(1/\alpha))}\Big]<\infty;
\end{aligned}
\tag{23}
$$

that is, $\mathscr{T}u\in\mathbb{B}$, which implies that operator $\mathscr{T}:\mathbb{B}\to\mathbb{B}$.

Step 2. For any $u,v\in\mathbb{B}$ and $t\ge 0$, it follows from Hölder's inequality, Lemma 1, and (H2) that

$$
\begin{aligned}
&E\left|\mathscr{T}_1u(t,x)-\mathscr{T}_1v(t,x)\right|^2\\
&\quad=E\left|\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\left(f(u_s)-f(v_s)\right)dy\,ds\right|^2\\
&\quad\le C\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\,dy\,ds\\
&\qquad\times\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\,E\left|f(u_s)-f(v_s)\right|^2dy\,ds\\
&\quad\le CK\int_0^t\sup_{y\in\mathbb{R}}E\Big(\left|u(s,y)-v(s,y)\right|^2\\
&\qquad\qquad+\int_{-r}^0\left|u(s+\eta,y)-v(s+\eta,y)\right|^2d\eta\Big)ds\\
&\quad\le C\Big(\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+\int_{-r}^0\int_{-r}^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\,d\eta\Big)\\
&=C\Big((r+1)\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+r\int_{-r}^0\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\Big).
\end{aligned}
\tag{24}
$$

By Burkholder-Davis-Gundy inequality, Lemma 1 and (H2), we have

$$
\begin{aligned}
&E\left|\mathscr{T}_2u(t,x)-\mathscr{T}_2v(t,x)\right|^2\\
&\quad=E\left|\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)\left(h(u_s)-h(v_s)\right)W(dy\,ds)\right|^2\\
&\quad\le C\int_0^t\int_{\mathbb{R}}G_\alpha^2(t-s,x-y)\,E\left|h(u_s)-h(v_s)\right|^2dy\,ds\\
&\quad\le CK\int_0^t(t-s)^{-(1/\alpha)}\\
&\qquad\times\sup_{y\in\mathbb{R}}E\Big(\left|u(s,y)-v(s,y)\right|^2\\
&\qquad\qquad+\int_{-r}^0\left|u(s+\eta,y)-v(s+\eta,y)\right|^2d\eta\Big)ds\\
&\quad\le C\Big(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+\int_{-r}^0\int_{-r}^t\left|t+\eta-s\right|^{-1/\alpha}\\
&\qquad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\,d\eta\Big)\\
&\quad\le C\Big(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+\int_{-r}^{t+\eta}\int_{-r}^0(t+\eta-s)^{-1/\alpha}\\
&\qquad\qquad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2d\eta\,ds\\
&\qquad+\int_{t+\eta}^t\int_{-r}^0(s-\eta-t)^{-1/\alpha}\\
&\qquad\qquad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2d\eta\,ds\Big)\\
&\quad\le C\Big(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+\int_{-r}^0(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+\int_0^{t+\eta}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\\
&\qquad+r^{1-(1/\alpha)}\int_{t+\eta}^t\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds\Big),
\end{aligned}
$$

$$
\begin{aligned}
&E\left|\mathscr{T}_3u(t,x)-\mathscr{T}_3v(t,x)\right|^2\\
&\quad=E\Big|\int_0^t\iint_{\mathbb{R}}G_\alpha(t-s,x-y)\\
&\qquad\qquad\times\left(g(u_s)-g(v_s)\right)zN(ds,dy,dz)\Big|^2\\
&\quad\le C\int_0^t\int_{\mathbb{R}}G_\alpha^2(t-s,x-y)\\
&\qquad\qquad\times E\Big(\int_{\mathbb{R}}\left|g(u_s)-g(v_s)\right|\left|z\right|\nu(dz)\Big)^2dy\,ds\\
&\quad\le CK\int_0^t(t-s)^{-1/\alpha}\\
&\qquad\times\sup_{y\in\mathbb{R}}E\Big(\left|u(s,y)-v(s,y)\right|^2\\
&\qquad\qquad+\int_{-r}^0\left|u(s+\eta,y)-v(s+\eta,y)\right|^2d\eta\Big)ds\\
&\quad\le C\Big(\int_0^t(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^2ds
\end{aligned}
$$

$$+\int_{-r}^{0}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds$$
$$+\int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds$$
$$+r^{1-(1/\alpha)}\int_{t+\eta}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg). \tag{25}$$

Thus, it follows that

$$\begin{aligned}
&E\left|\mathcal{T}u(t,x)-\mathcal{T}v(t,x)\right|^{2}\\
&\quad\le C\Bigg((r+1)\int_{0}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+r\int_{-r}^{0}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{t}(t-s)^{-1/\alpha}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{-r}^{0}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+r^{1-(1/\alpha)}\int_{t+\eta}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg).
\end{aligned} \tag{26}$$

Finally, direct computation implies that

$$\begin{aligned}
&\left|\mathcal{T}u(t,x)-\mathcal{T}v(t,x)\right|_{\lambda}^{2}\\
&\quad=\int_{0}^{T}e^{-\lambda t}\sup_{y\in\mathbb{R}}E\left|\mathcal{T}u(t,x)-\mathcal{T}v(t,x)\right|^{2}\\
&\quad\le C\int_{0}^{T}e^{-\lambda t}\Bigg((r+1)\\
&\qquad\times\int_{0}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+r\int_{-r}^{0}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{t}(t-s)^{-1/\alpha}\\
&\qquad\quad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{-r}^{0}(t-s)^{1-(1/\alpha)}\\
&\qquad\quad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{t+\eta}(t-s)^{1-(1/\alpha)}\\
&\qquad\quad\times\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+r^{1-(1/\alpha)}\\
&\qquad\times\int_{t+\eta}^{t}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg)dt\\
&\quad\le C\Bigg[\int_{0}^{\infty}e^{-\lambda t}dt\\
&\qquad\times\Bigg((r+1)\int_{0}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad\quad+r\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg)\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}t^{-1/\alpha}dt\\
&\qquad\times\int_{0}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}t^{1-(1/\alpha)}dt\\
&\qquad\times\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}t^{1-(1/\alpha)}dt\\
&\qquad\times\int_{0}^{T+\eta}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\int_{0}^{\infty}e^{-\lambda t}dt\times r^{1-(1/\alpha)}\\
&\qquad\times\int_{T+\eta}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\Bigg]\\
&\quad\le C\Bigg[\left(\frac{r+1}{\lambda}+\frac{\Gamma(1-(1/\alpha))}{\lambda^{1-(1/\alpha)}}\right)\\
&\qquad\times\int_{0}^{T}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\left(\frac{r}{\lambda}+\frac{\Gamma(2-(1/\alpha))}{\lambda^{2-(1/\alpha)}}\right)\\
&\qquad\times\int_{-r}^{0}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds\\
&\qquad+\frac{\Gamma(2-(1/\alpha))}{\lambda^{2-(1/\alpha)}}\\
&\qquad\times\int_{0}^{T+\eta}e^{-\lambda s}\sup_{y\in\mathbb{R}}E\left|u(s,y)-v(s,y)\right|^{2}ds
\end{aligned}$$

$$+\frac{r^{1-(1/\alpha)}}{\lambda}\int_{T+\eta}^{T} e^{-\lambda s}\sup_{y\in\mathbb{R}} E\left|u(s,y)-v(s,y)\right|^2 ds\Bigg]$$
$$\leq \frac{C_1}{\lambda^{K_1}}\left(\int_0^T e^{-\lambda s}\sup_{y\in\mathbb{R}} E\left|u(s,y)-v(s,y)\right|^2 ds\right.$$
$$\left.+\int_{-r}^{0} e^{-\lambda s}\sup_{y\in\mathbb{R}} E\left|u(s,y)-v(s,y)\right|^2 ds\right)$$
$$=\frac{C_1}{\lambda^{K_1}}|u-v|_\lambda^2, \tag{27}$$

where $C_1 = 2C\cdot\max\{r+1, \Gamma(1-(1/\alpha)), \Gamma(2-(1/\alpha)), r^{1-(1/\alpha)}\}$ and $K_1 = 1-(1/\alpha)$.

Let λ that large enough such that

$$\frac{C_1}{\lambda^{K_1}} < 1, \tag{28}$$

which implies that the operator $\mathcal{T} : \mathbb{B} \to \mathbb{B}$ is contraction. By the Banach fixed point theorem, there exists a unique fixed point in $\mathbb{B}$. Moreover, the fixed point is the unique mild solution of (8). □

Remark 5. If there are no delays, Theorem 3 can be solved in the following working function space:

$$|u|_\lambda^2 = \int_0^T e^{-\lambda t}\sup_{x\in\mathbb{R}} E|u(t,x)|^2 dt, \tag{29}$$

where $u \in \mathbb{B}$, which implies that the delays affect the working function space.

4. The Regularity of the Mild Solution

In this section, we will show the time regularity and space regularity of the mild solution for (8). In order to prove the regularity, we need the following assumptions:

(H3) there exists some $\gamma < 1/2$

$$\sup_{x\in\mathbb{R}} E\left(\left|u_0(x+z)-u_0(x)\right|^2\right) < c|z|^{2\gamma}; \tag{30}$$

(H4) for $\theta > 0$, let $|\phi(\eta, x)| < \infty$ and there exists some $\pi < 1$ such that

$$\sup_{x\in\mathbb{R}}\int_{-r}^{-\theta} E\left|u(\eta+\theta,x)-u(\eta,x)\right|^2 d\eta < c|\theta|^{2\pi}; \tag{31}$$

(H5) $|f(t_1, y, u_{t_1}) - f(t_2, y, u_{t_2})|^2 \leq c(|t_1 - t_2|^2 + |u(t_1) - u(t_2)|^2 + \int_{-r}^{0} |u(t_1+\eta) - u(t_2+\eta)|^2 d\eta)$.

To the end, we will give an important lemma from [7].

Lemma 6. (1) *For* $1 < n < \alpha+1$, $\int_0^\infty\int_{\mathbb{R}} |G_\alpha(1+v,z) - G_\alpha(v,z)|^n dz\, dv < \infty$.

(2) *For* $(\alpha+1)/2 < n < \alpha+1$, $\int_0^\infty\int_{\mathbb{R}} |G_\alpha(v,z+1) - G_\alpha(v,z)|^n dz\, dv < \infty$.

