Complexity

Fuzzy Calculus Theory and Its Applications

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Editorial

Fuzzy Calculus Theory and Its Applications

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Fuzzy calculus is the study of theory and applications of integrals and derivatives of uncertain functions. This branch of mathematical analysis, extensively investigated in recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena.

Based on the wide applications in engineering and sciences such as physics, mechanics, chemistry, and biology, research on fuzzy ordinary and partial differential equations and other relative topics is active and extensive around the world. In the past few years, the growth of the subject is manifested by hundreds of research papers, several monographs, and many international conferences.

This special issue contains 8 papers, the contents of which are summarized as follows.

The paper "Fuzzy Fixed Point Results For Φ Contractive Mapping with Applications" by H. Humaira et al. establishes common fuzzy fixed point results for Φ contractive mappings involving control functions as coefficients of contractions in the setting of complex-valued metric space by using rational type contractions.

The paper "On Fuzzy Portfolio Selection Problems: A Parametric Representation Approach" by O. S. Fard and M. Ramezanzadeh investigates the constrained fuzzy-valued optimization problems with regard to the features of the parametric representation of fuzzy numbers.

In "Parameter Optimization of MIMO Fuzzy Optimal Model Predictive Control By APSO," A. Taieb et al introduce a new development for designing a multi-input multioutput fuzzy optimal model predictive control using the adaptive particle swarm optimization algorithm.

In "The Karush-Kuhn-Tucker Optimality Conditions for the Fuzzy Optimization Problems in the Quotient Space of Fuzzy Numbers," N. Yu and D. Qiu propose the solution concepts for the fuzzy optimization problems in the quotient space of fuzzy numbers.

In "Random Fuzzy Differential Equations with Impulses" by H. Vu, the random fuzzy differential equations (RFDEs) with impulses are considered. Using Picard method of successive approximations, the existence and uniqueness of solutions under suitable conditions are proved and some properties of solution are studied.

The paper "Methods in Ranking Fuzzy Numbers: A Unified Index and Comparative Reviews" by T.-L. Nguyen proposes a unified index that multiplies weighted-mean and weighted-area discriminatory components of a fuzzy number, respectively, called centroid value and attitude-incorporated left-and-right area.

In "The Portfolio Balanced Risk Index Model and Analysis of Examples of Large-Scale Infrastructure Project," W. Gao and K. Hong focus on a three-dimensional portfolio balanced risk index model for large-scale infrastructure project risk evaluation. Taking subjectivity utility and complex evaluation motivation into consideration, a method of combinational equilibrium evaluation is built using the index form to reflect whole loss changes of risk.

The paper "Different solution strategy for solving epidemic model in imprecise Environment" by A. Mahata

et al. discusses the different solution strategy for solving epidemic model in different imprecise environment, that is, a susceptible-infected-susceptible model in imprecise environment.

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Research Article

Different Solution Strategies for Solving Epidemic Model in Imprecise Environment

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We study the different solution strategy for solving epidemic model in different imprecise environment, that is, a Susceptible-Infected-Susceptible (SIS) model in imprecise environment. The imprecise parameter is also taken as fuzzy and interval environment. Three different solution procedures for solving governing fuzzy differential equation, that is, fuzzy differential inclusion method, extension principle method, and fuzzy derivative approaches, are considered. The interval differential equation is also solved. The numerical results are discussed for all approaches in different imprecise environment.

1. Introduction

1.1. Modeling with Impreciseness. The aim of mathematical modeling is to imitate some real world problems as far as possible. The presence of imprecise variable and parameters in practical problems in the field of biomathematical modeling became a new area of research in uncertainty modeling. So, the solution procedure of such problems is very important. If the solution of said problems with uncertainty is developed, then, many real life models in different fields with imprecise variable can be formulated and solved easily and accurately.

1.2. Fuzzy Set Theory and Differential Equation. Differential equations may arise in the mathematical modeling of real world problems. But when the impreciseness comes to it, the behavior of the differential equation is altered. The solution procedures are taken in different way. In this paper we take two types of imprecise environments, fuzzy and interval, and find their exact solution. In 1965, Zadeh [1] published the first of his papers on the new theory of Fuzzy Sets and Systems. After that Chang and Zadeh [2] introduced the concept of fuzzy numbers. In the last few years researchers have been giving their great contribution on the topic of fuzzy number research [3–5]. As for the application of the fuzzy set theory applied in fuzzy equation [6], fuzzy differential equation [7], fuzzy integrodifferential equation [8–10], fuzzy integral equation [11], and so on were developed.

1.3. Different Approaches for Solving Fuzzy Differential Equation. The application of differential equations has been widely explored in various fields like engineering, economics, biology, and physics. For constructing different types of problems in real life situation the fuzzy set theory plays an important role. The applicability of nonsharp or imprecise concept is very useful for exploring different sectors for its applicability. A differential equation can be called fuzzy differential equation if (1) only the coefficient (or coefficients) of the differential equation is fuzzy valued number, (2) only

Table 1

	Name of the theory	Some references
Fuzzy differential equation	Fuzzy differential inclusion	Baidosov [12], Hüllermeier [13]
	Zadeh's Extension principle	Oberguggenberger and Pittschmann [14], Buckley and Feuring [15]
	Approach using derivative of fuzzy valued functions	
	Dubois-Prade derivative	Dubois and Prade [16]
	Puri-Ralescu derivative	Puri and Ralescu [17]
	Goetschel-Voxman derivative	Goetschel Jr. and Voxman [18]
	Friedman-Ming-Kandel derivative	Friedman et al. [19]
	Seikkala derivative	Seikkala [20]
	SGH derivative	Bede and Gal [21]
	Same-order and reverse-order derivative	Zhang and Wang [22]
	π-derivative	Chalco-Cano et al. [23]
	gH-derivative	Stefanini and Bede [24]
	g-derivative	Bede and Stefanini [25]
	H_2-derivative	Mazandarani and Najariyan [26]
	Interactive derivative	de Barros and Santo Pedro [27]
	gr-derivative	Mazandarani et al. [28]
	Approach using fuzzy bunch of real valued functions instead of fuzzy valued functions	Gasilov et al. [29–32], Amrahov et al. [33]

the initial value (or values) or boundary value (or values) is fuzzy valued number, (3) the forcing term is fuzzy valued function, and (4) all the conditions (1), (2), and (3) or their combination is present on the differential equation.

There exist two types of strategies for solving the FDEs, which are as follows:

(a) Zadeh's extension principle method.

(b) Differential inclusion method.

(c) Approach using derivative of fuzzy valued functions.

(d) Approach using fuzzy bunch of real valued functions instead of fuzzy valued functions.

Now we look on some different procedure and concepts of derivation in Table 1.

There exist different numerical techniques [34–36] for solving the fuzzy differential equation. The techniques are not fully similar to any differential equation solving techniques.

In this paper we only study the first three approaches.

1.4. Interval Differential Equation. An interval number is itself an imprecise parameter. Because the value is not a crisps number, the value lies between two crisp numbers. When we take any parameter, may be coefficients or initial condition or both, of a differential equation then the interval differential equation comes. The basic behaviors of that number are different from a crisp number. Hence, the calculus of those types numbers valued functions is different. So we need to study the differential equation in these environments. From the time that Moore [37] and Markov [38] as the pioneers introduced the interval analysis and related notions, several monographs and papers were devoted to connect the fuzzy analysis and interval analysis [39], but, the later one was not well-realized and applicable to model dynamical systems. After introducing generalized Hukuhara differentiability, different perspectives, which leaded to nice schemes and strategies to achieve the solutions, were discussed in the literature [40–43]. Lupulescu in [44] developed the notions of RL- and Caputo-types derivatives for interval-valued functions. Salahshour et al. [45, 46] proposed a nonsingular kernel and conformable fractional derivative for interval differential equations of fractional order. Recently interval differential equation is studied by da Costa et al. [47] and Gasilov and Emrah Amrahov [48].

1.5. Work Done Using Fuzzy Differential Equation and Interval Differential Equation on Biomathematical Problem. Fuzzy differential equation and biomathematics are not new topics. A lot of research was done in this field. For instance, check [49–68]. Many authors consider interval parameter with differential equation in biomathematical model. For presence of interval parameter the equation becomes interval differential equation. Using the property of interval number they solve the concerned model and discuss the behavior. Pal and Mahapatra [62] consider a bioeconomic modeling of two-prey and one-predator fishery model with optimal harvesting policy through hybridization approach in interval environment. Similarly, optimal harvesting of prey–predator system with interval biological parameters is discussed in [63]. Sharma and Samanta consider optimal harvesting of

a two species competition model with imprecise biological parameters in [69]. Although Barros et al. [70] studied SIS model in fuzzy environment using fuzzy differential inclusion still we can study the model in different environments by different approaches.

1.6. Motivation. Impreciseness comes in every model for biological system. The necessity for taking some parameter as imprecise in a model is an important topic today. There are so many works done on biological model with imprecise data. Sometimes parameters are taken as fuzzy and sometimes it is an interval. Our main aim is modeled as a biological problem associated with differential equation with some imprecise parameters. Thus fuzzy differential equation and imprecise differential equation are important. Now we can concentrate some previous works on biological modeling in imprecise environments:

1.7. Novelties. Although some developments are done, some new and interesting research works have been done by ourselves, which are mentioned as follows:

(i) SIS model is studied in imprecise environment.

(ii) The fuzzy and interval environments are taken for analyses in the model.

(iii) The governing fuzzy differential equation is solved by three approaches: fuzzy differential inclusion, extension principal, and fuzzy derivative approaches.

(iv) The SIS model is solved by reducing the dimension of the model for fuzzy cases. For these reasons we use completely correlated fuzzy number.

(v) Numerical examples are taken for showing the comparative view of different approaches.

Moreover, we can say all developments can help for the researchers who are engaged with uncertainty modeling, differential equation, and biological system if fuzzy parameters are assumed in the models. One can model and find the solution on any biological model with fuzzy and differential equation by the same approaches.

2. Basic Definitions

2.1. Definition

Definition 1 (fuzzy set). Let $\widetilde{F}$ be a fuzzy set which is defined by a pair $(U, \mu_{\widetilde{F}}(x))$, where U is a nonempty universal set and

$$\mu_{\widetilde{F}}(x) : U \longrightarrow [0,1]. \tag{1}$$

For each $x \in U$, $\mu_{\widetilde{F}}(x)$ is the grade of membership function of $\widetilde{F}$.

Definition 2 (fuzzy number in trapezoidal form). A fuzzy number in trapezoidal form represented by three points like as $\widetilde{K} = (K_1, K_2, K_3, K_4)$ and the presentation can be illustrated as membership function as

$$\mu_{\widetilde{K}}(y) = \begin{cases} 0, & y \le K_1 \\ \dfrac{y - K_1}{K_2 - F_1}, & K_1 \le y \le K_2 \\ 1, & y = F_2 \\ \dfrac{K_4 - y}{K_4 - K_3}, & K_3 \le y \le K_4 \\ 0, & y \ge K_4. \end{cases} \tag{2}$$

Definition 3 (α-cut of a fuzzy set). The α-cut of $\widetilde{K} = (K_1, K_2, K_3, K_4)$ is given by

$$K_\alpha = [K_1 + \alpha(K_2 - K_1), K_4 - \alpha(K_4 - K_3)], \quad \forall \alpha \in [0,1]. \tag{3}$$

Definition 4 (correlated fuzzy number: [71]). Let $\widetilde{A}_f$ and $\widetilde{B}_f$ are two fuzzy sets whose membership function is written as follows: $\mu_{\widetilde{A}_f}(x)$ and $\mu_{\widetilde{B}_f}(x)$. Then there exist $d, e \in R$ with $q \neq 0$ such that their joint possibility distribution is given by

$$\begin{aligned} \mu_{\widetilde{C}_f}(x,y) &= \mu_{\widetilde{A}_f}(x)\, \chi_{\{dx+e=y\}}(x,y) \\ &= \mu_{\widetilde{B}_f}(x)\, \chi_{\{dx+e=y\}}(x,y), \end{aligned} \tag{4}$$

where $\chi_{\{dx+e=y\}}(x,y) = \{1, \text{if } dx+e = y; 0, \text{if } dx+e \neq y\}$ is the characteristic function of the line $\{(x,y) \in R^2 : dx + e = y\}$.

In this case we have $(\widetilde{C}_f)_\alpha = \{(x, dx + e = y) \in R^2 : x = (1-l)A_{01}(\alpha) + lA_{02}(\alpha),\ l \in [0,1]\}$, where $(\widetilde{A}_f)_\alpha = [A_{01}(\alpha), A_{02}(\alpha)]$, $(\widetilde{B}_f)_\alpha = d(\widetilde{A}_f)_\alpha + e$, for any $\alpha \in [0,1]$. Moreover if $d \neq 0$,

$$\mu_{\widetilde{B}_f}(x) = \mu_{\widetilde{A}_f}\left(\frac{x-e}{d}\right), \quad \forall x \in R. \tag{5}$$

Definition 5 (correlated trapezoidal fuzzy number). Two trapezoidal fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ are said to be correlated if $a_1 \triangle b_1 = a_2 \triangle b_2 = a_3 \triangle b_3 = a_4 \triangle b_4 = q$, where $\triangle$ is arbitrary operation and $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, q$ is constant.

Example 6. Let a trapezoidal fuzzy number be like $\widetilde{A} = (1/5, 2/5, 3/5, 4/5)$. Now we have to find another trapezoidal fuzzy number $\widetilde{B}$ that is correlated to $\widetilde{A}$.

Let $\widetilde{B}$ be of the form (b_1, b_2, b_3, b_4).

So clearly we have $b_1 + 1/5 = b_2 + 2/5 = b_3 + 3/5 = b_4 + 4/5 = 1$

So, $b_1 = 4/5, b_2 = 3/5, b_3 = 2/5, b_4 = 1/5$.

So we can write $\widetilde{B} = (4/5, 3/5, 2/5, 1/5)$, but here $4/5 \nless 3/5 \nless 2/5 \nless 1/5$.

We can write it in modified form as $\widetilde{B} = (1/5, 2/5, 3/5, 4/5)$.

Note 7 (use of correlated fuzzy number). There can be a basic question arising here, which is why we take correlated fuzzy

variables. Fuzzy number can be employed and applied in various fields for various models. Sometimes for simplification of a model, we give a transformation so that the operation between two variables becomes unit. Now, if the initial condition or the solution is defined as a fuzzy parameter then the certain operation on this quantity is obviously a unit number. Otherwise, the importance of using a correlated fuzzy number is to take the data in fewer amounts, which can be very helpful for calculation.

Definition 8 (strong and weak solution of fuzzy differential equation). Consider the first order fuzzy differential equation $dx/dt = f(k, x(t))$ with $(t_0) = x_0$. Here k or (and) x_0 is fuzzy number(s).

Let the solution (the solution comes from any method) of the above FDE be $\tilde{x}(t)$ and its α-cut be $x(t,\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$.

If $x_1(t,\alpha) \le x_2(t,\alpha)$ $\forall\alpha \in [0,1]$ then $\tilde{x}(t)$ is called strong solution; otherwise $\tilde{x}(t)$ is called weak solution and in that case the α-cut of the solution is given by

$$x(t,\alpha) = [\min\{x_1(t,\alpha), x_2(t,\alpha)\}, \max\{x_1(t,\alpha), x_2(t,\alpha)\}]. \tag{6}$$

Definition 9 (interval number). An interval number I is represented by closed interval $[I_l, I_u]$ and defined by $I = [I_l, I_u] = \{x : I_l \le x \le I_u, x \in \mathfrak{R}\}$, where $\mathfrak{R}$ is the set of real numbers and I_l and I_u are the left and right boundary of the interval number, respectively.

3. Method for Solving Fuzzy Differential Equation

Let us consider the differential equation

$$x'(t) = f(t,k,x(t)), \quad x(t_0) = x_0, \ a \le t \le b, \tag{7}$$

where k is constant, x_0 is initial condition, and $f(t,k,x(t))$ is the function which may be linear or nonlinear.

The differential equation (7) can be fuzzy differential equation if

(i) x_0, that is, initial condition, is fuzzy number.

(ii) k, that is, coefficient, is a fuzzy number.

(iii) x_0 and k, that is, initial condition and coefficient, are both fuzzy numbers.

3.1. Differential Inclusion Method for Solving Fuzzy Differential Equation. There are the papers where the concept of fuzzy differential equations is understood as the family of differential inclusions. For details see Agarwal et al. [72, 73], Diamond [74, 75], Laksmikantham et al. [76], and Lakshmikantham and Tolstonogov [77]. This new approach allowed considering some interesting aspects of fuzzy differential equations such as periodicity, Lyapunov stability, regularity of solution sets, attraction, and variation of constants formula (see [74, 75, 78, 79]). Also the numerical methods for FDEs have been developed in Hüllermeier [13, 80] and Ma et al. [81].

Let us assume the following differential inclusion is of the form

$$u'(t) \in g(t, u(t)) \tag{8}$$

with $u(0) = u_0 \in U_0$.

$g : [0,T] \times R^n \to F^n$ is a set valued function and $U_0 \in F^n$ (here F^n is space of fuzzy numbered valued functions). We have to solve $u(\cdot, u_0)$ of (8) with $u_0 \in U_0$ provided:

(a) The function $u(\cdot, u_0)$ is absolutely continuous in $[0,T]$.

(b) The function $u(\cdot, u_0)$ satisfies (8) for $t \in [0,T]$.

Now we denote the attainable set at time $t \in [0,T]$ which is subset of R^n associated with the problem (8) defined by $A(t, U_0) = \{u(t, u_0) : u(\cdot, u_0)$ which is solution of (8) $\}$.

In fuzzy environment dynamical system the problem (8) can be formed as

$$u'(t) \in \tilde{g}(t, u(t)) \quad u(0) = u_0 \in \tilde{U}_0, \tag{9}$$

where $\tilde{g} : [0,T] \times R^n \to F^n$ is a fuzzy set valued function and $U_0 \in F^n$.

According to Hüllermeier [13] the fuzzy initial value problem can be formed as family of differential inclusion given as

$$u'_\alpha(t) \in g(t, u_\alpha(t)) \quad u_\alpha(0) \in U_{0\alpha} \text{ with } \alpha \in [0,1], \tag{10}$$

where $g(t, u_\alpha(t))$ are the α-cuts of fuzzy set $\tilde{g}(t, u(t))$.

Here the attainable sets related to the problem (10) can be defined by $A_\alpha(t, U_{0\alpha}, \cdot) = \{u_\alpha(t) : u_\alpha(\cdot, u_\alpha)$ which is a solution of (10) in $[0,T]\}$.

Hence there is fuzzy interval $U(t) = A(t, U_0, \cdot)$ which is a fuzzy solution of (10) via differential inclusion if for all $t \in [0,T]$ the collection of α-cuts $\{A_\alpha(t, U_{0\alpha}, \cdot)\}_{\alpha\in[0,1]}$ satisfies the condition of the following theorem.

Theorem 10 (see [71]). *Let $\{A_\alpha \subseteq R \mid 0 \le \alpha \le 1\}$ be family of sets satisfying the following:*

(i) *A_α is a compact and convex interval, for all $0 \le \alpha \le 1$;*

(ii) *$A_\beta \subseteq A_\alpha$ for $0 \le \alpha \le \beta \le 1$;*

(iii) *$A_\alpha = \bigcap_{i=1} A_{\alpha_i}$ for any nondecreasing sequence $\alpha_i \to \alpha$ in $[0,1]$.*

Then there is a unique fuzzy interval $u \in \mathcal{F}_c$ such that $[u]^\alpha = Y_\alpha$. Conversely, the α-cuts sets $[u]^\alpha$ for any $u \in \mathcal{F}_c$ satisfy these conditions.

Therefore, we have the α solution $u_\alpha : [0,T] \to R^n$ of (9) if it is a solution of (10).

Theorem 11 (see [71]). *Suppose $E, F \in \varepsilon(R)$ are completely correlated fuzzy numbers; let G be their joint possibility distribution and $f : R^2 \to R^2$ be a continuous function; then $[f_C(E,F)]^\alpha = f([C]^\alpha)$.*

Theorem 12 (see [71]). *For all $(u_0, v) \in R^2$ there exists a unique solution to (10) in $[0, T_0]$. Then the solution of the problem (7)*

via extension principle when U_0 and Y are noninteractive, and when U_0 and Y are completely correlated satisfies the following relation $[(L_t)_c(U_0, V)]^\alpha \subseteq [(L_t)_{J_p}(U_0, V)]^\alpha$, for all $\alpha \in [0, 1]$, where $J_p(u_0, v) = \min(\mu_{U_0}(u_0), \mu_V(v))$, meaning J_p is the joint possibility distribution of the noninteractive fuzzy numbers U_0, V.

3.2. Extension Principle for Solving Fuzzy Differential Equation. Extension principle is a method by which we can easily find the solution of a fuzzy differential equation. Some researchers considered this method to find the solution of fuzzy differential equations [82–84].

Definition 13 (extension principle on fuzzy sets). Suppose that we have some usual sets X_R and choose some fuzzy sets $\widetilde{A} \in FS(X_R)$.

The extension principle for fuzzy sets states that if $F(\widetilde{A}) \in FS(Y_R)$ such that $y \in X_R$,

$$\mu_{F(\widetilde{A})}(y) = \begin{cases} \sup\left\{\mu_{\widetilde{A}}(x) : x \in F^{-1}\{y\}\right\}, & \text{if } y \in \text{Range}(F) \\ 0, & \text{if } y \notin \text{Range}(F) \end{cases} \quad (11)$$

and for every $\widetilde{B} \in FS(Y)$, $F^{-1}(\widetilde{B})$ is defined in the following way

$$\mu_{F^{-1}(\widetilde{B})}(x) = \mu_{\widetilde{B}}(F(x)) \quad (12)$$

for every $x \in X_R$.

Example 14. Let $\widetilde{A}$ be a fuzzy set where membership function is written as

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & \text{if } x \le 3 \\ x - 3 & \text{if } 3 \le x < 4 \\ 1 & \text{if } x = 4 \\ \dfrac{6 - x}{2} & \text{if } 4 < x \le 6 \\ 0 & \text{if } x \ge 6. \end{cases} \quad (13)$$

Let us choose a function $F(x) = 2x + 3$.

Now by Zadeh's extension principle, $F(\widetilde{A})$ can be determined and its membership function is written as

$$\mu_{F(\widetilde{A})}(y) = \begin{cases} 0 & \text{if } x \le 9 \\ \dfrac{y - 9}{2} & \text{if } 9 \le y < 11 \\ 1 & \text{if } y = 11 \\ \dfrac{15 - y}{4} & \text{if } 11 < y \le 15 \\ 0 & \text{if } y \ge 15. \end{cases} \quad (14)$$

Method 15 (solution of fuzzy differential equation using extension principle). Let us consider the fuzzy initial value problem (FIVP)

$$u'(t) = g(t, u(t)), \quad \widetilde{u}(t_0) = \widetilde{u}_0, \ a \le t \le b. \quad (15)$$

If we denote

$$\begin{aligned} &[\widetilde{u}(t)]^\alpha = \left[u_1^\alpha(t), u_2^\alpha(t)\right], \\ &[\widetilde{u}_0]^\alpha = \left[u_{0,1}^\alpha, u_{0,2}^\alpha\right], \\ &[f(t, \widetilde{x}(t))]^\alpha \\ &\quad = \left[g_1^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right), g_2^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right)\right]. \end{aligned} \quad (16)$$

By using the extension principle we have the membership function

$$g(t, u(t))(s) = \sup\{u(t)(\tau) \mid s = g(t, \tau)\}, \quad s \in R. \quad (17)$$

The result $g(t, x(t))$ is a fuzzy function.

And

$$\begin{aligned} &g_1^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right) \\ &\quad = \min\left\{g(t, u) \mid u \in \left[u_1^\alpha(t), u_2^\alpha(t)\right]\right\}, \\ &g_2^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right) \\ &\quad = \max\left\{g(t, u) \mid u \in \left[u_1^\alpha(t), u_2^\alpha(t)\right]\right\}. \end{aligned} \quad (18)$$

3.3. Fuzzy Derivative and Solution of Fuzzy Differential Equation by Fuzzy Derivative Approach. Bede and Gal [85] presented a concept of generalized Hukuhara differentiability of fuzzy valued map-pings, which permits them to obtain the solutions of FDEs with a diminishing diameter of solutions values. This was followed up in the literature [85–91]. This comprehensive definition allows us to resolve the disadvantages of the previous fuzzy derivatives. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy number valued functions in the case of the Hukuhara derivative.

Before going to the fuzzy differential equation approach we first know the following definition.

Definition 16 (generalized Hukuhara difference). The generalized Hukuhara difference of two fuzzy numbers $p, q \in \mathfrak{R}_{\mathcal{F}}$ is defined as follows:

$$p \ominus_g q = r \Longleftrightarrow \begin{cases} \text{(i)} \ \ p = q \oplus r \\ \text{or (ii)} \ \ p = q \oplus (-1) r. \end{cases} \quad (19)$$

Consider $[r]_\alpha = [r_1(\alpha), r_1(\alpha)]$; then $r_1(\alpha) = \min\{p_1(\alpha) - q_1(\alpha), p_2(\alpha) - q_2(\alpha)\}$ and $r_1(\alpha) = \max\{p_1(\alpha) - q_1(\alpha), p_2(\alpha) - q_2(\alpha)\}$.

Here the parametric representation of a fuzzy valued function $G : [a, b] \to \mathfrak{R}_{\mathcal{F}}$ is expressed by

$$[G(t)]_\alpha = \left[G_1(t, \alpha), G_2(t, \alpha)\right], \quad t \in [a, b], \ \alpha \in [0, 1]. \quad (20)$$

Definition 17 (generalized Hukuhara derivative on a fuzzy function). The generalized Hukuhara derivative of a fuzzy valued function $G : (a, b) \to \mathfrak{R}_{\mathcal{F}}$ at t_0 is defined as

$$G'(t_0) = \lim_{h \to 0} \frac{G(t_0 + h) \ominus_g G(\ominus_g)}{h}. \tag{21}$$

If $G'(t_0) \in \mathfrak{R}_{\mathcal{F}}$ satisfying (21) exists, we say that G is generalized Hukuhara differentiable at t_0.

Also we say that $G(t)$ is (i)-gH differentiable at t_0 if

$$\left[G'(t_0)\right]_\alpha = \left[G_1'(t_0, \alpha), G_2'(t_0, \alpha)\right] \tag{22}$$

and $G(t)$ is (ii)-gH differentiable at t_0 if

$$\left[G'(t_0)\right]_\alpha = \left[G_2'(t_0, \alpha), G_1'(t_0, \alpha)\right]. \tag{23}$$

Method 18 (solution of fuzzy differential equation using fuzzy differential equation approach). Consider the fuzzy differential equation taking in (15).

We have the following two cases.

Case 1. If we consider $u'(t)$ in the first from (i), then we have to solve the following system of ODEs:

$$\begin{aligned} \frac{d}{dt}\left(u_1^\alpha(t)\right) &= g_1^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right), \quad u_1^\alpha(t_0) = u_{0,1}^\alpha \\ \frac{d}{dt}\left(u_2^\alpha(t)\right) &= g_2^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right), \quad u_2^\alpha(t_0) = u_{0,2}^\alpha. \end{aligned} \tag{24}$$

Case 2. If we consider $u'(t)$ in the first from (ii), then we have to solve the following system of ODEs:

$$\begin{aligned} \frac{d}{dt}\left(u_1^\alpha(t)\right) &= g_2^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right), \quad u_1^\alpha(t_0) = u_{0,1}^\alpha \\ \frac{d}{dt}\left(u_2^\alpha(t)\right) &= g_1^\alpha\left(t, u_1^\alpha(t), u_2^\alpha(t)\right), \quad u_2^\alpha(t_0) = u_{0,2}^\alpha. \end{aligned} \tag{25}$$

In both cases, we should ensure that the solution $[u_1^\alpha(t), u_2^\alpha(t)]$ is valid level sets of a fuzzy number valued function and $[(d/dt)(u_1^\alpha(t)), (d/dt)(u_2^\alpha(t))]$ are valid level sets of a fuzzy valued function.

4. Model Formulation on Epidemic

There are so many mathematical models in biology; SIS model is an important model of them. In a given species population at time t, let $S(t)$ be the number of susceptible, which means the number of those who can be infected, and $I(t)$ be the number of infected persons in the species population. In this model, a susceptible species can become infected at a rate proportional to $S(t)I(t)$ and an infected species can recover and become susceptible again at a rate of $\gamma I(t)$ so that the model can be formulated as follows:

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t) I(t) + \gamma I(t) \\ \frac{dI(t)}{dt} &= \beta S(t) I(t) - \gamma I(t), \end{aligned} \tag{26}$$

where $S(t) = S_0 I(t) = I_0$ at $t = 0$ is the initial condition.

Here a susceptible $S(t)$ can become infected $I(t)$ at rate proportional of SI and on infected can recover and become susceptible again at a rate proportional to I.

$S(0) + I(0) = N(0)$ (total number of population).

Now taking $S(t)/N(t) = s(t)$, $I(t)/N(t) = i(t)$, the model can be written as

$$\begin{aligned} \frac{ds(t)}{dt} &= -ms(t) i(t) + \gamma i(t) \\ \frac{di(t)}{dt} &= ms(t) i(t) - \gamma i(t), \end{aligned} \tag{27}$$

where $m = \beta N(t)$ with initial condition $s(t) + i(t) = 1$.

Note 19 (dimension less of a model). Sometimes for a mathematical model, it is critical to find the dynamical behavior. However, the dependent variables in the model are connected with another dependent variable, which makes the finding of the behavior complicated. In this regard, there is some criterion in which we can eliminate the conditions and make the model more simple and which is very easy to solve. According to these circumstances, we reduce the dimension of the above model.

The crisp solution of the above system of equations is written in two different cases.

Case 1 (when $p = m - \gamma \neq 0$). In this case the solution can be written as

$$\begin{aligned} s(t) &= \frac{i_0 e^{pt}(m - p) + p - i_0 m}{i_0 m (e^{pt} - 1) + p}, \\ i(t) &= \frac{i_0 p e^{pt}}{i_0 m (e^{pt} - 1) + p}. \end{aligned} \tag{28}$$

Case 2 (when $p = m - \gamma = 0$). In this case the solution can be written as

$$\begin{aligned} s(t) &= \frac{1 + i_0(mt - 1)}{1 + mti_0}, \\ i(t) &= \frac{i_0}{1 + mti_0}. \end{aligned} \tag{29}$$

Note 20. May be someone will ask why do we take SIS model for comparing different solution strategy for solving in uncertain environment? Basically we take the particular SIS model and apply the different techniques in uncertain environment. Once one can be familiar with it, anyone can take one of the strategies which best fits their model.

Table 2

	$i_{e1}(t)$	$i_{e2}(t)$
$\frac{\partial p_1(i_0)}{\partial i_0} > 0$	$\frac{i_{01}(\alpha)\cdot pe^{pt}}{i_{01}(\alpha)m(e^{pt}-1)+p}$	$\frac{i_{02}(\alpha)pe^{pt}}{i_{02}(\alpha)m(e^{pt}-1)+p}$
$\frac{\partial p_1(i_0)}{\partial i_0} < 0$	$\frac{i_{02}(\alpha)pe^{pt}}{i_{02}(\alpha)m(e^{pt}-1)+p}$	$\frac{i_{01}(\alpha)pe^{pt}}{i_{01}(\alpha)m(e^{pt}-1)+p}$

5. Solution of the above SIS Model in Fuzzy Environment by Different Strategy

5.1. Solution of Fuzzy SIS Model via Differential Inclusion.

$$\left(\frac{ds(t)}{dt}, \frac{di(t)}{dt}\right) = \left(-ms(t)i(t)+\gamma i(t), ms(t)i(t)-\gamma i(t)\right) \quad (s_0, i_0) \in C. \tag{30}$$

The solution of the problem (30) using differential inclusion is obtained from the solution of the auxiliary

$$\left(\frac{ds(t)}{dt}, \frac{di(t)}{dt}\right) = \left(-ms(t)i(t)+\gamma i(t), ms(t)i(t)-\gamma i(t)\right) \quad (s_0, i_0) \in C(\alpha), \tag{31}$$

where $C(\alpha) = \{(1-i_0, i_0) \in R^2 : i_0 = (1-l)i_{01}(\alpha)+li_{02}(\alpha),\ l \in [0,1]\}$.

The attainable sets of the problem of (31) can be written as

$A_t(C(\alpha)) = \{u(t, s_0, i_0) : u(\cdot, s_0, i_0)$, solution of (31)$\} = \{u(t, s_0, i_0) : u'(t, s_0, i_0) = (-msi+\gamma i, msi-\gamma i), (s_0, i_0) \in C(\alpha)\}$.

Case 1 (when $p = m-\gamma \neq 0$).

$$A_t(C(\alpha)) = \left\{\left(\frac{i_0e^{pt}(m-p)+p-i_0m}{i_0m(e^{pt}-1)+p}, \frac{i_0pe^{pt}}{i_0m(e^{pt}-1)+p}\right) : i_0 = (1-l)i_{01}(\alpha)+li_{02}(\alpha),\ l \in [0,1]\right\}. \tag{32}$$

Case 2 (when $p = m-\gamma = 0$).

$$A_t(C(\alpha)) = \left\{\left(\frac{1+i_0(mt-1)}{1+mti_0}, \frac{i_0}{1+mti_0}\right) : i_0 = (1-l)i_{01}(\alpha)+li_{02}(\alpha),\ l \in [0,1]\right\}. \tag{33}$$

5.2. Solution of Fuzzy SIS Model by Extension Principle Method. Let $[i_{e1}(t), i_{e2}(t)]$ and $[s_{e1}(t), s_{e2}(t)]$ be the solution by extension principle method.

Now different cases arise.

Case 1 (when $p = m-\gamma \neq 0$). In this case the solution can be written as

$$i_{e1}(t) = \max\left\{\frac{i_0pe^{pt}}{i_0m(e^{pt}-1)+p} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\},$$
$$i_{e2}(t) = \min\left\{\frac{i_0pe^{pt}}{i_0m(e^{pt}-1)+p} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\}. \tag{34}$$

The solution depends on the function $p_1(i_0) = i_0pe^{pt}/(i_0m(e^{pt}-1)+p)$ whether it is increasing or decreasing. The solution can be written as in Table 2.

Here,

$$\frac{\partial p_1(i_0)}{\partial i_0} = \frac{i_0pe^{pt}}{\{i_{01}(\alpha)m(e^{pt}-1)+p\}^2}. \tag{35}$$

So, it depends upon p, whether it is negative or positive. If we take $p > 0$ then

$$i_{e1}(t) = \frac{i_{01}(\alpha)\cdot pe^{pt}}{i_{01}(\alpha)m(e^{pt}-1)+p},$$
$$i_{e2}(t) = \frac{i_{02}(\alpha)pe^{pt}}{i_{02}(\alpha)m(e^{pt}-1)+p} \tag{36}$$

and also

$$s_{e1}(t) = \max\left\{\frac{i_0e^{pt}(m-p)+p-i_0m}{i_0m(e^{pt}-1)+p} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\},$$
$$s_{e2}(t) = \min\left\{\frac{i_0e^{pt}(m-p)+p-i_0m}{i_0m(e^{pt}-1)+p} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\}. \tag{37}$$

The solution depends on the function $q_1(i_0) = (i_0e^{pt}(m-p)+p-i_0m)/(i_0m(e^{pt}-1)+p)$ whether it is increasing or decreasing. The solution can be written as in Table 3.

Here,

$$\frac{\partial q_1(i_0)}{\partial i_0} = -\frac{pm(1+e^{pt})}{\{i_0m(e^{pt}-1)+p\}^2} < 0. \tag{38}$$

Table 3

	$s_{e1}(t)$	$s_{e2}(t)$
$\frac{\partial q_1(i_0)}{\partial i_0} > 0$	$\frac{i_{01}(\alpha)e^{pt}(m-p)+p-i_{01}(\alpha)m}{i_{01}(\alpha)m(e^{pt}-1)+p}$	$\frac{i_{02}(\alpha)e^{pt}(m-p)+p-i_{02}(\alpha)m}{i_{02}(\alpha)m(e^{pt}-1)+p}$
$\frac{\partial q_1(i_0)}{\partial i_0} < 0$	$\frac{i_{02}(\alpha)e^{pt}(m-p)+p-i_{02}(\alpha)m}{i_{02}(\alpha)m(e^{pt}-1)+p}$	$\frac{i_{01}(\alpha)e^{pt}(m-p)+p-i_{01}(\alpha)m}{i_{01}(\alpha)m(e^{pt}-1)+p}$

Table 4

	$i_{e1}(t)$	$i_{e2}(t)$
$\frac{\partial p_2(i_0)}{\partial i_0} > 0$	$\frac{i_{01}(\alpha)}{1+mti_{01}(\alpha)}$	$\frac{i_{02}(\alpha)}{1+mti_{02}(\alpha)}$
$\frac{\partial p_2(i_0)}{\partial i_0} < 0$	$\frac{i_{02}(\alpha)}{1+mti_{02}(\alpha)}$	$\frac{i_{01}(\alpha)}{1+mti_{01}(\alpha)}$

Table 5

	$s_{e1}(t)$	$s_{e2}(t)$
$\frac{\partial q_2(i_0)}{\partial i_0} > 0$	$\frac{1+i_{01}(\alpha)(mt-1)}{1+mti_{01}(\alpha)}$	$\frac{1+i_{02}(\alpha)(mt-1)}{1+mti_{02}(\alpha)}$
$\frac{\partial q_2(i_0)}{\partial i_0} < 0$	$\frac{1+i_{02}(\alpha)(mt-1)}{1+mti_{02}(\alpha)}$	$\frac{1+i_{01}(\alpha)(mt-1)}{1+mti_{01}(\alpha)}$

So, the solution is given by

$$s_{e1}(t) = \frac{i_{02}(\alpha)e^{pt}(m-p)+p-i_{02}(\alpha)m}{i_{02}(\alpha)m(e^{pt}-1)+p},$$
$$s_{e2}(t) = \frac{i_{01}(\alpha)e^{pt}(m-p)+p-i_{01}(\alpha)m}{i_{01}(\alpha)m(e^{pt}-1)+p}. \tag{39}$$

Case 2 (when $p = m - \gamma = 0$). In this case the solution can be written as

$$i_{e1}(t) = \max\left\{\frac{i_0}{1+mti_0} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\},$$
$$i_{e2}(t) = \min\left\{\frac{i_0}{1+mti_0} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\}. \tag{40}$$

The solution depends on the function $p_2(i_0) = i_0/(1 + mti_0)$ whether it is increasing or decreasing. The solution can be written as in Table 4.

Here,

$$\frac{\partial p_2(i_0)}{\partial i_0} = \frac{1}{\{1+mti_{01}(\alpha)\}^2} > 0. \tag{41}$$

Hence the solution is

$$i_{e1}(t) = \frac{i_{01}(\alpha)}{1+mti_{01}(\alpha)},$$
$$i_{e2}(t) = \frac{i_{02}(\alpha)}{1+mti_{02}(\alpha)} \tag{42}$$

and also

$$s_{e1}(t) = \max\left\{\frac{1+i_0(mt-1)}{1+mti_0} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\},$$
$$s_{e2}(t) = \min\left\{\frac{1+i_0(mt-1)}{1+mti_0} \mid i_0 \in [i_{01}(\alpha), i_{02}(\alpha)]\right\}. \tag{43}$$

The solution depends on the function $q_2(i_0) = (1 + i_0(mt - 1))/(1 + mti_0)$ whether it is increasing or decreasing. The solution can be written as in Table 5.

Here,

$$\frac{\partial q_2(i_0)}{\partial i_0} = -\frac{1}{\{1+mti_0\}^2} < 0. \tag{44}$$

Hence the solution is

$$s_{e1}(t) = \frac{1+i_{02}(\alpha)(mt-1)}{1+mti_{02}(\alpha)},$$
$$s_{e2}(t) = \frac{1+i_{01}(\alpha)(mt-1)}{1+mti_{01}(\alpha)}. \tag{45}$$

5.3. Solution of Fuzzy SIS Model by Fuzzy Differential Equation Approach. Let $[i_1(t,\alpha), i_2(t,\alpha)]$ and $[s_1(t,\alpha), s_2(t,\alpha)]$ be the solution using generalized Hukuhara derivative approach.

Now different cases can be found as follows.

Case 1 ($s(t)$ and $i(t)$ is (i)-gH differentiable). In this case the differential equation transforms to

$$\begin{aligned}\frac{ds_1(t,\alpha)}{dt} &= -ms_2(t,\alpha)\,i_2(t,\alpha) + \gamma i_1(t,\alpha)\\ \frac{ds_2(t,\alpha)}{dt} &= -ms_1(t,\alpha)\,i_1(t,\alpha) + \gamma i_2(t,\alpha)\\ \frac{di_1(t,\alpha)}{dt} &= ms_1(t,\alpha)\,i_1(t,\alpha) - \gamma i_2(t,\alpha)\\ \frac{di_2(t,\alpha)}{dt} &= ms_2(t,\alpha)\,i_2(t,\alpha) - \gamma i_1(t,\alpha),\end{aligned} \tag{46}$$

with initial conditions $s_1(0,\alpha) = s_{01}(\alpha)$, $s_2(0,\alpha) = s_{02}(\alpha)$, $i_1(0,\alpha) = i_{01}(\alpha)$, and $i_2(0,\alpha) = i_{02}(\alpha)$.

Case 2 ($s(t)$ is (i)-gH and $i(t)$ is (ii)-gH differentiable). In this case the differential equation transforms to

$$\begin{aligned}\frac{ds_1(t,\alpha)}{dt} &= -ms_2(t,\alpha)\,i_2(t,\alpha) + \gamma i_1(t,\alpha)\\ \frac{ds_2(t,\alpha)}{dt} &= -ms_1(t,\alpha)\,i_1(t,\alpha) + \gamma i_2(t,\alpha)\\ \frac{di_2(t,\alpha)}{dt} &= ms_1(t,\alpha)\,i_1(t,\alpha) - \gamma i_2(t,\alpha)\\ \frac{di_1(t,\alpha)}{dt} &= ms_2(t,\alpha)\,i_2(t,\alpha) - \gamma i_1(t,\alpha),\end{aligned} \tag{47}$$

with initial conditions $s_1(0,\alpha) = s_{01}(\alpha)$, $s_2(0,\alpha) = s_{02}(\alpha)$, $i_1(0,\alpha) = i_{01}(\alpha)$, and $i_2(0,\alpha) = i_{02}(\alpha)$.

Case 3 ($s(t)$ is (ii)-gH and $i(t)$ is (i)-gH differentiable). In this case the differential equation transforms to

$$\begin{aligned}\frac{ds_2(t,\alpha)}{dt} &= -ms_2(t,\alpha)\,i_2(t,\alpha) + \gamma i_1(t,\alpha)\\ \frac{ds_1(t,\alpha)}{dt} &= -ms_1(t,\alpha)\,i_1(t,\alpha) + \gamma i_2(t,\alpha)\\ \frac{di_1(t,\alpha)}{dt} &= ms_1(t,\alpha)\,i_1(t,\alpha) - \gamma i_2(t,\alpha)\\ \frac{di_2(t,\alpha)}{dt} &= ms_2(t,\alpha)\,i_2(t,\alpha) - \gamma i_1(t,\alpha),\end{aligned} \tag{48}$$

with initial conditions $s_1(0,\alpha) = s_{01}(\alpha)$, $s_2(0,\alpha) = s_{02}(\alpha)$, $i_1(0,\alpha) = i_{01}(\alpha)$, and $i_2(0,\alpha) = i_{02}(\alpha)$.

Case 4 ($s(t)$ and $i(t)$ is (ii)-gH differentiable). In this case the differential equation transforms to

$$\begin{aligned}\frac{ds_2(t,\alpha)}{dt} &= -ms_2(t,\alpha)\,i_2(t,\alpha) + \gamma i_1(t,\alpha)\\ \frac{ds_1(t,\alpha)}{dt} &= -ms_1(t,\alpha)\,i_1(t,\alpha) + \gamma i_2(t,\alpha)\\ \frac{di_2(t,\alpha)}{dt} &= ms_1(t,\alpha)\,i_1(t,\alpha) - \gamma i_2(t,\alpha)\\ \frac{di_1(t,\alpha)}{dt} &= ms_2(t,\alpha)\,i_2(t,\alpha) - \gamma i_1(t,\alpha),\end{aligned} \tag{49}$$

with initial condition $s_1(0,\alpha) = s_{01}(\alpha)$, $s_2(0,\alpha) = s_{02}(\alpha)$, $i_1(0,\alpha) = i_{01}(\alpha)$, and $i_2(0,\alpha) = i_{02}(\alpha)$

6. Modeling SIS in Interval Environment

The problem in interval environment is

$$\begin{aligned}\frac{ds(t,\lambda)}{dt} &= -ms(t;\lambda)\,i(t;\lambda) + \gamma i(t;\lambda)\\ \frac{di(t,\lambda)}{dt} &= ms(t;\lambda)\,i(t;\lambda) - \gamma i(t;\lambda),\end{aligned} \tag{50}$$

where $m = \beta N(t)$ with initial condition $i(0;\lambda) = (i_{0l})^{1-\lambda}(i_{0u})^{\lambda}$ at $t = 0$, $s(t) + i(t) = 1$.

We get the solution for two cases as follows.

Case 1 (when $p \neq 0$). The solution is written as

$$\begin{aligned}i(t;\lambda) &= \frac{(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}\,pe^{pt}}{(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}\,m(e^{pt}-1)+p},\\ s(t;\lambda) &= 1 - \frac{(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}\,pe^{pt}}{(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}\,m(e^{pt}-1)+p}.\end{aligned} \tag{51}$$

Case 2 (when $p = 0$). The solution is written as

$$\begin{aligned}i(t,\lambda) &= \frac{(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}}{1+mt(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}},\\ s(t;\lambda) &= 1 - \frac{(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}}{1+mt(i_{0u})^{1-\lambda}(i_{0v})^{\lambda}}.\end{aligned} \tag{52}$$

7. Numerical Examples

7.1. Numerical Example on Fuzzy Cases. Find the solution after $t = 10$ when $\widetilde{S}_0 = (0.80, 0.85, 0.90, 0.95)$ and $\widetilde{I}_0 = (0.05, 0.10, 0.15, 0.20)$, when $m = 0.3$ and $\gamma = 0.005$.

Solution by differential inclusion and extension principle and fuzzy differential equation is given by

$$\begin{aligned}&[(s_{i1}(t,\alpha), s_{i2}(t,\alpha)); (i_{i1}(t,\alpha), i_{i2}(t,\alpha))],\\ &[(s_{e1}(t,\alpha), s_{e2}(t,\alpha)); (i_{e1}(t,\alpha), i_{e2}(t,\alpha))],\\ &[(s_1(t,\alpha), s_2(t,\alpha)); (i_1(t,\alpha), i_2(t,\alpha))].\end{aligned} \tag{53}$$

7.1.1. Solution by Differential Inclusion.

Case 1 (when $p \neq 0$).

$$\begin{aligned}A_t(C(\alpha)) = \Bigg\{\Bigg(&\frac{0.005i_0e^{0.295t} + 0.295 - 0.305i_0}{0.3i_0(e^{0.295t}-1)+0.295},\\ &\frac{0.295i_0e^{0.295t}}{0.3i_0(e^{0.295t}-1)+0.295}\Bigg) : i_0 = (1-l)(0.05\\ &+ 0.05\alpha) + l(0.20 - 0.05\alpha),\ l \in [0,1]\Bigg\}.\end{aligned} \tag{54}$$

Table 6: Solution boundary for $t = 10$.

α	$s_{i1}(t,\alpha)$	$s_{i2}(t,\alpha)$	$i_{i1}(t,\alpha)$	$i_{i2}(t,\alpha)$
0	0.5022	0.1832	0.4974	0.8160
0.2	0.4549	0.1923	0.5446	0.8070
0.4	0.4152	0.2022	0.5843	0.7971
0.6	0.3814	0.2129	0.6181	0.7864
0.8	0.3523	0.2246	0.6471	0.7747
1	0.3270	0.2375	0.6724	0.7618

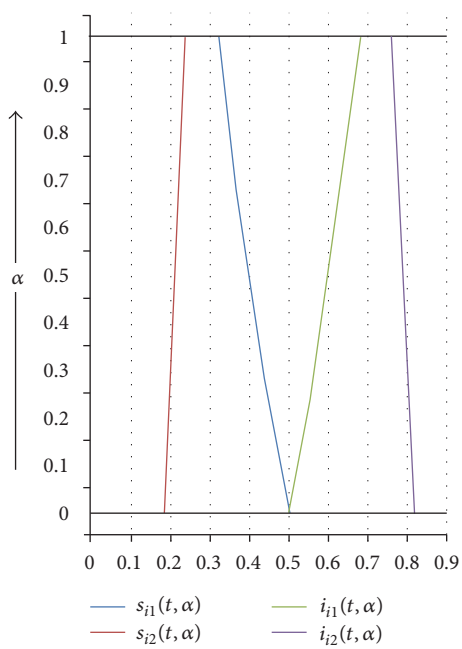

Figure 1: Solution boundary for $t = 10$.

The boundary of the solution is given by

$$\begin{aligned} s_{i1}(t,\alpha) &= \frac{0.005(0.05+0.05\alpha)e^{0.295t}+0.295-0.305(0.05+0.05\alpha)}{0.3(0.05+0.05\alpha)(e^{0.295t}-1)+0.295} \\ s_{i2}(t,\alpha) &= \frac{0.005(0.20-0.05\alpha)e^{0.295t}+0.295-0.305(0.20-0.05\alpha)}{0.3(0.20-0.05\alpha)(e^{0.295t}-1)+0.295} \\ i_{i1}(t,\alpha) &= \frac{0.295(0.05+0.05\alpha)e^{0.295t}}{0.3(0.05+0.05\alpha)(e^{0.295t}-1)+0.295} \\ i_{i2}(t,\alpha) &= \frac{0.295(0.20-0.05\alpha)e^{0.295t}}{0.3(0.20-0.05\alpha)(e^{0.295t}-1)+0.295}. \end{aligned} \tag{55}$$

Remarks 21. From Figure 1 and Table 6 it shows that $s_{i1}(t,\alpha)$ is decreasing and $s_{i2}(t,\alpha)$ is increasingwhereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the boundary of the solution. The solution for $\tilde{s}(t)$ gives the natural weak solution but $i(t)$ gives the natural strong solution.

Case 2 (when $p = 0$). The boundary of the solutions is

$$\begin{aligned} A_t(C(\alpha)) = \Bigg\{ &\left(\frac{1+i_0(0.3t-1)}{1+0.3ti_0}, \frac{i_0}{1+0.3ti_0}\right) : i_0 \\ &= (1-l)(0.05+0.05\alpha)+l(0.20-0.05\alpha),\ l \\ &\in [0,1] \Bigg\}. \end{aligned} \tag{56}$$

Remarks 22. From Figure 2 and Table 7 it shows that $s_{i1}(t,\alpha)$ is decreasing and $s_{i2}(t,\alpha)$ is increasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the boundary of the solution. The solution for $\tilde{s}(t)$ gives the natural weak solution but $i(t)$ gives the natural strong solution.

7.2. Solution by Extension Principle.

Case 1 (when $p \neq 0$). Here the solutions are given by

$$\begin{aligned} s_{i1}(t,\alpha) &= \frac{0.005(0.20-0.05\alpha)e^{0.295t}+0.295-0.305(0.20-0.05\alpha)}{0.3(0.20-0.05\alpha)(e^{0.295t}-1)+0.295} \\ s_{i2}(t,\alpha) &= \frac{0.005(0.05+0.05\alpha)e^{0.295t}+0.295-0.305(0.05+0.05\alpha)}{0.3(0.05+0.05\alpha)(e^{0.295t}-1)+0.295} \\ i_{i1}(t,\alpha) &= \frac{0.295(0.05+0.05\alpha)e^{0.295t}}{0.3(0.05+0.05\alpha)(e^{0.295t}-1)+0.295} \\ i_{i2}(t,\alpha) &= \frac{0.295(0.20-0.05\alpha)e^{0.295t}}{0.3(0.20-0.05\alpha)(e^{0.295t}-1)+0.295}. \end{aligned} \tag{57}$$

Remarks 23. From Figure 3 and Table 8 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\tilde{s}(t)$ gives the natural strong solution but $i(t)$ gives the natural strong solution.

Table 7: Solutions boundary for $t = 10$.

α	$s_{i1}(t,\alpha)$	$s_{i2}(t,\alpha)$	$i_{i1}(t,\alpha)$	$i_{i2}(t,\alpha)$
0	0.9565	0.8750	0.0435	0.1250
0.2	0.9492	0.8790	0.0508	0.1210
0.4	0.9421	0.8831	0.0579	0.1169
0.6	0.9355	0.8874	0.0645	0.1126
0.8	0.9291	0.8919	0.0709	0.1081
1	0.9231	0.8966	0.0769	0.1034

Table 8: Solution for $t = 10$.

α	$s_{e1}(t,\alpha)$	$s_{e2}(t,\alpha)$	$i_{e1}(t,\alpha)$	$i_{e2}(t,\alpha)$
0	0.1832	0.5022	0.4974	0.8160
0.2	0.1923	0.4549	0.5446	0.8070
0.4	0.2022	0.4152	0.5843	0.7971
0.6	0.2129	0.3814	0.6181	0.7864
0.8	0.2246	0.3523	0.6471	0.7747
1	0.2375	0.3270	0.6724	0.7618

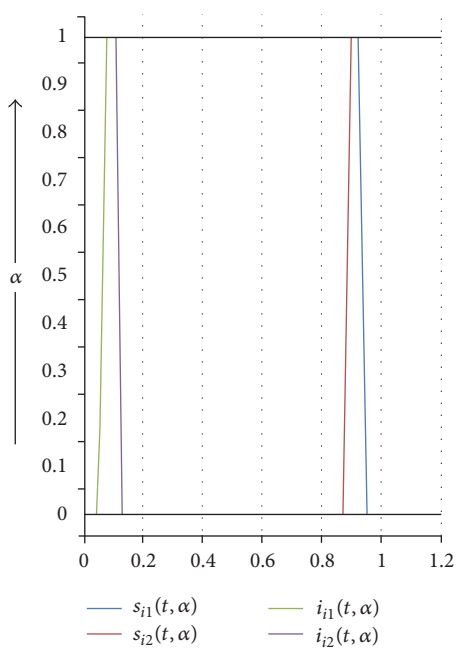

Figure 2: Solutions boundary for $t = 10$.

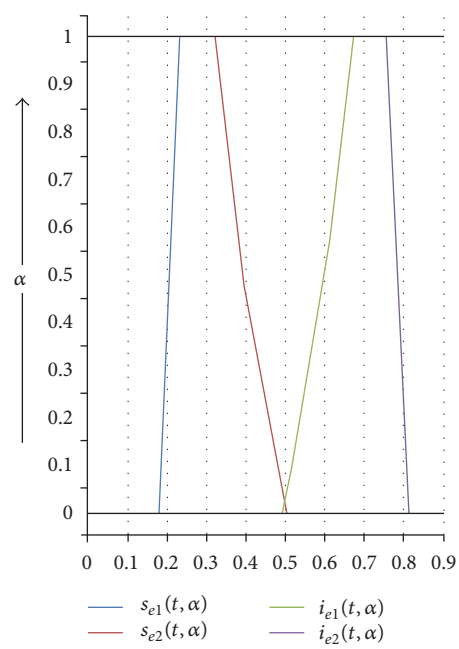

Figure 3: Solution for $t = 10$.

