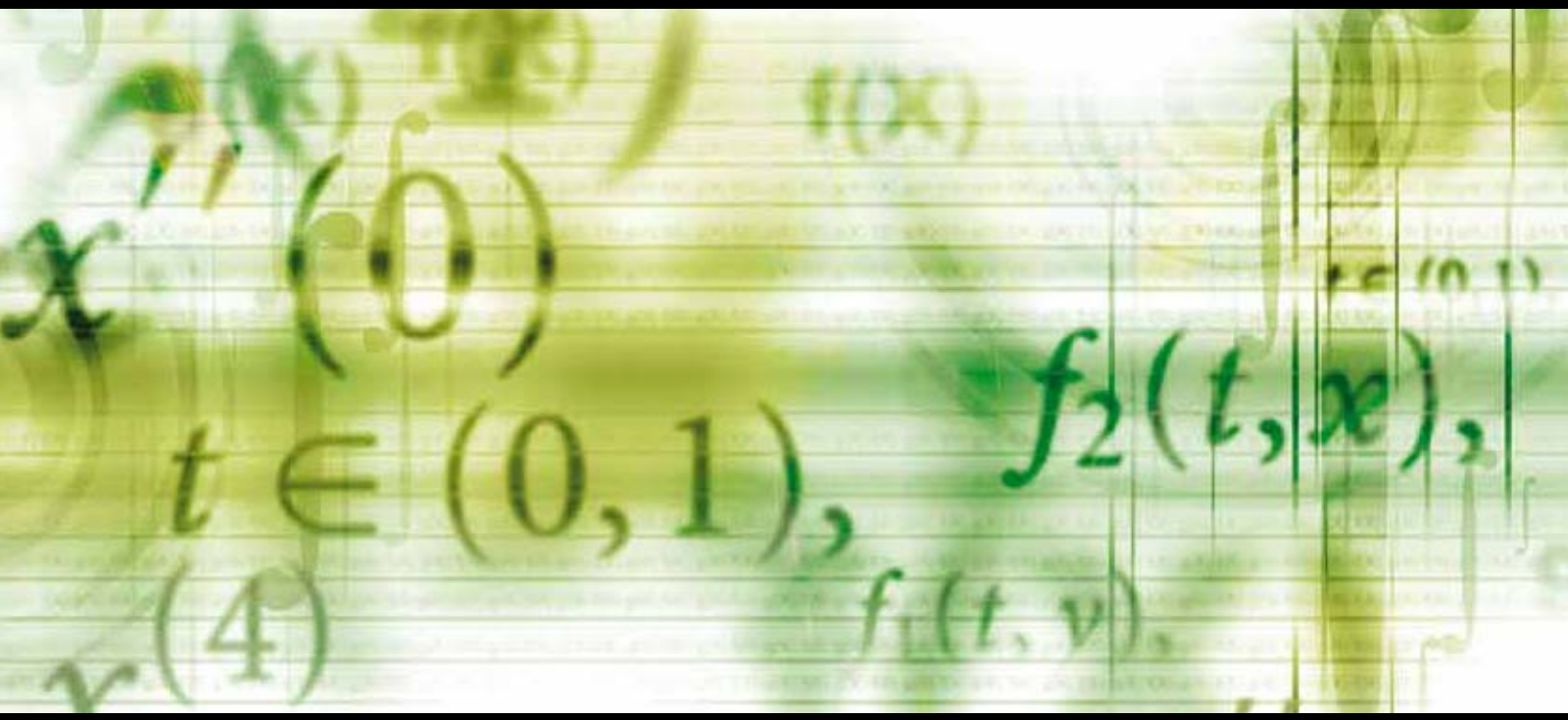
International Journal of Differential Equations

Fractional Differential Equations 2011

Guest Editors: Fawang Liu, Om P. Agrawal, Shaher Momani, Nikolai N. Leonenko, and Wen Chen



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Editorial

Fractional Differential Equations 2011

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It is our pleasure to bring this special issue of the International Journal of Differential Equations dedicated to fractional differential equations (FDEs).

In recent years, a growing number of work by many authors from various fields of science and engineering deal with dynamical systems described by fractional partial differential equations. Due to the extensive applications of FDEs in engineering and science, research in this area has grown significantly all around the world.

This special issue of Fractional Differential Equations consists of 13 original articles covering various aspects of FDEs and their applications by some of the prominent researchers in the field. These papers could be broadly grouped into three categories, namely, numerical and approximate schemes to solve fractional dynamical models (1st to 5th paper), existence and uniqueness of solutions of fractional differential equations and other theoretical results (6th to 11th paper), and application of fractional differential equations in various fields (12th and 13th paper). Other papers could also have been considered in this last category. However, because of their emphasis, they have been included in the first two categories.

The first paper introduces a new modified step variational iteration method for solving biochemical reaction model. The second and third papers use homotopy analysis method for solving a space- and time-fractional foam drainage equation and nonlinear coupled equations with parameters derivative, respectively. The fourth paper develops a new application of Mittag-Leffler function method and extends the application of the method to fractional linear differential equations. The fifth paper proposes an explicit numerical method for the

fractional Cable equation together with a stability and convergence analysis of the numerical method by means of a kind of von Neumann method.

The sixth paper investigates Malliavin calculus of Bismut type for fractional powers of Laplacians in semigroup theory. The seventh paper derives a sufficient condition on asymptotical stability of nonlinear fractional differential system with Caputo derivative. The eighth paper studies existence and uniqueness theorem of fractional mixed Volterra-Fredholm integro-differential equation with integral boundary conditions. The ninth paper proves the existence of positive solution for fractional differential equation with nonlocal boundary consider. The tenth paper gives existence of solutions for a nonlinear fractional multipoint boundary value problem at resonance. The eleventh paper proves the uniqueness of the Gellerstedt problem by energy integral method and the existence by reducing it to the ordinary differential equations.

The twelfth paper studies slip effects on fractional viscoelastic fluids. The final paper addresses antisynchronization phenomena in nonidentical fractional-order chaotic systems using active control.

Thus, this special issue provides a wide spectrum of current research in the area of FDEs, and we hope that experts in this and related fields would find it useful.

Fawang Liu
Om P. Agrawal
Shaher Momani
Nikolai N. Leonenko
Wen Chen

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Research Article

Modified Step Variational Iteration Method for Solving Fractional Biochemical Reaction Model

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A new method called the modification of step variational iteration method (MoSVIM) is introduced and used to solve the fractional biochemical reaction model. The MoSVIM uses general Lagrange multipliers for construction of the correction functional for the problems, and it runs by step approach, which is to divide the interval into subintervals with time step, and the solutions are obtained at each subinterval as well adopting a nonzero auxiliary parameter $\hbar$ to control the convergence region of series' solutions. The MoSVIM yields an analytical solution of a rapidly convergent infinite power series with easily computable terms and produces a good approximate solution on enlarged intervals for solving the fractional biochemical reaction model. The accuracy of the results obtained is in a excellent agreement with the Adam Bashforth Moulton method (ABMM).

1. Introduction

The mathematical modelling of numerous phenomena in various areas of science and engineering using fractional derivatives naturally leads, in most cases, to what is called fractional differential equations (FDEs). Although the fractional calculus has a long history and has been applied in various fields in real life, the interest in the study of FDEs and their applications has attracted the attention of many researchers and scientific societies beginning only in the last three decades [1, 2]. Since the exact solutions of most of the FDEs cannot be found easily, thus analytical and numerical methods must be used. For example,

the ABMM is one of the most used methods to solve fractional differential equations [3–5]. Several of the other numerical analytical methods for solving fractional problems are the Adomian decomposition method (ADM), the homotopy perturbation method (HPM) and the homotopy analysis method (HAM). For example, Ray [6] and Abdulaziz et al. [7] used ADM to solve fractional diffusion equations and solve linear and nonlinear fractional differential equations, respectively. Hosseinnia et al. [8] presented an enhanced HPM to obtain an approximate solution of FDEs, and Abdulaziz et al. [9] extended the application of HPM to systems of FDEs. The HAM was applied to fractional KDV-Burgers-Kuromoto equations [10], systems of nonlinear FDEs [11], and fractional Lorenz system [12].

Another powerful method which can also give explicit form for the solution is the variational iteration method (VIM). It was proposed by He [13, 14], and other researchers have applied VIM to solve various problems [15–17]. For example, Song et al. [18] used VIM to obtain approximate solution of the fractional Sharma-Tasso-Olever equations. Yulita Molliq et al. [19, 20] solved fractional Zhakanov-Kuznetsov and fractional heat-and wave-like equations using VIM to obtain the approximate solution have shown the accuracy and efficiently of VIM. Nevertheless, VIM is only valid for short-time interval for solving the fractional system.

In this paper, we propose a modification of VIM to overcome this weakness of VIM. In particular, motivated by the work of [12] the procedure of dividing the time interval of solution in VIM to subintervals with the same step size Δt and the solution at each subinterval must necessary to satisfy the initial condition at each of the subinterval has been considered. Unfortunately, this idea does not give a good approximate solution when compared to the ABMM. Therefore, to obtain a good approximate solution which has a good agreement with ABMM, another idea is used: motivated by HAM, a nonzero auxiliary parameter $\hbar$ is considered into the correction functional in VIM. This parameter $\hbar$ was inserted to adjust and control the convergence region of the series solutions. In general, it is straightforward to choose a proper value of $\hbar$ from the so-called $\hbar$-curve. We call this modification involving time step and auxiliary parameter $\hbar$ the MoSVIM. Strictly speaking MoSVIM is a modification of our earlier proposed method—step variational iteration method—which is still under review [21].

As an application, this paper investigates for the first time the applicability and effectiveness of MoSVIM to obtain the approximate solutions of the fractional version of the biochemical reaction model as studied in [22] for interval $[0, T]$. The fractional biochemical reaction model (shortly called FBRM) is considered in the following form:

$$\begin{aligned} \frac{\mathrm{d}^{\theta} u}{\mathrm{d}t} &= -u + (\beta - \alpha)\, v + uv, \\ \frac{\mathrm{d}^{\theta} v}{\mathrm{d}t} &= \frac{1}{\mu}(u - \beta v - uv), \end{aligned} \tag{1.1}$$

subject to initial conditions

$$u(0) = 1, \qquad v(0) = 0, \tag{1.2}$$

where θ is a parameter describing the order of the fractional derivative $(0 < \theta \leq 1)$, α, β, and μ are dimensionless parameters.

Our objective is to provide an alternative analytical method to achieve the solution and also highlight the limitations of solutions using VIM, MoVIM, and SVIM for solving the fractional biochemical reaction model when compared to ABMM.

2. Basic Definitions

Fractional calculus unifies and generalizes the notions of integer-order differentiation and n-fold integration [1, 2]. We give some basic definitions and properties of fractional calculus theory which will be used in this paper.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space $\mathcal{C}_\mu$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in \mathcal{C}[0, \infty)$, and it is said to be in the space $\mathcal{C}_\mu^q$ if and only if $f^{(q)} \in \mathcal{C}_\mu$, $q \in \mathbf{N}$.

The Riemann-Liouville fractional integral operator is defined as follows.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\theta \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_0^x (x-t)^{\theta-1} f(t)\, \mathrm{d}t, \quad \theta > 0,\ x > 0,$$
$$J^0 f(x) = f(x). \tag{2.1}$$

In this paper only real and positive values of θ will be considered.
Properties of the operator J^θ can be found in [2], and we mention only the following:
For $f \in C_\mu$, $\mu \geq -1, \theta, \eta \geq 0$, and $\gamma \geq -1$,

(1) $J^\theta J^\eta f(x) = J^{\theta+\eta} f(x)$,

(2) $J^\theta J^\eta f(x) = J^\eta J^\theta f(x)$,

(3) $J^\theta x^\gamma = (\Gamma(\gamma+1)/\Gamma(\theta+\gamma+1)) x^{\theta+\gamma}$.

The Reimann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with FDEs. Therefore, we will introduce a modified fractional differential operator D_*^θ proposed by Caputo in his work on the theory of viscoelasticity [23].

Definition 2.3. The fractional derivative of $f(x)$ in Caputo sense is defined as

$$\begin{aligned} D_*^\theta f(x) &= J^{q-\theta} D^q f(x) \\ &= \frac{1}{\Gamma(q-\theta)} \int_0^x (x-\xi)^{q-\theta-1} f^{(q)}(\xi)\mathrm{d}\xi, \end{aligned} \tag{2.2}$$
$$\text{for } q-1 < \theta \leq q,\ q \in \mathbb{N},\ x > 0,\ f \in C_{-1}^q.$$

In addition, we also need the following property.

Lemma 2.4. *If $q-1<\theta\le q$, $q\in\mathbb{N}$, and $f\in C_{\mu}^{q}$, $\mu\ge -1$, then*

$$D_{*}^{\theta}J^{\theta}f(x)=f(x),$$
$$J^{\theta}D_{*}^{\theta}f(x)=f(x)-\sum_{i=0}^{q-1}f^{(i)}(0^{+})\frac{x^{i}}{i!},\quad x>0. \tag{2.3}$$

The Caputo differential derivative is considered here because the initial and boundary conditions can be included in the formulation of the problems [1]. The fractional derivative is taken in the Caputo sense as follows.

Definition 2.5. For m to be the smallest integer that exceeds θ, the Caputo fractional derivative operator of order $\theta>0$ is defined as

$$D_{t}^{\theta}u(t)=\begin{cases}\dfrac{1}{\Gamma(q-\theta)}\displaystyle\int_{0}^{t}(t-\xi)^{q-\theta-1}\frac{\partial^{q}u(\xi)}{\partial\xi^{q}}\mathrm{d}\xi, & \text{for } q-1<\theta<q,\\ \dfrac{\partial^{q}u(t)}{\partial t^{q}}, & \text{for } \theta=q\in\mathbb{N}.\end{cases} \tag{2.4}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult [1, 2].

3. Step Variational Iteration Method

The approximate solutions of fractional biochemical reaction model will be obtained in this paper. A simple way of ensuring validity of the approximations is solving under arbitrary initial conditions. In this case, $[0,T]$ is regarded as interval. From idea of Alomari et al. [12], the $[0,T]$ interval is divided to subintervals with time step Δt, and the solution at each subinterval was obtained. So it is necessary to satisfy the initial condition at each of the subinterval. Thus the step technique can describe as the following formula:

$$u_{i,n+1}(t)=u_{i,n}(t)+\int_{0}^{t-t^{*}}\lambda_{i}(\xi)\left[Lu_{i,n}(\xi)+N\tilde{u}_{i,n}(\xi)-g_{i}(\xi)\right]\mathrm{d}\xi, \tag{3.1}$$

where λ_i, for $i=1,\ldots,m$, is a general Lagrange multiplier, L is linear operator, N is nonlinear operator, and g is inhomogeneous term. As knowledge, the optimal general Lagrange multiplier is obtained by constructing the correction functional as in VIM which is $\tilde{u}_{i,n}$ is considered as restricted variations, that is, $\delta\tilde{u}_{i,n}=0$.

Accordingly, the initial values $u_{1,0},u_{2,0},\ldots,u_{m,0}$ will be changed for each subinterval, that is, $u_1(t^{*})=c_1^{*}=u_{1,0}$, $u_2(t^{*})=c_2^{*}=u_{2,0},\ldots,u_m(t^{*})=c_m^{*}=u_{m,0}$, and it should be satisfied through the initial conditions $u_{i,n}(t^{*})=0$ for all $n\ge 1$, so

$$u_{i}(t)\simeq u_{i,n}(t-t^{*}),\quad i=0,\ 1,\ldots,m, \tag{3.2}$$

where t^* starting from $t_0 = 0$ until $t_J = T$, J is number of subinterval. To carry out the solution on every subinterval of equal length Δt, the values of the following initial conditions are shown below:

$$c_i^* = u_i(t^*), \quad i = 0, 1, \ldots, m. \tag{3.3}$$

In general, we do not have this information at our clearance except at the initial point $t^* = t_0 = 0$, but these values can be obtained by assuming that the new initial condition is the solution in previous interval (i.e., if the solution in interval $[t_j, t_{j+1}]$ is necessary, then the initial conditions of this interval will be as follows:

$$c_i = u_i(t) \simeq u_{i,n}(t_j - t_{j-1}), \tag{3.4}$$

where c_i, $i = 0,\ 1, \ldots, m$ are the initial conditions in the interval $[t_j, t_{j+1}]$).

4. Modified Step Variational Iteration Method

Furthermore, to implement the modification of SVIM, we consider $\hbar \neq 0$, a nonzero auxiliary parameter. Multiply $\hbar$ by correction functional in (3.1), yield

$$u_{i,n+1}(t) = u_{i,n}(t) + \hbar \int_0^{t-t^*} \lambda_i(\xi) \left[L u_{i,n}(\xi) + N \tilde{u}_{i,n}(\xi) - g_i(\xi) \right] \mathrm{d}\xi, \tag{4.1}$$

where $i = 0,\ 1,\ 2, \ldots, m$, $m \in \mathbb{N}$ and $\hbar$ is the convergence-control parameter which ensures that this assumption can be satisfied. The subscript n denotes the nth iteration.

Accordingly, the successive approximations $u_n(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained by selecting initial approximation u_0 that at least satisfies the initial conditions. The computations and plotting of figures for the algorithm, has been done using Maple package.

5. Application

In this section, we demonstrate the efficiency of MoSVIM od fractional biochemical reaction model in (1.1). The correction functionals for the system (1.1) can be approximately constructed as used by VIM and (2.4) to find the general Lagrange multiplier in the following forms:

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_0^t \lambda_1(\xi) \left[\frac{\mathrm{d}^q u_n}{\mathrm{d}\xi^q} + u_n - (\beta - \alpha) \tilde{v}_n - \widetilde{u_n v_n} \right] \mathrm{d}\xi, \\ v_{n+1}(t) &= v_n(t) + \int_0^t \lambda_2(\xi) \left[\frac{\mathrm{d}^q v_n}{\mathrm{d}\xi^q} - \frac{1}{\mu} (\tilde{u}_n - \beta v_n - \widetilde{u_n v_n} \right] \mathrm{d}\xi, \end{aligned} \tag{5.1}$$

where λ_1 and λ_2 are general Lagrange multipliers which can be identified optimally via variational theory. n denotes the nth iteration. $\widetilde{u_n}$, $\widetilde{v_n}$, and $\widetilde{u_n v_n}$ denote restricted variations,

that is, $\delta\widetilde{u_n} = 0$, $\delta\widetilde{v_n} = 0$, and $\delta\widetilde{u_n v_n} = 0$. In this case, the general Lagrange multiplier can be easily determined by choosing the number of order q, that is, $q = 1$. Thus, the following sets of stationary conditions was obtained as follows:

$$\begin{gathered} 1 + \lambda_1(t)|_{\xi=t} = 0, \quad \lambda_1(\xi) - \lambda_1'(\xi) = 0, \\ 1 + \lambda_2(t)|_{\xi=t} = 0, \quad \beta\lambda_2(\xi)\mu - \lambda_2'(\xi) = 0. \end{gathered} \tag{5.2}$$

Therefore, the general Lagrange multipliers can be easily identified as

$$\begin{gathered} \lambda_1(\xi) = -e^{(\xi-t)}, \\ \lambda_2(\xi) = -e^{\beta(\xi-t)/\mu}. \end{gathered} \tag{5.3}$$

Here, the general Lagrange multiplier in (5.3) is expanded by Taylor series and is chosen only one term in order to calculate, the general Lagrange multiplier can write as follows

$$\begin{gathered} \lambda_1(\xi) = -1, \\ \lambda_2(\xi) = -\frac{\beta}{\mu}. \end{gathered} \tag{5.4}$$

Substituting the general Lagrange multipliers in (5.4) into the correction functional in (5.1) results in the following iteration formula:

$$\begin{gathered} u_{n+1}(t) = u_n(t) - \int_0^{t-t^*} \left[\frac{\mathrm{d}^\theta u_n}{\mathrm{d}\xi} + u_n - (\beta - \alpha) v_n - u_n v_n \right] \mathrm{d}\xi, \\ v_{n+1}(t) = v_n(t) - \int_0^{t-t^*} \frac{\beta}{\mu} \left[\frac{\mathrm{d}^\theta v_n}{\mathrm{d}\xi} - \frac{1}{\mu}(u_n - \beta v_n - u_n v_n) \right] \mathrm{d}\xi. \end{gathered} \tag{5.5}$$

Furthermore, we multiply the nonzero auxiliary parameter $\hbar$ by (5.5) which yields:

$$\begin{gathered} u_{n+1}(t) = u_n(t) - \hbar \int_0^{t-t^*} \left[\frac{\mathrm{d}^\theta u_n}{\mathrm{d}\xi} + u_n - (\beta - \alpha) v_n - u_n v_n \right] \mathrm{d}\xi, \\ v_{n+1}(t) = v_n(t) - \hbar \int_0^{t-t^*} \frac{\beta}{\mu} \left[\frac{\mathrm{d}^\theta v_n}{\mathrm{d}\xi} - \frac{1}{\mu}(u_n - \beta v_n - u_n v_n) \right] \mathrm{d}\xi. \end{gathered} \tag{5.6}$$

Then, the interval $[0, 2]$ is divided into subintervals with time step Δt, and we get the solution at each subinterval. In this case, the initial condition is regarded as initial approximation,

which is necessary satisfied at each of the subinterval, that is, $u(t^*) = c_1^* = u_0$, $v(t^*) = c_2^* = v_0$, and the initial conditions should be satisfied $u_n(t^*) = 0$, $v_n(t^*) = 0$ for all $n \geq 1$, so

$$
\begin{aligned}
u_1 &= c_1 - \hbar\left[c_1 - \frac{5}{8}\ c_2 - c_1c_2\right](t-t^*),\\
v_1 &= c_2 - 100\hbar[-c_1 + \ c_2 + c_1c_2](t-t^*),\\
u_2 &= c_1 - \hbar\left[c_1(t-t^*) - \frac{5}{8}c_2(t-t^*) - c_1c_2(t-t^*)\right]\\
&\quad - \hbar\left[-\frac{30553}{37952}\hbar c_1(t-t^*)^{7/5} + \frac{9897}{19670}\hbar c_2(t-t^*)^{7/5} + c_1(t-t^*)\right.\\
&\quad - \frac{127}{4}\hbar c_1(t-t^*)^2 + \frac{505}{16}\hbar c_2(t-t^*)^2 + \frac{329}{4}\hbar c_1(t-t^*)^2c_2\\
&\quad + \frac{30553}{37952}\hbar c_1c_2(t-t^*)^{7/5} - \frac{5}{8}\ c_2(t-t^*) + \frac{100}{3}\ \hbar^2(t-t^*)^3c_1^2\\
&\quad - \frac{200}{3}\ \hbar^2(t-t^*)^3c_1^2c_2 + \frac{125}{6}\ \hbar^2(t-t^*)^3c_2^2 + \frac{325}{6}\hbar^2(t-t^*)^3c_1c_2^2\\
&\quad + \frac{100}{3}\hbar^2(t-t^*)^3c_1^2c_2^2 - 50(t-t^*)^2\hbar c_1^2 + 50(t-t^*)^2\hbar c_1^2c_2\\
&\quad \left.-\frac{1}{2}(t-t^*)^2\hbar c_1c_2^2 - c_1c_2(t-t^*)\right],\\
v_2 &= c_2 - \hbar[-100c_1(t-t^*) + 100c_2(t-t^*) + 100c_1c_2(t-t^*)]\\
&\quad - \hbar\left[\frac{1583520}{1967}\hbar c_1(t-t^*)^{7/5} - \frac{1583520}{1967}\hbar c_2(t-t^*)^{7/5} - 100c_1(t-t^*)\right.\\
&\quad - \frac{1583520}{1967}\hbar c_1c_2(t-t^*)^{7/5} + 5050\hbar c_1(t-t^*)^2 - \frac{20125}{4}\hbar c_2(t-t^*)^2\\
&\quad - 10100\hbar c_1(t-t^*)^2c_2 - \frac{10000}{3}\hbar^2(t-t^*)^3c_1^2 + \frac{16250}{3}\hbar^2(t-t^*)^3c_1c_2\\
&\quad + \frac{20000}{3}\hbar^2(t-t^*)^3c_1^2c_2 - \frac{6250}{3}\hbar^2(t-t^*)^3c_2^2 - \frac{16250}{3}\hbar^2(t-t^*)^3c_1c_2^2\\
&\quad - \frac{10000}{3}\hbar^2(t-t^*)^3c_1^2c_2^2 + 5000(t-t^*)^2\hbar c_2^2 - 5000(t-t^*)^2\hbar c_1^2c_2\\
&\quad + \frac{125}{4}(t-t^*)^2\hbar c_2^2 + 50(t-t^*)^2\hbar c_1c_2^2 + 100c_1c_2(t-t^*)\\
&\quad \left.+100c_2(t-t^*)\right].
\end{aligned}
\tag{5.7}
$$

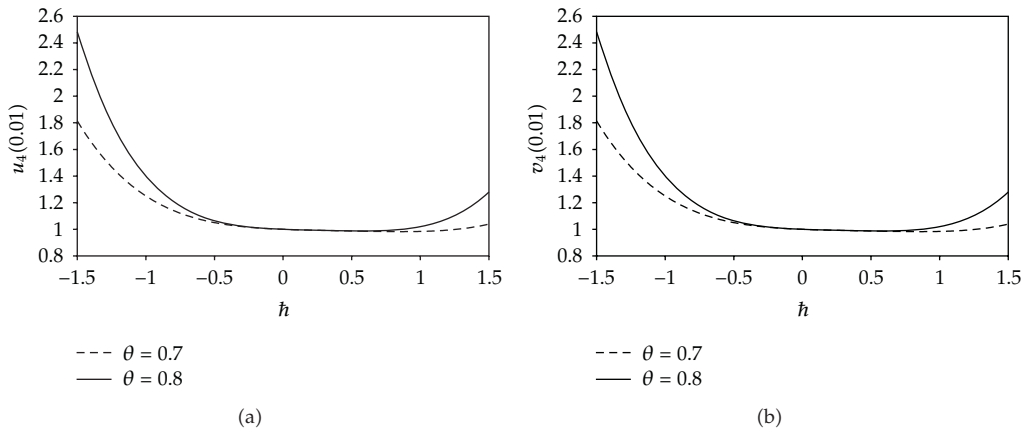

Figure 1: $\hbar$-curve for fractional biochemical reaction model using the third iteration MoSVIM with different value of θ, that is, $(0.7, 0.8)$.

Here, the iteration was chosen from previously research by Goh et al. [24]. Thus, the solution will be as follows:

$$\begin{aligned} u(t) &\simeq u_5(t - t^*), \\ v(t) &\simeq v_5(t - t^*), \end{aligned} \tag{5.8}$$

where t^* start from $t_0 = 0$ until $t_J = T = 2$. To carry out the solution on every subinterval of equal length Δt, the values of the following initial conditions is presented below:

$$c_1 = u(t^*), \qquad c_2 = v(t^*). \tag{5.9}$$

In general, we do not have this information at our clearance except at the initial point $t^* = t_0 = 0$, but we can obtain these values by assuming that the new initial condition is the solution in the previous interval (i.e., If we need the solution in interval $[t_j, t_{j+1}]$ then the initial conditions of this interval will be as

$$\begin{aligned} c_1 = u(t) &\simeq u_5(t_j - t_{j-1}), \\ c_2 = v(t) &\simeq v_5(t_j - t_{j-1}), \end{aligned} \tag{5.10}$$

where $c_1,\ c_2$ are the initial conditions in the interval $[t_j, t_{j+1}]$).

6. Result and Discussion

To investigate the influence of $\hbar$ on convergence of the solution series, we plot the $\hbar$-curves of $u_4(0.01)$ and $v_4(0.01)$ using the fifth iteration of MoSVIM when $\theta = 0.7$, and $\theta = 0.8$ as shown in Figure 1. We found that the range of values for $\hbar$ is between 0.1 and 0.7. Because the accuracy and efficiency, $\Delta t = 0.001$ was chosen as the benchmark for comparison between MoSVIM and ABMM. The constants $\mu = 0.1$, $\beta = 1$, $\tau = 0.375$ were fixed, as was chosen

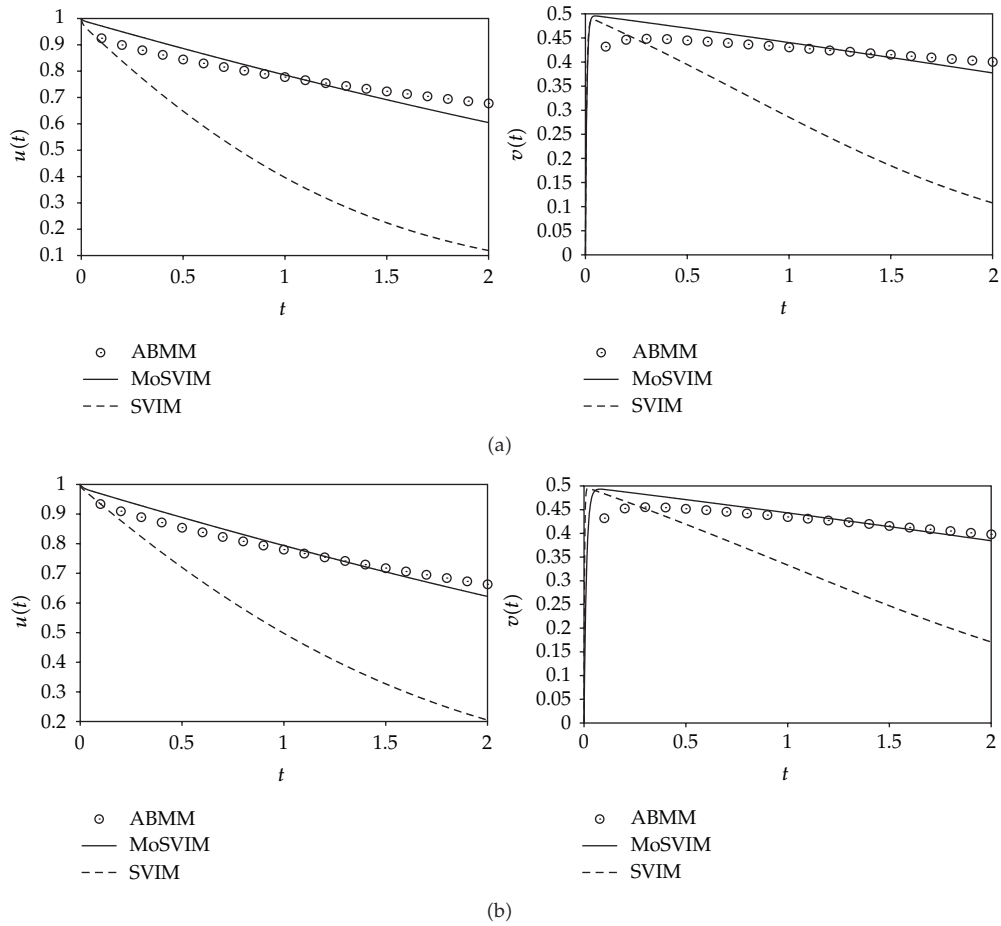

Figure 2: Approximate solution of fractional biochemical reaction model via the fifth iterate MoSVIM, SVIM and ABMM with different value of $\hbar = 0.25$; (a) $\theta = 0.7$, (b) $\theta = 0.8$.

by Hashim et al. [25]. In this case, the computational algorithms for the system in (1.1) are written using the Maple software. A good solutions of fractional biochemical reaction model when $\hbar = 0.25$ and $\theta = 0.7$ and $\theta = 0.8$ was presented in Tables 1 and 2, respectively. From the tables, MoSVIM is more accurate than SVIM in different value of θ, that is, $\theta = 0.7$ and $\theta = 0.8$. Figure 2 shows comparison of MoSVIM and SVIM. From the figure, MoSVIM solution is more closer to ABMM solution if it compare to SVIM solution. The comparison of MoSVIM, VIM and MoVIM is shown to exhibit the accuracy of MoSVIM, see Figure 3. From the figure, MoSVIM solutions is more accurate than the VIM and MoVIM solutions, and also is in good agreement with that of ABMM with $\Delta t = 0.001$.

7. Conclusions

In this paper, an algorithm of fractional biochemical reaction model (FBRM) using step modified variational iteration method (MoSVIM) was developed. For computations and plots, the Maple package were used. We found that MoSVIM is a suitable technique to

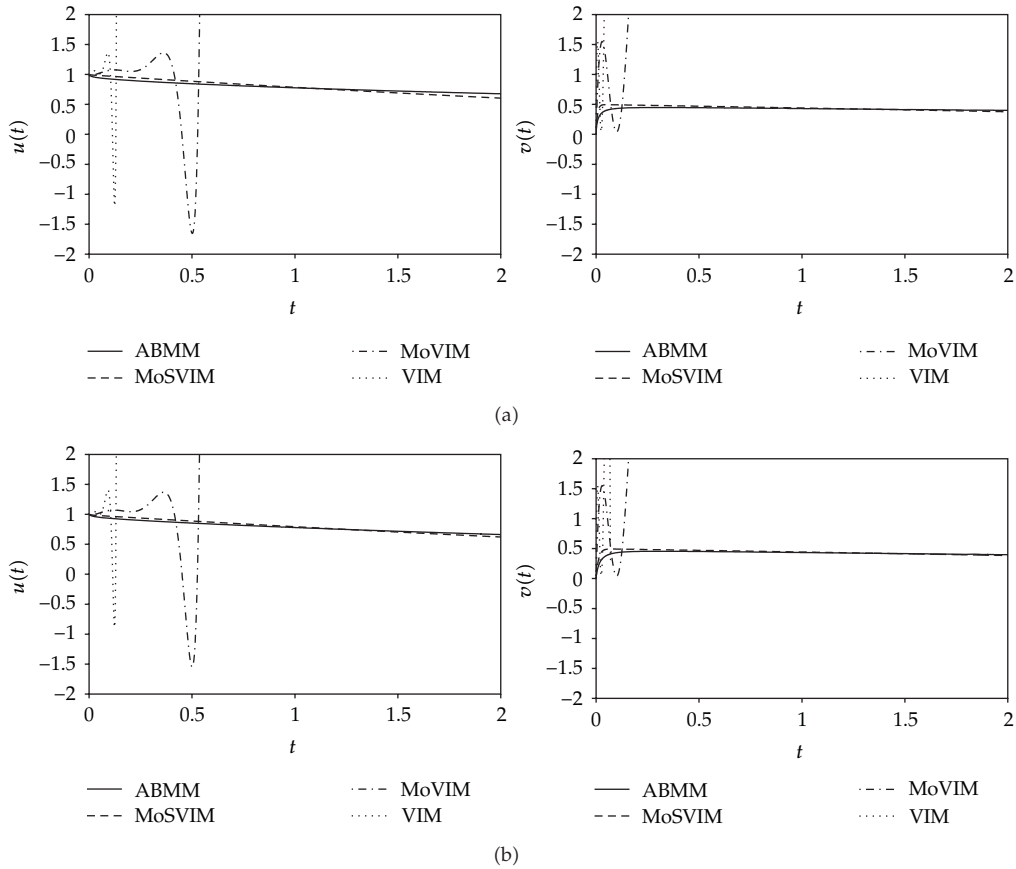

Figure 3: Approximate solution of fractional biochemical reaction model via the fifth iterate MoSVIM, VIM, MOVIM and ABMM with different value $\hbar = 0.25$; (a) $\theta = 0.7$, (b) $\theta = 0.8$.

Table 1: Approximate solution of fractional biochemical reaction model for $\theta = 0.7$, $\hbar = 0.25$ using fifth iterate of SVIM and MoSVIM, respectively, and ABMM in comparison with $\Delta t = 0.001$.

	$u(t)$			$v(t)$		
t	SVIM	MoSVIM $\hbar = 0.25$	ABMM	SVIM	MoSVIM $\hbar = 0.25$	ABMM
0.2	0.8386059622	0.94971579713	0.8997902940	0.4573898218	0.4875079322	0.4460892838
0.4	0.7085282553	0.90710678781	0.8613048034	0.4161477755	0.4760660497	0.4472272725
0.6	0.5912666037	0.86552788930	0.8298424157	0.3731707422	0.4643967818	0.4425233401
0.8	0.4871830355	0.82499870384	0.8023145667	0.3293135117	0.4525101582	0.4366487301
1.0	0.3963375974	0.78553782917	0.7774913063	0.28564459059	0.4404180963	0.4304899916
1.2	0.3184413078	0.74716268585	0.7547144581	0.2433500553	0.4281344807	0.4243110704
1.4	0.2528478376	0.70988933887	0.7335763272	0.2035911835	0.4156752212	0.4182067957
1.6	0.1985917817	0.67373231562	0.7138006533	0.1673493331	0.4030582851	0.4122134576
1.8	0.1544686371	0.63870442274	0.6951883968	0.1353025692	0.3903036993	0.4063440765
2.0	0.1191395751	0.60481656498	0.6775895839	0.1077686026	0.3774335172	0.4006016629

Table 2: Approximate solution of fractional biochemical reaction model for $\theta = 0.8$, $\hbar = 0.25$ using fifth iterate of SVIM and MoSVIM, respectively, and ABMM in comparison with $\Delta t = 0.001$.

	$u(t)$			$v(t)$		
t	SVIM	MoSVIM $\hbar = 0.25$	ABMM	SVIM	MoSVIM $\hbar = 0.25$	ABMM
0.2	0.8774940768	0.9483122837	0.9097501570	0.4679040865	0.4873856458	0.4527160444
0.4	0.7705371139	0.9082752827	0.8712461769	0.4357928339	0.4766431773	0.4544461158
0.6	0.6714218302	0.8691504634	0.83806131857	0.4023638935	0.4657006445	0.4488806205
0.8	0.5804145345	0.8309540052	0.80798209220	0.3679711022	0.4545663318	0.4419163408
1.0	0.4976827145	0.7937012985	0.78014908465	0.3330695561	0.4432499550	0.4345614413
1.2	0.4232727663	0.7574068186	0.75409601514	0.2982002140	0.4317049001	0.4271103147
1.4	0.3570935610	0.7220839956	0.72952829585	0.2639592227	0.4201174892	0.4196723265
1.6	0.2989092929	0.6877450824	0.70624217621	0.2309539383	0.4083285633	0.4122937832
1.8	0.2483439988	0.6544010212	0.68408785674	0.1997517207	0.3964119332	0.4049957868
2.0	0.2048982187	0.6220613114	0.66295019827	0.1708306193	0.3843851419	0.3977883047

solve the fractional problem. This modified method yields an analytical solution in iterations of a rapid convergent infinite power series with enlarged intervals. Comparison between MoSVIM, MoVIM and ABMM were made; the MoSVIM was found to be more accurate than the MoVIM. MoSVIM is easier in calculation yet powerful method and also is readily applicable to the more complex cases of fractional problems which arise in various fields of pure and applied sciences.

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Research Article

Homotopy Analysis Method for Solving Foam Drainage Equation with Space- and Time-Fractional Derivatives

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The analytical solution of the foam drainage equation with time- and space-fractional derivatives was derived by means of the homotopy analysis method (HAM). The fractional derivatives are described in the Caputo sense. Some examples are given and comparisons are made; the comparisons show that the homotopy analysis method is very effective and convenient. By choosing different values of the parameters α, β in general formal numerical solutions, as a result, a very rapidly convergent series solution is obtained.

1. Introduction

Many phenomena in engineering, physics, chemistry, and other science can be described very successfully by models using the theory of derivatives and integrals of fractional order. Interest in the concept of differentiation and integration to noninteger order has existed since the development of the classical calculus [1–3]. By implication, mathematical modeling of many physical systems are governed by linear and nonlinear fractional differential equations in various applications in fluid mechanics, viscoelasticity, chemistry, physics, biology, and engineering.

Since many fractional differential equations are nonlinear and do not have exact analytical solutions, various numerical and analytic methods have been used to solve these equations. The Adomian decomposition method (ADM) [4], the homotopy perturbation method (HPM) [5], the variational iteration method (VIM) [6], and other methods have been used to provide analytical approximation to linear and nonlinear problems [7, 8]. However, the convergence region of the corresponding results is rather small.

In 1992, Liao [9–13] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely, Homotopy Analysis Method

(HAM). This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [14], the KdV-type equations [15], higher-dimensional initial boundary value problems of variable coefficients [16], and finance problems [17]. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy and requires large computer memory and time. This computational method yields analytical solutions and has certain advantages over standard numerical methods. The HAM method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time.

The study of foam drainage equation is very significant for that the equation is a simple model of the flow of liquid through channels (Plateau borders [18]) and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [19]. It has been studied by many authors [20–22]. The study for the foam drainage equation with time and space-fractional derivatives of this form

$$D_t^{\alpha} u = \frac{1}{2} u u_{xx} - 2u^2 D_x^{\beta} u + \left(D_x^{\beta} u\right)^2, \quad 0 < \alpha,\ \beta \leq 1,\ x > 0, \tag{1.1}$$

has been investigated by the ADM and VIM method in [23, 24]. The fractional derivatives are considered in the Caputo sense. When $\alpha = \beta = 1$, the fractional equation reduces to the foam drainage equation of the form

$$u_t = \frac{1}{2} u u_{xx} - 2u^2 u_x + (u_x)^2. \tag{1.2}$$

In this paper, we extend the application of HAM to obtain analytic solutions to the space- and time-fractional foam drainage equation. Two cases of special interest such as the time-fractional foam drainage equation and the space-fractional foam drainage equation are discussed in details. Further, we give comparative remarks with the results obtained using ADM and VIM method (see [23, 24]).

The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. In Section 4 we extend the method to solve the space- and time-fractional foam drainage equation. Discussion and conclusions are presented in Section 5.

2. Description on the Fractional Calculus

Definition 2.1. A real function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1 \in (0, \infty)$, and it is said to be in the space C_n^μ l if and only if $h(n) \in C_\mu, n \in N$. Clearly $C_\mu \subset C_\nu$ if $\nu \leq \mu$.

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0,\ x > 0,$$
$$J^0 f(x) = f(x). \tag{2.1}$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$,

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$
$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$
$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \tag{2.2}$$

Definition 2.3. For the concept of fractional derivative, there exist many mathematical definitions [2, 25–28]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse operator Riemann-Liouville integral are adopted. That is because Caputo fractional derivative [2] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{2.3}$$

Here, we also need two basic properties about them:

$$D^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{2.4}$$

Definition 2.4. The MittagLeffler function $E_\alpha(z)$ with $a > 0$ is defined by the following series representation, valid in the whole complex plane:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0,\ z \in C. \tag{2.5}$$

3. Basic Idea of HAM

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N\left[D_t^{\alpha} u(x,t)\right] = 0, \quad t > 0, \tag{3.1}$$

where N is nonlinear operator, D_t^{α} stand for the fractional derivative and is defined as in (2.3), x, t denotes independent variables, and $u(x,t)$ is an unknown function, respectively.

By means of generalizing the traditional homotopy method, Liao [9] constructs the so-called zero-order deformation equation

$$(1-q)L\left[\phi(x,t,q) - u_0(x,t)\right] = qhH(t)N\left[D_t^{\alpha}\phi(x,t,q)\right], \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is initial guesse of $u(x,t)$, and $\phi(x,t,q)$ is unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds that

$$\phi(x,t,0) = u_0(x,t), \phi(x,t,1) = u(x,t), \tag{3.3}$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi(x,t,q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t,q)$ in Taylor series with respect to q, we have

$$\phi(x,t,q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m, \tag{3.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\left.\frac{\partial^m \phi(x,t,q)}{\partial q^m}\right|_{q=0}. \tag{3.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (3.4) converges at $q = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t), \tag{3.6}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [11]. As $h = -1$ and $H(t) = 1$, (3.2) becomes

$$(1-q)L\left[\phi_1(x,t,q) - u_0(x,t)\right] + qN\left[D_t^{\alpha}\phi(x,t,q)\right] = 0, \tag{3.7}$$

which is used mostly in the homotopy perturbation method [29], whereas the solution obtained directly, without using Taylor series. According to definition (3.5), the governing equation can be deduced from the zero-order deformation equation (3.2). Define the vector

$$\overrightarrow{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{3.8}$$

Differentiating (3.2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = hH(t)R_m\left(\overrightarrow{u}_{m-1}\right), \tag{3.9}$$

where

$$\begin{gathered} R_m\left(\overrightarrow{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N\left[D_t^{\alpha}\phi(x,t,q)\right]}{\partial q^{m-1}}\bigg|_{q=0}, \\ \chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \end{gathered} \tag{3.10}$$

Applying the Riemann-Liouville integral operator J^{α} on both side of (3.9), we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{i}(0^+)\frac{t^i}{i!} + hH(t)J^{\alpha}R_m\left(\overrightarrow{u}_{m-1}\right). \tag{3.11}$$

It should be emphasized that $u_m(x,t)$ for $m \geqslant 1$ is governed by the linear equation (3.9) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work.

Liao [10] proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of exact solutions. So, it is important to ensure that the solution series is convergent. Note that the solution series contain the auxiliary parameter h, which we can choose properly by plotting the so-called h-curves to ensure solution series converge.

Remark 3.1. The parameters α and β can be arbitrarily chosen as, integer or fraction, bigger or smaller than 1. When the parameter is bigger than 1, we will need more initial and boundary conditions such as $u_0'(x,0), u_0''(x,0), \ldots$ and the calculations will become more complicated correspondingly. In order to illustrate the problem and make it convenient for the readers, we only confine the parameter to $[0,1]$ to discuss.

4. Application

In this section we apply this method for solving foam drainage equation with time- and space-fractional derivatives.

Example 4.1. Consider the following form of the time-fractional equation:

$$D_t^{\alpha} u = \frac{1}{2} u u_{xx} - 2u^2 u_x + u_x^2, \quad 0 < \alpha \leq 1, \ x > 0, \tag{4.1}$$

with the initial condition

$$u(x,0) = -\sqrt{c}\tanh\left(\sqrt{c}x\right), \tag{4.2}$$

where c is the velocity of wavefront [15].

The exact solution of (4.1) for the special case $\alpha = \beta = 1$ is

$$u(x,t) = \begin{cases} -\sqrt{c}\tanh\left(\sqrt{c}(x-ct)\right), & x \leqslant ct, \\ 0, & x > ct. \end{cases} \tag{4.3}$$

For application of homotopy analysis method, in view of (4.1) and the initial condition given in (4.2), it in convenient to choose

$$u_0(x,t) = -\sqrt{c}\tanh\left(\sqrt{c}x\right), \tag{4.4}$$

as the initial approximate. We choose the linear operator

$$L\left[\phi(x,t;q)\right] = D_t^{\alpha}, \tag{4.5}$$

with the property $L(c) = 0$, where c is constant of integration. Furthermore, we define a nonlinear operator as

$$\begin{aligned} N\left[\phi(x,t,q)\right] = {} & D_t^{\alpha}\phi(x,t,q) - \frac{1}{2}\phi(x,t,q)\phi_{xx}(x,t,q) + 2\left(\phi(x,t,q)\right)^2\phi_x(x,t,q) \\ & - \left(\phi_x(x,t,q)\right)^2. \end{aligned} \tag{4.6}$$

We construct the zeroth-order and the mth-order deformation equations where

$$\begin{aligned} R_m\left(\vec{u}_{m-1}\right) = {} & D_t^{\alpha} u_{m-1} - \frac{1}{2}\sum_{k=0}^{m-1} u_k (u_{m-1-k})_{xx} + 2\sum_{k=0}^{m-1}\sum_{j=0}^{k} u_j u_{k-j} (u_{m-1-k})_x \\ & - \sum_{k=0}^{m-1} (u_k)_x (u_{m-1-k})_x. \end{aligned} \tag{4.7}$$

We now successively obtain

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{\Gamma(\alpha+1)}\left[h\left(-1+\tanh\left(\sqrt{c}x\right)^2\right)t^\alpha c^2\right],\\
u_2(x,t) &= \frac{1}{\Gamma(\alpha+1)^2}\Big[\sqrt{c}\tanh(\sqrt{c}x)\Gamma(\alpha+1)^2 + h\sqrt{c}\tanh(\sqrt{c}x)\Gamma(\alpha+1)^2\\
&\quad - ht^\alpha c^2\Gamma(\alpha+1) + ht^\alpha c^2\Gamma(\alpha+1)\tanh\left(\sqrt{c}x\right)^2 - h^2t^\alpha c^2\Gamma(\alpha+1)\\
&\quad + h^2t^\alpha c^2\Gamma(\alpha+1)\tanh\left(\sqrt{c}x\right)^2 + h^2t^{2a}c^{7/2}\tanh(\sqrt{c}x)\\
&\quad -2h^2t^{2a}c^{7/2}\tanh\left(\sqrt{c}x\right)^3\Big],\\
&\vdots
\end{aligned}
\tag{4.8}
$$

By taking $\alpha = 1, h = -1$, we reproduce the solution of problem as follows:

$$
\begin{aligned}
u(x,t) = \frac{1}{\Gamma(\alpha+1)^3}\Big[&-\sqrt{c}\tanh(\sqrt{c}x)\Gamma(\alpha+1)^3 + t^\alpha c^2\Gamma(\alpha+1)^2\\
&- t^\alpha c^2\Gamma(\alpha+1)^2\tanh\left(\sqrt{c}x\right)^2 + 2t^{2\alpha}c^{7/2}\tanh(\sqrt{c}x)\Gamma(\alpha+1)\\
&- 2t^{2\alpha}c^{7/2}\tanh\left(\sqrt{c}x\right)^3\Gamma(\alpha+1) + 13t^{3\alpha}c^5\tanh\left(\sqrt{c}x\right)^2\\
&-13t^{3\alpha}c^5\tanh\left(\sqrt{c}x\right)^4 + 3t^{3\alpha}c^5\tanh\left(\sqrt{c}x\right)^6 - 3t^{3\alpha}c^5\Big].
\end{aligned}
\tag{4.9}
$$

Figures 1 and 2 show the HAM and exact solutions of time-fractional foam drainage equation with $h = -1, n = 3, \alpha = 1$. It is obvious that, when $\alpha = 1$, the solution is nearly identical with the exact solution. Figures 3 and 4 show the approximate solutions of time-fractional foam drainage equation with $h = -1, n = 3, \alpha = 0.5$ and $\alpha = 0.75$, respectively.

Remark 4.2. This example has been solved using ADM and VIM in [23, 24]. The graphs drawn and Tables by $h = -1$ are in excellent agreement with that graphs drawn with ADM and VIM.

Example 4.3. Considering the operator form of the space-fractional equation

$$
u_t = \frac{1}{2}uu_{xx} - 2u^2D_x^\beta u + \left(D_x^\beta u\right)^2, \quad 0 < \beta \le 1,\ x > 0, \tag{4.10}
$$

with the initial condition

$$
u(x,0) = x^2. \tag{4.11}
$$

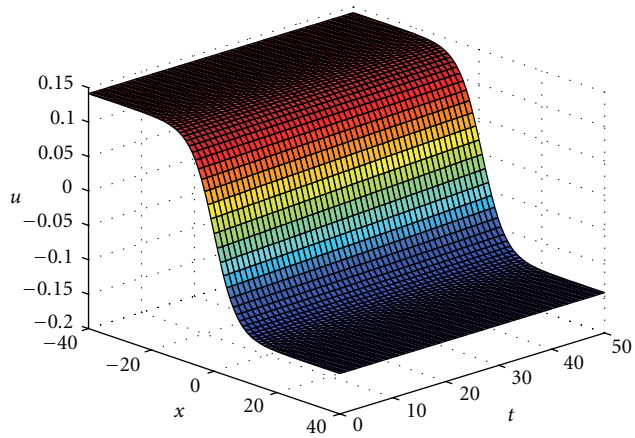

Figure 1: HAM solution with $\alpha = 1$.