Theorem 7. *Assume that the conditions (H1)–(H5) are satisfied; then for* $\alpha \in (1,2]$ *and* $t \geq 0$, *there exists a continuous modification* $u(t,x)$, *which is* β*-Hölder continuous in* t, *where* $\beta = \min\{\gamma/\alpha, \pi, (1/2)-(1/2\alpha)\}$.

Proof. For $t \geq 0$, it follows that, for any $x \in \mathbb{R}$ and $\theta > 0$,

$$\begin{aligned}
&|u(t+\theta,x)-u(t,x)| \\
&\leq \left|\int_{\mathbb{R}} \left(G_\alpha(t+\theta,x-y)-G_\alpha(t,x-y)\right)u_0(y)\,dy\right| \\
&\quad+\left|\int_0^{t+\theta}\int_{\mathbb{R}} \left(G_\alpha(t+\theta-s,x-y)\right.\right. \\
&\qquad\left.\left.-G_\alpha(t-s,x-y)\right) f(u_s)\,dy\,ds\right| \\
&\quad+\left|\int_0^{t}\int_{\mathbb{R}} \left(G_\alpha(t+\theta-s,x-y)\right.\right. \\
&\qquad\left.\left.-G_\alpha(t-s,x-y)\right) h(u_s)\,W(ds\,dy)\right| \\
&\quad+\left|\int_t^{t+\theta}\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\,h(u_s)\,W(ds\,dy)\right| \\
&\quad+\left|\int_0^{t}\iint_{\mathbb{R}} \left(G_\alpha(t+\theta-s,x-y)-G_\alpha(t-s,x-y)\right)\right. \\
&\qquad\left.\times g(u_s)\,zN(ds,dz)\,dy\right| \\
&\quad+\left|\int_t^{t+\theta}\iint_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\,g(u_s)\,zN(ds,dz)\,dy\right| \\
&=\phi_\theta^0+\phi_\theta^1+\phi_\theta^2+\phi_\theta^3+\phi_\theta^4+\phi_\theta^5.
\end{aligned} \tag{32}$$

Next, we will estimate each term ϕ_θ^j $(j = 0,1,\ldots,5)$, respectively.

Combining Hölder inequality, Lemma 1 with (H3) yields

$$\begin{aligned}
&E\left|\phi_\theta^0\right|^2 \\
&=E\left|\int_{\mathbb{R}} G_\alpha(\theta,z)\left[\int_{\mathbb{R}} G_\alpha(t,x-y)\right.\right. \\
&\qquad\left.\left.\times\left(u_0(y-z)-u_0(y)\right)dy\right]dz\right|^2 \\
&\leq CE\left(\int_{\mathbb{R}} G_\alpha(\theta,z)\left[\int_{\mathbb{R}} G_\alpha(t,x-y)\right.\right. \\
&\qquad\left.\left.\times\left(u_0(y-z)-u_0(y)\right)dy\right]^2 dz\right) \\
&\quad\times\left(\int_{\mathbb{R}} G_\alpha(\theta,z)\,dz\right)
\end{aligned}$$

$$\begin{aligned}
&\le C\int_{\mathbb{R}} G_\alpha(\theta,z)\sup_{y\in\mathbb{R}}E\left|u_0(y-z)-u_0(y)\right|^2dz\\
&\le C\int_{\mathbb{R}} G_\alpha(\theta,z)\left|z\right|^{2\gamma}dz = C\int_{\mathbb{R}}\theta^{2\gamma/\alpha}G_\alpha(1,z)\,z^{2\gamma}dz\\
&\le C\theta^{2\gamma/\alpha}\int_{\mathbb{R}}\frac{\left|z\right|^{2\gamma}}{1+\left|z\right|^{1+\alpha}}dz \le C\theta^{2\gamma/\alpha}.
\end{aligned} \tag{33}$$

Next, we consider ϕ_θ^1. Let $s' = s-\theta$; then,

$$\begin{aligned}
\left|\phi_\theta^1\right| \le{}& \left|\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\right.\\
&\left.\times\left(f(s+\theta,y,u_{s+\theta})-f(s,y,u_s)\right)dy\,ds\right|\\
&+\left|\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\right.\\
&\left.\times f(s,y,u_s)\,dy\,ds\right|\\
={}& \phi_\theta^{1.1}+\phi_\theta^{1.2}.
\end{aligned} \tag{34}$$

By Hölder inequality, Lemma 1, (H4), and (H5), we have

$$\begin{aligned}
&E\left|\phi_\theta^{1.1}\right|^2\\
&\le C\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\,dy\,ds\\
&\quad\times\int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\\
&\qquad\times E\left|f(s+\theta,y,u_{s+\theta})-f(s,y,u_s)\right|^2dy\,ds\\
&\le C\int_0^t\left(\theta^2+\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2\right.\\
&\qquad+\int_{-r}^0\sup_{y\in\mathbb{R}}E\left|u(s+\theta+\eta,y)\right.\\
&\qquad\left.\left.-u(s+\eta,y)\right|^2d\eta\right)ds\\
&\le C\left(T\theta^2+(r+1)\right.\\
&\qquad\times\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2ds\\
&\qquad+r\int_{-r}^{-\theta}\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2ds\\
&\qquad\left.+\int_{-\theta}^0\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2ds\right)\\
&\le C\left(\theta^2+\theta^{2\pi}+\theta\right.\\
&\qquad\left.+\int_0^t\sup_{y\in\mathbb{R}}E\left|u(s+\theta,y)-u(s,y)\right|^2ds\right),
\end{aligned}$$

$$\begin{aligned}
&E\left|\phi_\theta^{1.2}\right|^2\\
&\le C\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\,dy\,ds\\
&\quad\times\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\,E\left|f(u_s)\right|^2dy\,ds\\
&\le C\theta\int_0^\theta\int_{\mathbb{R}} G_\alpha(t+\theta-s,x-y)\,E\left|f(u_s)\right|^2dy\,ds\\
&\le C\theta\int_0^\theta\sup_{y\in\mathbb{R}}E\left(\left|u(s,y)\right|^2+\int_{-r}^0\left|u(s+\eta,y)\right|^2d\eta\right)ds\\
&\le C\theta\left((r+1)\int_0^\theta\sup_{y\in\mathbb{R}}E\left|u(s,y)\right|^2ds\right.\\
&\qquad\left.+r\int_{-r}^0\sup_{y\in\mathbb{R}}E\left|u(s,y)\right|^2ds\right)\\
&\le C\left(\theta^2+\theta\right).
\end{aligned} \tag{35}$$

Taking the transformation $s=\theta v$, $y=\theta^{1/\alpha}z$, by Burkholder-Davis-Gundy inequality, Lemmas 1 and 6, and (H2), we obtain

$$\begin{aligned}
&E\left|\phi_\theta^2\right|^2\\
&\le C\int_0^t\int_{\mathbb{R}} E\left(G_\alpha(t+\theta-s,x-y)\right.\\
&\qquad\left.-G_\alpha(t-s,x-y)\right)^2h^2(u_s)\,ds\,dy\\
&\le CK\int_0^t\int_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)\right.\\
&\qquad\left.-G_\alpha(t-s,x-y)\right)^2\\
&\qquad\times E\left(\left|u(s)\right|^2+\int_{-r}^0\left|u(s+\eta)\right|^2d\eta\right)dy\,ds\\
&\le C\int_0^t\int_{\mathbb{R}}\left(G_\alpha(t+\theta-s,x-y)-G_\alpha(t-s,x-y)\right)^2\\
&\qquad\times E\left(\left|u(s)\right|^2+\int_{-r}^0\left|u(\eta)\right|^2d\eta\right.\\
&\qquad\left.+\int_0^t\left|u(\eta')\right|^2d\eta'\right)dy\,ds\\
&\le C\left((T+1)\sup_{[0,T]\times\mathbb{R}}E\left|u(s,y)\right|^2+\sup_{y\in\mathbb{R}}\int_{-r}^0\left|u(\eta,y)\right|^2d\eta\right)\\
&\quad\times\int_0^t\int_{\mathbb{R}}\left(G_\alpha(s+\theta,y)-G_\alpha(s,y)\right)^2dy\,ds\\
&\le C\left(\theta^{-1/\alpha}\theta\int_0^\infty\int_{\mathbb{R}}\left(G_\alpha(v+1,z)-G_\alpha(v,z)\right)^2dz\,dv\right)\\
&\le C\theta^{1-(1/\alpha)},
\end{aligned}$$

$$
\begin{aligned}
&E\left|\phi_\theta^3\right|^2 \\
&\quad\le C\int_t^{t+\theta}\int_{\mathbb{R}} E\left(G_\alpha\left(t+\theta-s,x-y\right)\right. \\
&\qquad\qquad \left.-G_\alpha\left(t-s,x-y\right)\right)^2\left|h\left(u_s\right)\right|^2 ds\,dy \\
&\quad\le C\int_t^{t+\theta}\int_{\mathbb{R}} E\left(G_\alpha\left(t+\theta-s,x-y\right)\right. \\
&\qquad\qquad \left.-G_\alpha\left(t-s,x-y\right)\right)^2 \\
&\qquad\qquad \times\left(\left|u\left(s,y\right)\right|^2+\int_{-r}^0\left|u\left(s+\eta\right)\right|^2 d\eta\right) dy\,ds \\
&\quad\le C\int_t^{t+\theta}\int_{\mathbb{R}} E\left(G_\alpha\left(t+\theta-s,x-y\right)\right. \\
&\qquad\qquad \left.-G_\alpha\left(t-s,x-y\right)\right)^2 \\
&\qquad \times\left(\left|u\left(s,y\right)\right|^2+\int_{-r}^{t+\theta}\left|u\left(\eta'\right)\right|^2 d\eta'\right) dy\,ds \\
&\quad\le C\left((T+1+\theta)\sup_{[0,T+\theta]\times\mathbb{R}} E\left|u\left(s,y\right)\right|^2\right. \\
&\qquad\qquad \left.+\sup_{y\in\mathbb{R}}\int_{-r}^0\left|u\left(\eta,y\right)\right|^2 d\eta\right) \\
&\qquad \times\int_0^\theta\int_{\mathbb{R}} E\left(G_\alpha\left(s+\theta,y\right)-G_\alpha\left(s,y\right)\right)^2 dy\,ds \\
&\quad\le C\left(1+\theta\right)\theta^{1-(1/\alpha)} \\
&\qquad \times\int_0^1\int_{\mathbb{R}} E\left(G_\alpha\left(1+v,z\right)-G_\alpha\left(v,z\right)\right)^2 dz\,dv \\
&\quad\le C\left(\theta^{1-(1/\alpha)}+\theta^{2-(1/\alpha)}\right).
\end{aligned}
\tag{36}
$$

