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Editorial Board

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Editorial

Categorification in Representation Theory

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Categorification, a term coined by Louis Crane and Igor Frenkel, is the process of realizing mathematical structures as shadows of higher mathematics. The original motivation was the idea that one should be able to construct four-dimensional quantum field theories by categorifying the representation theory of quantum groups. Work of many people over the past 15 years has made great progress in clarifying and developing this vision. In that time, it has become increasingly clear that categorification is a broad mathematical phenomenon with applications extending far beyond these original motivations.

This special issue focuses on categorification in the context of representation theory of quantum groups and Hecke algebras, in the spirit of the examples of categorification coming from geometric representation theory and low-dimensional topology. A special emphasis is placed on the diagrammatic calculus created by Elias and Khovanov to describe the category of Seorgel bimodules in terms of planar diagrams.

One reason for the prominence of quantum groups and Hecke algebras in categorification is that they provide a bridge between representation theory and low-dimensional topology. Indeed, quantum groups and Hecke algebras can be viewed as the basic algebraic input giving rise to knot invariants such as the Jones polynomial, colored Jones polynomial, HOMFLYPT polynomial, and Witten-Reshetikhin-Turaev 3-manifold invariants. Khovanov showed that the Jones polynomial could be understood as the graded Euler characteristic of a graded homology theory for knots and links, thus opening the door to a new chapter of interaction between representation theory and low-dimensional topology. Besides being a strictly stronger knot invariant than its Euler characteristic, many of the knot homologies developed since Khovanov homology are functorial: cobordisms between tangles give rise to maps between homologies. This functoriality is explained by the rich structure of morphisms and natural transformations which exists in categorified quantum groups and Hecke algebras.

That representation theory has proven to be an especially fertile ground for categorification is a fact that owes much to the geometric methods pervading the subject. For example, constructions of natural bases with positivity and integrality properties are a central part of geometric representation theory. The categorifications that sit above such geometric constructions provide rich explanations for the existence of these bases: in the categorification, basis vectors are reinterpreted as indecomposable objects in a category, while structure constants become decomposition numbers or multiplicities. From this point of view, positivity and integrality are manifest. A particularly important object in geometric representation theory is the category of Soergel bimodules, which was used by Soergel to give a categorification of the Hecke algebra. The Elias-Khovanov description of the Soergel category also suggests a natural categorification of a quotient of the Hecke algebra known as the Temperley-Lieb algebra. Such a categorification is studied in the papers in this issue by B. Elias and B. Elias-Khovanov.

Constructions of knot homology theories associated to quantum $\mathfrak{sl}_n$ can be given using certain singular surfaces called foams. The papers by P. Vaz and M. Mackaay-P. Vaz in this issue construct representations of the Soergel category on the category of foams. These constructions clarify the representation theoretic meaning of foams in the construction of $\mathfrak{sl}_n$ knot homology theories. D. Krasner, in his paper, constructs knot homology from braid group actions built from the diagrammatic Soergel category, obtaining integral versions of HOMFLY-PT and $\mathfrak{sl}_n$-link homology theories.

J. Sussan and D. Hill's paper connects the Khovanov-Lauda diagrammatic categorification of $U_q(\mathfrak{sl}_n)$ with a previous categorification of the adjoint representation constructed by Khovanov-Huerfano. In particular, they obtain a categorification of the irreducible representations of highest weight $2\omega_k$, where the ω_k are the fundamental weights. The paper by A. Ram and P. Tingley uses the Shapovalov determinant for a universal Verma module to explain a connection between structure constants appearing in the Misra-Miwa Fock space and Weyl modules.

One theme common to several of the papers in this special issue is the use of a graphical calculus of planar diagrams in categorification. In these graphical presentations, the algebraic relations amongst morphisms are described by local geometry in the plane. While these diagrammatic constructions can be rigorously translated into a purely algebraic formulations, planar diagrams describe categorifications in a geometric and often quite intuitive manner.

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Research Article

Diagrammatics for Soergel Categories

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The monoidal category of Soergel bimodules can be thought of as a categorification of the Hecke algebra of a finite Weyl group. We present this category, when the Weyl group is the symmetric group, in the language of planar diagrams with local generators and local defining relations.

1. Introduction

In this paper [1], Soergel gave a combinatorial description of a certain category of Harish-Chandra bimodules over a simple Lie algebra $\mathfrak{g}$. This category was and continues to be of primary interest in the infinite-dimensional representation theory of simple Lie algebras. Soergel discovered a functor from this category to a full subcategory of bimodules over a certain ring R, the objects of which are now commonly called *Soergel bimodules*. The category of Soergel bimodules is additive and monoidal, unlike the original category which is abelian, but it still has sufficient information to describe the original category. Soergel constructed an isomorphism between the Grothendieck ring of his category and the integral form of the Hecke algebra of the Weyl group W of $\mathfrak{g}$. Hence, Soergel's construction provides a categorification of the Hecke algebra.

Given a k-dimensional $\mathbb{C}$-vector space V and a generic $q \in \mathbb{C}$, there are commuting actions of the quantum group $U_q(\mathfrak{sl}_k)$ and the Hecke algebra H of the symmetric group S_n on $V^{\otimes n}$. These actions turn the quotient $U_q(\mathfrak{sl}_k)/J_1$ of the quantum group and H/J_2 of the Hecke algebra by the kernels of these action into a dual pair. A categorical realization of the triple $(V^{\otimes n}, U_q(\mathfrak{sl}_k)/J_1, H/J_2)$ was given by Grojnowski and Lusztig [2] via categories of perverse sheaves on products of flag and partial flag varieties, also see [3–6].

Many foundational ideas about categorification were put forward by Igor Frenkel in the early 90s (a small fraction of these ideas formed a part of the paper [7]). In particular, Frenkel conjectured [8] that quantum groups and not just their finite-dimensional quotients $U_q(\mathfrak{sl}_k)/J_1$ can be categorified. These conjectures remained open until recently, when categorifications of quantum $\mathfrak{sl}_2$ and $\mathfrak{sl}_k$ were discovered in [9, 10], with a related but

different approach developed in [11, 12]. In the categorifications [9, 10] of quantum groups, 2 morphisms are given by linear combinations of planar diagrams, modulo local relations.

The parallel objective would be to categorify the Hecke algebra in the same spirit, using planar diagrams. Soergel had already provided a categorification, so it remains to ask whether his category can be rephrased diagrammatically. Diagrammatics should also provide a presentation of the category by generators and relations. A similar question was recently posed by Libedinsky [13], who essentially produced such a description for categorifications of Hecke algebras associated to "right-angled" Coxeter systems.

Here, we answer this question positively in the case of the Hecke algebra associated to the symmetric group. This is, of course, the Hecke algebra that appears in the Schur-Weyl duality for $V^{\otimes n}$. For notational convenience, we use $n+1$, not n, as our parameter and define a diagrammatical version $\mathfrak{DC}$ of the category of Soergel bimodules that categorifies the Hecke algebra of the symmetric group S_{n+1}.

In some sense, diagrammatic categorifications are very "low-tech," in that they can be described easily and do not rely on heavy machinery. While one can prove that Soergel bimodules categorify the Hecke algebra using only elaborate commutative algebra (see [14], although it is never stated explicitly), showing that indecomposable bimodules descend to the Kazhdan-Lusztig basis of the Hecke algebra utilizes Kazhdan-Lusztig theory [15, 16]. This, in turn, is related to fundamental developments in geometric representation theory like D-modules on flag manifolds [17, 18] and perverse sheaves [19]. One hopes that a diagrammatic approach will help to visualize and work with these sophisticated constructions, in the same way that the categorifications of quantum groups [9, 10] have led to an improved understanding of perverse sheaves on quiver varieties (see [20]). One also hopes that this approach can shed light on categorifications of representations of the Hecke algebra coming from the context of category $\mathcal{O}$ (see [21]).

We start with an intermediate category $\mathfrak{DC}_1$ whose objects are finite sequences $\underline{\mathbf{i}} = i_1 \cdots i_d$ of numbers between 1 and n. An object is represented graphically by marking d points in the standard position (say, having coordinates $1, \ldots, d$) on the x-axis and assigning labels $i_1, \ldots, i_d$ to marked points from left to right. Morphisms between $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ are given by linear combinations (with coefficients in a ground field $\Bbbk$) of planar diagrams embedded in the strip $\mathbb{R} \times [0,1]$. These diagrams are decorated planar graphs, where edges may extend to the boundary $\mathbb{R} \times \{0,1\}$. Each edge carries a label between 1 and n, so that the induced labellings of the lower and upper boundaries are $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$, respectively. In the interior of the strip, we allow

(1) vertices of valence 1,

(2) vertices of valence 3 with all 3 edges carrying the same label,

(3) vertices of valence 4 seen as intersections of i and j-labelled lines with $|i-j| > 1$,

(4) vertices of valence 6 with the edge labelling $i, i+1, i, i+1, i, i+1$, reading clockwise around the vertex,

(5) boxes labelled by numbers between 1 and $n+1$ which float in the regions of the graph.

We impose a set of local relations on linear combinations of these diagrams, including invariance of diagrams under all isotopies. A subset of the relations says that i is a Frobenius object in the category $\mathfrak{DC}_1$.

The space of morphisms in $\mathfrak{DC}_1$ between $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ is naturally a graded vector space. Allowing grading shifts and direct sums of objects, then restricting to grading-preserving

morphisms, and finally passing to the Karoubian closure of the category results in a graded $\Bbbk$-linear additive monoidal category $\mathfrak{DC}$. Our main result (Theorem 4.22 in Section 4) is an explicit equivalence between this category and the category $\mathcal{SC}$ of Soergel bimodules.

The category $\mathfrak{DC}$ is monoidal, and can be viewed as a 2-category with a single object. It may be easier to tackle the diagrammatics after reading an introductory reference on diagrammatics for 2-categories. Such an introduction can be found in [9]. This may make it easier to explore similarities with the categorifications of quantum groups in [9, 10], where regions of diagrams are labelled by integers in [9] and integral weights of $\mathfrak{sl}_n$ in [10]. Boxes floating in regions are superficially analogous to floating bubbles of [9, 10]. Unlike the diagrammatic categorifications in [9, 10], our lines do not carry dots and are not oriented.

There is another way to view our diagrammatics, which is not developed in this paper. Rouquier [22, 23] defined an action of the Coxeter braid group associated to W on the category of complexes of Soergel bimodules up to homotopies, which is related to a braid group action using Harish-Chandra bimodules that had been known for some time. These complexes were later used in an alternative construction [24] of a triply graded link homology theory [25] categorifying the HOMFLY-PT polynomial [25–28]. In this approach, a product Soergel bimodule $B_{i_1} \otimes \cdots \otimes B_{i_d}$ is depicted by a planar diagram given by concatening elementary planar diagrams lying in the xy-plane that consist of $n+1$ strands going up, with i and $i+1$-st strands merging and splitting, see [24, Figures 2 and 3]. Morphisms between product bimodules can be realized by linear combinations of foams—decorated two-dimensional CW-complexes embedded in $\mathbb{R}^3$ with suitable boundary conditions. Foams have been implicit throughout papers on triply graded link homology (see [29] for instance, where various arrows between planar diagrams can be implemented by foams), and explicitly appear in the papers on their doubly-graded cousins, see [30, 31] and references therein.

Foams are 3-dimensional objects; they are two-dimensional CW-complexes embedded in $\mathbb{R}^3$ that produce cobordisms along the z-axis direction between planar objects corresponding to product Soergel bimodules. The planar diagrams of our paper are two-dimensional encodings of these foams, essentially projections of the foams onto the yz-plane along the x-axis.

It was shown in [32] that the action of the braid group on the homotopy category of Soergel bimodules extends to a (projective) action of the category of braid cobordisms. Thus, the homotopy category of $\mathfrak{DC}$ produces invariants of braid cobordisms, so that our planar diagrammatics carry information about four-dimensional objects. This informational density indicates the efficiency of such encodings.

Addendum 1. *Since this paper first appeared, the diagrammatics developed here have led to several developments which we briefly mention here. In [33] it is shown that Rouquier's braid group action lifts functorially to the braid cobordism category. In [34, 35], a functor is given from the category $\mathfrak{DC}$ to categories of foams used in link homology. Together, these papers show that the encodings mentioned above are more than simply heuristic. In [36], the Temperley-Lieb algebra is categorified as a quotient of $\mathfrak{DC}$. Additional statements relating this paper to either newer papers or to previous versions of this paper are found sparsely under a similar "Addendum" heading.*

2. Preliminaries

Henceforth, we will fix a positive integer n. Indices i, j, and k will range over $1, \ldots, n$ if not otherwise specified. Finite ordered sequences of such indices (allowing repetition) will be denoted $\underline{\mathbf{i}} = i_1 \cdots i_d$, as well as $\underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$. The length of the sequence will be denoted $d = d(\underline{\mathbf{i}})$.

For sequences of length $d = 1$ where the single entry is i, we use i and $\underline{\mathbf{i}}$ interchangeably. Occasionally $i+1$ will also be used as an index, and whenever this occurs we make the tacit assumption that $i \leq n-1$, so that all indices used remain between 1 and n. The same goes for $i-1$, $i+2$, and the like. We denote the length 0 sequence by the empty set symbol $\emptyset$.

We work over a field $\Bbbk$, usually assuming that Char $\Bbbk \neq 2$, and sometimes specializing it to $\mathbb{c}$.

Given a noetherian ring R, the category R-molf-R is the full subcategory of R-bimodules consisting of objects which are finitely generated as left R-modules. If R is graded, the category R-molf$_{\mathbb{Z}}$-R is the analogous subcategory of graded R-bimodules and grading-preserving homomorphisms.

2.1. Hecke Algebra

Let $(W, \mathcal{S})$ be a Coxeter system of a finite Weyl group W, with length function $l : W \to \mathbb{n} = \{0, 1, 2, \ldots\}$, and let $e \in W$ be the identity. The Hecke algebra H is an algebra over $\mathbb{Z}[v, v^{-1}]$ (we follow Soergel's use [37] of the variable v; related variables are denoted in the literature by $t = v^{-1}$ and $q = t^2$), which is free as a module with basis T_w, $w \in W$. Multiplication in this basis is given by $T_x T_w = T_{xw}$ when $l(v) + l(w) = l(vw)$, and $T_s^2 = (v^{-2} - 1)T_s + v^{-2}T_e$ for $s \in \mathcal{S}$. T_e is the identity element in H and will often be written as 1.

In the case we are interested in presently, $W = S_{n+1}$, and $\mathcal{S}$ consists of the transpositions $s_i = (i, i+1)$ for $i = 1, \ldots, n$. The element T_{s_i} will be denoted T_i. The Hecke algebra has a presentation over $\mathbb{Z}[v, v^{-1}]$, being generated by T_i subject to the relations

$$\begin{gathered} T_i^2 = \left(v^{-2} - 1\right)T_i + v^{-2}, \\ T_i T_j = T_j T_i \quad \text{for } |i - j| \geq 2, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}. \end{gathered} \tag{2.1}$$

Clearly then, H is also generated as an algebra by $b_i \overset{\text{def}}{=} C'_{s_i} = v(T_i + 1)$, $1 \leq i \leq n$, and the relations above transform into

$$\begin{gathered} b_i^2 = \left(v + v^{-1}\right)b_i, \\ b_i b_j = b_j b_i \quad \text{for } |i - j| \geq 2, \\ b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i. \end{gathered} \tag{2.2}$$

We often write the monomial $b_{i_1} b_{i_2} \cdots b_{i_d}$ as $b_{\underline{\mathbf{i}}}$ where $\underline{\mathbf{i}} = i_1 \cdots i_d$. Notice that $b_\emptyset = 1$.

Let $a \mapsto \overline{a}$ be the involution of $\mathbb{Z}[v, v^{-1}]$ determined by $\overline{v} = v^{-1}$. It extends to an involution of H given by

$$\overline{\sum a_w T_w} = \sum \overline{a_w} T_{w^{-1}}^{-1}. \tag{2.3}$$

In particular, $\overline{T_i} = T_i^{-1} = v^2 T_i + v^2 - 1$.

Kazhdan and Lusztig [15] defined a pair of bases $\{C_w\}_{w \in W}$ and $\{C'_w\}_{w \in W}$ for H, which immediately proved to be of fundamental importance for representation theory and

combinatorics. The two bases are related via a suitable involution of H, and the elements of the second Kazhdan-Lusztig basis $\{C'_w\}_w$ are determined by the two properties

$$\overline{C'_w} = C'_w,$$
$$C'_w = v^{l(w)} \sum_{y \leq w} P_{y,w} T_y, \tag{2.4}$$

where $P_{y,w} \in \mathbb{Z}[v^{-2}]$ has negative v-degree strictly less than $l(w) - l(y)$ for $y < w$ and $P_{w,w} = 1$. There is no simple formula expressing C'_w in terms of T_y, but observe that $C'_e = 1$ and $C'_{s_i} = b_i = v(T_i + 1)$. For a good introduction to the Kazhdan-Lusztig basis, see [37].

Let $\varepsilon : H \to \mathbb{Z}[v, v^{-1}]$ be the $\mathbb{Z}[v, v^{-1}]$-linear map given by $\varepsilon(T_e) = 1$ and $\varepsilon(T_w) = 0$ if $w \neq e$. Thus, ε simply picks up the coefficient of T_e in x. The easily checked property $\varepsilon(T_i x) = \varepsilon(x T_i)$ for any $x \in H$ implies that $\varepsilon(xy) = \varepsilon(yx)$, for all $x, y \in H$, so that ε is a trace map and turns H into a symmetric Frobenius $\mathbb{Z}[v, v^{-1}]$-algebra.

Denote by ω a v-antilinear anti-involution $\omega : H \to H$ defined uniquely by $\omega(b_i) = b_i$. The anti-involution and v-antilinearity conditions say that $\omega(xy) = \omega(y)\omega(x)$ and $\omega(ax) = \overline{a}\omega(x)$, for $x, y \in H$ and $a \in \mathbb{Z}[v, v^{-1}]$.

Consider the pairing $(,) : H \times H \to \mathbb{Z}[v, v^{-1}]$ of $\mathbb{Z}$-modules given by

$$(x, y) = \varepsilon(\omega(x) y). \tag{2.5}$$

It satisfies the following properties:

(1) the pairing is *semi-linear*, that is, $(ax, y) = \overline{a}(x, y)$ while $(x, ay) = a(x, y)$, for $a \in \mathbb{Z}[v, v^{-1}]$,

(2) b_i is self-adjoint, that is, $(x, b_i y) = (b_i x, y)$ and $(x, y b_i) = (x b_i, y)$,

(3) if $\underline{\mathbf{i}} = i_1 \cdots i_d$ with $i_1 < i_2 < \cdots < i_d$ then $(1, b_{\underline{\mathbf{i}}}) = v^d$. Such a monomial $b_{\underline{\mathbf{i}}}$ is called an *increasing monomial*, and $\underline{\mathbf{i}}$ an *increasing sequence*. When $d = 0$, the sequence $\underline{\mathbf{i}}$ is empty and $(1, 1) = 1$.

Remark 2.1. It is not difficult to observe that $(,)$ is the unique form satisfying these three properties. This is because the Hecke algebra has a spanning set over $\mathbb{Z}[t, t^{-1}]$ consisting of monomials $b_{\underline{\mathbf{i}}}$, and every monomial may be reduced, by cycling the last b_i to the beginning and by applying the Hecke algebra relations, to an increasing monomial. This is a simple combinatorial argument that we leave to the reader.

2.2. Soergel Bimodules

In [1], Soergel introduced a category of bimodules which categorified the Hecke algebra, and later generalized his construction to any Coxeter group W [14]. Within the category R-molf$_{\mathbb{Z}}$-R, for R a certain graded $\Bbbk$-algebra ($\Bbbk$ an infinite field of characteristic $\neq 2$), he identified indecomposable modules B_w for $w \in W$, such that the only indecomposable summands of tensor products of B_w's are $B_{w'}$ for $w' \in W$. Thus, the subcategory of R-molf$_{\mathbb{Z}}$-R generated additively by the B_w has a tensor product, and its Grothendieck ring is isomorphic to H, under the isomorphism sending C'_s to $[B_s]$. Moreover, every B_w shows up as a summand of some tensor product of various B_s for $s \in \mathcal{S}$. While the general B_w may be difficult to describe, B_s has an easy description.

It is conjectured in [14] that this isomorphism sends $[B_w]$ to C'_w for all $w \in W$, and it is proven for $\Bbbk = \mathbb{C}$ and W a Weyl group in [1], using geometric methods.

Henceforth we specialize to the case where $W = S_{n+1}$ and $\mathcal{S} = \{s_i\}$. We also make one additional change from Soergel's conventions.

Remark 2.2. Soergel defines R to be the coordinate ring of the n-dimensional reflection representation V of S_{n+1}, while we find it easier to consider R', the coordinate ring of the $n+1$-dimensional standard representation V'. This is akin to treating $\mathfrak{gl}_n$ instead of $\mathfrak{sl}_n$, and a similar convention is adopted in [24]. The bimodules B_w are defined in [14] to be the coordinate rings of unions of "twisted diagonals" in $V \times V$, and B'_w can be defined analogously for V'. Now $R' \cong R \otimes_{\Bbbk} \Bbbk[y]$ and the entire story of B_w translates to B'_w by base extension. Conversely, R is a quotient of R' by the first elementary symmetric polynomial e_1, which is a central element of our category, so that the entire story of B'_w translates easily to B_w under the quotient. We will interest ourselves with the B'_w story below because the ring R' is slightly more intuitive, and mention briefly the changes required to deal with R in Section 4.6. Since we only use B'_w and R' below, we will denote them as B_w and R instead to avoid an apostrophe catastrophe.

With these conventions, we now make the story explicit.

Notation 2.3. Let $R = \Bbbk[x_1, x_2, \ldots, x_{n+1}]$ be the ring of polynomials in $n+1$ variables, with the natural action of S_{n+1}. The ring R is graded, with $\deg(x_i) = 2$. If W is the subgroup of S_{n+1} generated by transpositions $\{s_{i_1}, \ldots, s_{i_r}\}$, then we denote the ring of invariants under W as $R^{i_1,\ldots,i_r}$ or R^W. Thus R^i are the invariants under the transposition $(i, i+1)$.

Since R is an R^W-algebra, $\otimes_{R^W}$ is a bifunctor sending two R-bimodules to an R-bimodule. Henceforth, $\otimes$ with no subscript denotes tensoring over R, while $\otimes_{i_1,\ldots,i_r}$ denotes tensoring over the subring $R^{i_1,\ldots,i_r}$. Most commonly we will just use $\otimes_i$ for various indices i.

Definition 2.4. The Soergel bimodule B_i is $R \otimes_i R\{-1\}$, where $\{m\}$ denotes a grading shift.

Notation 2.5. We denote by $B_{\underline{\mathbf{i}}}$ the tensor product $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_d}$.

Note that $B_\emptyset = R$ and

$$B_{\underline{\mathbf{i}}} = (R \otimes_{i_1} R\{-1\}) \otimes (R \otimes_{i_2} R\{-1\}) \otimes \cdots = R \otimes_{i_1} R \otimes_{i_2} \cdots \otimes_{i_d} R\{-d\}. \tag{2.6}$$

We reiterate this important point: a typical element of a tensor product of d generators B_i can be expressed (up to linear combination) by a choice of $d+1$ polynomials, one in each slot separated by the tensors. Multiplication by an element of R in any particular slot is an R-bimodule endomorphism.

For each i there is a map of graded vector spaces $\partial_i : R \to R^i\{2\}$, called the *Demazure operator*, sending $f \mapsto (f - s_i(f))/(x_i - x_{i+1})$. This map is R^i-linear. Since $\partial_i(f)$ is s_i-invariant, it is not hard to see that $P_i(f) = f - x_i \partial_i(f)$ is also s_i-invariant. Since $f = P_i(f) + x_i \partial_i(f)$, this implies that R is a free graded R^i-module of rank two, with homogeneous generators 1 and x_i. In other words, there is an isomorphism $R \cong R^i \oplus R^i\{2\}$ of graded R^i-modules, sending $f \mapsto (P_i(f), \partial_i(f))$, with inverse $(f, g) \mapsto f + g x_i$.

From the isomorphism $R \cong R^i \oplus R^i\{2\}$ of graded R^i-modules just illustrated, one can deduce other isomorphisms. For instance, $B_i = R \otimes_i R\{-1\} \cong R\{-1\} \oplus R\{1\}$ as graded left (or right) R-modules. Repeating this, we see that $B_{\underline{\mathbf{i}}}$ is a free left R-module of rank $2^{d(\underline{\mathbf{i}})}$, and

properly belongs in R-molf$_{\mathbb{Z}}$-R. Finally, we can deduce an isomorphism $B_i \otimes_i B_i \cong B_i\{1\} \oplus B_i\{-1\}$, which unlike the previous isomorphisms is actually an isomorphism of R-bimodules.

Remark 2.6. Let us make this slightly more explicit. To give the isomorphism of left R-modules $B_i \cong R\{-1\} \oplus R\{1\}$, note that

$$f \otimes g = f \otimes P_i(g) + f \otimes x_i \partial_i(g) = P_i(g) f \otimes 1 + \partial_i(g) f \otimes x_i. \tag{2.7}$$

Rewriting a term $1 \otimes g$ as a sum of terms like above will happen often, and we refer to it as *forcing* the polynomial g across the tensor. If g is s_i-invariant then it may be slid across leaving nothing behind, while an arbitrary g when forced leaves terms with either 1 or x_i behind (alternatively, we may choose to leave 1 and x_{i+1} behind, if it is more convenient). We consistently use the term "slide" instead of "force" when the polynomial g is invariant so it can be moved across without any ado.

Inside $B_i \otimes B_i$, taking an element $f \otimes g \otimes h$ and forcing g to the left (or right) is now an R-bilinear operation, since multiplication on the left or right will only affect f or h, not g. This gives the isomorphism $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$ of R-bimodules, via $f \otimes g \otimes h \mapsto (f P_i(g) \otimes h, f \partial_i(g) \otimes h)$ with inverse $(f_1 \otimes h_1, f_2 \otimes h_2) \mapsto f_1 \otimes 1 \otimes h_1 + f_2 \otimes x_i \otimes h_2$. These maps are R-bimodule morphisms, since the only polynomials which can slide from f_i to h_i in both the source and the target of the map are those polynomials in R^i.

Remark 2.7. We also remark on spanning sets for $B_{\underline{\mathbf{i}}}$ as R-bimodules. For instance, we've seen that $B_i \otimes B_i$ has a spanning set $\{1 \otimes 1 \otimes 1,\ 1 \otimes x_i \otimes 1\}$. The bimodule $B_i \otimes B_j$ for $j \neq i$ has a spanning set $\{1 \otimes 1 \otimes 1\}$, since any polynomial in the middle can be forced to the left leaving at most x_i behind (or x_{i+1}, which we choose when $j = i - 1$), and that can be slid to the right; thus $1 \otimes 1 \otimes 1$ generates it as a R-bimodule. The bimodule $B_i \otimes B_j \otimes B_k \otimes B_i$ has a spanning set $\{1 \otimes 1 \otimes 1 \otimes 1 \otimes 1,\ 1 \otimes x_i \otimes 1 \otimes 1 \otimes 1\}$. This is because all polynomials anywhere between the two i tensors may be slid out, leaving x_i somewhere in-between. As an exercise, the reader may generalize this argument to an arbitrary $B_{\underline{\mathbf{i}}}$ and find a spanning set as a R-bimodule, consisting of $2^{m(\underline{\mathbf{i}})}$ terms, where $m(\underline{\mathbf{i}})$ is the number of pairs $1 \leq r < s \leq d$ such that $i_r = i_s$ and $i_t \neq i_s$ for t between r and s. Between such a pair, one either places a linear "unslideable" term like x_{i_r}, or just 1. Note that $m(\underline{\mathbf{i}})$ is equal to $d(\underline{\mathbf{i}})$ minus the number of distinct indices in $\underline{\mathbf{i}}$.

For more information on Soergel bimodules and their applications we refer the reader to [1, 22, 24, 38–44] and references therein.

2.3. The Soergel Categorification

Let us summarize the Soergel categorification of the Hecke algebra H of the symmetric group S_{n+1} and the various structures on it, following [1, 14, 22, 38, 41].

Several subcategories of R-molf-R and R-molf$_{\mathbb{Z}}$-R will play a role in what follows. Let $\mathcal{SC}_1$ be the full subcategory of R-molf-R whose objects consist of $B_{\underline{\mathbf{i}}}$ for all sequences of indices $\underline{\mathbf{i}}$; these are called *Bott-Samelson bimodules*. Since R is a commutative ring, the Hom spaces in $\mathcal{SC}_1$ are in fact enriched in R-molf$_{\mathbb{Z}}$-R. Let $\mathcal{SC}_2$ be the subcategory of R-molf$_{\mathbb{Z}}$-R whose objects are finite direct sums of various graded shifts of objects in $\mathcal{SC}_1$ and the morphisms are all grading-preserving bimodule homomorphisms. Finally, let $\mathcal{SC}$ be the Karoubi envelope of $\mathcal{SC}_2$, a category equivalent to the full subcategory of R-molf$_{\mathbb{Z}}$-R which contains all summands of objects of $\mathcal{SC}_2$

$$\mathcal{SC}_1 \xrightarrow{\text{Grading shifts and direct sums}} \mathcal{SC}_2 \xrightarrow{\text{Karoubi envelope}} \mathcal{SC}. \tag{2.8}$$

In general, the Karoubi envelope is the category which formally includes all "summands," where a summand of an object M is identified by an idempotent $p \in \mathrm{End}(M)$ corresponding to projection to that summand. Since R-molf$_{\mathbb{Z}}$-R is abelian, it is idempotent-closed, and the Karoubi envelope of $\mathcal{SC}_2$ can be realized as a subcategory in R-molf$_{\mathbb{Z}}$-R. We refer the reader to [45] for basic information about Karoubi envelopes.

This final category $\mathcal{SC}$ (the category of *Soergel bimodules*) is a $\Bbbk$-linear additive monoidal category with the Krull-Schmidt property. Soergel showed that, when $\Bbbk$ is an infinite field of characteristic other than 2, the indecomposable bimodules in this category are enumerated by elements of the Weyl group and grading shifts (Theorem 6.16 in [14]). They are denoted by $B_w\{j\}$ for $w \in W$ and $j \in \mathbb{Z}$. An indecomposable B_w is determined by the condition that it appears as a direct summand of $B_{\underline{\mathbf{i}}}$, where $\underline{\mathbf{i}} = i_1 \cdots i_d$ and $s_{i_1} \cdots s_{i_d}$ is a reduced presentation of w, and does not appear as direct summand of any $B_{\underline{\mathbf{i}}}$, for sequences $\underline{\mathbf{i}}$ of length less than $l(w)$.

The Hecke algebra H is canonically isomorphic to $K_0(\mathcal{SC})$, the Grothendieck group of $\mathcal{SC}$. Multiplication by v corresponds to the grading shift: $[M\{d\}] = v^d[M]$. Multiplication in the Hecke algebra corresponds to the tensor product of bimodules:

$$[M] \cdot [N] := [M \otimes_R N]. \tag{2.9}$$

The isomorphism takes $[B_i]$ to b_i and $[B_{\underline{\mathbf{i}}}]$ to $b_{\underline{\mathbf{i}}}$.

Remark 2.8. Nothing prevents one from defining a category $\mathcal{SC}_{\mathbb{Z}}$ where the field $\Bbbk$ is replaced with $\mathbb{Z}$ in the definitions of the previous section. Thus one could define the category for any ring. However, one does not have control over the size of the Grothendieck group in this instance. When defined over a field $\Bbbk$ of characteristic $\neq 2$, we may use Theorem 6.16 of [14] to classify indecomposables and get results about the Grothendieck group. When $\Bbbk = \mathbb{C}$, Soergel has shown that the Kazhdan-Lusztig basis $\{C'_w\}$ satisfies $C'_w = [B_w]$. This is unknown in general.

Relation (2.2) lifts to isomorphisms of graded bimodules in $\mathcal{SC}$

$$B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}, \tag{2.10}$$

$$B_i \otimes B_j \cong B_j \otimes B_i \quad \text{for } |i-j| \geq 2, \tag{2.11}$$

$$(B_i \otimes B_{i+1} \otimes B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes B_i \otimes B_{i+1}) \oplus B_i. \tag{2.12}$$

It is important to note that these isomorphisms take place in $\mathcal{SC}_2$, not $\mathcal{SC}_1$, since the latter does not have grading shifts or direct sums of objects. However, the same information can be encapsulated in inclusion and projection morphisms of various degrees, which do live in $\mathcal{SC}_1$. This will be explored in Section 4.5.

The first isomorphism has already been made explicit in Remark 2.6. We have chosen a specific isomorphism; other choices were possible. The second and third isomorphisms come from the following isomorphisms in R-molf$_{\mathbb{Z}}$-R:

$$B_i \otimes B_j \cong R \otimes_{i,j} R\{-2\} \cong B_j \otimes B_i \quad \text{for } |i-j| \geq 2, \tag{2.13}$$

$$B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus (R \otimes_{i,i+1} R\{-3\}), \tag{2.14}$$

$$B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus (R \otimes_{i,i+1} R\{-3\}). \tag{2.15}$$

These isomorphisms will be made explicit in Section 4.5.

The ring $\mathbb{Z}[v, v^{-1}]$ is canonically isomorphic to the Grothendieck group of the category R-fmod of finitely generated graded free R-modules. Under this isomorphism

$$K_0(R\text{-fmod}) \cong \mathbb{Z}\left[v, v^{-1}\right] \tag{2.16}$$

$[R]$ goes to 1 and $[R\{d\}]$ to v^d. In particular, given any graded free R-module M, its image in the Grothendieck group will be its *graded rank*, calculated by choosing a homogeneous R-basis $\{y_j\}$ of M and letting $\operatorname{grk} M \overset{\text{def}}{=} \sum_j v^{\deg y_j}$. The bar involution on $\mathbb{Z}[v, v^{-1}]$ lifts to the contravariant equivalence that takes $M \in R$-fmod to $\operatorname{Hom}_R(M, R)$, the latter naturally viewed as an R-module.

In the category $\mathcal{SC}_1$, given any objects M, N, the space $\operatorname{Hom}_{\mathcal{SC}_1}(M, N)$ is a graded free finitely generated left R-module. By extension, the same is true of the module $\oplus_{m\in\mathbb{Z}}\operatorname{Hom}_{\mathcal{SC}}(M\{m\}, N)$. Shifting the grading of N will shift the grading of this Hom space in the same direction, while shifting M will shift the Hom space in the opposite direction. Therefore, the bifunctor

$$\operatorname{Hom}_{\mathcal{SC}}(\cdot,\cdot) : \mathcal{SC}^{\text{op}} \times \mathcal{SC} \longrightarrow R\text{-fmod} \tag{2.17}$$

categorifies a semilinear form $H \times H \to \mathbb{Z}[v, v^{-1}]$ which sends

$$([M], [N]) \longmapsto \operatorname{grk}(\oplus_{m\in\mathbb{Z}}\operatorname{Hom}_{\mathcal{SC}}(M\{m\}, N)). \tag{2.18}$$

The bimodule B_i self-biadjoint, that is, that $\operatorname{Hom}_{\mathcal{SC}_1}(M \otimes B_i, N) = \operatorname{Hom}_{\mathcal{SC}_1}(M, N \otimes B_i)$ and $\operatorname{Hom}_{\mathcal{SC}_1}(B_i \otimes M, N) = \operatorname{Hom}_{\mathcal{SC}_1}(M, B_i \otimes N)$ via some adjunction maps. This will become explicit in Section 3.1. In fact, every bimodule M in $\mathcal{SC}$ has a biadjoint bimodule $\Omega(M)$ such that tensoring with M on the left (resp., right) is biadjoint to tensoring with $\Omega(M)$ on the left (resp., right). Due to a cyclicity property (see the next section) any homomorphism $f : M \to N$ of bimodules dualizes to a canonical homomorphism $\Omega(f) : \Omega(N) \to \Omega(M)$, so that Ω can be made into an antiequivalence of $\mathcal{SC}$, lifting the anti-involution ω. Notice that Ω takes B_i to itself and $B_{\underline{\mathbf{i}}}$ to $B_{\underline{\mathbf{j}}}$ where $\underline{\mathbf{j}}$ is given by reading $\underline{\mathbf{i}}$ from right to left.

Unsurprisingly, the semilinear product on H above (induced by the Hom bifunctor) agrees with the one defined in Section 2.1. To check this, following Remark 2.1 and using the self-biadjointness of B_i, we only need to show the following claim.

Claim 2.9. When $\underline{\mathbf{i}}$ is an increasing sequence, $\operatorname{Hom}_{\mathcal{SC}_1}(R, B_{\underline{\mathbf{i}}})$ is a free left R-module of rank 1, generated by a morphism of degree d, the length of $\underline{\mathbf{i}}$.

Proof. We only sketch this result. An R-bimodule map from R to B_i is clearly determined by an element of B_i on which right and left multiplication by polynomials in R are identical. Any element of B_i is of the form $m = f \otimes 1 + g \otimes x_i$, and clearly $x_j m = m x_j$ for $j \neq i, i+1$, and $(x_i + x_{i+1})m = m(x_i + x_{i+1})$. Hence m can be the image of 1 under a bimodule map from R if and only if $x_i m = m x_i$.

$$m x_i = f \otimes x_i + g \otimes x_i^2 = f \otimes x_i + g(x_i + x_{i+1}) \otimes x_i - g(x_i x_{i+1}) \otimes 1. \tag{2.19}$$

Thus m can be an image if and only if $g(x_i x_{i+1}) = -f x_i$ and $f + g(x_i + x_{i+1}) = g x_i$, if and only if $f = -g x_{i+1}$. Such elements form a left R-module generated by the case $g = 1$, $f = -x_{i+1}$, or in other words by $m = 1 \otimes x_i - x_{i+1} \otimes 1$. The element m has degree 1 in B_i, so we deduce that $(R, B_i) = v$. Let us call φ_i the corresponding map $R \to B_i$, $1 \mapsto m$.

One may use the same argument for the general case. Suppose $\underline{\mathbf{i}}$ is increasing and has length d, then $B_{\underline{\mathbf{i}}}$ is a free left R-module of degree 2^d, with generators $\{1 \otimes_{i_i} f_1 \cdots \otimes_{i_k} f_k\}$ ranging over terms where either $f_l = 1$ or $f_l = x_{i_l}$. As an exercise, the reader may find the criteria for a general element to be ad-invariant under x_i, and verify that the only possible bimodule maps $R \to B_{\underline{\mathbf{i}}}$ are R-multiples of the following iterated version of φ_i:

$$R \longrightarrow B_{i_1} \cong B_{i_1} \otimes R \longrightarrow B_{i_1} \otimes B_{i_2} \cong \cdots . \tag{2.20}$$

The first map is φ_{i_1}, the second map is $\mathrm{Id} \otimes \varphi_{i_2}$, and so forth. This generator is a map of degree d, so that $\mathrm{Hom}_{\mathcal{SC}_1}(R, B_{\underline{\mathbf{i}}}) = R\{d\}$ and $([R], [B_{\underline{\mathbf{i}}}]) = v^d$. □

Remark 2.10. We have swept the calculation under the rug, so the dependence of this claim on the fact that $\underline{\mathbf{i}}$ is increasing is unclear. In general, when $\underline{\mathbf{i}}$ has a repeated index there will be additional maps from R to $B_{\underline{\mathbf{i}}}$. Roughly speaking, certain symmetry conditions are placed upon polynomials in order for them to slide across certain tensors. The duplication of an index will yield a redundant symmetry condition that places fewer constraints on an ad-invariant element than would be expected from the length of the sequence. We suggest the reader try to find all the maps from R to $B_{\underline{\mathbf{i}}}$ in the length 2 case, first when $\underline{\mathbf{i}} = ij$ has no repeated index and then when $\underline{\mathbf{i}} = ii$ has a repeated index. This should illustrate the main idea.

Because it is a crucial statement which we use again and again, we restate the overall result and give a reference.

Proposition 2.11. *Given two sequences $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$, $Hom_{\mathcal{SC}}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}})$ is a free graded left (or right) R-module of rank $\varepsilon(\omega(b_{\underline{\mathbf{i}}})b_{\underline{\mathbf{j}}})$, where ε is the standard trace on H defined in Section 2.1.*

Proof. This is deduced from the above discussion. For Soergel's proof, see [14], although it is somewhat obscured. Theorem 5.15 in that paper and especially its proof together state this result, once one unravels exactly what h_Δ means. Propositions 5.7 and 5.9 state that $h_\Delta(B_{\underline{\mathbf{i}}}) = b_{\underline{\mathbf{i}}}$, since h_Δ sends $B_i \otimes \cdot$ to $b_i \cdot$ and sends R to 1. □

The facts below will not be used in this paper.

For a Soergel bimodule M the space of bimodule homomorphisms $\mathrm{Hom}_{\mathcal{SC}_1}(R, M)$ is just the 0th Hochschild cohomology $\mathrm{HH}^0(R, M)$ of M. Thus, unraveling the definitions,

$$\varepsilon([M]) := \mathrm{grk}\Big(\mathrm{HH}^0(R, M)\Big). \tag{2.21}$$

Calculations in Hochschild cohomology can be used to provide a proof of the claim above. One could also define a trace map $\tau(x) = \overline{\varepsilon(\omega(x))}$ which is the decategorification of the functor HH_0 of taking the 0th Hochschild homology.

Hecke algebra H has a trace more sophisticated than ε or τ, called the Ocneanu trace [46], which describes the HOMFLY-PT polynomial. The categorification of the Ocneanu trace utilizes all Hochschild homology groups rather than just HH_0, see [24, 43, 47].

The Rouquier complexes mentioned in the introduction are described here. Invertible elements T_i that satisfy the braid relations become [22] invertible complexes

$$0 \longrightarrow R \longrightarrow B_i\{-1\} \longrightarrow 0, \tag{2.22}$$

in the homotopy category of the Soergel category (with R sitting in cohomological degree -1). This aligns with the fact that $T_i = v^{-1}b_i - 1$ in the Hecke quotient of the braid group. Their inverses T_i^{-1} become inverse complexes

$$0 \longrightarrow B_i\{1\} \longrightarrow R \longrightarrow 0, \tag{2.23}$$

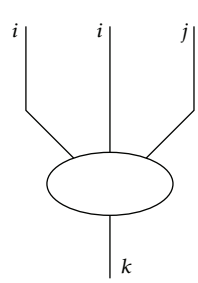
with R in cohomological degree 1, agreeing with $T_i^{-1} = vb_i - 1$. The homomorphism from the braid group into the Hecke algebra is categorified by a projective functor from the category of braid cobordisms between $(n+1)$-stranded braids to the category of endofunctors of the homotopy category of the Soergel category [32].

Remark 2.12. Note that this convention for Rouquier complexes is opposite that found in [22], which is to say that we have flipped T_i with T_i^{-1}. Presumably this arises because we are using v as a parameter for the grading shift, and not $t = v^{-1}$. The choice of v is more natural for the calculation of graded ranks of Hom spaces.

2.4. Diagrammatic Calculus for Bimodule Maps

We follow the standard rules for the diagrammatic calculus of bimodules, or more generally for the diagrammatic calculus of a monoidal category. An excellent and thorough explanation of these rules can be found in [9], so we will provide a quick summary. A planar diagram will represent a morphism of R-bimodules, with the following conventions. A horizontal slice or line segment in this diagram will represent an object (an R-bimodule). A rectangle inside the plane will represent a morphism from its bottom horizontal line segment to its top horizontal line segment.

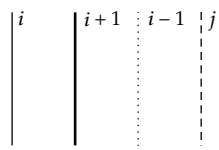
The R-bimodule B_i is denoted by a point (on a horizontal line segment) labelled i. The tensor product of bimodules is depicted by a sequence of labelled points on a horizontal line segment, so that tensor products are formed "horizontally". A vertical line labelled i denotes the identity endomorphism of B_i, and similarly labelled lines placed side by side denote the identity endomorphism of the tensor product. More general bimodule maps are represented by some symbols connecting the appropriate lines, and are composed "vertically", and tensored "horizontally". All diagrams are read from bottom to top, so that the following diagram represents a bimodule map from B_k to $B_i \otimes B_i \otimes B_j$:



A horizontal line segment which does not contain any marked points represents R as a bimodule over itself, the monoidal identity. The empty rectangle represents the identity endomorphism of R. Planar diagrams without top and bottom endpoints (without boundary) represent more general endomorphisms of R.

The structure of bimodule categories (or more generally strict 2-categories) guarantees that a planar diagram will unambiguously denote a morphism of bimodules.

We will be using so many such pictures that it will become cumbersome to continuously label each line by an index. Generally, the calculations we do will work independently for each i, and can be expressed with diagrams that use lines labelled i, $i+1$ and the like. In these circumstances, when there is no ambiguity, we will fix an index i and draw a line labelled i with one style, a line labelled $i+1$ with a different style, and so forth, maintaining the same conventions throughout the paper. We use different styles of lines because most printers are black and white, but we recommend that you do your calculations at home in colored pen or pencils instead; we even refer to the labels as "colors" throughout this paper.



We use the styles above when referring to indices i, $i+1$, $i-1$, and j, where j will be used unambiguously for any index which is "far away" from any other indices in the picture (in other words, when drawing a picture only involving i-colored strands, we require $|i-j|>1$, while for a picture involving both i and $i+1$ we require $j<i-1$ or $j>i+2$).

2.5. Methodology

Proposition 2.13. *Suppose one chooses the subset of the morphisms in $\mathcal{SC}_1$, including the identity morphism of each object, as well as the following morphisms:*

(1) *the generating morphism from R to B_i,*

(2) *some isomorphisms that yield the Hecke algebra relations, as well as the respective projections to and inclusions from each summand in* (2.10) *and* (2.12),

(3) *the unit and counit of adjunction that make B_i into a self-biadjoint bimodule.*

Consider $\mathcal{C}$ the subcategory generated monoidally over the left action of R by these morphisms, that is, it includes left R-linear combinations, compositions, and tensors of all its morphisms, then $\mathcal{C}$ is a full subcategory, and thus it is actually $\mathcal{SC}_1$.

Proof. For any objects M, N in $\mathcal{SC}_1$, there is an inclusion $\mathrm{Hom}_{\mathcal{C}}(M,N) \subset \mathrm{Hom}_{\mathcal{SC}_1}(M,N)$ of graded left R-modules (since it is clearly an inclusion of R-modules, and all generating morphisms are homogeneous). One can define grk for any graded left R-module M by choosing generators of $M/(R^+M)$, where R^+ is the ideal of positively graded elements, and it is a simple argument that a submodule of a free graded R-module with the same graded rank is in fact the entire module. So we need only show that Hom spaces in $\mathcal{C}$ have the same graded rank.

We can define a semilinear form on the free $\mathbb{Z}[v, v^{-1}]$-algebra generated by b_i by the formula $(b_{\underline{\mathbf{i}}}, b_{\underline{\mathbf{j}}}) = \operatorname{grk} \operatorname{Hom}_{\mathcal{C}}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}})$. The existence of isomorphisms and projection maps will give us the direct sum decompositions (2.10)–(2.12) in $\mathcal{C}$, with the resulting implications for Hom spaces. Therefore the Hecke algebra relation (2.2) are in the kernel of this semilinear form, so it descends to a form on the Hecke algebra. Each b_i will be self-adjoint. When $\underline{\mathbf{i}}$ is increasing, $\operatorname{Hom}_{\mathcal{C}}(R, B_{\underline{\mathbf{i}}})$ contains the generator of the free rank one R-module $\operatorname{Hom}_{\mathcal{SC}_1}(R, B_{\underline{\mathbf{i}}})$, since that generator is the tensor of the generating morphisms from R to various B_i (see the proof of Claim 2.9). Hence it is in fact the entire module, so $\operatorname{Hom}_{\mathcal{C}}(R, B_{\underline{\mathbf{i}}}) \cong R\{d\}$, and $(1, b_{\underline{\mathbf{i}}}) = v^d$.

By unicity, this inner product agrees with our earlier inner product on the Hecke algebra. In particular, the graded ranks agree, and the inclusion is full. □

Below we will construct a category $\mathfrak{D}\mathcal{C}_1$ of diagrams via generators and local relations, where the Hom spaces will be graded R-bimodules. We will construct a functor F_1 from $\mathfrak{D}\mathcal{C}_1$ to $\mathcal{SC}_1$, showing that our diagrams give graphical presentation of morphisms in $\mathcal{SC}_1$. The morphisms in the image of F_1 will include all the morphisms enumerated in Proposition 2.13, hence the functor will be full. Calculating the Hom spaces in $\mathfrak{D}\mathcal{C}_1$ between certain objects (corresponding to R, $B_{\underline{\mathbf{i}}}$ for $\underline{\mathbf{i}}$ increasing), we may use a similar argument to the above proposition to show that they are free R-modules of the same graded rank as the Hom spaces in $\mathcal{SC}_1$, then the functor F_1 will be faithful, and an equivalence of categories. This describes $\mathcal{SC}_1$ in terms of generators and relations.

Let $\mathfrak{D}\mathcal{C}_2$ be the category whose objects are finite direct sums of formal grading shifts of objects in $\mathfrak{D}\mathcal{C}_1$, but whose morphisms only include degree 0 maps. Finally, let $\mathfrak{D}\mathcal{C} = \operatorname{Kar}(\mathfrak{D}\mathcal{C}_2)$ be the Karoubi envelope of $\mathfrak{D}\mathcal{C}_2$. The functor F_1 lifts to functors F_2 and F, as in the picture below, with all three horizontal arrows being equivalences of categories.

$$\begin{tikzcd}
\mathfrak{D}\mathcal{C}_1 \arrow[r, "\mathcal{F}_1"] \arrow[d] & \mathcal{SC}_1 \arrow[d, "\text{Grading shifts and direct sums}"] \\
\mathfrak{D}\mathcal{C}_2 \arrow[r, "\mathcal{F}_2"] \arrow[d] & \mathcal{SC}_2 \arrow[d, "\text{Karoubi envelope}"] \\
\mathfrak{D}\mathcal{C} \arrow[r, "\mathcal{F}"] & \mathcal{SC}
\end{tikzcd} \tag{2.24}$$

We will define the category $\mathfrak{D}\mathcal{C}_1$ originally without reference to isotopy, in order to make the definition of the functor F_1 entirely straightforward, using the standard rules for diagrammatics for bimodules. The category would be entirely unchanged if one used different pictures to represent each morphism. However, when the "correct" pictures are chosen for the generators, then every morphism can actually be viewed as a *planar graph*, and moreover two embedded graphs linked by isotopy represent the same morphism. One could very well define $\mathfrak{D}\mathcal{C}_1$ originally using graphs, but this would obscure the definition of F_1.

The most difficult part of the proof will be showing the faithfulness of F_1, which involves a calculation of certain Hom spaces in the diagrammatic category. This calculation will be made possible by the planar graphs interpretation of $\mathfrak{D}\mathcal{C}_1$, wherein some relatively simple graph theory can be applied to simplify pictures.

3. Definition of $\mathfrak{D}\mathcal{C}$

This section contains a piecemeal definition of $\mathfrak{D}\mathcal{C}$ and $\mathfrak{D}\mathcal{C}_1$. For pedagogical reasons, we prefer to provide commentary as we go, instead of defining the category all at once (in fact, some relations do not make sense without the commentary). We also provide some redundant relations in the first pass, because they help make the category more intuitive. However, we repeat the definition all in one place in Section 3.4, without redundant relations, where we also explicitly define the functor F_1.

3.1. The Category $\mathfrak{D}\mathcal{C}_1$: Zero Colors and One Color

This section and the next several will hold the definition of the category $\mathfrak{D}\mathcal{C}_1$, which will be a $\Bbbk$-linear additive monoidal category, with $\mathbb{Z}$-graded Hom spaces. Shortly it will become clear that Hom spaces are actually graded R-bimodules. It is generated monoidally by n objects $i = 1, \ldots, n$, whose tensor products will be denoted $\underline{\mathbf{i}} = i_1 \cdots i_d$.

Morphisms will be given by (linear combinations of) diagrams inside the strip $\mathbb{R} \times [0,1]$, constructed out of lines colored by an index i, and certain other planar diagrams, modulo local relations. The intersection of the diagram with $\mathbb{R} \times \{0\}$, called the *lower boundary*, will be a sequence $\underline{\mathbf{i}}$ of colored endpoints, the source of the map, and the *upper boundary* $\underline{\mathbf{j}}$ will be the target. A vertical line colored i represents the identity map from i to i. The monoidal structure consists of placing diagrams side by side, and composition consists of placing diagrams one above the other, in the standard fashion for diagrammatic categories.

We present the generators and relations in an order based on the number of colors they use. The one-color generators and relations will be sufficient to describe the category for $n = 1$, the two-color ones for $n = 2$, and the three-color ones for the general case. The set of all relations is invariant under all color changes that preserve adjacency, so we only display each generator for a single color i, using the conventions described in Section 2.4. However, the generator exists for each index i.

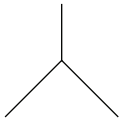
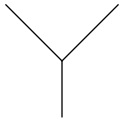
All the relations we will give are homogeneous with respect to the grading on generators stated.

The first class of generators, which use no colors, are the following endomorphisms of the monoidal identity $\emptyset$:

$\boxed{i}$

There is one such generator for each $i = 1, \ldots, n+1$. It is a map of degree 2, which we call *multiplication by* x_i. After we apply the functor F_1, this will actually correspond to the endomorphism of R given by multiplication by x_i. Together, these generators are called *boxes*. A morphism from $\emptyset$ to $\emptyset$ consisting of a sum of disjoint unions of boxes will be called a *polynomial*. Since the composition of multiplication by x_i and multiplication by x_j is multiplication by $x_i x_j$, such a sum of products of boxes will obviously correspond under the functor to multiplication by an element $f \in R$. As a shorthand we draw such a morphism as a box with the corresponding element f inside. As a map from $\emptyset$ to $\emptyset$, and thus a closed diagram, a polynomial may be placed in any region of another diagram. Placing boxes in the rightmost and leftmost regions of a diagram will define the R-bimodule structure on Hom spaces in $\mathfrak{D}\mathcal{C}_1$.

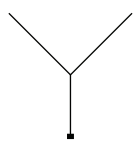
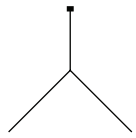
The generating morphisms which use only one color are

Symbol	Degree	Name
	1	EndDot
	1	StartDot
	−1	Merge
	−1	Split

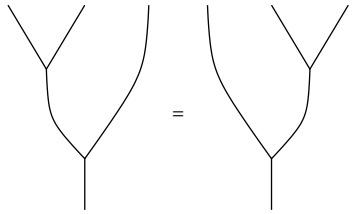
For the beginner, the maps are, respectively, a map from i to $\emptyset$, a map from $\emptyset$ to i, a map from ii to i, a map from i to ii.

Remember, there is one such set of generators for each color i. We give these maps names, but the names are temporary. Once we explore the meaning of isotopy invariance, we will stop distinguishing between Merge and Split, and call them both *trivalent vertices*. Similarly we will stop distinguishing between StartDot and EndDot, and call them both *dots*.

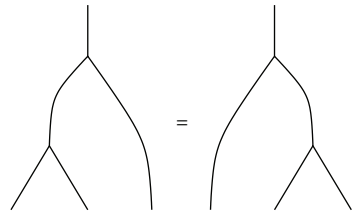
We also use a shorthand for the following compositions:

Symbol			Degree	Name
	$\overset{\text{def}}{=}$		0	Cup
	$\overset{\text{def}}{=}$		0	Cap

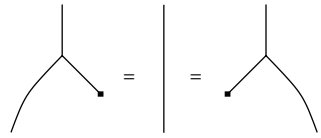
We now list a series of relations using only one color, the *one-color relations*, dividing them into several types of relations for ease of reference. The first set we refer to as the *Frobenius relations*, since they imply that i is a Frobenius object in $\mathfrak{D}\mathcal{C}_1$ (see [48, 49] for more on Frobenius algebras). Once we define the functor F_1, this will imply that B_i is a Frobenius object in $\mathcal{SC}_1$. Remember that the cups and caps appearing below can actually be rewritten in terms of the generators.



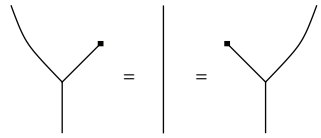
(3.1)



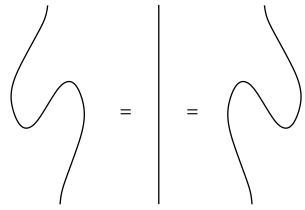
(3.2)



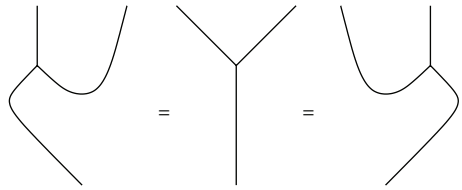
(3.3)



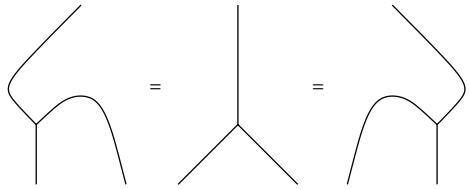
(3.4)



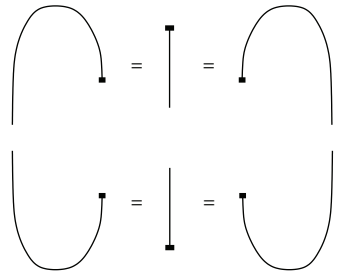
(3.5)



(3.6)



(3.7)

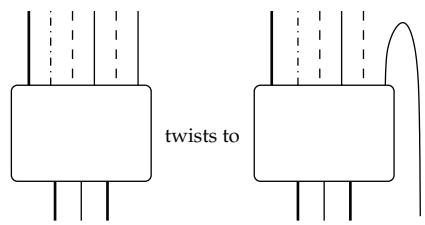


(3.8)

(3.9)

For quick reference, we refer to these relations by their Frobenius algebra names. The first two are the associativity of Merge and the coassociativity of Split. The next two are the unit and counit relations. Relation (3.5) is the biadjunction relation, and the final four are cyclicity relations.

Remark 3.1. For readers not well versed in cyclicity properties and their implications towards isotopy invariance, let us quickly discuss the topic, using the easily visualized notion of a *twist*. Given a morphism, one can twist it by taking a line which goes to the upper boundary and adding a cap, letting the line go to the other boundary instead. An example is given below,



One can also twist a downward line back up, or twist lines on the left as well. Two morphisms are twists of each other if they are related by a series of these simple twists, using cups and caps on the right and left side. For instance, relations (3.6) and (3.7) state that the Merge is a simple twist of the Split, twisting on the left or right. If one applies the same twist to every term in a relation, one gets a twist of that relation. For instance, relation (3.4) is actually a twist of the definition of the cup.

Because of biadjointness (3.5), twisting a line down and then back up will do nothing to the morphism. Once biadjointness is shown, all twists of a relation are equivalent, because twisting in the reverse direction we get the original relation back. When a morphism has a total of m inputs and outputs, twisting a single strand will often be referred to as rotation by $180/m$ degrees.

The above relations imply that twisting any of the above generators by 360 degrees will do nothing. A morphism is said to be *cyclic* with respect to a fixed set of adjunctions (i.e., cups and caps) if 360 rotation does not change the morphism. Cyclicity is useful because of the following proposition.

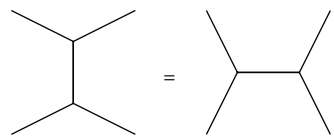
Proposition 3.2. *Fix adjunctions of each object, which are drawn as caps and cups. If every generating morphism in a diagram is cyclic with respect to those adjunctions, then so is the entire diagram, and the morphism represented by that diagram is invariant under isotopy of the diagram.*

For more on diagrammatics of biadjointness and the cyclicity property we refer the reader to [9, 48, 50–52].

Merge and Split are 60 rotations of each other, and each is invariant under 120 degree rotation, so we may represent them isotopy-unambiguously with pictures that satisfy the same properties. A similar statement holds for StartDot and EndDot. We will refer to these morphisms as dots and trivalent vertices from now on, because these terms encapsulate the picture up to rotation.

Remark 3.3. Henceforth, we can take more liberties in our drawings. We can draw a horizontal line colored i, and even though this can not be constructed using our generators, it is isotopy equivalent to a cup or cap which can be so constructed. We can allow a diagram to have a boundary not just on the top or bottom, but also on the side. While this does not represent a morphism in our category, the line running to the side boundary can be twisted either up or down to represent a genuine morphism. A relation drawn using diagrams with side boundaries does unambiguously give a relation in $\mathfrak{D}\mathcal{C}_1$.

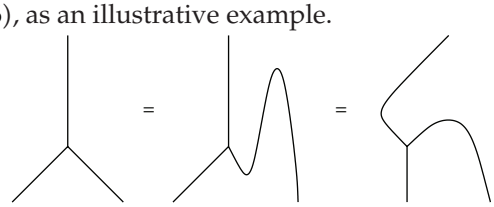
Associativity and coassociativity are twists of each other. This relation is written in a rotation-invariant form below, and will be crucial in the sequel. We refer to this relation, which permits one to "slide" one trivalent vertex over another, as *one-color associativity*.



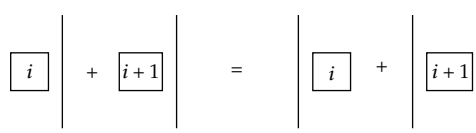
(3.10)

We refer to either picture above as an "H". The horizontal line in the right picture is exactly such a liberty as in Remark 3.3.

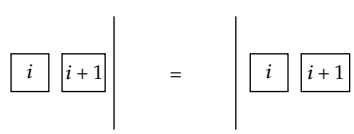
Note that relations (3.1), (3.5), (3.6), and (3.8) are sufficient to imply the other Frobenius relations, because of the remarks about twisting made above. Here is the proof of half of (3.7) using (3.6), as an illustrative example.