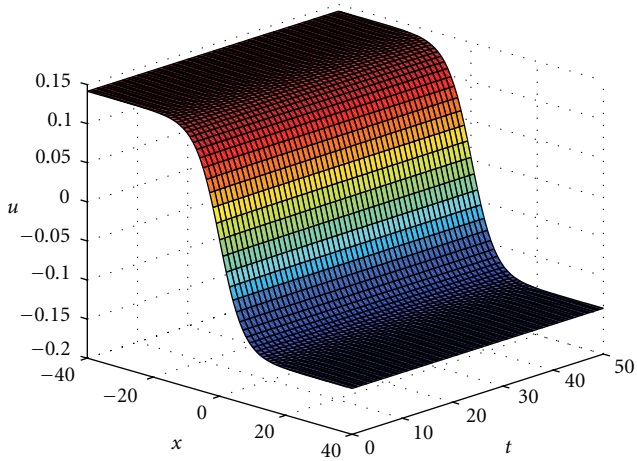

Figure 2: Exact solution.

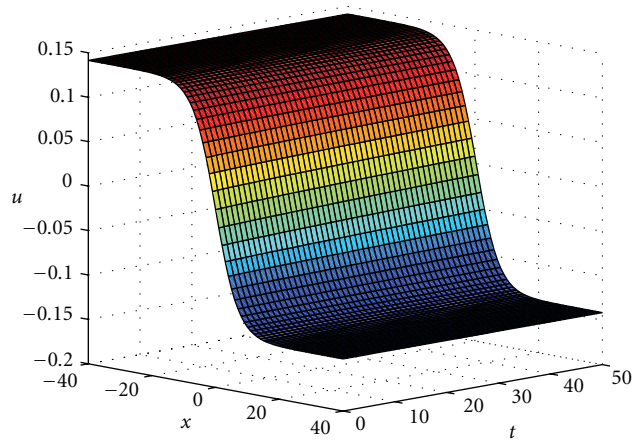

Figure 3: HAM solution with $\alpha = \frac{1}{2}$.

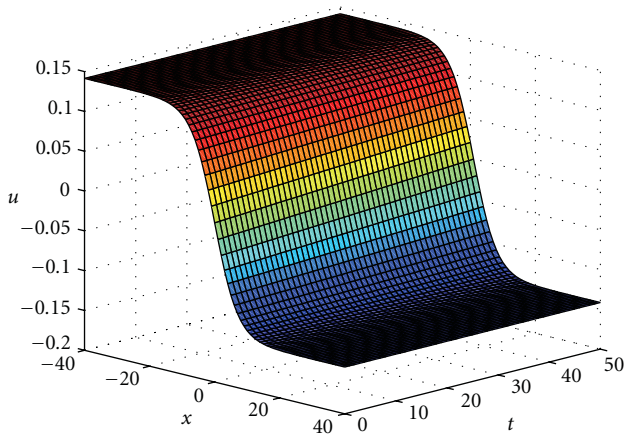

Figure 4: HAM solution with $\alpha = \frac{3}{4}$.

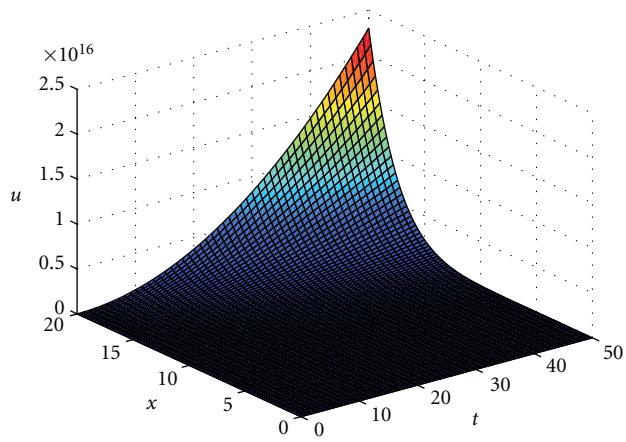

Figure 5: HAM solution with $\beta = 1/2$.

For application of homotopy analysis method, in view of (4.10) and the initial condition given in (4.2), it is inconvenient to choose

$$u_0(x,t) = x^2. \tag{4.12}$$

Initial condition has been taken as the above polynomial to avoid heavy calculation of fractional differentiation.
We choose the linear operator

$$L[\phi(x,t,q)] = \frac{\partial \phi(x,t,q)}{\partial t}, \tag{4.13}$$

with the property $L(c) = 0$, where c is constant of integration. Furthermore, we define a nonlinear operator as

$$N[\phi(x,t,q)] = \phi_t(x,t,q) - \frac{1}{2}\phi(x,t,q)\phi_{xx}(x,t,q) + 2(\phi(x,t,q))^2 D_x^\beta \phi(x,t,q) - \left(D_x^\beta \phi(x,t,q)\right)^2. \tag{4.14}$$

We construct the zeroth-order and the mth-order deformation equations where

$$R_m(\vec{u}_{m-1}) = (u_t)_{m-1} - \frac{1}{2}\sum_{k=0}^{m-1} u_k (u_{m-1-k})_{xx} + 2\sum_{k=0}^{m-1}\sum_{j=0}^{k} u_j u_{k-j} D_x^\beta u_{m-1-k} - \sum_{k=0}^{m-1} D_x^\beta u_k D_x^\beta u_{m-1-k}. \tag{4.15}$$

We now successively obtain

$$\begin{aligned}
u_1(x,t) &= -hx^2t + 4\frac{hx^{6-\beta}t}{\Gamma(3-\beta)} - 4\frac{hx^{4-2\beta}t}{\Gamma(3-\beta)^2},\\
u_2(x,t) &= \frac{1}{\Gamma(3-\beta)^2}\Bigg[-14h^2t^2x^{4-2\beta}\beta + 4h^2t^2x^{4-2\beta}\beta^2 + 18hx^{4-2\beta}ht^2\\
&\qquad - 4h^2tx^{4-2\beta} - 4htx^{4-2\beta}\\
&\qquad + \frac{h^2x^{8-3\beta}\pi^{1/2}(-4+\beta)(-5+\beta)(-6+\beta)t^2 4^\beta}{8\Gamma((7/2)-\beta)}\\
&\qquad + \frac{128h^2x^{6-4\beta}4^{-\beta}t^2\Gamma((5/2)-\beta)}{\pi^{1/2}\Gamma(5-3\beta)}\Bigg]\\
&\quad + \frac{1}{\Gamma(3-\beta)}\Bigg[\frac{-h^2x^{10-2\beta}\pi^{1/2}(-4+\beta)(-5+\beta)(-6+\beta)t^2 4^\beta}{16\Gamma((7/2)-\beta)}\\
&\qquad - \frac{64h^2x^{8-3\beta}4^{-\beta}t^2\Gamma((5/2)-\beta)}{\pi^{1/2}\Gamma(5-3\beta)} + 4htx^{6-\beta} + 4h^2tx^{6-\beta}\\
&\qquad -38h^2t62x^{6-\beta} + 11h^2t^2\beta^2x^{6-\beta}\Bigg]\\
&\qquad - \frac{16h^2t^2x^{8-3\beta}}{\Gamma(3-\beta)^3} - hx^2t - h^2x^2t + t^2x^2h^2.\\
&\ \vdots
\end{aligned} \tag{4.16}$$

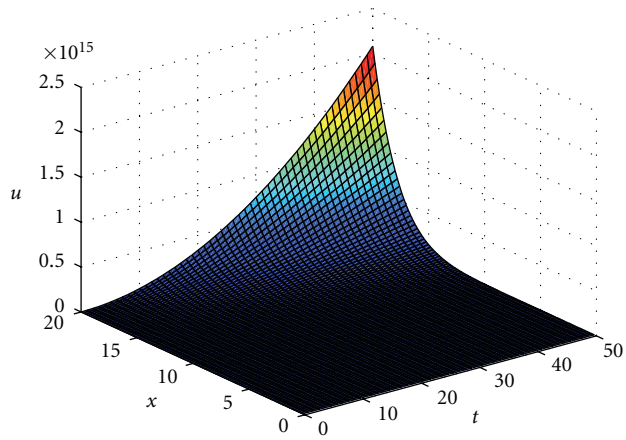

Figure 6: HAM solution with $\beta = 1$.

Figures 5 and 6 show the HAM solutions of space-fractional foam drainage equation with $h = -1, n = 3, \beta = 0.5$ and $\beta = 1$, respectively.

Remark 4.4. This example has been solved using ADM and VIM in [23, 24]. The graphs drawn and Tables by $h = -1$ are in excellent agreement with that graphs drawn with ADM and VIM.

5. Conclusion

In this paper, we have successfully developed HAM for solving space- and time-fractional foam drainage equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting h, which is a fundamental qualitative difference in analysis between HAM and other methods. The obtained results demonstrate the reliability of the HAM and its wider applicability to fractional differential equation. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems.

Matlab has been used for computations in this paper.

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Research Article

Solving Famous Nonlinear Coupled Equations with Parameters Derivative by Homotopy Analysis Method

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The homotopy analysis method (HAM) is employed to obtain symbolic approximate solutions for nonlinear coupled equations with parameters derivative. These nonlinear coupled equations with parameters derivative contain many important mathematical physics equations and reaction diffusion equations. By choosing different values of the parameters in general formal numerical solutions, as a result, a very rapidly convergent series solution is obtained. The efficiency and accuracy of the method are verified by using two famous examples: coupled Burgers and mKdV equations. The obtained results show that the homotopy perturbation method is a special case of homotopy analysis method.

1. Introduction

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. The differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [1, 2]. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. Most nonlinear fractional equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The Adomain decomposition method [3], the homotopy perturbation method [4], the variational iteration

method [5], and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small. In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely, homotopy analysis method [6–10]. This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [11], the KdV-type equations [12], finance problems [13], fractional Lorenz system [14], and delay differential equation [15]. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy and requires large computer memory and time. This computational method yields analytical solutions and has certain advantages over standard numerical methods. The HAM method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time.

In this paper, we extend the application of HAM to discuss the explicit numerical solutions of a type of nonlinear-coupled equations with parameters derivative in this form:

$$\begin{aligned} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} &= L_1(u,v) + N_1(u,v), \quad t > 0, \\ \frac{\partial^{\beta} v}{\partial t^{\beta}} &= L_2(u,v) + N_2(u,v), \quad t > 0, \end{aligned} \tag{1.1}$$

where L_i and N_i $(i = 1, 2)$ are the linear and nonlinear functions of u and v, respectively, α and β are the parameters that describe the order of the derivative. Different nonlinear coupled systems can be obtained when one of the parameters α or β varies. The study of (1.1) is very necessary and significant because when α and β are integers, it contains many important mathematical physics equations.

The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. In Section 4 applying HAM for two famous coupled examples: Burgers and mKdV equations. Discussion and conclusions are presented in Section 5.

2. Description on the Fractional Calculus

Definition 2.1. A real function $f(t), t > 0$ is said to be in the space C_{μ}, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^{p} f_1(t)$ where $f_1 \in (0, \infty)$, and it is said to be in the space C_n^{μ} l if and only if $h(n) \in C_{\mu}$, $n \in N$. Clearly $C_{\mu} \subset C_{\nu}$ if $\nu \leq \mu$.

Definition 2.2. The Riemann-Liouville fractional integral operator (J^{α}) of order $\alpha \geq 0$, of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$\begin{gathered} J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > 0. \\ J^0 f(x) = f(x). \end{gathered} \tag{2.1}$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows.

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+\beta} f(x), \\ J^\alpha J^\beta f(x) &= J^\beta J^\alpha f(x), \\ J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned} \tag{2.2}$$

Definition 2.3. For the concept of fractional derivative, there exist many mathematical definitions [1, 16–19]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse operator Riemann-Liouville integral are adopted. That is because Caputo fractional derivative [1] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$\begin{aligned} D_t^\alpha u(x,t) &= \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \\ &= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \end{aligned} \tag{2.3}$$

Here, we also need two basic properties about them:

$$\begin{aligned} D^\alpha J^\alpha f(x) &= f(x), \\ J^\alpha D^\alpha f(x) &= f(x) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \end{aligned} \tag{2.4}$$

Definition 2.4. The Mittag-Leffler function $E_\alpha(z)$ with $a > 0$ is defined by the following series representation, valid in the whole complex plane:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \ z \in \mathrm{C}. \tag{2.5}$$

3. Basic Idea of HAM

To describe the basic ideas of the HAM, we consider the operator form of (1.1):

$$\begin{aligned} N\left[D_t^\alpha u(x,t)\right] &= 0, \quad t > 0, \\ N\left[D_t^\beta v(x,t)\right] &= 0, \quad t > 0, \end{aligned} \tag{3.1}$$

where N is nonlinear operator, D_t^α and D_t^β stand for the fractional derivative and are defined as in (2.3), t denotes an independent operator, and $u(x,t)$, $v(x,t)$ are unknown functions.

By means of generalizing the traditional homotopy method, Liao [6] constructs the so-called zero-order deformation equations:

$$(1-q)L\left[\phi_1(x,t,q)-u_0(x,t)\right]=q\ hH(t)N\left[D_t^{\alpha}\phi_1(x,t,q)\right], \tag{3.2}$$

$$(1-q)L\left[\phi_2(x,t,q)-v_0(x,t)\right]=q\ hH(t)N\left[D_t^{\beta}\phi_2(x,t,q)\right], \tag{3.3}$$

where $q\in[0,1]$ is the embedding parameter, $h\neq 0$ is a non-zero auxiliary parameter, $H(t)\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$, $v_0(x,t)$ are initial guesses of $u(x,t)$, $v(x,t)$ and $\phi_1(x,t,q)$, $\phi_2(x,t,q)$ are two unknown functions, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q=0$ and $q=1$, the following holds:

$$\begin{aligned}\phi_1(x,t,0)&=u_0(x,t), \qquad \phi_1(x,t,1)=u(x,t),\\ \phi_2(x,t,0)&=v_0(x,t), \qquad \phi_2(x,t,1)=v(x,t),\end{aligned} \tag{3.4}$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi_1(x,t,q)$, $\phi_2(x,t,q)$ varies from the initial guess $u_0(x,t)$, $v_0(x,t)$ to the solution $u(x,t)$, $v(x,t)$. Expanding $\phi_1(x,t,q)$, $\phi_2(x,t,q)$ in Taylor series with respect to q, we have

$$\begin{aligned}\phi_1(x,t,q)&=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t)q^m,\\ \phi_2(x,t,q)&=v_0(x,t)+\sum_{m=1}^{+\infty}v_m(x,t)q^m,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}u_m(x,t)&=\frac{1}{m!}\frac{\partial^m\phi_1(x,t,q)}{\partial q^m}\Big|_{q=0},\\ v_m(x,t)&=\frac{1}{m!}\frac{\partial^m\phi_2(x,t,q)}{\partial q^m}\Big|_{q=0}.\end{aligned} \tag{3.6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (3.5) converges at $q=1$, then we have

$$\begin{aligned}u(x,t)&=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t),\\ v(x,t)&=v_0(x,t)+\sum_{m=1}^{+\infty}v_m(x,t),\end{aligned} \tag{3.7}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [8]. As $h = -1$ and $H(\mathrm{t}) = 1$, (3.2) and (3.3) become

$$\begin{aligned}(1-q)L[\phi_1(x,t,q) - u_0(x,t)] + qN[\phi_1(x,t,q)] = 0,\\ (1-q)L[\phi_2(x,t,q) - u_0(x,t)] + qN[\phi_2(x,t,;q)] = 0,\end{aligned} \tag{3.8}$$

which is used mostly in the homotopy perturbation method [20], where as the solution obtained directly, without using Taylor series. According to the definition (3.6), the governing equation can be deduced from the zero-order deformation equation (3.2). Define the vector

$$\overrightarrow{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}, \qquad \overrightarrow{v}_n = \{v_0(x,t), v_1(x,t), \ldots, v_n(x,t)\}. \tag{3.9}$$

Differentiating equations (3.2) and (3.3) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation:

$$\begin{aligned}L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(t)R_{1,m}(\overrightarrow{u}_{m-1}),\\ L[v_m(x,t) - \chi_m v_{m-1}(x,t)] = hH(t)R_{2,m}(\overrightarrow{v}_{m-1}),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}R_{1,m}(\overrightarrow{u}_{m-1}) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}D_t^{\alpha}[\phi_1(x,t,q)]}{\partial q^{m-1}}\Big|_{q=0},\\ R_{2,m}(\overrightarrow{v}_{m-1}) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}D_t^{\beta}[\phi_2(x,t,q)]}{\partial q^{m-1}}\Big|_{q=0},\\ \chi_m &= \begin{cases} 0, & m \leqslant 1,\\ 1, & m > 1.\end{cases}\end{aligned} \tag{3.11}$$

Applying the Riemann-Liouville integral operator J^{α}, J^{β} on both side of (3.10), we have

$$\begin{aligned}u_m(x,t) &= \chi_m u_{m-1}(x,t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{i}(0^+)\frac{t^i}{i!} + hH(t)J^{\alpha}R_{1,m}(\overrightarrow{u}_{m-1}),\\ v_m(x,t) &= \chi_m v_{m-1}(x,t) - \chi_m \sum_{i=0}^{n-1} v_{m-1}^{i}(0^+)\frac{t^i}{i!} + hH(t)J^{\beta}R_{2,m}(\overrightarrow{v}_{m-1}).\end{aligned} \tag{3.12}$$

It should be emphasized that $u_m(x,t)$, $v_m(x,t)$ for $m \geqslant 1$ is governed by the linear equation (3.10), under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as MATLAB. For the convergence

of the above method we refer the reader to Liao's work. Liao [7] proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of exact solutions. So, it is important to ensure that the solution series is convergent. Note that the solution series contain the auxiliary parameter h, which we can choose properly by plotting the so-called h-curves to ensure solution series converge.

Remark 3.1. The parameters α and β can be arbitrarily chosen as, integer or fraction, bigger or smaller than 1. When the parameters are bigger than 1, we will need more initial and boundary conditions such as $u_0'(x,0)$, $u_0''(x,0),\ldots$ and the calculations will become more complicated correspondingly. In order to illustrate the problem and make it convenient for the readers, we only confine the parameters to $[0,1]$ to discuss.

4. Application

4.1. The Nonlinear Coupled Burgers Equations with Parameters Derivative

In order to illustrate the method discussed above, we consider the nonlinear coupled Burgers equations with parameters derivative in an operator form:

$$\begin{aligned} D_t^{\alpha} u - L_{xx}u - 2uL_x u + L_x uv = 0, \quad (0 < \alpha \leq 1), \\ D_t^{\beta} v - L_{xx}v - 2vL_x v + L_x uv = 0, \quad (0 < \beta \leq 1), \end{aligned} \tag{4.1}$$

where $t > 0$, $L_x = \partial/\partial x$ and the fractional operators D_t^{α} and D_t^{β} are defined as in (2.3). Assuming the initial value as

$$u(x,0) = \sin x, \qquad v(x,0) = \sin x. \tag{4.2}$$

The exact solutions of (4.1) for the special case: $\alpha = \beta = 1$ are

$$u(x,t) = e^{-t}\sin x, \qquad v(x,t) = e^{-t}\sin x. \tag{4.3}$$

For application of homotopy analysis method, in view of (4.1) and the initial condition given in (4.2), it is convenient to choose

$$u_0(x,t) = \sin x, \qquad v_0(x,t) = \sin x, \tag{4.4}$$

as the initial approximate of (4.1). We choose the linear operators

$$\begin{aligned} L_1[\phi_1(x,t,q)] &= D_t^{\alpha}[\phi_1(x,t,q)], \\ L_2[\phi_2(x,t,q)] &= D_t^{\beta}[\phi_2(x,t,q)], \end{aligned} \tag{4.5}$$

with the property $L(c) = 0$ where c is constant of integration. Furthermore, we define a system of nonlinear operators as

$$\begin{aligned} N_1[\phi_i(x,t,q)] &= D_t^{\alpha}[\phi_1(x,t,q)] - \frac{\partial^2\phi_1(x,t,q)}{\partial x^2} - 2\phi_1(x,t,q)\frac{\partial\phi_1(x,t,q)}{\partial x} \\ &\quad + \frac{\partial[\phi_1(x,t,q)\phi_2(x,t,q)]}{\partial x} \\ N_2[\phi_i(x,t,q)] &= D_t^{\beta}[\phi_2(x,t,q)] - \frac{\partial^2\phi_2(x,t,q)}{\partial x^2} - 2\phi_2(x,t,q)\frac{\partial\phi_2(x,t,q)}{\partial x} \\ &\quad + \frac{\partial[\phi_1(x,t,q)\phi_2(x,t,q)]}{\partial x}. \end{aligned} \tag{4.6}$$

We construct the zeroth-order and the mth-order deformation equations where

$$\begin{aligned} R_{1,m}(\vec{u}_{m-1}) &= D_t^{\alpha}[u_{m-1}] - (u_{m-1})_{xx} - 2\sum_{k=0}^{m-1} u_k(u_{m-1-k})_x + \left(\sum_{k=0}^{m-1} u_k v_{m-k-1}\right)_x, \\ R_{2,m}(\vec{v}_{m-1}) &= D_t^{\beta}[v_{m-1}] - (v_{m-1})_{xx} - 2\sum_{k=0}^{m-1} v_k(v_{m-1-k})_x + \left(\sum_{k=0}^{m-1} u_k v_{m-k-1}\right)_x. \end{aligned} \tag{4.7}$$

We start with an initial approximation $u(x,0) = \sin(x)$, $v(x,0) = \sin(x)$, thus we can obtain directly the other components as

$$\begin{aligned} u_1 &= \frac{ht^a \sin(x)}{\Gamma(a+1)}, \\ u_2 &= \frac{-\sin x}{a^2\Gamma(b+1)\Gamma(a)^2\Gamma(a+(1/2))} \\ &\quad \times \Bigg[a^2\Gamma(b+1)\Gamma(a)^2\Gamma\left(a+\frac{1}{2}\right) - a^2\Gamma(b+1)\Gamma(a)^2 \\ &\qquad \Gamma\left(a+\frac{1}{2}\right)h + ha\Gamma(a)\Gamma(2a+1)t^a\Gamma(b+1) + h^2a\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)t^a\Gamma(b+1) + 2h^2a\Gamma(a) \\ &\qquad \Gamma\left(a+\frac{1}{2}\right)t^{(b+a)}\cos(x) - 2h^2t^{(2a)}\cos(x)\Gamma(b+1)\Gamma\left(a+\frac{1}{2}\right) \\ &\qquad + h^2t^{(2a)}a^2\Gamma(b+1)\Gamma(a)^2\Bigg] \\ &\quad \vdots \end{aligned}$$

$$
\begin{aligned}
v_1 &= \frac{ht^b \sin(x)}{\Gamma(b+1)}, \\
v_2 &= \frac{-\sin x}{\left(b^2\Gamma(a+1)\Gamma(b)^2\Gamma(b+(1/2))\right)} \\
&\quad \times \Big[b^2\Gamma(a+1)\Gamma(b)^2\Gamma(2b+1)h + hb\Gamma(b), \\
&\qquad \Gamma(2b+1)t^b\Gamma(a+1) + h^2b\Gamma(b)\Gamma(2b+1)t^b\Gamma(b+1) + 2h^2b\Gamma(b)\Gamma(2b+1), \\
&\qquad t^{(a+b)}\cos(x) - 2h^2t^{(2b)}\cos(x)\Gamma(a+1)\Gamma(2b+1) + h^2\left(\frac{1}{t}\right)^{(-b)} t^b b^2\Gamma(a+1), \\
&\qquad \Gamma(b)^2\Big] \\
&\vdots
\end{aligned}
\tag{4.8}
$$

The absolute error of the 6th-order HAM and exact solution with $h = -1$ as shown in Figure 1. Also the absolute errors $|u(t) - \phi_6(t)|$ have been calculated in Table 1. Figure 2 shows the numerical solutions of the nonlinear coupled Burgers equations with parameters derivative with $h = -1$, $\alpha = \beta = 1$. Figure 3 shows the explicit numerical solutions with $h = -1$, $\alpha = 1/4$, and $\beta = 1/3$ at $t = 0.02$.

As suggested by Liao [7], the appropriate region for h is a horizontal line segment. We can investigate the influence of h on the convergence of the solution series gevin by the HAM, by plotting its curve versus h, as shown in Figure 4.

Remark 4.1. This example has been solved using homotopy perturbation method [21]. The graphs drawn and tables by $h = -1$ are in excellent agreement with that graphs drawn with HPM.

4.2. The Nonlinear Coupled mKdV Equations with Parameters Derivative

In order to illustrate the method discussed above, we consider the nonlinear coupled mKdV equations with parameters derivative in an operator form:

$$
\begin{aligned}
D_t^{\alpha} u - \frac{1}{2}u_{xxx} + 3u^2u_x - \frac{3}{2}v_{xx} - 3(uv)_x + 3\lambda u_x = 0, \\
D_t^{\beta} v + v_{xxx} + 3vv_x + 3u_xv_x - 3u^2v_x - 3\lambda v_x = 0,
\end{aligned}
\tag{4.9}
$$

with the initial conditions,

$$
u(x,0) = \frac{b}{2k} + k\tanh(kx), \qquad v(x,0) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(kx). \tag{4.10}
$$

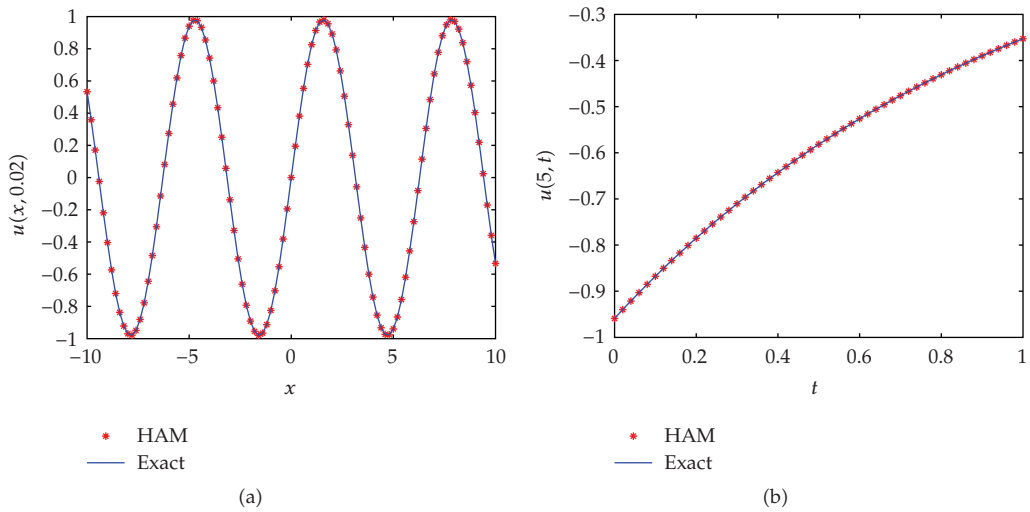


Figure 1: The comparison of the 6th-order HAM and exact solution with $h = -1$, $\alpha = \beta = 1$.

Table 1: The comparison of the results of the HAM ($h = -1$) and exact solution for the $u(x,t)$, $\alpha = \beta = 1$.

x	t	ϕ_6	$u(x,t)$	Realtive error
10	0.01	−5.3861e-001	−0.9002497662	$4.5253e \times 10^{-13}$
10	0.02	−5.3325e-001	−0.8912921314	1.4459×10^{-11}
10	0.03	−5.2794e-001	−0.8824236265	1.0962×10^{-10}
5	0.01	−0.9493828183	−0.9493828187	7.9781×10^{-13}
5	0.02	−0.9399363019	−0.9399363019	2.5486×10^{-11}
5	0.03	−0.9305837792	−0.9305837793	1.9322×10^{-10}
-2	0.01	−0.5386080102	−0.5386080104	7.5651×10^{-13}
-2	0.02	−0.5332487712	−0.5332487712	2.4167×10^{-11}
-2	0.03	−0.5279428571	−0.5279428572	1.8322×10^{-10}

As we know, when $\alpha = \beta = 1$ (4.9) has the kink-type soliton solutions

$$\begin{aligned} u(x,t) &= \frac{b}{2k} + k\tanh(k\xi), \\ v(x,t) &= \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(k\xi), \end{aligned} \tag{4.11}$$

constructed by Fan [22], where $\xi = x + (1/4)(-4k^2 - 6\lambda + 6k\lambda/b + 3b^2/k^2)t$, $k \neq 0$, and $b \neq 0$. For application of homotopy analysis method, in view of (4.9) and the initial condition given in (4.10), it in convenient to choose

$$u(x,0) = \frac{b}{2k} + k\tanh(kx), \qquad v(x,0) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(kx), \tag{4.12}$$

Table 2: The comparison of the results of the HAM ($h = -1$) and exact solution for the $u(x,t)$, $\alpha = \beta = 1$.

x	t	$\lvert \sum_{i=0}^{5} u(i) - \text{Exact}\rvert$	$\lvert \sum_{i=0}^{5} v(i) - \text{Exact}\rvert$
−15	0.002	8.0672×10^{-8}	2.4426×10^{-8}
−12	0.002	5.9539×10^{-7}	1.8038×10^{-7}
−6	0.002	3.0257×10^{-6}	1.3270×10^{-6}
6	0.002	3.0263×10^{-6}	1.3481×10^{-6}
12	0.002	5.9550×10^{-7}	1.7722×10^{-7}
15	0.002	8.0686×10^{-8}	2.3998×10^{-8}

as the initial approximate of (4.10). We choose the linear operators

$$\begin{aligned} L_1[\phi_1(x,t,q)] &= D_t^{\alpha}[\phi_1(x,t,q)], \\ L_2[\phi_2(x,t,q)] &= D_t^{\beta}[\phi_2(x,t,q)], \end{aligned} \tag{4.13}$$

with the property $L(c) = 0$ where c is constant of integration. Furthermore, we define a system of nonlinear operators as

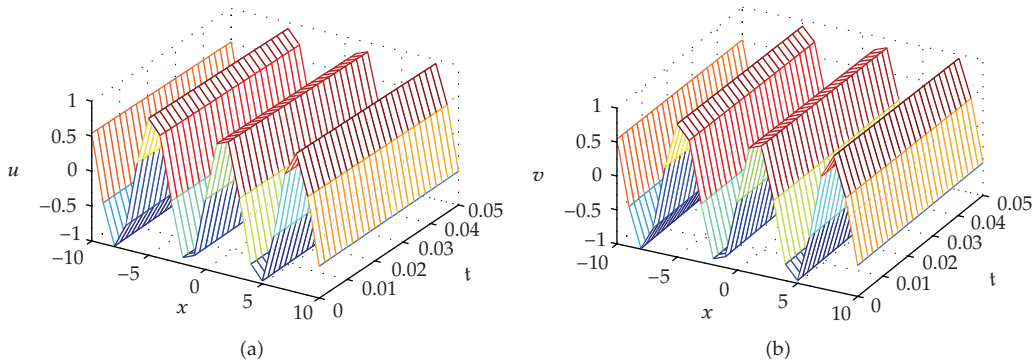
$$\begin{aligned} N_1[\phi_i(x,t,q)] &= D_t^{\alpha}[\phi_1(x,t,q)] - \frac{1}{2}\frac{\partial^3\phi_1(x,t,q)}{\partial x^3} + 3\phi_1(x,t,q)^2\frac{\partial\phi_1(x,t,q)}{\partial x}, \\ &\quad - \frac{3}{2}\frac{\partial^2\phi_2(x,t,q)}{\partial x^2} - 3\frac{\partial[\phi_1(x,t,q)\phi_2(x,t,q)]}{\partial x} + 3\lambda\frac{\partial\phi_1(x,t,q)}{\partial x}, \\ N_2[\phi_i(x,t,q)] &= D_t^{\beta}[\phi_2(x,t,q)] + \frac{\partial^3\phi_2(x,t,q)}{\partial x^3} + 3\phi_2(x,t,q)\frac{\partial\phi_2(x,t,q)}{\partial x}, \\ &\quad + 3\frac{\partial\phi_1(x,t,q)}{\partial x}\frac{\partial\phi_2(x,t,q)}{\partial x} - 3\phi_1(x,t,q)^2\frac{\partial\phi_2(x,t,q)}{\partial x} - 3\lambda\frac{\partial\phi_2(x,t,q)}{\partial x}. \end{aligned} \tag{4.14}$$

We construct the zeroth-order and the mth-order deformation equations where

$$\begin{aligned} R_{1,m}(\vec{u}_{m-1}) &= D_t^{\alpha}[u_{m-1}] - \frac{1}{2}(u_{m-1})_{xxx} + 3\sum_{i=0}^{m-1}u_i\sum_{k=0}^{m-1-i}u_k(v_{m-1-i-k})_x, \\ &\quad - \frac{3}{2}(v_{m-1})_{xx} - 3\left(\sum_{k=0}^{m-1}u_k v_{m-k-1}\right)_x + 3\lambda(u_{m-1})_x, \\ R_{2,m}(\vec{v}_{m-1}) &= D_t^{\beta}[v_{m-1}] + (v_{m-1})_{xxx} + 3\sum_{k=0}^{m-1}v_k(v_{m-1-k})_x + 3\sum_{k=0}^{m-1}(u_k)_x(v_{m-k-1})_x, \\ &\quad - 3\sum_{i=0}^{m-1}u_i\sum_{k=0}^{m-1-i}u_k(v_{m-1-i-k})_x - 3\lambda(v_{m-1})_x. \end{aligned} \tag{4.15}$$

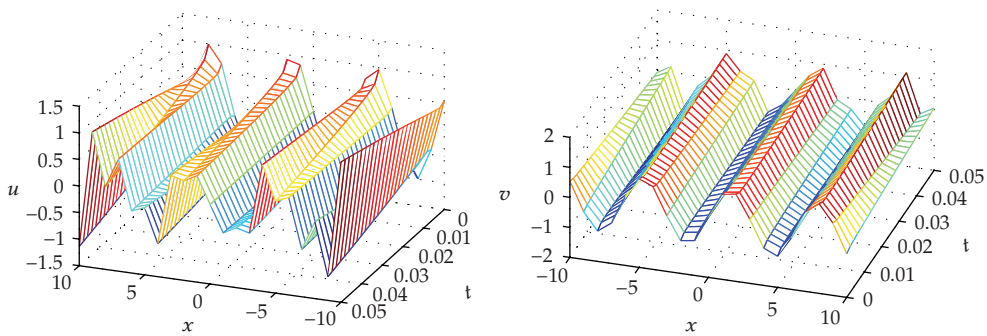
We start with an initial approximation $u(x,0) = (b/2k) + k\tanh(kx)$, $v(x,0) = (\lambda/2)(1 + (k/b)) + b\tanh(kx)$, with $k = 0.1$, $b = 1$, $k = 1/3$, thus we can obtain directly the other components as follows:

$$
\begin{aligned}
u_1 &= \frac{1177}{1620} h t^a \frac{-1 + \tanh\left((1/3)x\right)^2}{\Gamma(a+1)},\\
u_2 &= \frac{1}{437400} - 656100\Gamma(a+b+1)a^2\Gamma(b+1)\Gamma(a)^2\Gamma(2a+1) - 145800,\\
&\quad \Gamma(a+b+1)a^2\Gamma(b+1)\Gamma(a)^2\Gamma(2\mathrm{a}+1)\ \tanh\left(\frac{1}{3}x\right) - 656100\Gamma(a+b+1)a^2,\\
&\quad \Gamma(b+1)\Gamma(a)^2\Gamma(2a+1)h - 145800\Gamma(a+b+1)a^2\Gamma(b+1)\Gamma(a)^2\Gamma(2a+1),\\
&\quad h\tanh\left(\frac{1}{3}x\right) - 327510h^2t^{(b+a)}a^2\Gamma(b+1)\Gamma(a)^2\Gamma(2a+1) + 1310040h^2t^{(b+a)},\\
&\quad a^2\Gamma(b+1)\Gamma(a)^2\Gamma(2a+1)\ \tanh\left(\frac{1}{3}x\right)^2 - 982530h^2t^{(b+a)}a^2\Gamma(b+1)\Gamma(a)^2,\\
&\quad \Gamma(2a+1)\tanh\left(\frac{1}{3}x\right)^4 - 317790h\Gamma(a+b+1)a\Gamma(a)\Gamma(2a+1)t^a\Gamma(b+1),\\
&\quad + \cdots\\
&\ \vdots\\
v_1 &= \frac{1213}{540} h t^b \frac{-1 + \tanh\left((1/3)x\right)^2}{\Gamma(b+1)},\\
v_2 &= \frac{1}{145800} - 145800b^2\Gamma(a+1)\Gamma(b)^2\Gamma(2b+1)h\tanh\left(\frac{1}{3}x\right) - 145800b^2,\\
&\quad \Gamma(a+1)\Gamma(b)^2\Gamma(2b+1)\tanh\left(\frac{1}{3}x\right) - 9720b^2\Gamma(a+1)\Gamma(b)^2\Gamma(2b+1)h,\\
&\quad - 327510hb\Gamma(b)\Gamma(2b+1)t^b\Gamma(a+1) - 327510h^2b\Gamma(b)\Gamma(2b+1)t^b\Gamma(a+1),\\
&\quad + 317790h^2b\Gamma(b)\Gamma(2b+1)t^{(b+a)} - 635580h^2b\Gamma(b)\Gamma(2b+1)t^{(b+a)},\\
&\quad \tanh\left(\frac{1}{3}x\right)^2 - 282480h^2b\Gamma(b)\Gamma(2b+1)t^{(b+a)}\tanh\left(\frac{1}{3}x\right)^3 + 141240h^2b,\\
&\quad \tanh\left(\frac{1}{3}x\right)^4 + \cdots\\
&\ \vdots
\end{aligned}
\tag{4.16}
$$



(a) (b)

Figure 2: Explicit numerical solutions with $h = -1$, $\alpha = \beta = 1$.



(a) (b)

Figure 3: Explicit numerical solutions with $h = -1$, $\alpha = 1/4$, and $\beta = 1/3$.

The absolute error of the 6th-order HAM and exact solution with $h = -1$ as shown in Figure 5. Also the absolute errors $|u(t) - \phi_6(t)|$ have been calculated for in Table 2. Figure 6 shows the numerical solutions of the nonlinear coupled Burgers equations with parameters derivative with $h = -1$, $\alpha = \beta = 1$. Figure 7 shows the explicit numerical solutions with $h = -1$, $\alpha = 1/2$, and $\beta = 2/3$ at $t = 0.002$.

As suggested by Liao [7], the appropriate region for h is a horizontal line segment. We can investigate the influence of h on the convergence of the solution series gevin by the HAM, by plotting its curve versus h, as shown in Figure 8.

Remark 4.2. This example has been solved using homotopy perturbation method [21]. The graphs drawn and tables by $h = -1$ are in excellent agreement with those graphs drawn with HPM.

5. Conclusion

In this paper, based on the symbolic computation MATLAB, the HAM is directly extended to derive explicit and numerical solutions of the nonlinear coupled equations with parameters

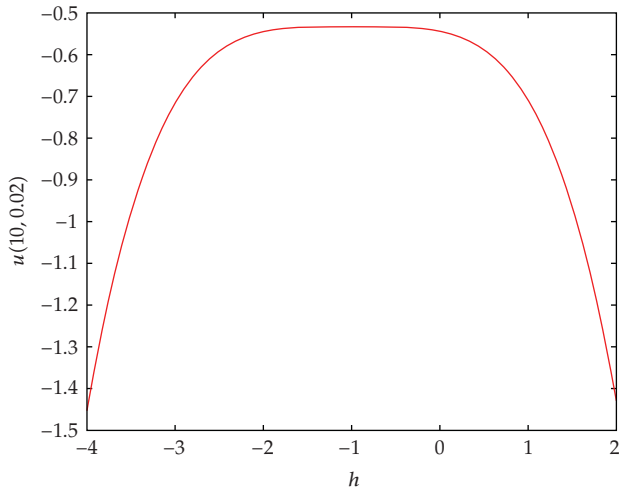


Figure 4: The h-curves obtained from the 5-order HAM approximate solution.

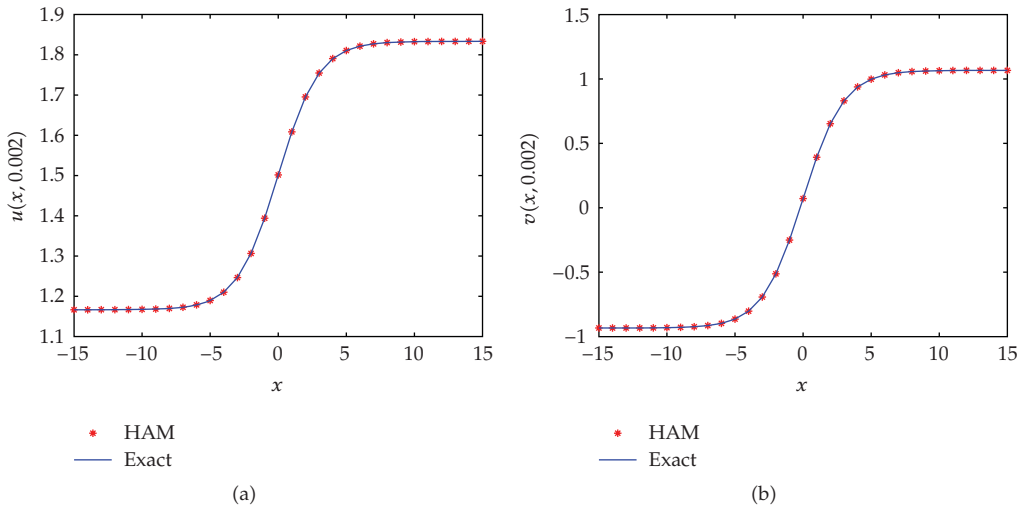
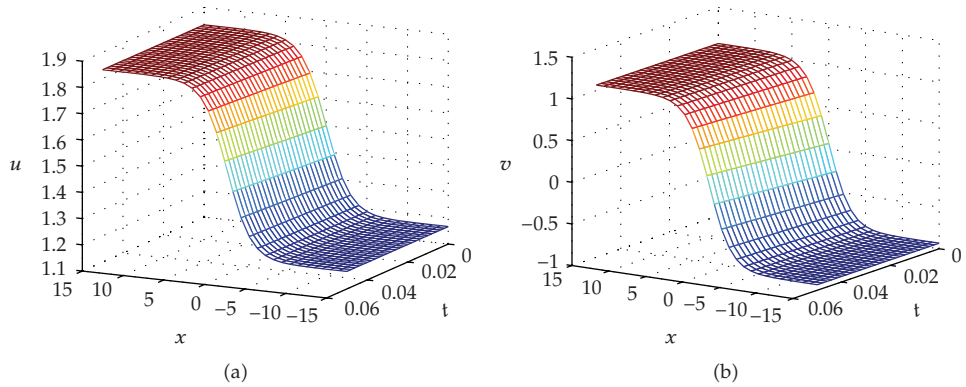


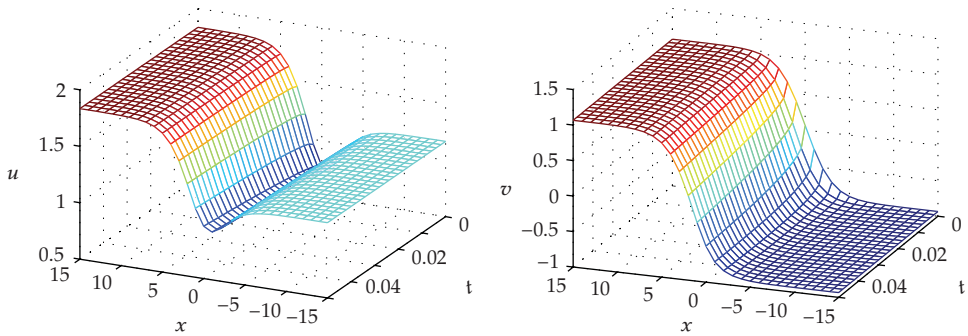
Figure 5: The comparison of the 6th-order HAM and exact solution with $= -1$, $\alpha = \beta = 1$, $\lambda = 0.1$, $b = 1$, and $k = 1/3$.

derivative. HAM provides us with a convenient way to control the convergence of approximation series by adapting h, which is a fundamental qualitative difference in analysis between HAM and other methods. The obtained results demonstrate the reliability of the HAM and its wider applicability to fractional differential equation. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems. MATLAB has been used for computations in this paper.



(a) (b)

Figure 6: Explicit numerical solutions with $h = -1$, $\alpha = \beta = 1$, $\lambda = 0.1$, $b = 1$, and $k = 1/3$.



(a) (b)

Figure 7: Explicit numerical solutions with $h = -1$, $\alpha = 1/2$, $\beta = 2/3$, $\lambda = 0.1$, $b = 1$, and $k = 1/3$.

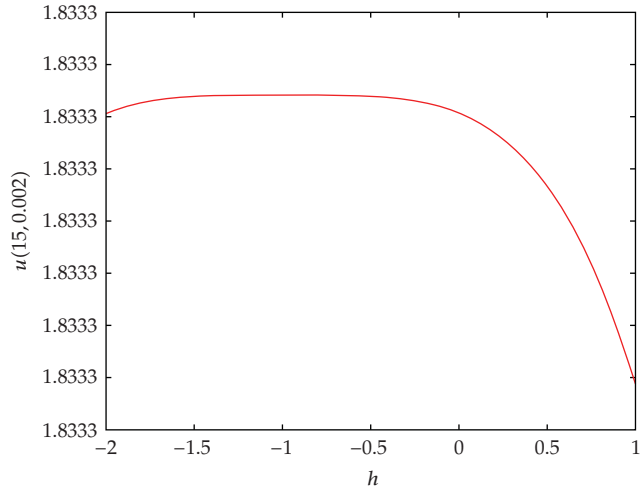


Figure 8: The h-curves obtained from the 5-order HAM approximate solution.

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Research Article

New Method for Solving Linear Fractional Differential Equations

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We develop a new application of the Mittag-Leffler Function method that will extend the application of the method to linear differential equations with fractional order. A new solution is constructed in power series. The fractional derivatives are described in the Caputo sense. To illustrate the reliability of the method, some examples are provided. The results reveal that the technique introduced here is very effective and convenient for solving linear differential equations of fractional order.

1. Introduction

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and engineering (see, e.g., [1–6]). In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors: the iteration method in [7], the series method in [8], the Fourier transform technique in [9, 10], special methods for fractional differential equations of rational order or for equations of special type in [11–16], the Laplace transform technique in [3–6, 16, 17], and the operational calculus method in [18–23]. Recently, several mathematical methods including the Adomian decomposition method [18–25], variational iteration method [23–26] and homotopy perturbation method [27, 28] have been developed to obtain the exact and approximate analytic solutions. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations, and some other methods give the solution in a series form which converges to the exact solution.

The reason of using fractional order differential (FOD) equations is that FOD equations are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results

derived from the fractional system are of a more general nature. Respectively, solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

The concept of fractional or noninteger order derivation and integration can be traced back to the genesis of integer order calculus itself [29]. Almost all of the mathematical theory applicable to the study of noninteger order calculus was developed through the end of the 19th century. However, it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation techniques in some cases meet the requirement of physical reality. The use of fractional differentiation for the mathematical modeling of real-world physical problems has been widespread in recent years, for example, the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, and measurement of viscoelastic material properties. Applications of fractional derivatives in other fields and related mathematical tools and techniques could be found in [30–41]. In fact, real-world processes generally or most likely are fractional order systems.

The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

2. Fractional Calculus

There are several approaches to the generalization of the notion of differentiation to fractional orders, for example, the Riemann-Liouville, Grünwald-Letnikov, Caputo, and generalized functions approach [42]. The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real-world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [42]. Unlike the Riemann-Liouville approach, which derives its definition from repeated integration, the Grünwald-Letnikov formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the upcoming paper and play the most important role in the theory of differential and integral equation of fractional order.

The main advantages of Caputo approach are the initial conditions for fractional differential equations with the Caputo derivatives taking on the same form as for integer order differential equations.

Definition 2.1. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$\begin{aligned} D^{\alpha} f(x) &= I^{m-\alpha} D^{m} f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha+1} f^{(m)}(t)\,dt \end{aligned} \tag{2.1}$$

for $m-1 < \alpha \leq m$, $m \in N$, $x > 0$.

For the Caputo derivative we have $D^{\alpha}C = 0$, C is constant,

$$D^{\alpha}t^{n} = \left\{ \begin{array}{ll} 0, & (n \leq \alpha - 1), \\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, & (n > \alpha - 1). \end{array} \right\}. \tag{2.2}$$

Definition 2.2. For m to be the smallest integer that exceeds α, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$$\begin{aligned} D^{\alpha}u(x,t) &= \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} \\ &= \left\{ \begin{array}{ll} \dfrac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha+1} \dfrac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{array} \right\}. \end{aligned} \tag{2.3}$$

3. Analysis of the Method

The Mittag-Leffler (1902–1905) functions E_{α} and $E_{\alpha,\beta}$ [42], defined by the power series

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0, \tag{3.1}$$

have already proved their efficiency as solutions of fractional order differential and integral equations and thus have become important elements of the fractional calculus theory and applications.

In this paper, we will explain how to solve some of differential equations with fractional level through the imposition of the generalized Mittag-Leffler function $E_{\alpha}(z)$. The generalized Mittag-Leffler method suggests that the linear term $y(x)$ is decomposed by an infinite series of components:

$$y = E_{\alpha}(ax^{\alpha}) = \sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{3.2}$$

We will use the following definitions of fractional calculus:

$$D^{\alpha}y = \sum_{n=1}^{\infty} a^n \frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha + 1)}, \tag{3.3}$$

$$D^{2\alpha}y = \sum_{n=2}^{\infty} a^n \frac{x^{(n-2)\alpha}}{\Gamma((n-2)\alpha + 1)}. \tag{3.4}$$

This is based on the Caputo fractional is derivatives. The convergence of the Mittag Leffler function discussed in [42].

4. Applications and Results

In this section, we consider a few examples that demonstrate the performance and efficiency of the generalized Mittag-Leffler function method for solving linear differential equations with fractional derivatives.