Then, by the same method, we have

$$
\begin{aligned}
E\left|\phi_\theta^4\right|^2 &\le \int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t+\theta-s,x-y\right)-G_\alpha\left(t-s,x-y\right)\right)^2 \\
&\qquad\qquad \times E\left(\int_{\mathbb{R}}\left|g\left(u_s\right)\right|\left|z\right|\nu\left(dz\right)\right)^2 ds\,dy \\
&\le CK\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t+\theta-s,x-y\right)\right. \\
&\qquad\qquad \left.-G_\alpha\left(t-s,x-y\right)\right)^2 \\
&\qquad\qquad \times E\left(\left|u\left(s\right)\right|^2+\int_{-r}^0\left|u\left(s+\eta\right)\right|^2 d\eta\right) dy\,ds \\
&\le C\theta^{1-(1/\alpha)}, \\
E\left|\phi_\theta^5\right|^2 &\le C\left(\theta^{1-(1/\alpha)}+\theta^{2-(1/\alpha)}\right).
\end{aligned}
\tag{37}
$$

Thus, from the previous estimates, let $\beta=\min\{\gamma/\alpha,\pi,(1/2)-(1/2\alpha)\}$:

$$
\begin{aligned}
&E\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|^2 \\
&\quad\le C\left[\theta^{2\beta}+\int_0^t\sup_{y\in\mathbb{R}}E\left|u\left(s+\theta,y\right)-u\left(s,y\right)\right|^2 ds\right].
\end{aligned}
\tag{38}
$$

Hence, it follows from Gronwall's Lemma that

$$
E\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|^2\le C\theta^{2\beta}. \tag{39}
$$

Then, for $t\ge 0$, we have

$$
\begin{aligned}
&\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|_\lambda^p \\
&\quad=\left(\int_0^T e^{-\lambda t}\sup_{x\in\mathbb{R}}E\left|u\left(t+\theta,x\right)-u\left(t,x\right)\right|^2 ds\right)^{p/2} \\
&\quad\le C\theta^{\beta p}.
\end{aligned}
\tag{40}
$$

□

Finally, we study the space regularity of the mild solution for (8).

Theorem 8. *Assume that the conditions (H1)–(H3) are satisfied; then for $\alpha\in(1,2]$ and $t\ge 0$, there exists a continuous modification $u(t,x)$, which is ρ-Hölder continuous in x, where $\rho=\min\{\gamma,\vartheta,\alpha-1\}$.*

Proof. It follows that, for any $t\in[0,T]$ and $\zeta>0$,

$$
\begin{aligned}
&\left|u\left(t,x+\zeta\right)-u\left(t,x\right)\right| \\
&\quad\le\left|\int_{\mathbb{R}}\left(G_\alpha\left(t,x+\zeta-y\right)-G_\alpha\left(t,x-y\right)\right)u_0\left(y\right)dy\right| \\
&\qquad+\left|\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\right. \\
&\qquad\qquad\left.\left.-G_\alpha\left(t-s,x-y\right)\right)f\left(u_s\right)dyds\right| \\
&\qquad+\left|\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\right. \\
&\qquad\qquad\left.\left.-G_\alpha\left(t-s,x-y\right)\right)h\left(u_s\right)W\left(dsdy\right)\right| \\
&\qquad+\left|\int_0^t\iint_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\right. \\
&\qquad\qquad\left.\left.-G_\alpha\left(t-s,x-y\right)\right)g\left(u_s\right)zN\left(ds,dz\right)dy\right| \\
&\quad:=\sum_{j=0}^3\phi_\zeta^j.
\end{aligned}
\tag{41}
$$

By (H3) and Lemma 1, we have

$$\begin{aligned} E\left|\phi_\zeta^0\right|^2 &= E\left|\int_{\mathbb{R}} G_\alpha(t,x-y)\left[u_0(y+\zeta)-u_0(y)\right]dy\right|^2 \\ &\le \sup_{y\in\mathbb{R}} E\left(\left|u_0(y+\zeta)-u_0(y)\right|^2\right) \\ &\quad\times\left(\int_{\mathbb{R}} G_\alpha(t,x-y)\,dy\right)^2 \\ &\le C\zeta^{2\gamma}. \end{aligned} \tag{42}$$

By (H2), Hölder's inequality and Lemma 1, we set $\epsilon = (1/2)(\alpha+1)-\delta$ (δ is small enough) and $\vartheta\in(0,1)$ and we can derive

$$\begin{aligned} &E\left|\phi_\zeta^1\right|^2 \\ &\le C\left((1+T)\sup_{[0,T]\times\mathbb{R}} E|u(t,x)|^2+\sup_{y\in\mathbb{R}}\int_{-r}^0\left|u(\eta,y)\right|^2 d\eta\right) \\ &\quad\times\left|\int_0^t\int_{\mathbb{R}}\left(G_\alpha(s,y+\zeta)-G_\alpha(s,y)\right)dy\,ds\right|^2 \\ &\le C\left|\int_0^t\int_{\mathbb{R}} s^{-1/\alpha}\left(G_\alpha\left(1,s^{-1/\alpha}(y+\zeta)\right)\right.\right. \\ &\qquad\left.\left.-G_\alpha(1,s^{-1/\alpha}y)\right)dy\,ds\right|^2 \\ &= C\left|\int_0^t\int_{\mathbb{R}} s^{-\epsilon/\alpha}\left(G_\alpha\left(1,s^{-1/\alpha}(y+\zeta)\right)\right.\right. \\ &\qquad\left.-G_\alpha\left(1,s^{-1/\alpha}y\right)\right)^{(1-\vartheta)} \\ &\qquad\times s^{-((1-\epsilon)/\alpha)}\left(G_\alpha\left(1,s^{-1/\alpha}(y+\zeta)\right)\right. \\ &\qquad\left.\left.-G_\alpha\left(1,s^{-1/\alpha}y\right)\right)^{\vartheta}dy\,ds\right|^2 \\ &\le C\left(\int_0^t\int_{\mathbb{R}} s^{-2\epsilon/\alpha}\left|G_\alpha\left(1,s^{-1/\alpha}(y+\zeta)\right)\right.\right. \\ &\qquad\left.\left.-G_\alpha\left(1,s^{-1/\alpha}y\right)\right|^{2(1-\vartheta)}dy\,ds\right) \\ &\quad\times\left(\int_0^t\int_{\mathbb{R}} s^{-2(1-\epsilon)/\alpha}\left|G_\alpha\left(1,s^{-1/\alpha}(y+\zeta)\right)\right.\right. \\ &\qquad\left.\left.-G_\alpha\left(1,s^{-1/\alpha}y\right)\right|^{2\vartheta}dy\,ds\right) \\ &:= cI\times II. \end{aligned} \tag{43}$$

By Lemma 1, recall that $2\epsilon<\alpha+1$, and we have

$$\begin{aligned} I &\le C\left(\int_0^t\int_{\mathbb{R}} s^{-(2\epsilon-1)/\alpha}\left(G_\alpha(1,y)\right)^{2(1-\vartheta)}dy\,ds\right) \\ &\le C\int_0^t s^{-(2\epsilon-1)/\alpha}ds<\infty. \end{aligned} \tag{44}$$

Then, by the mean value of the theorem and Lemma 1, we have that

$$\begin{aligned} II &= \int_0^t\int_{\mathbb{R}} s^{-2(1-\epsilon)/\alpha}\left(\frac{\partial}{\partial y}G_\alpha\left(1,s^{-1/\alpha}(y+\varepsilon)\right)\right. \\ &\qquad\left.\times s^{-1/\alpha}\zeta\right)^{2\vartheta}dy\,ds \quad (\varepsilon\in(0,\zeta)) \\ &= \zeta^{2\vartheta}\int_0^t\int_{\mathbb{R}} s^{-(2(1-\epsilon+\vartheta)-1)/\alpha} \\ &\qquad\times\left(\frac{\partial}{\partial y}G_\alpha(1,y)\right)^{2\vartheta}dy\,ds \\ &\le C\zeta^{2\vartheta}\int_0^t s^{-(2(1-\epsilon+\vartheta)-1)/\alpha}ds. \end{aligned} \tag{45}$$

Choosing $\vartheta<\alpha-\delta$ such that $(2(1-\epsilon+\vartheta)-1)/\alpha<1$, then

$$II<C\zeta^{2\vartheta}. \tag{46}$$

Combining (44) and (46), we have

$$E\left|\phi_\zeta^1\right|^2\le C\zeta^{2\vartheta}. \tag{47}$$

Taking the change of variable $s=\zeta^\alpha v$, $y=z\zeta$, and by Burkholder-Davis-Gundy inequality, Lemmas 1 and 6, (H1), and (H2), we derive

$$\begin{aligned} E\left|\phi_\zeta^2\right|^2 &\le \left(\int_0^t\int_{\mathbb{R}} E\left(G_\alpha(t-s,x+\zeta-y)\right.\right. \\ &\qquad\left.-G_\alpha(t-s,x-y)\right)^2 \\ &\qquad\left.\times E\left|h(u_s)\right|^2 dy\,ds\right) \\ &\le C\left((T+1)\sup_{[0,T]\times\mathbb{R}} E|u(t,x)|^2\right. \\ &\qquad\left.+\sup_{y\in\mathbb{R}}\int_{-r}^0\left|u(\eta,y)\right|^2 d\eta\right) \\ &\quad\times\left(\int_0^t\int_{\mathbb{R}}\left(G_\alpha(s,y+\zeta)-G_\alpha(s,y)\right)^2 dy\,ds\right) \\ &\le C\left(\zeta^{-2}\zeta\zeta^\alpha\int_0^\infty\int_{\mathbb{R}}\left(G_\alpha(v,z+1)\right.\right. \\ &\qquad\left.\left.-G_\alpha(v,z)\right)^2 dz\,dv\right) \\ &\le C\zeta^{2(\alpha-1)}, \end{aligned}$$

$$\begin{aligned} E\left|\phi_\zeta^3\right|^2 &\le C\int_0^t\int_{\mathbb{R}}\left(G_\alpha\left(t-s,x+\zeta-y\right)\right.\\ &\qquad\left.-G_\alpha\left(t-s,x-y\right)\right)^2\\ &\qquad\times E\left(\int_{\mathbb{R}}\left|g\left(u_s\right)\right|z\nu\left(dz\right)\right)^2 ds\,dy\\ &\le C\zeta^{2(\alpha-1)}. \end{aligned} \tag{48}$$

Combining (42)–(48), we have

$$E\left|u\left(t,x+\zeta\right)-u\left(t,x\right)\right|^2 \le C\left(\zeta^{2\gamma}+\zeta^{2\vartheta}+\zeta^{2(\alpha-1)}\right). \tag{49}$$

Then, we have, for $t\in[0,T]$,

$$\left|u\left(t,x+\zeta\right)-u\left(t,x\right)\right|_\lambda^p \le C\left(\zeta^{\gamma p}+\zeta^{\vartheta p}+\zeta^{(\alpha-1)p}\right)\le C\zeta^{\rho p}, \tag{50}$$

where $\rho=\min\{\gamma,\vartheta,\alpha-1\}$. □

Remark 9. Theorems 7 and 8 show that the regularity of initial value and the order of fractional operator can affect both time regularity and space regularity of the mild solution for (1). In particular, the time regularity is affected by the regularity of initial value with delays.