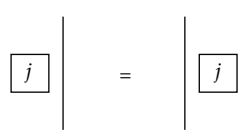
The next set of relations are known as *polynomial slides*, which have obvious analogies in the definitions of the modules B_i.



(3.11)



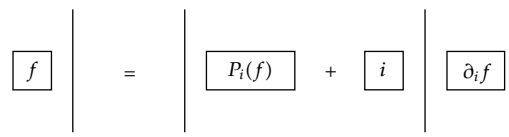
(3.12)



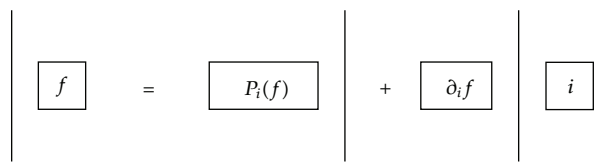
(3.13)

The j appearing in the box in the last relation can be any index not equal to i or $i+1$. Together, these relations imply precisely that any polynomial which is invariant under s_i can be slid across a line colored i, since R^i is generated by $x_i + x_{i+1}$, $x_i x_{i+1}$, and x_j for $j \neq i, i+1$. Therefore, for an arbitrary polynomial f, we have the following immediate consequence (see Remark 2.6).

Proposition 3.4. *One may force a polynomial to the other side of a line, leaving at most x_i behind, as follows:*



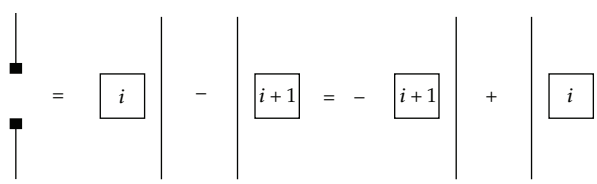
(3.14)



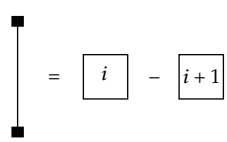
(3.15)

Proof. This is proven in the same way as (2.7). □

Now these are the final one-color relations. First, the *dot relations*

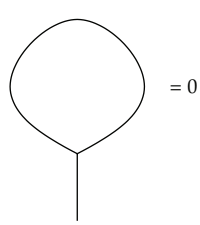


(3.16)



(3.17)

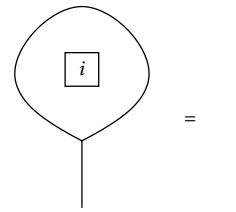
The second equality in (3.16) is just the relation (3.11). Now, is the *needle relation*:



(3.18)

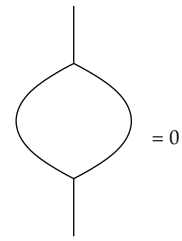
It is important to realize that such a relation does *not* apply if there is anything inside the eye of the needle, as can be seen in the following examples.

Example 3.5. Combining these relations, we have a number of simple but important consequences, which we leave as easy exercises to get the reader used to the diagrammatic calculus



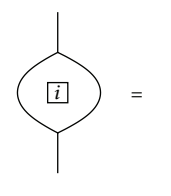
(3.19)

Use the first dot relation, then the needle relation and the unit relation



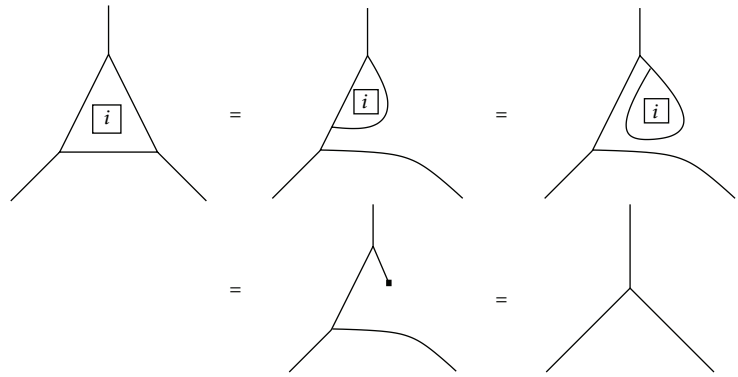
(3.20)

Use the needle relation and associativity



(3.21)

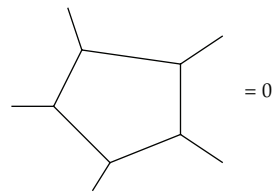
As above, with the unit relation



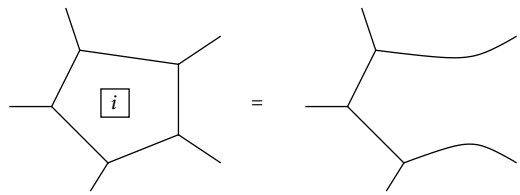
This is effectively the same example again, with more uses of associativity.

As the examples demonstrate, and the following proposition proves, we may remove cycles of this nice form from a one-color graph.

Proposition 3.6. *The following relations hold:*

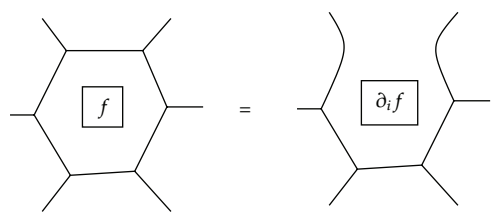


(3.22)



(3.23)

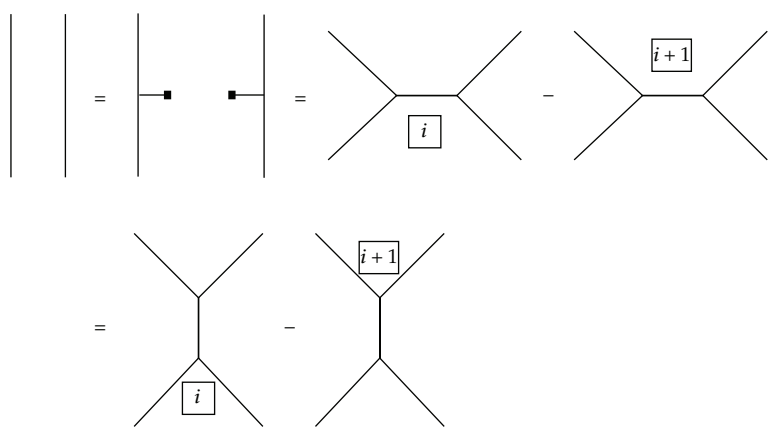
More generally, for any polynomial f, one has



(3.24)

Proof. This is a simple consequence of (3.4), along with the needle, associativity, and unit relations. □

There is another relation which is equivalent (given the others) to the first equality in (3.16)



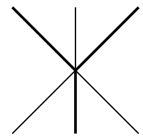
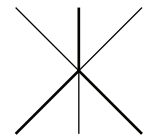
(3.25)

This relation quickly leads to the decomposition $B_i \otimes B_i = B_i\{1\} \oplus B_i\{-1\}$, see Section 4.5.

For a single color and two variables x_1, x_2, the category above, modulo the relation $x_1 = x_2$, is equivalent to the category considered by Libedinsky [13] in the case of a single label r. Morphisms given by dot, Merge, Split, and Cap correspond to morphisms $\widehat{\varepsilon}_r$, $\widehat{m}_r$, $\widehat{p}_r$, $\widehat{j}_r$, and $\widehat{\alpha}_r$ in [13, section 2.4]. Planar graphical notation, of paramount importance to us, is implicit in [13]. From here on, we diverge from Libedinsky's work, by generalizing to the case of the Weyl group S_{n+1}, while Libedinsky [13] investigates the right-angled case.

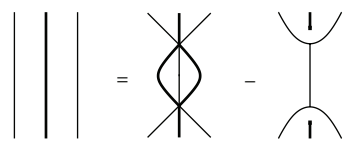
3.2. The Category $\mathfrak{D}\mathcal{C}_1$: Adjacent Colors

We now add some generators which mix adjacent colors, which we call *6-valent vertices*. Remember that the thick lines represent $i+1$, and the thin lines represent i,

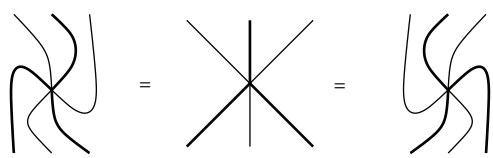
Symbol	Degree	Symbol	Degree
	0		0

For the beginner, these maps are, respectively, a map from $i(i+1)i$ to $(i+1)i(i+1)$, a map from $(i+1)i(i+1)$ to $i(i+1)i$.

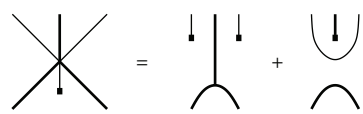
Below are the relations which deal with our new generators. In addition to the relations below, we also impose the same relations with the colors switched. The two color variants in general do not imply each other. However, it is better to think of the two colors as being arbitrary adjacent colors, rather than one being i and the other $i+1$; then one views these relations as generic for adjacent colors



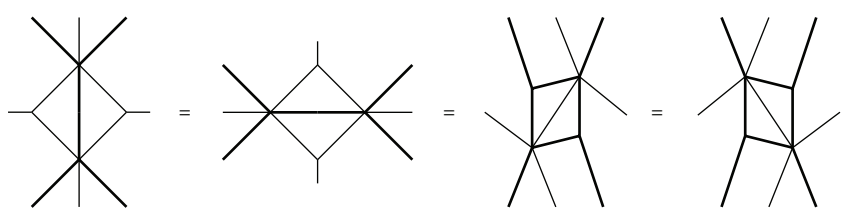
(3.26)



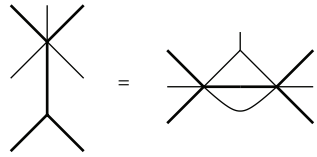
(3.27)



(3.28)



(3.29)



(3.30)

It will be shown in Section 4.5 that the first relation is related to the isomorphism $(B_i \otimes B_{i+1} \otimes B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes B_i \otimes B_{i+1}) \oplus B_i$.

The second relation shows that the 6-valent vertex is cyclic, that drawing it as a 6-valent vertex is unambiguous, and that isotopy classes of diagrams built out of our local generators will still unambiguously designate a morphism. See Remark 3.1 for details. Because of this, we have used the liberties of Remark 3.3 when writing the last two relations. Note that (3.27) does in fact imply the color-switched version of that same relation, using (3.5).

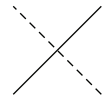
The relation (3.29) contains a number of equalities, and it is clear that the last equality is merely a rotation of the color switch of the first equality. In fact, there are numerous redundancies amongst (3.29) and (3.30). It is a worthwhile exercise for the reader at this point to check the following statement.

Example 3.7. Assume the relation (3.28) and those before it, then any pair of equalities from (3.29) will imply both color variants of (3.30) as well as the remainder of the equalities from (3.29). Hint: adding a dot to the relation (3.29) allows one to recover (3.30), while the latter may be applied twice within the former.

An important feature to notice is that the 6-valent vertex can be visualized as two trivalent vertices, one of each color, that overlap. If one takes a graph constructed out of dots, trivalent vertices, and 6-valent vertices (our generators so far), then the subgraph formed by all edges of a specific color i will have only univalent and trivalent vertices. We use the term *two-color (overlap) associativity for* $i+1$ to refer to the transformation performed by either (3.30) or the first equality of (3.29), because when viewed as an operation on the "thick"-colored graph, these operations mimic one-color associativity (3.10). Note that, under the same transformations, the "thin"-colored graph (labelled i) is transformed in a different way. However, the color-switched relations will give two-color associativity for i instead.

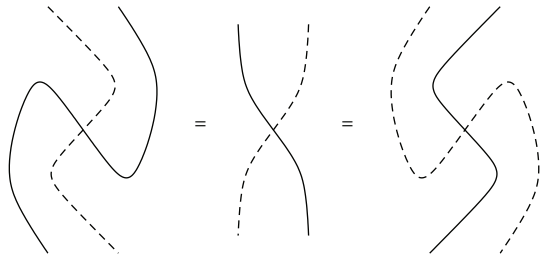
3.3. The Category $\mathfrak{D}\mathcal{C}_1$: Distant Colors

Fix j, an index which is not adjacent to i. In pictures involving both i and $i+1$, we also assume j is not adjacent to $i+1$. Remember that j is represented by a dashed line. This new generator is called a *4-valent vertex*, or a *crossing*.

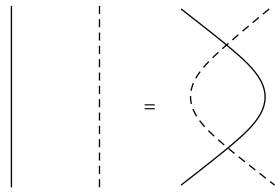
Symbol	Degree
	0

Note that this definition also covers the same picture with the colors reversed. The colors i and j can be switched freely since the only requirement was that they were distant from each other.

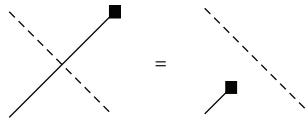
Now, for relations involving the new generator.



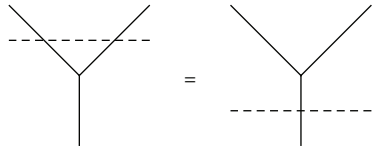
(3.31)



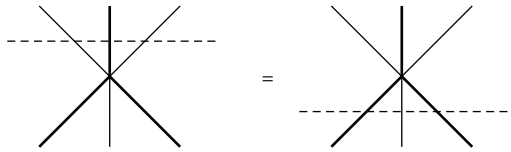
(3.32)



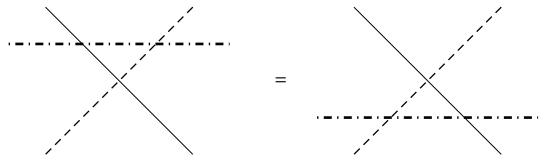
(3.33)



(3.34)



(3.35)



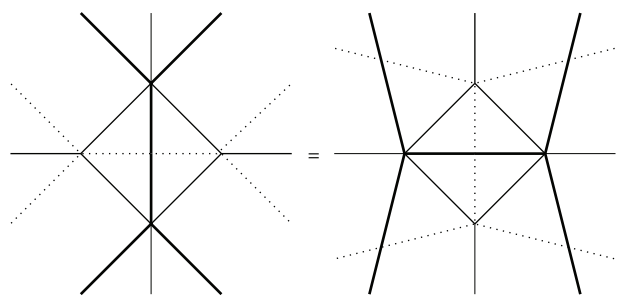
(3.36)

Relation (3.35) holds when you switch i and $i+1$, but one color variant will follow quickly from the other by twisting and applying the first relation. In the final relation, the new color represents an index k which is distant from both i and j. We will refer to (3.32) and (3.36) as the R2 and R3 moves, respectively, because of the obvious analogy to knot theory. The R2 relation is essentially the isomorphism $B_i \otimes B_j \cong B_j \otimes B_i$, see Section 4.5.

The same statements about cyclicity and drawing diagrams with sideways boundaries apply from before (see Remarks 3.1 and 3.3). Once again, the 4-valent vertices are drawn so that morphisms are isotopy invariant.

The relations (3.32)–(3.36) imply that a j-colored strand can just be pulled underneath any morphism only using colors distant from j, since it can be pulled under any generating morphism, whether it be a line, a dot, a trivalent vertex, or a 6-valent vertex. In fact, thanks to (3.4), the R2 move follows from (3.33) and (3.34).

We have now listed all the generators of our subcategory: trivalent, 4-valent, and 6-valent vertices, and dots. There is one final relation, coming from the fact that $i+1$ and $i-1$ may not interact, but they do jointly interact with i. The final relation will be called *three-color (overlap) associativity for* $i \pm 1$:



(3.37)

In the above diagram, dotted lines carry label $i-1$, thick lines $i+1$ and thin solid lines i. Rotating this relation by 90 degrees, we get the same relation except with $i+1$ and $i-1$ switched, so that only color variant is needed to imply both. The "thick"-colored graph undergoes the associativity transformation. The same is true (symmetrically) with the "dotted"-colored graph.

This concludes the definition of the category $\mathfrak{D}\mathcal{C}_1$.

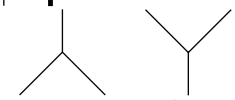
3.4. The Complete Definition and the Functor $\mathcal{F}_1$

In order to put everything in one place with no redundancy, let us define the category again.

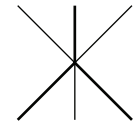
Definition 3.8. The category $\mathfrak{D}\mathcal{C}_1$ has objects given by sequences $\underline{\mathbf{i}}$ of indices in $\{1,\ldots,n\}$, with a monoidal structure given by concatenation. Fix two sequences $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$. Consider the set of all diagrams in $\mathbb{R}\times[0,1]$, constructed out of vertical lines colored by indices, and out of the generating pictures below, such that the intersection of the diagram with $\mathbb{R}\times\{0\}$ is the sequence of points $\underline{i}$, and the intersection with $\mathbb{R}\times\{1\}$ is $\underline{\mathbf{j}}$. This set is graded, where the generators have the degree indicated, then the space $\mathrm{Hom}_{\mathfrak{D}\mathcal{C}_1}(\underline{\mathbf{i}},\underline{\mathbf{j}})$ is defined to be the $\Bbbk$-linear span of this set of diagrams, modulo the homogeneous local relations below.

Generators:

(1) For each color, pictures of degree 1: 

(2) For each color, pictures of degree −1: 

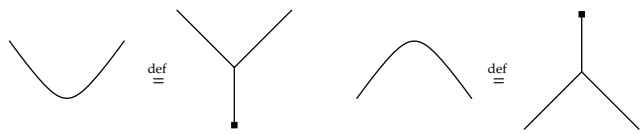
(3) For each pair of distant colors, a picture of degree 0: 

(4) For each pair of adjacent colors, a picture of degree 0: 

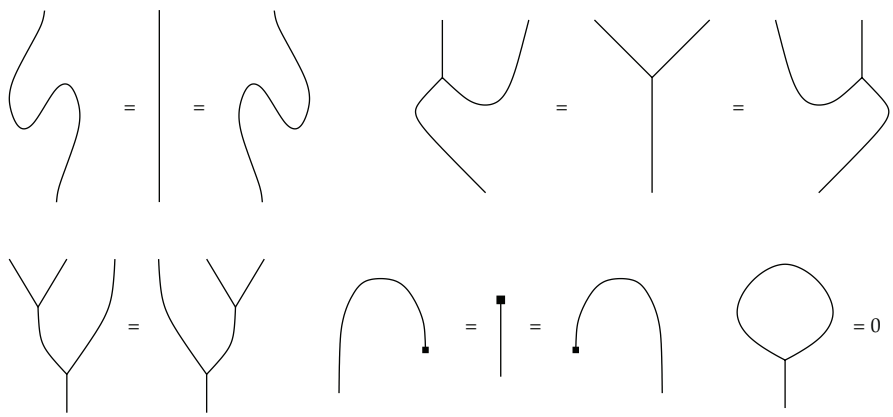
(5) For each $i \in \{1, \ldots, n+1\}$, a picture of degree 2: $\boxed{i}$

Relations:

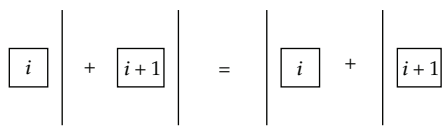
Some relations are drawn using the liberties of Remark 3.3. We also use the definition of the cap and cup,

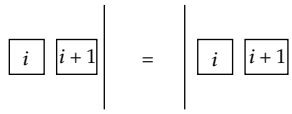


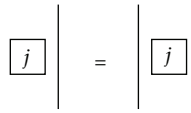
For each color,

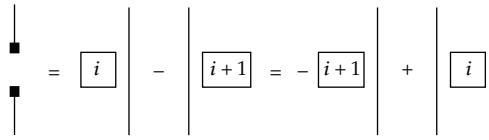


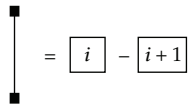
Here, i is the color of the line, and j is a color $\neq i, i+1$,



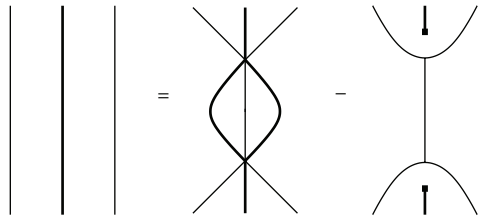


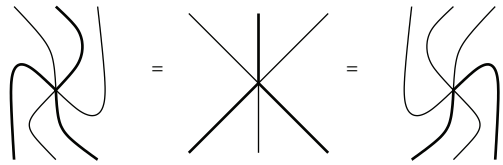


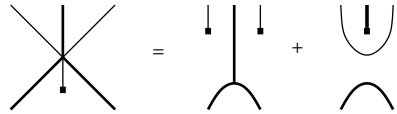


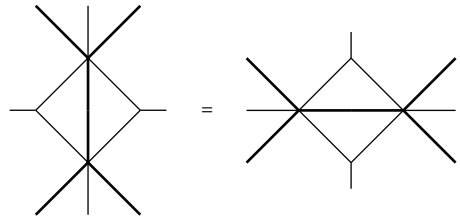


For any two adjacent colors,

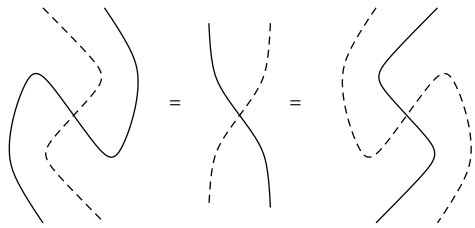


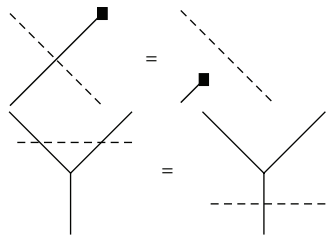




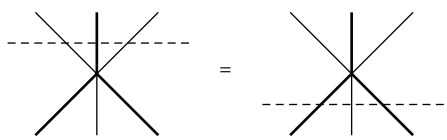


For any two distant colors,

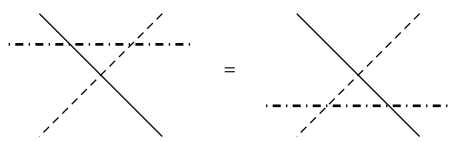




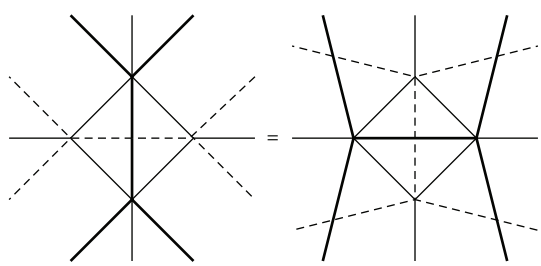
For two adjacent colors and a third, distant to both,



For three mutually distant colors,



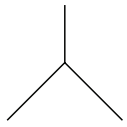
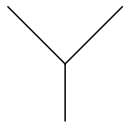
For three colors with the same adjacency as $\{1,2,3\}$,



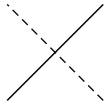
Definition 3.9. Let F_1 be the functor from $\mathfrak{D}\mathcal{C}_1$ to $\mathcal{SC}_1$ specified as follows. On objects, $F_1(\underline{\mathbf{i}}) = B_{\underline{\mathbf{i}}}$. We define the functor on generating morphisms and extend it monoidally to all morphisms.

In doing so, we *always* use the isomorphism (2.6) to identify $B_{\underline{\mathbf{i}}}$ with the R-bimodule spanned by a choice of $d(\underline{\mathbf{i}}) + 1$ polynomials. If one thinks of $B_{\underline{\mathbf{i}}}$ diagrammatically as d vertical lines, then a spanning element of $B_{\underline{\mathbf{i}}}$ is a choice of polynomial for each empty region delineated by the lines (and polynomials with the appropriate symmetry may slide across the lines). We write the map explicitly for a general element when it is easy enough to do so, or we write it for a spanning set as an R-bimodule (see Remark 2.7).

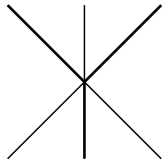
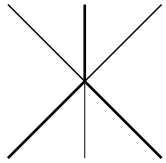
For a line colored i,

Symbol	$\mathcal{F}_1$	
	fg $\uparrow$ $f \otimes g$	
	$x_i \otimes 1 - 1 \otimes x_{i+1}$ $\uparrow$ 1	
	0 $\uparrow$ $f \otimes 1 \otimes g$	$f \otimes g$ $\uparrow$ $f \otimes x_i \otimes g$
	$f \otimes 1 \otimes g$ $\uparrow$ $f \otimes g$	

For lines colored i and j distant,

	$f \otimes 1 \otimes g$ $\uparrow$ $f \otimes 1 \otimes g$

For a thin line colored i and a thick line colored $i + 1$,

	$1 \otimes 1 \otimes 1 \otimes 1$ $\uparrow$ $1 \otimes 1 \otimes 1 \otimes 1$	$(x_i + x_{i+1}) \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_i + 2$ $\uparrow$ $1 \otimes x_i \otimes 1 \otimes 1$
	$1 \otimes 1 \otimes 1 \otimes 1$ $\uparrow$ $1 \otimes 1 \otimes 1 \otimes 1$	$1 \otimes 1 \otimes 1 \otimes (x_{i+1} + x_{i+2}) - x_i \otimes 1 \otimes 1 \otimes 1$ $\uparrow$ $1 \otimes x_{i+2} \otimes 1 \otimes 1$

For any $1 \leq i \leq n+1$,

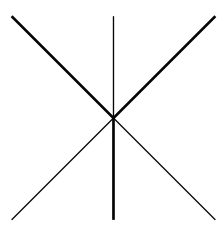
$$\boxed{i} \quad \begin{array}{c} x_i \\ \uparrow \\ 1 \end{array}$$

Claim 3.10. The above maps are R-bimodule maps.

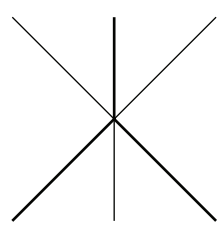
Proof. This is obviously true for EndDot, since the resulting map is no more than multiplication. StartDot is sent precisely to the generator φ_i of $\mathrm{Hom}(R, B_i)$ discussed in Section 2.3. Split and Merge have already been seen as inclusion and projection maps in the isomorphism $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$, see Remark 2.6. For the 4-valent vertex, the only polynomials which slide all the way across $B_i \otimes B_j$ or $B_j \otimes B_i$ are in $R^{i,j}$, so that the map $f \otimes 1 \otimes g \to f \otimes 1 \otimes g$ is a bimodule map (that $f \otimes 1 \otimes g$ spans it was observed in Remark 2.7).

Only the 6-valent vertices remain to be checked. Consider the first of the two variants. The generating set $\{1 \otimes 1 \otimes 1 \otimes 1, 1 \otimes x_i \otimes 1 \otimes 1\}$ as an R-bimodule was chosen because x_i can be slid freely between the second and third slots. We have defined the R-bimodule map on generators before showing that the map is an R-bimodule map at all, which is akin to putting the cart before the horse. Let us explicitly define the map on a $\Bbbk$ spanning set by the following algorithm: given $f \otimes g \otimes h \otimes k \in B_i \otimes B_{i+1} \otimes B_i$, first we force h to the right and slide the "remainder" to the left, that is $f \otimes g \otimes h \otimes k = f \otimes g \otimes 1 \otimes kP_i(h) + f \otimes gx_i \otimes 1 \otimes k\partial_i(h)$; then we force the terms in the second slot to the left, yielding $fP_i(g) \otimes 1 \otimes 1 \otimes kP_i(h) + f\partial_i(g) \otimes x_i \otimes 1 \otimes kP_i(h) + fP_i(gx_i) \otimes 1 \otimes 1 \otimes k\partial_i(h) + f\partial_i(gx_i) \otimes x_i \otimes 1 \otimes k\partial_i(h)$. Finally, each term can be evaluated using the given definition of F_1 on generators. This gives an explicit formula for the image of $f \otimes g \otimes h \otimes k$, which we only need check is invariant under: sliding an element of R^i from f to g, or from h to k; sliding an element of R^{i+1} from g to h. Sliding elements of R^i does not pose a problem, since we defined the map by forcing h to k and g to f, which fully respects such slides. Checking invariance under slides from g to h is nontrivial. However, the bulk of the work is encapsulated in the following discussion, which is useful for calculations in general.

By adding and subtracting x_{i+2}, the image of $1 \otimes x_i \otimes 1 \otimes 1$ under the first 6-valent vertex (see above) can be written more symmetrically as $(x_i + x_{i+1} + x_{i+2})(1 \otimes 1 \otimes 1 \otimes 1) - x_{i+2} \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_{i+2}$. The first term is a polynomial symmetric in all the relevant variables and thus can be slid anywhere. In the other two terms, x_{i+2} can not be slid freely under a line labelled $i+1$, so it is stuck in its respective position. In contrast, $1 \otimes x_{i+1} \otimes 1 \otimes 1$ and $1 \otimes 1 \otimes x_{i+1} \otimes 1$ are not equal, since x_{i+1} can not be slid over a line labelled $i+1$, but the images of both these elements are easier to remember, and are shown below,



$$\begin{array}{c} 1 \otimes 1 \otimes 1 \otimes x_{i+2} \\ \uparrow \\ 1 \otimes x_{i+1} \otimes 1 \otimes 1 \end{array} \qquad \begin{array}{c} x_{i+2} \otimes 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes x_{i+1} \otimes 1 \end{array}$$



$$\begin{array}{c} 1 \otimes 1 \otimes 1 \otimes x_i \\ \uparrow \\ 1 \otimes x_{i+1} \otimes 1 \otimes 1 \end{array} \qquad \begin{array}{c} x_i \otimes 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes x_{i+1} \otimes 1 \end{array}$$

The way to remember these formulae is that the variable which cannot be slid is sent to the variable which cannot be slid, from the middle on one side to the exterior on the other. It is easy to see that these calculations were done according to the algorithm above, forcing x_{i+1} to the outside first and then evaluating on the leftover x_i.

Now, we do the consistency check for the simplest cases. We wish to show that $1 \otimes (x_{i+1} + x_{i+2}) \otimes 1 \otimes 1$ and $1 \otimes 1 \otimes (x_{i+1} + x_{i+2}) \otimes 1$ are sent to the same element by the algorithm. However, this is rather easy, for in both cases, the x_{i+2} term slides immediately to the exterior, and the x_{i+1} term is evaluated as above, so both are sent to $x_{i+2} \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes x_{i+2}$. Similarly, both $1 \otimes x_{i+1}x_{i+2} \otimes 1 \otimes 1$ and $1 \otimes 1 \otimes x_{i+1}x_{i+2} \otimes 1$ are sent to $x_{i+2} \otimes 1 \otimes 1 \otimes x_{i+2}$. The general case is not significantly more difficult than this; we leave the details to the reader. □

Proposition 3.11. *The functor F_1 is well-defined. That is, the relations of $\mathfrak{DC}_1$ hold between morphisms of R-bimodules in $\mathcal{SC}_1$.*

Checking that the relations hold is a series of simple but tedious calculations that is postponed until Section 5.1. We assume this result henceforth. In addition, we note once and for all that (as one can easily check) all relations in the definition of $\mathfrak{DC}_1$ are homogeneous, and F_1 preserves the degree of the generators.

Addendum 2. *It may strike the reader as unusual that the definition of the functor seems lopsided, while the definition of $\mathfrak{DC}_1$ is invariant under right-left reflection, or under reversing the order of the colors $n, n-1, \ldots, 1$. For instance, F_1 applied to StartDot yields the element $x_i \otimes 1 - 1 \otimes x_{i+1}$, which is actually invariant under right-left reflection but not immediately so. Had this element been rewritten as $(x_i - x_{i+1})/2 \otimes 1 + 1 \otimes (x_i - x_{i+1})/2$, perhaps the calculations would be more natural despite having more fractions. A worse offender is the forcing rule $1 \otimes g = (g - \partial_i(g)x_i) \otimes 1 + \partial_i(g) \otimes x_i$, which should be rewritten $1 \otimes g = (g + s_i(g))/2 \otimes 1 + \partial_i(g) \otimes (x_i - x_{i+1})/2$. In general, the elements $x_i - x_{i+1}$ are more natural than x_i or x_{i+1}, coming from the reflection representation rather than the standard representation of S_{n+1} (see Remark 2.2 and Section 4.6).*

Previous versions of this paper, however, used the style above, and so we feel compelled to stick with it to maintain consistency. Also, checking that F_1 is a functor may be easier with the current notation.

4. Consequences

4.1. Terminology

We will spend the next few sections classifying the homomorphisms in $\mathfrak{DC}_1$. For many of the results, proofs will be postponed until Section 5.2.

We will we using the fact, extensively discussed in the previous sections, that a morphism can be viewed unambiguously as an isotopy class of graphs with polynomials in the regions (or rather, a linear combination of these). Henceforth, the term *graph* only refers to colored finite graphs with boundary (embedded in the planar strip) which can be constructed out of univalent, trivalent, 4-valent, and 6-valent vertices as above. Remember that these graphs do have edges which run to the boundary, which we call *boundary lines*, and may have edges which meet neither the boundary nor any vertex, and thus must necessarily form a circle. We say a graph *has a boundary* if it has at least one boundary line. A graph divides the planar strip into regions, and there are two distinguished regions: the *lefthand* and *righthand* regions, which contain $-\infty$ and ∞, respectively.

We call a *boundary dot* any connected component of a graph which consists entirely of an edge starting at the boundary and ending in a dot. We call a *double dot* any connected

component of a graph which consists entirely of an edge with a dot on both ends. Cutting an edge in a diagram and replacing it with two dots we call *breaking* the edge (see, for instance, relation (3.16)).

Given a set S of graphs and a morphism φ in $\mathfrak{D}\mathcal{C}_1$, we say that S *underlies* φ if φ can be written as a linear combination of morphisms, each of which is given by a graph $\Gamma \in S$ with polynomials in regions. We say that a graph Γ*reduces* to S if S underlies every morphism that Γ underlies. Clearly reduction is transitive, in that if Γ reduces to S, and every graph in S reduces to S', then Γ reduces to S'. Our goal will be to find a nice set of graphs to which all other graphs reduce. We will do this by finding *reduction moves*, which are local moves on graphs, sending a graph to a set of graphs to which it reduces.

Example 4.1. The relation (3.25) implies that || reduces to  . In other words,  underlies both the terms on the right side of (3.25). This can be applied as a local reduction move within any graph.

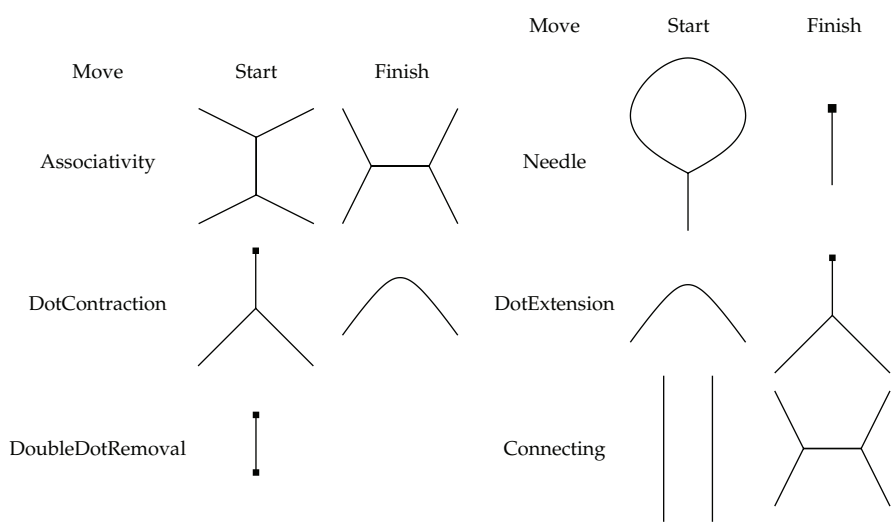
Let T be a subset of $\{1, \ldots, n\}$. The T-*graph* of a graph will be the subgraph consisting of all edges colored i for $i \in T$. Some 6-valent vertices in the original graph may become trivalent vertices in the T-graph. Similarly, some 4-valent vertices in the original graph may become 2-valent vertices in the T-graph, which we ignore, connecting the incoming edges into a single edge. The T-graph is itself a graph by our above definition. Most often we will just consider the i-graph for a single color (i.e., $T = \{i\}$). Typically, our reduction moves will be designed to simplify the i-graph for a particular i, allowing us to simplify the graph one color at a time.

Remark 4.2. The rest of this paper will have numerous calculations, but they will mostly be calculations with the underlying graphs, not keeping track of polynomials, so they do not reflect how morphisms actually behave in $\mathfrak{D}\mathcal{C}_1$. For lots of examples of computations in the graphical calculus, see [33].

4.2. One Color Reductions

In this section, we assume all graphs consist of a single color i.

Definition 4.3. Consider the following "moves", or transformations. They take a subdiagram looking like Start, and replace that subdiagram with Finish. We call these the *basic moves*.



Remember, these are moves on graphs, not graphs with polynomials. Note that the needle move, by adding a dot on the bottom, yields a reduction from the circle to a double dot. The only moves which change the connectivity of a graph are double dot removal, which deletes a connected component, and the connecting move, which has the potential to link two components into one.

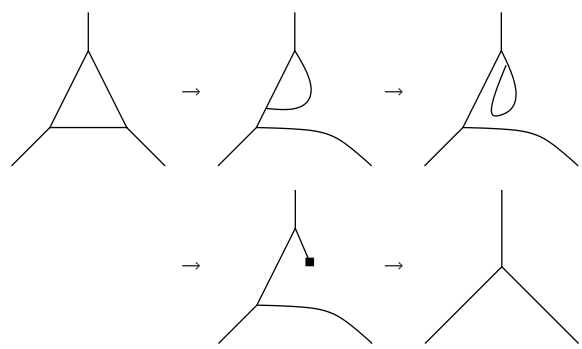
Claim 4.4. All of these moves are reduction moves in $\mathfrak{D}\mathcal{C}_1$.

Proof. The associativity move follows from (3.10). That is, even if there are polynomials in the regions of the graph, the relation (3.10) can still be applied. These polynomials, being in external regions, do not interfere with the application of relations. Similarly, dot contraction/extension follow from (3.4), dot removal follows from (3.17), and the connecting move follows from (3.25).

The needle move remains. Suppose we have an arbitrary polynomial f in the eye of the needle. We may use (3.24), generalizing (3.19), to replace the diagram with a dot accompanied by $\partial_i(f)$. □

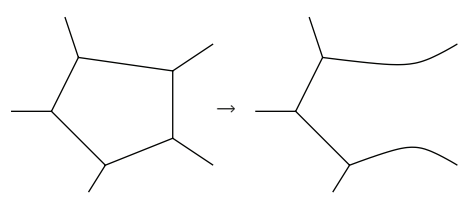
The following example of reduction should be familiar.

Example 4.5.



In order, this is done with associativity, associativity, needle, and dot contraction moves.

Example 4.6.



This is meant to indicate an arbitrary length cycle of this form, and reduction is done with associativity, needle, and dot contraction moves.

Definition 4.7. A *simple tree* T with m boundary lines is a connected one-color graph with boundary, whose form depends on m:

(1) if $m \geq 2$, then T is a trivalent tree with $m-2$ vertices connecting all the boundary lines. Note that any two such trees are equivalent under the associativity move;

(2) if $m = 1$, then T is a single boundary dot;

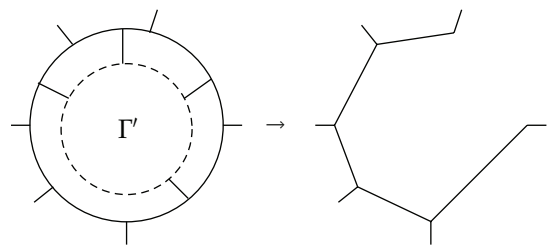
(3) if $m = 0$, then T is the empty graph.

Definition 4.8. A *cycle* in a one-color graph is either a circle or a path from a vertex to itself which does not repeat any edges. Any cycle splits the plane into two parts, the inside and outside of the cycle. A cycle is *minimal* if the inside of the cycle consists of a single region.

By counting vertices, it is clear that any connected purely trivalent graph with no cycles is a simple tree. Any graph with a cycle has a minimal cycle.

We will now give a precise inductive algorithm to reduce a graph to a disjoint union of simple trees, by reducing minimal cycles.

Proposition 4.9. *Consider a minimal cycle in a one-color graph* Γ. *Using the associativity, needle, dot removal, and dot contraction/extension moves, we may reduce a neighborhood of the cycle (including the inside region) to a simple tree (see* (4.1)*).*



(4.1)

Proposition 4.10. *Using the associativity, needle, dot removal and dot contraction/extension moves, one can reduce any one-color graph* Γ *to a disjoint union of simple trees. During this process, each component with no boundary lines will be replaced with the empty graph, and after this is done, no further connected components are created or destroyed or merged.*

We prove both propositions together, in several steps.

Proof of Proposition 4.10 for a Graph with No Cycles. Suppose there are no cycles in Γ. If there is a dot, the edge coming from that dot must connect to either the boundary, another dot, or a trivalent vertex. Our simplification algorithm is as follows.

(1) Remove any dot connected to a trivalent vertex, using dot contraction. Repeat Step 1 until no such dots remain. This does not alter the connected components.

(2) Replace any double dot with the empty set, using double dot removal. Because Step 1 did not alter the connected components, this could only be applied to double dots which arose from components which had no boundary lines.

Boundary dots are in their own connected component. Any other connected component is purely trivalent and has no cycles, so it must be a simple tree. □

Proof of Proposition 4.9, Assuming Proposition 4.10 for Graphs with No Cycles. Let γ denote a minimal cycle of Γ. Consider a neighborhood of γ in Γ, and let Γ' be the subgraph consisting of the interior of the cycle, as in (4.1). Let us call the boundary lines of Γ'*spokes*, since they run into γ from the inside, like spokes hitting the wheel of a bicycle. Now Γ' may not have any cycles, or else γ would not be a minimal cycle. Moreover, the spokes of Γ' must be in distinct

connected components of Γ', or else they would create additional regions and γ would not be a minimal cycle. Our simplification algorithm is as follows:

(1) Apply Proposition 4.10 to Γ', replacing Γ' with a disjoint union of boundary dots, one for each spoke.

(2) Use dot contraction to remove all the spokes. We are now in the situation of Example 4.6.

(3) Apply associativity as in Example 4.5, reducing the length of γ by one. Repeat until γ is a needle or a circle.

(4) If γ is a needle, apply the needle move to replace it with a dot. If associativity moves were performed in Step 3, use dot contraction to contract this dot into one of the trivalent vertices, as in Example 4.5.

(5) If γ is a circle, apply dot extension to replace it with a needle attached to a dot, then apply needle reduction to obtain a double dot, and double dot removal to obtain the empty graph.

It is a simple observation that the result of this procedure is a simple tree, and that the only alteration of connected components which occurred was the removal of components which had no boundary lines to begin with. □

Proof of Proposition 4.10 in the General Case. Suppose we have an arbitrary graph Γ. Our simplification algorithm is as follows:

(1) If Γ has a cycle, apply Proposition 4.9 to replace its neighborhood with a simple tree. Repeat this process until Γ has no cycles.

(2) Apply the procedure for the case of no cycles above.

Note that Step 1 will terminate, which can be shown by induction on the number of internal regions (regions which do not meet the boundary of the planar strip). Each application of Proposition 4.9 reduces the number of internal regions by 1. □

Corollary 4.11. *Using the connecting move in addition, we can reduce the graph to a single simple tree.*

Proof. It is an easy observation that when one uses the connecting move on a simple tree with m boundary lines and a simple tree with m' boundary lines, one gets a simple tree with $m+m'$ boundary lines, after possibly removing extraneous dots if either m or m' equals 1. □

Remark 4.12. There are two useful sets of one-color graphs with m boundary lines, to which all others reduce. The first set just contains the simple tree with m boundary lines, and the latter is the collection of all disjoint unions of simple trees whose number of boundary lines add up to m. The former is useful because there is a single graph, so we have fewer cases to deal with. The latter is useful because it does not require the connecting move.

More importantly, these sets behave differently when we introduce polynomials into the equation. Let us assume that all m boundary lines are on the top boundary, so that we are looking at a morphism in $\mathrm{Hom}(\emptyset, \underline{\mathbf{i}})$ where $\underline{\mathbf{i}}$ is $iiiiiiii$ (m times). The following statements will not be used in this paper, and can be more easily proven after the calculation of Hom spaces.

Claim 4.13. Consider diagrams which are a simple tree, with an arbitrary monomial in the lefthand region, and either 1 or x_i in each other region. This is a basis for $\mathrm{Hom}(\emptyset, \underline{\mathbf{i}})$ over $\Bbbk$.

Claim 4.14. Consider diagrams which are disjoint unions of simple trees, with an arbitrary monomial in the lefthand region, and no other polynomials. These are a spanning set for $\mathrm{Hom}(\emptyset, \underline{\mathbf{i}})$ over $\Bbbk$.

The second claim is easy to see, given Proposition 4.10. Given a disjoint union of simple trees, with arbitrary polynomials, we may use relation (3.16) and the polynomial slides to force all the polynomials to the left, at the cost of potentially breaking some lines. This breakage is not a problem, since one can reduce again to a simple tree without adding more polynomials, using (3.10) and (3.4). We do not get a basis this way: consider the three different ways to break a line in a trivalent vertex diagram; there is a linear dependence relation between these diagrams and the trivalent vertex with a polynomial in the lefthand region.

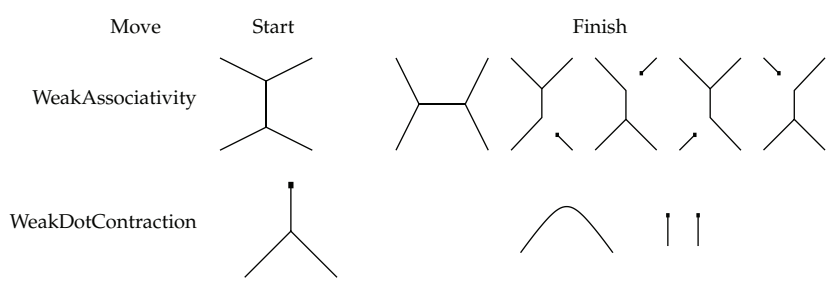
Relations (3.25) and (3.11) essentially allow us to get from the second spanning set to the first, showing that the first is at least a spanning set. That it is a basis is immediate from counting the graded dimension of the Hom space, one we prove that the dimension of Hom spaces in $\mathfrak{D}\mathcal{C}_1$ conforms with a certain semilinear form.

The connecting move is less important than the others in the proofs, and was introduced primarily to make these remarks. It can generally be ignored below.

4.3. Broken One-Color Reductions

The reductions of the previous section do apply, as stated, to any one color graph. However, we would like to apply these moves to the i-colored graph of a multicolor graph, where the moves above do not extend trivially to reduction moves in $\mathfrak{D}\mathcal{C}_1$. In this section, we quickly generalize the results of the previous section to a weaker set of moves.

Definition 4.15. Consider the following reductions for one-color graphs, which take the graph on the left and replace it with the set of graphs on the right:



We call the moves above (together with the basic moves that have no weak analog) the *broken* or *weak basic moves*.

These moves behave like the basic moves of the same name, except that they may also replace the original diagram with a broken version of itself, that is, a version with some edges broken. To distinguish the original basic moves, we may call them the *strict basic moves*.

What is important is that we have analogs of Propositions 4.9 and 4.10, despite only being able to use weak moves.

Proposition 4.16. *Consider a minimal cycle in a one-color graph* Γ. *Using the weak associativity, needle, dot removal, weak dot contraction, and dot extension moves, we may reduce a neighborhood of the cycle (including the inside region) to a disjoint union of simple trees.*

Proposition 4.17. *Using the weak associativity, needle, dot removal, weak dot contraction, and dot extension moves, one can reduce any one-color graph* Γ *to a disjoint union of simple trees.*

Of course, two distinct simple trees are no longer equivalent under the weak associativity move, but this is really irrelevant for us.

Proof. Breaking a line will never create a cycle or increase the number of trivalent vertices. Because of this, the proofs of the previous section go through almost verbatim (ignoring any statements about connected component, and occasionally replacing "a simple tree" with "a disjoint union of simple trees"). The only significant alterations that need to be made come in the proof of Proposition 4.9. In Step 4 or Step 5 one may need to remove an additional double dot. In Step 2 or Step 3, weak dot contraction and weak associativity have multiple outcomes, but each outcome that does not agree with strict dot contraction or strict associativity will have broken the cycle already, allowing us to complete the proof using the no-cycle algorithm of Proposition 4.10.

Alternatively, one could also prove these statements by induction on the number of trivalent vertices. Each weak move is equivalent to a strong move modulo diagrams with fewer trivalent vertices. The only part of the proof that ever created additional trivalent vertices was the single use of dot extension in Step 5 of the proof of Proposition 4.9. It is easy to see how Step 5 does not actually cause a problem, however, since after dot extension is applied, the needle move and double dot removal will do the trick in the same way regardless. □

Addendum 3. *The overall proof using weak one-color moves is slightly different than the treatment in previous versions of this paper, but it is cleaner and more straightforward.*

4.4. i-Colored Moves

Now we list the moves which allow us to simplify multicolor graphs.

Definition 4.18. Consider a graph Γ whose i-graph looks like one of the pictures in the start column of Definition 4.3. Let S be the set of all graphs whose i-graph looks like the corresponding picture in the finish column. Let W be the set of all graphs whose i-graph looks like any of the corresponding pictures in the finish column of definition weakbasicmoves. The *strict i-colored move* replaces Γ with the set S. The *weak i-colored move* replaces Γ with the set W.

For instance, strict i-colored associativity will replace any graph Γ whose i-graph is  with the set S of graphs whose i-graph is  . This set S is enormous, for other colors can interfere, and the j-graph for some other j can be arbitrarily complicated. The i-colored vertices, seemingly trivalent, could come from 6-valent vertices in Γ. In general, an i-colored move will behave nicely on the i-graph, but may significantly complicate the full graph.

Proposition 4.19. *The weak i-colored basic moves are reduction moves in $\mathfrak{DC}_1$, so long as they are applied to graphs which do not contain either the color $i-1$ or $i+1$.*

A color i will be called *extremal* for a graph Γ if it appears in Γ but either $i-1$ or $i+1$ does not appear. Clearly, any nonempty graph will have an extremal color, such as the minimal color present.

The proof of this proposition is found in Section 5.2, as well as more precise details on what can be done. The power of the proposition can be seen immediately:

Corollary 4.20. *Any graph Γ without boundary lines can be reduced to the empty graph.*

Proof. We induct on the set of colors present in the graph Γ. If no colors are present, then Γ is the empty graph and we are done. Else, choose an extremal color i. By Proposition 4.19 we may apply the weak i-colored basic moves and use Proposition 4.17 to replace every connected component of the i-graph with a disjoint union of simple trees with no boundary lines. Since a simple tree with no boundary lines is the empty set, Γ reduces to a set of graphs which do not include the color i. By induction, Γ now reduces to the empty graph. □

One can apply a similar procedure to a graph Γ with boundary lines. Choose an extremal color i, and reduce the i-graph to a disjoint union of simple trees, then, within each region delimited by the i-graph, the colors $i+1$ and $i-1$ are now extremal, and we can reduce those. One can repeat this procedure, however, it will not produce a very simple graph in all cases. If the color i has at least 3 boundary lines, the i-graph may have trivalent vertices, and the graph Γ itself may have 6-valent vertices in their place. These 6-valent vertices will produce more $i+1$ or $i-1$ colored boundary lines inside the regions delimited by the i-graph. Nonetheless, we have the following simple case.

Corollary 4.21. *Any graph whose boundary has at most one line of each color can be reduced to a disjoint union of boundary dots.*

Proof. We know we can reduce the i-graph, for i an extremal color, to a disjoint union of simple trees. A simple tree with at most one boundary line is either the empty set or a boundary dot. Therefore the i-graph is now either the empty set or a boundary dot, depending on whether or not i appears in the boundary. The dot need not be a boundary dot in the entire graph Γ, but it can encounter only 4-valent vertices en route to the boundary. Since a dot can be slid under a 4-valent vertex by relation (3.33), we may turn the dot into a boundary dot (its own connected component). The remaining connected components form a subgraph (also viewable in the planar strip) without the color i. Induction now concludes the proof. □

4.5. $\mathcal{F}_1$ Is Fully Faithful

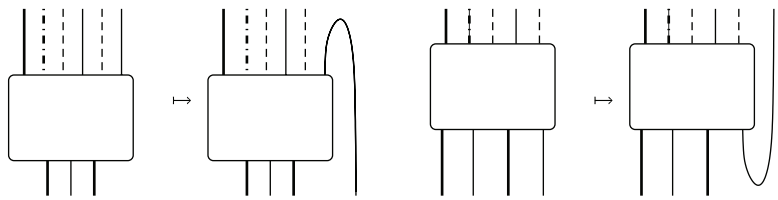
In this section, modulo the proofs of previous sections which were delayed until Section 5, we prove our main theorem.

Theorem 4.22. *The functor F_1 from $\mathfrak{DC}_1$ to $\mathcal{SC}_1$ is an equivalence of $\Bbbk$-linear monoidal categories with Hom spaces enriched in R-molf$_{\mathbb{Z}}$-R.*

We know F_1 is a functor by Proposition 3.11, and inspection of the objects in both categories shows immediately that it is essentially surjective. To show F_1 is full, we use Proposition 2.13, which motivates the next few statements.

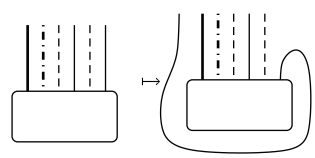
Corollary 4.23. *For any index i, the object i in $\mathfrak{D}\mathcal{C}_1$ is self-biadjoint. This means that for any sequences $\underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$, there are natural isomorphisms $Hom_{\mathfrak{D}\mathcal{C}_1}(\underline{\mathbf{k}}, \underline{\mathbf{j}}i) \to Hom_{\mathfrak{D}\mathcal{C}_1}(\underline{\mathbf{k}}i, \underline{\mathbf{j}})$ and $Hom_{\mathfrak{D}\mathcal{C}_1}(\underline{\mathbf{k}}, i\underline{\mathbf{j}}) \to Hom_{\mathfrak{D}\mathcal{C}_1}(i\underline{\mathbf{k}}, \underline{\mathbf{j}})$.*

Proof. The first isomorphism and its inverse are shown below. That these maps compose to be the identity is exactly the relation (3.5)



The second isomorphism and its inverse are the left-right mirror of the maps above. □

Note that when $\underline{\mathbf{k}} = \emptyset$ in the corollary above, these isomorphisms combine to yield an isomorphism $\mathrm{Hom}_{\mathfrak{D}\mathcal{C}_1}(\emptyset, \underline{\mathbf{j}}i) \to \mathrm{Hom}_{\mathfrak{D}\mathcal{C}_1}(\emptyset, i\underline{\mathbf{j}})$, drawn as below



At this point, one could construct a semilinear product on the free algebra generated by b_i, $i = 1, \ldots, n$, via $(b_{\underline{\mathbf{i}}}, b_{\underline{\mathbf{j}}}) = \mathrm{grk}\,\mathrm{Hom}_{\mathfrak{D}\mathcal{C}_1}(\underline{\mathbf{i}}, \underline{\mathbf{j}})$, and b_i would be self-adjoint. If we show that the Hecke algebra relations are in the kernel of this semilinear product then it will descend to the Hecke algebra. We have several methods by which we could do this.

(1) Look in $\mathfrak{D}\mathcal{C}_2$, where we have direct sums and grading shifts, and prove the isomorphisms (2.10)–(2.12).

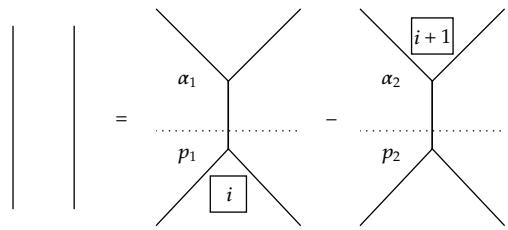
(2) Look in the Karoubi envelope $\mathfrak{D}\mathcal{C}$, find idempotents corresponding to the auxiliary modules in (2.13) and friends, and show those isomorphisms.

(3) Work entirely within $\mathfrak{D}\mathcal{C}_1$ and show the isomorphisms (2.12) only after applying the Hom functor. For instance, showing that $\mathrm{Hom}(ii, \underline{\mathbf{j}}) \cong \mathrm{Hom}(i, \underline{\mathbf{j}})\{1\} \oplus \mathrm{Hom}(i, \underline{\mathbf{j}})\{-1\}$ will be sufficient.

All these tactics are primarily the same. We illustrate the third method, although we do explore the auxiliary modules of the second method.

The relation (3.25) precisely descends to $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$. We decompose the identity of ii into the sum of two idempotents, and obtain orthogonal projections from ii to i

of degrees 1 and −1, respectively,

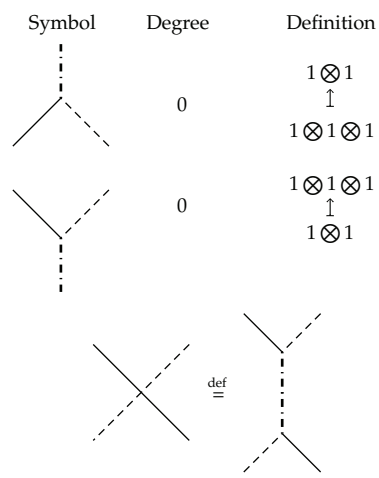


(4.2)

Stated very explicitly, we have two projection maps $p_1, p_2 : ii \to i$, and two inclusion maps $\alpha_1, \alpha_2 : i \to ii$, as indicated in the diagram above, which is just relation (3.25) divided in half. Include the minus sign on the right picture into the map α_2, then one can quickly check $p_1\alpha_1 = 1_i$, $p_2\alpha_2 = 1_i$, $p_1\alpha_2 = 0$, $p_2\alpha_1 = 0$, and $1_{ii} = \alpha_1 p_1 + \alpha_2 p_2$. So these must be projections to and inclusions of summands, and they are maps of the correct degree.

Now, we look at i and j which are distant. We have the isomorphism $B_i \otimes B_j \cong R\otimes_{R^{i,j}} R\{-2\} \cong B_j \otimes B_i$. It is clear that we may construct for any $\underline{\mathbf{k}}$ isomorphisms $\mathrm{Hom}(ij, \underline{\mathbf{k}}) \to \mathrm{Hom}(ji, \underline{\mathbf{k}})$ or vice versa merely by precomposing with the appropriate 4-valent vertex, and (3.32) shows that these maps are inverses.

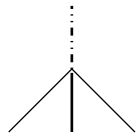
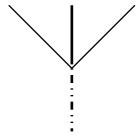
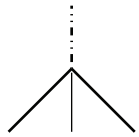
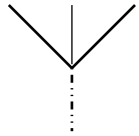
Because we can, let us discuss how we might extend our diagrammatic calculus to include additional modules from R-molf-R, like $R\otimes_{R^{i,j}} R\{-2\}$. Let us (temporarily) allow a new color of line in our pictures, call it w, and extend the functor F_1 so that it sends w to $R\otimes_{R^{i,j}} R\{-2\}$. We add new generators to our diagrammatics, and specify the image under the extended functor, as below

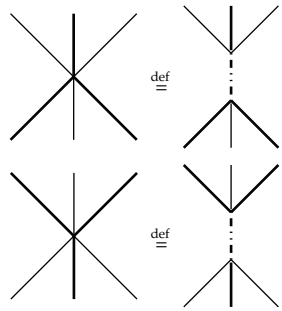


The definition of these bimodule maps, and the proof that they are in fact bimodule maps, is exactly akin to the discussion in Section 3.4. It is clear that composing these morphisms to get an endomorphism of ij will yield the identity map, and composing them to get an endomorphism of w will also yield the identity map (these would be relations in the extended diagrammatic calculus). Thus we get isomorphisms $ij \cong w \cong ji$ in $\mathfrak{D}\mathcal{C}$.

We now combine these techniques to deal with adjacent colors. We have isomorphisms $B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus (R\otimes_{R^{i,i+1}} R\{-3\})$ and $B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus (R\otimes_{R^{i,i+1}} R\{-3\})$. If we allow the new color of line, again called w, to represent the bimodule $R\otimes_{R^{i,i+1}} R\{-3\}$, then we may

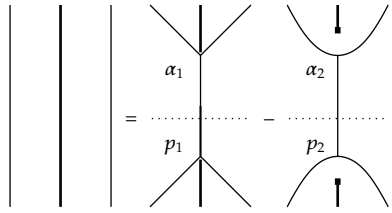
define the following maps:

Symbol	Degree	Definition	
	0	$1\otimes 1$ $\updownarrow$ $1\otimes 1\otimes 1\otimes 1$	$(x_i + x_{i+1})\otimes 1 - 1\otimes 1\otimes x_{i+2}$ $\updownarrow$ $1\otimes x_i\otimes 1\otimes 1$
	0	$1\otimes 1\otimes 1\otimes 1$ $\updownarrow$ $1\otimes 1$	
	0	$1\otimes 1$ $\updownarrow$ $1\otimes 1\otimes 1\otimes 1$	$1\otimes(x_{i+1} + x_{i+2}) - x_i\otimes 1$ $\updownarrow$ $1\otimes x_{i+2}\otimes 1\otimes 1$
	0	$1\otimes 1\otimes 1\otimes 1$ $\updownarrow$ $1\otimes 1$	



Again, checking that we have well-defined bimodule maps is akin to Section 3.4. Composing the two maps that go through $i(i+1)i$ and w to get an endomorphism of w will yield the identity map of w, and the same is true, respectively, of $(i+1)i(i+1)$.