Case 2 (when $p = 0$). The solutions are given by

$$\begin{aligned} s_{i1}(t,\alpha) &= \frac{1 + (0.20 - 0.05\alpha)(0.3t - 1)}{1 + 0.3t(0.20 - 0.05\alpha)} \\ s_{i2}(t,\alpha) &= \frac{1 + (0.05 + 0.05\alpha)(0.3t - 1)}{1 + 0.3t(0.05 + 0.05\alpha)} \\ i_{i1}(t,\alpha) &= \frac{(0.05 + 0.05\alpha)}{1 + 0.3t(0.05 + 0.05\alpha)} \\ i_{i2}(t,\alpha) &= \frac{(0.20 - 0.05\alpha)}{1 + 0.3t(0.20 - 0.05\alpha)}. \end{aligned} \tag{58}$$

Remarks 24. From Figure 4 and Table 9 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\tilde{s}(t)$ gives the natural strong solution and $i(t)$ gives the natural strong solution.

7.3. Solution by Fuzzy Differential Equation Approach. Now the solutions for different cases are given by the following.

Case 1 ($s(t)$ and $i(t)$ is (i)-gH differentiable).

Remarks 25. From Figure 5 and Table 10 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is

Table 9: Solution for $t = 10$.

α	$s_{e1}(t,\alpha)$	$s_{e2}(t,\alpha)$	$i_{e1}(t,\alpha)$	$i_{e2}(t,\alpha)$
0	0.8750	0.9565	0.0435	0.1250
0.2	0.8790	0.9492	0.0508	0.1210
0.4	0.8831	0.9421	0.0579	0.1169
0.6	0.8874	0.9355	0.0645	0.1126
0.8	0.8919	0.9291	0.0709	0.1081
1	0.8966	0.9231	0.0769	0.1034

Table 10: Solutions for $t = 10$.

α	$s_1(t,\alpha)$	$s_2(t,\alpha)$	$i_1(t,\alpha)$	$i_2(t,\alpha)$
0	0.4471	0.9046	0.0954	0.5529
0.2	0.4845	0.8810	0.1190	0.5155
0.4	0.5209	0.8564	0.1436	0.4791
0.6	0.5564	0.8308	0.1692	0.4436
0.8	0.5908	0.8043	0.1957	0.4092
1	0.6243	0.7768	0.2232	0.3757

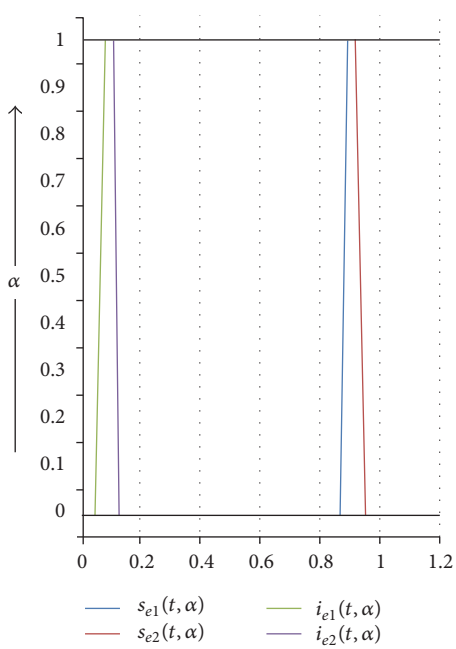

Figure 4: Solution for $t = 10$.

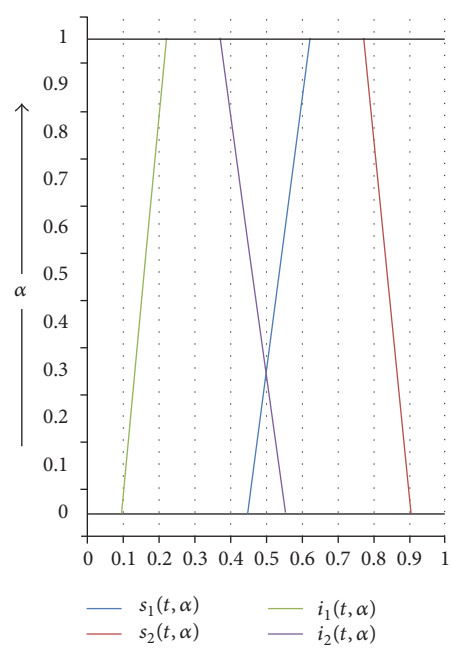

Figure 5: Figure for $t = 10$.

increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\tilde{s}(t)$ gives the natural strong solution and $\tilde{i}(t)$ gives the natural strong solution.

Case 2 ($s(t)$ is (i)-gH and $i(t)$ is (ii)-gH differentiable).

Remarks 26. From Figure 6 and Table 11 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\tilde{s}(t)$ gives the natural weak solution but $\tilde{i}(t)$ gives the natural strong solution.

Case 3 ($s(t)$ is (ii)-gH and $i(t)$ is (i)-gH differentiable).

Remarks 27. From Figure 7 and Table 12 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\tilde{s}(t)$ gives the natural weak solution but $\tilde{i}(t)$ gives the natural strong solution.

Table 11: Solution for $t = 10$.

α	$s_1(t,\alpha)$	$s_2(t,\alpha)$	$i_1(t,\alpha)$	$i_2(t,\alpha)$
0	0.7403	0.6601	0.1097	0.4899
0.2	0.7359	0.6662	0.1341	0.4638
0.4	0.7313	0.6722	0.1587	0.4378
0.6	0.7266	0.6782	0.1834	0.4118
0.8	0.7218	0.6840	0.2082	0.3860
1	0.7168	0.6898	0.2332	0.3602

Table 12: Solution for $t = 10$.

α	$s_1(t,\alpha)$	$s_2(t,\alpha)$	$i_1(t,\alpha)$	$i_2(t,\alpha)$
0	0.7403	0.6601	0.1097	0.4899
0.2	0.7359	0.6662	0.1341	0.4638
0.4	0.7313	0.6722	0.1587	0.4378
0.6	0.7266	0.6782	0.1834	0.4118
0.8	0.7218	0.6840	0.2082	0.3860
1	0.7168	0.6898	0.2332	0.3602

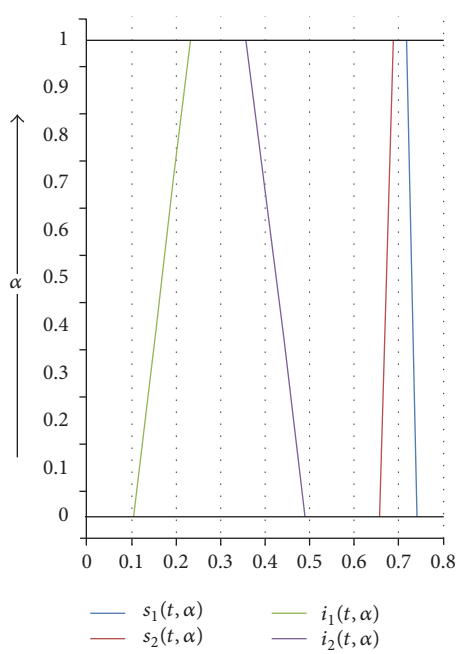

Figure 6: Figure for $t = 10$.

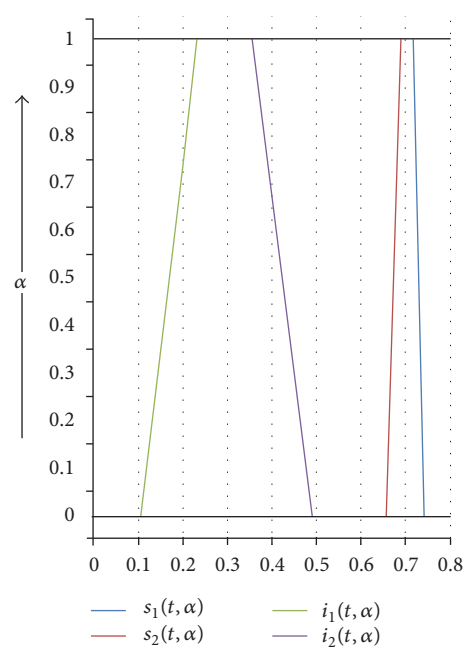

Figure 7: Solution for $t = 10$.

Case 4 ($s(t)$ and $i(t)$ is (ii)-gH differentiable).

Remarks 28. From Figure 8 and Table 13 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\widetilde{s}(t)$ gives the natural strong solution and $i(t)$ gives the natural strong solution.

7.4. Numerical Example on Interval Cases. Find the solution after $t = 10$ when $\widehat{S} = [0.80, 0.95]$ and $\widehat{I} = [0.05, 0.20]$, when $m = 0.3$ and $\gamma = 0.005$.

Case 1 (when $p \neq 0$).

$$s(t;\lambda) = 1 - \frac{0.295\,(0.05)^{1-\lambda}\,(0.20)^{\lambda}\,e^{0.295t}}{0.3\,(0.05)^{1-\lambda}\,(0.20)^{\lambda}\left(e^{0.295t}-1\right)+0.295},$$
$$i(t;\lambda) = \frac{0.295\,(0.05)^{1-\lambda}\,(0.20)^{\lambda}\,e^{0.295t}}{0.3\,(0.05)^{1-\lambda}\,(0.20)^{\lambda}\left(e^{0.295t}-1\right)+0.295}. \quad (59)$$

Table 13: Solution for $t = 10$.

α	$s_1(t,\alpha)$	$s_2(t,\alpha)$	$i_1(t,\alpha)$	$i_2(t,\alpha)$
0	0.6855	0.7376	0.2624	0.3145
0.2	0.6872	0.7317	0.2683	0.3128
0.4	0.6883	0.7254	0.2746	0.3117
0.6	0.6875	0.7171	0.2829	0.3125
0.8	0.6914	0.7143	0.2857	0.3086
1	0.6950	0.7114	0.2886	0.3050

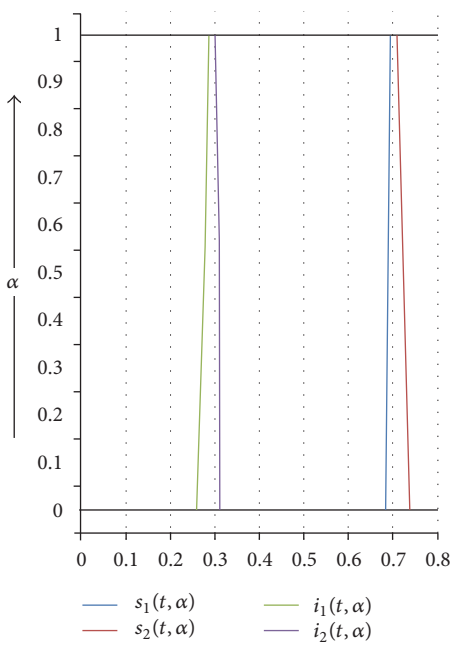

Figure 8: Figure for $t = 10$.

Table 14: Solution for $t = 10$.

λ	$s(t;\lambda)$	$i(t;\lambda)$
0	0.5026	0.4974
0.2	0.4309	0.5691
0.4	0.3610	0.6390
0.6	0.2955	0.7045
0.8	0.2361	0.7639
1	0.1840	0.8160

Table 15: Solution at $t = 10$.

λ	$s(t;\lambda)$	$i(t,\lambda)$
0	0.9565	0.0435
0.2	0.9449	0.0551
0.4	0.9310	0.0690
0.6	0.9146	0.0854
0.8	0.8958	0.1042
1	0.8750	0.1250

Case 2 (when $p = 0$).

$$s(t;\lambda) = 1 - \frac{(0.05)^{1-\lambda}(0.20)^{\lambda}}{1 + 0.3t(0.05)^{1-\lambda}(0.20)^{\lambda}},$$
$$i(t,\lambda) = \frac{(0.05)^{1-\lambda}(0.20)^{\lambda}}{1 + 0.3t(0.05)^{1-\lambda}(0.20)^{\lambda}}. \quad (60)$$

Remarks 29. From Figures 9 and 10 and Tables 14 and 15 it shows that $s_{i1}(t,\alpha)$ is increasing and $s_{i2}(t,\alpha)$ is decreasing whereas $i_{i1}(t,\alpha)$ is increasing and $i_{i2}(t,\alpha)$ is decreasing. The figure demonstrates the solution of the problem. The solution for $\tilde{s}(t)$ gives the natural strong solution but $i(t)$ gives the natural strong solution.

8. Conclusion

In this paper we study the different solution strategies for analyzing fuzzy differential equation and application in mathematical biology model, namely, SIS model, which is considered to be an important area of research in biological research. The approaches regarding fuzzy differential inclusion, extension principle, and fuzzy differential equation were applied to find the fuzzy solutions of the given model. The whole paper is concluded as follows:

(i) Demonstrating SIS model with fuzzy numbers which enabled meeting the uncertain parameters as well, which is appreciatively helpful for the decision makers to investigate the situation in a more precise manner.

(ii) The different approaches having significant place in fuzzy calculus efficiently made it possible to obtain the fuzzy solution of the governing model by different methods.

(iii) The use of correlated fuzzy number in the said model is for finding the fuzzy solution.

Thus in the future we seek to apply these concepts to different types of differential equation models in fuzzy environments.

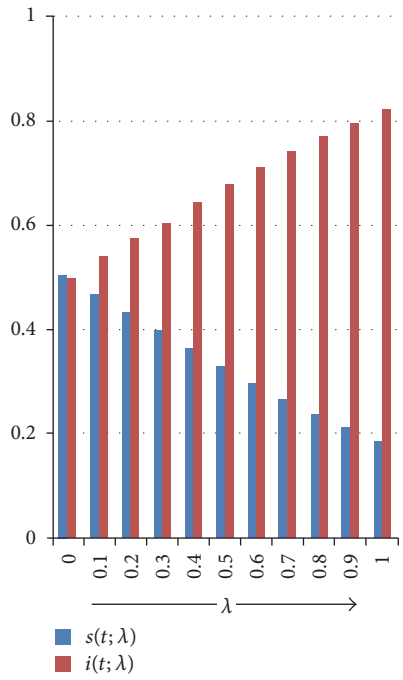

Figure 9: Figure at $t = 10$.

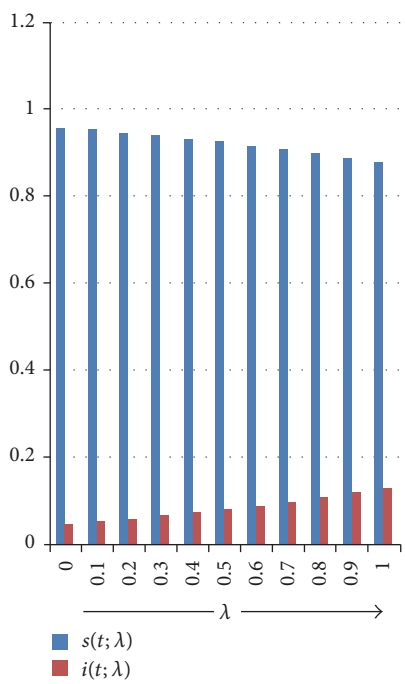

Figure 10: Figure at $t = 10$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fuzzy Fixed Point Results For Φ Contractive Mapping with Applications

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In this paper, using rational type contractions, common fuzzy fixed point result for Φ contractive mappings involving control functions as coefficients of contractions in the setting of complex-valued metric space is established. The derived results generalizes some result in the existing literature. To show the validity of the derived results an appropriate example and applications are also discussed.

1. Introduction

Fixed point theory is considered to be the most interesting and dynamic area of research in the development of nonlinear analysis. In this area, Banach contraction principal [1] is an initiative for researchers during last few decades. This principal plays an important and key role in investigating the existence and uniqueness of solution to various problems in mathematics, physics, engineering, medicines, and social sciences which leads to mathematical models design by system of nonlinear integral equations, functional equations, and differential equations. Banach contraction principal has been generalized in different directions by changing the condition of contraction or by the underlying space. For instance, we refer to [2–8]. Particularly Dass and Gupta [9] extended the Banach contraction principal for rational type inequality and obtained fixed point results in metric space, which is further extended to different spaces by many authors. In the meanwhile researchers realized that where division occurs in cone metric spaces, the concept of rational type contraction is not meaningful.

To overcome this problem a new metric space was recently established by Azam et al. [10], known as complex-valued metric space, where the author obtained fixed point results via rational type contractive condition. This work was further extended by Sitthikul and Saejung [11]. Afterwards Rouzkard and Imdad [12] extended the aforementioned results of Azam et al. by obtaining common fixed point results which satisfies certain rational contractions in complex-valued metrics spaces. Consequently in [13, 14], the authors extended common fixed point results for multivalued mappings in complex-valued metric space. In addition, Sintunavarat and Kumam [15] derived common fixed point results by substituting the constant coefficients in contractive condition by control functions.

Heilpern [16] established the concept of fuzzy mappings and obtained fixed point results in metric linear space. He generalized the results of [1, 17], under the consideration of fuzzy mappings in complete metric linear spaces. Several mathematicians extended the work of Heilpern in different metric spaces for linear contraction. For instance, we refer to [18–24]. While in [25], the author investigated for fuzzy common fixed point with rational contractive condition. The concept of fuzziness is helpful in solving such real world problems where uncertainty occurs and many authors solve such problems by mathematical modeling in terms of fuzzy differential equations. For instance in [26], the author investigated the existence of solution for fuzzy differential equations. Nieto [27] worked on Cauchy problems for continuous fuzzy differential equations. Song et al. studied the global existence of solutions to fuzzy differential equation [28]. Moreover, the existence of fuzzy solution of first order initial value problem

was studied in [29], which is lately extended to integrodifferential equations [30]. Recently Long et al. [31] combined the matrix convergent to zero technique with calculations of fuzzy-valued functions, which is quite a new approach to study the system of differential and partial differential equations (PDE's) in generalized fuzzy metric spaces. In [32] Long et al. improved different results existing in the literature on the existence of coincidence points for a pair of mappings and studied applications to partial differential equations with uncertainty. After the wide study of fuzziness in the system of differential equations, it has now been studied in fractional differential equations to obtain the existence and uniqueness of fuzzy solution under Caputo generalized Hukuhara differentiability; for instance, see [33].

In the current work, using rational type contraction, common fuzzy fixed point results for Φ contractive mappings are studied. The established results generalizes some results from the exiting literature particularly the result of Joshi et al. [34] for fuzzy mappings. Applications and appropriate example are also provided.

2. Preliminaries

Definition 1 (see [10]). Assume $\mathfrak{C}$ is the set of complex numbers. For $\varepsilon_1, \varepsilon_2 \in \mathfrak{C}$ we define a partial order $\precsim$ on $\mathfrak{C}$ as follows:

(Ci) $\varepsilon_1 \precsim \varepsilon_2 \Leftrightarrow \mathrm{Re}(\varepsilon_1) \leq \mathrm{Re}(\varepsilon_2)$ and $\mathrm{Im}(\varepsilon_1) \leq \mathrm{Im}(\varepsilon_2)$;

(Cii) $\varepsilon_1 \prec \varepsilon_2 \Leftrightarrow \mathrm{Re}(\varepsilon_1) < \mathrm{Re}(\varepsilon_2)$ and $\mathrm{Im}(\varepsilon_1) < \mathrm{Im}(\varepsilon_2)$;

(Ciii) $\varepsilon_1 \precnsim \varepsilon_2 \Leftrightarrow \mathrm{Re}(\varepsilon_1) = \mathrm{Re}(\varepsilon_2)$ and $\mathrm{Im}(\varepsilon_1) < \mathrm{Im}(\varepsilon_2)$;

(Civ) $\varepsilon_1 = \varepsilon_2 \Leftrightarrow \mathrm{Re}(\varepsilon_1) = \mathrm{Re}(\varepsilon_2)$ and $\mathrm{Im}(\varepsilon_1) = \mathrm{Im}(\varepsilon_2)$.

Clearly if $a \leq b, \Rightarrow az \precsim bz$, for all $z \in \mathfrak{C}$ and for all $a, b \in \mathscr{R}$. Note that if $\varepsilon_1 \neq \varepsilon_2$ and one of (Ci), (Cii) and (Ciii) is satisfied then $\varepsilon_1 \precnsim \varepsilon_2$, and we write $\varepsilon_1 = \varepsilon_2$ if only (Civ) is satisfied. Note that

(i) $0 \precsim \varepsilon_1 \precnsim \varepsilon_2 \Rightarrow |\varepsilon_1| < |\varepsilon_2|, \ \forall \varepsilon_1, \varepsilon_2 \in \mathfrak{C}$;

(ii) $\varepsilon_1 \precsim \varepsilon_2$ and $\varepsilon_2 \prec \varepsilon_3 \ \Rightarrow \varepsilon_1 \prec \varepsilon_3, \ \forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathfrak{C}$.

Definition 2 (see [10]). Let $\mathscr{X}$ be a nonempty set and $d : \mathscr{X} \times \mathscr{X} \to \mathfrak{C}$ be a mapping which satisfies the following conditions:

(1) $0 \precsim \rho(z, w)$, for all $z, w \in \mathscr{X}$ and $\rho(z, w) = 0$ if and only if $z = w$;

(2) $\rho(z, w) = \rho(w, z)$, for all $z, w \in \mathscr{X}$;

(3) $\rho(z, w) \precsim \rho(z, z_1) + \rho(z_1, w)$, for all $z, z_1, w \in \mathscr{X}$.

Then $(\mathscr{X}, \rho)$ is called a complex-valued metric space.

Definition 3 (see [10]). A point $z \in \mathscr{X}$ is known as an interior point of a set $Z \subseteq \mathscr{X}$, if we find $0 \prec \epsilon \in \mathfrak{C}$ such that

$$\mathfrak{B}(z, \epsilon) = \{w \in \mathscr{X} : \rho(z, w) \prec \epsilon\} \subseteq Z. \tag{1}$$

A point $z \in Z$ is known as the limit point of Z, if there exists an open ball $\mathfrak{B}(z, \epsilon)$ such that

$$\mathfrak{B}(z, \epsilon) \cap (Z \setminus \{z\}) \neq \phi, \tag{2}$$

where $0 \prec \epsilon \in \mathfrak{C}$. A subset Z of $\mathscr{X}$ is said to be open if each point of Z is an interior point of Z. Furthermore, Z is said to be closed if it contain all its limit points.
The family

$$\mathscr{B} = \{\mathfrak{B}(z, \epsilon) : z \in \mathscr{X}, 0 \prec \epsilon\} \tag{3}$$

is a subbasis for a Hausdorff topology $\mathfrak{T}$ on $\mathscr{X}$.

Now recall some definitions from [13, 14].

Let $(\mathscr{X}, \rho)$ be a complex-valued metric space. Throughout this paper we denoted the family of all nonempty closed bounded subsets of complex-valued metric space $\mathscr{X}$ by $\mathscr{CB}(\mathscr{X})$. For $\nu \in \mathfrak{C}$ we represent

$$s(\nu) = \{z \in \mathfrak{C} : \nu \precsim z\} \tag{4}$$

and for $w \in \mathscr{X}$ and $B \in \mathscr{CB}(\mathscr{X})$.

$$s(w, B) = \bigcup_{b \in B} s(\rho(w, b)) = \bigcup_{b \in B} \{z \in \mathfrak{C} : \rho(w, b) \precsim z\}. \tag{5}$$

For $A, B \in \mathscr{CB}(\mathscr{X})$, we denote

$$s(A, B) = \left(\bigcap_{p \in A} s(p, B)\right) \cap \left(\bigcap_{q \in B} s(q, A)\right). \tag{6}$$

Let τ be a multivalued mapping from $\mathscr{X}$ into $\mathscr{CB}(\mathscr{X})$; for $z \in \mathscr{X}$ and $Q \in \mathscr{CB}(\mathscr{X})$ we define

$$\mathscr{W}_z(Q) = \{\rho(z, q) : q \in Q\}. \tag{7}$$

Thus for $z, w \in \mathscr{X}$

$$\mathscr{W}_z(\tau w) = \{\rho(z, u) : u \in \tau w\}. \tag{8}$$

Lemma 4 (see [35]). *Let $(\mathscr{X}, \rho)$ be complex-valued metric space.*

(i) *Let $z, w \in \mathfrak{C}$. If $z \precsim w$, then $s(z) \subset s(w)$.*

(ii) *Let $z \in \mathscr{X}$ and $D \in \mathbb{N}(\mathscr{X})$. If $\delta \in s(z, D)$, then $z \in D$.*

(iii) *Let $w \in \mathfrak{C}, P, Q \in \mathscr{CB}(\mathscr{X})$ and $p \in P$. If $v \in s(P, Q)$, then $z \in s(p, Q)$ for all $p \in P$ or $z \in s(P, q)$ for all $q \in Q$.*

Definition 5 (see [10]). Let $\{w_r\}$ be a sequence in complex-valued metric space $(\mathscr{X}, \rho)$ and $w \in \mathscr{X}$; then

(i) w is a limit point of $\{w_r\}$ if for each $0 \prec \epsilon \in \mathfrak{C}$ there exists $r_0 \in N$ such that $\rho(w_r, w) \precsim \epsilon$ for all $r \succsim r_0$ and it is written as $\lim_{r \to \infty} w_r = w$.

(ii) $\{w_r\}$ is a Cauchy sequence if for any $0 \prec \epsilon \in \mathfrak{C}$ there exists $r_0 \in N$ such that $\rho(w_r, w_{r+t}) \prec \epsilon$ for all $r \succ r_0$ where $t \in N$.

(iii) we say that $(\mathscr{X}, \rho)$ is complete complex-valued metric space if every Cauchy sequence in $\mathscr{X}$ converges to a point in $\mathscr{X}$.

Definition 6 (see [18]). Let (V, ρ) be a metric space. A fuzzy set B is characterized by its membership function $f_B : V \to [0,1]$. A set of elements of V along with its grade of membership is called a fuzzy set. For simplicity we denote $f_B(u)$ by $B(u)$. The α-level set of a fuzzy set B is mentioned by $[B]_\alpha$ and is defined as follows:

$$\begin{aligned} [B]_\alpha &= \{u : B(u) \geq \alpha\} \quad \text{if } \alpha \in (0,1], \\ [B]_0 &= \overline{\{u : B(u) > 0\}}. \end{aligned} \tag{9}$$

Definition 7 (see [18]). Let $\mathfrak{L}(\mathcal{X})$ be the family of all fuzzy sets in a metric space $\mathcal{X}$. For $G, H \in \mathfrak{L}(\mathcal{X}), G \subset H$ means $G(z) \leq H(z)$ for each $z \in \mathcal{X}$.

Definition 8 (see [16]). Assume $\mathcal{X}$ is an arbitrary set and Y is a metric space. A mapping G is called a fuzzy mapping if $G : \mathcal{X} \to \mathfrak{L}(Y)$. A fuzzy mapping G is a fuzzy subset on $\mathcal{X} \times Y$ with a membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of y in $G(x)$.

Definition 9 (see [20]). Assume that $(\mathcal{X}, \rho)$ is complex-valued metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ are fuzzy mappings. A point $w \in \mathcal{X}$ is a fuzzy fixed point of G_1 if $w \in [G_1 w]_\alpha$ where $\alpha \in [0,1]$ and a common fuzzy fixed point of G_1, G_2 if $w \in [G_1 w]_\alpha \cap [G_2 w]_\alpha$. If $\alpha = 1$ then w is known as common fixed point of fuzzy mappings.

Definition 10 (see [14]). Suppose $(\mathcal{X}, \rho)$ is complex-valued metric space; the fuzzy mapping $G_1 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ enjoys the greatest lower bound property (glb property) on $(\mathcal{X}, \rho)$, if, for any $w \in \mathcal{X}$ and $\alpha \in (0,1]$, the greatest lower bound of $W_w([G_1 y]_\alpha)$ exists in $\mathcal{C}$ for all $w, y \in \mathcal{X}$. Here we mention $\rho(w, [G_1 y]_\alpha)$ by the glb of $W_w([G_1 y]_\alpha)$. That is,

$$\rho(w, [G_1 y]_\alpha) = \inf\{\rho(w, u) : u \in [G_1 y]_\alpha\}. \tag{10}$$

Remark 11 (see [13]). Let $(\mathcal{X}, \rho)$ be a complex-valued metric space. If $\mathfrak{C} = \mathbb{R}$, then $(\mathcal{X}, \rho)$ is a metric space. Furthermore $H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by ρ, where $A, B \in \mathcal{CB}(\mathcal{X})$.

Definition 12 (see [34]). Suppose Ψ is a collection of nondecreasing functions, $\Phi : \mathfrak{C} \to \mathfrak{C}$, such that $\Phi(0) = 0$ and $\Phi(t) \prec t$, when $0 \prec t$.

3. Main Result

In this section we present our main results. To present the main results we need the lemmas given below.

Lemma 13. *Let $(\mathcal{X}, \rho)$ be complex-valued metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ be fuzzy mappings, such that for each $w \in \mathcal{X}$ and some $\alpha \in (0,1]$ there exists $[G_1 w]_\alpha, [G_2 w]_\alpha$, nonempty closed and bounded subsets of $\mathcal{X}$. Let $w_0 \in \mathcal{X}$ and define the sequence $\{w_k\}$ by*

$$\begin{aligned} w_{2k+1} &\in [G_1 w_{2k}]_\alpha, \\ w_{2k+2} &\in [G_2 w_{2k+1}]_\alpha, \\ &\forall k = 0, 1, 2, \ldots. \end{aligned} \tag{11}$$

Assume that there exists a mapping $\phi : \mathcal{X} \to [0,1)$ such that $\phi(u) \preceq \phi(w)$ for all $u \in [G_1 w]_\alpha$ and $\phi(v) \preceq \phi(w)$ for all $v \in [G_2 w]_\alpha$. Then $\phi(w_{2k}) \preceq \phi(w_0)$ and $\phi(w_{2k+1}) \preceq \phi(w_1)$.

Proof. Suppose $w \in \mathcal{X}$ and $k = 0, 1, 2, \ldots$. Then we have

$$\begin{aligned} \phi(w_{2k}) &\preceq \phi(w_{2k-2}) \quad \text{for } w_{2k-1} \in [G_1 w_{2k-2}]_\alpha, \\ &\preceq \phi(w_{2k-4}) \quad \text{for } w_{2k-2} \in [G_1 w_{2k-4}]_\alpha, \\ &\preceq \cdots \preceq \phi(w_0). \end{aligned} \tag{12}$$

Similarly we have

$$\phi(w_{2k+1}) \preceq \phi(w_1). \tag{13}$$

□

Theorem 14. *Suppose $(\mathcal{X}, \rho)$ is a complete complex-valued metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ are fuzzy mappings satisfying glb property. Assume that for each $y \in \mathcal{X}$ and some $\alpha \in (0,1]$ there exist $[G_1 y]_\alpha, [G_2 y]_\alpha$ which are nonempty closed bounded subsets of $\mathcal{X}$. Suppose there exist mappings $\psi_i : \mathcal{X} \to [0,1)$, $i = 1, \ldots, 7$ such that*

(i) *$\psi_i(u) \preceq \psi_i(y)$, $i = 1, \ldots, 7$ for all $u \in [G_1 y]_\alpha$ and $y \in \mathcal{X}$;*

(ii) *$\psi_i(v) \preceq \psi_i(y)$, $i = 1, \ldots, 7$ for all $v \in [G_2 y]_\alpha$ and $y \in \mathcal{X}$;*

(iii) *$\sum \psi_i(y) + 2\psi_4(y) < 1$, $i = 1, 2, 3, 6, 7$ $\forall y \in \mathcal{X}$; and*

$$\begin{aligned} \Phi\Bigg(&\psi_1(y)\rho(y, w) + \psi_2(y)\rho(y, [G_1 y]_\alpha) \\ &+ \psi_3(y)\rho(w, [G_2 w]_\alpha) \\ &+ \psi_4(y)\frac{\rho(y, [G_1 y]_\alpha)\rho(y, [G_2 w]_\alpha)}{1 + \rho(y, w)} \\ &+ \psi_5(y)\frac{\rho(w, [G_1 y]_\alpha)\rho(w, [G_2 w]_\alpha)}{1 + \rho(y, w)} \\ &+ \psi_6(y)\frac{\rho(y, [G_1 y]_\alpha)\rho(w, [G_2 w]_\alpha)}{1 + \rho(y, w)} \\ &+ \psi_7(y)\frac{\rho(w, [G_1 y]_\alpha)\rho(y, [G_2 w]_\alpha)}{1 + \rho(y, w)}\Bigg) \\ &\in s([G_1 y]_\alpha, [G_2 w]_\alpha) \end{aligned} \tag{14}$$

for some $\Phi \in \Psi$ and for all $y, w \in \mathcal{X}$. Then G_1 and G_2 have a common fuzzy fixed point.

Proof. Let $y_0 \in \mathcal{X}$ and $y_1 \in [G_1 y_0]_\alpha$. Using (14) with $y = y_0$ and $w = y_1$ we get

$$\begin{aligned}
&\Phi\Big(\psi_1(y_0)\rho(y_0,y_1) + \psi_2(y_0)\rho(y_0,[G_1y_0]_\alpha) \\
&\quad + \psi_3(y_0)\rho(y_1,[G_2y_1]_\alpha) \\
&\quad + \psi_4(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_5(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_6(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_7(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)}\Big) \\
&\quad \in s([G_1y_0]_\alpha,[G_2y_1]_\alpha).
\end{aligned} \tag{15}$$

By Lemma 4(iii) we have

$$\begin{aligned}
&\Phi\Big(\psi_1(y_0)\rho(y_0,y_1) + \psi_2(y_0)\rho(y_0,[G_1y_0]_\alpha) \\
&\quad + \psi_3(y_0)\rho(y_1,[G_2y_1]_\alpha) \\
&\quad + \psi_4(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_5(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_6(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_7(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)}\Big) \\
&\quad \in s(y_1,[G_2y_1]_\alpha).
\end{aligned} \tag{16}$$

By definition there exists some $y_2 \in [G_2y_1]_\alpha$, such that

$$\begin{aligned}
&\Phi\Big(\psi_1(y_0)\rho(y_0,y_1) + \psi_2(y_0)\rho(y_0,[G_1y_0]_\alpha) \\
&\quad + \psi_3(y_0)\rho(y_1,[G_2y_1]_\alpha) \\
&\quad + \psi_4(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_5(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_6(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_7(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)}\Big) \\
&\quad \in s(\rho(y_1,y_2)).
\end{aligned} \tag{17}$$

Therefore

$$\begin{aligned}
\rho(y_1,y_2) &\preceq \Phi\Big(\psi_1(y_0)\rho(y_0,y_1) \\
&\quad + \psi_2(y_0)\rho(y_0,[G_1y_0]_\alpha) \\
&\quad + \psi_3(y_0)\rho(y_1,[G_2y_1]_\alpha) \\
&\quad + \psi_4(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_5(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_6(y_0)\frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_7(y_0)\frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)}\Big) \\
&\quad \prec \psi_1(y_0)\rho(y_0,y_1) + \psi_2(y_0)\rho(y_0,[G_1y_0]_\alpha) \\
&\quad + \psi_3(y_0)\rho(y_1,[G_2y_1]_\alpha) + \psi_4(y_0) \\
&\quad \cdot \frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} + \psi_5(y_0) \\
&\quad \cdot \frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} + \psi_6(y_0) \\
&\quad \cdot \frac{\rho(y_0,[G_1y_0]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)} + \psi_7(y_0) \\
&\quad \cdot \frac{\rho(y_1,[G_1y_0]_\alpha)\rho(y_0,[G_2y_1]_\alpha)}{1+\rho(y_0,y_1)}.
\end{aligned} \tag{18}$$

Using the glb property of G_1 and G_2 we have

$$\begin{aligned}
\rho(y_1,y_2) &\preceq \psi_1(y_0)\rho(y_0,y_1) + \psi_2(y_0)\rho(y_0,y_1) \\
&\quad + \psi_3(y_0)\rho(y_1,y_2) \\
&\quad + \psi_4(y_0)\frac{\rho(y_0,y_1)\rho(y_0,y_2)}{1+\rho(y_0,y_1)} \\
&\quad + \psi_5(y_0)\frac{\rho(y_1,y_1)\rho(y_1,y_2)}{1+\rho(y_0,y_1)}
\end{aligned}$$

$$+\psi_6(y_0)\frac{\rho(y_0,y_1)\rho(y_1,y_2)}{1+\rho(y_0,y_1)}+\psi_7(y_0)\frac{\rho(y_1,y_1)\rho(y_0,y_2)}{1+\rho(y_0,y_1)}. \tag{19}$$

It implies that

$$\begin{aligned}\rho(y_1,y_2)&\preceq\psi_1(y_0)\rho(y_0,y_1)+\psi_2(y_0)\rho(y_0,y_1)\\&\quad+\psi_3(y_0)\rho(y_1,y_2)+\psi_4(y_0)\rho(y_0,y_2)\\&\quad+\psi_6(y_0)\rho(y_1,y_2)\\&\preceq\psi_1(y_0)\rho(y_0,y_1)+\psi_2(y_0)\rho(y_0,y_1)\\&\quad+\psi_3(y_0)\rho(y_1,y_2)+\psi_4(y_0)\rho(y_0,y_1)\\&\quad+\psi_4(y_0)\rho(y_1,y_2)+\psi_6(y_0)\rho(y_1,y_2).\end{aligned} \tag{20}$$

Finally we get

$$\begin{aligned}\rho(y_1,y_2)&\preceq\mu\rho(y_0,y_1)\\|\rho(y_1,y_2)|&\le\mu|\rho(y_0,y_1)|,\end{aligned} \tag{21}$$

where

$$\mu=\frac{\psi_1(y_0)+\psi_2(y_0)+\psi_4(y_0)}{1-(\psi_3(y_0)+\psi_4(y_0)+\psi_6(y_0))}<1. \tag{22}$$

Now for $y_2\in[G_2y_1]_\alpha$, consider

$$\begin{aligned}&\Phi\Big(\psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,[G_1y_2]_\alpha)\\&\quad+\psi_3(y_2)\rho(y_1,[G_2y_1]_\alpha)\\&\quad+\psi_4(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_5(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_6(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_7(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\Big)\\&\quad\in s([G_1y_2]_\alpha,[G_2y_1]_\alpha).\end{aligned} \tag{23}$$

Using Lemma 4(iii) we get

$$\begin{aligned}&\Phi\Big(\psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,[G_1y_2]_\alpha)\\&\quad+\psi_3(y_2)\rho(y_1,[G_2y_1]_\alpha)\\&\quad+\psi_4(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_5(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_6(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_7(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\Big)\\&\quad\in s([G_1y_2]_\alpha,y_2).\end{aligned} \tag{24}$$

By definition there exists $y_3\in[G_1y_2]_\alpha$, such that

$$\begin{aligned}&\Phi\Big(\psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,[G_1y_2]_\alpha)\\&\quad+\psi_3(y_2)\rho(y_1,[G_2y_1]_\alpha)\\&\quad+\psi_4(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_5(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_6(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_7(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\Big)\\&\quad\in s(\rho(y_3,y_2)).\end{aligned} \tag{25}$$

Therefore

$$\begin{aligned}\rho(y_3,y_2)&\preceq\Phi\Big(\psi_1(y_2)\rho(y_2,y_1)\\&\quad+\psi_2(y_2)\rho(y_2,[G_1y_2]_\alpha)\\&\quad+\psi_3(y_2)\rho(y_1,[G_2y_1]_\alpha)\\&\quad+\psi_4(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_5(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_6(y_2)\frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\\&\quad+\psi_7(y_2)\frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}\Big)\\&\prec\psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,[G_1y_2]_\alpha)\\&\quad+\psi_3(y_2)\rho(y_1,[G_2y_1]_\alpha)+\psi_4(y_2)\end{aligned}$$

$$\cdot \frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}+\psi_5(y_2)$$
$$\cdot \frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}+\psi_6(y_2)$$
$$\cdot \frac{\rho(y_2,[G_1y_2]_\alpha)\rho(y_1,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}+\psi_7(y_2)$$
$$\cdot \frac{\rho(y_1,[G_1y_2]_\alpha)\rho(y_2,[G_2y_1]_\alpha)}{1+\rho(y_2,y_1)}. \tag{26}$$

Again utilizing the greatest lower bound property of G_1 and G_2 we get

$$\begin{aligned}\rho(y_3,y_2) \preceq{}& \psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,y_3)\\ &+\psi_3(y_2)\rho(y_1,y_2)\\ &+\psi_4(y_2)\frac{\rho(y_2,y_3)\rho(y_2,y_2)}{1+\rho(y_2,y_1)}\\ &+\psi_5(y_2)\frac{\rho(y_1,y_3)\rho(y_1,y_2)}{1+\rho(y_2,y_1)}\\ &+\psi_6(y_2)\frac{\rho(y_2,y_3)\rho(y_1,y_2)}{1+\rho(y_2,y_1)}\\ &+\psi_7(y_2)\frac{\rho(y_1,y_3)\rho(y_2,y_2)}{1+\rho(y_2,y_1)}.\end{aligned} \tag{27}$$

It implies that

$$\begin{aligned}\rho(y_3,y_2) \preceq{}& \psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,y_3)\\ &+\psi_3(y_2)\rho(y_1,y_2)+\psi_5(y_2)\rho(y_1,y_3)\\ &+\psi_6(y_2)\rho(y_2,y_3)\\ \preceq{}& \psi_1(y_2)\rho(y_2,y_1)+\psi_2(y_2)\rho(y_2,y_3)\\ &+\psi_3(y_2)\rho(y_1,y_2)+\psi_5(y_2)\rho(y_1,y_2)\\ &+\psi_5(y_2)\rho(y_2,y_3)+\psi_6(y_2)\rho(y_2,y_3).\end{aligned} \tag{28}$$

Applying Lemma 13 we get

$$\begin{aligned}\rho(y_3,y_2) \preceq{}& \psi_1(y_0)\rho(y_2,y_1)+\psi_2(y_0)\rho(y_2,y_3)\\ &+\psi_3(y_0)\rho(y_1,y_2)+\psi_5(y_0)\rho(y_1,y_2)\\ &+\psi_5(y_0)\rho(y_2,y_3)+\psi_6(y_0)\rho(y_2,y_3).\end{aligned} \tag{29}$$

Finally we get

$$\begin{aligned}&\rho(y_3,y_2) \preceq \nu\rho(y_2,y_1)\\ &|\rho(y_2,y_3)| \le \nu|\rho(y_1,y_2)|,\end{aligned} \tag{30}$$

where

$$\nu=\frac{\psi_1(y_0)+\psi_3(y_0)+\psi_5(y_0)}{1-(\psi_2(y_0)+\psi_5(y_0)+\psi_6(y_0))}<1. \tag{31}$$

Inductively we can obtain a sequence $\{y_n\}$ in $\mathcal{X}$ such that $y_{2r+1}\in[G_1x_{2r}]_\alpha$, $y_{2r+2}\in[G_2y_{2r+1}]_\alpha$ for $r=0,1,2\ldots$.

$$\begin{aligned}&|\rho(y_{2r+1},y_{2r+2})| \le \mu|\rho(y_{2r},y_{2r+1})|\\ &|\rho(y_{2r+2},y_{2r+3})| \le \nu|\rho(y_{2r+1},y_{2r+2})|.\end{aligned} \tag{32}$$

It implies that

$$\begin{aligned}|\rho(y_{2r+1},y_{2r+2})| &\le \mu|\rho(y_{2r},y_{2r+1})|\\ &\le \mu\nu|\rho(y_{2r-1},y_{2r})|\\ &\le \mu\nu\mu|\rho(y_{2r-2},y_{2r-1})| \le \cdots\\ &\le \mu(\mu\nu)^r|\rho(y_0,y_1)|,\\ |\rho(y_{2r+2},y_{2r+3})| &\le \nu|\rho(y_{2r+1},y_{2r+2})| \le \cdots\\ &\le (\mu\nu)^{r+1}|\rho(y_0,y_1)|.\end{aligned} \tag{33}$$

Then for $s<t$, we have

$$\begin{aligned}\rho(y_{2s+1},y_{2t+1}) \preceq{}& \rho(y_{2s+1},y_{2s+2})+\rho(y_{2s+2},y_{2s+3})\\ &+\rho(y_{2s+3},y_{2s+4})+\cdots\\ &+\rho(y_{2t},y_{2t+1}),\end{aligned} \tag{34}$$

which implies that

$$\begin{aligned}&|\rho(y_{2s+1},y_{2t+1})|\\ &\quad\le |\rho(y_{2s+1},y_{2s+2})|+|\rho(y_{2s+2},y_{2s+3})|\\ &\qquad+|\rho(y_{2s+3},y_{2s+4})|+\cdots+|\rho(y_{2t},y_{2t+1})|\\ &\quad\le \left[\mu\sum_{k=s}^{t-1}(\mu\nu)^k+\sum_{k=s+1}^{t}(\mu\nu)^k\right]|\rho(y_0,y_1)|.\end{aligned} \tag{35}$$

Similarly we obtain

$$\begin{aligned}&\rho(y_{2s},y_{2t+1}) \le \left[\sum_{k=s}^{t}(\mu\nu)^k+\mu\sum_{k=s}^{t-1}(\mu\nu)^k\right]|\rho(y_0,y_1)|,\\ &\rho(y_{2s},y_{2t}) \le \left[\sum_{k=s}^{t-1}(\mu\nu)^k+\mu\sum_{k=s}^{t-1}(\mu\nu)^k\right]|\rho(y_0,y_1)|,\\ &\rho(y_{2s+1},y_{2t})\\ &\quad\le \left[\mu\sum_{k=s}^{t-1}(\mu\nu)^k+\sum_{k=s+1}^{t-1}(\mu\nu)^k\right]|\rho(y_0,y_1)|.\end{aligned} \tag{36}$$

Since $(\mu\nu)<1$, therefore $\{y_r\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete so there exists $l\in\mathcal{X}$ such that $y_r\to l$

when $r \to \infty$. Now we have to prove that $l \in [G_1 l]_\alpha$ and $l \in [G_2 l]_\alpha$. From (14) with $y = y_{2r}$ and $w = l$ we get

$$\begin{aligned}
&\Phi\Bigg(\psi_1(y_{2r})\rho(y_{2r}, l) + \psi_2(y_{2r})\rho(y_{2r}, [G_1 y_{2r}]_\alpha) \\
&\quad + \psi_3(y_{2r})\rho(l, [G_2 l]_\alpha) \\
&\quad + \psi_4(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_5(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_6(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_7(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)}\Bigg) \\
&\quad \in s([G_1 y_{2r}]_\alpha, [G_2 l]_\alpha).
\end{aligned} \tag{37}$$

Since $y_{2r+1} \in [G_1 y_{2r}]_\alpha$, so by Lemma 4(iii) we have

$$\begin{aligned}
&\Phi\Bigg(\psi_1(y_{2r})\rho(y_{2r}, l) + \psi_2(y_{2r})\rho(y_{2r}, [G_1 y_{2r}]_\alpha) \\
&\quad + \psi_3(y_{2r})\rho(l, [G_2 l]_\alpha) \\
&\quad + \psi_4(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_5(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_6(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_7(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)}\Bigg) \\
&\quad \in s(y_{2r+1}, [G_2 l]_\alpha).
\end{aligned} \tag{38}$$

By definition there exists some $w_r \in [G_2 l]_\alpha$, such that

$$\begin{aligned}
&\Phi\Bigg(\psi_1(y_{2r})\rho(y_{2r}, l) + \psi_2(y_{2r})\rho(y_{2r}, [G_1 y_{2r}]_\alpha) \\
&\quad + \psi_3(y_{2r})\rho(l, [G_2 l]_\alpha) \\
&\quad + \psi_4(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_5(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_6(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&\quad + \psi_7(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)}\Bigg) \\
&\quad \in s(\rho(y_{2r+1}, w_r)).
\end{aligned} \tag{39}$$

Therefore

$$\begin{aligned}
\rho(y_{2r+1}, w_r) \preceq \Phi\Bigg(&\psi_1(y_{2r})\rho(y_{2r}, l) \\
&+ \psi_2(y_{2r})\rho(y_{2r}, [G_1 y_{2r}]_\alpha) \\
&+ \psi_3(y_{2r})\rho(l, [G_2 l]_\alpha) \\
&+ \psi_4(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&+ \psi_5(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&+ \psi_6(y_{2r})\frac{\rho(y_{2r}, [G_1 y_{2r}]_\alpha)\rho(l, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)} \\
&+ \psi_7(y_{2r})\frac{\rho(l, [G_1 y_{2r}]_\alpha)\rho(y_{2r}, [G_2 l]_\alpha)}{1+\rho(y_{2r}, l)}\Bigg).
\end{aligned} \tag{40}$$

By using the greatest lower bound property of G_1 and G_2, we have

$$\begin{aligned}
\rho(y_{2r+1}, w_r) \preceq \Phi\Bigg(&\psi_1(y_{2r})\rho(y_{2r}, l) \\
&+ \psi_2(y_{2r})\rho(y_{2r}, y_{2r+1}) + \psi_3(y_{2r})\rho(l, w_r) \\
&+ \psi_4(y_{2r})\frac{\rho(y_{2r}, y_{2r+1})\rho(y_{2r}, w_r)}{1+\rho(y_{2r}, l)} \\
&+ \psi_5(y_{2r})\frac{\rho(l, y_{2r+1})\rho(l, w_r)}{1+\rho(y_{2r}, l)} \\
&+ \psi_6(y_{2r})\frac{\rho(y_{2r}, y_{2r+1})\rho(l, w_r)}{1+\rho(y_{2r}, l)} \\
&+ \psi_7(y_{2r})\frac{\rho(l, y_{2r+1})\rho(y_{2r+1}, w_r)}{1+\rho(y_{2r}, l)}\Bigg) \preceq \psi_1(y_{2r}) \\
&\cdot \rho(y_{2r}, l) + \psi_2(y_{2r})\rho(y_{2r}, y_{2r+1}) + \psi_3(y_{2r}) \\
&\cdot \rho(l, w_r) + \psi_4(y_{2r})\frac{\rho(y_{2r}, y_{2r+1})\rho(y_{2r}, w_r)}{1+\rho(y_{2r}, l)} \\
&+ \psi_5(y_{2r})\frac{\rho(l, y_{2r+1})\rho(l, w_r)}{1+\rho(y_{2r}, l)} + \psi_6(y_{2r}) \\
&\cdot \frac{\rho(y_{2r}, y_{2r+1})\rho(l, w_r)}{1+\rho(y_{2r}, l)} + \psi_7(y_{2r}) \\
&\cdot \frac{\rho(l, y_{2r+1})\rho(y_{2r+1}, w_r)}{1+\rho(y_{2r}, l)}.
\end{aligned} \tag{41}$$

Now by using triangular inequality, we get

$$\begin{aligned}\rho(l,w_r) &\preceq \rho(l,y_{2r+1}) + \rho(y_{2r+1},w_r)\\ &\preceq \rho(l,y_{2r+1}) + \psi_1(y_{2r})\rho(y_{2r},l)\\ &\quad + \psi_2(y_{2r})\rho(y_{2r},y_{2r+1})\\ &\quad + \psi_3(y_{2r})\rho(l,w_r)\\ &\quad + \psi_4(y_{2r})\frac{\rho(y_{2r},y_{2r+1})\rho(y_{2r},w_r)}{1+\rho(y_{2r},l)}\\ &\quad + \psi_5(y_{2r})\frac{\rho(l,y_{2r+1})\rho(l,w_r)}{1+\rho(y_{2r},l)}\\ &\quad + \psi_6(y_{2r})\frac{\rho(y_{2r},y_{2r+1})\rho(l,w_r)}{1+\rho(y_{2r},l)}\\ &\quad + \psi_7(y_{2r})\frac{\rho(l,y_{2r+1})\rho(y_{2r+1},w_r)}{1+\rho(y_{2r},l)}.\end{aligned} \tag{42}$$

Applying Lemma 13 we get

$$\begin{aligned}\rho(l,w_r) &\preceq \rho(l,y_{2r+1}) + \rho(y_{2r+1},w_r)\\ &\preceq \rho(l,y_{2r+1}) + \psi_1(y_0)\rho(y_{2r},l)\\ &\quad + \psi_2(y_0)\rho(y_{2r},y_{2r+1}) + \psi_3(y_0)\rho(l,w_r)\\ &\quad + \psi_4(y_0)\frac{\rho(y_{2r},y_{2r+1})\rho(y_{2r},w_r)}{1+\rho(y_{2r},l)}\\ &\quad + \psi_5(y_0)\frac{\rho(l,y_{2r+1})\rho(l,w_r)}{1+\rho(y_{2r},l)}\\ &\quad + \psi_6(y_0)\frac{\rho(y_{2r},y_{2r+1})\rho(l,w_r)}{1+\rho(y_{2r},l)}\\ &\quad + \psi_7(y_0)\frac{\rho(l,y_{2r+1})\rho(y_{2r+1},w_r)}{1+\rho(y_{2r},l)},\end{aligned} \tag{43}$$

which, on $r \to \infty$, reduced to

$$\begin{aligned}\rho(l,w_r) &\preceq \psi_3(y_0)\rho(l,w_r)\\ |\rho(l,w_r)| &\le \psi_3(y_0)|\rho(l,w_r)|.\end{aligned} \tag{44}$$

Since $\psi_3(y_0) < 1$, so $|\rho(l,w_r)| \to 0$ as $r \to \infty$. so we have $w_r \to l$ as $r \to \infty$. Since $[G_2 l]_\alpha$ is closed, so $l \in [G_2 l]_\alpha$. Similarly, it follows that $l \in [G_1 l]_\alpha$. Thus we obtain that G_1 and G_2 have common fixed points. □

Corollary 15. *Let $(\mathcal{X},\rho)$ be a complete complex-valued metric space and $G_1 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ be fuzzy mapping with glb property. For each $y \in \mathcal{X}$ there exists $\alpha \in (0,1]$ such that $[G_1 y]_\alpha$ is nonempty closed bounded subset of $\mathcal{X}$. Then there exists mappings $\psi_i : \mathcal{X} \to [0,1)$, $i = 1,\ldots,7$ with*

(i) *$\psi_i(u) \preceq \psi_i(y)$, $i = 1,2,\ldots,7$ for all $u \in [G_1 y]_\alpha$ and $\forall y \in \mathcal{X}$;*

(ii) *$\sum \psi_i(y) + 2\psi_4(y) < 1$, where $i = 1,2,3,6,7$ $\forall y \in \mathcal{X}$; and*

$$\begin{aligned}&\Phi\Big(\psi_1(y)\rho(y,w) + \psi_2(y)\rho(y,[G_1 y]_\alpha)\\ &\quad + \psi_3(y)\rho(w,[G_1 w]_\alpha)\\ &\quad + \psi_4(y)\frac{\rho(y,[G_1 y]_\alpha)\rho(y,[G_1 w]_\alpha)}{1+\rho(y,w)}\\ &\quad + \psi_5(y)\frac{\rho(w,[G_1 y]_\alpha)\rho(w,[G_1 w]_\alpha)}{1+\rho(y,w)}\\ &\quad + \psi_6(y)\frac{\rho(y,[G_1 y]_\alpha)\rho(w,[G_1 w]_\alpha)}{1+\rho(y,w)}\\ &\quad + \psi_7(y)\frac{\rho(w,[G_1 y]_\alpha)\rho(y,[G_1 w]_\alpha)}{1+\rho(y,w)}\Big)\\ &\quad \in s([G_1 y]_\alpha,[G_1 w]_\alpha),\end{aligned} \tag{45}$$

for some $\Phi \in \Psi$ and for all $y,w \in \mathcal{X}$. Then G_1 has a fuzzy fixed point.