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Research Article

A Half-Inverse Problem for Impulsive Dirac Operator with Discontinuous Coefficient

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An inverse problem for Dirac differential operators with discontinuity conditions and discontinuous coefficient is studied. It is shown by Hochstadt and Lieberman's method that if the potential function $p(x)$ in $\Omega(x)$ is prescribed over the interval $(\pi/2, \pi)$, then a single spectrum suffices to determine $p(x)$ on the interval $(0, \pi)$ and it is also shown here that $\rho(x)$ is uniquely determined by a spectrum.

1. Introduction

In this paper, we are concerned with the Dirac operator L generated by equation

$$\ell(y) := By'(x) + \Omega(x)\, y(x) = \lambda \rho(x)\, y(x), \qquad x \in I := \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \tag{1}$$

with

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \qquad \rho(x) = \begin{cases} 1, & 0 \le x < \dfrac{\pi}{2} \\ \alpha, & \dfrac{\pi}{2} < x \le \pi, \end{cases} \tag{2}$$

where $1 < \alpha \in \mathbb{R}^+$, $y(x) = \left(\begin{smallmatrix} y_1(x) \\ y_2(x) \end{smallmatrix}\right)$, subject to the boundary conditions

$$y_1(0) = 0, \tag{3}$$

$$y_2(\pi) = 0, \tag{4}$$

and discontinuity conditions

$$y\left(\frac{\pi}{2} + 0\right) = Ay\left(\frac{\pi}{2} - 0\right), \tag{5}$$

where $p(x)$ and $q(x)$ are real valued functions in $L_2(0, \pi)$, λ is a spectral parameter, and $A = \left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{smallmatrix}\right)$, $\beta \in \mathbb{R}^+ \setminus \{1\}$.

Inverse problems of spectral analysis implicate the reconstruction of a linear operator from its spectral data (e.g., see [1–5]). Half inverse problem for a Dirac operator consists in reconstruction of this operator from its spectrum and half of the potential.

The first result on the half-inverse problem is due to Hochstadt and Lieberman [6]. After that, Hald [7] proved that if the potential is known over half of the interval and if one boundary conditions is given, then the potential and the other boundary condition are uniquely determined by the eigenvalues. In [8, 9], Malamud and Gesztesy and Simon obtained some new uniqueness results in inverse spectral analysis with partial information on the potential for scalar and matrix Sturm-Liouville equations, respectively. In 2001, Sakhnovich [10] studied the existence of solution to the half-inverse problem. In [11], necessary and sufficient condition for solvability of the half-inverse spectral problem for Sturm-Liouville operators with singular operator was taken. In [12], by using the Hochstadt and Lieberman's method, half-inverse problem was solved for diffusion operators. In [13], the authors presented half-inverse problem for the Sturm-Liouville equation with eigenparameter-dependent boundary conditions.

On the other hand, the fundamental and detailed results about Dirac operators were given in [14]. Moreover, direct or

inverse spectral problems for Dirac operators are largely well studied in [1, 8, 15].

There are also some studies about the interior inverse problems. Arutyunyan [16] proved that the eigenvalues λ_n, $n = 0, 1, \ldots$ and normalizing coefficients $\alpha_n = \|y_n\|_{\{L^2(0,1)\}^2}$, $n = 0, 1, \ldots$ uniquely determined the potential $Q(x)$. Malamud [8] proved an analog of Borg theorem [17]; he showed that the spectra of two boundary value problems for an operator with different boundary conditions at one end uniquely determined the potential $Q(x)$. He also proved an analog of the theorem of Hochstadt and Lieberman [6]; one spectrum and a potential on the interval $(0, 1/2)$ uniquely determined the potential $Q(x)$ on the whole interval $[0, 1]$. On the other hand, in 2001, Del Rio and Grébert [18] proved that in the case where φ is a priori known on $[a, 1]$, then only a part (depending on a) of two spectra determined φ on $[0, 1]$. Furthermore, inverse problems for interior spectral data of the Sturm-Liouville and Dirac operators were studied in [19–22].

The jump conditions like (5) appear in-some important physical problems. The work in [7] is a well-known work about discontinuous inverse eigenvalue problems. Direct and inverse problems for Dirac operators with discontinuities inside the interval were investigated in [23].

In this paper, by using the Hochstadt and Lieberman's method in [6], we discuss the half-inverse problem for Dirac operator with discontinuity conditions and discontinuous coefficients (1)–(5). Furthermore, the potential $p(x)$ and discontinuous coefficient $\rho(x)$ are uniquely determined.

2. Statement of Results

Let the function $\varphi(\cdot, \lambda) : I \to R^2$ be solution of (1) which satisfies the initial conditions

$$\varphi(0, \lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{6}$$

and the jump conditions (5).

It is shown in [14] that, for the solution $\varphi(x, \lambda)$, the following representation holds:

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x K(x, t)\, \varphi_0(t, \lambda)\, dt, \quad \text{for } 0 < x < \frac{\pi}{2}, \tag{7}$$

where $\varphi_0(x, \lambda) = (\varphi_{01}(x, \lambda), \varphi_{02}(x, \lambda))^T$ has the form

$$\varphi_{01}(x, \lambda) = \begin{cases} \sin \lambda x, & 0 \le x < \dfrac{\pi}{2} \\ \beta^+ \sin \lambda \mu(x) & \\ \quad + \beta^- \sin \lambda (\pi - \mu(x)), & \dfrac{\pi}{2} < x \le \pi, \end{cases} \tag{8}$$

$$\varphi_{02}(x, \lambda) = \begin{cases} -\cos \lambda x, & 0 \le x < \dfrac{\pi}{2} \\ -\beta^+ \cos \lambda \mu(x) & \\ \quad + \beta^- \cos \lambda (\pi - \mu(x)), & \dfrac{\pi}{2} < x \le \pi, \end{cases} \tag{9}$$

$\beta^{\pm} = (1/2)(\beta \pm \beta)$, $K(x, t) = (K_{ij}(x, t))_{i,j=1,2}$, and $K_{ij}(x, t)$ are real valued continuous functions for $i, j = 1, 2$ and for each

$$x, \ \mu(x) = \begin{cases} x, & 0 \le x < \pi/2 \\ \alpha x - \alpha(\pi/2) + (\pi/2), & \pi/2 < x \le \pi. \end{cases} \tag{10}$$

Next, we define the function

$$\Delta(\lambda) = \varphi_2(\pi, \lambda). \tag{11}$$

The zeros of $\Delta(\lambda)$ which is called the characteristic function of the problem (1)–(5) are the eigenvalues of L.

From the equalities (7)–(11), we have

$$\Delta(\lambda) = \Delta_0(\lambda) + o(\exp \tau\mu(\pi)), \tag{12}$$

where $\Delta_0(\lambda) = -\beta^+ \cos \lambda\mu(\pi) + \beta^- \cos \lambda(\pi - \mu(\pi))$ and $\tau = |\operatorname{Im} \lambda|$.

Theorem 1. *(i) The problem L has denumerable many eigenvalues such that all of them are real and simple.*

(ii) The eigenvalues λ_n are expressed by the following asymptotic formula:

$$\lambda_n = \lambda_n^0 + O(1), \tag{13}$$

where λ_n^0 are the zeros of $\Delta_0(\lambda)$ and $\lambda_n^0 = n\pi/\mu(\pi) + h_n$, $\sup_n |h_n| < \infty$.

Proof. (i) Since $\Delta(\lambda)$ is entire function, it has denumerable many zeros. Moreover, from [23], zeros of $\{\lambda_n\}$ are real and simple.

(ii) It is shown in [24] that $\lambda_n^0 = n\pi/\mu(\pi) + O(1)$. It is obvious that $|\Delta_0(\lambda)| \ge C_\delta \exp \tau\mu(\pi)$ for $\lambda \in G_\delta := \{\lambda : |\lambda - \lambda_n| > \delta\}$ and $\Delta(\lambda) - \Delta_0(\lambda) = o(\exp \tau\mu(\pi))$. □

Therefore, it follows from the Rouche's theorem that the functions $\Delta_0(\lambda)$ and $\Delta(\lambda)$ have the same number of zeros inside the contour $\gamma_\varepsilon := \{\lambda : |\lambda| = |\lambda_n^0| - \varepsilon\}$; that is, the eigenvalues λ_n are given by the following asymptotic formula:

$$\lambda_n = \frac{n\pi}{\mu(\pi)} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{14}$$

Consider a second operator $\widetilde{L}$ generated by the differential equation

$$\begin{gathered} By'(x) + \widetilde{\Omega}(x)\, y(x) = \lambda \widetilde{\rho}(x)\, y(x), \\ x \in I := \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \end{gathered} \tag{15}$$

subject to the same boundary conditions (3) and (4) and discontinuity condition (5). Here, $\widetilde{\Omega}(x) = \begin{pmatrix} \widetilde{p}(x) & q(x) \\ q(x) & -\widetilde{p}(x) \end{pmatrix}$ with a real valued function $\widetilde{p}(x) \in L_2(0, \pi)$.