Example 4.1. Consider the following fractional differential equation [43]:

$$\frac{d^{\alpha}y}{dx^{\alpha}} = Ay. \tag{4.1}$$

By using (3.3) into (4.1) we find

$$\sum_{n=1}^{\infty} a^n \frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} - A\sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \tag{4.2}$$

Combining the alike terms and replacing (n) by $(n+1)$ in the first sum, we assume the form

$$\begin{gathered}\sum_{n=0}^{\infty} a^{n+1} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} - A\sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0,\\ \sum_{n=0}^{\infty} \left(a^{n+1} - Aa^n\right) \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0.\end{gathered} \tag{4.3}$$

With the coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recursive

$$\begin{aligned} &a^{n+1} - Aa^n = 0 \Longrightarrow a^{n+1} = Aa^n,\\ &\text{at } n = 0, \quad a^1 = Aa^0 = A,\\ &\text{at } n = 1, \quad a^2 = Aa^1 \Longrightarrow a^2 = A^2,\\ &\text{at } n = 2, \quad a^3 = Aa^2 \Longrightarrow a^3 = A^3. \end{aligned} \tag{4.4}$$

Substituting into (3.2)

$$\begin{aligned} y(x) &= a^0 + a^1 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + a^2 \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + a^3 \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots,\\ y(x) &= 1 + A \frac{x^{\alpha}}{\Gamma(\alpha+1)} + A^2 \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + A^3 \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \end{aligned} \tag{4.5}$$

The general solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{A^n x^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{4.6}$$

We can write the general solution in the Mittag-Leffler function form as

$$y(x) = E_\alpha(A^n x^\alpha). \tag{4.7}$$

As $\alpha = 1$, we have the exact solution:

$$y(x) = \sum_{n=0}^{\infty} \frac{(Ax)^n}{\Gamma(n+1)} = e^{Ax}, \tag{4.8}$$

which is the exact solution of the standard form.

Example 4.2. Consider the fractional differential equation [44]

$$\frac{d^{2\alpha} y}{dx^{2\alpha}} - y = 0. \tag{4.9}$$

By using (3.2) and (3.4) into (4.9) we find

$$\sum_{n=2}^{\infty} a^n \frac{x^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)} - \sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \tag{4.10}$$

Combining the alike terms and replacing (n) by $(n+2)$ in the first sum, we assume the form

$$\begin{gathered} \sum_{n=0}^{\infty} a^{n+2} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} - \sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \\ \sum_{n=0}^{\infty} \left(a^{n+2} - a^n\right) \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \end{gathered} \tag{4.11}$$

With the Coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recursive

$$a^{n+2} = a^n. \tag{4.12}$$

Substituting into (3.2), we find that:

$$y(x) = 1 + a\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + a^2 \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \tag{4.13}$$

If $a = 1$, we can write the general solution in the Mittag-Leffler function form as

$$y(x) = \sum_{n=0}^{\infty} \frac{x^\alpha}{\Gamma(n\alpha+1)} = E_\alpha(x^\alpha) \tag{4.14}$$

which is the exact solution of the linear fractional differential equation (4.9).

Example 4.3. Consider the fractional differential equation [43]

$$\frac{d^{2\alpha}y}{dx^{2\alpha}} + \frac{d^{\alpha}y}{dx^{\alpha}} - 2y = 0. \tag{4.15}$$

By using (3.2) and (3.4) into (4.15) we find

$$\sum_{n=2}^{\infty} a^n \frac{x^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)} + \sum_{n=1}^{\infty} a^n \frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} - 2\sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \tag{4.16}$$

Combining the alike terms and replacing (n) by $(n+2)$ in the first sum, we assume the form

$$\begin{gathered}\sum_{n=0}^{\infty} a^{n+2} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} + \sum_{n=0}^{\infty} a^{n+1} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} - 2\sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \\ \sum_{n=0}^{\infty} \left(a^{n+2} + a^{n+1} - 2a^n\right) \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0\end{gathered} \tag{4.17}$$

With the coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recursive

$$a^{n+2} = 2a^n - a^{n+1}. \tag{4.18}$$

Substituting into (3.2), we find that:

$$y(x) = 1 + a\frac{x^{\alpha}}{\Gamma(\alpha+1)} + (2-a)\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + (a-2)\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \tag{4.19}$$

If $a = 1$, we can write the general solution in the Mittag-Leffler function form as

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha}}{\Gamma(n\alpha+1)} = E_{\alpha}(x^{\alpha}) \tag{4.20}$$

which is the solution of the linear fractional differential equation (4.15).

5. Conclusions

A new generalization of the Mittag-Leffler function method has been developed for linear differential equations with fractional derivatives. The new generalization is based on the Caputo fractional derivative. It may be concluded that this technique is very powerful and efficient in finding the analytical solutions for a large class of linear differential equations of fractional order.

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Research Article

An Explicit Numerical Method for the Fractional Cable Equation

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An explicit numerical method to solve a fractional cable equation which involves two temporal Riemann-Liouville derivatives is studied. The numerical difference scheme is obtained by approximating the first-order derivative by a forward difference formula, the Riemann-Liouville derivatives by the Grünwald-Letnikov formula, and the spatial derivative by a three-point centered formula. The accuracy, stability, and convergence of the method are considered. The stability analysis is carried out by means of a kind of von Neumann method adapted to fractional equations. The convergence analysis is accomplished with a similar procedure. The von-Neumann stability analysis predicted very accurately the conditions under which the present explicit method is stable. This was thoroughly checked by means of extensive numerical integrations.

1. Introduction

Fractional calculus is a key tool for solving some relevant scientific problems in physics, engineering, biology, chemistry, hydrology, and so on [1–6]. A field of research in which the fractional formalism has been particularly useful is that related to anomalous diffusion processes [1, 7–13]. This kind of process is singularly abundant and important in biological media [14–16]. In this context, the electrodiffusion of ions in neurons is an anomalous diffusion problem to which the fractional calculus has recently been applied. The precise origin of the anomalous character of this diffusion process is not clear (see [17] and references therein), but in any case the consideration of anomalous diffusion in the modeling of electrodiffusion of ions in neurons seems pertinent. This problem has been addressed recently by Langlands et al. [17, 18]. An equation that plays a key role in their analysis is the following fractional cable (or telegrapher's or Cattaneo) equation (model II):

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}}\left(K\frac{\partial^2 u}{\partial x^2}\right) - \mu^2 \frac{\partial^{1-\gamma_2} u}{\partial t^{1-\gamma_2}}, \tag{1.1}$$

where

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}} f(t) \equiv \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{d\tau^n} \int_0^t d\tau \frac{f(\tau)}{(t-\tau)^{1+\gamma-n}}, \tag{1.2}$$

with $n-1 < \gamma < n$ and $n =$ integer, is the Riemann-Liouville fractional derivative. Here u is the difference between the membrane potential and the resting membrane potential, γ_1 is the exponent characterizing the anomalous flux of ions along the nerve cell, and γ_2 is the exponent characterizing the anomalous flux across the membrane [17, 18]. Some earlier fractional cable equations were discussed in [19, 20].

A variety of analytical and numerical methods to solve many classes of fractional equations have been proposed and studied over the last few years [10, 21–30]. Of the numerical methods, finite difference methods have been particularly fruitful [31–38]. These methods can be broadly classified as explicit or implicit [39]. An implicit method for dealing with (1.1) has recently been considered by Liu et al. [38]. Although implicit methods are more cumbersome than explicit methods, they usually remain stable over a larger range of parameters, especially for large timesteps, which makes them particularly suitable for fractional diffusion problems. Nevertheless, explicit methods have some features that make them widely appreciated [32, 39]: flexibility, simplicity, small computational demand, and easy generalization to spatial dimensions higher than one. Unfortunately, they can become unstable in some cases, so that it is of great importance to determine the conditions under which these methods are stable. In this paper we will discuss an explicit finite difference scheme for solving the fractional cable equation, which is close to the methods studied in [32, 33]. We shall address two main questions: (i) whether this kind of method can cope with fractional equations involving different fractional derivatives, such as the fractional cable equation; (ii) whether the von Neumann stability analysis put forward in [32, 34] is suitable for this kind of equation.

2. The Numerical Method

Henceforth, we will use the notation $x_j = j\Delta x$, $t_m = m\Delta t$, and $u(x_j, t_m) = u_j^{(m)} \simeq U_j^{(m)}$, where $U_j^{(m)}$ is the numerical estimate of the exact solution $u(x,t)$ at $x = x_j$ and $t = t_m$.

In order to get the numerical difference algorithm, we discretize the continuous differential and integro-differential operators as follows. For the discretization of the fractional Riemann-Liouville derivative we use the Grünwald-Letnikov formula

$$\frac{d^{1-\gamma}}{dt^{1-\gamma}} u(x,t) = \Delta_t^{1-\gamma} u(x, t_m) + O(\Delta t) \tag{2.1}$$

with

$$\Delta_t^{\alpha} f(t_m) = \frac{1}{(\Delta t)^{\alpha}} \sum_{k=0}^{m} \omega_k^{(\alpha)} f(t_{m-k}), \tag{2.2}$$

$$\omega_k^{(\alpha)} = \left(1 - \frac{1+\alpha}{k}\right) \omega_{k-1}^{(\alpha)}, \tag{2.3}$$

and $\omega_0^{(\alpha)} = 1$. These coefficients come from the generating function [40]

$$(1-z)^{\alpha} = \sum_{k=0}^{\infty} \omega_k^{(\alpha)} z^k. \tag{2.4}$$

To discretize the integer derivatives we use standard formulas: for the second-order spatial derivative we employ the three-point centered formula

$$\frac{\partial^2}{\partial x^2} u(x_j, t_m) = \Delta_x^2 u_j^{(m)} + O(\Delta x)^2 \tag{2.5}$$

with

$$\Delta_x^2 u_j^{(m)} = \frac{u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)}{(\Delta x)^2}, \tag{2.6}$$

and for the first-order time derivative we use the forward derivative

$$\frac{\partial}{\partial t} u(x_j, t_m) = \delta_t u_j^{m+1} + O(\Delta t), \tag{2.7}$$

where

$$\delta_t u_j^{m+1} = \frac{u(x_j, t_{m+1}) - u(x_j, t_m)}{\Delta t}. \tag{2.8}$$

Inserting (2.1), (2.5), and (2.7) into (1.1), one gets

$$\delta_t u_j^{(m+1)} - K \Delta_t^{1-\gamma_1} \left(\Delta_x^2 u_j^{(m)} \right) + \mu^2 \Delta_t^{1-\gamma_2} u_j^{(m)} = T(x,t), \tag{2.9}$$

where, as can easily be proved, the truncating error $T(x,t)$ is

$$T(x,t) = O(\Delta t) + O(\Delta x)^2. \tag{2.10}$$

Neglecting the truncating error we get the finite difference scheme we are seeking:

$$\delta_t U_j^{(m+1)} - K \Delta_t^{1-\gamma_1} \ \Delta_x^2 U_j^{(m)} + \mu^2 \Delta_t^{1-\gamma_2} U_j^{(m)} = 0, \tag{2.11}$$

that is,

$$U_j^{(m+1)} = U_j^{(m)} + S \sum_{k=0}^{m} \omega_k^{1-\gamma_1} \left(U_{j+1}^{(m-k)} - 2U_j^{(m-k)} + U_{j-1}^{(m-k)} \right) - \mu^2 (\Delta t)^{\gamma_2} \sum_{k=0}^{m} \omega_k^{1-\gamma_2} U_j^{(m-k)}, \tag{2.12}$$

where

$$S = K\frac{(\Delta t)^{\gamma_1}}{(\Delta x)^2}. \tag{2.13}$$

To test this algorithm, we solved (1.1) in the interval $-L/2 \le x \le L/2$, with absorbing boundary conditions, $u(x = -L/2, t) = u(x = L/2, t) = 0$, and initial condition given by a Dirac's delta function centered at $x = 0$: $u(x, 0) = \delta(x)$. The exact solution of this problem for $L \to \infty$ is [17]

$$u(x,t) = \frac{1}{\sqrt{4t^{\gamma_1}\pi}}\sum_{k=0}^{\infty}\frac{\left(-\mu^2 t^{\gamma_2}\right)^k}{k!}H_{1,2}^{2,0}\left[\frac{x^2}{4t^{\gamma_1}}\left|\begin{matrix}\left(1-\frac{\gamma_1}{2}+\gamma_2 k, \gamma_1\right)\\ (0,1), \left(\frac{1}{2}+k, 1\right)\end{matrix}\right.\right], \tag{2.14}$$

where H denotes the Fox H function [10, 41]. In our numerical procedure, the exact initial condition $u(x, 0) = \delta(x)$ is approximated by

$$u(x_j, 0) = \begin{cases} \dfrac{1}{\Delta x}, & j = 0, \\ 0, & j \neq 0. \end{cases} \tag{2.15}$$

The explicit difference scheme (2.12) is tested by comparing the analytical solution with the numerical solution for several cases of the problem described following (2.13) with different values of γ_1 and γ_2. We have computed the analytical solution by means of (2.14) truncating the series at $k = 20$. The corresponding Fox H function was evaluated by means of the series expansion described in [10, 41] truncating the infinite series after the first 50 terms. In Figures 1 and 2 we show the analytical and numerical solution for two values of γ_1 and γ_2 at $x = 0$ and $x = 0.5$. The differences between the exact and the numerical solution are shown in Figures 3 and 4. One sees that, except for very short times, the agreement is quite good. The large value of the error for small times is due in part to the approximation of the Dirac's delta function at $x = 0$ by (2.15). This is clearly appreciated when noticing the quite different scales of Figures 3 and 4: the error is much smaller for $x = 0.5$ than for $x = 0$. For the cases with $\gamma_1 = 1/2$ we used a smaller value of Δt and, simultaneously, a larger value of Δx than for the cases with $\gamma_1 = 1$ in order to keep the numerical scheme stable. This issue will be discussed in Section 3.

3. Stability

As usual for explicit methods, the present explicit difference scheme (2.12) is not unconditionally stable, that is, for any given set of values of γ_1, γ_2, μ, and K there are choices of Δx and Δt for which the method is unstable. Therefore, it is important to determine the conditions under which the method is stable. To this end, here we shall employ the fractional von Neumann stability analysis (or Fourier analysis) put forward in [32] (see also [33–35]). The question we address is to what extent this procedure is valid for fractional diffusion equations that involve fractional derivatives of different order.

Proceeding as [32], we start by recognizing that the solution of our problem can be written as the linear combination of subdiffusive modes, $u_j^{(m)} = \sum_q \zeta_q^{(m)} e^{iqj\Delta x}$, where the

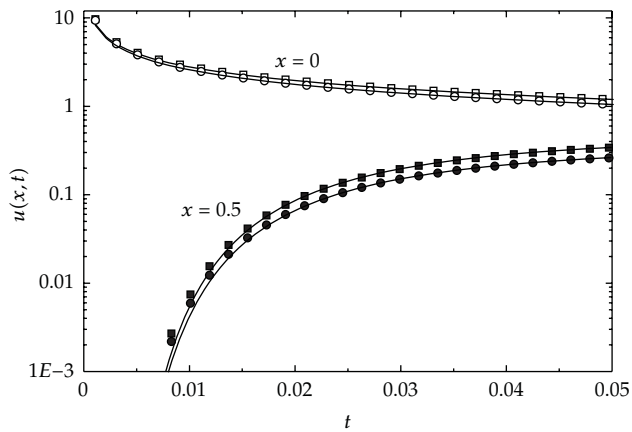

Figure 1: Numerical solution at the mid-point $x = 0$ (hollow symbols) and $x = 0.5$ (filled symbols) of the fractional cable equation for $\gamma_1 = 1$ and $\gamma_2 = 1$ (squares) and $\gamma_2 = 1/2$ (circles) with $\Delta x = 1/20$, $\Delta t = 10^{-4}$, $K = \mu = 1$, and $L = 5$. Lines are the exact solutions given by (2.14).

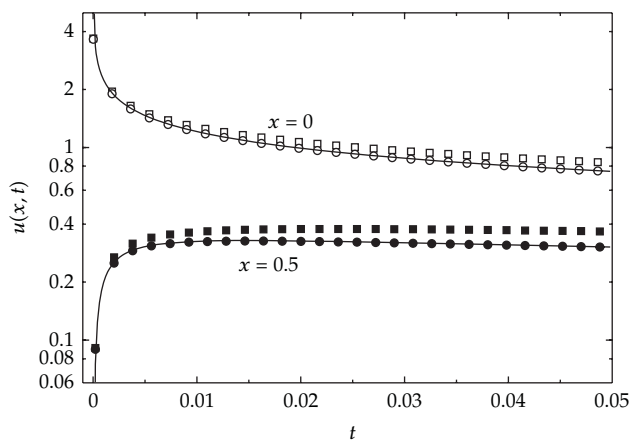

Figure 2: Numerical solution at the mid-point $x = 0$ (hollow symbols) and $x = 0.5$ (filled symbols) of the fractional cable equation for $\gamma_1 = 1/2$ and $\gamma_2 = 1$ (squares) and $\gamma_2 = 1/2$ (circles) with $\Delta x = 1/10$, $\Delta t = 10^{-5}$, $K = \mu = 1$, and $L = 5$. Lines are the exact solutions given by (2.14).

sum is over all the wave numbers q supported by the lattice. Therefore, following the von Neumann ideas, we reduce the problem of analyzing the stability of the solution to the problem of analyzing the stability of a single generic subdiffusion mode, $\zeta^{(m)} e^{iqj\Delta x}$. Inserting this expression into (2.12) one gets

$$\zeta^{(m+1)} = \zeta^{(m)} + S\sum_{k=0}^{m}\omega_k^{(1-\gamma_1)}\left(e^{iq\Delta x} - 2 + e^{-iq\Delta x}\right)\zeta^{(m-k)} - \mu^2(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{(1-\gamma_2)}\zeta^{(m-k)}. \tag{3.1}$$

The stability of the mode is determined by the behavior of $\zeta^{(m)}$. Writing

$$\zeta^{(m+1)} = \xi\zeta^{(m)} \tag{3.2}$$

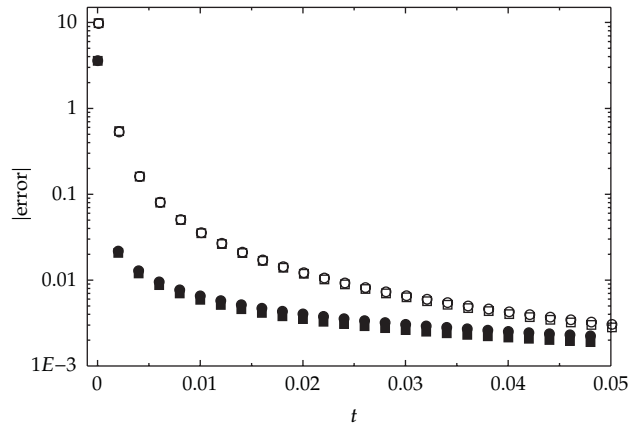

Figure 3: Absolute error $|U_j^{(m)} - u_j^{(m)}|$ of the numerical method for the problems considered in Figures 1 and 2 at $x = 0$. Squares: $\gamma_2 = 1$; circles: $\gamma_2 = 1/2$; hollow symbols: $\gamma_1 = 1$; filled symbols: $\gamma_1 = 1/2$.

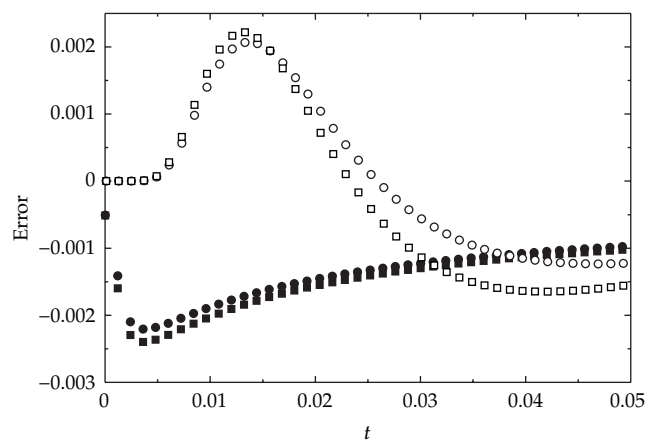

Figure 4: Error $U_j^{(m)} - u_j^{(m)}$ of the numerical method for the problem considered in Figures 1 and 2 at $x = 0.5$. Squares: $\gamma_2 = 1$; circles: $\gamma_2 = 1/2$; hollow symbols: $\gamma_1 = 1$; filled symbols: $\gamma_1 = 1/2$.

and assuming that the amplification factor ξ of the subdiffusive mode is independent of time, we get

$$\xi = 1 + S\sum_{k=0}^{m}\omega_k^{(1-\gamma_1)}\left(e^{iq\Delta x} - 2 + e^{-iq\Delta x}\right)\xi^{-k} - \mu^2(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{(1-\gamma_2)}\xi^{-k}. \tag{3.3}$$

If $|\xi| > 1$ for some q, the temporal factor of the solution grows to infinity [c.f., (3.2)], and the mode is unstable. Considering the extreme value $\xi = -1$, one gets from (3.3) that the numerical method is stable if this inequality holds:

$$\overline{S} \le S_\times^m = \frac{-2 + \mu^2(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{(1-\gamma_2)}(-1)^k}{-4\sum_{k=0}^{m}\omega_k^{(1-\gamma_1)}(-1)^k}, \tag{3.4}$$

where

$$\overline{S} = S\sin^2\left(\frac{q\Delta x}{2}\right). \tag{3.5}$$

If one defines $S_\times = \lim_{m\to\infty} S_\times^m$, one gets

$$\overline{S} \le S_\times = \frac{-2 + \mu^2(\Delta t)^{\gamma_2}\sum_{k=0}^{\infty}\omega_k^{(1-\gamma_2)}(-1)^k}{-4\sum_{k=0}^{\infty}\omega_k^{1-\gamma_1}(-1)^k}. \tag{3.6}$$

But from (2.4) with $z = -1$ one sees that $\sum_{k=1}^{\infty}(-1)^k\omega_k^{(1-\gamma)} = 2^{1-\gamma}$, so that

$$S_\times = \frac{2^{\gamma_2} - \mu^2(\Delta t)^{\gamma_2}}{2^{2+\gamma_2-\gamma_1}}. \tag{3.7}$$

Therefore, because $S \le \overline{S}$, we find that a sufficient condition for the present method to be stable is that $S \le S_\times$. In Figures 5 and 6 we show two representative examples of the problem considered in Figure 2 but for two values of S, respectively, larger and smaller than the stability bound provided by (3.7). One sees that the value of S that one chooses is crucial: when S is smaller than $S_\times$ one is inside the stable region and gets a sensible numerical solution (Figure 5); otherwise one gets an evidently unstable and nonsensical solution (Figure 6).

4. Numerical Check of the Stability Analysis

In this section we describe a comprehensive check of the validity of our stability analysis by using many different values of the parameters γ_1, γ_2, Δt, and Δx and testing whether the stability of the numerical method is as predicted by (3.7). Without loss of generality, we assume $\mu = K = 1$ in all cases. We proceed in the following way. First, we choose a set of values of γ_1, γ_2, Δx, and S and integrate the corresponding fractional cable equation. If

$$\left|U_j^{m-1} - U_j^m\right| > \lambda \tag{4.1}$$

for $\lambda = 10$ within the first 1000 integrations, then we say the method is unstable; otherwise, we label the method as stable. We generated Figure 7 by starting the integration for values of S well below the theoretical stability limit given by (3.7) and kept increasing its value by 0.001 until condition (4.1) was first reached. The last value for which the method was stable is recorded and plotted in Figure 7. The limit value $\lambda = 10$ is arbitrary, but choosing any other reasonable value does not significantly change these plots.

5. Convergence Analysis

In this section we show that the present numerical method is convergent, that is, that the numerical solution converges towards the exact solution when the size of the spatiotemporal

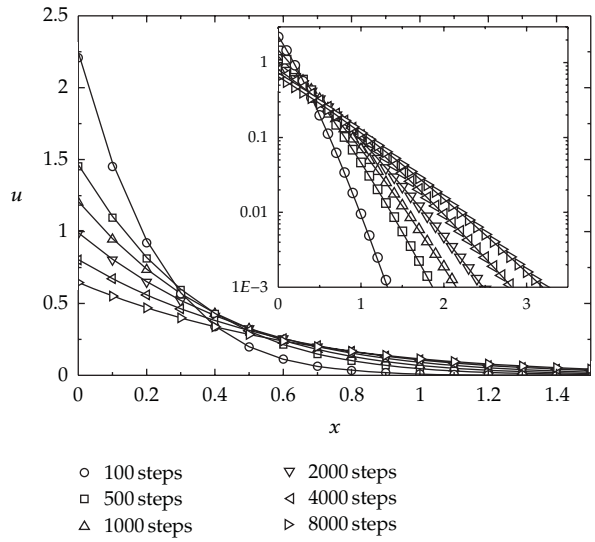

Figure 5: Exact solution (lines) and numerical solution (symbols) provided by our method for the fractional cable equation with $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ for different numbers of timesteps when $\Delta x = 1/10$, $\Delta t = 10^{-5}$, $K = \mu = 1$, $L = 5$, and $S = (\Delta t)^{\gamma_1}/(\Delta x)^2 = 0.316$. This case is inside the stability region because S is smaller than the stability limit $S_\times = (2^{\gamma_2} - \mu^2(\Delta t)^{\gamma_2})/(2^{2+\gamma_2-\gamma_1}) \simeq 0.352\ldots$ provided by (3.7). The inset shows the results on logarithmic scale.

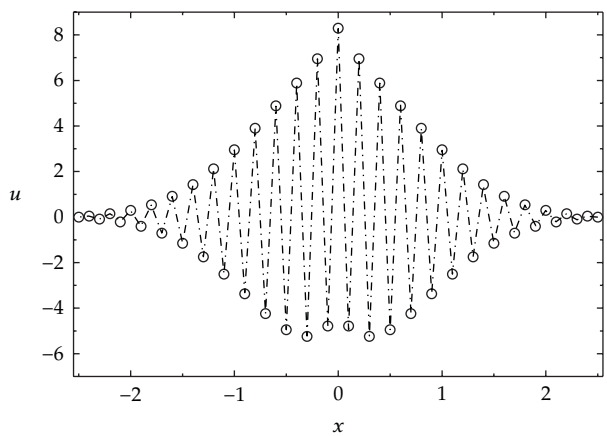

Figure 6: Numerical solution (circles) provided by our explicit method for the fractional cable equation with $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ after 100 timesteps when $\Delta x = 1/10$, $\Delta t = 1.3 \times 10^{-5}$, $K = \mu = 1$, $L = 5$, and $S = (\Delta t)^{\gamma}/(\Delta x)^2 = 0.36$. Note that this value is larger than the stability limit $S_\times = (2^{\gamma_2} - \mu^2(\Delta t)^{\gamma_2})/(2^{2+\gamma_2-\gamma_1}) \simeq 0.352\ldots$ provided by (3.7). The broken line is to guide the eye.

discretization goes to zero. Let us define $e_j^{(k)}$ as the difference between the exact and numerical solutions at the point (x_j, t_m): $e_j^{(k)} = u_j^{(k)} - U_j^{(k)}$. Taking into account (2.9) and (2.11), one gets the equation that describes how this difference evolves:

$$\begin{aligned} &e_j^{(m+1)} - e_j^{(m)} - S\sum_{k=0}^{m}\omega_k^{(1-\gamma_1)}\left(e_{j+1}^{(m-k)} - 2e_j^{(m-k)} + e_{j-1}^{(m-k)}\right) + \mu^2(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{(1-\gamma_2)}e_j^{(m-k)} \\ &\quad = T(x_j, t_k) \equiv T_j^{(m)}. \end{aligned} \tag{5.1}$$

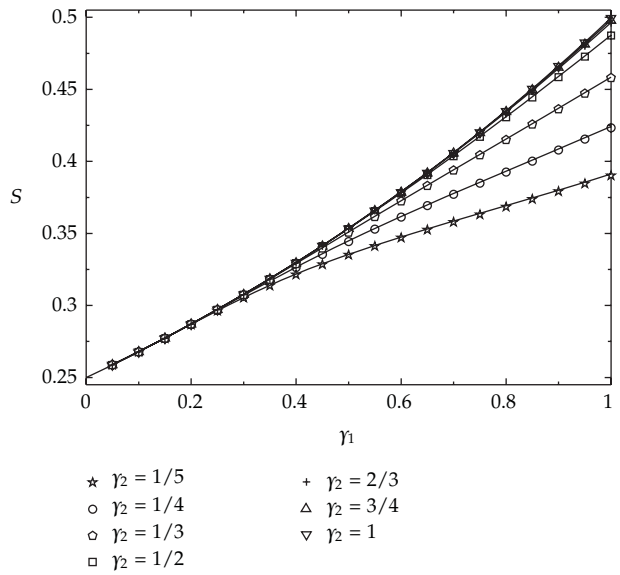

Figure 7: Stability bound S versus γ_1 for several values of γ_2 where $\Delta x = 1/20$, and $K = \mu = 1$. Symbols are numerical estimates. Lines correspond to the theoretical prediction (3.7).

As we did in the previous section for $U_j^{(k)}$, we write $e_j^{(k)}$ and $T_j^{(m)}$ as a combination of (sub) diffusion modes, $e_j^{(k)} = \sum_q \zeta_q^{(k)} e^{iqj\Delta x}$ and $T_j^{(m)} = \sum_q \chi_q^{(m)} e^{iqj\Delta x}$, and analyze the convergence of a single but generic q-mode [36, 39, 42]. Therefore, replacing $e_j^{(k)}$ by $\zeta^{(k)} e^{iqj\Delta x}$ and $T_j^{(m)}$ by $\chi^{(m)} e^{iqj\Delta x}$ in (5.1), we get

$$\zeta^{(m+1)} = \zeta^{(m)} + \overline{S}\sum_{k=0}^{m} \omega_k^{(1-\gamma_1)} \zeta^{(m-k)} + \mu^2 (\Delta t)^{\gamma_2} \sum_{k=0}^{m} \omega_k^{(1-\gamma_2)} \zeta^{(m-k)} + \chi^{(m)}. \tag{5.2}$$

Now we will prove by induction that $|\zeta^{(m)}| = O(\Delta t) + O(\Delta x)^2$ for all m. To start, $U_j^{(0)}$ satisfies the initial condition by construction, so that $e_j^{(0)} = 0$. This means that $\zeta^{(0)} = 0$. Therefore, from (5.2) one gets $\zeta^{(1)} = \chi^{(0)}$. But from (2.10) one knows that $|T_j^{(0)}| = |\chi^{(0)}| = O(\Delta t) + O(\Delta x)^2$, so that $|\zeta^{(1)}| = O(\Delta t) + O(\Delta x)^2$. Let us now assume that $|\zeta^{(k)}| = O(\Delta t) + O(\Delta x)^2$ holds for $k = 1, \ldots, m$. Then we will prove that $|\zeta^{(m+1)}| = O(\Delta t) + O(\Delta x)^2$. From (5.2) we obtain

$$\left|\zeta^{(m+1)}\right| \le \left|\chi^{(m)}\right| + \left|\zeta^{(m)}\right| + \overline{S}\left|\zeta^{\{m\}}\right| \left|\sum_{k=0}^{m} \left|\omega_k^{(1-\gamma_1)}\right| - \mu^2 (\Delta t)^{\gamma_2} \left|\zeta^{\{m\}}\right| \sum_{k=0}^{m} \left|\omega_k^{(1-\gamma_2)}\right|\right|, \tag{5.3}$$

where $|\zeta^{\{m\}}|$ is the maximum value of $|\zeta^{(k)}|$ for $k = 0, \ldots, m$. Taking into account (2.4), using the value $z = 1$, and because $\omega_0^{(\alpha)} = 1$, it is easy to see that $\sum_{k=1}^{\infty} \omega_k^{(\alpha)} = -1$ or, equivalently, $\sum_{k=1}^{\infty} |\omega_k^{(\alpha)}| = 1$ since $\omega_k^{(\alpha)} < 0$ for $k \ge 1$ (see (2.3)). Therefore $\sum_{k=0}^{m} |\omega_k^{(1-\gamma)}|$ is bounded (in

fact, it is smaller than 2). Using this result in (5.3), together with $|\zeta^{(k)}| \leq C(\Delta t + (\Delta x)^2)$ and $|\chi^{(k)}| \leq C(\Delta t + (\Delta x)^2)$, we find that

$$\left|\zeta^{(m+1)}\right| \leq C\left(\Delta t + (\Delta x)^2\right). \tag{5.4}$$

Therefore the amplitude of the subdiffusive modes goes to zero when the spatiotemporal mesh goes to zero. Employing the Parseval relation, this means that the norm of the error $\|e^{(k)}\|^2 \equiv \sum_j |e_j^{(k)}|^2 = \sum_q |\zeta_q^{(k)}|^2$ goes to zero when Δt and Δx go to zero. This is what we aimed to prove.

6. Conclusions

An explicit method for solving a kind of fractional diffusion equation that involves several fractional Riemann-Liouville derivatives, which are approximated by means of the Grünwald-Letnikov formula, has been considered. The method was used to solve a class of equations of this type (fractional cable equations) with free boundary conditions, Dirac's delta initial condition, and different fractional exponents. The error of the numerical method is compatible with the truncating error, which is of order $O(\Delta t) + O(\Delta x)^2$. It was also proved that the method is convergent. Besides, it was also found that a fractional von-Neumann stability analysis, which provides very precise stability conditions for standard fractional diffusion equations, leads also to a very accurate estimate of the stability conditions for cable equations.

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Research Article

Malliavin Calculus of Bismut Type for Fractional Powers of Laplacians in Semi-Group Theory

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We translate into the language of semi-group theory Bismut's Calculus on boundary processes (Bismut (1983), Lèandre (1989)) which gives regularity result on the heat kernel associated with fractional powers of degenerated Laplacian. We translate into the language of semi-group theory the marriage of Bismut (1983) between the Malliavin Calculus of Bismut type on the underlying diffusion process and the Malliavin Calculus of Bismut type on the subordinator which is a jump process.

1. Introduction

Let $X_0^1, X_1^1, \ldots, X_1^m, X_0^2, X_1^2, \ldots, X_2^m$ be $2m+2$ vector fields on $\mathbb{R}^d$ with bounded derivatives at each order. Let

$$\mathbb{L}^1 = \frac{\partial}{\partial s} + X_0^1 + \frac{1}{2}\sum_{i>0}\left(X_i^1\right)^2 \tag{1.1}$$

be an Hoermander's type operator on $\mathbb{R}^{1+d}$. Let

$$\mathbb{L}^2 = \frac{\partial}{\partial s} + X_0^2 + \frac{1}{2}\sum_{i>0}\left(X_i^2\right)^2 \tag{1.2}$$

be a second Hoermander's operator on $\mathbb{R}^{1+d}$. Bismut [1] considers the generator

$$\mathbb{A} = -\frac{1}{2}\sqrt{-2\mathbb{L}^1} - \frac{1}{2}\sqrt{-2\mathbb{L}^2} \tag{1.3}$$

and the Markov semi-group $\exp[t\mathbb{A}]$. This semi-group has a probabilistic representation. We consider a Brownian motion $t \to z_t$ independent of the others Brownian motions B_t^i. Bismut introduced the solution of the stochastic differential equation starting at x in Stratonovitch sense:

$$\begin{aligned} dx_t(x) = {} & \mathbb{I}_{z_t<0}\left(X_0^1(x_t(x))dt + \sum_{i>0} X_i^1(x_t(x))dB_t^i\right) \\ & + \mathbb{I}_{z_t>0}\left(X_0^2(x_t(x))dt + \sum_{i>0} X_i^2(x_t(x))dB_t^i\right), \end{aligned} \tag{1.4}$$

where $t \to B_t^i$ are m independent Brownian motions.

Let us introduce the local time $t \to L_t$ associated with $t \to z_t$ and its right inverse $t \to A_t$ (see [2, 3]). Then,

$$\exp[t\mathbb{A}]f(0,x) = E\left[f(A_t, x_{A_t}(x))\right]. \tag{1.5}$$

Such operator is classically related to the Dirichlet Problem [3].

Classically [4],

$$\exp\left[t\mathbb{L}^1\right]f(x) = E\left[f\left(x_t^1(x)\right)\right], \tag{1.6}$$

where $x_t^1(x)$ is the solution of the Stratonovitch differential equation starting at x:

$$dx_t^1(x) = X_0^1\left(x_t^1(x)\right)dt + \sum X_i^1\left(x_t^1(x)\right)dB_t^i, \tag{1.7}$$

The question is as following: is there an heat-kernel associated with the semi-group $\exp[t\mathbb{L}^1]$? This means that

$$\exp\left[t\mathbb{L}^1\right]f(x) = \int_{\mathbb{R}^d} f(y)p_t(x,y)dy. \tag{1.8}$$

There are several approaches in analysis to solve this problem, either by using tools of microlocal analysis or tools of harmonic analysis. Malliavin [5] uses the probabilistic representation of the semi-group. Malliavin uses a heavy apparatus of functional analysis (number operator on Fock space or equivalently Ornstein-Uhlenbeck operator on the Wiener space, Sobolev spaces on the Wiener space) in order to solve this problem.

Bismut [6] avoids using this machinery to solve this hypoellipticity problem. In particular, Bismut's approach can be adapted immediately to the case of the Poisson process [7]. The main difficulty to treat in the case of a Poisson process is the following: in general the solution of a stochastic differential equation with jumps is not a diffeomorphism when the starting point is moving (see [8–10]).

The main remark of Bismut in [1] is that if we consider the jump process $t \to x_{A_t}^1(x)$, then it is a diffeomorphism almost surely in x. So, Bismut mixed the tools of the Malliavin

Calculus for diffusion (on the process $t \to x_t^1(x)$) and the tools of the Malliavin Calculus for Poisson process (on the jump process $t \to A_t$) in order to show that this is the problem if

$$E\left[f(A_t, x_{A_t}(x))\right] = \int_{\mathbb{R}^{1+d}} q_t(s,y) f(s,y)\, ds\, dy. \tag{1.9}$$

Developments on Bismut's idea was performed by Léandre in [9, 11]. Let us remark that this problem is related to study the regularity of the Dirichlet problem (see [1, page 598]) (see [12–14] for related works).

Recently, we have translated into the language of semi-group theory the Malliavin Calculus of Bismut type for diffusion [15]. We have translated in semi-group theory a lot of tools on Poisson processes [16–22]. Especially, we have translated the Malliavin Calculus of Bismut type for Poisson process in semi-group theory in [17]. It should be tempting to translate in semi-group theory Bismut's Calculus on boundary process. It is the object of this work.

On the general problematic on this work, we refer to the review papers of Léandre [23–25]. It enters in the general program to introduce stochastic analysis tools in the theory of partial differential equation (see [26–28]).

2. Statements of the Theorems

Let us recall some basis on the study of fractional powers of operators [29]. Let $\mathbb{L}$ be a generator of a Markovian semi-group P_s. Then,

$$-\sqrt{-\mathbb{L}} = C \int_0^\infty \mathrm{s}^{-3/2}(P_s - \mathbb{I})ds. \tag{2.1}$$

The results of this paper could be extended to generators of the type

$$\mathbb{A} = \int_0^\infty g(s)(P_s - \mathbb{I})ds, \tag{2.2}$$

where $\int_0^\infty g(t) \wedge 1 dt < \infty$ and $g \geq 0$, but we have chosen the operator of the type (1.3) to be more closely related to the original intuition on Bismut's Calculus on boundary process. Let be $\mathbb{E}_d = \mathbb{R}^{1+d} \times \mathbb{G}_d \times \mathbb{M}_\mathrm{d}$ where $\mathbb{G}_d$ is the space of invertible matrices on $\mathbb{R}^d$ and $\mathbb{M}_d$ the space of symmetric matrices on $\mathbb{R}^d$. (s, x, U, V) is the generic element of $\mathbb{E}_d$. V is called the Malliavin matrix.

On $\mathbb{E}_d$, we consider the vector fields:

$$\begin{aligned} \widehat{X}_i^1 &= \left(0, X_i, DX_i^1(x)U, 0\right), \\ \widehat{Y}^1 &= \left(0, 0, 0, \sum_{i=1}^m \left\langle U^{-1}X_i, \cdot \right\rangle^2\right). \end{aligned} \tag{2.3}$$

We consider the Malliavin generator $\widehat{\mathbb{L}}^1$ on $\mathbb{E}_d$:

$$\widehat{L}^1 = \frac{\partial}{\partial s} + \widehat{X}_0^1 + \frac{1}{2}\sum_{i>0}\left(\widehat{X}_i^1\right)^2 + \widehat{Y}^1. \tag{2.4}$$

We consider the Malliavin semi-group $\widehat{P}_t^1$ associated and $\sqrt{-\widehat{L}^1}$.

We perform the same algebraic considerations on $\mathbb{L}^2$. We get $\widehat{L}^2$, $\widehat{P}_t^2$, and $\sqrt{-\widehat{L}^2}$. Let us consider the total generator

$$\widehat{\mathbb{A}} = -\sqrt{-\widehat{L}^1} - \sqrt{-\widehat{L}^2} \tag{2.5}$$

and the Malliavin semi-group $\exp[t\widehat{\mathbb{A}}]$.

We get a theorem which enters in the framework of the Malliavin Calculus for heat-kernel.

Theorem 2.1. *Let one suppose that the Malliavin condition in x is checked:*

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[\det V^{-p}\right](0,x,I,0) < \infty \tag{2.6}$$

holds for all p, then

$$\exp[t\mathbb{A}]f(0,x) = \int_{\mathbb{R}^{1+d}} f(s,y)q_t(s,y)\,ds\,dy, \tag{2.7}$$

where $q_t(s,y)$ is the density of a probability measure on $\mathbb{R}^{1+d}$.

Theorem 2.2. *If the quadratic form*

$$\sum_{i>0}\left\langle X_i^1(x),\cdot\right\rangle^2 + \sum_{i>0}\left\langle X_i^2(x),\cdot\right\rangle^2 \tag{2.8}$$

is invertible in x, then the Malliavin condition holds in x.

Remark 2.3. We give simple statements to simplify the exposition. It should be possible by the method of this paper to translate the results of [9, part III], got by using stochastic analysis as a tool.

3. Integration by Parts on the Underlying Diffusion

We consider the vector fields on $\mathbb{R}^{1+d+1}$,

$$X_{i,s,t}^{j,1} = \left(0, X_i^j(x), Z_{i,s,t}^j\right), \tag{3.1}$$

where $Z^j_{i,s,t} = \langle \phi(x), h^j_{s,t} \rangle_i$ ($\phi(x)$ is a convenient matrix on $\mathbb{R}^m$ which depends smoothly on x and whose derivatives at each order are bounded. $(s,t) \to h^j_{s,t}$ does not depend on x, and $h^j_{s,t}$ belong to $\mathbb{R}^m$). Let $\tilde{f}$ be a smooth function on $\mathbb{R}^{1+d+1}$, $\tilde{D}\tilde{f}$ denotes its gradient, and $\tilde{D}^2\tilde{f}$ denotes its Hessian.

We consider the generator $\mathbb{L}^{j,1}_{s,t}$ acting on smooth functions on $\mathbb{R}^{1+d+1}$,

$$\begin{aligned}\mathbb{L}^{j,1}_{s,t}\tilde{f} = \frac{\partial}{\partial s}\tilde{f} + \left\langle X^j_0(x), \tilde{D}\tilde{f}\right\rangle + \frac{1}{2}\sum_{i>0}\left\langle DX^j_i(x)X^j_i(x), \tilde{D}\tilde{f}\right\rangle \\ + \frac{1}{2}\sum_{i>0}\left\langle X^{j,1}_{i,s,t}, \tilde{D}^2\tilde{f}, X^{j,1}_{i,s,t}\right\rangle.\end{aligned} \tag{3.2}$$

In (3.2), the generator is written under Itô's form. It generates a time inhomogeneous in the parameter s semi-group $P^{j,1}_{s,t}$. We can consider

$$-\sqrt{-\mathbb{L}^{j,1}_{\cdot,t}} = C\int_0^\infty s^{-3/2}\left(P^{j,1}_{s,t} - \mathbb{I}\right)ds. \tag{3.3}$$

We put

$$\mathbb{A}^1_t = -\sqrt{-\mathbb{L}^{j,1}_{\cdot,t}} - \sqrt{-\mathbb{L}^{j,2}_{\cdot,t}}. \tag{3.4}$$

It generates a semi-group P^1_t.

Let us consider the Hoermander's type generator associated with the smooth Lipschitz vector fields on $\mathbb{R}^{1+d+d}$ ((s,x,U) on $\mathbb{R}^{1+d+d}$):

$$\begin{aligned}X^{j,2}_i &= \left(0, X^j_i, DX^j_i U\right), \\ Y^{j,2}_{0,s,t} &= \left(0, 0, \sum X^j_i(x)Z^j_{i,s,t}\right) = \left(0, 0, Y^j_{i,s,t}\right), \\ \mathbb{L}^{j,2}_{s,t} &= X^{j,2}_0 + \frac{1}{2}\sum_{i>1}\left(X^{j,2}_i\right)^2 + Y^{j,2}_{0,s,t}.\end{aligned} \tag{3.5}$$

We consider the heat semi-group associated with $\mathbb{L}^{j,2}_{s,t}$

$$\frac{\partial}{\partial s}P^{j,2}_{s,t}\tilde{f} = \mathbb{L}^{j,2}_{s,t}P^{j,2}_{s,t}\tilde{f}. \tag{3.6}$$

Let us recall [15, Theorem 2.2] that

$$P^{j,1}_{s,t}[uf](s_0, x_0, 0) = P^{j,2}_{s,t}[\langle Df, U\rangle](s_0, x_0, 0), \tag{3.7}$$

where f depends only on (s,x). In the left-hand side of (3.7), we apply the enlarged semi-group to the test function $(s,x,u) \to f(s,x)u$ and in the right-hand side we apply the semi-group to the test function $(s,x,U) \to \langle Df, U\rangle$. u belongs to $\mathbb{R}$ and U belongs to $\mathbb{R}^d$. From this, we deduce the following.

Lemma 3.1. *One has the relation*

$$-\mathbb{L}_{\cdot,t}^{j,1}[uf](s_0,x_0,0) = -\mathbb{L}_{\cdot,t}^{j,2}[\langle Df, U\rangle](s_0,x_0,0). \tag{3.8}$$

Let us consider the semi-group P_t^2 associated with

$$\mathbb{A}_t^2 = -\sqrt{-\mathbb{L}_{\cdot,t}^{1,2}} - \sqrt{-\mathbb{L}_{\cdot,t}^{2,2}}. \tag{3.9}$$

We get, with the same notations for (s,x,u,U) the following.

Theorem 3.2. *For f bounded continuous with compact support in (s,x), one has the following relation:*

$$P_t^2[\langle Df, U\rangle](s_0,x_0,0) = P_t^1[fu](s_0,x_0,0). \tag{3.10}$$

Proof. For the integrability conditions, we refer to the appendix.

We remark that $\partial/\partial u$ commute with $\mathbb{A}_t^1$, therefore with P_t^1. We deduce that

$$P_t^1[fu](s_0,x_0,u_0) = u_0 \exp[t\mathbb{A}][f](s_0,x_0) + P_t^1[fu](s_0,x_0,0). \tag{3.11}$$

By the method of variation of constants,

$$P_t^1[fu](s_0,x_0,0) = \int_0^t \exp[(t-s)\mathbb{A}]\Big[\mathbb{A}_s^1[u \exp[s\mathbb{A}][f](\cdot,\cdot,0)]\Big](s_0,x_0)ds. \tag{3.12}$$

In order to show that, we follow the lines of (2.17) and (2.18) in [15]. We apply $\mathbb{A}_t^1$ to (3.11).

By Lemma 3.1,

$$\mathbb{A}_s^1[u \exp[s\mathbb{A}][f](\cdot,\cdot)](s_1,x_1,0) = \mathbb{A}_s^2[\langle D(\exp[s\mathbb{A}]), U\rangle](s_1,x_1,0). \tag{3.13}$$

Let us consider the vector fields on $\mathbb{R}^{1+d} \times \mathbb{G}_d$,

$$X_i^{j,3} = \left(0, X_i^j, DX_i^j U\right). \tag{3.14}$$

We consider the Hoermander's type operator associated with these vector fields:

$$\mathbb{L}^{j,3} = X_0^{j,3} + \frac{1}{2}\sum_{i>0}\left(X_i^{j,3}\right)^2. \tag{3.15}$$

We consider the generator

$$\mathbb{A}_t^3 = -\sqrt{-\mathbb{L}_{s,t}^{1,3}} - \sqrt{-\mathbb{L}_{s,t}^{2,3}}. \tag{3.16}$$

It generates a semi-group P_t^3. By lemma 3.2 of [15], we have

$$D\ \exp[s\mathbb{A}][f](s_1, x_1) = P_s^3[DfV](s_1, x_1, I). \tag{3.17}$$

By [15, Equation (3.18)],

$$P_{s,t}^{j,2}\left[P_t^3[DfU](\cdot, I)V\right](s_1, x_1, 0) = \sum_i \int_0^s P_{s-v,t}^j \left[\sum_i \left\langle Y_{i,v,t}^j, P_{v,t}^{j,3}\left[P_t^3[DfU](\cdot, I)\right]\right\rangle\right](s_1, x_1, 0). \tag{3.18}$$

In [15, Equation (3.18)], we consider the semi-group $\overline{P}_t^j$ instead of the semi-group $P_{s,t}^{j,2}$ and the test function Df instead as of the test function $P_t^3[DfU](\cdot, I)$ here. $Y_{i,v,t}^j$ is considered as an element of $\mathbb{R}$ and not as a one-order differential operator:

$$\frac{\partial}{\partial s} P_{s,t}^{j,3} \tilde{f} = \mathbb{L}_{s,t}^{j,3} P_{s,t}^{j,3} \tilde{f}. \tag{3.19}$$

Therefore,

$$\begin{aligned} &\mathbb{A}_t^2\left[P_t^3[DfV](\cdot, I)V\right](s_1, x_1, 0) \\ &\quad = \sum_{i,j} C \int_0^\infty s^{-3/2} \int_0^s P_{s-v,t}^j \left[\left\langle \sum_i Y_{i,v,t}^j, P_{v,t}^{j,3}\left[P_t^3[DfU](\cdot, I)\right]\right\rangle\right](s_1, x_1, 0)\, dv\, ds. \end{aligned} \tag{3.20}$$

We write

$$\mathbb{A}_t^2 = \mathbb{A}_t^3 + \tilde{A}_t^3, \tag{3.21}$$

where

$$\tilde{A}_t^3[fU](s_0, x_0, U_0) = \sum_j C \int_0^\infty s^{-3/2}\left(P_{s,t}^{j,2} - P_{s,t}^{j,3}\right)[fU](s_0, x_0, U_0)\, ds. \tag{3.22}$$

The Volterra expansion (see [15, Equation (3.17)]) if it converges gives the following formula:

$$\begin{aligned} P_{s,t}^{j,2}[fU](s_0, x_0, U_0) = &\sum \int_{0<s_1<s_2<\cdots<s_n<t} ds_1 \cdots ds_n P_{s_1}^{j,3} \sum Y_{i,s_1,t}^j \cdots P_{s_n - s_{n-1}}^{j,3} \\ &\times \sum Y_{i,s_n,t}^j \cdots P_{t-s_n}^{j,3}[fU](s_0, x_0, U_0). \end{aligned} \tag{3.23}$$

But $u_0 \to P^{j,3}_{s,t}[fU](s_0, x_0, U_0)$ is linear in u_0. Therefore:

$$P^{j,2}_{s,t}[fU](s_0, x_0, U_0) = P^{j,3}_{s,t}[fU](s_0, x_0, U_0) + \int_0^s P^j_v \left\langle \sum_i Y^j_{i,v,t}, P^{j,3}_{s-v,t}[fU] \right\rangle (s_0, x_0, U_0) dv. \tag{3.24}$$

In this last formula, $Y^j_{i,s,t}$ are considered as differential operators.