Proof. Proof is immediate on setting $G_1 = G_2$ in Theorem 14. □

Corollary 16. *Let $(\mathcal{X},\rho)$ be a complete complex-valued metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ be fuzzy mappings with glb property. For each $y \in \mathcal{X}$ there exists some $\alpha \in (0,1]$ such that $[G_1 y]_\alpha, [G_2 y]_\alpha$, nonempty closed bounded subsets of $\mathcal{X}$. Then there exist mappings $\varrho,\sigma,\gamma,\xi,\zeta,\lambda,\eta : \mathcal{X} \to [0,1)$ with*

(i) *$\varrho(u) \preceq \varrho(y)$, $\sigma(u) \preceq \sigma(y)$, $\gamma(u) \preceq \gamma(y)$, $\xi(u) \preceq \xi(y)$, $\zeta(u) \preceq \zeta(y)$, $\lambda(u) \preceq \lambda(y)$, $\eta(u) \preceq \eta(y)$ for all $u \in [G_1 y]_\alpha$ and $y \in \mathcal{X}$;*

(ii) *$\varrho(v) \preceq \varrho(w)$, $\sigma(v) \preceq \sigma(w)$, $\gamma(v) \preceq \gamma(w)$, $\xi(v) \preceq \xi(w)$, $\zeta(v) \preceq \zeta(w)$, $\lambda(v) \preceq \lambda(w)$, $\eta(v) \preceq \eta(w)$ for all $v \in [G_2 w]_\alpha$ and $w \in \mathcal{X}$;*

(iii) *$\varrho(y)+\sigma(y)+\gamma(y)+2\xi(y)+\zeta(y)+\lambda(y)+\eta(y) \preceq 1$ $\forall y \in \mathcal{X}$; and*

$$\begin{aligned}&\varrho(y)\rho(y,w) + \sigma(y)\rho(y,[G_1 y]_\alpha)\\ &\quad + \gamma(y)\rho(w,[G_2 w]_\alpha)\\ &\quad + \xi(y)\frac{\rho(y,[G_1 y]_\alpha)\rho(y,[G_2 w]_\alpha)}{1+\rho(y,w)}\\ &\quad + \zeta(y)\frac{\rho(w,[G_1 y]_\alpha)\rho(w,[G_2 w]_\alpha)}{1+\rho(y,w)}\\ &\quad + \lambda(y)\frac{\rho(y,[G_1 y]_\alpha)\rho(w,[G_2 w]_\alpha)}{1+\rho(y,w)}\\ &\quad + \eta(x)\frac{\rho(w,[G_1 y]_\alpha)\rho(y,[G_2 w]_\alpha)}{1+\rho(y,w)}\\ &\quad \in s([G_1 y]_\alpha,[G_2 w]_\alpha),\end{aligned} \tag{46}$$

$\forall y,w \in \mathcal{X}$; then G_1 and G_2 have a common fuzzy fixed point.

Proof. It can be easily proven by letting $\Phi(t) = pt$ where $p \in (0,1)$ in Theorem 14 with $\varrho(y) = p\psi_1(y)$, $\sigma(y) = p\psi_2(y)$, $\gamma(y) = p\psi_3(y)$, $\xi(y) = p\psi_4(y)$, $\zeta(y) = p\psi_5(y)$, $\lambda(y) = p\psi_6(y)$, $\eta(y) = p\psi_7(y)$. □

Corollary 17. *Suppose $(\mathcal{X}, \rho)$ is a complete complex-valued metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ are fuzzy mappings enjoying glb property. For each $y \in \mathcal{X}$ there exists some $\alpha \in (0,1]$ such that $[G_1 y]_\alpha, [G_2 y]_\alpha$, nonempty closed bounded subsets of $\mathcal{X}$ with*

$$\begin{aligned} &\varrho\rho(y,w) + \sigma\rho(y,[G_1y]_\alpha) + \gamma\rho(w,[G_2w]_\alpha) \\ &\quad + \xi\frac{\rho(y,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\quad + \zeta\frac{\rho(w,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\quad + \lambda\frac{\rho(y,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\quad + \eta\frac{\rho(w,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\in s([G_1y]_\alpha, [G_2w]_\alpha), \end{aligned} \tag{47}$$

for all $y, w \in \mathcal{X}$ and $\varrho, \sigma, \gamma, \xi, \zeta, \lambda, \eta$ are nonnegative reals with $\varrho + \sigma + \gamma + 2\xi + \zeta + \lambda + \eta < 1$. Then G_1 and G_2 have a common fuzzy fixed point.

Proof. It can be easily proven by setting $\varrho(y) = \varrho$, $\sigma(y) = \sigma$, $\gamma(y) = \gamma$, $\xi(y) = \xi$, $\zeta(y) = \zeta$, $\lambda(y) = \lambda$, $\eta(y) = \eta$ in Corollary 16 with $\varrho, \sigma, \gamma, \xi, \zeta, \lambda, \eta$ being nonnegative reals such that $\varrho + \sigma + \gamma + 2\xi + \zeta + \lambda + \eta < 1$. □

Using Remark 11 we get the following corollaries from Theorem 14.

Corollary 18. *Suppose $(\mathcal{X}, \rho)$ is a complete metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ are fuzzy mappings with glb property. For each $y \in \mathcal{X}$ related to some $\alpha \in (0,1]$ there exists $[G_1 y]_\alpha, [G_2 y]_\alpha$, nonempty closed bounded subsets of $\mathcal{X}$. Then there exist mappings $\psi_i : \mathcal{X} \to [0,1)$, $i = 1,2,\ldots,7$ such that*

(i) *$\psi_i(u) \le \psi_i(y)$, for all $u \in [G_1 y]_\alpha$ and $\forall y \in \mathcal{X}$;*

(ii) *$\psi_i(v) \le \psi_i(y)$, for all $v \in [G_2 y]_\alpha$ and $\forall y \in \mathcal{X}$;*

(iii) *$\sum \psi_i(y) + 2\psi_4(y) < 1$, $i = 1,2,3,5,6,7$ $\forall y \in \mathcal{X}$; and*

$$\begin{aligned} H([G_1y]_\alpha, [G_2w]_\alpha) \le \Phi\Bigg(&\psi_1(y)\rho(y,w) \\ &+ \psi_2(y)\rho(y,[G_1y]_\alpha) + \psi_3(y)\rho(w,[G_2w]_\alpha) \\ &+ \psi_4(y)\frac{\rho(y,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &+ \psi_5(y)\frac{\rho(w,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &+ \psi_6(y)\frac{\rho(y,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &+ \psi_7(y)\frac{\rho(w,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)}\Bigg), \end{aligned} \tag{48}$$

for some $\Phi \in \Psi$ and for all $y, w \in \mathcal{X}$. Then G_1 and G_2 have a common fuzzy fixed point.

Corollary 19. *Suppose $(\mathcal{X}, \rho)$ is complete metric space and $G_1, G_2 : \mathcal{X} \to \mathfrak{L}(\mathcal{X})$ is fuzzy mappings with glb property. For each $y \in \mathcal{X}$ there exists some $\alpha \in (0,1]$ such that $[G_1 y]_\alpha, [G_2 y]_\alpha$, nonempty closed bounded subsets of $\mathcal{X}$ such that*

$$\begin{aligned} &H([G_1y]_\alpha, [G_2w]_\alpha) \\ &\quad \le \varrho\rho(y,w) + \sigma\rho(y,[G_1y]_\alpha) + \gamma\rho(w,[G_2w]_\alpha) \\ &\quad + \xi\frac{\rho(y,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\quad + \zeta\frac{\rho(w,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\quad + \lambda\frac{\rho(y,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)} \\ &\quad + \eta\frac{\rho(w,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)}, \end{aligned} \tag{49}$$

for all $y, w \in \mathcal{X}$ and $\varrho, \sigma, \gamma, \xi, \zeta, \lambda, \eta$ are nonnegative reals with $\varrho + \sigma + \gamma + 2\xi + \zeta + \lambda + \eta < 1$. Then G_1 and G_2 have a common fuzzy fixed point.

Proof. By putting $\varrho(y) = \psi_1(y)$, $\sigma(y) = \psi_2(y)$, $\gamma(y) = \psi_3(y)$, $\xi(y) = \psi_4(y)$, $\zeta(y) = \psi_5(y)$, $\lambda(y) = \psi_6(y)$, $\eta(y) = \psi_7(y)$ in Corollary 18, it can be easily proven. □

4. Application

Theorem 20. *Let $(\mathcal{X}, \rho)$ be a complete complex-valued metric space and $G_1, G_2 : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be multivalued mapping with glb property. If there exist mappings $\psi_i : \mathcal{X} \to [0,1)$, $i = 1,2,\ldots,7$ such that*

(i) *$\psi_i(u) \preceq \psi_i(y)$ for all $u \in G_1 y$ and $\forall y \in \mathcal{X}$;*

(ii) *$\psi_i(v) \preceq \psi_i(y)$ for all $v \in G_2 y$ and $\forall y \in \mathcal{X}$;*

(iii) *$\psi_i(y) + 2\psi_4(y) < 1$, $i = 1,2,3,5,6,7$ $\forall y \in \mathcal{X}$; and*

$$\begin{aligned} \Phi\Bigg(&\psi_1(y)\rho(y,w) + \psi_2(y)\rho(y,G_1y) \\ &+ \psi_3(y)\rho(w,G_2w) \\ &+ \psi_4(y)\frac{\rho(y,G_1y)\rho(y,G_2w)}{1+\rho(y,w)} \end{aligned}$$

$$+\psi_5(y)\frac{\rho(w,G_1y)\rho(w,G_2w)}{1+\rho(y,w)}$$
$$+\psi_6(y)\frac{\rho(y,G_1y)\rho(w,G_2w)}{1+\rho(y,w)}$$
$$+\psi_7(y)\frac{\rho(w,G_1y)\rho(y,G_2w)}{1+\rho(y,w)}\Bigg)\in s(G_1y,G_2w), \tag{50}$$

for some $\Phi\in\Psi$ and for all $y,w\in\mathcal{X}$, then G_1 and G_2 have a common fixed point.

Proof. Let the fuzzy mapping $S,T:\mathcal{X}\to\mathfrak{L}(\mathcal{X})$ be defined by

$$Sy=\begin{cases}\alpha & \text{if } y\in G_1y\\ 0 & \text{if } y\notin G_1y.\end{cases}$$
$$Ty=\begin{cases}\alpha & \text{if } y\in G_2y\\ 0 & \text{if } y\notin G_2y.\end{cases} \tag{51}$$

Then for any $\alpha\in(0,1,Sy]_\alpha=G_1y$ and $[Ty]_\alpha=G_2y$.

Since for every $y,w\in\mathcal{X}$, $s([Sy]_\alpha,[Tw]_\alpha)=s(G_1y,G_2w)$, therefore Theorem 14 can be applied to obtain some points $u\in\mathcal{X}$ such that $u\in G_1(u)\cap G_2(u)$. □

Corollary 21. *Let $(\mathcal{X},\rho)$ be a complete complex-valued metric space and $G_1,G_2:\mathcal{X}\to\mathcal{CB}(\mathcal{X})$ be multivalued mapping with glb property. Suppose there exist mappings $\psi_i:\mathcal{X}\to[0,1)$, $i=1,2,\ldots,7$ such that*

(i) *$\psi_i(u)\preceq\psi_i(y)$ for all $u\in G_1y$ and $\forall y\in\mathcal{X}$;*

(ii) *$\psi_i(v)\preceq\psi_i(y)$ for all $v\in G_2y$ and $\forall y\in\mathcal{X}$;*

(iii) *$\psi_i(y)+2\psi_4(y)<1$, $i=1,2,3,5,6,7$ $\forall y\in\mathcal{X}$; and*

$$\begin{aligned}&\psi_1(y)\rho(y,w)+\psi_2(y)\rho(y,G_1y)\\&+\psi_3(y)\rho(w,G_2w)\\&+\psi_4(y)\frac{\rho(y,G_1y)\rho(y,G_2w)}{1+\rho(y,w)}\\&+\psi_5(y)\frac{\rho(w,G_1y)\rho(w,G_2w)}{1+\rho(y,w)}\\&+\psi_6(y)\frac{\rho(y,G_1y)\rho(w,G_2w)}{1+\rho(y,w)}\\&+\psi_7(y)\frac{\rho(w,G_1y)\rho(y,G_2w)}{1+\rho(y,w)}\\&\in s(G_1y,G_2w).\end{aligned} \tag{52}$$

Then G_1 and G_2 have a common fixed point.

Proof. It can be proven by the same way as Corollary 16. □

Remark 22. (i) Theorem 20 is actually Theorem 2.3 of [34].

(ii) By setting $\psi_2(y)=\psi_3(y)=\psi_4(y)=\psi_5(y)=0$, $\psi_1(y)=\varrho$, $\psi_6(y)=\lambda,\psi_7(y)=\eta$ in Corollary 21, we get Theorem 9 of [13].

(iii) By setting $\psi_1(y)=\psi_4(y)=\psi_5(y)=\psi_7=0$, $\psi_2(y)=\sigma$, $\psi_3(y)=\gamma$, $\psi_6(y)=\lambda$ in Corollary 21 we obtain Theorem 15 of [13].

(iv) By setting $\psi_2(y)=\psi_3(y)=0$, $\psi_1(y)=\varrho$, $\psi_4(y)=\xi$, $\psi_5(y)=\zeta$, $\psi_6(y)=\lambda,\psi_7(y)=\eta$ in Corollary 21 we get Theorem 9 of [36].

(v) By setting $\sigma=\gamma=\xi=\zeta=0$ in Corollary 17, we get Theorem 12 of [25].

(vi) By setting $\varrho=\xi=\zeta=\eta=0$ in Corollary 17, we get Theorem 19 of [25].

Example 23. Let $\mathcal{X}=[0,1]$ and $\rho:\mathcal{X}\times\mathcal{X}\to\mathcal{C}$ be complex-valued metric space defined by $\rho(y,w)=|y-w|e^{i(\pi/12)}$, for all $y,w\in\mathcal{X}$.

Let $\alpha\in(0,1]$ and $G_1,G_2:\mathcal{X}\to\mathfrak{L}(\mathcal{X})$ be fuzzy mappings defined by

$$G_1(0)(r)=\begin{cases}1 & \text{if } r=0\\ \frac{1}{2} & \text{if } 0<r\le\frac{y}{50}\\ 0 & \text{if } \frac{y}{50}<r\le 1,\end{cases}$$
$$G_2(0)(r)=\begin{cases}1 & \text{if } r=0\\ \frac{1}{6} & \text{if } 0<r\le\frac{y}{150}\\ 0 & \text{if } \frac{y}{150}<r\le 1,\end{cases} \tag{53}$$

if $y\neq 0$,

$$G_1(y)(r)=\begin{cases}\alpha & \text{if } 0\le r\le\frac{y}{75}\\ \frac{\alpha}{3} & \text{if } \frac{y}{75}<r\le\frac{y}{10}\\ \frac{\alpha}{4} & \text{if } \frac{y}{10}<r\le 1,\end{cases}$$
$$G_2(y)(r)=\begin{cases}\alpha & \text{if } 0\le r\le\frac{y}{40}\\ \frac{\alpha}{2} & \text{if } \frac{y}{40}<r\le\frac{y}{20}\\ \frac{\alpha}{5} & \text{if } \frac{y}{20}<r\le 1.\end{cases} \tag{54}$$

Let $\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7:\mathcal{X}\to[0,1)$ be defined by $\psi_1(y)=(y+1)/74$, $\psi_2(y)=y/10$, $\psi_3(y)=y/20$, $\psi_4(y)=y/30$, $\psi_5(y)=y/60$, $\psi_6(y)=y/40$.

Then for $y=0$, $[G_10]_1=[G_20]_1=\{0\}$ and $\forall y,w\neq 0$ $[G_1y]_\alpha=[0,y/75]$ and $[G_2y]_\alpha=[0,y/40]$.

And then

$$\mathcal{W}_w([G_1y]_\alpha)=\left\{\rho(w,u):u\in\left[0,\frac{y}{75}\right]\right\}$$
$$\mathcal{W}_w([G_2y]_\alpha)=\left\{\rho(w,u):u\in\left[0,\frac{y}{40}\right]\right\}. \tag{55}$$

Let $\rho(w,[G_1w]_\alpha)$ and $\rho(w,[G_2w]_\alpha)$ be the greatest lower bound of $\mathscr{W}_w([G_1y]_\alpha)$ and $\mathscr{W}_w([G_2y]_\alpha)$. Then

$$\rho(w,[G_1y]_\alpha)=\begin{cases}0 & \text{if } w\le \dfrac{y}{75}\\ \left(w-\dfrac{y}{75}\right)e^{i(\pi/12)} & \text{if } w>\dfrac{y}{75},\end{cases}$$
$$\rho(y,[G_2w]_\alpha)=\begin{cases}0 & \text{if } y\le \dfrac{y}{40}\\ \left(y-\dfrac{w}{40}\right)e^{i(\pi/12)} & \text{if } x>\dfrac{w}{40};\end{cases} \tag{56}$$

also $\rho(y,[G_1y]_\alpha)=(74y/75)e^{i(\pi/12)}$, and $\rho(w,[G_2w]_\alpha)=(39w/40)e^{i(\pi/12)}$.

It can be easily verified that $\psi_i(u)\le\psi_i(w)$, $\forall u\in[G_1y]_\alpha$ and $\psi_i(v)\le\psi_i(w)$, $\forall v\in[G_2y]_\alpha$. Moreover if $\varpi_{yw}\in\mathfrak{C}$ such that

$$\varpi_{yw}=\left|\frac{y}{75}-\frac{w}{40}\right|\sqrt{2}e^{i(\pi/12)}, \tag{57}$$

then

$$s([G_1y]_\alpha,[G_2w]_\alpha)=\{\omega\in\mathfrak{C}:\varpi_{yw}\preceq\omega\}. \tag{58}$$

Consider

$$\begin{aligned}\Phi\Big(&\psi_1(y)|\rho(y,w)|+\psi_2(y)|\rho(y,[G_1y]_\alpha)|\\&+\psi_3(y)|\rho(w,[G_2w]_\alpha)|\\&+\psi_4(y)\frac{|\rho(y,[G_1y]_\alpha)||\rho(y,[G_2w]_\alpha)|}{1+|\rho(y,w)|}\\&+\psi_5(y)\frac{|\rho(w,[G_1y]_\alpha)||\rho(w,[G_2w]_\alpha)|}{1+|\rho(y,w)|}\\&+\psi_6(y)\frac{|\rho(y,[G_1y]_\alpha)||\rho(w,[G_2w]_\alpha)|}{1+|\rho(y,w)|}\\&+\psi_7(y)\frac{|\rho(w,[G_1y]_\alpha)||\rho(y,[G_2w]_\alpha)|}{1+|\rho(y,w)|}\Big);\end{aligned} \tag{59}$$

then clearly for $\psi_1(y)=(y+1)/74$, $\psi_2(y)=y/10$, $\psi_3(y)=y/20$, $\psi_4(y)=y/30$, $\psi_5(y)=y/60$, $\psi_6(y)=y/40$, $\psi_7=y/25$ and $\Phi(t)=74t/75$.

$$\begin{aligned}\left|\frac{y}{75}-\frac{w}{40}\right|\le\frac{74}{75}\Big(&\frac{y+1}{74}|y-w|+\frac{y}{10}\left|\frac{74y}{75}\right|\\&+\frac{y}{20}\left|\frac{39w}{40}\right|+\frac{y}{30}\frac{|74y/75||y-w/40|}{1+|y-w|}\\&+\frac{y}{60}\frac{|w-y/75||39w/40|}{1+|y-w|}\\&+\frac{y}{40}\frac{|74y/75||39w/40|}{1+|y-w|}\\&+\frac{y}{75}\frac{|w-y/75||y-w/40|}{1+|y-w|}\Big)\end{aligned} \tag{60}$$

which can be easily calculated by

$$\begin{aligned}\frac{74}{75}\left(\frac{y+1}{74}|y-w|\right)&=\frac{74}{25}\left(\frac{1}{74}\left|y^2+y-w-yw\right|\right)\\&\ge\frac{1}{75}(|y-w-yw|)\\&=\left|\frac{y}{75}-\frac{w(1+y)}{75}\right|\ge\left|\frac{y}{75}-\frac{w}{75}\right|\\&\ge\left|\frac{y}{75}-\frac{w}{40}\right|.\end{aligned} \tag{61}$$

The remaining terms of (59) are nonzero reals. Consequently we can obtain

$$\begin{aligned}\Phi\Big(&\psi_1(y)\rho(y,w)+\psi_2(y)\rho(y,[G_1y]_\alpha)\\&+\psi_3(y)\rho(w,[G_2w]_\alpha)\\&+\psi_4(y)\frac{\rho(y,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)}\\&+\psi_5(y)\frac{\rho(w,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)}\\&+\psi_6(y)\frac{\rho(y,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)}\\&+\psi_7(y)\frac{\rho(w,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)}\Big)\succeq\varpi_{yw}.\end{aligned} \tag{62}$$

Therefore

$$\begin{aligned}\Phi\Big(&\psi_1(y)\rho(y,w)+\psi_2(y)\rho(y,[G_1y]_\alpha)\\&+\psi_3(y)\rho(w,[G_2w]_\alpha)\\&+\psi_4(y)\frac{\rho(y,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)}\\&+\psi_5(y)\frac{\rho(w,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)}\\&+\psi_6(y)\frac{\rho(y,[G_1y]_\alpha)\rho(w,[G_2w]_\alpha)}{1+\rho(y,w)}\\&+\psi_7(y)\frac{\rho(w,[G_1y]_\alpha)\rho(y,[G_2w]_\alpha)}{1+\rho(y,w)}\Big)\\&\in s([G_1y]_\alpha,[G_2w]_\alpha).\end{aligned} \tag{63}$$

Hence all conditions of Theorem 14 are satisfied by G_1,G_2; therefore there exists $0\in\mathscr{X}$ such that $0\in[G_10]_\alpha\cap[G_20]_\alpha$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Parameter Optimization of MIMO Fuzzy Optimal Model Predictive Control By APSO

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This paper introduces a new development for designing a Multi-Input Multi-Output (MIMO) Fuzzy Optimal Model Predictive Control (FOMPC) using the Adaptive Particle Swarm Optimization (APSO) algorithm. The aim of this proposed control, called FOMPC-APSO, is to develop an efficient algorithm that is able to have good performance by guaranteeing a minimal control. This is done by determining the optimal weights of the objective function. Our method is considered an optimization problem based on the APSO algorithm. The MIMO system to be controlled is modeled by a Takagi-Sugeno (TS) fuzzy system whose parameters are identified using weighted recursive least squares method. The utility of the proposed controller is demonstrated by applying it to two nonlinear processes, Continuous Stirred Tank Reactor (CSTR) and Tank system, where the proposed approach provides better performances compared with other methods.

1. Introduction

Predictive control is a member of advanced discrete-time process control algorithms. This control algorithm is based on the use of an explicit process model to predict the manipulated variables and thus the future control actions are optimized throughout a finite horizon. To obtain a good performance, a process model describing the effects of all the different inputs on all the outputs must be developed. Although the linear model predictive control is suitable for processes that are not highly nonlinear, this strategy has been applied in the control of nonlinear systems, whether for SISO systems [1–3] or for MIMO systems [4–7]. But many industrial processes have strong nonlinearities and predictive control is applied in order to provide satisfactory control results. Two problems have appeared because of the introduction of nonlinearities in the predictive control.

(i) The first of the problems is that the modeling of processes is much more difficult and complex than the linear case. fuzzy logic is among the most used strategies in all areas [4]. Despite the fact that this strategy has been developed in the last few years, the fuzzy models of the TS type remain among the most used methods to deal with the MPC control for nonlinear systems (NMPC), because of their capacity to give an accurate approximation of the complex nonlinear MIMO systems.

(ii) The second important problem in nonlinear predictive control is the solving of the optimization problem. Conventional optimization methods, such as the gradient search method, used for designing the state feedback controller are restricted to the eigenvalues of the linear system matrix that not only increases the difficulty but also takes long time to find the global optimum solution [8]. Hence, evolutionary computation (EC) can be considered an alternative method to solve this type of optimization problem. In literature, plenty of works have been reported to solve the control optimization problems using EC techniques because they do not require explicit gradient information for optimization.

Particle Swarm Optimization (PSO), introduced by [9], is a population based metaheuristic search (MS) algorithm,

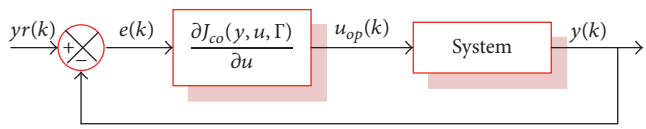

Figure 1: Problem principle of optimal control.

which emulates the collaborative behaviour of bird flocking and fish schooling in searching for foods. In addition, unlike other heuristic optimization methods, PSO has a flexible and well-balanced mechanism to enhance the global and local exploration abilities. Since the introduction of PSO, many works based on MPC have been solved using PSO because it is not largely affected by the size and nonlinearity of the problem. Reference [10] introduces an approach for designing an adaptive fuzzy model predictive control using the PSO algorithm (AFMPC-PSO). In [11], a type of MPC is proposed using Chaos Particle Swarm Optimization (CPSO). First, for the modeling phase, the TS fuzzy model is employed to approximate the nonlinear system. Second, we introduce CPSO into MPC using a modified performance criterion in order to provide less computational controller's expression.

Although these methods have represented effective solutions to the problem of the MPC control of nonlinear systems, there is the problem of choice of control parameters. Several studies have shown the influence of these parameters on the quality of the responses of the systems treated. To overcome this problem, this paper presents a method of tuning the weight parameters of the performance function according to the output quantities detected from the system. One of the challenges in MPC is how control parameters can be tuned for various target systems, and the use of APSO for automatic tuning is one of the solutions. Firstly, for the modeling phase, the TS fuzzy model is employed to approximate the nonlinear system. In the second step, we used the principle of optimal control to calculate the control law of each linear subsystem. Then, we introduce APSO algorithm to determine the best weight parameters of the performance function using a performance criterion in order to improve the quality of the response with a minimum of energy.

The rest of the paper is organized as follows. Section 2 presents the influence of weights existing in the objective function on the quality of system performances. Section 3 reviewed the TS fuzzy model and the OMPC design method. The main contribution of this paper is presented in Section 4. In order to show the good performance of the proposal approach, simulation results are given in Section 5. Finally, Section 6 concludes.

2. Statement

The MPC control is a method of designing process control systems with feedback. This designing is carried out by the online repetition of a procedure that includes inputting data to a system of the initial input values. The principle of calculating the control law is the resolution of an optimal control problem using the present output values. So, the MPC control is a special case of the optimal control whose horizon is finite. It is recalled that this command minimizes the function described above. However, when the horizon is infinite, we speak of optimal control. The objective function is written as follows:

$$J_{co} = \sum_{i=1}^{n_y} \Gamma_{yi} \left(yr_i - y_i\right)^2 + \sum_{i=1}^{n_u} \Gamma_{ui} \left(u_i - ur_i\right)^2, \tag{1}$$

where Γ_{yi}, Γ_{ui} are the weight values of the outputs and inputs. ur_i are the control instructions that are defined beforehand based on an expertise of the system. The partial derivative of the objective function described by (1) is as follows:

$$\frac{\partial J_{co}}{\partial u} = \frac{\partial \sum_{i=1}^{n_y} \Gamma_{yi} \left(yr_i - y_i\right)^2 + \sum_{i=1}^{n_u} \Gamma_{ui} \left(u_i - ur_i\right)^2}{\partial u}. \tag{2}$$

By applying the principle of the optimal control, we obtain the following:

$$u_{op}(k) = \frac{\partial J_{co}\left(y, u, \Gamma_{yi}, \Gamma_{ui}\right)}{\partial u}. \tag{3}$$

The structure of the loop system based on OMPC illustrating this method is shown in Figure 1. According to (1), the criterion J_{co} depends strongly on the weights $\Gamma_i = [\Gamma y_i, \Gamma u_i]$. These weights directly affect the performances index (Pi) of the system studied, such as overshoot (Ov%), settling time (Ts), rise time (Tr), and static error (Es).

To show the importance of the choice of these parameters on the response of the system in closed loop, we consider the following a multivariable linear system. This system is described by the following equations:

$$\begin{aligned}
x_{11}(k) &= 0.9401 x_{11}(k-1) + u_1(k-1) \\
x_{12}(k) &= 0.9524 x_{12}(k-1) + u_2(k-1) \\
x_{21}(k) &= 0.9083 x_{21}(k-1) + u_1(k-1) \\
x_{22}(k) &= 0.9306 x_{22}(k-1) + u_2(k-1) \\
y_1(k) &= -0.7664 x_{11}(k) + 0.9 x_{12}(k) \\
y_2(k) &= -0.6055 x_{21}(k) + 1.3472 x_{22}(k).
\end{aligned} \tag{4}$$

The criterion to be minimized is:

$$J_{co} = \sum_{i=1}^{2} \Gamma y_i \left(y_i - yr_i\right)^2 + \sum_{i=1}^{2} \Gamma u_i \left(u_i - ur_i\right)^2; \quad (i = 1, 2) \tag{5}$$

with $yr_1 = 6.1$, $yr_2 = 12.8$, $ur_1 = 1$, and $ur_2 = 1$.

In a first step, the minimization of the objective function is obtained by canceling the gradient of J_{co} with respect to u_i; the expressions are obtained as a function of x_{ij} of (4) and Γ:

$$\begin{aligned} u_1(k) &= 2\Gamma_{u1} + \Gamma y_2\left(1.63\Gamma y_2(k-1) - 0.66x_{21}(k-1) + 1.52x_{22}(k-1) - 15,50\right) \\ &\quad + \frac{\Gamma y_1\left(\left(1.37u_2(k-1) - 1.10x_{11}(k-1) + 1.13x_{22}(k-1) - 9.35\right)\right)}{\left(1.17\Gamma y_1 + 0.73\Gamma y_2 + 2\Gamma_{u1}\right)}. \\ u_2(k) &= 2\Gamma_{u_2} + \Gamma y_1\left(1,37u_1(k-1) + 1.29x_{11}(k-1) - 1.54x_{12}(k-1)\right) + 10.98 \\ &\quad + \frac{\Gamma y_2\left(1.36u_1(k-1) + 1,48x_{21}(k-1) - 3.37x_{22}(k-1) + 34.48\right)}{\left(1.62\Gamma y_1 + 3.62\Gamma y_2 + 2\Gamma_{u2}\right)} \end{aligned} \tag{6}$$

Table 1 shows how the overshoot, settling time, rise time, and static error as the performance indices vary with the weight parameters $(\Gamma y_1, \Gamma y_2, \Gamma u_1, \Gamma u_2)$ of the performance function when outputs transitions y_1 and y_2 of the optimal control are used. As can be seen from the table, the transient characteristics depend strongly on the weight parameters $(\Gamma y_1, \Gamma y_2, \Gamma u_1, \Gamma u_2)$ of evaluation function (4).

The example presented shows the influence of the choice of the values of the weights existing in the expression of the optimal control law. So, the weight parameters directly affect Ov%, Es, ts, and tr.

3. MIMO TS Fuzzy Model and OMPC Design

Takagi and Sugeno proposed the well-known TS fuzzy model in [12] to describe the complicated nonlinear system. The TS fuzzy models are universal approximators capable of approximating any continuous function with certain level of accuracy [13]. Since the MIMO system can be divided into multiple input-single output (MISO) systems, we take MISO systems instead, to identify MIMO systems. It is assumed that an MISO system $F(x, y)$ is the system that needs identification, where x is the system input with $x \in R^M$ and y is the system output with $y \in R$.

The rules of TS fuzzy models, used in this work, have the following form:

$$\begin{aligned} R_g\text{: IF} \quad & Z_{i1} \text{ est } \Omega_{ig1} \text{ et} \ldots Z_{iM} \text{ est } \Omega_{igM} \\ \text{THEN} \quad & y_{ig}(k) = A_{ig}y(k-1) + B_{ig}u(k-1), \\ & i = 1,\ldots,ny, \;\; g = 1,2,\ldots,c_i, \end{aligned} \tag{7}$$

where R_g represents the gth rule, c_i is the number of rules for the ith subsystem, M is the dimension of the input vector, Ω_{ig} is the fuzzy subset of the g-th rule, $Z_i = [Z_{i1}, \ldots, Z_{iM}] \in R^M$ is the input vector, and $A_g = [A_{g1}, \ldots, A_{gny}]$ and $B_g = [B_{g1}, \ldots, B_{gnu}]$ are two polynomial vectors. The final output is calculated as the average of the outputs corresponding to the rules R_r, weighted by the normalized degree of completion (membership), as follows:

$$\hat{y}_i(k) = \frac{\sum_{g=1}^{c_i} \mathsf{w}_{ig}(Z_i)\, y_{ig}(k)}{\sum_{g=1}^{c_i} \mathsf{w}_{ig}(Z_i)} \tag{8}$$

$$\mathsf{w}_{ig}(Z_i) = \prod_{j=1}^{M} \mathsf{A}_{igj}(Z_{ij}). \tag{9}$$

The standardized degree of completion is described in the following form:

$$\mu_{ig}(Z_i) = \frac{\mathsf{w}_{ig}(Z_i)}{\sum_{g=1}^{c_i} \mathsf{w}_{ig}(Z_i)}. \tag{10}$$

The standard degree of achievement is actually the degree of activation of the corresponding local model in the region where the system evolves. The fuzzy subsets are generally Gaussian, triangular or sigmoid and must satisfy the following properties:

$$\begin{aligned} &\sum_{g=1}^{c_i} \mu_{ir}(Z_i) = 1, \quad \forall g = 1,2,\ldots,c_i \\ &0 \le \mu_{ig}(Z_i) \le 1. \end{aligned} \tag{11}$$

The MPC control is a method of designing process control systems with feedback. This designing is carried out by the online repetition of a procedure that includes inputting data to a system of the initial input values. The MPC control is a special case of the optimal control. It is recalled that this control minimizes the function described above. However, when the horizon is infinite, we speak of optimal control for each linear subsystem.

The concept of the OMPC technique for TS system is utilized to design fuzzy controller. In this concept, the fuzzy controller rule shares the same premise part as the fuzzy system (8) and use same number of fuzzy rules.

The fuzzy controller is inferred as follows:

$$U_i = \frac{\sum_{g=1}^{c_i} \mathsf{w}_{ig}(Z_i)\, u_{ig}}{\sum_{g=1}^{c_i} \mathsf{w}_{ig}(Z_i)}. \tag{12}$$

In the OMPC, the cost index to be minimized is expressed as follows:

$$J_g = \sum_{i=1}^{n_y} \Gamma_{y_g}\left(yr_{ig} - y_{ig}\right)^2 + \sum_{i=1}^{n_u} \Gamma_{u_g}\left(u_{ig} - ur_{ig}\right)^2, \tag{13}$$

Table 1: Weight parameter Γ_i dependence of transient characteristic.

Pi	$\left\{\begin{matrix}1,1\\1,1\end{matrix}\right\}$	$\left\{\begin{matrix}1,1\\0.2,0.2\end{matrix}\right\}$	$\left\{\begin{matrix}\Gamma y_1,\Gamma y_2\\\Gamma u_1,\Gamma u_2\end{matrix}\right\}$ $\left\{\begin{matrix}1,0.1\\1,0.1\end{matrix}\right\}$	$\left\{\begin{matrix}0.3,0.3\\1,0.1\end{matrix}\right\}$
Es1	0.000	0.003	0.015	0.000
Es2	0.000	0.001	0.02	0.000
Ov1 (%)	40.41	32.09	09.33	26.33
Ov2 (%)	57.3988	36.36	25.33	53.20
Ts1	12.00	10.50	8.00	12.00
Ts2	11.00	13.00	16.00	17.00
Tr1	4.869	5.4701	4.6000	5.5637
Tr2	7.00	6.50	20.00	21.00

where Γ_{y_g}, Γ_{u_g} are the weight values of the outputs and inputs. ur_{ig} are the control instructions that are defined beforehand based on an expertise of the system. The general structure of the gth controller is then as follows:

$$\begin{aligned} R_g\text{: IF} \quad & Z_{i1} \text{ is } \Omega_{ig1}, \ldots, Z_{iM} \text{ is } \Omega_{igM} \\ \text{THEN} \quad & u_{ig}(k) = \frac{\partial J_{ig}}{\partial u_{ig}}. \end{aligned} \tag{14}$$

The previous solution shows that the Γ weights directly affect the performance of the system. Indeed, despite the fact that the weight values Γ chosen can give good performance, we can not conclude that these are the best values.

4. Design of the Proposed FOMPC Controller

In this section, we describe a new method for determining the weight values of the objective function J_g of each local system.

So, an objective function is given, and the optimal control law of each local system is calculated. Then, the APSO algorithm intervenes to determine the optimal values of the weight. The next step is to design the global control law such that the proposed OFMPC-APSO presents the desired dynamic characteristics. The proposed FOMPC based on APSO combines both of the advantages of FOMPC and APSO algorithm. APSO algorithm is used to search and to fine tune the weight vector Γ_y and Γ_u of MPC controller. So, the expression of control law by the principle of optimal control is calculated. Once the latter has been established, the next step is to calculate the optimal weights such that the proposed algorithm presents the desired dynamic characteristics.

4.1. Design of the Linear Control Law. We consider parameter optimization problem as a simple problem of tuning only the weight Γ. Now let us assume that the performance function Φ for each output evaluates the (Ov%), (Ts), (Tr), and (Es). Let us define the performance function Φ as proposed in the following:

$$\Phi_g = q_1 \mathrm{Es} + q_2 \mathrm{Ov\%} + q_3 \mathrm{Ts} + q_4 \mathrm{Tr}. \tag{15}$$

Here (q_1, q_2, q_3 et q_4) are the weights of the respective performance indices. So, we obtain the optimal values of the weights of criterion (13) while respecting the following constraint:

$$\min_{\Gamma_{\min}<\Gamma<\Gamma_{\max}} \Phi_g(\Gamma). \tag{16}$$

With $\Gamma_{\min}$ and $\Gamma_{\max}$ the minimum and maximum limits are chosen.

The optimization problem given by (15) is a constrained nonlinear and nonconvex optimization problem, the solution of which is difficult and generally expensive in computing time. Different approaches were investigated to solve this problem, such as the numerical optimization techniques [14, 15], the metaheuristic based optimization algorithms [16–18], the linearization of the process fuzzy model [19], and the use of particular model structures to obtain a convex form for the cost function [20].

4.2. APSO Algorithm. The PSO algorithm is a type of stochastic global optimization algorithm for improving candidate solutions [9]. This algorithm iteratively explores a multidimensional search space with a swarm of individuals (referred to as "particles"), looking for the global optima [21, 22]. PSO has memory to store the knowledge of good solutions by all particles; in addition, particles in the swarm share information with each other. Therefore, due to the simple concept, easy implementation, and quick convergence, nowadays PSO has gained much attention and wide applications in solving continuous nonlinear optimization problems [23–26].

It is initialized with a population of random solutions, called particles, to find the optimal result. Each particle has a position and a velocity, representing a possible solution to the optimization problem and a search direction in the search space. In each iterative process, the particle adjusts the velocity and position according to the best experiences that are called the $p$best, found by itself, and $g$best, found by all its neighbors [27]. For every generation, the velocity and position can be updated by the following equations:

$$\begin{aligned} V_p^d(\mathrm{t}+1) = {} & wV_p^d(\mathrm{t}) + r_1 C_1 \left(p\mathrm{best}_p^d - X_p^d(\mathrm{t})\right) \\ & + r_2 C_2 \left(p\mathrm{best}_g^d - X_p^d(\mathrm{t})\right) \end{aligned} \tag{17}$$

$$X_p^d(\mathrm{t}+1) = X_p^d(\mathrm{t}) + V_p^d(\mathrm{t}+1), \tag{18}$$

where $i = 1, 2, \ldots, Np$ and Np is the number of particles, t is the number of iterations, and r_1 and r_2 are two random

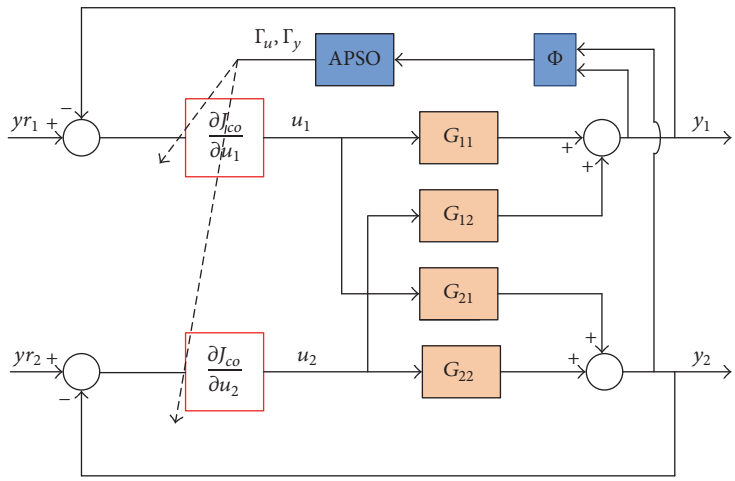

Figure 2: The control scheme.

Pahse 1. Identification of the model
Step 1. The nonlinear MIMO system is modeled by a TS fuzzy inference system.
Step 2. Identify of parameters using WRLS method.
Pahse 2. Application of Algorithm 1 *for each local model*
Step 1. The weight parameter Γ is specified.
Step 2. Give the objective function in the form of Eq. (13).
Step 3. Calculate the control law u_i using Eq. (3).
Step 4. Compute Γ^{Best} that minimize the restricted function (15) using the procedure of Algorithm 2 and find the best particle labeled as $\Gamma = [\Gamma_{y_i}^{\text{Best}}, \Gamma_{u_i}^{\text{Best}}]$.
Step 5. Calculate the control law u_i using (3) According to the optimum values of Γ^{Best}
Pahse 3. Design the control law U via equation (12)

Algorithm 1: FOMPC-APSO algorithm.

numbers in the interval $[0, 1]$. C_1 and C_2 are positive constants. w is the inertia weight, a parameter used to control the impact of the previous velocities on the current velocity. If it is chosen properly, the particle will have the balanced ability of exploitation and development. w is updated as follows:

$$w = w_{\max}\left(\frac{w_{\max} - w_{\min}}{\mathrm{t}_{\max}}\right)\mathrm{t}, \tag{19}$$

where $w_{\min}$ and $w_{\max}$ are minimum and maximum values of w which are taken as 0.4 and 0.9, respectively.

In conventional PSO, the velocity of each particle in the next search is updated using the knowledge of its past velocity and personal and global best positions. However, the performance of PSO greatly depends on its parameters; it often suffers from being trapped in local optimum [28, 29]. Indeed, the inertia weight is the most important parameter to balance the local search ability and global search ability [30]. This balancing is a key role to improve the performance of PSO. However, the method of selecting inertia weight is difficult. And experiments show that particles can accumulate at point in local searching area. So, to avoid all these difficulties, an improved version of PSO has appeared; it is the APSO algorithm.

The basic idea of APSO is that the global best and the personal best position of particle always change over iteration and tend to the similar value if the swarm has approached the solution. The values of $p$best and $g$best are taken and are used to adjust the value of inertia weight by using feedback mechanism. In this condition, the inertia weight should be set to larger value. So, the balancing between local-global can be controlled based on the swarm condition.

The modified inertia weight is modified as follows:

$$\omega = \left(\omega_0 - \frac{p_{g_{\text{best}}}}{p_{p_{\text{best}}}}\right), \tag{20}$$

where χ_0 is the initial value of inertia weight.

The controller structure is illustrated in Figure 2. This Figure represents the case of a system with two inputs (u_1 and u_2) and two outputs (y_1 and y_2). yr_1 and yr_2 represent the references of each output, respectively.

4.3. FOMPC-APSO Algorithm. The general design steps of the FOMPC-APSO algorithm are summarized in Algorithm 1.

5. Simulation Study

In order to show the considerable contribution in the performance of the proposed control scheme, two highly nonlinear

S.1. Choose the weighting matrices $\Gamma = [\Gamma_{y_i} > 0, \Gamma_{u_i} > 0]$, $\Gamma_{\max}$, $\Gamma_{\min}$, the number of particles NP. Initialize the position and velocity of each particle; fix learning factors $C1$ and $C2$; ω_0 and the number of iterations $t_{\max}$.
S.2. For t = 1 to $t_{\max}$ do
for each particle do
(1) Calculate the fittness value of each particle by minimizing the following Eq. (15)
(2) Find the individual best pbest for each particle and the global best $g$best.
(3) Update the velocity and the position of each particle using equations (17) and (18), respectively.
end for
end for
S.3. Find the best particle labeled as $\Gamma = [\Gamma_{y_i}^{\text{Best}}, \Gamma_{u_i}^{\text{Best}}]$

Algorithm 2: Procedure of weight parameter optimization.

Table 2: Specification of the surge tank.

Parameter	Description	Normal operation condition
H_0	Initial value of tank level	0.15 [m]
H_s	Initial value of the output channel level	0.015 [m]
a	Section of the channel output	0.0001 [m^2]
A	Section of the tank	0.04 [m^2]
Q_0	The initial flow	0.00013 [$m^3 s^{-1}$]
k_0	Constant	1
k_1	Constant	0.1

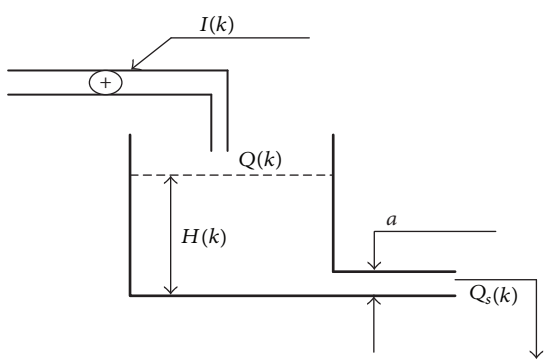

Figure 3: The surge tank system.

systems are selected. The first example is the surge tank. The second example is the CSTR. We compare our results with those obtained by other existing methods such as NMPC [31] and FMPC using the APSO algorithm [32].

In this paper, the Tr, Ts, Ov%, pic, and Es are used as the performance indexes.

5.1. Surge Tank System. The behaviour of the surge tank system, shown in Figure 3, is fed by a pump driven by a current $I(k)$ [10].

In Figure 3, $Q(k)$ is the feed rate, $I(k)$ is the supply current of the pump, H is the liquid level in the tank, Q_s is the output flow, a is the section of the output channel, A is the section of the tank, and H_s is the water level in the output channel. The mathematical model of this reactor is

(i) Model of the valve is as follows:

$$Q(k) = Q(k-1) + Te\left(-k_0 Q(k-1) + k_1 I(k-1)\right) \quad (21)$$

(ii) The change in water level in the tank is given by the following:

$$\begin{aligned} V(k) &= AH(k) \\ &= H(k-1) + Te\left(Q(k-1) - Q_s(k-1)\right), \end{aligned} \quad (22)$$

where $Q_s(k) = 0.6a\sqrt{2g(H(k) - H_s)}$.

The values of the constant parameters of this system are grouped in the Table 2.

Fuzzy modeling: 1000 pairs of data are used to identify the model using the FCM algorithm. For a good approximation of the plant, we suppose that the subsystems are in the third order. The model consists of two rules of the form:

$$\begin{aligned} R^1\text{: If} \quad & I_1 \text{ is } Q^1 \\ \text{THEN} \quad & H^1(k) \\ & = a_{11}H(k-1) + a_{12}H(k-2) \\ & \quad + a_{13}H(k-3) + b_{11}I_1(k-1) \\ & \quad + b_{12}I_1(k-2) \\ R^2\text{: IF} \quad & I_1 \text{ is } Q^2 \\ \text{THEN} \quad & H^2(k) \\ & = a_{21}H(k-1) + a_{22}H(k-2) \\ & \quad + a_{23}H(k-3) + b_{21}I_1(k-1) \\ & \quad + b_{22}I_1(k-2). \end{aligned} \quad (23)$$

Table 3: Pi performances obtained by the different algorithms.

Pi	Algorithms		
	NMPC	FMPC-APSO	*FOMPC-APSO*
Tr	0.8111	0.9820	*0.7402*
Ts	5.6402	11.0788	*1.9936*
Ov%	6.0804	2.6922	*0.5017*
Pic	1.26	0.2055	*0.2162*
Es	0.00	0.00	*0.00*
W%	53.4	54.86	—

The vector of parameters of gth rule is obtained by using the WRLS. This fuzzy model is used to represent the process model in the controller:

$$\begin{aligned} a_{11} &= 1.952, \\ a_{21} &= 0.0487, \\ a_{31} &= 0.0522, \\ b_{11} &= 0.1131, \\ b_{21} &= 0.1106, \\ a_{21} &= 0.8421, \\ a_{22} &= 0.0808, \\ a_{23} &= 0.0764, \\ b_{21} &= 0.1668, \\ b_{22} &= 0.1697. \end{aligned} \tag{24}$$

Once the estimated model is obtained, we will investigate the optimal parameters of the FOMPC control law using the APSO algorithm. This gives the best results with these settings: $t_{\max} = 350$, $Np = 35$, $C_1 = C_2 = 2.05$, $r_2 = 0.2$, and $w_0 = 1.4$. The fitness function of the APSO algorithm is defined by the following:

$$J_i = \Gamma_{yi}\left(H_i - H_r\right)^2 + \Gamma_{Ii}\left(I_i\right)^2, \quad i = 1, 2. \tag{25}$$

Table 3 shows the performances obtained by these methods. In each interval time, we have changed the set point for evaluating each method to control a highly nonlinear system. The proposed method can generate a high quality solution within shorter calculation time and it tends to converge very fast compared to other methods. The comparison shows some interesting results. It is important to observe that, with FOMPC-APSO, the Ts has been reduced almost 2 times comparing with that obtained from the NMPC and has been reduced almost 5 times comparing with that obtained from the FMPC-APSO. The same observation can be made for the Ov%, where in the FOMPC-APSO case we notice a reduction of nearly 4 times compared with that obtained from FMPC-APSO and more than 10 times with NMPC without any increase in Tr. So, the proposed method is able to keep better stability with less control effort applied.

In fact, the proposed technique is able to achieve good performance using *53.4%* of total control energy consumed by FMPC-APSO as well as *54.68%* of FMPC-APSO. As it is presented in Table 3, the accuracy of our model outperforms that of other methods. To confirm these results further, Figure 4 shows the variations in liquid level in the tank when a step change is applied at 30, 60, and 90, respectively, by FOMPC-APSO, FMPC-APSO, and NMPC.

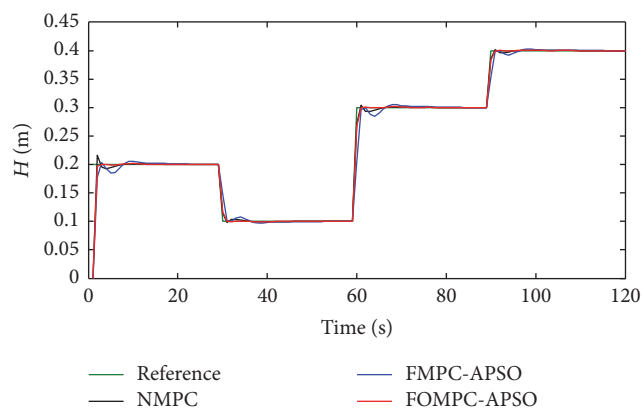

Figure 4: Evolution of liquid level H.

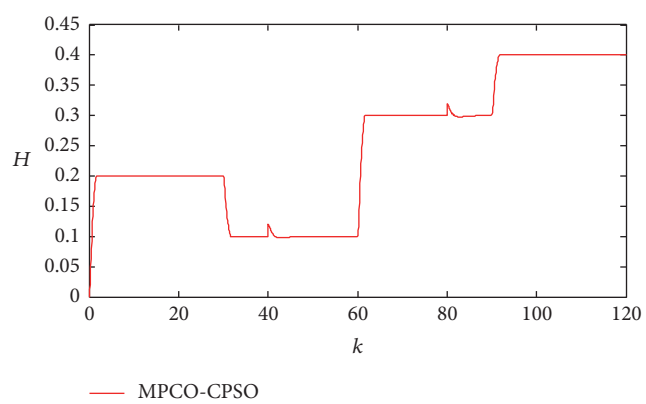

Figure 5: Evolution of liquid level H in the presence of disturbance.

Figure 5 shows the manipulated responses when a disturbance is applied to the feed rate, at 40 and 80, respectively.

In conclusion, the proposed controller shows the best performance for both set point tracking and regulatory conditions for the entire range of the process as compared to the other controllers.

Table 4: CSTR model parameters.

Description	Parameter	Nominal value
Product concentration	Ca	0.1 [mol L^{-1}]
Reactor temperature	T	438.51 [K]
Coolant flow rate	q_c	103.41 [min^{-1}]
Process flow rate	q	100 [L min^{-1}]
Feed concentration	C_{Af}	1 [mol L^{-1}]
Feed temperature	T_f	350 [K]
Inlet coolant temp	T_{cf}	350 [K]
Reaction volume	v	100 [L]
Heat transfer coefficient	h_a	$7 * 10^5$ [cal min^{-1} K]
Reaction rate constant	k_0	$7.2 * 10^{10}$ [min^{-1}]
Activation energy term	E/R	$1 * 10^4$ [K]
Heat of reaction	ΔH	$2 * 10^5$ [cal mol^{-1}]
Liquid densities	ℓ_c, ℓ	$1 * 10^3$ [g L^{-1}]
Specific heats	C_p, C_{pc}	1 [cal g^{-1} K^{-1}]

In fact, the proposed technique is able to achieve good performance using 53.4% of total control energy consumed by NMPC as well as 54.86% of FMPC-APSO. As it is presented in Table 3, the accuracy of our model outperforms that of other methods.