We denote eigenvalues by λ_n and the corresponding eigenfunctions by $\varphi_n(x) = \varphi(x, \lambda_n)$ of the problem L and denote eigenvalues by $\widetilde{\lambda}_n$ and the corresponding eigenfunctions by $\widetilde{\varphi}_n(x) = \varphi(x, \widetilde{\lambda}_n)$ of the problem $\widetilde{L}$.

Lemma 2. *If $\lambda_n = \tilde{\lambda}_n$, then $\alpha = \tilde{\alpha}$ that is, $\rho(x) = \tilde{\rho}(x)$.*

Proof. Since $\lambda_n = (n\pi/\mu(\pi))(1 + O(1/n))$, then $\tilde{\lambda}_n = (n\pi/\tilde{\mu}(\pi))(1+O(1/n)), (n\pi/\mu(\pi))(1+O(1/n)) = (n\pi/\tilde{\mu}(\pi))(1+O(1/n))$. Letting $n \to \infty$, then we conclude that $\mu(\pi) = \tilde{\mu}(\pi)$. Moreover, since $\mu(\pi) = (\pi/2)(\alpha + 1)$, then $\alpha = \tilde{\alpha}$. So $\rho(x) = \tilde{\rho}(x)$. □

Theorem 3. *If $\lambda_n = \tilde{\lambda}_n$, for all $n \in \mathbb{N}$ and $p(x) = \tilde{p}(x)$, for $x \in (\pi/2, \pi)$, then $p(x) = \tilde{p}(x)$ almost everywhere on $(0, \pi)$.*

Proof. Let us write (1) for the solutions φ and $\tilde{\varphi}$ and take into account Lemma 2 as

$$\begin{aligned} B\varphi'(x,\lambda) + \Omega(x)\varphi(x,\lambda) &= \lambda\rho(x)\varphi(x,\lambda), \\ B\tilde{\varphi}'(x,\lambda) + \tilde{\Omega}(x)\tilde{\varphi}(x,\lambda) &= \lambda\rho(x)\tilde{\varphi}(x,\lambda). \end{aligned} \tag{16}$$

If we multiply these equalities by $\tilde{\varphi}(x,\lambda)$ and $\varphi(x,\lambda)$, respectively, and subtract, then we obtain

$$\begin{aligned} &\frac{d}{dx}\{\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\} \\ &\quad = \left[\Omega(x) - \tilde{\Omega}(x)\right]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda). \end{aligned} \tag{17}$$

Integrating the last equality from 0 to π with respect to x, the equality

$$\begin{aligned} &\{\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\}\Big|_0^\pi \\ &\quad = \int_0^\pi \left[\Omega(x) - \tilde{\Omega}(x)\right]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx \\ &\quad = \int_0^\pi \left[p(x) - \tilde{p}(x)\right] J\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx \end{aligned} \tag{18}$$

is obtained where $J := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Applying the initial condition (6) and the assumption $p(x) = \tilde{p}(x)$, $x \in (\pi/2, \pi)$ in hypothesis, we get

$$\begin{aligned} &\{\varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\}\Big|_0^\pi \\ &\quad = \int_0^{\pi/2} \left[\Omega(x) - \tilde{\Omega}(x)\right]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx \\ &\quad = \int_0^{\pi/2} \left[p(x) - \tilde{p}(x)\right] J\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx. \end{aligned} \tag{19}$$

Define

$$F(\lambda) := \int_0^{\pi/2} \left[p(x) - \tilde{p}(x)\right] J\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx, \tag{20}$$

where

$$\begin{aligned} J\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) &= -\cos 2\lambda x + \int_0^x K_1(x,t)\cos 2\lambda t\,dt \\ &\quad + \int_0^x K_2(x,t)\sin 2\lambda t\,dt \end{aligned} \tag{21}$$

and $K_i(x,t)$, $i = 1, 2$ depend only on x, t.

Then we get from the boundary condition (4) that

$$F(\lambda_n) = 0 \tag{22}$$

for all n.

Now, define

$$\chi(\lambda) := \frac{F(\lambda)}{\Delta(\lambda)}. \tag{23}$$

$\chi(\lambda)$ is an entire function. Since $F(\lambda) = O(\exp \tau\pi)$ and $|\Delta(\lambda)| \geq C_\delta \exp \tau\mu(\pi)$ for $\lambda \in G_\delta := \{\lambda : |\lambda - \lambda_n| > \delta\}$ where $\mu(\pi) = (\pi/2)(\alpha + 1)$, then $\chi(\lambda)$ is constant from the Liouville's theorem. Furthermore,

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \chi(\lambda) = 0 \tag{24}$$

from the equalities (7), (9), and (21) and the Riemann-Lebesgue lemma. Thus, $\chi(\lambda) = 0$ on the whole λ-plane.

It follows from (20) and (21) that

$$\begin{aligned} &\int_0^{\pi/2} Q(x)\left\{\cos 2\lambda x - \int_0^x K_1(x,t)\cos 2\lambda t\,dt \right. \\ &\quad \left. - \int_0^x K_2(x,t)\sin 2\lambda t\,dt\right\} dx = 0 \end{aligned} \tag{25}$$

for all λ where $Q(x) := [p(x) - \tilde{p}(x)]$. This can be rewritten as

$$\begin{aligned} &\int_0^{\pi/2} \cos 2\lambda\tau \left[Q(\tau) + \int_\tau^{\pi/2} Q(x)K_1(x,t)\,dx\right] dt \\ &\quad + \int_0^{\pi/2} \sin 2\lambda t \int_\tau^{\pi/2} Q(x)K_2(x,t)\,dx\,dt = 0. \end{aligned} \tag{26}$$

From the completeness of the functions $(\cos 2\lambda\tau, \sin 2\lambda t)^T$ in $L_2(0,\pi) \oplus L_2(0,\pi)$, we have

$$Q(\tau) + \int_\tau^{\pi/2} Q(x)K_1(x,t)\,dx = 0, \quad \text{for } 0 < \tau < \frac{\pi}{2}. \tag{27}$$

It follows that $Q(x) = 0$; that is, $p(x) = \tilde{p}(x)$ almost everywhere for $x \in (0, \pi)$. □

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Research Article

Duffing-Type Oscillator with a Bounded from above Potential in the Presence of Saddle-Center Bifurcation and Singular Perturbation: Frequency Control

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We analyze the dynamics of the forced singularly perturbed differential equations of Duffing's type with a potential that is bounded from above. We explain the appearance of the large frequency nonlinear oscillations of the solutions. It is shown that the frequency can be controlled by a small parameter at the highest derivative.

1. Introduction

Duffing's equation is regarded as one of the most important differential equations because it appears in various physical and engineering problems. For example, the periodically forced Duffing oscillator

$$y'' + \delta y' + \alpha y + \beta y^3 = \gamma \cos \omega t \tag{1}$$

exhibits a wide variety of interesting phenomena which are fundamental to the behavior of nonlinear dynamical systems, such as regular and chaotic motions (see, e.g., [1–5] and the references therein; we also refer to the classical book of Nayfeh and Mook [6]). In this context, usually two-well potential of an unperturbed system was considered ($\beta > 0$; see also [7–9]) by using analytical methods and numerical simulations.

In this case, the undamped ($\delta = 0$) and unperturbed ($\gamma = 0$) Duffing's oscillator can basically exhibit two distinct types of steady-state oscillations, namely,

(i) in-well, small orbit dynamics, where the system state remains within the potential well centred at a stable equilibrium point (center);

(ii) cross-well, large orbit dynamics, whose trajectories surround the three equilibrium points (saddle between two centers).

In both cases, under periodic external excitation, a chaotic motion can be observed when the control parameters are changed.

On the contrary, in this paper we consider that $\beta < 0$ and a potential tends to $-\infty$ for $|y| \to \infty$, so the object can escape to infinity because of the bounded from above potential. There exist several atomic or subatomic situations in quantum physics where the total energy governing the particles contains an approximately square-well potential which is bounded from above; see, for example, [10] and discussion in [11]. For example, recently it has been found that the meson spectroscopy is better described by "confining" potential which is bounded from above; for details and references, see [10].

This paper concentrates on the mathematical aspects of systems with a potential that is bounded from above; more concretely, we focus our attention on the existence of nonlinear oscillations in the context of saddle-center bifurcation in the dynamical system describing the singularly

perturbed forced oscillator of Duffing's type with a nonlinear restoring and a nonperiodic external driving force

$$\epsilon^2\left(a^2(t)\,y'\right)' + f(y) = m(t), \quad 0 < \epsilon \ll 1 \tag{2}$$

or rewriting to an equivalent set of the three first-order autonomous equations:

$$\begin{aligned} \epsilon y' &= \frac{w}{a(t)}, \\ \epsilon w' &= \frac{m(t)}{a(t)} - \frac{f(y)}{a(t)} - \epsilon\frac{a'(t)}{a(t)}w, \\ t' &= 1 \end{aligned} \tag{3}$$

with potential with two-barrier structure. Here a and m are the C^1 functions on the interval $\langle t_B, t_E\rangle$, a is positive and f is a C^1 function on $\mathbb{R}$.

We show that the singular perturbation parameter ϵ play role modeling tool for the frequency control of the nonlinear oscillations arising in these systems (relationship (33)). Finally we prove that under some assumptions the solutions of (2) will rapidly oscillate, with the frequency of the oscillations increasing unboundedly as $\epsilon \to 0^+$.

System (3) is an example of a singularly perturbed system, because in the limit $\epsilon \to 0^+$, it does not reduce to a differential equation of the same type, but to an algebraic-differential reduced system:

$$\begin{aligned} 0 &= \frac{w}{a(t)}, \\ 0 &= \frac{m(t)}{a(t)} - \frac{f(y)}{a(t)}, \\ t' &= 1. \end{aligned} \tag{4}$$

Another way to study the singular limit $\epsilon \to 0^+$ is by introducing the new independent variable $\tau = t/\epsilon$ which transforms (3) to the system

$$\begin{aligned} \frac{dy}{d\tau} &= \frac{w}{a(t)}, \\ \frac{dw}{d\tau} &= \frac{m(t)}{a(t)} - \frac{f(y)}{a(t)} - \epsilon\frac{a'(t)}{a(t)}w, \\ \frac{dt}{d\tau} &= \epsilon. \end{aligned} \tag{5}$$

Taking the limit $\epsilon \to 0^+$, we obtain the so-called associated system ([12])

$$\frac{dy}{d\tau} = \frac{w}{a(t)}, \tag{6}$$

$$\frac{dw}{d\tau} = \frac{m(t)}{a(t)} - \frac{f(y)}{a(t)}, \tag{7}$$

$$\frac{dt}{d\tau} = 0; \quad \text{that is}, t = t^* = \text{constant}, \tag{8}$$

in which t plays the role of a parameter.