Therefore, $\widetilde{A}^3_t[fU](s_0, x_0, U_0)$ does not depend on U_0 and is equal to

$$\sum_{i,j} C \int_0^\infty s^{-3/2} \int_0^s P^j_v \left\langle \sum_i Y^j_{i,v,t}, P^{j,3}_{s-v,t}[fU](s_0, x_0, I) \right\rangle ds\, dv, \tag{3.25}$$

where $Y^j_{i,s,t}$ are considered as elements of $\mathbb{R}^d$. We deduce as in [15, Equation (3.17)],

$$P^2_t[fU](s_0, x_0, 0) = \int_0^t \exp[(t-s)\mathbb{A}] \widetilde{A}^3_s P^3_s[fU](s_0, x_0, 0) ds. \tag{3.26}$$

But $U_0 \to P^3_s[fU](s_0, x_0, U_0)$ is linear. Therefore,

$$\widetilde{A}^3_t P^3_t[fU](s_0, x_0, 0) = \sum_{i,j} C \int_0^\infty s^{-3/2} \int_0^s P^j_v \left\langle \sum_i Y^j_{i,v,t}, P^{j,3}_{s-v,t}\left[P^3_t[fU]\right](s_0, x_0, I) \right\rangle ds\, dv. \tag{3.27}$$

It remains to replace f by Df in this last equation and to compare (3.26) with (3.13) and (3.20). □

We consider the Malliavin generator $\widehat{\mathbb{A}}$. We can perform the same algebraic construction as in Theorem 3.2. We get two semi-groups $\widehat{P}^2_t$ and $\widehat{P}^1_t$. $\widehat{Y}^j_{i,s,t}$ and $\widehat{Z}^j_{i,s,t}$ are smooth with bounded derivatives in $\widehat{x} = (x, U, U^{-1}, V)$. We get by the same procedure the following.

Theorem 3.3. *If $\widehat{f}$ is bounded with bounded derivatives and with compact support in s, then one gets*

$$\widehat{P}^2_t\left[\left\langle D\widehat{f}, \widehat{U}\right\rangle\right](s_0, \widehat{x}, 0) = \widehat{P}^1_t\left[\widehat{f}\widehat{u}\right](s_0, \widehat{x}, 0), \tag{3.28}$$

where one take does not derivative in the direction of s in $D\widehat{f}$.

We can perform the same improvements as in [15, page 512]. We define on $\mathbb{R}^d \times \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$ some vectors fields:

$$X^{j,\mathrm{tot}}_i = \left(X^{j,1}_i(x_1), \ldots, X^{j,l}_i(x_1, \ldots, x_l), X^{j,k}_i(x_1, \ldots, x_k)\right), \tag{3.29}$$

where

$$X_i^{j,l}\left(x^1,\ldots,x^l\right) = X_{1,i}^{j,l}\left(x^1,\ldots,x^{l-1}\right)x^l\frac{\partial}{\partial x^l} + X_{2,i}^{l}\left(x^1,\ldots,x^l\right)\frac{\partial}{\partial x^l} + X_{3,i}^{l}\left(x^1,\ldots,x^{l-1}\right) \tag{3.30}$$

where $X_{1,i}^{j,l}$, $X_{2,i}^{j,l}$ have derivatives bounded at each order and $X_{3,i}^{j,l}$ has derivative with polynomial growth.

We can consider the generator $\widehat{\mathbb{A}}^{\text{tot}}$ associated with these vector fields and perform the same algebraic computations as in Theorem 3.2. We get two semi-groups $\widehat{P}_t^{2,\text{tot}}$ and $\widehat{P}_t^{1,\text{tot}}\cdot\widehat{Y}_{i,s,t}^{j}$ and $\widehat{Z}_{i,s,t}^{j}$ are smooth with bounded derivatives in $\widehat{x} = (x, U, U^{-1}, V)$. We get by the same procedure the following.

Theorem 3.4. *If $\widehat{f}^{\text{tot}}$ is bounded with bounded derivatives and with compact support in s, then one gets*

$$\widehat{P}_t^{2,\text{tot}}\left[\left\langle D\widehat{f}^{\text{tot}},\widehat{U}\right\rangle\right](s_0,\widehat{x}^{\text{tot}},0) = \widehat{P}_t^{1,\text{tot}}\left[\widehat{f}^{\text{tot}}\widehat{u}\right](s_0,\widehat{x}^{\text{tot}},0), \tag{3.31}$$

where $D\widehat{f}^{\text{tot}}$ does not include derivative in the direction of s.

We refer to the appendix for the proof and the subsequent estimates.

Remark 3.5. Let us show from where come these identities, by using (1.4): we consider a time interval $[A_{t-}, A_t]$. On this random time interval, we do the following translation on the leading Brownian motion B_s^i:

(i) if $z_s > 0$ on this time interval, then dB_s^i is transformed in $dB_s^i + \lambda\langle\phi(x_s), h_{s,t}^2\rangle_i ds$ for a small parameter λ,

(ii) if $z_s < 0$ on this time interval, then dB_s^i is transformed in $dB_s^i + \lambda\langle\phi(x_s), h_{s,t}^1\rangle_i ds$ for a small parameter λ.

According to the fact that f has compact support (this means that we consider bounded values of A_t), the transformed Brownian motion has an equivalent law through the Girsanov exponential to the original Brownian motions. The term in u in Theorem 3.2 come that from the fact we take the derivative in $\lambda = 0$ of the Girsanov exponential. When we do this transformation, we get a random process $x_t^\lambda(x)$. Derivation of it in $\lambda = 0$ is done classically according to the stochastic flow theorem, which leads to the study of generators of the type $\mathbb{L}_{s,t}^{j,2}$ and of the type $\mathbb{L}^{j,3}$.

4. Integration by Parts on the Subordinator

Let us consider diffusion type generator of the previous part:

$$\begin{aligned}\mathbb{L} &= Y_0 + \frac{1}{2}\sum Y_i^2,\\ \mathbb{L}^{\sqrt{t}} &= \left(\sqrt{t}\right)^2 Y_0 + \frac{1}{2}\sum_{i>0}\left(\sqrt{t}Y_i\right)^2.\end{aligned} \tag{4.1}$$

Let us consider the semi-group

$$\frac{\partial}{\partial t} P_t = \mathbb{L} P_t \tag{4.2}$$

and the semi-group

$$\frac{\partial}{\partial s} P_s^{\sqrt{t}} = \mathbb{L}^{\sqrt{t}} P_s^{\sqrt{t}}. \tag{4.3}$$

We have classically

$$P_t = P_1^{\sqrt{t}}, \tag{4.4}$$

where the smooth vector fields are Lipschitz.

Therefore, we can write

$$-\sqrt{-\mathbb{L}} = C \int_0^\infty s^{-3/2} \left(P_1^{\sqrt{s}} - \mathbb{I} \right) ds. \tag{4.5}$$

We consider a diffeomorphsim $f_\lambda(s)$ of $[0, \infty[$ with bounded derivative of first order in λ equal to s if $s < \epsilon$ and equals to s if $s > 2$ (we suppose λ small). We can write

$$\sqrt{-\mathbb{A}^\lambda} = C \int_0^\infty \left(f_\lambda(s))^{-3/2} P_1^{\sqrt{f_\lambda(s)}} - s^{-3/2} \mathbb{I} \right) ds. \tag{4.6}$$

We do this operation on the two operators on $\mathbb{R}^{1+d}$ giving $\mathbb{A}$. We get a generator $\mathbb{A}^\lambda$.

According the line of stochastic analysis, we consider a generator $\mathbb{A}^{\lambda,1}$ on $\mathbb{R}^{1+d+1}$. If $\mathbb{L}^1$ is a generator on $\mathbb{R}^{1+d}$ with associated semi-group P_s, then we consider $\mathbb{A}^{\lambda,1}$ the generator on $\mathbb{R}^{1+d+1}$,

$$\mathbb{A}^{\lambda,1} f(s_0, x_0, u_0) = \sum_j C \int_0^\infty \left(f_\lambda(s)^{-3/2} \left[P_1^{j,\sqrt{f_\lambda(s)}} f(s_0, x_0, u_0 J_\lambda(s)) \right] - s^{-3/2} f(s_0, x_0, u_0) \right) ds, \tag{4.7}$$

where $J_\lambda(s)$ is the Jacobian of the transformation $s \to f_\lambda(s)$. By doing this procedure in (1.3), we deduce a global generator $\mathbb{A}^{\lambda,1}$ and a semi-group $P_t^{\lambda,1}$ associated with it.

It is not clear that $P_t^{\lambda,1}$ is a Markovian semi-group. We decompose

$$\mathbb{A}^{\lambda,1} = \mathbb{A}^{\lambda,1,\epsilon} + \mathbb{A}^{\lambda,1,\epsilon^c}, \tag{4.8}$$

where

$$\mathbb{A}^{\lambda,1,\epsilon} f(s_0, x_0, u_0) = \sum_j C \int_0^\epsilon \left(s^{-3/2} P_1^{j,\sqrt{s}} f(s_0, x_0, u_0) - s^{-3/2} f(s_0, x_0, u_0) \right) ds. \tag{4.9}$$

$\mathbb{A}^{\lambda,1,\epsilon}$ generates a Markovian semi-group $P_t^{\lambda,1,\epsilon}$. But $\mathbb{A}^{\lambda,1,\epsilon^c}$ is a bounded operator on the set of bounded continuous functions on $\mathbb{R}^{1+d+1}$ endowed with the uniform norm. The Volterra expansion converges on this set:

$$P_t^{\lambda,1} f(s_0, x_0, u_0) = P_t^{\lambda,1,\epsilon} f(s_0, x_0, u_0) + \sum_n \int_{0<s_1<\cdots<s_n<t} P_{s_1}^{\lambda,1,\epsilon} \mathbb{A}^{\lambda,1,\epsilon^c} P_{s_2-s_1}^{\lambda,1,\epsilon} \cdots \mathbb{A}^{\lambda,1,\epsilon^c} P_{t-s_n}^{\lambda,1,\epsilon} ds_1 \cdots ds_n. \tag{4.10}$$

Theorem 4.1 (Girsanov). *For f with compact support in (s, x), one has*

$$P_t^{\lambda,1}[uf](s_0, x_0, 1) = \exp[t\mathbb{A}][f](s_0, x_0). \tag{4.11}$$

Proof. By linearity,

$$P_t^{\lambda,1}[uf](s_0, x_0, u_0) = u_0 P_t^{\lambda,1}[uf](s_0, x_0, 1). \tag{4.12}$$

But by an elementary change of variable,

$$\mathbb{A}^{\lambda,1}\left[uP_t^{\lambda,1}[uf](\cdot,\cdot,1)\right] = \mathbb{A}\left[P_t^{\lambda,1}[uf](\cdot,\cdot,1)\right]. \tag{4.13}$$

The result holds by the unicity of the solution of the parabolic equation associated with $\mathbb{A}$. To state the integrability of u, we refer to [16]. □

Remark 4.2. Let us show from where this formula comes. In the previous part, we have done a perturbation of the leading Brownian motion B_t^i. Here, we do a perturbation of ΔA_s into $f_\lambda(\Delta A_s) = \Delta A_s^\lambda$. By standard result on Levy processes, the law of the Levy process A_t^λ is equivalent to the law of A_t. Moreover, A_t^λ and B_t^i are independents. Therefore, the result.

Bismut's idea to state hypoellipticity result is to take the derivative in λ of

$$P_t^{\lambda,1}[uf](s_0, x_0, 1) = \exp[t\mathbb{A}][f](s_0, x_0) \tag{4.14}$$

in order to get an integration by parts.

First of all, let us compute $(\partial/\partial\lambda)P_t^{\sqrt{f_\lambda(s)}} f$ in $\lambda = 0$. It is fulfilled by the next considerations. Let us consider a generator written under Hoermander's form:

$$\mathbb{L}^\lambda = g_\lambda Y_0 + \frac{1}{2} g_\lambda^2 \sum_{i>0} Y_i^2, \tag{4.15}$$

where g_λ are smooth and where the vector fields Y_i are smooth Lipschitz on $\mathbb{R}^{\tilde{d}}$. We consider the semi-group $P_t^{\lambda,\cdot}$ associated with it. Let us introduce the vector fields on $\mathbb{R}^{\tilde{d}+\tilde{d}}$:

$$Y_i^{\lambda,1} = \left(g_\lambda Y_i, g_\lambda DY_i U + \frac{d}{d\lambda} g_\lambda Y_i\right). \tag{4.16}$$

Let us consider the diffusion generator

$$\mathbb{L}^{\lambda,1} = Y_0^{\lambda,1} + \frac{1}{2}\sum_{i>0}\left(Y_i^{\lambda,1}\right)^2. \tag{4.17}$$

Associated with it there is a semi-group $P_t^{\lambda,1,\cdot}$.

Proposition 4.3. *For f smooth with compact support, one has*

$$\frac{\partial}{\partial\lambda}P_t^{\lambda,\cdot}[f](\tilde{x}) = P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0). \tag{4.18}$$

Proof. Let us introduce the vector fields on $\mathbb{R}^{\tilde{d}+\tilde{d}}$:

$$Y_i^{\lambda,2} = (g_\lambda Y_i, g_\lambda DY_i U) \tag{4.19}$$

and the generator

$$\mathbb{L}^{\lambda,2} = Y_0^{\lambda,2} + \frac{1}{2}\sum_{i>0}\left(Y_i^{\lambda,2}\right)^2. \tag{4.20}$$

Associated with it there is a semi-group $P_t^{\lambda,2,\cdot}$.

If the Volterra expansion converges, then

$$\begin{aligned} P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0) = \sum\int_{0<s_1<\cdots<s_n<t} & ds_1\cdots ds_n P_{s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right) \\ & \times P_{s_2-s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right)\cdots\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right)P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},0). \end{aligned} \tag{4.21}$$

But $\tilde{U}_0 \to P_t^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},\tilde{U}_0)$ is linear in $\tilde{U}_0$ and therefore the quantity $(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2})P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},\tilde{U}_0)$ does not depend on $\tilde{U}_0$. Therefore the Volterra expansion reads

$$P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0) = \int_0^t P_s^{\lambda}\left(\mathbb{L}^{\lambda,1}-\mathbb{L}^{\lambda,2}\right)P_{t-s}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x},0). \tag{4.22}$$

Let us compute $\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}$. It is equal to

$$\sum_{i>0} g_\lambda g_\lambda' \langle DY_i Y_i, D_U\rangle + \sum_{i>0} g_\lambda g_\lambda' \langle Y_i, D_U D_X, Y_i\rangle + g_\lambda' \langle Y_0, D_U\rangle. \tag{4.23}$$

We use the relation (see [15, Lemma 3.2])

$$D_X P_t^{\lambda,\cdot} f(\tilde{x}) = \left\langle P_t^{\lambda,2,\cdot}[\langle Df, U\rangle](\tilde{x},I), \cdot\right\rangle \tag{4.24}$$

and the relation

$$D_U P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{X},0\right) = P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{X},\tilde{\mathbb{I}}\right). \tag{4.25}$$

Therefore,

$$\begin{aligned}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right)P_t^{\lambda,2,\cdot}[\langle Df, U\rangle](\tilde{x}_0, U_0) &= g'_\lambda\left\langle Y_0, DP_t^{\lambda,\cdot}\right\rangle + \sum_{i>0} g_\lambda g'_\lambda\left\langle DY_i Y_i, DP_t^{\lambda,\cdot}\right\rangle \\ &\quad + \sum_{i>0} g_\lambda g'_\lambda\left\langle Y_i, D^2 P_t^{\lambda,\cdot}, Y_i\right\rangle.\end{aligned} \tag{4.26}$$

We insert this formula in the right-hand side of (4.23) and we see that $P_t^{\lambda,1,\cdot}(\langle Df, U\rangle)(\tilde{x},0)$ satisfies the same parabolic equation as $(\partial/\partial\lambda)P_t^{\lambda,\cdot}f(x)$. □

Remark 4.4. Let us show from where this formula comes. Classically,

$$P_t^{\lambda,\cdot}[f](x) = E\left[f\left(x_t^\lambda(x)\right)\right], \tag{4.27}$$

where x_t^λ is the solution of the Stratonovitch equation starting at x:

$$dx_s^\lambda(x) = g_\lambda Y_0\left(x_s^\lambda(s)\right)ds + \sum_{i>0} g_\lambda Y_i\left(x_s^\lambda(s)\right)dB_s^i. \tag{4.28}$$

Therefore, $U_s = (\partial/\partial s)x_s^\lambda(x)$ is solution starting at 0 of the Stratonovitch differential equation:

$$\begin{aligned}dU_s &= g'_\lambda Y_0\left(x_s^\lambda(s)\right)ds + \sum_{i>0} g'_\lambda Y_i\left(x_s^\lambda(s)\right)dB_s^i \\ &\quad + g_\lambda\left\langle DY_0(x_s\lambda(x)), U_s^\lambda\right\rangle ds + \sum_{i>0} g_\lambda\left\langle DY_i\left(x_s^\lambda(x)\right), U_s^\lambda\right\rangle dB_s^i\end{aligned} \tag{4.29}$$

which can be solved classically by using the method of variation of constant [4].

Let us introduce the generator on $\mathbb{R}^{1+d+1+1+d}\mathbb{A}^{\lambda,2}$:

$$\begin{aligned}&\mathbb{A}^{\lambda,2}f(s_0, x_0, u_0, v_0, U_0) \\ &\quad = \sum_j C\int_0^\infty\left(f_\lambda(s)^{-3/2}P_1^{j,\sqrt{f_\lambda(s)},2,\cdot}f(s_0, u_0, u_0 J_\lambda(s), v_0, U_0) - s^{-3/2}f(s_0, x_0, u_0, v_0, U_0)\right)ds.\end{aligned} \tag{4.30}$$

It generates a semi-group $P_t^{\lambda,2}$. In order to see that, we split the generator by keeping the values od $s\langle\epsilon$ or $s\rangle\epsilon$ and we proceed as for $\mathbb{A}^{\lambda,1}$ (see (4.10)).

We get the following.

Theorem 4.5. *For f smooth with compact support in s and with derivatives of each order bounded, one has the relation if one takes only derivatives in (s_0, x_0) of the considered expressions:*

$$DP_t^{\lambda,1}[f](s_0, x_0, u_0) = P_t^{\lambda,2}[\langle Df, v, U\rangle](s_0, x_0, u_0, 1, \mathbb{I}). \tag{4.31}$$

Proof. We have

$$\frac{\partial}{\partial t} DP_t^{\lambda,1} = \sum_j C \int_0^\infty \left(f_\lambda(s)^{-3/2} DP_1^{j,\sqrt{f_\lambda(s)}} P_t^{\lambda,1}[f](s_0, u_0, u_0 J_\lambda(s)) - s^{-3/2} DP_t^{\lambda,1} f(s_0, x_0, u_0) \right) ds. \tag{4.32}$$

But by [15, Lemma 3.2.]:

$$DP_1^{j,\sqrt{f_\lambda(s)}} f(s_0, u_0, u_0 J_\lambda(s)) = P_1^{j,\sqrt{f_\lambda(s)},2,\cdot}[\langle Df, v, U\rangle](s_0, x_0, u_0 J_\lambda(s), 1, \mathbb{I}) \tag{4.33}$$

Therefore $DP_t^{\lambda,1}$ satisfies the parabolic equation associated with $P_t^{\lambda,2}[\langle df, v, U\rangle](s_0, x_0, u_0, 1, \mathbb{I})$. Only the integrability of U puts any problem. It is solved by the appendix since f has compact support in s. □

Theorem 4.6. *For f with compact support in $\tilde{x}$ in $\mathbb{R}^{\tilde{d}}$.*

$$P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x}_0, \tilde{U}_0\right) = P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x}_0, \tilde{U}_0\right) + P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x}_0, \tilde{0}\right) \tag{4.34}$$

if $\tilde{U}$, $\tilde{U}_0$ belong to $\mathbb{R}^{\tilde{d}}$.

Proof. If the Volterra expansion converges, then

$$\begin{aligned} P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x}, \tilde{U}_0\right) &= P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x}, \tilde{U}_0\right) \\ &\quad + \sum \int_{0<s_1<\cdots<s_n<t} ds_1 \cdots ds_n P_{s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right) \\ &\quad \times P_{s_2-s_1}^{\lambda,2,\cdot}\left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right) \cdots \left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right) P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x}, \tilde{U}_0\right). \end{aligned} \tag{4.35}$$

But $\tilde{U}_0 \to P_t^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x}, \tilde{U}_0)$ is linear in $\tilde{U}_0$ and therefore the quantity $(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}) P_{t-s_n}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x}, \tilde{U}_0)$ do not depend of $\tilde{U}_0$. Therefore the Volterra expansion reads

$$\begin{aligned} P_t^{\lambda,1,\cdot}[\langle df, U\rangle]\left(\tilde{x}, \tilde{U}_0\right) &= P_t^{\lambda,2,\cdot}[\langle df, U\rangle]\left(\tilde{x}, \tilde{U}_0\right) \\ &\quad + \int_0^t P_s^\lambda \left(\mathbb{L}^{\lambda,1} - \mathbb{L}^{\lambda,2}\right) P_{t-s}^{\lambda,2,\cdot}[\langle df, U\rangle](\tilde{x}, 0) ds \end{aligned} \tag{4.36}$$

But the last term in the right-hand side of (4.26) is equal to $P_t^{\lambda,1,\cdot}[\langle df, U\rangle](\tilde{x},0)$ by the end of the proof of the Proposition 4.3. □

Remark 4.7. Analogous formula works for $D\ \exp[t\mathbb{A}]f$.

Let us compute $\alpha_t = (\partial/\partial\lambda)P_t^{0,1}[uf](s_0,x_0,1)$. We remark that

$$P_1^{j,\sqrt{f_\lambda(s)}}[uf](s_0,x_0,u_0J_\lambda(s)) = P_1^{j,\sqrt{f_\lambda(s)}}[f](s_0,x_0)u_0J_\lambda(s). \tag{4.37}$$

Namely, the generator of $P_t^{j,\sqrt{f_\lambda(s)}}$ does not act on the u_0 component such that the two sides of (4.37) satisfy the same parabolic equality.

Therefore,

$$\begin{aligned} d_t\alpha_t &= \mathbb{A}\alpha_t + \sum_j C\int_0^\infty f_0'(s)s^{-5/2}P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]ds + \sum_j C\int_0^\infty s^{-3/2}J_0'(s)P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]ds \\ &\quad + \sum_j C\int_0^\infty s^{-3/2}\frac{\partial}{\partial\lambda}P_1^{j,\sqrt{s}}[\exp[t\mathbb{A}][f]]ds \\ &= \mathbb{A}\alpha_t + a_1(t) + a_2(t) + a_3(t), \end{aligned} \tag{4.38}$$

where $J_0'(s) = (\partial/\partial\lambda)J_0(s)$, Therefore,

$$\alpha_t = \int_0^t \exp[(t-s)\mathbb{A}](a_1(s)+a_2(s)+a_3(s))ds. \tag{4.39}$$

$a_3(t)$ in the previous expression is the only term which contains a derivative of f, because by Proposition 4.3,

$$\frac{\partial}{\partial\lambda}P_1^{j,\sqrt{s}}[\exp\ t\mathbb{A}][f](s_0,x_0) = P_1^{j,\sqrt{s},1,\cdot}[\langle D\ \exp[t\mathbb{A}][f],u,U\rangle](s_0,x_0,0,0). \tag{4.40}$$

Let $\mathbb{A}^3$ be the generator on $\mathbb{R}^{1+d+1+d}$:

$$\mathbb{A}^3 f(s_0,x_0,u_0,U_0) = C\sum_j\int_0^\infty s^{-3/2}\left(P_1^{j,\sqrt{s},1,\cdot}[f](s_0,x_0,u_0,U_0) - f(s_0,x_0,u_0,U_0)\right)ds. \tag{4.41}$$

It generates a semi-group, P_t^3. We get the following.

Theorem 4.8. *For f with compact support in s and with bounded derivatives at each order, we have*

$$P_t^3\left[\langle df,u,U\rangle\right](s_0,x_0,0,0) = \int_0^t \exp[(t-v)\mathbb{A}]\left[\sum_j C\int_0^\infty s^{-3/2}P_1^{j,\sqrt{s},1,\cdot}\left[\langle D\exp[v\mathbb{A}][f],u,U\rangle\right](s_0,x_0,0,0)ds\right]dv. \tag{4.42}$$

Proof. If the Volterra expansion converges, then

$$P_t^3\left[\langle df,U\rangle\right](s_0,x_0,0,0) = \sum_n \int_{0<s_1<\cdots<s_n<t} ds_1\cdots ds_n P_{s_1}^2\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdots P_{s_n-s_{n-1}}^2\left(\mathbb{A}^3-\mathbb{A}^2\right) \times P_{t-s_n}^2\left[\langle df,u,U\rangle\right](s_0,x_0,0,0). \tag{4.43}$$

But $P_{t-s_n}^2[\langle df,u,U\rangle](s_0,x_0,u_0,U_0)$ is linear in (u_0,U_0). Let us explain the details of that. We have to compute

$$\left(P_1^{j,\sqrt{s},1,\cdot}-P_1^{j,\sqrt{s},2,\cdot}\right)P_{t-s_n}^2\left[\langle df,u,U\rangle\right](s_0,x_0,u_0,U_0). \tag{4.44}$$

By the technique of the beginning of the proof of Proposition 4.3, it does not depend on (u_0,U_0). Therefore, the Volterra expansion reads:

$$P_t^3\left[\langle df,U\rangle\right](s_0,x_0,0,0) = \int_{0<v<t} P_v^2\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdot P_{t-v}^2\left[\langle df,u,U\rangle\right](s_0,x_0,0,0)dv. \tag{4.45}$$

But

$$\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdot P_{t-v}^2\left[\langle df,u,U\rangle\right](s_0,x_0,u_0,U_0). \tag{4.46}$$

does not depend on (u_0,U_0). Therefore, the right-hand side of formula (4.45) is equal to

$$\int_{0<v<t} \exp[v\mathbb{A}]\left(\mathbb{A}^3-\mathbb{A}^2\right)\cdot P_{t-v}^2\left[\langle df,u,U\rangle\right](s_0,x_0,0,0)dv. \tag{4.47}$$

But

$$\mathbb{A}^2\cdot P_{t-v}^2\left[\langle df,u,U\rangle\right](s_0,x_0,0,0) = \sum_j C\int_0^\infty s^{-3/2}P_s^{j,\sqrt{2},2,0}P_{t-v}^2\left[\langle df,u,U\rangle\right](s_0,x_0,0,0) = O \tag{4.48}$$

because $(u_0, U_0) \to P^j_{t-v}[\langle df, u, U\rangle](s_0, x_0, u_0, U_0)$ is linear in (u_0, U_0) and because the vector fields which give the generator of $P_s^{j,\sqrt{s},2,\cdot}$ are linear in u_0, U_0. Therefore,

$$\begin{aligned}
&\int_{0<v<t} \exp[v\mathbb{A}]\left(\mathbb{A}^3 - \mathbb{A}^2\right) \cdot P^2_{t-v}\left[\langle df, u, U\rangle\right](s_0, x_0, 0, 0)dv \\
&\quad = \int_{0<v<t} \exp[v\mathbb{A}]dv \sum_j C \int_0^\infty s^{-3/2} P_1^{j,\sqrt{s},1,\cdot}\left[P^2_{t-v}\left[\langle df, u, U\rangle\right]\right](s_0, x_0, 0, 0)ds.
\end{aligned} \tag{4.49}$$

But by an analog of Theorem 4.5,

$$P^2_{t-v}\left[\langle df, u, U\rangle\right](s_0, x_0, u_0, U_0) = \langle D\ \exp[t\mathbb{A}], u_0, U_0\rangle. \tag{4.50}$$

□

We can summarize the previous considerations in the next theorem.

Theorem 4.9. *If $f_\lambda(s)$ is a diffeomorphism of $[0, \infty[$ equal to s if $s \in [0, \epsilon[$ and if $s > 1$, then one has the following integration by part formula if f is with compact support in s, bounded with bounded derivatives at each order:*

$$\begin{aligned}
0 &= \sum_j C \int_0^t du\ \exp[(t-u)\mathbb{A}]\left[\int_0^\infty f_0'(s) s^{-5/2} P_1^{j,\sqrt{s}}\left[\exp[t\mathbb{A}][f]\right]\right](s_0, x_0) \\
&\quad + \sum_j C \int_0^t du\ \exp[(t-u)\mathbb{A}]\left[\int_0^\infty J_0'(s) s^{-3/2} P_1^{j,\sqrt{s}}\left[\exp[t\mathbb{A}][f]\right]\right](s_0, x_0) \\
&\quad + P_t^3\left[\langle df, u, U\rangle\right](s_0, x_0, 0, 0),
\end{aligned} \tag{4.51}$$

where $J_0'(s) = (\partial/\partial\lambda) J_0(s)$.

Theorem 4.10. *Let one suppose that $f_\lambda(s) = s + \lambda s^5$ near 0. Then,* (4.51) *is still true.*

Proof. It is enough to show that wecan approximate $f_\lambda(s)$ by a function $f_\lambda^\epsilon(s)$ equal to s if $s < \epsilon$. Let us give some details on this approximation. We consider a smooth function g from $\mathbb{R}$ into $[0, 1]$ equal to zero if $s \le 1/2$ and equal to 1 if $s > 1$. We put

$$\begin{aligned}
f_\lambda(s) &= s + g\left(\frac{s}{\epsilon}\right)\lambda s^5, \\
\frac{\partial}{\partial\lambda} f_0^\epsilon(s) &= g\left(\frac{s}{\epsilon}\right) s^5, \\
\frac{\partial}{\partial\lambda} J_0^\epsilon(s) &= g'\left(\frac{s}{\epsilon}\right)\frac{s^5}{\epsilon} + 5g\left(\frac{s}{\epsilon}\right)s^4.
\end{aligned} \tag{4.52}$$

We remark that

(i) if $s \leq \epsilon/2$, then $g'(s/\epsilon)s^5/\epsilon = 0$,

(ii) if $s > \epsilon$, then $g'(s/\epsilon)s^5/\epsilon = 0$,

(iii) if $s \in [\epsilon/2, \epsilon]$, then $|g'(s/\epsilon)s^5/\epsilon| \leq Cs^4$.

$P_t^{3,\epsilon}$ is the semi-group associated with $\mathbb{A}^{3,\epsilon}$ where we replace in the construction of (4.41) $f_\lambda(s)$ by $f_\lambda^\epsilon(s)$:

$$P_t^{3,\epsilon}\left[\langle df, u, U\rangle\right](s_0, x_0, 0, 0) \longrightarrow P_t^3\left[\langle df, u, U\rangle\right](s_0, x_0, 0, 0). \tag{4.53}$$

By the appendix,

$$P_s^{3,\epsilon}\left[\left(|u|^p + |U|^p\right)h\right](s, x, u_0, U_0) < \infty \tag{4.54}$$

if h is compact support in s. Let us consider the generator $A^{3,\epsilon}$ associated with f_λ^ϵ. If $g = \langle df, u, U\rangle$, then we have by Duhamel principle

$$P_1^3[g](s_0, x_0, 0, 0) = P_1^{3,\epsilon}[g](s_0, x_0, 0, 0) + \int_0^1 P_s^{3,\epsilon}\left[\left(\mathbb{A} - \mathbb{A}^{3,\epsilon}\right)P_{1-s}^3[g]\right](s_0, x_0, 0, 0). \tag{4.55}$$

By the proof of Theorem 4.8, $P_{1-s}^3[g](s_0, x_0, u_0, U_0)$ is affine in (u_0, U_0). Namely, in the proof of this theorem, we have removed the $P_{1-s}^2[g](s_0, x_0, u_0, U_0)$ which is equal to zero in $u_0 = 0, U_0 = 0$ because this expression is linear in u_0, U_0. Its component in (u_0, U_0) is smooth with bounded derivatives at each order. By Theorem 4.6, $(\mathbb{A}^{3,\epsilon} - \mathbb{A})P_{1-s}^3[g](s_0, x_0, u_0, U_0)$ is still affine in (u_0, U_0) and its components in (u_0, U_0) are smooth with bounded derivatives at each order. Moreover, if g^1 is affine in (u_0, U_0) with components in (u_0, U_0) smooth with bounded derivatives at each order, then we get that, for $s \leq 1$,

$$\sup_{s_0, x_0}\left|\left(P_1^{j,\sqrt{s},1,0} - P_1^{\epsilon,j,\sqrt{s},1,\cdot}\right)\left[g^1\right](s_0, x_0, u_0, U_0)\right| \leq C(\epsilon)s(|u_0| + |U_0|), \tag{4.56}$$

where $C(\epsilon) \to 0$ when $\epsilon \to 0$. This can be seen as an appliation of the Duhamel formula applied to the two semi-groups $t \to P_t^{j,\sqrt{s},1,\cdot}[g^1]$ and $t \to P_t^{\epsilon,j,\sqrt{s},1,\cdot}[g^1]$. Then, the result arises from the Duhamel formula (4.55). □

We can consider vector fields at the manner of (3.30) and $f_\lambda(s) = s + \lambda s^5$ in a neighborhood of 0. We get a generator $\mathbb{A}^{\text{tot}}$ and semi-groups $P_s^{j,\sqrt{s},\text{tot}}$ and $P_t^{3,\text{tot}}$. We have with the extension of Theorem 4.10 the following.

Theorem 4.11 (Bismut). *If $f_\lambda(s) = s + \lambda s^5$ in a neighborhood of 0 and is equal to 1 if $s > 1$, then one has the following integration by parts: let f^{tot} be a function with compact support in s, bounded with bounded derivatives at each order. Then,*

$$\begin{aligned} 0 = {} & \sum_j C \int_0^t du \, \exp\left[(t-u)\mathbb{A}^{\text{tot}}\right] \left[\int_0^\infty f_0'(s) s^{-5/2} P_1^{j,\sqrt{s},\text{tot}}\left[\exp\left[t\mathbb{A}^{\text{tot}}\right]\left[f^{\text{tot}}\right]\right]\right](s_0, x_0^{\text{tot}}) \\ & + \sum_j C \int_0^t du \, \exp\left[(t-u)\mathbb{A}^{\text{tot}}\right] \left[\int_0^\infty J_0'(s) s^{-3/2} P_1^{j,\sqrt{s},\text{tot}}\left[\exp\left[t\mathbb{A}^{\text{tot}}\right]\left[f^{\text{tot}}\right]\right]\right](s_0, x_0^{\text{tot}}) \\ & + P_t^{3,\text{tot}}\left[\langle df^{\text{tot}}, u, U\rangle\right](s_0, x_0^{\text{tot}}, 0, 0). \end{aligned} \tag{4.57}$$

5. The Abstract Theorem

The proof of Theorem 2.1 follows the idea of Malliavin [5]. If there exist C_l such that, for function f with compact support in $[0,1] \times [0,l]^d$,

$$\left|\exp[t\mathbb{A}]\left[Df\right](0,x)\right| \le C_l \|f\|_\infty \tag{5.1}$$

then the heat kernel $q_t(s,y)$ exists.

There are two partial derivatives to treat:

(i) the partial derivative in the time of the subordinator s,

(ii) the partial derivatives in the space of the underlying diffusion x.

Let us begin by the most original part of Bismut's Calculus on boundary process, that is, the integration by parts in the time s.

We look at (4.42). We remark (see the next part) that

$$P_t^{3,\text{tot}}\left[u^{-p}\right](0,x,0,0) < \infty \tag{5.2}$$

for all p. So, we take $f^{\text{tot}}(s,x,u) = f(s,x)1/u$ and we apply (4.42) for this convenient semigroup. We get

$$\exp[t\mathbb{A}]\left[\frac{\partial}{\partial s} f\right](0,x) = -P_t^{3,\text{tot}}\left[\langle D_x f, U\rangle \frac{1}{u}\right](0,x,0,0) + R \tag{5.3}$$

R can be estimated by using the appendix by $C_l\|f\|_\infty$ for f with compact support in $[0,l] \times [0,l]^d$ and by (5.2).

Lemma 5.1. *For a conveniently enlarged semi-group in the manner of Theorem 3.4, one has for f with compact support in s*

$$P_t^{2,\mathrm{tot}}\left[\langle Df^{\mathrm{tot}}, U\rangle\right](s_0, x_0^{\mathrm{tot}}, 0) = \exp\left[t\widehat{\mathbb{A}}^{\mathrm{tot}}\right]\left[\langle Df^{\mathrm{tot}}, UV\rangle\right](s_0, x_0^{\mathrm{tot}}, I, 0). \tag{5.4}$$

Proof. If $\widetilde{f}$ is a function with compact support depending only of s, x^{tot} and V, we have

$$P_t^{2,\mathrm{tot}}\left[\widetilde{f}\right](s_0, x_0^{\mathrm{tot}}, 0) = \exp\left[t\widehat{\mathbb{A}}^{\mathrm{tot}}\right]\left[\widetilde{f}(\cdot, UV)\right](s_0, x_0^{\mathrm{tot}}, I, 0). \tag{5.5}$$

We do the change of variable $U \to U$ and $V \to UV$ on the Malliavin generator $\widehat{\mathbb{A}}^{\mathrm{tot}}$. By using Lemma 3.7 of [15], it is transformed in $\mathbb{A}^{2,\mathrm{tot}}\widetilde{f}(s, x^{\mathrm{tot}}, U, V)$ where for $\mathbb{A}^{2,\mathrm{tot}}$ we consider the same type of operator as $\mathbb{A}^2$ but with the modified vector fields:

$$\begin{aligned} X_i^{j,2} &= \left(0, X_i^{j,\mathrm{tot}}, DX_i^j U, DX_i^j V\right), \\ Y_0^{j,2} &= \left(0, 0, 0, \sum \left(X_i^j\right)^t \left(U^{-1} X_i^j\right)\right). \end{aligned} \tag{5.6}$$

It remains to use the appendix to show the Lemma. □

We consider $Z_i^j = {}^t(U^{-1}X_i^j)$. By the previous Lemma and Malliavin hypothesis,

$$P_t^{2,\mathrm{tot}}\left[\det V^{-p} g\right](0, x_0^{\mathrm{tot}}, I, 0) < \infty \tag{5.7}$$

for all p if $g(s)$ has compact support (V is a matrix). After we consider a test function of the type of Bismut, we consider the component u_i of U in (5.3). We consider the Bismut function $fV^{-1}(u_i/u)$. We integrate by parts as in Theorem 3.4. We deduce under Malliavin assumption that

$$\left|P_t^{3,\mathrm{tot}}\left[\langle D_x f, U\rangle \frac{1}{u}\right](0, x, 0, 0)\right| \le C_l \|f\|_\infty \tag{5.8}$$

if f has compact support in $[0, l] \times [0, l]^d$.

By the same way, we deduce that if f has compact support in $[0, l] \times [0, l]^d$ then

$$\left|\exp[t\mathbb{A}]\left[D_x f\right](0, x, 0, 0)\right| \le C_l \|f\|_\infty. \tag{5.9}$$

Therefore, the result is obtained .

Remark 5.2. We could do integration by parts to each order in order to show that the semi-group $\exp[t\mathbb{A}]$ has a smooth heat-kernel under Malliavin assumption.

6. Inversion of the Malliavin Matrix

Proof of Theorem 2.2. Let $s_1 < s_2$ and let ξ be of modulus 1. Then,

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,s_1]}\mathbb{I}_{V(\xi)\le\epsilon}\right](0,x,I,0) \ge \exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,s_1]}\mathbb{I}_{V(\xi)\le\epsilon}\right](0,x,I,0). \tag{6.1}$$

These two quantities are equal in $t = 0$ when we consider the semi-group $\exp[t\widehat{\mathbb{A}}]$. Let us compute their derivative in time t. The derivative of the left-hand side is bigger than the derivative of the right-hand side because

$$\widehat{\mathbb{A}}\left[\mathbb{I}_{[0,s_1]}\mathbb{I}_{V(\xi)\le\epsilon}\right](s,x,U,V) \ge \widehat{\mathbb{A}}\left[\mathbb{I}_{[0,s_2]}\mathbb{I}_{V(\xi)\le\epsilon}\right](s,x,U,V). \tag{6.2}$$

(These two quantities are negative.)

By the result of the appendix,

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,t]}\left\{\mathbb{I}_{|U^{-1}-I|>C} + \mathbb{I}_{|U-I|>C} + \mathbb{I}_{|\cdot-x|>C} + \mathbb{I}_{V>C}\right\}\right](0,x,I,0) \le C(p)t^p \tag{6.3}$$

for all p. □

Lemma 6.1. *If $|\xi| = 1$, then there exist C and C_0 independent of ξ such that*

$$\exp\left[\epsilon\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{|V\xi|<C_0\epsilon}\mathbb{I}_{[0,\epsilon]}\right](0,x,I,0) < 1 - C\epsilon^{1/2}. \tag{6.4}$$

Proof. We consider a convex function decreasing from $[0,\infty[$ into $[0,1]$ equal to 1 in 0 and tending to 0 at infinity. Let us introduce

$$\alpha_s = \exp\left[s\widehat{\mathbb{A}}\right]\left[g\left(\frac{|V\xi|}{\epsilon}\right)\mathbb{I}_{[0,\epsilon]}\right](0,x,I,0). \tag{6.5}$$

In order to consider the derivative in s of α_s, we study the expression

$$\beta_\epsilon = \widehat{\mathbb{A}}\left[g\left(\frac{|V\xi|}{\epsilon}\right)\mathbb{I}_{[0,\epsilon]}\right](s',x',U',V'). \tag{6.6}$$

We have only to consider by (6.3) the case where s' is small enough, $|x' - x|$ is small enough, $|U - I|$ is small enough, and the positive matrix V' is small enough. For that we have to estimate

$$\gamma_u = \sum_j\left(P_u^{j,2}\left[g\left(\frac{|V\xi|}{\epsilon}\right)\right](s',x',U',V') - g\left(\frac{|V'\xi|}{\epsilon}\right)\right) \tag{6.7}$$

for u between 0 and ϵ. The first derivative of γ_u has an equivalent $-C\epsilon^{-1}$ when $\epsilon \to 0$, and its second derivative has a bound $C\epsilon^{-2}$ when $\epsilon \to 0$. Therefore,

$$0 \geq \gamma_u \geq -\frac{Cu}{\epsilon} \tag{6.8}$$

on $[0, \epsilon]$ and

$$\beta_\epsilon \geq -\frac{C}{\epsilon}\int_0^\epsilon s^{-1/2} ds = -C\epsilon^{-1/2}. \tag{6.9}$$

We deduce from that that

$$\alpha_\epsilon \leq 1 - C\epsilon^{1/2}. \tag{6.10}$$

□

Remark 6.2. We could improve (6.4) by showing that

$$\exp\left[\epsilon\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{|V\xi|<C_0\epsilon}\mathbb{I}_{[0,\epsilon]}\right](s', x', U', V') < 1 - C\epsilon^{1/2} \tag{6.11}$$

if s' is small enough, $|x' - x|$ is small enough, $|U' - I|$ is small enough, and the positive matrix V' is small enough.

We consider a very small α. We slice the time interval $[0, \epsilon^\alpha]$ in $\epsilon^{\alpha-1}$ intervals of length ϵ. We have

$$\begin{aligned}
\exp\left[t\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,l]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x, I, 0) &\leq \exp\left[\epsilon^\alpha\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,l]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x, I, 0) \\
&\leq \exp\left[\epsilon^\alpha\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,\epsilon]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x, I, 0) \\
&\leq \left\{\sup_{|x'-x|\leq C_0,\, |U-I|\leq C_0} \exp\left[\epsilon\widehat{\mathbb{A}}\right]\left[\mathbb{I}_{[0,\epsilon]}\mathbb{I}_{V(\xi)\leq\epsilon}\right](0, x', U', 0)\right\}^{\epsilon^{\alpha-1}} + C\epsilon^p
\end{aligned} \tag{6.12}$$

for a small C_0. This last quantity is smaller than $C\epsilon^p$ for all p by the previous lemma if α is small enough. The proof of Theorem 2.2 follows from

$$\exp\left[t\widehat{\mathbb{A}}\right]\left[V^p\mathbb{I}_{[0,l]}\right](0, x, I, 0) \leq \infty \tag{6.13}$$

for all p by using the result of the appendix. The result follows by standard methods (see [15, Equations (4.8) and (4.9)].

It remains to show the following.

Theorem 6.3. *For all $p > 0$,*

$$P_t^{3,\mathrm{tot}}\left[u^{-p}\right](0,x,0,0) \leq \infty. \tag{6.14}$$

Proof. We remark that if we consider only functions of u, then

$$P_t^{3,\mathrm{tot}}\left[f\right](0,x,u,0) = P_t^4\left[f\right](u), \tag{6.15}$$

where P_t^4 is a Lévy semi-group with generator

$$\mathbb{A}^4 f(u) = C\int_0^\infty \frac{ds}{s^{3/2}}\left(f\left(s^5 g(s)+u\right) - f(u)\right), \tag{6.16}$$

where $g(s) = 1$ on a neighborhood of 0, is with compact support and is positive. The result follows from the adaptation in [17, 18] of the proof of [7] in semi-group theory. We remark that

$$P_t^4\left[u^{-p}\right](0) = C\int_0^\infty \beta^{p-1} P_t^4\left[\exp\left[-\beta u\right]\right](0)d\beta. \tag{6.17}$$

By using the adaptation in semi-group theory of the exponential martingales of Levy process of [17, 18], we have

$$P_t^4\left[\exp\left[-\beta u\right]\right](0) = \exp\left[t\int_0^\infty \left[\exp\left[-\beta s^5 g(s)\right] - 1\right)\frac{ds}{s^{3/2}}\right]. \tag{6.18}$$

The result holds from the Tauberian theorem of [7, 17, 18]. □

Appendix

Burkholder-Davies-Gundy Inequality

Theorem A.4. *Let $s_0 > 0$ and $p \in \mathbb{N}$. Then,*

$$\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}\left|x^{\mathrm{tot}}\right|^{2p}\right]\left(0,x_0^{\mathrm{tot}},0\right) < \infty. \tag{A.1}$$

Proof. Following the idea of [17, Appendix], we consider the auxiliary function

$$F_C\left(x^{\mathrm{tot}}\right) = \frac{\left|x^{\mathrm{tot}}\right|^{2p}+1}{1+\left|x^{\mathrm{tot}}\right|^{2k}/C}. \tag{A.2}$$

We get

$$\begin{aligned}
&\frac{d}{dt}\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\mathrm{tot}}\right)\right]\left(0,x_0^{\mathrm{tot}},0\right)\\
&\quad=\widehat{P}_t^{2,\mathrm{tot}}\left[\int_0^{s_0-s}\frac{du}{u^{3/2}}\sum_j\left(P_u^{j,2,\mathrm{tot}}\left[F_C\right]\left(x^{\mathrm{tot}}\right)-F_C\left(x^{\mathrm{tot}}\right)\right)\right]\left(0,x_0^{\mathrm{tot}},0\right).
\end{aligned}\tag{A.3}$$

Let us consider an improvement of the Gronwall lemma: if $|x_s - x_0| \le \int_0^s |x_u|du$, then $|x_t - x_0| \le Kt|x_0|$ if $t \in [0,1]$.

We remark that

$$\left|L^{j,2,\mathrm{tot}}F_C\left(x^{\mathrm{tot}}\right)\right| \le KF_C\left(x^{\mathrm{tot}}\right)\tag{A.4}$$

for K independent of C. Then, by the modified Gronwall lemma,

$$\begin{aligned}
&\left|P_u^{j,2,\mathrm{tot}}|F_C|\left(x^{\mathrm{tot}}\right)-F_C\left(x^{\mathrm{tot}}\right)\right| \le KuF_C\left(x^{\mathrm{tot}}\right),\\
&\left|\frac{d}{dt}\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\mathrm{tot}}\right)\right]\left(0,x_0^{\mathrm{tot}},0\right)\right| \le KF_C\left(x^{\mathrm{tot}}\right)+K\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\mathrm{tot}}\right)\right]\left(0,x_0^{\mathrm{tot}},0\right),
\end{aligned}\tag{A.5}$$

where K does not depend on C.

By Gronwall lemma,

$$\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}F_C\left(x^{\mathrm{tot}}\right)\right]\left(0,x_0^{\mathrm{tot}},0\right) \le K < \infty,\tag{A.6}$$

where K does not depend on C. The result arises by doing $C \to \infty$. □

By the same procedure, we get the following.

Theorem A.5. *Let be $s_0 > 0$ and $p \in \mathbb{N}$. Then*

$$\widehat{P}_t^{2,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}|U|^{2p}\right]\left(0,x_0^{\mathrm{tot}},U_0\right) < \infty\tag{A.7}$$

and we get the following.

Theorem A.6. *Let $s_0 > 0$ and $p \in \mathbb{N}$:*

$$P_t^{3,\mathrm{tot}}\left[\mathbb{I}_{[0,s_0]}\left(\left|x^{\mathrm{tot}}\right|+|u|+|U|\right)^{2p}\right]\left(0,x_0^{\mathrm{tot}},u_0,U_0\right) < \infty.\tag{A.8}$$

Remark A.7. We can show (6.3) by the same way.

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Research Article

Asymptotical Stability of Nonlinear Fractional Differential System with Caputo Derivative

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This paper deals with the stability of nonlinear fractional differential systems equipped with the Caputo derivative. At first, a sufficient condition on asymptotical stability is established by using a Lyapunov-like function. Then, the fractional differential inequalities and comparison method are applied to the analysis of the stability of fractional differential systems. In addition, some other sufficient conditions on stability are also presented.

1. Introduction

Fractional calculus is more than 300 years old, but it did not attract enough interest at the early stage of development. In the last three decades, fractional calculus has become popular among scientists in order to model various physical phenomena with anomalous decay, such as dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, and viscoelastic systems [1]. Recent advances in fractional calculus have been reported in [2].

Recently, stability of fractional differential systems has attracted increasing interest. In 1996, Matignon [3] firstly studied the stability of linear fractional differential systems with the Caputo derivative. Since then, many researchers have done further studies on the stability of linear fractional differential systems [4–11]. For the nonlinear fractional differential systems, the stability analysis is much more difficult and only a few are available.

Some authors [12, 13] studied the following nonlinear fractional differential system:

$$ {}_{C}D_{0,t}^{q}x(t) = f(t, x(t)), \tag{1.1} $$

with initial values $x(0) = x_0^{(0)}, \ldots, x^{(m-1)}(0) = x_0^{(m-1)}$, where $m-1 < q \leq m$. They discussed the continuous dependence of solution on initial conditions and the corresponding structural stability by applying Gronwall's inequality. In [14] the authors dealt with the following fractional differential system:

$$\mathfrak{D}_{0,t}^{q} x(t) = f(t, x(t)), \tag{1.2}$$

where $0 < q \leq 1$, $\mathfrak{D}_{0,t}^{q}$ denotes either the Caputo, or the Riemann-Liouville fractional derivative operator. They proposed fractional Lyapunov's second method and firstly extended the exponential stability of integer order differential systems to the Mittag-Leffler stability of fractional differential systems. Moreover, the pioneering work on the generalized Mittag-Leffler stability and the generalized fractional Lyapunov direct method was proposed in [15].