5.2. Continuous Stirred Tank Reactor (CSTR). The efficiency and the control accuracy of the proposed algorithm were investigated and compared to other control strategies by considering the control of a highly nonlinear MIMO process, namely, a Continuous Stirred Tank Reactor, where the model is presented in [33] and described by the following differentials equations:

$$\begin{aligned} C_a(k+1) &= C_a(k) + T_e\left(\frac{1}{v}q(k)\left(C_{Af} - C_a(k)\right)\right. \\ &\quad \left. - k_0 C_a(k)\, e^{(-E/RT(k))}\right) \\ T(k+1) &= T(k) + T_e\left(\frac{1}{v}q(k)\left(T_f - T(k)\right)\right. \\ &\quad + k_1 C_a(k)\, e^{-(E/RT(k))} \\ &\quad \left. + k_2 q_c(k)\left(1 - e^{-(k_3/q_c(k))}\right)\left(T_{cf} - T(k)\right)\right), \end{aligned} \tag{26}$$

where $k_1 = -\Delta H k_0/\ell C_p$, $k_2 = \ell_c C_{pc}/\ell C_p v$, and $k_3 = h_a/\ell_c C_{pc}$.

The process describes the reaction that converts the product A into a new product B, the concentration C_a is the concentration of product A, and T is the temperature of the mixture. The reaction is exothermic and it is controlled by a coolant flow whose rate is represented by q_c. The temperature is controlled by changing the coolant and by controlling the temperature, and the concentration is also controlled. Ca_0 is the inlet feed concentration, q is the process flow rate, and T_f and Tc_f are the inlet feed and coolant temperatures. All these values are assumed constant at nominal values. In the same way, k_0, E/R, v, k_1, k_2, and k_3 are thermodynamic and chemical constants. The numerical values of these parameters are given in Table 4.

Fuzzy modeling: the above nonlinear model is used to produce input-output time data. The sampling time is set to 0.01 min. $[Ca(k-1),\ Ca(k-2),\ q_c(k-1),\ q_c(k-2),\ q(k-1)]$ and $[T(k-1),\ T(k-2),\ q(k-1),\ q(k-2),\ q_c(k-1)]$ are selected as the input vector. The rule numbers of the identified fuzzy models are four.

The APSO algorithm parameters are chosen as follows: the maximum number of APSO iterations $t_{\max} = 110$, number of particles $Np = 10$, $r_1 = 0.2$, $r_2 = 0.2$, $C_1 = 1.5$, $C_2 = 2.5$, and $w_0 = 1.4$.

The fitness function is selected as follows:

$$\begin{aligned} J_i &= \Gamma y_{1i}\left(Ca_i - Ca_r\right)^2 + \Gamma q_i\left(q_{ci}\right)^2 + \Gamma y_{2i}\left(T_i - T_r\right)^2 \\ &\quad + \Gamma_{q_i}\left(q_i\right)^2 \quad i = 1, 2, 3, 4. \end{aligned} \tag{27}$$

The reference signals applied to the system are as follows:

$$\begin{aligned} Ca_r &= \begin{cases} 0.1 & 0 \le k < 50 \\ 0.05 & 50 \le k < 100 \\ 0.15 & 100 \le k < 150 \\ 0.25 & 150 \le k \end{cases} \\ T_r &= \begin{cases} 480 & 0 \le k < 50 \\ 365 & 50 \le k < 100 \\ 495 & 100 \le k < 150 \\ 465 & 150 \le k. \end{cases} \end{aligned} \tag{28}$$

Tables 5 and 6 contain the performance indices obtained for the outputs Ca and T by the NMPC [10], FMPC-APSO [11], and FOMPC-APSO algorithms. It summarizes the results of this example in terms of the Ov%, Tr, Ts, and Es. As seen

Table 5: Pi found by different methods.

Pi	Algorithms		
	NMPC	FMPC-APSO	*FOMPC-APSO*
Tr	1.4965	4.4693	*0.4135*
Ts	6.8561	7.2779	*3.5480*
Ov%	5.0973	0.1215	*0.00*
Pic	1.0510	1.0012	*1.0000*
Es	0.00	0.00	*0.00*
W%	48.94	47.28	—

Table 6: Pi found by different methods.

Pi	Algorithms		
	NMPC	FMPC-APSO	*FOMPC-APSO*
Tr	1.3711	3.0459	*0.4116*
Ts	9.2185	4.5471	*3.2626*
Ov%	5.4003	1.6309	*0.0011*
Pic	1.0540	1.0163	*1.0000*
ES	0.00	0.00	*0.00*
W%	48.14	49.07	—

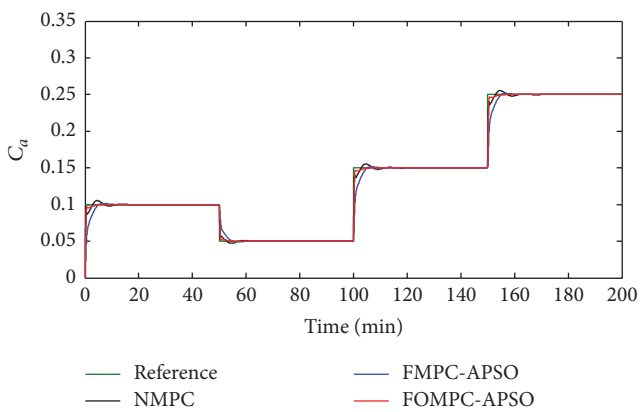

Figure 6: Evolution of concentration *Ca*.

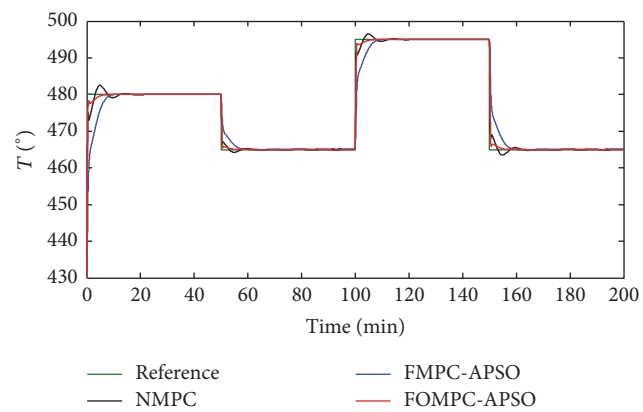

Figure 7: Evolution of temperature *T*.

in these tables, we can note that our method gives the best performance of all compared techniques.

They applied NMPC algorithm to control the concentration of product and it has the Tr and Ts values of 1.4965 min and 6.8561 min, while the FMPC-APSO approach has 4.4693 min and 7.2779 min. However, the corresponding Tr and Ts values for the same problem were 0.4135 min and 3.5480 and with no Ov%. This indicates that the proposed controller is able to perform faster than the other methods in real application environment. Figures 6 and 7 show the evolution of the *Ca* and *T* outputs from the three methods. From these figures, there is a perfect continuation of the signal of the setpoint whose FOMPC-APSO method has ensured good performances. We also note that our method provides more acceptable control effort regarding Figures 8 and 9.

In a second test, the disturbances on the system output in different times have been applied to validate the tracking of the reactor concentration. Thus, a disturbance of 0.002 mol/l at time $t = 82$ min and time $t = 130$ min is added. Figure 10 illustrates the disturbance rejection performance of the FOMPC-APSO controller. The results show that the adaptive controller has the ability to keep the process stable and regulate the outlet concentration at its desired set point value.

6. Conclusion

In this paper, we have introduced the FOMPC-APSO controller applied to highly nonlinear systems. An approach of determining the optimum weights is developed by minimizing a chosen performance criterion using APSO algorithm. The proposed approach is based on the advantage of the TS fuzzy system in the modeling phase and the metaheuristic optimization APSO algorithm in a new structure predictive controller. The advantage of this structure is its ability to handle highly nonlinear systems regardless and keep a good stability in terms of overshoot, rise time, and settling time including disturbances. We have achieved these objectives

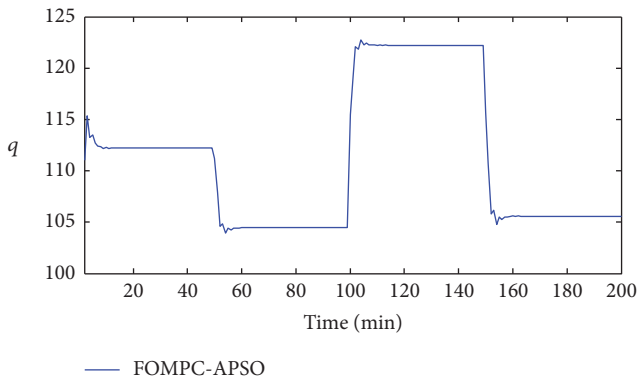

Figure 8: Evolution of control q.

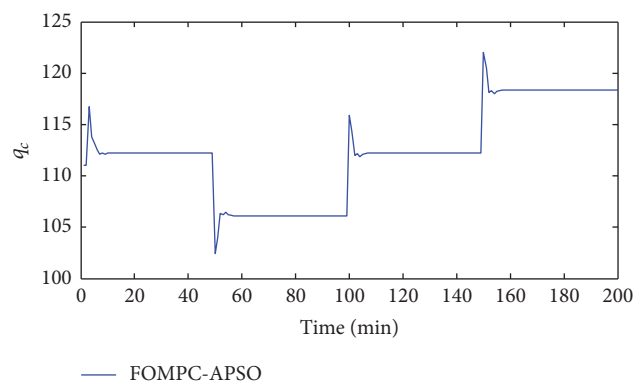

Figure 9: Evolution of control q_c.

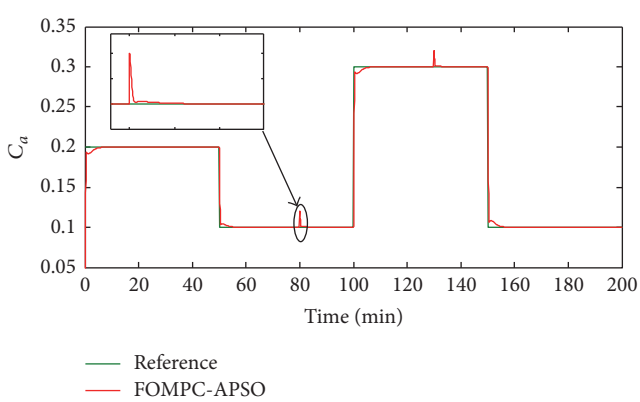

Figure 10: Evolution of concentration with external disturbances.

without any obligation increase in the control signal since we have injected the phenomenon of optimal control in the synthesis of our controller. Compared with other similar existing methods, the FOMPC-APSO algorithm enhances the convergence and accuracy of the controller optimization, which is much easier for implementation in real time.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Fuzzy Portfolio Selection Problems: A Parametric Representation Approach

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Fuzzy portfolio selection problem is a major issue in the financial field and a special case of constrained fuzzy-valued optimization problems (CFOPs). In this respect, the present paper aims to investigate the CFOP with regard to the features of the parametric representation of fuzzy numbers named as convex constraint function (CCF) which is proposed by Chalco-Cano et al. in 2014. Furthermore, relying on this parametric representation, some proper conditions are provided for the existence of solutions to a CFOP. To this end, by the increasing representation of CCF, the main problem is converted to a parametric multiobjective programming problem and some solution concepts from a similar framework in the multiobjective programming are proposed for the CFOP. Eventually to illustrate the proposed results, the fuzzy portfolio selection problem is discussed.

1. Introduction

In fact, because of using the experimental and empirical data for modeling a real world phenomenon, a deterministic mathematical model may not be a perfectly realistic representation. There are several approaches to deal with such real world phenomena, for example, fuzzy techniques, stochastic models, and interval analysis, which differ by their advantages and disadvantages [1, 2]. However, in many practical situations, the uncertainties are not of the statistical or interval type; more precisely, this situation happens mainly through the modeling in terms of linguistic expressions that depend on the human judgment. In other words, an expert perceives exactly which values and parameters are possible and which are not. Therefore, the set of all possible values and parameters can be naturally described as fuzzy numbers by the expert's knowledge.

Historically, fuzzy set theory was proposed by Zadeh in [3] and developed considerably by many other researchers. This theory provides conceptually powerful techniques to handle the imperfect information related to vagueness and imprecision.

Nowadays, the fuzzy optimization problem is effective in a lot of different disciplines related to optimization such as operations research engineering, economics, and artificial intelligence [4–7]. It can be said that the fuzzy optimization problem provides an appropriate choice for considering the vagueness and ambiguousness into the formulation and solutions of the multitude of optimization problems. Indeed, there are several motivations to apply fuzzy optimization model; first, it deals with some practical optimization problems more conveniently than conventional optimization model; also, fuzzy optimization model efficiently reduces information loss arising from the traditional optimization model; moreover, it allows the designer to implement linguistic constraints that may not be easily defined using more conventional optimization algorithms; finally, this model may permit managers to have not only one solution but also a set of them, so that the most suitable solution can be applied according to the state of existing decision of the production process at a given time and without increasing delay. Furthermore, accessing a set of solutions enables user to investigate and analyze the system information in more detail.

On the other hand, most of the common portfolio selection models deal with the uncertainty via probabilistic approaches, where those probabilistic approaches only partly

capture the reality. In addition, there are some other techniques that manage the uncertainty of the financial markets as the fuzzy set theory. It is noteworthy that the fuzzy portfolio selection model integrates the quantitative and qualitative analysis, experts' knowledge, and the investors' opinion in a better manner [7]. Therefore, in this paper, the portfolio selection problem under fuzzy environment based on the constrained fuzzy optimization problem is going to be studied. Many modern computing methodologies can be seen for various fuzzy systems, for example, [8–14]. It is also worth mentioning that there are several results associated with parametric representations of fuzzy numbers [15–17]. Recently, Chalco-Cano et al. [15] have proposed two parametric representations for interval numbers named as "*increasing/decreasing convex constraint function*" and then explicitly extended the proposed representations to the fuzzy case. The representations have the advantage of allowing flexible and easy-to-control shapes of the fuzzy numbers and it is very simple to implement [17]. Accordingly, this point of view and its increasing parametric representation motivate us here to study the *fuzzy portfolio selection problem* as an application of the constrained fuzzy-valued optimization problem. To this end, the arithmetic of fuzzy numbers and the calculus of fuzzy-valued functions are developed based on this parametric representation. Then, by parametric representation of fuzzy-valued function, the constrained fuzzy optimization problem is transformed into a deterministic multiobjective problem. Besides, some solution concepts from a similar framework in the multiobjective programming problem are proposed for the constrained fuzzy-valued optimization problem, by converting it to a general constrained optimization problem. Finally, it has been demonstrated that the solution of the general optimization is related to the solution of the main problem.

The rest of the paper is organized as follows. Section 2 is devoted to giving the definitions of fuzzy numbers and some arithmetic that are used later in the development of results in fuzzy environment. The fuzzy-valued functions in the parametric form and their properties, calculus and convexity, are studied in Section 3. In Section 4, the constrained fuzzy-valued optimization problem is discussed and, as an application, the proposed method is considered to the fuzzy-valued quadratic programming problem. In Section 5, two numerical examples are established to confirm the efficacy of the proposed approach; more particularly one of them reveals how to solve the *fuzzy portfolio selection problem*. At the end, the conclusion is made in Section 6.

2. Fuzzy Numbers and Their Arithmetic

In this section, some basic notations and results on the fuzzy arithmetic are presented; however, it is assumed that the reader is familiar with the fuzzy theory.

Definition 1 (see [18]). Let $\widetilde{a} : \mathbb{R} \to [0,1]$ be a fuzzy set on the set of real numbers $\mathbb{R}$. The fuzzy set $\widetilde{a}$ is a fuzzy number if it is a normal, convex, upper semicontinuous, and compactly supported.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $\mathscr{F}(\mathbb{R})$. For all $\alpha \in (0,1]$, α-level set $[\widetilde{a}]^\alpha$ of any $\widetilde{a} \in \mathscr{F}(\mathbb{R})$ is defined as $[\widetilde{a}]^\alpha = \{x \in \mathbb{R};\ \widetilde{a}(x) \geq \alpha\}$. The 0-level set $[\widetilde{a}]^0$ is defined as the closure of the set $\{x \in \mathbb{R};\ \widetilde{a}(x) > 0\}$. By Definition 1, for any $\widetilde{a} \in \mathscr{F}(\mathbb{R})$ and for each $\alpha \in (0,1]$, $[\widetilde{a}]^\alpha$ is a compact and convex subset of $\mathbb{R}$ and $[\widetilde{a}]^\alpha = [\underline{a}^\alpha, \overline{a}^\alpha]$. $\widetilde{a} \in \mathscr{F}(\mathbb{R})$ can be recovered from its α-level by a well-known decomposition theorem, which states that

$$\widetilde{a} = \bigcup_{\alpha \in [0,1]} \alpha\, [\widetilde{a}]^\alpha, \tag{1}$$

where $\alpha[\widetilde{a}]^\alpha$ denotes the algebraic product of a scalar α with the α-level set $[\widetilde{a}]^\alpha$ and union on the right-hand side is the standard fuzzy union.

As previously mentioned, fuzzy numbers and their arithmetic can be expressed in terms of parameters in the several models [15–17]. Here, from increasing parametric representation [15], each α-level of an arbitrary fuzzy number $\widetilde{a} \in \mathscr{F}(\mathbb{R})$ is represented alternatively by its bounds as follows:

$$\begin{aligned}[\widetilde{a}]^\alpha &= [\underline{a}^\alpha, \overline{a}^\alpha] \\ &= \{a(t,\alpha) = \underline{a}^\alpha + t(\overline{a}^\alpha - \underline{a}^\alpha) \mid t, \alpha \in [0,1]\},\end{aligned} \tag{2}$$

which is based on the convex combination of upper and lower bounds. Moreover, by the parametric representation (2), the α-level of a k-dimensional fuzzy vector $\widetilde{\mathbf{C}}_\nu^k \in (\mathscr{F}(\mathbb{R}))^k$ and a fuzzy matrix $\widetilde{\mathbf{A}}_m \in (\mathscr{F}(\mathbb{R}))^{p\times q}$ can be represented as the set of real-valued vectors and matrices, respectively; that is,

$$\begin{aligned}\left[\widetilde{\mathbf{C}}_\nu^k\right]^\alpha &= \left[\left(\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_k\right)^T\right]^\alpha = \Big\{c(\mathbf{t},\alpha) \mid c(\mathbf{t},\alpha) \\ &= \left(c_1(t_1,\alpha), c_2(t_2,\alpha), \ldots, c_k(t_k,\alpha)\right)^T,\ c_i(t_i,\alpha) \\ &= \underline{c}_i^\alpha + t_i\left(\overline{c}_i^\alpha - \underline{c}_i^\alpha\right),\ \mathbf{t} = (t_1, \ldots, t_k)^T,\ 0 \leq t_i \leq 1,\ i \\ &= 1, \ldots, k,\ 0 \leq \alpha \leq 1\Big\},\end{aligned} \tag{3}$$

$$\begin{aligned}\left[\widetilde{\mathbf{A}}_m\right]^\alpha &= \Big\{A(\mathbf{t},\alpha) \mid A(\mathbf{t},\alpha) \\ &= \left(a_{ij}(t_{ij},\alpha)\right)_{p\times q},\ a_{ij}(t_{ij},\alpha) = \underline{a}_{ij}^\alpha \\ &+ t_{ij}\left(\overline{a}_{ij}^\alpha - \underline{a}_{ij}^\alpha\right),\ 0 \leq t_{ij} \leq 1,\ i = 1, 2, \ldots, p,\ j \\ &= 1, 2, \ldots, q,\ 0 \leq \alpha \leq 1\Big\}.\end{aligned} \tag{4}$$

The parametric representation (2) helps us to build the fuzzy arithmetic, immediately. Nevertheless, the binary operations between two arbitrary fuzzy number can be defined in terms of parameter as follows.

Definition 2. For $\widetilde{a}, \widetilde{b} \in \mathscr{F}(\mathbb{R})$, the algebraic operations can be defined as

$$\left[\widetilde{a} \circledast \widetilde{b}\right]^\alpha = \{a(t_1,\alpha) * b(t_2,\alpha) \mid t_1, t_2, \alpha \in [0,1]\}, \tag{5}$$

$$\left[\tilde{a} \oslash \tilde{b}\right]^{\alpha} = \left\{ \frac{a(t_1,\alpha)}{b(t_2,\alpha)} \mid t_1, t_2, \alpha \in [0,1],\ b(t_2,\alpha) \neq 0 \right\}, \tag{6}$$

$$[k\tilde{a}]^{\alpha} = \{ka(t,\alpha) \mid t, \alpha \in [0,1]\}, \tag{7}$$

where $* \in \{+,-,\cdot\}$ and $k \in \mathbb{R}$.

Remark 3. It is clear that $\tilde{a} \ominus \tilde{b} \neq \tilde{a} \ominus_H \tilde{b}$ in general, where $\ominus_H$ is the Hukuhara difference. However, it can be deduced that $\tilde{a} \ominus_H \tilde{b} = \tilde{a} \ominus \tilde{b}$ if, in (5), $t = t_1 = t_2$ and $c(t,\alpha) = a(t,\alpha) - b(t,\alpha)$ is a nondecreasing function for all $t, \alpha \in [0,1]$.

Definition 4. The product of k-dimensional fuzzy vector $\widetilde{\mathbf{C}}_{\nu}^{k} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k)^T \in (\mathscr{F}(\mathbb{R}))^k$ and a k-dimensional real vector $d = (d_1, d_2, \ldots, d_k)^T \in \mathbb{R}^k$ is defined as $(\widetilde{\mathbf{C}}_{\nu}^{k})^T \diamond d = \sum_{j=1}^{k} \tilde{c}_j d_j$, where it is a fuzzy number.

It is noteworthy that fuzzy numbers are frequently partial ordered. In fact, there are many ways to define the fuzzy order among the set of all fuzzy numbers [19–22]. For example, Ramík and ímánek [22] proposed a partial order relation called the fuzzy-max order; Molinari [20] considered a new criterion of choice between generalized triangular fuzzy numbers and so on. In this paper, two specific partial ordering relations on fuzzy numbers using parametric representation are introduced.

Definition 5. For two arbitrary $\tilde{a}, \tilde{b} \in \mathscr{F}(\mathbb{R})$, with parametric representations $[\tilde{a}]^{\alpha} = \{a(t,\alpha) \mid t, \alpha \in [0,1]\}$ and $[\tilde{b}]^{\alpha} = \{b(t,\alpha) \mid t, \alpha \in [0,1]\}$, it can be deduced that

(i) $\tilde{a} \preceq \tilde{b}$ if $a(t_1,\alpha) \leq b(t_2,\alpha)$, $\forall t_1, t_2, \alpha \in [0,1]$,

(ii) $\tilde{a} \preceq_w \tilde{b}$ if $a(t,\alpha) \leq b(t,\alpha)$, $\forall t, \alpha \in [0,1]$.

It is easy to see that $\preceq$ and $\prec_w$ are partial order relations on $\mathscr{F}(\mathbb{R})$.

3. Fuzzy-Valued Function and Its Differential Calculus

A fuzzy-valued function is a function with fuzzy values, as $\tilde{f} : X \to \mathscr{F}(\mathbb{R})$, where X is a subset of the vector space $\mathbb{R}^n$. Here, the fuzzy-valued functions with fuzzy coefficients are considered which allow us to express their α-levels as a set of classical functions, using the parametric representation (2). To this end, let $\widetilde{\mathbf{C}}$ denote the set of all coefficients present in the fuzzy-valued function $\tilde{f}$, respectively. Without loss of generality, one can consider $\widetilde{\mathbf{C}}$ as an ordered set with respect to its presence in the fuzzy-valued function $\tilde{f}$ (or as fuzzy vector like $\widetilde{\mathbf{C}}_{\nu}^{k}$). Then, for a given fuzzy vector $\widetilde{\mathbf{C}}_{\nu}^{k}$, the α-level of fuzzy-valued function $F_{\widetilde{\mathbf{C}}_{\nu}^{k}} : \mathbb{R}^n \to \mathscr{F}(\mathbb{R})$ can be considered as

$$\left[F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\mathbf{x})\right]^{\alpha} = \left\{ f_{c(\mathbf{t},\alpha)}(\mathbf{x}) \mid f_{c(\mathbf{t},\alpha)} : \mathbb{R}^n \to \mathbb{R},\ c(\mathbf{t},\alpha) \in \left[\widetilde{\mathbf{C}}_{\nu}^{k}\right]^{\alpha} \right\}. \tag{8}$$

For every fixed $\mathbf{x}$ and $\alpha \in [0,1]$, $f_{c(\mathbf{t},\alpha)}(\mathbf{x})$ is continuous in $\mathbf{t}$; consequently, $\min_{c(\mathbf{t},\alpha) \in [\widetilde{\mathbf{C}}_{\nu}^{k}]^{\alpha}} f_{c(\mathbf{t},\alpha)}(\mathbf{x}) = \min_{\mathbf{t} \in [0,1]^k} f_{c(\mathbf{t},\alpha)}(\mathbf{x})$ and $\max_{c(\mathbf{t},\alpha) \in [\widetilde{\mathbf{C}}_{\nu}^{k}]^{\alpha}} f_{c(\mathbf{t},\alpha)}(\mathbf{x}) = \max_{\mathbf{t} \in [0,1]^k} f_{c(\mathbf{t},\alpha)}(\mathbf{x})$ exist and

$$\left[F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\mathbf{x})\right]^{\alpha} = \left[\min_{\mathbf{t} \in [0,1]^k} f_{c(\mathbf{t},\alpha)}(\mathbf{x}),\ \max_{\mathbf{t} \in [0,1]^k} f_{c(\mathbf{t},\alpha)}(\mathbf{x}) \right]. \tag{9}$$

Example 6. Consider the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_{\nu}^{3}} : \mathbb{R}^2 \to \mathscr{F}(\mathbb{R})$, where $F_{\widetilde{\mathbf{C}}_{\nu}^{3}}(x_1, x_2) = \tilde{c}_1 \odot x_1^2 \oplus \tilde{c}_2 \odot \cos(\tilde{c}_3 \odot x_2)$, $\widetilde{\mathbf{C}}_{\nu}^{3} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) = (\tilde{5}, \tilde{2}, \tilde{3})$, and $\tilde{5} = \langle 1,5,6 \rangle$, $\tilde{2} = \langle 0,2,4 \rangle$, and $\tilde{3} = \langle 1,3,5 \rangle$. For every $\alpha \in [0,1]$, by the parametric representation (3), we have

$$\begin{aligned} \left[\widetilde{\mathbf{C}}_{\nu}^{3}\right]^{\alpha} = \Big\{ & c(\mathbf{t},\alpha) \mid c(\mathbf{t},\alpha) = \big(1 + 4\alpha + t_1(5 - 5\alpha), 2\alpha \\ & + t_2(4 - 4\alpha), 1 + 2\alpha + t_3(4 - 4\alpha)\big)^T,\ \mathbf{t} \\ & = (t_1, t_2, t_3)^T,\ 0 \leq t_i \leq 1,\ i = 1,2,3 \Big\} \end{aligned} \tag{10}$$

and so

$$\begin{aligned} \left[F_{\widetilde{\mathbf{C}}_{\nu}^{3}}\right]^{\alpha} = \Big\{ & f_{c(\mathbf{t},\alpha)}(x_1, x_2) \mid f_{c(\mathbf{t},\alpha)}(x_1, x_2) \\ & = \big(1 + 4\alpha + t_1(5 - 5\alpha)\big) x_1^2 + \big(2\alpha + t_2(4 - 4\alpha)\big) \\ & \cdot \cos\big(\big(1 + 2\alpha + t_3(4 - 4\alpha)\big) x_2\big),\ \mathbf{t} = (t_1, t_2, t_3)^T,\ 0 \\ & \leq t_i \leq 1,\ i = 1,2,3 \Big\}. \end{aligned} \tag{11}$$

Because of the continuity of $f_{c(\mathbf{t},\alpha)}(x)$ at $\mathbf{t}$, for every x and $\alpha \in [0,1]$, (9) provides that

$$\begin{aligned} \left[F_{\widetilde{\mathbf{C}}_{\nu}^{3}}\right]^{\alpha} = \big[& (1 + 4\alpha) x_1^2 \\ & + 2\alpha \cos\big((1 + 2\alpha) x_2\big), (6 - \alpha) x_1^2 \\ & + (4 - 2\alpha) \cos\big((5 - 2\alpha) x_2\big) \big]. \end{aligned} \tag{12}$$

Definition 7. Let $F_{\widetilde{\mathbf{C}}_{\nu}^{k}} : \Omega \subseteq \mathbb{R}^n \to \mathscr{F}(\mathbb{R})$ be a fuzzy-valued function, where Ω is a convex subset of $\mathbb{R}^n$. $F_{\widetilde{\mathbf{C}}_{\nu}^{k}}$ is called convex on Ω with respect to $\preceq$ if

$$F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \preceq \lambda F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\mathbf{x}_1) \oplus (1 - \lambda) F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\mathbf{x}_2), \tag{13}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$ and $0 \leq \lambda \leq 1$. Moreover, the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_{\nu}^{k}}$ is convex with respect to $\preceq_w$, if (13) is valid for $\preceq_w$.

Remark 8. By Definitions 5 and 7, the convexity of fuzzy-valued function $F_{\widetilde{\mathbf{C}}_{\nu}^{k}}$ with respect to $\preceq$ or $\preceq_w$ can be deduced by

$$\begin{aligned} & f_{c(\mathbf{t}',\alpha)}(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \\ & \quad \leq \lambda f_{c(\mathbf{t}'',\alpha)}(\mathbf{x}_1) + (1 - \lambda) f_{c(\mathbf{t}'',\alpha)}(\mathbf{x}_2), \\ & \qquad \forall \alpha \in [0,1],\ \mathbf{t}', \mathbf{t}'' \in [0,1]^k, \end{aligned}$$

$$f_{c(\mathbf{t},\alpha)}\left(\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2\right) \le \lambda f_{c(\mathbf{t},\alpha)}\left(\mathbf{x}_1\right)+(1-\lambda)f_{c(\mathbf{t},\alpha)}\left(\mathbf{x}_2\right), \quad \forall\alpha\in[0,1],\ \mathbf{t}\in[0,1]^k, \tag{14}$$

respectively, for all $\mathbf{x}_1,\mathbf{x}_2\in\Omega$ and $0\le\lambda\le1$.

For any two arbitrary fuzzy vectors $\widetilde{\mathbf{C}}_\nu^k,\widetilde{\mathbf{D}}_\nu^k\in(\mathscr{F}(\mathbb{R}))^k$, the definition of algebraic operations between fuzzy-valued functions can be expressed, based on the parametric representation (8), as

$$\begin{aligned}
&\text{(i)}\quad \left[F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x})\oplus F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{y})\right]^\alpha=\left\{f_{c(\mathbf{t},\alpha)}(\mathbf{x})*f_{c(\mathbf{t},\alpha)}(\mathbf{y})\mid f_{c(\mathbf{t},\alpha)}:\mathbb{R}^n\longrightarrow\mathbb{R},\ c(\mathbf{t},\alpha)\in\left[\widetilde{\mathbf{C}}_\nu^k\right]^\alpha\right\},\\
&\text{(ii)}\quad \left[F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x})\oplus F_{\widetilde{\mathbf{D}}_\nu^k}(\mathbf{x})\right]^\alpha=\left\{f_{c(\mathbf{t},\alpha)}(\mathbf{x})*f_{d(\mathbf{t}'',\alpha)}(\mathbf{x})\mid f_{c(\mathbf{t},\alpha)},f_{d(\mathbf{t}'',\alpha)}:\mathbb{R}^n\longrightarrow\mathbb{R},\ c(\mathbf{t},\alpha)\in\left[\widetilde{\mathbf{C}}_\nu^k\right]^\alpha,\ d\left(\mathbf{t}'',\alpha\right)\in\left[\widetilde{\mathbf{D}}_\nu^k\right]^\alpha\right\},
\end{aligned}\tag{15}$$

where $*\in\{+,-,\cdot,/\}$.

It is obvious that a metric to define the distance between two arbitrary fuzzy numbers is required to introduce the differential calculus of a fuzzy-valued function in the parametric form. Hereinafter, for two arbitrary fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$, the quantity

$$D(\widetilde{u},\widetilde{v})=\sup_{0\le\alpha\le1}\left\{d\left([\widetilde{u}]^\alpha,[\widetilde{v}]^\alpha\right)\right\} \tag{16}$$

describes the distance between $\widetilde{u}$ and $\widetilde{v}$, where

$$\begin{aligned}
d\left([\widetilde{u}]^\alpha,[\widetilde{v}]^\alpha\right)=\max\Bigg\{&\max_{\mathbf{t}'}\min_{\mathbf{t}''}\left|u\left(\mathbf{t}',\alpha\right)-v\left(\mathbf{t}'',\alpha\right)\right|,\\
&\max_{\mathbf{t}''}\min_{\mathbf{t}'}\left|u\left(\mathbf{t}',\alpha\right)-v\left(\mathbf{t}'',\alpha\right)\right|\Bigg\}.
\end{aligned}\tag{17}$$

It is easy to see that the upper metric is equivalent to the well-known Hausdorff metric [23].

Definition 9 (limit of fuzzy-valued function [24]). Let $F_{\widetilde{\mathbf{C}}_\nu^k}:\mathbb{R}^n\to\mathscr{F}(\mathbb{R})$ be a fuzzy-valued function and $\mathbf{a}\in\mathbb{R}^n$, $\widetilde{c}\in\mathscr{F}(\mathbb{R})$. The limit of $F_{\widetilde{\mathbf{C}}_\nu^k}$ as $\mathbf{x}$ approaches $\mathbf{a}$ is the fuzzy number $\widetilde{c}$ and we write $\lim_{\mathbf{x}\to\mathbf{a}}F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x})=\widetilde{c}$, if for every $\varepsilon>0$ there exists $\delta>0$ such that $D(F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}),\widetilde{c})<\varepsilon$, whenever $\|\mathbf{x}-\mathbf{a}\|<\delta$. Here, $\|\cdot\|$ is the usual (Euclidean) norm in $\mathbb{R}^n$.

Moreover, the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^k}:\mathbb{R}^n\to\mathscr{F}(\mathbb{R})$ is continuous at $\mathbf{x}^*\in\mathbb{R}^n$ if and only if, for every $\varepsilon>0$, there exists $\delta=\delta(\mathbf{x}^*,\varepsilon)>0$ such that $\lim_{\mathbf{x}\to\mathbf{x}^*}F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x})=F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}^*)$.

Proposition 10 (see [24]). *The limit of fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x})$ exists at $\mathbf{x}^*$, if $\lim_{\mathbf{x}\to\mathbf{x}^*}f_{c(\mathbf{t},\alpha)}(\mathbf{x})$ exists for every $c(\mathbf{t},\alpha)\in[\widetilde{\mathbf{C}}_\nu^k]^\alpha$ and*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{x}^*}\left[F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x})\right]^\alpha&=\left[F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}^*)\right]^\alpha\\
&=\left[\min_{\mathbf{t}}\lim_{\mathbf{x}\to\mathbf{x}^*}f_{c(\mathbf{t},\alpha)}(\mathbf{x}),\max_{\mathbf{t}}\lim_{\mathbf{x}\to\mathbf{x}^*}f_{c(\mathbf{t},\alpha)}(\mathbf{x})\right].
\end{aligned}\tag{18}$$

Moreover, $F_{\widetilde{\mathbf{C}}_\nu^k}$ is continuous at $\mathbf{x}^*$ if and only if $f_{c(\mathbf{t},\alpha)}$ is continuous at $\mathbf{x}^*$ for every $\mathbf{t}\in[0,1]^k$ and $\alpha\in[0,1]$.

One of the first definitions of differentiability for fuzzy-valued functions is the Hukuhara differentiability, which suffers disadvantages in particular to the point that the inverse subtraction does not exist. The generalized Hukuhara derivative has been attempted to clear these difficulties, which is more general than Hukuhara derivative. Finally, the generalized derivative is proposed based on the generalized difference [23]. Roughly speaking, all these derivatives vary with respect to their corresponding differences. Here, based on Definition 2, the following derivative can be defined in terms of parameter.

Definition 11 (differentiability of fuzzy-valued function). The fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^k}$ is said to be differentiable at $\mathbf{x}^*$, if $f_{c(\mathbf{t},\alpha)}$ is differentiable at $\mathbf{x}^*$ for every $\mathbf{t}\in[0,1]^k$ and $\alpha\in[0,1]$ and $\{f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*+\mathbf{h})-f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*),\ 0\le\alpha\le1\}$ satisfies the assumptions of Stacking Theorem [23].

In addition, the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^k}$ is said to be differentiable on $\mathbb{R}^n$ if it is differentiable for all $\mathbf{x}\in\mathbb{R}^n$.

Proposition 12 (see [24]). *If the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^k}$ is differentiable at $\mathbf{x}^*$, then there exists a fuzzy number $F'_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}^*)$ such that*

$$\begin{aligned}
\left[F'_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}^*)\right]^\alpha&=\lim_{\mathbf{h}\to\mathbf{0}}\frac{\left[F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}^*+\mathbf{h})\ominus F_{\widetilde{\mathbf{C}}_\nu^k}(\mathbf{x}^*)\right]^\alpha}{\|\mathbf{h}\|}\\
&=\left[\min_{\mathbf{t}\in[0,1]^k}\lim_{\mathbf{h}\to\mathbf{0}}\frac{f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*+\mathbf{h})-f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\|\mathbf{h}\|},\right.\\
&\quad\left.\max_{\mathbf{t}\in[0,1]^k}\lim_{\mathbf{h}\to\mathbf{0}}\frac{f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*+\mathbf{h})-f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\|\mathbf{h}\|}\right].
\end{aligned}\tag{19}$$

Remark 13. Consider the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^2}:[-2,5]\to\mathscr{F}(R)$ defined as

$$F_{\widetilde{\mathbf{C}}_\nu^2}(x)=-\widetilde{1}\odot x\oplus-\widetilde{2}\odot\sinh(x+2), \tag{20}$$

where $-\widetilde{1}=\langle-2,-1,3\rangle$ and $-\widetilde{2}=\langle-3,-2,-1\rangle$. Using gH-derivative and Definition 11, we have

$$\left[F'_{\widetilde{\mathbf{C}}^2_\nu}(x)\right]^\alpha = \begin{cases} [(3-4\alpha)+(-3+\alpha)\cosh(x+2),(-2+\alpha)+(-1-\alpha)\cosh(x+2)], & -2\le x\le 0, \\ [(-2+\alpha)+(-3+\alpha)\cosh(x+2),(3-4\alpha)+(-1-\alpha)\cosh(x+2)], & 0<x\le 5, \end{cases}$$
$$\left[F'_{\widetilde{\mathbf{C}}^2_\nu}(x)\right]^\alpha = [(-2+\alpha)+(-3+\alpha)\cosh(x+2),(3-4\alpha)+(-1-\alpha)\cosh(x+2)], \tag{21}$$

respectively, which are different. In fact, the sign of the independent variable $\mathbf{x}$ is not considered in Definition 11, while the gH-derivative depends on the sign of $\mathbf{x}$. Therefore, in general, it cannot be expected that the derivatives of a fuzzy-valued function be equal.

The partial derivative of $F_{\widetilde{\mathbf{C}}^k_\nu}$ in the direction x_i at the point $\mathbf{x}^*$ can be defined in terms of its α-level as

$$\left[\frac{\partial F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)}{\partial x_i}\right]^\alpha = \left\{\frac{\partial f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\partial x_i} \mid \forall c(\mathbf{t},\alpha)\in\left[\widetilde{\mathbf{C}}^k_\nu\right]^\alpha\right\} = \left[\min_{\mathbf{t}\in[0,1]^k}\frac{\partial f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\partial x_i}, \max_{\mathbf{t}\in[0,1]^k}\frac{\partial f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\partial x_i}\right], \tag{22}$$

provided that, for every $\alpha\in[0,1]$, $\partial f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)/\partial x_i$, $i=1,2,\ldots,n$, exist and they are the α-level of a fuzzy number. Moreover, the gradient of fuzzy-valued function $F_{\widetilde{\mathbf{C}}^k_\nu}$ (i.e., the partial derivatives $\partial F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)/\partial x_i$ at the point $\mathbf{x}^*$) is defined as a fuzzy vector as follows:

$$\nabla F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*) = \left(\frac{\partial F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)}{\partial x_1}, \frac{\partial F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)}{\partial x_2}, \ldots, \frac{\partial F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)}{\partial x_n}\right)^T. \tag{23}$$

Definition 14. Suppose that the second-order partial derivatives $\partial^2 f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)/\partial x_i\partial x_j$, $i,j=1,2,\ldots,n$, at the point $\mathbf{x}^*$ exist and are α-levels of fuzzy number (i.e., satisfying the assumption of Stacking Theorem). Then, the Hessian matrix of $F_{\widetilde{\mathbf{C}}^k_\nu}$ at the given point is given by

$$\nabla^2 F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*) = \left(\frac{\partial^2 F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)}{\partial x_i\partial x_j}\right)_{n\times n}, \quad \forall i,j=1,2,\ldots,n, \tag{24}$$

where

$$\left[\frac{\partial^2 F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)}{\partial x_i\partial x_j}\right]^\alpha = \left\{\frac{\partial^2 f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\partial x_i\partial x_j} \mid \forall c(\mathbf{t},\alpha)\in\left[\mathbf{C}^k_\nu\right]^\alpha\right\} = \left[\min_{\mathbf{t}\in[0,1]^k}\frac{\partial^2 f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\partial x_i\partial x_j}, \max_{\mathbf{t}\in[0,1]^k}\frac{\partial^2 f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)}{\partial x_i\partial x_j}\right], \quad i,j=1,2,\ldots,n. \tag{25}$$

It is apparent that

$$\left[\nabla^2 F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)\right]^\alpha = \left\{\nabla^2 f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*) \mid \forall f_{c(\mathbf{t},\alpha)}\in\left[F_{\widetilde{\mathbf{C}}^k_\nu}\right]^\alpha\right\}. \tag{26}$$

Definition 15. The fuzzy-valued function $F_{\widetilde{\mathbf{C}}^k_\nu}$ is said to be twice continuously differentiable at $\mathbf{x}^*$, if its Hessian matrix at that point (i.e., $\nabla^2 F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}^*)$) exists and all of its components are continuous functions. Also, $F_{\widetilde{\mathbf{C}}^k_\nu}$ is twice continuously differentiable on $\mathbb{R}^n$, if it is twice continuously differentiable for all $\mathbf{x}\in\mathbb{R}^n$.

Furthermore, using Definition 14 and Proposition 10, it can be shown that the fuzzy-valued function $F_{\widetilde{\mathbf{C}}^k_\nu}$ inherits the twice continuous differentiability of $f_{c(\mathbf{t},\alpha)}$ at $\mathbf{x}^*$, when $\partial^2 f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)/\partial x_i\partial x_j$, $i,j=1,2,\ldots,n$, are α-levels of a fuzzy number.

Definition 16. A fuzzy matrix $\widetilde{\mathbf{A}}_m$ is said to be symmetric if each of its α-levels is symmetric. Moreover, by the parametric representation (4), a fuzzy matrix $\widetilde{\mathbf{A}}_m$ is positive definite (or positive semidefinite) if every $A(\mathbf{t},\alpha)$ is positive definite (or positive semidefinite).

Theorem 17 (see [24]). *Let $F_{\widetilde{\mathbf{C}}^k_\nu}$ be a twice continuously differentiable fuzzy-valued function on the open convex set $\Omega\subseteq\mathbb{R}^n$. The function $F_{\widetilde{\mathbf{C}}^k_\nu}$ is convex with respect to $\preceq_w$ if and only if its fuzzy Hessian matrix is positive semidefinite for all $\mathbf{x}\in\Omega$.*

4. Constrained Fuzzy-Valued Optimization Problem

Consider the following constrained fuzzy-valued optimization problem:

$$\begin{aligned} \text{(CFOP)}\quad \min\quad & F_{\widetilde{\mathbf{C}}^k_\nu}(\mathbf{x}) \\ \text{s.t.}\quad & G_{j\widetilde{\mathbf{D}}^{m_j}_\nu}(\mathbf{x}) \preceq (\text{or } \preceq_w)\,\widetilde{B}_j \\ & j=1,2,\ldots,p, \end{aligned} \tag{27}$$

where $\widetilde{B}_j\in\mathscr{F}(\mathbb{R})$ for $j=1,2,\ldots,p$ and $F_{\widetilde{\mathbf{C}}^k_\nu}, G_{j\widetilde{\mathbf{D}}^{m_j}_\nu}:\mathbb{R}^n\to\mathscr{F}(\mathbb{R})$ are fuzzy-valued functions with the parametric representations

$$\left[F_{\widetilde{\mathbf{C}}^k_\nu}(x)\right]^\alpha = \left\{f_{c(\mathbf{t},\alpha)}(\mathbf{x}) \mid f_{c(\mathbf{t},\alpha)}:\mathbb{R}^n\longrightarrow\mathbb{R},\ c(\mathbf{t},\alpha)\in\left[\widetilde{\mathbf{C}}^k_\nu\right]^\alpha\right\},$$

$$\left[G_{j\widetilde{\mathbf{D}}_{\nu}^{m_j}}(\mathbf{x})\right]^{\alpha} = \left\{g_{jd(\mathbf{t}_j,\alpha)}(\mathbf{x}) \mid g_{jd(\mathbf{t}_j,\alpha)}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R},\ d\left(\mathbf{t}_j,\alpha\right) \in \left[\widetilde{\mathbf{D}}_{\nu}^{m_j}\right]^{\alpha}\right\}. \tag{28}$$

According to the partial orderings as discussed in Definition 5 and the parametric representations (3) and (4), the feasible region of the CFOP can be expressed as

$$\begin{aligned}
\mathbf{F} &= \left\{x \in \mathbb{R}^n : G_{j\widetilde{\mathbf{D}}_{\nu}^{m_j}}(x) \preceq (\text{or } \preceq_w)\, \widetilde{B}_j \ \ j = 1,2,\ldots,p\right\} \\
&= \begin{cases} \left\{\mathbf{x} \in \mathbb{R}^n : g_{jd(\mathbf{t}'_j,\alpha)}(\mathbf{x}) \le b_j\left(\mathbf{t}''_j,\alpha\right),\ \forall\alpha \in [0,1],\ \forall j \in 1,2,\ldots,p\right\} & \text{if } G_{j\widetilde{\mathbf{D}}_{\nu}^{m_j}}(\mathbf{x}) \preceq \widetilde{B}_j, \\ \left\{\mathbf{x} \in \mathbb{R}^n : g_{jd(\mathbf{t}_j,\alpha)}(\mathbf{x}) \le b_j\left(\mathbf{t}_j,\alpha\right),\ \forall\alpha \in [0,1],\ \forall j \in 1,2,\ldots,p\right\} & \text{if } G_{j\widetilde{\mathbf{D}}_{\nu}^{m_j}}(\mathbf{x}) \preceq_w \widetilde{B}_j, \end{cases} \\
&= \begin{cases} \left\{\mathbf{x} \in \mathbb{R}^n : \max\limits_{\mathbf{t}_j \in [0,1]^{m_j}} g_{jd(\mathbf{t}_j,\alpha)}(\mathbf{x}) \le b_j(0,\alpha),\ \forall\alpha \in [0,1], \forall j\right\} & \text{if } G_{j\widetilde{\mathbf{D}}_{\nu}^{m_j}}(\mathbf{x}) \preceq \widetilde{B}_j, \\ \left\{\mathbf{x} \in \mathbb{R}^n : \max\limits_{\mathbf{t}_j \in [0,1]^{m_j}} g_{jd(\mathbf{t}_j,\alpha)}(\mathbf{x}) \le b_j(1,\alpha),\ \min\limits_{\mathbf{t}_j \in [0,1]^{m}} g_{jd(t_j,\alpha)}(\mathbf{x}) \le b_j(0,\alpha),\ \forall\alpha \in [0,1], \forall j\right\} & \text{if } G_{j\widetilde{\mathbf{D}}_{\nu}^{m_j}}(\mathbf{x}) \preceq_w \widetilde{B}_j. \end{cases}
\end{aligned} \tag{29}$$

4.1. Solution Concepts and Optimality Conditions. In this section, a novel solution methodology was presented on the CFOP, which has depended on the definition of the corresponding optimal solution. Accordingly, it has been tried to define this concept based on the proposed partial orders. The feasible point $\mathbf{x}^*$ is said to be an optimal solution of the CFOP with respect to $\preceq_w$, if and only if

$$F_{\widetilde{\mathbf{C}}_{\nu}^{k}}\left(\mathbf{x}^*\right) \preceq_w F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\mathbf{x}) \tag{30}$$

for all $\mathbf{x} \in \mathbf{F}$. Nevertheless, by the definition of the partial order $\preceq_w$, the CFOP can be handled via the following multiobjective problem:

$$(\text{COP})_{\mathbf{t}} \ \min_{\mathbf{x}\in\mathbf{F}} \ f_{c(\mathbf{t},\alpha)}(\mathbf{x}). \tag{31}$$

So, the solution of the CFOP can be interpreted as the solution of $(\text{COP})_{\mathbf{t}}$, which is conforming to the concept of an efficient solution of a multiobjective problem. Consequently, the solution concept of the CFOP can be determined based on the thought of dominance.

Definition 18. Let $\mathbf{F} \subseteq \mathbb{R}^n$ be the feasible set of the CFOP

(i) A point $\mathbf{x}^* \in \mathbf{F}$ is an efficient solution of the CFOP if there is no $\mathbf{x} \in \mathbf{F}$, where

$$f_{c(\mathbf{t},\alpha)}(\mathbf{x}) \le f_{c(\mathbf{t},\alpha)}\left(\mathbf{x}^*\right) \quad \forall\mathbf{t} \in [0,1]^k,\ F_{\widetilde{\mathbf{C}}_{\nu}^{k}}(\mathbf{x}) \ne F_{\widetilde{\mathbf{C}}_{\nu}^{k}}\left(\mathbf{x}^*\right). \tag{32}$$

(ii) A point $\mathbf{x}^* \in \mathbf{F}$ is said to be a properly efficient solution of the CFOP, if it is an efficient solution and there is a real number $\mu > 0$ such that there exists at least one $\mathbf{t}' \in [0,1]^k$, $\mathbf{t} \ne \mathbf{t}'$, with $f_{c(\mathbf{t},\alpha)}(\mathbf{x}) > f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)$, whereas

$$\frac{f_{c(\mathbf{t},\alpha)}\left(\mathbf{x}^*\right) - f_{c(\mathbf{t},\alpha)}(\mathbf{x})}{f_{c(\mathbf{t},\alpha)}(\mathbf{x}) - f_{c(\mathbf{t},\alpha)}\left(\mathbf{x}^*\right)} \le \mu, \tag{33}$$

for some $\mathbf{t} \in [0,1]^k$ and every $\mathbf{x} \in \mathbf{F}$ with $f_{c(\mathbf{t},\alpha)}(\mathbf{x}) < f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)$.

One of the main advantages of efficient solutions is to enable the decision maker to select one optimal solution that is matched best to his demand. In order to enhance the usefulness, the proposed solution concepts can be typically expanded as follows:

(1) CFOP has a weak efficient solution at $\mathbf{x}^* \in \mathbf{F}$, whenever relation (32) is established for some $\alpha \in [0,1]$.

(2) CFOP has a strong efficient solution at $\mathbf{x}^* \in \mathbf{F}$, if, for all $\alpha \in [0,1]$, relation (32) is valid.

(3) CFOP has a strong independent efficient solution at $\mathbf{x}^* \in \mathbf{F}$, when $\mathbf{x}^*$ is an efficient solution and it is independent of α.

(4) CFOP has no efficient solution, if there is no $\mathbf{x}^* \in \mathbf{F}$ such that relation (32) is satisfied for any $\alpha \in [0,1]$.

Likewise, the concepts of weak, strong, strong independent, and no properly efficient solutions for the CFOP can be defined.

The fundamental idea to handle the considered multiobjective problem with infinity objective $(\text{COP})_{\mathbf{t}}$ is to convert it to the following constrained single-objective optimization problem:

$$(\text{COP}) \ \min_{\mathbf{x}\in\mathbf{F}} \ \int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x})\, dt_1 dt_2 \cdots dt_k, \tag{34}$$

where $w(\mathbf{t}) = w(t_1,t_2,\ldots,t_k)$ is a weight function $w : [0,1]^k \to (0,+\infty)$ and $t_1,t_2,\ldots,t_k$ are mutually independent. It can be shown that the solutions of the COP can be related to the CFOP ones.

Theorem 19. *If $\mathbf{x}^* \in \mathbf{F}$ is an optimal solution of the COP, then it is a properly efficient solution of the CFOP.*

Proof. By contradiction, assume that $\mathbf{x}^* \in \mathbf{F}$ is not a properly efficient solution of the CFOP. Therefore, $\mathbf{x}^*$ is not an efficient solution of the CFOP or the second part of Definition 18

(ii) is violated. Anyway, for some $\mathbf{t} \in [0,1]^k$ and $\mathbf{x} \in \mathbf{F}$ with $f_{c(\mathbf{t},\alpha)}(\mathbf{x}) < f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)$, pick out a continuous weight function $w : [0,1]^k \to (0,+\infty)$. Then, by choosing $\mu = \max_{\{\mathbf{t}\neq\mathbf{t}',\ \mathbf{t},\mathbf{t}'\in[0,1]^k,\ w(\mathbf{t})>0\}}\{w(\mathbf{t}')/w(\mathbf{t})\}$, we have

$$\frac{f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*) - f_{c(\mathbf{t},\alpha)}(\mathbf{x})}{f_{c(\mathbf{t},\alpha)}(\mathbf{x}) - f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)} > \mu \tag{35}$$

for all $\mathbf{t}' \in [0,1]^k$ with $f_{c(\mathbf{t},\alpha)}(\mathbf{x}) > f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)$. Consequently,

$$\begin{aligned} f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*) - f_{c(\mathbf{t},\alpha)}(\mathbf{x}) &> \mu\left(f_{c(\mathbf{t},\alpha)}(\mathbf{x}) - f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)\right) \\ &> \frac{w(\mathbf{t}')}{w(\mathbf{t})}\left(f_{c(\mathbf{t},\alpha)}(\mathbf{x}) - f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)\right), \\ w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*) - w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}) &> w(\mathbf{t}')\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}) - w(\mathbf{t}')\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*). \end{aligned} \tag{36}$$

By integrating with respect to $t_1, t_2, \ldots, t_k$, we have

$$\begin{aligned} &\int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)\, dt_1 dt_2\cdots dt_k \\ &\quad - \int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x})\, dt_1 dt_2\cdots dt_k \\ &> \int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t}')\, f_{c(\mathbf{t},\alpha)}(\mathbf{x})\, dt_1' dt_2'\cdots dt_k' \\ &\quad - \int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t}')\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)\, dt_1' dt_2'\cdots dt_k'; \end{aligned} \tag{37}$$

therefore

$$\begin{aligned} &\int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x}^*)\, dt_1 dt_2\cdots dt_k \\ &\quad > \int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t})\, f_{c(\mathbf{t},\alpha)}(\mathbf{x})\, dt_1 dt_2\cdots dt_k, \end{aligned} \tag{38}$$

which contradicts the assumption that $\mathbf{x}^*$ is an optimal solution of the COP. □

It is noteworthy that a CFOP is said to be a constrained fuzzy-valued convex programming problem if $F_{\widetilde{\mathbf{C}}_\nu^k}$, $G_{j\widetilde{\mathbf{D}}_\nu^{mj}}$ are convex functions with respect to $\preceq_w$ or $\preceq$.