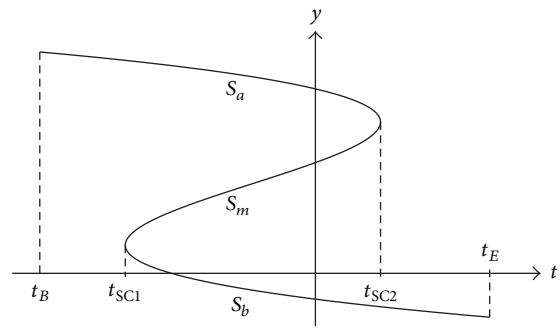


Figure 1: The critical manifold S.

Both scalings agree on the level of phase space structure when $\epsilon \neq 0$ but offer very different perspectives since they differ radically in the limit when $\epsilon = 0$. The main goal of singular perturbation theory is to use these limits to understand structure in the full system when $\epsilon \neq 0$.

The critical manifold S is defined as a solution of the reduced system; that is,

$$S := \{(t, y, w) : t \in \langle t_B, t_E\rangle,\ f(y) = m(t),\ w = 0\} \tag{9}$$

which corresponds to a set of equilibria for the associated system (6), (7), and (8).

2. Saddle-Center Bifurcations of Associated System

We assume the following.

($A1$) The critical manifold is S-shaped curve with two folds; that is, it can be written in the form $t = \varphi(y)$, $t \in \langle t_B, t_E\rangle$, and the function φ has precisely two critical points, one nondegenerate minimum y_{SC1} and one nondegenerate maximum y_{SC2}; let $y_{SC1} < y_{SC2}$. Thus, the critical manifold can be broken up into three pieces S_b, S_m, and S_a, separated by the minimum and maximum (Figure 1). These three pieces are defined as follows:

$$\begin{aligned} S_b &= \{(y, \varphi(y), 0) : y < y_{SC1}\}, \\ S_m &= \{(y, \varphi(y), 0) : y_{SC1} < y < y_{SC2}\}, \\ S_a &= \{(y, \varphi(y), 0) : y_{SC2} < y\}. \end{aligned} \tag{10}$$

($A2$) Consider $\varphi'(y) \neq 0$ for $y \neq y_{SC1}, y_{SC2}$.

($A3$) Consider $(df/dy)(y) > 0$ for every $(t, y, 0) \in S_m$ and $(df/dy)(y) < 0$ for every $(t, y, 0) \in S_a \cup S_b$.

Due to the assumption ($A3$), the situation considered here substantially differs from the situation in [13], where two pieces of critical manifold, namely, S_a and S_b, are not normally hyperbolic, and large orbit oscillations that encircle all the three pieces of critical manifold were studied (for the definition of a normal hyperbolicity of critical manifold see, e.g., [12]). In this paper, the pieces S_a and S_b of the critical

manifold S are normally hyperbolic, and thus the system under consideration allows another type of nonlinear oscillations, namely, small orbit oscillations around the middle piece S_m of critical manifold S. For comparison, see Figure 3(a) and Figure 5.

Let $t_{SC1} = \varphi(y_{SC1})$, $t_{SC2} = \varphi(y_{SC2})$. Denote by

$$\begin{aligned} u_1(t) = \varphi^{-1}(t): &\quad t \in \langle t_B, t_{SC2}\rangle, \ y_{SC2} \le u_1(t), \\ u_2(t) = \varphi^{-1}(t): &\quad t \in \langle t_{SC1}, t_{SC2}\rangle, \ y_{SC1} \le u_2(t) \le y_{SC2}, \\ u_3(t) = \varphi^{-1}(t): &\quad t \in \langle t_{SC1}, t_E\rangle, \ u_3(t) \le y_{SC1}. \end{aligned} \tag{11}$$

The equations

$$f(y) = m(t), \qquad w = 0 \tag{12}$$

have three solutions for y, w if $t^* \in (t_{SC1}, t_{SC2})$ and one if $t^* \in \langle t_B, t_{SC1})$ and $t^* \in (t_{SC2}, t_E\rangle$. Thus the associated system (6), (7), $t = t^* =$ constant has three equilibria (two saddles, one center) for $t^* \in \langle t_B, t_{SC1})$ and one equilibrium (saddle) for $t^* \in \langle t_B, t_{SC1})$ and $t^* \in (t_{SC2}, t_E\rangle$; the eigenvalues of the jacobian are

$$\lambda_{1,2}(t^*, y, w) = \pm a^{-1}(t^*)\sqrt{-\frac{df}{dy}(y)}. \tag{13}$$

Thus, the pieces S_a and S_b of critical manifold S are the normally hyperbolic manifolds. The points $(t_{SC1}, y_{SC1}, 0)$ and $(t_{SC2}, y_{SC2}, 0)$ of S are the cusps of the codimension two ([14, 15]), and the corresponding bifurcation is known as saddle-center bifurcation ([16]) (Figure 2).

More precisely, at the point $t^* = t_{SC1}$, the birth of the saddle and the center occurs. At $t^* = t_{SC2}$, the left side center and the saddle coalesce. There is a unique $t_0^* \in (t_{SC1}, t_{SC2})$, such that between hyperbolic points (saddles) there is a heteroclinic connection. The homoclinic loop of one hyperbolic point surrounds the corresponding elliptic (center) one for every $t^* \in (t_{SC1}, t_{SC2})$, $t^* \ne t_0^*$.

We use the level surfaces $H^\epsilon(t, y, w) = H^\epsilon(t)$ of the energy function H^ϵ,

$$\begin{aligned} H^\epsilon(t, y, w) &= \frac{1}{2}w^2 + V(t, y), \\ V(t, y) &= \int_0^y f(s)\,ds - m(t)\,y, \end{aligned} \tag{14}$$

to characterize the trajectories of (3). These surfaces in (t, y, w)-space are defined by

$$w = \pm\left(2\left(H^\epsilon(t) - V(t, y)\right)\right)^{1/2} \tag{15}$$

extending it as long as w remains real. On the intervals $\langle t_B, t_{SC1})$ and $(t_{SC2}, t_E\rangle$, there is a motion across a single potential barrier, and on the interval (t_{SC1}, t_{SC2}), there is double barrier with a well in between.

The derivative of $H^\epsilon(t)$ along any solution path of (3) is

$$\begin{aligned} H^{\epsilon'}(t) &= w^\epsilon w^{\epsilon'} + f(y^\epsilon)\,y^{\epsilon'} - [m(t)\,y^\epsilon]' \\ &= w^\epsilon\left[-\frac{f(y^\epsilon)}{\epsilon a} + \frac{m(t)}{\epsilon a} - \frac{a'}{a}w^\epsilon\right] \\ &\quad + f(y^\epsilon)\,y^{\epsilon'} - [m(t)\,y^\epsilon]' \\ &= -\frac{a'(t)}{a(t)}(w^\epsilon)^2 - m'(t)\,y^\epsilon. \end{aligned} \tag{16}$$

The main objective of this paper is to prove the strongly nonlinear oscillations on the interval (t_{SC1}, t_{SC2}) as possible scenario of behavior of the solutions for problem (2). For this reason, we rewrite the differential equation (2) in the system form

$$\begin{aligned} y' &= \frac{w}{\epsilon a(t)}, \\ (a(t)\,w)' &= \frac{m(t)}{\epsilon} - \frac{f(y)}{\epsilon}. \end{aligned} \tag{17}$$

Then, we make a change of variables from rectangular coordinates (y, w) to the dynamic polar coordinates (r, γ) centered at $(u_2(t), 0)$ defined by the equations

$$y = u_2(t) + r\cos\gamma, \qquad a(t)\,w = -r\sin\gamma, \tag{18}$$

and let us consider the system (17) on the interval (t_{SC1}, t_{SC2}) in these new coordinates. The function $u_2(t)$ acting in the first polar transform equation corresponds to the middle piece S_m of critical manifold S. At first, we derive the differential equation for polar angle γ, which is crucial for an analysis of nonlinear oscillations in this system. Dividing formally the second transform equation by the first, we get

$$\tan\gamma = -\frac{aw}{y - u_2}. \tag{19}$$

Differentiating this equation with respect to t, we consecutively have

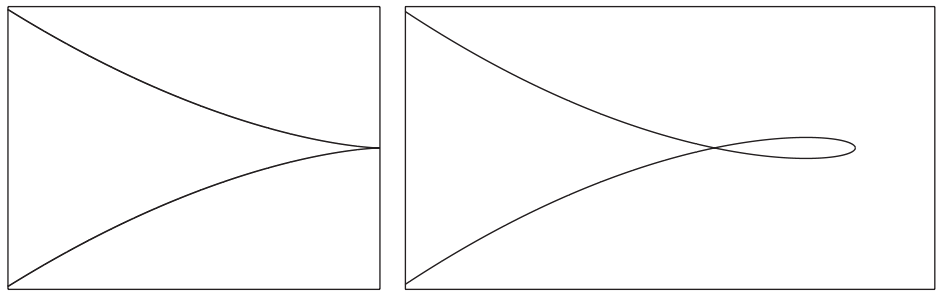
$$\frac{1}{\cos^2\gamma}\gamma' = -\left[\frac{aw}{y - u_2}\right]' = \frac{awy'}{(y - u_2)^2} - \frac{(aw)'}{y - u_2} - \frac{awu_2'}{(y - u_2)^2}. \tag{20}$$

Finally, using (17) and (18), we obtain the following differential equation for γ:

$$\gamma' = \frac{1}{\epsilon}\left[\frac{1}{a^2(t)}\sin^2\gamma + \overline{f}(t, y)\cos^2\gamma + \frac{\epsilon u_2'(t)}{r^\epsilon(t)}\sin\gamma\right], \tag{21}$$