In this paper, we further study the stability of nonlinear fractional differential systems with Caputo derivative by utilizing a Lyapunov-like function. Taking into account the relation between asymptotical stability and generalized Mittag-Leffler stability, we are able to weaken the conditions assumed for the Lyapunov-like function. In addition, based on the comparison principle of fractional differential equations [16, 17], we also study the stability of nonlinear fractional differential systems by utilizing the comparison method. Our contribution in this paper is that we have relaxed the condition of the Lyapunov-like function and that we have further studied the stability. The present paper is organized as follows. In Section 2, some definitions and lemmas are introduced. In Section 3, sufficient conditions on asymptotical stability and generalized Mittag-Leffler stability are given. The comparison method is applied to the analysis of the stability of fractional differential systems in Section 4. Conclusions are included in the last section.

2. Preliminaries and Notations

Let us denote by $\mathbb{R}_+$ the set of nonnegative real numbers, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{Z}_+$ the set of positive integer numbers. Let $0 < q < 1$ and set $C_q([t_0, T], \mathbb{R}) = \{f \in C((t_0, T], \mathbb{R}), (t-t_0)^q f(t) \in C([t_0, T], \mathbb{R})\}$, and $C_q([t_0, T] \times \Omega, \mathbb{R}) = \{f(t, x(t)) \in C((t_0, T] \times \Omega, \mathbb{R}), (t-t_0)^q f(t, x(t)) \in C([t_0, T] \times \Omega, \mathbb{R})\}$, where $C((t_0, t], \mathbb{R})$ denotes the space of continuous functions on the interval $(t_0, t]$.

Let us first introduce several definitions, results, and citations needed here with respect to fractional calculus which will be used later. As to fractional integrability and differentiability, the reader may refer to [18].

Definition 2.1. The fractional integral with noninteger order $q \geq 0$ of function $x(t)$ is defined as follows:

$$D_{t_0,t}^{-q} x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\tau)^{q-1} x(\tau) d\tau, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The Riemann-Liouville derivative with order q of function $x(t)$ is defined as follows:

$$ {}_{\mathrm{RL}}D^{q}_{t_0,t}x(t) = \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{m-q-1}x(\tau)d\tau, \tag{2.2} $$

where $m-1 \le q < m$ and $m \in \mathbb{Z}_+$.

Definition 2.3. The Caputo derivative with noninteger order q of function $x(t)$ is defined as follows:

$$ {}_{C}D^{q}_{t_0,t}x(t) = \frac{1}{\Gamma(m-q)}\int_{t_0}^{t}(t-\tau)^{m-q-1}x^{(m)}(\tau)d\tau, \tag{2.3} $$

where $m-1 < q < m$ and $m \in \mathbb{Z}_+$.

Definition 2.4. The Mittag-Leffler function is defined by

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{2.4} $$

where $\alpha > 0$, $z \in \mathbb{R}$. The two-parameter Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{2.5} $$

where $\alpha > 0$ and $\beta \in \mathbb{R}$, $z \in \mathbb{R}$.

Clearly $E_{\alpha}(z) = E_{\alpha,1}(z)$. The following definitions are associated with the stability problem in the paper.

Definition 2.5. The constant x_{eq} is an equilibrium of fractional differential system $\mathfrak{D}^{q}_{t_0,t}x(t) = f(t,x)$ if and only if $f(t,x_{\mathrm{eq}}) = \mathfrak{D}^{q}_{t_0,t}x(t)|_{x(t)=x_{\mathrm{eq}}}$ for all $t > t_0$, where $\mathfrak{D}^{q}_{t_0,t}$ means either the Caputo or the Riemann-Liouville fractional derivative operator.

Throughout the paper, we always assume that $x_{\mathrm{eq}} = 0$.

Definition 2.6 (see [15]). The zero solution of $\mathfrak{D}^{q}_{t_0,t}x(t) = f(t,x(t))$ with order $q \in (0,1)$ is said to be stable if, for any initial value x_0, there exists an $\varepsilon > 0$ such that $\|x(t)\| \le \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to +\infty$.

Definition 2.7. Let $\mathbb{B} \subset \mathbb{R}^n$ be a domain containing the origin. The zero solution of $\mathfrak{D}^{q}_{t_0,t}x(t) = f(t,x(t))$ is said to be Mittag-Leffler stable if

$$ \|x(t)\| \le \left\{m(x_0)E_q\left(-\lambda(t-t_0)^q\right)\right\}^b, \tag{2.6} $$

where t_0 is the initial time and x_0 is the corresponding initial value, $q \in (0,1)$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \subset \mathbb{R}^n$ with the Lipschitz constant $\mathcal{L}_0$.

Definition 2.8. Let $\mathbb{B} \subset \mathbb{R}^n$ be a domain containing the origin. The zero solution of $\mathfrak{D}^q_{t_0,t}x(t) = f(t, x(t))$ is said to be generalized Mittag-Leffler stable if

$$\|x(t)\| \leq \left\{ m(x_0)(t-t_0)^{-\gamma} E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right) \right\}^b, \tag{2.7}$$

where t_0 is the initial time and x_0 is the corresponding initial value, $q \in (0,1)$, $-q < \gamma \leq 1-q$, $\lambda \geq 0, b > 0, m(0) = 0, m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \subset \mathbb{R}^n$ with the Lipschitz constant $\mathcal{L}_0$.

Remark 2.9. Mittag-Leffler stability and generalized Mittag-Leffler stability both belong to algebraical stability, which also imply asymptotical stability (see [15]).

Definition 2.10. A function $\alpha(r)$ is said to belong to class-$\mathcal{K}$ if $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function such that $\alpha(0) = 0$ and it is strictly increasing.

Definition 2.11 (see [19]). The class-$\mathcal{K}$ functions $\alpha(r)$ and $\beta(r)$ are said to be with local growth momentum at the same level if there exist $r_1 > 0$, $k_i > 0$ $(i = 1,2)$ such that $k_1\alpha(r) \geq \beta(r) \geq k_2\alpha(r)$ for all $r \in [0, r_1]$. The class-$\mathcal{K}$ functions $\alpha(r)$ and $\beta(r)$ are said to be with global growth momentum at the same level if there exist $k_i > 0$ $(i = 1,2)$ such that $k_1\alpha(r) \geq \beta(r) \geq k_2\alpha(r)$ for all $r \in \mathbb{R}_+$.

It is useful to recall the following lemmas for our developments in the sequel.

Lemma 2.12 (see [20]). *Let $v, w \in C_{1-q}([t_0, T], \mathbb{R})$ be locally Hölder continuous for an exponent $0 < q < \nu \leq 1$, $h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and*

(i) ${}_{RL}D^q_{t_0,t}v(t) \leq h(t, v(t))$,

(ii) ${}_{RL}D^q_{t_0,t}w(t) \geq h(t, w(t))$, $t_0 < t \leq T$,

with nonstrict inequalities (i) and (ii), where $v_0 = \Gamma(q)v(t)(t-t_0)^{1-q}|_{t=t_0}$ and $w_0 = \Gamma(q)w(t)(t-t_0)^{1-q}|_{t=t_0}$. Suppose further that h satisfies the standard Lipschitz condition

$$h(t,x) - h(t,y) \leq \mathcal{L}(x-y), \quad x \geq y,\ \mathcal{L} > 0. \tag{2.8}$$

Then, $v_0 \leq w_0$ implies $v(t) \leq w(t)$, $t_0 < t \leq T$.

Remark 2.13. In Lemma 2.12, if we replace ${}_{\mathrm{RL}}D^q_{t_0,t}$ by ${}_{C}D^q_{t_0,t}$, but other conditions remain unchanged, then the same result holds.

Lemma 2.14 (see [16]). *Let $v, w \in C_{1-q}([t_0, T], \mathbb{R})$, $h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and*

(i) $v(t) \leq (v_0/\Gamma(q))(t-t_0)^{q-1} + (1/\Gamma(q)) \int_{t_0}^{t} (t-s)^{q-1} h(s, v(s))ds$,

(ii) $w(t) \geq (w_0/\Gamma(q))(t-t_0)^{q-1} + (1/\Gamma(q)) \int_{t_0}^{t} (t-s)^{q-1} h(s, w(s))ds$,

where $t_0 < t \leq T$, $v_0 = \Gamma(q)v(t)(t-t_0)^{1-q}|_{t=t_0}$, $w_0 = \Gamma(q)w(t)(t-t_0)^{1-q}|_{t=t_0}$, and $0 < q < 1$. Assume that both inequalities are nonstrict and $h(t,x)$ is nondecreasing in x for each t. Further, suppose that h satisfies the standard Lipschitz condition

$$h(t,x) - h(t,y) \leq \mathcal{L}(x-y), \quad x \geq y,\ \mathcal{L} > 0. \tag{2.9}$$

Then, $v_0 \leq w_0$ implies $v(t) \leq w(t)$, $t_0 < t \leq T$.

Remark 2.15. In Lemmas 2.12 and 2.14, T can be $+\infty$.

3. Stability of Nonlinear Fractional Differential Systems

Let us consider the following nonlinear fractional differential system [14, 15]:

$$ {}_C D^q_{t_0,t} x(t) = f(t, x(t)), \tag{3.1} $$

with the initial condition $x_0 = x(t_0)$, where $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and $\Omega \subset \mathbb{R}^n$ is a domain that contains the equilibrium point $x_{\text{eq}} = 0$, $0 < q < 1$. Here and throughout the paper, we always assume there exists a unique solution $x(t) \in C^1[t_0, \infty)$ to system (3.1) with the initial condition $x(t_0)$.

Recently, Li et al. [14, 15] investigated the Mittag-Leffler stability and the generalized Mittag-Leffler stability (the asymptotic stability) of system (3.1) by using the fractional Lyapunov's second method, where the following theorem has been presented.

Theorem 3.1. *Let $x_{eq} = 0$ be an equilibrium point of system* (3.1) *with $t_0 = 0$, and let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0, \infty) \times \mathbb{D} \to \mathbb{R}_+$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$ \alpha_1 \|x\|^a \le V(t, x(t)) \le \alpha_2 \|x\|^{ab}, \tag{3.2} $$

$$ {}_C D^p_{0,t} V(t, x(t)) \le -\alpha_3 \|x\|^{ab}, \tag{3.3} $$

where $t \ge 0$, $x \in \mathbb{D}$, $p \in (0, 1)$, and α_1, α_2, α_3, a, and b are arbitrary positive constants. Then $x_{eq} = 0$ is Mittag-Leffler stable (locally asymptotically stable). If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally Mittag-Leffler stable (globally asymptotically stable).

In the following, we give a new proof for Theorem 3.1.

Proof of Theorem 3.1. From (3.2) and (3.3), we can get

$$ {}_C D^p_{0,t} V(t, x(t)) \le -\frac{\alpha_3}{\alpha_2} V(t, x(t)). \tag{3.4} $$

Obviously, for the initial value $V(0, x(0))$, the linear fractional differential equation

$$ {}_C D^p_{0,t} V(t, x(t)) = -\frac{\alpha_3}{\alpha_2} V(t, x(t)) \tag{3.5} $$

has a unique solution $V(t, x(t)) = V(0, x(0)) E_p((-\alpha_3/\alpha_2) t^p)$.

Taking into account Remark 2.13 and the relationship between (3.4) and (3.5), we obtain

$$ V(t, x(t)) \le V(0, x(0)) E_p\left(-\frac{\alpha_3}{\alpha_2} t^p\right), \tag{3.6} $$

where $E_p((-\alpha_3/\alpha_2)t^p)$ is a nonnegative function [21]. Substituting (3.6) in (3.2) yields

$$\|x(t)\| \leq \left[\frac{V(0,x(0))}{\alpha_1} E_p\left(-\frac{\alpha_3}{\alpha_2} t^p\right)\right]^{1/a}, \tag{3.7}$$

where $E_p((-\alpha_3/\alpha_2)t^p) \to 0$ $(t \to +\infty)$ from the asymptotic expansion of Mittag-Leffler function [22]. Hence the proof is completed. □

According to the above results, we have the following theorem.

Theorem 3.2. *Let $x_{eq} = 0$ be an equilibrium point of system* (3.1)*, and let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a continuously differentiable function $V(t,x(t)) : [t_0,\infty) \times \mathbb{D} \to \mathbb{R}_+$ and class-$\mathcal{K}$ function α satisfying*

$$V(t,x(t)) \geq \alpha(\|x\|), \tag{3.8}$$

$${}_C D^p_{t_0,t} V(t,x(t)) \leq 0, \tag{3.9}$$

where $x \in \mathbb{D}$, $p \in (0,1)$. Then $x_{eq} = 0$ is locally stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally stable.

Proof. Proceeding the same way as that in the proof of Theorem 3.1, it follows from (3.9) that $V(t,x(t)) \leq V(t_0,x(t_0))$. Again taking into account (3.8), one can get

$$\|x(t)\| \leq \alpha^{-1}(V(t_0,x(t_0))), \tag{3.10}$$

where $t \geq t_0$. Therefore, the equilibrium point $x_{\mathrm{eq}} = 0$ is stable. So the proof is finished. □

In the above two theorems, the stronger requirements on function V have been assumed to ensure the existence of ${}_C D^p_{t_0,t} V(t,x(t))$. This undoubtedly increases the difficulty in choosing the function $V(t,x(t))$. In fact, we can weaken the continuously differential function $V(t,x(t))$ as $V(t,x(t)) \in C_{1-p}([t_0,\infty) \times \mathbb{D}, \mathbb{R}_+)$. Here we give the corresponding results.

Theorem 3.3. *Let $x_{eq} = 0$ be an equilibrium point of system* (3.1)*, and let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin, $V(t,x(t)) \in C_{1-p}([t_0,\infty) \times \mathbb{D}, \mathbb{R}_+)$. Assume there exists a class-$\mathcal{K}$ function α such that*

$$V(t,x(t)) \geq \alpha(\|x\|), \tag{3.11}$$

$${}_{RL} D^p_{t_0,t} V(t,x(t)) \leq 0, \tag{3.12}$$

where $t > t_0 \geq 0$, $x \in \mathbb{D}$, and $p \in (0,1)$. Then $x_{eq} = 0$ is locally asymptotically stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally asymptotically stable.

Proof. Note that the linear fractional differential equation

$$ {}_{\mathrm{RL}}D^{p}_{t_0,t} V(t, x(t)) = 0 \tag{3.13} $$

has a unique solution $V(t, x(t)) = (V_0/\Gamma(p))(t - t_0)^{p-1}$ for initial value $V_0 = \Gamma(p)V(t, x(t))(t - t_0)^{1-p}|_{t=t_0}$.

Taking into account Lemma 2.12 and the relationship between (3.12) and (3.13), we obtain

$$ V(t, x(t)) \leq \frac{V_0}{\Gamma(p)}(t - t_0)^{p-1}. \tag{3.14} $$

Substituting (3.14) into (3.11) gives

$$ \|x(t)\| \leq \alpha^{-1}\left(\frac{V_0}{\Gamma(p)}(t - t_0)^{p-1}\right) \longrightarrow 0 \quad (t \longrightarrow +\infty), \tag{3.15} $$

from the definition of class-$\mathcal{K}$. This completes the proof. □

Corollary 3.4. *Let $x_{eq} = 0$ be an equilibrium point of system (3.1), let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin, and let $V(t, x(t)) \in C_{1-p}([t_0, \infty) \times \mathbb{D}, \mathbb{R}_+)$ be locally Lipschitz with respect to x. Assume $V(t, 0) = 0$,*

$$ V(t, x(t)) \geq a\|x\|^b, \qquad {}_{RL}D^{p}_{t_0,t} V(t, x(t)) \leq 0, \tag{3.16} $$

where $t > t_0 \geq 0$, $x \in \mathbb{D}$, $p \in (0,1)$, and a, b are arbitrary positive constants. Then $x_{eq} = 0$ is generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally generalized Mittag-Leffler stable.

Proof. In Theorem 3.3, by replacing $\alpha(\|x\|)$ by $a\|x\|^b$, we can get

$$ \|x(t)\| \leq \left\{\frac{V_0}{a}(t - t_0)^{p-1} E_{p,p}\left(0 \cdot (t - t_0)^p\right)\right\}^{1/b}, \tag{3.17} $$

so the conclusion holds. □

Theorem 3.5. *Let $x_{eq} = 0$ be an equilibrium point of system (3.1), let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin, and let $V(t, x(t)) \in C_{1-p}([t_0, \infty) \times \mathbb{D}, \mathbb{R}_+)$ be locally Lipschitz with respect to x. Assume*

(i) *there exist class-$\mathcal{K}$ functions α_i $(i = 1, 2, 3)$ having global growth momentum at the same level and satisfying*

$$\begin{aligned} \alpha_1(\|x\|) &\le V(t, x(t)) \le \alpha_2(\|x\|), \\ {}_{RL}D^{p}_{t_0,t} V(t, x(t)) &\le -\alpha_3(\|x\|), \end{aligned} \tag{3.18}$$

(ii) *there exists $a > 0$ such that $\alpha_1(r)$ and r^a have global growth momentum at the same level,*

where $t > t_0 \ge 0$, $x \in \mathbb{D}$, and $p \in (0, 1)$. Then $x_{eq} = 0$ is locally generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally generalized Mittag-Leffler stable.

Proof. It follows from condition (i) that there exists $k_1 > 0$ such that

$$\begin{aligned} {}_{\mathrm{RL}}D^{p}_{t_0,t} V(t, x(t)) &\le -\alpha_3(\|x\|) \\ &\le -k_1 \alpha_2(\|x\|) \\ &\le -k_1 V(t, x(t)). \end{aligned} \tag{3.19}$$

On the other hand, the linear fractional differential equation

$${}_{\mathrm{RL}}D^{p}_{t_0,t} V(t, x(t)) = -k_1 V(t, x(t)) \tag{3.20}$$

has a unique solution

$$V(t, x(t)) = \frac{V_0}{\Gamma(p)} (t - t_0)^{p-1} \cdot E_{p,p}\left(-k_1 (t - t_0)^p\right), \tag{3.21}$$

for the initial value $V_0 = \Gamma(p) V(t, x(t)) (t - t_0)^{1-p}|_{t=t_0}$.

Using (3.19), (3.20), and Lemma 2.12, we obtain

$$\alpha_1(\|x\|) \le V(t, x(t)) \le \frac{V_0}{\Gamma(p)} (t - t_0)^{p-1} \cdot E_{p,p}\left(-k_1 (t - t_0)^p\right), \tag{3.22}$$

where $E_{p,p}(-k_1 (t - t_0)^p)$ is a nonnegative function [23, 24].

In addition, using condition (ii), one gets

$$(k_2 \|x\|)^a \le \alpha_1(\|x\|), \tag{3.23}$$

for all $x \in \mathbb{D}$, where $k_2 > 0$.

Substituting (3.23) into (3.22), we finally obtain

$$\|x(t)\| \le \left\{ \frac{V_0}{k_2^a \Gamma(p)} (t - t_0)^{p-1} E_{p,p}\left(-k_1 (t - t_0)^p\right) \right\}^{1/a}. \tag{3.24}$$

Hence, the zero solution of system (3.1) is locally generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{\text{eq}} = 0$ is globally generalized Mittag-Leffler stable. The proof is completed. □

Remark 3.6. The nonnegative function $E_{p,p}(-k_1(t-t_0)^p)$ tends to zero as t approaches infinity from the asymptotic expansion of two-parameter Mittag-Leffler function [22], so the zero solution of system (3.1) satisfying the conditions of Theorem 3.5 is also asymptotically stable.

4. The Comparison Results on the Stability

It is well known that the comparison method is an effective way in judging the stability of ordinary differential systems. In this section, we will discuss similar results on the stability of fractional differential systems by using the comparison method.

In what follows, we consider system (3.1) with $f(t,0) = 0$ and the scalar fractional differential equation

$$ {}_{\text{RL}}D^{q}_{t_0,t}u(t) = g(t,u), \qquad u_0 = \Gamma(q)u(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \tag{4.1}$$

where the initial value $u_0 \in \mathbb{R}_+$, $u(t) \in C_{1-q}([t_0,\infty),\mathbb{R})$, $g \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ is Lipschitz in u and $g(t,0) = 0$, $0 < q < 1$. Also, we assume there exists a unique solution $u(t)$ $(t \geq t_0)$ for system (4.1) with the initial value u_0.

Theorem 4.1. *For system* (3.1), *let $x_{eq} = 0$ be an equilibrium point of system* (3.1), *and let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a Lyapunov-like function $V \in C_{1-q}([t_0,\infty)\times\Omega,\mathbb{R}_+)$ and a class-$\mathcal{K}$ function α such that $V(t,0) = 0$, $V(t,x) \geq \alpha(\|x\|)$, and $V(t,x)$ satisfies the inequality*

$$ {}_{RL}D^{q}_{t_0,t}V(t,x) \leq g(t,V(t,x)), \quad (t,x) \in [t_0,\infty)\times\Omega. \tag{4.2}$$

Suppose further that $g(t,x)$ is nondecreasing in x for each t.

(i) *If the zero solution of* (4.1) *is stable, then the zero solution of system* (3.1) *is stable;*

(ii) *if the zero solution of* (4.1) *is asymptotically stable, then the zero solution of system* (3.1) *is asymptotically stable, too.*

Proof. Let $x(t) = x(t,t_0,x_0)$ denote the solution of system (3.1) with initial value $x_0 \in \Omega$. Along the solution curve $x(t)$, $V(t,x(t)$ can be written as $V(t)$ and

$$ V(t) \leq \frac{V_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,V(s))ds, \tag{4.3}$$

where $V_0 = \Gamma(q)V(t)(t-t_0)^{1-q}|_{t=t_0}$. Applying the fractional integral operator $D^{-q}_{t_0,t}$ to both sides of (4.1) leads to

$$ u(t) = \frac{u_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,u(s))ds. \tag{4.4}$$

Now, taking $u_0 = V_0$ and applying Lemma 2.14 to inequalities (4.3) and (4.4), one has $V(t) \leq u(t)$, $t > t_0$.

(i) If the zero solution of (4.1) is stable, then for any initial value $u_0 \geq 0$, there exists $\epsilon > 0$ such that $|u(t)| < \epsilon$ for all $t > t_0$. Therefore, taking into account $V(t, x(t)) \geq \alpha(\|x\|)$, one gets

$$\alpha(\|x(t)\|) \leq V(t, x) \leq u(t) < \epsilon, \tag{4.5}$$

that is, $\|x(t)\| < \alpha^{-1}(\epsilon)$, and the zero solution of system (3.1) is stable.

(ii) One can directly derive

$$\alpha(\|x(t)\|) \leq V(t, x) \leq u(t) < \epsilon \tag{4.6}$$

from the same argument in (i). Then, taking the limit to both sides of (4.6) and combining with the definition of class-$\mathcal{K}$ function, one can obtain $\lim_{t\to+\infty}\|x(t)\| = 0$.

The proof is thus finished. □

Remark 4.2. In Theorem 4.1 and system (4.1), if we replace order q by $p \in (0, 1)$, but other conditions remain unchanged, then the result in Theorem 4.1 still holds.

Especially, if the class-$\mathcal{K}$ function $\alpha(\|x\|)$ in Theorem 4.1 and $\|x\|^a$ have global growth momentum at the same level, then we can have similar comparison result on the generalized Mittag-Leffler stability as follows.

Theorem 4.3. *For system* (3.1), *let $x_{eq} = 0$ be an equilibrium of system* (3.1), *and let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exists a Lyapunov-like function $V \in C_{1-q}([t_0, \infty) \times \Omega, \mathbb{R}_+)$ such that $V(t, 0) = 0$, $V(t, x) \geq k\|x\|^a$, and $V(t, x)$ is locally Lipschitz in x and satisfies the inequality*

$${}_{RL}D^q_{t_0,t}V(t, x) \leq g(t, V(t, x)), \quad (t, x) \in [t_0, \infty) \times \Omega, \tag{4.7}$$

where $k > 0$, $a > 0$. Suppose further that $g(t, x)$ is nondecreasing in x for each t. Then the zero solution of system (3.1) *is also locally generalized Mittag-Leffler stable if the zero solution of* (4.1) *is locally generalized Mittag-Leffler stable. In addition, if the assumptions hold globally on $\mathbb{R}^n$, then the globally generalized Mittag-Leffler stability of zero solution of* (4.1) *implies the globally generalized Mittag-Leffler stability of zero solution of system* (3.1).

Proof. First, from Definition 2.8, if the zero solution of (4.1) is generalized Mittag-Leffler stable, then there exist $\lambda \geq 0$, $b > 0$, $-q < \gamma \leq 1 - q$ such that

$$|u(t)| \leq \left\{m(u_0)(t - t_0)^{-\gamma} E_{q,1-\gamma}\left(-\lambda(t - t_0)^q\right)\right\}^b, \tag{4.8}$$

where $m(0) = 0$, $m(x) \geq 0$ and $m(x)$ is locally Lipschitz in x with Lipschitz constant $\mathcal{L}_0$.

Taking $u_0 = V_0 = \Gamma(q)V(t,x)(t-t_0)^{1-q}|_{t=t_0}$ and noting that $V(t,x) \le u(t)$ holds from Theorem 4.1, then taking into account (4.8) and $V(t,x) \ge k\|x\|^a$, we obtain

$$k\|x(t)\|^a \le V(t,x) \le \left\{m(u_0)(t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^b. \tag{4.9}$$

Furthermore,

$$\|x(t)\| \le \left\{\frac{m\left(\Gamma(q)V(t,x(t_0))(t-t_0)^{1-q}\Big|_{t=t_0}\right)}{k^{1/b}} \cdot (t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^{b/a}. \tag{4.10}$$

Let $M(x) = m(\Gamma(q)V(t,x)(t-t_0)^{1-q}|_{t=t_0})/k^{1/b}$. Then it follows that

$$\|x(t)\| \le \left\{M(x(t_0))(t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^{b/a}, \tag{4.11}$$

where $M(0) = m(\Gamma(q)V(t,0)(t-t_0)^{1-q}|_{t=t_0})/k^{1/b} = 0$ due to $V(t,0) = 0$. It is obvious that $M(x)$ is a nonnegative function from $m(x), V(t,x) \ge 0$ and $k > 0$. In addition, $M(x)$ is locally Lipschitz in x since $m(x)$ and $V(t,x)$ are locally Lipschitz in x. So, the zero solution of system (3.1) is generalized Mittag-Leffler stable. The proof is completed. □

5. Conclusion

In this paper, we have studied the stability of the zero solution of nonlinear fractional differential systems with the Caputo derivative and the commensurate order $0 < q < 1$ by using a Lyapunov-like function. Compared to [15], we weaken the continuously differential function $V(t,x)$ as $V(t,x) \in C_{1-p}([t_0,\infty)\times\mathbb{D},\mathbb{R}_+)$. Sufficient conditions on generalized Mittag-Leffler stability and asymptotical stability are derived. Meanwhile, comparison method is applied to the analysis of the stability of fractional differential systems by fractional differential inequalities.

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Research Article

Existence and Uniqueness Theorem of Fractional Mixed Volterra-Fredholm Integrodifferential Equation with Integral Boundary Conditions

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We study the existence and uniqueness of the solutions of mixed Volterra-Fredholm type integral equations with integral boundary condition in Banach space. Our analysis is based on an application of the Krasnosel'skii fixed-point theorem.

1. Introduction

In the last century, notable contributions have been made to both the theory and applications of the fractional differential equations. For the theory part, Momani and Hadid have investigated the local and global existence theorem of both fractional differential equation and fractional integrodifferential equations; see [1–6]. Fractional-order differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering.

Integrodifferential equations with integral boundary conditions are often encountered in various applications; it is worthwhile mentioning the applications of those conditions in the study of population dynamics and cellular systems. For a detailed description of the integral boundary conditions, we refer the reader to a recent paper [7]. In [8], Tidke studied the problem of existence of global solutions to nonlinear mixed Volttera-Fredholm integrodifferential equations with nonlocal condition.

Ahmad and Nieto [9] studied some existence results for boundary value problem involving a nonlinear integrodifferential equation of fractional order with integral equation.

Very recently N'Guérékata [10] discussed the existence of solutions of fractional abstract differential equations with nonlocal initial condition. Anguraj et al. [11] studied the existence and uniqueness theorem for the nonlinear fractional mixed Volterra-Fredholm integrodifferential equation with nonlocal initial condition.

Motivated by these works, we study in this paper the existence of solution of boundary value problem for fractional integrodifferential equations (in the case $1 < \alpha \leq 2$) in Banach spaces by using Banach and Krasnosel'skii fixed-point theorems.

2. Preliminaries

First of all, we recall some basic definitions; see [12–15].

Definition 2.1. For a function f given on the interval $[a,b]$, the Caputo fractional order derivative of f is defined by

$$ {}_a^tD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \tag{2.1} $$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.2. *Let $\alpha > 0$, then*

$$ {}_a^tD^{-\alpha}\, {}_a^tD^\alpha y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.2} $$

for some $c_i \in R$, $i = 0, 1, \ldots, n-1$, $n = [\alpha] + 1$.

Definition 2.3. Let f be a function which is defined almost everywhere (a.e) on $[a,b]$, for $\alpha > 0$, we define

$$ {}_a^bD^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t)dt, \tag{2.3} $$

provided that the integral (Lebesgue) exists.

Theorem 2.4 (Krasnosel'skii fixed point theorem). *Let M be a closed-convex bounded nonempty subset of a Banach space X. Let A and B be two operators such that*

(i) *$Ax + By = M$, whenever $x, y \in M$,*

(ii) *A is compact and continuous;*

(iii) *B is a contraction mapping,*

then there exists $z \in M$ such that $z = Az + Bz$.

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = ([0,T], X)$ be Banach space ofall

continuous functions $\varphi : [0,T] \to X$, *with supermum norm* $\|\varphi\| = \sup\{\|\varphi(s)\| : s \in [0,T]\}$. *Consider the fractional mixed Volttera-Fredholm integrodifferential equation with boundary conditions, which has the form*

$$D^{\alpha}y(t) = f\left(t, y(t), \int_0^t k(t,s,y(s))ds, \int_0^T h_1(t,s,y(s))ds\right), \tag{2.4}$$

$$y(0) - y'(0) = \int_0^T g(y(s))ds, \qquad y(T) - y'(T) = \int_0^T h(y(s))ds, \tag{2.5}$$

where $1 < \alpha \le 2$, D^{α} *is the Caputo fractional derivative and the nonlinear functions* $f : [0,T] \times X \times X \times X \to X$, $k, h_1 : [0,T] \times [0,T] \times X \to X$ *and* $g, h : X \to X$ *satisfy the following hypotheses:*

(H1) *there exists constants* G_1, G_2 *such that* $\|h(y)\| \le G_1, \|g(y)\| \le G_2$ *for* $y \in X$,

(H2) *there exists constants* b_1, b_2 *such that* $\| h(x) - h(y)\| \le b_1\| x - y \|$ *and*

$$\| g(x) - g(y)\| \le b_2\| x - y \|, \quad \forall x, y \in X, \tag{2.6}$$

(H3) *there exists continuous functions* $p : [0,T] \to R^+ = [0,\infty)$ *and* $p_1 : [0,T] \to R^+$ *such that* $\| \int_0^t (k(t,s,x) - k(t, s, y))ds\| \le p(t) \|x - y \|$ *and* $\| \int_0^t k(t, s, y)ds\| \le p_1(t) \|y\|$, *for every* $t, s \in [0,T]$ *and* $x, y \in X$,

(H4) *there exists continuous functions* $q : [0,T] \to R^+$ *and* $q_1 : [0,T] \to R^+$ *such that* $\| \int_0^T (h_1(t,s,x) - h_1(t,s,y)ds\| \le q(t) \|x - y \|$ *and* $\| \int_0^T h_1(t, s, y)ds\| \le q_1(t)\|y \| for\ every\ t, s \in [0,T] and x, y \in X$

(H5) *there exists continuous function* $L : [0,T] \to R^+$, *and* N_1 *is positive constant such that* $\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \le L(t)K(\|x_1 - x_2\| + \|y_1 - y_2 \| + \|z_1 - z_2\|)$ *and* $N_1 = \sup_{t\in[0,T]}\|f(t,0,0,0)\|$, *for every* $t \in [0,T]$ *and* $x_1, y_1, z_1, x_2, y_2, z_2 \in X$, *where* $K : R^+ \to (0,\infty)$ *is continuous nondecreasing function satisfying* $K(\gamma(t)x) \le \gamma(t)K(x)$, *where* γ *is a continuous function* $\gamma : [0,T] \to R^+$.

Lemma 2.5. *Let* $1 < \alpha \le 2$ *and* $f : J \times X \to X$, *where* $J = [0,T]$, *be a continuous function, then the solution of fractional differential equation* (2.4) *with the boundary condition* (2.5) *is*

$$\begin{aligned} y(t) = {} & \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1 - \frac{(1+t)}{T}\right)\int_0^T g(y(s))ds \\ & - \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) \\ & + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds. \end{aligned} \tag{2.7}$$

Proof. By Lemma 2.2, we reduce the problem (2.4)-(2.5) to an equivalent integral equation

$$\begin{aligned} y(t) &= {}_0^tI^{\alpha} f + C_1 + C_2 t, \\ y(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds + C_1 + C_2 t. \end{aligned} \tag{2.8}$$

In view of the relations ${}^cD^{\alpha} I^{\alpha} y(t) = y(t)$ and $I^{\alpha} I^{\beta} y(t) = I^{\alpha+\beta} y(t)$, for $\alpha, \beta > 0$, we obtain

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds + C_2. \tag{2.9}$$

Applying the boundary condition (2.5), we find that

$$\begin{aligned} y(0) &= C_1, \qquad y(T) = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds \\ &\qquad + C_1 + C_2 T, \\ y'(0) &= C_2, \qquad y'(T) = \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds \\ &\qquad + C_2, \end{aligned} \tag{2.10}$$

that is,

$$\begin{aligned} C_2 &= \frac{1}{T}\int_0^T h(y(s))ds - \frac{1}{T}\int_0^T g(y(s))ds \\ &\quad - \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds \\ &\quad + \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds, \\ C_1 &= \frac{1}{T}\int_0^T h(y(s))ds + \left(1 - \frac{1}{T}\right) \int_0^T g(y(s))ds \\ &\quad - \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds \\ &\quad + \frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right) ds. \end{aligned} \tag{2.11}$$

Therefore the solution of (2.4)-(2.5) is

$$\begin{aligned} y(t) = {} & \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds \\ & - \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\ & + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds, \end{aligned} \tag{2.12}$$

which completes the proof. □

3. The Main Result

Theorem 3.1. *If the hypotheses (H1)–(H5) are satisfied, then the fractional integrodifferential equation* (2.4)-(2.5) *has a unique solution on* J.

Proof. Define $F : C \to C$ by

$$\begin{aligned} Fy(t) = {} & \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds \\ & - \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\ & + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds. \end{aligned} \tag{3.1}$$

We show that F has a fixed point on Br. This fixed point is then a solution of (2.4)-(2.5). Firstly, we show that $FBr \subset Br$, where $Br = \{y \in C : \|y\| \le r\}$. For $y \in Br$, we have

$$\begin{aligned} \|Fy(t)\| \le {} & \frac{(1+t)}{T}\int_0^T \|h(y(s))\|\, ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(s))\|ds \\ & + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\| ds \end{aligned}$$

$$+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\|ds$$

$$+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\|ds,$$

$$\begin{aligned}
\|Fy(t)\| \leq{}& \frac{(1+t)}{T}\int_0^T\|h(y(s))\|ds+\left(1-\frac{(1+t)}{T}\right)\int_0^T\|g(y(s))\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)+f(s,0,0,0)\right\|ds\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)+f(s,0,0,0)\right\|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)+f(s,0,0,0)\right\|ds,
\end{aligned}$$

$$\begin{aligned}
\|Fy(t)\| \leq{}& \frac{(1+t)}{T}G_1T+\left(1-\frac{(1+t)}{T}\right)G_2T\\
&+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)\right\|ds+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,0,0,0)\|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)\right\|ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,0,0,0)\|ds\\
&+\left(\frac{1+t}{T}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,y(s),\int_0^S k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right.\\
&\qquad\left.-f(s,0,0,0)\right\|ds+\left(\frac{1+t}{T}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,0,0,0)\|ds\\
\leq{}& \frac{(1+t)}{T}\,G_1T+\left(1-\frac{(1+t)}{T}\right)G_2\,T+\frac{(1+t)}{T}\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)
\end{aligned}$$

$$
\begin{aligned}
&K\left(\|y(s)\| + \left\|\int_0^S k(s,\tau,y(\tau))d\tau\right\| + \left\|\int_0^T h_1(s,\tau,y(\tau))d\tau)\right\|\right)ds \\
&+ \frac{(1+t)N_1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s) \\
&K\left(\|y(s)\| + \left\|\int_0^s k(s,\tau,y(\tau))d\tau\right\| + \left\|\int_0^T h_1(s,\tau,y(\tau))d\tau)\right\|\right)ds \\
&+ \frac{(1+t)N_1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s) \\
&K\left(\|y(s)\| + \left\|\int_0^s k(s,\tau,y(\tau))d\tau\right\| + \left\|\int_0^T h_1(s,\tau,y(\tau))d\tau)\right\|\right)ds \\
&+ N_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\le (1+t)G_1 + \left(1-\frac{(1+t)}{T}\right)G_2T + \frac{(1+t)N_1}{T} \\
&\left(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right) + N_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)K(\|y\| + p_1(s)\|y\| + q_1(s)\|y\|)ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|y\| + p_1(s)\|y\| + q_1(s)\|y\|)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|y\| + p_1(s)\|y\| + q_1(s)\|y\|)ds, \\
\|Fy(t)\| \le{}& (1+t)G_1 + \left(1-\frac{(1+t)}{T}\right)G_2T \\
&+ \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)(1+p_1(s)+q_1(s))K(\|y\|)ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)(1+p_1(s)+q_1(s))K(\|y\|)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)(1+p_1(s)+q_1(s))K(\|y\|)ds.
\end{aligned}
\tag{3.2}
$$

Since we have $M_1 = \sup\{L(t)(\ 1+p_1(t)+q_1(t));\ t \in [0,T]\}$, and $(1-((1+t)/T)) < (1-(1/T))$, we get

$$
\begin{aligned}
&\leq (1+t)G_1 + \left(1-\frac{1}{T}\right) G_2\ T + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(1+t)M_1}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}K(\|y\|)ds + \frac{(1+t)M_1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}K(\|y\|)ds \\
&\quad + M_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}K(\|y\|)ds \\
&\leq (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2\ T + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(1+t)M_1K(r)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \frac{(1+t)M_1K(r)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + M_1K(r)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\leq (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2T + \frac{(1+t)N_1}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + N_1\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(1+t)M_1K(r)}{T}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) + \frac{M_1K(r)T^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq (1+t)G_1 + (T-1)G_2\ + \frac{(1+t)}{T}(N_1+M_1K(r))\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&\quad + (N_1+M_1K(r))\frac{T^{\alpha}}{\Gamma(\alpha+1)}, \\
\|Fy(t)\| &\leq (1+T)G_1 + (T-1)G_2 + \frac{(1+T)}{T}(N_1+M_1K(r))\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&\quad + (N_1+M_1K(r))\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq G_1(1+T) + G_2(T-1) + \frac{C_0(N_1+M_1K(r))}{\Gamma(\alpha+1)T^{2-\alpha}},
\end{aligned}
\tag{3.3}
$$

where $C_0 = 2T^2 + T + \alpha(T+1)$.

Now, take $x,\ y \in C$ and for each $t \in [0,T]$, we obtain

$$
\begin{aligned}
\|Fx(t) - Fy(t)\| &\leq \frac{(1+t)}{T}\int_0^T \|h(x)-h(y)\|ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|\ g(x)-g(y)\|ds \\
&\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&\left\| f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)\right. \\
&\quad \left. -f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\| ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\left\| f\left(s,\ x(s), \int_0^s k(s,\tau,x(\tau))d\tau\ , \int_0^T h_1(s,\tau,x(\tau))d\tau\right)\right. \\
&\quad \left. -f\left(s,\ y(s), \int_0^s k(s,\tau,\ y(\tau))d\tau, \int_0^T h_1(s,\ \tau,\ y(\tau))d\tau\right)\right\| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\left\| f\left(s,\ x(s), \int_0^s k(s,\tau,x(\tau))d\tau\ , \int_0^T h_1(s,\tau,x(\tau))d\tau\right)\right. \\
&\quad \left. -f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\| ds,
\end{aligned}
\tag{3.4}
$$

by using (H1)–(H5), we get

$$
\begin{aligned}
\|Fx(t) - Fy(t)\| \leq{}& \frac{b_1(1+t)}{T}\int_0^T \|x-y\| ds\ + b_2\left(1 - \frac{(1+t)}{T}\right)\int_0^T \|x-y\|\ ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} L(s) \\
&K\left(\|x(s)-y(s)\| + \left\|\int_0^s (k(s,\tau,x(\tau)) - k(s,\tau,y(\tau)))d\tau\ \right\|\right. \\
&\quad \left. + \left\|\int_0^T (h_1(s,\tau,x(\tau)) - h_1(s,\tau,y(\tau)))d\tau\right\|\right) ds \\
&+ \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} L(s) \\
&K\left(\|x(s)-y(s)\| + \left\|\int_0^s (k(s,\tau,x(\tau)) - k(s,\tau,y(\tau)))d\tau\right\|\right. \\
&\quad \left. + \left\|\int_0^T (h_1(s,\tau,x(\tau)) - h_1(s,\tau,y(\tau)))d\tau\right\|\right) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L(s)
\end{aligned}
$$

$$
\begin{aligned}
& K\Bigg(\|x(s)-y(s)\| + \left\|\int_0^s \big(k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\big)d\tau\right\| \\
& \quad + \left\|\int_0^T \big(h_1(s,\tau,x(\tau)) - h_1(s,\tau,y(\tau))\big)d\tau\right\|\Bigg)ds \\
\leq\ & \frac{b_1(1+t)}{T}\int_0^T \|x-y\|ds + b_2\left(1-\frac{(1+t)}{T}\right)\int_0^T \|x-y\|ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)K(\|x-y\| + p(s)\|x-y\| + q(s)\|x-y\|)ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|x-y\| + p(s)\|x-y\| + q(s)\|x-y\|)ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)K(\|x-y\| + p(s)\|x-y\| + q(s)\|x-y\|)ds \\
\leq\ & \frac{b_1(1+t)}{T}\int_0^T \|x-y\|ds + b_2\left(1-\frac{1}{T}\right)\int_0^T \|x-y\|ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)(1+p(s)+q(s))K(\|x-y\|)ds \\
& + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)(1+p(s)+q(s))K(\|x-y\|)ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ L(s)(1+p(s)+q(s))K(\|x-y\|)ds\ .
\end{aligned}
\tag{3.5}
$$

Since we have $M(t) = L(t)(1+p(t)+q(t)$, $M^* = \sup\{M(t) : t \in [0,T]\}$, and, Let $K(\|x-y\|) \leq w\|x-y\|$, $(w > 0)$, then

$$
\begin{aligned}
\|Fx(t) - Fy(t)\| \leq\ & \frac{b_1(1+t)}{T}\int_0^T \|x-y\|ds + b_2\left(1-\frac{1}{T}\right)\int_0^T \|x-y\|ds \\
& + \frac{wM^*(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|x-y\|ds \\
& + \frac{w\ M^*(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|x-y\|ds \\
& + wM^*\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ \|x-y\|ds \\
\leq\ & \left[b_1(1+T) + b_2(T-1) + \frac{wM^*C_1(1+T)}{\Gamma(\alpha+1)\ T^{2-\alpha}}\right]\|\ x-y\ \|,
\end{aligned}
\tag{3.6}
$$

where $C_1 = 2T^2 + T + \alpha(1+T)$.

As $b_1(1+T)+b_2(T-1)+(wM^*C_1(1+T))/(\Gamma(\alpha+1)T^{2-\alpha}) < 1$, therefore f is a contraction. Thus, the conclusion of the theorem is followed by the contraction mapping principle. □

Theorem 3.2. *Assume that (H1)–(H5) hold with*

$$\left\| f\left(t, y(t), \int_0^t k(t,\tau,y(\tau))\,d\tau, \int_0^T h_1(t,\tau,y(\tau))\,d\tau\right)\right\| \le \psi(t), \quad \text{where } \psi(t) \in L_1(J). \tag{3.7}$$

Then the boundary value problem (2.4)-(2.5) *has at least one element on* $[0,T]$.

Proof. Consider $Br = \{y \in C : \|y\| \le r\}$. We define the operators A and B as

$$\begin{aligned}
(Ax)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(t, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds,\\
(Bx)(t) &= \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\quad f\left(s, x(s), \int_0^s k(s,\ \tau,\ x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\\
&\quad + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds.
\end{aligned} \tag{3.8}$$

Let us observe that if $x,\ y \in Br$, then $Ax + By \in Br$,

$$\begin{aligned}
\|Ax+By\| = \Bigg\| &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\\
&+ \frac{(1+t)}{T}\int_0^T h(y(s))ds + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(s))ds + \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau\ , \int_0^T h_1(s,\ \tau,\ y(\tau))d\tau\right)ds + \frac{(1+t)}{T}\\
&\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^T h_1(s,\tau,y(\tau))d\tau\right)ds\Bigg\|\\
\le &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\| f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)\right\|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1+t)}{T}\int_0^T \|h(y(s))\|ds+\left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(s))\|ds\\
&+\frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\left\|f\left(s,y(s),\int_0^s k(s,\tau,y(\tau))d\tau,\int_0^T h_1(s,\tau,y(\tau))d\tau\right)\right\|ds\\
&+\frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\left\|f\left(s,\ y(s),\ \int_0^s k(s,\ \tau,\ y(\tau))d\tau,\ \int_0^T h_1(s,\ \tau,\ y(\tau))d\tau\right)\right\|ds\\
&\le \|\psi\|_{L_1}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+(1+t)G_1\\
&+\left(1-\frac{(1+t)}{T}\right)G_2T+\frac{(1+t)\|\psi\|_{L_1}}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\\
&+\frac{(1+t)\|\psi\|_{L_1}}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\le \frac{\|\psi\|_{L_1}T^{\alpha}}{\Gamma(\alpha+1)}+(1+t)G_1+\left(1-\frac{1}{T}\right)G_2T+\frac{(1+t)T^{\alpha-1}\|\psi\|_{L_1}}{T\Gamma(\alpha)}+\frac{(1+t)T^{\alpha}\|\psi\|_{L_1}}{T\Gamma(\alpha+1)}\\
&\le \frac{\|\psi\|_{L_1}T^{\alpha}}{\Gamma(\alpha+1)}+\frac{(1+t)T^{\alpha-1}\|\psi\|_{L_1}}{T\Gamma(\alpha)}+\frac{(1+t)T^{\alpha}\|\psi\|_{L_1}}{T\Gamma(\alpha+1)}+(1+T)G_1+(T-1)G_2\\
&\le G_1(1+T)+G_2(T-1)+\frac{C_2T^{\alpha-2}}{\Gamma(\alpha+1)}\ \|\psi\|_{L_1},
\end{aligned}
\tag{3.9}
$$

where $C_2 = 2T^2 + T(\alpha+1) + T$.

Now we prove that Bx is contraction mapping,

$$
\begin{aligned}
\|Bx_1 - Bx_2\| \le{}& \frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\ \left\|f\left(s,x_1(s),\int_0^s k(s,\tau,x_1(\tau))d\tau\ ,\ \int_0^T h_1(s,\tau,x_1(\tau))d\tau\right)\right.\\
&\left.-f\left(s,x_2(s)\int_0^s k(s,\tau,x_2(\tau))d\tau\ ,\int_0^T h_1(s,\tau,x_2(\tau))d\tau\right)\right\|ds\\
&+\frac{(1+t)}{T}\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s,x_1(s),\int_0^s k(s,\tau,x_1(\tau))d\tau,\int_0^T h_1(s,\tau,x_1(\tau))d\tau\right)\right.\\
&\left.-f\left(s,x_2(s),\int_0^s k(s,\tau,x_2(\tau))d\tau\ ,\int_0^T h_1(s,\tau,x_2(\tau))d\tau\right)\right\|ds
\end{aligned}
$$

$$\leq \frac{(1+t)}{T}\left[\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\ L(s)\big(1+p(s)+q(s)\big)K(\|x_1-x_2\|)ds \right.$$
$$\left. + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} L(s)\big(\ 1+p(s)+q(s)\big)K(\|x_1-x_2\|)ds\right]. \tag{3.10}$$

Let $K(\|x_1-x_2\|) \leq w\|x_1-x_2\|$, we obtain

$$\|Bx_1-Bx_2\| \leq \frac{(1+t)wM^*}{T}\|\ x_1-x_2\ \|\left[\ \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\ +\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right],$$
$$\|Bx_1-Bx_2\| \leq \frac{(1+T)wM^*}{T}\left[\ \frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\|\ x_1-x_2\ \|$$
$$\leq \frac{wM^*(1+T)(\alpha+T)}{\Gamma(\alpha+1)T^{2-\alpha}}\|\ x_1-x_2\ \|. \tag{3.11}$$

It is clear that B is contraction mapping, since $x(t)$ is continuous, then Ax is continuous

$$\|Ax(t)\| = \left\|\frac{1}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right\|$$
$$\leq \|\psi\|_{L_1}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ ds \tag{3.12}$$
$$\|Ax(t)\| \leq \frac{T^{\alpha}\|\psi\|_{L_1}}{\Gamma(\alpha+1)}.$$

Hence, A is uniformly bounded on Br. Now, let us prove that $Ax(t)$ is equicontinuous, let $t_1,\ t_2 \in [0,\ T]$ and $x \in Br$. Using the fact that f is bounded on the compact set $J \times Br$, thus $\sup_{(t,s)\in J\times Br}\|f(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))d\tau, \int_0^T h_1(s,\ \tau,\ x(\tau))d\tau)\| = c_0 < \infty$,we get

$$\|Ax(t_1)-Ax(t_2)\| = \left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1} f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau,\right.\right.$$
$$\left.\int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\ \int_0^{t_2}(t_2-s)^{\alpha-1}$$
$$\left. f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right\|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]\right.$$

$$
\begin{aligned}
&f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\Bigg\| \\
&+\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\ , \int_0^T h_1(s,\tau,x(\tau))d\tau\right)ds\right\| \\
&\leq \frac{c_0}{\Gamma(\alpha+1)}\ \left[2(t_2-t_1)^{\alpha}+(t_1{}^{\alpha}-t_2{}^{\alpha})\right].
\end{aligned}
\tag{3.13}
$$

So A is relatively compact. By Arzela-Ascoli theorem, A is compact. Now we conclude the result of the theorem of Krasnosel'skii theorem. □

Example 3.3. Consider the following fractional mixed Volterra-Fredholm integrodifferential equation:

$$
y^{(1.5)}(t) = \frac{1}{10}+\frac{1}{10+|y(t)|}+\int_0^t \frac{|y(t)|}{10\ e^{|y(t)|}+t}dt\ +\int_0^1 \frac{|y(t)|e^{-t}}{10\ +|y(t)|^2}dt, \tag{3.14}
$$

with integral boundary conditions

$$
y(0)-y'(0)=\int_0^1 \frac{1}{10+|y(t)|}dt, \qquad y(1)-y'(1)=\int_0^1 \frac{1}{10+e^{-|y(t)|}}dt. \tag{3.15}
$$

Here,

$$
\begin{gathered}
\|g(y(t))\| = \left\|\frac{1}{10+|y(t)|}\right\| \leq \frac{1}{10}, \qquad \|g(x)-g(y)\| \leq \frac{1}{100}\|x-y\|, \\
\|h(y(t))\| = \left\|\frac{1}{10+e^{-|y(t)|}}\right\| \leq \frac{1}{10}, \qquad \|h(x)-h(y)\| \leq \frac{1}{100}\|x-y\|, \\
\left\|\int_0^t (k(t,s,x)-k(t,s,y))ds\right\| \leq \frac{1}{10e^t}\|x-y\ \|, \qquad \left\|\int_0^t k(t,s,y)ds\right\| \leq \frac{1}{10+t}\|y(t)\|, \\
\left\|\int_0^t (h_1(t,s,x)-h_1(t,s,y))ds\right\| \leq \frac{1}{10e^t}\|x-y\ \|, \qquad \left\|\int_0^t h_1(t,s,y)ds\right\| \leq \frac{1}{10+t}\|y(t)\|, \\
\|f(t,x_1y_1,z_1)-f(t,x_2,y_2,z_2)\| \leq \frac{1}{10\ +t}(\|x_1-x_2\ \|+\|y_1-y_2\ \|+\|z_1-z_2\ \|), \\
f(t,0,0,0)=\frac{1}{10}.
\end{gathered}
\tag{3.16}
$$

Hence, the conditions (H1)–(H5) hold with $G_1 = G_2 = 0.1$, $b_1 = b_2 = 0.01$, $M_1^* = 0.12$, $w = 0.1$, $C_o = 6$, $N_1 = 0.1$, $M^* = 0.12$, and $C_1 = 6$, thus

$$b_1(1+T) + b_2(T-1) + \frac{wM^*C_1(1+T)}{\Gamma(\alpha+1)T^{2-\alpha}} < 1 \Longleftrightarrow 0.01(2) + \frac{(0.1)(0.12)6(2)}{\Gamma(2.5)} < 1. \tag{3.17}$$

We conclude from the above example that the integrodifferential equation has unique solution.