Theorem 20. *If the CFOP is a constrained fuzzy-valued convex programming problem, then the COP is a constrained convex programming problem.*

Proof. It is the same as the proof of Theorem 3 of [25]. □

4.2. Constrained Fuzzy-Valued Quadratic Programming Problem. The constrained fuzzy-valued quadratic programming (CFQP) problem is a special case of the CFOP, when the fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^k}$ is quadratic and the constraints $G_{j\widetilde{\mathbf{D}}_\nu^{mj}}$ are linear in $\mathbf{x} \in \mathbb{R}^n$. Generally, the problem can be formulated as follows:

$$\begin{aligned} \text{(CFQP)}\ \min\quad & \widetilde{\mathbf{C}}_\nu^n \diamond \mathbf{x} \oplus \frac{1}{2} \odot \mathbf{x}^T \diamond \widetilde{\mathbf{Q}}_m \diamond \mathbf{x} \\ \text{s.t.}\quad & \widetilde{\mathbf{A}}_m \mathbf{x} \preceq (\text{or } \preceq_w)\, \widetilde{\mathbf{B}}_\nu^p, \quad \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \in \mathbb{R}^n, \end{aligned} \tag{39}$$

where $\widetilde{\mathbf{Q}}_m = (\widetilde{\mathbf{Q}}_{ij})_{n\times n} \in (\mathscr{F}(\mathbb{R}))^{n\times n}$ is a symmetric fuzzy matrix, $\widetilde{\mathbf{A}}_m = (\widetilde{\mathbf{A}}_{ij})_{p\times n} \in (\mathscr{F}(\mathbb{R}))^{p\times n}$, $\widetilde{\mathbf{C}}_\nu^n \in (\mathscr{F}(\mathbb{R}))^n$, and $\widetilde{\mathbf{B}}_\nu^p \in (\mathscr{F}(\mathbb{R}))^p$. According to (29), the feasible region of the CFQP can be obtained by the set

$$\begin{aligned} \mathbf{F} &= \left\{\mathbf{x} \in \mathbb{R}^n : \widetilde{\mathbf{A}}_m \mathbf{x} \preceq (\text{or } \preceq_w)\, \widetilde{\mathbf{B}}_\nu^p, \mathbf{x} \geq \mathbf{0}\right\} \\ &= \begin{cases} \{\mathbf{x} \in \mathbb{R}^n : A(1,\alpha)\mathbf{x} \leq b(0,\alpha),\ \mathbf{x} \geq \mathbf{0},\ \forall j,\alpha\} & \text{if } \widetilde{\mathbf{A}}_m \mathbf{x} \preceq \widetilde{\mathbf{B}}_\nu^p, \\ \{\mathbf{x} \in \mathbb{R}^n : A(1,\alpha)\mathbf{x} \leq b(1,\alpha),\ A(0,\alpha)\mathbf{x} \leq b(0,\alpha),\ \mathbf{x} \geq \mathbf{0},\ \forall j,\alpha,\} & \text{if } \widetilde{\mathbf{A}}_m \mathbf{x} \preceq_w \widetilde{\mathbf{B}}_\nu^p. \end{cases} \end{aligned} \tag{40}$$

It is self-evident that $\mathbf{F}$ is a convex set. Moreover, if the fuzzy Hessian matrix $\widetilde{\mathbf{Q}}_m$ is positive semidefinite then, by Theorem 17, the objective function will also be a fuzzy-valued convex function with respect to $\preceq_w$. Consequently, the CFQP will be a convex optimization problem. On the other hand, its corresponding optimization problem with a weight function $w : [0,1]^{n^2+n} \to (0,+\infty)$ can be denoted by

$$\text{(CQP)}\ \min_{\mathbf{x}\in\mathbf{F}}\quad \int_0^1\int_0^1\cdots\int_0^1 w(\mathbf{t})\left\{c(\mathbf{t}',\alpha)^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T Q(\mathbf{t}'',\alpha)\mathbf{x}\right\} d\mathbf{t}' d\mathbf{t}'', \tag{41}$$

where $c(\mathbf{t}',\alpha) \in [\widetilde{\mathbf{C}}_\nu^n]^\alpha$, $Q(\mathbf{t}'',\alpha) \in [\widetilde{\mathbf{Q}}_m]^\alpha$, $d\mathbf{t}' = dt_1' dt_2'\cdots dt_n'$, and $d\mathbf{t}'' = dt_{ij}''$, $i,j = 1,2,\ldots,m$, $\mathbf{t} = (\mathbf{t}',\mathbf{t}'')^T$.

By Theorem 20, the optimization problem (41) is also a constrained convex quadratic programming problem. Therefore,

using KKT optimality conditions, its corresponding optimal solution can be obtained which is a properly efficient solution for (39).

Consider the feasible region $\mathbf{F} = \{\mathbf{x} \in \mathbb{R}^n : A(1,\alpha)\,\mathbf{x} \leq b(0,\alpha),\ \mathbf{x} \geq \mathbf{0}\}$; accordingly, the Lagrange function is obtained as

$$L(\mathbf{x},\alpha,\lambda,\mu) = h(\mathbf{x},\alpha) + \lambda^T (A(1,\alpha)\,\mathbf{x} - b(0,\alpha)) - \mu^T \mathbf{x}, \tag{42}$$

where

$$h(\mathbf{x},\alpha) = \int_0^1 \int_0^1 \cdots \int_0^1 w(\mathbf{t}) \cdot \left\{ c(\mathbf{t}',\alpha)^T \mathbf{x} + \frac{1}{2}\mathbf{x}^T Q(\mathbf{t}'',\alpha)\,\mathbf{x} \right\} d\mathbf{t}' d\mathbf{t}'', \tag{43}$$

and $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^n$, $\lambda \geq 0$, and $\mu \geq 0$. So, the KKT optimality conditions are

$$\begin{aligned} &\nabla_{\mathbf{x}} L(\mathbf{x},\alpha,\lambda,\mu) \\ &\quad = \int_0^1 \int_0^1 \cdots \int_0^1 w(t) \left\{ c(\mathbf{t},\alpha)^T + Q(t'',\alpha)\,\mathbf{x} \right\} \mathbf{dt}'\mathbf{dt}'' \\ &\qquad + \lambda^T A(1,\alpha) = \mu^T, \\ &\lambda^T (A(1,\alpha)\,\mathbf{x} - b(0,\alpha)) = 0, \\ &\mu^T \mathbf{x} = 0, \end{aligned} \tag{44}$$

where $\lambda \geq 0$, $\mu \geq 0$, and $\mathbf{x} \in \mathbf{F}$.

Furthermore, if the feasible region $\mathbf{F}$ takes another form in (40), then the KKT optimality conditions can be determined in a similar way.

5. Numerical Examples

In this section, two examples are given to illustrate the efficiency of the proposed approach. In the first example, the various solutions are discussed in detail and in the second one a special problem, namely, the *constrained portfolio selection problem*, is expressed.

Example 1. Consider the following constrained fuzzy-valued quadratic programming problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^2} \quad & F_{\widetilde{\mathbf{C}}_\nu^5}(x_1,x_2) \\ & = \tilde{6} \odot x_1^2 \oplus \tilde{4} \odot x_1 x_2 \oplus \widetilde{10} \odot x_2^2 \oplus \tilde{1} \odot x_1 \oplus \tilde{2} \\ & \quad \odot x_2 \\ \text{s.t.} \quad & \widetilde{-2} \odot x_1 \oplus \widetilde{-1} \odot x_2 \preceq \widetilde{-1}, \\ & \tilde{1} \odot x_1 \oplus \tilde{1} \odot x_2 \preceq \tilde{4}, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{45}$$

where $\tilde{6} = \langle 4,6,7\rangle$, $\tilde{4} = \langle 3,4,5\rangle$, $\widetilde{10} = \langle 9,10,12\rangle$, $\tilde{1} = \langle 0,1,2\rangle$, $\tilde{2} = \langle 0,2,4\rangle$, $\widetilde{-2} = \langle -4,-2,0\rangle$, and $\widetilde{-1} = \langle -2,-1,2\rangle$. The corresponding optimization problem with respect to $w : [0,1]^5 \to (0,+\infty)$ is

$$\begin{aligned} \text{(CQP)} \ \min_{\mathbf{x}\in\mathbf{F}} \quad & \int_0^1 \int_0^1 \cdots \int_0^1 w(\mathbf{t}) \\ & \cdot \left\{ (4 + 2\alpha + t_1(3-3\alpha))\,x_1^2 \right. \\ & + (3 + \alpha + t_2(2-2\alpha))\,x_1 x_2 \\ & + (9 + \alpha + t_3(3-3\alpha))\,x_2^2 \\ & + (\alpha + t_4(2-2\alpha))\,x_1 \\ & \left. + (2\alpha + t_5(4-4\alpha))\,x_2 \right\} d\mathbf{t}, \\ \mathbf{F} = \{ & (x_1,x_2) : -2\alpha x_1 + (2-3\alpha)\,x_2 \\ & \leq -2+\alpha,\ (2-\alpha)\,x_1 + (2-\alpha)\,x_2 \\ & \leq 3+\alpha,\ x_1,x_2 \geq 0 \}, \end{aligned} \tag{46}$$

where $d\mathbf{t} = dt_1 dt_2 dt_3 dt_4 dt_5$. The fuzzy-valued function $F_{\widetilde{\mathbf{C}}_\nu^5}$ is convex (see Example 2.2 [24]). Therefore, the constrained quadratic programming problem (46) is a convex programming problem, by Theorem 20. Accordingly, using the obtained result of Section 4.2, the KKT conditions for CQP are

$$\begin{aligned} &\int_0^1 \int_0^1 \cdots \int_0^1 w(t) \left\{ 2(4 + 2\alpha + t_1(3-3\alpha))\,x_1 + (3 + \alpha + t_2(2-2\alpha))\,x_2 + (\alpha + t_4(2-2\alpha)) \right\} \mathbf{dt} - 2\alpha\lambda_1 + (2-\alpha)\,\lambda_2 \\ &\quad = \mu_1, \\ &\int_0^1 \int_0^1 \cdots \int_0^1 w(t) \left\{ (3 + \alpha + t_2(2-2\alpha))\,x_1 + 2(9 + \alpha + t_3(3-3\alpha))\,x_2 + (2\alpha + t_5(4-4\alpha)) \right\} \mathbf{dt} + (2-3\alpha)\,\lambda_1 + (2-\alpha) \\ &\quad \cdot \lambda_2 = \mu_2, \\ &\lambda_1 (-2\alpha x_1 + (2-3\alpha)\,x_2 - (-2+\alpha)) = 0, \\ &\lambda_2 ((2-\alpha)\,x_1 + (2-\alpha)\,x_2 - (3+\alpha)) = 0, \\ &\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ \lambda_1,\lambda_2,\mu_1,\mu_2 \geq 0,\ (x_1,x_2) \in \mathbf{F}. \end{aligned} \tag{47}$$

For a particular weight function $w(\mathbf{t}) = t_1 + t_3$, the above system can be simplified as

$$\begin{aligned}
&\frac{23}{2}x_1 + \frac{1}{2}x_1\alpha + 4x_2 + 1 - 2\alpha\lambda_1 + (2-\alpha)\lambda_2 = \mu_1,\\
&4x_1 + \frac{43}{2}x_2 - \frac{3}{2}x_2\alpha + 2 + (2-3\alpha)\lambda_1 + (2-\alpha)\lambda_2\\
&\quad = \mu_2,\\
&\lambda_1\left(-2\alpha x_1 + (2-3\alpha)x_2 - (-2+\alpha)\right) = 0,\\
&\lambda_2\left((2-\alpha)x_1 + (2-\alpha)x_2 - (3+\alpha)\right) = 0,\\
&\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0,\ (x_1, x_2) \in \mathbf{F}.
\end{aligned} \tag{48}$$

For each $\alpha \in [0.3852, 1]$, the solution of problem is $(x_1^*, x_2^*) = (-0.5(\alpha - 2)/\alpha, 0)$. Therefore, the fuzzy-valued optimization problem (45) has a weak properly efficient solution at $(-0.5(\alpha-2)/\alpha, 0)$.

Now, by considering some changes in the fuzzy-valued optimization problem (45), it can be shown that the solution of problem changes.

Case 1. Set $\widetilde{\mathbf{B}'}_\nu^2 = [\widetilde{2}; \widetilde{4}]$ instead of $\widetilde{\mathbf{B}}_\nu^2 = [\widetilde{-1}; \widetilde{4}]$. In this case, the following system may be obtained:

$$\begin{aligned}
&\frac{23}{2}x_1 + \frac{1}{2}x_1\alpha + 4x_2 + 1 - 2\alpha\lambda_1 + (2-\alpha)\lambda_2 = \mu_1,\\
&4x_1 + \frac{43}{2}x_2 - \frac{3}{2}x_2\alpha + 2 + (2-3\alpha)\lambda_1 + (2-\alpha)\lambda_2\\
&\quad = \mu_2,\\
&\lambda_1\left(-2\alpha x_1 + (2-3\alpha)x_2 - 2\alpha\right) = 0,\\
&\lambda_2\left((2-\alpha)x_1 + (2-\alpha)x_2 - (3+\alpha)\right) = 0,\\
&-2\alpha x_1 + (2-3\alpha)x_2 \leq 2\alpha,\\
&(2-\alpha)x_1 + (2-\alpha)x_2 \leq (3+\alpha),\\
&\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0.
\end{aligned} \tag{49}$$

Consequently, the problem has a strong independent properly efficient solution at $(x_1^*, x_2^*) = (0, 0)$.

Case 2. Set $\widetilde{\mathbf{B}'}_\nu^2 = [\widetilde{-1}; \widetilde{-4}]$ instead of $\widetilde{\mathbf{B}}_\nu^2 = [\widetilde{-1}; \widetilde{4}]$, where $\widetilde{-4} = \langle -5, -4, -2\rangle$ is a triangular fuzzy number; we have

$$\begin{aligned}
&\frac{23}{2}x_1 + \frac{1}{2}x_1\alpha + 4x_2 + 1 - 2\alpha\lambda_1 + (2-\alpha)\lambda_2 = \mu_1,\\
&4x_1 + \frac{43}{2}x_2 - \frac{3}{2}x_2\alpha + 2 + (2-3\alpha)\lambda_1 + (2-\alpha)\lambda_2\\
&\quad = \mu_2,\\
&\lambda_1\left(-2\alpha x_1 + (2-3\alpha)x_2 - (-2+\alpha)\right) = 0,\\
&\lambda_2\left((2-\alpha)x_1 + (2-\alpha)x_2 - (-5+\alpha)\right) = 0,\\
&-2\alpha x_1 + (2-3\alpha)x_2 \leq -2+\alpha,\\
&(2-\alpha)x_1 + (2-\alpha)x_2 \leq -5+\alpha,\\
&\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0,
\end{aligned} \tag{50}$$

which has no solution, and so the problem has no properly efficient solution.

Case 3. Set $\widetilde{\mathbf{A}}'_m = [\widetilde{-4}, \widetilde{2}; \widetilde{-1}, \widetilde{-4}]$ instead of $\widetilde{\mathbf{A}}_m = [\widetilde{-2}, \widetilde{-1}; \widetilde{-1}, \widetilde{-1}]$. Therefore, the corresponding system is

$$\begin{aligned}
&\frac{23}{2}x_1 + \frac{1}{2}x_1\alpha + 4x_2 + 1 + (-2-2\alpha)\lambda_1 + (2-3\alpha)\lambda_2\\
&\quad = \mu_1,\\
&4x_1 + \frac{43}{2}x_2 - \frac{3}{2}x_2\alpha + 2 + (4-2\alpha)\lambda_1 + (2-3\alpha)\lambda_2\\
&\quad = \mu_2,\\
&\lambda_1\left((-2-2\alpha)x_1 + (4-2\alpha)x_2 - (-2+\alpha)\right) = 0,\\
&\lambda_2\left((2-3\alpha)x_1 + (2-3\alpha)x_2 - (3+\alpha)\right) = 0,\\
&(-2-2\alpha)x_1 + (4-2\alpha)x_2 \leq -2+\alpha,\\
&(2-3\alpha)x_1 + (2-3\alpha)x_2 \leq 3+\alpha,\\
&\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0,
\end{aligned} \tag{51}$$

where, for each $\alpha \in [0, 1]$, the pair $(x_1^*, x_2^*) = (-0.5(\alpha-2)/(\alpha+1), 0)$ is the solution of the above system. Thus, the problem has a strong properly efficient solution at $(-0.5(\alpha-2)/(\alpha+1), 0)$.

To complete the discussion, it is interesting to explain the results by giving a special example named as portfolio selection problem. With respect to the mathematical programming problems including randomness and fuzziness, it is necessary to consider a certain optimization criterion so as to transform these problems into well-defined problems [26]. Therefore, in this paper, we consider fuzzy portfolio selection problem.

Example 2 (fuzzy portfolio selection problem [27]). The portfolio selection problem consists in selecting a portfolio of assets (or securities) that provides the investor with a given expected return and minimizes the risk. Mean-Variance optimization is probably the most popular approach to portfolio selection, which takes the variance of the portfolio as the measure of risk. It was introduced more than 50 years ago in the pioneering work by Markowitz [28].

Suppose that there are n assets indexed by $i = 1, 2, \ldots, n$. Each asset i is characterized by its random rate of return r_i, and its covariances with the rates of return of other assets are σ_{ij} for $j = 1, 2, \ldots, n$. The matrix $\sigma_{n\times n}$ is symmetric and

each diagonal element σ_{ii} represents the variance of asset i. A positive value R represents at least the desired rate of return.

The portfolio problem is to allocate total available wealth among these n assets, allocating a fraction x_i of wealth to the asset i. The value $\sum_{i,j=1}^{n} x_i\sigma_{ij}x_j$ represents the variance of the portfolio, and it is considered as the measure of the risk associated with the portfolio. Consequently, the problem is to minimize the overall variance, still ensuring the rate of return R. Thus, the problem can be formulated as follows:

$$\begin{aligned} \min_{x_1,x_2,\ldots,x_n} \quad & \sum_{i,j=1}^{n} x_i\sigma_{ij}x_j \\ \text{s.t.} \quad & \sum_{i=1}^{n} r_i x_i \geq R \\ & \sum_{i=1}^{n} x_i = 1. \end{aligned} \tag{52}$$

There may be the further restriction that each $x_i \geq 0$ which would imply that the assets must not be shorted. For $n = 3$, we have

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^3} \quad & \frac{1}{2}\mathbf{x}^T Q_m \mathbf{x} \\ \text{s.t.} \quad & -r_1x_1 - r_2x_2 - r_3x_3 \leq -R \\ & x_1 + x_2 + x_3 = 1 \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3)^T, \\ Q_m &= \begin{pmatrix} 2\sigma_{11} & \sigma_{12}+\sigma_{21} & \sigma_{13}+\sigma_{31} \\ \sigma_{12}+\sigma_{21} & 2\sigma_{22} & \sigma_{23}+\sigma_{32} \\ \sigma_{13}+\sigma_{31} & \sigma_{23}+\sigma_{32} & 2\sigma_{33} \end{pmatrix}. \end{aligned} \tag{54}$$

Since each asset is characterized by its random rate of return, then, for a closer look, we consider that the coefficients σ_{ij}, r_i, and R become imprecise numbers. Thus, interpretation makes it flexible and allows us to have a class of solutions and also it helps us to improve the prediction and simulation and better assess the problem. In other words, the purpose is to introduce a model that, considering the uncertainty, the basket offers the best way to meet the demands of investors.

To have a typical application of this model, let the fuzzy optimization problem be of the following form:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^3} \quad & \frac{1}{2} \odot \mathbf{x}^T \diamond \widetilde{\mathbf{Q}}_m \diamond \mathbf{x} \\ \text{s.t.} \quad & -\widetilde{r}_1x_1 - \widetilde{r}_2x_2 - \widetilde{r}_3x_3 \preceq -\widetilde{R} \\ & x_1 + x_2 + x_3 = 1 \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{55}$$

where

$$\begin{aligned} (\widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3) &= (\widetilde{1}, \widetilde{3}, \widetilde{2}), \\ \widetilde{R} &= \widetilde{3}, \\ \widetilde{\mathbf{Q}}_m &= \begin{pmatrix} \widetilde{6} & \widetilde{1} & \widetilde{4} \\ \widetilde{1} & \widetilde{9} & \widetilde{2} \\ \widetilde{4} & \widetilde{2} & \widetilde{16} \end{pmatrix}, \end{aligned} \tag{56}$$

and $\widetilde{1} = \langle 0,1,2\rangle$, $\widetilde{2} = \langle 1,2,4\rangle$, $\widetilde{3} = \langle 1,3,6\rangle$, $\widetilde{4} = \langle 3,4,5\rangle$, $\widetilde{6} = \langle 4,6,9\rangle$, $\widetilde{9} = \langle 6,9,11\rangle$, and $\widetilde{16} = \langle 12,16,20\rangle$. One can easily check that $(1/2) \odot \mathbf{x}^T \diamond \widetilde{\mathbf{Q}}_m \diamond \mathbf{x}$ is a fuzzy-valued convex function from Definition 7 or Theorem 20. Using the method proposed in Section 4.2, the corresponding constrained convex quadratic programming, with the weight function $w : [0,1]^6 \to (0,+\infty)$, is

$$\begin{aligned} \min_{\mathbf{F}} \quad & \int_0^1\int_0^1\cdots\int_0^1 \frac{1}{2} w(\mathbf{t}) \\ & \cdot \{(4 + 2\alpha + t_1(5-5\alpha))x_1^2 \\ & + (6 + 3\alpha + t_2(5-5\alpha))x_2^2 \\ & + (12 + 4\alpha + t_3(8-8\alpha))x_3^2 \\ & + 2(\alpha + t_4(2-2\alpha))x_1x_2 \\ & + 2(3 + \alpha + t_5(2-2\alpha))x_1x_3 \\ & + 2(1 + \alpha + t_6(3-3\alpha))x_2x_3\}\, d\mathbf{t}, \\ \mathbf{F} = \{ & (x_1,x_2,x_3) : -(2-\alpha)x_1 - (6-3\alpha)x_2 \\ & - (4-2\alpha)x_3 \leq -(1+2\alpha), \\ & x_1 + x_2 + x_3 = 1,\ x_1,x_2,x_3 \geq 0\}, \end{aligned} \tag{57}$$

where $\mathbf{dt} = dt_1dt_2\cdots dt_6$. For a particular weight function $w(\mathbf{t}) = 1$, one can obtain the following systems from KKT conditions:

$$\begin{aligned} & \frac{1}{2}(13x_1 + 2x_2 + 8x_3 - \alpha x_1) - (2-\alpha)\lambda_1 + \lambda_2 = \mu_1, \\ & \frac{1}{2}(2x_1 + 17x_2 + 5x_3 + \alpha x_2 - \alpha x_3) - (6-3\alpha)\lambda_1 \\ & \quad + \lambda_2 = \mu_2, \\ & \frac{1}{2}(8x_1 + 5x_2 + 32x_3 - \alpha x_2) - (4-2\alpha)\lambda_1 + \lambda_2 = \mu_3, \\ & \lambda_1(-(2-\alpha)x_1 - (6-3\alpha)x_2 - (4-2\alpha)x_3 \\ & \quad + (1+2\alpha)) = 0, \\ & \mu_1x_1 = 0,\ \mu_2x_2 = 0,\ \mu_3x_3 = 0,\ \lambda_1,\mu_1,\mu_2,\mu_3 \geq 0, \\ & \quad (x_1,x_2,x_3) \in \mathbf{F}. \end{aligned} \tag{58}$$

For each $\alpha \in [0.9455, 1]$, the solution of problem is $(x_1^*, x_2^*) = (0, -(4*\alpha-3)/(\alpha-2))$. Then, the problem has weak properly efficient solutions.

6. Conclusion

This study identified that a specific parametric representation for the fuzzy number can clarify the fuzzy arithmetic and calculus of fuzzy-valued function which had several acceptable properties as flexibility, easy-to-control shapes, and applicability in practice. Furthermore, the various solution concepts associated with constrained fuzzy-valued optimization problem were outlined. More precisely, the constrained fuzzy-valued optimization problem with both fuzzy-valued objective function and constraints was converted to a general constrained optimization problem, based on its underlying fuzzy-valued functions. The ability of the proposed approach might help to consider more realistic modeling efforts in the real world, such as *fuzzy portfolio selection problem* as a prominent problem in the financial field.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

The Karush-Kuhn-Tucker Optimality Conditions for the Fuzzy Optimization Problems in the Quotient Space of Fuzzy Numbers

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We propose the solution concepts for the fuzzy optimization problems in the quotient space of fuzzy numbers. The Karush-Kuhn-Tucker (KKT) optimality conditions are elicited naturally by introducing the Lagrange function multipliers. The effectiveness is illustrated by examples.

1. Introduction

The fuzzy set theory was introduced initially in 1965 by Zadeh [1]. After that, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. The fuzziness occurring in the optimization problems is categorized as the fuzzy optimization problems. Bellman and Zadeh [2] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there come out a lot of works dealing with the fuzzy optimization problems.

Zimmermann and Rödder initially applied fuzzy sets theory to the linear programing problems and linear multiobjective programing problems by using the aspiration level approach [3–6]. Durea and Tammer [7] derived the Lagrange multiplier rules for fuzzy optimization problems using the concept of abstract subdifferential. Bazine et al. [8] developed some fuzzy optimality conditions for fractional multiobjective optimization problems. In 2013, the solution approach for the lower level fuzzy optimization problem and the fuzzy bilevel optimization problem was investigated by Budnitzki [9]. Panigrahi et al. [10] extended and generalized these concepts to fuzzy mappings of several variables using the approach due to Buckley and Feuring [11] for fuzzy differentiation and derived the KKT conditions for the constrained fuzzy minimization problems. Wu [12, 13] presented the KKT conditions for the optimization problems with convex constraints and fuzzy-valued objective functions on the class of all fuzzy numbers by considering the concepts of Hausdorff metric and Hukuhara difference. Chalco-Cano et al. [14] discussed the KKT optimality conditions for a class of fuzzy optimization problems using strongly generalized differentiable fuzzy-valued functions, which is a concept of differentiability for fuzzy mappings more general than the Hukuhara differentiability.

These above results of fuzzy optimization are based on well-known and widely used algebraic structures of fuzzy numbers and the differentiability of fuzzy mappings was based on the concept of Hukuhara difference. However these operations can have some disadvantages for both theory and practical application. In [15], Qiu et al. intuitively showed a method of finding the inverse operation in the quotient space of fuzzy numbers based on the Mareš equivalence relation [16, 17], which have the desired group properties for the addition operation [18–20] midpoint function. As an application of the main results, it is shown that if we identify every fuzzy number with the corresponding equivalence class, there would be more differentiable fuzzy functions than what is found in the literature. In [21] Qiu et al. further

investigated the differentiability properties of such functions in the quotient space of fuzzy numbers. In this paper, the KKT optimality conditions for the constrained fuzzy optimization problems in the quotient space of fuzzy numbers are derived.

2. Preliminaries

We start this section by recalling some pertinent concepts and key lemmas from the function of bounded variation, fuzzy numbers, and fuzzy number equivalence classes which will be used later.

Definition 1 (see [22]). Let $f : [a,b] \to \mathbb{R}$ be a function. f is said to be of bounded variation if there exists a $C > 0$ such that

$$\sum_{i=1}^{n} \left| f(x_{i-1}) - f(x_{i-1}) \right| \le C \tag{1}$$

for every partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ on $[a,b]$. The set of all functions of bounded variation on $[a,b]$ is denoted by $\mathrm{BV}[a,b]$.

Definition 2 (see [22]). Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation. The total variation of f on $[a,b]$, denoted by $V_a^b(f)$, is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} \left| f(x_{i-1}) - f(x_i) \right|, \tag{2}$$

where p represents all partitions of $[a,b]$.

Lemma 3 (see [22]). *Let $f, g \in \mathrm{BV}[a,b]$, and then we have the following:*

(1) $cf + dg \in \mathrm{BV}[a,b]$ *and*

$$V_a^b(cf + dg) \le |c| V_a^b(f) + |d| V_a^b(g) \tag{3}$$

for any contents $c, d \in \mathbb{R}$.

(2) $f \cdot g \in \mathrm{BV}[a,b]$ *and*

$$\begin{aligned} V_a^b(f \cdot g) \le{}& V_a^b(f) \sup_{x\in[a,b]} |g(x)| \\ &+ V_a^b(g) \sup_{x\in[a,b]} |f(x)|. \end{aligned} \tag{4}$$

Lemma 4 (see [22]). *Every monotonic function $f : [a,b] \to \mathbb{R}$ is of bounded variation and*

$$V_a^b(f) = |f(a) - f(b)|. \tag{5}$$

Any mapping $\tilde{x} : \mathbb{R} \to [0,1]$ will be called a fuzzy set $\tilde{x}$ on $\mathbb{R}$. Its α-level set of $\tilde{x}$ is $[\tilde{x}]^\alpha = \{x \in \mathbb{R} : \tilde{x}(x) \ge \alpha\}$ for each $\alpha \in (0,1]$. Specifically, for $\alpha = 0$, the set $[\tilde{x}]^0$ is defined by $[\tilde{x}]^0 = \mathrm{cl}\{x \in \mathbb{R} : \tilde{x}(x) > 0\}$, where $\mathrm{cl}A$ denotes the closure of a crisp set A. A fuzzy set $\tilde{x}$ is said to be a fuzzy number if it is normal, fuzzy convex, and upper semicontinuous and the set $[\tilde{x}]^0$ is compact.

Let F be the set of all fuzzy numbers on $\mathbb{R}$. Then for an $\tilde{x} \in F$ it is well known that the α-level set $[\tilde{x}]^\alpha = [\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)]$ is a nonempty bounded closed interval in $\mathbb{R}$ for all $\alpha \in [0,1]$, where $\tilde{x}_L(\alpha)$ denotes the left-hand end point of $[\tilde{x}]^\alpha$ and $\tilde{x}_R(\alpha)$ denotes the right one. For any $\tilde{x}, \tilde{y} \in F$ and $\lambda \in \mathbb{R}$, owing to Zadeh's extension principle [23], the addition and scalar multiplication can be, respectively, defined for any $x \in \mathbb{R}$ by

$$\begin{aligned} (\tilde{x} + \tilde{y})(x) &= \sup_{x_1, x_2 : x_1 + x_2 = x} \min\{\tilde{x}(x_1), \tilde{x}(x_2)\}, \\ \lambda\tilde{x}(x) &= \begin{cases} \tilde{x}\left(\dfrac{x}{\lambda}\right), & \lambda \ne 0, \\ 0, & \lambda = 0. \end{cases} \end{aligned} \tag{6}$$

We say that a fuzzy number $\tilde{s} \in F$ is symmetric if $\tilde{s} = -\tilde{s}$ [16]. We denote the set of all symmetric fuzzy numbers by φ.

Definition 5 (see [15]). Let $\tilde{x} \in F$, and we define a function $\tilde{x}_M : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\tilde{x}_M(\alpha)$ for all $\alpha \in [0,1]$; that is,

$$\tilde{x}_M(\alpha) = \frac{\tilde{x}_L(\alpha) + \tilde{x}_R(\alpha)}{2}. \tag{7}$$

Then the function $\tilde{x}_M : [0,1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\tilde{x}$.

Lemma 6 (see [15]). *For any $\tilde{x} \in F$, the midpoint function $\tilde{x}_M$ is continuous from the right at* 0 *and continuous from the left on* $[0,1]$. *Furthermore, it is a function of bounded variation on* $[0,1]$.

Definition 7 (see [24]). Let $\tilde{x}, \tilde{y} \in F$, and we say that $\tilde{x}$ is equivalent to $\tilde{y}$, if there exist two symmetric fuzzy numbers $\tilde{s}_1, \tilde{s}_2 \in \varphi$ such that $\tilde{x} + \tilde{s}_1 = \tilde{y} + \tilde{s}_2$ and then we denote this by $\tilde{x} \sim \tilde{y}$.

It is easy to verify that the equivalence relation defined above is reflexive, symmetric, and transitive [16]. Let $\langle\tilde{x}\rangle$ denote the fuzzy number equivalence class containing the element $\tilde{x}$ and denote the set of all fuzzy number equivalence classes by F/φ.

Definition 8 (see [17]). Let $\tilde{x} \in F$ and let $\hat{x}$ be a fuzzy number such that $\tilde{x} = \hat{x} + \tilde{s}$ for some $\tilde{s} \in \varphi$, and if $\hat{x} = \tilde{y} + \tilde{s}_1$ for some $\tilde{y} \in F$ and $\tilde{s}_1 \in \varphi$, then $\tilde{s}_1 = \tilde{0}$. Then the fuzzy number $\hat{x}$ will be called the Mareš core of the fuzzy number $\tilde{x}$.

Definition 9 (see [21]). Let $\langle\tilde{x}\rangle \in F/\varphi$, and we define the midpoint function $M_{\langle\tilde{x}\rangle} : [0,1] \to \mathbb{R}$ by

$$M_{\langle\tilde{x}\rangle}(\alpha) = \hat{x}_M(\alpha) \tag{8}$$

for all $\alpha \in [0,1]$, where $\hat{x}$ is the Mareš core of $\tilde{x}$.

Definition 10 (see [21]). Let $\langle\tilde{x}\rangle, \langle\tilde{y}\rangle \in F/\varphi$, and we define the sum of this two fuzzy number equivalence classes as a fuzzy equivalence class $\langle\tilde{z}\rangle \in F/\varphi$, which satisfies the condition

$$M_{\langle\tilde{x}\rangle}(\alpha) + M_{\langle\tilde{y}\rangle}(\alpha) = M_{\langle\tilde{z}\rangle}(\alpha) \tag{9}$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle \tilde{x} \rangle + \langle \tilde{y} \rangle = \langle \tilde{x} + \tilde{y} \rangle = \langle \tilde{z} \rangle . \quad (10)$$

Remark 11. The addition operation defined by Definition 10 is a group operation over the set of fuzzy number equivalence classes F/φ up to the equivalence relation in Definition 7. For the details of the discussion, please see [25, 26].

Definition 12 (see [15]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$, and we say that $\langle \tilde{z} \rangle \in F/\varphi$ is the product of $\langle \tilde{x} \rangle$ and $\langle \tilde{y} \rangle$ if their midpoint functions satisfy

$$M_{\langle \tilde{x} \rangle}(\alpha) \cdot M_{\langle \tilde{y} \rangle}(\alpha) = M_{\langle \tilde{z} \rangle}(\alpha) \quad (11)$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle \tilde{x} \rangle \cdot \langle \tilde{y} \rangle = \langle \tilde{z} \rangle . \quad (12)$$

Definition 13 (see [21]). For any $\langle \tilde{x} \rangle \in F/\varphi$ and $\lambda \in \mathbb{R}$, we define $\lambda \cdot \langle \tilde{x} \rangle = \lambda \langle \tilde{x} \rangle$ by

$$\lambda \langle \tilde{x} \rangle = \langle \tilde{x} \rangle \lambda = \langle \lambda \tilde{x} \rangle . \quad (13)$$

It is obvious that $M_{\lambda \langle \tilde{x} \rangle}(\alpha) = M_{\langle \lambda \tilde{x} \rangle}(\alpha) = \lambda M_{\langle \tilde{x} \rangle}(\alpha)$ for all $\alpha \in [0,1]$.

Definition 14 (see [15]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$, and we define $d_{\sup} : F/\varphi \times F/\varphi \to \mathbb{R}^+ \cup \{0\}$ by

$$d_{\sup}(\langle \tilde{x} \rangle, \langle \tilde{y} \rangle) = \sup_{\alpha \in [0,1]} \left| M_{\langle \tilde{x} \rangle}(\alpha) - M_{\langle \tilde{y} \rangle}(\alpha) \right| . \quad (14)$$

It is easy to see that $(F/\varphi, d_{\sup})$ is a metric space [15].

3. The Karush-Kuhn-Tucker Optimality Conditions

In this paper, we always suppose that the range of fuzzy mappings is the set of all fuzzy number equivalence classes.

Definition 15 (see [21]). Let $F : T \to F/\varphi$ be a fuzzy mapping, where $T = [a,b] \subseteq \mathbb{R}$. Then F is said to be differentiable at $t \in T$ if there exists an $F'(t) \in F/\varphi$ such that

$$\lim_{h \to 0} d_{\sup}\left(\frac{F(t+h) - F(t)}{h}, F'(t) \right) = 0. \quad (15)$$

If $t = a$ (or b), then we consider only $h \to 0^+$ (or $h \to 0^-$).

Lemma 16 (see [21]). *$F : T \to F/\varphi$ is differentiable on T if and only if*

(1) *$M_{F(t)}(\alpha)$ is differentiable with respect to $t \in T$ for all $\alpha \in [0,1]$. That is, $(\partial/\partial t) M_{F(t)}(\alpha)$ exists and is of bounded variation with respect to $\alpha \in [0,1]$ for all $t \in T$;*

(2) *the mappings $\{M_{F(t)}(\alpha)\}_{\alpha \in [0,1]}$ are uniformly differentiable with the derivatives $(\partial/\partial t) M_{F(t)}(\alpha)$. That is, for each $t \in T$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\left| \frac{M_{F(t+h)}(\alpha) - M_{F(t)}(\alpha)}{h} - \frac{\partial}{\partial t} M_{F(t)}(\alpha) \right| < \varepsilon \quad (16)$$

for all $|h| \in (0, \delta)$ and $\alpha \in [0,1]$.

Definition 17 (see [27]). Let $\langle \tilde{a} \rangle = (\langle \tilde{a}_1 \rangle, \langle \tilde{a}_2 \rangle, \ldots, \langle \tilde{a}_n \rangle)^T \in (F/\varphi)^n$ and $t = (t_1, t_2, \ldots, t_n)^T \in \mathbb{R}^n$ be an n-dimensional fuzzy number equivalence class vector and n-dimensional real vector, respectively. We define their product as

$$\langle \tilde{a} \rangle^T t = \sum_{i=1}^{n} \langle \tilde{a}_i \rangle t_i = \langle \tilde{a}_1 \rangle t_1 + \langle \tilde{a}_2 \rangle t_2 + \cdots + \langle \tilde{a}_n \rangle t_n, \quad (17)$$

which is a fuzzy number equivalence class.

Definition 18 (see [27]). Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is an open subset in $\mathbb{R}^n$. We say that F has a partial derivative at $t = (t_1, t_2, \ldots, t_n)^T \in \Omega$ with respect to the ith variable t_i if there exists an $(\partial/\partial t_i) F(t) \in F/\varphi$ such that

$$\lim_{h \to o} d_{\sup}\left(\frac{F(t + he^i) - F(t)}{h}, \frac{\partial}{\partial t_i} F(t) \right) = 0, \quad (18)$$

where e^i stands for the unit vector that the ith component is 1 and the others are 0.

Definition 19 (see [27]). Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is an open subset in $\mathbb{R}^n$. We say that F is differentiable at $t = (t_1, t_2, \ldots, t_n)^T \in \Omega$ if F has continuous partial derivatives $(\partial/\partial t_i) F(t)$ with respect to ith variable t_i $(i = 1, 2, \ldots, n)$ and satisfies

$$F(t+h) = F(t) + \tilde{\nabla} F(t)^T h + o(\|h\|), \quad h = (h_1, h_2, \ldots, h_n)^T \in \mathbb{R}^n, \quad (19)$$

where $\tilde{\nabla} F(t) \in (F/\varphi)^n$ is an n-dimensional fuzzy number equivalence class vector defined by

$$\tilde{\nabla} F(t) = \left(\frac{\partial F(t)}{\partial t_1}, \frac{\partial F(t)}{\partial t_2}, \ldots, \frac{\partial F(t)}{\partial t_n} \right)^T, \quad (20)$$

and $\|h\|$ is the usual Euclid norm of h and $o : [0, +\infty) \to F/\varphi$ is a fuzzy mapping that satisfies

$$\lim_{t \to 0} d_{\sup}\left(\frac{o(t)}{t}, \langle \tilde{0} \rangle \right) = 0. \quad (21)$$

Then we call $\tilde{\nabla} F(t)$ the gradient of the fuzzy mappings F at t.

Definition 20 (see [27]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$.

(1) We say that $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ if $M_{\langle \tilde{x} \rangle}(\alpha) \le M_{\langle \tilde{y} \rangle}(\alpha)$ for all $\alpha \in [0,1]$.

(2) We say that $\langle \tilde{x} \rangle \prec \langle \tilde{y} \rangle$ if $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ and there exists at least one $\alpha_0 \in [0,1]$ such that $M_{\langle \tilde{x} \rangle}(\alpha_0) < M_{\langle \tilde{y} \rangle}(\alpha_0)$.

(3) If $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ and $\langle \tilde{y} \rangle \preceq \langle \tilde{x} \rangle$ then $\langle \tilde{x} \rangle = \langle \tilde{y} \rangle$.

Sometimes we may write $\langle \tilde{y} \rangle \succeq \langle \tilde{x} \rangle$ instead of $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ and write $\langle \tilde{y} \rangle \succ \langle \tilde{x} \rangle$ instead of $\langle \tilde{x} \rangle \prec \langle \tilde{y} \rangle$. Note that $\preceq$ is a partial order relation on F/φ.

Definition 21. Let $\langle \tilde{a} \rangle \in F/\varphi$, and we say that $\langle \tilde{a} \rangle$ is nonnegative if $\langle \tilde{a} \rangle \succeq \langle \tilde{0} \rangle$; that is, $M_{\langle \tilde{a} \rangle}(\alpha) \ge 0$ for all $\alpha \in [0,1]$.

Let $F : \mathbb{R}^n \to F/\varphi$ be a fuzzy mapping. Consider the following optimization problem:

$$\begin{aligned} \min \quad & F(t) = F(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & t = (t_1, t_2, \ldots, t_n)^T \in \Omega \subseteq \mathbb{R}^n, \end{aligned} \tag{22}$$

where the feasible set Ω is assumed to be convex subset of $\mathbb{R}^n$. Since $\preceq$ is a partial order relation on F/φ, we may follow the similar solution concept (the nondominated solution) used in multiobjective programing problems to interpret the meaning of minimization in problem (22).

Definition 22. Let t^* be a feasible solution of problem (22); that is, $t^* \in \Omega$.

(1) We say that t^* is a local nondominated solution of problem (22) if there exists an $\varepsilon > 0$ and there does not exist any $t \in N_\varepsilon(t^*) \cap \Omega$ such that $F(t) \prec F(t^*)$, where $N_\varepsilon(t^*)$ is an ε-neighborhood around t^*.

(2) We say that t^* is a (global) nondominated solution of problem (22) if there exists no $t \in \Omega$ such that $F(t) \prec F(t^*)$.

Definition 23. Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is a nonempty convex subset in $\mathbb{R}^n$. F is said to be convex on Ω if, for any $s, t \in \Omega$ and $\lambda \in (0, 1)$, we always have $F(\lambda s + (1-\lambda)t) \preceq \lambda F(s) + (1-\lambda)F(t)$. F is said to be concave if $-F$ is convex.

Theorem 24. *Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is a nonempty convex subset in $\mathbb{R}^n$. Then F is convex on Ω if and only if $M_{F(t)}(\alpha)$ is convex with respect to $t \in \Omega$ for all $\alpha \in [0, 1]$.*

Proof. The result follows from Definitions 20 and 23 immediately.

Let $f, g_j : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions. Consider the following optimization problem:

$$\begin{aligned} \min \quad & f(t) = f(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & g_j(t) \le 0, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{23}$$

Suppose that the constraint functions g_j are convex on $\mathbb{R}^n$ for all $j = 1, 2, \ldots, m$, and then the feasible set $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ is a convex subset of $\mathbb{R}^n$. The well-known KKT optimality conditions for problem (23) are stated as below. □

Theorem 25 (see [28, 29]). *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (23). Suppose that the objective function f and constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist nonnegative Lagrange multipliers $u_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, such that*

(1) $\nabla f(t^*) + \sum_{j=1}^m u_j \nabla g_j(t^*) = 0$,

(2) $u_j g_j(t^*) = 0$ *for all* $j = 1, 2, \ldots, m$,

then t^* is nondominated solution of problem (23).

Let $F : \mathbb{R}^n \to F/\varphi$ be a fuzzy mapping and $g_j : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions, $j = 1, 2, \ldots, m$. Now we consider the following optimization problem:

$$\begin{aligned} \min \quad & F(t) = F(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & g_j(t) \le 0, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{24}$$

If we suppose that the constraint functions g_j are convex on $\mathbb{R}^n$ for all $j = 1, 2, \ldots, m$, then we can see that problem (24) follows from problem (22) by taking the convex feasible set as $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$.

Now we are in a position to present the KKT optimality conditions for nondominated solutions of problem (24).

Theorem 26. *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (24). Suppose that the fuzzy-valued objective function F and real-valued constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist nonnegative real-valued Lagrange function multipliers u_j for $j = 1, 2, \ldots, m$ defined on $[0, 1]$ such that*

(1) $M_{\bar{\nabla} F(t^*)}(\alpha) + \sum_{j=1}^m u_j(\alpha) \nabla g_j(t^*) = 0$ *for all* $\alpha \in [0, 1]$,

(2) $u_j(\alpha) g_j(t^*) = 0$ *for all* $\alpha \in [0, 1]$ *and* $j = 1, 2, \ldots, m$,

then t^* is a nondominated solution of problem (24).

Proof. Suppose that conditions (1) and (2) are satisfied and t^* is not a nondominated solution of problem (24). Then there exists a $\bar{t} \in \Omega$ such that $F(\bar{t}) \prec F(t^*)$; that is, for some $\alpha^* \in [0, 1]$ we have that $M_{F(\bar{t})}(\alpha^*) < M_{F(t^*)}(\alpha^*)$. We now define a real-valued function f by $f(t) = M_{F(t)}(\alpha^*)$. Then we have

$$f(\bar{t}) < f(t^*). \tag{25}$$

Since the fuzzy mapping F is convex on $\mathbb{R}^n$ and continuously differentiable at t^*, by Theorem 24 and Lemma 16 we see that f is also convex on $\mathbb{R}^n$ and continuously differentiable at t^*. Furthermore, we have $\nabla f(t) = \nabla M_{F(t)}(\alpha^*) = M_{\bar{\nabla} F(t)}(\alpha^*)$. Since conditions (1) and (2) are satisfied, we can obtain the following two new conditions for any fixed $\alpha^* \in [0, 1]$:

($1'$) $\nabla f(t^*) + \sum_{j=1}^m u_{j\alpha^*} \cdot \nabla g_j(t^*) = 0$;

($2'$) $u_{j\alpha^*} \cdot g_j(t^*) = 0$ for all $j = 1, 2, \ldots, m$,

where $u_{j\alpha^*} = u_j(\alpha^*) \ge 0$ for $j = 1, 2, \ldots, m$. Now we consider the following constrained optimization problem:

$$\begin{aligned} \min \quad & f(t) = f(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & g_j(t) \le 0, \quad j = 1, 2, \ldots, m \end{aligned} \tag{26}$$

which has the same constraints of problem (24). By Theorem 25, conditions ($1'$) and ($2'$) are the KKT conditions of problem (26). Therefore, we have that t^* is an optimal solution of problem (26) with the real-valued objective function f; that is, $f(t^*) \le f(t)$ for all $t \in \Omega$, which contradicts inequality (25). Then we get that t^* is indeed a nondominated solution of problem (24). □

Theorem 27. *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (24). Suppose that the fuzzy-valued objective function F and real-valued constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist nonnegative fuzzy number equivalent class Lagrange multipliers $\langle \tilde{x}_j \rangle \in F/\varphi$ for $j = 1, 2, \ldots, m$ such that*

(1) $\tilde{\nabla} F(t^*) + \sum_{j=1}^{m} \langle \tilde{x}_j \rangle \cdot \nabla g_j(t^*) = \langle \tilde{0} \rangle$,

(2) $\langle \tilde{x}_j \rangle \cdot g_j(t^*) = \langle \tilde{0} \rangle$ *for all* $j = 1, 2, \ldots, m$,

then t^* is a nondominated solution of problem (24).

Proof. Since conditions (1) and (2) are satisfied, taking the midpoint function of (1) and (2), we obtain the following new conditions:

(1′) $M_{\tilde{\nabla} F(t^*)}(\alpha) + \sum_{j=1}^{m} M_{\langle \tilde{x}_j \rangle}(\alpha) \cdot \nabla g_j(t^*) = 0$ for all $\alpha \in [0, 1]$.

(2′) $M_{\langle \tilde{x}_j \rangle}(\alpha) \cdot g_j(t^*) = 0$ for all $\alpha \in [0, 1]$ and $j = 1, 2, \ldots, m$.

Since the fuzzy number equivalence classes $\langle \tilde{x}_j \rangle$ are nonnegative for all $j = 1, 2, \ldots, m$, then we can get that $M_{\langle \tilde{x}_j \rangle}$ are nonnegative real-valued functions defined on $[0, 1]$ for all $j = 1, 2, \ldots, m$. So, (1′) and (2′) verify the KKT optimality conditions (1) and (2) of Theorem 26, respectively. Therefore, we get that t^* is a nondominated solution of problem (24). □

Lemma 28 (see [28]). *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be a feasible set and $t^* \in \Omega$. Assume that g_j are differentiable at t^* for all $j = 1, 2, \ldots, m$. Let $J = \{j : g_j(t) = 0\}$ be the index set for the active constraints. Then we have*

$$D \subseteq \left\{ d \in \mathbb{R}^n : \nabla g_j(t^*)^T d \le 0\ \forall j \in J \right\}, \tag{27}$$

where D is the cone of feasible directions of Ω at t^* defined by

$$\begin{aligned} D = \{ & d \in \mathbb{R}^n : d \ne 0,\ \text{there exists a } \delta \\ & > 0 \text{ such that } t^* + \eta d \in \Omega\ \forall \eta \in (0, \delta) \}. \end{aligned} \tag{28}$$

Lemma 29 (see [28]). *Let A and C be two matrices. Exactly one of the following systems has a solution:*

System I: $Ax \le 0,\ Ax \ne 0,\ Cx \le 0$ *for some* $x \in \mathbb{R}^n$.

System II: $A^T\lambda + C^T u = 0$ *for some* $(\lambda, u),\ \lambda > 0,\ u \ge 0$.

Theorem 30. *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (24). Suppose that the fuzzy-valued objective function F is differentiable and strictly pseudoconvex on Ω, and the real-valued constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist a $\alpha^* \in [0, 1]$ and nonnegative Lagrange multipliers $u_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$ such that*

(1) $M_{\tilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j=1}^{m} u_j \cdot \nabla g_j(t^*) = 0$,

(2) $u_j \cdot g_j(t^*) = 0$ *for all* $j = 1, 2, \ldots, m$,

then t^* is a strongly nondominated solution of problem (24).

Proof. Suppose that conditions (1) and (2) are satisfied and t^* is not a strongly nondominated solution of problem (24). Then there exists a $\bar{t} \in \Omega$ with $\bar{t} \ne t^*$ such that $F(\bar{t}) \preceq F(t^*)$. Since F is differentiable and strictly pseudoconvex on Ω, we have

$$\tilde{\nabla} F(t^*)^T (\bar{t} - t^*) \prec \langle \tilde{0} \rangle; \tag{29}$$

that is,

$$M_{\tilde{\nabla} F(t^*)}(\alpha^*)^T (\bar{t} - t^*) < 0. \tag{30}$$

Let $d = \bar{t} - t^*$. Since Ω is a convex set and $\bar{t}, t^* \in \Omega$, we have

$$t^* + \eta d = t^* + \eta(\bar{t} - t^*) = \eta \bar{t} + (1 - \eta) t^* \in \Omega \tag{31}$$

for any $\eta \in (0, 1)$. By Lemma 28 we get that $d \in D$, which means that

$$\nabla g_j(t^*)^T d \le 0 \quad \forall i \in J, \tag{32}$$

where D is the cone of feasible directions of Ω at t^* and $J = \{j : g_j(t) = 0\}$ is the index set for the active constraints. Now let $A = M_{\tilde{\nabla} F(t^*)}(\alpha^*)^T$ and C be the matrix whose rows are $\nabla g_j(t^*)^T$ for $j \in J$. We consider the following two systems:

System I: $Ax \le 0,\ Ax \ne 0,\ Cx \le 0$ for some $x \in \mathbb{R}^n$.

System II: $A^T\lambda + C^T u = 0$ for some $(\lambda, u),\ \lambda > 0,\ u \ge 0$.