Then the equation (3.26), which is a decomposition of the identity of $i(i+1)i$, actually follows from this relation in $\mathfrak{DC}$



(4.3)

There is a more explicit statement to be derived from this relation, completely analogous to the two-line case, with projection and inclusion maps $p_1, \alpha_1, p_2, \alpha_2$ (include the

minus sign on the right picture in α_2). Similarly we get a decomposition of $(i+1)i(i+1)$ via the same relation with the colors switched. The auxiliary module used here is in fact the indecomposable Soergel bimodule B_w where $w = s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

To state the result without extending the calculus to the Karoubi envelope, one may merely observe that for any $\underline{\mathbf{k}}$ there is a map $\mathrm{Hom}(i(i+1)i, \underline{\mathbf{k}}) \to \mathrm{Hom}((i+1)i(i+1), \underline{\mathbf{k}})$ and vice versa given by precomposing with the appropriate 6-valent vertex; there is another map $\mathrm{Hom}(i(i+1)i, \underline{\mathbf{k}}) \to \mathrm{Hom}(i, \underline{\mathbf{k}})$ given by precomposing with α_2, and a map backwards given by precomposing with p_2, then (3.26) exactly yields that $\mathrm{Hom}(i(i+1)i, \underline{\mathbf{k}}) \oplus \mathrm{Hom}((i+1), \underline{\mathbf{k}}) \cong \mathrm{Hom}((i+1)i(i+1), \underline{\mathbf{k}}) \oplus \mathrm{Hom}(i, \underline{\mathbf{k}})$.

Thus we have shown the following claim:

Claim 4.24. The isomorphisms (2.10) through (2.12) hold in $\mathfrak{D}\mathcal{C}_1$ after applying any Hom functor.

We have now satisfied the requirements of Proposition 2.13, which implies the fullness of F_1.

Claim 4.25. The functor F_1 is full.

Finally, our graphical reductions give us a classification of certain Hom spaces.

Corollary 4.26. *The space $\mathrm{Hom}_{\mathfrak{D}\mathcal{C}_1}(\emptyset, \underline{\mathbf{i}})$, where $\underline{\mathbf{i}}$ is a length d increasing sequence, is a free left (or right) R-module of rank 1, generated by a homogeneous morphism of degree d.*

Proof. Let φ be the morphism consisting entirely of boundary dots. This corresponds, under the functor, to the generator of degree d discussed in Claim 2.9. By Corollary 4.21, φ viewed as a graph underlies every morphism in $\mathfrak{D}\mathcal{C}_1$. This graph has a single region, so any morphism that it underlies will be generated by φ under the action of R on the right or left, which puts a polynomial into that single region. This gives a surjective map from R to the Hom space. After applying F_1, we know that the morphisms must surject onto $\mathrm{Hom}_{\mathcal{SC}_1}(R, B_{\underline{\mathbf{i}}})$, which is a free R-module (see Claim 2.9). Therefore, the Hom space in $\mathfrak{D}\mathcal{C}_1$ must also be free. □

Corollary 4.27. *The semilinear form on H induced by Hom spaces in $\mathfrak{D}\mathcal{C}_1$ agrees with the form defined in Section 2.1, so it agrees with the form induced by $\mathcal{SC}_1$. Therefore F_1 is faithful.*

Proof. This is now immediate, from Remark 2.1. □

In conclusion, F_1 is an equivalence of categories, and Theorem 4.22 is proven.

We also get for free the following corollary, which is difficult to prove purely diagrammatically.

Corollary 4.28. *Hom spaces in $\mathfrak{D}\mathcal{C}_1$ are free as left or right R-modules.*

Remark 4.29. It is worth reiterating what is proven diagrammatically, and what is proven using the functor F_1 to Soergel's category. Diagrammatically, we can prove Proposition 4.19 and Corollary 4.21, which implies that R surjects onto $\mathrm{Hom}_{\mathfrak{D}\mathcal{C}_1}(\emptyset, \underline{\mathbf{i}})$ for $\underline{\mathbf{i}}$ increasing. However, without the full functor F_1 and the nondiagrammatic knowledge of Hom spaces in Soergel's category, we do not know how to prove that the Hom space is free. We do not have a fully diagrammatic proof that the Hom spaces in $\mathfrak{D}\mathcal{C}_1$ are what they are, we only have a diagrammatic proof of an upper bound on the size of the Hom spaces.

4.6. The e_1 Quotient

In the remainder of this chapter, we provide some sketched statements about generalizations and variations of $\mathfrak{DC}_1$. We do not use these results elsewhere in the paper, and while the proofs are only sketched, they are fairly obvious and can be fleshed out without too much work.

As described in Remark 2.2, one usually constructs Soergel bimodules with respect to the n-dimensional fundamental or geometric representation, instead of the $n+1$-dimensional standard representation. This amounts to working over the ring $R_1 = \Bbbk[x_1, x_2, \ldots, x_{n+1}]/(x_1 + x_2 + \cdots + x_{n+1})$, which is a quotient of R by the first elementary symmetric function e_1. Since e_1 is symmetric, it is in the center of the category $\mathfrak{DC}_1$ (it slides freely under all lines, tensor-commuting with all morphisms), and one may easily take the quotient category, setting $e_1 = 0$, without changing any of the diagrammatics. There is a functor from this quotient category to the appropriate category of Soergel bimodules in R_1-molf-R_1, which is an equivalence of categories, so this diagrammatical quotient category also categorifies the Hecke algebra.

One advantage to passing to the quotient $e_1 = 0$ is that, after inverting a suitable integer, we may remove the boxes from our list of generators. According to relation (3.17), the double dot colored i is equal to $x_i - x_{i+1}$. Thus linear combinations of double dots of colors $i = 1, \ldots, n$ will give us the $\Bbbk$-span of $x_1 - x_2, x_2 - x_3, \ldots$ inside the space of linear polynomials (these are the simple roots). This span will not include x_i in R, which is why we require at least one box as an additional generator. However, it is easy to check that, if $n+1$ is invertible in $\Bbbk$, then x_i is in the $\Bbbk$-span after passing to the quotient R_1. As an example, when $n = 1$, the double dot is equal to $x_1 - x_2$, and passing to $x_1 + x_2 = 0$, we get $x_1 = (1/2)(x_1 - x_2)$. Thus, if $n+1$ is invertible in $\Bbbk$, one can eliminate boxes from the quotient calculus altogether, replacing them with linear combinations of double dots.

Addendum 4. *All the relations which involve boxes can be rewritten so that they only involve double dots. This is not done here, but can be found in any of the papers [33, 34, 36].*

Replacing boxes with double dots is much more natural, since it emphasizes that the boxes themselves should have a color, and that the set of polynomials depends in a natural way upon the coxeter group S_{n+1}. Viewing them as boxes may help make the proofs in this paper more intuitive, however.

4.7. Color Elimination

We have already shown that $\mathfrak{DC}_1 \cong \mathcal{SC}_1$, without needing to investigate in depth any morphisms except those from $\emptyset$ to $\underline{\mathbf{i}}$ an increasing sequence. Because this pins down the size of all Hom spaces, we can deduce some additional facts about general morphisms. The following result is not used elsewhere in this paper, but may come in handy when constructing the analogous category for arbitrary Coxeter systems.

Proposition 4.30. *Let Γ be a graph where the color j does not appear in the boundary, then Γ can be reduced to graphs not containing the color j at all.*

We can already show this when j is extremal, since by Proposition 4.19 we may apply the weak j-colored basic moves and reduce the color j to the empty graph. Unfortunately, as in Remark 4.29, we do not have a diagrammatic result of this proposition in general, but use the known size of Hom spaces to prove it.

For $X \subset \{1, \ldots, n\}$, (i.e., for a Coxeter subgraph of A_n) we let $\mathfrak{DC}_1[X]$ be the category defined analogously but where edges can only be labelled by colors in X. If colors which do

not appear on the boundary are not needed in the graph after reduction, then the natural inclusion of $\mathfrak{DC}_1[X]$ into $\mathfrak{DC}_1$ is fully faithful, which one would expect. The full faithfulness of this inclusion is equivalent to the above proposition.

However, as discussed in the previous section, the boxes which are allowed to appear should also depend on X. For the rest of this section, we assume we are working in the e_1 quotient. Let $\mathfrak{DC}_1(X)$ be the category defined analogously, where edges are labelled in X, and where the only polynomials appearing are in the subring $R(X)$ generated by double dots colored in X. This is really the category we should look at. Given a morphism in $\mathfrak{DC}_1[X]$, we can use polynomial forcing rules to guarantee that the only polynomials not in $R(X)$ appear in the lefthand region. We do not prove it here, $\mathfrak{DC}_1[X]$ is simply $\mathfrak{DC}_1(X)\otimes_{R(X)}R$; that is, the objects are unchanged, and morphism spaces undergo base change.

Proposition 4.31. *Let X be a Coxeter subgraph of A_n, and let $W(X)$, $H(X)$, $\mathcal{SC}_1(X)$ designate the corresponding constructions for this Coxeter graph, then there is a functor $F_1(X)$ from $\mathfrak{DC}_1(X)$ to $\mathcal{SC}_1(X)$ which is an equivalence of categories.*

Proof. We will only sketch this result. The set X will be a disjoint union of various subgraphs X_l, each isomorphic to A_{m_l} for some $m_l \leq n$. For each X_l we know that $\mathfrak{DC}_1(X_l) \cong \mathcal{SC}_1(X_l)$, and we have all the results aforementioned. Moreover, for any $l \neq l'$ and objects $M \in \mathfrak{DC}_1(X_l)$ and $N \in \mathfrak{DC}_1(X_l')$, we have natural isomorphisms $M \otimes N \cong N \otimes M$ constructed with 4-valent crossings. Thanks to the distant sliding rules, these natural isomorphisms commute in the proper way with all morphisms in $\mathfrak{DC}_1(X_l)$ or $\mathfrak{DC}_1(X_l')$. The category $\mathfrak{DC}_1(X)$ can be constructed via some universal "symmetric monoidal product" construction, by taking the product over the categories $\mathfrak{DC}_1(X_l)$. The same holds true for $\mathcal{SC}_1(X)$, which will yield the result. □

Given the sketchy result above, we can calculate the rank of Hom spaces in $\mathfrak{DC}_1(X)$ by using the standard trace map on $H(X)$. This trace map commutes with the standard trace on H under the inclusion $H(X) \subset H$. Thus we see that, for objects $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ in $\mathfrak{DC}_1(X)$, the space $\mathrm{Hom}_{\mathfrak{DC}_1}(\underline{\mathbf{i}}, \underline{\mathbf{j}})$ is a free R module of some rank r, and the space $\mathrm{Hom}_{\mathfrak{DC}_1(X)}(\underline{\mathbf{i}}, \underline{\mathbf{j}})$ is a free $R(X)$ module of the same rank r. Therefore $\mathrm{Hom}_{\mathfrak{DC}_1[X]}(\underline{\mathbf{i}}, \underline{\mathbf{j}})$ is a free R module of rank r as well, and the inclusion of $\mathfrak{DC}_1[X]$ in $\mathfrak{DC}_1$ is fully faithful.

We quickly sketch an alternative proof of Proposition 4.30, which assumes that we know that Hom spaces in $\mathfrak{DC}_1$ conform to the standard trace. We can think of any graph with boundary as a morphism in $\mathfrak{DC}_1$ from $\emptyset$ to $\underline{\mathbf{j}}$. We have already "inductively calculated" the space of such morphisms, along the lines of Remark 2.1 and Proposition 2.13. Namely, if $\underline{\mathbf{j}}$ has no repeated colors, then every morphism is a polynomial with boundary dots. If $\underline{\mathbf{j}}$ has repeated colors, we use idempotent decompositions to consider the Hom space as the sum of the Hom spaces of its summands, or apply biadjunction to cycle the sequence $\underline{\mathbf{j}}$, until we have expressed the Hom space in terms of $\mathrm{Hom}(\emptyset, \underline{\mathbf{k}})$ for various $\underline{\mathbf{k}}$ nonrepeating. Since neither biadjunctions nor idempotent decompositions add any new colors to the graph, we have just constructed a generating set of morphisms in a way which does not involve any colors which were not in $\underline{\mathbf{j}}$ to begin with.

5. Proofs

It remains to prove Proposition 3.11, which we do in the first section, and Proposition 4.19, which we do in the second.

5.1. $\mathcal{F}_1$ Is a Functor

We need now to check that each of our relations holds true for Soergel bimodules, after application of F_1. The relations are listed in full in Section 3.4. These checks are entirely tedious and straightforward, and require little imagination. With a little imagination, however, there are some tricks which make checking most relations trivial.

Again, we always use relation (2.6) to identify elements of $B_{\underline{\mathbf{i}}}$ with $d(\underline{\mathbf{i}}) + 1$ tensors of polynomials.

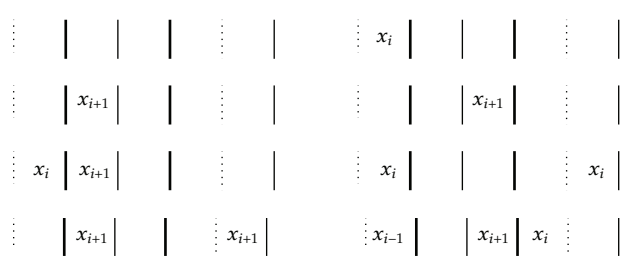
For brevity, we will call any element which is a tensor product $1 \otimes 1 \otimes 1 \otimes \cdots$ a *1-tensor*. EndDot, Split, and the 4-valent and 6-valent vertices all send 1-tensors to 1-tensors. This simple fact is already enough to prove relations (3.1), (3.32), (3.33), (3.34), and (3.36), since the bottom bimodule in each of these pictures is generated by the 1-tensor; we call this the *1-tensor trick*. Caps and Merges kill a 1-tensor, while Cups and StartDots send it to a sum of two-linear terms.

There are several choices to make when checking the various relations. Once the twisting relations are shown, one is free to prove any twist of the other relations. Also, one may choose which set of generators of the source bimodule to check equality on. Finally, whenever a Cup or a StartDot appears, there are two equivalent ways to write the result, since $x_i \otimes 1 - 1 \otimes x_{i+1} = 1 \otimes x_i - x_{i+1} \otimes 1$ in B_i, and one might be easier than the other to evaluate.

The first trick is to choose the set of generators which makes the calculation the easiest. Let us remind ourselves of the arguments of Remark 2.7. An arbitrary module $B_{\underline{\mathbf{i}}}$ of length d will have 2^{d-m} generators as an R-bimodule, where m is the number of different colors that appear. Let us use the term *i-pair* to denote two instances of the index i in $\underline{\mathbf{i}}$, separated only by colors $\neq i$. Let X denote the set of all i-pairs for all i; the size of X is $d - m$. As we force polynomials to the right or left, a variable x_i (or alternatively x_{i+1}) might get stuck between the two i-colored lines of an i-pair, and this independently of each other i-pair or j-pair for $j \neq i$. The following claim is easy to show from the forcing rules.

Claim 5.1. Each $B_{\underline{\mathbf{i}}}$ will be generated as an R-bimodule by any set Y of 2^{d-m} linearly independent tensors, for which we have a bijection between Y and the power set of X, satisfying the following property: If a tensor $y \in Y$ corresponds to a subset $X_y \subset X$, then each i-pair in X_y corresponds to a distinct linear factor of y which is either x_i or x_{i+1} somewhere inside the i-pair.

It is a mouthful, but an example clarifies it. The following is an example of a set of generators for $B_{(i-1)(i+1)i(i+1)(i-1)i}$, where $d - m = 3$ so we need 8 generators:

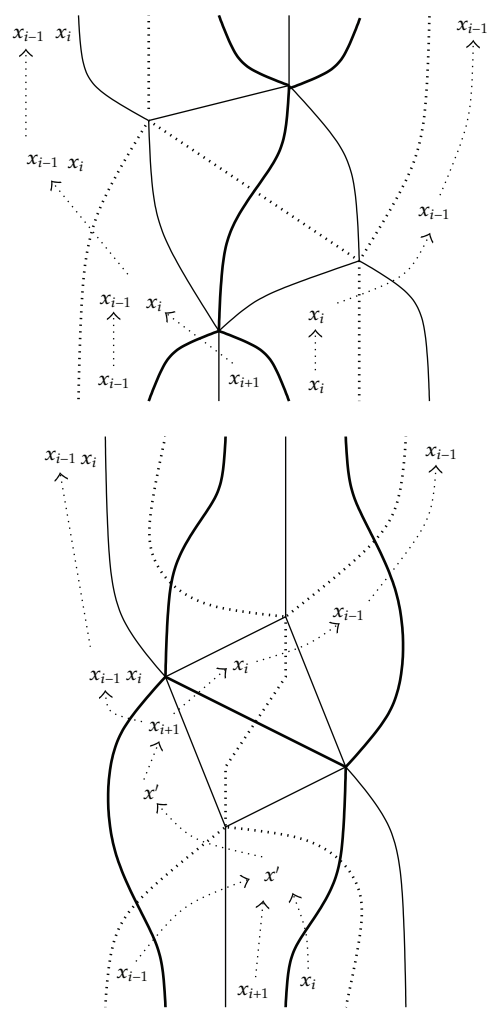


Each picture represents a tensor, where by convention a blank area is filled with a 1-tensor. Reading across, the first picture corresponds to the empty subset of X. The second picture corresponds to the $i - 1$-pair, the third to the $i + 1$-pair, and the fourth to the i-pair. Since these three are linearly independent, they take care of all the linear generators. Clearly

the fourth picture could have also worked for the $i+1$-pair, but if we had chosen it as such, we would have had to choose a new linearly independent vector for the i-pair. The fifth generator corresponds to the i − 1-pair and the $i+1$-pair, the sixth generator to the $i-1$-pair and the i-pair, and the seventh generator to the i-pair and the $i+1$-pair. These three are also linearly independent. The final generator corresponds to the entire set X.

A clever choice of which generators to use may greatly simplify a calculation by reducing the number of terms in intermediate steps. There are two main instances when this occurs. Either 6-valent vertex with strands i and $i+1$ will send $1\otimes x_{i+1}\otimes 1\otimes 1$ or $1\otimes 1\otimes x_{i+1}\otimes 1$ to a single tensor, while it may send $1\otimes x_i\otimes 1\otimes 1$ to the sum of two tensors, thus doubling the work we need to do in the remainder of the calculation. Also, $x_i x_{i+1}$ entering a 6-valent vertex $i(i+1)i$ in the second slot may by moved across the i-strand to the left for a simple calculation, while x_i^2 leads to a more complicated solution.

The second trick will be a useful diagrammatic way of evaluating homomorphisms in $\mathcal{SC}_1$, which really only works well when applied to a well-chosen set of generators. In particular, the choice of generators above makes the verification of triple overlap associativity (3.37) rather straightforward. We demonstrate the graphical method in the most difficult case, the highest degree generator

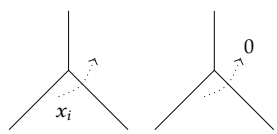


Here $x' = x_{i-1}x_ix_{i+1}$.

We keep track of the image at every stage in the calculation, which is one term as shown, with all blank spaces being filled by 1-tensors. An arrow indicates either bringing a symmetric polynomial through a line, or applying a 6-valent vertex to a tensor. The lack of an arrow indicates that a 1-tensor is sent to a 1-tensor. This works well only because every intermediate term is a single tensor, not a sum of tensors.

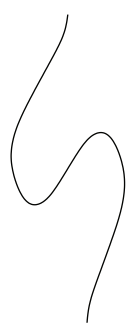
We leave it as an exercise to the reader to verify, using this graphical method of calculation, that both sides of the triple overlap associativity relation, as displayed above, send the other 7 generators to the same elements. Precisely, both sides send the 1-tensor to a 1-tensor, and the other generators to 1-tensors with polynomials in various slots: the 2nd and 4th generators to x_{i+2} in the last slot, the third generator to x_{i-1} in the last slot, the 5th to $x_{i-1}x_{i+2}$ in the last slot, the 6th to x_ix_{i-1} in the first slot, the 7th to $x_{i+1}x_{i+2}$ in the first slot, and the 8th as shown above to x_ix_{i-1} in the first slot and x_{i-1} in the last slot. We promise that the graphical method will work for this set of generators.

One can extend this graphical method easily to handle other morphisms. As an example, we show the Merge map below



When x_{i+1} enters a Merge, it is sent to -1. Almost the same pictures can be drawn for the Cap. Split and EndDot send 1-tensors to 1-tensors, and no arrows are needed.

When there is a cup or a StartDot, a 1-tensor is sent to a sum of two terms, each of which must be evaluated separately. However, the sum can be written as either $x_i \otimes 1 - 1 \otimes x_{i+1}$ or $1 \otimes x_i - x_{i+1} \otimes 1$, so we may choose the one whichever is more convenient. Often the problem of having two terms is very temporary. For example, consider what happens to the 1-tensor under the map



The cup creates two terms, the first with a 1 under the cap and the second with x_i under the cap. But the first term is annihilated immediately by the cap, and the cap eats the x_i from the other term, to return back a 1-tensor. So long as a cup or a StartDot appears right next to a cap or a merge, one of the two terms is always immediately annihilated. This is the case for (3.5), (3.6), (3.8), which all send 1-tensors to 1-tensors as a result. After a quick polynomial slide, the same is true for (3.31), and using the more convenient choice so the x_{i+1} ends up underneath the 6-valent vertex, a quick calculation shows it for (3.27) as well. We call this the *cupcap trick*.

We now list and prove the necessary relations, following the same order as in Section 3.4. One could print out this list and hold it next to that section, where the relations are and the functor is defined, to make this ordeal easier.

(1) Biadjointness (3.5) follows from the cupcap trick.

(2) Trivalent twisting (3.6) follows from the cupcap trick.

(3) Associativity (3.1) follows from the 1-tensor trick.

(4) Dot twisting (3.8) follows from the cupcap trick.

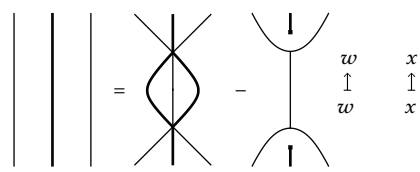
(5) The needle consists of a split, sending a 1-tensor to a 1-tensor, and a cap, killing a 1-tensor. Hence the needle relation (3.18) follows.

(6) The polynomial slide relations clearly hold for Soergel bimodules, by definition.

(7) For the broken line relation (3.16), both sides clearly send the 1-tensor to $x_i \otimes 1 - 1 \otimes x_{i+1}$.

(8) For the double dot relation (3.17), both sides clearly send the 1-tensor to $x_i - x_{i+1}$.

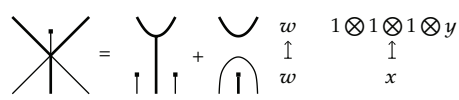
(9) The three-line decomposition relation (3.26) is a calculation. We need to show it for both color variants, whether thin is i and thick $i+1$ or vice versa. Let w be the 1-tensor and $x = 1 \otimes x_{i+1} \otimes 1 \otimes 1$, regardless of which variant we are in, then it is easy to check that



Consider what happens to x, under the first color variant. The double 6-valent vertex sends x to $1 \otimes 1 \otimes 1 \otimes x_{i+2} = 1 \otimes 1 \otimes x_{i+2} \otimes 1$, while the rightmost picture sends x to $1 \otimes x_{i+1} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_{i+2} \otimes 1$. The sum of these two is just x again. We leave similarly easy calculations to the reader in the future.

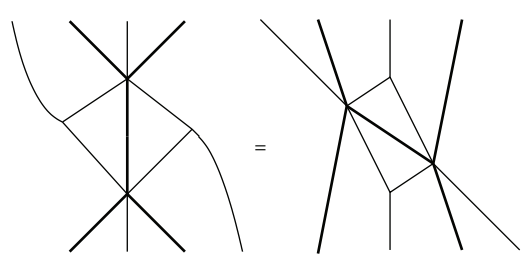
(10) 6-valent twisting (3.27) follows from the cupcap trick.

(11) Adding a dot to a 6-valent vertex (3.28) needs to be checked for both color variants. Define w and x as before, and it is easy to check that



Here $y = x_i$ for one color variant, $y = x_{i+2}$ for the other.

(12) Two-color associativity (3.29) is a calculation. We recommend using the following twist:



and using the four generators: a 1-tensor, $1 \otimes 1 \otimes x_{i+1} \otimes 1 \otimes 1$, $1 \otimes x_{i+1} \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes x_{i+1} \otimes 1 \otimes x_{i+1} \otimes 1$. Evaluate using the graphical calculus. The first two generators are killed by both maps thanks to merge maps. The third generator is send to a 1-tensor by both maps. The final generator is also killed by both maps, as symmetric polynomials are slid out of the way and merge maps eat the remaining 1-tensor.

(13) 4-valent twisting (3.31) follows from the cupcap trick.

(14) Sliding a dot or a trivalent vertex through a distant line both follow from the 1-tensor trick.

(15) Sliding a 6-valent vertex through a distant line is easily checked on both generators.

(16) Sliding a 4-valent vertex through a distant line follows from the 1-tensor trick.

(17) Three-color associativity will be a calculation, using the generators and pictures described above, that we leave to the reader.

5.2. Graphical Proofs

In this section, we provide graphical proofs for a series of propositions which collectively prove Proposition 4.19. Recall the definitions of the strict and weak i-colored moves from Section 4.4. Remember that an extremal color i is one which appears in a graph, but for which either $i-1$ or $i+1$ does not appear. Each of the results states what reduction moves the relations of $\mathfrak{D}\mathcal{C}_1$ allow. Because more than three colors may be required for some proofs, we use extra line styles that designate other arbitrary colors, and are very explicit about what colors they are allowed to be. A thin line will still always represent i and a thick line $i+1$, but we will be lax about other colors.

Lemma 5.2. *One may strictly pull a distant line under any other graph, like in the relations* (3.32)–(3.36). *That is, these relations can be viewed as graph reduction moves.*

Proof. The only significant part of this lemma is that we can still apply the relations mentioned, even in the presence of arbitrary polynomials in each region. However, it is an immediate observation that any polynomial inside a region bounded by at least two-lines of distant colors may be slid entirely out of the region, which does not affect the underlying graph. □

The above lemma highlights the fact that we must deal with polynomials when they appear. However, the only significant polynomials are those which appear in internal regions (regions not touching the boundary of the planar strip) since all our computations will be local, so polynomials in external regions can be moved outside the calculation. We will mention whenever a polynomial is relevant, and the reader can check that whenever we do not mention it, there are only external regions.

Proposition 5.3. *One may apply the (strict) i-colored double dot removal move to any graph* Γ.

Proof. Suppose the i-graph of Γ contains two dots connected by an edge, then in Γ, the "edge" connecting them can only meet series of 4-valent vertices with various j-strands for j distant from i. Since dots can be slid across distant-colored strands by (3.33), we may slide the dots until they are connected directly by an edge, and just use (3.17) to reduce the graph. □

Proposition 5.4. *One may apply the weak i-colored dot contraction move to any graph* Γ.

Proof. Suppose the i-graph of Γ contains a trivalent vertex connected to a dot. In Γ, the trivalent vertex is either a trivalent or 6-valent vertex v, and the edge connected the dot to the vertex may have numerous 4-valent vertices. Again, in Γ the dot may be slid across distant-colored strands until it is connected directly to v. If v is trivalent then (3.4) allows strict dot contraction, while if v is 6-valent, (3.28) sends Γ to the sum of two graphs allowed by weak dot contraction. □

Proposition 5.5. *One may apply the (strict) i-colored dot extension move to any graph Γ.*

Proof. This is trivial, by (3.4). □

The following lemma is needed to prove the remainder of the basic i-colored moves.

Lemma 5.6. *Let Γ be a sequence of lines $\underline{\mathbf{j}}$ (i.e., the identity map of $\underline{\mathbf{j}}$), and let i be an extremal color in $\underline{\mathbf{j}}$, then Γ is equal in $\mathfrak{DC}_1$ to a sum of idempotents, such that each idempotent factors through a sequence of lines $\underline{\mathbf{k}}$ where i appears no more than once, and such that the idempotents do not introduce any new colors not present in $\underline{\mathbf{j}}$.*

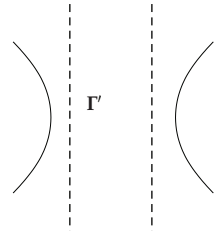
Proof. This proof is akin to the combinatorial argument that the relations defining the Hecke algebra or the symmetric algebra are enough to take any complicated word in b_j or s_j to a reduced form. We use induction on the length of $\underline{\mathbf{j}}$. If $k(k+1)k$ appears in $\underline{\mathbf{j}}$ for some k, then using (3.26), we may factor through $(k+1)k(k+1)$ instead, modulo morphisms that factor through shorter length sequences. If any color appears twice consecutively in $\underline{\mathbf{j}}$ then we may apply (3.25) to replace the two adjacent lines with an "H", which factors through a sequence of lines of shorter length. If jk appears for j distant from k, then applying the R2 move we can factor through kj. None of these procedures added any new colors. So if we consider the sequence $\underline{\mathbf{j}}$ as a word in the symmetric group $s_{j_1}s_{j_2}\cdots s_{j_d}$, then any nonreduced words will reduce to smaller length sequences, and any two reduced words for the same element will factor through each other, modulo smaller length sequences.

We may now use the easily observable fact that any element of the symmetric group can be represented by a reduced word which uses s_1 at most once, or s_n at most once (although not both!). It follows from this that any element in a parabolic subgroup of S_{n+1} can be represented by a reduced word which uses an extremal index at most once. Thus, by the process above, we can replace $\underline{\mathbf{j}}$ by idempotents factoring through reduced expressions $\underline{\mathbf{k}}$ which use i at most once. □

The usefulness of this lemma can be seen in the following proof.

Proposition 5.7. *One may apply the (strict) i-colored connecting move to any graph Γ, so long as i is extremal.*

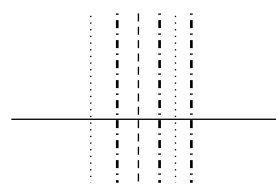
Proof. Assume $i-1$ does not appear, although the $i+1$ case is analogous. Induct on the number of colors present in the graph: the base case of one color is clear, then we are attempting to apply the i-colored connecting move to some region of the graph which looks like this:



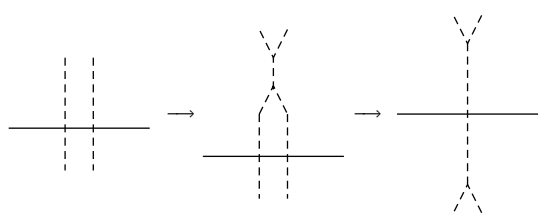
Γ' is some graph composed entirely of the colors $1, \ldots, i-2$ and $i+1, \ldots, n$, so that $i+1$ is extremal or nonexistent. In some neighborhood Γ' is just a sequence of lines $\mathbb{1}_{\underline{\mathbf{j}}}$. By the previous lemma, we may replace this sequence of lines with a sum of idempotents factoring through sequences $\underline{\mathbf{k}}$ where $i+1$ appears at most once, and $i-1$ appears not at all. Localizing to a neighborhood around $\mathbb{1}_{\underline{\mathbf{k}}}$, we may as well assume that $\underline{\mathbf{j}} = \underline{\mathbf{k}}$. Now apply the R2 move to bring the i-strands past all the other colors so that they are separated by either nothing or a single $i+1$ strand. We have not yet altered the i-graph at all, then equation (3.25) or equation (3.26) (respectively) will allow the strict i-colored connecting move. □

When we have an edge in an i-graph, a neighborhood of that edge in the full graph will be nothing more than a sequence of 4-valent vertices as the i-strand crosses distant strands. The next corollary allows us to place restrictions on which distant strands the i-strand need cross.

Corollary 5.8. *Suppose we are given a graph Γ which is a neighborhood of an i-colored strand consisting only of 4-valent vertices crossing the i-strand (see the picture below), then Γ is equivalent in $\mathfrak{D}\mathcal{C}_1$ to a linear combination of graphs Γ' which have the same i-graph, and for which the sequence of lines crossing the i-strand in Γ' contains $i+2$ and $i-2$ each at most once.*

$\Gamma =$ 

Proof. Let $\underline{\mathbf{j}}$ be the sequence of lines crossing the i-strand in Γ; clearly i, $i-1$, and $i+1$ do not appear in $\underline{\mathbf{j}}$. According to the above lemma, we may replace $\mathbb{1}_{\underline{\mathbf{j}}}$ with a sum of idempotents which factor through "nice" sequences $\underline{\mathbf{k}}$ where $i+2$ and $i-2$ each appear at most once. Moreover, for any idempotent appearing in this sum, i is distant from every color appearing in the idempotent (as well as the polynomials which may appear in the idempotent). So replacing $\mathbb{1}_{\underline{\mathbf{j}}}$ with this sum immediately above the i-strand, we may then slide the i-strand in each idempotent so that it passes through $\underline{\mathbf{k}}$. An example is given below, ignoring polynomials which can also be slid.

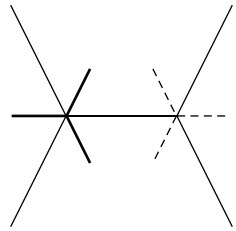


□

The last two moves will take the rest of the paper to prove.

Proposition 5.9. *One may apply the (strict) i-colored needle move to any graph Γ.*

Proposition 5.10. *One may apply the weak i-colored associativity move to any i-colored "H" in any graph Γ, so long as the "H" does not look like the following picture:*

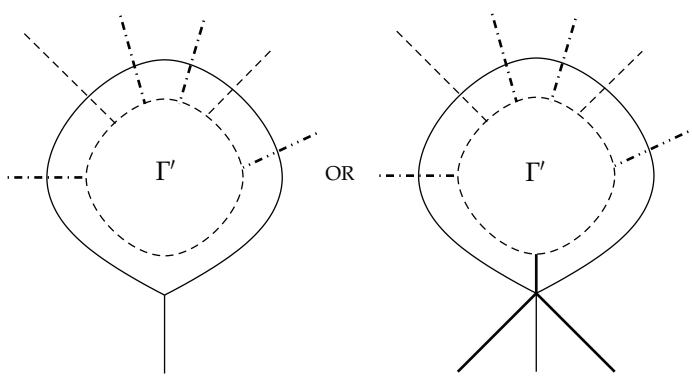


where the two vertices are 6-valent in Γ with different colors $i+1$ and $i-1$.

In particular, we may apply the i-colored associativity move to any graph Γ when i is extremal.

Proof Setup. We apply induction on the set of colors present in the graph to prove these two propositions simultaneously. Both propositions hold when i is the only color in the graph, by (3.10), (3.18), (3.24), where the latter is needed because an arbitrary polynomial may be within the eye of the needle. The inductive hypothesis then implies, with the previous propositions, that for any graph with fewer colors, we may apply *all* the weak k-colored basic moves for an extremal color k. In this case Proposition 4.17 says that we may reduce the k-graph to a disjoint union of simple trees. Within each region of Γ delimited by the i-graph the color i is absent, so there are fewer total colors and we may reduce both the $i-1$-graph and the $i+1$-graph inside this region to a union of simple trees. This application of the inductive hypothesis was the only reason to consider the two propositions together; we now treat them separately.

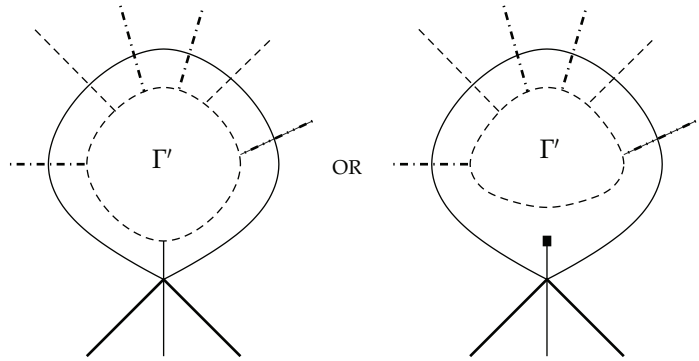
Needle Move: Modulo Induction Hypothesis. Suppose the i-colored graph contains a needle. The trivalent vertex of the needle is either trivalent or 6-valent in Γ, and the edge of the needle may contain a series of 4-valent vertices. So a neighborhood of the needle in Γ looks like one of the pictures below.



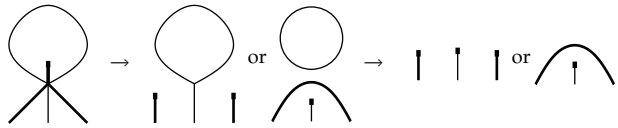
The lines entering the needle around the top are colored with various j all distant from i. We assume without loss of generality that, in the second case, the 6-valent vertex has second color $i+1$.

We may now reduce the $i+1$ and $i-1$ graph of Γ', since Γ' does not contain the color i. In the first case, the $i \pm 1$ graph has no boundary, so both reduce to the empty graph. In the second case, $i+1$ has a single boundary line so it reduces to a boundary dot, and $i-1$ has none so it reduces to the empty graph. Within the reduced Γ', the possible $i+1$ dot may be

slid under other lines until no 4-valent vertices separate it from the 6-valent vertex (as in the proof of weak dot contraction). Thus we may assume that, after reduction, our neighborhood looks like

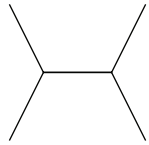


where now Γ' does not contain any of the colors $i-1$, i, $i+1$. Note that Γ' may have arbitrary polynomials in its various regions. But then by Lemma 5.2 we can pull the line forming the needle through all of Γ', thus completely ignoring Γ' from the picture! There still may be a polynomial in the eye of the needle. We have effectively reduced to the 2-color case on the right, or the one color case on the left. We know the one color case works. To check the 2-color case, we use (3.28) followed by other reduction moves.

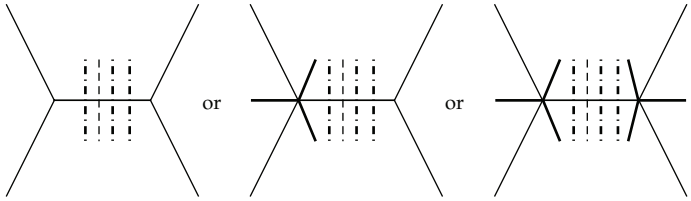


In these final graphs, the i-graph is a dot, as desired. So we may apply the strict i-colored needle move.

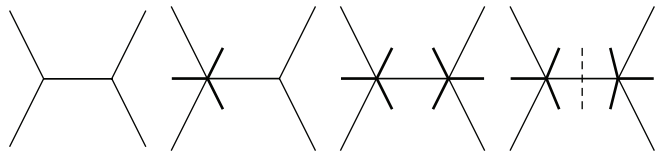
Associativity Move: Modulo Induction Hypothesis. We would like to apply associativity to the following subgraph of the i-graph.



Each trivalent vertex of the i-graph may be either trivalent or 6-valent in Γ. We are forbidding the case when one is 6-valent with $i+1$ and the other with $i-1$, so without loss of generality with assume that a 6-valent vertex has $i+1$ as the other color. The neighborhood of this subgraph in Γ looks like one of the following cases:



All polynomials may be assumed to lie outside the neighborhood. We may use Corollary 5.8 to assume that the sequence of lines passing through the middle strand contains at most one instance of $i+2$ and $i-2$. In the first two cases, all such lines may be slid one by one to the right over the trivalent vertex, removing them from the neighborhood. In the third case, all lines not labelled $i+2$ can be slid to the right or to the left, and since there is at most one line labelled $i+2$, then at most one line remains. (If there had been multiple lines labelled $i+2$, then additional lines may have been stuck between them, but thankfully Corollary 5.8 eliminates this possibility.) Four cases remain:



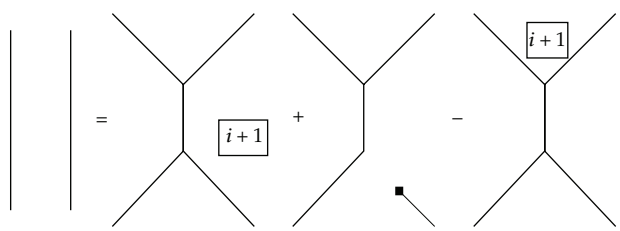
In the fourth case, the additional line is $i+2$. So far, the i-graph has been unchanged.

Case 1. One color associativity allows the strict i-colored associativity move.

Case 2. Double overlap associativity (3.30) allows the strict i-colored associativity move.

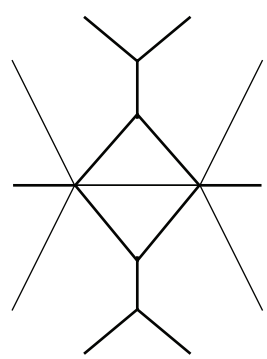
The remaining two cases use the same trick: they replace the interior lines on the top and bottom of the graph with the corresponding sum of idempotents, and then resolve each one with double or triple overlap associativity. The remaining two cases will not allow the strict associativity move, only the weak one.

Case 3. We rewrite equation (3.25), using (3.16), so that there is no polynomial on the bottom.



Applying this to the thick edges on top of Case 3, and symmetrically to the bottom, we get a difference of graphs of two kinds.

For terms which do not involve a dot, we get a graph which looks like the following, with *no* polynomial in any interior region:

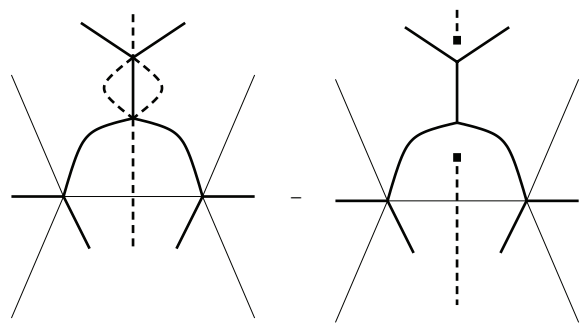


We may apply double overlap associativity (3.29) to a subgraph of this diagram (ignoring the top and bottom thick trivalent vertices), which will apply associativity to the i-graph.

Consider a term where a dot connects to a 6-valent vertex. Resolving the dot using (3.28), we get a difference of two further terms: one which looks like Case 2, and the other which is one of the alternate graphs allowed by the weak associativity move.

Thus we may apply weak i-colored associativity to each term.

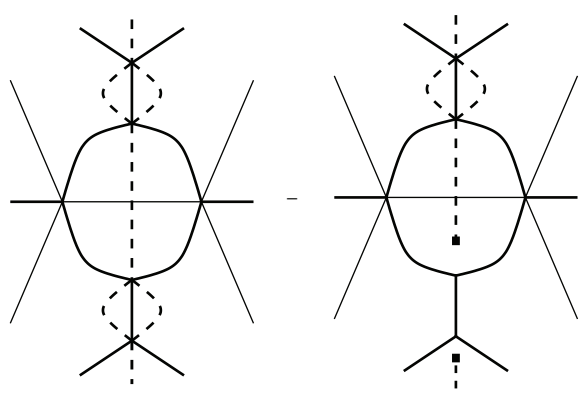
Case 4. Applying equation (3.26) to the $i+1$, $i+2$, $i+1$ sequence on top of the graph, we get the difference



Now we need to show that we can apply associativity to each of these.

For the second picture, we may drag the dot on $i+2$ through the i-strand, and then a smaller neighborhood of the X looks like Case 3.

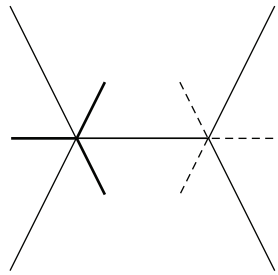
For the first picture, we once again apply (3.26) to the bottom of the graph to get the difference



Again, for the second picture we may drag the dot through and reduce using Case 3.

For the first picture, there are *no* polynomials in any internal region, because these internal regions were just created by relations. We may now apply triple overlap associativity (3.37) to the subgraph of the this picture which ignores the top and bottom 6-valent vertices. This has the effect of applying weak i-colored associativity.

Remark 5.11. It does not seem possible to apply i-colored associativity to the following graph.



Consider switching the i-graph to the other version of an "H", and look at the triangular region delimited by the i-graph on top. It has at most one 6-valent vertex on its perimeter, so that its boundary lines consist of an $i-1$ line, an $i+1$ line, and at most one other $i \pm 1$ line. This means that one of the two colors $i-1$ or $i+1$ must reduce to a boundary dot. It is not possible to obtain a degree 0 morphism if this happens.

Because of the failure of i-colored associativity in this context, our algorithm towards reducing graphs has always been to treat extremal colors and use induction. Despite this pessimism, Proposition 4.30 (color elimination) implies that we can still do a lot. However, any purely graphical proof of color elimination is complicated by the failure of i-colored associativity for nonextremal colors.

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Research Article

The Diagrammatic Soergel Category and $sl(2)$ and $sl(3)$ Foams

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We define two functors from Elias and Khovanov's diagrammatic Soergel category, one targeting Clark-Morrison-Walker's category of disoriented $sl(2)$ cobordisms and the other targeting the category of (universal) $sl(3)$ foams.

1. Introduction

In this paper we define functors between the Elias-Khovanov diagrammatic version of the Soergel category $\mathcal{SC}$ defined in [1] and the categories of universal $sl(2)$ and $sl(3)$ foams defined in [2, 3].

The Soergel category provides a categorification of the Hecke algebra and was used by Khovanov in [4] to construct a triply-graded link homology categorifying the HOMFLYPT polynomial. Elias and Khovanov constructed in [1] a category defined diagrammatically by generators and relations and showed it to be equivalent to $\mathcal{SC}$.

The $sl(2)$ and $sl(3)$ foams were introduced in [2, 5] and in [3, 6], respectively, to give topological constructions of the $sl(2)$ and $sl(3)$ link homologies.

This paper can be seen as a first step towards the construction of a family of functors between $\mathcal{SC}$ and the categories of $sl(N)$-foams for all $N \in \mathbb{Z}_+$, to be completed in a subsequent paper [7]. The functors $\mathcal{F}_{sl(2),n}$ and $\mathcal{F}_{sl(3),n}$ are not faithful. In [7] we will extend the construction of these functors to all N. The whole family of functors is faithful in the following sense: if for a morphism f in $\mathcal{SC}_1$ we have $\mathcal{F}_{sl(N),n}(f) = 0$ for all N, then $f = 0$. With these functors one can try to give a graphical interpretation of Rasmussen's [8] spectral sequences from the HOMFLYPT link homology to the $sl(N)$-link homologies.

The plan of the paper is as follows. In Section 2 we give a brief description of Elias and Khovanov's diagrammatic Soergel category. In Section 3 we describe the category $\mathbf{Foam}_2$

of $sl(2)$ foams and construct a functor from $\mathcal{SC}$ to $\mathbf{Foam}_2$. Finally in Section 4 we give the analogue of these results for the case of $sl(3)$ foams.

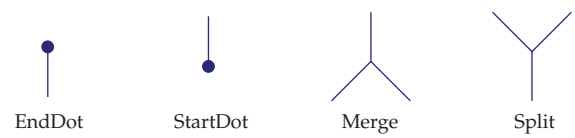
We have tried to keep this paper reasonably self-contained. Although not mandatory, some acquaintance with [1–3, 9] is desirable.

2. The Diagrammatic Soergel Category Revisited

This section is a reminder of the diagrammatics for Soergel categories introduced by Elias and Khovanov in [1]. Actually we give the version which they explained in [1, Section 4.5] and which can be found in detail in [9].

Fix a positive integer n. The category $\mathcal{SC}_1$ is the category whose objects are finite length sequences of points on the real line, where each point is colored by an integer between 1 and n. We read sequences of points from left to right. Two colors i and j are called adjacent if $|i-j| = 1$ and distant if $|i-j| > 1$. The morphisms of $\mathcal{SC}_1$ are given by generators modulo relations. A morphism of $\mathcal{SC}_1$ is a $\mathbb{C}$-linear combination of planar diagrams constructed by horizontal and vertical gluings of the following generators (by convention no label means a generic color j).

(i) Generators involving only one color are as follows:



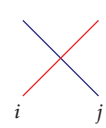
(2.1)

It is useful to define the cap and cup as

(2.2)

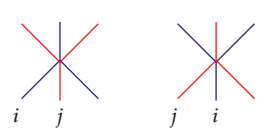
(ii) Generators involving two colors are as follows:

- The 4-valent vertex, with distant colors,



(2.3)

- and the 6-valent vertex, with adjacent colors i and j

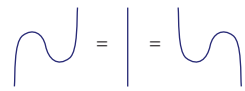


(2.4)

read from bottom to top. In this setting a diagram represents a morphism from the bottom boundary to the top. We can add a new colored point to a sequence and this endows $\mathcal{SC}_1$ with a monoidal structure on objects, which is extended to morphisms in the obvious way. Composition of morphisms consists of stacking one diagram on top of the other.

We consider our diagrams modulo the following relations.

"Isotopy" Relations.



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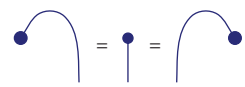
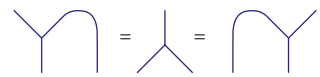
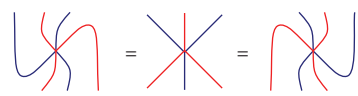


(2.6)



(2.7)

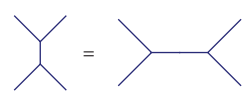
(2.8)



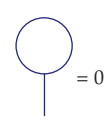
(2.9)

The relations are presented in terms of diagrams with generic colorings. Because of isotopy invariance, one may draw a diagram with a boundary on the side, and view it as a morphism in $\mathcal{SC}_1$ by either bending the line up or down. By the same reasoning, a horizontal line corresponds to a sequence of cups and caps.

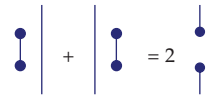
One Color Relations.



(2.10)

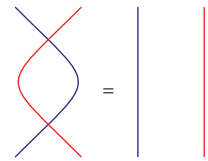


(2.11)

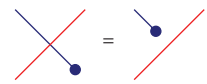


(2.12)

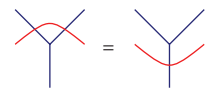
Two Distant Colors.



(2.13)

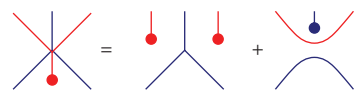


(2.14)

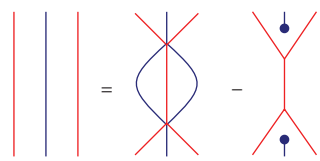


(2.15)

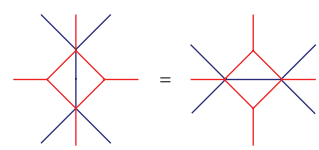
Two Adjacent Colors.



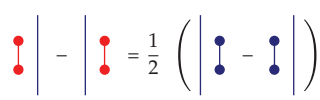
(2.16)



(2.17)

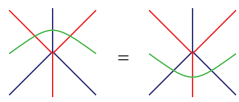


(2.18)

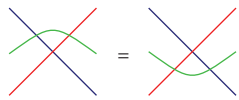


(2.19)

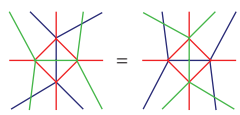
Relations Involving Three Colors: (Adjacency is determined by the vertices which appear)



(2.20)



(2.21)



(2.22)

Introduce a q-grading on $\mathcal{SC}_1$ declaring that dots have degree 1, trivalent vertices have degree -1 and 4-, and 6-valent vertices have degree 0.

Definition 2.1. The category $\mathcal{SC}_2$ is the category containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$ and whose morphisms are the grading-preserving morphisms from $\mathcal{SC}_1$.

Definition 2.2. The category $\mathcal{SC}$ is the Karoubi envelope of the category $\mathcal{SC}_2$.

Elias and Khovanov's main result in [1] is the following theorem.

Theorem 2.3 (Elias-Khovanov). *The category $\mathcal{SC}$ is equivalent to the Soergel category in [10].*

From Soergel's results from [10] we have the following corollary.

Corollary 2.4. *The Grothendieck algebra of $\mathcal{SC}$ is isomorphic to the Hecke algebra.*

Notice that $\mathcal{SC}$ is an additive category but not abelian and we are using the (additive) split Grothendieck algebra.

In Sections 3 and 4 we will define functors from $\mathcal{SC}_1$ to the categories of $sl(2)$ and $sl(3)$ foams. These functors are grading preserving, so they obviously extend uniquely to $\mathcal{SC}_2$. By the universality of the Karoubi envelope, they also extend uniquely to functors between the respective Karoubi envelopes.

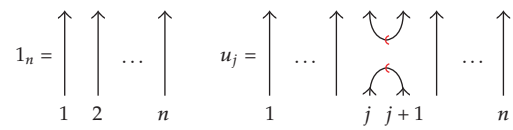
3. The $sl(2)$ Case

3.1. Clark-Morrison-Walker's Category of Disoriented $sl(2)$ Foams

In this subsection we review the category $\mathbf{Foam}_2$ of $sl(2)$ foams following Clark et al. construction in [2]. This category was introduced in [2] to modify Khovanov's link homology theory making it properly functorial with respect to link cobordisms. Actually we will use

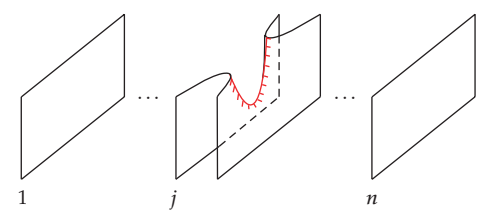
the version with dots of Clark-Morrison-Walker's original construction in [2]. Recall that we obtain one from the other by replacing each dot by 1/2 times a handle.

A *disoriented arc* is an arc composed by oriented segments with oppositely oriented segments separated by a mark pointing to one of these segments. A *disoriented diagram* consists of a collection D of disoriented arcs in the strip in $\mathbb{R}^2$ bounded by the lines $y = 0,\ 1$ containing the boundary points of D. We allow diagrams containing oriented and disoriented circles. Disoriented diagrams can be composed vertically, which endows $\mathbf{Foam}_2$ with a monoidal structure on objects. For example, the diagrams 1_n and u_j for $(1 < j < n)$ are disoriented diagrams:



(3.1)

A *disoriented cobordism* between disoriented diagrams is a $2D$ cobordism which can be decorated with dots and with seams separating differently oriented regions and such that the vertical boundary of each cobordism is a set (possibly empty) of vertical lines. Disorientation seams can have one out of two possible orientations which we identify with a fringe. We read cobordisms from bottom to top. For example,



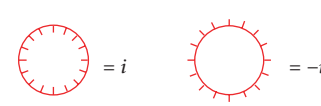
(3.2)

is a disoriented cobordism from 1_n to u_j.

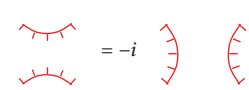
Cobordism composition consists of placing one cobordism on top of the other and the monoidal structure is given by vertical composition which corresponds to placing one cobordism behind the other in our pictures. Let $\mathbb{C}[t]$ be the ring of polynomials in t with coefficients in $\mathbb{C}$.

Definition 3.1. The category $\mathbf{Foam}_2$ is the category whose objects are disoriented diagrams, and whose morphisms are $\mathbb{C}[t]$-linear combinations of isotopy classes of disoriented cobordisms, modulo some relations:

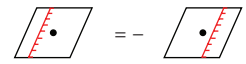
(i) the disorientation relations



(3.3)



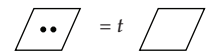
(3.4)



(3.5)

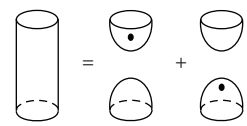
where i is the imaginary unit,

(ii) and the Bar-Natan (BN) relations



(3.6)

(3.7)



(3.8)

which are only valid away from the disorientations.

The universal theory for the original Khovanov homology contains another parameter h, but we have to put $h = 0$ in the Clark-Morrison-Walker's cobordism theory over a field of characteristic zero. Suppose that we have a cylinder with a transversal disoriented circle. Applying (3.8) on one side of the disorientation circle followed by the disoriented relation (3.3) gives a cobordism that is independent of the side chosen to apply (3.8) only if $h = 0$ over a field of characteristic zero.

Define a q-grading on $\mathbb{C}[t]$ by $q(1) = 0$ and $q(t) = 4$. We introduce a q-grading on $\mathbf{Foam}_2$ as follows. Let f be a cobordism with $|\bullet|$ dots and $|b|$ vertical boundary components. The q-grading of f is given by

$$q(f) = -\chi(f) + 2|\bullet| + \frac{1}{2}|b|, \tag{3.9}$$

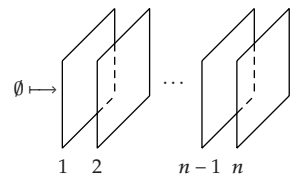
where χ is the Euler characteristic. For example, the degree of a saddle is 1 while the degree of a cap or a cup is -1. The category $\mathbf{Foam}_2$ is additive and monoidal. More details about $\mathbf{Foam}_2$ can be found in [2].

3.2. The Functor $\mathcal{F}_{sl(2),n}$

In this subsection we define a monoidal functor $\mathcal{F}_{sl(2),n}$ between the categories $\mathcal{SC}$ and $\mathbf{Foam}_2$.
On Objects. $\mathcal{F}_{sl(2),n}$ sends the empty sequence to 1_n and the one-term sequence (j) to u_j with $\mathcal{F}_{sl(2),n}(jk)$ given by the vertical composite $u_j u_k$.

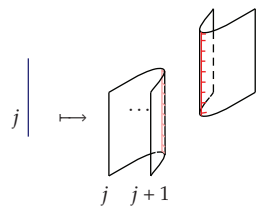
On Morphisms

(i) The empty diagram is sent to n parallel vertical sheets:



(3.10)

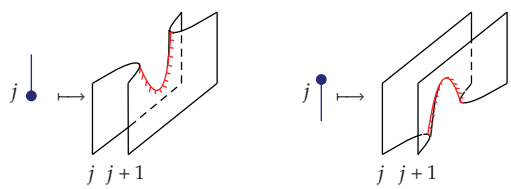
(ii) The vertical line colored j is sent to the identity cobordism of u_j:



(3.11)

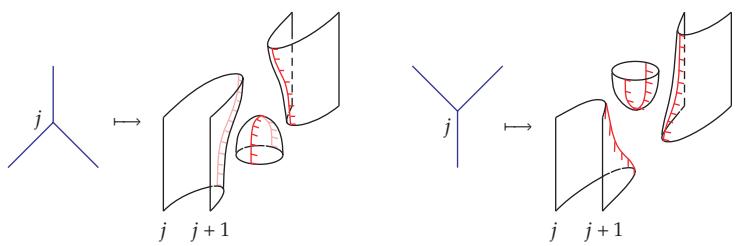
The remaining $n-2$ vertical parallel sheets on the r.h.s. are not shown for simplicity, a convention that we will follow from now on.

(iii) The *StartDot* and *EndDot* morphisms are sent to saddle cobordisms:



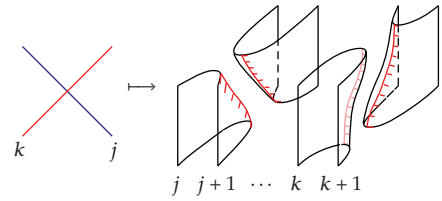
(3.12)

(iv) *Merge* and *Split* are sent to cup and cap cobordisms:



(3.13)

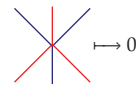
(v) The *4-valent vertex* with distant colors is given as follows. For $j+1<k$ we have



(3.14)

The case $j>k+1$ is given by reflection in a horizontal plane.

(vi) The *6-valent vertices* are sent to zero:



(3.15)

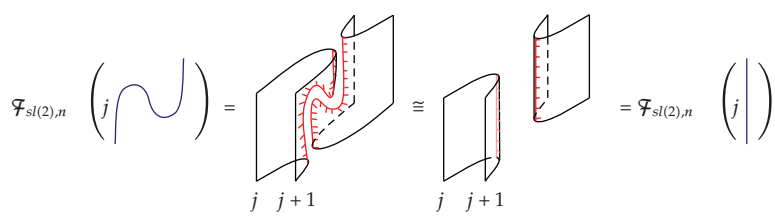
Notice that $\mathcal{F}_{sl(2),n}$ respects the gradings of the morphisms. Taking the quotient of $\mathcal{SC}$ by the 6-valent vertex gives a diagrammatic category $\mathcal{TL}$ categorifying the Temperley-Lieb algebra. According to [11] relations (2.16) and (2.17) can be replaced by a single relation in $\mathcal{TL}$. The functor $\mathcal{F}_{sl(2),n}$ descends to a functor between $\mathcal{TL}$ and $\mathbf{Foam}_2$.

Proposition 3.2. *$\mathcal{F}_{sl(2),n}$ is a monoidal functor.*

Proof. The assignment given by $\mathcal{F}_{sl(2),n}$ clearly respects the monoidal structures of $\mathcal{SC}_1$ and $\mathbf{Foam}_2$. So we only need to show that $\mathcal{F}_{sl(2),n}$ is a functor, that is, it respects the relations (2.5) to (2.22) of Section 2.