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Research Article

Existence of Positive Solutions for Fractional Differential Equation with Nonlocal Boundary Condition

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By using the fixed point theorem, existence of positive solutions for fractional differential equation with nonlocal boundary condition $D_{0+}^{\alpha}u(t) + a(t)f(t,u(t)) = 0$, $0 < t < 1$, $u(0) = 0$, $u(1) = \sum_{i=1}^{\infty}\alpha_i u(\xi_i)$ is considered, where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $\xi_i \in (0,1)$, $\alpha_i \in [0,\infty)$ with $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} < 1$, $a(t) \in C([0,1],[0,\infty))$, $f(t,u) \in C([0,1]\times[0,\infty),[0,\infty))$.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering. For details, see [1–6] and references therein.

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions [6–8]. Recently, there are some papers that deal with the existence and multiplicity of solution (or positive solution) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems, Leray-Shauder theory, etc.); see [9–17].

Recently, Bai and Lü [15] studied the existence of positive solutions of nonlinear fractional differential equation

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In this paper, we study the existence of positive solutions for fractional differential equation with nonlocal boundary condition

$$\begin{gathered} D_{0+}^{\alpha} u(t) + a(t) f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i), \end{gathered} \tag{1.2}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $\xi_i \in (0,1)$, $\alpha_i \in [0,\infty)$ with $\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1} < 1$, $a(t) \in C([0,1],[0,\infty))$, $f(t,u) \in C([0,1] \times [0,\infty),[0,\infty))$.

We assume the following conditions hold throughout the paper:

(H1) $\xi_i \in (0,1)$, $\alpha_i \in [0,\infty)$ is both constants with $\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1} < 1$,

(H2) $a(t) \in C([0,1],[0,\infty))$, $a(t) \not\equiv 0$ on $[a,b] \subset (0,1)$,

(H3) $f(t,u) \in C([0,1] \times [0,\infty),[0,\infty))$.

Remark 1.1. To our knowledge, there are no results about the existence of positive solutions for problem (1.2).

2. The Preliminary Lemmas

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.3. The map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E, provided that $\theta : P \to [0,\infty)$ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y), \tag{2.3}$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Remark 2.4. As a basic example, we quote for $\lambda > -1$,

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}, \tag{2.4}$$

giving in particular $D_{0+}^{\alpha} t^{\alpha-m} = 0,\ m = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

From Definition 2.2 and Remark 2.4, we then obtain the following.

Lemma 2.5. *Let $\alpha > 0$. If one assumes $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0 \tag{2.5}$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},\ C_i \in R,\ i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α, as unique solutions.

Lemma 2.6. *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{2.6}$$

for some $C_i \in R,\ i = 1, 2, \ldots, N$.

Lemma 2.7 (see [15]). *Given $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0 \end{gathered} \tag{2.7}$$

is

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.8}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.9}$$

Lemma 2.8. *Suppose $(H1)$ holds. Given $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i) \end{gathered} \tag{2.10}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)ds + B(y)t^{\alpha-1}, \tag{2.11}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.12}$$

$$B(y) = \frac{\sum_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)y(s)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}.$$

Proof. By applying Lemmas 2.6 and 2.7, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds + C_1t^{\alpha-1} + C_2t^{\alpha-2}. \tag{2.13}$$

Because

$$\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)ds = \frac{\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}(1-\xi_i)}{\alpha\Gamma(\alpha)}, \quad \alpha_i\xi_i^{\alpha-1}(1-\xi_i) < \alpha_i\xi_i^{\alpha-1}, \tag{2.14}$$

by (H1), $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}(1-\xi_i)$ is converge. Therefore, $\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)ds$ is converge. $y(t)$ is continuous function on $[0,1]$, so $\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds$ is converge.

By $u(0)=0$, $u(1)=\sum_{i=1}^{\infty}\alpha_i u(\xi_i)$, there are $C_2=0$, $C_1=(\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds)/(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})$. Therefore,

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)ds + B(y)t^{\alpha-1}, \\ B(y) &= \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}. \end{aligned} \tag{2.15}$$

□

Lemma 2.9 (see [15]). *The function $G(t,s)$ defined by* (2.9) *satisfies the following conditions:*

(1) $G(t,s) > 0$, *for* $t,s \in (0,1)$,

(2) *there exists a positive function* $\gamma \in C(0,1)$ *such that*

$$\min_{(1/4)\le t\le(3/4)} G(t,s) \ge \gamma(s)\max_{0\le t\le 1} G(t,s) = \gamma(s)G(s,s), \quad 0 < s < 1. \tag{2.16}$$

Lemma 2.10 (see [18]). *Let E be a Banach space, $P \subseteq E$ a cone and $\Omega_1,\ \Omega_2$ two bounded open sets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|, x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|, x \in P \cap \partial\Omega_2$

holds. Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.11 (see [19]). *Let P be a cone in real Banach space $E, P_c = \{x \in P \mid \|x\| \le c\}, \theta$ a nonnegative continuous concave functional on P such that $\theta(x) \le \|x\|$, for all $x \in \overline{P}_c$, and $P(\theta, b, d) = \{x \in P \mid b \le \theta(x),\ \|x\| \le d\}$. Suppose that $A : \overline{P}_c \to \overline{P}_c$ is completely continuous, and there exist constants $0 < a < b < d \le c$ such that*

(C1) $\{x \in P(\theta, b, d) \mid \theta(x) > b\} \ne \emptyset$, *and* $\theta(Ax) > b,\ x \in P(\theta, b, d)$,

(C2) $\|Ax\| \le a$, *for* $x \le a$,

(C3) $\theta(Ax) > b$ *for* $x \in P(\theta, b, c)$ *with* $\|Ax\| > d$.

Then, A has at least three fixed points $x_1,\ x_2,\ x_3$ with

$$\|x_1\| < a, \qquad b < \theta(x_2), \qquad a < \|x_3\|, \qquad \theta(x_3) < b. \tag{2.17}$$

Remark 2.12. If there holds $d = c$, then condition (C1) of Lemma 2.11 implies condition (C3) of Lemma 2.11.

3. The Main Results

Let $E = C[0,1]$ be endowed with the ordering $u \le v$ if $u(t) \le v(t)$ for all $t \in [0,1]$, and the maximum norm, $\|u\| = \max_{0\le t\le 1}|u(t)|$. Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \ge 0\}$.

Let the nonnegative continuous concave functional θ on the cone P be defined by $\theta(u) = \min_{(1/4)\le t\le(3/4)}|u(t)|$.

Lemma 3.1 (see [15]). *Let $T : P \to E$ be the operator defined by $Tu(t) := \int_0^1 G(t,s)f(s,u(s))ds$, then $T : P \to P$ is completely continuous.*

Lemma 3.2. *Let $A : P \to E$ be the operator defined by*

$$Au(t) := \int_0^1 G(t,s)a(s)f(s,u(s))ds + B\big(a(\cdot)f(\cdot,u(\cdot))\big)t^{\alpha-1}, \tag{3.1}$$

then $A : P \to P$ is completely continuous.

Proof. The proof is similar to Lemma 3.1, so we omit.

Denote

$$M = \left(\int_0^1 G(s,s)a(s)ds + \frac{\sum_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)a(s)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \right)^{-1},$$
$$N = \left(\int_{1/4}^{3/4} \gamma(s)G(s,s)a(s)ds \right)^{-1}. \tag{$*$}$$

□

Theorem 3.3. *Assume (H1)–(H3) hold, and there exist two positive constants $r_2 > r_1 > 0$ such that*

(1) $f(t,u) \le Mr_2$, *for all* $t \in [0,1]$, $u \in [0,r_2]$,

(2) $f(t,u) \ge Nr_1$, *for all* $t \in [0,1]$, $u \in [0,r_1]$, *where* M, N *is defined in* $(*)$,

then problem (1.2) *has at least one positive solution* u *such that* $r_1 \le \|u\| \le r_2$.

Proof. By Lemmas 2.8 and 3.2, we know $A : P \to P$ is completely continuous, and problem (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Au$. In order to apply Lemma 2.10, we separate the proof into the following two steps.

Step 1. Let $\Omega_2 = \{u \in P \mid \|u\| \le r_2\}$. For $u \in \partial\Omega_2$, we have $0 \le u(t) \le r_2$ for all $t \in [0,1]$. It follows from (1) that for $t \in [0,1]$,

$$\begin{aligned}
\|Au\| &\le \int_0^1 G(s,s)a(s)f(s,u(s))ds + \frac{\sum_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\
&\le Mr_2 \left[\int_0^1 G(s,s)a(s)ds + \frac{\sum_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)a(s)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \right] \\
&= r_2 = \|u\|.
\end{aligned} \tag{3.2}$$

Therefore,

$$\|Au\| \le \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.3}$$

Step 2. Let $\Omega_1 = \{u \in P \mid \|u\| \le r_1\}$. For $u \in \partial\Omega_1$, we have $0 \le u(t) \le r_1$ for all $t \in [0,1]$. By assumption (2), for $t \in [1/4, 3/4]$, there is

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\
&\ge \int_0^1 \gamma(s)G(s,s)a(s)f(s,u(s))ds \\
&\ge Nr \int_{1/4}^{3/4} \gamma(s)G(s,s)a(s)ds \\
&= r_1 = \|u\|.
\end{aligned} \tag{3.4}$$

So,

$$\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.5}$$

Therefore, by (ii) of Lemma 2.10, we complete the proof. □

Example 3.4. Consider the problem

$$\begin{gathered} D_{0+}^{3/2}u(t) + u^2 + \frac{\sin t}{4} + \frac{1}{5} = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i), \end{gathered} \tag{3.6}$$

where $\sum_{i=1}^{\infty} \alpha_i \xi_i^{1/2} = 1/5$.

A simple computation showed $M \geq 1.4, N \approx 13.6649$. Choosing $r_1 = (1/70), r_2 = (9/10)$, we have

$$\begin{gathered} f(t,u) = u^2 + \frac{\sin t}{4} + \frac{1}{5} \leq 1.2207 \leq Mr_2, \quad (t,u) \in [0,1] \times \left[0, \frac{9}{10}\right], \\ f(t,u) = u^2 + \frac{\sin t}{4} + \frac{1}{5} \geq \frac{1}{5} \geq Nr_1, \quad (t,u) \in [0,1] \times \left[0, \frac{1}{70}\right]. \end{gathered} \tag{3.7}$$

With the use of Theorem 3.3, problem (3.6) has at least one positive solutions u such that $(1/70) \leq \|u\| \leq (9/10)$.

Theorem 3.5. *Assume (H1)–(H3) hold, and there exist constants $0 < a < b < c$ such that the following assumptions hold:*

(A1) $f(t,u) < Ma$ *for* $(t,u) \in [0,1] \times [0,a]$,

(A2) $f(t,u) \geq Nb$ *for* $(t,u) \in [1/4, 3/4] \times [b,c]$,

(A3) $f(t,u) \leq Mc$ *for* $(t,u) \in [0,1] \times [0,c]$, *where* M, N *is defined in* $(*)$.

Then, the boundary value problem (1.2) *has at least three positive solutions* u_1, u_2, u_3 *with*

$$\begin{gathered} \|u_1\| < a, \qquad b < \min_{(1/4) \leq t \leq (3/4)} |u_2| < \|u_2\| \leq c, \\ a < \|u_3\| \leq c, \qquad \min_{(1/4) \leq t \leq (3/4)} |u_3| < b. \end{gathered} \tag{3.8}$$

Proof. We show that all the conditions of Lemma 2.9 are satisfied.

If $u \in \overline{P}_c$, then $\|u\| \leq c$. Assumption (A3) implies $f(t,u(t)) \leq Mc$ for $0 \leq t \leq 1$. Consequently,

$$\begin{aligned}
\|Au\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)a(s)f(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\right| \\
&\leq \int_0^1 G(s,s)a(s)f(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\
&\leq \left[\int_0^1 G(s,s)a(s)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)a(s)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\right]\|u\| \\
&\leq \|u\|.
\end{aligned} \tag{3.9}$$

Hence, $A : \overline{P}_c \to \overline{P}_c$. In the same way, if $u \in \overline{P}_a$, then assumption (A1) yields $f(t,u(t)) < Ma$, $0 \leq t \leq 1$. Therefore, condition (C2) of Lemma 2.11 is satisfied.

To check condition (C1) of Lemma 2.11, we choose $u(t) = (b+c)/2$, $0 \leq t \leq 1$. It is easy to see that $u(t) = (b+c)/2 \in P(\theta,b,c)$, $\theta(u) = (\theta(b+c))/2 > b$, and consequently, $\{u \in P(\theta,b,d) \mid \theta(u) > b\} \neq \emptyset$ Hence, if $u \in P(\theta,b,c)$, then $b \leq u(t) \leq c$ for $(1/4) \leq t \leq (3/4)$. From assumption (A2), we have $f(t,u(t)) \geq Nb$ for $(1/4) \leq t \leq (3/4)$. So,

$$\begin{aligned}
\theta(Au) &= \min_{(1/4)\leq t\leq(3/4)}|(Au)(t)| \\
&\geq \int_0^1 \gamma(s)G(s,s)a(s)f(s,u(s))ds \\
&> Nb\int_{1/4}^{3/4}\gamma(s)G(s,s)a(s)ds \\
&= b = \|u\|
\end{aligned} \tag{3.10}$$

$\theta(Au) > b$, for all $u \in P(\theta,b,c)$.

This shows that condition (C1) of Lemma 2.11 is also satisfied.

By Lemma 2.11 and Remark 2.12, the boundary value problem (1.2) has at least three positive solutions $u_1, u_2,$ and u_3 with

$$\begin{aligned}
&\|u_1\| < a, \qquad b < \min_{(1/4)\leq t\leq(3/4)}|u_2|, \\
&a < \|u_3\|, \qquad \min_{(1/4)\leq t\leq(3/4)}|u_3| < b.
\end{aligned} \tag{3.11}$$

The proof is complete. □

Example 3.6. Consider the problem

$$D_{0+}^{3/2}u(t) + f(t,u) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i), \tag{3.12}$$

where $\sum_{i=1}^{\infty} \alpha_i \xi_i^{1/2} = (1/5)$,

$$f(t,u) = \begin{cases} \left(\dfrac{t}{40}\right) + 14u^2, & u \le 1, \\ 13 + \left(\dfrac{t}{40}\right) + u, & u > 1. \end{cases} \tag{3.13}$$

We have $M \ge 1.4$, $N \approx 13.6649$. Choosing $a = (1/14)$, $b = 1$, $c = 36$, there hold

$$\begin{aligned} f(t,u) &= \frac{t}{40} + 14u^2 \le 0.097 \le Ma, \quad (t,u) \in [0,1] \times \left[0, \frac{1}{14}\right], \\ f(t,u) &= 13 + \frac{t}{40} + u \ge 14.025 \ge Nb \approx 13.7, \quad (t,u) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [1,36], \\ f(t,u) &\le 13 + \frac{t}{40} + u \le 48.025 \le Mc \approx 50.4, \quad (t,u) \in [0,1] \times [0,36]. \end{aligned} \tag{3.14}$$

With the use of Theorem 3.5, problem (3.12) has at least three positive solutions u_1, u_2, and u_3 with

$$\begin{aligned} &\max_{0\le t\le 1}|u_1(t)| < \frac{1}{14}, \qquad 1 < \min_{(1/4)\le t\le(3/4)}|u_2(t)| < \max_{0\le t\le 1}|u_2(t)| \le 36, \\ &\frac{1}{14} < \max_{0\le t\le 1}|u_3(t)| \le 36, \qquad \min_{(1/4)\le t\le(3/4)}|u_3(t)| < 1. \end{aligned} \tag{3.15}$$

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Research Article

The Existence of Solutions for a Nonlinear Fractional Multi-Point Boundary Value Problem at Resonance

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We discuss the existence of solution for a multipoint boundary value problem of fractional differential equation. An existence result is obtained with the use of the coincidence degree theory.

1. Introduction

In this paper, we study the multipoint boundary value problem

$$D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\right) + e(t), \quad 0 < t < 1, \tag{1.1}$$

$$I_{0+}^{3-\alpha}u(0) = 0, \qquad D_{0+}^{\alpha-2}u(0) = \sum_{j=1}^{n}\beta_j D_{0+}^{\alpha-2}u(\xi_j), \qquad u(1) = \sum_{i=1}^{m}\alpha_i u(\eta_i), \tag{1.2}$$

where $2 < \alpha \le 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, $n \ge 1$, $0 < \eta_1 < \cdots < \eta_m < 1$, $m \ge 2$, $\alpha_i, \beta_j \in \mathbb{R}$,

$$\sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-1} = \sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-2} = 1, \qquad \sum_{j=1}^{n}\beta_j\xi_j = 0, \quad \sum_{j=1}^{n}\beta_j = 1, \tag{1.3}$$

$f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfying the Carathéodory conditions, $e \in L^1[0,1]$. D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville derivative and integral, respectively. We assume, in addition, that

$$\begin{aligned} R = {} & \frac{\Gamma(\alpha)^2\Gamma(\alpha-1)}{\Gamma(2\alpha)\Gamma(\alpha+1)}\sum_{j=1}^{n}\beta_j\xi_j^{\alpha}\left(1-\sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-1}\right) \\ & -\frac{\Gamma(\alpha)^2\Gamma(\alpha-1)}{\Gamma(\alpha+2)\Gamma(2\alpha-1)}\sum_{j=1}^{n}\beta_j\xi_j^{\alpha+1}\left(1-\sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-2}\right) \\ & \neq 0, \end{aligned} \tag{1.4}$$

where Γ is the Gamma function. Due to condition (1.3), the fractional differential operator in (1.1), (1.2) is not invertible.

Fractional differential equation can describe many phenomena in various fields of science and engineering. Many methods have been introduced for solving fractional differential equations, such as the popular Laplace transform method, the iteration method. For details, see [1, 2] and the references therein.

Recently, there are some papers dealing with the solvability of nonlinear boundary value problems of fractional differential equation, by use of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, etc.), see, for example, [3–6]. But there are few papers that consider the fractional-order boundary problems at resonance. Very recently [7], Y. H. Zhang and Z. B. Bai considered the existence of solutions for the fractional ordinary differential equation

$$D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-(n-1)}u(t), \ldots, D_{0+}^{\alpha-1}u(t)\right) + e(t), \quad 0 < t < 1, \tag{1.5}$$

subject to the following boundary value conditions:

$$I_{0+}^{n-\alpha}u(0) = D_{0+}^{\alpha-(n-1)}u(0) = \cdots = D_{0+}^{\alpha-2}u(0) = 0, \qquad u(1) = \sigma u(\eta), \tag{1.6}$$

where $n > 2$ is a natural number, $n-1 < \alpha \leq n$ is a real number, $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and $e \in L^1[0,1]$, $\sigma \in (0,\infty)$, and $\eta \in (0,1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$. D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville derivative and integral, respectively. By the conditions, the kernel of the linear operator is one dimensional.

Motivated by the above work and recent studies on fractional differential equations [8–18], in this paper, we consider the existence of solutions for multipoint boundary value problem (1.1), (1.2) at resonance. Note that under condition (1.3), the kernel of the linear operator in (1.1), (1.2) is two dimensional. Our method is based upon the coincidence degree theory of Mawhin [18].

Now, we will briefly recall some notation and abstract existence result.

Let Y, Z be real Banach spaces, let $L : \operatorname{dom}(L) \subset Y \to Z$ be a Fredholm map of index zero, and let $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im}(P) = \operatorname{Ker}(P)$, $\operatorname{Ker}(Q) = \operatorname{Im}(L)$, and $Y = \operatorname{Ker}(L) \oplus \operatorname{Ker}(P)$, $Z = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$. It follows that $L|_{\operatorname{dom}(L)\cap\operatorname{Ker}(P)} : \operatorname{dom}(L) \cap \operatorname{Ker}(P) \to \operatorname{Im}(L)$ is invertible. We denote the inverse of the map by

K_P. If Ω is an open-bounded subset of Y such that $\mathrm{dom}(L)\cap\Omega\neq\emptyset$, the map $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

The theorem that we used is Theorem 2.4 of [18].

Theorem 1.1. *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(\mathrm{dom}(L) \setminus \mathrm{Ker}(L)) \cap \partial\Omega] \times (0,1)$,

(ii) $Nx \notin \mathrm{Im}(L)$ *for every* $x \in \mathrm{Ker}(L) \cap \partial\Omega$,

(iii) $\deg(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, *where* $Q : Z \to Z$ *is a projection as above with* $\mathrm{Im}(L) = \mathrm{Ker}(Q)$, *and* $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ *is any isomorphism,*

then the equation $Lx = Nx$ *has at least one solution in* $\mathrm{dom}(L) \cap \overline{\Omega}$.

The rest of this paper is organized as follows. In Section 2, we give some notation and Lemmas. In Section 3, we establish an existence theorem of a solution for the problem (1.1), (1.2).

2. Background Materials and Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions for fractional calculus theory, and these definitions can be found in the recent literature [1, 2].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$. And we let $I_{0+}^{0} y(t) = y(t)$ for every continuous $y : (0,\infty) \to \mathbb{R}$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3 (see [3]). *Assume that $u \in C(0,1) \cap L^1[0,1]$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1[0,1]$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{2.3}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

We use the classical space $C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$. Given $\mu > 0$ and $N = [\mu] + 1$, one can define a linear space

$$C^\mu[0,1] := \left\{u \mid u(t) = I_{0+}^{\mu}x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_{N-1}t^{\mu-(N-1)},\ t \in [0,1]\right\}, \tag{2.4}$$

where $x \in C[0,1]$ and $c_i \in \mathbb{R}$, $i = 1,2,\ldots,N-1$. By means of the linear function analysis theory, one can prove that with the norm $\|u\|_{C^\mu} = \|D_{0+}^{\mu}u\|_\infty + \cdots + \|D_{0+}^{\mu-(N-1)}u\|_\infty + \|u\|_\infty$, $C^\mu[0,1]$ is a Banach space.

Lemma 2.4 (see [7]). *$F \subset C^\mu[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here, uniformly bounded means that there exists $M > 0$ such that for every $u \in F$,*

$$\|u\|_{C^\mu} = \left\|D_{0+}^{\mu}u\right\|_\infty + \cdots + \left\|D_{0+}^{\mu-(N-1)}u\right\|_\infty + \|u\|_\infty < M, \tag{2.5}$$

and equicontinuous means that for all $\varepsilon > 0$, $\exists\delta > 0$ such that

$$\begin{gathered} |u(t_1) - u(t_2)| < \varepsilon, \quad (\forall t_1, t_2 \in [0,1],\ |t_1 - t_2| < \delta,\ \forall u \in F), \\ \left|D_{0+}^{\alpha-i}u(t_1) - D_{0+}^{\alpha-i}u(t_2)\right| < \varepsilon, \quad (t_1, t_2 \in [0,1], |t_1 - t_2| < \delta,\ \forall u \in F,\ \forall i \in \{0,\ldots,N-1\}). \end{gathered} \tag{2.6}$$

Let $Z = L^1[0,1]$ with the norm $\|g\|_1 = \int_0^1 |g(s)|ds$. $Y = C^{\alpha-1}[0,1] = \{u \mid u(t) = I_{0+}^{\alpha-1}x(t) + ct^{\alpha-2}, t \in [0,1]\}$, where $x \in C[0,1]$, $c \in \mathbb{R}$, with the norm $\|u\|_{C^{\alpha-1}} = \|D_{0+}^{\alpha-1}u\|_\infty + \|D_{0+}^{\alpha-2}u\|_\infty + \|u\|_\infty$, and Y is a Banach space.

Definition 2.5. By a solution of the boundary value problem (1.1), (1.2), we understand a function $u \in C^{\alpha-1}[0,1]$ such that $D_{0+}^{\alpha-1}u$ is absolutely continuous on $(0,1)$ and satisfies (1.1), (1.2).

Definition 2.6. We say that the map $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions with respect to $L^1[0,1]$ if the following conditions are satisfied:

(i) for each $z \in \mathbb{R}$, the mapping $t \to f(t,z)$ is Lebesgue measurable,

(ii) for almost every $t \in [0,1]$, the mapping $z \to f(t,z)$ is continuous on $\mathbb{R}$,

(iii) for each $r > 0$, there exists $\rho_r \in L^1([0,1],\mathbb{R})$ such that, for a.e., $t \in [0,1]$ and every $|z| \le r$, we have $|f(t,z)| \le \rho_r(t)$.

Define L to be the linear operator from $\operatorname{dom}(L) \cap Y$ to Z with

$$\begin{gathered} \operatorname{dom}(L) = \left\{u \in C^{\alpha-1}[0,1] \mid D_{0+}^{\alpha}u \in L^1[0,1],\ u \text{ satisfies (1.2)}\right\}, \\ Lu = D_{0+}^{\alpha}u, \quad u \in \operatorname{dom}(L). \end{gathered} \tag{2.7}$$

We define $N : Y \to Z$ by setting

$$Nu(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\right) + e(t). \tag{2.8}$$

Then boundary value problem (1.1), (1.2) can be written as

$$Lu = Nu. \tag{2.9}$$

Lemma 2.7. *Let condition* (1.3) *and* (1.4) *hold, then* $L : \mathrm{dom}(L) \cap Y \to Z$ *is a Fredholm map of index zero.*

Proof. It is clear that $\mathrm{Ker}(L) = \{at^{\alpha-1} + bt^{\alpha-2} \mid a, b \in \mathbb{R}\} \cong \mathbb{R}^2$.

Let $g \in Z$ and

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{2.10}$$

then $D_{0+}^{\alpha} u(t) = g(t)$ a.e., $t \in (0,1)$ and, if

$$\begin{gathered} \int_0^1 (1-s)^{\alpha-1} g(s)ds - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} g(s)ds = 0, \\ \sum_{j=1}^{n} \beta_j \int_0^{\xi_j} (\xi_j - s) g(s)ds = 0 \end{gathered} \tag{2.11}$$

hold. Then $u(t)$ satisfies the boundary conditions (1.2), that is, $u \in \mathrm{dom}(L)$, and we have

$$\{g \in Z \mid g \text{ satisfies } (2.11)\} \subseteq \mathrm{Im}(L). \tag{2.12}$$

Let $u \in \mathrm{dom}(L)$, then for $D_{0+}^{\alpha} u \in \mathrm{Im}(L)$, we have

$$u(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.13}$$

which, due to the boundary value condition (1.2), implies that $D_{0+}^{\alpha} u$ satisfies (2.11). In fact, from $I_{0+}^{3-\alpha} u(0) = 0$, we have $c_3 = 0$, from $u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i)$, we have

$$\int_0^1 (1-s)^{\alpha-1} D_{0+}^{\alpha} u(s)ds - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} D_{0+}^{\alpha} u(s)ds = 0, \tag{2.14}$$

and from $D_{0+}^{\alpha-2} u(0) = \sum_{j=1}^{n} \beta_j D_{0+}^{\alpha-2} u(\xi_j)$, we have

$$\sum_{j=1}^{n} \beta_j \int_0^{\xi_j} (\xi_j - s) D_{0+}^{\alpha} u(s)ds = 0. \tag{2.15}$$

Hence,

$$\mathrm{Im}(L) \subseteq \{g \in Z \mid g \text{ satisfies } (2.11)\}. \tag{2.16}$$

Therefore,

$$\operatorname{Im}(L) = \{g \in Z \mid g \text{ satisfies } (2.11)\}. \tag{2.17}$$

Consider the continuous linear mapping $Q_1 : Z \to Z$ and $Q_2 : Z \to Z$ defined by

$$\begin{gathered} Q_1 g = \int_0^1 (1-s)^{\alpha-1} g(s)\,ds - \sum_{i=1}^{m} \alpha_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} g(s)\,ds, \\ Q_2 g = \sum_{j=1}^{n} \beta_j \int_0^{\xi_j} (\xi_j - s) g(s)\,ds. \end{gathered} \tag{2.18}$$

Using the above definitions, we construct the following auxiliary maps $R_1,\ R_2 : Z \to Z$:

$$\begin{gathered} R_1 g = \frac{1}{R}\left[\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha} Q_1 g(t) - \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right) Q_2 g(t)\right], \\ R_2 g = -\frac{1}{R}\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha+1} Q_1 g(t) - \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\right) Q_2 g(t)\right]. \end{gathered} \tag{2.19}$$

Since the condition (1.4) holds, the mapping $Q : Z \to Z$ defined by

$$(Qy)(t) = (R_1 g(t)) t^{\alpha-1} + (R_2 g(t)) t^{\alpha-2} \tag{2.20}$$

is well defined.

Recall (1.4) and note that

$$\begin{aligned} R_1\left(R_1 g t^{\alpha-1}\right) &= \frac{1}{R}\left[\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha} Q_1\left(R_1 g t^{\alpha-1}\right)\right. \\ &\qquad \left. - \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right) Q_2\left(R_1 g t^{\alpha-1}\right)\right] \\ &= R_1 g \frac{1}{R}\left[\frac{\Gamma(\alpha-1)\Gamma(\alpha^2)}{\Gamma(\alpha+1)\Gamma(2\alpha)} \sum_{j=1}^{n} \beta_j \xi_j^{\alpha}\left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\right)\right. \\ &\qquad \left. - \frac{\Gamma(\alpha-1)\Gamma(\alpha^2)}{\Gamma(2\alpha-1)\Gamma(\alpha+2)}\left(1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right) \sum_{j=1}^{n} \beta_j \xi_j^{\alpha+1}\right] \\ &= R_1 g, \end{aligned} \tag{2.21}$$

and similarly we can derive that

$$\begin{aligned} R_1\left(R_2 g t^{\alpha-2}\right) &= 0, \\ R_2\left(R_1 g t^{\alpha-1}\right) &= 0, \\ R_2\left(R_2 g t^{\alpha-2}\right) &= R_2 g. \end{aligned} \tag{2.22}$$

So, for $g \in Z$, it follows from the four relations above that

$$\begin{aligned} Q^2 g &= R_1\left(R_1 g t^{\alpha-1} + R_2 g t^{\alpha-2}\right)t^{\alpha-1} + R_2\left(R_1 g t^{\alpha-1} + R_2 g t^{\alpha-2}\right)t^{\alpha-2} \\ &= R_1\left(R_1 g t^{\alpha-1}\right)t^{\alpha-1} + R_1\left(R_2 g t^{\alpha-2}\right)t^{\alpha-1} + R_2\left(R_1 g t^{\alpha-1}\right)t^{\alpha-2} + R_2\left(R_2 g t^{\alpha-2}\right)t^{\alpha-2} \\ &= R_1 g t^{\alpha-1} + R_2 g t^{\alpha-2} \\ &= Qg, \end{aligned} \tag{2.23}$$

that is, the map Q is idempotent. In fact, Q is a continuous linear projector.

Note that $g \in \operatorname{Im}(L)$ implies $Qg = 0$. Conversely, if $Qg = 0$, then we must have $R_1 g = R_2 g = 0$; since the condition (1.4) holds, this can only be the case if $Q_1 g = Q_2 g = 0$, that is, $g \in \operatorname{Im}(L)$. In fact, $\operatorname{Im}(L) = \operatorname{Ker}(Q)$.

Take $g \in Z$ in the form $g = (g - Qg) + Qg$, so that $g - Qg \in \operatorname{Im}(L) = \operatorname{Ker}(Q)$ and $Qg \in \operatorname{Im}(Q)$. Thus, $Z = \operatorname{Im}(L) + \operatorname{Im}(Q)$. Let $g \in \operatorname{Im}(L) \cap \operatorname{Im}(Q)$ and assume that $g(s) = as^{\alpha-1} + bs^{\alpha-2}$ is not identically zero on $[0,1]$, then, since $g \in \operatorname{Im}(L)$, from (2.11) and the condition (1.4), we derive $a = b = 0$, which is a contradiction. Hence, $\operatorname{Im}(L) \cap \operatorname{Im}(Q) = \{0\}$; thus, $Z = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$.

Now, $\dim \operatorname{Ker}(L) = 2 = \operatorname{co}\dim \operatorname{Im}(L)$, and so L is a Fredholm operator of index zero. □

Let $P : Y \to Y$ be defined by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2} u(0) t^{\alpha-2}, \quad t \in [0,1]. \tag{2.24}$$

Note that P is a continuous linear projector and

$$\operatorname{Ker}(P) = \left\{ u \in Y \mid D_{0+}^{\alpha-1} u(0) = D_{0+}^{\alpha-2} u(0) = 0 \right\}. \tag{2.25}$$

It is clear that $Y = \operatorname{Ker}(L) \oplus \operatorname{Ker}(P)$.

Note that the projectors P and Q are exact. Define $K_P : \operatorname{Im}(L) \to \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ by

$$K_P g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,ds = I_{0+}^{\alpha} g(t). \tag{2.26}$$

Hence, we have

$$D_{0+}^{\alpha-1}(K_P g)(t) = \int_0^t g(s)ds, \qquad D_{0+}^{\alpha-2}(K_P g)(t) = \int_0^t (t-s)g(s)ds, \tag{2.27}$$

then $\|K_P g\|_\infty \le (1/\Gamma(\alpha))\|g\|_1$, $\|D_{0+}^{\alpha-1}(K_P g)\|_\infty \le \|g\|_1$, $\|D_{0+}^{\alpha-2}(K_P g)\|_\infty \le \|g\|_1$, and thus

$$\|K_P g\|_{C^{\alpha-1}} \le \left(2 + \frac{1}{\Gamma(\alpha)}\right)\|g\|_1. \tag{2.28}$$

In fact, if $g \in \mathrm{Im}(L)$, then $(LK_P)g = D_{0+}^{\alpha} I_{0+}^{\alpha} g = g$. Also, if $u \in \mathrm{dom}(L) \cap \mathrm{Ker}(P)$, then

$$(K_P L g)(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} g(t) = g(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.29}$$

from boundary value condition (1.2) and the fact that $u \in \mathrm{dom}(L) \cap \mathrm{Ker}(P)$, we have $c_1 = c_2 = c_3 = 0$. Thus,

$$K_P = \left(L|_{\mathrm{dom}(L)\cap \mathrm{Ker}(P)}\right)^{-1}. \tag{2.30}$$

Using (2.19), we write

$$\begin{gathered} QNu(t) = (R_1 Nu)t^{\alpha-1} + (R_2 Nu)t^{\alpha-2}, \\ K_P(I-Q)Nu(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}[Nu(s) - QNu(s)]ds. \end{gathered} \tag{2.31}$$

With arguments similar to those of [7], we obtain the following Lemma.

Lemma 2.8. *$K_{P(I-Q)}N : Y \to Y$ is completely continuous.*

3. The Main Results

Assume that the following conditions on the function $f(t, x, y, z)$ are satisfied:

(H1) there exists a constant $A > 0$, such that for $u \in \mathrm{dom}(L) \setminus \mathrm{Ker}(L)$ satisfying $|D_{0+}^{\alpha-1}u(t)| + |D_{0+}^{\alpha-2}u(t)| > A$ for all $t \in [0, 1]$, we have

$$Q_1 Nu(t) \ne 0 \qquad \text{or } Q_2 Nu(t) \ne 0, \tag{3.1}$$

(H2) there exist functions $a, b, c, d, r \in L^1[0,1]$ and a constant $\theta \in [0,1]$ such that for all $(x, y, z) \in \mathbb{R}^3$ and a.e., $t \in [0,1]$, one of the following inequalities is satisfied:

$$\begin{aligned}
\left|f(t,x,y,z)\right| &\le a(t)|x| + b(t)\left|y\right| + c(t)|z| + d(t)|z|^{\theta} + r(t),\\
\left|f(t,x,y,z)\right| &\le a(t)|x| + b(t)\left|y\right| + c(t)|z| + d(t)\left|y\right|^{\theta} + r(t),\\
\left|f(t,x,y,z)\right| &\le a(t)|x| + b(t)\left|y\right| + c(t)|z| + d(t)|x|^{\theta} + r(t),
\end{aligned} \tag{3.2}$$

(H3) there exists a constant $B > 0$ such that for every $a, b \in \mathbb{R}$ satisfying $a^2 + b^2 > B$, then either

$$aR_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bR_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right) < 0, \tag{3.3}$$

or else

$$aR_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bR_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right) > 0. \tag{3.4}$$

Remark 3.1. $R_1N(at^{\alpha-1} + bt^{\alpha-2})$ and $R_2N(at^{\alpha-1} + bt^{\alpha-2})$ from (H3) stand for the images of $u(t) = at^{\alpha-1} + bt^{\alpha-2}$ under the maps R_1N and R_2N, respectively.

Theorem 3.2. *If (H1)–(H3) hold, then boundary value problem* (1.1)-(1.2) *has at least one solution provided that*

$$\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{1}{\tau}, \tag{3.5}$$

where $\tau = 5 + 2/\Gamma(\alpha) + 1/\Gamma(\alpha - 1)$.

Proof. Set

$$\Omega_1 = \{u \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L) \mid Lu = \lambda Nu \text{ for some } \lambda \in [0,1]\}, \tag{3.6}$$

then for $u \in \Omega_1$, $Lu = \lambda Nu$; thus, $\lambda \neq 0$, $Nu \in \operatorname{Im}(L) = \operatorname{Ker}(Q)$, and hence $QNu(t) = 0$ for all $t \in [0,1]$. By the definition of Q, we have $Q_1Nu(t) = Q_2Nu(t) = 0$. It follows from (H1) that there exists $t_0 \in [0,1]$ such that $|D_{0+}^{\alpha-1}u(t_0)| + |D_{0+}^{\alpha-2}u(t_0)| \le A$. Now,

$$\begin{aligned}
D_{0+}^{\alpha-1}u(t) &= D_{0+}^{\alpha-1}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha}u(s)ds,\\
D_{0+}^{\alpha-2}u(t) &= D_{0+}^{\alpha-2}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha-1}u(s)ds,
\end{aligned} \tag{3.7}$$

so

$$\begin{aligned}
\left|D_{0+}^{\alpha-1}u(0)\right| &\le \left\|D_{0+}^{\alpha-1}u(t)\right\|_{\infty} \\
&\le \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u\right\|_1 \\
&\le A + \|Lu\|_1 \\
&\le A + \|Nu\|_1, \\
\left|D_{0+}^{\alpha-2}u(0)\right| &\le \left\|D_{0+}^{\alpha-2}u(t)\right\|_{\infty} \\
&\le \left|D_{0+}^{\alpha-2}u(t_0)\right| + \left\|D_{0+}^{\alpha-1}u\right\|_{\infty} \\
&\le \left|D_{0+}^{\alpha-2}u(t_0)\right| + \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u\right\|_1 \\
&\le A + \|Lu\|_1 \\
&\le A + \|Nu\|_1.
\end{aligned} \tag{3.8}$$

Now by (3.8), we have

$$\begin{aligned}
\|Pu\|_{C^{\alpha-1}} &= \left\|\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2}\right\|_{C^{\alpha-1}} \\
&= \left\|\frac{1}{\Gamma(\alpha)}D_{0}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0}^{\alpha-2}u(0)t^{\alpha-2}\right\|_{\infty} \\
&\quad + \left\|D_{0+}^{\alpha-1}u(0)\right\|_{\infty} + \left\|D_{0+}^{\alpha-1}u(0)t + D_{0+}^{\alpha-2}u(0)\right\|_{\infty} \\
&\le \left(2 + \frac{1}{\Gamma(\alpha)}\right)\left|D_{0+}^{\alpha-1}u(0)\right| + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)\left|D_{0+}^{\alpha-2}u(0)\right| \\
&\le \left(2 + \frac{1}{\Gamma(\alpha)}\right)(A + \|Nu\|_1) + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)(A + \|Nu\|_1).
\end{aligned} \tag{3.9}$$

Note that $(I-P)u \in \operatorname{Im}(K_P) = \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ for $u \in \Omega_1$, then, by (2.28) and (2.30),

$$\begin{aligned}
\|(I-P)u\|_{C^{\alpha-1}} &= \|K_P L(I-P)\|_{C^{\alpha-1}} \\
&\le \left(2 - \frac{1}{\Gamma(\alpha)}\right)\|L(I-P)u\|_1 \\
&= \left(2 - \frac{1}{\Gamma(\alpha)}\right)\|Lu\|_1 \\
&\le \left(2 - \frac{1}{\Gamma(\alpha)}\right)\|Nu\|_1.
\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10), we obtain

$$\begin{aligned}
\|u\|_{C^{\alpha-1}} &= \|Pu + (I-P)u\|_{C^{\alpha-1}} \\
&\le \|Pu\|_{C^{\alpha-1}} + \|(I-P)u\|_{C^{\alpha-1}} \\
&\le \left(2 + \frac{1}{\Gamma(\alpha)}\right)(A + \|Nu\|_1) + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)(A + \|Nu\|_1) + \left(2 + \frac{1}{\Gamma(\alpha)}\right)\|Nu\|_1 \\
&= \left(5 + \frac{2}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right)\|Nu\|_1 + \left(3 + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right)A \\
&= \tau\|Nu\|_1 + C_1,
\end{aligned} \tag{3.11}$$

where $C_1 = (3 + 1/\Gamma(\alpha) + 1/\Gamma(\alpha-1))A$ is a constant. This is for all $u \in \Omega_1$,

$$\|u\|_{C^{\alpha-1}} \le \tau\|Nu\|_1 + C_1. \tag{3.12}$$

If the first condition of (H2) is satisfied, then we have

$$\begin{aligned}
&\max\left\{\|u\|_\infty, \left\|D_{0+}^{\alpha-1}u\right\|_\infty, \left\|D_{0+}^{\alpha-2}u\right\|_\infty\right\} \\
&\quad \le \|u\|_{C^{\alpha-1}} \le \tau\left(\|a\|_1\|u\|_\infty + \|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty\right. \\
&\qquad \left. + \|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right) + C_1,
\end{aligned} \tag{3.13}$$

and consequently,

$$\begin{aligned}
\|u\|_\infty &\le \frac{\tau}{1 - \|a\|_1\tau}\left(\|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right) \\
&\quad + \frac{C_1}{1 - \|a\|_1\tau},
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\left\|D_{0+}^{\alpha-1}u\right\|_\infty &\le \frac{\tau}{1 - \|a\|_1\tau - \|b\|_1\tau}\left(\|c\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right) \\
&\quad + \frac{C_1}{1 - \|a\|_1\tau - \|b\|_1\tau},
\end{aligned} \tag{3.15}$$

$$\left\|D_{0+}^{\alpha-1}u\right\|_\infty \le \frac{\tau\|d\|_1\left\|D_{0+}^{\alpha-2}u\right\|_\infty^\theta}{1 - \|a\|_1\tau - \|b\|_1\tau - \|c\|_1\tau} + \frac{\tau(\|r\|_1 + \|e\|_1) + C_1}{1 - \|a\|_1\tau - \|b\|_1\tau - \|c\|_1\tau}. \tag{3.16}$$

Note that $\theta \in [0,1)$ and $\|a\|_1 + \|b\|_1 + \|c\|_1 < 1/\tau$, so there exists $M_1 > 0$ such that $\|D_{0+}^{\alpha-1}u\|_\infty \leq M_1$ for all $u \in \Omega_1$. The inequalities (3.14) and (3.15) show that there exist $M_2, M_3 > 0$ such that $\|D_{0+}^{\alpha-1}u\|_\infty \leq M_2$, $\|u\|_\infty \leq M_3$ for all $u \in \Omega_1$. Therefore, for all $u \in \Omega_1$, $\|u\|_{C^{\alpha-1}} = \|u\|_\infty + \|D_{0+}^{\alpha-1}u\|_\infty + \|D_{0+}^{\alpha-2}u\|_\infty \leq M_1 + M_2 + M_3$, that is, Ω_1 is bounded given the first condition of (H2). If the other conditions of (H2) hold, by using an argument similar to the above, we can prove that Ω_1 is also bounded.

Let

$$\Omega_2 = \{u \in \operatorname{Ker}(L) \mid Nu \in \operatorname{Im}(L)\}. \tag{3.17}$$

For $u \in \Omega_2$, $u \in \operatorname{Ker}(L) = \{u \in \operatorname{dom}(L) \mid u = at^{\alpha-1} + bt^{\alpha-2}, a, b \in \mathbb{R}, t \in [0,1]\}$, and $QN(at^{\alpha-1} + bt^{\alpha-2}) = 0$; thus, $R_1N(at^{\alpha-1} + bt^{\alpha-2}) = R_2N(at^{\alpha-1} + bt^{\alpha-2}) = 0$. By (H3), $a^2 + b^2 \leq B$, that is, Ω_2 is bounded.

We define the isomorphism $J : \operatorname{Im}(Q) \to \operatorname{Ker}(L)$ by

$$J\left(at^{\alpha-1} + bt^{\alpha-2}\right) = at^{\alpha-1} + bt^{\alpha-2}, \quad a, b \in \mathbb{R}. \tag{3.18}$$

If the first part of (H3) is satisfied, let

$$\Omega_3 = \left\{u \in \operatorname{Ker} L : -\lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\right\}. \tag{3.19}$$

For every $at^{\alpha-1} + bt^{\alpha-2} \in \Omega_3$,

$$\lambda\left(at^{\alpha-1} + bt^{\alpha-2}\right) = (1-\lambda)\left[\left(R_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right)t^{\alpha-1} + \left(R_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right)t^{\alpha-2}\right]. \tag{3.20}$$

If $\lambda = 1$, then $a = b = 0$, and if $a^2 + b^2 > B$, then by (H3),

$$\lambda\left(a^2 + b^2\right) = (1-\lambda)\left[aR_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bR_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right] < 0, \tag{3.21}$$

which, in either case, obtain a contradiction. If the other part of (H3) is satisfied, then we take

$$\Omega_3 = \left\{u \in \operatorname{Ker} L : \lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\right\}, \tag{3.22}$$

and, again, obtain a contradiction. Thus, in either case,

$$\begin{aligned}
\|u\|_{C^{\alpha-1}} &= \|u\|_\infty + \left\|D_{0+}^{\alpha-1}u\right\|_\infty + \left\|D_{0+}^{\alpha-2}u\right\|_\infty \\
&= \left\|at^{\alpha-1} + bt^{\alpha-2}\right\|_{C^{\alpha-1}} \\
&= \left\|at^{\alpha-1} + bt^{\alpha-2}\right\|_\infty + \|a\Gamma(\alpha)\|_\infty + \|a\Gamma(\alpha)t + b\Gamma(\alpha-1)\|_\infty \\
&\leq (1 + 2\Gamma(\alpha))|a| + (1 + \Gamma(\alpha-1))|b| \\
&\leq (2 + 2\Gamma(\alpha) + \Gamma(\alpha-1))|a|,
\end{aligned} \tag{3.23}$$

for all $u \in \Omega_3$, that is, Ω_3 is bounded.

In the following, we will prove that all the conditions of Theorem 1.1 are satisfied. Set Ω to be a bounded open set of Y such that $U_{i=1}^{3}\overline{\Omega_i} \subset \Omega$. by Lemma 2.8, the operator $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact; thus, N is L-compact on $\overline{\Omega}$, then by the above argument, we have

(i) $Lu \neq \lambda Nx$, for every $(u,\lambda) \in [(\mathrm{dom}(L) \setminus \mathrm{Ker}\, L) \cap \partial\Omega] \times (0,1)$,

(ii) $Nu \notin \mathrm{Im}(L)$, for every $u \in \mathrm{Ker}(L) \cap \partial\Omega$.

Finally, we will prove that (iii) of Theorem 1.1 is satisfied. Let $H(u,\lambda) = \pm Iu + (1-\lambda)JQNu$, where I is the identity operator in the Banach space Y. According to the above argument, we know that

$$H(u,\lambda) \neq 0, \quad \forall u \in \partial\Omega \cap \mathrm{Ker}(L), \tag{3.24}$$

and thus, by the homotopy property of degree,

$$\begin{aligned}
&\deg\Big(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0\Big) \\
&\quad = \deg(H(\ldots,0), \Omega \cap \mathrm{Ker}(L), 0) \\
&\quad = \deg(H(\ldots,1), \Omega \cap \mathrm{Ker}(L), 0) \\
&\quad = \deg(\pm I, \Omega \cap \mathrm{Ker}(L), 0) \\
&\quad = \pm 1 \neq 0,
\end{aligned} \tag{3.25}$$

then by Theorem 1.1, $Lu = Nu$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$, so boundary problem (1.1), (1.2) has at least one solution in the space $C^{\alpha-1}[0,1]$. The proof is finished. □

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Research Article

Boundary Value Problems with Integral Gluing Conditions for Fractional-Order Mixed-Type Equation

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Analogs of the Tricomi and the Gellerstedt problems with integral gluing conditions for mixed parabolic-hyperbolic equation with parameter have been considered. The considered mixed-type equation consists of fractional diffusion and telegraph equation. The Tricomi problem is equivalently reduced to the second-kind Volterra integral equation, which is uniquely solvable. The uniqueness of the Gellerstedt problem is proven by energy integrals' method and the existence by reducing it to the ordinary differential equations. The method of Green functions and properties of integral-differential operators have been used.