Then by (30) and (32) we get that System I has a solution $d = \bar{t} - t^*$. Further, by Lemma 29 System II has no solutions, which means that there exist no multipliers $0 < \lambda \in \mathbb{R}$ and $0 \le u_j \in \mathbb{R}$ for $j \in J$ such that

$$\lambda M_{\tilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j \in J} u_j \cdot \nabla g_j(t^*) = 0. \tag{33}$$

Since $\lambda > 0$, dividing (33) by λ and denoting $\eta_j = u_j/\lambda$ for $j \in J$, we have that

$$M_{\tilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j \in J} \eta_j \cdot \nabla g_j(t^*) = 0. \tag{34}$$

Since J is the index set for the active constraints, we have $g_j(t^*) < 0$ for $j \notin J$. Further, if $u_j \cdot g_j(t^*) = 0$ for all $j = 1, 2, \ldots, m$, we can get that $\eta_j = 0$ for $j \notin J$; that is,

$$\sum_{j \in J} \eta_j \cdot \nabla g_j(t^*) = \sum_{j=1}^{m} \eta_j \cdot \nabla g_j(t^*). \tag{35}$$

From (34) and (35), there exist no multipliers $0 \le \eta_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$ such that

$$(1') \; M_{\widetilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j=1}^{m} \eta_j \nabla g_j(t^*) = 0,$$
$$(2') \; u_j \cdot g_j(t^*) = 0 \text{ for all } j = 1, 2, \ldots, m,$$

which contradicts conditions (1) and (2) for the existence of multipliers $0 \le u_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$. Hence, we have that t^* is indeed a strongly nondominated solution of problem (24). □

Example 31. Define a fuzzy mapping $F : \mathbb{R}^3 \to F/\varphi$ by

$$\begin{aligned} F(t) &= \langle \widetilde{a} \rangle^T t + \|t\|^2 = \sum_{i=1}^{3} \left(\langle \widetilde{a}_i \rangle t_i + t_i^2 \right) \\ &= \langle \widetilde{a}_1 \rangle t_1 + \langle \widetilde{a}_2 \rangle t_2 + \langle \widetilde{a}_3 \rangle t_3 + t_1^2 + t_2^2 + t_3^2 \end{aligned} \tag{36}$$

for all $t = (t_1, t_2, t_3)^T \in \mathbb{R}^3$, where $\langle \widetilde{a} \rangle = (\langle \widetilde{a}_1 \rangle, \langle \widetilde{a}_2 \rangle, \langle \widetilde{a}_3 \rangle)^T \in (F/\varphi)^3$ and we define $\langle \widetilde{a}_i \rangle$ by the level sets of its Mareš core $[\widehat{a}_1]^\alpha = [-6, -12\alpha + 6]$, $[\widehat{a}_2]^\alpha = [-1, -2\alpha + 1]$, and $[\widehat{a}_3]^\alpha = [-4, -8\alpha + 4]$ for all $\alpha \in [0, 1]$ and $i = 1, 2, 3$, respectively. Thus, we have

$$\begin{aligned} M_{F(t_1,t_2,t_3)}(\alpha) &= -6\alpha t_1 + t_1^2 - \alpha t_2 + t_2^2 - 4\alpha t_3 + t_3^2 \\ &= \alpha(-6t_1 - t_2 - 4t_3) + t_1^2 + t_2^2 + t_3^2 \end{aligned} \tag{37}$$

for all $\alpha \in [0, 1]$ and $t = (t_1, t_2, t_3)^T \in \mathbb{R}^3$. It is obvious that $M_{F(t)}(\alpha)$ is continuous from the right at 0 and continuous from the left on $[0, 1]$ with respect to α. Now we consider the following optimization problem:

$$\begin{aligned} \min \quad & F(t) = F(t_1, t_2, t_3), \\ \text{subject to} \quad & g_1(t_1, t_2, t_3) = 4t_1 - t_2 + 2t_3 - 8 \le 0, \\ & g_2(t_1, t_2, t_3) = 3t_1 + 2t_2 - t_3 - 1 \le 0, \\ & g_j(t_1, t_2, t_3) = -t_{j-2} \le 0, \quad \text{for } j = 3, 4, 5, \\ & g_j(t_1, t_2, t_3) = t_{j-5} - 2 \le 0, \quad \text{for } j = 6, 7, 8. \end{aligned} \tag{38}$$

It is obvious that the constraint functions g_j are convex on $\mathbb{R}^3$ for all $j = 1, 2, \ldots, 8$, and then we know that the feasible set $\Omega = \{t \in \mathbb{R}^3 : g_j(t) \le 0, \; j = 1, 2, \ldots, 8\}$ is convex. Since $M_{F(t)}(\alpha)$ is decreasing with respect to α for all $t \in \Omega$, we get that

$$\begin{aligned} V_0^1(M_{F(t)}) &= |M_{F(t)}(1) - M_{F(t)}(0)| = |6t_1 + t_2 + 4t_3| \\ &\le 6|t_1| + |t_2| + 4|t_3| \le 22. \end{aligned} \tag{39}$$

Thus, we find that $M_{F(t)}(\alpha)$ is of bounded variation with respect to α for all $t = (t_1, t_2, t_3)^T \in \Omega \subseteq \mathbb{R}^3$. It is easy to verify that F is differentiable and strictly pseudoconvex on Ω, and g_j are convex on $\mathbb{R}^3$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, 8$. Then we obtain

$$\begin{aligned} M_{\widetilde{\nabla} F(t_1,t_2,t_3)}(\alpha) &= (2t_1 - 6\alpha, 2t_2 - \alpha, 2t_3 - 4\alpha)^T, \\ \nabla g_1(t_1, t_2, t_3) &= (4, -1, 2)^T, \\ \nabla g_2(t_1, t_2, t_3) &= (3, 2, -1)^T, \\ \nabla g_3(t_1, t_2, t_3) &= (-1, 0, 0)^T, \\ \nabla g_4(t_1, t_2, t_3) &= (0, -1, 0)^T, \\ \nabla g_5(t_1, t_2, t_3) &= (0, 0, -1)^T, \\ \nabla g_6(t_1, t_2, t_3) &= (1, 0, 0)^T, \\ \nabla g_7(t_1, t_2, t_3) &= (0, 1, 0)^T, \\ \nabla g_8(t_1, t_2, t_3) &= (0, 0, 1)^T, \end{aligned} \tag{40}$$

for all $\alpha \in [0, 1]$ and $t = (t_1, t_2, t_3)^T \in \Omega$. Now we consider the point $t^* = (t_1^*, t_2^*, t_3^*)^T = (1, 0, 2)^T \in \Omega$. Since

$$\begin{aligned} g_3(t^*) &\ne 0, \\ g_5(t^*) &\ne 0, \\ g_6(t^*) &\ne 0, \\ g_7(t^*) &\ne 0, \end{aligned} \tag{41}$$

from condition (2) in Theorem 30, we get that

$$u_3 = u_5 = u_6 = u_7 = 0. \tag{42}$$

Now, applying condition (2) of Theorem 30 at the point t^*, we obtain

$$\begin{aligned} & M_{\widetilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j=1}^{8} u_j \cdot \nabla g_j(t^*) \\ &= \begin{bmatrix} 2 - 6\alpha^* + 4u_1 + 3u_2 \\ -\alpha^* - u_1 + 2u_2 - u_4 \\ 4 - 4\alpha^* + 2u_1 - u_2 + u_8 \end{bmatrix} = 0. \end{aligned} \tag{43}$$

After these algebraic calculations, we obtain that there exist a $\alpha^* = 1 \in [0, 1]$ and nonnegative Lagrange multipliers

$$\begin{aligned} u_1 &= \frac{1}{4}, \\ u_2 &= 1, \\ u_4 &= \frac{3}{4}, \\ u_8 &= \frac{1}{2}, \\ u_j &= 0, \quad j = 3, 5, 6, 7, \end{aligned} \tag{44}$$

which satisfied conditions (1) and (2) of Theorem 30. Hence, we get that $t^* = (t_1^*, t_2^*, t_3^*)^T = (1, 0, 2)^T \in \Omega$ is a strongly nondominated solution of problem (38).

4. Conclusions

In this present investigation, the KKT optimality conditions are elicited naturally by introducing the Lagrange function multipliers, and we also provided some examples to illustrate the main results. The research on the quotient space of fuzzy numbers can be traced back to the works of Mareš [16, 17]. Hong and Do [24] improved this result and proposed a more refined equivalence relation. This equivalence relation can be used to partition the set of fuzzy numbers into equivalence class having the desired group properties for the addition operation. Since the quotient space of fuzzy numbers is characterized by the midpoint functions, there are more differentiable fuzzy mappings. As a matter of fact, there are still many other types of the KKT optimality conditions that can be derived using the similar techniques discussed in this paper on the quotient space of fuzzy numbers. However, for the nondifferentiable fuzzy optimization problem, we can follow the approach proposed by Ruziyeva and Dempe [30] to derive the necessary and sufficient optimality conditions in the quotient space of fuzzy numbers. In addition, Fuzzy sets and fuzzy optimization problems have several appropriate applications to today's world. But there are no sufficient examples and applications of the topics discussed in this paper. Therefore, we will develop the contribution of this research to practical problems in future studies.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Methods in Ranking Fuzzy Numbers: A Unified Index and Comparative Reviews

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Fuzzy set theory, extensively applied in abundant disciplines, has been recognized as a plausible tool in dealing with uncertain and vague information due to its prowess in mathematically manipulating the knowledge of imprecision. In fuzzy-data comparisons, exploring the general ranking measure that is capable of consistently differentiating the magnitude of fuzzy numbers has widely captivated academics' attention. To date, numerous indices have been established; however, counterintuition, less discrimination, and/or inconsistency on their fuzzy-number rating outcomes have prohibited their comprehensive implementation. To ameliorate their manifested ranking weaknesses, this paper proposes a unified index that multiplies weighted-mean and weighted-area discriminatory components of a fuzzy number, respectively, called centroid value and attitude-incorporated left-and-right area. From theoretical proof of consistency property and comparative studies for triangular, triangular-and-trapezoidal mixed, and nonlinear fuzzy numbers, the unified index demonstrates conspicuous ranking gains in terms of intuition support, consistency, reliability, and computational simplicity capability. More importantly, the unified index possesses the consistency property for ranking fuzzy numbers and their images as well as for symmetric fuzzy numbers with an identical altitude which is a rather critical property for accurate matching and/or retrieval of information in the field of computer vision and image pattern recognition.

1. Introduction

It has been well recognized that uncertainty inevitably exists in several real-world phenomena due to the inherent errors or impreciseness of measurement tools, methods, and uncontrollable conditions [1, 2]. In managing the uncertainty and vagueness, the fuzzy set theory has been widely considered as a powerful tool [3, 4]. And many scholars have made special efforts in proposing more and more effective approaches to deal with practical problems in the fuzzy environment. Since the inception of the fuzzy set theory, Soliman and Mantawy [5] showed that five major strongly connected branches have been developed, including fuzzy mathematics, fuzzy logic and artificial intelligence, fuzzy systems, uncertainty and information, and fuzzy decision-making. Their subbranches have also been established; for example, fuzzy differential equations [6–14] and fuzzy integrodifferential equations [15–22] are of fuzzy mathematics while fuzzy-number ranking, the focus of this paper, is of fuzzy decision-making. Specifically, based on its feasible mathematical capacity for representing the imprecise information in practice, we have observed many successful cases spreading in disparate disciplines, such as robot selection [23], supplier selection [24], logistics center allocation [25], facility location determination [26], choosing mining methods [27], manufacturing process monitoring [1, 2, 28–31], cutting force prediction [32], firm-environmental knowledge management [33, 34], green supply-chain operation [35], and weapon procurement decision [36]. Apparently, to find their best alternative, those decisive problems are evaluated under resource constraints and with to some extent linguistic preference of multiattribute, which is realized from users' perspectives, as well as subjective quantification of multiple characteristics, which is assessed from decision-makers [2, 3, 37–39]. In these cases, fuzzy-data comparisons and rankings are inevitable.

As the fuzzy data (fuzzy numbers) can overlap with each other and are represented by possibility distributions, their comparison and ordering, not akin to that of real numbers which can be linearly ordered, become challenging and cumbersome. Generally, to rank fuzzy quantities, a set of

fuzzy numbers, through a specific defuzzification measure, is converted into real numbers, where a natural order between them is definitive [40]. However, even when ordering for a set of single fuzzy numbers, this defuzzification procedure does lose a certain amount of fuzziness/imprecision information existing in the original data [1, 40–47], not to mention the ordering for problems of multicriteria decision-making, where sets of fuzzy numbers have experienced some mathematical operations [48]; therefore, much endeavor has been attempted to minimize loss of information, a fundamental problem for fuzzy-data analysis.

Jain [49] in 1977 first launched a fuzzy set rating procedure for multiple-aspect decision-making. Since then, exploring a general ranking measure, capable of consistently differentiating the magnitude of fuzzy numbers, has widely captivated academics' attention [50]. Nowadays, a majority of diverse improved approaches/indices established from wide-range perspectives focus on either compensating their predecessors' failures in certain reasonable properties for ordering of fuzzy quantities [43, 44] or resolving the counterintuitive, indiscriminate, and/or inconsistent rating outcomes among certain types of fuzzy numbers [42, 51–54].

In general, the existing ranking measures can be classified into two main categories:

(i) Indices that value the fuzzy number itself such as center-, area-, and deviation-driven ordering measures

(ii) Indices that not only evaluate the fuzzy number itself, but also gauge decision-maker's attitude in regard to specific purposes such as confidence and risk

In category one, Yager [55] and Lee and Li [56] first borrowed statistical *center-oriented* measures for assessing fuzzy numbers, where the former constructed a centroid (weighted mean) index and the latter developed mean and standard deviation indices; however, Cheng [57] pointed out their inefficient manipulation of the fuzzy numbers that possesses unusually large or small data (outliers) and mean-and-spread values. To cope with the inefficiencies, R. Saneifard and R. Saneifard [58], Zhang et al. [59], Bodjanova [60, 61], and Yamashiro [62] suggested a median index, a resistant measure of the center, to take into account data located on the tails; Cheng [57] proposed coefficient-of-variation and distance indices; but both indices were later criticized for some inconsistent ordering among specific types of fuzzy numbers [63]. Based on the area between the centroid point and the original point, Chu and Tsao [63] succeeded in establishing an *area-driven* ranking index; unfortunately, because of its inherent computation flaw, the area index was questioned by Wang and Lee [64] who illustrated some numerical examples to show its counterintuitive results and further provided a compelling revised index to resolve the problem. Nonetheless, Wang and Lee's area index does have its own deficiency of ordering correctness when encountering fuzzy numbers with identical centroid points [65]. By defining fuzzy-number maximal and minimal reference sets, Wang et al. [66] first introduced a *deviation-driven* ordering index by combining right-and-left deviation degree with the coefficient of relative variation; not surprisingly, this index was argued (1) bearing mathematical incapability with zero value in the denominator [53] and pointed out (2) leaving substantial room for improvement under some special occasions such as fuzzy numbers with the same left, right, and total utilities [39] as well as ranking fuzzy numbers' images [46].

Emphatically, the aforementioned drawbacks plagued on this *deviation-driven* ordering index have somewhat reignited the development of *category two*, initially proposed by Liou and Wang [67] in 1992, and contrived ranking measures that not only evaluate the fuzzy number itself, but also consider decision-maker's attitude in relation to specific purposes. The evidence can be seen in the most recent works; for example, to remove shortages of Wang et al.'s deviation-degree index [66], Wang and Luo [39] incorporated decision-maker's attitude towards risk into left-and-right area between fuzzy-number points and the positive-and-negative ideal points; to improve Liou and Wang's index [67], Yu and Dat [48] incorporated decision-maker's attitude regarding confidence into left-right-total integral value subjected to fuzzy-number median value. More recently, Das and Guha [68] proposed a new ranking approach by computing the centroid point of trapezoidal intuitionistic fuzzy numbers (TrIFN) and applied it to solve multicriteria decision-making problems in combination with expert's degree of satisfaction. However, their formulas fail to effectively work when their TrIFN (a, b, c, d) becomes either (a, a, c, d) or (a, b, c, c) or the satisfaction/dissatisfaction degree takes a value of zero. In addition, as shown in Table 1, certain shortcomings such as counterintuition, less reliability, inconsistency, complex/laborious computation, and indecisive ranking results have been found to be existing in several current ranking approaches.

Ostensibly, as opposed to the prolific ranking indices to date that have been presented in *category one*, the established ranking indices related to *category two* are still few, leaving a wide range of topics for further investigation. Based on the integration of the two categories, this paper proposes a unified index that multiplies weighted mean and weighted area, two discriminatory components of a fuzzy number, respectively, called centroid value *(the category one measurement)* and attitude-incorporated left-and-right area *(the category two measurement)*. According to comprehensively comparative studies from triangular, triangular-and-trapezoidal mixed, and nonlinear fuzzy numbers, the unified index demonstrates obtrusive ranking benefits with respect to intuition support, computational easiness, consistency, and reliability capability.

Aside from the Introduction, the remainder of this paper is organized into four sections as follows. Section 2 provides preliminary definitions and remarks for the research. The proposed unified index is described in Section 3, whose comparative studies with some existing ranking indices are done with several literature-exemplary fuzzy numbers in Section 4. Summary and conclusions make up the last section.

2. Preliminaries

The following definitions and remarks are mainly adopted from Zimmermann [69] and Lee [70].

Definition 1 (fuzzy subset). Let $\mathbb{R}$ be a nonempty set. The fuzzy subset $\widetilde{A}$ of $\mathbb{R}$ is defined by a function $\xi_{\widetilde{A}} : \mathbb{R} \to [0,1]$. $\xi_{\widetilde{A}}$ is called a *membership function*.

Definition 2 (α-cut set). The α-cut set of $\widetilde{A}$, denoted by $\widetilde{A}_{\alpha c}$, is defined by $\widetilde{A}_{\alpha c} = \{x \in \mathbb{R} : \xi_{\widetilde{A}}(x) \geq \alpha\}$ for all $\alpha \in (0,1]$. The 0-cut set $\widetilde{A}_{0c}$ is defined as the closure of the set $\{x \in \mathbb{R} : \xi_{\widetilde{A}}(x) > 0\}$.

Definition 3 (α-level set). The α-level set of $\widetilde{A}$, denoted by $\widetilde{A}_{\alpha}$, is defined by $\widetilde{A}_{\alpha} = \{x \in \mathbb{R} : \xi_{\widetilde{A}}(x) = \alpha\}$ for all $\alpha \in [0,1]$.

Definition 4 (fuzzy number). A fuzzy number $\widetilde{A} = (a,b,c,d;w)$ is described as any fuzzy subset of the real line $\mathbb{R}$ with the membership function $\xi_{\widetilde{A}}(x)$ which is given by

$$\xi_{\widetilde{A}}(x) = \begin{cases} \xi^L_{\widetilde{A}}(x), & a \leq x < b \\ w, & b \leq x \leq c \\ \xi^R_{\widetilde{A}}(x), & c < x \leq d \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $0 \leq w \leq 1$ is a constant and $\xi^L_{\widetilde{A}}(x), \xi^R_{\widetilde{A}}(x)$ are continuous functions on $[0,1]$.

A fuzzy number has the following properties:

(i) $\widetilde{A}$ is normal if there exists an $x \in \mathbb{R}$ such that $\xi_{\widetilde{A}}(x) = 1$; that is, $w = 1$.

(ii) $\xi_{\widetilde{A}}(x)$ is fuzzy convex; that is, $\xi_{\widetilde{A}}(tx + (1-t)y) \geq \min\{\xi_{\widetilde{A}}(x), \xi_{\widetilde{A}}(y)\}$ for $t \in [0,1]$.

(iii) $\xi_{\widetilde{A}}(x)$ is upper semicontinuous; that is, $\{x \in \mathbb{R} : \xi_{\widetilde{A}}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0,1]$.

(iv) The 0-level set $\widetilde{A}_0$ is a closed and bounded subset of $\mathbb{R}$.

Since $\widetilde{A}_{\alpha} \subset \widetilde{A}_0$ for each $\alpha \in (0,1]$, condition (iv) shows that the α-level sets $\widetilde{A}_{\alpha}$ are bounded subsets of $\mathbb{R}$ for all $\alpha \in (0,1]$. It is well known that condition (ii) is satisfied if and only if the α-level set $\widetilde{A}_{\alpha}$ is a convex subset of $\mathbb{R}$. Therefore, from conditions (i)–(iv), it is implied that if $\widetilde{A}$ is a fuzzy number, then the α-level set of $\widetilde{A}$ is a closed, bounded, and convex subset of $\mathbb{R}$, that is, a closed interval in $\mathbb{R}$, denoted by $\widetilde{A}_{\alpha} = [\widetilde{A}^L_{\alpha}, \widetilde{A}^U_{\alpha}]$.

Remark 5. Let $\widetilde{A}$ be a fuzzy number. Then, the following statements hold true:

(i) $\widetilde{A}^L_{\alpha} \leq \widetilde{A}^U_{\alpha}$ for all $\alpha \in [0,1]$.

(ii) $\widetilde{A}^L_{\alpha}$ is increasing with respect to $\alpha \in [0,1]$; that is, $\widetilde{A}^L_{\alpha} \leq \widetilde{A}^L_{\beta}$ for $0 \leq \alpha < \beta \leq 1$.

(iii) $\widetilde{A}^U_{\alpha}$ is decreasing with respect to $\alpha \in [0,1]$; that is, $\widetilde{A}^U_{\alpha} \geq \widetilde{A}^U_{\beta}$ for $0 \leq \alpha < \beta \leq 1$.

Remark 6. Let $\widetilde{A}$ be a fuzzy number such that its membership function is strictly increasing on interval $[a,b]$ and strictly decreasing on interval $[c,d]$. From the fact of strict monotonicity, $\xi^L_{\widetilde{A}}(x)$ and $\xi^R_{\widetilde{A}}(x)$ are continuous functions on $[0,1]$. This implies that $\widetilde{A}$ is also a real fuzzy number.

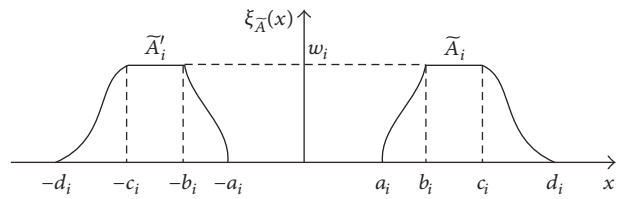

FIGURE 1: $\widetilde{A}'_i$ is the image of $\widetilde{A}_i$.

Definition 7 (the image of a fuzzy number [4]). Let n fuzzy numbers be $\widetilde{A}_i = (a_i, b_i, c_i, d_i; w_i)$ $(i = \overline{1,n})$. Then, the image of $\widetilde{A}_i$ is $\widetilde{A}'_i = (-d_i, -c_i, -b_i, -a_i; w_i)$, as shown in Figure 1.

3. A Unified Index

Based on integration of the two aforementioned categories for ranking fuzzy numbers, a unified index, which combines centroid value (weighted mean) and attitude-incorporated left-and-right area (weighted area), is proposed in this section.

Definition 8 (centroid value (a center-driven measure that belongs to category one)). Centroid value of a fuzzy number $\widetilde{A}_i = (a_i, b_i, c_i, d_i; w_i)$ for $i = \overline{1,n}$, symbolized by CV_i, is defined as [3, 4, 38, 63, 65, 71]

$$\mathrm{CV}_i = \frac{\int_{a_i}^{d_i} x \xi_{\widetilde{A}_i}(x)\, dx}{\int_{a_i}^{d_i} \xi_{\widetilde{A}_i}(x)\, dx}. \tag{2}$$

From the statistical point of view, it is the weighted mean of $\widetilde{A}_i$, meaning that when $\widetilde{A}_i = (a,a,a,a;w_i)$, we can accordingly have $\mathrm{CV}_i = a$.

Definition 9 (left-and-right areas (an area-driven measure that belongs to category one)). Left-and-right areas of a fuzzy number $\widetilde{A}_i$ for $i = \overline{1,n}$, denoted by S^L_i and S^R_i, are given by

$$\begin{aligned} S^L_i &= \left| \int_0^{w_i} g^L_{\widetilde{A}_i}(y)\, dy \right|, \\ S^R_i &= \left| \int_0^{w_i} g^R_{\widetilde{A}_i}(y)\, dy \right|, \end{aligned} \tag{3}$$

where $g^L_{\widetilde{A}_i}(y)$ and $g^R_{\widetilde{A}_i}(y)$ stand for inverse functions of the left-and-right membership functions, $\xi^L_{\widetilde{A}_i}(x)$ and $\xi^R_{\widetilde{A}_i}(x)$, respectively, and visual views of S^L_i and S^R_i are shown in Figure 2 [72].

Now, a fuzzy-number measure belonging to category two is presented. It also contemplates decision-maker's attitude as regards data revelation, called attitude-incorporated left-and-right area, signified by AA^{λ}_i.

$$\mathrm{AA}^{\lambda}_i = \lambda S^R_i + (1-\lambda) S^L_i, \tag{4}$$

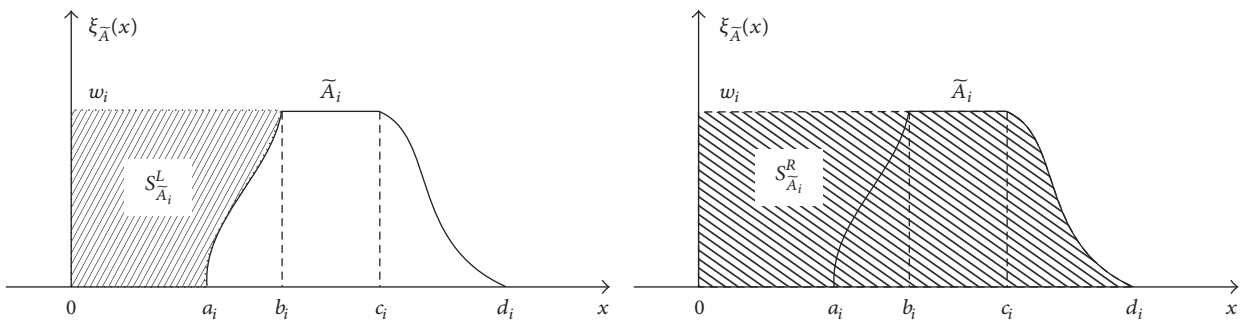

FIGURE 2: Left area S_i^L and right area S_i^R.

where $\lambda \in [0,1]$ is level of optimism reflecting a data-revelation optimism degree of a decision-maker, where the larger the λ set by the decision-maker is, the more optimistic attitude the decision-maker has on the data revelation. Two extreme cases are $\lambda = 0$, meaning the decision-maker is completely pessimistic, and $\lambda = 1$, meaning the decision-maker is completely optimistic. Case $\lambda = 1/2$ reflects a neutral decision attitude. From the mathematical viewpoint, (4) can be seen as a weighted-area value of $\widetilde{A}_i$.

For boosting the fuzzy-number discrimination power, let us consider an index named UI_i^λ by multiplying two size-discriminatory values of a fuzzy number; that is,

$$\mathrm{UI}_i^\lambda = \left(\mathrm{CV}_i + \varepsilon_i\right)\left[\lambda S_i^R + (1-\lambda) S_i^L\right]. \tag{5}$$

UI_i^λ is called *unified index*. And, ε_i initially takes a very small real number which is quantifiable and rational for comparing the targeted fuzzy numbers whose centroid values take a value of zero, $\mathrm{CV}_i = 0$. It is used to provide consistent ranking power when $\mathrm{CV}_i = 0$. Particularly, this paper suggests using $\varepsilon_i = w_i \times 10^{-9}$ so that we can efficiently rank fuzzy numbers that have similar centroids but different height.

Remark 10. Consider the ranking of two fuzzy numbers, $\widetilde{A}_i$ and $\widetilde{A}_j$. Given the data-optimistic level λ, from (5), we obtain their realized unified indices, UI_i^λ and UI_j^λ. Then, the following decisions can be made:

(i) At the data-optimistic level λ, if $\mathrm{UI}_i^\lambda > \mathrm{UI}_j^\lambda$, then $\widetilde{A}_i \succ \widetilde{A}_j$.

(ii) At the data-optimistic level λ, if $\mathrm{UI}_i^\lambda < \mathrm{UI}_j^\lambda$, then $\widetilde{A}_i \prec \widetilde{A}_j$.

(iii) At the data-optimistic level λ, if $\mathrm{UI}_i^\lambda = \mathrm{UI}_j^\lambda$, then $\widetilde{A}_i \simeq \widetilde{A}_j$.

Now, we will prove the unified index's consistency property when ranking fuzzy numbers and their images. Without loss of generality, $\mathrm{CV}_i \neq 0$ is considered in the following.

Proposition 11. *Let $\widetilde{A}_i' = (-d_i, -c_i, -b_i, -a_i; w_i)$ be the image of a fuzzy number $\widetilde{A}_i = (a_i, b_i, c_i, d_i; w_i)$ for $i = \overline{1,n}$. Its centroid value is $CV_{i'} = -CV_i$, left-and-right areas are $S_{i'}^R = S_i^L$ and $S_{i'}^L = S_i^R$, attitude-incorporated left-and-right area is $AA_{i'}^\lambda = AA_i^{1-\lambda}$ and $AA_{i'}^{1-\lambda} = AA_i^\lambda$, and unified index is $UI_{i'}^\lambda = -UI_i^{1-\lambda}$ and $UI_{i'}^{1-\lambda} = -UI_i^\lambda$.*

Proof. From (2),

$$\mathrm{CV}_{i'} = \frac{\int_{-d_i}^{-a_i} x\xi_{\widetilde{A}_i'}(x)\,dx}{\int_{-d_i}^{-a_i} \xi_{\widetilde{A}_i'}(x)\,dx} = -\frac{\int_{a_i}^{d_i} x\xi_{\widetilde{A}_i}(x)\,dx}{\int_{a_i}^{d_i} \xi_{\widetilde{A}_i}(x)\,dx} = -\mathrm{CV}_i. \tag{6}$$

Based on (3),

$$\begin{aligned} S_{i'}^L &= \left|\int_0^{w_i} g_{\widetilde{A}_i'}^L(y)\,dy\right| = \left|\int_0^{w_i} g_{\widetilde{A}_i}^R(y)\,dy\right| = S_i^R \\ S_{i'}^R &= \left|\int_0^{w_i} g_{\widetilde{A}_i'}^R(y)\,dy\right| = \left|\int_0^{w_i} g_{\widetilde{A}_i}^L(y)\,dy\right| = S_i^L. \end{aligned} \tag{7}$$

According to (4) and with the above results, $S_{i'}^R = S_i^L$ and $S_{i'}^L = S_i^R$, we further have

$$\mathrm{AA}_{i'}^\lambda = \lambda S_{i'}^R + (1-\lambda) S_{i'}^L = \lambda S_i^L + (1-\lambda) S_i^R = \mathrm{AA}_i^{1-\lambda}. \tag{8}$$

Similarly,

$$\mathrm{AA}_{i'}^{1-\lambda} = \left(1 - \lambda S_{i'}^R\right) + \lambda S_{i'}^L = (1-\lambda) S_i^L + \lambda S_i^R = \mathrm{AA}_i^\lambda. \tag{9}$$

Finally, regarding (5) and the aforementioned outcomes, we can simply obtain

$$\begin{aligned} \mathrm{UI}_{i'}^\lambda &= \mathrm{CV}_{i'}\left[\lambda S_{i'}^R + (1-\lambda) S_{i'}^L\right] = -\mathrm{UI}_i^{1-\lambda}, \\ \mathrm{UI}_{i'}^{1-\lambda} &= \mathrm{CV}_{i'}\left[(1-\lambda) S_{i'}^R + \lambda S_{i'}^L\right] = -\mathrm{UI}_i^\lambda. \end{aligned} \tag{10}$$

We complete the proof. □

Proposition 12. *Let a set of fuzzy numbers be $\widetilde{A}_k = (a_k, b_k, c_k, d_k; w_k)$ and their images $\widetilde{A}_k' = (-d_k, -c_k, -b_k, -a_k; w_k)$, $k = \overline{1,n}$. For a pairwise comparison of $\widetilde{A}_i$ and $\widetilde{A}_j$ for $i, j \in k$, two statements hold true: (1) $UI_i^\lambda > UI_j^\lambda$ if and only if $UI_{i'}^{1-\lambda} < UI_{j'}^{1-\lambda}$ and (2) $UI_i^\lambda < UI_j^\lambda$ if and only if $UI_{i'}^{1-\lambda} > UI_{j'}^{1-\lambda}$.*

Proof. Consider $\mathrm{UI}_i^\lambda > \mathrm{UI}_j^\lambda$. From Proposition 11, we have the results $\mathrm{UI}_i^\lambda = -\mathrm{UI}_{i'}^{1-\lambda}$ and $\mathrm{UI}_j^\lambda = -\mathrm{UI}_{j'}^{1-\lambda}$. Thus, $\mathrm{UI}_{i'}^{1-\lambda} < \mathrm{UI}_{j'}^{1-\lambda}$. On the other hand, consider $\mathrm{UI}_{i'}^{1-\lambda} < \mathrm{UI}_{j'}^{1-\lambda}$. According to Proposition 11, $\mathrm{UI}_{i'}^{1-\lambda} = -\mathrm{UI}_i^\lambda$ and $\mathrm{UI}_{j'}^{1-\lambda} = -\mathrm{UI}_j^\lambda$. Hence, $\mathrm{UI}_i^\lambda > \mathrm{UI}_j^\lambda$. Overall, the proof is completed. □

Remark 13. Let a set of fuzzy numbers be $\widetilde{A}_k = (a_k, b_k, c_k, d_k; w_k)$ and their images $\widetilde{A}'_k = (-d_k, -c_k, -b_k, -a_k; w_k)$, $k = \overline{1,n}$. As regards Remark 10 and Propositions 11 and 12, the following decisions can be made for a pairwise comparison of $\widetilde{A}_i$ and $\widetilde{A}_j$, for $i, j \in k$.

(i) At the data-optimistic level λ, if $\mathrm{UI}_i^\lambda > \mathrm{UI}_j^\lambda$, which is equivalent to $\mathrm{UI}_{i'}^{1-\lambda} < \mathrm{UI}_{j'}^{1-\lambda}$, then $\widetilde{A}_i \succ \widetilde{A}_j$, which is equivalent to $\widetilde{A}'_i \prec \widetilde{A}'_j$.

(ii) At the data-optimistic level λ, if $\mathrm{UI}_i^\lambda < \mathrm{UI}_j^\lambda$, which is equivalent to $\mathrm{UI}_{i'}^{1-\lambda} > \mathrm{UI}_{j'}^{1-\lambda}$, then $\widetilde{A}_i \prec \widetilde{A}_j$, which is equivalent to $\widetilde{A}'_i \succ \widetilde{A}'_j$.

(iii) At the data-optimistic level λ, if $\mathrm{UI}_i^\lambda = \mathrm{UI}_j^\lambda$, which is equivalent to $\mathrm{UI}_{i'}^{1-\lambda} = \mathrm{UI}_{j'}^{1-\lambda}$, then $\widetilde{A}_i \simeq \widetilde{A}_j$, which is equivalent to $\widetilde{A}'_i \simeq \widetilde{A}'_j$.

Finally, the following theory is very useful for ranking "symmetric" fuzzy numbers with an identical altitude.

Theorem 14. *Consider a set of "symmetric" fuzzy numbers, $\widetilde{A}_k = (a_k, b_k, c_k, d_k; w_k)$, and their images $\widetilde{A}'_k = (-d_k, -c_k, -b_k, -a_k; w_k)$, $k = \overline{1,n}$. By using the unified index, the pairwise comparison of $\widetilde{A}_i$ and $\widetilde{A}_j$ for $i, j \in k$ is $\lambda = 0.5$, $\widetilde{A}_i \simeq \widetilde{A}_j$ ($\widetilde{A}'_i \simeq \widetilde{A}'_j$), $\lambda \in [0, 0.5)$, $\widetilde{A}_i \prec \widetilde{A}_j$ ($\widetilde{A}'_i \succ \widetilde{A}'_j$), and $\lambda \in (0.5, 1]$, $\widetilde{A}_i \succ \widetilde{A}_j$ ($\widetilde{A}'_i \prec \widetilde{A}'_j$).*

Proof. (i) Since $\widetilde{A}_i = (a_i, b_i, c_i, d_i; w)$ and $\widetilde{A}_j = (a_j, b_j, c_j, d_j; w)$ for $i, j = \overline{1,n}$ are symmetric, we have $a_i + d_i = a_j + d_j$. Moreover, from (2),

$$\mathrm{CV}_i = \frac{\int_{a_i}^{d_i} x\xi_{\widetilde{A}_i}(x)\,dx}{\int_{a_i}^{d_i} \xi_{\widetilde{A}_i}(x)\,dx} = \frac{a_i + d_i}{2},$$
$$\mathrm{CV}_j = \frac{\int_{a_j}^{d_j} x\xi_{\widetilde{A}_j}(x)\,dx}{\int_{a_j}^{d_j} \xi_{\widetilde{A}_j}(x)\,dx} = \frac{a_j + d_j}{2}. \tag{11}$$

Therefore, $\mathrm{CV}_i = \mathrm{CV}_j$.

(ii) According to (3) and (4), we have

$$\mathrm{AA}_i^\lambda = \lambda S_i^R + (1-\lambda) S_i^L$$
$$\mathrm{AA}_j^\lambda = \lambda S_j^R + (1-\lambda) S_j^L, \tag{12}$$

where

$$S_i^L = \left|\int_0^w g_{\widetilde{A}_i}^L(y)\,dy\right|,$$
$$S_i^R = \left|\int_0^w g_{\widetilde{A}_i}^R(y)\,dy\right|$$
$$S_j^L = \left|\int_0^w g_{\widetilde{A}_j}^L(y)\,dy\right|,$$
$$S_j^R = \left|\int_0^w g_{\widetilde{A}_j}^R(y)\,dy\right|. \tag{13}$$

Due to the symmetry, we have $\mathrm{AA}_i^\lambda < \mathrm{AA}_j^\lambda$ for $\lambda \in [0, 0.5)$, $\mathrm{AA}_i^\lambda = \mathrm{AA}_j^\lambda$ when $\lambda = 0.5$, and vice versa.

(iii) From (i), (ii), and (5), we have

(i) $\lambda \in [0, 0.5)$, $\mathrm{UI}_i^\lambda < \mathrm{UI}_j^\lambda$,

(ii) $\lambda = 0.5$, $\mathrm{UI}_i^\lambda = \mathrm{UI}_j^\lambda$,

(iii) $\lambda \in (0.5, 1]$, $\mathrm{UI}_i^\lambda > \mathrm{UI}_j^\lambda$.

Finally, according to Remark 13, we complete the proof. □

4. Comparative Studies

In this section, several fuzzy-number examples, which are popular in the literature for a wide range of fuzzy-number comparative studies, are used to compare ranking performance between the unified index and some up-to-date representative indices from the publications. To make it easier to follow the whole discussion of comparison, Table 1 briefly shows the evaluated types of fuzzy numbers, reference sources, and critical shortcomings of the references. Detailed explanations about performance shortages for existing indices in contrast with the proposed index are subsequently described in Examples 15~22.

It can be noted that, based on Propositions 11 and 12 and Remark 13, the unified index fulfills the consistency property for ranking the fuzzy numbers and their partnered images; for conciseness, in several examples, the consistency of image-ranking results is not mentioned or shown on the result tables.

4.1. Ranking of Normal Triangular Fuzzy Numbers. This subsection focuses on the ranking of normal triangular fuzzy numbers with some special shape which are recognizably difficult to discriminate in the literature. First, a case with two congruent fuzzy numbers is employed for checking index's computation easiness; then, the work is extended on three similar fuzzy numbers for contrasting indices' ranking consistency and intuition satisfaction; finally, an example, which includes a slight move-away fuzzy number and two fuzzy numbers with an identical center value and geometric enlargement relationship, is examined with respect to ranking indices' reliability and consistency.

Example 15. Rank two fuzzy numbers $\widetilde{A}_1 = (1, 4, 5)$ and $\widetilde{A}_2 = (2, 3, 6)$ as shown in Figure 3 [48], which are congruent,

Table 1: The ranking performance assessments for some representative indices as opposed to the unified index.

Section	Example	Evaluated fuzzy numbers	Compared references	Shortcomings (cf. the index)
Section 4.1	Example 15	$\widetilde{A}_1 = (1, 4, 5)$ $\widetilde{A}_2 = (2, 3, 6)$	Yu & Dat [48]	More laborious in computation
Section 4.1	Example 16	$\widetilde{A}_1 = (5, 6, 7)$ $\widetilde{A}_2 = (5.9, 6, 7)$ $\widetilde{A}_3 = (6, 6, 7)$	Chu & Tsao [63] Cheng [57] Yu & Dat [48]	Counterintuition Counterintuition Less reliability
Section 4.1	Example 17	$\widetilde{A}_1 = (1, 3, 5)$ $\widetilde{A}_2 = (2, 3, 4)$ $\widetilde{A}_3 = (1, 4, 6)$	Liou & Wang [67], Yu & Dat [48]	Inconsistency Counterintuition at $\lambda = 0$
Section 4.2	Example 18	$\widetilde{A}_1 = (1, 5, 5)$ $\widetilde{A}_2 = (2, 3, 5, 5)$	Zhang et al. [73]	Computation complexity Inconsistency
Section 4.2	Example 19	$\widetilde{A}_1 = (0, 3, 6)$ $\widetilde{A}_2 = (-1, 0, 2)$ $\widetilde{A}_3 = (0, 2, 4, 6)$	Ky Phuc et al. [38], Asady [46]	Computation complexity Indecisive ranking for $(\widetilde{A}_1, \widetilde{A}_3)$
Section 4.2	Example 20	$\widetilde{A}_1 = (-12, 1, 2)$ $\widetilde{A}_2 = (-23/12, 1/12, 13/12)$ $\widetilde{A}_3 = (-6, 0, 1, 1)$	Abbasbandy & Hajjari [74], Nasseri & Sohrabi [75]	Counterintuition
Section 4.3	Example 22	$\widetilde{A}_1 = (1, 2, 5)$ $\widetilde{A}_2 = (1, 2, 2, 4)$	Ky Phuc et al. [38], Asady [46], Zhang et al. [73]	More elaborate in computation

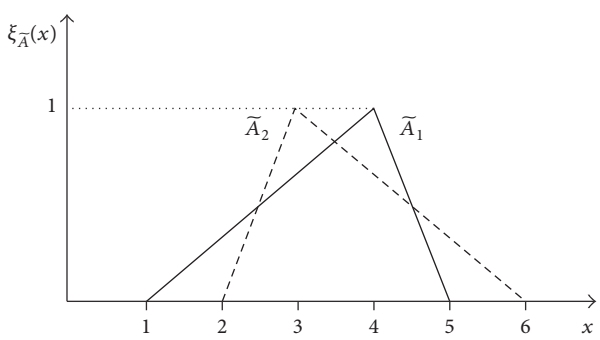

Figure 3: Fuzzy numbers $\widetilde{A}_1$ and $\widetilde{A}_2$ in Example 15.

Table 2: Ranking results for Example 15.

λ	UI_1^λ	UI_2^λ	Ranking result
0.0	8.333	9.167	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.1	9.000	9.900	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.2	9.667	10.633	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.3	10.333	11.367	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.4	11.000	12.100	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.5	11.667	12.833	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.6	12.333	13.567	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.7	13.000	14.300	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.8	13.667	15.033	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
0.9	14.333	15.767	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$
1.0	15.000	16.500	$\widetilde{A}_1 \prec \widetilde{A}_2$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2$

but overlapping after flipping and sliding movement. Here, the proposed unified index is contrasted with the most recent work published by Yu and Dat [48] in 2014 as regards computation simpleness.

According to the unified index in (5), we simply have the results shown in Table 2, $\widetilde{A}_1 \prec \widetilde{A}_2$ ($\widetilde{A}'_1 \succ \widetilde{A}'_2$) at any arbitrary level-of-optimism attitude of data revelation from the decision-maker, $\lambda \in [0, 1]$. Yu and Dat [48] advocated the identical ranking result in this case; however, their computation of median values before ranking these two fuzzy numbers is procedure-laborious in practice as reported by some predecessors [58–62].

By the same token, when comparing two normal triangular fuzzy numbers $\widetilde{B}_1 = (0.1, 0.6, 0.8)$ and $\widetilde{B}_2 = (0.2, 0.5, 0.9)$, taken from [76] and based on the proposed approach, we always have $\widetilde{B}_1 \prec \widetilde{B}_2$, which is coherent with that in [57, 63, 77–80]. However, the approaches by R. Chutia and B. Chutia [81] and Deng [82] lead to a counterintuitive result $\widetilde{B}_2 \prec \widetilde{B}_1$.

Example 16. Consider three triangle fuzzy numbers, $\widetilde{A}_1 = (5, 6, 7)$, $\widetilde{A}_2 = (5.9, 6, 7)$, and $\widetilde{A}_3 = (6, 6, 7)$ [39], which are similar and covered with the same right-hand side as displayed in Figure 4. By human instinct, they are easily being discriminated; that is, for the fuzzy numbers and their images, the intuitive and consistent rankings are $\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$. Therefore, this example is capable of judging the indices' performance if intuition- and consistency-satisfied.

We first check the unified index. Based on (5), Propositions 11 and 12, and Remark 13, the ranking results, listed

Table 3: Ranking results for Example 16.

λ	UI_1^λ	UI_2^λ	UI_3^λ	Ranking result
0.0	33.000	37.485	38.000	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.1	33.600	37.831	38.317	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.2	34.200	38.178	38.633	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.3	34.800	38.524	38.950	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.4	35.400	38.871	39.267	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.5	36.000	39.217	39.583	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.6	36.600	39.564	39.900	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.7	37.200	39.910	40.217	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.8	37.800	40.257	40.533	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.9	38.400	40.603	40.850	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
1.0	39.000	40.950	41.167	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$

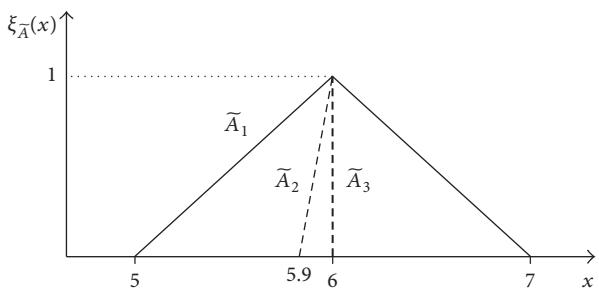

Figure 4: Fuzzy numbers $\widetilde{A}_1$, $\widetilde{A}_2$, and $\widetilde{A}_3$ in Example 16.

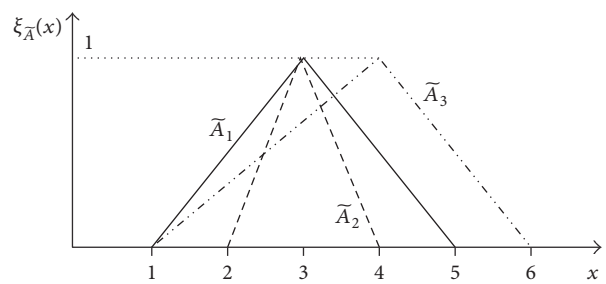

Figure 5: Fuzzy numbers $\widetilde{A}_1$, $\widetilde{A}_2$, and $\widetilde{A}_3$ in Example 17.

in Table 3 for the fuzzy numbers and their images, affirm the intuitive and consistent outcomes, $\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$.

In the literature, while many support the intuitive results for ranking the fuzzy numbers [39, 46, 57, 66, 83, 84], Chen [85] and Chu and Tsao [63] provide a different consequence as $\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$ and Cheng [57] gives $\widetilde{A}_3 \prec \widetilde{A}_2 \prec \widetilde{A}_1$, so their counterintuitions are apparent.

Moreover, due to scarcity of methods in the literature for consistently ranking their images, a recent work from Yu and Dat [48] claimed to bridge the gap. Unfortunately, when $\lambda = 1$, their approach leads to a disparate ranking, $\widetilde{A}_1 \simeq \widetilde{A}_2 \simeq \widetilde{A}_3$ ($\widetilde{A}'_1 \simeq \widetilde{A}'_2 \simeq \widetilde{A}'_3$), indicating that their index as a whole somewhat lacks reliability.

Example 17. Again, examine three fuzzy numbers, $\widetilde{A}_1 = (1, 3, 5)$, $\widetilde{A}_2 = (2, 3, 4)$, and $\widetilde{A}_3 = (1, 4, 6)$, as shown in Figure 5. Visibly, $\widetilde{A}_3 = (1, 4, 6)$ is right way out $\widetilde{A}_1$ and $\widetilde{A}_2$, so there is no dispute that a capable index should rate $\widetilde{A}_3$ ($\widetilde{A}'_3$) as the largest (smallest). The challenging one is to distinguish $\widetilde{A}_1$ and $\widetilde{A}_2$ ($\widetilde{A}'_1$ and $\widetilde{A}'_2$) due to their symmetry with respect to $x = 3$, identical centroid value, and their geometric enlargement relationship. Actually, majority of the existing ranking measures in category one (evaluating the fuzzy number itself) rank $\widetilde{A}_1 \simeq \widetilde{A}_2$, and their image ranking is not available. Therefore, this example is to compare the proposed unified index with the category two ranking measures (not only evaluating the fuzzy number itself, but also gauging decision-maker's attitude in regard to specific purposes such as confidence and risk), initiated by Wang and Luo [39], Yu and Dat [48], Yu et al. [65], and Liou and Wang [67], in terms of ranking indices' reliability and consistency.

First, we check the unified index's results in Table 4. Regardless of $\lambda \in [0, 1]$, $\widetilde{A}_3$ ($\widetilde{A}'_3$) is always the largest (smallest), which confirms human intuition. For the ranking of $\widetilde{A}_1$ and $\widetilde{A}_2$, dividing from $\lambda = 0.5$, $\widetilde{A}_1 \simeq \widetilde{A}_2$ ($\widetilde{A}'_1 \simeq \widetilde{A}'_2$); the upper part $\lambda \in [0, 0.5)$, $\widetilde{A}_1 \prec \widetilde{A}_2$ ($\widetilde{A}'_1 \succ \widetilde{A}'_2$); the lower part $\lambda \in (0.5, 1]$, $\widetilde{A}_1 \succ \widetilde{A}_2$ ($\widetilde{A}'_1 \prec \widetilde{A}'_2$). Although this result has been proved in Theorem 14, there are still some insightful conclusions to be addressed.

First, this finding is consistent with that of Wang and Luo [39] and Yu et al. [65]. In fact, with respect to the unified index, these results are reasonable because the chosen λ value manifests the decision-maker's optimism towards revelation of left- and right-area data. $\lambda \in (0.5, 1]$ implies that the right-area data is more preferred by the decision-maker; $\lambda \in [0, 0.5)$ represents the notion that the decision-maker is more optimistic regarding the left-area data; $\lambda = 0.5$ indicates that the decision-maker is neutral towards preference of data location.

Then, we evaluate the indices proposed by Yu and Dat [48] and Liou and Wang [67]. While Yu and Dat's work confirms most of the results in Table 4, it does exhibit an apparent counterintuition issue at $\lambda = 0$, where it suggests that $\widetilde{A}_3$ does not dominate $\widetilde{A}_2$; that is, $\widetilde{A}_2 \simeq \widetilde{A}_3$ ($\widetilde{A}'_2 \simeq \widetilde{A}'_3$). Moreover, Liou and Wang's index [67] not only afflicts the same shortage of Yu and Dat's index, but also has

Table 4: Ranking results at different optimism levels in Example 17.

λ	UI_1^λ	UI_2^λ	UI_3^λ	Ranking result
0.0	6.000	7.500	9.167	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.1	6.600	7.800	10.083	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.2	7.200	8.100	11.000	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.3	7.800	8.400	11.917	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.4	8.400	8.700	12.833	$\widetilde{A}_1 \prec \widetilde{A}_2 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.5	**9.000**	**9.000**	**13.750**	$\widetilde{\mathbf{A}}_1 \simeq \widetilde{\mathbf{A}}_2 \prec \widetilde{\mathbf{A}}_3$ and $\widetilde{\mathbf{A}}'_1 \simeq \widetilde{\mathbf{A}}'_2 \succ \widetilde{\mathbf{A}}'_3$
0.6	9.600	9.300	14.667	$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.7	10.200	9.600	15.583	$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.8	10.800	9.900	16.500	$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
0.9	11.400	10.200	17.417	$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$
1.0	12.000	10.500	18.333	$\widetilde{A}_2 \prec \widetilde{A}_1 \prec \widetilde{A}_3$ and $\widetilde{A}'_1 \succ \widetilde{A}'_2 \succ \widetilde{A}'_3$

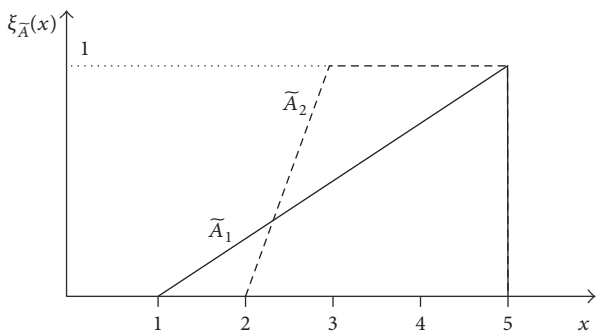

Figure 6: Fuzzy numbers $\widetilde{A}_1$ and $\widetilde{A}_2$ in Example 18.

shown inconsistent results for ranking the fuzzy numbers and their images due to the index's limited definition and generalization.

4.2. Ranking for Normal Triangular-and-Trapezoid Mixed Fuzzy Numbers. Here, the proposed unified index is used to broaden the ranking comparisons to normal triangular-and-trapezoid mixed fuzzy numbers. The cases from the literature that have one trapezoid mixed with one triangular fuzzy number, followed by two examples with two triangular fuzzy numbers, are investigated.

Example 18. Compare a triangular fuzzy number $\widetilde{A}_1 = (1, 5, 5)$ overlapping with a trapezoidal fuzzy number $\widetilde{A}_2 = (2, 3, 5, 5)$, as shown in Figure 6. Of ten existing measures that have been studied in this case, three (30%) support $\widetilde{A}_1 \prec \widetilde{A}_2$ [30, 66, 86] and seven (70%) stand for $\widetilde{A}_1 \succ \widetilde{A}_2$ [47, 53, 63, 73, 74, 83, 87]. Clearly, this stark contrast outcome is intriguing for further investigation. Therefore, in this example, we first attempt to explain the predecessors' conflicting consequence by using the unified index. Then, the index itself will be compared with the recent work proposed by Zhang et al. in 2014 [73] to lay out their result similarity as well as their performance with regard to computation easiness and image consistency.

Table 5 is the ranking results of using the unified index, where $\lambda \in [0, 0.8]$, $\widetilde{A}_1 \succ \widetilde{A}_2$ and $\lambda \in [0.9, 1]$, $\widetilde{A}_1 \prec \widetilde{A}_2$. Once more, the chosen λ value manifests the decision-maker's optimism towards revelation of the left-and-right area of fuzzy data. From the λ-probability point of view, around 80% support $\widetilde{A}_1 \succ \widetilde{A}_2$ and 20% favor $\widetilde{A}_1 \prec \widetilde{A}_2$. In fact, this result, providing a level-of-optimism attitude-based explanation for conflicts among the comparison, is interesting to be approximate with aforementioned percentages obtained from the literature conclusions. Moreover, it is also similar to Zhang et al.'s [73] result who uses a preference-probability relation to explain the uncertainty level of the comparison; with seven intricate and somewhat complicated steps, they concluded $\widetilde{A}_1 \succ \widetilde{A}_2$ with a confidence degree of 73% and $\widetilde{A}_1 \prec \widetilde{A}_2$ with 27%.

Table 5: Ranking results at different optimism levels in Example 18.

λ	UI_1^λ	UI_2^λ	Ranking result
0.0	11.000	9.333	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.1	11.733	10.267	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.2	12.467	11.200	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.3	13.200	12.133	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.4	13.933	13.067	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.5	14.667	14.000	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.6	15.400	14.933	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.7	16.133	15.867	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.8	16.867	16.800	$\widetilde{A}_1 \succ \widetilde{A}_2$
0.9	17.600	17.733	$\widetilde{A}_1 \prec \widetilde{A}_2$
1.0	18.333	18.667	$\widetilde{A}_1 \prec \widetilde{A}_2$

Finally, it is worth mentioning that as opposed to the unified index, Zhang et al.'s [73] seven-step algorithm for ranking fuzzy numbers not only suffers a computation-complexity problem, but also lacks capacity for ranking the fuzzy-number image.

Example 19. Taken from [38] and shown in Figure 7, one trapezoid fuzzy number, $\widetilde{A}_3 = (0, 2, 4, 6)$, mingled with two triangular fuzzy numbers, $\widetilde{A}_1 = (0, 3, 6)$ and $\widetilde{A}_2 = (-1, 0, 2)$, is considered in this example. Noticeably, $\widetilde{A}_2$ left distances away from $\widetilde{A}_1$ and $\widetilde{A}_3$, so there is no argument that a reliable

Table 6: Ranking results of the three fuzzy numbers in Example 19.