"Isotopy Relations"

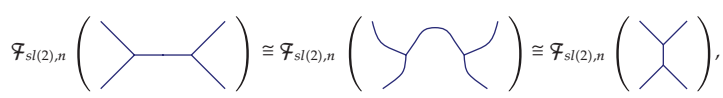
Relations (2.5) to (2.8) are straightforward to check and correspond to isotopies of their images under $\mathcal{F}_{sl(2),n}$ which respect the disorientations. Relation (2.9) is automatic since $\mathcal{F}_{sl(2),n}$ sends all terms to zero. For the sake of completeness we show the first equality in (2.5). We have



(3.16)

One Color Relations

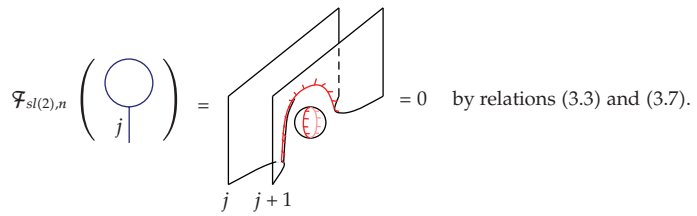
For relation (2.10) we have



(3.17)

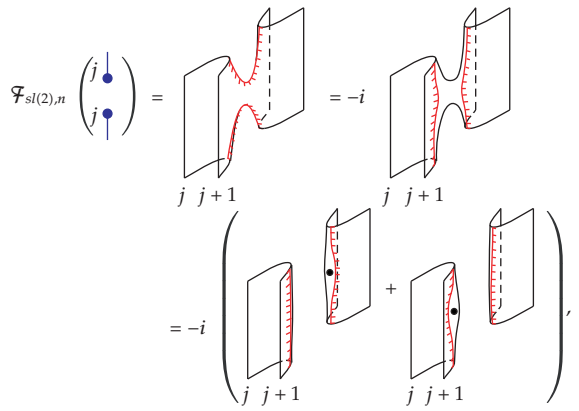
where the first equivalence follows from relations (2.5) and (2.7) and the second from isotopy of the cobordisms involved.

For relation (2.11) we have



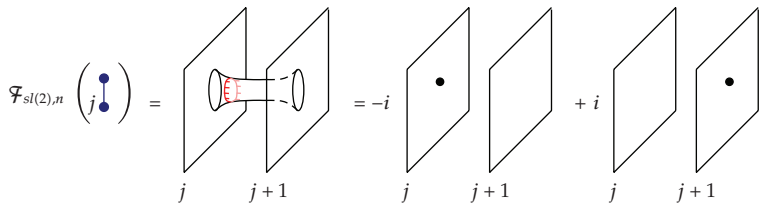
(3.18)

Relation (2.12) requires some more work. We have



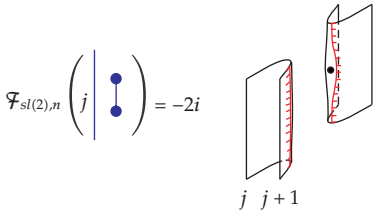
(3.19)

where the second equality follows from the disoriented relation (3.4) and the third follows from the BN relation (3.8). We also have

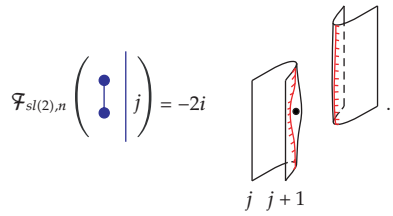


(3.20)

and therefore

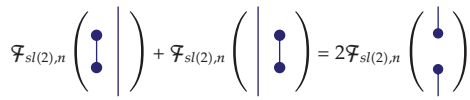


(3.21)



(3.22)

We thus have that



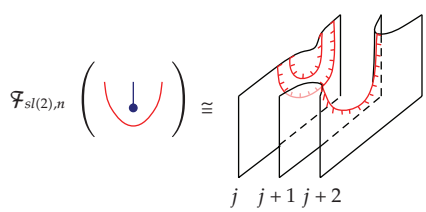
(3.23)

Two Distant Colors

Relations (2.13) to (2.15) correspond to isotopies of the cobordisms involved and are straightforward to check.

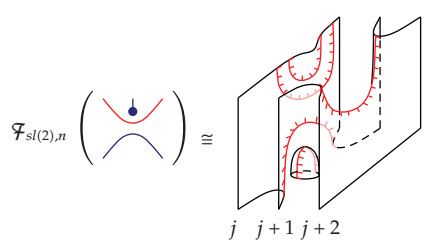
Adjacent Colors

We prove the case where "blue" corresponds to j and "red" corresponds to $j+1$. The relations with colors reversed are proved the same way. To prove relation (2.16) we first notice that



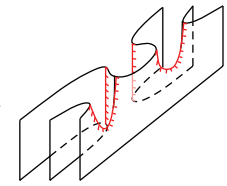
(3.24)

which means that



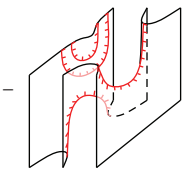
(3.25)

On the other side we have

$\mathcal{F}_{sl(2),n}\Big($  $\Big) \cong$ 

(3.26)

which, using isotopies and the disorientation relation (3.4) twice, can be seen to be equivalent to

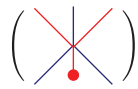
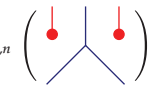
$-$ 

(3.27)

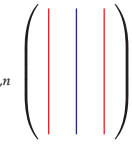
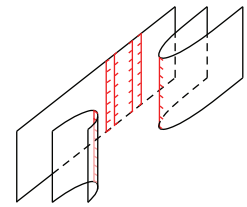
which equals

$-\mathcal{F}_{sl(2),n}\Big($  $\Big).$ (3.28)

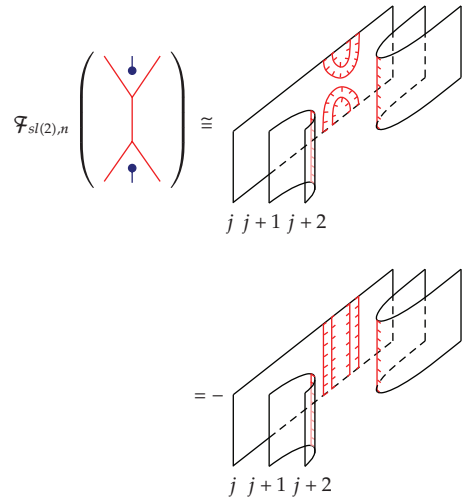
This implies that

$0 = \mathcal{F}_{sl(2),n}\Big($  $\Big) = \mathcal{F}_{sl(2),n}\Big($  $\Big) + \mathcal{F}_{sl(2),n}\Big($  $\Big).$ (3.29)

We now prove relation (2.17). We have isotopy equivalences

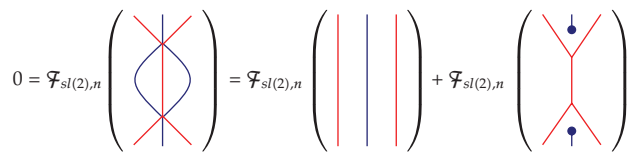
$\mathcal{F}_{sl(2),n}\Big($  $\Big) \cong$ 

(3.30)



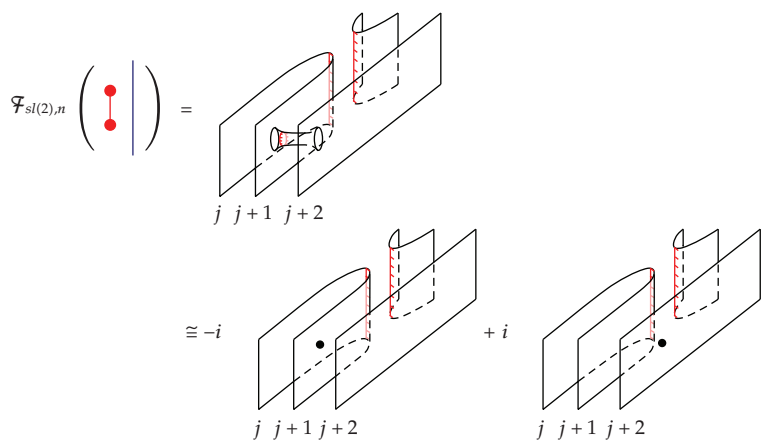
(3.31)

Therefore we see that



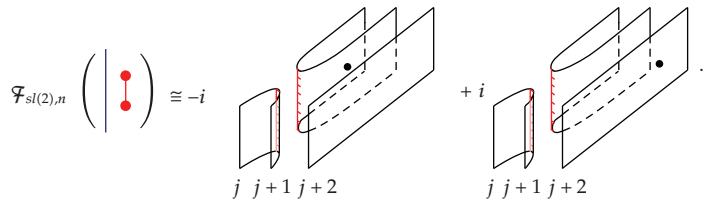
(3.32)

The functor $\mathcal{F}_{sl(2),n}$ sends both sides of relation (2.18) to zero and so there is nothing to prove here. To prove relation (2.19) we start with the equivalence



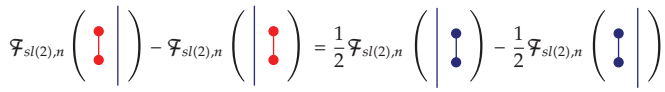
(3.33)

which is a consequence of the neck-cutting relation (3.8) and the disorientation relations (3.3) and (3.5). We also have



(3.34)

Comparing with (3.21) and (3.22) and using the disoriented relation (3.5), we get



(3.35)

Relations Involving Three Colors

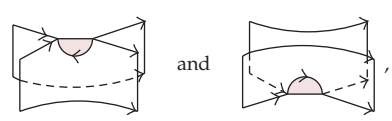
Functor $\mathcal{F}_{sl(2),n}$ sends to zero both sides of relations (2.20) and (2.22). Relation (2.21) follows from isotopies of the cobordisms involved. □

4. The $sl(3)$ Case

4.1. The Category Foam$_3$ of $sl(3)$ Foams

In this subsection we review the category **Foam**$_3$ of $sl(3)$ foams introduced by the author and Mackaay in [3]. This category was introduced to universally deform Khovanov's construction in [6] leading to the $sl(3)$-link homology theory.

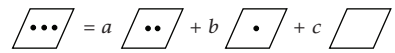
We follow the conventions and notation of [3]. Recall that a *web* is a trivalent planar graph, where near each vertex all edges are oriented away from it or all edges are oriented towards it. We also allow webs without vertices, which are oriented loops. A *pre-foam* is a cobordism with singular arcs between two webs. A singular arc in a prefoam f is the set of points of f which has a neighborhood homeomorphic to the letter Y times an interval. Singular arcs are disjoint. Interpreted as morphisms, we read prefoams from bottom to top by convention; foam composition consists of placing one prefoam on top of the other. The orientation of the singular arcs is by convention as in the *zip* and the *unzip*:



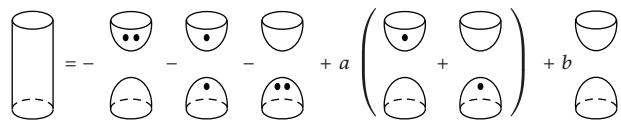
(4.1)

respectively. Pre-foams can have dots which can move freely on the facet to which they belong but are not allowed to cross singular arcs. A *foam* is an isotopy class of pre-foams. Let $\mathbb{C}[a,b,c]$ be the ring of polynomials in a,b,c with coefficients in $\mathbb{C}$.

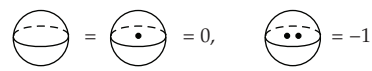
We impose the set of relations $\ell = (3D, CN, S, \Theta)$ on foams, as well as the *closure relation*, which are explained below.



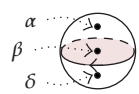
(3D)



(CN)



(S)

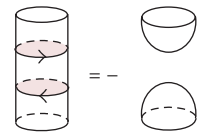


$$\begin{cases} 1 & (\alpha,\beta,\delta) = (1,2,0) \text{ or a cyclic permutation} \\ -1 & (\alpha,\beta,\delta) = (2,1,0) \text{ or a cyclic permutation} \\ 0 & \text{else} \end{cases}$$

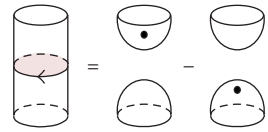
(Θ)

The *closure relation* says that any $\mathbb{C}[a,b,c]$-linear combination of foams, all of which having the same boundary, is equal to zero if and only if any common way of closing these foams yields a $\mathbb{C}[a,b,c]$-linear combination of closed foams whose evaluation is zero.

Using the relations ℓ, one can prove the identities below (for detailed proofs see [3]).

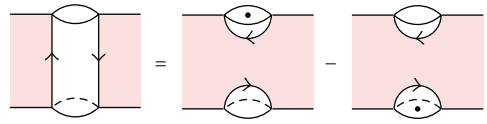


(Bamboo)

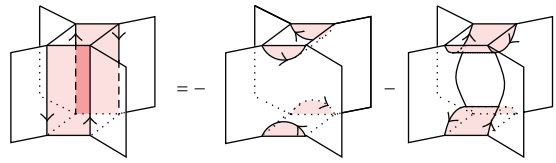


(RD)

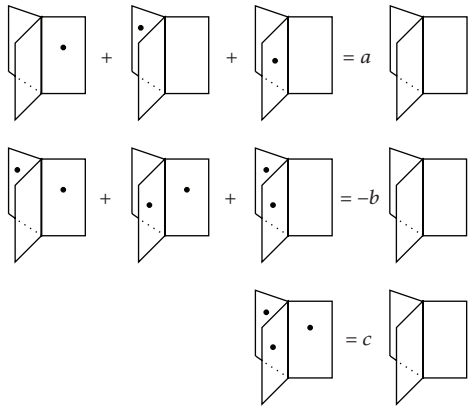
(Bubble)



(DR)



(SqR)

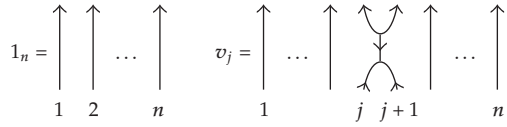


(Dot Migration)

In this paper we will work with open webs and open foams.

Definition 4.1. **Foam**$_3$ is the category whose objects are webs Γ inside a horizontal strip in $\mathbb{R}^2$ bounded by the lines $y = 0, 1$ containing the boundary points of Γ and whose morphisms are $\mathbb{C}[a, b, c]$-linear combinations of foams inside that strip times the unit interval such that the vertical boundary of each foam is a set (possibly empty) of vertical lines.

For example, the diagrams 1_n and v_j are objects of **Foam**$_3$:



(4.2)

The category **Foam**$_3$ is additive and monoidal, with the monoidal structure given as in **Foam**$_2$. The category **Foam**$_3$ is also additive and graded. The q-grading in $\mathbb{C}[a, b, c]$ is defined as

$$q(1) = 0, \qquad q(a) = 2, \qquad q(b) = 4, \qquad q(c) = 6 \tag{4.3}$$

and the degree of a foam f with $|\bullet|$ dots and $|b|$ vertical boundary components is given by

$$q(f) = -2\chi(f) + \chi(\partial f) + 2|\bullet| + |b|, \tag{4.4}$$

where χ denotes the Euler characteristic and ∂f is the boundary of f.

4.2. The Functor $\mathcal{F}_{sl(3),n}$

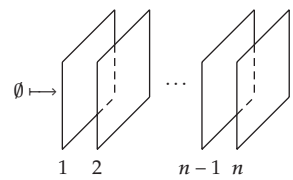
In this subsection we define a monoidal functor $\mathcal{F}_{sl(3),n}$ between the categories $\mathcal{SC}$ and $\mathbf{Foam}_3$.

On Objects

$\mathcal{F}_{sl(3),n}$ sends the empty sequence to 1_n and the one-term sequence (j) to v_j with $\mathcal{F}_{sl(3),n}(jk)$ given by the vertical composite $v_j v_k$.

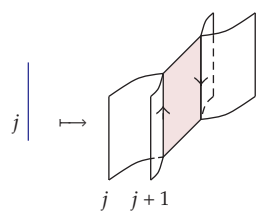
On Morphisms

(i) As before the empty diagram is sent to n parallel vertical sheets:



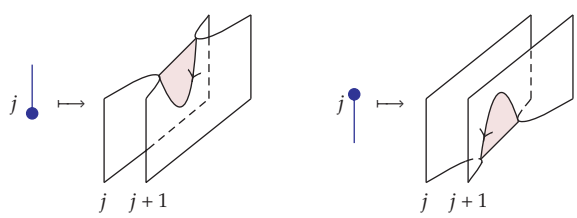
(4.5)

(ii) The vertical line colored j is sent to the identity foam of v_j:



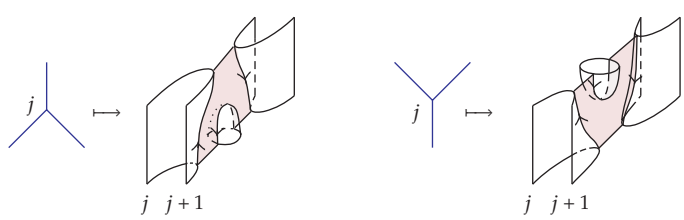
(4.6)

(iii) The *StartDot* and *EndDot* morphisms are sent to the zip and the unzip, respectively:



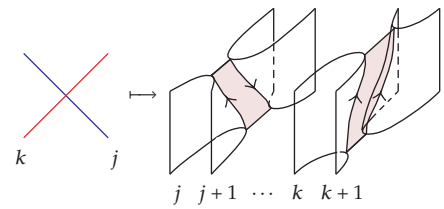
(4.7)

(iv) *Merge* and *Split* are sent to the digon annihilation and creation, respectively:



(4.8)

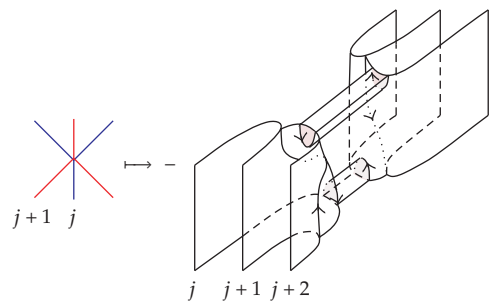
(v) The *4-valent vertex* with distant colors is showen as follows. For $j+1<k$ we have.



(4.9)

The case $j>k+1$ is given by reflection around a horizontal plane.

(vi) For the *6-valent vertex* we have



(4.10)

The case with the colors switched is given by reflection in a vertical plane.
Notice that $\mathcal{F}_{sl(3),n}$ respects the gradings of the morphisms.

Proposition 4.2. *$\mathcal{F}_{sl(3),n}$ is a monoidal functor.*

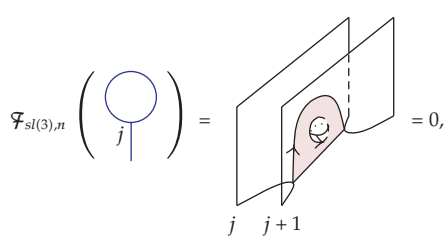
Proof. The assignment given by $\mathcal{F}_{sl(3),n}$ clearly respects the monoidal structures of $\mathcal{SC}_1$ and $\mathbf{Foam}_3$. To prove that it is a monoidal functor we need only to show that it is actually a functor, that is, it respects relations (2.5) to (2.22) of Section 2.

Isotopy Relations

Relations (2.5) to (2.9) correspond to isotopies of their images under $\mathcal{F}_{sl(3),n}$, and we leave its check to the reader.

One-Color Relations

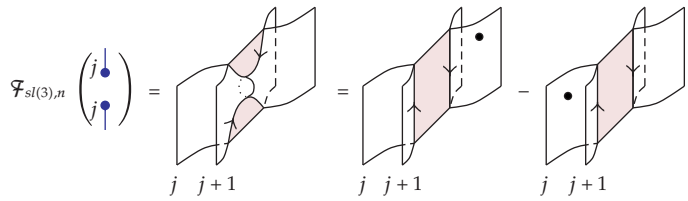
Relation (2.10) is straightforward and left to the reader. For relation (2.11) we have



(4.11)

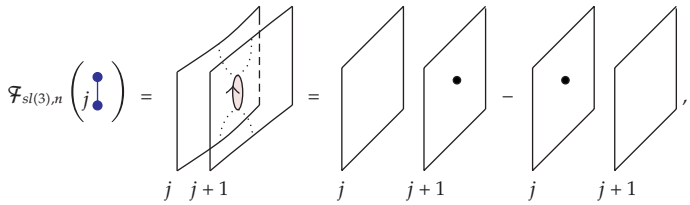
the last equality following from the (Bubble) relation.

For relation (2.12) we have



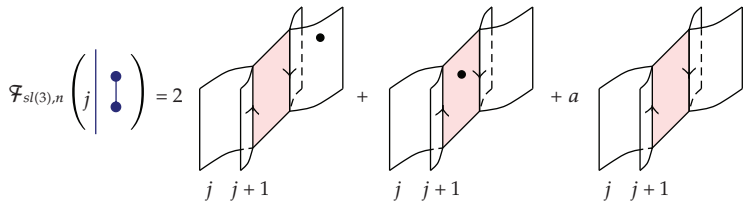
(4.12)

where the second equality follows from the (DR) relation. We also have

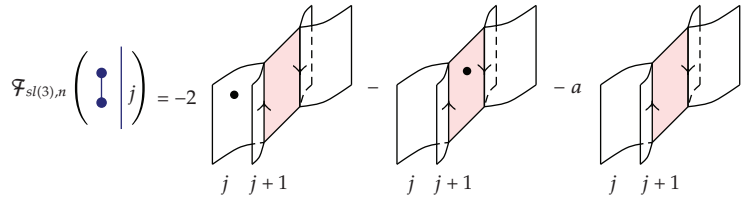


(4.13)

which is given by (RD). Using (Dot Migration) one obtains

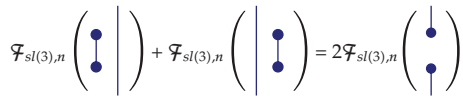


(4.14)



(4.15)

and therefore, we have that



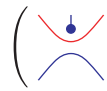
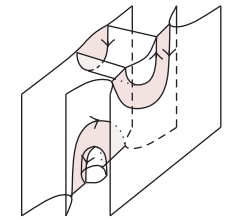
(4.16)

Two Distant Colors

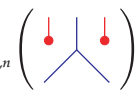
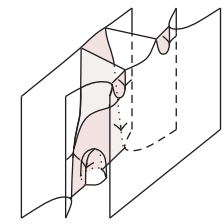
Relations (2.13) to (2.15) correspond to isotopies of the foams involved and are straightforward to check.

Adjacent Colors

We prove the case where "blue" corresponds to j and "red" corresponds to $j+1$. The relations with colors reversed are proved the same way. To prove relation (2.16) we first notice that

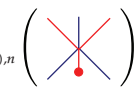
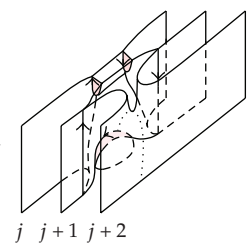
$\mathcal{F}_{sl(3),n}$  = 

(4.17)

$\mathcal{F}_{sl(3),n}$  = 

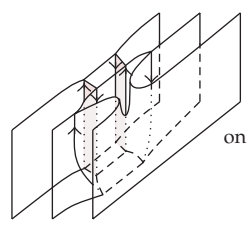
(4.18)

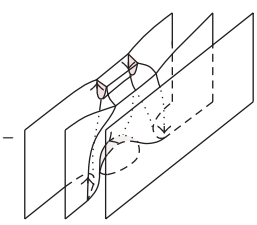
We also have an isotopy equivalence

$\mathcal{F}_{sl(3),n}$  $\cong -$ 

(4.19)

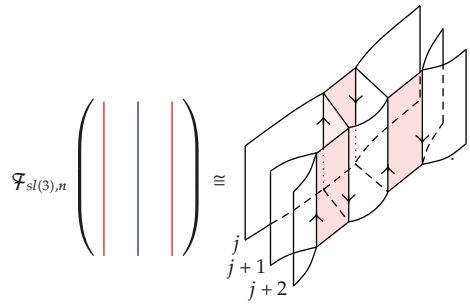
which in turn is isotopy equivalent to the foam obtained by putting



$-$ 
(4.20)

T he common boundary of these two foams contains two squares. Putting (SqR) on the square on the right glued with the identity foam everywhere else gives two terms, one isotopic to $\mathcal{F}_{sl(3),n}$  and the other isotopic to $\mathcal{F}_{sl(3),n}$.

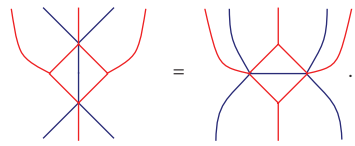
We now prove relation (2.17). We have



(4.21)

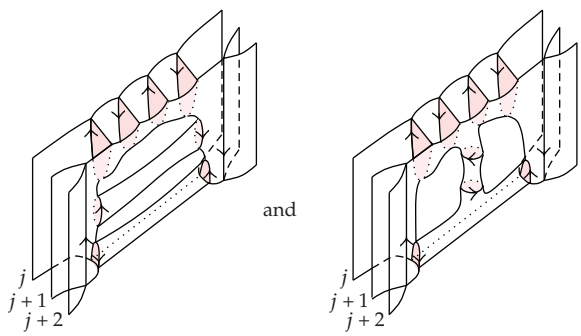
Applying (SqR) to the middle square we obtain two terms. One is isotopic to $-\mathcal{F}_{sl(3),n}$ and the other gives $\mathcal{F}_{sl(3),n}$ after using the (Bamboo) relation.

We now prove relation (2.18) in the form



(4.22)

The images of the l.h.s. and r.h.s. under $\mathcal{F}_{sl(3),n}$ are isotopic to



(4.23)

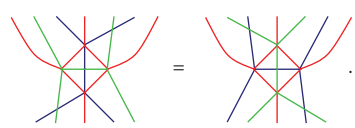
respectively, and both give the same foam after applying the (Bamboo) relation.

Relation (2.19) follows from a straightforward computation and is left to the reader.

Relations Involving Three Colors

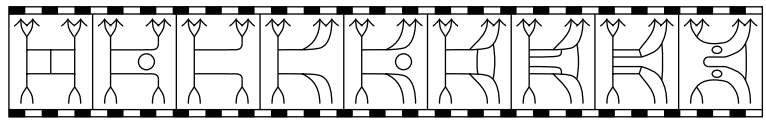
Relations (2.20) and (2.21) follow from isotopies of the cobordisms involved.

We prove relation (2.22) in the form

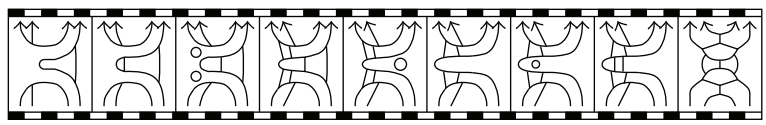


(4.24)

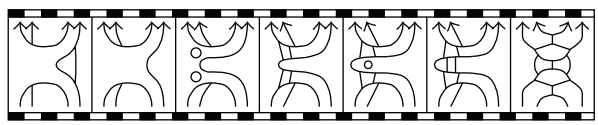
We claim that $\mathcal{F}_{sl(3),n}$ sends both sides to zero. Since the images of both sides of (4.24) can be obtained from each other using a symmetry relative to a vertical plane placed between the sheets labelled $j+1$ and $j+2$, it suffices to show that one side of (4.24) is sent to zero. The foams involved are rather complicated and hard to visualize. To make the computations easier we use movies (two dimensional diagrams) for the whole foam and implicitly translate some bits to three-dimensional foams to apply isotopy equivalences or relations from Section 4.1. The r.h.s. corresponds to

$f_1 =$  (4.25)

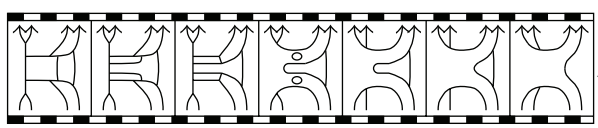
followed by

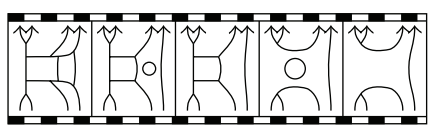
$f_2 =$  . (4.26)

The foam f_2 is isotopic to

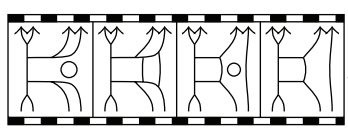
 . (4.27)

Using this, we can also see that the foams corresponding with

 , (4.28)

 (4.29)

are isotopic. We see that the foam we have contains

 , (4.30)

which corresponds to a foam containing  , which is zero by the (Bubble) relation. □

Acknowledgments

The author would like to thank Mikhail Khovanov and Ben Elias for valuable comments on a previous version of this paper. The author also thanks Ben Elias for sharing reference [11]. The author was financially supported by FCT (Portugal), postdoc Grant no. SFRH/BPD/46299/2008.

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Research Article

The Diagrammatic Soergel Category and $sl(N)$-Foams, for $N \geq 4$

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For each $N \geq 4$, we define a monoidal functor from Elias and Khovanov's diagrammatic version of Soergel's category of bimodules to the category of $sl(N)$ foams defined by Mackaay, Stošić, and Vaz. We show that through these functors Soergel's category can be obtained from the $sl(N)$ foams.

1. Introduction

In [1] Soergel categorified the Hecke algebra using bimodules. Just as the Hecke algebra is important for the construction of the HOMFLY-PT link polynomial, so is Soergel's category for the construction of Khovanov and Rozansky's HOMFLY-PT link homology [2], as explained by Khovanov in [3]. Elias and Khovanov [4] constructed a diagrammatic version of the Soergel category with generators and relations, which Elias and Krasner [5] used for a diagrammatic construction of Rouquier's complexes associated to braids.

In [6] Bar-Natan gave a new version of Khovanov's [7] original link homology, also called the $sl(2)$ link homology, using 2d-cobordisms modulo certain relations, which we will call $sl(2)$ foams. Using 2d-cobordisms with a particular sort of singularity modulo certain relations, which we will call $sl(3)$ foams, Khovanov constructed the $sl(3)$ link homology [8]. Khovanov and Rozansky [9] then constructed the $sl(N)$ link homologies, for any $N \geq 1$, using matrix factorizations. These link homologies are closely related to the HOMFLY-PT link homology by Rasmussen's spectral sequences [10], with E_1-page isomorphic to the HOMFLY-PT homology and converging to the $sl(N)$ homology, for any $N \geq 1$. In [11] Mackaay et al. gave an alternative construction of these $sl(N)$ link homologies, for $N \geq 4$, using $sl(N)$ foams, which are 2d-cobordisms with two types of singularities satisfying relations

determined by a formula from quantum field theory, originally obtained by Kapustin and Li [12] and later adapted by Khovanov and Rozansky [13].

Khovanov and Rozansky in [2, 9] and Rasmussen in [10] used matrix factorizations for their constructions. Therefore, the question arises whether their results can be understood in diagrammatic terms and what could be learned from that. In [14] Vaz constructed functors from Elias and Khovanov's diagrammatic version of Soergel's category to the categories of $sl(2)$ and $sl(3)$ foams. In this paper we construct the analogous functors from the same version of Soergel's category to the category of $sl(N)$ foams for $N \geq 4$. To complete the picture, one would like to construct the analogues of Rasmussen's spectral sequences in this setting. However for this, one would first have to understand the Hochschild homology of bimodules in diagrammatic terms, which has not been accomplished yet. Hochschild homology plays an integral part of the construction. Nevertheless, there is an interesting result which can already be shown using the functors in this paper. In a certain technical sense, which we will make precise in Proposition 4.2, Soergel's category can be obtained from the $sl(N)$ foams, and therefore from the Kapustin-Li formula, using our functors. This result should be compared to Rasmussen's Theorem 1 in [10].

We thank Catharina Stroppel for pointing out the connection of our work to results in [15]. We quote her directly: In [15] a categorification of "special trivalent" graphs modulo the MOY relations was constructed by exact functors acting between certain blocks of parabolic category $\mathcal{O}$. Using Soergel's functor passing from Lie theory to the combinatorial bimodule category the construction in [15] in fact produces an action of the diagrammatic Soergel category on these various category $\mathcal{O}$s.

We have tried to make the paper as self-contained as possible, but the reader should definitely leaf through [4, 5, 11, 14] before reading the rest of this paper.

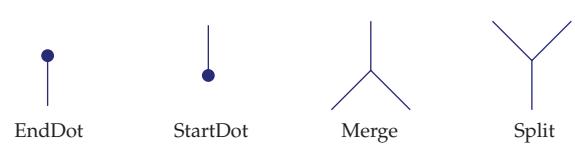
In Section 2 we recall Elias and Khovanov's version of Soergel's category. In Section 3 we review $sl(N)$ foams, as defined by Mackaay, Stošic', and Vaz. Section 4 contains the new results: the definition of our functors, the proof that they are indeed monoidal, and a statement on faithfulness in Proposition 4.2.

2. Elias and Khovanov's Version of Soergel's Category

This section is a reminder of the diagrammatics for Soergel categories introduced by Elias and Khovanov in [4]. Actually we give the version which they explained in [4, Section 4.5] and which can be found in detail in [5].

Fix a positive integer n. The category $\mathcal{SC}_1$ is the category whose objects are finite length sequences of points on the real line, where each point is colored by an integer between 1 and n. We read sequences of points from left to right. Two colors i and j are called adjacent if $|i-j| = 1$ and distant if $|i-j| > 1$. The morphisms of $\mathcal{SC}_1$ are given by generators modulo relations. A morphism of $\mathcal{SC}_1$ is a $\mathbb{C}$-linear combination of planar diagrams constructed by horizontal and vertical gluings of the following generators (by convention no label means a generic color j)

(i) Generators involving only one color are



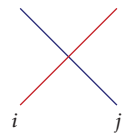
(2.1)

It is useful to define the cap and cup as

(2.2)

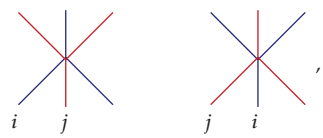
(ii) Generators involving two colors are

–The 4-valent vertex, with distant colors,



(2.3)

and the 6-valent vertex, with adjacent colors i and j,

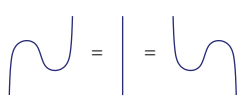


(2.4)

read from bottom to top. In this setting a diagram represents a morphism from the bottom boundary to the top. We can add a new colored point to a sequence and this endows $\mathcal{SC}_1$ with a monoidal structure on objects, which is extended to morphisms in the obvious way. Composition of morphisms consists of stacking one diagram on top of the other.

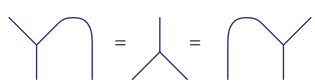
We consider our diagrams modulo the following relations.

"Isotopy" relations are

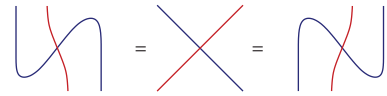


(2.5)

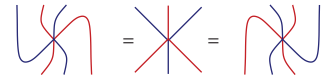
(2.6)



(2.7)



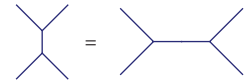
(2.8)



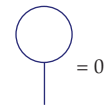
(2.9)

The relations are presented in terms of diagrams with generic colorings. Because of isotopy invariance, one may draw a diagram with a boundary on the side, and view it as a morphism in $\mathcal{SC}_1$ by either bending the line up or down. By the same reasoning, a horizontal line corresponds to a sequence of cups and caps.

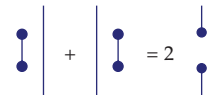
One color relations are



(2.10)

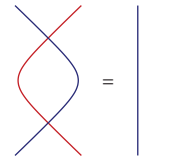


(2.11)

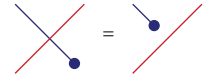


(2.12)

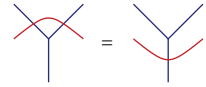
Relations involvingtwo distant colors are



(2.13)

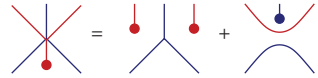


(2.14)

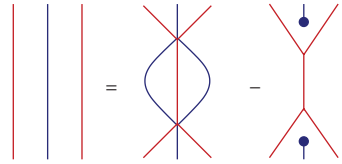


(2.15)

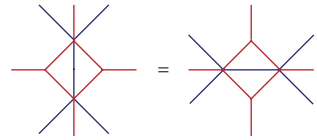
Relations involving two adjacent colors are



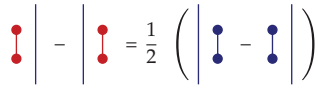
(2.16)



(2.17)

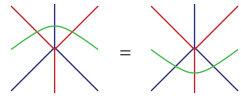


(2.18)

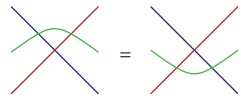


(2.19)

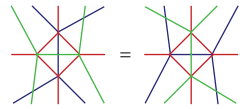
Relations involving three colors are (adjacency is determined by the vertices which appear)



(2.20)

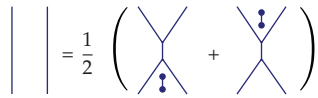


(2.21)



(2.22)

Furthermore, we also have a useful implication of relation (2.12) as follows:



(2.23)

Introduce a q-grading on $\mathcal{SC}_1$ declaring that dots have degree 1, trivalent vertices have degree -1, and 4- and 6-valent vertices have degree 0.

Definition 2.1. The category $\mathcal{SC}_2$ is the category containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$ and whose morphisms are the grading-preserving morphisms from $\mathcal{SC}_1$.

Definition 2.2. The category $\mathcal{SC}$ is the Karoubi envelope of the category $\mathcal{SC}_2$.

Elias and Khovanov's main result in [4] is the following theorem.

Theorem 2.3 (Elias-Khovanov). *The category $\mathcal{SC}$ is equivalent to the Soergel category in [1].*

From Soergel's results from [1] we have the following corollary.

Corollary 2.4. *The Grothendieck algebra of $\mathcal{SC}$ is isomorphic to the Hecke algebra.*

Notice that $\mathcal{SC}$ is an additive category but not abelian and we are using the (additive) split Grothendieck algebra.

In Section 4 we will define a a family of functors from $\mathcal{SC}_{1,n}$ to the category of $sl(N)$ foams, one for each $N \geq 4$. These functors are grading preserving, so they obviously extend uniquely to $\mathcal{SC}_{2,n}$. By the universality of the Karoubi envelope, they also extend uniquely to functors between the respective Karoubi envelopes.

3. Foams

3.1. Prefoams

In this section we recall the basic facts about foams. For the definition of the Kapustin-Li formula, for proofs of the relations between foams, and for other details, see [11, 16]. The foams in this paper are composed of three types of facets: simple, double, and triple facets. The double facets are coloured and the triple facets are marked to show the difference. Intersecting such a foam with a generic plane results in a web, as long as the plane avoids the singularities where six facets meet, such as on the right in Figure 1.

Definition 3.1. Let s_γ be a finite oriented closed 4-valent graph, which may contain disjoint circles and loose endpoints. We assume that all edges of s_γ are oriented. A cycle in s_γ is defined to be a circle or a closed sequence of edges which form a piecewise linear circle. Let Σ be a compact orientable possibly disconnected surface, whose connected components are simple, double, or triple, denoted by white, coloured, or marked. Each component can have a boundary consisting of several disjoint circles and can have additional decorations which we discuss below. A closed *prefoam* u is the identification space Σ/s_γ obtained by gluing boundary circles of Σ to cycles in s_γ such that every edge and circle in s_γ are glued to exactly three boundary circles of Σ and such that for any point $p \in s_\gamma$,

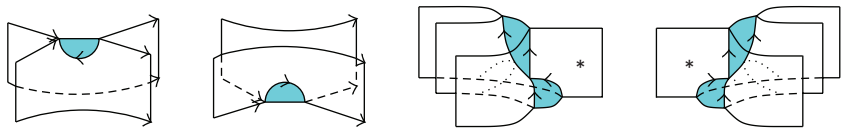


Figure 1: Some elementary prefoams.

(1) if p is an interior point of an edge, then p has a neighborhood homeomorphic to the letter Y times an interval with exactly one of the facets being double, and at most one of them being triple; for an example see Figure 1,

(2) if p is a vertex of s_γ, then it has a neighborhood as shown in Figure 1.

We call s_γ the *singular graph*, its edges and vertices *singular arcs* and *singular vertices*, and the connected components of $u - s_\gamma$ the *facets*.

Furthermore the facets can be decorated with dots. A simple facet can only have black dots ($\bullet$), a double facet can also have white dots ($\circ$), and a triple facet besides black and white dots can have double dots ($\odot$). Dots can move freely on a facet but are not allowed to cross singular arcs.

Note that the cycles to which the boundaries of the simple and the triple facets are glued are always oriented, whereas the ones to which the boundaries of the double facets are glued are not, as can be seen in Figure 1. Note also that there are two types of singular vertices. Given a singular vertex v, there are precisely two singular edges which meet at v and bound a triple facet: one oriented toward v, denoted as e_1, and one oriented away from v, denoted as e_2. If we use the "left-hand rule", then the cyclic ordering of the facets incident to e_1 and e_2 is either $(3,2,1)$ or $(3,1,2)$, respectively, or the other way around. We say that v is of type I in the first case and of type II in the second case. When we go around a triple facet, we see that there have to be as many singular vertices of type I as there are of type II for the cyclic orderings of the facets to match up. This shows that for a closed prefoam the number of singular vertices of type I is equal to the number of singular vertices of type II.

We can intersect a prefoam u generically by a plane W in order to get a closed web, as long as the plane avoids the vertices of s_γ. The orientation of s_γ determines the orientation of the simple edges of the web according to the convention in Figure 2.

Suppose that, for all but a finite number of values $i \in]0,1[$, the plane $W \times i$ intersects u generically. Suppose also that $W \times 0$ and $W \times 1$ intersect u generically and outside the vertices of s_γ. Furthermore, suppose that $D \subset W$ is a disc in W and $C \subset D$ its boundary circle such that $C \times [0,1] \cap u$ is a disjoint union of vertical line segments. This means that we are assuming that s_γ does not intersect $C \times [0,1]$. We call $D \times [0,1] \cap u$ an *open* prefoam between the *open* webs $D \times \{0\} \cap u$ and $D \times \{1\} \cap u$. Interpreted as morphisms, we read open prefoams from bottom to top, and their composition consists of placing one prefoam on top of the other, as long as their boundaries are isotopic and the orientations of the simple edges coincide.

Definition 3.2. Let **Pfoam** be the category whose objects are webs and whose morphisms are $\mathbb{Q}$-linear combinations of isotopy classes of prefoams with the obvious identity prefoams and composition rule.

We now define the q-degree of a prefoam. Let u be a prefoam, u_1, u_2, and u_3 the disjoint union of its simple, double, and marked facets, respectively, and $s_\gamma(u)$ its singular

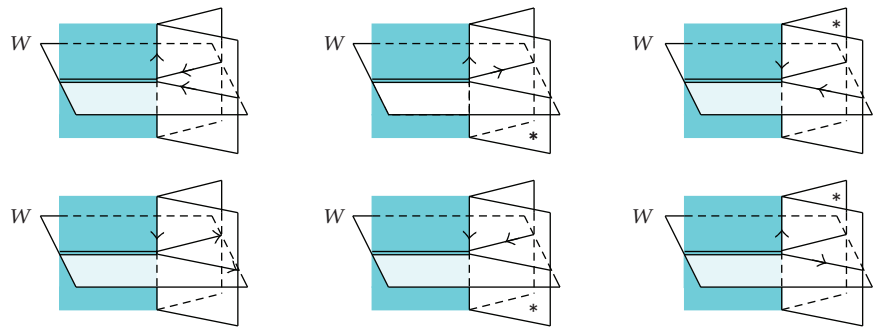


Figure 2: Orientations near a singular arc.

graph. Furthermore, let b_1, b_2, and b_3 be the number of simple, double, and marked vertical boundary edges of u, respectively. Define the partial q-gradings of u as

$$\begin{aligned} q_i(u) &= \chi(u_i) - \frac{1}{2}\chi(\partial u_i \cap \partial u) - \frac{1}{2}b_i, \quad i = 1,2,3, \\ q_{s_\gamma}(u) &= \chi(s_\gamma(u)) - \frac{1}{2}\chi(\partial s_\gamma(u)), \end{aligned} \tag{3.1}$$

where χ is the Euler characteristic and ∂ denotes the boundary.

Definition 3.3. Let u be a prefoam with $d_\bullet$ dots of type $\bullet$, $d_\circ$ dots of type $\circ$, and $d_\odot$ dots of type $\odot$. The q-grading of u is given by

$$q(u) = -\sum_{i=1}^{3} i(N-i)q_i(u) - 2(N-2)q_{s_\gamma}(u) + 2d_\bullet + 4d_\circ + 6d_\odot. \tag{3.2}$$

The following result is a direct consequence of the definitions.

Lemma 3.4. *$q(u)$ is additive under the gluing of prefoams.*

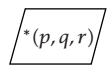
We denote a simple facet with i dots by

(3.3)

Recall that the two-variable Schur polynomial $\pi_{k,m}$ can be expressed in terms of the elementary symmetric polynomials $\pi_{1,0}$ and $\pi_{1,1}$. By convention, the latter correspond to $\bullet$ and $\circ$ on a double facet, respectively, so that

(3.4)

is defined to be the linear combination of dotted double facets corresponding to the expression of $\pi_{k,m}$ in terms of $\pi_{1,0}$ and $\pi_{1,1}$. Analogously we can express the three-variable Schur polynomial $\pi_{p,q,r}$ in terms of the elementary symmetric polynomials $\pi_{1,0,0}$, $\pi_{1,1,0}$, and $\pi_{1,1,1}$. By convention, the latter correspond to $\bullet$, $\circ$, and $\odot$ on a triple facet, respectively, so we can make sense of



(3.5)

3.2. Foams

In [11, 16] we gave a precise definition of the Kapustin-Li formula, following Khovanov and Rozansky's work [13]. We will not repeat that definition here, since it is complicated and unnecessary for our purposes in this paper. The only thing one needs to remember is that the Kapustin-Li formula associates a number to any closed prefoam and that those numbers have very special properties, some of which we will recall below. By $\langle u \rangle_{KL}$, we denote the Kapustin-Li evaluation of a closed prefoam u.

Definition 3.5. The category $\mathbf{Foam}_N$ is the quotient of the category **Pfoam** by the kernel of $\langle\,\rangle_{KL}$, that is, by the following identifications: for any webs Γ, Γ' and finite sets $f_i \in \mathrm{Hom}_{\mathbf{Pfoam}}(\Gamma, \Gamma')$ and $c_i \in \mathbb{Q}$ we impose the relations

$$\sum_i c_i f_i = 0 \Longleftrightarrow \sum_i c_i \langle \overline{f_i} \rangle_{KL} = 0, \tag{3.6}$$

for any fixed way of closing the f_i, denoted by $\overline{f_i}$. By "fixed" we mean that all the f_i are closed in the same way. The morphisms of $\mathbf{Foam}_N$ are called *foams*.

In the next proposition we recall those relations in $\mathbf{Foam}_N$ that we need in the sequel. For their proofs and other relations we refer to [11].

Proposition 3.6. *The following identities hold in* $\mathbf{Foam}_N$

The dot conversion relations are

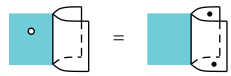
$= 0 \quad \text{if } i \geq N,$ (3.7)

$= 0 \quad \text{if } k \geq N-1,$ (3.8)

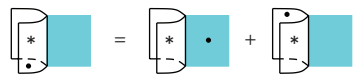
$= 0 \text{ if } p \geq N-2.$ (3.9)

The dot migration relations are

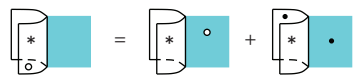
(3.10)



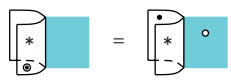
(3.11)



(3.12)

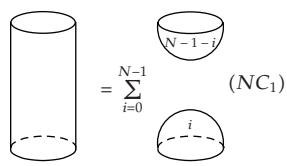


(3.13)

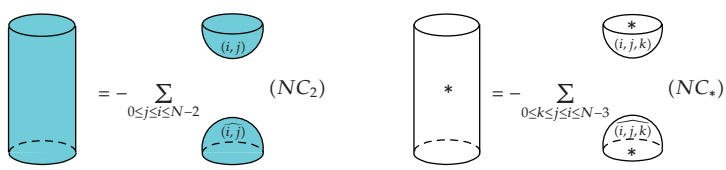


(3.14)

The neck cutting relations are (these were called cutting neck relations in [11, 16])

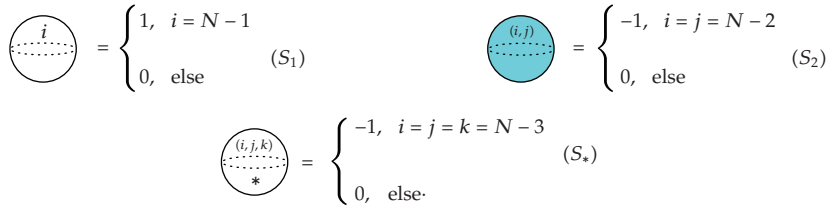


(3.15)



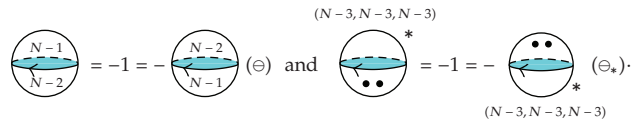
(3.16)

The sphere relations are



(3.17)

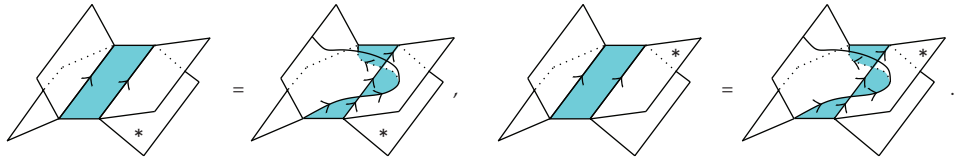
The Θ-foam relations are



(3.18)

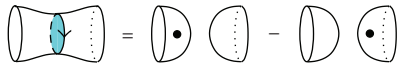
Inverting the orientation of the singular circle of $(\ominus_*)$ *inverts the sign of the corresponding foam. A theta-foam with dots on the double facet can be transformed into a theta-foam with dots only on the other two facets, using the dot migration relations.*

The Matveev-Piergalini relation is

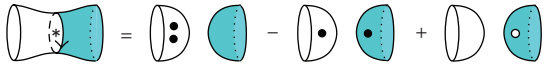


(MP)

The disc removal relations are

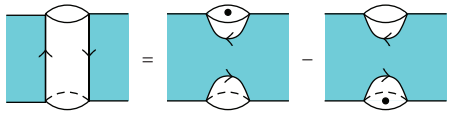


(RD$_1$)

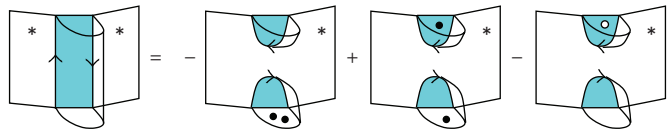


(RD$_2$)

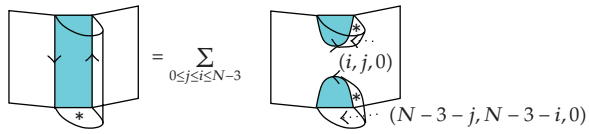
The digon removal relations are



(DR$_1$)

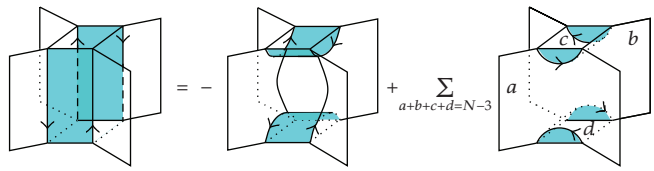


(DR_{3_1})

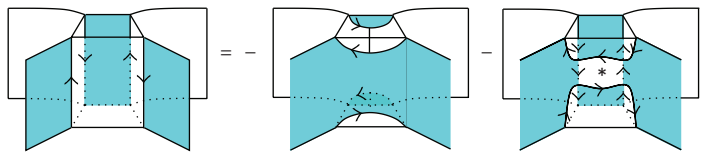


(DR_{3_2})

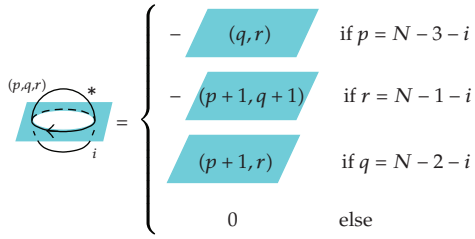
The square removal relations are



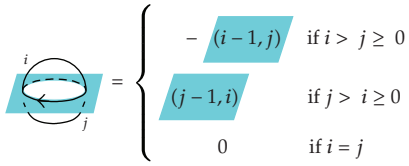
(SqR_1)



(SqR_2)



(3.19)

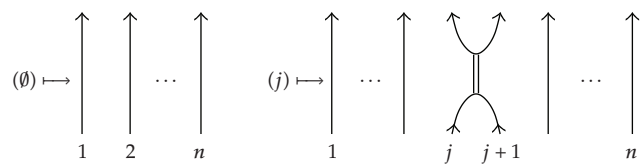


(3.20)

4. The Functors $\mathcal{F}_{N,n}$

Let $n \geq 1$ and $N \geq 4$ be arbitrary but fixed. In this section we define a monoidal functor $\mathcal{F}_{N,n}$ between the categories $\mathcal{SC}_{1,n}$ and $\mathbf{Foam}_N$.

On Objects. $\mathcal{F}_{N,n}$ sends the empty sequence to 1_n and the one-term sequence (j) to w_j:

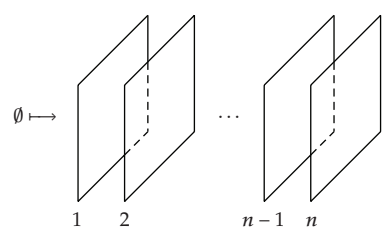


(4.1)

with $\mathcal{F}_{N,n}(jk)$ given by the vertical composite $w_j w_k$.

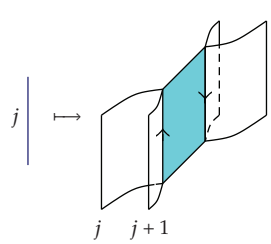
On Morphisms.

(i) The empty diagram is sent to n parallel vertical sheets:



(4.2)

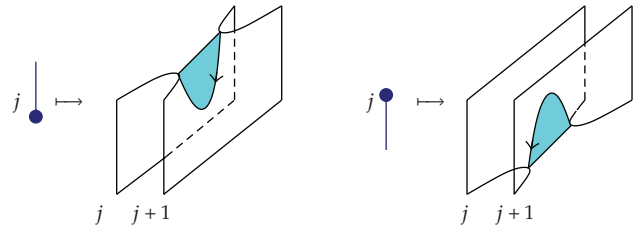
(ii) The vertical line coloured j is sent to the identity cobordism of w_j:



(4.3)

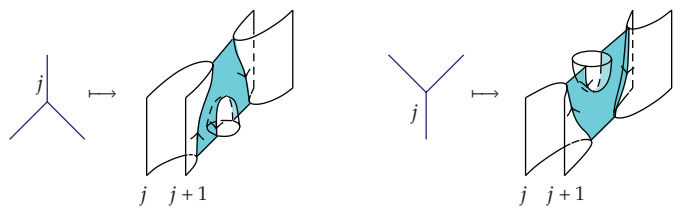
The remaining $n-2$ vertical parallel sheets on the r.h.s. are not shown for simplicity, a convention that we will follow from now on.

(iii) The *StartDot* and *EndDot* morphisms are sent to the zip and the unzip, respectively:



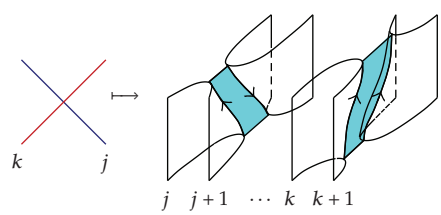
(4.4)

(iv) *Merge* and *Split* are sent to cup and cap cobordisms:



(4.5)

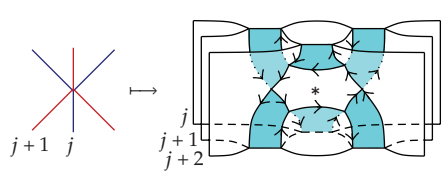
(v) The *4-valent vertex* with distant colors. For $j+1<k$ we have



(4.6)

The case $j>k+1$ is given by reflexion around a horizontal plane.

(vi) For the *6-valent vertices* we have



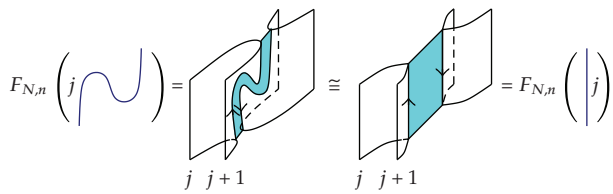
(4.7)

The case with the colors switched is given by reflection in a vertical plane. Notice that $\mathcal{F}_{N,n}$ respects the gradings of the morphisms.

Proposition 4.1. *$\mathcal{F}_{N,n}$ is a monoidal functor.*

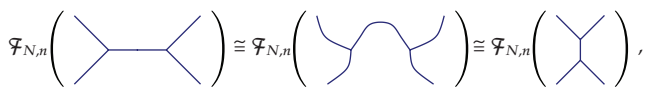
Proof. The assignment given by $\mathcal{F}_{N,n}$ clearly respects the monoidal structures of $\mathcal{SC}_{1,n}$ and $\mathbf{Foam}_N$. So we only need to show that $\mathcal{F}_{N,n}$ is a functor, that is, it respects the relations (2.5) to (2.22) of Section 2.

"*Isotopy Relations*". Relations (2.5) to (2.9) are straightforward to check and correspond to isotopies of their images under $\mathcal{F}_{N,n}$. For the sake of completeness we show the first equality in (2.5). We have



(4.8)

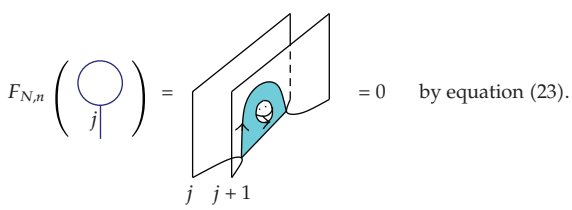
One Color Relations. For relation (2.10) we have



(4.9)

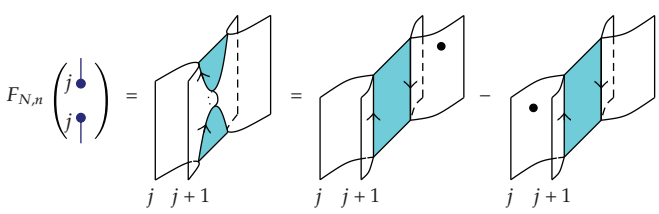
where the first equivalence follows from relations (2.5) and (2.7) and the second from isotopy of the foams involved.

For relation (2.11) we have



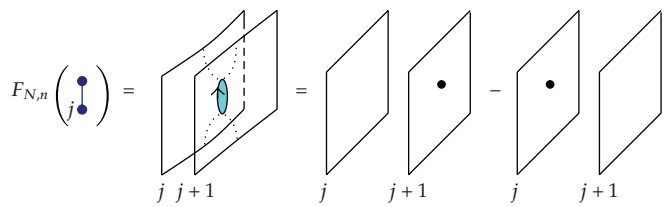
(4.10)

Relation (2.12) requires some more work. We have



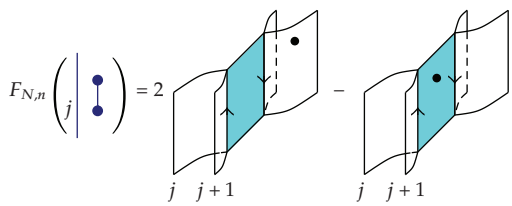
(4.11)

where the second equality follows from the (DR_1) relation. We also have

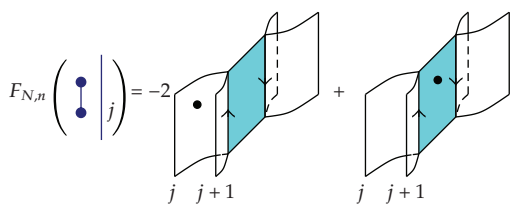


(4.12)

Using (3.12), we obtain

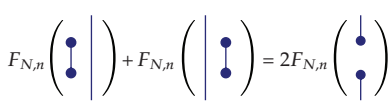


(4.13)



(4.14)

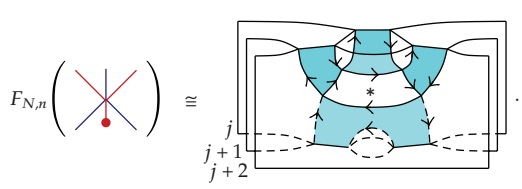
and, therefore, we have that



(4.15)

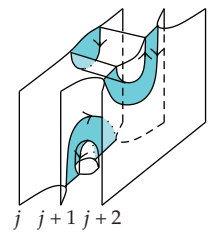
Two Distant Colors. Relations (2.13) to (2.15) correspond to isotopies of the foams involved and are straightforward to check.

Adjacent Colors. We prove the case where "blue" corresponds to j and "red" corresponds to $j+1$. The relations with colors reversed are proved the same way. To prove relation (2.16) we first notice that using the (MP) move we get



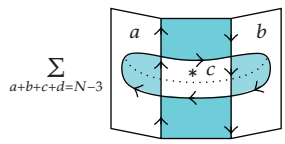
(4.16)

Apply (SqR_1) to the simple-double square tube perpendicular to the triple facet to obtain two terms. The first term contains a double-triple digon tube which is the left-hand side of the (DR_{3_2}) relation rotated by 180^0 around a vertical axis. Next apply the (DR_{3_2}) relation and use (MP) to remove the four singular vertices, which results in simple-triple bubbles (with dots) in the double facets. Using (3.19) to remove these bubbles gives



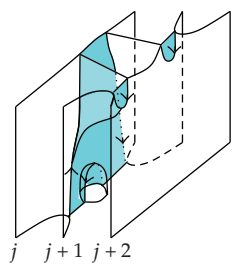
(4.17)

which is $\mathcal{F}_{N,n}$ ( ). The second term contains



(4.18)

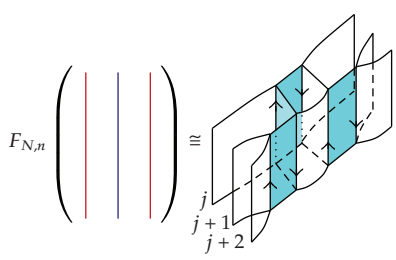
behind a simple facet with d dots (notice that all dots are on simple facets). Using the (MP) relation to get a simple-triple bubble in the double facet, followed by (RD_2) and (S_1) we obtain



(4.19)

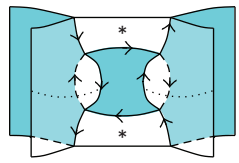
which equals $\mathcal{F}_{N,n}$ ( ).