1. Introduction

Mathematical model of the movement of gas in a channel surrounded by a porous environment was described by parabolic-hyperbolic equation. This was done in the fundamental work of Gel'fand [1]. Modeling of heat transfer processes in composite environment with finite and infinite velocities leads to boundary value problems (BVPs) for parabolic-hyperbolic equations [2]. Omitting the huge amount of works devoted to studying these kinds of equations, we refer the readers to [3, 4].

We would like to note works [5–10], devoted to the studying of BVPs for parabolic-hyperbolic equations, involving fractional derivatives. In turn, applications of Fractional-order differential equations can be found in the monographs [11–15]. We also note some recent papers [16–18], related to the fractional diffusion and diffusion-wave equations.

BVP for parabolic-hyperbolic equations with integral gluing condition for the first time was investigated by Kapustin and Moiseev [19] and was generalized for this kind of equation,

but with parameters, in the work [20]. Another motivation of the usage of integral gluing conditions comes from the appearance of them in heat exchange processes [21].

The consideration of equations with parameters was interesting because of the possibility of studying some multidimensional analogues of the main BVP via reducing them by Fourier transformation to the BVP for equations with parameters. On the other hand, consideration of equations with parameters will give possibility to study some spectral properties of BVPs for this kind of equations such as the existence of nontrivial solutions for corresponding homogeneous problem at some values of parameters [22].

2. Analog of the Tricomi Problem

Consider an equation

$$u_{xx} - D_{0y}^{\alpha H(x)+2H(-x)} u = \lambda u \tag{2.1}$$

in the domain $\Omega = \Omega_1 \cup AA_0 \cup \Omega_2$. Here $\Omega_1 = \{(x,y) : \ 0 < x < 1, \ 0 < y < 1\}$, Ω_2 is characteristic triangle with endpoints $A(0,0)$, $A_0(0,1)$, $C(-1/2,1/2)$, $H(x)$ is Heaviside function,

$$D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s) ds \tag{2.2}$$

is the αth Riemann-Liouville fractional-order derivative of a function f given on interval $[a,b]$, where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α, and $\Gamma(\cdot)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0. \tag{2.3}$$

For $\lambda > 0$ and $0 < \alpha \leq 1$ given, we formulate the following problem called the analog of the Tricomi problem.

Problem AT

To find a solution of (2.1), which belongs to the class of functions

$$W_1 = \left\{ u : D_{0y}^{\alpha-1} u \in C\left(\overline{\Omega_1}\right), \ u_{xx}, D_{0y}^{\alpha} u \in C(\Omega_1), \ u_x\left(0^{\pm}, y\right) \in H(0;1), u \in C\left(\overline{\Omega_2}\right) \cap C^2(\Omega_2) \right\}, \tag{2.4}$$

satisfying the initial condition

$$\lim_{y \to 0} y^{1-\alpha} u(x,y) = \omega(x), \quad 0 \leq x \leq 1 \tag{2.5}$$

together with the boundary conditions

$$\begin{gathered} u(-y/2, y/2) = \varphi_1(y), \quad 0 \leq y \leq 1, \\ u(1,y) = \varphi_2(y), \quad 0 \leq y \leq 1, \end{gathered} \tag{2.6}$$

and the gluing conditions

$$u(0^-,y) = \frac{1}{\Gamma(1-\alpha)}\int_0^y u(0^+,t)(y-t)^{-\alpha}dt, \quad 0<y\le 1,$$
$$\int_0^y u_x(0^-,t)J_0\left[\sqrt{\lambda}(y-t)\right]dt = \frac{1}{\Gamma(1-\alpha)}\int_0^y u_x(0^+,t)(y-t)^{-\alpha}dt, \quad 0<y<1. \tag{2.7}$$

Here $\omega(x)$, $\varphi_i(y)$ $(i=1,2)$ are given functions such as $\lim_{y\to 0} y^{1-\alpha}\varphi_1(y) = \omega(0)$.

Solution of the Cauchy problem for (2.1) in Ω_2 defined as

$$u(x,y) = \frac{1}{2}\left\{\tau^-(y+x)+\tau^-(y-x)+\int_{y-x}^{y+x}\nu^-(t)J_0\left[\sqrt{\lambda\left[(y-t)^2-x^2\right]}\right]dt \right.$$
$$\left. +\lambda x\int_{y-x}^{y+x}\tau^-(t)\frac{J_1\left[\sqrt{\lambda\left[(y-t)^2-x^2\right]}\right]}{\sqrt{\lambda\left[(y-t)^2-x^2\right]}}dt\right\}, \tag{2.8}$$

where $J_k[\cdot]$ is the first-kind Bessel function of the order k, $\tau^-(y) = u(0^-,y)$, $\nu^-(y) = u_x(0^-,y)$.

We calculate $u(-y/2,y/2)$ in order to use condition (2.5):

$$u(-y/2,y/2)$$
$$= \frac{1}{2}\left\{\tau^-(0)+\tau^-(y)-\int_0^y \nu^-(t)J_0\left[\sqrt{\lambda t(t-y)}\right]dt+\lambda\frac{y}{2}\int_0^y \tau^-(t)\frac{J_1\left[\sqrt{\lambda t(t-y)}\right]}{\sqrt{\lambda t(t-y)}}dt\right\}. \tag{2.9}$$

Considering the condition (2.5) and the following integral operator [23]

$$B_{mx}^{n,\sqrt{\lambda}}[f(x)] = f(x)+\int_m^x f(t)\left(\frac{x-m}{t-m}\right)^{1-n}\frac{\partial}{\partial x}J_0\left[\sqrt{\lambda(t-m)(t-x)}\right]dt, \quad m,n=0,1, \tag{2.10}$$

equality (2.9) can be written as follows

$$\varphi_1(y) = \frac{1}{2}\left\{\varphi_1(0)+B_{0y}^{0,\sqrt{\lambda}}[\tau^-(y)]-\int_0^y B_{0t}^{1,\sqrt{\lambda}}[\nu^-(t)]dt\right\}. \tag{2.11}$$

Now we use an integral operator

$$A_{mx}^{n,\sqrt{\lambda}}[f(x)] = f(x)-\int_m^x f(t)\left(\frac{t-m}{x-m}\right)^{n}\frac{\partial}{\partial t}J_0\left[\sqrt{\lambda(x-m)(x-t)}\right]dt, \quad m,n=0,1, \tag{2.12}$$

which is mutually inverse with the operator (2.10). Applying the operator (2.12) to both sides of (2.11), we obtain

$$A_{0y}^{0,\sqrt{\lambda}}[\varphi_1(y)] = \frac{1}{2}\left\{\varphi_1(0) + A_{0y}^{0,\sqrt{\lambda}}\left\{B_{0y}^{0,\sqrt{\lambda}}[\tau^-(y)]\right\} - A_{0y}^{0,\sqrt{\lambda}}\left\{\int_0^y B_{0t}^{1,\sqrt{\lambda}}[\nu^-(t)]\,dt\right\}\right\}. \tag{2.13}$$

Considering the following properties of operators (2.10) and (2.12)

$$A_{0y}^{0,\sqrt{\lambda}}\left\{B_{0y}^{0,\sqrt{\lambda}}[f(y)]\right\} = f(y), \qquad A_{0y}^{0,\sqrt{\lambda}}\left\{\int_0^y B_{0t}^{1,\sqrt{\lambda}}[f(t)]\,dt\right\} = \int_0^y f(t)J_0\left[\sqrt{\lambda}(y-t)\right]dt, \tag{2.14}$$

we derive

$$2A_{0y}^{0,\sqrt{\lambda}}[\varphi_1(y)] = \varphi_1(0) + \tau^-(y) - \int_0^y \nu^-(t)J_0\left[\sqrt{\lambda}(y-t)\right]dt. \tag{2.15}$$

Taking gluing conditions (2.7) into account, we have

$$D_{0y}^{\alpha-1}\nu^+(y) = D_{0y}^{\alpha-1}\tau^+(y) - 2A_{0y}^{0,\sqrt{\lambda}}[\varphi_1(y)] + \varphi_1(0). \tag{2.16}$$

Applying operator $D_{0y}^{1-\alpha}$ to both sides of (2.16) and considering the following composition rule [11]:

$$D_{at}^{\alpha}D_{at}^{\beta}f(t) = D_{at}^{\alpha+\beta}f(t), \quad \beta \le 0, \tag{2.17}$$

we get

$$\tau^+(y) = \nu^+(y) + \varphi_1^*(y), \quad 0 < y < 1, \tag{2.18}$$

where $\varphi_1^*(y) = D_{0y}^{1-\alpha}\{2A_{0y}^{0,\sqrt{\lambda}}[\varphi_1(y)] - \varphi_1(0)\}$.

Let us consider the following auxiliary problem:

$$\begin{gathered} u_{xx} - D_{0y}^{\alpha}u - \lambda u = 0, \\ u_x(0,y) = \nu^+(y), \qquad u(1,y) = \varphi_2(y), \qquad \lim_{y\to 0} y^{1-\alpha}u(x,y) = \omega(x). \end{gathered} \tag{2.19}$$

Solution of this problem can be defined as [24]

$$\begin{aligned} u(x,y) = {} & \int_0^1 \omega(\xi)G(x,y,\xi,0)\,d\xi - \int_0^y \nu^+(\eta)G(x,y,0,\eta)\,d\eta \\ & + \int_0^y \varphi_2(\eta)G_\xi(x,y,1,\eta)\,d\eta - \lambda\int_0^1\int_0^y u(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta, \end{aligned} \tag{2.20}$$

where

$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\beta-1}}{2}\sum_{n=-\infty}^{\infty}\left[e_{1,\beta}^{1,\beta}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\beta}}\right)+e_{1,\beta}^{1,\beta}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\beta}}\right)\right] \tag{2.21}$$

is the Green function of the problem (2.19),

$$e_{1,\beta}^{1,\beta}(z)=\Phi(-\beta,\beta,z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(-\beta n+\beta)} \tag{2.22}$$

is the function of Wright [25], $\beta=\alpha/2$.

Considering (2.20) as an integral equation regarding the function $u(x,y)$, we write solution via resolvent of the kernel $\lambda G(x,y,\xi,\eta)$:

$$u(x,y)=P(x,y)-\int_0^y \nu^+(\eta)K_1(x,y,\eta)d\eta, \tag{2.23}$$

where

$$\begin{aligned}P(x,y)&=\int_0^1\omega(\xi)G(x,y,\xi,0)d\xi+\int_0^y\int_0^1\int_0^1\omega(\xi)G(s,t,\xi,0)R(x,y,\xi,0)d\xi\,ds\,dt\\&\quad+\int_0^y\varphi_2(\eta)\left[G(x,y,1,\eta)+\int_\eta^y\int_0^1 G(s,t,1,\eta)R(x,y,1,\eta)ds\,dt\right]d\eta,\end{aligned} \tag{2.24}$$

$$K_1(x,y,\eta)=G(x,y,0,\eta)+\int_\eta^y\int_0^1 G(s,t,0,\eta)R(x,y,0,\eta)ds\,dt,$$

$R(x,y,\xi,\eta)$ is a resolvent of the kernel $\lambda G(x,y,\xi,\eta)$.

From (2.23), tending x to 0^+, we obtain

$$u(0^+,y)=\tau^+(y)=P(0^+,y)-\int_0^y \nu^-(\eta)K_1(0^+,y,\eta)d\eta. \tag{2.25}$$

Considering functional relation (2.18), from (2.25) we get

$$\nu^+(y)+\int_0^y \nu^+(\eta)K_1(y,\eta)d\eta=\varphi_1^*(y)-P(0,y). \tag{2.26}$$

Equality (2.26) is the second-kind Volterra-type integral equation regarding the function $\nu^+(y)$. Since kernel $K_1(y,\eta)$ has weak singularity and functions on the right-hand side are continuous, we can conclude that (2.26) is uniquely solvable [26], and solution can be represented as

$$\nu^+(y)=\Psi(y)+\int_0^y \Psi(\eta)K_2(y,\eta)d\eta, \tag{2.27}$$

where $\Psi(y)=\varphi_1^*(y)-P(0,y)$, $K_2(y,\eta)$ is the resolvent of the kernel $K_1(y,\eta)$.

Once we have obtained $\nu^+(y)$, considering (2.18) or (2.25) we find function $\tau^+(y)$. Then using gluing conditions (2.7) we find functions $\tau^-(\mathrm{y})$, $\nu^-(\mathrm{y})$. Finally, we can define solution of the considered problem by the formula (2.23) in the domain Ω_1, by formula (2.8) in the domain Ω_2.

Hence, we prove the following theorem.

Theorem 2.1. *If*

$$\omega(x) \in C^2[0,1], \qquad \varphi_i(y) \in C^1[0,1] \cap C^2(0,1) \quad (i = 1,2), \tag{2.28}$$

then there exists unique solution of the Problem AT and is defined by formulas (2.23) *and* (2.8) *in the domains* Ω_1, Ω_2, *respectively.*

3. Analog of the Gellerstedt Problem

We would like to note some related works. Regarding the consideration of Gellerstedt problem for parabolic-hyperbolic equations with constant coefficients we refer the readers to [3] and for loaded parabolic-hyperbolic equations work by Khubiev [27], and also for Lavrent'ev-Bitsadze equation [28].

Consider an equation

$$0 = \begin{cases} u_{xx} - D_{0y}^{\alpha} u - \lambda u, & \Phi_0 \\ u_{xx} - u_{yy} + \lambda u, & \Phi_i, \ (i = 1,2) \end{cases} \tag{3.1}$$

in the domain $\Phi = (\bigcup_{k=0}^{2} \Phi_k) \cup I_0$, where Φ_0 is a domain, bounded by segments AA_0, BB_0, A_0B_0 of straight lines $x = 0$, $x = 1$, $y = 1$, respectively; Φ_1 is a domain, bounded by the segment AE of the axe x and by characteristics of (3.1) $AC_1 : \ x + y = 0$, $EC_1 : \ x - y = r$; Φ_2 is a domain, bounded by the segment EB of the axe x and by characteristics of (3.1) $EC_2 : \ x - y = r$, $BC_2 : \ x - y = 1$; I_0 is an interval $0 < x < 1$, I_1 is an interval $0 < x < r$, and I_2 is an interval $r < x < 1$.

Problem AG

To find a solution of (3.1) from the class of functions

$$W_2 = \left\{ u : D_{0y}^{\alpha-1} u \in C\left(\overline{\Phi_0}\right),\ u_{xx}, D_{0y}^{\alpha} u \in C(\Phi_0),\ u_y(x, 0^{\pm}) \in H(I_0),\ u \in C\left(\overline{\Phi_i}\right) \cap C^2(\Phi_i)\ (i = 1,2) \right\}, \tag{3.2}$$

satisfying boundary conditions

$$u(0,y) = \varphi_1(y), \quad u(1,y) = \varphi_2(y), \quad 0 \le y \le 1, \tag{3.3}$$

$$u\,|_{AC_1} = u\left(\frac{x}{2}, -\frac{x}{2}\right) = \varphi_3(x), \quad 0 \le x \le r, \tag{3.4}$$

$$u\,|_{EC_2} = u\left(\frac{(x+r)}{2}, \frac{(r-x)}{2}\right) = \varphi_4(x), \quad r \le x \le 1, \tag{3.5}$$

together with gluing conditions

$$\lim_{y\to 0^+} y^{1-\alpha} u(x,y) = \lim_{y\to 0^+} u(x,y), \quad x \in \overline{I_0}, \tag{3.6}$$

$$\lim_{y\to 0^+}\left[y^{1-\alpha}\left(y^{1-\alpha}u(x,y)\right)_y\right] = \int_0^x \lim_{y\to 0^+} u_y(t,y) J_0\left[\sqrt{\alpha}(x-t)\right]dt, \quad x \in I_0 \setminus \{\mathrm{r}\}. \tag{3.7}$$

Here $\varphi_j(\cdot)$ $(j=\overline{1,4})$ are given functions such as $\lim_{y\to 0^+} y^{1-\alpha}\varphi_1(y) = \varphi_3(0)$, $\lim_{\mathrm{y}\to 0^+} y^{1-\alpha}u(r,y) = \varphi_4(r)$.

Theorem 3.1. *If the following conditions*

$$\lambda \ge 0, \quad \varphi_i(y) \in C^1[0,1] \cap C^2(0,1), \quad \varphi_j(x) \in C^1\left(\overline{I_i}\right) \cap C^2(I_i) \quad (i = 1,2;\ j = 3,4) \tag{3.8}$$

are fulfilled, then the Problem AG has a unique solution.

Proof. Introduce the following designations:

$$\begin{gathered}\lim_{y\to 0^+} y^{1-\alpha}u(x,y) = \tau^+(x), \quad \lim_{y\to 0^-} u(x,y) = \tau^-(x), \quad x \in \overline{I_0}, \\ \lim_{y\to 0^+}\left[y^{1-\alpha}\left(y^{1-\alpha}u(x,y)\right)_y\right] = \nu^+(x), \quad \lim_{y\to 0^-} u_y(x,y) = \nu^-(x), \quad x \in I_0.\end{gathered} \tag{3.9}$$

Solution of the Cauchy problem for (3.1) in the domain Φ_i $(i = 1,2)$ in case, when $\lambda \ge 0$ has a form

$$\begin{aligned} u(x,y) = \frac{1}{2}\Bigg\{ & \tau^-(x+y) + \tau^-(x-y) + \int_{x-y}^{x+y} \nu^-(t) J_0\left[\sqrt{\lambda\left[(x-t)^2 - y^2\right]}\right]dt \\ & + \lambda y \int_{x-y}^{x+y} \tau^-(t) \frac{J_1\left[\sqrt{\lambda\left[(x-t)^2 - y^2\right]}\right]}{\sqrt{\lambda\left[(x-t)^2 - y^2\right]}}dt \Bigg\}. \end{aligned} \tag{3.10}$$

Using boundary conditions (3.4), (3.5), and gluing conditions (3.6), (3.7), from (3.10) we obtain

$$\nu^{+}(x) = \tau^{+}(x) + \varphi_3^{*}(x), \quad x \in I_1, \tag{3.11}$$

$$\nu^{+}(x) = \tau^{+}(x) + \varphi_4^{*}(x), \quad x \in I_2, \tag{3.12}$$

where

$$\varphi_3^{*}(x) = \varphi_3(0) - A_{0x}^{0,\sqrt{\lambda}}\left[2\varphi_3(x)\right], \qquad \varphi_4^{*}(x) = \varphi_4(r) - A_{rx}^{0,\sqrt{\lambda}}\left[2\varphi_4(x)\right]. \tag{3.13}$$

According to [10], tending y to $+0$, from (3.1) we get

$$\nu^{+}(y) = \frac{1}{\Gamma(1+\alpha)}\left[\tau^{+''}(x) - \lambda\tau^{+}(x)\right]. \tag{3.14}$$

In order to prove the uniqueness of the solution for the Problem AG, we need estimate the following integral:

$$\mathbb{I} = \int_0^1 \tau^{+}(x)\nu^{+}(x)dx. \tag{3.15}$$

Considering homogeneous case of the condition (3.3) and taking designation (3.9) into account, after some evaluations we derive

$$\mathbb{I} = -\int_0^1 \left\{\left[\tau^{+'}(x)\right]^2 + \lambda[\tau^{+}(x)]^2\right\}dx. \tag{3.16}$$

If $\lambda \geq 0$, then $\mathbb{I} \leq 0$. On the other hand, if we consider homogeneous cases of (3.11) and (3.12), one can easily be sure that $\mathbb{I} \geq 0$. Hence, we get that $\mathbb{I} \equiv 0$. Based on (3.16) we can conclude that $\tau^{+}(x) = 0$ for all $x \in \overline{I_0}$. Due to the solution of the first boundary problem [24] we can conclude that $u(x,y) \equiv 0$ in $\overline{\Phi_0}$. Further, according to the gluing conditions and the solution of Cauchy problem, we have $u(x,y) \equiv 0$ in $\overline{\Phi}$.

Considering functional relations (3.11)–(3.14) and conditions (3.3)–(3.5), we get the following problems:

$$\begin{gathered}\tau^{+''}(x) - (\lambda + \Gamma(1+\alpha))\tau^{+}(x) = \varphi_3^{*}(x)\Gamma(1+\alpha),\\ \tau^{+}(0) = \varphi_3(0), \qquad \tau^{+}(r) = \varphi_4(r), \quad x \in \overline{I_1},\end{gathered} \tag{3.17}$$

$$\begin{gathered}\tau^{+''}(x) - (\lambda + \Gamma(1+\alpha))\tau^{+}(x) = \varphi_4^{*}(x)\Gamma(1+\alpha),\\ \tau^{+}(r) = \varphi_4(r), \qquad \tau^{+}(1) = \lim_{y\to+0} y^{1-\alpha}\varphi_2(y), \quad x \in \overline{I_1}.\end{gathered} \tag{3.18}$$

The problems (3.17) and (3.18) are model problems and can be solved directly. After the finding function $\tau^{+}(x)$ for all $x \in \overline{I_0}$, functions $\nu^{+}(x)$ and $\tau^{-}(x)$, $\nu^{-}(x)$ can be defined by formulas (3.14) and (3.6), (3.7), respectively. Finally, solution of the Problem AG can be recovered by formulas (3.10) and (2.23) in the domains Φ_i $(i = 1, 2)$ and Φ_0, respectively, but only with some changes in (2.23), precisely, Green function $G(x, y, \xi, \eta)$ should be replaced by

$$G^{*}(x, y, \xi, \eta) = \frac{(y-\eta)^{\beta-1}}{2} \sum_{n=-\infty}^{\infty} \left[e_{1,\beta}^{1,\beta}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\beta}} \right) - e_{1,\beta}^{1,\beta}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\beta}} \right) \right], \tag{3.19}$$

which is the Green function of the first boundary problem for the (3.1) in Φ_0 [24].

Theorem 3.1 is proved. □

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Research Article

Slip Effects on Fractional Viscoelastic Fluids

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Unsteady flow of an incompressible Maxwell fluid with fractional derivative induced by a sudden moved plate has been studied, where the no-slip assumption between the wall and the fluid is no longer valid. The solutions obtained for the velocity field and shear stress, written in terms of Wright generalized hypergeometric functions ${}_p\Psi_q$, by using discrete Laplace transform of the sequential fractional derivatives, satisfy all imposed initial and boundary conditions. The no-slip contributions, that appeared in the general solutions, as expected, tend to zero when slip parameter is $\theta \to 0$. Furthermore, the solutions for ordinary Maxwell and Newtonian fluids, performing the same motion, are obtained as special cases of general solutions. The solutions for fractional and ordinary Maxwell fluid for no-slip condition also obtained as limiting cases, and they are equivalent to the previously known results. Finally, the influence of the material, slip, and the fractional parameters on the fluid motion as well as a comparison among fractional Maxwell, ordinary Maxwell, and Newtonian fluids is also discussed by graphical illustrations.

1. Introduction

There are many fluids in industry and technology whose behavior cannot be explained by the classical linearly viscous Newtonian model. The departure from the Newtonian behavior manifests itself in a variety of ways: non-Newtonian viscosity (shear thinning or shear thickening), stress relaxation, nonlinear creeping, development of normal stress differences, and yield stress [1]. The Navier-Stokes equations are inadequate to predicted the behavior of such type of fluids; therefore, many constitutive relations of non-Newtonian fluids are proposed [2]. These constitutive relations give rise to the differential equations, which, in general, are more complicated and higher order than the Navier-Stokes equations. Therefore, it is difficult to obtain exact analytical solutions for non-Newtonian fluids [3]. Modeling of the equation governing the behaviors of non-Newtonian fluids in different

circumstance is important from many points of view. For examples, plastics and polymers are extensively handeled by the chemical industry, whereas biological and biomedical devices like hemodialyser make use of the rheological behavior [4]. In general, the analysis of the behavior of the fluid motion of non-Newtonian fluids tends to be much more complicated and subtle in comparison with that of the Newtonian fluids [5].

The fractional calculus, almost as old as the standard differential and integral one, is increasingly seen as an efficient tool and subtle frame work within which useful generalization is quite long and arguments almost yearly. It includes fractal media, fractional wave diffusion, fractional Hamiltonian dynamics, and biopolymer dynamics as well as many other topics in physics. Fractional calculus is useful in the field of biorheology and bioengineering, in part, because many tissue-like materials (polymers, gels, emulsions, composites, and suspensions) exhibit power-law responses to an applied stress or strain [6, 7]. An example of such power-law behavior in elastic tissue was observed recently for viscoelastic measurements of the aorta, both in vivo and in vitro [8, 9], and the analysis of these data was most conveniently performed using fractional order viscoelastic models. The starting point of the fractional derivative model of non-Newtonian model is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville/Caputo fractional calculus operators. This generalization allows one to define precisely noninteger order integrals or derivatives. In general, fractional model of viscoelastic fluids is derived from well-known ordinary model by replacing the ordinary time derivatives, to fractional order time derivatives and this plays an important role to study the valuable tool of viscoelastic properties. We include here some investigation [10–18] in which the fractional calculus approach has been adopted for the flows of non-Newtonian fluids. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of some polymers in the glass transition and the glass state [19]. In other cases, it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories [20]. The use of fractional derivatives within the context of viscoelasticity was firstly proposed by Germant [21]. Later, Bagley and Torvik [22] demonstrated that the theory of viscoelasticity of coiling polymers predicts constitutive relations with fractional derivatives, and Makris et al. [23] achieved a very good fit of the experimental data when the fractional derivative Maxwell model has been used instead of the Maxwell model for the silicon gel fluid. Furthermore, it is worth pointing out that Palade et al. [24] developed a fully objective constitutive equation for an incompressible fluidreducible to the linear fractional derivative Maxwell model under small deformations hypothesis.

A general view of the literature shows that the slip effects on the flows of non-Newtonian fluids has been given not much attention. Especially, polymer melts exhibit a macroscopic wall slip. The fluids exhibiting a boundary slip are important in technological applications, for example, the polishing of artificial heart valves, rarefied fluid problems, and flow on multiple interfaces. In the study of fluid-solid surface interactions, the concept of slip of a fluid at a solid wall serves to describe macroscopic effects of certain molecular phenomena. When the molecular mean free path length of the fluid is comparable to the distance between the plates as in nanochannels or microchannels, the fluid exhibits non-continuum effects such as slipflow, as demonstrated experimentally by Derek et al. [25]. Experimental observations show that [26–28] non-Newtonian fluids, such as polymer melts, often exhibit macroscopic wall slip, which, in general, is described by a nonlinear and nonmonotone relation between the wall slip velocity and the traction. A more realistic class of slip flows are those in which the magnitude of the shear stress reaches some critical value,

here called the slip yield stress, before slip occurs. In fact, some experiments show that the onset slip and slip velocity may also depend on the normal stress at the boundary [26, 29]. Much of the research involving slip presumes that the slip velocity depends on the shear stress. The slip condition is an important factor in sharskin, spurt, and hysteresis effects, but the existing theory for non-Newtonian fluids with wall slippage is scant. We mention here some recent attempt regarding exact analytical solutions of non-Newtonian fluids with slip effects [30–36].

The objective of this paper is twofold. Firstly, is to give few more exact analytical solutions for viscoelastic fluids with fractional derivative approach, which is more natural and appropriate tool to describe the complex behavior of such fluids. Secondly, is to study the slip effects on viscoelastic fluid flows, which is important due to their practical applications. More precisely, our aim is to find the velocity field and the shear stress corresponding to the motion of a Maxwell fluid due to a sudden moved plate, where no-slip assumption is no longer valid. However, for completeness, we will determine exact solutions for a larger class of such fluids. Consequently, motivated by the above remarks, we solve our problem for Maxwell fluids with fractional derivatives. The general solutions are obtained using the discrete Laplace transforms. They are presented in series form in terms of the Wright generalized hypergeometric functions ${}_p\Psi_q$ and presented as sum of the slip contribution and the corresponding no-slip contributions. The similar solutions for ordinary Maxwell fluids can easily be obtained as limiting cases of general solutions. The Newtonian solutions are also obtained as special cases of fractional and ordinary Maxwell fluids. Furthermore, the solutions for fractional and ordinary Maxwell fluid for no-slip condition also obtained as a special cases, and they are similar with previously known results in the literature. Finally, the influence of the material, slip and fractional parameters on the motion of fractional and ordinary Maxwell fluids is underlined by graphical illustrations. The difference among fractional Maxwell, ordinary Maxwell, and Newtonian fluid models is also highlighted.

2. The Differential Equations Governing the Flow

The equations governing the flow of an incompressible fluid include the continuity equation and the momentum equation. In the absence of body forces, they are

$$\nabla \cdot \mathbf{V} = 0, \qquad \nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V} \cdot \nabla)\mathbf{V}, \tag{2.1}$$

where ρ is the fluid density, $\mathbf{V}$ is the velocity field, t is the time, and ∇ represents the gradient operator. The Cauchy stress $\mathbf{T}$ in an incompressible Maxwell fluid is given by [10, 11, 14–17]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda\left(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T\right) = \mu\mathbf{A}, \tag{2.2}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extrastress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericsen tensor, μ is the dynamic viscosity of the fluid, λ is relaxation time, the superscript T indicates the transpose operation, and the superposed dot indicates the material time derivative. The model characterized by the constitutive equations (2.2) contains as special

case the Newtonian fluid model for $\lambda \to 0$. For the problem under consideration, we assume a velocity field $\mathbf{V}$ and an extrastress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)\mathbf{i}, \qquad \mathbf{S} = \mathbf{S}(y,t), \tag{2.3}$$

where $\mathbf{i}$ is the unit vector along the x-coordinate direction. For these flows, the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(y,0) = \mathbf{0}, \qquad \mathbf{S}(y,0) = \mathbf{0}, \tag{2.4}$$

and (2.1)–(2.3) yield the meaningful equation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = -\frac{1}{\rho}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u(y,t)}{\partial y^2}, \qquad \left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t) = \mu\frac{\partial u(y,t)}{\partial y}, \tag{2.5}$$

where $\tau(y,t) = S_{xy}(y,t)$ is the nonzero shear stress and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The governing equations corresponding to an incompressible Maxwell fluid with fractional derivatives, performing the same motion in the absence of a pressure gradient in the flow direction, are (cf. [4, 15, 17])

$$(1+\lambda^\alpha D_t^\alpha)\frac{\partial u(y,t)}{\partial t} = \nu\frac{\partial^2 u(y,t)}{\partial y^2}, \qquad (1+\lambda^\alpha D_t^\alpha)\tau(y,t) = \mu\frac{\partial u(y,t)}{\partial y}, \tag{2.6}$$

where α is the fractional parameter and the fractional differential operator so-called Caputo fractional operator D_t^α is defined by [37, 38]

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}d\tau, & 0<\alpha<1, \\ \dfrac{df(t)}{dt}, & \alpha = 1, \end{cases} \tag{2.7}$$

and $\Gamma(\bullet)$ is the Gamma function. In the following, the system of fractional partial differential equations (2.6), with appropriate initial and boundary conditions, will be solved by means of Fourier sine and Laplace transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used [10–18].

3. Statement of the Problem

Consider an incompressible Maxwell fluid with fractional derivatives occupying the space lying over an infinitely extended plate which is situated in the (x,z) plane and perpendicular to the y-axis. Initially, the fluid is at rest, and at the moment $t = 0^+$, the plate is impulsively brought to the constant velocity U in its plane. Here, we assume the existence of slip boundary between the velocity of the fluid at the wall $u(0,t)$ and the speed of the wall, and

the relative velocity between $u(0,t)$ and the wall is assumed to be proportional to the shear rate at the wall. Due to the shear, the fluid above the plate is gradually moved. Its velocity is of the form $(2.3)_1$ while the governing equations are given by (2.6). The appropriate initial and boundary conditions are [39]

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0; \quad \tau(y,0) = 0, \quad y > 0, \tag{3.1}$$

$$u(0,t) = UH(t) + \theta H(t) \frac{\partial u(y,t)}{\partial y}\bigg|_{y=0}; \qquad t \geq 0, \tag{3.2}$$

where $H(t)$ is the Heaviside function and θ is the slip strength or slip coefficient. If $\theta = 0$, then the general assumed no-slip boundary condition is obtained. If θ is finite, fluid slip occurs at the wall, but its effect depends upon the length scale of the flow. Furthermore, the natural conditions

$$u(y,t), \quad \frac{\partial u(y,t)}{\partial y} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \ t > 0 \tag{3.3}$$

have to be also satisfied. They are consequences of the fact that the fluid is at rest at infinity, and there is no shear in the free stream.

4. Solution of the Problem

4.1. Calculation of the Velocity Field

Applying the Laplace transform to $(2.6)_1$, using the Laplace transform formula for sequential fractional derivatives [37, 38], and taking into account the initial conditions $(3.1)_{1,2}$, we find that

$$\frac{\partial \overline{u}(y,q)}{\partial y^2} - \frac{q(1+\lambda^\alpha q^\alpha)}{\nu}\overline{u}(y,q) = 0, \tag{4.1}$$

subject to the boundary conditions

$$\overline{u}(0,q) = \frac{U}{q} + \theta \frac{\partial \overline{u}(y,q)}{\partial y}\bigg|_{y=0}, \quad \overline{u}(y,q), \frac{\partial \overline{u}(y,q)}{\partial y} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \tag{4.2}$$

where $\overline{u}(y,q)$ is the image function of $u(y,t)$ and q is a transform parameter. Solving (4.1) and (4.2), we get

$$\overline{u}(y,q) = \frac{U}{q\left\{1+\theta\left[q(1+\lambda^\alpha q^\alpha)/\nu\right]^{1/2}\right\}} \exp\left\{-\left[\frac{q(1+\lambda^\alpha q^\alpha)}{\nu}\right]^{1/2} y\right\}. \tag{4.3}$$

In order to obtain $u(y,t) = \mathcal{L}^{-1}\{\overline{u}(y,q)\}$ and to avoid the lengthy and burdensome calculations of residues and contours integrals, we apply the discrete inverse Laplace transform method [10–18]. However, for a suitable presentation of the velocity field, we firstly rewrite (4.3) in series form

$$\overline{u}(y,q) = \frac{U}{q} + U\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k}\sum_{m=0}^{\infty}\frac{\Gamma(m-(k/2))(-\lambda^{-\alpha})^{m}}{m!\Gamma(-k/2)}\frac{1}{q^{-(\alpha+1)(k/2)+\alpha m+1}}$$
$$+U\sum_{k=0}^{\infty}\theta^{k}\sum_{m=1}^{\infty}\frac{y^{m}}{m!}\left(-\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k+m}\sum_{n=0}^{\infty}\frac{\Gamma(n-((k+m)/2))(-\lambda^{-\alpha})^{n}}{n!\Gamma(-(k+m)/2)}\frac{1}{q^{-(\alpha+1)(k+m/2)+\alpha n+1}}, \tag{4.4}$$

where we use the fact that

$$(-1)^{k}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)}. \tag{4.5}$$

Inverting (4.4) by means of discrete inverse Laplace transform, we find that

$$u(y,t) = UH(t) + UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k}t^{-(\alpha+1)(k/2)}\sum_{m=0}^{\infty}\frac{\Gamma(m-(k/2))(-t^{\alpha}/\lambda^{\alpha})^{m}}{m!\Gamma(-k/2)\Gamma(-(k/2)(\alpha+1)+\alpha n+1)}$$
$$+UH(t)\sum_{k=0}^{\infty}\theta^{k}\sum_{m=1}^{\infty}\frac{y^{m}}{m!}\left(-\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k+m}t^{-(\alpha+1)((k+m)/2)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(n-((k+m)/2))(-t^{\alpha}/\lambda^{\alpha})^{n}}{n!\Gamma(-(k+m)/2)\Gamma(-((k+m)/2)(\alpha+1)+\alpha n+1)}. \tag{4.6}$$

In term of Wright generalized hypergeometric function [40], we rewrite the above equation as a simpler form

$$u(y,t) = UH(t) + UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k}t^{-(\alpha+1)k/2}{}_{1}\Psi_{2}\left[-\frac{t^{\alpha}}{\lambda^{\alpha}}\middle|\begin{matrix}(-k/2,1)\\(-k/2,0),(-(k/2)(\alpha+1)+1,\alpha)\end{matrix}\right]$$
$$+UH(t)\sum_{k=0}^{\infty}\theta^{k}\sum_{m=1}^{\infty}\frac{y^{m}}{m!}\left(-\sqrt{\frac{\lambda^{\alpha}}{\nu}}\right)^{k+m}t^{-(\alpha+1)(k+m/2)}{}_{1}\Psi_{2}\left[-\frac{t^{\alpha}}{\lambda^{\alpha}}\middle|\begin{matrix}(-(k+m)/2,1)\\(-(k+m)/2,0),(-((k+m)/2)(\alpha+1)+1,\alpha)\end{matrix}\right], \tag{4.7}$$

where the Wright generalized hypergeometric function ${}_p\Psi_q$ is defined as

$$ {}_p\Psi_q\left[z\Big|_{(b_1,B_1),\ldots(b_q,B_q)}^{(a_1,A_1),\ldots(a_p,A_p)}\right] = \sum_{n=0}^{\infty}\frac{(z)^n\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\prod_{j=1}^{q}\Gamma(b_j+B_jn)}. \tag{4.8} $$

In order to justify the initial conditions $(3.1)_{1,2}$, we use the initial value theorem of Laplace transform [41]

$$ \begin{aligned} u(y,0) &= \lim_{t\to 0}u(y,t) = \lim_{q\to\infty}\left[q\overline{u}(y,q)\right] = 0, \\ \partial_t u(y,0) &= \lim_{t\to 0}\partial_t\, u(y,t) = \lim_{q\to\infty}\left[q^2\overline{u}(y,q) - qu(y,0)\right] = 0. \end{aligned} \tag{4.9} $$

Furthermore, to justify the boundary condition (3.2), we have

$$ \begin{aligned} \theta\frac{\partial u(y,t)}{\partial y}\bigg|_{y=0} &= UH(t)\sum_{k=0}^{\infty}\left(-\theta\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+1} t^{-(\alpha+1)((k+1)/2)}\ {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\Big|_{(-(k+1)/2,0),(-((k+1)/2)(\alpha+1)+1,\alpha)}^{(-(k+1)/2,1)}\right] \\ &= UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k} t^{-(\alpha+1)(k/2)}\ {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\Big|_{(-(k+1)/2,0),(-((k+1)/2)(\alpha+1)+1,\alpha)}^{(-(k+1)/2,1)}\right], \\ u(0,t) &= UH(t) + UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k} t^{-(\alpha+1)(k/2)}{}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\Big|_{(-k/2,0),(-(k/2)(\alpha+1)+1,\alpha)}^{(-(k/2),1)}\right]. \end{aligned} \tag{4.10} $$

It is easy to see that the exact solution (4.7) satisfies the boundary condition (3.2).

4.2. Calculation of the Shear Stress

Applying the Laplace transform to $(2.6)_2$ and using the initial condition $(3.1)_3$, we find that

$$ \overline{\tau}(y,q) = \frac{\mu}{1+\lambda^\alpha q^\alpha}\frac{\partial\overline{u}(y,q)}{\partial y}, \tag{4.11} $$

where $\overline{\tau}(y,q)$ is the Laplace transform of $\tau(y,t)$. Using (4.3) in (4.11), we find that

$$ \overline{\tau}(y,q) = -\frac{\mu U\left[q(1+\lambda^\alpha q^\alpha)\right]^{-1/2}}{\sqrt{\nu}\left\{1+\theta\left[q(1+\lambda^\alpha q^\alpha)/\nu\right]^{1/2}\right\}}\exp\left\{-\left[\frac{q(1+\lambda^\alpha q^\alpha)}{\nu}\right]^{1/2}y\right\}, \tag{4.12} $$

in order to obtain $\tau(y,t)$ under the suitable form, we write (4.12) in series form

$$\begin{aligned}\overline{\tau}(y,q) &= -\frac{\mu U}{\nu}\sum_{k=0}^{\infty}\theta^k\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+m-1}\\ &\quad\times\sum_{n=0}^{\infty}\frac{\Gamma(n-(k+m-1)/2)(-\lambda^{-\alpha})^n}{n!\Gamma(-(k+m-1)/2)}\frac{1}{q^{-(\alpha+1)((k+m-1)/2)+\alpha n}}.\end{aligned} \tag{4.13}$$

Inverting (4.13) by means of the discrete inverse Laplace transform, we find the shear stress $\tau(y,t)$ under simple form

$$\begin{aligned}\tau(y,t) &= -\rho UH(t)\sum_{k=0}^{\infty}\theta^k\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+m-1}t^{-(\alpha+1)((m+k-1)/2)-1}\\ &\quad\times\sum_{n=0}^{\infty}\frac{\Gamma(n-(k+m-1)/2)(-t^\alpha/\lambda^\alpha)^n}{n!\Gamma(-(k+m-1)/2)\Gamma(-(\alpha+1)(k+m-1)/2+\alpha n)},\end{aligned} \tag{4.14}$$

or equivalently

$$\begin{aligned}\tau(y,t) &= -\rho UH(t)\sum_{k=0}^{\infty}\theta^k\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{k+m-1}t^{-(\alpha+1)((m+k-1)/2)-1}\\ &\quad\times {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\middle|\begin{matrix}(-(k+m-1)/2,1)\\(-(k+m-1)/2,0),(-((k+m-1)/2)(\alpha+1),\alpha)\end{matrix}\right].\end{aligned} \tag{4.15}$$

5. The Special Cases

5.1. Ordinary Maxwell Fluid with Slip Effects

Making $\alpha \to 1$ into (4.7) and (4.15), we obtain the velocity field

$$\begin{aligned}u(y,t) &= UH(t)+UH(t)\sum_{k=1}^{\infty}\left(-\theta\sqrt{\frac{\lambda}{\nu}}\right)^{k}t^{-k}\,{}_1\Psi_2\left[-\frac{t}{\lambda}\middle|\begin{matrix}(-k/2,1)\\(-k/2,0),(-k+1,1)\end{matrix}\right],\\ &\quad+UH(t)\sum_{k=0}^{\infty}\theta^k\sum_{m=1}^{\infty}\frac{y^m}{m!}\left(-\sqrt{\frac{\lambda}{\nu}}\right)^{k+m}t^{-(k+m)}\,{}_1\Psi_2\left[-\frac{t}{\lambda}\middle|\begin{matrix}(-(k+m)/2,1)\\(-(k+m)/2,0),(-(k+m)+1,1)\end{matrix}\right],\end{aligned} \tag{5.1}$$

and the associated shear stress

$$\tau(y,t) = -\rho U H(t) \sum_{k=0}^{\infty} \theta^k \sum_{m=0}^{\infty} \frac{y^m}{m!} \left(-\sqrt{\frac{\lambda}{\nu}}\right)^{k+m} t^{-(k+m)} \, {}_1\Psi_2\left[-\frac{t}{\lambda}\middle|\begin{matrix}(-(k+m-1)/2,1)\\(-(k+m-1)/2,0),(-(k+m)+1,1)\end{matrix}\right], \quad (5.2)$$

corresponding to an ordinary Maxwell fluid performing the same motion.

5.2. Fractional Maxwell Fluid without Slip Effects

Making $\theta \to 0$ into (4.7) and (4.15), we obtain the solutions for velocity field

$$u(y,t) = UH(t) + UH(t) \sum_{m=1}^{\infty} \frac{y^m}{m!} \left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{m} t^{-(\alpha+1)(m/2)} \, {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\middle|\begin{matrix}(-m/2,1)\\(-m/2,0),(-(m/2)(\alpha+1)+1,\alpha)\end{matrix}\right] \quad (5.3)$$

and the associated shear stress

$$\tau(y,t) = -\rho U H(t) \sum_{m=0}^{\infty} \frac{y^m}{m!} \left(-\sqrt{\frac{\lambda^\alpha}{\nu}}\right)^{m-1} t^{-(\alpha+1)((m-1)/2)-1} \, {}_1\Psi_2\left[-\frac{t^\alpha}{\lambda^\alpha}\middle|\begin{matrix}(-(m-1)/2,1)\\(-(m-1)/2,0),(-((m-1)/2)(\alpha+1),\alpha)\end{matrix}\right], \quad (5.4)$$

they are equivalent to the known solutions obtained in [42, 43] for Sokes' first problem of fractional Maxwell fluid.

5.3. Ordinary Maxwell Fluid without Slip Effects

Making $\alpha \to 1$ into (5.3) and (5.4), we recover the solutions for velocity field shear stress for Stokes' first problem of ordinary Maxwell fluid.

5.4. Newtonian Fluid with Slip Effects

Finally, making $\lambda \to 0$ into (4.3) and (4.12), the solutions for a Newtonian fluid with slip effects are obtained

$$\begin{aligned} u(y,t) &= U W_{-1/2,1}\left(-\frac{y}{\sqrt{\nu t}}\right) + U \sum_{k=1}^{\infty} \left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k/2)+1}\left(-\frac{y}{\sqrt{\nu t}}\right), \\ \tau(y,t) &= -\frac{\mu U}{\sqrt{\nu t}} W_{-1/2,-1/2}\left(-\frac{y}{\sqrt{\nu t}}\right) - \frac{\mu U}{\sqrt{\nu t}} \sum_{k=1}^{\infty} \left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k-1)/2}\left(-\frac{y}{\sqrt{\nu t}}\right), \end{aligned} \quad (5.5)$$

in which

$$W_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(an+b)}, \qquad z \in C \quad (5.6)$$

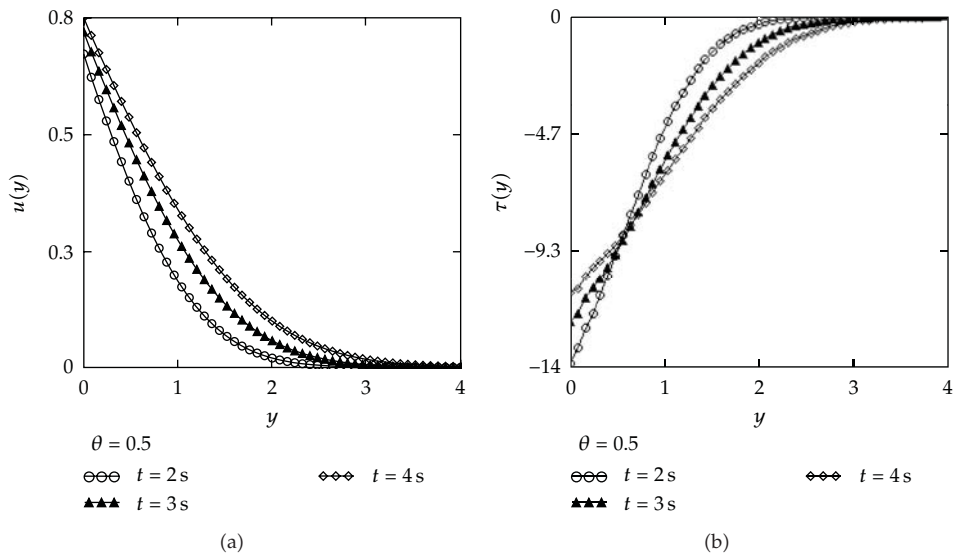

Figure 1: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.5$, and different values of t.

is the Wright function [40]. Using the definition of Wright function and the series expression of error function, we can easily prove that

$$W_{-1/2,1}(-z) = \operatorname{erfc}\left(\frac{z}{2}\right), \qquad W_{-1/2,1/2}(-z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right). \tag{5.7}$$

Substituting (5.7) into (5.5), we can reduce to

$$\begin{aligned} u(y,t) &= u_N(y,t) + U \sum_{k=1}^{\infty} \left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k/2)+1}\left(-\frac{y}{\sqrt{\nu t}}\right), \\ \tau(y,t) &= \tau_N(y,t) - \frac{\mu U}{\sqrt{\nu t}} \sum_{k=1}^{\infty} \left(-\frac{\theta}{\sqrt{\nu t}}\right)^k W_{-1/2,-(k-1)/2}\left(-\frac{y}{\sqrt{\nu t}}\right), \end{aligned} \tag{5.8}$$

where

$$u_N(y,t) = U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \qquad \tau_N(y,t) = -\frac{\mu U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t}\right) \tag{5.9}$$

are classical solutions for Stokes' first problem of Newtonian fluid [42, 44].

6. Numerical Results and Conclusions

In this paper, the unsteady flow of fractional Maxwell fluid over an infinite plate, where the no-slip assumption between the wall and the fluid is no longer valid, is studied by means

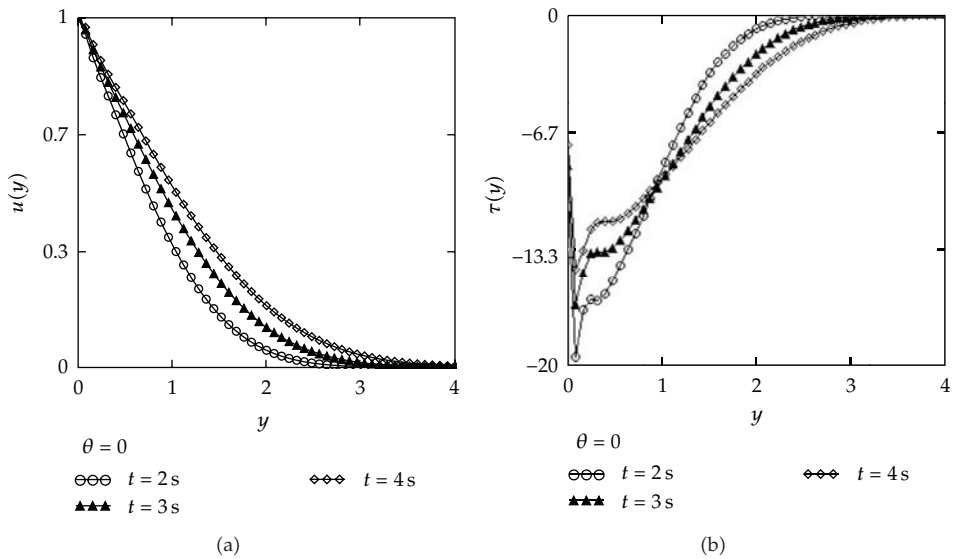

Figure 2: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.0$, and different values of t.