λ	UI_1^λ	UI_2^λ	UI_3^λ	Ranking result
0.0	4.500	0.167	3.000	$\widetilde{A}_1 \succ \widetilde{A}_3 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_3 \prec \widetilde{A}'_2$
0.1	5.400	0.183	4.200	$\widetilde{A}_1 \succ \widetilde{A}_3 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_3 \prec \widetilde{A}'_2$
0.2	6.300	0.200	5.400	$\widetilde{A}_1 \succ \widetilde{A}_3 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_3 \prec \widetilde{A}'_2$
0.3	7.200	0.217	6.600	$\widetilde{A}_1 \succ \widetilde{A}_3 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_3 \prec \widetilde{A}'_2$
0.4	8.100	0.233	7.800	$\widetilde{A}_1 \succ \widetilde{A}_3 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_3 \prec \widetilde{A}'_2$
0.5	**9.000**	**0.250**	**9.000**	$\widetilde{\mathbf{A}}_1 \simeq \widetilde{\mathbf{A}}_3 \succ \widetilde{\mathbf{A}}_2$ and $\widetilde{\mathbf{A}}'_1 \simeq \widetilde{\mathbf{A}}'_3 \prec \widetilde{\mathbf{A}}'_2$
0.6	9.900	0.267	10.200	$\widetilde{A}_3 \succ \widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_3 \prec \widetilde{A}'_1 \prec \widetilde{A}'_2$
0.7	10.800	0.283	11.400	$\widetilde{A}_3 \succ \widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_3 \prec \widetilde{A}'_1 \prec \widetilde{A}'_2$
0.8	11.700	0.300	12.600	$\widetilde{A}_3 \succ \widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_3 \prec \widetilde{A}'_1 \prec \widetilde{A}'_2$
0.9	12.600	0.317	13.800	$\widetilde{A}_3 \succ \widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_3 \prec \widetilde{A}'_1 \prec \widetilde{A}'_2$
1.0	13.500	0.333	15.000	$\widetilde{A}_3 \succ \widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_3 \prec \widetilde{A}'_1 \prec \widetilde{A}'_2$

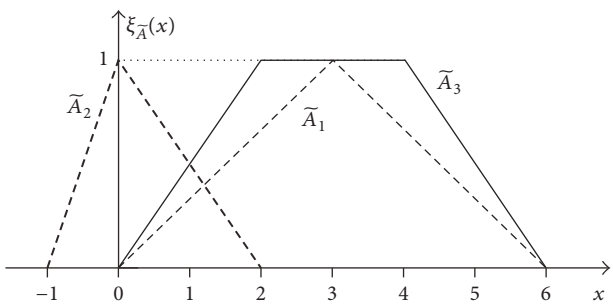

Figure 7: Fuzzy numbers $\widetilde{A}_1$, $\widetilde{A}_2$, and $\widetilde{A}_3$ in Example 19.

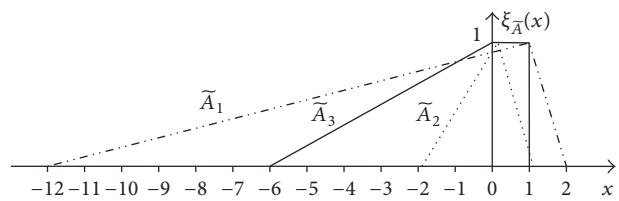

Figure 8: Fuzzy numbers $\widetilde{A}_1$, $\widetilde{A}_2$, and $\widetilde{A}_3$ in Example 20.

index should discriminate $\widetilde{A}_2$ ($\widetilde{A}'_2$) as the smallest (largest). The question is the rating result of the triangular fuzzy number $\widetilde{A}_1$ and the trapezoid fuzzy number $\widetilde{A}_3$ and their images. Therefore, this example is to compare the unified index with the recent works of Asady in 2010 and Ky Phuc et al. [38] in 2012 who proposed a deviation-degree ranking measure.

First, we check the unified index's results in Table 6. Regardless of $\lambda \in [0,1]$, $\widetilde{A}_2$ ($\widetilde{A}'_2$) is always the smallest (largest), which confirms human intuition. For the ranking of $\widetilde{A}_1$ and $\widetilde{A}_3$, dividing from $\lambda = 0.5$, $\widetilde{A}_1 \simeq \widetilde{A}_3$ ($\widetilde{A}'_1 \simeq \widetilde{A}'_3$); the upper part $\lambda \in [0, 0.5)$, $\widetilde{A}_1 \succ \widetilde{A}_3$ ($\widetilde{A}'_1 \prec \widetilde{A}'_3$); the lower part $\lambda \in (0.5, 1]$, $\widetilde{A}_1 \prec \widetilde{A}_3$ ($\widetilde{A}'_1 \succ \widetilde{A}'_3$). Literally, this finding (refer to Theorem 14) is consistent with two fuzzy numbers with the same attitude and symmetry, shown in Example 17.

Then, we evaluate Ky Phuc et al.'s [38] and Asady's [46] deviation-degree index. Despite the exhausted computation, its capability can only provide the partial result, "$\widetilde{A}_1 \succ \widetilde{A}_2$" and "$\widetilde{A}_3 \succ \widetilde{A}_2$," leaving undecided ranking for $\widetilde{A}_1$ and $\widetilde{A}_3$. Actually, as mentioned in Section 1, the deviation-degree index, belonging to the category one ranking measure, has the limitation for ranking the fuzzy numbers akin to $\widetilde{A}_1$ and $\widetilde{A}_3$ that are overlapping and each has axis-of-symmetry property.

Example 20. Additionally, let us consider one trapezoidal fuzzy number, $\widetilde{A}_3 = (-6, 0, 1, 1)$, blended with two triangular fuzzy numbers, $\widetilde{A}_1 = (-12, 1, 2)$ and $\widetilde{A}_2 = (-23/12, 1/12, 13/12)$, which are adapted from [66] and shown in Figure 8. Unlike the previous challenging one that is with a symmetric and triangle-embedded trapezoid shape, they are all left-skewed fuzzy numbers and easy to be distinguished by human perception; that is, $\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$. Hence, for this subsection, this example is capable of judging the indices' performance if intuition-satisfied.

The result in Table 7, obtained with the unified index, clearly shows that $\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$, which is identical to previous works in [46, 47, 63, 64, 66, 87]. However, counter results are claimed by Abbasbandy and Hajjari [74] who ranked them as $\widetilde{A}_1 \simeq \widetilde{A}_2 \simeq \widetilde{A}_3$ and Nasseri and Sohrabi [75] who suggested $\widetilde{A}_2 \prec \widetilde{A}_3 \prec \widetilde{A}_1$. Both works' counterintuition is obvious.

Example 21. Now, two special cases taken from R. Chutia and B. Chutia [81] are considered. The first set includes $\widetilde{A}_1 = (0.1, 0.1, 0.1, 0.1; 0.8)$ and $\widetilde{A}_2 = (-0.1, -0.1, -0.1, -0.1; 1.0)$ which were ranked as $\widetilde{A}_1 > \widetilde{A}_2$; and the second one includes $\widetilde{B}_1 = (1, 1, 1, 1; 0.5)$ and $\widetilde{B}_2 = (1, 1, 1, 1; 1.0)$ which were ranked as $\widetilde{B}_1 < \widetilde{B}_2$. The proposed unified index also leads to similar conclusions as in [37, 78, 81, 88–90], indicating that the index can effectively work with crisp numbers as well.

4.3. Ranking for Nonlinear Fuzzy Numbers. Finally, although empirical phenomenon and human perception are rather unlikely to gather the nonlinear fuzzy numbers, this more general type can be justifiable for investigating the index's computation easiness as well as adaptability.

Table 7: Ranking results of the three fuzzy numbers in Example 20.

λ	UI_1^λ	UI_2^λ	UI_3^λ	Ranking result
0.0	−18.333	−0.229	−4.125	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.1	−16.833	−0.221	−3.850	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.2	−15.333	−0.213	−3.575	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.3	−13.833	−0.204	−3.300	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.4	−12.333	−0.196	−3.025	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.5	−10.833	−0.188	−2.750	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.6	−9.333	−0.179	−2.475	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.7	−7.833	−0.171	−2.200	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.8	−6.333	−0.163	−1.925	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
0.9	−4.833	−0.154	−1.650	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$
1.0	−3.333	−0.146	−1.375	$\widetilde{A}_1 \prec \widetilde{A}_3 \prec \widetilde{A}_2$

Table 8: Ranking results at different optimism levels in Example 22.

λ	UI_1^λ	UI_2^λ	Ranking result
0.0	6.750	2.945	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.1	7.650	3.516	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.2	8.550	4.087	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.3	9.450	4.658	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.4	10.350	5.230	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.5	11.250	5.801	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.6	12.150	6.372	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.7	13.050	6.943	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.8	13.950	7.515	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
0.9	14.850	8.086	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$
1.0	15.750	8.657	$\widetilde{A}_1 \succ \widetilde{A}_2$ and $\widetilde{A}'_1 \prec \widetilde{A}'_2$

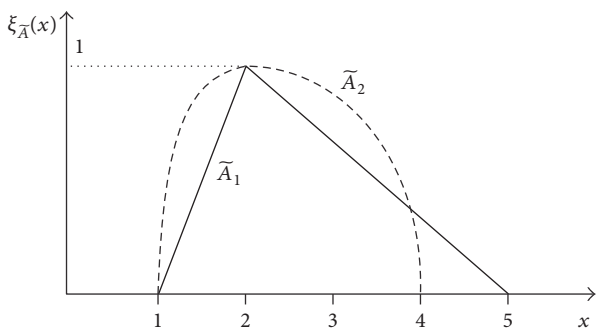

Figure 9: Fuzzy numbers $\widetilde{A}_1$ and $\widetilde{A}_2$ in Example 22.

Example 22. Let us consider two fuzzy numbers shown in Figure 9, adapted from Liou and Wang [67]: $\widetilde{A}_1 = (1, 2, 5)$ and $\widetilde{A}_2 = (1, 2, 2, 4)$ with a nonlinear membership function

$$f_{\widetilde{A}_2}(x) = \begin{cases} \left[1 - (x-2)^2\right]^{1/2}, & 1 \le x \le 2, \\ \left[1 - \frac{1}{4}(x-2)^2\right]^{1/2}, & 2 \le x \le 4, \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

In this nonlinear case, by using the unified index, the conclusions in Table 8, $\widetilde{A}_1 \succ \widetilde{A}_2$ ($\widetilde{A}'_1 \prec \widetilde{A}'_2$) for $\lambda \in [0, 1]$, do not add much complexity for the computation. Obviously, previous proposed measures in [53, 63, 66, 67, 76] possess the same conclusion and computation easiness. However, in recent works, Ky Phuc et al. [38], Asady [46], and Zhang et al. [73], their indices become more complicated and elaborate for ranking the nonlinear fuzzy numbers as well as their images.

5. Conclusions

Numerous indices for fuzzy-data comparisons and rankings have been widely implemented to resolve decisive problems that are evaluated under resources constraint and with to some extent linguistic preference of multiattribute, realized from users' perspectives, as well as subjective quantification of multiple characteristics, assessed from decision-makers. However, counterintuition, computation complexity, less reliability, and/or inconsistency on their fuzzy-number rating outcomes have hampered their comprehensive implementation. To lessen their exhibited ranking weaknesses, this paper develops a unified index that multiplies weighted-mean and weighted-area discriminatory components of a fuzzy number, respectively, called centroid value (a measure that values the fuzzy number itself) and attitude-incorporated left-and-right area (a fuzzy-number measure that also reflects on the decision-maker's attitude as regards data revelation). From theoretical proofs and comparative studies, this unified index has demonstrated four advantages for ranking fuzzy numbers.

First, ranking results of the unified index support the human-intuition judgement. Secondly, it shows computation easiness regardless of different types of fuzzy numbers. It can be noted that this computation simplicity becomes crucial for multiagents-multicriteria decision-making problems, which normally involve numerous comparisons and analyses of fuzzy numbers. Thirdly, the unified index can provide a level-of-optimism attitude-based explanation for ranking conflicts among the literature. Most importantly, the unified index possesses the consistency property for ranking fuzzy numbers and their images as well as for symmetric fuzzy numbers with an identical altitude. Literally, in fields of computer vision and image pattern recognition, this property has been a rather critical one for accurate matching and/or retrieval of information.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Random Fuzzy Differential Equations with Impulses

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We consider the random fuzzy differential equations (RFDEs) with impulses. Using Picard method of successive approximations, we shall prove the existence and uniqueness of solutions to RFDEs with impulses under suitable conditions. Some of the properties of solution of RFDEs with impulses are studied. Finally, an example is presented to illustrate the results.

1. Introduction

Impulsive differential equations (IDEs) are a new branch of differential equations. IDEs can find numerous applications in different branches of optimal control, electronics, economics, physics, chemistry, and biological sciences. We refer to [1–4] and the references therein. As we know, the real systems are often faced with two kinds of uncertainties (fuzziness and randomness). Therefore, this topic has extensively been studied by mathematicians in recent years. Investigations of dynamic systems with fuzziness have been developed in connection with fuzzy differential equations (FDEs). Evidence of FDEs for such areas as control theory, differential inclusions, and fuzzy differential equations can be found in the papers of [5–8], the books and monographs [9], and references therein. In [10], Lakshmikantham and McRae combined the theories of impulsive differential equations and fuzzy differential equations. There are a few papers on the latter topic; see [10–12].

Moreover, the class of random fuzzy differential equations (RFDEs) could be applicable in the investigation of numerous engineering and economics problems where the phenomena are simultaneously subjected to two kinds of uncertainties, that is, fuzziness and randomness, simultaneously (see, e.g., Malinowski [13–16], Feng [17, 18], and Fei [19, 20]). Feng [17] introduced the concepts by the mean-square derivative and mean-square integral of second-order fuzzy stochastic processes. Using the results, the author [18] investigated the properties of solutions of the fuzzy stochastic differential systems, including the existence and uniqueness of solution, the dependence of the solution of the initial condition, and the continuity and the boundedness of solution of systems when there are perturbations of the coefficients and the initial conditions. In [19, 20], Fei proved the existence and uniqueness of solution of fuzzy random differential equation (FRDE). The author also discussed the dependence of solution to FRDE on initial values. Finally, the nonconfluence property of the solution for FRDE is studied.

In [13], Malinowski considered the following random fuzzy differential equations:

$$\begin{aligned} D_H x(t,\omega) &\overset{[t_0,t_0+p],\mathbb{P}.1}{=} f(t, x(t,\omega)), \\ x(t_0,\omega) &\overset{\mathbb{P}.1}{=} x_0(\omega) \in E^d, \end{aligned} \tag{1}$$

where $f : \Omega \times [t_0, t_0 + p] \times E^d \to E^d$ and the symbol D_H denotes the fuzzy Hukuhara derivative. The author proved the existence and uniqueness of the solution for RFDEs under Lipschitz condition. Malinowski [14, 15] studied two kinds of solutions to the RFDEs with two kinds of fuzzy derivatives. For both cases the author established the existence and uniqueness of local solutions to RFDEs. In addition, the author also presented some examples being simple illustrations of the theory of RFDEs.

Inspired and motivated by Fei [19], Feng [18], Malinowski [14], and other authors as in [3, 10, 21], in this paper, we consider the RFDEs with impulses under Hukuhara derivative. The paper is organized as follows: in Section 2, we summarize

some preliminary facts and properties of the fuzzy set space, fuzzy differentiation, and integration. We also recall the notions of fuzzy random variable and fuzzy stochastic process. In Section 3, we discuss the RFDEs with impulses. Under suitable conditions, we prove the existence and uniqueness of solutions to RFDEs with impulses. In Section 4, we give some examples to illustrate these results.

2. Preliminaries

In this section, we give some definitions and properties and introduce the necessary notation which will be used throughout the paper. We denote $E^d = \{u : \mathbb{R}^d \to [0,1] \mid u$ satisfies (i)–(iv) stated below$\}$, where

(i) u is normal; that is, there exists an $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex; that is, for $0 \le \lambda \le 1$, $u(\lambda x_1 + (1-\lambda)x_2) \ge \min\{u(x_1), u(x_2)\}$, for any $x_1, x_2 \in \mathbb{R}^d$;

(iii) u is upper semicontinuous;

(iv) $cl\{x \in \mathbb{R}^d : u(z) > 0\}$ is compact set.

Then E^d is called the space of fuzzy numbers.

For $0 < \alpha \le 1$, we denote $[u]^\alpha = \{x \in \mathbb{R}^d \mid u(x) \ge \alpha\}$ and $[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}$. For $d = 1$ and from conditions (i)–(iv), we infer that the α-level cut of u, denoted by $[u]^\alpha$, is a bounded closed interval for any $\alpha \in [0,1]$ and $u \in E^d$, and $[u]^\alpha = [u^l_\alpha, u^r_\alpha]$, where u^l_α and u^r_α are the lower and upper branches of u.

For $u, v \in E^d$, the Hausdorff distance between u and v is defined by

$$d_\infty(x,y) = \sup_{\alpha\in[0,1]} \max\{d_H([u]^\alpha), d_H([v]^\alpha)\} \tag{2}$$

and (E^d, d_∞) is a complete metric space.

If we define $D : E^d \times E^d \to \mathbb{R}_+$ by the expression

$$D(u,v) = \sup_{t\in[a,b]} d_\infty(u(t), v(t)), \tag{3}$$

then it is well-known that D is metric in E^d and (E^d, D) is also a complete metric space.

Some properties are well-known for the metric Hausdorff D defined on E^d as follows:

$$\begin{gathered} D(u+w, v+w) = D(u,v), \\ D(\lambda u, \lambda v) = \lambda D(u,v), \\ D(u,v) \le D(u,w) + D(v,w), \end{gathered} \tag{4}$$

for every $u, v, w \in E^d$ and $\lambda \in \mathbb{R}_+$.

Definition 1 (see [22]). Let $u, v \in E^d$. If there exists $w \in E^d$ such that $u = v + w$, then w is called the Hukuhara difference of u, v and it is denoted by $u \ominus v$.

Definition 2 (see [22]). Let $f : (a,b) \to E^d$ and $t \in (a,b)$. We say that f is differentiable at t if there exists an element $D_H f(t) \in E^d$ such that the limits

$$\lim_{h\to 0^+} \frac{f(t+h) \ominus f(t)}{h} = \lim_{h\to 0^+} \frac{f(t) \ominus f(t-h)}{h} \tag{5}$$

exist and are equal to $D_H f(t)$.

Definition 3 (see [22]). Let $f : (a,b) \to E^d$. The integral of f on (a,b), denoted by $\int_a^b f(t)dt$, is defined levelwise by the equation

$$\begin{aligned} \left[\int_a^b f(t)\,dt\right]^\alpha &= \int_a^b [f(t)]^\alpha\,dt \\ &= \left\{\int_a^b \widetilde{f}(t)\,dt \mid \widetilde{f} : (a,b) \right. \\ &\quad \left. \longrightarrow \mathbb{R} \text{ is a measurable selection for } [f(\cdot)]^\alpha\right\}, \end{aligned} \tag{6}$$

for all $\alpha \in [0,1]$.

Definition 4 (see [22]). A fuzzy mapping $f : (a,b) \to E^d$ is integrable if f is integrable bounded and strongly measurable.

The following are some properties of integrability of fuzzy mapping (see [22]):

(a) If $f : (a,b) \to E^d$ is continuous then it is integrable.

(b) If $f : (a,b) \to E^d$ is integrable and $c \in (a,b)$ then $\int_a^b f(s)ds = \int_a^c f(s)ds + \int_c^b f(s)ds$.

(c) Let $f, g : (a,b) \to E^d$ be integrable and $\lambda > 0$. Then

(i) $\int_a^b (f(s) + g(s))ds = \int_a^b f(s)ds + \int_a^b g(s)ds$,

(ii) $\int_a^b \lambda f(s)ds = \lambda \int_a^b f(s)ds$,

(iii) $D(f,g)$ is integrable and $D(\int_a^b f(s)ds, \int_a^b g(s)ds) \le \int_a^b D(f(s), g(s))ds$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A function $x : \Omega \to E^d$ is called a fuzzy random variable, if the set-valued mapping $[x(\cdot)]^\alpha : \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ is a measurable multifunction for all $\alpha \in [0,1]$; that is,

$$\{\omega \in \Omega \mid [x(\omega)]^\alpha \cap B \ne \emptyset\} \in \mathcal{F} \tag{7}$$

for every closed set $B \subset \mathbb{R}^d$.

Definition 5 (see [13]). A mapping $x : [a,b] \times \Omega \to E^d$ is said to be a fuzzy stochastic process if $x(\cdot, \omega)$ is a fuzzy-set-valued function with any fixed $\omega \in \Omega$ and $x(t, \cdot)$ is a fuzzy random variable for any fixed $t \in [a,b]$.

Definition 6 (see [13]). A fuzzy stochastic process $x : [a,b] \times \Omega \to E^d$ is called continuous if there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and such that, for every $\omega \in \Omega_0$, the trajectory $x(\cdot, \omega)$ is a continuous function on $[a,b]$ with respect to the metric D.

For convenience, from now on, we shall write $x(\omega) \overset{\mathbb{P}.1}{=} y(\omega)$ to replace $\mathbb{P}(\{\omega \mid x(\omega) = y(\omega)\}) = 1$ for short, where x, y are random elements, and similarly for inequalities. Also we shall write $x(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} y(t,\omega)$ to replace $\mathbb{P}(\{\omega \mid x(t,\omega) = y(t,\omega)\}, \forall t \in [a,b]) = 1$ for short, where x, y are some stochastic processes, and similarly for inequalities.

3. Existence and Uniqueness for RFDEs with Impulses

In this section, we consider the following random fuzzy differential equation with impulses:

$$\begin{aligned} D_H x(t,\omega) &\overset{\mathbb{P}.1}{=} f(t, x(t,\omega), \omega), \\ &\qquad t \in J := [t_0, t_0 + p], \ t \neq t_k, \\ x(t_k^+, \omega) &\overset{\mathbb{P}.1}{=} I_k(x(t_k,\omega), \omega), \quad k = \overline{1,m}, \ t = t_k, \\ x(t_0^+, \omega) &\overset{\mathbb{P}.1}{=} x_0(\omega) \in E^d, \end{aligned} \tag{8}$$

where $f : J \times E^d \times \Omega \to E^d$, $I_k : E^d \times \Omega \to E^d$ is continuous with $\mathbb{P}.1$, and t_k, $k = \overline{1,m}$, are points of impulses such that $t_0 \leq \cdots < t_k < t_{k+1} \leq t_0 + p$ and $x_0 : \Omega \to E^d$ is fuzzy random variable.

Lemma 7. *Let $x : J \times \Omega \to E^d$ be a fuzzy stochastic process. Then x is the solution of problem (8) if and only if x is a continuous fuzzy stochastic process and satisfy the following random impulsive fuzzy integral equation:*

$$\begin{aligned} x(t,\omega) = x_0(\omega) &+ \int_{t_0}^{t} f(s, x(s,\omega), \omega)\, ds \\ &+ \sum_{i=1}^{k} I_i(x(t_i,\omega), \omega). \end{aligned} \tag{9}$$

Proof. We divide the proof into two steps.

Step 1. If $x(t)$ satisfies problem (8), then it will be expressed as (9). Indeed, for every $t \in [t_0, t_1)$ we have

$$D_H x(t,\omega) \overset{\mathbb{P}.1}{=} f(t, x(t,\omega), \omega). \tag{10}$$

By Lemma 3.1 in [13], we obtain

$$x(t_1,\omega) \overset{[t_0,t_1),\mathbb{P}.1}{=} x_0(\omega) + \int_{t_0}^{t_1} f(s, x(s,\omega), \omega)\, ds. \tag{11}$$

If $t \in [t_1, t_2)$ and by Lemma 3.1 in [13], we have

$$\begin{aligned} x(t,\omega) &\overset{\mathbb{P}.1}{=} x(t_1^+, \omega) + \int_{t_1}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\overset{\mathbb{P}.1}{=} I_1(x_1(t,\omega), \omega) + \int_{t_1}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\overset{\mathbb{P}.1}{=} I_1(x_1(t,\omega), \omega) + x_0(\omega) \\ &\quad + \int_{t_0}^{t_1} f(s, x(s,\omega), \omega)\, ds \\ &\quad + \int_{t_1}^{t} f(s, x(s,\omega), \omega)\, ds. \end{aligned} \tag{12}$$

If we assume that

$$\begin{aligned} x(t,\omega) &\overset{[t_{k-1},t_k),\mathbb{P}.1}{=} x_0(\omega) + \int_{t_0}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\quad + \sum_{i=1}^{k-1} I_i(x(t_i,\omega), \omega), \end{aligned} \tag{13}$$

then we have

$$\begin{aligned} x(t,\omega) &\overset{[t_k,t_{k+1}],\mathbb{P}.1}{=} x(t_k^+, \omega) + \int_{t_k}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\overset{[t_k,t_{k+1}],\mathbb{P}.1}{=} I_1(x_k(t,\omega), \omega) + \int_{t_k}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\qquad \vdots \\ &\overset{[t_k,t_{k+1}],\mathbb{P}.1}{=} \sum_{i=1}^{k} I_i(x(t_i,\omega), \omega) + x_0(\omega) \\ &\quad + \int_{t_0}^{t} f(s, x(s,\omega), \omega)\, ds. \end{aligned} \tag{14}$$

It follows by mathematical induction that (13) holds for any $k \geq 1$.

Step 2. Conversely, if a fuzzy stochastic process x satisfies the random fuzzy integral equation (9), then it is equivalent to problem (8). Indeed, if $t \in [t_0, t_1)$ we easily see that $x(t_0,\omega) \overset{\mathbb{P}.1}{=} x_0(\omega)$ and the Hukuhara difference $x_0(\omega) + \int_{t_0}^{t} f(s, u(s,\omega), \omega) ds$ exists, with $\mathbb{P}.1$. By Lemma 3.2 in [13] we have

$$D_H x(t,\omega) \overset{[t_0,t_1),\mathbb{P}.1}{=} f(t, x(t,\omega), \omega). \tag{15}$$

Let $h > 0$ small enough such that $t - h \in [t_1, t_2)$ for every $t \in [t_1, t_2)$; we have

$$\begin{aligned} x(t,\omega) \ominus x(t-h,\omega) &\overset{\mathbb{P}.1}{=} \int_{t_1}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\quad \ominus \int_{t_1}^{t-h} f(s, x(s,\omega), \omega)\, ds \\ &\overset{\mathbb{P}.1}{=} \int_{t-h}^{t} f(s, x(s,\omega), \omega)\, ds. \end{aligned} \tag{16}$$

Similarly, let $h > 0$ small enough such that $t + h \in (t_1, t_2)$ for every $t \in (t_1, t_2)$; we obtain

$$\begin{aligned} x(t+h,\omega) \ominus x(t,\omega) &\overset{\mathbb{P}.1}{=} \int_{t_1}^{t} f(s, x(s,\omega), \omega)\, ds \\ &\quad \ominus \int_{t_1}^{t+h} f(s, x(s,\omega), \omega)\, ds \\ &\overset{\mathbb{P}.1}{=} \int_{t}^{t+h} f(s, x(s,\omega), \omega)\, ds. \end{aligned} \tag{17}$$

Multiplying both sides of (16) and (17) by $1/h$ and passing to the limit with $h \to 0^+$, we obtain

$$\begin{aligned} &\lim_{h\to 0^+} \frac{x(t,\omega) \ominus x(t-h,\omega)}{h} \\ &\quad = \lim_{h\to 0^+} \frac{1}{h} \int_{t-h}^{t} f(s, x(s,\omega), \omega)\, ds = D_H x(t,\omega), \\ &\lim_{h\to 0^+} \frac{x(t+h,\omega) \ominus x(t,\omega)}{h} \\ &\quad = \lim_{h\to 0^+} \frac{1}{h} \int_{t}^{t+h} f(s, x(s,\omega), \omega)\, ds = D_H x(t,\omega). \end{aligned} \tag{18}$$

This allows us to claim that x is differentiable on $(t_1, t_2]$ and consequently

$$D_H x(t,\omega) \overset{[t_1,t_2),\mathbb{P}.1}{=} f(t, x(t,\omega), \omega). \tag{19}$$

By mathematical induction, if $t \in (t_k, t_{k+1}]$, $k = \overline{1,m}$, we get

$$D_H x(t,\omega) \overset{(t_k,t_{k+1}],\mathbb{P}.1}{=} f(t, x(t,\omega), \omega). \tag{20}$$

Also, we can easily show that

$$\Delta x(t_k,\omega) \overset{\mathbb{P}.1}{=} I_k(x(t_k,\omega), \omega), \quad k = \overline{1,m}. \tag{21}$$

The proof is complete. □

Lemma 8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A : \Omega \to \mathbb{R}_+$, $B_i : \Omega \to \mathbb{R}_+$, $i = 0, 1, 2, \ldots$, and stochastic processes $X, Y : J \times \Omega \to \mathbb{R}$ be such that*

(i) *$X(\cdot,\omega)$ is nonnegative and continuous with $\mathbb{P}.1$ and t_i are the points of discontinuity of the first of $X(\cdot,\omega)$ with $\mathbb{P}.1$,*

(ii) *$Y(\cdot,\omega)$ is locally Lebesgue integrable with $\mathbb{P}.1$.*

If

$$\begin{aligned} X(t,\omega) &\overset{J,\mathbb{P}.1}{\leq} A(\omega) + \int_{t_0}^{t} X(s,\omega) Y(s,\omega)\, ds \\ &\quad + \sum_{t_0 \leq t_i < t} B_i(\omega) X(t_i,\omega), \end{aligned} \tag{22}$$

then we have

$$\begin{aligned} &X(t,\omega) \\ &\quad \overset{J,\mathbb{P}.1}{\leq} A(\omega) \prod_{t_0 \leq t_i < t} (1 + B_i(\omega)) \exp\left(\int_{t_0}^{t} Y(s,\omega)\, ds\right). \end{aligned} \tag{23}$$

Now, we show the main results of this paper.

Theorem 9. *Let the mapping $f : J \times E^d \times \Omega \to E^d$ be continuous with $\mathbb{P}.1$ and $I_k : E^d \times \Omega \to E^d$. Assume the following conditions hold:*

(A1) *There exists a nonnegative constant L_1 such that $D[f(t,\varphi,\omega), f(t,\phi,\omega)] \leq L_1 D[\varphi,\phi]$, for every $t \in J$ and $\varphi, \phi \in E^d$ with $\mathbb{P}.1$.*

(A2) *There exists a nonnegative constant $L_{2,k}$ such that $D[I_k(\varphi,\omega), I_k(\phi,\omega)] \leq L_{2,k} D[\varphi,\phi]$, for $k = 0, 1, 2, \ldots, m$, for every $t \in J$ and $\varphi, \phi \in E^d$ with $\mathbb{P}.1$.*

(A3) *There exists a nonnegative constant M_1 such that $D[f(t, x_0(\omega), \omega), \widehat{0}] \leq M_1$, for every $t \in J$ and $x_0 \in E^d$ with $\mathbb{P}.1$.*

Then the random fuzzy differential equation with impulses (9) has a unique solution, provided that

$$\frac{L_1 p^n}{n!} + \sum_{i=1}^{k} L_{2,i} < 1, \quad \text{for any } n \in \mathbb{N}. \tag{24}$$

Proof. Define a sequence of the functions $x_n : J \times \Omega \to E^d$, $n = 0, 1, 2, \ldots$ as follows: for every $\omega \in \Omega$ let us put

$$\begin{aligned} x_0(t,\omega) &= x_0(\omega), \\ x_n(t,\omega) &= x_0(\omega) + \int_{t_0}^{t} f(s, x_{n-1}(s,\omega), \omega)\, ds \\ &\quad + \sum_{i=1}^{k} I_i(x_{n-1}(t_i,\omega), \omega). \end{aligned} \tag{25}$$

For every $t \in J$ and $\omega \in \Omega$, we have

$$\begin{aligned} & d_\infty\left[x_1(t,\omega), x_0(t,\omega)\right] \\ & = d_\infty\left[\int_{t_0}^{t} f\left(s, x_0(s,\omega), \omega\right) ds, \hat{0}\right] \\ & \quad + d_\infty\left[\sum_{i=1}^{k} I_i\left(x_0\left(t_i,\omega\right),\omega\right), \hat{0}\right] \\ & \le \int_{t_0}^{t} d_\infty\left[f\left(s, x_0(\omega), \omega\right), \hat{0}\right] ds \\ & \quad + \sum_{i=1}^{k} d_\infty\left[I_i\left(x_0(\omega),\omega\right), \hat{0}\right] \\ & \le M_1\left(t-t_0\right) + \sum_{i=1}^{k} d_\infty\left[I_i\left(x_0(\omega),\omega\right), \hat{0}\right] \\ & \le pM_1 + \sum_{i=1}^{k} d_\infty\left[I_i\left(x_0(\omega),\omega\right), \hat{0}\right] := M_0; \end{aligned} \tag{26}$$

it follows that $d_\infty[x_1(t,\omega), x_0(t,\omega)] \overset{J,\mathbb{P}.1}{\le} M_0$. Furthermore, by assumptions (A1)-(A2) and (25), we can find that

$$\begin{aligned} & d_\infty\left[x_n(t,\omega), x_{n-1}(t,\omega)\right] \\ & \overset{J,\mathbb{P}.1}{\le} \int_{t_0}^{t} d_\infty\left[f\left(s, x_{n-1}(s,\omega), \omega\right),\right. \\ & \left. f\left(s, x_{n-2}(s,\omega), \omega\right)\right] ds \\ & + \sum_{i=1}^{k} d_\infty\left[I_i\left(x_{n-1}\left(t_i,\omega\right),\omega\right), I_i\left(x_{n-2}\left(t_i,\omega\right),\omega\right)\right] \\ & \overset{J,\mathbb{P}.1}{\le} L_1 \int_{t_0}^{t} d_\infty\left[x_{n-1}(s,\omega), x_{n-2}(s,\omega)\right] ds \\ & + \sum_{i=1}^{k} L_{2,i} d_\infty\left[x_{n-1}\left(t_i,\omega\right), x_{n-2}\left(t_i,\omega\right)\right] \\ & \overset{J,\mathbb{P}.1}{\le} L_1 \int_{t_0}^{t} \sup_{s\in J} d_\infty\left[x_{n-1}(s,\omega), x_{n-2}(s,\omega)\right] ds \\ & + \sum_{i=1}^{k} L_{2,i} d_\infty\left[x_{n-1}\left(t_i,\omega\right), x_{n-2}\left(t_i,\omega\right)\right] \overset{J,\mathbb{P}.1}{\le} \left(L_1\right. \\ & \left. \cdot \frac{\left(t-t_0\right)^{n-1}}{(n-1)!} + \sum_{i=1}^{k} L_{2,i}\right) D\left[x_{n-1}(\omega), x_{n-2}(\omega)\right], \end{aligned} \tag{27}$$

which implies that

$$\begin{aligned} & D\left[x_n(t,\omega), x_{n-1}(t,\omega)\right] \\ & \overset{J,\mathbb{P}.1}{\le} \left(L_1 \frac{\left(t-t_0\right)^{n-1}}{(n-1)!} + \sum_{i=1}^{k} L_{2,i}\right) \\ & \cdot D\left[x_{n-1}(\omega), x_{n-2}(\omega)\right]. \end{aligned} \tag{28}$$

Now, we need to prove that for all $t \in J$ with $\mathbb{P}.1$ the following inequality holds: for any $n = 1, 2, \ldots$,

$$\begin{aligned} & D\left[x_n(t,\omega), x_{n-1}(t,\omega)\right] \\ & \overset{J,\mathbb{P}.1}{\le} \left(L_1 \frac{\left(t-t_0\right)^{n-1}}{(n-1)!} + \sum_{i=1}^{k} L_{2,i}\right) \\ & \cdot D\left[x_{n-1}(\omega), x_{n-2}(\omega)\right]. \end{aligned} \tag{29}$$

Indeed, inequality (29) holds for $n = 1$. Further, if inequality (29) is true for any $n = m \ge 1$, then using (25) and assumptions (A1)-(A2), we have

$$\begin{aligned} & D\left[x_{m+1}(t,\omega), x_m(t,\omega)\right] \overset{J,\mathbb{P}.1}{\le} \int_{t_0}^{t} \sup_{s\in J} d_\infty \\ & \cdot \left[f\left(s, x_m(s,\omega), \omega\right), f\left(s, x_{m-1}(s,\omega), \omega\right)\right] ds \\ & + \sum_{i=1}^{k} d_\infty\left[I_i\left(x_m\left(t_i,\omega\right),\omega\right), I_i\left(x_{m-1}\left(t_i,\omega\right),\omega\right)\right] \\ & \overset{J,\mathbb{P}.1}{\le} L_1 \int_{t_0}^{t} \sup_{s\in J} d_\infty\left[x_m(s,\omega), x_{m-1}(s,\omega)\right] ds \\ & + \sum_{i=1}^{k} L_{2,i} d_\infty\left[x_m\left(t_i,\omega\right), x_{m-1}\left(t_i,\omega\right)\right] \\ & \overset{J,\mathbb{P}.1}{\le} L_1 \int_{t_0}^{t} \sup_{s\in J} d_\infty\left[x_m(s,\omega), x_{m-1}(s,\omega)\right] ds \\ & + \sum_{i=1}^{k} L_{2,i} d_\infty\left[x_m\left(t_i,\omega\right), x_{m-1}\left(t_i,\omega\right)\right] \overset{J,\mathbb{P}.1}{\le} \left(L_1\right. \\ & \left. \cdot \frac{\left(t-t_0\right)^{m}}{m!} + \sum_{i=1}^{k} L_{2,i}\right) D\left[x_m(\omega), x_{m-1}(\omega)\right]. \end{aligned} \tag{30}$$

Thus, inequality (29) is true for every $t \in J$ with $\mathbb{P}.1$.

Next, we see that $x_0(t,\omega)$ does not depend on t and for the right-side continuity of $x_1(\cdot,\omega)$, one obtains

$$\begin{aligned} & D\left[x_1(t+h,\omega), x_1(t,\omega)\right] \\ & \overset{[t_0,t_0+p),\mathbb{P}.1}{\le} \int_{t}^{t+h} D\left[f\left(s, x_0(s,\omega), \omega\right), \hat{0}\right] ds \\ & + \sum_{i=1}^{k} D\left[I_i\left(x_0\left(t_i+h,\omega\right),\omega\right), I_i\left(x_0\left(t_i,\omega\right),\omega\right)\right]. \end{aligned} \tag{31}$$

From the assumption (A3) and $D[I_i(x_0(t_i+h,\omega),\omega), I_i(x_0(t_i,\omega),\omega)] \to 0$ as $h \to 0^+$ with $\mathbb{P}.1$, we imply that $d_\infty[x_1(t+h,\omega), x_1(t,\omega)] \to 0$ as $h \to 0^+$ with $\mathbb{P}.1$.

For every $n \geq 2$, we deduce that

$$\begin{aligned} &D\left[x_n(t+h,\omega), x_n(t,\omega)\right] \\ &\overset{[t_0,t_0+p),\mathbb{P}.1}{\leq} \int_t^{t+h} \Bigg(D\left[f\left(s, x_0(s,\omega),\omega\right), \hat{0}\right] \\ &+ \sum_{q=1}^{n-1} D\left[f\left(s, x_q(s,\omega),\omega\right), f\left(s, x_{q-1}(s,\omega),\omega\right)\right]\Bigg) ds \\ &+ \sum_{i=1}^{k} D\left[I_i\left(x_0\left(t_i+h,\omega\right),\omega\right), I_i\left(x_0\left(t_i,\omega\right),\omega\right)\right]. \end{aligned} \tag{32}$$

Using inequality (29) and assumption (A3), we get

$$D\left[x_n(t+h,\omega), x_n(t,\omega)\right] \longrightarrow 0 \quad \text{as } h \longrightarrow 0^+ \text{ with } \mathbb{P}.1. \tag{33}$$

Similar for the left-side continuity, we have $d_\infty[x_n(t-h,\omega), x_n(t,\omega)] \to 0$ as $h \to 0^+$. Hence the functions $x_n(\cdot,\omega)$, $n \geq 2$, are continuous with $\mathbb{P}.1$.

For $n \in \mathbb{N}$ and $t \in J$ the function $x_n(t,\cdot)$ defined by (25) is fuzzy random variable. Indeed, $[x_n(\cdot)]^\alpha$ is measurable multifunction for every $\alpha \in [0,1]$; it remains to show the same for the mapping $\omega \mapsto [\int_{t_0}^t f(s, x_{n-1}(s,\omega),\omega)ds + \sum_{i=1}^k I_i(x_{n-1}(t_i,\omega),\omega)]^\alpha$ which is a measurable multifunction with every $\alpha \in [0,1]$, $n \in \mathbb{N}$, and $t \in J$. Let $\alpha \in [0,1]$ be fixed. By virtue of the definition of fuzzy integral and theorem of Nguyen [23] we obtain

$$\begin{aligned} &\left[\int_{t_0}^t f\left(s, x_{n-1}(s,\omega),\omega\right) ds + \sum_{i=1}^k I_i\left(x_{n-1}\left(t_i,\omega\right),\omega\right)\right]^\alpha \\ &= \int_{t_0}^t f\left(s, \left[x_{n-1}(s,\omega)\right]^\alpha, \omega\right) ds \\ &+ \sum_{i=1}^k I_i\left(\left[x_{n-1}\left(t_i,\omega\right)\right]^\alpha, \omega\right). \end{aligned} \tag{34}$$

As the integrand is a multifunction continuous in s and measurable in ω, with any $t \in J$, the mapping

$$\begin{aligned} \omega \longmapsto &\int_{t_0}^t f\left(s, \left[x_{n-1}(s,\omega)\right]^\alpha, \omega\right) ds \\ &+ \sum_{i=1}^k I_i\left(\left[x_{n-1}\left(t_i,\omega\right)\right]^\alpha, \omega\right) \end{aligned} \tag{35}$$

is a measurable multifunction for $n \in \mathbb{N}$. Therefore, for every $t \in J$, the sequence $\{x_n(t,\cdot)\}$ is a sequence of fuzzy random variable. Consequently, $\{x_n(t,\omega)\}$ is a sequence of fuzzy stochastic process.

In the sequel, for any $n \in \mathbb{N}$, we shall prove that the sequence $\{x_n(t,\omega)\}$ is a Cauchy sequence uniformly on the variable t with $\mathbb{P}.1$ and then $\{x_n(\cdot,\omega)\}$ is uniformly convergent with $\mathbb{P}.1$.

For any $n \in \mathbb{N}$ and by inequality (29), we obtain

$$\begin{aligned} D\left[x_{n+1}(t,\omega), x_n(t,\omega)\right] &\overset{J,\mathbb{P}.1}{\leq} MD\left[x_n(\omega), x_{n-1}(\omega)\right] \\ &\overset{J,\mathbb{P}.1}{\leq} M^n D\left[x_1(\omega), x_0(\omega)\right]. \end{aligned} \tag{36}$$

Notice now that, for every $m > n > 0$, we have

$$\begin{aligned} &D\left[x_m(t,\omega), x_n(t,\omega)\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \sum_{l=n}^{m-1} D\left[x_{l+1}(t,\omega), x_l(t,\omega)\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \left(M^n + M^{n+1} + \cdots + M^{m-1}\right) D\left[x_1(\omega), x_0(\omega)\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \frac{M^n}{1-M} D\left[x_1(\omega), x_0(\omega)\right]. \end{aligned} \tag{37}$$

For $m > n > 0$ large enough, it follows from the above inequalities with $M < 1$ that

$$D\left[x_m(t,\omega), x_n(t,\omega)\right] \overset{\mathbb{P}.1}{\longrightarrow} 0. \tag{38}$$

Since (E^d, D) is a complete metric space and (38) holds, then $D[x_n(t,\omega), x(t,\omega)] \overset{\mathbb{P}.1}{\longrightarrow} 0$, which means that there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega) = 1$ and for every $\omega \in \Omega_0$ the sequence $\{x_n(\cdot,\omega)\}$ is uniformly convergent.

In the following, we shall show that $x(t,\omega)$ is solution of the random impulsive fuzzy integral equation (8). Let $n \in \mathbb{N}$. Observe that

$$\begin{aligned} &D\left[\int_{t_0}^t f\left(s, x_{n-1}(s,\omega),\omega\right) ds, \int_{t_0}^t f\left(s, x(s,\omega),\omega\right) ds\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \int_{t_0}^t D\left[f\left(s, x_{n-1}(s,\omega),\omega\right), f\left(s, x(s,\omega),\omega\right)\right] ds \\ &\overset{J,\mathbb{P}.1}{\leq} L_1 \int_{t_0}^t D\left[x_{n-1}(s,\omega), x(s,\omega)\right] ds. \end{aligned} \tag{39}$$

Since the sequence $x_n(t,\omega)$ converges uniformly to $x(t,\omega)$ on the variable $t \in J$ with $\mathbb{P}.1$ as $n \to +\infty$, Thus for any $\varepsilon > 0$ there is $n_0 > 0$ large enough such that, for all $n > n_0$, we derive

$$\begin{aligned} &D\left[x_{n-1}(t,\omega), x(t,\omega)\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \min\left\{\frac{(n-1)!}{L_1 p^{n-1}}\varepsilon, \left(\sum_{i=1}^k L_{2,i}\right)^{-1}\varepsilon\right\}. \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} &D\left[\int_{t_0}^t f\left(s, x_{n-1}(s,\omega),\omega\right) ds, \int_{t_0}^t f\left(s, x(s,\omega),\omega\right) ds\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \varepsilon, \\ &D\left[\sum_{i=1}^k I_i\left(x_{n-1}\left(t_i,\omega\right),\omega\right), \sum_{i=1}^k I_i\left(x\left(t_i,\omega\right),\omega\right)\right] \\ &\overset{J,\mathbb{P}.1}{\leq} \sum_{i=1}^k L_{2,i} D\left[x_{n-1}\left(t_i,\omega\right), x\left(t_i,\omega\right)\right] \overset{J,\mathbb{P}.1}{\leq} \varepsilon. \end{aligned} \tag{41}$$

On the other hand, we have

$$\begin{aligned}
&D\left[x(t,\omega), x_0(\omega)+\int_{t_0}^{t} f(s,x(s,\omega),\omega)\,ds\right.\\
&\quad\left.+\sum_{i=1}^{k} I_i(x(t_i,\omega),\omega)\right] \overset{J,\mathbb{P}.1}{\leq} D[x(t,\omega), x_n(t,\omega)]\\
&\quad+d_\infty\left[x_n(t,\omega), x_0(\omega)\right.\\
&\quad+\int_{t_0}^{t} f(s,x_{n-1}(s,\omega),\omega)\,ds\\
&\quad\left.+\sum_{i=1}^{k} I_i(x_{n-1}(t_i,\omega),\omega)\right]\\
&\quad+D\left[\int_{t_0}^{t} f(s,x_{n-1}(s,\omega),\omega)\,ds,\right.\\
&\quad\left.\int_{t_0}^{t} f(s,x(s,\omega),\omega)\,ds\right]+D\left[\sum_{i=1}^{k} I_i(x_{n-1}(t_i,\omega),\omega),\right.\\
&\quad\left.\sum_{i=1}^{k} I_i(x(t_i,\omega),\omega)\right].
\end{aligned} \tag{42}$$

Thus, in view of the convergence of the two previous equations and (41), one obtains that

$$\begin{aligned}
&D\left[x(t,\omega), x_0(\omega)+\int_{t_0}^{t} f(s,x(s,\omega),\omega)\,ds\right.\\
&\quad\left.+\sum_{i=1}^{k} I_i(x(t_i,\omega),\omega)\right] \overset{J,\mathbb{P}.1}{=} 0.
\end{aligned} \tag{43}$$

It means the fuzzy stochastic process $x(t,\omega)$ is solution of problem (8).

To prove the uniqueness, let us assume that $x, y : J\times\Omega \to E^d$ are the two continuous fuzzy stochastic processes which are solutions of problem (8). Note that

$$\begin{aligned}
&D[x(t,\omega), y(t,\omega)] \overset{J,\mathbb{P}.1}{=} D\left[\int_{t_0}^{t} f(s,x(s,\omega),\omega)\,ds,\right.\\
&\quad\left.\int_{t_0}^{t} f(s,y(s,\omega),\omega)\,ds\right]+D\left[\sum_{i=1}^{k} I_i(x(t_i,\omega),\omega),\right.\\
&\quad\left.\sum_{i=1}^{k} I_i(y(t_i,\omega),\omega)\right] \overset{J,\mathbb{P}.1}{\leq} \left(\frac{L_1 p^n}{n!}+\sum_{i=1}^{k} L_{2,i}\right) D[x(\omega),\\
&\quad y(\omega)].
\end{aligned} \tag{44}$$

By Lemma 8, we get

$$D[x(t,\omega), y(t,\omega)] \overset{J,\mathbb{P}.1}{\leq} 0. \tag{45}$$

The uniqueness is proved. The proof is complete. □

4. Some of the Properties of Solution of RFDEs with Impulses

Theorem 10. *Suppose that the mappings $f : J\times E^d\times\Omega \to E^d$ and $I_k : E^d\times\Omega \to E^d$ satisfy all the conditions of Theorem 9. Then we have*

$$\begin{aligned}
D[x(t,\omega),\hat{0}] &\overset{J,\mathbb{P}.1}{\leq} \left(D[x_0(\omega),\hat{0}]+(t-t_0)M_1\right)\\
&\quad\cdot\prod_{t_0\leq t_i<t}(1+L_{2,i})\\
&\quad\cdot\exp(L_1(t-t_0)),
\end{aligned} \tag{46}$$

where L_1, $L_{2,i}$ are constants nonnegative for any $i = 0,1,2,3,\ldots$.

Proof. Let $x(t,\omega)$ be solution of problem (8). For every $t \in [t_0,t_1)$ and $\omega\in\Omega$, we have

$$\begin{aligned}
&D[x(t,\omega),\hat{0}]\\
&\quad\leq D[x_0(\omega),\hat{0}]+D\left[\int_{t_0}^{t} f(s,x(s,\omega),\omega)\,ds,\hat{0}\right]\\
&\quad\leq D[x_0(\omega),\hat{0}]+\int_{t_0}^{t} D[f(s,x(s,\omega),\omega),\hat{0}]\,ds\\
&\quad\leq D[x_0(\omega),\hat{0}]+\int_{t_0}^{t} D[f(s,\hat{0},\omega),\hat{0}]\\
&\qquad+D[f(s,x(s,\omega),\omega), f(s,\hat{0},\omega)]\ )\,ds\\
&\quad\leq D[x_0(\omega),\hat{0}]+(t-t_0)D[f(s,\hat{0},\omega),\hat{0}]\\
&\qquad+L_1\int_{t_0}^{t} D[x(s,\omega),\hat{0}]\,ds\\
&\quad\leq D[x_0(\omega),\hat{0}]+(t-t_0)M_1\\
&\qquad+\int_{t_0}^{t} L_1 D[x(s,\omega),\hat{0}]\,ds.
\end{aligned} \tag{47}$$

For every $t\in[t_k,t_{k+1})$, $k = 1,2,3,\ldots$, and $\omega\in\Omega$, we have

$$\begin{aligned}
&D[x(t,\omega),\hat{0}] \leq d_\infty[x_0(\omega),\hat{0}]\\
&\quad+D\left[\int_{t_0}^{t} f(s,x(s,\omega),\omega)\,ds,\hat{0}\right]\\
&\quad+D\left[\sum_{i=1}^{k} I_i(x(t_i,\omega),\omega),\hat{0}\right] \leq D[x_0(\omega),\hat{0}]\\
&\quad+\int_{t_0}^{t}\left(D[f(s,\hat{0},\omega),\hat{0}]\right.\\
&\quad\left.+D[f(s,x(s,\omega),\omega), f(s,\hat{0},\omega)]\right)ds
\end{aligned}$$

$$+\sum_{i=1}^{k} D\left[I_i\left(x\left(t_i,\omega\right),\omega\right),\widehat{0}\right] \le D\left[x_0\left(\omega\right),\widehat{0}\right] + \left(t - t_0\right) M_1 + \int_{t_0}^{t} L_1 D\left[x\left(s,\omega\right),\widehat{0}\right] ds + \sum_{i=1}^{k} L_{2,k} D\left[x\left(t_i,\omega\right),\widehat{0}\right]. \tag{48}$$

If we let $\xi(t,\omega) = D[x(t,\omega),\widehat{0}]$, $t \in [t_k, t_{k+1})$, and $k = 0, 1, 2, 3, \ldots$, then we have

$$\xi\left(t,\omega\right) \overset{\mathbb{P}.1}{\le} \xi_0\left(\omega\right) + \left(t - t_0\right) M_1 + \int_{t_0}^{t} L_1 \xi\left(s,\omega\right) ds + \sum_{i=1}^{k} L_{2,i} \xi\left(t_i,\omega\right). \tag{49}$$

By virtue of Lemma 8, one obtains

$$\xi\left(t,\omega\right) \overset{\mathbb{P}.1}{\le} \left(\xi_0\left(\omega\right) + \left(t - t_0\right) M_1\right) \prod_{t_0 \le t_i < t} \left(1 + L_{2,i}\right) \cdot \exp\left(L_1\left(t - t_0\right)\right). \tag{50}$$

The proof is complete. □

Theorem 11. *Suppose that the mappings $f : J \times E^d \times \Omega \to E^d$ and $I_k : E^d \times \Omega \to E^d$ satisfy all the conditions of Theorem 9. Then we have*

$$D\left[x\left(t,\omega\right), y\left(t,\omega\right)\right] \overset{J,\mathbb{P}.1}{\le} D\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] \cdot \prod_{t_0 \le t_i < t} \left(1 + L_{2,i}\right) \exp\left(L_1\left(t - t_0\right)\right), \tag{51}$$

where L_1, $L_{2,i}$ are constants nonnegative for any $i = 0, 1, 2, 3, \ldots$.