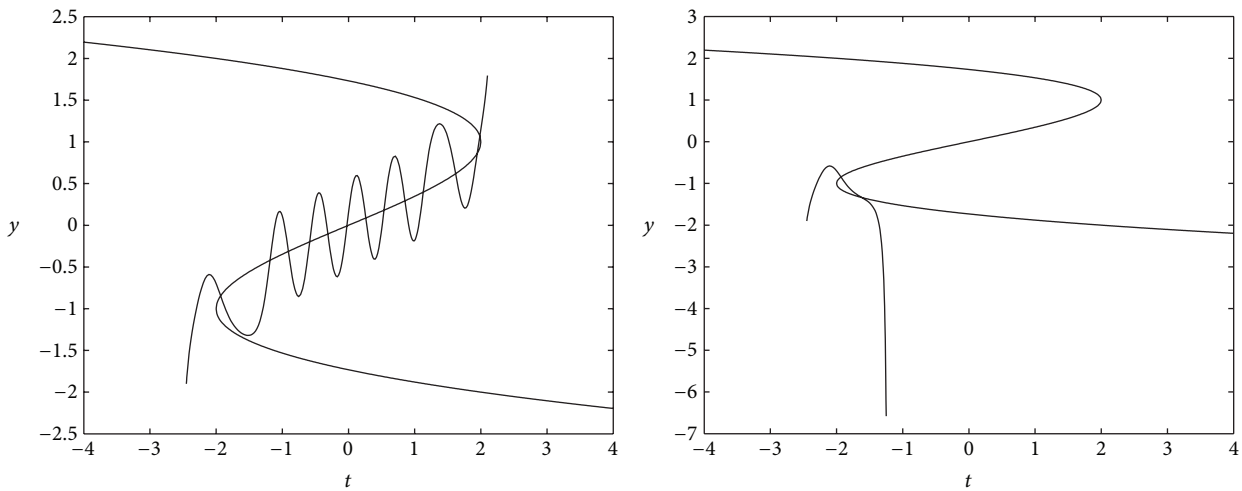
where the radius is as follows:

$$\begin{aligned} r^\epsilon(t) &= \sqrt{(y - u_2)^2 + (a(t)\,w)^2}, \\ \overline{f}(t, y) &= \frac{f(y) - m(t)}{y - u_2(t)}, \\ \overline{f}(t, u_2(t)) &\overset{\text{def}}{=} \lim_{y \to u_2}\frac{f(y) - m(t)}{y - u_2(t)} = \frac{df}{dy}(u_2(t)). \end{aligned} \tag{22}$$



(a) (b)

Figure 2: Creation and extinction (in mirror mode) of a separatrix loop in the saddle-center bifurcation at $t^* = t_{SC1}$ and $t^* = t_{SC2}$, respectively.



(a) (b)

Figure 3: Solution of (27), (28), $\epsilon^2 = 0.0224$ on $\langle -2.45, 2.1\rangle$ (a); solution of (27), (28), $\epsilon^2 = 0.0225$ on $\langle -2.45, -1.25\rangle$ (b).

Now let K be a compact subset of (t_{SC1}, t_{SC2}). On the periodic orbits (for fixed t), we define the minimal radius:

$$r^{\epsilon}_{\min}(K) \stackrel{\text{def}}{=} \min_{t\in K} r^{\epsilon}(t) = \min_{t\in K}\left\{u_2(t) - y^{\epsilon}_L(t), y^{\epsilon}_R(t) - u_2(t), \sqrt{2}a(t)\sqrt{H^{\epsilon}(t) - V(t, u_2(t))}\right\}, \quad (23)$$

where $y^{\epsilon}_L(t)$ and $y^{\epsilon}_R(t)$ ($y^{\epsilon}_L < y^{\epsilon}_R$) are the solutions of the equation

$$H^{\epsilon}(t) - V(t, y) = 0, \quad t \in (t_{SC1}, t_{SC2}) \quad (24)$$

lying on the periodic orbit.

Obviously, $y^{\epsilon}_i(t) \to u_2(t_{SC1})$ for $t \to t^+_{SC1}$ and $y^{\epsilon}_i(t) \to u_2(t_{SC2})$ for $t \to t^-_{SC2}$, $i = L, R$.

Let $\delta(t)$ be a positive function such that

$$\delta(t) + V(t, u_2(t)) < \min\{V(t, u_1(t)), V(t, u_3(t))\} \quad (25)$$

on K.

Now we make the following additional assumption.

($A4$) The total energy $H^{\epsilon}(t)$ of motion described by (2) satisfies

$$\delta(t) + V(t, u_2(t)) \le H^{\epsilon}(t) < \min\{V(t, u_1(t)), V(t, u_3(t))\} \quad (26)$$

on a compact subset K of (t_{SC1}, t_{SC2}).

If a total energy of motion described by (2) satisfies the assumption ($A4$) on every compact subset K of (t_{SC1}, t_{SC2}), then $y^{\epsilon}_i(t) \to u_2(t_{SC1})$ for $t \to t^+_{SC1}$, $t \in K$ and $y^{\epsilon}_i(t) \to u_2(t_{SC2})$ for $t \to t^-_{SC2}$, $t \in K$, and $i = L, R$.

We precede the main result on the existence of nonlinear oscillations of the solutions for (2) on the interval (t_{SC1}, t_{SC2}) by important example.

Example 1. Consider Duffing's oscillator with linear excitation

$$\epsilon^2 y'' + 3y - y^3 = t \quad (27)$$

for $\epsilon^2 = 0.0224$ subject to the initial conditions

$$y^{\epsilon}(-2.45) = -1.9, \qquad y^{\epsilon'}(-2.45) = 11.16 \quad (28)$$

on the interval $\langle -2.45, 2.1\rangle$.

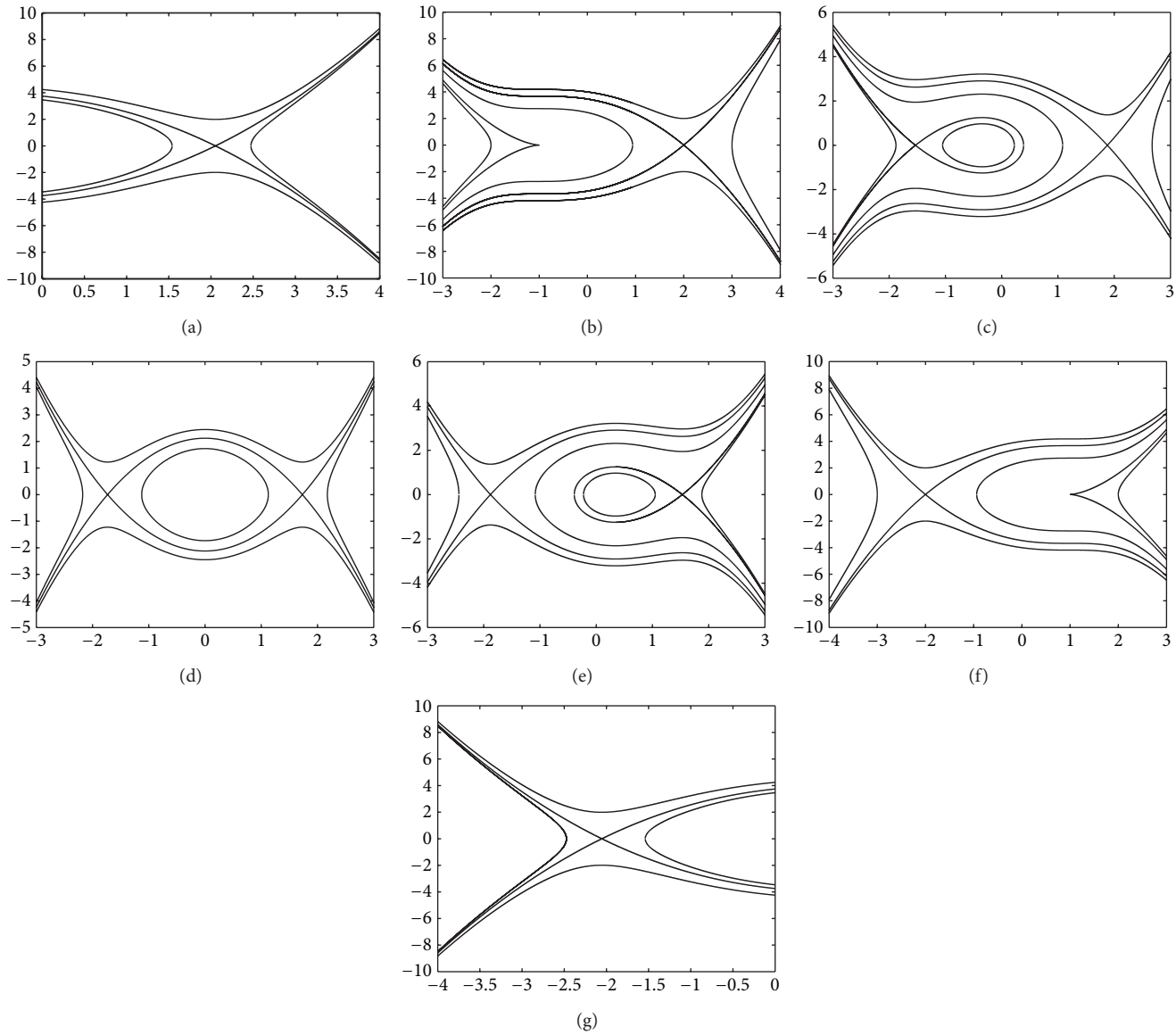


Figure 4: Phase portraits for t = (from top-left to bottom-right): -2.5 ($t < t_{SC1}$); -2 ($t = t_{SC1}$); -1 ($t_{SC1} < t < t_{SC2}$); 0 (heteroclinic connections between two saddles of associated system, $t_0^* = 0$); 1; 2 ($t = t_{SC2}$); 2.5 ($t > t_{SC2}$).

In our case $t_B = -2.45$, $t_E = 2.1$, $a \equiv 1$, $f(y) = 3y - y^3$, $m(t) = t$, $t_{SC1} = -2$, $t_{SC2} = 2$, $y_{SC1} = -1$, $y_{SC2} = 1$, and it is not difficult to check that the assumptions $(A1)$–$(A3)$ hold. Numerical results obtained from (27) subject to the initial conditions (28) using the software package MATLAB 7 are shown in Figure 3(a). These oscillations are very sensitive on the value of singular perturbation parameter ϵ. Figure 3(b) shows the solution of (27), (28) for $\epsilon^2 = 0.0225$.