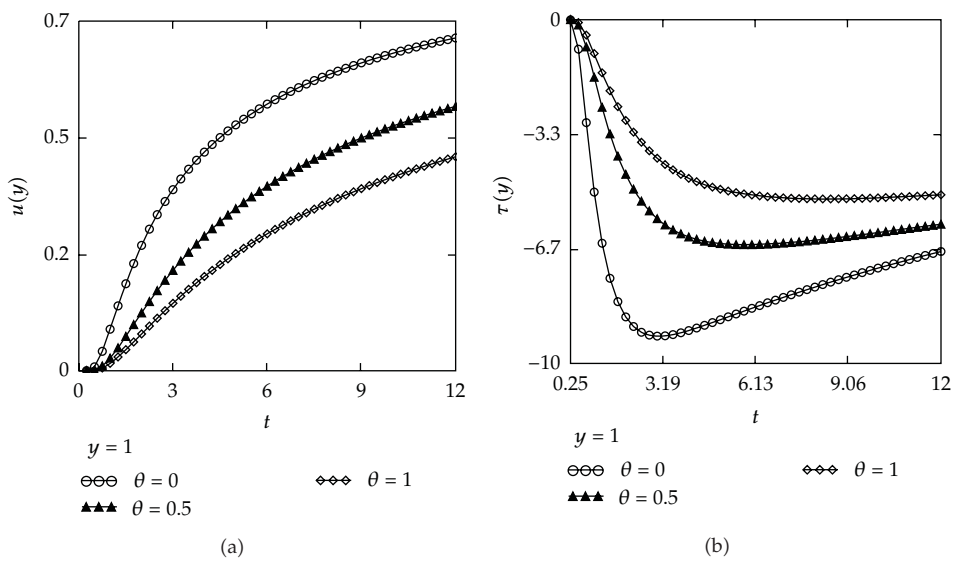

Figure 3: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, and different values of θ.

of the discrete Laplace transforms. The motion of the fluid is due to the plate that at time $t = 0^+$ is suddenly moved with a constant velocity U in its plane. Closed-form solutions are obtained for the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ in series form in terms of the Wright generalized hypergeometric functions. These solutions, presented as a sum of the slip contribution and the corresponding no-slip contributions, satisfy all imposed initial and boundary conditions. The corresponding solutions for ordinary Maxwell fluids are also

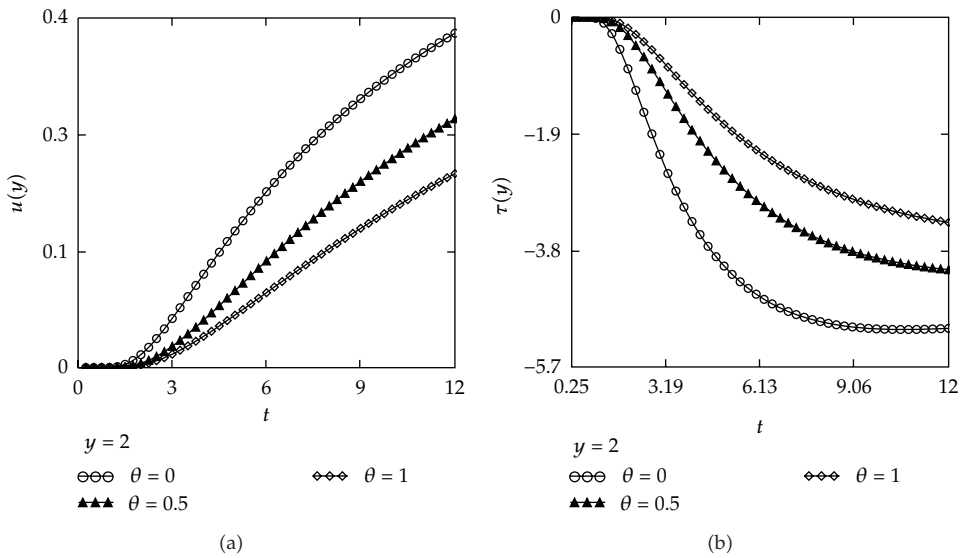

Figure 4: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 2$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, and different values of θ.

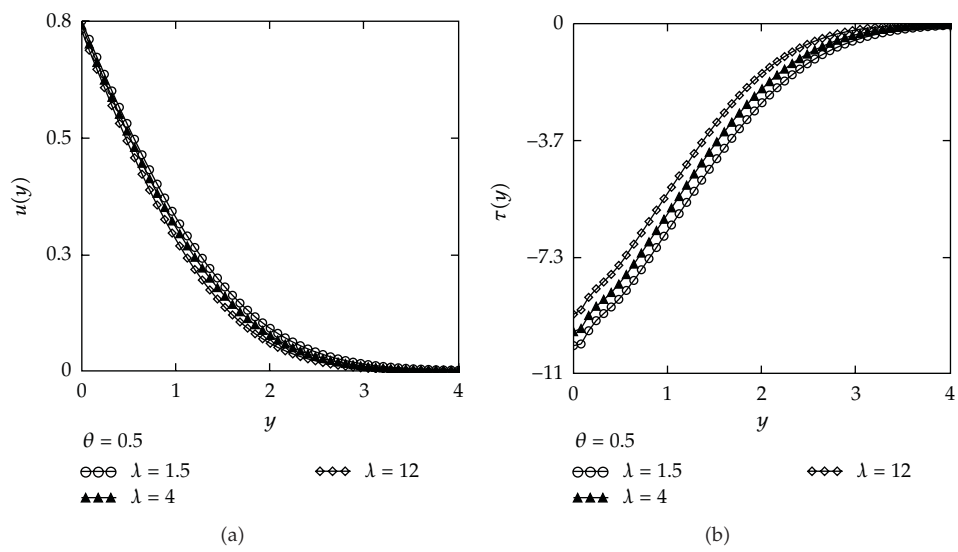

Figure 5: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\alpha = 0.2$, $\theta = 0.5$, $t = 5\,\mathrm{s}$, and different values of λ.

obtained from general solutions for $\alpha \to 1$. In the special case when $\theta \to 0$, the general solution reduces to previously known results for Stokes' first problem.

In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ have been drawn against y and t for different values of t, material constants ν, λ, slip parameter θ, and fractional parameter α. From all figures, it is clear that increasing the slip parameter at the wall the velocity decreases at the wall. Figures 1 and 2 are prepared to show the effect of time on velocity and shear

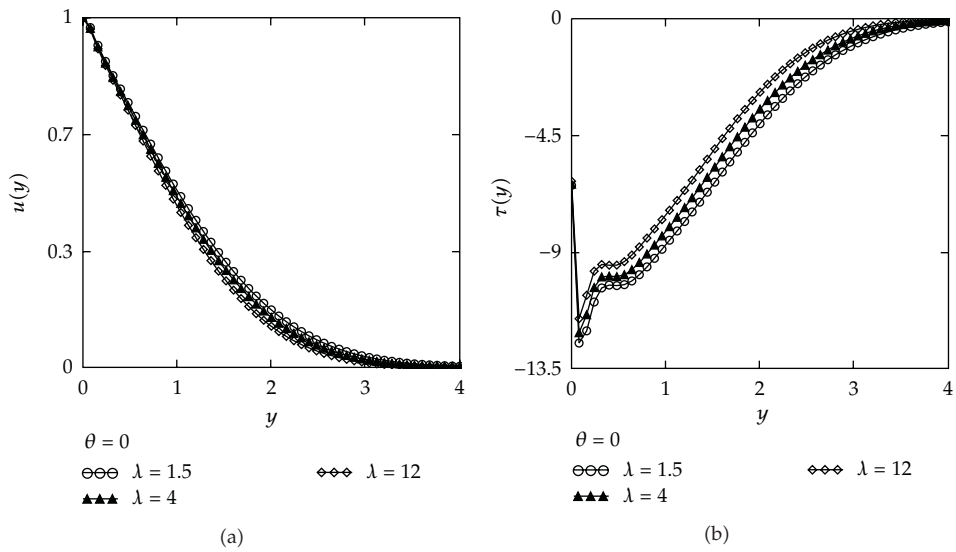

Figure 6: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\alpha = 0.2$, $\theta = 0.0$, $t = 5\,\mathrm{s}$, and different values of λ.

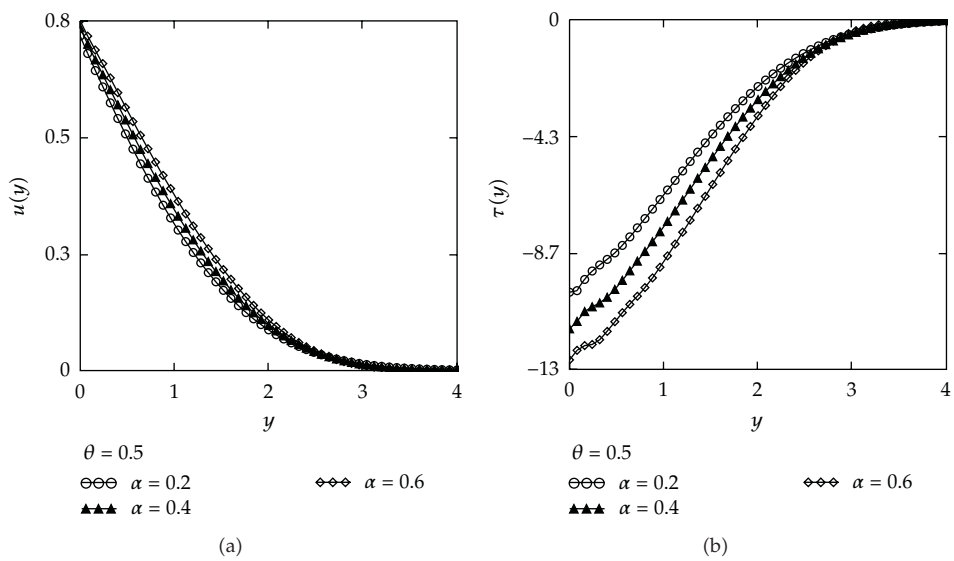

Figure 7: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\theta = 0.5$, $t = 5\,\mathrm{s}$, and different values of α.

stress profiles with and without slip effects. It is clear that velocity and shear stress (in absolute value) are smaller when slip parameter is nonzero. It is also noted that velocity on the whole flow domain while the shear stress (in absolute value) on large part of flow domain are increasing functions of time t. Figures 3 and 4 are sketched to see the influence of slip effects on fluid motion for two different values of y. It is noted that velocity and shear stress (in absolute value), as expected are decrease when slip parameter θ and y increase.

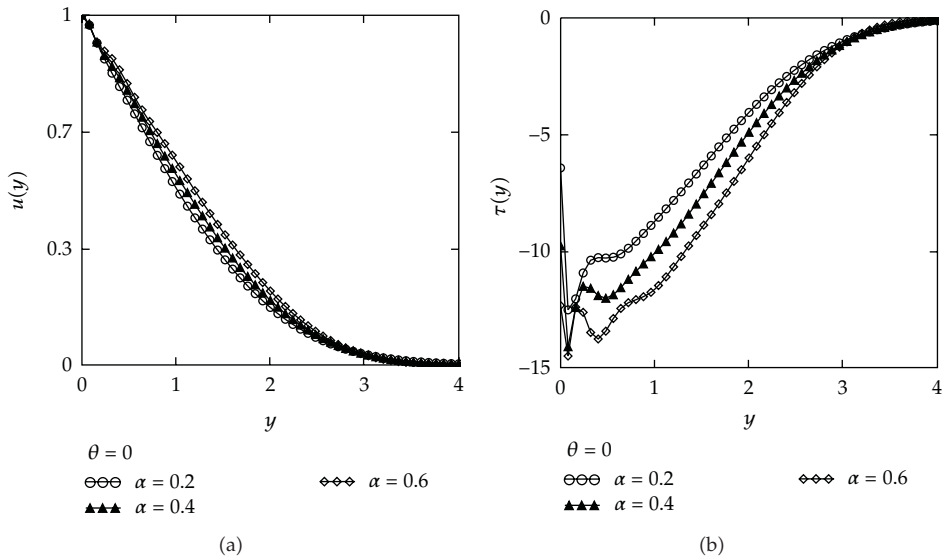

Figure 8: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\theta = 0.0$, $t = 5\,\mathrm{s}$, and different values of α.

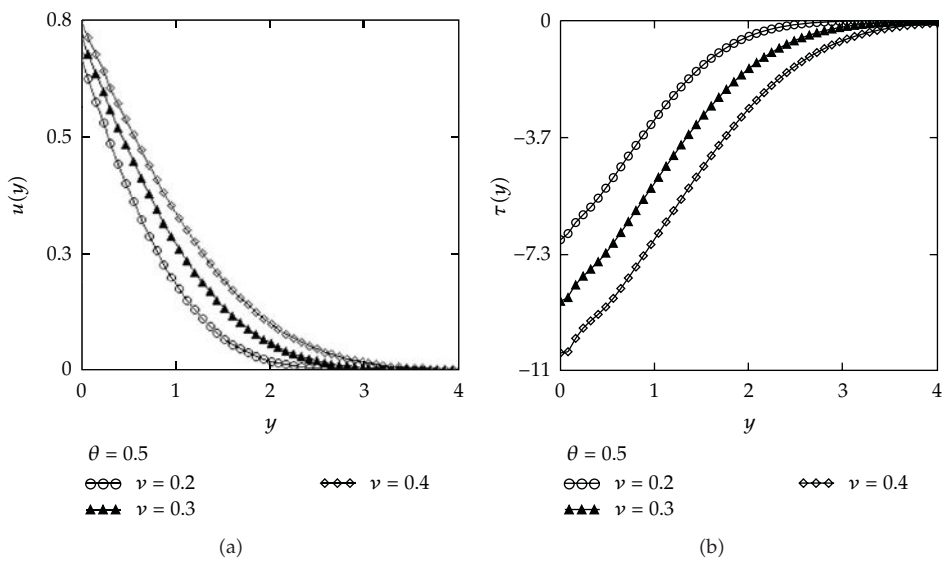

Figure 9: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\rho = 87.071$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.5$, $t = 5\,\mathrm{s}$, and different values of ν.

The influence of relaxation time and kinematic viscosity ν on fluid motion are presented in Figures 5–8. As expected, the two material parameter have opposite effects on fluid motion. For instance, the velocity and shear stress decreases with respect to λ. More important for us is to see the effects of fractional and slip parameter on fluid motion. It is observed that velocity and shear stress either slip effects present or not are increasing functions of fractional parameter, as shown in Figures 9 and 10. The effect of slip parameter is clear from Figure 11.

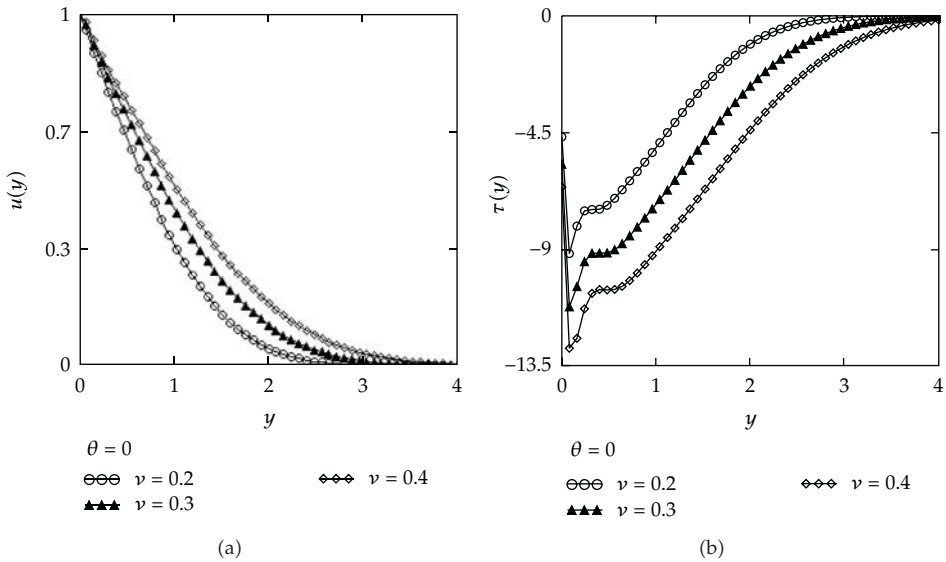

Figure 10: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (5.3) and (5.4), for $U = 1$, $y = 1$, $\rho = 87.071$, $\lambda = 1.5$, $\alpha = 0.2$, $\theta = 0.0$, $t = 5\,\text{s}$, and different values of ν.

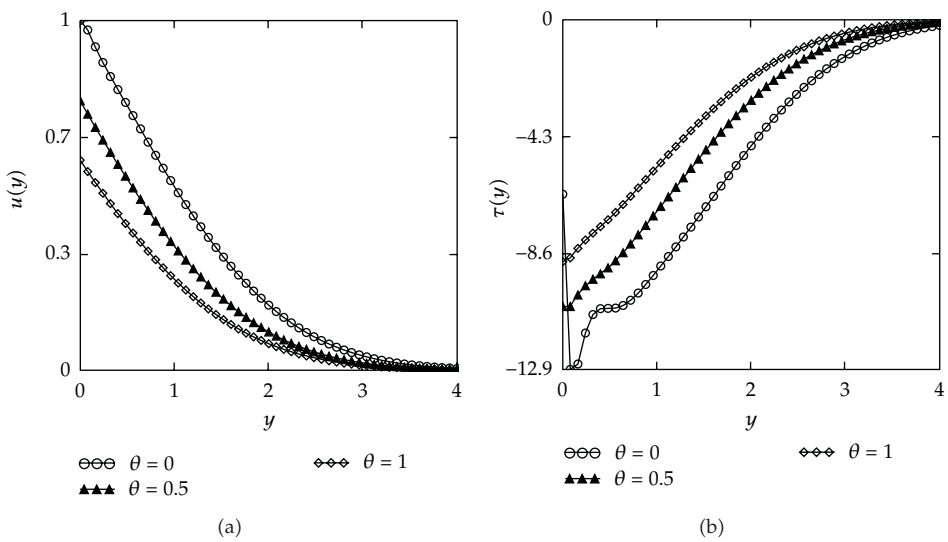

Figure 11: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell fluid given by (4.7) and (4.15), for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 5\,\text{s}$, and different values of θ.

Finally, for comparison, the velocity field and the shear stress corresponding to the three models (fractional Maxwell, ordinary Maxwell, and Newtonian) are together depicted in Figures 12–14 for three different values of slip parameter and the same values of t and of the material constants. It is clearly seen from Figures 12 and 13 that the ordinary Maxwell fluid swiftest and the fractional Maxwell fluid is the slowest near the moving plate for slip parameters $\theta = 0.0$ and $\theta = 0.2$. However, the monotonicity is change on large part of the flow domain. The shear stress corresponding to ordinary Maxwell fluid is highest near the

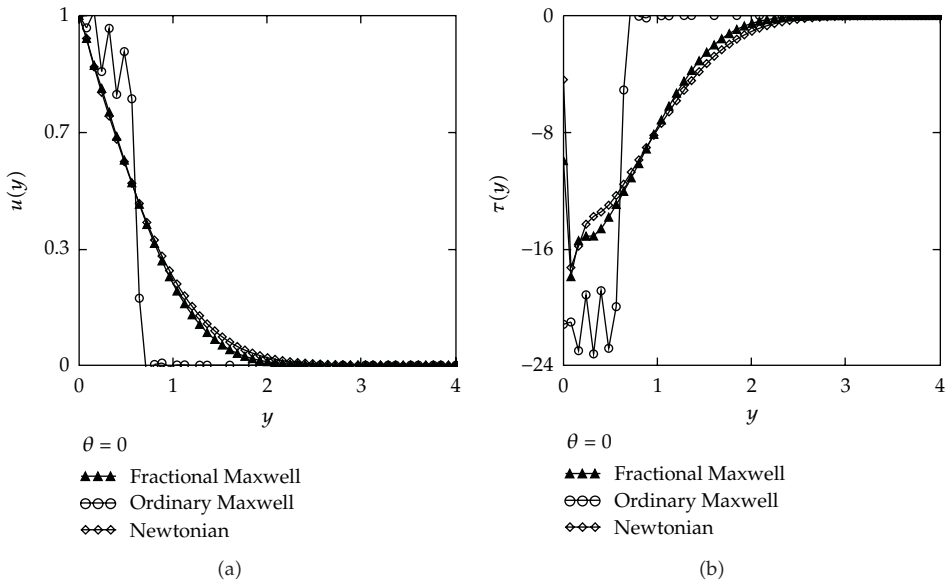

Figure 12: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 2\,\text{s}$, and $\theta = 0.0$.

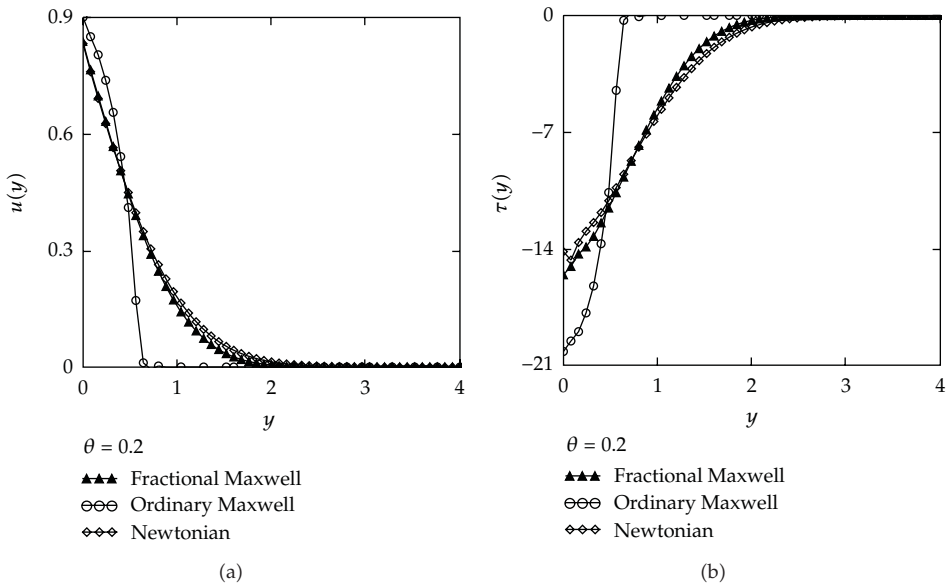

Figure 13: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 2\,\text{s}$, and $\theta = 0.2$.

moving plate. For higher values of slip parameter the fractional Maxwell fluid is swiftest and the ordinary Maxwell is slowest and shear stress corresponding to fractional Maxwell fluid is largest on the whole flow domain as it is clear from Figure 14. It is important to note the difference between fractional and ordinary Maxwell fluid that, when slip effect is not present, the ordinary Maxwell fluid have oscillating behavior near the moving plate as

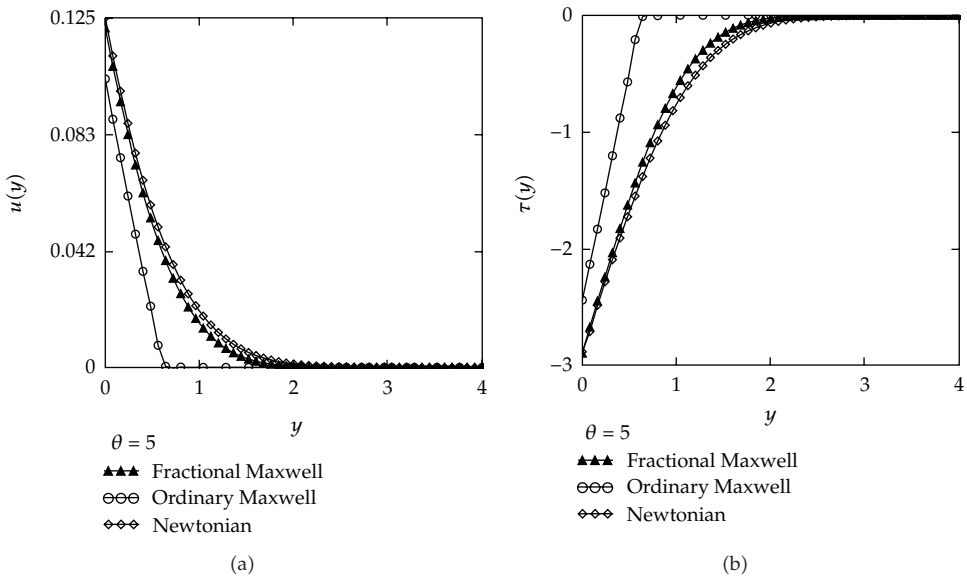

Figure 14: Profiles of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $U = 1$, $y = 1$, $\nu = 0.379$, $\mu = 33$, $\lambda = 1.5$, $\alpha = 0.2$, $t = 2\,\mathrm{s}$, and $\theta = 5.0$.

shown in Figure 12, which is the natural one, ordinary Maxwell fluid being the viscoelastic fluid. However the fractional Maxwell fluid have no oscillation. The units of the material constants in all figures are SI units.

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Research Article

Antisynchronization of Nonidentical Fractional-Order Chaotic Systems Using Active Control

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Antisynchronization phenomena are studied in nonidentical fractional-order differential systems. The characteristic feature of antisynchronization is that the sum of relevant state-variables vanishes for sufficiently large value of time variable. Active control method is used first time in the literature to achieve antisynchronization between fractional-order Lorenz and Financial systems, Financial and Chen systems, and Lü and Financial systems. The stability analysis is carried out using classical results. We also provide numerical results to verify the effectiveness of the proposed theory.

1. Introduction

In their pioneering work [1, 2], Pecora and Carroll have shown that chaotic systems can be synchronized by introducing appropriate coupling. The notion of synchronization of chaos has further been explored in secure communications of analog and digital signals [3] and for developing safe and reliable cryptographic systems [4]. For the synchronization of chaotic systems, a variety of approaches have been proposed which include nonlinear feedback [5], adaptive [6, 7], and active controls [8, 9].

Antisynchronization (AS) is a phenomenon in which the state vectors of the synchronized systems have the same amplitude but opposite signs to those of the driving system. Hence the sum of two signals converges to zero when AS appears. Antisynchronization has applications in lasers [10], in periodic oscillators, and in communication systems. Using AS to lasers, one may generate not only drop-outs of the intensity but also short pulses of high intensity, which results in the pulses of special shapes.

Active control method is used to AS for two identical integer order systems by Ho et al. [11] and for nonidentical systems by Li and Zhou [12]. Nonlinear control scheme was used by Li et al. [13] to study AS. Al-Sawalha [14] have reported AS between Chua's system and Nuclear spin generator (NSG) system. Recently AS between Lorenz system, Lü system, and Four-scroll system is investigated by Elabbasy and El-Dessoky [15].

Fractional calculus deals with derivatives and integration of arbitrary order [16–18] and has deep and natural connections with many fields of applied mathematics, engineering, and physics. Fractional calculus has a wide range of applications in control theory [19], viscoelasticity [20], diffusion [21–27], turbulence, electromagnetism, signal processing [28, 29], and bioengineering [30]. Analysis of fractional-order dynamical systems involving Riemann-Liouville as well as Caputo derivatives has been dealt with by present authors [31, 32].

Synchronization of fractional-order chaotic systems was first studied by Deng and Li [33] who carried out synchronization in case of the fractional Lü system. Further they have investigated synchronization of fractional Chen system [34]. Li and Deng have summarized the theory and techniques of synchronization in [35]. The theory for synchronization problems in an ω-symmetrically coupled fractional differential systems have been studied by Zhou and Li [36]. Since then many fractional systems have been investigated by various researchers. A few examples in this regards are Li et al. [37] (Chua system), Wang et al. [38] (Chen system), Wang and Zhang [39] (unified system), Wang and He [40] (unified system), Yu and Li [41] (Rossler hyperchaos system), and Tavazoei and Haeri [42] (Lü system and Chen system). Of late Matouk [43] has synchronized fractional Lü system with fractional Chen system and fractional Chen system with fractional Lorenz system. Hu et al. [44] have synchronized fractional Lorenz and fractional Chen systems. Further Bhalekar and Daftardar-Gejji [45] have investigated the interrelationship between the (fractional) order and synchronization in different chaotic dynamical systems. However, it seems that there are no previous results on AS of two nonidentical fractional-order chaotic systems.

In the present paper, we study the antisynchronization of the following fractional systems using active control method: (i) Lorenz with Financial, (ii) Financial with Chen, and (iii) Lü with Financial.

2. Preliminaries

2.1. Fractional Calculus

Basic definitions and properties of fractional derivative/integrals are given below [16, 17, 46].

Definition 2.1. A real function $f(t),\ t > 0$, is said to be in space $C_\alpha,\ \alpha \in \Re$ if there exists a real number $p\ (> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty)$.

Definition 2.2. A real function $f(t),\ t > 0$, is said to be in space $C_\alpha^m,\ m \in \mathbb{N} \cup \{0\}$ if $f^{(m)} \in C_\alpha$.

Definition 2.3. Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann-Liouville integral of order $\mu,\ \mu > 0$ is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > 0. \tag{2.1}$$

Definition 2.4. The (left-sided) Caputo fractional derivative of f, $f \in C_{-1}^{m}$, $m \in \mathbb{N} \cup \{0\}$, is defined as

$$\begin{aligned} D^{\mu} f(t) &= \frac{d^m}{dt^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{d^m f(t)}{dt^m}, \quad m-1 < \mu < m, \ m \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Note that for $m-1 < \mu \leq m$, $m \in \mathbb{N}$,

$$\begin{aligned} I^{\mu} D^{\mu} f(t) &= f(t) - \sum_{k=0}^{m-1} \frac{d^k f}{dt^k}(0) \frac{t^k}{k!}, \\ I^{\mu} t^{\nu} &= \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \end{aligned} \tag{2.3}$$

2.2. Numerical Method for Solving Fractional Differential Equations

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDEs). A modification of Adams-Bashforth-Moulton algorithm is proposed by Diethelm et al. in [47–49] to solve FDEs.

Consider for $\alpha \in (m-1, m]$ the initial value problem (IVP)

$$\begin{aligned} D^{\alpha} y(t) &= f(t, y(t)), \quad 0 \leq t \leq T, \\ y^{(k)}(0) &= y_0^{(k)}, \quad k = 0, 1, \ldots, m-1. \end{aligned} \tag{2.4}$$

The IVP (2.4) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{2.5}$$

Consider the uniform grid $\{t_n = nh/n = 0, 1, \ldots, N\}$ for some integer N and $h := T/N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j)$, $j = 1, 2, \ldots, n$, and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_h^P(t_{n+1})\right) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_n(t_j)), \tag{2.6}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha} & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & \text{if } 1 \leq j \leq n, \\ 1 & \text{if } j = n+1. \end{cases} \tag{2.7}$$

The preliminary approximation $y_h^P(t_{n+1})$ is called predictor and is given by

$$y_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_n(t_j)), \tag{2.8}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right). \tag{2.9}$$

Error in this method is

$$\max_{j=0,1,\ldots,N} \left| y(t_j) - y_h(t_j) \right| = O(h^p), \tag{2.10}$$

where $p = \min(2, 1+\alpha)$.

3. System Description

The fractional-order Lorenz system [50, 51] is described by

$$\begin{aligned} D^\alpha x &= \sigma(y - x), \\ D^\alpha y &= rx - y - xz, \\ D^\alpha z &= xy - \mu z, \end{aligned} \tag{3.1}$$

where $\sigma = 10$ is the Prandtl number, $r = 28$ is the Rayleigh number over the critical Rayleigh number, and $\mu = 8/3$ gives the size of the region approximated by the system. The minimum effective dimension for this system is 2.97 [51].

In [52] Chen proposed the financial system to fractional-order

$$\begin{aligned} D^\alpha x &= z + (y - a)x, \\ D^\alpha y &= 1 - by - x^2, \\ D^\alpha z &= -x - cz, \end{aligned} \tag{3.2}$$

where $a = 3$, $b = 0.1$, and $c = 1$. The minimum effective dimension for which the system exhibits chaos is given by 2.32 [52].

Li and Peng [53] studied chaos in fractional-order Chen system

$$\begin{aligned} D^\alpha x &= a_1(y - x), \\ D^\alpha y &= (c_1 - a_1)x - xz + c_1 y, \\ D^\alpha z &= xy - b_1 z, \end{aligned} \tag{3.3}$$

where $a_1 = 35$, $b_1 = 3$, and $c_1 = 27$. The minimum effective dimension reported is 2.92 [53].

Fractional-order Lü system is the lowest-order chaotic system among all the chaotic systems reported in the literature [54]. The minimum effective dimension reported is 0.30. The system is given by

$$\begin{aligned} D^{\alpha}x &= a_2(y-x), \\ D^{\alpha}y &= c_2 y - xz, \\ D^{\alpha}z &= xy - b_2 z, \end{aligned} \tag{3.4}$$

where $a_2 = 35$, $b_2 = 3$, and $c_2 = 28$.

4. Antisynchronization between Fractional-Order Lorenz and Financial System

In this section, we study the antisynchronization between Lorenz and Financial systems. Assuming that the Lorenz system drives the Financial system, we define the drive (master) and response (slave) systems as follows:

$$\begin{aligned} D^{\alpha}x_1 &= \sigma(y_1 - x_1), \\ D^{\alpha}y_1 &= rx_1 - y_1 - x_1 z_1, \\ D^{\alpha}z_1 &= x_1 y_1 - \mu z_1, \end{aligned} \tag{4.1}$$

$$\begin{aligned} D^{\alpha}x_2 &= z_2 + (y_2 - a)x_2 + u_1(t), \\ D^{\alpha}y_2 &= 1 - by_2 - x_2^2 + u_2(t), \\ D^{\alpha}z_2 &= -x_2 - cz_2 + u_3(t). \end{aligned} \tag{4.2}$$

The unknown terms u_1, u_2, u_3 in (4.2) are active control functions to be determined. Define the error functions as

$$e_1 = x_1 + x_2, \qquad e_2 = y_1 + y_2, \qquad e_3 = z_1 + z_2. \tag{4.3}$$

Equation (4.3) together with (4.1) and (4.2) yields the error system

$$\begin{aligned} D^{\alpha}e_1 &= (a-\sigma)x_1 + \sigma y_1 + x_1 y_1 - z_1 - ae_1 - y_1 e_1 - x_1 e_2 + e_1 e_2 + e_3 + u_1(t), \\ D^{\alpha}e_2 &= 1 + rx_1 - x_1^2 + (b-1)y_1 - x_1 z_1 + 2x_1 e_1 - e_1^2 - be_2 + u_2(t), \\ D^{\alpha}e_3 &= x_1 + (c-\mu)z_1 + x_1 y_1 - e_1 - ce_3 + u_3(t). \end{aligned} \tag{4.4}$$

We define active control functions $u_i(t)$ as

$$\begin{aligned} u_1(t) &= V_1(t) - (a-\sigma)x_1 - \sigma y_1 - x_1 y_1 + z_1 + y_1 e_1 + x_1 e_2 - e_1 e_2, \\ u_2(t) &= V_2(t) - 1 - r x_1 + x_1^2 - (b-1) y_1 + x_1 z_1 - 2 x_1 e_1 + e_1^2, \\ u_3(t) &= V_3(t) - x_1 - (c-\mu) z_1 - x_1 y_1. \end{aligned} \tag{4.5}$$

The terms $V_i(t)$ are linear functions of the error terms $e_i(t)$. With the choice of $u_i(t)$ given by (4.5), the error system (4.5) becomes

$$\begin{aligned} D^\alpha e_1 &= -a e_1 - e_3 + V_1(t), \\ D^\alpha e_2 &= -b e_2 + V_2(t), \\ D^\alpha e_3 &= -e_1 - c e_3 + V_3(t). \end{aligned} \tag{4.6}$$

The control terms $V_i(t)$ are chosen so that the system (4.6) becomes stable. There is not a unique choice for such functions. We choose

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{4.7}$$

where A is a 3×3 real matrix, chosen so that for all eigenvalues λ_i of the system (4.6) the condition

$$\left|\arg(\lambda_i)\right| > \frac{\alpha\pi}{2} \tag{4.8}$$

is satisfied. (The stability condition (4.8) is discussed in the literature [55–57]). If we choose

$$A = \begin{pmatrix} a-1 & 0 & -1 \\ 0 & -1+b & 0 \\ 1 & 0 & c-1 \end{pmatrix}, \tag{4.9}$$

then the eigenvalues of the linear system (4.6) are -1, -1, and -1. Hence the condition (4.8) is satisfied for $\alpha < 2$. Since we consider only the values $\alpha \leq 1$, we get the required antisynchronization.

4.1. Simulation and Results

Parameters of the Lorenz system are taken as $\sigma = 10$, $r = 28$, $\mu = 8/3$ and Financial system as $a = 3$, $b = 0.1$, $c = 1$. The fractional-order α is taken to be 0.99 for which both the systems are chaotic. The initial conditions for drive and response system are $x_1(0) = 10$, $y_1(0) = 5$,

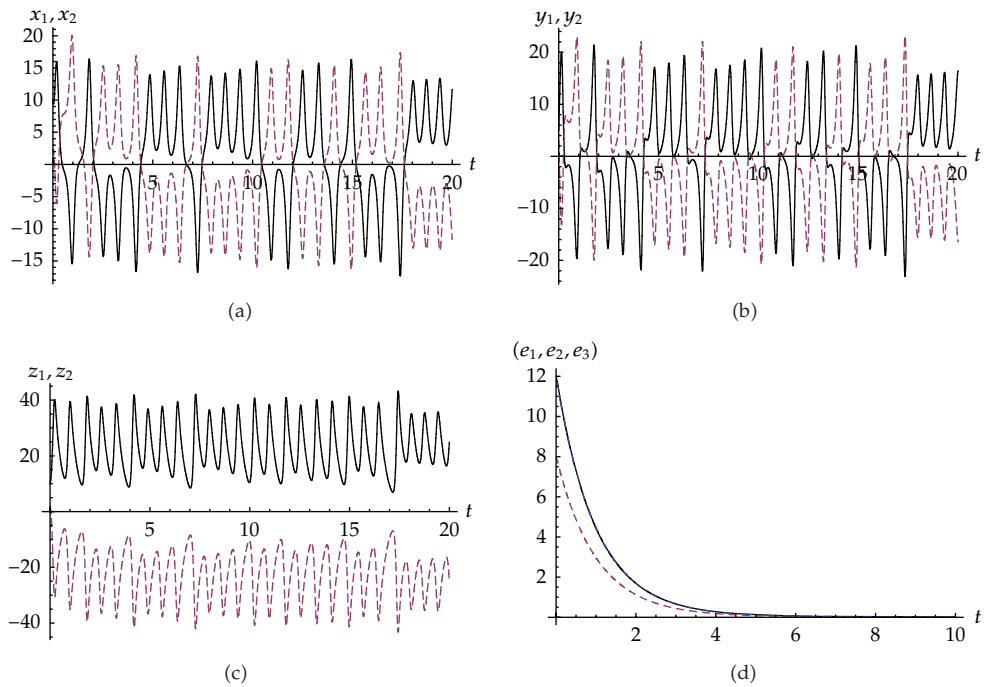

Figure 1: (a) Signals x_1, x_2, (b) Signals y_1, y_2, (c) Signals z_1, z_2, and (d) Error system.

$z_1(0) = 10$ and $x_2(0) = 2$, $y_2(0) = 3$, $z_2(0) = 2$, respectively. Initial conditions for the error system are thus $e_1(0) = 12$, $e_2(0) = 8$, and $e_3(0) = 12$. Figures 1(a)–1(c) show the antisynchronization between Lorenz and Financial system; the response system is given by dashed line. The errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line) and $e_3(t)$ (dot-dashed line) in the antisynchronization are shown in Figure 1(d).

5. Antisynchronization between Financial and Chen Systems of Fractional Order

Assuming that Chen system is antisynchronized with Financial system; define the drive system as

$$\begin{aligned} D^{\alpha}x_1 &= z_1 + (y_1 - a)x_1, \\ D^{\alpha}y_1 &= 1 - by_1 - x_1^2, \\ D^{\alpha}z_1 &= -x_1 - cz_1 \end{aligned} \tag{5.1}$$

and the response system as

$$\begin{aligned} D^{\alpha}x_2 &= a_1(y_2 - x_2) + u_4, \\ D^{\alpha}y_2 &= (c_1 - a_1)x_2 - x_2z_2 + c_1y_2 + u_5, \\ D^{\alpha}z_2 &= x_2y_2 - b_1z_2 + u_6. \end{aligned} \tag{5.2}$$

Let $e_1 = x_1 + x_2,\ e_2 = y_1 + y_2$, and $e_3 = z_1 + z_2$ be error functions. For antisynchronization, it is essential that the errors $e_i \to 0$ as $t \to \infty$. Note that

$$\begin{aligned} D^\alpha e_1 &= (a_1 - a)x_1 - a_1 y_1 + x_1 y_1 + z_1 - a_1 e_1 + a_1 e_2 + u_4(t), \\ D^\alpha e_2 &= 1 + (a_1 - c_1)x_1 - x_1^2 - (b + c_1)y_1 - x_1 z_1 \\ &\quad + (c_1 - a_1)e_1 + z_1 e_1 + c_1 e_2 + x_1 e_3 - e_1 e_3 + u_5(t), \\ D^\alpha e_3 &= -x_1 + x_1 y_1 + (b_1 - c)z_1 - y_1 e_1 - x_1 e_2 + e_1 e_2 - b_1 e_3 + u_6(t). \end{aligned} \tag{5.3}$$

The control functions are chosen as

$$\begin{aligned} u_4 &= V_4 - (a_1 - a)x_1 + a_1 y_1 - x_1 y_1 - z_1, \\ u_5 &= V_5 - 1 - (a_1 - c_1)x_1 + x_1^2 + (b + c_1)y_1 + x_1 z_1 - z_1 e_1 - x_1 e_3 + e_1 e_3, \\ u_6 &= V_6 + x_1 - x_1 y_1 - (b_1 - \mathrm{c})z_1 + y_1 e_1 + x_1 e_2 - e_1 e_2. \end{aligned} \tag{5.4}$$

The linear functions $V_4,\ V_5,\ V_6$ are given by

$$\begin{aligned} V_4 &= (a_1 - 1)e_1 - a_1 e_2, \\ V_5 &= -(a_1 - c_1)e_1 - (c_1 + 1)e_2, \\ V_6 &= (b_1 - 1)e_3. \end{aligned} \tag{5.5}$$

With the values given in (5.4) and (5.5), the error system (5.3) becomes

$$\begin{pmatrix} D^\alpha e_1 \\ D^\alpha e_2 \\ D^\alpha e_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{5.6}$$

It can be observed that the coefficient matrix of the error system (5.6) has eigenvalues -1, $-1,\ -1$. So the system is stable and antisynchronization is achieved.

5.1. Simulations and Results

We take parameters for fractional-order Chen system as $a_1 = 35,\ b_1 = 3,\ c_1 = 27$. Parameters for the Financial system are same as given in Section 4.1. Experiments are done for fixed value of fractional-order $\alpha = 0.95$, which is same for drive and response system (5.1) and (5.2). The initial conditions for the systems (5.1) and (5.2) are $x_1(0) = 2,\ y_1(0) = 3,\ z_1(0) = 2$ and $x_2(0) = 10,\ y_2(0) = 25,\ z_2(0) = 36$, respectively. For the error system (5.6), the initial conditions turns out to be $e_1(0) = 12,\ e_2(0) = 28,\ e_3(0) = 38$. The simulation results are summarized in Figure 2. Antisynchronization between fractional Financial and Chen system is shown in Figure 2(a) (signals $x_1,\ x_2$), Figure 2(b) (signals $y_1,\ y_2$), and Figure 2(c) (signals $z_1,\ z_2$). Note that the drive systems are shown by solid lines, whereas response systems are

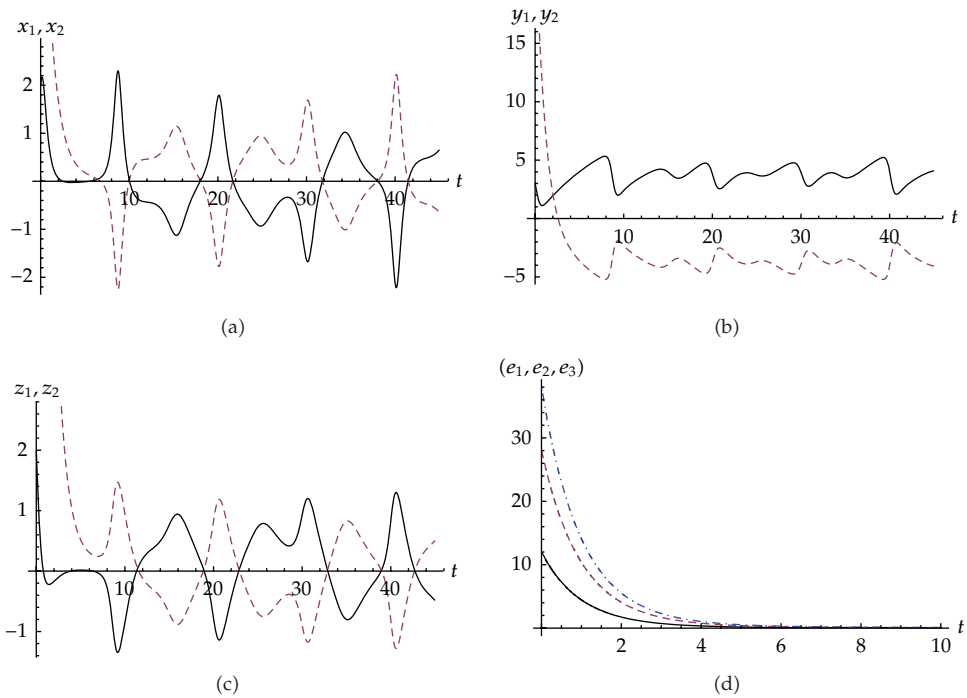

Figure 2: (a) Signals x_1, x_2, (b) Signals y_1, y_2, (c) Signals z_1, z_2, and (d) Error system.

shown by dashed lines. The errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line), and $e_3(t)$ (dot-dashed line) in the antisynchronization are shown in Figure 2(d).

6. Antisynchronization between Fractional Lü and Financial System

In this case, consider Lü system as the drive system

$$\begin{aligned} D^{\alpha} x_1 &= a_2(y_1 - x_1), \\ D^{\alpha} y_1 &= c_2 y_1 - x_1 z_1, \\ D^{\alpha} z_1 &= x_1 y_1 - b_2 z_1, \end{aligned} \tag{6.1}$$

and the response system as the Financial system

$$\begin{aligned} D^{\alpha} x_2 &= z_2 + (y_2 - a) x_2 + u_7, \\ D^{\alpha} y_2 &= 1 - b y_2 - x_2^2 + u_8, \\ D^{\alpha} z_2 &= -x_2 - c z_2 + u_9. \end{aligned} \tag{6.2}$$

Let $e_1 = x_1+x_2,\ e_2 = y_1+y_2$, and $e_3 = z_1+z_2$ be error functions. For antisynchronization, it is essential that the errors $e_i \to 0$ as $t \to \infty$. To achieve this one should choose the control terms $u_7,\ u_8,\ u_9$ properly. The error system thus becomes

$$\begin{aligned} D^\alpha e_1 &= (a-a_2)x_1 + a_2y_1 + x_1y_1 - z_1 - ae_1 - y_1e_1 - x_1e_2 + e_1e_2 + e_3 + u_7, \\ D^\alpha e_2 &= 1 - x_1^2 + (b+c_2)y_1 - x_1z_1 + 2x_1e_1 - e_1^2 - be_2 + u_8, \\ D^\alpha e_3 &= x_1 + x_1y_1 + (c-b_2)z_1 - e_1 - ce_3 + u_9. \end{aligned} \tag{6.3}$$

The control functions are chosen as

$$\begin{aligned} u_7 &= V_7 - (a-a_2)x_1 - a_2y_1 - x_1y_1 + z_1 + y_1e_1 + x_1e_2 - e_1e_2, \\ u_8 &= V_8 - 1 + x_1^2 - (b+c_2)y_1 + x_1z_1 - 2x_1e_1 + e_1^2, \\ u_9 &= V_9 - x_1 - x_1y_1 - (c-b_2)z_1. \end{aligned} \tag{6.4}$$

The linear functions $V_7,\ V_8,\ V_9$ are given by

$$\begin{aligned} V_7 &= (a-1)e_1 - e_3, \\ V_8 &= (-1+b)e_2, \\ V_9 &= e_1 + (c-1)e_3. \end{aligned} \tag{6.5}$$

With the values given in (6.4) and (6.5), the error system (6.3) becomes

$$\begin{pmatrix} D^\alpha e_1 \\ D^\alpha e_2 \\ D^\alpha e_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{6.6}$$

It can be observed that the coefficient matrix of the error system (6.6) has eigenvalues -1, $-1,\ -1$. So the system is stable and antisynchronization is achieved.

6.1. Simulations and Results

Parameters for the Lü system are $a_2 = 35,\ b_2 = 3,\ c_2 = 28$, whereas parameters for Financial system are unaltered. The initial conditions for drive system are $x_1(0) = 0.2,\ y_1(0) = 0,\ z_1(0) = 0.5$, whereas the initial conditions for response system are $x_2(0) = 2,\ y_2(0) = 3,\ z_2(0) = 2$. Hence the initial conditions for the error system (6.6) are $e_1(0) = 2.2,\ e_2(0) = 3,\ e_3(0) = 2.5$. We perform the numerical simulations for fractional order α, namely, 0.91 of the drive system (6.1) and response system (6.2). Figures 3(a), 3(b), and 3(c) show antisynchronization between fractional Lü and Financial system for $\alpha = 0.91$. Figure 3(d) shows the errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line), and $e_3(t)$ (dot-dashed line) in the antisynchronization for $\alpha = 0.91$.

Mathematica 7 has been used for computations in the present paper.

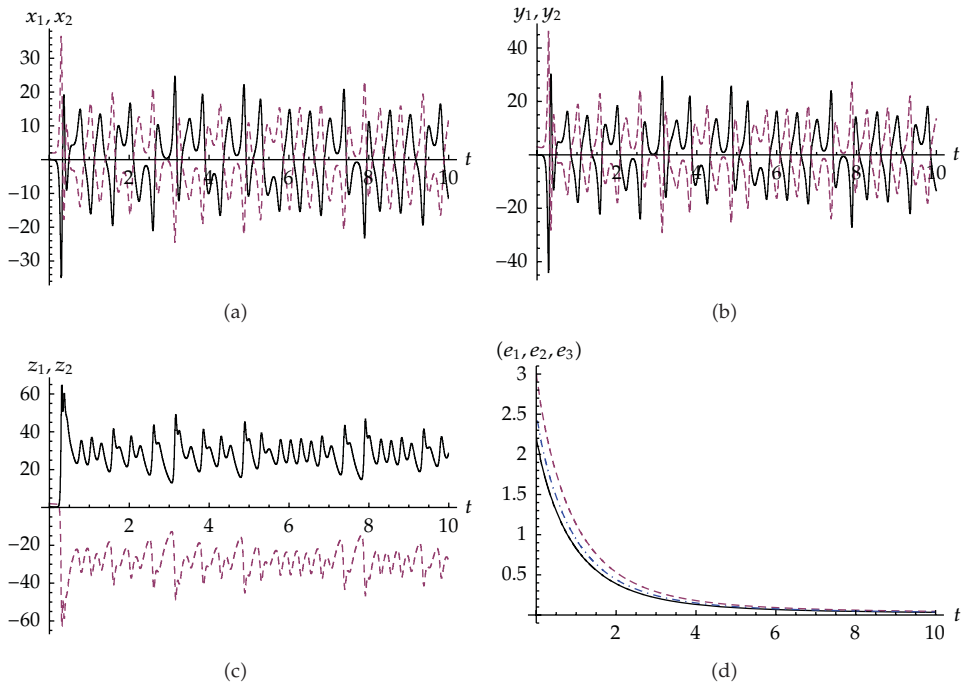

Figure 3: (a) $\alpha = 0.91$, Signals x_1, x_2, (b) $\alpha = 0.91$, Signals y_1, y_2, (c) $\alpha = 0.91$, Signals z_1, z_2, and (d) $\alpha = 0.91$, Error system.

7. Conclusions

Antisynchronization of nonidentical fractional-order chaotic systems has been done first time in the literature using active control. The fractional Financial system is controlled by fractional Lorenz system, the fractional Chen system is controlled by fractional Financial system, and the fractional Financial system is controlled by fractional Lü system.

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