Proof. Let $x(t,\omega)$ and $y(t,\omega)$ be solutions of problem (8). For every $t \in [t_0, t_1)$ and $\omega \in \Omega$, we have

$$\begin{aligned} D\left[x\left(t,\omega\right), y\left(t,\omega\right)\right] &\le d_\infty\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] + D\left[\int_{t_0}^{t} f\left(s, x\left(s,\omega\right),\omega\right) ds, \int_{t_0}^{t} f\left(s, y\left(s,\omega\right),\omega\right) ds\right] \le D\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] \\ &+ \int_{t_0}^{t} D\left[f\left(s, x\left(s,\omega\right),\omega\right), f\left(s, y\left(s,\omega\right),\omega\right)\right] ds \\ &= D\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] + L_1 \int_{t_0}^{t} D\left[x\left(s,\omega\right), y\left(s,\omega\right)\right] ds. \end{aligned} \tag{52}$$

For every $t \in [t_k, t_{k+1})$, $k = 1, 2, 3, \ldots$, and $\omega \in \Omega$, we have

$$\begin{aligned} D\left[x\left(t,\omega\right), y\left(t,\omega\right)\right] &\le d_\infty\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] + D\left[\int_{t_0}^{t} f\left(s, x\left(s,\omega\right),\omega\right) ds, \int_{t_0}^{t} f\left(s, y\left(s,\omega\right),\omega\right) ds\right] + D\left[\sum_{i=1}^{k} I_i\left(x\left(t_i,\omega\right),\omega\right), \sum_{i=1}^{k} I_i\left(y\left(t_i,\omega\right),\omega\right)\right] \le D\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] \\ &+ \int_{t_0}^{t} D\left[f\left(s, x\left(s,\omega\right),\omega\right), f\left(s, y\left(s,\omega\right),\omega\right)\right] ds + \sum_{i=1}^{k} d_\infty\left[I_i\left(x\left(t_i,\omega\right),\omega\right), I_i\left(y\left(t_i,\omega\right),\omega\right)\right] \\ &= D\left[x_0\left(\omega\right), y_0\left(\omega\right)\right] + \int_{t_0}^{t} L_1 D\left[x\left(s,\omega\right), y\left(s,\omega\right)\right] ds + \sum_{i=1}^{k} L_{2,i} D\left[x\left(t_i,\omega\right), y\left(t_i,\omega\right)\right]. \end{aligned} \tag{53}$$

If we let $\xi(t,\omega) = D[x_0(\omega), y_0(\omega)]$, $t \in [t_k, t_{k+1})$, and $k = 0, 1, 2, 3, \ldots$, then we have

$$\xi\left(t,\omega\right) \overset{\mathbb{P}.1}{\le} \xi_0\left(\omega\right) + \int_{t_0}^{t} L_1 \xi\left(s,\omega\right) ds + \sum_{i=1}^{k} L_{2,i} \xi\left(t_i,\omega\right). \tag{54}$$

By virtue of Lemma 8, one obtains

$$\xi\left(t,\omega\right) \overset{\mathbb{P}.1}{\le} \xi_0\left(\omega\right) \prod_{t_0 \le t_i < t} \left(1 + L_{2,i}\right) \exp\left(L_1\left(t - t_0\right)\right). \tag{55}$$

The proof is complete. □

5. Illustrative Examples

In this section, we shall consider two examples. First, we give an example to illustrate the existence and uniqueness results obtained in Section 3. Second, we will find explicit representation of solutions RFDEs with impulses.

Example 1. Let $\Omega = (0,1)$, $\mathscr{F}$-Borel σ-algebra of subsets of Ω, and $\mathbb{P}$-Lebesgue measure on $(\Omega, \mathscr{F})$. Let us consider the problem as follows:

$$D_H x\left(t,\omega\right) \overset{[0,1],\mathbb{P}.1}{=} \frac{\exp\left(-t\right)}{\left(5 + \exp\left(t\right)\right)\left(1 + x\left(t,\omega\right)\right)}, \quad t \ne t_k,$$

$$x\left(t_k^+,\omega\right) \overset{\mathbb{P}.1}{=} \frac{x\left(t_k,\omega\right)}{2+x\left(t_k,\omega\right)}, \quad t=t_k,\ k=0,1,2,\ldots,m,$$
$$x\left(0,\omega\right) \overset{\mathbb{P}.1}{=} \left(-1,0,1,2\right)\omega \in E^1, \tag{56}$$

where $x:[0,1]\times\Omega\to E^1$ is a fuzzy stochastic process.

Set

$$f\left(t,x\left(t,\omega\right),\omega\right) = \frac{\exp\left(-t\right)}{\left(5+\exp\left(t\right)\right)\left(1+x\left(t,\omega\right)\right)}, \quad \text{for every } t\in[0,1],\ t\neq t_k,\ k=0,1,2,\ldots,m,$$
$$I_k\left(x_k\left(t,\omega\right),\omega\right) = \frac{x\left(t_k,\omega\right)}{2+x\left(t_k,\omega\right)}, \quad \text{for every } t\in[0,1],\ t=t_k,\ k=0,1,2,\ldots,m. \tag{57}$$

For every $t\in[0,1]$, $t\neq t_k$, $k=0,1,2,\ldots,m$, we have

$$\begin{aligned}
&d_\infty\left[f\left(t,x\left(t,\omega\right),\omega\right),f\left(t,y\left(t,\omega\right),\omega\right)\right]\\
&\quad= d_\infty\left[\frac{\exp\left(-t\right)}{\left(5+\exp\left(t\right)\right)\left(1+x\left(t,\omega\right)\right)},\right.\\
&\quad\left.\frac{\exp\left(-t\right)}{\left(5+\exp\left(t\right)\right)\left(1+y\left(t,\omega\right)\right)}\right] \overset{[0,1],\mathbb{P}.1}{=} \frac{\exp\left(-t\right)}{5+\exp\left(t\right)}\\
&\quad\cdot d_\infty\left[\frac{1}{1+x\left(t,\omega\right)},\frac{1}{1+y\left(t,\omega\right)}\right]\\
&\quad\overset{[0,1],\mathbb{P}.1}{=} \frac{\exp\left(-t\right)}{5+\exp\left(t\right)}\\
&\quad\cdot \sup_{\alpha\in[0,1]}\max\left\{\left|\frac{1}{1+x_{l\alpha}\left(t,\omega\right)}-\frac{1}{1+y_{l\alpha}\left(t,\omega\right)}\right|,\right.\\
&\quad\left.\left|\frac{1}{1+x_{r\alpha}\left(t,\omega\right)}-\frac{1}{1+y_{r\alpha}\left(t,\omega\right)}\right|\right\}\\
&\quad\overset{[0,1],\mathbb{P}.1}{=} \frac{\exp\left(-t\right)}{5+\exp\left(t\right)}\\
&\quad\cdot \sup_{\alpha\in[0,1]}\max\left\{\left|\frac{x_{l\alpha}\left(t,\omega\right)-y_{l\alpha}\left(t,\omega\right)}{\left(1+x_{l\alpha}\left(t,\omega\right)\right)\left(1+y_{l\alpha}\left(t,\omega\right)\right)}\right|,\right.\\
&\quad\left.\left|\frac{x_{r\alpha}\left(t,\omega\right)-y_{r\alpha}\left(t,\omega\right)}{\left(1+x_{r\alpha}\left(t,\omega\right)\right)\left(1+y_{r\alpha}\left(t,\omega\right)\right)}\right|\right\} \overset{[0,1],\mathbb{P}.1}{\leq} \frac{1}{6}\\
&\quad\cdot \sup_{\alpha\in[0,1]}\max\left\{\left|x_{l\alpha}\left(t,\omega\right)-y_{l\alpha}\left(t,\omega\right)\right|,\right.\\
&\quad\left.\left|x_{r\alpha}\left(t,\omega\right)-y_{r\alpha}\left(t,\omega\right)\right|\right\} \overset{[0,1],\mathbb{P}.1}{\leq} \frac{1}{6}d_\infty\left[x,y\right],
\end{aligned} \tag{58}$$

where $L_1=\sup_{t\in[0,1]}(\exp(-t)/(5+\exp(t)))=1/6$.

Using a similar calculation as above, for every $t\in[0,1]$, $t=t_k$, $k=0,1,2,\ldots,m$, we obtain

$$\begin{aligned}
&d_\infty\left[I_k\left(x_k\left(t,\omega\right),\omega\right),I_k\left(y_k\left(t,\omega\right),\omega\right)\right]\\
&\quad\overset{[0,1],\mathbb{P}.1}{\leq} \frac{1}{2}d_\infty\left[x,y\right], \quad \text{where } L_{2,k}=\frac{1}{2}.
\end{aligned} \tag{59}$$

By a direct calculation, one obtains that

$$\begin{aligned}
&d_\infty\left[\frac{\exp\left(-t\right)}{\left(5+\exp\left(t\right)\right)\left(1+x\left(t,\omega\right)\right)},\widehat{0}\right]\\
&\quad\overset{[0,1],\mathbb{P}.1}{=} \frac{\exp\left(-t\right)}{5+\exp\left(t\right)}d_\infty\left[\frac{1}{1+x\left(t,\omega\right)},\widehat{0}\right]\\
&\quad\overset{[0,1],\mathbb{P}.1}{=} \frac{\exp\left(-t\right)}{5+\exp\left(t\right)}\\
&\quad\cdot \sup_{\alpha\in[0,1]}\max\left\{\left|\frac{1}{1+x_{l\alpha}\left(t,\omega\right)}\right|,\left|\frac{1}{1+x_{r\alpha}\left(t,\omega\right)}\right|\right\}\\
&\quad\overset{[0,1],\mathbb{P}.1}{\leq} \frac{1}{6}
\end{aligned} \tag{60}$$

and for any $n=1,2,\ldots$,

$$\frac{L_1p^n}{n!}+\sum_{i=1}^{k}L_{2,i}=\frac{1}{6^nn!}+\frac{1}{2}<1. \tag{61}$$

We can see that conditions (A1)–(A4) are satisfied. Hence, by Theorem 9, problem (56) has a solution defined on $[0,1]$.

Example 2. Let $\Omega=(0,1)$, $\mathscr{F}$-Borel σ-algebra of subsets of Ω, and $\mathbb{P}$-Lebesgue measure on $(\Omega,\mathscr{F})$. Consider the RFDEs with impulses as follows:

$$\begin{aligned}
&D_Hx\left(t,\omega\right) \overset{[0,T],\mathbb{P}.1}{=} \lambda\left(\omega\right)x\left(t,\omega\right), \quad t\neq t_k,\ k=0,1,2,\ldots,m,\\
&x\left(t_k^+,\omega\right) \overset{\mathbb{P}.1}{=} x_k\left(t_k,\omega\right)+I_k\left(x_k\left(t,\omega\right),\omega\right), \quad t=t_k,\ k=0,1,2,\ldots,m,\\
&x\left(0,\omega\right) \overset{\mathbb{P}.1}{=} x_0\left(\omega\right),
\end{aligned} \tag{62}$$

where $\lambda:\Omega\to\mathbb{R}_+$ is a random variable and $x:[0,1]\times\Omega\to E^1$ is a fuzzy stochastic process. In this example, we suppose that $t\in[0,2]$ and $\lambda(\omega)=1$ with $\mathbb{P}.1$ and for every $\alpha\in[0,1]$

$$\begin{aligned}
&\left[x\left(t,\omega\right)\right]^\alpha=\left[x_{l\alpha}\left(t,\omega\right),x_{r\alpha}\left(t,\omega\right)\right]\\
&\left[x_k\left(t_k,\omega\right)\right]^\alpha=\left[\left(\omega,2\omega,3\omega\right)\right]^\alpha\\
&\quad=\left[\left(1+\alpha\right)\omega,\left(3-\alpha\right)\omega\right], \quad t=k,\ k=1,2,
\end{aligned} \tag{63}$$

and initial conditions $[x_0(\omega)]^\alpha=[(-\omega,0,\omega)]^\alpha=[(\alpha-1)\omega,(1-\alpha)\omega]$, where $x_{l\alpha},x_{r\alpha}:[0,\infty)\times\Omega\to\mathbb{R}$ are the crisp stochastic process.

Problem (62) can translate this into the following system of random differential equation with impulses:

$$\begin{aligned} x'_{l\alpha}(t,\omega) &= x_{l\alpha}(t,\omega), \quad t \in [0,2], \ t \neq t_k, \\ x'_{r\alpha}(t,\omega) &= x_{r\alpha}(t,\omega), \quad t \in [0,2], \ t \neq t_k, \\ x_{l\alpha}(t_k^+,\omega) &= (1+\alpha)\,\omega, \quad t = k, \ k = 1,2, \\ x_{r\alpha}(t_k^+,\omega) &= (3-\alpha)\,\omega, \quad t = k, \ k = 1,2, \\ x_{l\alpha}(0,\omega) &= (\alpha-1)\,\omega, \\ x_{r\alpha}(0,\omega) &= (1-\alpha)\,\omega. \end{aligned} \tag{64}$$

Solving system (64) on $[0,2]$, we obtain

$$\begin{aligned} x_{l\alpha}(t,\omega) &= \begin{cases} (\alpha-1)\,\omega\exp(t), & \text{for } t \in [0,1), \\ (1+\alpha)\,\omega\exp(t-1), & \text{for } t \in [0,2), \end{cases} \\ x_{r\alpha}(t,\omega) &= \begin{cases} (1-\alpha)\,\omega\exp(t), & \text{for } t \in [0,1), \\ (3-\alpha)\,\omega\exp(t-1), & \text{for } t \in [0,2). \end{cases} \end{aligned} \tag{65}$$

It is easy to see that the diameter of solution $x(t,\omega)$ of (62) is an increasing function with $\mathbb{P}.1$ for every $t \in [0,2]$. Hence we infer that the solution $x : [0,2] \times \Omega \to E^1$ to (62) is as follows:

$$\begin{aligned} &x(t,\omega) \\ &= \begin{cases} [(\alpha-1),(1-\alpha)]\,\omega\exp(t), & \text{for } t \in [0,1), \\ [(1+\alpha),(3-\alpha)]\,\omega\exp(t-1), & \text{for } t \in [0,2) \end{cases} \end{aligned} \tag{66}$$

or

$$x(t,\omega) = \begin{cases} (-1,0,1)\,\omega\exp(t), & \text{for } t \in [0,1), \\ (1,2,3)\,\omega\exp(t-1), & \text{for } t \in [0,2). \end{cases} \tag{67}$$

Note that the existence of a unique solution is guaranteed. Therefore, this procedure can be continued to be the solution on each $[m, m+1]$, for every $m \in \mathbb{N}$, $m \geq 3$.

6. Conclusion

Under suitable conditions, we investigated the existence and uniqueness of solutions to random fuzzy differential equation with impulses by using the method of successive approximations. Moreover, we studied some of the properties of solution of RFDEs with impulses. Finally, some examples are given to illustrate the main theorems. In the future, we shall study the class of random fuzzy differential equations in the quotient space of fuzzy numbers, introduced by Qiu et al. in [24].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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Research Article

The Portfolio Balanced Risk Index Model and Analysis of Examples of Large-Scale Infrastructure Project

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This paper focuses on a three-dimensional portfolio balanced risk index model for large-scale infrastructure project risk evaluation as a hot topic of current research. Taking subjectivity utility and complex evaluation motivation into consideration, a method of combinational equilibrium evaluation is built using the index form to reflect whole loss changes of risk. For risk index evaluation and measurement issues, this paper first constructs a risk evaluation index system and three risk coefficients of single factor by questionnaire survey and fuzzy evaluation. Then we calculate the risk index of single factor, which arrives at the classification and combination risk index through AHP method. Finally, we verify the index validity by analysis of examples. With this research we expand the evaluation dimension and provide a new analytical tool for risk monitoring and warning.

1. Introduction

In the near future, major infrastructure project investment will prove to be a pivotal means of improving people's livelihood and promoting a healthy and sustainable development of China's national economy. How to scientifically and reasonably carry on the dynamic evaluation will influence the control, solution, and even the success of the project. Project risk refers to the deviation between the final result and expected subject or the loss due to the existence of uncertain factors. In the joint action of internal and external environment change and multiple subject game, major infrastructure project risk is a complex and evolved dynamic making project risk evaluation more difficult. Project risk measurement methods in the existing literature include variance, standard deviation, the product of probability and loss [1], and average value of personal injury and property losses [2].

Among them, using dynamic, comparative, integrative, and average characteristics, a project risk index is a more intuitive approach reflecting the project risk in numerical form on the basis of unified dimension [3–6]; for example, coast corrosion risk index can be divided into stable, low, medium, and high level according to corrosion rate led by coastal vulnerability and development degree (Guido Benassai, et al., 2015). In order to better conduct evaluation of risk parameters, based on existing fuzzy mathematical models, like fuzzy differential equations, integrodifferential equations, group decision-making [7], interval number [8], and so forth, this paper builds a portfolio balance index model (PBIM) for project risk to reflect the multidimensional combination value losses by combined dynamic equilibrium index and provides analysis tools for early warning and monitoring. The innovation points can be listed as follows: first, it changes the common practice of the individual project subject evaluation and lets all item subjects participate in the project risk evaluation; second, psychological utility factors are added in the project risk severity evaluation, which breaks through the two-dimensional evaluation method of probability multiplying the objective loss; third, group intuitionistic fuzzy evaluation and statistical analysis are used to determine the risk parameters, which can reflect all the will and preferences of the project.

Research ideas and methods of this article are as follows: (1) to filter main risk indicators through typical case analysis, questionnaire investigation, and statistical analysis, (2) to

construct risk evaluation index system and design index weight by analytic hierarchy process, (3) to build a project risk portfolio balance evaluation model and add main psychological effect factors based on traditional two-dimensional evaluation, which can make up the limitations of the evaluation of project risk material loss and at the same time make the project risk assessment have subjective and objective characteristics, and (4) to do questionnaire survey in fuzzy evaluation of project risk parameters. The average data reflects a collection of plural value preference, which can effectively make up disadvantages of single subject evaluation and make project risk assessment into a group of decision-making behaviors. A project risk index can not only make value loss explicit and comparable, but can also provide evidence to analyze and offer reasons for project changes and risk strategy.

2. The Theoretical Basis of Major Infrastructure Project Portfolio Balance Risk Index Model Construction

The risk portfolio balance index model of major infrastructure projects must be based on multidimensional value impact and evaluation of complex decision-making motivation analysis, because they jointly determine game design ideas of model parameters and a combined calculation method of risk index.

2.1. The Subjective and Objective Value of Project Risk Impact Analysis. General project risk evaluation only considers the probability of occurrence and the objective severity; however, project risk bears the characteristics of subjectivity and objectivity. Risk is the result of subjective evaluation wherein even the same project risk has different implications for different subject [9]. As a result, subject factors must be considered in risk assessment. The aim of rational behavior subject is to pursue comprehensive utility maximization [10] instead of simple profit maximization.

Project risk evaluation should not only consider objective value losses, but also measure the damage to the subject itself and others' well-being and satisfaction [11]. Therefore, a three-dimensional evaluation method used in project risk assessment (probability + objective severity + subjective utility loss) is a more scientific approach than the usual two-dimensional evaluation method (probability + objective severity). Objective loss refers to material loss caused by problems to project value such as a decrease of project quality, construction schedule delay, safety performance degradation, profit reduction, and rising costs. Subjective utility cost refers to all types of emotional damage and the subjective judgment value decline led by the risk to project subject. Emotional damage includes psychological fear, anxiety, frustration, discontent, injustice, and impatience.

2.2. The Complex Psychological Motivation Analysis of Evaluation Subject. Behavioral economics describes associated subjects of major infrastructure projects as "complex economic men." They have many preferences including risk aversion, altruism, and fairness preference. When evaluating project risk, they not only consider their own risk losses, but also take the risk of other associated subjects into account. They not only consider the material loss brought on by the risk, but also consider psychology utility loss. A project subject is based on a complex collection of preferences when making risk assessment decisions [12, 13]. In order to validate the above viewpoint, the questionnaire shown in Table 1 investigates the complex evaluation motivation of a project subject.

Issuing 100 questionnaires to related subjects of major infrastructure project in a mobile Internet platform, we had 96 valid questionnaires returned, indicating that 96% of respondents would both consider the social, environmental, and ecological value loss, while only 4% of respondents reported they would only consider the economic loss. Of respondents, 89% stated they would also consider personal emotion, risk capacity, and psychological disutility, and only 11% of the respondents would merely consider material loss. Of respondents, 82% said they would consider the interest and feelings of other related subjects, while only 18% reported they would consider only their own interest loss and subjective feeling. 98% of respondents indicated they would both consider other related subjects' behavior strategy reactions, and only 2% of the respondents would only consider their own behavior strategy. The survey suggests that most related subjects would consider economic, social, environmental, and ecological value loss, nonmaterial psychological utility cost, and behavior strategy reactions of correlation subjects. Therefore, building a framework, which contains factors such as risk probability, objective severity, and psychological utility loss makes the interactive equilibrium evaluation a more reasonable and scientific approach.

3. Construction of a Project Risk Portfolio Balance Evaluation Index System of a Major Infrastructure Project

The construction of an evaluation index system is the key to calculating a project risk index and to screen and sort risk factors within the whole life cycle of a major infrastructure project based on many case studies and questionnaires. The objective is to use the principles of importance, representativeness, and conciseness and to select appropriate main risk factors in order to form a project risk evaluation index system according to the nature of the classification.

(1) Questionnaire and Statistical Analysis of Project Risk Factors Screening. First, the main risk factors based on many case studies throughout the life cycle of major infrastructure projects are listed [14]. Then, the project risk factors list is sent to the related subjects for their additional inclusions or modifications in order to determine the main project risk factors which are established after 2-3 rounds of feedback and changes. Finally, respondents would score the occurrence probability and severity of risk factors by five-mark scoring. For the occurrence probability of risk factors, five-mark scoring ranges are as follows: 1 = *extremely unlikely*, 2 = *slim chance*, 3 = *certain possibility*, 4 = *high possibility*, and 5 = *most*

Table 1: Subject risk evaluation motivation questionnaire of major infrastructure projects.

Answer choices	Survey questions
□ Only consider the former □ Consider all factors	(1) Whether you only consider the economic loss or you consider the social, environmental and ecological value loss when evaluating project risk?
□ Only consider the former □ Consider all factors	(2) Whether you only consider the material loss or you consider the personal emotion, risk capacity and psychological disutility when evaluating project risk?
□ Only consider the former □ Consider both	(3) Whether you consider only your own interest loss and subjective feeling or both the interest loss and feelings of other related subjects when evaluating project risk?
□ Only consider the former □ Consider both	(4) If you do the project risk evaluation, do you prefer to consider your own behavior strategy, or consider other related subject behavior strategy reactions caused by your own evaluation behavior?

likely. For the severity of the risk factors, five-mark scoring ranges as follows: 1 = *mild*, 2 = *milder,* 3 = *generally serious*, 4 = *serious*, and 5 = *very serious*. Using SPSS17.0 software to statistically analyze 96 questionnaires, the mean, median, mode, and standard deviation [15] of all the risk factors' probability and severity evaluation are as shown in Table 2.

(2) The Importance Sequence of Project Risk Factors. According to the results shown in Table 2, we are required to rank the importance of the project risk factors based on the product size of probability and severity average. Results are shown in Table 3.

(3) The Choice and Grouping of Major Project Risk Factors. According to the sequence of risk factors shown in Table 3, we selected the top 50% of risk factors to construct a major infrastructure project risk evaluation index system, shown in Table 4. Owing to the uniqueness of each major infrastructure project, the indicators in Table 4 can be properly adjusted to a specific project risk assessment.

4. Construction of a Combinational Balanced Risk Index Model

The construction idea of a portfolio balanced risk index model is based on clear index connotation and principle. This includes calculating individual risk index through the base model and then calculating the project classification and balanced risk index by using the method of weighted portfolio addition [16].

4.1. Construction Principles of a Combinational Balanced Risk Index

4.1.1. Portfolio Addition Principle. The construction principle of portfolio balanced risk index is as follows: the project overall risk index is composed of six secondary indexes including (1) management risk index, (2) technology risk index, (3) economic risk index, (4) social risk index, (5) legal risk index, and (6) natural risk index; in addition, a secondary index is made up of several three-level indexes. In the process of portfolio addition, expert evaluation method is used to analyze the importance level of the index in order to determine the weight of each index so as to reach the weighted synthesis step by step.

4.1.2. Balance Reflects Risk Value Preference Principle of Multiple Subjects. The portfolio balanced risk index is a comprehensive reflection of the value of multivariate subjects' preferences and interests. Through the questionnaire survey of probability of project risk factors, objective severity and subjective opinions and preferences of all projects can be reflected in the project risk parameters.

4.1.3. Dynamic Comparable Principle. The portfolio balanced risk index reflects the size of the project risk and comparability between different periods of project risk. For example, if the projects' risks index in *t3* is 0.3 and in *t2* is 0.2, then the project risk in *t2* is smaller than *t3*.

4.2. The Basic Model of Portfolio Balanced Risk Index. The portfolio balanced risk index is the function of three variables: risk probability, objective severity, and subject negative feelings as shown in (1). The project risk coefficient value is in the range of 0~1; the greater the value, the greater the project risk is [17, 18].

$$\mathrm{RI}_i = f\left(P_i, V_i, F_i\right) = P_i' \times V_i' \times F_i'. \quad (1)$$

Index definition: RI_i is single factor risk index; P_i' is probability coefficient; V_i' is objective gravity coefficient; F_i' is subjective feeling coefficient; $P_i' = \overline{P}_i/5$ (probability coefficient is equal to probability scoring mean of all the main projects concerning risk i and the ratio of the maximum possible value); $V_i' = \overline{V}_i/5$ (objective gravity coefficient is equal to objective gravity mean of all the main projects concerning risk i and the ratio of the maximum possible value); $F_i' = \overline{F}_i/3$ (subjective feeling coefficient is equal to

Table 2: Survey results of main risk factors through the whole life cycle of a major infrastructure project.

Stages	Sequence number	Risk factors	Occurrence probability				Severity			
			Mean	Median	Mode	Standard deviation	Mean	Median	Mode	S.D
Decision stage	1	Incorrect project orientation	3.188	3.0	3.0	0.734	4.083	4.0	4.0	0.767
	2	Incorrect market demand forecasting	3.958	4.0	3.0	0.824	3.646	4.0	4.0	0.911
	3	Thoughtless about inflation impact	3.083	3.0	3.0	0.739	3.195	3.0	3.0	0.798
	4	Incorrect estimation about project investment	3.25	3.0	3.0	0.526	3.229	3.0	3.0	0.515
	6	Incorrect estimation aboutInvestment return	3.167	3.0	3.0	0.559	3.125	3.0	3.0	0.64
	6	Thoughtless about financing difficulty	3.25	3.0	3.0	0.526	3.083	3.0	3.0	0.739
	7	Government policy changes	3.417	3.0	3.0	0.767	3.5	3.0	3.0	0.744
	8	Lack of external experts consultation	3.167	3.0	3.0	0.695	3.229	3.0	3.0	0.515
	9	Thoughtless about project impact	3.167	3.0	3.0	0.559	3.125	3.0	3.0	0.64
	10	Wrong decision-making process or method	3.688	3.0	3.0	0.829	3.396	3.0	3.0	0.792
Design stage	11	Inability or inexperience of design team	3.167	3.0	3.0	0.559	3.125	3.0	3.0	0.64
	12	Lack of field investigation and not adjusting measures to local conditions	3.25	3.0	3.0	0.526	3.229	3.0	3.0	0.515
	13	Insufficient communication between designer and owner	3.167	3.0	3.0	0.559	3.021	3.0	3.0	0.699
	14	Lack of innovation and applicability of design plan	3.646	3.0	3.0	0.812	3.833	4.0	3.0	0.834
	15	Lack of designers' full participation	3.02	3.0	3.0	0.699	3.229	3.0	3.0	0.515
	16	Instability of safety equipment performance	3.166	3.0	3.0	0.695	3.124	3.0	3.0	0.64
	17	Financing difficulty or rising costs	3.25	3.0	3.0	0.526	3.229	3.0	3.0	0.515
	18	Improved environmental protection requirements on construction site	3.164	3.0	3.0	0.559	3.123	3.0	3.0	0.64
Construction stage	19	Materials and equipment supply not in time	3.25	3.0	3.0	0.526	3.229	3.0	3.0	0.515
	20	Lack of experienced construction personnel	3.25	3.0	3.0	0.526	3.146	3.0	3.0	0.772
	21	Inability or irresponsibility of contractors	3.333	3.0	3.0	0.519	3.271	3.0	3.0	0.536
	22	Inability or irresponsibility of supervisors	3.25	3.0	3.0	0.526	3.229	3.0	3.0	0.515
	23	Nontimely funding	3.833	4.0	3.0	0.883	3.896	4.0	4.0	0.831
	24	Material price increase	3.375	3.0	3.0	0.489	3.292	3.0	3.0	0.504
	26	Lack of scientific construction process and method	3.771	4.0	4.0	0.857	3.833	4.0	3.0	0.808
	26	Legal disputes of related subject	3.374	3.0	3.0	0.489	3.291	3.0	3.0	0.504
	27	Worsening social order of project area	3.333	3.0	3.0	0.519	3.271	3.0	3.0	0.536
	28	Opposition and obstruction to project construction	3.125	3.0	3.0	0.703	3.958	4.0	3.0	0.824
	29	Lack of good communication and cooperation among subjects	3.917	4.0	3.0	0.821	3.75	4.0	3.0	0.863
	30	Bad weather or major natural disasters	2.958	3.0	3.0	0.713	3.875	4.0	3.0	0.841
Operation stage	31	Trial operation effect can not meet the design requirements	3.083	3.0	3.0	0.739	3.125	3.0	3.0	0.64
	32	Instability of equipment performance	3.25	3.0	3.0	0.526	3.229	3.0	3.0	0.515
	33	Speedy technology and equipment renewal	3.373	3.0	3.0	0.489	3.290	3.0	3.0	0.504
	34	Technology and equipment maintenance is not timely	3.333	3.0	3.0	0.519	3.271	3.0	3.0	0.536
	36	Rising operating costs	3.083	3.0	3.0	0.739	3.188	3.0	3.0	0.798
	36	Sudden events	2.979	3.0	3.0	0.699	3.292	3.0	3.0	0.504
	36	Major natural disasters	2.958	3.0	3.0	0.713	3.917	4.0	3.0	0.846
	37	Lack of a clear accountability system	3.371	3.0	3.0	0.489	3.289	3.0	3.0	0.504
	38	Lack of operation management or experiences	3.333	3.0	3.0	0.519	3.271	3.0	3.0	0.536

Table 3: Main risk factors sequence through the whole life cycle of a major infrastructure project.

Importance sequence	Project risk factors	Probability average	Severity average	Probability average × Severity average
1	Incorrect project orientation	3.833	3.896	14.93
4	Incorrect market demand forecasting	3.958	3.646	14.43
34	Thoughtless about inflation impact	3.083	3.195	9.85
20	Incorrect estimation about project investment	3.25	3.229	10.49
29	Incorrect estimation about return on investment (ROI)	3.167	3.125	9.90
28	Thoughtless about financing difficulty	3.25	3.083	10.02
9	Incorrect estimation about investment return	3.417	3.5	11.96
26	Thoughtless about financing difficulty	3.167	3.229	10.23
30	Government policy changes	3.167	3.125	9.90
7	Lack of external experts consultation	3.688	3.396	12.52
31	Thoughtless about project impact	3.167	3.125	9.90
21	Lack of field investigation and not adjusting measures to local conditions	3.25	3.229	10.49
39	Insufficient communication between designer and owner	3.167	3.021	9.57
5	Lack of innovation and applicability of design plan	3.646	3.833	13.98
37	Lack of designers' full participation	3.02	3.229	9.75
32	Instability of safety equipment performance	3.166	3.124	9.89
22	Financing difficulty or rising costs	3.25	3.229	10.49
33	Improved environmental protection requirements on construction site	3.164	3.123	9.87
23	Materials and equipment supply not on time	3.25	3.229	10.49
27	Lack of experienced construction personnel	3.25	3.146	10.22
16	Inability or irresponsibility of contractors	3.333	3.271	10.9
24	Inability or irresponsibility of supervisors	3.25	3.229	10.49
6	Nontimely funding	3.188	4.083	13.02
12	Material price increase	3.375	3.292	11.11
3	Lack scientific construction process and method	3.771	3.833	14.45
13	Legal disputes of related subject	3.374	3.291	11.10
17	Worsening social order of project area	3.333	3.271	10.90
8	Opposition and obstruction to project construction	3.125	3.958	12.37
2	Lack good communication and cooperation among subjects	3.917	3.75	14.69
11	Bad weather or major natural disasters	2.958	3.875	11.46
38	Trial operation effect can not meet the design requirements	3.083	3.125	9.63
25	Instability of equipment performance	3.25	3.229	10.49
14	Speedy technology and equipment renewal	3.373	3.290	11.09
18	Technology and equipment maintenance is not timely	3.333	3.271	10.90
35	Rising operating costs	3.083	3.188	9.83
36	Sudden events	2.979	3.292	9.81
10	Major natural disasters	2.958	3.917	11.59
15	Lack of a clear accountability system	3.371	3.289	11.07
19	Lack of operation management or experiences	3.333	3.271	10.90

Table 4: Portfolio balanced risk index system of a major infrastructure project.

The target layer	The primary Risk	The secondary risk
The combinational balanced risk	Technical risk (TR)	Lack scientific construction process and method (TR_1)
		Poor creativity and applicability of design plan (TR_2)
		Speedy technology and equipment (TR_3)
		Inability or irresponsibility of contractors (TR_4)
		Technology and equipment maintenance is not timely (TR_5)
	Economic risk (ER)	Nontimely funding (ER_1)
		Material price increase (ER_2)
		Incorrect market demand forecasting (ER_3)
		Financing difficulty or rising costs (ER_4)
	Social risk (SR)	Opposition and obstruction to project construction (SR1)
		Government policy changes (SR_2)
		Worsening social order of project area (SR_3)
	Natural risk (NR)	Major natural disasters (NR_1)
		Bad weather (NR_2)
	Management risk (MR)	Lack good communication and cooperation among subjects (MR_1)
		Incorrect project orientation (MR_2)
		Wrong decision-making procedure or method (MR_3)
		Lack of a clear accountability system (MR_4)
	Legal risk (LR)	Contract inadequacy (LR_1)
		Low contracture capability of cooperative enterprise (LR_2)

subjective feeling mean of all the main projects concerning risk i and the ratio of the intermediate value; if the subject of subjective evaluation is over the median 3, the psychological effect is amplified or the project risk is narrowed and vice versa).

The source data of $\overline{P}_i$, $\overline{V}_i$, $\overline{F}_i$ are scoring risk probability, objective severity, and subjective feeling on a related project by five-mark scoring to obtain the scores $\{P_i\}$, $\{V_i\}$, $\{F_i\}$, and then use SPSS software to calculate the mean $\overline{P}_i$, $\overline{V}_i$, $\overline{F}_i$.

The fuzzy evaluation principle is as follows: for risk probability, five-mark scoring ranges from 1 = *extremely unlikely*, 2 = *slim chance*, 3 = *certain possibility*, 4 = *high possibility*, and 5 = *most likely*; for objective severity, five-mark scoring ranges from 1 = *the influence of the objective value of the project can be ignored*, 2 = *slightly*, 3 = *generally serious*, 4 = *serious*, and 5 = *very serious;* for subjective feelings, five-mark scoring ranges from 1 = *psychological negative impact is very small and completely tolerable*, 2 = *psychological negative influence is small and tolerable*, 3 = *appropriate psychological negative influence which can be withstood*, 4 = *psychological negative impact is larger and can barely be afforded*, and 5 = *psychological negative effect is very serious and hard to bear* [19, 20].

4.3. The Calculation of Risk Classification. Risk classification index is calculated by weighted average method of each single index; the formula is in

$$\mathrm{CRI} = \sum_{i=1}^{m} \mathrm{RI}_i \times w_i, \tag{2}$$

where CRI is portfolio balanced project risk index; RI_i is single factor portfolio balanced project risk index; w_i are single factor weights.

4.4. The Calculation of Portfolio Risk Index. The portfolio risk index is calculated by weighted average method of natural, social, legal, economic, management, and technology classification. The calculation formula is in

$$\mathrm{PRI} = \sum_{l=1}^{n} w_l \cdot \sum_{i=1}^{m} \mathrm{RI}_i \times w_i. \tag{3}$$

4.5. Establishment of a Project Risk Index Weight. Based on AHP method, this includes forming a discriminant matrix first by the importance of the comparison between two indicators at the same level and then calculating the index weight. The specific process is as follows.

4.5.1. To Build a Project Risk Hierarchical Structure. The project risk hierarchical structure of risk evaluation index system is shown in Figure 1.

4.5.2. To Build the Discriminant Matrix and Assignment. The importance scales and their meaning are shown in Table 5.

The discriminant matrix, after soliciting opinions from experts, is shown in Table 6.

4.5.3. The Calculation and Test of Weight. Using the sum method to calculate the weight and to get the arithmetic mean

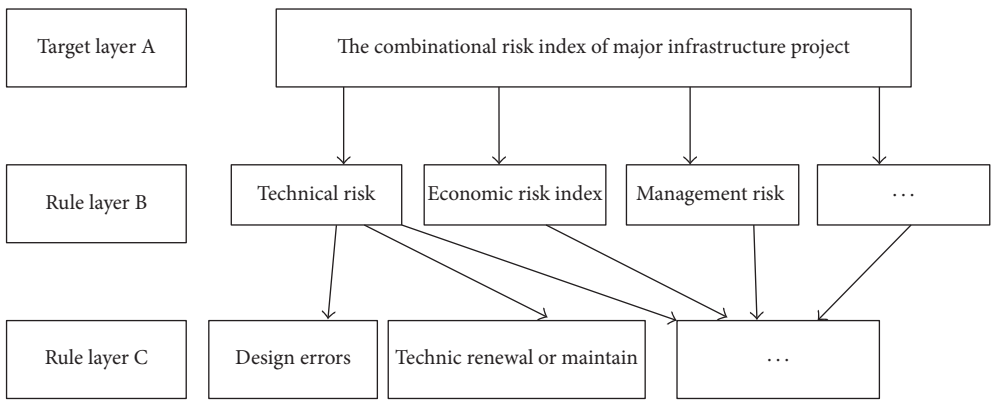

Figure 1: The hierarchy of project risk index.

Table 5: The Importance scale.

Importance scale	Meaning
1	Comparison between two factors, the former is equally as important as the latter
3	Comparison between two factors, the former is slightly more important than the latter
5	Comparison between two factors, the former is obviously more important than the latter
7	Comparison between two factors, the former is strongly more important than the latter
9	Comparison between two factors, the former is extremely more important than the latter
2, 4, 6, 8	Median value of above judgment
Inverse	If the importance percentage of factor I and factor j is a_{ij}, then the importance percentage of factor j and factor I is $1/a_{ij}$

Table 6: The discriminant matrix.

A	B_1	B_2	B_3	B_4	B_5	B_6
B_1	1	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
B_2	a_{21}	1	a_{23}	a_{24}	a_{25}	a_{26}
B_3	a_{31}	a_{32}	1	a_{34}	a_{35}	a_{36}
B_4	a_{41}	a_{42}	a_{43}	1	a_{45}	a_{46}
B_5	a_{51}	a_{52}	a_{53}	a_{54}	1	a_{56}
B_6	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	1

of the column vectors as the final weight is shown in (4). In addition, to limit the deviation of discriminant matrix in a certain range, we needed to undertake a consistency check with CR; when CR < 0.1, the consistency of discriminant matrix is acceptable

$$W_i = \frac{1}{n}\sum_{j=1}^{n}\frac{a_{ij}}{\sum_{k=1}^{n} a_{kl}}. \tag{4}$$

5. Example Analyses

The Hong Kong-Zhuhai-Macau Bridge (HZMB) is an oversized bridge-tunnel project linking Hong Kong, Zhuhai, and Macau, with a total length of 49.968 kilometers and a total investment of 72.9 billion Yuan. It is a world-class sea-crossing passageway of national strategic significance. With project construction projected to be 7 years, construction began in December 2009 and will be completed in 2017. It will be the world's longest six-lane driving immersed tunnel and in distance the world's longest sea-crossing bridge-tunnel road. Next we will evaluate and analyze the HZMB using a combinational balanced risk index model.

5.1. The Main Project Risk Identification in Construction Phase. In addition to the common features every large project generally shares such as large scale, tight construction period, high level of difficulty, and heightened risk, the HZMB also contends with the characteristics of high social attention, coconstruction, and coadministration by three distinct governments and complicated navigation environment constraints such as white dolphin conservation. On the basis of investigation and access to second-hand data, it is concluded that the HZMB construction stages of the main project risks are as follows.

5.1.1. Technical Risk

(1) Risk of Being Poorly Designed. The sea areas of the HZMB are the world's most important trade channel with extensive air- and waterways. The design height of the bridge cannot be too low because of the normal passage of tonnage ships. At the same time, the height of the bridge deck and bridge

tower cannot be too high or it will affect the normal takeoff and landing of planes.

(2) Technology Innovation Risk. The HZMB project is the construction of the world's longest immersed deep-water tunnel, requiring numerous technological innovations. For example, the connection between the bridge and the tunnel requires an artificial island to be constructed using a grouping of giant round steel cylinders fixed directly onto the seabed and then filled with earth in order to form the man-made island. For Chinese engineers this is a first endeavor at creating this type of structure and, therefore, it includes high levels of uncertainty.

5.1.2. Economic Risk

(1) Risk of Nontimely Funding. With an investment of over 70 billion Yuan, the financing of this project has been the subject of much debate including issues such as who will invest and how to allocate investment proportion from the decision-making stage to the time when the bridge officially started. The principal financing risk is whether all involved parties can provide project construction funds at the appropriate time.

(2) Risk of Rising Costs. On one hand, inflation causes a rise in the price of materials; even if it specifies the value adjusting formula in the contract terms, it is hard to fully compensate the loss caused by the rising cost of raw materials in the future. On the other hand, the frequent changes of complicated construction conditions will cause the rise of cost control risk.

5.1.3. Social Risk

(1) Risk of Regional System Differences. The HZMB belongs to the coconstruction and coadministration of three distinct governments and involves the policy of "one country, two systems." The interest orientation of all governments, laws and regulations, administrative systems, management procedures, and technical standard requirements vary thereby creating innumerable challenges and difficulties in coordination efforts.

(2) Public-Against-Project Risks. The project has a significant impact on local natural ecological environment and the lives of the public making it easy to trigger social dissatisfaction and opposition if mishandled.

5.1.4. Natural Risk

(1) Typhoon Risk. Typhoons are common in the Lingdingyang Bay and pass through the South China Sea every year with more than 200 days a year reporting a wind speed of 6 magnitude. Consequently, the wind action will naturally move the steel with the same frequency which can produce resonance and cause destructive effects on the bridge.

(2) Earthquake Risk. Construction of the HZMB faces a serious challenge in the form of an earthquake. It is difficult to predict earthquake risk because of complex seabed structure. An earthquake would cause horizontal and vertical deformation and destruction of the tunnel and differences in movement and rotation in the tunnel socket joints, after which the project would be a total loss.

(3) Chloride Salt Corrosion Risk. Experiments show that the reinforced concrete will rust under the action of chlorine salt corrosion and eventually result in cracking and peeling of the concrete. How to ensure a service life of 120 years for the bridge is uncertain.

5.1.5. Management Risk

(1) Schedule Control Risk. The main body construction began in December 2009 and was projected to be completed by the end of 2016. However, whether the project can be completed smoothly has become a great challenge due to hydrological and meteorological factors as well as less effective working days.

(2) Quality Management Risk. The construction project is difficult with many operation points, long duration, synchronous operation, and crossover operation processes. The complicated meteorology in Lingdingyang Bay can easily lead to negligence in the quality of management.

(3) Safety Management Risk. The construction environment is very poor due to many factors including a large tidal range, quick water flow, various flow directions, high waves, deep scour, thick, soft ground, and frequent typhoons that endanger the safety of the workers and the construction creating an environment where injuries and property losses are probable.

5.1.6. Legal Risk. The nature of the project attracts an international financial clique desiring to invest in the form of BOT. However, in view of the different legal systems of Mainland China, Hong Kong, and Macao, this may involve some legal conflict and blind areas. If legal blind areas are used by financial clique and some funds are reserved in the contract, the bridge construction may fall into endless legal disputes.

5.2. The Construction of Project Risk Evaluation Index System in Construction Stage. Based on the above project risks, to construct a project risk evaluation index system according to the AHP method, as shown in Table 7, some appropriate adjustments of indexes are made in line with the project.

5.3. Calculation of Portfolio Balanced Risk Index in Initial Construction Stage

5.3.1. Probability and Severity Investigation of Project Risk Factors. Invite 30 related subjects from the project construction unit, as well as investors and government departments in order to score the probability, severity, objectivity, and subjectivity of single risk factor in the HZMB construction stage, and then calculate the average by SPSS17.0.

Table 7: Risk evaluation index system in construction stage.

Primary index	Secondary index	Third-grade index
The CTR of the HZMB in construction stage	Technical risk (R_1)	Risk of being poorly designed (R_{11})
		Innovation risk (R_{12})
	Economic risk (R_2)	Risk of nontimely funding (R_{21})
		The risk of rising costs (R_{22})
	Social risk (R_3)	The risk of regional system differences (R_{31})
		The public against risks (R_{32})
	Nature risk (R_4)	Typhoon risk (R_{41})
		Earthquake risk (R_{42})
		Chloride salt corrosion risk (R_{43})
	Management risk (R_5)	Schedule control risk (R_{51})
		Quality management risk (R_{52})
		Safety management risk (R_{53})
	Legal risk (R_6)	Legal conflict or blind area risk (R_{61})

Table 8: Three-dimensional evaluation data of single risk factor in the HZMB initial construction stage.

Stage	Sequence	Risk factors	Average value of probability	Average value of object severity	Average value of subject feeling
Initial construction stage	1	Risk of being poorly designed (R_{11})	2.17	3.65	3.42
	2	Innovation risk (R_{12})	2.52	3.48	3.07
	3	Risk of nontimely funding (R_{21})	2.08	4.16	3.66
	4	The risk of rising costs (R_{22})	3.21	3.23	2.77
	5	The risk of regional system differences (R_{31})	2.78	2.35	2.47
	6	The public against risks (R_{32})	1.56	3.72	3.98
	7	Typhoon risk (R_{41})	3.45	4.03	3.86
	8	Earthquake risk (R_{42})	1.06	4.75	4.67
	9	Chloride salt corrosion risk (R_{43})	2.72	3.25	3.09
	10	Schedule control risk (R_{51})	2.91	2.76	2.92
	11	Quality management risk (R_{52})	1.85	4.49	4.33
	12	Safety management risk (R_{53})	2.04	4.16	4.08
	13	Legal conflict or blind area risk (R_{61})	2.11	3.53	3.16

5.3.2. Calculation of Single Risk Factor Parameter. Calculate, respectively, probability, objective severity, and subjective feeling coefficient of a single factor according to the survey results in Table 8 and the equation in Section 4.2, as shown in Table 9.

5.3.3. Calculation of Classification and Portfolio Risk Index. Use weighted addition to obtain project risk classification index according to single risk factor index shown in Table 10 and obtain portfolio risk index in the same way; then calculate index weight by AHP method; the final results are shown in Table 10.

5.4. The Calculation of Risk Index in Medium-Term Construction Stage. Calculate the classification and portfolio risk index of the HZMB in medium-term according to the above-mentioned method to reorganize investigation and collect basic data, as shown in Table 11.

5.5. Comparative Analysis of Project Risk Index in Early- and Mid-Construction. The portfolio balanced project risk index in early 2010 and middle 2012 is obtained through the above-mentioned calculation, as shown in Table 12. It is clear that the portfolio balanced project risk index fall is very obvious, and technology risk index and management risk index decrease dramatically, which is in accord with our intuitive understanding.

The combinational balanced risk index of this project in 2010 was 0.38, showing if all kinds of risk factors were not well controlled or changed; about 38% expected value will

Table 9: Single risk factor parameter and coefficient of the HZMB in initial construction stage.

Stages	Sequence	Risk factor	Probability coefficient	Objective severity coefficient	Subjective feeling coefficient	Single risk factor parameter
Initial construction stage	1	Risk of being poorly designed (R_{11})	0.43	0.73	1.14	0.36
	2	Innovation risk (R_{12})	0.50	0.70	1.02	0.36
	3	Risk of nontimely funding(R_{21})	0.42	0.83	1.22	0.43
	4	The risk of rising costs (R_{22})	0.64	0.65	0.92	0.38
	5	The risk of regional system differences (R_{31})	0.56	0.47	0.82	0.22
	6	The public against risks (R_{32})	0.31	0.74	1.33	0.31
	7	Typhoon risk (R_{41})	0.69	0.81	1.29	0.72
	8	Earthquake risk (R_{42})	0.21	0.95	1.56	0.31
	9	Chloride salt corrosion risk (R_{43})	0.54	0.65	1.03	0.36
	10	Schedule control risk (R_{51})	0.58	0.55	0.97	0.31
	11	Quality management risk (R_{52})	0.37	0.90	1.44	0.48
	12	Safety management risk (R_{53})	0.41	0.83	1.36	0.46
	13	Legal conflict or blind area risk (R_{61})	0.42	0.71	1.05	0.31

Table 10: The classification and portfolio risk index of the HZMB in initial construction stage.

Primary index	Secondary index			Third-grade index		
	Risk name	Index	Weight	Risk name	Index	Weight
Portfolio risk index of the HZMB in initial construction stage PRI = 0.38	Technical risk	0.36	0.212	Risk of being poorly designed (R_{11})	0.36	0.667
				Innovation risk (R_{12})	0.36	0.333
	Economic risk	0.41	0.137	Risk of nontimely funding (R_{21})	0.43	0.667
				The risk of rising costs (R_{22})	0.38	0.333
	Social risk	0.27	0.162	The risk of regional system differences (R_{31})	0.22	0.50
				The public against risks (R_{32})	0.31	0.50
	Nature risk	0.48	0.116	Typhoon risk (R_{41})	0.72	0.387
				Earthquake risk (R_{42})	0.31	0.412
				Chloride salt corrosion risk (R_{43})	0.36	0.201
	Management risk	0.44	0.265	Schedule control risk (R_{51})	0.31	0.227
				Quality management risk (R_{52})	0.48	0.538
				Safety management risk (R_{53})	0.46	0.235
	Legal risk	0.31	0.108	Legal conflict or blind area risk (R_{61})	0.31	1.00

be lost after their interaction. The combinational balanced risk index in 2012 was 0.15, showing only about 15% projects could not achieve expected value. The main reason for these dramatic declines is that related subjects accumulate substantial knowledge and experience and significantly improve the knowledge level and behavior ability during the construction process. In addition, after a running-in period, the cooperation relationship between subjects is effectively improved. Moreover, the subjects actively explore and innovate in areas such as technology research and development, plan design, government cooperation mechanism, and international BOT financing and formulate a series of effective countermeasures such as (1) largely eliminating and weakening the force between the seismic energy by using polymer rubber materials; (2) developing high-performance concrete to resist the erosion from chlorine salt on concrete in sea water; (3) devising a creative installation method to ensure that the crane tower height is less than 120 meters; (4) establishing a coordination team led by the National Development and Reform Commission to eliminate organizational difficulties

Table 11: Classification and portfolio risk index of the HZMB in medium-term construction stage.

Primary index	Secondary index			Third-grade index		
	Risk name	Index	Weight	Risk name	Index	Weight
Portfolio risk index of the HZMB in medium-term construction stage PRI = 0.15	Technical risk	0.17	0.212	Risk of being poorly designed (R_{11})	0.18	0.667
				Innovation risk (R_{12})	0.14	0.333
	Economic risk	0.30	0.137	Risk of nontimely funding (R_{21})	0.28	0.667
				The risk of rising costs (R_{22})	0.35	0.333
	Social risk	0.15	0.162	The risk of regional system differences (R_{31})	0.12	0.50
				The public against risks (R_{32})	0.18	0.50
	Nature risk	0.32	0.116	Typhoon risk (R_{41})	0.48	0.387
				Earthquake risk (R_{42})	0.25	0.412
				Chloride salt corrosion risk (R_{43})	0.14	0.201
	Management risk	0.20	0.265	Schedule control risk (R_{51})	0.29	0.227
				Quality management risk (R_{52})	0.18	0.538
				Safety management risk (R_{53})	0.16	0.235
	Legal risk	0.18	0.108	Legal conflict or blind area risk (R_{61})	0.18	1.00

Table 12: The portfolio balanced risk comparison of the HZMB in initial and medium-term construction stage.

	PR						
T	Technical risk index	Economic risk index	Social risk index	Nature risk index	Management risk index	Law risk index	Portfolio risk index
2010	0.36	0.41	0.27	0.48	0.44	0.31	0.38
2012	0.17	0.30	0.15	0.32	0.20	0.18	0.15

arising from three governments and the risks brought on by varying legal demands and management systems; and (5) inviting lawyers familiar with international BOT legal business to study the contract details, risk control, and so on.

5.6. Early Warning Analysis of Project Risk Index. In order to dynamically monitor and analyze early warning project risks, we can set the risk index threshold through the investigation of the risk bearing capacity and the degree of acceptance of the risk by the subjects, combined with the risk loss, and divide different levels of risk early warning intervals and set up corresponding risk countermeasures [21] as shown in Table 13, in order to ensure the appropriate measures be initiated according to the level of risk.

This study shows that the portfolio balanced risk index of the HZMB reaches above the orange line in initial construction stage, while it drops below the orange line and enters a relative safety area in the median-term construction.

6. Conclusion and Discussion

In this article, a combination of behavioral science, questionnaire method, statistical analysis, and fuzzy evaluation is used to construct a portfolio balanced index model in order to dynamically evaluate the risk factors of major infrastructure projects and to measure the combined loss of project risk to project subjects. PBIM is an effective and powerful tool for risk evaluation and monitoring of major infrastructure projects. Our conclusions are as follows.

(1) From the perspective of project entity utility, the risk of major infrastructure projects is not only related to the probability of occurrence of project risks, loss of objective value caused by risks, but also to the risk bearing capacity, emotional factors, and psychological utility of the project subjects. These factors need to be systematically balanced, considered, and measured in combination so as to fully evaluate the overall value loss of the project risk.

(2) A project risk index constructed on the basis of portfolio balanced evaluation has strong inclusiveness. This is accomplished through questionnaires and scenario investigation of the multiproject related subjects, the selection of project risk index and the design of relevant parameters reflecting the collective value preference of multiproject subjects, multiple psychological utility, and behavioral strategy interaction factors, and eliminating the limitations of a single subject closed evaluation of project risk.

Table 13: Early warning level of project risk index.

Portfolio balanced risk index	Qualitative description of the project risk severity	Early warning level and coping strategy
Above 0.40	The risk is very serious and the overall value may result in significant loss	Red warning interval, first-grade powerful measures are taken to control and resolve risk
[0.30, 0.40]	The risk is comparatively serious and the overall value may result in great loss	Orange warning interval, secondary measures are taken to control and resolve risk
[0.20, 0.30]	The risk is generally serious and the overall value may result in great loss	Yellow warning interval, three-grade measures are taken to control and resolve risks
[0.10, 0.20]	The risk is comparatively mild and the chance that the overall value deviating from the expected goal is minimal	A relatively safe interval, analyze the cause of risks and verify risk control measures
Below 0.1	The risk is comparatively mild and the chance of the overall value deviating from the expected goal is very minimal	Safe interval, analyze the cause of risks

(3) The combined equilibrium risk index is simple and intuitive for reflecting the size of the project risk, which can directly compare the risks of different projects and the same project in different periods, not only to determine the relevance of the main project feasibility and the size of the potential risk in order to provide an effective analysis tool, but also, according to the Early Warning Interval of Project risk index, to help the project in relation to the main control and resolving of risk.

(4) The main body of the major infrastructure project has complex risk evaluation decision motive; this includes avoidance of their own risk for self-interest motive needs, but also an interactive fairness and altruism motive, whereby the motivation to evaluate project risk is a complex preference set. This preference set affects the evaluation decision of the subject. For this reason, the preference set of the multiple project subjects can be displayed by means of group survey.

It should be noted that we principally used fuzzy mathematics and the questionnaire method to evaluate the risk factors of major infrastructure projects, and these methods have certain imprecision and subjectivity. However, this procedure is consistent with the characteristics of major infrastructure project risk and behavior decision-making and is also a scientific approach. In the next study, we will shorten the observation time for specific projects, extract more comprehensive data, and conduct a more detailed study of the evolution of a project risk index.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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