We now prove relation (2.17). We have an isotopy equivalence



(4.20)

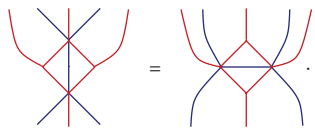
Notice that $\mathcal{F}_{N,n}$ (|||) is the l.h.s. of the (SqR_2) relation. The first term on the r.h.s. of (SqR_2) is isotopic to $-\mathcal{F}_{N,n}$ (⊗). For the second term on the r.h.s. of (SqR_2) we notice that $\mathcal{F}_{N,n}$ (⊗) contains



(4.21)

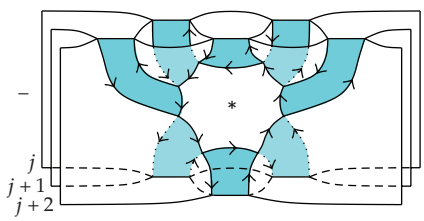
Applying (DR_{3_1}) followed by (MP) to remove the singular vertices creating simple-simple bubbles on the two double facets and using (3.20) to remove these bubbles, we conclude that $\mathcal{F}_{N,n}$ (⊗) is the second term on the r.h.s. of (SqR_2).

We now prove relation (2.18) in the form



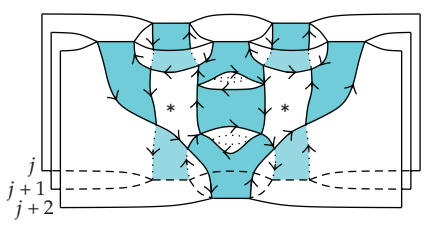
(4.22)

The image of the l.h.s. also contains a bit like the one in (4.21). Simplifying it like we did in the proof of (2.17), we obtain that $\mathcal{F}_{N,n}$ reduces to



(4.23)

For the r.h.s. we have



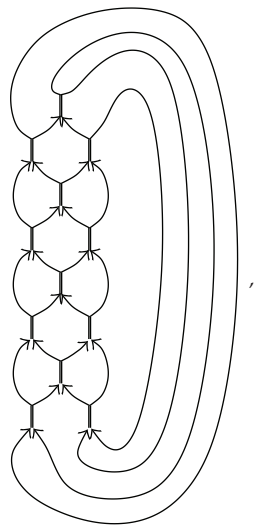
(4.24)

Using (DR_{3_1}) on the vertical digon, followed by (MP) and the Bubble relation (3.20), we obtain (4.23).

Relation (2.19) follows from straightforward computation and is left to the reader.

Relations Involving Three Colors. Relations (2.20) and (2.21) follow from isotopies of the foams involved. To show that $\mathcal{F}_{N,n}$ respects relation (2.22), we use a different type of argument. First of all, we note that the images under $\mathcal{F}_{N,n}$ of both sides of relation (2.22) are multiples of each other, because the graded vector space of morphisms in $\mathbf{Foam}_N$

between the bottom and top webs has dimension one in degree zero. Verifying this only requires computing the coefficient of $q^{-(4N-4)}$ (this includes the necessary shift!) in the MOY polynomial associated to the web



(4.25)

which is a standard calculation left to the reader. To see that the multiplicity coefficient is equal to one, we close both sides of relation (2.22) simply by putting a dot on each open end. Using relations (2.14) and (2.16) to reduce these closed diagrams, we see that both sides give the same nonzero sum of disjoint unions of coloured *StartDot-EndDot* diagrams. Note that we have already proved that $\mathcal{F}_{N,n}$ respects relations (2.14) and (2.16). By applying foam relation (4.12) to the images of all nonzero terms in the sum, we obtain a nonzero sum of dotted sheets. This implies that both sides of (2.22) have the same image under $\mathcal{F}_{N,n}$. □

We have now proved that $\mathcal{F}_{N,n}$ is a monoidal functor for all $N \geq 4$. Our main result about the whole family of these functors, that is, for all $N \geq 4$ together, is the proposition below. It implies that all the defining relations in Soergel's category can be obtained from the corresponding relations between $sl(N)$ foams, when all $N \geq 4$ are considered, and that there are no other independent relations in Soergel's category corresponding to relations between foams.

Proposition 4.2. *Let $\underline{i}$, $\underline{j}$ be two arbitrary objects in $\mathcal{SC}_{1,n}$ and let $f \in Hom(\underline{i}, \underline{j})$ be arbitrary. If $\mathcal{F}_{N,n}(f) = 0$ for all $N \geq 4$, then $f = 0$.*

Proof. Let us first suppose that $\underline{\mathbf{i}} = \underline{\mathbf{j}} = \emptyset$. Suppose also that f has degree $2d$ and that $N \geq \max\{4, d+1\}$. Recall that, as shown in [5, Corollary 3], we know that $\mathrm{Hom}(\emptyset, \emptyset)$ is the free commutative polynomial ring generated by the StartDot-EndDots of all possible colors. So f is a polynomial in StartDot-EndDots, and therefore a sum of monomials. Let m be one of these monomials, no matter which one, and let m_j denote the power of the StartDot-EndDot with color j in m. Close $\mathcal{F}_{N,n}(f)$ by gluing disjoint discs to the boundaries of all open simple facets (i.e., the vertical ones with corners in the pictures). For each color j, put $N-1-m_j$ dots on the left simple open facet corresponding to j and also put $N-1$ dots on the rightmost simple open facet. Note that, after applying (RD_1), we get a linear combination of dotted simple spheres.

Only one term survives and is equal to ± 1, because only in that term each sphere has exactly $N-1$ dots. This shows that $\mathcal{F}_{N,n}(f) \neq 0$, because it admits a nonzero closure.

Now let us suppose that $\underline{\mathbf{i}} = \emptyset$ and $\underline{\mathbf{j}}$ is arbitrary. By [4, Corollaries 4.11 and 4.12], we know that $\mathrm{Hom}(\emptyset, \underline{\mathbf{j}})$ is the free $\mathrm{Hom}(\emptyset, \emptyset)$-module of rank one generated by the disjoint union of StartDots coloured by $\underline{\mathbf{j}}$. Closing off the StartDots by putting dots on all open ends gives an element of $\mathrm{Hom}(\emptyset, \emptyset)$, whose image under $\mathcal{F}_{N,n}$ is nonzero for N big enough by the above. This shows that the generator of $\mathrm{Hom}(\emptyset, \underline{\mathbf{j}})$ has nonzero image under $\mathcal{F}_{N,n}$ for N big enough, because $\mathcal{F}_{N,n}$ is a functor.

Finally, the general case, for $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ arbitrary, can be reduced to the previous case by [4, Corollary 4.12]. □

Acknowledgments

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Research Article

A Diagrammatic Temperley-Lieb Categorification

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The monoidal category of Soergel bimodules categorifies the Hecke algebra of a finite Weyl group. In the case of the symmetric group, morphisms in this category can be drawn as graphs in the plane. We define a quotient category, also given in terms of planar graphs, which categorifies the Temperley-Lieb algebra. Certain ideals appearing in this quotient are related both to the 1-skeleton of the Coxeter complex and to the topology of 2D cobordisms. We demonstrate how further subquotients of this category will categorify the irreducible modules of the Temperley-Lieb algebra.

1. Introduction

A goal of the categorification theorist is to replace interesting endomorphisms of a vector space with interesting endofunctors of a category. The question is what makes these functors interesting? In the pivotal paper of Chuang and Rouquier [1], a fresh paradigm emerged. They noticed that by specifying structure on the natural transformations (morphisms) between these functors one obtains more useful categorifications (in this case, the added utility is a certain derived equivalence). The categorification of quantum groups by Rouquier [2], Lauda [3], and Khovanov and Lauda [4] has shown that categorifying an algebra A itself (with a category $\mathcal{A}$) will specify what this additional structure should be for a categorification of any representation of that algebra: a functor from $\mathcal{A}$ to an endofunctor category. That their categorifications $\mathcal{A}$ provide the "correct" extra structure is confirmed by the facts that existing geometric categorifications conform to it (see [5]) and that irreducible representations of A can be categorified in this framework (see [6, 7]). The salient feature of these categorifications is that, instead of being defined abstractly, the morphisms are presented by generators and relations, making it straightforward to define functors out of $\mathcal{A}$.

In the case of the Hecke algebra $\mathcal{H}$, categorifications have existed for some time, in the guise of category $\mathcal{O}$ or perverse sheaves on the flag variety. In [8] Soergel rephrased these categorifications in a more combinatorial way, constructing an additive categorification of $\mathcal{H}$ by a certain full monoidal subcategory $\mathcal{HC}$ of graded R-bimodules, where R is a polynomial

ring. Objects in this full subcategory are called *Soergel bimodules*. There are deep connections between Soergel bimodules, representation theory, and geometry, and we refer the reader to [8–11] for more details. Categorifications using category $\mathcal{O}$ and variants thereof are common in the literature, and often Soergel bimodules are used to aid calculations (see, e.g., [12–14]).

In [15], Elias and Khovanov provides (in type A) a presentation of $\mathcal{HC}$ by generators and relations, where morphisms can be viewed diagrammatically as decorated graphs in a plane. To be more precise, the diagrammatics are for a smaller category $\mathcal{HC}_1$, the (ungraded) category of *Bott-Samelson bimodules*, described in Section 2.1. Soergel bimodules are obtained from $\mathcal{HC}_1$ by taking the graded Karoubi envelope. This is in exact analogy with the procedures of Khovanov and Lauda in [4] and related papers.

The Temperley-Lieb algebra $\mathcal{TL}$ is a well-known quotient of $\mathcal{H}$, and it can be categorified by a quotient $\mathcal{TLC}$ of $\mathcal{HC}$, as this paper endeavors to show. Thus, we have a naturally arising categorification by generators and relations, and we expect it to be a useful one. Objects in $\mathcal{TLC}$ can no longer be viewed as R-bimodules (though their Hom spaces will be R-bimodules), so that diagrammatics provide the simplest way to define the category.

The most complicated generator of $\mathcal{HC}$ is killed in the quotient to $\mathcal{TLC}$, making $\mathcal{TLC}$ easy to describe diagrammatically in its own right. Take a category where objects are sequences of indices between 1 and n (denoted $\underline{\mathbf{i}}$). Morphisms will be given by (linear combinations of) collections of graphs Γ_i embedded in $\mathbb{R} \times [0,1]$, one for each index $i \in \{1,\ldots,n\}$, such that the graphs have only trivalent or univalent vertices, and such that Γ_i and Γ_{i+1} are disjoint. Each graph will have a degree, making Hom spaces into a graded vector space. The intersection of the graphs with $\mathbb{R}\times\{0\}$ and $\mathbb{R}\times\{1\}$ determines the source and target objects, respectively. Finally, some local graphical relations are imposed on these morphisms. This defines $\mathcal{TLC}_1$, and we take the graded Karoubi envelope to obtain $\mathcal{TLC}$.

The proof that $\mathcal{TLC}$ categorifies $\mathcal{TL}$ uses a method similar to that in [15]. We show first that $\mathcal{TLC}_1$ is a *potential categorification* of $\mathcal{TL}$, in the sense described in Section 2.2. Categorifications and potential categorifications define a pairing on $\mathcal{TL}$ given by $([M],[N]) = \mathrm{gdimHom}_{\mathcal{TLC}_1}(M,N)$, the graded dimension which takes values in $\mathbb{Z}[[t,t^{-1}]]$. Equivalently, it defines a trace on $\mathcal{TL}$ via $\varepsilon([M]) = \mathrm{gdimHom}(\mathbb{1},M)$ where $\mathbb{1}$ is the monoidal identity (see Section 2.1). The difficult part is to prove the following lemma.

Lemma 1.1. *The trace induced on $\mathcal{TL}$ from $\mathcal{TLC}_1$ is the map ε_{cat} defined in Section 2.2.*

Given this lemma, it is surprisingly easy (see Section 3.3) to show the main theorem.

Theorem 1.2. *Let $\mathcal{TLC}_2$ be the graded additive closure of $\mathcal{TLC}_1$, and let $\mathcal{TLC}$ be the graded Karoubi envelope of $\mathcal{TLC}_1$. Then $\mathcal{TLC}_2$ is Krull-Schmidt and idempotent closed, so $\mathcal{TLC}_2 \cong \mathcal{TLC}$, and $\mathcal{TLC}$ categorifies $\mathcal{TL}$.*

To prove the lemma, we note that there is a convenient set of elements in $\mathcal{TL}$, the *nonrepeating* monomials, whose values determine any pairing; hence, there is a convenient set of objects whose Hom spaces will determine all Hom spaces. If $\underline{\mathbf{i}}$ is a nonrepeating sequence, the Hom space we must calculate is (up to shift) a quotient of R by a two-sided ideal $I_{\underline{\mathbf{i}}}$. We use graphical methods to determine these rings explicitly, giving generators for the ideals in R which define them. As an interesting side note, these ideals also occur elsewhere in nature.

Proposition 1.3. *Let V be the reflection representation of S_{n+1}, and identify R with its coordinate ring. Let Z be the union of all the lines in V which are intersections of reflection-fixed hyperplanes, and let $I \subset R$ be the ideal which gives the reduced scheme structure on Z. Then Hom spaces in $\mathcal{TLC}$ are*

R/I-bimodules, and the ideals $I_{\underline{\mathbf{i}}}$ cut out subvarieties of Z given by lines with certain transverseness properties (see Section 3.7 for details).

Also in Section 3.7, we give a topological interpretation of the ideals $I_{\underline{\mathbf{i}}}$, using a functor defined by Vaz [16].

Now, let $\mathcal{TL}_{J_i}$ be the parabolic subalgebra of $\mathcal{TL}$ given by ignoring the index i, and let V^i be the induced (right) representation from the sign representation of $\mathcal{TL}_{J_i}$. Such an induced representation is useful because it is a quotient of $\mathcal{TL}$ and also contains an irreducible module L^i of $\mathcal{TL}$ as a submodule. All irreducibles can be constructed this way.

We provide a diagrammatic categorification of V^i as a quotient $\mathcal{V}^i$ of $\mathcal{TLC}$, and a categorification of L^i as a full subcategory $\mathcal{L}^i$ of $\mathcal{V}^i$, in a fashion analogous to quantum group categorifications. Having found a diagrammatic categorification $\mathcal{C}$ of the positive half U^+ of the quantum group, Khovanov and Lauda in [17] conjectured that the highest weight modules (naturally quotients of U^+) could be categorified by quotients of $\mathcal{C}$ by the appearance of certain pictures on the *left*. This approach was proven correct by Lauda and Vazirani [6] (for the U^+-module structure), and then used by Webster to categorify tensor products [18]. Similarly, to obtain $\mathcal{V}^i$, we mod out $\mathcal{TLC}$ by diagrams where any index except i appears on the left. The proof that this works is similar in style to the proof of Theorem 1.2: one calculates the dimension of all Hom spaces by calculating enough Hom spaces to specify a unique pairing on V^i and then uses simple arguments to identify the Grothendieck group.

Theorem 1.4. *The category $\mathcal{V}^i$ is idempotent closed and Krull-Schmidt. Its Grothendieck group is isomorphic to V^i. Letting $\mathcal{L}^i$ be the full subcategory generated by indecomposables which decategorify to elements of L^i, one has that $\mathcal{L}^i$ is idempotent closed and Krull-Schmidt, with Grothendieck group isomorphic to L^i.*

A future paper will categorify all representations induced from the sign and trivial representations of parabolic subalgebras of $\mathcal{H}$ and $\mathcal{TL}$. Induced representations were categorified more generally in [13] in the context of category $\mathcal{O}$, although not diagrammatically. We believe that our categorification should describe what happens in [13] after applying Soergel's functor.

Soergel bimodules are intrinsically linked with braids, as was shown by Rouquier in [19, 20], who used them to construct braid group actions (these braid group actions also appear in the category $\mathcal{O}$ context, see [21]). As such, morphisms between Soergel bimodules should correspond roughly to movies, and the graphs appearing in the diagrammatic presentation of the category $\mathcal{HC}$ should be (heuristically) viewed as 2-dimensional holograms of braid cobordisms. This is studied in [22]. The Temperley-Lieb quotient is associated with the representation theory of $U_q(\mathfrak{sl}_2)$, for which all braids degenerate into 1-manifolds, and braid cobordisms degenerate into surfaces with disorientations. There is a functor $\mathcal{F}$ from $\mathcal{TLC}$ to the category of disorientations constructed by Vaz [16]. The functor $\mathcal{F}$ is faithful (though certainly not full), as we remark in Section 3.7. This in turn yields a topological motivation of the variety Z and its subvarieties Z'. Because $\mathcal{F}$ is not full, there might be actions of $\mathcal{TLC}$ that do not extend to actions of disoriented cobordisms. Cobordisms have long been a reasonable candidate for morphisms in Temperley-Lieb categorifications, although we hope $\mathcal{TLC}$ will provide a useful substitute, with more explicit and computable Hom spaces.

Categorification and the Temperley-Lieb algebra have a long history. Khovanov in [23] constructed a categorification of $\mathcal{TL}$ using a TQFT, which was slightly generalized by Bar-Natan in [24]. This was then used to categorify the Jones polynomial. Bernstein

et al. in [25] provide a categorical action of the Temperley-Lieb algebra by Zuckerman and projective functors on category $\mathcal{O}$. Stroppel [14] showed that this categorical action extends to the full tangle algebroid, and also investigated the natural transformations between projective functors. Recent work of Brundan and Stroppel [26] connects these Temperley-Lieb categorifications to Khovanov-Lauda-Rouquier algebras, among other things. We hope that our diagrammatics will help to understand the morphisms in these categorifications.

The organization of this paper is as follows. Section 2 will provide a quick overview of the Hecke and Temperley-Lieb algebras, and the diagrammatic definition of the category $\mathcal{HC}$. Section 3 begins by defining the quotient category diagrammatically in its own right (which makes a thorough understanding of the diagrammatic calculus for $\mathcal{HC}$ unnecessary). Section 3.3 proves Theorem 1.2, modulo Lemma 1.1 which requires all the work. The remaining sections of that chapter do all the work, and starting with Section 3.6 one will not miss any important ideas if one skips the proofs. Section 4 begins with a discussion of cell modules for $\mathcal{TL}$ and certain other modules, and then goes on to categorify these modules, requiring only very simple diagrammatic arguments.

This paper is reasonably self-contained. We do not require familiarity with [15] and do not use any results other than Corollary 2.20. We do quote some results for motivational reasons, but the difficult graphical arguments of that paper can often be drastically simplified for the Temperley-Lieb setting, so that we provide easier proofs for the results we need. Familiarity with diagrammatics for monoidal categories with adjunction would be useful, and [3] provides a good introduction. More details on preliminary topics can be found in [15].

2. Preliminaries

Notation 1. Fix $n \in \mathbb{N}$, and let $I = 1, \ldots, n$ index the vertices of the Dynkin diagram A_n. We use the word index for an element of I, and the letters i, j always represent indices. Indices $i \neq j$ are *adjacent* if $|i - j| = 1$, and *distant* if $|i - j| \geq 2$, and questions of *adjacency* always refer to the Dynkin diagram, not the position of indices in a word or picture.

Notation 2. Let $W = S_{n+1}$ with simple reflections $s_i = (i, i+1)$. Let $\Bbbk$ be a field of characteristic not dividing $2(n+1)$; all vector spaces will be over this field. Let $R = \Bbbk[x_1, \ldots, x_{n+1}]/e_1$, where $e_1 = x_1 + x_2 + \cdots + x_{n+1}$; it is a graded ring, with $\deg(x_i) = 2$. We will abuse notation and refer to elements of $\Bbbk[x_1, \ldots, x_{n+1}]$ and their images in R in the same way, and will refer to both as *polynomials*. Note that $R = \Bbbk[f_1, \ldots, f_n]$, where $f_i = x_i - x_{i+1}$, since $x_1 = (nf_1 + (n-1)f_2 + \cdots + f_n)/(n+1)$ modulo e_1. The ring R arises as the coordinate ring of V, the reflection representation of W (the span of the root system), and f_i are the simple coroots.

There is an obvious action of S_{n+1} on R, which permutes the generators x_i. For each index we have a *Demazure operator* ∂_i, a map of degree -2 from R to the invariant subring R^{s_i}, which is R^{s_i}-linear and sends R^{s_i} to 0. Explicitly, $\partial_i(f) = (f - s_i(f))/(x_i - x_{i+1})$.

Notation 3. Let $\overline{(\cdot)}$ be the $\mathbb{Z}$-linear involution of $\mathbb{Z}[t, t^{-1}]$ switching t and t^{-1}. Given a $\mathbb{Z}$-linear map β of $\mathbb{Z}[t, t^{-1}]$ modules, we call it *antilinear* if it is $\mathbb{Z}[t, t^{-1}]$-linear after twisting by $\overline{(\cdot)}$, or in other words if $\beta(tm) = t^{-1}\beta(m)$. We write $[2] \overset{\text{def}}{=} t + t^{-1}$.

Let A be a $\mathbb{Z}[t, t^{-1}]$-algebra. In this paper we always use the word *trace* to designate a $\mathbb{Z}[t, t^{-1}]$-linear map $\varepsilon : A \to \mathbb{Z}[[t, t^{-1}]]$ satisfying $\varepsilon(xy) = \varepsilon(yx)$. We use the word *pairing* or *semilinear pairing* to denote a $\mathbb{Z}$-linear map $A \times A \to \mathbb{Z}[[t, t^{-1}]]$ which is $\mathbb{Z}[t, t^{-1}]$-linear in the second factor and $\mathbb{Z}[t, t^{-1}]$-antilinear in the first factor.

2.1. The Hecke Algebra and the Soergel Categorification

We state here without proof a number of basic facts about the Hecke algebra, its traces, and Soergel's categorification. For more background, see Soergel's original definition of his categorification [8], or an easier version [11]. A similar overview with more discussion can be found in [15]. A more in-depth introduction, connecting Soergel bimodules to other parts of representation theory, can be found in [13].

Definition 2.1. Denote by $\mathcal{H}$ the *Hecke algebra* for S_{n+1}. It is a $\mathbb{Z}[t,t^{-1}]$-algebra, specified here by its *Kazhdan-Lusztig presentation*: it has generators b_i, $i \in I$ and relations

$$b_i^2 = \left(t + t^{-1}\right) b_i \tag{2.1}$$

$$b_i b_j = b_j b_i \quad \text{for distant } i, j \tag{2.2}$$

$$b_i b_j b_i + b_j = b_j b_i b_j + b_i \quad \text{for adjacent } i, j. \tag{2.3}$$

Definition 2.2. Given two objects in a graded $\Bbbk$-linear (possibly additive) category $\mathcal{C}$, where $\{1\}$ denotes the grading shift, the *graded hom space* between them is the graded vector space $\mathrm{HOM}(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M, N\{n\})$. Given a class of objects $\{M_\alpha\}$ in $\mathcal{C}$, we can define a category with morphisms enriched in graded vector spaces, whose objects are $\{M_\alpha\}$ and whose morphisms are $\mathrm{HOM}(M_\alpha, M_\beta)$. Let us call this an *enriched full subcategory*, which we often shorten to the adjective *enriched*. While the enriched subcategory is neither additive nor graded, it has enough information to recover the hom spaces between grading shifts and direct sums of objects M_α in $\mathcal{C}$.

Let R-bim denote the category of finitely-generated graded (resp., ungraded) R-bimodules. Then HOM spaces in R-bim will be graded R-bimodules. For $i \in I$, let $B_i \in R$-bim be defined by $B_i = R \bigotimes_{R^{s_i}} R\{-1\}$, where R^{s_i} is the invariant subring. A *Bott-Samelson bimodule* is a tensor product $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_d}$ in R-bim, where here and henceforth $\otimes$ denotes the tensor product over R. Let $\mathcal{HC}_1$ be the enriched full subcategory generated by the Bott-Samelson bimodules; it is a monoidal category, but is neither additive nor graded. Let $\mathcal{HC}_2$ denote the full subcategory of R-bim given by all (finite) direct sums of grading shifts of Bott-Samelson bimodules; it is monoidal, additive, and graded. Finally, let $\mathcal{HC}$ denote the category of *Soergel bimodules* or *special bimodules*, the full subcategory of R-bim given by all (finite) direct sums of grading shifts of *summands* of Bott-Samelson bimodules; it is monoidal, additive, graded, and idempotent closed.

One can observe that all bimodules in $\mathcal{HC}$ are free and finitely generated when viewed as either left R-modules or right R-modules, and therefore the same is true of any HOM space. The following proposition parallels the Kazhdan-Lusztig presentation for $\mathcal{H}$.

Proposition 2.3. *The category $\mathcal{HC}_2$ is generated (as an additive, monoidal category) by objects B_i, $i \in I$ which satisfy*

$$B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}, \tag{2.4}$$

$$B_i \otimes B_j \cong B_j \otimes B_i \quad \textit{for distant } i, j, \tag{2.5}$$

$$B_i \otimes B_j \otimes B_i \oplus B_j \cong B_j \otimes B_i \otimes B_j \oplus B_i \quad \textit{for adjacent } i, j. \tag{2.6}$$

From this we might expect the next result.

Proposition 2.4. *The Grothendieck ring* $[\mathcal{HC}_2]$ *of* $\mathcal{HC}_2$ *is isomorphic to* $\mathcal{H}$*, with* $[B_i]$ *being sent to* b_i*, and* $[R\{1\}]$ *being sent to* t*. The Grothendieck ring* $[\mathcal{HC}]$ *of* $\mathcal{HC}$ *is isomorphic to* $\mathcal{H}$ *as well.*

Remark 2.5. The proof of this statement is not immediately obvious. There is clearly a surjective morphism from $\mathcal{H}$ to $[\mathcal{HC}_2]$. When one takes the idempotent closure of a category, one adds new indecomposables and can potentially enlarge the Grothendieck group. Soergel showed, via a support filtration, that all the new indecomposables in $\mathcal{HC}$ have symbols in $[\mathcal{HC}]$ which can be reached from certain symbols in $[\mathcal{HC}_2]$ by a unitriangular matrix (see [11]). Therefore, the Grothendieck rings of $\mathcal{HC}$ and $\mathcal{HC}_2$ are equal. Since $\mathcal{HC}$ is idempotent closed and is embedded in R-bim, it has the Krull-Schmidt property and the Grothendieck group behaves as one would expect: it has a basis given by indecomposables. By classifying indecomposables and using the unitriangular matrix, Soergel showed that the map from $\mathcal{H}$ to $[\mathcal{HC}_2]$ is actually an isomorphism.

It is important to note that one does not know what the image of the indecomposables of $\mathcal{HC}$ in $\mathcal{H}$ is. The Soergel conjecture, still unproven in generality, proposes that the indecomposables of $\mathcal{HC}$ descend to the Kazhdan-Lusztig basis of $\mathcal{H}$ (see [11]).

Notation 4. We write the monomial $b_{i_1}b_{i_2}\cdots b_{i_d} \in \mathcal{H}$ as $b_{\underline{\mathbf{i}}}$, where $\underline{\mathbf{i}} = i_1\cdots i_d$ is a finite sequence of indices; by abuse of notation, we sometimes refer to this monomial simply as $\underline{\mathbf{i}}$. If $\underline{\mathbf{i}}$ is as above, we say the monomial has length $d = d(\underline{\mathbf{i}})$. We call a monomial nonrepeating if $i_k \neq i_l$ for $k \neq l$, and increasing if $i_1 < i_2 < \cdots$. The empty set is a sequence of length 0, and $b_\emptyset = 1$. Similarly, in $\mathcal{HC}_1$, write $B_{i_1} \otimes \cdots \otimes B_{i_d}$ as $B_{\underline{\mathbf{i}}}$. Note that $B_\emptyset = R$, the monoidal identity. For an arbitrary index i and sequence $\underline{\mathbf{i}}$, we write $i \in \underline{\mathbf{i}}$ if i appears in $\underline{\mathbf{i}}$.

Given two objects $M, N \in R$-bim we say they are *biadjoint* if $M \otimes -$ and $N \otimes -$ are left and right adjoints of each other, and the same for $- \otimes M$ and $- \otimes N$. If M and N are biadjoint, so are $M\{1\}$ and $N\{-1\}$. We often want to specify additional compatibility between various adjunction maps, but we pass over the details here (see [3] for more information on biadjunction).

Proposition 2.6. *Each object in* $\mathcal{HC}$ *(resp.,* $\mathcal{HC}_1$*,* $\mathcal{HC}_2$*) has a biadjoint, and* B_i *is self-biadjoint. Let* ω *be the* t*-antilinear anti-involution on* $\mathcal{H}$ *which fixes* b_i*, that is,* $\omega(t^a b_{\underline{\mathbf{i}}}) = t^{-a} b_{\sigma(\underline{\mathbf{i}})}$*, where* σ *reverses the order of a sequence. There is a contravariant functor on* $\mathcal{HC}$ *sending an object to its biadjoint, and it descends on the Grothendieck ring to* ω*.*

Definition 2.7. An *adjoint pairing* on $\mathcal{H}$ is a pairing where each b_i is self-adjoint, so that $(x, b_i y) = (b_i x, y)$ and $(x, y b_i) = (x b_i, y)$ for all $x, y \in \mathcal{H}$ and all $i \in I$. Equivalently, for any $m \in \mathcal{H}$, $(mx, y) = (x, \omega(m)y)$ and $(xm, y) = (x, y\omega(m))$.

There is a bijection between adjoint pairings $(,)$ and traces ε, defined by letting $(x, y) = \varepsilon(\omega(x)y)$, or conversely $\varepsilon(y) = (1, y)$. Adjoint pairings appear often in the literature, for instance [27] (although they are usually $\mathbb{Z}[t, t^{-1}]$-linear in both factors, unlike our current semilinear definition). Semilinear adjoint pairings will be crucially important, due to the following remark.

Remark 2.8. Let $\mathcal{C}$ be a monoidal category with objects B_i, such that B_i are self-biadjoint. We assume that $\mathcal{C}$ is additive and graded and has isomorphisms (2.4)–(2.6). We call such a category a *potential categorification* of $\mathcal{H}$. In this case, there is a map of rings from $\mathcal{H}$ to $[\mathcal{C}]$

sending b_i to $[B_i]$, and (under suitable finite-dimensionality conditions) we get an adjoint semilinear pairing on $\mathscr{H}$ via $(b_{\underline{\mathbf{i}}}, b_{\underline{\mathbf{j}}}) = \mathrm{gdimHOM}_{\mathcal{C}}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}}) \in \mathbb{Z}[[t, t^{-1}]]$, the graded dimension as a vector space. Denote the pairing and its associated trace map as $(,)_{\mathcal{C}}$ and $\varepsilon_{\mathcal{C}}$.

Instead, we may assume $\mathcal{C}$ is an enriched monoidal subcategory, containing objects B_i. The isomorphisms (2.4)–(2.6) typically have no meaning in this context, since there are no grading shifts or direct sums, but we can require that they *Yoneda-hold*; that is, they hold after the application of any $\mathrm{Hom}(-, X)$ functor (to graded vector spaces). There is no definition of a Grothendieck ring in this case, but we still get an induced adjoint semilinear pairing induced by Hom spaces. We call this an *enriched potential categorification*.

We may use pairings to distinguish between different potential categorifications. The next proposition allows us to specify the pairing induced by a categorification by only investigating certain HOM spaces.

Proposition 2.9. *Traces on $\mathscr{H}$ are uniquely determined by their values $\varepsilon(b_{\underline{\mathbf{i}}})$ on increasing monomials $\underline{\mathbf{i}}$. Equivalently, adjoint pairings are determined by $(1, b_{\underline{\mathbf{i}}})$ for increasing $\underline{\mathbf{i}}$. If $\underline{\mathbf{i}}$ is nonrepeating and $\underline{\mathbf{j}}$ is a permutation of $\underline{\mathbf{i}}$, then $\varepsilon(b_{\underline{\mathbf{i}}}) = \varepsilon(b_{\underline{\mathbf{j}}})$.*

We quickly sketch the proof. Moving an index from the beginning of a monomial to the end, or vice versa, will be called *cycling* the monomial. It is clear, using biadjointness or the definition of trace, that the value of ε is invariant under cycling. It is not difficult to show that any monomial in W (in the letters s_i) will reduce, using the Coxeter relations and cycling, to an increasing monomial. When the monomial is already nonrepeating, one needs only use cycling and $s_i s_j = s_j s_i$ for distant i, j. Finally, using induction on the length of the monomial, the same principle shows that any monomial in $\mathscr{H}$ reduces to a linear combination of increasing monomials, and therefore ε is determined by these.

The upshot is that, given a potential categorification, one knows the dimension of all $\mathrm{HOM}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}})$ so long as one knows the dimension of $\mathrm{HOM}(B_{\emptyset}, B_{\underline{\mathbf{i}}})$ for increasing $\underline{\mathbf{i}}$. Note that not every choice of $(1, b_{\underline{\mathbf{i}}})$ for all increasing $\underline{\mathbf{i}}$ will yield a well-defined trace map.

Consider the adjoint pairing given by $\varepsilon_{\mathrm{std}}(b_{\underline{\mathbf{i}}}) = (1, b_{\underline{\mathbf{i}}}) = t^d$ for nonrepeating $\underline{\mathbf{i}}$ of length d. This is the semilinear version of the pairing found in [27] which picks out the coefficient of the identity in the standard basis of $\mathscr{H}$ and is called the *standard pairing*. Soergel showed that $\mathrm{HOM}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}})$ is a free graded left (or right) R-module of rank $(b_{\underline{\mathbf{i}}}, b_{\underline{\mathbf{j}}})$ using this pairing. In particular, for increasing $\underline{\mathbf{i}}$, $\mathrm{HOM}(R, B_{\underline{\mathbf{i}}})$ is generated by a single element in degree $d(\underline{\mathbf{i}})$. Since the graded dimension of R is $1/(1-t^2)^n$ we have that $(1, b_{\underline{\mathbf{i}}})_{\mathscr{H}\mathcal{C}} = t^d/(1-t^2)^n$ is a rescaling of the standard pairing.

Now let ε be the quotient map $\mathscr{H} \to \mathbb{Z}[t, t^{-1}]$ by the ideal generated by all b_i. It is a homomorphism to a commutative algebra, so it is a trace. The corresponding pairing satisfies $(1, 1) = 1$ and $(x, y) = 0$ for monomials x, y if either monomial is not 1. We call this the *trivial pairing*, $\varepsilon_{\mathrm{triv}}$.

2.2. The Temperley-Lieb Algebra

Here again we state without proof some basic facts about Temperley-Lieb algebras. They were originally defined by Temperley and Lieb in [28], and were given a topological interpretation by Kauffman [29]. There are many good expositions for the topic, such as [30, 31].

Definition 2.10. The *Temperley-Lieb algebra* $\mathcal{TL}$ is the $\mathbb{Z}[t, t^{-1}]$-algebra generated by u_i, $i \in I$ with relations

$$u_i^2 = [2]u_i, \tag{2.7}$$

$$u_i u_j = u_j u_i \quad \text{for } |i - j| \geq 2, \tag{2.8}$$

$$u_i u_j u_i = u_i \quad \text{for adjacent } i, j. \tag{2.9}$$

Proposition 2.11. *For adjacent $i, j \in I$, consider the element of $\mathcal{H}$ defined by $c_{ij} \overset{\text{def}}{=} b_i b_j b_i - b_i = b_j b_i b_j - b_j$, where the equality arises from relation* (2.3). *There is a surjective map $\mathcal{H} \to \mathcal{TL}$ sending b_i to u_i for all $i \in I$, and whose kernel is generated by c_{ij} for adjacent $i, j \in I$.*

Once again, write $u_{\underline{\mathbf{i}}}$ for a monomial in the above generators, with all the same conventions as before. The map ω descends from $\mathcal{H}$ to $\mathcal{TL}$, and we define an *adjoint pairing* on $\mathcal{TL}$ in the same way, with u_i replacing b_i everywhere. The results of Proposition 2.9 apply equally to $\mathcal{TL}$.

Definition 2.12. A category $\mathcal{C}$ as in Remark 2.8 is a *potential categorification* of $\mathcal{TL}$ if it has objects U_i satisfying

$$\begin{gathered} U_i \otimes U_i \cong U_i\{1\} \oplus U_i\{-1\}, \\ U_i \otimes U_j \cong U_j \otimes U_i \quad \text{for distant } i, j, \\ U_i \otimes U_j \otimes U_i \cong U_i \quad \text{for adjacent } i, j. \end{gathered} \tag{2.10}$$

We call it an *enriched potential categorification* if it is an enriched category with objects U_i such that these isomorphisms Yoneda-hold.

A permutation $\sigma \in S_{n+1}$ is called *321-avoiding* if it never happens that, for $i < j < k$, $\sigma(i) > \sigma(j) > \sigma(k)$. It turns out that, using the Temperley-Lieb relations, every monomial $u_{\underline{\mathbf{j}}}$ is equal to a scalar times some $u_{\underline{\mathbf{i}}}$ where $\underline{\mathbf{i}}$ is *321-avoiding*; that is, if viewed as a word in the symmetric group, it represents a *reduced* expression for a 321-avoiding permutation. Moreover, between 321-avoiding monomials, the only further relations come from (2.8), and hence it is easy to pick out a basis from this spanning set. See [30] for more details.

The Temperley-Lieb algebra has a well-known topological interpretation where an element of $\mathcal{TL}$ is a linear combination of crossingless matchings (isotopy classes of embedded planar 1-manifolds) between $n + 1$ bottom points and $n + 1$ top points. Multiplication of crossingless matchings consists of vertical concatenation (where ab is a above b), followed by removing any circles and replacing them with a factor of [2]. In this picture, u_i becomes the following:

(2.11)

The basis of 321-avoiding monomials agrees with the basis of crossingless matchings. Any increasing monomial is 321-avoiding. Increasing monomials are easy to visualize

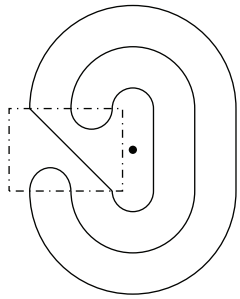


Figure 1: An example of the closure of a crossingless matching.

topologically, as they have only "right waves" and "simple cups and caps." For example:

$$u_1u_2u_3u_6u_7u_9 \longmapsto \tag{2.12}$$

As an example of a monomial which is not increasing:

$$u_4u_3u_1u_2 \longmapsto \tag{2.13}$$

Given a crossingless matching, its *closure* is a configuration of circles in the punctured plane obtained by wrapping the top boundary around the puncture to close up with the bottom boundary, as in Figure 1. Circle configurations have two topological invariants: the number of circles and the *nesting number* which is the number of circles which surround the puncture and is equal to $n+1-2l \geq 0$ for some $l \geq 0$. Given a scaling factor for each possible nesting number, one constructs a trace by letting $\varepsilon(u_{\underline{\mathbf{i}}}) = c_k[2]^m$ where m is the number of circles in the closure of $u_{\underline{\mathbf{i}}}$ and c_k is the scaling factor associated to its nesting number k. To calculate (x, y), we place y below an upside-down copy of x (or vice versa), and then take the closure. All pairings/traces on $\mathcal{TL}$ can be constructed this way, so they are all topological in nature.

The Temperley-Lieb algebra has a *standard pairing* of its own for which $c_k = 1$ for all nesting numbers k: $\varepsilon_{\text{std}}(u_{\underline{\mathbf{i}}}) = [2]^m$ as above. One can check that $\varepsilon_{\text{std}}(u_{\underline{\mathbf{i}}}) = [2]^{n+1-d(\underline{\mathbf{i}})}$ for an increasing monomial. This is *not* related to the standard pairing on $\mathcal{H}$, which does not descend to $\mathcal{TL}$. On the other hand, $\varepsilon_{\text{triv}}$ clearly does descend to a pairing *trivial pairing* on $\mathcal{TL}$, which only evaluates to a nonzero number when the nesting number is $n+1$.

It turns out that the pairing on $\mathcal{TL}$ arising from our categorification will satisfy $(1,1) = (t^n/(1-t^2))[2]^n - (t^2/(1-t^2))$ and $(1, u_{\underline{\mathbf{i}}}) = (t^n/(1-t^2))[2]^{n-d}$. We will call the associated trace ε_{cat}. Clearly $\varepsilon_{\text{cat}} = (t^n/(1-t^2)[2])\varepsilon_{\text{std}} - (t^2/(1-t^2))\varepsilon_{\text{triv}}$. In particular, on any monomial $x \neq 1$, our trace will agree with a rescaling of the standard trace. When $n = 1$, the algebras $\mathcal{TL}$ and $\mathcal{H}$ are already isomorphic, and ε_{cat} agrees with the rescaling of the standard trace on $\mathcal{H}$ discussed at the end of Section 2.1.

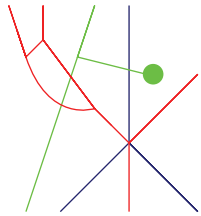


Figure 2: An example of a planar graph in the strip, with colored edges.

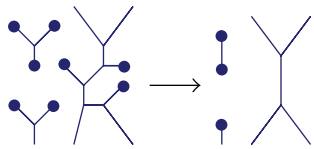


Figure 3: An example of tree reduction.

2.3. Definition of Soergel Diagrammatics

We now give a diagrammatic description of the category $\mathcal{HC}_1$, as discovered in [15]. Since the category to be defined will be equivalent to the category of Bott-Samelson bimodules, we will abuse notation temporarily and use the same names.

Definition 2.13. In this paper, a *planar graph in the strip* is a finite graph with boundary $(\Gamma, \partial\Gamma)$ embedded in $(\mathbb{R} \times [0,1], \mathbb{R} \times \{0,1\})$. In other words, all vertices of Γ occur in the interior $\mathbb{R} \times (0,1)$, and removing the vertices, we have a 1-manifold with boundary whose intersection with $\mathbb{R} \times \{0,1\}$ is precisely its boundary. This allows for edges which connect two vertices, edges which connect a vertex to the boundary, edges which connect two points on the boundary, and edges which form circles (closed 1-manifolds embedded in the plane).

We generally refer to $\mathbb{R} \times \{0,1\}$ as the *boundary*, which consists of two components, the *top boundary* $\mathbb{R} \times \{1\}$, and the *bottom boundary* $\mathbb{R} \times \{0\}$. We refer to a local segment of an edge which hits the boundary as a *boundary edge*; there is one boundary edge for each point on the boundary of the graph. We use the word *component* to mean a connected component of a graph with boundary.

This definition clearly extends to other subsets of the plane with boundary, so that we can speak of planar graphs in a disk or planar graphs in an annulus. The annulus has two boundary components, *inner* and *outer*. When we do not specify, we always mean a planar graph in the strip.

We will be drawing morphisms in $\mathcal{HC}_1$ as planar graphs with edges labelled in I. Instead of putting labels everywhere, we color the edges, assigning a color to each index in I. Henceforth, we use the term "color" and "index" interchangeably.

We now define $\mathcal{HC}_1$ anew. Let $\mathcal{HC}_1$ be the monoidal category, with hom spaces enriched over graded vector spaces, which is defined as follows.

Definition 2.14. An object in $\mathcal{HC}_1$ is given by a sequence of indices $\underline{\mathbf{i}}$, which is visualized as d points on the real line $\mathbb{R}$, labelled or "colored" by the indices in order from left to right. These objects are also called $B_{\underline{\mathbf{i}}}$. The monoidal structure on objects is concatenation of sequences.

Definition 2.15. Consider the set of *isotopy classes* of planar graphs in the strip whose edges are colored by indices in I such that only four types of vertices exist: univalent vertices or "dots", trivalent vertices with all three adjoining edges of the same color, 4-valent vertices whose adjoining edges alternate in colors between distant i and j, and 6-valent vertices whose adjoining edges alternate between adjacent i and j. This set has a grading, where the degree of a graph is +1 for each dot and -1 for each trivalent vertex; 4-valent and 6-valent vertices are of degree 0. The allowable vertices, which we call "generators," are pictured here:

(2.14)

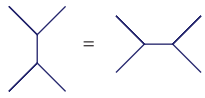
(2.15)

The intersection of a graph with the boundary yields two sequences of colored points on $\mathbb{R}$, the top boundary $\underline{\mathbf{i}}$ and the bottom boundary $\underline{\mathbf{j}}$. In this case, the graph is viewed as a morphism from $\underline{\mathbf{j}}$ to $\underline{\mathbf{i}}$. For instance, if "blue" corresponds to the index i and "red" to j, then the lower right generator is a degree 0 morphism from jij to iji. Although this paper is easiest to read in color, it should be readable in black and white: the colors appearing are typically either blue, red, green, or miscellaneous and irrelevant. We throughout use the convention that blue (the darker color) is always adjacent to red (the middle color) and distant from green (the lighter color).

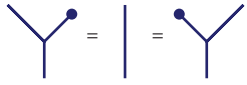
We let $\mathrm{Hom}_{\mathscr{UC}_1}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}})$ be the graded vector space with basis given by planar graphs as above which have the correct top and bottom boundary, modulo relations (2.16) through (2.30). As usual in a diagrammatic category, composition of morphisms is given by vertical concatenation (read from bottom to top), the monoidal structure is given by horizontal concatenation, and relations are to be interpreted monoidally (i.e., they may be applied locally inside any other planar diagram).

The relations are given in terms of colored graphs, but with no explicit assignment of indices to colors. They hold for *any* assignment of indices to colors, so long as certain adjacency conditions hold. We will specify adjacency for all pictures, although one can generally deduce it from the fact that 6-valent vertices only join adjacent colors, and 4-valent vertices only join distant colors.

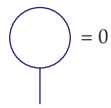
For example, these first four relations hold, with blue representing a generic index.



(2.16)



(2.17)



(2.18)

(2.19)

We will repeatedly call a picture looking like (2.18) by the name "needle." Note that a needle is not necessarily zero if there is something in the interior. Note that a circle is just a needle with a dot attached, by (2.17), so that an empty circle evaluates to 0.

Remark 2.16. It is an immediate consequence of relations (2.16) and (2.17) that any *tree* (connected graph with boundary without cycles) of one color is equal to

(i) if it has no boundary, two dots connected by an edge. Call the entire component a *double dot*.

(ii) if it has one boundary edge, a single dot connected by the edge to the boundary. Call the component a *boundary dot*.

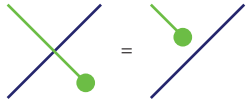
(iii) if it has more boundary edges, a tree with no dots and the fewest possible number of trivalent vertices needed to connect the boundaries. Moreover, any two such trees are equal. Call the component a *simple tree*.

We refer to this as *tree reduction*.

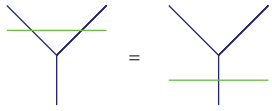
This applies only to components of a graph which are a single color. Even if the blue part of a graph looks like a tree, if other colors overlap, then we may not apply tree reduction in general.

In the following relations, the two colors are distant

(2.20)



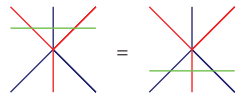
(2.21)



(2.22)

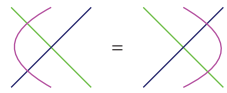
(2.23)

In this relation, two colors are adjacent, and both distant to the third color.



(2.24)

In this relation, all three colors are mutually distant.

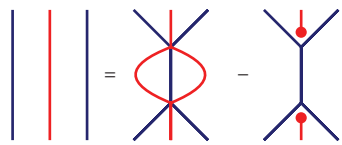


(2.25)

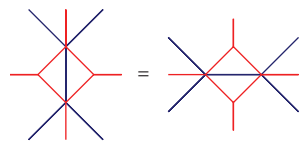
Remark 2.17. Relations (2.20) through (2.25) indicate that any part of the graph colored i and any part of the graph colored j "do not interact" for distant i and j, that is, one may visualize sliding the j-colored part past the i-colored part, and it will not change the morphism. We call this the *distant sliding property*.

In the following relations, the two colors are adjacent.

(2.26)



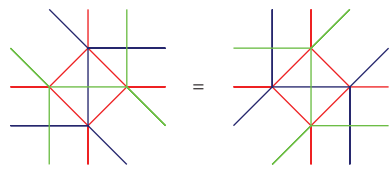
(2.27)



(2.28)

(2.29)

In this final relation, the colors have the same adjacency as $\{1, 2, 3\}$



(2.30)

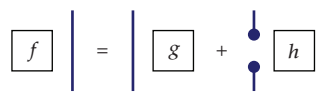
This concludes the list of relations defining $\mathcal{HC}_1$.

Remark 2.18. We chose here to describe $\mathcal{HC}_1$ in terms of planar graphs with relations, with the notion of isotopy built-in, rather than in terms of generators and relations. Note, however, that using isotopy and (2.17), we get  Therefore, all "cups" and "caps" can be expressed in terms of the generators. By adding new relations corresponding to isotopy, one could give a presentation of the category where the "generators" above (and their isotopy twists) are really generators. This is how the category is presented in [15].

We will occasionally use a shorthand to represent double dots. We identify a double dot colored i with the polynomial $f_i \in R$, and for a linear combination of disjoint unions of double dots in the same region of a graph, we associate the appropriate linear combination of products of f_i. For any polynomial $f \in R$, a square box with a polynomial f in a region will represent the corresponding linear combination of graphs with double dots.

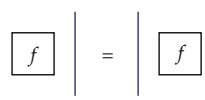
For instance,  $= \boxed{f_i^2 f_j}$.

Relations (2.19), (2.29), and (2.23) are referred to as *dot forcing rules*, because they describe at what price one can "force" a double dot to the other side of a line. The three relations imply that, given a line and an arbitrary collection of double dots on the left side of that line, one can express the morphism as a sum of diagrams where all double dots are on the right side, or where the line is "broken" (as illustrated next). Rephrasing this, for any polynomial f there exist polynomials g and h such that



(2.31)

The polynomials appearing can in fact be found using the Demazure operator ∂_i, and in particular, $h = \partial_i(f)$. One particular implication is that



(2.32)

whenever f is a polynomial invariant under s_i (and blue represents i). As an exercise, the reader can check that f_i^2 slides through a line colored i. These polynomial relations are easy to deduce, or one can refer to [15] (see page 7, pages 16-17, and relation 3.16).

We have an bimodule action of R on morphisms by placing boxes (i.e., double dots) in the leftmost or rightmost regions of a graph. Now we can formulate the main result of [15].

Theorem 2.19. *There is a functor from this diagrammatic category $\mathcal{HC}_1$ to the earlier definition in terms of Bott-Samelson bimodules. This functor sends $\underline{\mathbf{i}}$ to the bimodule $B_{\underline{\mathbf{i}}}$ and a planar graph to a map of bimodules, preserving the grading and the R-bimodule action on morphisms. This functor is an equivalence of categories.*

Corollary 2.20. *The R-bimodules* $\mathrm{Hom}_{\mathcal{HC}_1}(B_{\underline{\mathbf{i}}}, B_{\underline{\mathbf{j}}})$ *are free as left (or right) R-modules. In other words, placing double dots to the left of a graph is a torsion-free operation.*

Now, we have justified our abuse of notation. In this paper, we will never need to know explicitly what map of R-bimodules a planar graph corresponds to, so the interested reader can see [15] for details. In fact, we will not use Theorem 2.19 at all, preferring to work entirely with planar graphs. However, we do use Corollary 2.20, a fact which would be difficult to prove diagrammatically.

The proof of Theorem 2.19 can be quickly summarized: first, one explicitly constructs a functor from the diagrammatic category to the Bott-Samelson category. Then, using the observations of the next section, one shows that the diagrammatic category is a potential categorification of $\mathcal{H}$ and that the diagrammatic category, the Bott-Samelson category, and the image of the former in the latter all induce the same adjoint pairing on $\mathcal{H}$. Therefore, the functor is fully faithful.

2.4. Understanding Soergel Diagrammatics

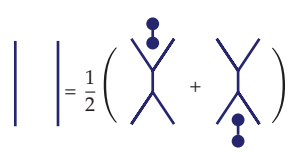
Let us explain diagrammatically why the category $\mathcal{H}\mathcal{C}_1$ is a potential categorification of $\mathcal{H}$, and induces the aforementioned adjoint pairing.

Definition 2.21. Given a category $\mathcal{C}$ whose morphism spaces are $\mathbb{Z}$-modules, one may take its *additive closure*, which formally adds direct sums of objects and yields an additive category. Given $\mathcal{C}$ whose morphism spaces are graded $\mathbb{Z}$-modules, one may take its *grading closure* which formally adds shifts of objects, but restricts morphisms to be homogeneous of degree 0. Given $\mathcal{C}$ an additive category, one may take the *idempotent completion* or *Karoubi envelope*, which formally adds direct *summands*. Recall that the Karoubi envelope has as objects pairs (B, e) where B is an object in $\mathcal{C}$ and e an idempotent endomorphism of B. This object acts as though it were the "image" of this projection e and behaves like a direct summand. When taking the Karoubi envelope of a graded category (or a category with graded morphisms) one restricts to homogeneous degree 0 idempotents. We refer in this paper to the entire process which takes a category $\mathcal{C}$, whose morphism spaces are graded $\mathbb{Z}$-modules, and returns the Karoubi envelope of its additive and grading closure as taking the *graded Karoubi envelope*. All these transformations interact nicely with monoidal structures. For more information on Karoubi envelopes see [32].

We let $\mathcal{H}\mathcal{C}_2$ be the graded additive closure of $\mathcal{H}\mathcal{C}_1$, and let $\mathcal{H}\mathcal{C}$ be the graded Karoubi envelope of $\mathcal{H}\mathcal{C}_1$.

We wish to show that the isomorphisms (2.4) through (2.6) hold in $\mathcal{H}\mathcal{C}_2$. Relation (2.20) immediately implies that $B_i \otimes B_j \cong B_j \otimes B_i$ for i, j distant, with the isomorphism being given by the 4-valent vertex.

We have the following equality:



(2.33)

To obtain this, use (2.17) to stretch two dots from the two lines into the middle, and then use (2.19) to connect them. The identity id_{ii} decomposes as a sum of two orthogonal idempotents, each of which is the composition of a "projection" and an "inclusion" map of degree ± 1, to

and from B_i (explicitly, $\mathrm{id}_{ii} = i_1p_1 + i_2p_2$ where $p_1i_1 = \mathrm{id}_i$, $p_2i_2 = \mathrm{id}_i$, $p_1i_2 = 0 = p_2i_1$). This implies that $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$ and is a typical example of how direct sum decompositions work in diagrammatic categories.

Similarly, the two color variants of relation (2.27) together express the direct sum decompositions in the Karoubi envelope

$$\begin{aligned} B_i \otimes B_{i+1} \otimes B_i &= C_{ij} \oplus B_i \\ B_{i+1} \otimes B_i \otimes B_{i+1} &= C_{ji} \oplus B_{i+1}. \end{aligned} \tag{2.34}$$

Again, the identity $\mathrm{id}_{i(i+1)i}$ is decomposed into orthogonal idempotents. The second idempotent factors through B_i, and the corresponding object in the Karoubi envelope will be isomorphic to B_i. The first idempotent, which we call a "doubled 6-valent vertex," corresponds to a new object C_{ij} in the idempotent completion. It turns out that the doubled 6-valent vertex C_{ij} for "blue red blue" is isomorphic in the Karoubi envelope to the doubled 6-valent vertex C_{ji} for "red blue red" (i.e., their images are isomorphic). We may abuse notation and call both of these new objects C_{ij}; it is a summand of both $i(i+1)i$ and $(i+1)i(i+1)$. The image of C_{ij} in the Grothendieck group is c_{ij}.

We can also understand the induced pairing on $\mathscr{H}$ using diagrammatic arguments. The theorems below are proven in [15], and we will not use them in this paper (except motivationally), proving their analogs in the Temperley-Lieb case directly.

Theorem 2.22 (Color Reduction). *Consider a morphism $\varphi : \emptyset \to \underline{\mathbf{i}}$, and suppose that the index i (blue) appears in $\underline{\mathbf{i}}$ zero times (resp.,: once). Then φ is in the -span of graphs which only contain blue in the form of double dots in the leftmost region of the graph (resp., as well as a single boundary dot). This result may be obtained simultaneously for multiple indices i.*

Corollary 2.23. *The space $\mathrm{Hom}_{\mathscr{H}C_1}(\emptyset, \emptyset)$ is precisely the graded ring R. In other words, it is freely generated (over double dots) by the empty diagram. The space $\mathrm{Hom}_{\mathscr{H}C_1}(\emptyset, \underline{\mathbf{i}})$ for $\underline{\mathbf{i}}$ nonrepeating is a free left (or right) R-module of rank 1, generated by the following morphism of degree $d(\underline{\mathbf{i}})$.*

(2.35)

The proof of the theorem does not use any sophisticated technology, only convoluted pictorial arguments. It comprises the bulk of [15]. The corollary implies that $\varepsilon_{\mathscr{H}C_1}(b_{\underline{\mathbf{i}}}) = t^d/(1-t^2)^n$ for nonrepeating $\underline{\mathbf{i}}$ of length d, as stated in Section 2.1.

2.5. Aside from Karoubi Envelopes and Quotients

Return to the setup of Definition 2.21. If $\mathcal{C}$ is a full subcategory of (graded) R-bimodules for some ring R, then the transformations described above behave as one would expect them to. In particular, the Karoubi envelope agrees with the full subcategory which includes all summands of the previous objects. The Grothendieck group of the Karoubi envelope is in some sense "under control" if one understands indecomposable R-bimodules already. On the other hand, the Karoubi envelope of an arbitrary additive category may be enormous, and to control the size of its Grothendieck group, one should understand and classify all

idempotents in the category, a serious task. Also, arbitrary additive categories need not have the Krull-Schmidt property, making their Grothendieck groups even more complicated.

The Temperley-Lieb algebra is obtained from the Hecke algebra by setting the elements c_{ij} to zero, for $i = 1, \ldots, n-1$. These elements lift in the Soergel categorification to objects C_{ij}. The obvious way one might hope to categorify $\mathcal{TL}$ would be to take the quotient of the category $\mathcal{HC}$ by each object C_{ij}.

To mod out an additive *monoidal* category $\mathcal{C}$ by an object Z, one must kill the *monoidal* ideal of id_Z in $\mathrm{Mor}(\mathcal{C})$, that is, the morphism space $\mathrm{Hom}(X, Y)$ in the quotient category is exactly $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ modulo the submodule of morphisms factoring through $V \otimes Z \otimes W$ for any V, W. If the category is drawn diagrammatically, one needs to only kill any diagram which has id_Z as a subdiagram.

We have not truly drawn $\mathcal{HC}$ diagrammatically, only $\mathcal{HC}_1$. The object we wish to kill is not an object in $\mathcal{HC}_1$; the closest thing we have is the corresponding idempotent, the doubled 6-valent vertex. However, this is not truly a problem, due to the following proposition, whose proof we leave to the reader.

Proposition 2.24. *Let $\mathcal{C}_1$ be an additive category, and let B be an object in $\mathcal{C}_1$, and let e be an idempotent in End(B). Let $\mathfrak{D}_1$ be the quotient of $\mathcal{C}_1$ by the morphism e. Let $\mathcal{C}$ and $\mathfrak{D}$ be the respective Karoubi envelopes. Finally, let $\mathfrak{D}'$ be the quotient of $\mathcal{C}$ by the identity of the object (B, e). Then, there is a natural equivalence of categories from $\mathfrak{D}$ to $\mathfrak{D}'$.*

The analogous statement holds when one considers graded Karoubi envelopes.

Remark 2.25. Note that $\mathfrak{D}'$ has more objects than $\mathfrak{D}$, but they are still equivalent. For instance, (B, e) and $(B, 0)$ are distinct (isomorphic) objects in $\mathfrak{D}'$, but are the same object in $\mathfrak{D}$.

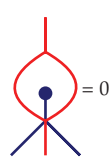
So to categorify $\mathcal{TL}$, one might wish to take the quotient of $\mathcal{HC}_1$ by the doubled 6-valent vertex, and then take the Karoubi envelope. This is easy to do diagrammatically, which is one advantage to the diagrammatic approach over the R-bimodule approach. The quotient of $\mathcal{HC}_1$ will no longer be a category which embeds nicely as a full subcategory of bimodules. One might worry that Krull-Schmidt fails, or that to understand its Karoubi envelope one must classify all idempotents therein. Thankfully, our calculation of HOM spaces will imply easily that its graded additive closure is Krull-Schmidt and is *already* idempotent closed, so it is equivalent to its own Karoubi envelope (see Section 3.3).

3. The Quotient Category $\mathcal{TLC}$

3.1. A Motivating Calculation

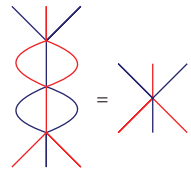
As discussed in the previous section, our desire is to take the quotient of $\mathcal{HC}_1$ by the doubled 6-valent vertex, and then take the graded Karoubi envelope.

An important consequence of relations (2.26) and (2.18) is that



(3.1)

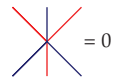
from which it follows, using (2.27), that



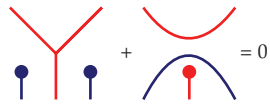
(3.2)

so the (monoidal) ideal generated in $\mathcal{HC}_1$ by a doubled 6-valent vertex is the same as the ideal generated by the 6-valent vertex.