In order to facilitate the understanding of the qualitative behaviors of this dynamical system, we draw the (y, w)-phase portraits at the relevant fixed value of time t (Figure 4). For comparison, Figure 5 shows the solution of initial value problem

$$\epsilon^2 y'' - 3y + y^3 = -t,$$
$$y^\epsilon(-4) = 3.1821, \qquad y^{\epsilon'}(-4) = 0 \tag{29}$$

with twin-(or single-) well potential for $\epsilon^2 = 0.00354$ on the interval $\langle t_B, t_E \rangle = \langle -4, 4 \rangle$. This type of problems has been studied in [13].

Now we will analyze the solution of (27), (28) after the time t_{SC2}. The total energy (14) for (27) is

$$H^\epsilon(t, y, w) = \frac{1}{2}w^2 + \frac{3}{2}y^2 - \frac{1}{4}y^4 + ty, \tag{30}$$

and for its derivative along the solution given by (16), we obtain

$$H^{\epsilon'}(t) = -y^\epsilon(t). \tag{31}$$

Due to the oscillations around $u_2(t)$ in left neighborhood of t_{SC2} between $u_1(t)$ and $u_3(t)$, that is, $y^\epsilon > 0$, the total energy decreases (dissipation) in right neighborhood of t_{SC2},

$$H^{\epsilon'}(t) = -y^\epsilon(t) < 0. \tag{32}$$

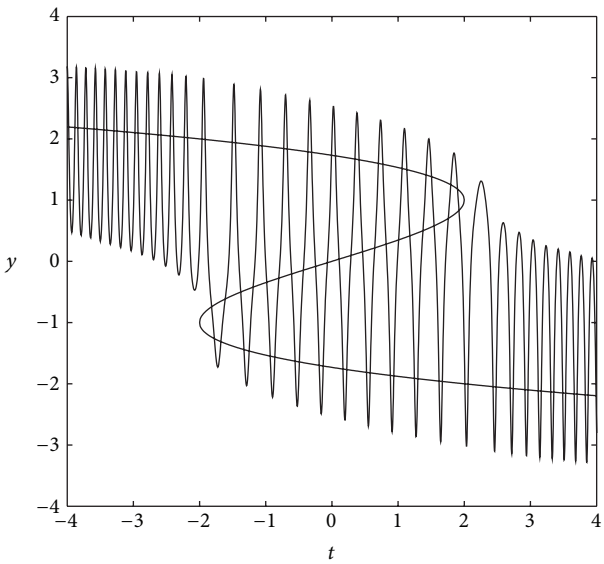


FIGURE 5: Solution of (29), $\epsilon^2 = 0.00354$ on $\langle -4, 4\rangle$.

As it follows from the shape of potential V and the phase portrait for $t > t_{SC2}$, $y^{\epsilon}(t^*) \to \infty$ for $\epsilon \to 0^+$ and $t^* \in (t_{SC2}, t_E\rangle$.

3. Analysis of Solutions Lying on the Periodic Orbits: Main Result

Now we formulate the theorem on nonlinear oscillations of solutions of (2) for the motion with total energy $H^{\epsilon}(t)$ satisfying the assumption $(A4)$. Moreover, we show that the parameter ϵ plays role modeling tool for the frequency control of the nonlinear oscillations.

Denote by s_{ϵ} the spacing between two successive zero numbers of the function $y^{\epsilon} - u_2$ on K, where y^{ϵ} is a solution of (2).

Theorem 2. *Under the assumptions (A1)–(A4), the solutions of problem* (2) *oscillate on the compact set* K, $K \subset (t_{SC1}, t_{SC2})$ *between* $u_1(t)$ *and* $u_3(t)$ *and*

$$\epsilon \frac{\pi}{\mu_2(K, \epsilon_0)} \le s_{\epsilon} \le \epsilon \frac{\pi}{\mu_1(K, \epsilon_0)}, \quad \epsilon \in (0, \epsilon_0], \tag{33}$$

where $\mu_1(K, \epsilon_0)$ *and* $\mu_2(K, \epsilon_0)$ *are the positive constants independent of the singular perturbation parameter* ϵ, $\epsilon \in (0, \epsilon_0]$.

Proof. To obtain the oscillations and the estimate of their frequencies, we analyze the differential equation (21); that is,

$$\gamma' = \frac{1}{\epsilon}\left[\frac{1}{a^2(t)}\sin^2\gamma + \overline{f}(t, y)\cos^2\gamma + \frac{\epsilon u_2'(t)}{r^{\epsilon}(t)}\sin\gamma\right]. \tag{34}$$

Taking into consideration the fact that $r^{\epsilon}_{\min}(K) \ge \Delta > 0$ independently of parameter ϵ due to the assumption $(A4)$, we can estimate that

$$\left|\frac{\epsilon u_2'(t)}{r^{\epsilon}(t)}\sin\gamma\right| \le \epsilon \frac{|u_2'(t)|}{r^{\epsilon}_{\min}(K)} \le \epsilon |u_2'(t)| \Delta \longrightarrow 0^+ \tag{35}$$

for $\epsilon \to 0^+$. Further for $t \in K$ and $y \in (u_3(t), u_1(t))$, we have $\overline{f}(t, y) > 0$. Thus for sufficiently small values of the singular perturbation parameter ϵ, say, $\epsilon \in (0, \epsilon_0]$, there exist the positive constants $\mu_1(K, \epsilon_0)$, $\mu_2(K, \epsilon_0)$, and $\mu_1(K, \epsilon_0) < \mu_2(K, \epsilon_0)$:

$$\begin{aligned}
&\mu_1(K, \epsilon_0) \\
&\quad = \min\left[\frac{1}{a^2(t)}\sin^2\gamma + \overline{f}(t, y)\cos^2\gamma + \frac{\epsilon_0 u_2'(t)}{r^{\epsilon}(t)}\sin\gamma;\right. \\
&\qquad \left. t \in K,\ y \in \left\langle y_L^0(t), y_R^0(t)\right\rangle,\ \gamma \in \langle 0, 2\pi\rangle\right], \\
&\mu_2(K, \epsilon_0) \\
&\quad = \max\left[\frac{1}{a^2(t)}\sin^2\gamma + \overline{f}(t, y)\cos^2\gamma + \frac{\epsilon_0 u_2'(t)}{r^{\epsilon}(t)}\sin\gamma;\right. \\
&\qquad \left. t \in K,\ y \in \left\langle y_L^0(t), y_R^0(t)\right\rangle,\ \gamma \in \langle 0, 2\pi\rangle\right],
\end{aligned} \tag{36}$$

where $y_L^0(t)$, and $y_R^0(t)$ are the roots of the equation

$$\min\{V(t, u_1(t)), V(t, u_3(t))\} - V(t, y) = 0, \tag{37}$$

lying on the periodic orbit.

Putting (35) into the definitions of constants $\mu_1(K, \epsilon_0)$, and $\mu_2(K, \epsilon_0)$, we obtain that

$$\begin{aligned}
&\mu_1(K, \epsilon_0) \\
&\quad \ge \min\left[\frac{1}{a^2(t)}\sin^2\gamma + \overline{f}(t, y)\cos^2\gamma - \epsilon_0 |u_2'(t)| \Delta;\right. \\
&\qquad \left. t \in K,\ y \in \left\langle y_L^0(t), y_R^0(t)\right\rangle,\ \gamma \in \langle 0, 2\pi\rangle\right] > 0, \\
&\mu_2(K, \epsilon_0) \\
&\quad \le \max\left[\frac{1}{a^2(t)}\sin^2\gamma + \overline{f}(t, y)\cos^2\gamma + \epsilon_0 |u_2'(t)| \Delta;\right. \\
&\qquad \left. t \in K,\ y \in \left\langle y_L^0(t), y_R^0(t)\right\rangle,\ \gamma \in \langle 0, 2\pi\rangle\right] > 0
\end{aligned} \tag{38}$$

for sufficiently small value of upper bound ϵ_0 of singular perturbation parameter ϵ.

Further, if $\epsilon_0^{(1)} < \epsilon_0^{(2)}$, then

$$\mu_1\left(K, \epsilon_0^{(1)}\right) > \mu_1\left(K, \epsilon_0^{(2)}\right), \tag{39}$$

and conversely

$$\mu_2\left(K, \epsilon_0^{(1)}\right) < \mu_2\left(K, \epsilon_0^{(2)}\right). \tag{40}$$

Thus we have the inequality

$$\mu_1(K, \epsilon_0)\epsilon^{-1} \le \gamma' \le \mu_2(K, \epsilon_0)\epsilon^{-1}. \tag{41}$$

Integrating this inequality with respect to the variable t between two successive zeros (jth and $(j+1)$th, $j = 1, 2, \ldots$) of $y^{\epsilon}(t) - u_2(t)$, $t \in K$, we obtain immediately the lower and upper bound of their spacing s_{ϵ}. Indeed,

$$\int_{\text{zero}(j)\text{th}}^{\text{zero}(j+1)\text{th}} \frac{\mu_2(K,\epsilon_0)}{\epsilon} dt \geq \int_{\text{zero}(j)\text{th}}^{\text{zero}(j+1)\text{th}} \gamma' dt \geq \int_{\text{zero}(j)\text{th}}^{\text{zero}(j+1)\text{th}} \frac{\mu_1(K,\epsilon_0)}{\epsilon} dt, \quad (42)$$

$$\frac{\mu_2(K,\epsilon_0)}{\epsilon} s_{\epsilon} \geq \pi \geq \frac{\mu_1(K,\epsilon_0)}{\epsilon} s_{\epsilon}.$$

Hence,

$$\epsilon \frac{\pi}{\mu_2(K,\epsilon_0)} \leq s_{\epsilon} \leq \epsilon \frac{\pi}{\mu_1(K,\epsilon_0)}, \quad \epsilon \in (0, \epsilon_0]. \quad (43)$$

□

4. Conclusion

The frequency of nonlinear oscillations of Duffing's type equations arising via saddle-center bifurcation in associated system may be controlled by the singular perturbation parameter ϵ. These oscillations are very sensitive on the initial conditions.

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