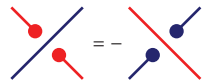
Claim 1. The following relations are all equivalent (the ideals they generate are equal)



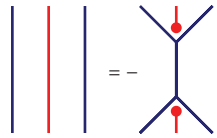
(3.3)



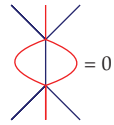
(3.4)



(3.5)



(3.6)



(3.7)

Proof. (3.3)⇒(3.4): add a dot, and use relation (2.26).

(3.4)⇒(3.5): add a dot to the top, and use (2.17).

(3.5)⇒(3.4): apply to the middle of the diagram.

(3.5)⇒(3.6): stretch dots from the blue strands towards the red strand using (2.17), and then apply (3.5) to the middle.

(3.6)⇒(3.7): use relation (2.27).

(3.7)⇒(3.3): use (3.2). □

Modulo 6-valent vertices, the relations (2.26) and (2.27) become (3.4) and (3.6) above. All other relations involving 6-valent vertices, namely, (2.28), (2.30), and (2.24), are sent to zero modulo 6-valent vertices. Relation (3.5) implies both (3.4) and (3.6) without reference to any graphs using 6-valent vertices. So, if we wish to rephrase our quotient in terms of graphs that never have 6-valent vertices, the sole necessary relation imposed by the fact that 6-valent vertices were sent to zero is the relation (3.5).

Suppose, we only allow ourselves univalent, trivalent, and 4-valent vertices, but no 6-valent vertices, in a graph Γ. Then, the i-graph of Γ, which consists of all edges colored i and all vertices they touch, will be disjoint from the $i+1$- and $i-1$-graphs of Γ. The distant sliding property implies that the i-graph and the j-graph of Γ do not interact effectively, when i and j are distant. This will motivate the definition in the next section.

3.2. Diagrammatic Definition of $\mathcal{TLC}$

Definition 3.1. We let $\mathcal{TLC}_1$ be the monoidal category, with hom spaces enriched over graded vector spaces, defined as follows. Objects will be sequences of colored points on the line $\mathbb{R}$, which we will call $\underline{\mathbf{i}}$ or $U_{\underline{\mathbf{i}}}$. Consider the set whose elements are described as follows:

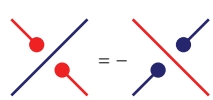
(1) for each $i \in I$, consider a planar graph Γ_i in the strip, which is drawn with edges colored i (see Definition 2.13);

(2) the only vertices in Γ_i are univalent vertices (dots) and trivalent vertices;

(3) the graphs Γ_i and Γ_{i+1} are disjoint. All graphs Γ_i are pairwise disjoint on the boundary;

(4) we consider isotopy classes of this data, so that one may apply isotopy to each Γ_i individually so long as it stays appropriately disjoint.

This set has a grading, where the degree of a graph is +1 for each dot and −1 for each trivalent vertex, and the degree of an element of this set is the sum of the degrees for each graph Γ_i. Just as in Definition 2.15, each element of the set has a top and bottom boundary which is an object in $\mathcal{TLC}$, and will be thought of as a map from the bottom boundary to the top. We let $\mathrm{Hom}_{\mathcal{TLC}_1}(U_{\underline{\mathbf{i}}}, U_{\underline{\mathbf{j}}})$ be the graded vector space with basis given by elements of the set above with bottom boundary $\underline{\mathbf{j}}$ and top boundary $\underline{\mathbf{i}}$, modulo the relations (2.16) through (2.19), (2.29), and the new relation (3.5). As a reminder, the new relation is given here again.



(3.8)

As before, composition of morphisms is given by vertical concatenation, the monoidal structure is given by horizontal concatenation, and relations are to be interpreted monoidally. This concludes the definition.

Phrasing the definition in this fashion eliminates the need to add distant sliding rules, for these are now built into the notion of isotopy. Note that as we have stated it here, Γ_i and Γ_j may have edges which are embedded in a tangent fashion, or even entirely overlapped. However, such embeddings are isotopic to graph embeddings with only transverse edge intersections, which arise as 4-valent vertices in our earlier viewpoint.

Proposition 3.2. *The category $\mathcal{TLC}_1$ is isomorphic to $\mathcal{HC}_1$ modulo the 6-valent vertex.*

Proof. Due to the observations of Section 3.1, this is obvious. □

Hom spaces in $\mathcal{TLC}_1$ are in fact enriched over graded R-bimodules, by placing double dots as before. However, they will no longer be free as left or right R-modules, as we will see.

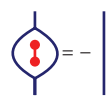
Remark 3.3. Note that tree reduction (see Remark 2.16) can now be applied to any tree of a single color in $\mathcal{TLC}$, regardless of what other colors are present, since the only colors which can intersect the tree are distant colors which do not actually interfere.

We denote by $\mathcal{TLC}$ the graded Karoubi envelope of $\mathcal{TLC}_1$, and by $\mathcal{TLC}_2$ the graded additive closure of $\mathcal{TLC}_1$. However, we will show that $\mathcal{TLC}_2$ is already idempotent closed, so that $\mathcal{TLC}_2$ and $\mathcal{TLC}$ are the same.

It is obvious that

$$\begin{gathered} U_i \otimes U_{i+1} \otimes U_i \cong U_i, \\ U_{i+1} \otimes U_i \otimes U_{i+1} \cong U_{i+1} \end{gathered} \tag{3.9}$$

in $\mathcal{TLC}_1$, and from the relation (3.6) and the simple calculation (using dot forcing rules) that



(3.10)

For the same reasons as in Section 2.4, we still have $U_i \otimes U_j \cong U_j \otimes U_i$ for distant i, j, and $U_i \otimes \mathrm{U}_i \cong U_i\{1\} \oplus U_i\{-1\}$ in $\mathcal{TLC}_2$. Therefore, $\mathcal{TLC}$ is a potential categorification of $\mathcal{TL}$, and induces an adjoint pairing and a trace map $\varepsilon_{\mathcal{TLC}}$ on $\mathcal{TL}$. At this point, we have not shown that the category $\mathcal{TLC}_1$ is nonzero, so this pairing could be 0.

3.3. *Using the Adjoint Pairing*

Proposition 3.4. *Let $\mathcal{C}_1$ be an enriched category which is a potential categorification of $\mathcal{TL}$, whose objects are $U_{\underline{\mathbf{i}}}$ for sequences $\underline{\mathbf{i}}$. Let $\mathcal{C}_2$ be its additive graded closure, and let $\mathcal{C}$ be its graded Karoubi envelope. Suppose that the induced trace map $\varepsilon_{\mathcal{C}_1}$ on $\mathcal{TL}$ is equal to ε_{cat}. Then, the set of $U_{\underline{\mathbf{i}}}\{n\}$ for $n \in \mathbb{Z}$ and 321-avoiding $\underline{\mathbf{i}}$ forms an exhaustive irredundant list of indecomposables in $\mathcal{C}_2$. In addition, $\mathcal{C}_2$ is Krull-Schmidt and idempotent closed (so $\mathcal{C}_2$ and $\mathcal{C}$ are equivalent), and $\mathcal{C}$ categorifies $\mathcal{TL}$.*

This proposition is an excellent illustration of the utility of the induced adjoint pairing. We prove it in a series of lemmas, which all assume the hypotheses above.

Lemma 3.5. *The object $U_{\underline{\mathbf{i}}}$ in $\mathcal{C}_1$ has no nontrivial (homogeneous) idempotents when $\underline{\mathbf{i}}$ is 321-avoiding. Moreover, if both $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ are 321-avoiding, then $U_{\underline{\mathbf{i}}} \cong U_{\underline{\mathbf{j}}}\{m\}$ in $\mathcal{C}_2$ if and only if $m = 0$ and $u_{\underline{\mathbf{i}}} = u_{\underline{\mathbf{j}}}$ in $\mathcal{TL}$.*

Proof. Two 321-avoiding monomials in $\mathcal{TL}$ are equal only if they are related by the relation (2.8). Since this lifts to an isomorphism $U_i \otimes U_j \cong U_j \otimes U_i$ in $\mathcal{C}_2$, we have $u_{\underline{\mathbf{i}}} = u_{\underline{\mathbf{j}}} \Rightarrow U_{\underline{\mathbf{i}}} \cong U_{\underline{\mathbf{j}}}$.

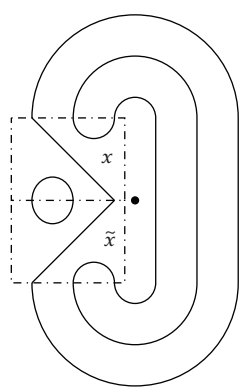
If an object has a 1-dimensional space of degree 0 endomorphisms, then it must be spanned by the identity map, and there can be no nontrivial idempotents. If an object has endomorphisms only in nonnegative degrees, then it can not be isomorphic to any nonzero degree shift of itself. If two objects X and Y are such that both $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Y,X)$ are concentrated in strictly positive degrees, then no grading shift of X is isomorphic to Y, since there can not be a degree zero map in both directions.

Therefore, we need only show that (for 321-avoiding monomials) $U_{\underline{\mathbf{i}}}$ has endomorphisms concentrated in nonnegative degree, with a 1-dimensional degree 0 part, and that when $u_{\underline{\mathbf{i}}} \neq u_{\underline{\mathbf{j}}}$, $\mathrm{Hom}(U_{\underline{\mathbf{i}}}, U_{\underline{\mathbf{j}}})$ is in strictly positive degrees. This question is entirely determined by the pairing on $\mathcal{TL}$, since it only asks about the graded dimension of Hom spaces.

When $\underline{\mathbf{i}}$ is empty, we already know that $(1,1) = (t^n/(1-t^2))[2]^n - (t^2/(1-t^2))$, which has degree 0 coefficient 1, and is concentrated in nonnegative degrees.

We know how to calculate (x,y) in $\mathcal{TL}$ when x and y are monomials, and either x or y is not 1 (see Section 2.2). We draw x as a crossingless matching, draw y upside-down and place it below x, and close off the diagram: if there are m circles in the diagram, then $(x,y) = t^n[2]^{m-1}/(1-t^2)$. In particular, if $m = n+1$, then the Hom space will be concentrated in nonnegative degrees, with 1-dimensional degree 0 part. If $m < n+1$, then the Hom space will be concentrated in strictly positive degrees.

We leave it as an exercise to show that, if x is a crossingless matching (i.e., a 321-avoiding monomial) then the closed diagram for (x,x) has exactly $n+1$ circles. The following example makes the statement fairly clear, where $\tilde{x}$ is x upside-down:



(3.11)

In this example x has all 3 kinds of arcs which appear in a crossingless matching: bottom to top, bottom to bottom, and top to top. Each of these corresponds to a single circle in the diagram closure.

Similarly, there are fewer than $n+1$ circles in the diagram for (x,y) whenever the crossingless matchings x, y are nonequal. Consider the diagram above but with the region x removed. One can see that no circles are yet completed, and each boundary point of x's region is matched to the other by an arc. The number of circles is maximized when you pair these boundary points to each other, and this clearly gives the matching x. For any other matching y, two arcs will become joined into one, and fewer than $n+1$ circles will be created. □

Lemma 3.6. *$\mathcal{C}_2$ is idempotent closed, and its indecomposables can all be expressed as grading shifts of $U_{\underline{\mathbf{i}}}$ for 321-avoiding $\underline{\mathbf{i}}$. It has the Krull-Schmidt property.*

Proof. Since the Temperley-Lieb relations allow one to reduce a general word to a 321-avoiding word, one can show that every $U_{\underline{\mathbf{i}}}$ is isomorphic to a direct sum of shifts of $U_{\underline{\mathbf{j}}}$ for 321-avoiding $\underline{\mathbf{j}}$, using isomorphisms and direct sum decompositions instead of the analogous Temperley-Lieb relations. Clearly these shifted $U_{\underline{\mathbf{j}}}$ are all indecomposable, since they have no nontrivial idempotents; these are then all the indecomposables. Since every indecomposable in $\mathcal{C}_2$ has a graded local endomorphism ring (with maximal ideal given by positively graded morphisms), $\mathcal{C}_2$ is idempotent closed and Krull-Schmidt (see [33], Section 2.2). □

The Krull-Schmidt property implies that isomorphism classes of indecomposables form a basis for the Grothendieck group.

Proof of Proposition 3.4. There is a $\mathbb{Z}[t, t^{-1}]$-linear map of rings $\mathcal{TL} \to [\mathcal{C}_2]$, which is evidently bijective because it sends the 321-avoiding basis to the 321-avoiding basis. Since $\mathcal{C} = \mathcal{C}_2$, we are done. □

This proposition shows that Lemma 1.1 implies Theorem 1.2.

Remark 3.7. In analogy to the paper [15], the bulk of the proof of Theorem 1.2 lies in proving that hom spaces induce a particular adjoint pairing. Beyond that, we have mostly stated the obvious. Let us note that what is obvious for $\mathcal{TL}$ and $\mathcal{TLC}$ is *not* obvious at all when dealing with $\mathcal{H}$ and $\mathcal{HC}$. In particular, if we are given a category $\mathcal{C}_1$ which is a potential categorification of $\mathcal{H}$ as in Proposition 3.4, we can not conclude that $\mathcal{C}$ categorifies $\mathcal{H}$. We summarize the differences here.

It is clear (for both Hecke and Temperley-Lieb) that the map $\mathcal{H} \to [\mathcal{HC}_2]$ is well defined and surjective. The two main subtleties are (1) the difference between $\mathcal{HC}_2$ and $\mathcal{HC}$, and (2) the injectivity of the map.

In general, one likes to examine the additive Grothendieck group only of idempotent closed categories with the Krull-Schmidt property, because this guarantees that indecomposables form a basis for the Grothendieck group. Thus it is convenient that $\mathcal{TLC}_2$ is already idempotent closed. Thankfully, we have a result of Soergel [11] that proves that $[\mathcal{HC}_2] \cong [\mathcal{HC}]$, as was discussed in Remark 2.5.

To show injectivity of the map in the $\mathcal{TL}$ case, we can identify a basis for $\mathcal{TL}$ which is sent to a complete set of indecomposables, and then we can evaluate the trace map to show that these indecomposables are pairwise nonisomorphic. For $\mathcal{HC}$, we do not currently know what the indecomposables (i.e., idempotents) are, nor do we know their preimage in $\mathcal{H}$. If we knew a class of indecomposables which decategorified to the Kazhdan-Lusztig basis, then we could use a similar argument to the above to show that they and their shifts form an exhaustive irredundant list of indecomposables in $\mathcal{HC}$, and therefore that the map $\mathcal{H} \to [\mathcal{HC}]$ is injective. Soergel discusses this in the last chapter of [11]. This is actually a deep question, shown by Soergel ([34], see also [8, 11]) to be equivalent to proving a version of the Kazhdan-Lusztig conjectures. In any case, the result depends on the base field $\Bbbk$, and no simple proof has been found. In particular, to prove that the graded Karoubi closure of the diagrammatic category $\mathcal{HC}_1$ categorifies the Hecke algebra (for certain $\Bbbk$), we must pass to the world of bimodules where Soergel's powerful geometric techniques will work. In particular, there is currently no proof of injectivity if one defines the category $\mathcal{HC}$ diagrammatically over $\Bbbk = \mathbb{Z}$.

It should be emphasized that the story of $\mathcal{TL}$ is a particularly easy one (as is its Kazhdan-Lusztig theory). No high-powered technical machinery is needed, and the proofs of idempotent closure and injectivity are self-contained and diagrammatic. In fact, the

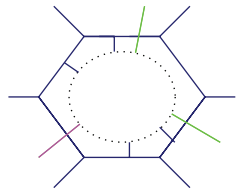


Figure 4: An arbitrary innermost blue cycle. The dotted line encapsulates the subgraph on the interior which may contain colors adjacent to blue.

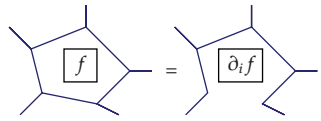
arguments in this paper *do* work entirely over $\mathbb{Z}[1/2]$, as can be checked. Dividing by two must be allowed in order to split the identity of U_{ii} into idempotents, as in (2.33); however, it is likely that the arguments would work over $\mathbb{Z}$ as well. Working over $\mathbb{Z}$ is discussed more extensively in [22].

Remark 3.8. A category $\mathcal{O}$ analog of the fact that 321-avoiding monomials lift to indecomposable Soergel bimodules, which remain indecomposable upon passage to the Temperley-Lieb quotient, can be found in Lemma 5.2 of [14].

3.4. Reductions

When we say that a graph or a morphism "reduces" to a set of other graphs, we mean that the morphism is in the $\Bbbk$-span of those graphs. We refer to a one-color graph, each of whose (connected) components is either a simple tree with respect to its boundary or a double dot, as a *simple forest with double dots*. If there are no double dots, it is a *simple forest without double dots*. Tree reduction implies that any graph Γ_i without cycles reduces to a simple forest with double dots. Note also that circles in a graph are equal to needles with a dot attached, and can be treated just like any other cycle.

If there were only one color, we could iterate the following rule (which is an implication of the dot forcing rules and (2.18)) to break cycles:



(3.12)

We do something similar for the general case.

Proposition 3.9. *In $\mathcal{TLC}_1$, any morphism reduces to one where, for each i, the i-graph is a simple forest with double dots. Moreover, we may assume that all double dots are in the lefthand region.*

Proof. We use induction on the total number of cycles (of any color) in the graph. Suppose there is a blue colored cycle: choose one so that it delineates a single region (i.e., there are no other cycles inside). There may be blue "spokes" going from this cycle into the interior, but no two spokes can meet, lest they create another region. By tree reduction on the spokes, we can assume that any blue appearing inside the cycle is in a different blue component than the cycle. Other colors may cross over the cycle, into the interior. If we view the interior of the cycle as a graph of its own, it has fewer total cycles so we may use induction. Since

the boundary of the interior contains no blue color or colors adjacent to blue, they may be assumed to appear in the interior only in the form of double dots next to the cycle. Using dot-forcing rules, we reduce to two graphs: one with the cycle broken, and one with all these double dots on the exterior of the cycle. The former reduces by induction. For the latter, only distant colors enter the cycle, so they can be slid out of the way to leave an empty blue cycle, which is 0 by the rule above.

We need to only do the base case, where the graph has no cycles. The dot-forcing rules imply that double dots may be moved to any region of the (multicolored) graph, at the cost of breaking a few lines. Breaking lines will never increase the number of cycles. Therefore, if we have a graph without cycles, tree reduction implies that we actually have a simple forest with double dots, and dot forcing allows us to move these double dots to the left. The breaking of lines may require more tree reduction, yielding more double dots, but this process is finite. □

Remark 3.10. This proposition and its proof will apply to graphs in any connected simply connected region in the plane.

Corollary 3.11. *For any nonrepeating* $\underline{\mathbf{i}}$, $\mathrm{Hom}_{\mathcal{TLC}_1}(\emptyset, \underline{\mathbf{i}})$ *is generated (as a left or right R-module) by a single element* $\varphi_{\underline{\mathbf{i}}}$ *of degree* $d(\underline{\mathbf{i}})$, *pictured below*

(3.13)

Proof. A simple forest with double dots and at most one boundary edge is no more than a boundary dot with double dots. Thus, any morphism reduces to a boundary dot for each color, accompanied by double dots. □

To show Lemma 1.1, we need to only investigate $\mathrm{Hom}(\emptyset, \underline{\mathbf{i}})$ for increasing $\underline{\mathbf{i}}$, since we have already shown that the values of $\varepsilon(u_{\underline{\mathbf{i}}})$ are determined by their values for increasing $\underline{\mathbf{i}}$. This space will be an R-bimodule where the left and right action are the same (since the lefthand and righthand regions are the same in any picture with no bottom boundary), so we view it as an R-module, and we have just shown that it is cyclic. Let $I_{\underline{\mathbf{i}}}$ be the ideal which is the kernel of the map $R \to \mathrm{HOM}(\emptyset, \underline{\mathbf{i}})$ sending $1 \mapsto \varphi_{\underline{\mathbf{i}}}$; we call it the *TL ideal* of $\underline{\mathbf{i}}$. Proving the Lemma 1.1 is to find $I_{\underline{\mathbf{i}}}$ and show that the graded dimension of $R/I_{\underline{\mathbf{i}}}\{d\}$ is $\varepsilon_{\mathrm{cat}}(u_{\underline{\mathbf{i}}})$.

Remark 3.12. Since the space $\mathrm{Hom}_{\mathcal{HC}_1}(\emptyset, \underline{\mathbf{i}})$ is a *free* R-module, all polynomials in $I_{\underline{\mathbf{i}}}$ must have arisen from reducing to some morphism which contained the relation (3.5) to a "nice form," that is, $\varphi_{\underline{\mathbf{i}}}$ plus double dots. In other words, letting α_i be the morphism pictured below, we want to plug α_i into a bigger graph, reduce it to a nice form, and see what we get.

(3.14)

Remember that α_i is actually just a 6-valent vertex with two dots attached (one red and one blue). This bigger graph, into which α_i is plugged, will actually be a graph on the *punctured plane* or *punctured disk* with specified boundary conditions on both the outer and inner boundaries. The difficult graphical proofs of this paper just consist in analyzing such

graphs. This is done by splitting the punctured plane into simply connected regions and using the above proposition.

3.5. Generators of the TL Ideal

The sequence $\underline{\mathbf{i}}$ is assumed to be nonrepeating.

Proposition 3.13. *The TL ideal of* $\emptyset$ *contains* $y_{i,j} \overset{\text{def}}{=} f_i f_j (f_i + 2f_{i+1} + 2f_{i+2} + \cdots + 2f_{j-1} + f_j)$ *over all* $1 \le i < j \le n$.

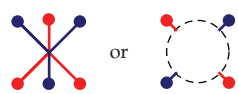
The TL ideal of $\underline{\mathbf{i}}$ *contains* $z_{i,j,\underline{\mathbf{i}}} \overset{\text{def}}{=} y_{i,j}/g_i g_j$, *where* $g_i = f_i$ *if* $i \in \underline{\mathbf{i}}$, $g_i = 1$ *otherwise.*

We will prove that these actually generate the ideal in Proposition 3.24 but postpone the proof as it is long and unenlightening.

Proof. Adding 4 dots to α_i, or 6 dots to a 6-valent vertex, we get

(3.15)

This is $y_{i,i+1} = f_i f_{i+1}(f_i + f_{i+1}) = (x_i - x_{i+1})\ (x_{i+1} - x_{i+2})\ (x_i - x_{i+2})$. Even though we are not allowing 6-valent vertices in our diagrams, we will sometimes express $y_{i,i+1}$ as



(3.16)

to avoid having to consider sums of graphs (it's easier for me to draw!).

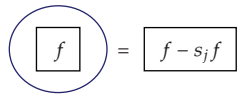
To obtain the other $y_{i,j}$, note the following equalities under the action of S_{n+1} on R:

$$\begin{gathered} s_i f_{i+1} = f_i + f_{i+1}, \\ s_{i+1} f_i = f_i + f_{i+1}, \\ s_i f_i = -f_i, \\ s_i f_j = f_j \quad \text{for } |i-j| > 1. \end{gathered} \tag{3.17}$$

From this it follows by explicit calculation that

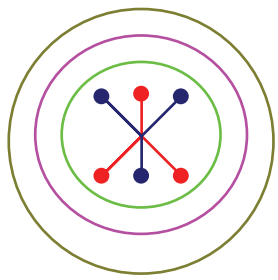
$$\begin{gathered} s_{i-1} y_{i,j} - y_{i,j} = y_{i-1,j}, \\ s_{j+1} y_{i,j} - y_{i,j} = y_{i,j+1}. \end{gathered} \tag{3.18}$$

Now, when we surround a polynomial f with a j-colored circle and use (3.12), we are left with a j-colored double dot times $\partial_j(f)$, so we get $f - s_j f = \partial_j(f) f_j$.



(3.19)

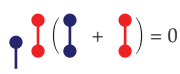
Combining this with the calculations we just made, we see that a $j+1$ circle around $y_{i,j}$ will yield $y_{i,j+1}$ up to sign, and so forth. We now have numerous ways to express $\pm y_{i,j}$: for any $i \le k \le j-1$, take α_k with 4 dots to get $y_{k,k+1}$, and then surround it with concentric circles whose colors, from inside to out, are $k+2, k+3, \ldots, j$ and then $k-1, k-2, \ldots, i$



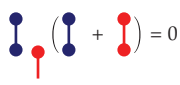
(3.20)

Clearly the colors of the increasing sequence and those of the decreasing sequence are distant, so a sequence like $k-1, k+2, k+3, k-2, \ldots$ is also okay, or any permutation which preserves the order of the increasing and the decreasing sequence individually.

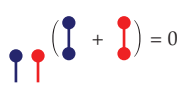
For very similar reasons, $z_{i,j,\underline{\mathbf{i}}}$ is in the TL ideal of $\underline{\mathbf{i}}$. Adding two or three dots to (3.5), we get several more equations.



(3.21)

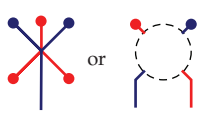


(3.22)



(3.23)

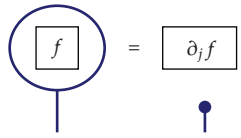
Again, for a variety of these pictures, we use shorthand like



(3.24)

These give you $z_{i,i+1,\underline{\mathbf{i}}}$ in the case where at least one of $i, i+1 \in \underline{\mathbf{i}}$. Again by (3.12), putting a polynomial f in the eye of a j-colored needle will yield $\partial_j(f) = (f - s_j f)/f_j$ next to

a j-colored boundary dot.

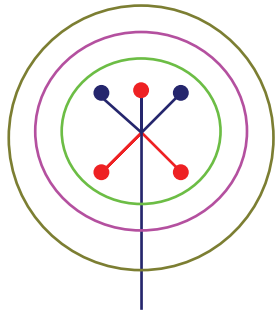


(3.25)

This gives us several ways to draw $z_{i,j,\underline{\mathbf{i}}}$.

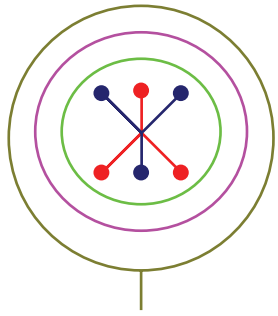
If neither i nor j are in $\underline{\mathbf{i}}$, then $z_{i,j,\underline{\mathbf{i}}} = y_{i,j}$ and is pictured as above, but with additional boundary dots being put below to account for $\varphi_{\underline{\mathbf{i}}}$. Since these extra dots are generally irrelevant, we often do not bother to draw them.

If $i \in \underline{\mathbf{i}}$ and $j \notin \underline{\mathbf{i}}$, we have two ways of drawing $z_{i,j,\underline{\mathbf{i}}}$. One can take α_i, connect one i input to the outer boundary, add dots, and surround it with circles colored $i+2, i+3, \ldots, j$



(3.26)

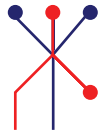
Alternatively, take some $i < k < j$, add dots to α_k, and surround it with circles forming an increasing sequence $k+2 \cdots j$ and a decreasing sequence $k-1 \cdots i$, *except* that the final i-colored circle is a needle



(3.27)

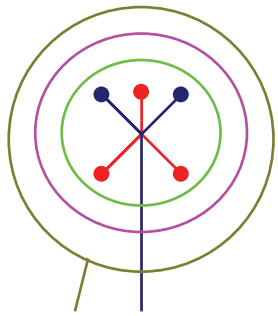
The case of $j \in \underline{\mathbf{i}}$ and $i \notin \underline{\mathbf{i}}$ is obvious.

If both $i, j \in \underline{\mathbf{i}}$, then we have several choices again. If $j = i+1$, then we must use



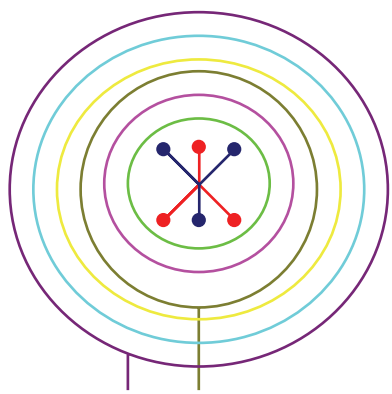
(3.28)

but in general, we may either repeat (3.26) with a j-needle instead of a j-circle



(3.29)

or repeat (3.27) with a j-needle instead of a j-circle



(3.30)

In any case, it is clear that the polynomials above are in the TL ideal, and the claim is proven. □

Let us quickly consider the redundancy in this generating set of the ideal. When $i > j$ let $y_{i,j} \overset{\text{def}}{=} y_{j,i}$ and $z_{i,j,\underline{\mathbf{i}}} \overset{\text{def}}{=} z_{j,i,\underline{\mathbf{i}}}$.

Corollary 3.14. *Suppose that $\underline{\mathbf{i}}$ is nonempty, and fix an index $k \in \underline{\mathbf{i}}$. Then, $I_{\underline{\mathbf{i}}}$ is generated by $z_{k,j,\underline{\mathbf{i}}}$ for $1 \le j \le n$, $j \ne k$. None of these generators are redundant.*

None of the generators $y_{i,j}$ of $I_\emptyset$ are redundant.

Proof. We leave the checks of irredundancy to the reader, but a proof will also arise as a byproduct in the next section (see Remark 3.17).

Suppose that $k \in \underline{\mathbf{i}}$ but $i, j \notin \underline{\mathbf{i}}$. If $k < i < j$, then $z_{i,j,\underline{\mathbf{i}}} = y_{i,j} = f_i z_{k,j,\underline{\mathbf{i}}} - f_j z_{k,i,\underline{\mathbf{i}}}$ so that $z_{i,j,\underline{\mathbf{i}}}$ is redundant. If $i < k < j$, then $z_{i,j,\underline{\mathbf{i}}} = f_i z_{k,j,\underline{\mathbf{i}}} + f_j z_{i,k,\underline{\mathbf{i}}}$. A similar statement holds for $i < j < k$. In the same vein, if $k, l \in \underline{\mathbf{i}}$ but $i \notin \underline{\mathbf{i}}$, then for a given $z_{k,l,\underline{\mathbf{i}}}$, only one of $z_{k,i,\underline{\mathbf{i}}}$ or $z_{l,i,\underline{\mathbf{i}}}$ is needed, and if $k, l, m \in \underline{\mathbf{i}}$, then any two of the three pairwise relations will imply the third. □

3.6. Graded Dimensions

In this section, fix a nonrepeating sequence $\underline{\mathbf{i}}$. We assume in this section that the generators of $I_{\underline{\mathbf{i}}}$ are precisely the polynomials described in Proposition 3.13.

Notation 5. An element of R can be written as a polynomial in f_i, so let $x = f_1^{a_1} \cdots f_n^{a_n}$ be a general monomial. Choose any $\underline{\mathbf{i}}$, possibly empty. Given a monomial x, let $J_x \subset \{1, \ldots, n\}$ be the subset containing $\underline{\mathbf{i}}$ and all indices j such that $a_j \neq 0$. For a fixed subset J, let R_J be the subset of all monomials x with $J_x = J$. This inherently depends on the choice of $\underline{\mathbf{i}}$.

Under the map $R \to \mathrm{HOM}(U_\emptyset, U_{\underline{\mathbf{i}}})$, the image of R_J will be graphs where the colors appearing are precisely J. Every color in $\underline{\mathbf{i}}$ appears as a boundary dot, and every f_j corresponds to a double dot of that color. The case $J = \emptyset$ only occurs when $\underline{\mathbf{i}} = \emptyset$ and $R_\emptyset = \{1\}$.

To find a basis for $R/I_{\underline{\mathbf{i}}}$, we will use the Bergman Diamond Lemma [35] for commutative rings.

Definition 3.15. Let A be a free commutative polynomial ring, where monomials are given a partial order with the DCC, compatible with multiplication in that $x < y \Rightarrow ax < ay$. Let I be an ideal generated by relations r of the form $x_r = y_r$, where x_r is a monomial and y_r is a linear combination of monomials which are each less than x_r in the partial order. A *reduction* is an application of a relation r to replace x_r with y_r, but not the other way around (a reduction always lowers the partial order on each term in a polynomial). One says a polynomial x *reduces* to y if y can be obtained from x by a series of reductions applied to monomials in x. A monomial is called *irreducible* if it does not have x_r as a factor for any relation r. An *inclusion ambiguity* is a monomial $x = ab$, where $x = x_r$ for some r, and $b = x_{r'}$ for some $r' \neq r$. An *overlap ambiguity* is a monomial $x = abc$, where $ab = x_r$ for some r and $bc = x_{r'}$ for some $r' \neq r$. Each ambiguity has two natural reductions, and one says the ambiguity is *resolvable* if the two reductions are then jointly reducible to the same element.

Lemma 3.16 (Bergman Diamond Lemma for Commutative Rings, [35]). *With these definitions in place, if every inclusion and overlap ambiguity is resolvable, then the images of the irreducible monomials form a basis for A/I.*

This process may become more transparent from the example below; in addition, Bergman's paper has a number of nice examples for the trickier, noncommutative version. We treat two separate cases, when $\underline{\mathbf{i}} = \emptyset$ and when $\underline{\mathbf{i}} \neq \emptyset$.

Claim 2. Let $\underline{\mathbf{i}} = \emptyset$. We place the lexicographic order on monomials in R, so that $f_1 < f_2 < \cdots$. The relation $y_{i,j} = 0$ for $i < j$ will be rewritten as $f_i f_j^2 = -f_i f_j (f_i + 2\sum_{i<k<j} f_k)$, which replaces $f_i f_j^2$ with a sum of monomials all lower in the order. For each $J \neq \emptyset$, the irreducible monomials in R_J are precisely $f_k^m \prod_{i \in J} f_i$, where k is the minimal index in J and $m \geq 0$ (note: the exponent of f_k is $m + 1 \geq 1$). When $J = \emptyset$, 1 is irreducible. Irreducibles form a basis for $R/I_\emptyset$.

Proof. A monomial is irreducible if $f_i f_j^2$ never appears as a factor for any $i < j$. Because of this, the classification of irreducible monomials in each R_J is obvious. There are no inclusion ambiguities between relations, since they are all homogeneous and degree 3. There are two kinds of overlap ambiguities, both labelled by a choice of $i < l < j$.

For the first ambiguity, one can reduce $x = f_i f_l f_j^2$ by either reducing $f_l f_j^2$ or $f_i f_j^2$. Applying the former reduction, $x \mapsto f_i f_l f_j (-f_l - 2\sum_{l<k<j} f_k)$ which has a term given by

$-f_i f_l^2 f_j$ that can be further reduced, yielding $f_i f_l f_j(f_i + 2\sum_{i<k<l} f_k - 2\sum_{l<k<j} f_k)$. Applying the latter reduction, $x \mapsto f_i f_l f_j(-f_i - 2\sum_{i<k<j} f_k) = f_i f_l f_j(-f_i - 2\sum_{i<k<l} f_k - 2f_l - 2\sum_{l<k<j} f_k)$, which has a term given by $-2f_i f_l^2 f_j$ that can be further reduced, yielding $f_i f_l f_j(-f_i - 2\sum_{i<k<l} f_k - 2\sum_{l<k<j} f_k + 2f_i + 4\sum_{i<k<l} f_k) = f_i f_l f_j(f_i + 2\sum_{i<k<l} f_k - 2\sum_{l<k<j} f_k)$. Since these agree, the ambiguity is resolvable.

For the second ambiguity, one can reduce $x = f_i f_l^2 f_j^2$ by either reducing $f_i f_l^2$ or $f_l f_j^2$. A very similar calculation shows that this ambiguity is resolvable as well. Therefore, the Bergman diamond lemma implies that irreducibles form a basis for the quotient. □

Remark 3.17. This also proves that none of the $y_{i,j}$ is redundant. Removing $y_{i,j}$ from the ideal, we may apply the same Bergman diamond lemma argument to say that irreducibles form a basis for the quotient. However, with no $y_{i,j}$, the monomial $f_i f_j^2$ is irreducible, and the quotient is larger than before. A similar statement can be made about the $z_{k,j,\underline{\mathbf{i}}}$ below.

When $J \neq \emptyset$, the graded rank of the irreducibles in R_J is $t^{2|J|}/(1-t^2)$. When J is empty, the only element of R_J is 1. So the graded rank of $R/I_\emptyset$ is $1 + \sum_{J\neq\emptyset}(t^{2|J|}/(1-t^2))$, but $\sum_J t^{2|J|} = (1+t^2)^n$ since every f_i may either appear or not appear, independently of every other. Hence $\sum_{J\neq\emptyset} t^{2|J|} = (1+t^2)^n - 1$. Putting it all together, the graded rank is $((1+t^2)^n - t^2)/(1-t^2) = (t^n[2]^n - t^2)/(1-t^2)$. Hence we have proven the following claim.

Claim 3. The graded dimension of $R/I_\emptyset$ is exactly $\varepsilon_{\mathrm{cat}}(u_\emptyset)$.

Claim 4. Let $\underline{\mathbf{i}} \neq \emptyset$, and fix $k \in \underline{\mathbf{i}}$. We choose a different order on indices, where $k < k+1 < k-1 < k+2 < k-2 < \cdots$, and then place the lexicographic order on monomials. The relation $z_{k,j,\underline{\mathbf{i}}}$ for $j \neq k$ will be rewritten in decreasing order format as either $f_j^2 = -f_j(f_k + 2\sum_l f_l)$ for $j \notin \underline{\mathbf{i}}$, or $f_j = -(f_k + 2\sum_l f_l)$ for $j \in \underline{\mathbf{i}}$, where the sum is over l between k and j. Then, the irreducible monomials in R_J are precisely $f_k^m \prod_{j\in J\setminus\underline{\mathbf{i}}} f_j$ for $m \geq 0$. Irreducibles form a basis for $R/I_{\underline{\mathbf{i}}}$.

Proof. An irreducible polynomial will be a polynomial which does not have f_j^2 as a factor, for $k \neq j \notin \underline{\mathbf{i}}$, and does not have f_j as a factor for $k \neq j \in \underline{\mathbf{i}}$. The classification of irreducibles in R_J is now obvious. There are no ambiguities whatsoever, so we are done by the Bergman diamond lemma. □

The graded rank of irreducibles in R_J is $t^{2|J|-2d}/(1-t^2)$, for d the length of $\underline{\mathbf{i}}$ (remember that $\underline{\mathbf{i}} \subset J$). Thus, the graded rank of $R/I_{\underline{\mathbf{i}}}$ is $\sum_{\underline{\mathbf{i}}\subset J}(t^{2|J|-2d}/(1-t^2)) = (1+t^2)^{n-d}/(1-t^2)$, and the graded rank of $R/I_{\underline{\mathbf{i}}}\{d\}$ is $t^d(1+t^2)^{n-d}/(1-t^2) = t^n[2]^{n-d}/(1-t^2)$. Hence, one considers the following.

Claim 5. The graded dimension of $R/I_{\underline{\mathbf{i}}}\{d(\underline{\mathbf{i}})\}$ is exactly $\varepsilon_{\mathrm{cat}}(u_{\underline{\mathbf{i}}})$.

This is clearly sufficient to prove Lemma 1.1, modulo Proposition 3.24.

3.7. Weyl Lines and Disoriented Tubes

We now give two alternate interpretations of the TL ideals $I_{\underline{\mathbf{i}}}$. We continue to assume that $\underline{\mathbf{i}}$ is nonrepeating and $z_{i,j,\underline{\mathbf{i}}}$ generates $I_{\underline{\mathbf{i}}}$.

Definition 3.18. Let V be the reflection representation of S_{n+1}, such that $R = \mathbb{C}[f_1, \ldots, f_n]$ is the coordinate ring of V. Note that the linear equations which cut out reflection-fixed

hyperplanes are precisely $w_{i,j} = f_i + f_{i+1} + \cdots + f_j = x_i - x_{j+1}$ for $i \leq j$. A *Weyl line* is a line in V through the origin which is defined by the intersection of reflection-fixed hyperplanes; it is given by a choice of $n-1$ transversely intersecting reflection-fixed hyperplanes. Given a nonrepeating sequence $\underline{\mathbf{i}}$, one says that a Weyl line is *transverse to* $\underline{\mathbf{i}}$ if it is transverse to (i.e., not contained in) the hyperplanes $f_k = 0$ for each $k \in \underline{\mathbf{i}}$.

Proposition 3.19. *The TL ideal of* $\underline{\mathbf{i}}$ *is the ideal associated with the union of all Weyl lines transverse to* $\underline{\mathbf{i}}$ *(with its reduced scheme structure).*

Example 3.20. Let $n = 3$. One can check that $f_1f_2(f_1+f_2) = f_2f_3(f_2+f_3) = f_1f_3(f_1+2f_2+f_3) = 0$ cuts out 7 lines in V, namely,

(1) $f_1 = f_2 = f_1 + f_2 = 0$,

(2) $f_1 = f_3 = 0$,

(3) $f_2 = f_3 = f_2 + f_3 = 0$,

(4) $f_1 = f_2 + f_3 = f_1 + f_2 + f_3 = 0$,

(5) $f_1 + f_2 = f_3 = f_1 + f_2 + f_3 = 0$,

(6) $f_2 = f_1 + f_2 + f_3 = 0$,

(7) $f_1 + f_2 = f_2 + f_3 = 0$,

These 7 lines are precisely the 7 lines cut out by the intersection of pairs of reflection-fixed hyperplanes. There are 6 reflection-fixed hyperplanes, given by equations f_1, f_2, f_3, $f_1 + f_2$, $f_2 + f_3$, and $f_1 + f_2 + f_3$, or alternatively, by $x_i - x_j$ for $4 \geq j > i \geq 1$. Intersecting pairs of hyperplanes will give a line, and occasionally this line is forced to lie in a third hyperplane, as in the list above. One can check that this list covers all pairs of hyperplanes which give distinct lines as their intersection.

Proof. This is not difficult to show, but since we have not seen it elsewhere, we provide a complete proof. First, we show by induction on n that the ideal $I_\emptyset$ cuts out the Weyl lines with the reduced scheme structure. The case $n = 1$ is trivial (and $n = 2$ is also obvious).

For any $1 \leq k \leq n$, consider the hyperplane $f_k = 0$ as an $n-1$-dimensional space V', with an action of $S_{n+1}/\langle s_k \rangle \cong S_n$. Giving S_n a Coxeter structure with simple reflections s_i for $i \neq k$ (note that $s_{k+1} = (k+1, k+2) = (k, k+2)$ in the quotient), it is quite easy to see that V' is the reflection representation of S_n. Moreover, the Weyl hyperplanes are cut out by $w'_{i,j} = f_i + f_{i+1} + \cdots + f_j$ (where $f_k = 0$ so it may be left out of the sum) for $i, j \neq k$, and the equivalent polynomials $y'_{i,j}$ also have the same formulae, and are indexed by $i, j \neq k$. Therefore, for $i, j \neq k$, the images of $w_{i,j}$ are just $w'_{i,j}$, and the same for $y_{i,j}$ and $y'_{i,j}$. Moreover, if either i or j equals k, then $y_{i,j} = 0$ on $f_k = 0$, and $w_{i,j}$ is redundant on $f_k = 0$, being equal to some $w_{i',j'}$. By induction, $y'_{i,j}$ cut out the Weyl lines with the reduced scheme structure on V', and therefore the vanishing set of $y_{i,j}$ agrees with the Weyl lines on $f_k = 0$.

If all $f_k \neq 0$, then it is easy to see that the $y_{i,j}$ cut out a single line with the reduced scheme structure, namely, $-f_1 = f_2 = -f_3 = \cdots = (-1)^n f_n$. This is a Weyl line, the intersection of all $w_{i,i+1}$. We wish to show that this is the only Weyl line transverse to all $f_k = 0$. We can show this by induction as well (again, the base case $n = 2$ is easy). Suppose we are given $n-1$ transverse hyperplanes $w_{i,j}$. If any two involve the index n, that is, $w_{i,n}$ and $w_{j,n}$, then we may replace the pair with $w_{i,n}$ and $w_{i,j}$ since they have the same intersection (and $w_{i,j}$ is not already in the set, or the intersection would not be transverse). So, we may assume that at most one of

the chosen hyperplanes involves the index n, but then we have $n-2$ transverse hyperplanes which only involve indices $\{1,\dots,n-1\}$, which must then be mutually transverse to $f_n = 0$. Letting V' be the hyperplane $f_n = 0$ viewed as a reflection representation as above, we have $n-2$ transverse hyperplanes which cut out a Weyl line transverse to $f_k = 0$ for all $1 \le k \le n-1$. By induction, that Weyl line is $-f_1 = f_2 = -f_3 = \cdots = (-1)^{n-1} f_{n-1}$ (which holds true modulo $f_n = 0$), but repeating the same argument for the index k instead, we leave out the kth term and get $-f_1 = f_2 = \cdots = \widehat{(-1)^k f_k} = \cdots = (-1)^n f_n$ modulo $f_k = 0$. Together, all these equalities imply that $-f_1 = f_2 = \cdots = (-1)^n f_n$ everywhere.

One might be worried, because of the restrictions used in the induction step, that $I_{\underline{\mathbf{i}}}$ does not give the reduced structure on the Weyl lines at the origin. However, $I_{\underline{\mathbf{i}}}$ is a homogeneous ideal which cuts out a reduced 0-dimensional subscheme of $\mathbb{P}(V)$, so that its vanishing on V is the cone of a reduced scheme, and hence is reduced. This concludes the proof that $I_\emptyset$ cuts out the Weyl lines with the reduced scheme structure.

For $\underline{\mathbf{i}} \neq \emptyset$, $I_\emptyset \subset I_{\underline{\mathbf{i}}}$ and the vanishing of $I_{\underline{\mathbf{i}}}$ is contained in that of $I_\emptyset$. Choose $k \in \underline{\mathbf{i}}$. If $f_k = 0$ then $z_{k,k+1,\underline{\mathbf{i}}}$ is equal to f_{k+1}^a, where $a = 1,2$ depending on whether $k+1 \in \underline{\mathbf{i}}$, but either way we get that $f_{k+1} = 0$. Then $z_{k,k+2} = f_{k+2}^a$ for $a = 1,2$, and so forth. Therefore $f_k = 0$ only intersects the vanishing of $I_{\underline{\mathbf{i}}}$ at the origin (as sets). It is clear that, on the open set where $f_k \neq 0$ for all $k \in \underline{\mathbf{i}}$, the polynomials $z_{i,j,\underline{\mathbf{i}}}$ and $y_{i,j}$ have the same vanishing (as schemes), since they differ by a unit. The same cone argument shows that $I_{\underline{\mathbf{i}}}$ gives the reduced structure at the origin. □

Remark 3.21. In particular, $I_\emptyset$ is contained in every ideal, and the category $\mathcal{TLC}_1$ is manifestly $R/I_\emptyset$-linear.

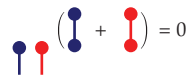
Remark 3.22. Let Z be the union of all Weyl lines in V. The previous results should lead one to guess that the Temperley-Lieb algebra should be connected to the geometry of the S_{n+1} action on Z via $\mathcal{TLC}$, in much the same way that the Hecke algebra is connected to the reflection representation via $\mathcal{HC}$ (see [11]). However, at the moment, we have no way to formulate the category $\mathcal{TLC}$ in terms of coherent sheaves on $Z \times Z$ (i.e., $R/I_\emptyset$-bimodules) or the derived category thereof. Describing $\mathcal{TLC}$ using sheaves on Z seems like an interesting question.

As an example of the difficulties, let U_i be the bimodule $R/I_i \otimes R/I_i\{-1\}$, where the tensor is over R^{s_i}; this should be the equivalent of the Soergel bimodule B_i. Then, there is a degree-1 map $R/I_\emptyset \to U_i$ sending 1 to $x_i \otimes 1 - 1 \otimes x_{i+1}$ (the boundary dot on the top), but there is no degree-1 map $U_i \to R/I_\emptyset$ (the boundary dot on the bottom); such a map should send $1 \otimes 1$ to 1. There is only a degree-3 map, sending $1 \otimes 1$ to f_i (the boundary dot with a double dot). A similar problem occurs again: the trivalent vertex seems to be defined only in one direction.

Now, we describe briefly the topological intuition associated with the category $\mathcal{TLC}$, and another way to view $I_{\underline{\mathbf{i}}}$. These remarks will not be used in the remainder of the paper, nor will we give a proof. The reader should be acquainted with the section on $\mathfrak{sl}_2$-foams in Vaz's paper [16].

Remark 3.23. Let $\mathcal{F}$ be the functor from $\mathcal{TLC}_1$ to the category of disoriented cobordisms Foam_2, as defined in Vaz's paper. If f_i is the double dot colored i, then one can easily see that $\mathcal{F}$ sends f_i to a tube connecting the ith sheet to the $(i+1)$th sheet, with a disorientation on it. If the double dot appears in a larger morphism φ, such that in $\mathcal{F}(\varphi)$, the ith sheet and the $(i+1)$th sheet are already connected by a saddle or tube, then adding another tube between them does nothing more than adding a disoriented handle to the existing surface. Note that the map $\varphi_{\underline{\mathbf{i}}}$ previously defined will connect the ith sheet to the $(i+1)$th sheet for any $i \in \underline{\mathbf{i}}$.

Suppose that the ith, $(i+1)$th, and $(i+2)$th sheets are all connected in a cobordism. Then f_i adds a handle on the left side of the $(i+1)$th sheet, f_{i+1} adds a handle on the right side, and these two disoriented surfaces are equal up to a minus sign in Foam_2. This fact is essentially the statement that:



(3.31)

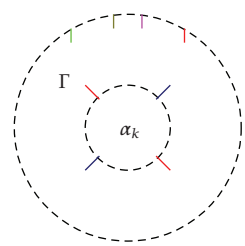
In other words, the algebra $\Bbbk[f_1,\ldots,f_n]$ maps to Foam_2, sending f_i to the disoriented tube between the ith and $(i+1)$th sheet. The ideal $I_{\underline{\mathbf{i}}}$ is clearly in the kernel of this action when applied to the cobordism $\mathcal{F}(\varphi_{\underline{\mathbf{i}}})$. In fact, it is precisely the kernel, using the argument of Proposition 4.2 in [36]: for any distinct monomials in a basis for $R/I_{\underline{\mathbf{i}}}$, their image in Foam_2 will have independent evaluations with respect to some closure of the cobordism. We do not do the calculation here. The usual arguments involving adjoint pairings imply that the faithfulness of the functor $\mathcal{F}$ can be checked on $\mathrm{Hom}(\emptyset,\underline{\mathbf{i}})$. Therefore, the functor $\mathcal{F}$ is faithful.

3.8. Proof of Generation

Proposition 3.24. *The TL ideal $I_\emptyset$ is generated by $y_{i,j} \overset{\text{def}}{=} f_i f_j (f_i + 2f_{i+1} + 2f_{i+2} + \cdots + 2f_{j-1} + f_j)$ over all $1 \le i < j \le n$.*

The TL ideal $I_{\underline{\mathbf{i}}}$ is generated by all $z_{i,j,\underline{\mathbf{i}}} \overset{\text{def}}{=} y_{i,j}/g_i g_j$, where $g_i = f_i$ if $i \in \underline{\mathbf{i}}$, $g_i = 1$ otherwise.

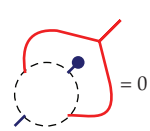
We wish to determine the ideal generated by α_k inside $\mathrm{HOM}(\emptyset,\underline{\mathbf{i}})$, for nonrepeating $\underline{\mathbf{i}}$. As discussed in Remark 3.12 (where α_k is defined), our goal is to take any graph Γ on the punctured plane, with $\underline{\mathbf{i}}$ as its outer boundary and $k(k+1)\,k(k+1)$ as its inner boundary, plug α_k into the puncture, and reduce it to something in the ideal generated by the pictures of Section 3.5.



(3.32)

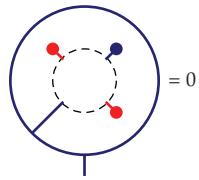
Our coloring conventions for this chapter will be that blue always represents the index k, red represents $k+1$, and other colors tend to be arbitrary (often, the number of other colors appearing is also arbitrary). However, it will often happen that colors will appear in increasing or decreasing sequences, and these will be annotated as such. Note that blue or red may appear in the outer boundary as well, but at most once each.

Let us study Γ, and not bother to plug in α_k. The only properties of α_k which we need are the following:



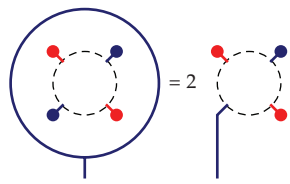
(3.33)

This follows from (3.1), or just from isotopy. The same holds with colors being switched.



(3.34)

This is because the diagram reduces to a k-colored needle, with $f = f_{k+1}(f_k + f_{k+1})$ inside, but f is fixed by s_k, so it slides out of the needle, and the empty needle is equal to 0. A similar equality holds with colors switched.



(3.35)

This follows from the above and the dot forcing rules.

The final property we use is that any graph only using colors $< k - 1$ or $> k + 2$ can slide freely across the puncture.

Note however that, say, an arbitrary $k - 3$ edge cannot automatically slide across the puncture, because a $k - 2$ edge might be in the way, and this could be in turn obstructed by a $k - 1$ edge, which cannot slide across the blue at all.

The one-color reduction results apply to any simply connected planar region, so we may assume (without even using the relation (3.5)) that in a simply connected region of our choice, the i-graph for each i is a simple forest with double dots. Any connected component of an i-graph that does not encircle the puncture will be contained in a simply connected region, and hence can be simplified; this will be the crux of the proof. The proof is simple, but has many cases.

Remark 3.25. We will still need to use relation (3.5) as we simplify graphs.

We will treat cases based on the "connectivity" of Γ that is, how many of the blue and red boundary lines in the inner and outer boundary are connected with each other. We will rarely perform an operation which makes the graph more connected. At each stage, we will reduce the graph to something known to be in the ideal or break edges to decrease the connectivity. We call an edge coming from the puncture an *interior line* and one coming from the outer boundary an *exterior line*.

Note also that any double dots that we can move to the exterior of the diagram become irrelevant, since the picture with those double dots is in the ideal generated by the picture without double dots. Also, any exterior boundary dots are irrelevant, since they are merely part of the map $\varphi_{\underline{i}}$ and do not interfere with the rest of the diagram at all.

Step 1. Suppose that the two interior red lines are in the same component of Γ_{k+1}. Then, there is some innermost red path from one to the other, such that the interior of this path (the region

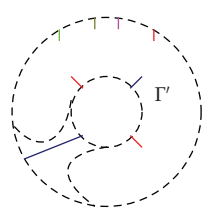
towards the puncture) is simply connected. Applying reductions, we may assume that the k-graph in this region consists of a blue boundary dot with double dots, and the $k+1$-graph and $k+2$-graph each consists only of double dots. We may assume all double dots occur right next to one of the red lines coming from the puncture. The current picture is exactly like that in (3.33), except that there may be double dots inside and other colors may be present (also, there could be more red spokes emanating from the red arc, but these can be ignored or eliminated using (2.16) and tree reduction). However, the double dots may be forced out of the red enclosure at the cost of potentially breaking the red edge, and breaking it will cause the two red interior lines to be no longer in the same connected component. If there are no double dots, then all the remaining colors (which are $< k-2$ or $> k+1$) may be slid across the red line and out of the picture. Hence, we are left with the exact picture of (3.33), which is zero.

Thus, we may assume that the two red lines coming from the puncture are not in the same component. The same holds for the blue lines.

Step 2. Suppose that the component of one of the interior blue lines wraps the puncture, creating an internal region (which contains the puncture). Again, reducing in that internal region, the other interior blue line cannot connect to the boundary so it must reduce to a boundary dot (with double dots), the reds may not connect to each other so each reduces to a boundary dot, and as before we are left in the picture of (3.34) except possibly with double dots and other colors. If there are no double dots, all other colors may be slid out, and the picture is zero by (3.34). Again, we can put the double dots near the exterior, and forcing them out will break the blue arc. It is still possible that some other cycle still allows that component to wrap the puncture; however, this process needs to only be iterated a finite number of times, and finitely many arcs broken, until that component no longer wraps the puncture.

So, we may assume that the component of any interior line, red or blue, does not wrap the puncture. That component is contained in a simply connected region, so it reduces to a simple tree. Hence, we may assume that the components of interior lines either end immediately in boundary dots, or connect directly to an external line of the same color (at most one as such exists of each color).

Step 3. Suppose that there is a blue edge connecting an internal line directly to an external one. Consider the region Γ':



(3.36)

Then Γ' is simply connected. Other colors in Γ may leave Γ' to cross through the blue line; however, the colors $k-1$, $k, k+1$ may not. Therefore, reducing within Γ', we may end the internal blue line in a boundary dot and eliminate all other instances of the color blue (since they become irrelevant double dots on the exterior), reduce red to a simple forest where the two interior lines are not connected (again, ignoring irrelevant double dots), and reduce $k-1$ to either the empty diagram or an external boundary dot (depending on whether $k+1 \in \underline{\mathbf{i}}$). Once this has been accomplished, the absence of the color $k-1$ implies that we

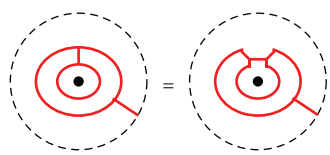
may slide $k-2$ freely across the puncture! The color $k-2$ can be dealt with in the entire disk, which is simply connected, so it reduces to the empty diagram or an external boundary dot (depending on whether $k+2 \in \underline{\mathbf{i}}$), with extraneous double dots. Then we may deal with color $k-3$, and so forth.

Thus, the existence of the blue edge implies that all colors $< k$ can be ignored: they appear in irrelevant double dots, in irrelevant boundary dots, or not at all. Similarly, the existence of a red edge allows us to ignore all colors $> k+1$.

Step 4. Let us consider only the components of graphs which do not meet the internal boundary.

Lemma 3.26. *Consider a component of a graph on a punctured disk which does not meet the internal boundary and which meets the external boundary at most once. Then it can be reduced to one of the following, with double dots on the exterior: the empty graph, a boundary dot, a circle around the puncture, a needle coming from the external boundary, with its eye around the puncture.*

Proof. Suppose that the component splits the punctured plane into m regions. If the component is contained in a simply connected part of the punctured plane, we are done. This is always true for $m = 1$. So we may suppose that $m \geq 2$ and, we have two distinguished regions: the external region, and the region containing the puncture. Any other region is one of two kinds, as illustrated in the following equality (due to (2.16):



(3.37)

On the right side we have a region which is contained in a simply connected part and thus can be eliminated by reduction (see Proposition 3.9). On the left side the region is not contained in a simply connected part nor does it contain the puncture. However, any such region can be altered, using (2.16) as in the heuristic example above, into a cycle of the first kind. Therefore, we may assume there are exactly 2 regions.

In the event that there are two regions, we have a cycle which surrounds around the puncture and may have numerous branches into both regions, internal and external. However, each branch must be a tree lest another region be created. These trees reduce in the usual fashion, and therefore the internal branches disappear, and the external branches either disappear or connect directly to the single exterior boundary. Thus, we have either a needle or a circle. Double dots, as usual, can be forced out of the way possibly at the cost of breaking the cycle, and reducing to the case $m = 1$. □

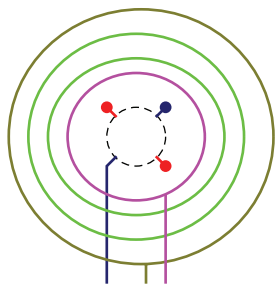
Let us now examine the remaining cases. We will ignore all parts of a graph which are double dots on the exterior, or are external boundary dots.

Case 1. Both a blue edge and a red edge connect an internal line to an external line. Then, as in Step 3, all other colors can be ignored, and the entire graph is

(3.38)

This, as explained in Section 3.5, is $z_{k,k+1,\underline{\mathbf{i}}}$.

Case 2. A blue edge connects an internal line to an external line, and both red internal lines end in boundary dots. As discussed in Step 3, we may ignore all colors $< k$, and both colors k and $k+1$ do not appear in a relevant fashion outside of what is already described. We may ignore the presence of any double dots. However, there may be numerous circles and needles colored $\geq k+2$ which surround the puncture and cross through the blue line, in an arbitrary order.

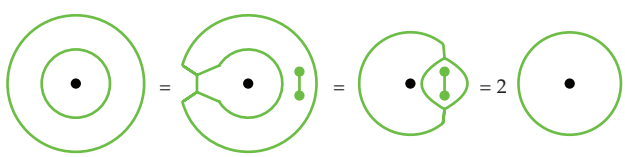


(3.39)

Claim 6. The sequence of circles and needles can be assumed to form an increasing sequence of colors, from $k+2, k+3, \ldots$ until the final color, and assuming that only the final color may be a needle.

Proof. If the innermost circle/needle is not colored $k+2$, then it may slide through the puncture, and will evaluate to zero by (2.18). So, suppose the innermost is $k+2$. If it is a needle, not a circle, then there can be no more $k+2$-colored circles, and no $k+3$-colored circles. Color $k+4$ can be pulled through the middle so resolved on the entire disk and hence can be ignored, and so too with $k+5$ and higher. This is the "needle" analogy to the conclusion of Step 3: the existence of an m-colored needle around the puncture and the lack of m or $m+1$ on the interior of the needle will allow us to ignore all colors $\geq m+1$.

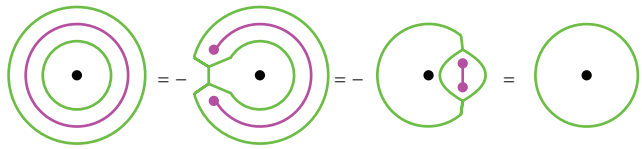
So, suppose it is a $k+2$-colored circle. If the next circle/needle is colored $\geq k+4$ then it slides through the $k+2$ circle and the puncture and evaluates to zero. If the next circle/needle is *also* colored $k+2$, then we may use the following calculation to ignore it. The calculation begins by using (2.33)



(3.40)

Thus, we may assume that the next circle/needle is colored $k+3$. Again, if it is a needle, then we can ignore all other colors, and our picture is complete.

Similarly, the next circle/needle can not be colored $\geq k+5$ lest it slide through, and it can not be colored $k+3$ lest we use (3.40). If it is colored $k+2$, then we may use the following calculation to ignore it. The calculation begins by using (3.5), and assumes green and purple are adjacent



(3.41)

Thus, we can assume the next circle/needle is colored $k+4$. If it is a needle, then all colors $k+5$ and higher can be ignored. Additional circles of color $k+2$ could run through the needle, but these could be slid inwards and reduced as before. So, if it is a needle, our picture is complete.

Finally, the next circle/needle can not be colored $\geq k+6$ lest it slide, $k+4$ lest we use (3.40), $k+3$ lest we use (3.41), or $k+2$ lest we slide it inside and reduce it as above. Hence it is colored $k+5$, and if it is a needle, we are done. This argument can now be repeated ad infinitum. □

Thus, our final picture yields $z_{k,j,\underline{i}}$ as in (3.26) or (3.29).

Note that the case of a red edge works the same way, with a decreasing sequence instead of an increasing sequence.

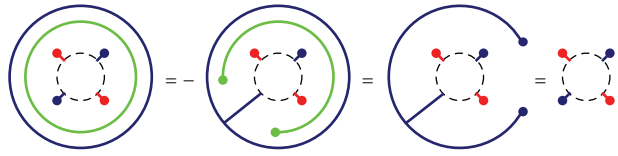
Case 3. All the internal lines end in boundary dots. We may assume that the remainder of the graph consists of circles/needles around this diagram, but we have no restrictions at the moment on which colors may appear.

Claim 7. We may assume that the colors in circles/needles form an increasing sequence from $k+2$ up, and a decreasing sequence from $k-1$ down (these sequences do not interact, so without loss of generality. we may assume the increasing sequence comes first, then the decreasing one). Only the highest and lowest color may be a needle.

Proof. The method of proof will be the same as the arguments of the previous case.

Consider the innermost circle/needle. If it is colored k or $k+1$, then we may use (3.35) to reduce the situation to a previous case. If it is colored $\geq k+3$ or $\leq k-2$ then it slides through the puncture. So we may assume it is $k+2$ or $k-1$. If it is a $k+2$-colored (resp., $k-1$-colored) needle, then the usual arguments imply that all colors $> k+2$ (resp., $< k-1$) can be ignored. This same argument with needles will always work, so we will not discuss the circle/needle question again, and speak as though everything is a circle.

Assume that the first colors appearing are an increasing sequence from $k+2$ to i and then a decreasing sequence from $k-1$ to j. Note that either sequence may be empty. If the next color appearing is $\leq j-2$ then it slides through the whole diagram and the puncture, and evaluates to zero. If the decreasing sequence is nonempty and the next color is j then we use (3.40); if it is $\geq j+1$ and $\leq k-1$ then we slide it as far in as it will go and use (3.41). If the decreasing sequence is nonempty and the next color is k then one can push it almost to the center, and use the following variant of (3.41):



(3.42)

In this picture, green is $k-1$ and is the only thing in the way of the blue circle. The first equality uses (3.5), and the second equality uses (2.29), and eliminates the terms which vanish due to (3.34).

Continuously, if the decreasing sequence is empty and the next color is k, then we may use (3.35) as above. Any colors which are $\geq k+1$ do not depend on the increasing sequence, and instead use the exact analogs for the increasing sequence.

Hence, in any case in which the next color appearing is not $i+1$, or $j-1$, or the beginning of a new increasing/decreasing series, we may simplify the diagram to ignore the new circle. Induction will now finish the proof. □

Therefore, the resulting diagram is equal to $z_{i,j,\underline{\mathrm{i}}}$, matching up either with (3.27) or (3.30).

Since every possible graph can be reduced to a form which is demonstrably in the ideal generated by $z_{i,j,\underline{\mathrm{i}}}$, we have proven that these elements do in fact generate the TL ideal $I_{\underline{\mathrm{i}}}$.

4. Irreducible Representations

In this section, we may vary the number of strands appearing in the Temperley-Lieb algebra. When $\mathcal{TL}$ appears, it designates the Temperley-Lieb algebra on $n+1$ strands, but $\mathcal{TL}_k$ designates the algebra on k strands.

4.1. Cell Modules

The Temperley-Lieb algebra has the structure of a cellular algebra, a concept first defined by Graham and Lehrer [37]. One feature of cellular algebras is that they are equipped with certain modules known as *cell modules*. Cell modules provide a complete set of nonisomorphic irreducible modules in many cases (such as $\mathcal{TL}$ in type A). Cell modules come equipped with a basis and a bilinear form, making them obvious candidates for categorification. We will not go into detail on cellular algebras here, or even use their general properties; instead we will describe the cell modules explicitly and pictorially for the case of $\mathcal{TL}$, where things are unusually simple. Nothing in this section or the next is particularly original, and we state some standard results without proof.

Notation 6. Consider a crossingless matching in the planar strip between n points on the bottom boundary and m points on the top. We call this briefly an (n,m) *diagram*. In the terminology of [30], there are two kinds of arcs in a diagram: *horizontal arcs* which connect two points on the top (let us call it a *top arc*), or two points on the bottom (*bottom arc*) and *vertical arcs* which connect a point on the top to one on the bottom. Elsewhere in the literature, vertical arcs are called through-strands. An (n,k) diagram with exactly k *through-strands* (and therefore no top arcs) has an isotopy representative with only "caps" (local maxima) and no "cups" (local minima) so it is called an (n,k) *cap diagram*. A (k,n) diagram with k through-strands is called a (k,n) *cup diagram*.

The set of all (n,m) diagrams can be partitioned by the number of through-strands. Any (n,m) diagram with k through-strands can be expressed as the concatenation of an (n,k) cap diagram with a (k,m) cup diagram in a unique way. For an illustration of this concept, see Figure 5.

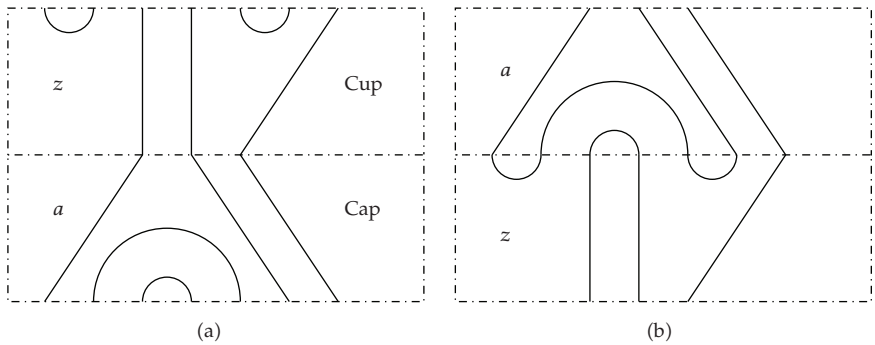


Figure 5: On the left side, a $(7,7)$ diagram with $k = 3$ through-strands and $l = 2$ top arcs (resp., bottom arcs) is decomposed into a $(7,3)$ cap diagram a-composed with a $(3,7)$ cup diagram z. On the right side, an element of $\mathcal{TL}_3$ is obtained by composing a and z in the opposite order.

In an (m,m) diagram the number l of top arcs equals the number of bottom arcs, and if k is the number of through-strands, then $k+2l = m$. We will typically use k and l to represent the number of through-strands and top arcs in an (m,m) diagram henceforth.

Notation 7. Let X be the set of all $(n+1,n+1)$ diagrams. Let ω be the endomorphism of X sending each diagram to its vertical flip. We will write the operation on diagrams of reduced vertical concatenation by $\circ$: $a \circ b$ places a above b and remove any circles. Let X_k be the set of crossingless matchings with exactly k through-strands. Let M_k be the set of all $(n+1,k)$ cap diagrams, so that $\omega(M_k)$ is the set of all $(k,n+1)$ cup diagrams.

Definition 4.1. Let L_k be the free $\mathbb{Z}[t,t^{-1}]$ module spanned by M_k, the $(n+1,k)$ cap diagrams. We place a right $\mathcal{TL}$-module structure on L_k by concatenation, where circles become factors of [2] as usual, and any resulting diagram with fewer than k through-strands is sent to 0. This is the *cell module* for cell k, and it is irreducible.

Example 4.2. The only diagram in X_{n+1} corresponds to the identity map in $\mathcal{TL}$. The cell module L_{n+1} has rank 1 over $\mathbb{Z}[t,t^{-1}]$, and its generator is killed by all u_i. We will take this as the definition of the *sign representation* of $\mathcal{TL}$.

Example 4.3. The next cell module L_{n-1} has rank n over $\mathbb{Z}[t,t^{-1}]$, having generators v_i, $i = 1\cdots n$ (see Figure 6), such that

$$v_j u_i = \begin{cases} [2]v_j & \text{if } i = j, \\ v_i & \text{if } i \text{ and } j \text{ are adjacent}, \\ 0 & \text{if } i \text{ and } j \text{ are distant}. \end{cases} \tag{4.1}$$

Given a $(n+1,k)$ cap diagram a and a $(k,n+1)$ cup diagram z, there are two things we can do: take the composition $z \circ a$ to obtain an element called $c_{z,a}$ of X_k or take the composition $a \circ z$ to get an element of $\mathcal{TL}_k$ (there may be additional circles created, and the final diagram may have fewer than k through-strands). Both compositions have the same closure on the

punctured plane. Note that $\omega(c_{z,a}) = c_{\omega(a),\omega(z)}$. The seemingly extraneous use of the notation $c_{\cdot,\cdot}$ is standard for cellular algebras.

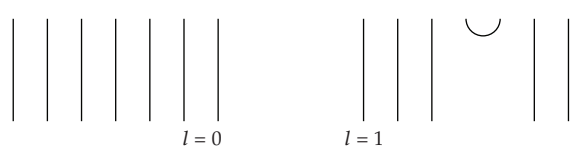
Proposition 4.4. *There is, up to rescaling, a unique pairing* $(,) : L_k \times L_k \to \mathbb{Z}[[t, t^{-1}]]$ *for which* u_i *is self-adjoint, that is,* $(au_i, b) = (a, bu_i)$. *Given cup diagrams* a *and* b *in* M_k, *one evaluates* (a, b) *by considering the closure of* $c_{\omega(a),b} \in \mathcal{TL}$, *or equivalently the closure of* $b \circ \omega(a) \in \mathcal{TL}_k$. *If the diagram has nesting number* k, *one returns a scalar times* $[2]$ *raised to the number of circles; if it has nesting number* $< k$, *one returns zero. This is precisely the evaluation* $\varepsilon(c_{\omega(a),b})$ *for some well-defined trace on* $\mathcal{TL}$ *supported on nesting number* k *(which are unique up to rescaling).*

4.2. Some Induced Sign Representations

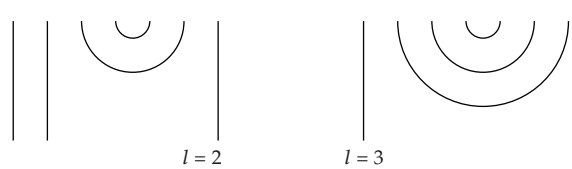
Cell modules are naturally subquotients of the cellular algebra itself, viewed as a free module (see [37]). For our purposes, we will describe the cell modules as subquotients of $\mathcal{TL}$ in a different way, which will be more convenient to diagrammatically categorify. Taking the inclusion $\mathcal{TL}_J \to \mathcal{TL}$ for some sub-Dynkin diagram J, we can induce the sign representation of $\mathcal{TL}_J$ up to $\mathcal{TL}$. This is the quotient of $\mathcal{TL}$ by the right ideal generated by $u_i, i \in J$. In a future paper, we will describe, for both the Hecke and Temperley-Lieb algebras, a diagrammatic way to categorify the induction of both the "sign" and "trivial" representations of sub-Dynkin diagrams, but for this paper we restrict to a specific case. For the sub-Dynkin diagram which contains every index except i, let I_i be the corresponding ideal (generated by u_j for $j \neq i$), and consider the induced sign representation $V^i = \mathcal{TL}/I_i$. Let $l_i = \min(i, n+1-i)$ and let $k_i = n + 1 - 2l_i$. It turns out that we can embed L_{k_i} inside V^i, as shown explicitly below, and we will categorify both modules accordingly. For this reason, we use L^i to denote L_{k_i}. Note that every possible L_k can be achieved as some L^i with the exception of L_{n+1}.

For the rest of this section, fix an index $i \in I$. We define a module V^i over $\mathcal{TL}$ abstractly, and then prove that this module is isomorphic to $\mathcal{TL}/I_i$.

Definition 4.5. For $0 \leq l \leq l_i$ (and letting $k = n + 1 - 2l$ as always), let a_k^i be the following $(k, n+1)$ cup diagram with l top arcs, where the innermost top arc always connects i to $i+1$:



(4.2)



(4.3)

Let $X_k^i \subset X_k$ consist of all matchings of the form $c_{a_k^i,b}$ for $b \in M_k$. Let X^i be the disjoint union of all X_k^i for $0 \leq l \leq l_i$, and let V^i be the free $\mathbb{Z}[t, t^{-1}]$-module with basis X^i. There is a distinguished element $\mathbb{1}$ of this basis, the unique member of X_{n+1}^i. Let $\mathcal{TL}$ act on V^i on the

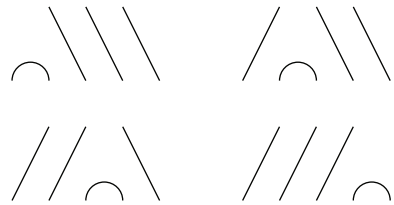


Figure 6: A basis for the cell module L_{n-1}, consisting of $(n+1, n-1)$ cap diagrams (here, $n = 4$).

right by viewing elements of V^i as though they were in $\mathcal{TL}$, using the standard multiplication rules, and then killing any terms whose diagrams are not in X^i.

The elements of X^i exhaust those elements of X where the only *simple* top arcs (those connecting j to $j+1$ for some j) connect i to $i+1$. Any crossingless matching with a simple top arc connecting j to $j+1$ has an expression in $\mathcal{TL}$ as a monomial $u_{\underline{i}}$ which begins with u_j. The converse is also true. Thus, X^i are the elements of X for which *every* expression of the matching begins with u_i. This motivates the definition.

While something does need to be checked to ensure that this defines a module action, it is entirely straightforward. In the Temperley-Lieb algebra, things are generally easy to prove because products of monomials always reduce to another monomial (with a scalar), not a linear combination of multiple monomials. Therefore, checking the associativity condition for being a module, say, involves showing that both sides of an equation are the same diagram in X^i, or that both sides are 0. This module is cyclic, generated by $\mathbb{1}$, and I_i is clearly in the annihilator of $\mathbb{1}$, so that $\mathcal{TL}/I_i$ surjects onto V^i. One could prove the following by bounding dimensions.

Claim 8. The modules V^i and $\mathcal{TL}/I_i$ are isomorphic.

There is a (cellular) filtration on V^i, given by the span of $X^i_{\leq k}$, diagrams with at most k through-strands (call it $V^i_{\leq k}$). Clearly, each subquotient in this filtration has a basis given by X^i_k, or in other words by the elements $c_{a^i_k,b}$ for $b \in M_k$. It is an easy exercise that this subquotient is isomorphic to the cell module L_k, under the map sending $b \in M_k$ to $c_{a^i_k,b}$. There is one subquotient for each $0 \leq l \leq l_i$.

Claim 9. The module L^i is a submodule of V^i.

Proof. Letting $l = l_i$ and $k = k_i$, the final term in the filtration is precisely $L^i \cong V^i_{k_i}$. □

Having explicitly defined the embedding $L^i \subset V^i$, we pause to investigate adjoint pairings on V^i.

Proposition 4.6. *Consider the $\mathbb{Z}[[t, t^{-1}]]$ module of semi-linear pairings on V^i where $(xu_j, y) = (x, yu_j)$ for all j. Consider the $l_i + 1$ functionals on this space, which send a pairing to $(\mathbb{1}, c_{a^i_k,\omega(a^i_k)})$ for various $k = n + 1 - 2l$, $0 \leq l \leq l_i$. Then these linear functionals are independent and yield an isomorphism between the space of pairings and a free module of rank $l_i + 1$.*

Note that, using adjunction, one can check that $[2]^l(\mathbb{1}, c_{a^i_k,\omega(a^i_k)}) = (c_{a^i_k,\omega(a^i_k)}, c_{a^i_k,\omega(a^i_k)})$.

Proof. Given diagrams $x, y \in X^i$, the self-adjointness of u_i implies that the value of (x, y) is an invariant of the diagram $y \circ \omega(x)$. In particular, $(x, y) = (\mathbb{1}, y\omega(x)) = (x\omega(y), \mathbb{1})$, where $y\omega(x)$ refers to the image of this diagram in the quotient $\mathcal{TL}/I_i$. Therefore, if either $y\omega(x)$ or $x\omega(y)$ is not in X^i then the value of (x, y) is zero. However, $X^i \cap \omega(X^i) = \{c_{a_k^i, \omega(a_k^i)}\}$ where this set runs over all k with $0 \le l \le l_i$. Thus the value of the pairing on all elements is clearly determined by the values of $(\mathbb{1}, c_{a_k^i, \omega(a_k^i)})$ for all such k.

Consider the following map $V^i \times V^i \to \mathbb{Z}[[t, t^{-1}]]$: fix k, and for basis elements x, y send (x, y) to $r \in \mathbb{Z}[t, t^{-1}]$ if $y\omega(x) = rc_{a_k^i, \omega(a_k^i)} \in \mathcal{TL}$, and send (x, y) to zero otherwise. Clearly this is a well-defined semi-linear map (being defined on a $\mathbb{Z}[t, t^{-1}]$-basis) and u_j is self-adjoint. Thus we have enough pairings to prove independence. □

Remark 4.7. Once again, all pairings are defined topologically. The closure of $c_{a_k^i, \omega(a_k^i)}$ has nesting number exactly k, which distinguishes the traces.

4.3. Categorifying Cell Modules

Categorifying the sign representation L_{n+1} is easy. If we take the quotient of $\mathcal{TLC}$ by all nonempty diagrams, we get a category where the only nonzero morphism space is the one-dimensional space $\mathrm{Hom}(\emptyset, \emptyset)$. This clearly categorifies L_{n+1}, and we will say no more.

Consider the quotient of the category $\mathcal{TLC}_1$ by all diagrams where any color not equal to i appears on the left. Call this quotient $\mathcal{V}_1^i$. As usual, we let $\mathcal{V}_2^i$ be its additive grading closure, and let $\mathcal{V}^i$ be its graded Karoubi envelope. We will show that $\mathcal{V}^i \cong \mathcal{V}_2^i$, so that we really may think of $\mathcal{V}^i$ entirely diagrammatically without worrying about idempotents. We claim that $\mathcal{V}^i$ categorifies V^i. Not only this, but the action of $\mathcal{TLC}$ on $\mathcal{V}^i$ by placing diagrams on the right will categorify the action of $\mathcal{TL}$ on V^i.

Any monomial $u_{\underline{\mathbf{i}}}$ which goes to zero in V^i is equal to a (scalar multiple of a) monomial $u_{\underline{\mathbf{j}}}$ where some index $j \neq i$ appears on the left. Therefore, the corresponding object $U_{\underline{\mathbf{i}}}$ will be isomorphic to $U_{\underline{\mathbf{j}}}$, whose identity morphism is sent to zero in $\mathcal{V}_1^i$ since it has a j-colored line on the left. There is an obvious map from V^i to the Grothendieck group of $\mathcal{V}_2^i$, and the action of $\mathcal{TLC}$, descended to the Grothendieck group, will commute with the action of $\mathcal{TL}$ on V^i.

Therefore, Hom spaces in $\mathcal{V}_1^i$ will induce a semi-linear pairing on V^i, which satisfies the property $(au_j, b) = (a, bu_j)$ because U_j is self-adjoint. As before, once we determine which pairing this is, our proof will be almost complete.

Lemma 4.8. *The pairing induced by $\mathcal{V}_1^i$ will satisfy $(\mathbb{1}, c_{a_k^i, \omega(a_k^i)}) = t^l/(1 - t^2)$, where $k = n + 1 - 2l$.*

Remark 4.9. Taking a $(n + 1, n + 1)$ diagram and closing it off on the punctured plane, if m is the number of circles and k is the nesting number, then the pairing comes from the trace on $\mathcal{TL}$ which sends this configuration to $[2]^{l+m-(n+1)}(t^l/(1 - t^2))$.

For a closure of an arbitrary diagram, $l + m < n + 1$ is possible. However, for any diagram in $X^i \cap \omega(X^i)$ (with extra circles thrown in), we have $l + m \ge n + 1$, since removing the circles yields precisely $c_{a_k^i, \omega(a_k^i)}$ for some k. This guarantees that evaluating the formula on an element of V^i yields a power series with *nonnegative* coefficients.

The proof of the lemma may be found shortly below. Temporarily assuming the lemma, the remainder of our results are easy.

Theorem 4.10. *$\mathcal{V}_2^i$ is idempotent closed and Krull-Schmidt, so that $\mathcal{V}^i \cong \mathcal{V}_2^i$. Its Grothendieck group is isomorphic to V^i.*

Proof. It is enough to check that for any $u_{\underline{\mathbf{i}}} \neq u_{\underline{\mathbf{j}}}$ corresponding to matchings in X^i, $\mathrm{Hom}(U_{\underline{\mathbf{i}}}, U_{\underline{\mathbf{i}}})$ is concentrated in nonnegative degrees with a 1-dimensional degree 0 part, and $\mathrm{Hom}(U_{\underline{\mathbf{i}}}, U_{\underline{\mathbf{j}}})$ is concentrated in strictly positive degrees (see the proof of Proposition 3.4). This is a calculation using the semi-linear pairing.

Letting m be the number of circles in a configuration on the punctured disk, and letting $k = n + 1 - 2l$ be the nesting number, then the evaluation will be in strictly positive degrees if $m < n + 1$, and will be in nonnegative degrees with a 1-dimensional degree 0 part if $m = n + 1$ exactly, but this was precisely the calculation in the proof of Lemma 3.5: for arbitrary crossingless matchings $u_{\underline{\mathbf{i}}}$ and $u_{\underline{\mathbf{j}}}$, the closure of $u_{\underline{\mathbf{i}}}\omega(u_{\underline{\mathbf{j}}})$ has fewer than $n + 1$ circles if $u_{\underline{\mathbf{i}}} \neq u_{\underline{\mathbf{j}}}$, and exactly $n + 1$ if they are equal. □

Corollary 4.11. *Let $\mathcal{L}^i$ be the full subcategory of $\mathcal{V}^i$ with objects consisting of (sums and grading shifts of) $U_{\underline{\mathbf{i}}}$ such that $u_{\underline{\mathbf{i}}}$ is an element of $V^i_{n+1-2l_i}$. This has an action of $\mathcal{TLC}$ on the right. On the Grothendieck group, this setup categorifies the cell module $L^i = V^i_{n+1-2l_i}$.*

Proof. That this subcategory is closed under the action of $\mathcal{TLC}$ is obvious, as is the existence of a map from L^i to the Grothendieck group. We already know the induced pairing, because the subcategory is full. Therefore, the same arguments imply that the Grothendieck group behaves as planned. □

Proof of Lemma 4.8. To calculate the pairing, we may calculate $(u_{\underline{\mathbf{i}}}, u_{\underline{\mathbf{i}}}) = \mathrm{gdimEnd}(U_{\underline{\mathbf{i}}})$ for the following choices of $\underline{\mathbf{i}}$: $\emptyset, i,\ i(i+1)\ (i-1), i(i+1)\ (i-1)(i+2)i(i-2),\ i(i+1)\ (i-1)(i+2)i(i-2)(i+3)(i+1)(i-1)(i-3)$, and so forth. These are pictured below.

(4.4)

(4.5)

These sequences are split into subsequences we call "tiers", where the mth sequence adds the mth tier. The following property of these sequences is easily verified: each sequence $\underline{\mathbf{i}}$ is in X^i, and remains in X^i if one removes any subset of the final tier, but ceases to be in X^i if one removes a single element from any other tier instead.

Fix nonempty $\underline{\mathbf{i}}$ in this sequence, and let $\underline{\mathbf{j}}$ be the subsequence with the final tier removed. It is a quick exercise to show that the lemma is equivalent to $\mathrm{gdimEnd}(U_{\underline{\mathbf{i}}}) = (1 + t^2)^l/(1 - t^2)$, where l is the number of elements in the final tier.

Now consider an element of the endomorphism ring. Using previous results, we may assume it is a simple forest, with all double dots on the far left. Any double dot colored $j \neq i$

will be sent to zero, so we have only an action of the ring $\mathbb{k}[f_i]$ on the left. This accounts for the $1/(1-t^2)$ appearing in all the formulae.

Suppose there is a boundary dot in the morphism on any line not in the final tier. Then the morphism factors through the sequence $U_{\underline{\mathbf{k}}}$ where $\underline{\mathbf{k}}$ is $\underline{\mathbf{i}}$ with that index removed. As discussed above, $u_{\underline{\mathbf{k}}}$ is not in X^i and therefore $U_{\underline{\mathbf{k}}}$ is isomorphic to the zero object. See the first picture below for an intuitive reason why such a morphism vanishes. Hence, the only boundary dots which can appear occur on the final tier. It is easy to check that the existence of a trivalent vertex joining three boundary lines will force the existence of a dot not on the final tier; see the second picture below. Both pictures are for the sequence $i(i+1)(i-1)(i+2)i(i-2)$, and blue will represent i in all pictures in this section.

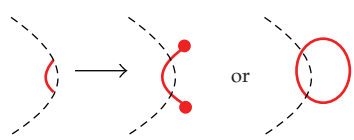
(4.6)

Therefore a nonzero endomorphism must be $\mathbb{1}_{U_{\underline{\mathbf{i}}}}$ accompanied on the right by either identity maps or broken lines (pairs of boundary dots), because all other simple forests yield a zero map. Identity maps have degree zero, while broken lines have degree 2. If these pictures form a basis (along with the action of blue double dots on the left), then the graded dimension will be exactly as desired. An example with $l = 3$, two broken lines, and one unbroken line is shown below.

(4.7)

This spanning set is linearly independent in $\mathcal{TLC}$ over $\mathbb{k}$, so any further dependencies must come from having a nonblue color on the left. Consider an arbitrary endomorphism, and reduce it using the $\mathcal{TLC}$ relations to a simple forest with all double dots on the left. The actual double dots appearing are ambiguous, since there are polynomial relations in $\mathcal{TLC}$, but it is easy (knowing the generators of the TL ideal) to note that these relations are trivial modulo nonblue colors. Hence the spanning set will be linearly independent if any diagram in $\mathcal{TLC}$ which started with a nonblue color on the left will still have a nonblue color (perhaps in a double-dot) on the left after reducing to a simple forest. This will be the case for any diagram with a boundary dot on $\underline{\mathbf{j}}$.

Let red indicate any other index, and suppose that red appears on the far left. Regardless of what index red is, unless there is a dot on $\underline{\mathbf{j}}$, the identity lines of $\underline{\mathbf{j}}$ block this leftmost red component from reaching any red on the boundary of the graph. Take a neighborhood of a red line segment which includes no other colors and goes to $-\infty$. Excising this neighborhood, we get a simply connected region where the only relevant red boundary lines are the two which connect to the ends of the segment. Red then will reduce to a simple forest with double dots on the left, which in this case yields either a red double dot or a red circle (potentially with more double dots).



(4.8)

However, no colors adjacent to red can interfere on the interior of a red circle, so the circle evaluates to zero. Therefore, the diagram evaluates to zero or has at least one red double dot on the left. We may ignore the red double dot and reduce the remainder of the diagram, and so regardless of what else is done, the final result will have a red double dot on the left. □

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Research Article

Integral HOMFLY-PT and sl(n)-Link Homology

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Using the diagrammatic calculus for Soergel bimodules, developed by Elias and Khovanov, as well as Rasmussen's spectral sequence, we construct an integral version of HOMFLY-PT and sl(n)-link homology.

1. Introduction

During the past half-decade, categorification and, in particular, that of topological invariants has flourished into a subject of its own right. It has been a study finding connections and ramifications over a vast spectrum of mathematics, including areas such as low-dimensional topology, representation theory, and algebraic geometry. Following the original work of Khovanov on the categorification of the Jones polynomial in [1], came a slew of link homology theories lifting other quantum invariants. With a construction that utilized matrix factorizations, a tool previously developed in an algebra-geometric context, Khovanov and Rozansky produced the sl(n) and HOMFLY-PT link homology theories. Albeit computationally intensive, it was clear from the onset that thick interlacing structure was hidden within. The most insightful and influential work in uncovering these innerconnections was that of Rasmussen in [2], where he constructed a spectral sequence from the HOMFLY-PT to the sl(n)-link homology. This was a major step in deconstructing the web of how these theories come together, yet many structural questions remained and still remain unanswered, waiting for a new approach. Close to the time of the original work, Khovanov produced an equivalent categorification of the HOMFLY-PT polynomial in [3], but this time using Hochschild homology of Soergel bimodules and Rouquier complexes of [4]. The latter proved to be more computation-friendly and was used by Webster to calculate many examples in [5].

In the meantime, a new flavor of categorification came into light. With the work of Khovanov and Lauda on the categorification of quantum groups in [6], a diagrammatic calculus originating in the study of 2 categories arrived into the foreground. This graphical approach proved quite fruitful and was soon used by Elias and Khovanov to rewrite the work of Soergel in [7], and en suite by Elias and the author to repackage Rouquier's complexes and to prove that they are functorial over braid cobordisms [8] (not just projectively functorial as was known before). We note the closely related independent construction of Chuang and Rouquier in [9, 10]. An immediate advantage to the diagrammatic construction was a comparative ease of calculation.

As there has yet to be seen an integral version of either HOMFLY-PT or sl(n)-link homology, with the original Khovanov homology being defined over $\mathbb{Z}$ and torsion playing an interesting role, a natural question arose as to whether this graphical calculus could be used to define these. The definition of such integral theories is precisely the purpose of this paper. The one immediate disadvantage to the graphical approach is that at the present moment there does not exist a diagrammatic calculus for the Hochschild homology of Soergel bimodules. Hence, to define integral HOMFLY-PT homology, our paper takes a rather roundabout way, jumping between matrix factorizations and diagrammatic Rouquier complexes whenever one is deemed more advantageous than the other. For the sl(n) version of the story, we add the Rasmussen spectral sequence into the mix and essentially repeat his construction in our context.

When choosing what to define in full and what to leave out, we assume the reader's familiarity with [8]. The organization of the paper is the following: in Section 2, we give a brief account of the necessary tools (matrix factorizations, Soergel bimodules, Hochschild homology, Rouquier complexes, and corresponding diagrammatics)—the emphasis here is brevity and we refer the reader to more original sources for particulars and details; in Sections 3 and 4, we describe the integral HOMFLY-PT complex and prove the Reidemeister moves, utilizing all of the background in Section 2; Section 5 is devoted to the Rasmussen spectral sequence and integral sl(n)-link homology. We conclude it with some remarks and questions.

Throughout the paper, we will refer to a positive crossing as the one labelled D_+ and negative crossing as the one labelled D_- in Figure 1. For resolutions of a crossing, we will refer to D_o and D_s of Figure 1 as the "oriented" and "singular" resolutions, respectively. We will use the following conventions for the HOMLFY-PT polynomial:

$$aP(D_-) - a^{-1}P(D_+) = \left(q - q^{-1}\right)P(D_o), \tag{1.1}$$

with P of the unknot being 1. Substituting $a = q^n$ we arrive at the quantum sl(n)-link polynomial.

2. The Toolkit

We will require some knowledge of matrix factorizations, Soergel bimodules, and Rouquier complexes, as well as the corresponding diagrammatic calculus of Elias and Khovanov [7]. In this section the reader will find a brief survery of the necessary tools, and for more details we refer him to the following papers: matrix factorizations [2, 11], Soergel bimodules and Rouquier complexes and diagrammatics [3, 4, 7, 8], and Hochschild homology [3, 12].

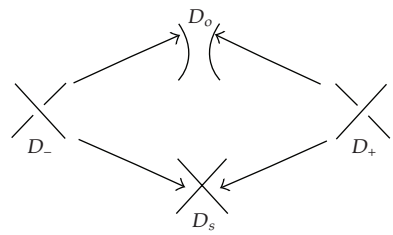


Figure 1: Crossings and resolutions (note: these are braid diagrams).

2.1. Matrix Factorizations

Definition 2.1. Let R be a Noetherian commutative ring, $w \in R$, and C^*, $* \in \mathbb{Z}$, a free graded R-module. A *$\mathbb{Z}$-graded matrix factorization with potential* $w \in R$ consists of C^* and a pair of differentials $d_\pm : C^* \to C^{*\pm 1}$, such that $(d_+ + d_-)^2 = wId_{C^*}$.

A morphism of two matrix factorizations C^* and D^* is a homomorphism of graded R-modules $f : C^* \to D^*$ that commutes with both d_+ and d_-. The tensor product $C^* \otimes D^*$ is taken as the regular tensor product of complexes, and is itself a matrix factorization with diffentials d_+ and d_-. A useful and easy exercise is the following.

Lemma 2.2. *Given two matrix factorizations C^* and D^* with potenials w_c and w_d, respectively, the tensor product $C^* \otimes D^*$ is a matrix factorization with potential $w_c + w_d$.*

Remark 2.3. Following Rasmussen [2], we work with $\mathbb{Z}$-graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded, matrix factorizations as in [11]. The $\mathbb{Z}$-grading implies that $(d_+ + d_-)^2 = wId_{C^*}$ is equivalent to

$$\begin{gathered} d_+^2 = d_-^2 = 0, \\ d_+d_- + d_-d_+ = wId_{C^*}. \end{gathered} \tag{2.1}$$

In the case that $w = 0$, we acquire a new $\mathbb{Z}/2\mathbb{Z}$-graded chain complex structure with differential $d_+ + d_-$.

Suppressing the underlying ring R and potential w, we will denote the category of graded matrix factorizations by mf.

We also need the notion of complexes of matrix factorizations. If we visualize a collection of matrix factorizations as sitting horizontally in the plane at each integer level, with differentials d_+ and d_- running right and left, respectively, we can think of morphisms $\{d_v\}$ between these as running in the vertical direction. All together, we have that

$$d_\pm : C^{i,j} \longrightarrow C^{i\pm 1,j}, \qquad d_v : C^{i,j} \longrightarrow C^{i,j+1}, \tag{2.2}$$

where we think of i as the *horizontal* grading and j as the *vertical* grading, and will denote these as gr_h and gr_v, respectively.

In addition we will be taking tensor products of complexes of matrix factorizations (in the obvious way) and, just to add to the confusion, we will also have homotopies of these complexes as well homotopies of matrix factorizations themselves. These notions will land us in different categories for which we now give some notation.

(i) hmf will denote the *homotopy category of matrix factorizations.*

(ii) $\mathcal{KOM}(mf)$ will denote the *category of complexes of matrix factorizations.*

(iii) $\mathcal{KOM}_h(mf)$ will denote *homotopy category of complexes of matrix factorizations.*

(iv) $\mathcal{KOM}_h(hmf)$ will denote the obvious conglomerate.

2.2. Diagrammatics of Soergel Bimodules

The category of Soergel bimodules $\mathcal{SC}_1$ is a monoidal category generated by objects B_i, where $i \in I$ is a finite indexing set, which satisfy

$$\begin{gathered} B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}, \\ B_i \otimes B_j \cong B_j \otimes B_i, \quad \text{for distant } i, j, \\ B_i \otimes B_j \otimes B_i \oplus B_j \cong B_j \otimes B_i \otimes B_j \oplus B_i, \quad \text{for adjacent } i, j. \end{gathered} \tag{2.3}$$

These objects B_i are graded and the notation $\{j\}$ refers a grading shift of $+j$. Technically speaking this should be called the category of Bott-Samuelson bimodules and the "real" category of Soergel bimodules is obtained as described at the end of this section. A key feature of this category is that the Grothendieck group of $\mathcal{SC}(I)$ is isomorphic to the Hecke algebra $\mathcal{H}$ of type A_∞ over the ring $\mathbb{Z}[t, t^{-1}]$. We refer the reader to [7, 8] for defenitions and relvant details.

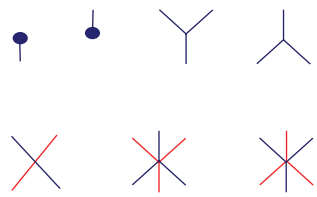
More concretely, the Soergel bimodule $B_i = R \bigotimes_{R^i} R\{-1\}$, where $R = \mathbb{Z}[x_1 - x_2, \ldots, x_{n-1} - x_n]$ with $\deg x_i = 2$, $\{m\}$ denotes the grading shift by m, and R^i is the subring of invariants corresponding to the permutation $(i, i+1)$ under the natural action of S_n on the variables. There is some flexibility as to the exact description of R, but we work with the most convenient for the constructions below (note that our grading shift of -1 in the definition of B_i is absent from the contruction of [3]). We have that $B_\emptyset = R$ itself, and $B_{\underline{i}} = B_{i_1} \bigotimes_R B_{i_2} \bigotimes_R \cdots \bigotimes_R B_{i_d}$, where $\underline{i}$ is denotes the sequence $\{i_1, i_2, \ldots, i_d\}$, that is,

$$\begin{aligned} B_{\underline{i}} &= \left(R \bigotimes_{R^{i_1}} R\{-1\} \right) \otimes \left(R \bigotimes_{R^{i_2}} R\{-1\} \right) \otimes \cdots \otimes \left(R \bigotimes_{R^{i_d}} R\{-1\} \right) \\ &= R \bigotimes_{R^{i_1}} R \otimes R \bigotimes_{R^{i_2}} R \otimes \cdots \otimes R \bigotimes_{R^{i_d}} R\{-d\} \\ &= R \bigotimes_{R^{i_1}} \otimes R \bigotimes_{R^{i_2}} R \bigotimes_{R^{i_3}} \cdots R \bigotimes_{R^{i_d}} R\{-d\}. \end{aligned} \tag{2.4}$$

One useful feature of this categorification is that it is easy to calculate the dimension of Hom spaces in each degree. Let $\mathrm{HOM}(M,N) \stackrel{\text{def}}{=} \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}(M,N\{m\})$ be the graded vector space (actually an R-bimodule) generated by homogeneous morphisms of all degrees. Then $\mathrm{HOM}(B_{\underline{i}},B_{\underline{j}})$ is a free left R-module, and its graded rank over R is given by a natural bilinear form $(b_{\underline{i}},b_{\underline{j}})$ defined on the Hecke algebra $\mathcal{H}$. For more information on this categorification and related topics we refer the reader to [7, 13].

The graphical counterpart, which we will also refer to as $\mathcal{SC}_1$ was given a diagrammatic presentation by generators and relations, allowing morphisms to be viewed as isotopy classes of certain graphs.

An object in $\mathcal{SC}_1$ is given by a sequence of indices $\underline{i}$, which is visualized as d points on the real line $\mathbb{R}$, labelled or "colored" by the indices in order from left to right. Sometimes these objects are also called $B_{\underline{i}}$. Morphisms are given by pictures embedded in the strip $\mathbb{R}\times[0,1]$ (modulo certain relations), constructed by gluing the following generators horizontally and vertically:



For instance, if "blue" corresponds to the index i and "red" to j, then the lower right generator is a morphism from jij to iji. The generating pictures above may exist in various colors, although there are some restrictions based on adjacency conditions.

We can view a morphism as an embedding of a planar graph, satisfying the following properties:

(1) edges of the graph are colored by indices from 1 to n;

(2) edges may run into the boundary $\mathbb{R}\times\{0,1\}$, yielding two sequences of colored points on $\mathbb{R}$, the top boundary $\underline{i}$, and the bottom boundary $\underline{j}$. In this case, the graph is viewed as a morphism from $\underline{j}$ to $\underline{i}$;

(3) only four types of vertices exist in this graph: univalent vertices or "dots," trivalent vertices with all three adjoining edges of the same color, 4-valent vertices whose adjoining edges alternate in colors between distant i and j, and 6-valent vertices whose adjoining edges alternate between adjacent i and j.

The degree of a graph is +1 for each dot and −1 for each trivalent vertex. 4-valent and 6-valent vertices are of degree 0. The term *graph* henceforth refers to such a graph embedding.

By convention, we color the edges with different colors, but do not specify which colors match up with which $i \in I$. This is legitimate, as only the various adjacency relations between colors are relevant for any relations or calculations. We will specify adjacency for all pictures, although one can generally deduce it from the fact that 6-valent vertices only join adjacent colors, and 4-valent vertices join only distant colors.

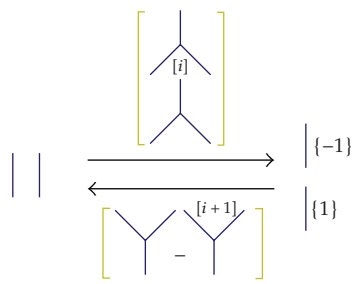
In addition to the bimodules $B_{\underline{i}}$ above, we will require the use of the bimodule $R\otimes_{R^{i,i+1}}R\{-3\}$, where $R^{i,i+1}$ is the ring of invariants under the transpositions $(i,i+1)$ and

$(i+1,\ i+2)$, and will use a black squiggly line, as in (2.7) below, to represent it. This bimodule comes into play in the isomorphisms:

$$\begin{aligned} B_i \otimes B_{i+1} \otimes B_i &\cong B_i \oplus \left(R \bigotimes_{R^{i,i+1}} R\{-3\} \right), \\ B_{i+1} \otimes B_i \otimes B_{i+1} &\cong B_{i+1} \oplus \left(R \bigotimes_{R^{i,i+1}} R\{-3\} \right), \end{aligned} \tag{2.5}$$

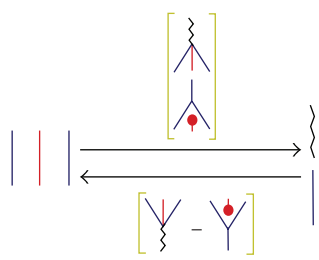
which we will use in the proof of Reidemeister move III. As usual in a diagrammatic category, composition of morphisms is given by vertical concatenation, and the monoidal structure is given by horizontal concatenation.

We then allow $\mathbb{Z}$-linear sums of graphs, and apply relations to obtain our category $\mathcal{SC}_1$. The relations come in three flavors: one color, two distant colors, two adjacent and one distant, and three mutually distant colors. We do not list all of them here, just the consequences necessary for the calculations at hand, and refer the reader to [7, 11] for a complete picture. Our graphs are invariant under isotopy and in addition, we have the following isomorphisms or "decompositions":



(2.6)

The vertical juxtapositions of diagrams corresponds to direct sums of morphisms and $[i]$ corresponds to the morphism induced by multiplication by the polynomial x_i. Note that this relation is precisely that of 1 described diagrammatically.



(2.7)

Here, we have the graphical counterpart of 4 and 5.

Remark 2.4. Primarily we will work in another category denoted $\mathcal{SC}_2$, the category formally containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$, but whose morphisms are forced to be degree 0. In addition, we let $\mathcal{SC}$ be the Karoubi envelope, or idempotent

completion, of the category $\mathcal{SC}_2$. Recall that the Karoubi envelope of a category $\mathcal{C}$ has as objects pairs (B, e) where B is an object in $\mathcal{C}$ and e an idempotent endomorphism of B. This object acts as though it were the "image" of this projection e, and in an additive category behaves like a direct summand. For more information on Karoubi envelopes, see Wikipedia. It is really here that the object $R\bigotimes_{R^{i,i+1}} R\{-3\}$ of 4 and 5 resides. In practice all our calculations will be done in $\mathcal{SC}_2$, but since $\mathcal{SC}_2$ includes fully faithfully into $\mathcal{SC}$ they will be valid there as well.

The important fact here is that there is a functor from $\mathcal{SC}$ to the category of R-bimodules, sending a line colored i to B_i and each generator to an appropriate bimodule map. The functor gives an equivalence of categories between this diagrammatic category and the subcategory $\mathcal{SC}_1$ of R-bimodules mentioned in the previous section, so the use of the same name is legitimate.

Our diagrammatic category has many wonderful properties, such as the self-adjointness of B_i, which permits us to "twist" morphisms around and view any morphism as one from or to the empty diagram. This allows for a very hands-on, explicit, understanding of hompaces between objects in $\mathcal{SC}_1$, which was key in proving functoriality in [8].

2.3. Hochschild (Co)Homology

Let A be a $\Bbbk$-algebra and M an A-bimodule, or equivalently a left $A \otimes A^{op}$-module or a right $A^{op} \otimes A$-module. The definitions of the *Hochschild (co)homology* groups $HH_*(A, M)$ $(HH^*(A, M))$ are the following:

$$HH_*(A, M) := \mathrm{Tor}_*^{A\otimes A^{op}}(M, A), \qquad HH^*(A, M) := \mathrm{Ext}^*_{A\otimes A^{op}}(A, M). \tag{2.8}$$

To compute them we take a projective resolution of the A-bimodule A, with the natural left and right action, by projective A-bimodules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \tag{2.9}$$

and tensor this with M over $A \otimes A^{op}$ to get

$$\cdots \longrightarrow P_2 \bigotimes_{A\otimes A^{op}} M \longrightarrow P_1 \bigotimes_{A\otimes A^{op}} M \longrightarrow P_0 \bigotimes_{A\otimes A^{op}} M \longrightarrow 0. \tag{2.10}$$

The homology of this complex is isomorphic to $HH_*(A, M)$.

Example 2.5. For any bimodule M, we have

$$HH_0(A, M) \cong \frac{M}{[A, M]}, \qquad HH^0(A, M) \cong M^A, \tag{2.11}$$

where $[A, M]$ is the subspace of M generated by all elements of the form $am - ma$, $a \in A$ and $m \in M$, and $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. We leave this as an exercise or refer the reader to [12].

For the polynomial algebra $A = \Bbbk[x_1, \ldots, x_n]$, with $\Bbbk$ commutative, we can use the Koszul resolution of A by free $A \otimes A$-modules, which is the tensor product of the following complexes:

$$0 \longrightarrow A \otimes A \xrightarrow{x_i \otimes 1 - 1 \otimes x_i} A \otimes A \longrightarrow 0, \tag{2.12}$$

for $1 \geq i \geq n$. This resolution has length n, and its total space is naturally isomorphic to the exterior algebra on n generators tensored with $A \otimes A$. Hence, we have that the complex which computes Hochschild homology of a bimodule M over A is made up of 2^n copies of M, with the differentials coming from multiplication by $x_i \otimes 1 - 1 \otimes x_i$. In other words

$$0 \longrightarrow C_n(M) \longrightarrow \cdots \longrightarrow C_1(M) \longrightarrow C_0(M) \longrightarrow 0, \tag{2.13}$$

with

$$C_j(M) = \bigoplus_{I \subset \{1,\ldots,n\}, |I| = j} M \bigotimes_{\mathbb{Z}} \mathbb{Z}[I], \tag{2.14}$$

where $\mathbb{Z}[I]$ is the rank 1 free abelian group generated by the symbol $[I]$ (i.e., it is there to keep track where exactly we are in the complex). Here, the differential takes the form

$$d(m \otimes [I]) = \sum_{i \in I} \pm (x_i m - m x_i) \otimes [I \setminus \{i\}], \tag{2.15}$$

and the sign is taken as negative if I contains an odd number of elements less than i.

Remark 2.6. For the polynomial algebra, the Hochschild homology and cohomology are isomorphic,

$$HH_i(A, M) \cong HH^{n-i}(A, M), \tag{2.16}$$

for any bimodule M. This comes from self-duality of the Koszul resolution for such algebras. Hence, we will be free to use either homology or cohomology groups in the constructions below.

For us, taking Hochschild homology will come into play when looking at closed braid diagrams. To a given resolution of a braid diagram we will assign a Soergel bimodule; "closing off" this diagram will correspond to taking Hochschild homology of the associated bimodule. More details are below in Section 3.2.

3. The Integral HOMFLY-PT Complex

3.1. The Matrix Factorization Construction

As stated above we will work with $\mathbb{Z}$-graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded, matrix factorizations and follow closely the conventions laid out in [2]. We begin by first assigning the appropriate complex to a single crossing and then extend this to general braids.

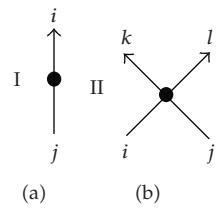


Figure 2

Gradings

Our complex will be triply graded, coming from the internal or "quantum" grading of the underlying ring, the homological grading of the matrix factorizations, and finally an overall homological grading of the entire complex. It will be convenient to visualize our complexes in the plane with the latter two homological gradings lying in the horizontal and vertical directions, respectively. We will denoted these gradings by $(i, j, k) = (q, 2gr_h, 2gr_v)$ and their shifts by curly brackets, that is, $\{a, b, c\}$ will indicate a shift in the quantum grading by a, in the horizontal grading by b, and in the vertical grading by c. Note that following the conventions in [2], we have doubled the latter two gradings, as illustrated in Figure 2.

Definition 3.1 (edge ring). Given a diagram D with vertices labelled by $x_1, \ldots, x_n$, define the *edge ring* of D as $R(D) := \mathbb{Z}[x_1, \ldots, x_n]/\langle \mathrm{rel}(v_i)\rangle$, where i runs over all internal vertices, or marks, with the defining relations being $x_i - x_j$ for type I and $x_k + x_l - x_i - x_j$ for type II vertices (see Figure 2).

Consider the two types of crossings D_+ and D_-, as in Figure 1, with outgoing edges labeled by k, l, and incoming edges labelled by i, j. Let

$$R_c := \frac{\mathbb{Z}[x_i, x_j, x_k, x_l]}{(x_k + x_l - x_i - x_j)} \cong \mathbb{Z}[x_i, x_j, x_k] \tag{3.1}$$

be the underlying ring associated to a crossing. The complex for the positive crossing D_+ is given by the following diagram:

$$\begin{tikzcd} R_c\{0,-2,0\} \arrow[r, "(x_k-x_i)"] & R_c\{0,0,0\} \\ R_c\{2,-2,-2\} \arrow[u, "(x_j-x_k)"] \arrow[r, "-(x_k-x_i)(x_k-x_j)"] & R_c\{0,0,-2\} \arrow[u, "1"'] \end{tikzcd} \tag{3.2}$$

The complex for the negative crossing D_+ is given by the following diagram:

$$\begin{tikzcd} R_c\{0,-2,2\} \arrow[r, "-(x_k-x_i)(x_k-x_j)"] & R_c\{-2,0,2\} \\ R_c\{0,-2,0\} \arrow[u, "1"] \arrow[r, "(x_k-x_i)"] & R_c\{0,0,0\} \arrow[u, "(x_j-x_k)"'] \end{tikzcd} \tag{3.3}$$

Remark 3.2. The horizontal and vertical differentials d_+ and d_v are homogeneous of degrees $(2,2,0)$ and $(0,0,2)$, respectively. For those more familiar with [11] and hoping to reconcile the differences, note that in R_c multiplication by $x_k x_l - x_i x_j = -(x_k - x_i)(x_k - x_j)$, so up to some grading shifts we are working with the same underlying complex as in the original construction, but over $\mathbb{Z}$, not $\mathbb{Q}$.

To write down the complex for a general braid we tensor the above for every crossing, keeping track of markings, replace the underlying ring with a copy of the edge ring $R(D)$ and replace d_v with $(-1)^i d_v$ to make it anticommute with d_h (here i is the degree if d_v). More precisely, given a diagram D of a braid let

$$C(D) := \bigotimes_{\text{crossings}} \left(C(D_c) \bigotimes_{R_c} R(D) \right). \tag{3.4}$$

Definition 3.3 (HOMFLY-PT homology). Given a braid diagram D of a link L we define its HOMFLY-PT homology to be the group

$$H(L) := H(H(C(D), d_+), d_v^*)\{-w + b, w + b - 1, w - b + 1\}, \tag{3.5}$$

where w and b are the writhe and the number of strands of D, respectively.

Remark 3.4. In [2], this is what Rasmussen calls the "middle HOMFLY homology." The relation between this link homology theory and the HOMFLY-PT polynomial is that for any link $L \subset S^3$

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim H^{i,j,k}(L) = \frac{-P(L)}{q - q^{-1}}. \tag{3.6}$$

The Reduced Complex

There is a natural subcomplex $\overline{C}(D) \subset C(D)$ defined as follows: let $\overline{R}(D) \subset R(D)$ to be the subring generated by $x_i - x_j$ where i, j run over all edges of D and let $\overline{C}(D)$ be the subcomplex gotten by replacing in $C(D)$ each copy of $R(D)$ by one of $\overline{R}(D)$. A quick glance at the complexes $C(D_+)$ and $C(D_-)$ will reassure the reader that this is indeed a subcomplex, as the coefficients of both d_v and d_+ lie in $\overline{R}(D)$. We will refer to $\overline{C}(D)$ as the *reduced complex* for D.

(i) If i is an edge of D we can also define the complex $\overline{C}(D, i) := C(D)/(x_i)$. It is not hard to see that $\overline{C}(D, i) \cong \overline{C}(D)$ and is, hence, independent of the choice of edge i. See [2, Section 2.8] for a discussion as well as [11].

Below we will work primarily with the reduced complex $\overline{C}(D)$, and will stick with the grading conventions of [2], which are different than that of [11].

Definition 3.5 (reduced homology). Given a braid diagram D of a link L we define its reduced HOMFLY-PT homology to be the group

$$\overline{H}(L) := H\left(H\left(\overline{C}(D), d_+\right), d_v^*\right)\{-w+b-1, w+b-1, w-b+1\}, \tag{3.7}$$

where w and b are the writhe and the number of strands of D, respectively.

Remark 3.6. For any link $L \subset S^3$, we have

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i \dim \overline{H}^{i,j,k}(L) = P(L). \tag{3.8}$$

We can look at the complex $C(D)$ in two essential ways: either as the tensor product, over appropriate rings, of $C(D_+)$ and $C(D_-)$ for every crossing in our diagram D (as described above), or as a tensor product of corresponding complexes over all resolutions of the diagram. Although this is really just a matter of point of view, the latter approach is what we find in the original construction of Khovanov and Rozansky, as well as in the Soergel bimodule construction to be described below. To clarify this approach, consider the oriented D_o and singular D_s resolution of a crossing as in Figure 1. Assign to D_o the complex

$$0 \longrightarrow R_c \xrightarrow{(x_k - x_i)} R_c \longrightarrow 0 \tag{3.9}$$

and to D_s the complex

$$0 \longrightarrow R_c \xrightarrow{-(x_k - x_i)(x_k - x_j)} R_c \longrightarrow 0. \tag{3.10}$$

Then, we have

$$\begin{aligned} C(D_+) &: 0 \longrightarrow C(D_s) \longrightarrow C(D_o) \longrightarrow 0, \\ C(D_-) &: 0 \longrightarrow C(D_o) \longrightarrow C(D_s) \longrightarrow 0, \end{aligned} \tag{3.11}$$

where the maps are given by d_v as defined above. (For simplicity we leave out the internal grading shifts.) Let a resolution of a link diagram D be a resolution of each crossing in either of the two ways above, and let the complex assigned to each resolution be the tensor product of the corresponding complexes for each resolved crossing. Then, modulo grading shifts, our total complex can be viewed as

$$C(D) = \bigoplus_{\text{resolutions}} C(D_{\text{res}}), \tag{3.12}$$

where D_{res} is the diagram of a given resolution. This closely mimics the "state-sum model" for the Jones polynomial, due to Kauffman [14], or the MOY calculus of [15] for other quantum polynomials.

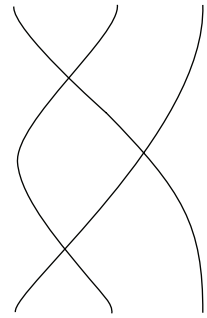


Figure 3: A braid diagram.

3.2. The Soergel Bimodule Construction

We now turn to the Soergel bimodule construction for the HOMFLY-PT homology found in [3]. Recall from Section 2.2 that the Soergel bimodule $B_i = R\bigotimes_{R^i} R\{-1\}$ where $R = \mathbb{Z}[x_1 - x_2, \ldots, x_{n-1} - x_n]$ is the ring generated by consecutive differences in variables $x_1, \ldots, x_n$ (n is the number of strands in the braid diagram), and $R^i \subset R$ is the subring of S_2-invariants corresponding to the permutation action $x_i \leftrightarrow x_{i+1}$. Furthermore define the map $B_i \to R$ by $1 \otimes 1 \mapsto 1$, and the map $R \to B_i$ by $1 \mapsto (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})$. We resolve a crossing in position $[i, i{+}1]$ in the either of two ways, as in Figure 1, assigning R to the oriented resolution and B_i to the singular resolution. For a positive crossing, we have the following complex:

$$C(D_+) : 0 \longrightarrow R\{2\} \longrightarrow B_i\{1\} \longrightarrow 0, \tag{3.13}$$

and for a negative crossing the complex

$$C(D_-) : 0 \longrightarrow B_i\{-1\} \longrightarrow R\{-2\} \longrightarrow 0. \tag{3.14}$$

We place B_i in homological grading 0 and increase/decrease by 1, that is, in the complex for D_+, $R\{2\}$ is in homological grading -1. Note, this grading convention differs from [3], and is the convention used in [8]. The complexes above are known as Rouquier complexes, due to Rouquier who studied braid group actions with relation to the category of Soergel bimodules; for more information we refer the reader to [3, 4, 8].

Given a braid diagram D we tensor the above complexes for each crossing, arriving at a total complex of length k, where k is the number of crossings of D, or equivalently the length of the corresponding braid word (Figure 3). Each entry in the complex can be thought of as a resolution of the diagram consisting of the tensor product of the appropriate Soergel bimodules. For example, to the graph in Section 3.2, we assign the bimodule $B_1 \otimes B_2 \otimes B_1$. That is, modulo grading shifts, we can view our total complex as

$$C(D) = \bigoplus_{\text{resolutions}} C(D_{\text{res}}). \tag{3.15}$$

To proceed, we take Hochschild homology $HH(C(D_{\text{res}}))$ for each resolution of D and arrive at the complex

$$HH(C(D)) = \bigoplus_{\text{resolutions}} HH(C(D_{\text{res}})), \tag{3.16}$$

with the induced differentials. Finally, taking homology of $HH(C(D))$ with respect to these differentials gives us our link homology.

Definition 3.7 (reduced homology). Given a braid diagram D of a link L we define its reduced HOMFLY-PT homology to be the group

$$H(HH(C(D))). \tag{3.17}$$

Of course, now that, we have defined reduced HOMFLY-PT homology in two different ways, it would be nice to reconcile the fact that they are indeed the same.

Claim 1. Up to grading shifts the two definitions of reduced HOMFLY-PT homology agree, that is, $H(H(\overline{C}(D), d_+), d_v^*) \cong H(HH(C(D)))$ for a diagram D of a link L.

Proof. The proof in [3] works without any changes for matrix factorizations and Soergel bimodules over $\mathbb{Z}$. We sketch it here for completeness and the fact that we will be referring to some of its details a bit later. Letus first look at the matrix factorization $C(D_s)$ (unreduced version) associated to a singular resolution D_s. Now $C(D_s)$ can be thought of as a Koszul complex of the sequence $(x_k + x_l - x_i - x_j,\ x_k x_l - x_i x_j)$ in the polynomial ring $\mathbb{Z}[x_i, x_j, x_k, x_l]$ (donot forget that in R_c multiplication by $x_k x_l - x_i x_j = -(x_k - x_i)(x_k - x_j)$). This sequence is regular so the complex has cohomology only in the right-most degree. The cohomology is the quotient ring

$$\frac{\mathbb{Z}[x_i, x_j, x_k, x_l]}{(x_i + x_j - x_k - x_l, x_k x_l - x_i x_j)}. \tag{3.18}$$

This is naturally isomorphic to the Soergel bimodule B_i' (notice that this is the "unreduced" Soergel bimodule) over the polynomial ring $R' = \mathbb{Z}[x_i, x_j]$. The left and right action of R' on B_i' corresponds to multiplication by x_i, x_j and x_k, x_l, respectively. Quotienting out by $x_k + x_l - x_i - x_j$ and $x_k x_l - x_i x_j$ agrees with the definition of B_i' as the tensor product $R' \otimes_{R_i'} R'$ over the subalgebra R_i' of symmetric polynomials in x_i, x_j.

Now letus consider a general resolution D_{res}. The matrix factorization for D_{res} is, once again, just a Koszul complex corresponding to a sequence of two types of elements. The first ones are as above, that is, they are of the form $x_k + x_l - x_i - x_j$ and $x_k x_l - x_i x_j$ and come from the singular resolutions D_s, and the remaining are of the form $x_i - x_j$ that come from "closing off" our braid diagram D, which in turn means equating the corresponding marks at the top and bottom the diagram. Now it is pretty easy to see that the polynomials of the first type, coming from the D_ss form a regular sequence and we can quotient out by them immediately, just like above. The quotient ring we get is naturally isomorphic to the Soergel bimodule $B'(D_{\text{res}})$ associated to the resolution D_{res}. At this point all, we have left is to deal with the remaining elements of the form $x_i - x_j$ coming from closing off D; to be more concrete,

the Koszul complex we started with for D_{res} is quasi-isomorphic to the Koszul complex of the ring $B'(D_{\text{res}})$ corresponding to these remaining elements. This in turn precisely computes the Hochschild homology of $B'(D_{\text{res}})$.

Finally if we downsize from B'_i to B_i and from $C(D_{\text{res}})$ to $\overline{C}(D_{\text{res}})$ we get the required isomorphism. For more details we refer the reader to [3]. □

Gradings et al.

We come to the usual rigmarole of grading conventions, which seems to be evepresent in link homology. Perhaps when using the Rouquier complexes above we could have picked conventions that more closely matched those of Section 3.1. However, we chose not to for a couple of reasons: first there would inevitably be some grading conversion to be done either way due to the inherent difference in the nature of the constructions, and second we use Rouquier complexes to aid us in just a few results (namely the proof of Reidemeister moves II and III), and leave them shortly after attaining these; hence, it is convenient for us, as well as for the reader familiar with the Soergel bimodule construction of [3] and the diagrammatic construction of [7], to adhere to the conventions of the former and the subsequent results in [8]. For completeness, we descibe the conversion rules. Recall that in the matrix factorization construction of Section 3.1 we denoted the gradings as $(i, j, k) = (q, 2gr_h, 2gr_v)$.

(i) To get the cohomological grading in the Soergel construction take $(j - k)/2$ from Section 3.1.

(ii) The Hochschild grading here matches the "horizontal" or j grading of Section 3.1.

(iii) To get the "quantum" grading i of Section 3.1 of an element x, take Hochschild grading of x minus $\deg(x)$, that is, $\deg(x) = j(x) - i(x)$.

3.2.1. Diagrammatic Rouquier Complexes

We now restate the last section in the diagrammatic language of [8] as outlined above in Section 2.2. The main advantage of doing this is the inherent ability of the graphical calculus developed by Elias and Khovanov in [7] to hide and, hence simplify, the complexity of the calculations at hand. Recall that we work in the integral version of Soergel category $\mathcal{SC}_2$ as defined in Section 2.3 of [8], which allows for constructions over $\mathbb{Z}$ without adjoining inverses (see Section 5.2 in [8] for a discussion of these facts). Recall, that an object of $\mathcal{SC}_2$ is given by a sequence of indices $\underline{i}$, visualized as d points on the real line and morhisms are given by pictures or graphs embedded in the strip $\mathbb{R} \times [0, 1]$. We think of the indices as "colors," and depict them accordingly. The Soergel bimodule B_i is represented by a vertical line of "color" i (i.e., by the identity morphism from B_i to itself) and the maps we find in the Rouquier complexes above, Section 3.2, are given by those referred to as "start dot" and "end-dot." More precisely, the complexes $C(D_-)$ and $C(D_+)$ become as illustrated in Figure 4.

For completeness and ease we remind the reader of the diagrammatic calculus rubric used to contruct Rouquier complexes for a given braid diagram.

3.2.2. Conventions

We use a colored circle to indicate the empty graph, but maintain the color for reasons of sanity. It is immediately clear that in the complex associated to a tensor product of d Rouquier

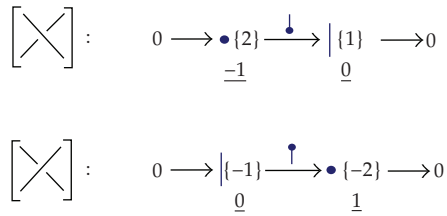


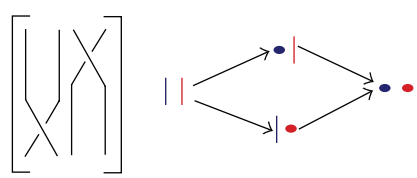
Figure 4: Diagrammatic Rouquier complex for right and left crossings.

complexes, each summand will be a sequence of k lines where $0 \leq k \leq d$ (interspersed with colored circles, but these represent the empty graph so could be ignored). Each differential from one summand to another will be a "dot" map, with an appropriate sign.

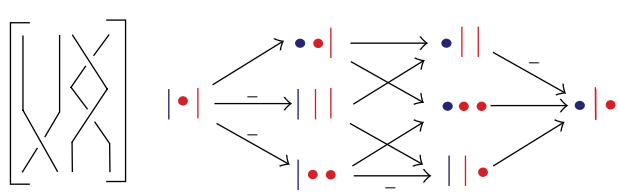
(1) The dot would be a map of degree 1 if B_i had not been shifted accordingly. In $\mathcal{SC}_2$, all maps must be homogeneous, so we could have deduced the degree shift in B_i from the degree of the differential. Because of this, it is not useful to keep track of various degree shifts of objects in a complex. Hence at times we will draw all the objects without degree shifts, and all differentials will therefore be maps of graded degree 1 (as well as homological degree 1). It follows from this that homotopies will have degree -1, in order to be degree 0 when the shifts are put back in. One could put in the degree shifts later, noting that $B_\emptyset$ always occurs as a summand in a tensor product exactly once, with degree shift 0.

(2) We will use blue for the index associated to the leftmost crossing in the braid, then red and dotted orange for other crossings, from left to right. The adjacency of these various colors is determined from the braid.

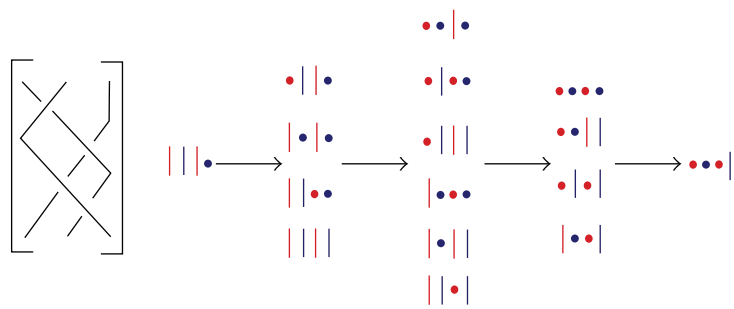
(3) We read tensor products in a braid diagram from bottom to top. That is, in the following diagram, we take the complex for the blue crossing, and tensor by the complex for the red crossing. Then we translate this into pictures by saying that tensors go from left to right. In other words, in the complex associated to this braid, blue always appears to the left of red.



(4) One can deduce the sign of a differential between two summands using the Liebnitz rule, $d(ab) = d(a)b + (-1)^{|a|}ad(b)$. In particular, since a line always occurs in the basic complex in homological dimension ± 1, the sign on a particular differential is exactly given by the parity of lines appearing to the left of the map. For example,



(5) When putting an order on the summands in the tensored complex, we use the following standardized order. Draw the picture for the object of smallest homological degree, which we draw with lines and circles. In the next homological degree, the first summand has the first color switched (from line to circle, or circle to line), the second has the second color switched, and so forth. In the next homological degree, two colors will be switched, and we use the lexicographic order: 1st and 2nd, then 1st and 3rd, then 1st and 4th,... then 2nd and 3rd, and so forth. This pattern continues.



4. Checking the Reidemeister Moves

We will use the matrix factorization construction of Section 3.1 to check Reidemeister move I, as it is not very difficult to verify even over $\mathbb{Z}$ that this goes through, and the diagrammatic calculus of Section 3.2.1 for the remaining moves. There are two main reasons for the interplay: first, checking Reidemeister II and III over $\mathbb{Z}$ using the matix factorization approach is rather computationally intensive (it was already quite so over $\mathbb{Q}$ in [11] with all the algebraic advantages of working over a field at hand); second, at this moment there does not exist a full diagrammatic description of Hochschild homology of Soergel bimodules, which prevents us from using a pictorial calculus to compute link homology from closed braid diagrams. Of course, for Reidemeister II and III we could have used the computations of [8], where we prove the stronger result that Rouquier complexes are functorial over braid cobordisms. Instead, the proofs we exhibit below use essentially the same strategy as the original paper [11], but are so much simpler and more concise that they underline well the usefulness of the diagrammatic calculus for computations.

A small lemma from linear algebra, which Bar-Natan refers to as "Gaussian Elimination for Complexes" in [16], will be very helpful to us.

Lemma 4.1. *If $\phi : B \to D$ is an isomorphism (in some additive category $\mathcal{C}$), then the four-term complex segment below*

$$\cdots [A] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}B\\ C\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix}D\\ E\end{bmatrix} \xrightarrow{(\mu \ \ \nu)} [F] \cdots \tag{4.1}$$

is isomorphic to the (direct sum) complex segment

$$\cdots [A] \xrightarrow{\begin{pmatrix} 0 \\ \beta \end{pmatrix}} \begin{bmatrix} B \\ C \end{bmatrix} \xrightarrow{\begin{pmatrix} \phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{bmatrix} D \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix} 0 & \nu \end{pmatrix}} [F] \cdots . \tag{4.2}$$

Both of these complexes are homotopy equivalent to the (simpler) complex segment

$$\cdots [A] \xrightarrow{(\beta)} [C] \xrightarrow{(\epsilon-\gamma\phi^{-1}\delta)} [E] \xrightarrow{(\nu)} [F] \cdots . \tag{4.3}$$

Here the capital letters are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters are arbitrary matrices representing morphisms (all the matrices are block matrices); $\phi : B \to D$ is an isomorphism, that is, it is invertible.

4.1. Reidemeister I

Proof. The complex $C(D_{I_a})$ for the left-hand side braid in Reidemester Ia, see Figure 5, has the form

$$\begin{tikzcd} \mathbb{Z}[x_1, x_2]\{0, -2, 0\} \arrow[r, "0"] & \mathbb{Z}[x_1, x_2]\{0, 0, 0\} \\ \mathbb{Z}[x_1, x_2]\{2, -2, -2\} \arrow[u, "(x_2-x_1)"] \arrow[r, "0"] & \mathbb{Z}[x_1, x_2]\{0, 0, -2\} \arrow[u, "1"'] \end{tikzcd} \tag{4.4}$$

Up to homotopy, the right-hand side of the complex dissapears and only the top left corner survives after quotienting out by the relation $x_2 - x_1$. Note that the overall degree shifts of the total complex make sure that the left-over entry sits in the correct trigrading.

Similarly, the complex $C(D_{I_b})$ for the left-hand side braid in Reidemester Ib, has the form

$$\begin{tikzcd} \mathbb{Z}[x_1, x_2]\{0, -2, 2\} \arrow[r, "0"] & \mathbb{Z}[x_1, x_2]\{-2, 0, 2\} \\ \mathbb{Z}[x_1, x_2]\{0, -2, 0\} \arrow[u, "1"] \arrow[r, "0"] & \mathbb{Z}[x_1, x_2]\{0, 0, 0\} \arrow[u, "(x_2-x_1)"'] \end{tikzcd} \tag{4.5}$$

The left-hand side is annihilated and the upper-right corner remains modulo the relation $x_2 - x_1$. □

4.2. Reidemeister II

Proof. Consider the braid diagrams for Reidemeister type IIa in Figure 5. Recall the decomposition $B_i \otimes B_i \cong B_i\{-1\} \oplus B_i\{1\}$ in $\mathcal{SC}_2$ and its pictorial counterpart 6. The complex we are interested in is, as illustrated in Figure 6.

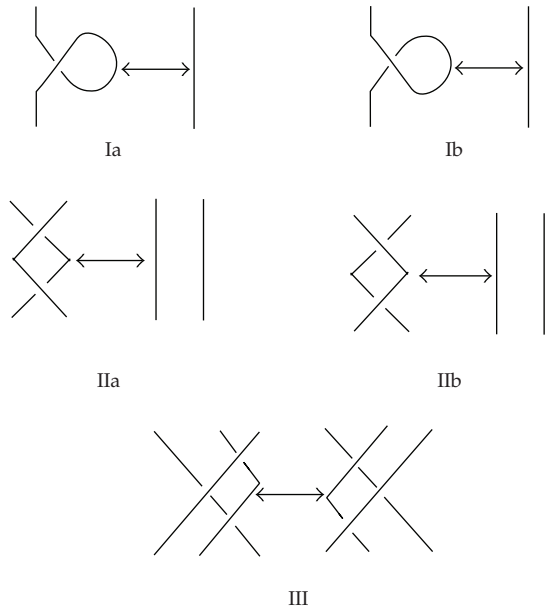


Figure 5: The Reidemeister moves.

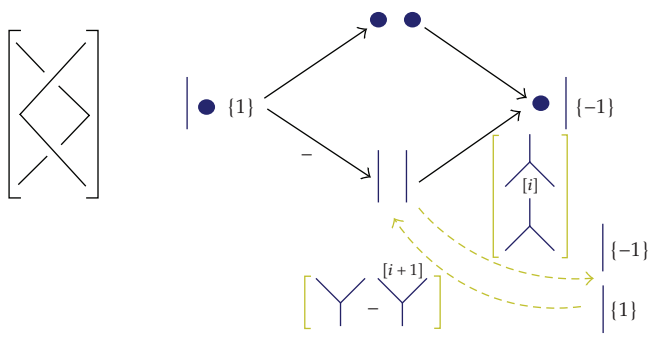


Figure 6: Reidemeister IIa complex with decomposition 6.

Inserting the decomposed $B_i \otimes B_i$ and the corresponding maps, we find two isomorphisms staring at us; we pick the left most one and mark it for removal, (see Figure 7).

After changing basis and removing the acyclic complex, as in Lemma 4.1, we arrive at the complex below with two more entries marked for removal, (see Figure 8.)

With the marked acyclic subcomplex removed, we arrive at our desired result, the complex assigned to the no crossing braid of two strands as in Figure 5. The computation for Reidemeister IIb is virtually identical. □

4.3. Reidemeister III

Proof. Luckily, we only have to check one version of Reidemeister move III, but as the reader will see below even that is pretty easy and not much harder than that of Reidemeister II above. We follow closely the structure of the proof in [11], utilizing the bimodule $R \bigotimes_{R^{i,i+1}} R\{-3\}$ and

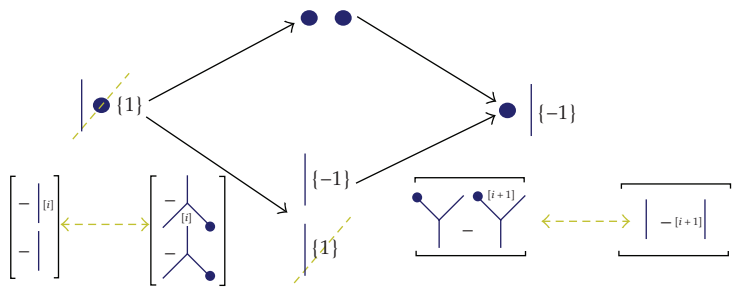


Figure 7: Reidemeister IIa complex, removing one of the acyclic subcomplexes.

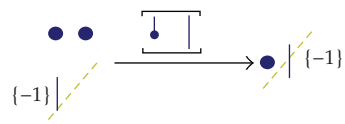


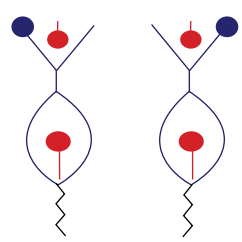
Figure 8: Reidemeister IIa complex, removing a second acyclic subcomplex.

decomposition 7 to reduce the complex for one of the RIII braids to that which is invariant under the move or, equivalently in our case, invariant under color flip. We start with the braid on the left-hand side of III in Figure 5; the corresponding complex, with decomposition 6 and 7 given by dashed/yellow arrows, is, (as illustrated in Figure 9).

We insert the decomposed bimodules and the appropriate maps; then we change bases as in Lemma 4.1 (the higher matrix of the two is before base-change, and the lower is after), (see Figure 10.)

We strike out the acyclic subcomplex and mark another one for removal; yet again we change bases (the lower matrix is the one after base change), (see Figure 11.)

Now we are almost done; if we can prove that the maps



are invariant under color change, we would arrive at a complex that is invariant under Reidemeister move III. To do this we must stop for a second, go back to the source and examine the original, algebraic, definitions of the morphisms in [7]; upon doing so we are relieved to see that the maps we are interested in are actually equal to zero (they are defined by sending $1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 \mapsto 0$). In all, we have arrived, (see Figure 12).

Repeating the calculation for the braid on the right-hand side of RIII, Figure 5, amounts to the above calculation with the colors switched—a quick glance will convince the reader that the end result is the same complex rotated about the x-axis. □

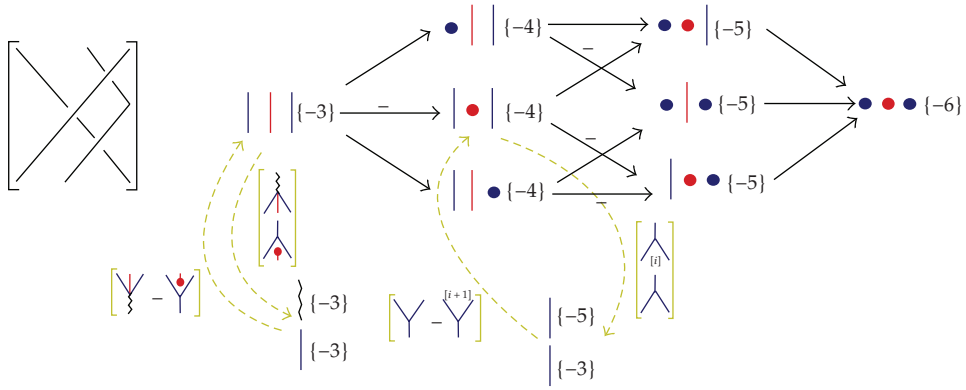


Figure 9: Reidemeister III complex with decompositions 6 and 7.

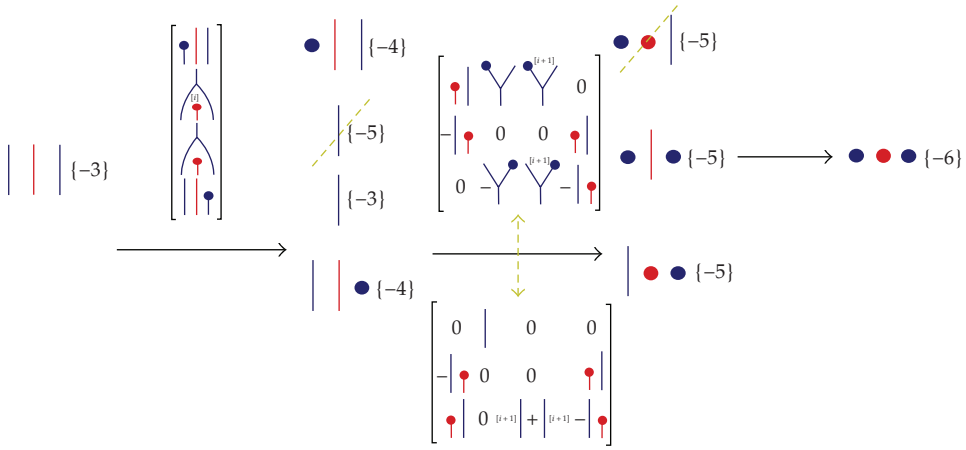


Figure 10: Reidemeister III complex, with an acyclic subcomplex marked for removal.

4.4. Observations

Having seen this interplay between the different constructions, perhaps it is a good moment to highlight exactly what categories we do need to work in so as to arrive at a genuine link invariant, or a braid invariant at that. Let us start with the latter: we can take the category of complexes of Soergel bimodules $\mathcal{KOM}(\mathcal{SC})$ (either the diagrammatic or "original" version) and construct Rouquier complexes; if we mod out by homotopies and work in $\mathcal{KOM}_h(\mathcal{SC})$, we arrive at something that is not only an invariant of braids but of braid cobordisms as well (over $\mathbb{Z}$ or $\mathbb{Q}$ if we wish). Now if we repeat the construction in the category of complexes of graded matrix factorizations $\mathcal{KOM}(mf)$, we have some choices of homotopies to quotient out by. First, we can quotient out by the homotopies in the category of graded matrix factorizations and work in $\mathcal{KOM}(hmf)$ and second, we can quotient in the category of the complexes and work in $\mathcal{KOM}_h(mf)$, or we can do both and work in $\mathcal{KOM}_h(hmf)$. It is immediate that working in $\mathcal{KOM}_h(mf)$ is necessary, but one could hope that it is also sufficient. A close look at the argument of Claim 1, where the two constructions are proven equivalent, shows that if we start with the Koszul complex associated to the resolution of

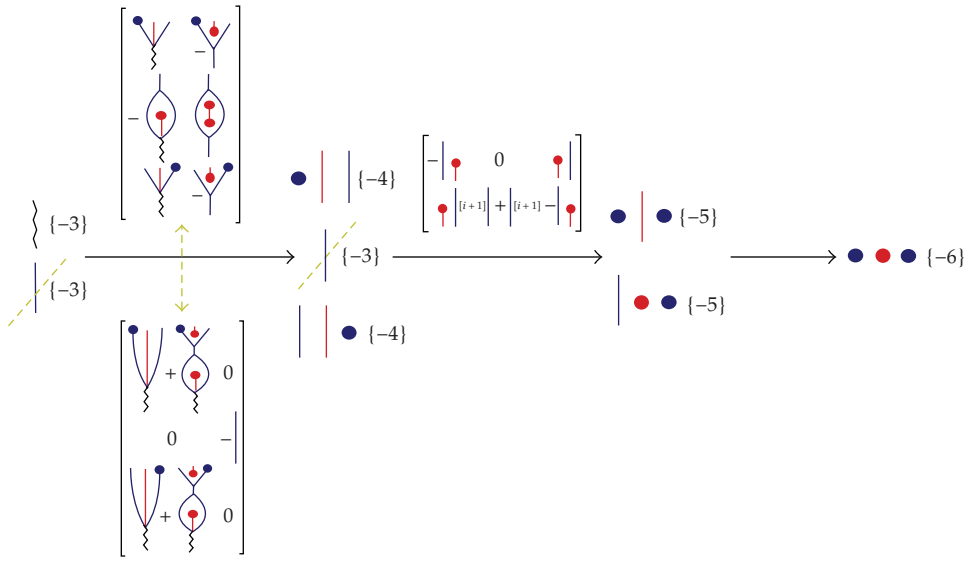


Figure 11: Reidemeister III complex, with another acyclic subcomplex marked for removal.

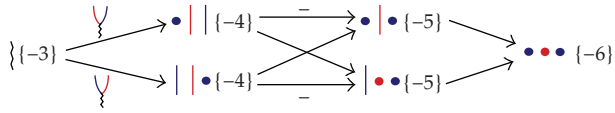


Figure 12: Reidemeister III complex—the end result, after removal of all acyclic subcomplexes.

a braid D_{res} the polynomial relations coming from the singular vertices in D_{res} form a regular sequence and, hence, the homology of this complex is the quotient of the edge ring $R(D_{\text{res}})$ by these relations and is supported in the right-most degree. It is this quotient that is isomorphic to the corresponding Soergel bimodule, that is, the Koszul complex is quasi-isomorphic, as a bimodule, to $B'(D_{\text{res}})$. Hence, we really do need to work in $\mathcal{KOM}_h(hmf)$, to have a braid invariant or an invariant of braid cobordisms, or a link invariant.

Anyone, who has suffered throught the proofs of, say, Reidemeister III in [11] would probably find the above a relief. Of course, much of the ease in computation using this pictorial language is founded upon the intimate understanding and knowledge of hom spaces between objects in $\mathcal{SC}$, which is something that is only available to us due to the labors of Elias and Khovanov in [7]. With that said, it would not be suprising if this diagrammatic calculus would aid other calculations of link homology in the future.

All in all, we have arrived at an integral version of HOMFLY-PT link homology; combining with the results of [8], we have the following.

Theorem 4.2. *Given a link $L \subset S^3$, the groups $H(L)$ and $\overline{H}(L)$ are invariants of L and when tensored with $\mathbb{Q}$ are isomorphic to the unreduced and reduced versions, respectively, of the Khovanov-Rozansky HOMFLY-PT link homology. Moreover, these integral homology theories give rise to functors from the category of braid cobordisms to the category of complexes of graded R-bimodules.*

5. Rasmussen's Spectral Sequence and Integral sl(n)-Link Homology

It is time for us to look more closely at Rasmussen's spectral sequence from HOMFLY-PT to sl(n)-link homology. For this we need an extra "horizontal" differential d_- in our complex, and here is the first time we encounter matrix factorizations with a nonzero potential; as before, to a link diagram D we will associate the tensor product of complexes of matrix factorizations with potential for each crossing. These will be complexes over the ring

$$R_c = \frac{\mathbb{Z}[x_i, x_j, x_k, x_l]}{(x_k + x_l - x_i - x_j)} \cong \mathbb{Z}[x_i, x_j, x_k], \tag{5.1}$$

with total potential

$$\begin{aligned} W_p[x_i, x_j, x_k, x_l] &= p(x_k) + p(x_l) - p(x_i) - p(x_j), \\ &= p(x_k) + p(x_i + x_j - x_k) - p(x_i) - p(x_j), \end{aligned} \tag{5.2}$$

where the $p(x) \in \mathbb{Z}[x]$. We do not specify the potential $p(x)$ at the moment as the spectral sequence works for any choice; later on when looking at sl(n)-link homology we will set $p(x) = x^{n+1}$.

To define d_-, let $p_i = W_p/(x_k - x_i)$ and $p_{ij} = -W_p/(x_k - x_i)(x_k - x_j)$ (recall that in R_c, $(x_k - x_i)(x_k - x_j) = x_i x_j - x_k x_l$, and note that substituting either $x_k = x_i$ or $x_k = x_j$ into W_p makes it vanish, so p_{ij} is indeed a polynomial in R_c).

To the positive crossing D_+, we assign the following complex:

$$\begin{tikzcd} R_c\{0,-2,0\} \arrow[r, shift left, "(x_k-x_i)"] & R_c\{0,0,0\} \arrow[l, shift left, "p_i"] \\ R_c\{2,-2,-2\} \arrow[u, "(x_j-x_k)"] \arrow[r, shift left, "-(x_k-x_i)(x_k-x_j)"] & R_c\{0,0,-2\} \arrow[u, "1"'] \arrow[l, shift left, "p_{ij}"] \end{tikzcd} \tag{5.3}$$

To the negative crossing D_-, we assign the following complex:

$$\begin{tikzcd} R_c\{0,-2,2\} \arrow[r, shift left, "-(x_k-x_i)(x_k-x_j)"] & R_c\{-2,0,2\} \arrow[l, shift left, "p_{ij}"] \\ R_c\{0,-2,0\} \arrow[u, "1"] \arrow[r, shift left, "(x_k-x_i)"] & R_c\{0,0,0\} \arrow[u, "(x_j-x_k)"'] \arrow[l, shift left, "p_i"] \end{tikzcd} \tag{5.4}$$

The total complex for a link L with diagram D will be defined analagously to the one above, that is,

$$C_p(D) := \bigotimes_{\text{crossings}} \left(C(D_c) \bigotimes_{R_c} R(D) \right), \tag{5.5}$$

as will be the reduced $\overline{H}_p(L, i)$ and unreduced $H_p(L)$ versions of link homology.

The main result of [2] is the following.

Theorem 5.1 (Rasmussen [2]). *Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p(x) \in \mathbb{Q}[x]$, there is a spectral sequence $E_k(p)$ with $E_1(p) \cong \overline{H}(L)$ and $E_\infty(p) \cong \overline{H}_p(L,i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L,i).*

In particular setting $p(x) = x^{n+1}$ one would arrive at a spectral sequence from the HOMFLY-PT to the sl(n)-link homology. Rasmussen's result pertains to rational link homology with matrix factorizations defined over the ring $\mathbb{Q}[x_1,\ldots,x_n]$ and potentials polynomials in $\mathbb{Q}[x]$. We will essentially repeat his construction in our setting and, for the benefit of those familiar with the results of [2], will stay as close as possible to the notation and conventions therein. This will be a rather condensed version of the story and we refer the reader to the original paper for more details.

We will work primarily with reduced link homology (although all the results follow through for both versions) and with closed link diagrams, where all three differentials d_v, d_+, and d_- anticommute. We have some choices as to the order of running the differentials, so let us define

$$\overline{H}^+(D,i) = H\left(\overline{C}(D,i), d_+\right). \tag{5.6}$$

Here, $\overline{H}^+(D,i)$ inherits a pair of anticommuting differentials d_-^* and d_v^*, where d_-^* lowers gr_h by 1 while preserving gr_v and d_v^* raises gr_v by 1 while preserving gr_h. Hence, $(\overline{H}_p^+(D,i), d_v^*, d_-^*)$ defines a double complex with total differential $d_{v-} := d_v^* + d_-^*$.

Definition 5.2. Let $E_k(p)$ be the spectral sequence induced by the horizontal filtration on the complex $(\overline{H}_p^+(D,i), d_{v-})$.

After shifting the triple grading of $E_k(p)$ by $\{-w+b-1, w+b-1, w-b+1\}$ it is immediate that the first page of the spectral sequence is isomorphic to $\overline{H}(L,i)$ (the part of the differential $d_v^* + d_-^*$ which preserves horizontal grading on $E_0(p) = \overline{H}^+(D,i)\{-w+b-1, w+b-1, w-b+1\}$ is precisely d_v^*, that is, $d_0(p) = d_v^*$ and

$$E_1(p) = H\left(\overline{H}^+(D,i), d_v^*\right)\{-w+b-1, w+b-1, w-b+1\} \cong \overline{H}(L,i), \tag{5.7}$$

where D is a diagram for L. It also follows that $d_k(p)$ is homogenous of degree $-k$ with respect to gr_h and degree $1-k$ with respect to gr_v, and in the case that $p(x) = x^{n+1}$ it is also homogeneous of degree $2nk$ with respect to the q-grading.

Claim 2. Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p(x) \in \mathbb{Z}[x]$, the spectral sequence $E_k(p)$ has $E_1(p) \cong \overline{H}(L,i)$ and $E_\infty(p) \cong \overline{H}_p(L,i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L,i).

Proof. We argue as in [2, Section 5.4]. Suppose that, we have two closed diagrams D_j and D_j' that are related by the jth Reidemeister move, and suppose that there is a morphism

$$\sigma_j : \overline{H}_p^+(D_j,i) \longrightarrow \overline{H}_p^+\left(D_j',i\right) \tag{5.8}$$

in the category $\mathcal{KOM}(mf)$ that extends to a homotopy equivalence in the category of modules over the edge ring R. Then σ_j induces a morphism of spectral sequences $(\sigma_j)_k : E_k(D_j, i, p) \to E_k(D'_j, i, p)$ which is an isomorphism for $k > 0$. See [2] for more details and discussion. Hence, in practice, we have to exhibit morphisms and prove invariance for the first page of the spectral sequence, that is, for the HOMLFY-PT homology, which is basically already done. However, we ought to be a bit careful, of course, as here we are working with $\overline{H}_p^+(D, i)$ and not with the complex $\overline{C}(D, i)$ defined in Section 4.

Reidemeister I is done, as in this case $d_+ = 0$ and, hence, the complex $\overline{H}_p^+(D, i) = \overline{C}_p(D, i)$ and the same argument as the one in Section 4.1 works here.

For Reidemesiter II and III, we have to observe that for a closed diagram, we have morphisms $\sigma_j : \overline{C}_p(D_j, i) \to \overline{C}_p(D'_j, i)$ for $j = \mathrm{II}, \mathrm{III}$, which are homotopy equivalences of complexes (these can be extrapolated from Section 4 above, or from [8], where all chain maps are exhibited concretely). Therefore we get induced maps $(\sigma_j)_k$ on the spectral sequence with the property that $(\sigma_j)_1 = \sigma_{j*}$ is an isomorphism.

To get the last part of the claim, that is, that the reduced homology depends only on the link component and not on the edge therein we refer the reader to [2], as the arguments from there are valid verbatum. □

Setting $p(x) = x^{n+1}$, we get that the differentials $d_k(p)$ preserve $q + 2ngr_h$ and, hence, the graded Euler characteristic of $H(\overline{H}_p^+(D, i), d_{v-})$ with respect to this quantity is the same as that of $E_1(x^{n+1})$. Tensoring with $\mathbb{Q}$, to get rid of torsion elements, and computing we see that the Euler characteristic of the $E_\infty(x^{n+1})$ is the quantum $\mathrm{sl}(n)$-link polynomial $P_L(q^n, q)$ of L. See [2, Section 5.1] for details. We have arrived at the following.

Theorem 5.3. *The $E_\infty(x^{n+1})$ of the spectral sequence defined in 11 is an invariant of L and categorifies the quantum* $\mathrm{sl}(n)$*-link polynomial $P_L(q^n, q)$.*

Remark 5.4. Well, we have a categorification over $\mathbb{Z}$ of the quantum $\mathrm{sl}(n)$-link polynomial, but what homology theory exactly are we dealing with? Is it isomorphic to $H(H(H(\overline{C}_{x^{n+1}}(D, i), d_+), d_-^*), d_v^*)$ or to $H(H(\overline{C}_{x^{n+1}}(D, i), d_+ + d_-), d_v^*)$ and are these two isomorphic here? The answer is not immediate. In [2], Rasmussen bases the corresponding results on a lemma that utilizes the Kunneth formula, which is much more manageable in this context when looked at over $\mathbb{Q}$. Of course, for certain classes of knots things are easier. For example, if we take the class of knots that are *KR-thin*, then the spectral sequence converges at the E_1 term, as this statement only depends on the degrees of the differentials, and we have that $E_\infty(x^{n+1}) \cong H(H(\overline{C}_{x^{n+1}}(D, i), d_+), d_v^*)$. However, that is a bit of a "red herring" as stated.

Acknowledgments

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Research Article

The Khovanov-Lauda 2-Category and Categorifications of a Level Two Quantum SL(N) Representation

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We construct 2-functors from a 2-category categorifying quantum sl(n) to 2-categories categorifying the irreducible representation of highest weight $2\omega_k$.

1. Introduction

Khovanov and Lauda introduced a 2-category whose Grothendieck group is $\mathcal{U}_q(\mathfrak{sl}_n)$ [1]. This work generalizes earlier work by Lauda for the $\mathcal{U}_q(\mathfrak{sl}_2)$ case [2]. Rouquier has independently produced a 2-category with similar generators and relations [3]. There have been several examples of categorifications of representations of $\mathcal{U}_q(\mathfrak{sl}_n)$ arising in various contexts. Khovanov and Lauda conjectured that their 2-category acts on various known categorifications via a 2-functor. For example, in their work they construct such a 2-functor to a category of graded modules over the cohomology of partial flag varieties. This 2-category categorifies the irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_n)$ of highest weight $n\omega_1$ where ω_1 is the first fundamental weight.

In this paper we construct this action for the categorification constructed by Huerfano and Khovanov in [4]. They categorify the irreducible representation $V_{2\omega_k}$ of highest weight $2\omega_k$, by a modification of a diagram algebra introduced in [5]. The objects of 2-category $\mathcal{HK}_{k,n}$ are categories $\mathcal{C}_\lambda$ which are module categories over the modified Khovanov algebra. We explicitly construct natural transformations between the functors in [4] and show that they satisfy the relations in the Khovanov-Lauda 2-category giving the following theorem.

Theorem 1.1. *Over a field of characteristic two, there exists a 2-functor* $\Omega_{k,n} : \mathcal{KL} \to \mathcal{HK}_{k,n}$.

The Huerfano-Khovanov categorification is based on categories used for the categorification of $\mathcal{U}_q(\mathfrak{sl}_2)$-tangle invariants. This hints that a categorification of $V_{2\omega_k}$ may also be obtained on maximal parabolic subcategories of certain blocks of category $\mathcal{O}(\mathfrak{gl}_{2k})$. More specifically, we construct a 2-category $\mathcal{P}_{k,n}$ whose objects are full subcategories ${}_{\mathbb{Z}}\mathcal{P}_{\mu}^{(k,k)}(\mathfrak{gl}_{2k})$ of graded category ${}_{\mathbb{Z}}\mathcal{O}_{\mu}^{(k,k)}(\mathfrak{gl}_{2k})$ whose set of objects are those modules which have projective presentations by projective-injective objects. The 1-morphisms of $\mathcal{P}_{k,n}$ are certain projective functors. We explicitly construct the 2-morphisms as natural transformations between the projective functors by the Soergel functor $\mathbb{V}$. We then prove the following.

Theorem 1.2. *There is a 2-functor* $\Pi_{k,n} : \mathcal{KL} \to \mathcal{P}_{k,n}$.

It should be possible to categorify $V_{N\omega_k}$ for $N \geq 1$ using categories which appear in various knot homologies. For $N \geq 2$, the module categories $\mathcal{C}_\lambda$ in the Huerfano-Khovanov construction should be replaced by suitable categories of matrix factorization based on Khovanov-Rozansky link homology. The categories of matrix factorizations must be generalized from those used in [6]. Khovanov and Rozansky suggest that the categories of matrix factorizations should be taken over tensor products of polynomial rings invariant under the symmetric group. These categories were studied in depth by Yonezawa and Wu [7, 8]. In fact, the isomorphisms of functors categorifying the $\mathcal{U}_q(\mathfrak{sl}_n)$ relations were defined implicitly in [8]. To check that there is a 2-representation of the Khovanov-Lauda 2-category, these isomorphisms would need to be made more explicit. The category $\mathcal{O}$ approach should be modified as well. Now the objects of the 2-category should be subcategories of parabolic subcategories corresponding to the composition $Nk = k + \cdots + k$ of blocks of $\mathcal{O}_\lambda(\mathfrak{gl}(Nk))$, and the stabilizer of the dominant integral weight μ is taken to be $\mathbb{S}_{\lambda_1} \times \cdots \times \mathbb{S}_{\lambda_n}$ where each $\lambda_i \in \{0, 1, \ldots, N\}$; compare, for example, Section 5 below. Note that a categorification of V_λ for arbitrary dominant integral λ, hence in particular of $V_{N\omega_k}$, is constructed in [9] using cyclotomic quotients of Khovanov-Lauda-Rouquier algebras.

While this paper was in preparation, two very relevant papers appeared. In [10], Brundan and Stroppel also defined the appropriate natural transformations and checked relations between them to establish a version of the first theorem above, but for Rouquier's 2-category from [3] rather than the Khovanov-Lauda 2-category. One of the advantages of their result is that they are able to work over an arbitrary field, while we work over a field of characteristic 2 in constructing the 2-functor to $\mathcal{HK}_{k,n}$. It is not immediately clear to us how to use their sign conventions to get an action of the full Khovanov-Lauda 2-category in characteristic zero, because they seem to lead to inconsistencies between Propositions 4.7, 4.8, 4.10, and 4.16. Additionally, Brundan and Stroppel categorify $V_{2\omega_k}$ using graded category $\mathcal{O}$. More precisely, they first categorify the classical limit of $V_{2\omega_k}$ at $q = 1$ using a certain parabolic category $\mathcal{O}$, without mentioning gradings. Then they establish an equivalence between this category and the (ungraded) diagrammatic category. Finally, they observe that both categories are Koszul (by [11] and [12], respectively) so, exploiting unicity of Koszul gradings, their categorification at $q = 1$ can be lifted to a categorification of the module $V_{2\omega_k}$ itself in terms of graded category $\mathcal{O}$. Our construction on the graded category $\mathcal{O}$ side is more explicit, relying heavily on the Soergel functor, the Koszul grading that $\mathcal{O}$ inherits from geometry, and explicit calculations on the cohomology of flag varieties made in [1]. In the other relevant paper, M. Mackaay [13] constructs an action of the Khovanov-Lauda 2-category on a category of foams which is the basis of an $\mathfrak{sl}_3$-knot homology.

2. The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_n)$

2.1. Root Data

Let $\mathfrak{sl}_n = \mathfrak{sl}_n(\mathbb{C})$ denote the Lie algebra of traceless $n \times n$-matrices with standard triangular decomposition $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of type A_{n-1} with simple system $\Pi = \{\alpha_i \mid i = 1, \ldots, n-1\}$. Let $(\cdot, \cdot)$ denote the symmetric bilinear form on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \alpha_j) = a_{ij}, \tag{2.1}$$

where $A = (a_{ij})_{1 \le i,j < n}$ is the Cartan matrix of type A_{n-1}:

$$a_{ij} = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } |j - i| = 1, \\ 0 & \text{if } |i - j| > 1. \end{cases} \tag{2.2}$$

Let Δ^+ be the set of simple roots relative to Π. Let $\omega_1, \ldots, \omega_{n-1} \in \mathfrak{h}^*$ be the elements satisfying $(\omega_i, \alpha_j) = \delta_{ij}$, and let

$$Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i, \qquad Q^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\ge 0}\alpha_i, \qquad P = \bigoplus_{i=1}^{n-1} \mathbb{Z}\omega_i, \qquad P^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\ge 0}\omega_i \tag{2.3}$$

denote the root lattice, positive root lattice, weight lattice, and dominant weight lattice, respectively.

Set $I = \{1, \ldots, n-1, -1, \ldots, -n+1\}$, $I^+ = \{1, \ldots, n-1\}$, and $I^- = -I^+$. Define $\alpha_{-i} = -\alpha_i$, and extend the definition of a_{ij} to all $i, j \in I$ accordingly. Finally, for $i \in I$, let $\operatorname{sgn}(i) = i/|i|$ be the sign of i.

The quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$ is the associative algebra over $\mathbb{Q}(q)$ with generators E_i, K_i, for $i \in I$, satisfying the following conditions:

(1) $K_i K_{-i} = K_{-i} K_i = 1$, and $K_i K_j = K_j K_i$ for $i, j \in I$,

(2) $K_i E_j = q^{a_{i,j}} E_j K_i$, $i, j \in I$,

(3) $E_i E_{-j} - E_{-j} E_i = \delta_{i,j}((K_i - K_{-i})/(q - q^{-1}))$, $i, j \in I^{\pm}$,

(4) $E_i E_j = E_j E_i$, $i, j \in I^{\pm}$, $|i - j| > 1$,

(5) $E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0$, $i, j \in I^{\pm}$, $|i - j| = 1$.

We fix a comultiplication $\Delta : \mathcal{U}_q(\mathfrak{sl}_n) \to \mathcal{U}_q(\mathfrak{sl}_n) \otimes \mathcal{U}_q(\mathfrak{sl}_n)$ given as follows for all $i \in I^+$:

$$\begin{aligned} \Delta(E_i) &= 1 \otimes E_i + E_i \otimes K_i, \\ \Delta(E_{-i}) &= K_{-i} \otimes E_{-i} + E_{-i} \otimes 1, \\ \Delta(K_{\pm i}) &= K_{\pm i} \otimes K_{\pm i}. \end{aligned} \tag{2.4}$$

Via Δ, a tensor product of $\mathcal{U}_q(\mathfrak{sl}_n)$-modules becomes a $\mathcal{U}_q(\mathfrak{sl}_n)$-module.

In this paper we are interested in the irreducible $\mathcal{U}_q(\mathfrak{sl}_n)$-modules, $V_{2\omega_k}$ with highest weight $2\omega_k$. Therefore, we will identify the weight lattice $P \cong \mathbb{Z}^{n-1} \subset \mathbb{Z}^n$ as follows. Assume that $\lambda = \sum_i a_i\omega_i$. For each $1 \le i < n$, set

$$\lambda_i = \frac{2k - a_1 - 2a_2 - \cdots - (i-1)a_{i-1} + (n-i)a_i + (n-i-1)a_{i+1} + \cdots + a_{n-1}}{n}. \tag{2.5}$$

Let $P(2\omega_k)$ denote the set of weights of $V_{2\omega_k}$. It is well known that under this identification each $\lambda \in P(2\omega_k)$ satisfies $\lambda_i \in \{0,1,2\}$ for all $1 \le i \le n$ and $\lambda_1 + \cdots + \lambda_n = 2k$.

3. The Khovanov-Lauda 2-Category

Let $\Bbbk$ be a field. The $\Bbbk$-linear 2-category $\mathcal{KL}$ defined here was originally constructed in [1].

Let $I_\infty = \bigcup_{n\ge 0} I^n$, $I_\infty^+ = \bigcup_{n\ge 0}(I^+)^n$ where I^n and $(I^+)^n$ denote n-fold Cartesian products. Given that $\underline{i} = (i_1, i_2, \ldots) \in I_\infty$, let

$$\operatorname{cont}(\underline{i}) = \sum_{i=1}^{n-1} c_i\alpha_i, \quad \text{where } c_i = \#\{j \mid i_j = i\} - \#\{j \mid i_j = -i\}. \tag{3.1}$$

Given that $\nu \in Q$, let $\operatorname{Seq}(\nu) = \{\underline{i} \in I_\infty \mid \operatorname{cont}(\underline{i}) = \nu\}$ and, for $\nu \in Q^+$, define $\operatorname{Seq}^+(\nu) = \{\underline{i} \in I_\infty^+ | \operatorname{cont}(\underline{i}) = \nu\}$. Finally, define

$$\operatorname{Seq} = \bigcup_{\nu\in Q} \operatorname{Seq}(\nu). \tag{3.2}$$

3.1. The Objects

The set of objects for this 2-category is the weight lattice, P.

3.2. The 1-Morphisms

For each $\lambda \in P$, let $\mathcal{I}_\lambda \in \operatorname{End}_{\mathcal{KL}}(\lambda)$ be the identity morphism and, for $\lambda, \lambda' \in P$, set $\mathcal{I}_\lambda\mathcal{I}_{\lambda'} = \delta_{\lambda,\lambda'}\mathcal{I}_\lambda$. For each $i \in I$, we define morphisms $\mathcal{E}_i\mathcal{I}_\lambda \in \operatorname{Hom}_{\mathcal{KL}}(\lambda, \lambda + \alpha_i)$. Evidently, we have $\mathcal{E}_i\mathcal{I}_\lambda = \mathcal{I}_{\lambda+\alpha_i}\mathcal{E}_i\mathcal{I}_\lambda$. For $\lambda, \lambda' \in P$, we have

$$\operatorname{Hom}_{\mathcal{KL}}(\lambda, \lambda') = \bigoplus_{\substack{\underline{i}\in\operatorname{Seq} \\ s\in\mathbb{Z}}} \mathcal{I}_{\lambda'}\mathcal{E}_{\underline{i}}\mathcal{I}_\lambda\{s\}, \tag{3.3}$$

where $\mathcal{E}_{\underline{i}} := \mathcal{E}_{i_1}\cdots\mathcal{E}_{i_r}$ if $\underline{i} = (i_1, \ldots, i_r) \in I_\infty$, and s refers to a *grading shift*. Observe that $\mathcal{I}_{\lambda'}\mathcal{E}_{\underline{i}}\mathcal{I}_\lambda = 0$ unless $\operatorname{cont}(\underline{i}) = \lambda' - \lambda$, and $\mathcal{I}_{\lambda+\operatorname{cont}(\underline{i})}\mathcal{E}_{\underline{i}}\mathcal{I}_\lambda = \mathcal{E}_{\underline{i}}\mathcal{I}_\lambda$.

3.3. The 2-Morphisms

The 2-morphisms are generated by

$$\begin{aligned} &Y_{i;\lambda} \in \mathrm{End}_{\mathcal{KL}}(\mathcal{E}_i\mathcal{O}_\lambda), \qquad \Psi_{i,j;\lambda} \in \mathrm{Hom}_{\mathcal{KL}}(\mathcal{E}_i\mathcal{E}_j\mathcal{O}_\lambda, \mathcal{E}_j\mathcal{E}_i\mathcal{O}_\lambda), \\ &\bigcup_{i;\lambda} \in \mathrm{Hom}_{\mathcal{KL}}(\mathcal{O}_\lambda, \mathcal{E}_{-i}\mathcal{E}_i\mathcal{O}_\lambda), \qquad \bigcap_{i;\lambda} \in Hom_{\mathcal{KL}}(\mathcal{E}_{-i}\mathcal{E}_i\mathcal{O}_\lambda, \mathcal{O}_\lambda), \end{aligned} \tag{3.4}$$

for $i, j \in I^{\pm}$. We define $\mathbf{1}_{i;\lambda} \in \mathrm{End}_{\mathcal{KL}}(\mathcal{E}_i\mathcal{O}_\lambda)$ to be the identity transformation.

For $\lambda \in P$, the degrees of the basic 2-morphisms are given by

$$\deg Y_{i;\lambda} = a_{ii}, \qquad \deg \Psi_{i,j;\lambda} = -a_{ij}, \qquad \deg \bigcup_{i;\lambda} = \deg \bigcap_{i;\lambda} = 1 + (\alpha_i, \lambda). \tag{3.5}$$

Let $\lambda+\mathrm{cont}(\underline{i}) = \lambda+\mathrm{cont}(\underline{j}) = \lambda+\mathrm{cont}(\underline{k}) = \lambda'$ and $\lambda'+\mathrm{cont}(\underline{i}') = \lambda+\mathrm{cont}(\underline{j}') = \lambda''$. Let $\Theta_1 \in \mathrm{Hom}_{\mathcal{KL}}(\mathcal{E}_{\underline{i}}\mathcal{O}_\lambda, \mathcal{E}_{\underline{j}}\mathcal{O}_\lambda)$ and $\Theta_2 \in \mathrm{Hom}_{\mathcal{KL}}(\mathcal{E}_{\underline{i}'}\mathcal{O}_{\lambda'}, \mathcal{E}_{\underline{j}'}\mathcal{O}_{\lambda'})$. Then denote the horizontal composition of these 2-morphisms by $\Theta_2\Theta_1$ which is an element of $\mathrm{Hom}_{\mathcal{KL}}(\mathcal{E}_{\underline{i}'}\mathcal{O}_{\lambda'}\mathcal{E}_{\underline{i}}\mathcal{O}_\lambda, \mathcal{E}_{\underline{j}'}\mathcal{O}_{\lambda'}\mathcal{E}_{\underline{j}}\mathcal{O}_\lambda)$. If $\Theta_3 \in \mathrm{Hom}_{\mathcal{KL}}(\mathcal{E}_{\underline{j}}\mathcal{O}_\lambda, \mathcal{E}_{\underline{k}}\mathcal{O}_\lambda)$, denote the vertical composition of Θ_3 and Θ_1 by $\Theta_3 \circ \Theta_1$.

For convenience of notation, we define the following 2-morphisms. If $\theta \in \mathrm{End}(\mathcal{E}_{\underline{i}}\mathcal{O}_\lambda)$, let $\theta^{[j]} = \underbrace{\theta \circ \cdots \circ \theta}_{j}$. For each $i \in I$, define the *bubble*

$$\bigcirc^{\bullet N}_{i;\lambda} = \bigcap_{i;\lambda} \circ (\mathbf{1}_{-i;\lambda+\alpha_i} Y_{i;\lambda})^{[N]} \circ \bigcup_{i;\lambda}. \tag{3.6}$$

Also, define *half-bubbles*

$$\overset{\bullet N}{\bigcup_{i;\lambda}} = (\mathbf{1}_{-i;\lambda+\alpha_i} Y_{i;\lambda})^{[N]} \circ \bigcup_{i;\lambda}, \qquad \overset{\bullet N}{\bigcup_{i;\lambda}} = \bigcap_{i;\lambda} \circ (Y_{-i;\lambda+\alpha_i} \mathbf{1}_{i,\lambda})^{[N]}. \tag{3.7}$$

We now define the relations satisfied by these basic 2 morphisms. In what follows, we omit the argument λ when the relation is independent of it.

(1) $\mathfrak{sl}_2$*Relations*

(a) For all $i \in I$,

$$\left(\bigcap_{-i} \mathbf{1}_i\right) \circ \left(\mathbf{1}_i \bigcup_{i}\right) = \mathbf{1}_i = \left(\mathbf{1}_i \bigcap_{i}\right) \circ \left(\bigcup_{-i} \mathbf{1}_i\right). \tag{3.8}$$

(b) For all $i \in I^+$,

$$Y_i = \left(\bigcap_{-i} \mathbf{1}_i\right) \circ (\mathbf{1}_i Y_{-i} \mathbf{1}_i) \circ \left(\mathbf{1}_i \bigcup_{i}\right) = \left(\mathbf{1}_i \bigcap_{i}\right) \circ (\mathbf{1}_i Y_{-i} \mathbf{1}_i) \circ \left(\bigcup_{-i} \mathbf{1}_i\right). \tag{3.9}$$

(c) Suppose that $i \in I$ and $(-\alpha_i, \lambda) > r + 1$, then

$$\bigcirc_{i;\lambda}^{\bullet r} = 0. \tag{3.10}$$

(d) Let $i \in I$. If $(\alpha_i, \lambda) \leq -1$,

$$\bigcirc_{i;\lambda}^{\bullet -(\alpha_i,\lambda)-1} = 1. \tag{3.11}$$

(e) Let $i \in I$. If $(\alpha_i, \lambda) \geq 1$, then

$$\begin{aligned} \mathbf{1}_{i;\lambda-\alpha_i}\mathbf{1}_{-i;\lambda} &= -\Psi_{-i,i;\lambda} \circ \Psi_{i,-i;\lambda} \\ &\quad + \sum_{f=0}^{(\alpha_i,\lambda)-1} \sum_{g=0}^{f} \bigcup_{-i;\lambda}^{\bullet[(\alpha_i,\lambda)-f-1]} \circ \bigcirc_{i;\lambda}^{\bullet[-(\alpha_i,\lambda)-1+g]} \circ \bigcap_{-i;\lambda}^{\bullet[f-g]}. \end{aligned} \tag{3.12}$$

(f) Let $i \in I^+$. If $(\alpha_i, \lambda) \leq 0$, then

$$\begin{aligned} &\left(\mathbf{1}_{i;\lambda} \bigcap_{-i;\lambda}\right) \circ (\Psi_{i,i;\lambda-\alpha_i}\mathbf{1}_{-i;\lambda}) \circ \left(\mathbf{1}_{i;\lambda} \bigcup_{-i;\lambda}\right) \\ &\quad = -\sum_{f=0}^{-(\alpha_i,\lambda)} \Upsilon_{i;\lambda}{}^{[-(\alpha_i,\lambda)-f]} \bigcirc_{-i;\lambda}^{\bullet[(\alpha_i,\lambda)-1+f]}. \end{aligned} \tag{3.13}$$

If $(\alpha_i, \lambda) \geq -2$, then

$$\begin{aligned} &\left(\bigcap_{i;\lambda} \mathbf{1}_{i;\lambda-\alpha_i}\right) \circ (\mathbf{1}_{-i;\lambda+\alpha_i}\Psi_{i,i;\lambda-\alpha_i}) \circ \left(\bigcup_{i;\lambda} \mathbf{1}_{i;\lambda-\alpha_i}\right) \\ &\quad = \sum_{g=0}^{(\alpha_i,\lambda)+2} \bigcirc_{i;\lambda}^{\bullet[-(\alpha_i,\lambda)-1+g]} \Upsilon_{i;\lambda-\alpha_i}{}^{[(\alpha_i,\lambda)-g]}. \end{aligned} \tag{3.14}$$

Remark 3.1. Note that in 1(e) above the exponent of the bubble may be negative, which is not defined. To make sense of this, for $i \in I^+$, define these symbols (referred to as *fake bubbles* in [1]) inductively by the formula

$$\left(\sum_{n\geq 0} \bigcirc_{i;\lambda}^{\bullet(\alpha_{-i},\lambda)-1+n} t^n\right)\left(\sum_{n\geq 0} \bigcirc_{-i;\lambda}^{\bullet(\alpha_{-i},\lambda)-1+n} t^n\right) = 1 \tag{3.15}$$

and $\bigcirc_{i;\lambda}^{\bullet -1} = 1$ whenever $(\alpha_i, \lambda) = 0$.

(2) *The nil-Hecke Relations*

(a) For each $i \in I^+$, $\Psi_{i,i}^{[2]} = 0$.

(b) For $i \in I^+$, $(\Psi_{i,i}\mathbf{1}_i) \circ (\mathbf{1}_i\Psi_{i,i}) \circ (\Psi_{i,i}\mathbf{1}_i) = (\mathbf{1}_i\Psi_{i,i}) \circ (\Psi_{i,i}\mathbf{1}_i) \circ (\mathbf{1}_i\Psi_{i,i})$.

(c) For $i \in I^+$, $(\mathbf{1}_i\mathbf{1}_i) = (\Psi_{i,i}) \circ (Y_i\mathbf{1}_i) - (\mathbf{1}_iY_i) \circ (\Psi_{i,i}) = (Y_i\mathbf{1}_i) \circ (\Psi_{i,i}) - (\Psi_{i,i}) \circ (\mathbf{1}_iY_i)$.

(d) For $j, i \in I^-$,

$$\begin{aligned}\Psi_{j,i} &= \left(\bigcap_{-j}\mathbf{1}_i\mathbf{1}_j\right) \circ \left(\mathbf{1}_j\bigcap_{-i}\mathbf{1}_{-j}\mathbf{1}_i\mathbf{1}_j\right) \circ \left(\mathbf{1}_j\mathbf{1}_i\Psi_{-j,-i}\mathbf{1}_i\mathbf{1}_j\right) \circ \left(\mathbf{1}_j\mathbf{1}_i\mathbf{1}_{-j}\bigcup_{i}\mathbf{1}_j\right) \circ \left(\mathbf{1}_j\mathbf{1}_i\bigcup_{j}\right) \\ &= \left(\mathbf{1}_i\mathbf{1}_j\bigcap_{i}\right) \circ \left(\mathbf{1}_i\mathbf{1}_j\mathbf{1}_{-i}\bigcap_{j}\mathbf{1}_i\right) \circ \left(\mathbf{1}_i\mathbf{1}_j\Psi_{-j,-i}\mathbf{1}_j\mathbf{1}_i\right) \circ \left(\mathbf{1}_i\bigcup_{-j}\mathbf{1}_{-i}\mathbf{1}_j\mathbf{1}_i\right) \circ \left(\bigcup_{-i}\mathbf{1}_j\mathbf{1}_i\right).\end{aligned} \tag{3.16}$$

Remark 3.2. For all $i, j \in I^{\pm}$, set $\Psi_{i,-j} = (\mathbf{1}_{-j}\mathbf{1}_i\bigcap_{-j}) \circ (\mathbf{1}_{-j}\Psi_{j,i}\mathbf{1}_{-j}) \circ (\bigcup_j \mathbf{1}_i\mathbf{1}_{-j})$.

(3) *The $R(\nu)$ Relations*

(a) For $i, j \in I^{\pm}$, $(\Psi_{-j,i}) \circ (\Psi_{i,-j}) = \mathbf{1}_i\mathbf{1}_{-j}$.

(b) For $i, j \in I^+$, $i \neq j$,

$$\Psi_{j,i} \circ \Psi_{i,j} = \begin{cases} \mathbf{1}_i\mathbf{1}_j & \text{if } |i-j| > 1, \\ (i-j)(Y_i\mathbf{1}_j - \mathbf{1}_iY_j) & \text{if } |i-j| = 1. \end{cases} \tag{3.17}$$

(c) For $i, j \in I^+$, $i \neq j$,

$$(\mathbf{1}_jY_i) \circ (\Psi_{i,j}) = (\Psi_{i,j}) \circ (Y_i\mathbf{1}_j), \qquad (Y_j\mathbf{1}_i) \circ (\Psi_{i,j}) = (\Psi_{i,j}) \circ (\mathbf{1}_iY_j). \tag{3.18}$$

(d) For $i, j, k \in I^+$,

$$\begin{aligned}&(\Psi_{j,k}\mathbf{1}_i) \circ (\mathbf{1}_j\Psi_{i,k}) \circ (\Psi_{i,j}\mathbf{1}_k) - (\mathbf{1}_k\Psi_{i,j}) \circ (\Psi_{i,k}\mathbf{1}_j) \circ (\mathbf{1}_i\Psi_{j,k}) \\ &\quad = \begin{cases} 0 & i \neq k \text{ or } |i-j| = 0, \\ (i-j)\mathbf{1}_i\mathbf{1}_j\mathbf{1}_i & i = k \text{ and } |i-j| = 1. \end{cases}\end{aligned} \tag{3.19}$$

4. The Huerfano-Khovanov 2-Category

4.1. The Khovanov Diagram Algebra

Let $\mathcal{A} = \mathbb{C}[x]/x^2$. This is a $\mathbb{Z}$-graded algebra with multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that $\deg 1 = -1$ and $\deg x = 1$. There is a comultiplication map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that $\Delta(1) = x \otimes 1 + 1 \otimes x$ and $\Delta(x) = x \otimes x$. There is a trace map $\mathrm{Tr} : \mathcal{A} \to \mathbb{C}$ such that $\mathrm{Tr}(x) = 1$ and $\mathrm{Tr}(1) = 0$. There is also a unit map $\iota : \mathbb{C} \to \mathcal{A}$ given by $\iota(1) = 1$. Also, let $\kappa : \mathcal{A} \to \mathcal{A}$ be given by $\kappa(1) = 0$, $\kappa(x) = 1$. This algebra gives rise to a two-dimensional TQFT $\mathfrak{F}$, which is a functor from the category of oriented $1+1$ cobordisms to the category of abelian groups. The functor $\mathfrak{F}$ sends a disjoint union of m copies of the circle $\mathbb{S}^1$ to $\mathcal{A}^{\otimes m}$. For a cobordism $\mathcal{C}_1$, from two circles to one circle, $\mathfrak{F}(\mathcal{C}_1) = m$. For a cobordism $\mathcal{C}_2$, from one circle to two circles,

Figure 1: Crossingless matches a and b for $r = 2$.

Figure 2: Concatenation $(Ra)b$.

$\mathfrak{F}(\mathcal{C}_2) = \Delta$. For a cobordism $\mathcal{C}_3$, from the empty manifold to $\mathbb{S}^1$, $\mathfrak{F}(\mathcal{C}_3) = \iota$. For a cobordism $\mathcal{C}_4$, from the empty manifold to $\mathbb{S}^1, \mathfrak{F}(\mathcal{C}_4) = \mathrm{Tr}$.

For any nonnegative integer r, consider $2r$ marked points on a line. Let CM_r be the set of nonintersecting curves up to isotopy whose boundary is the set of the $2r$ marked points such that all of the curves lie on one side of the line. Then there are $\binom{2r}{r}/r + 1$ elements in this set. The set of crossingless matches for $r = 2$ is given in Figure 1.

Let $a, b \in \mathrm{CM}_r$. Then $(Rb)a$ is a collection of circles obtained by concatenating $a \in \mathrm{CM}_r$ with the reflection Rb of $b \in \mathrm{CM}_r$ in the line. Then applying the two-dimensional TQFT $\mathfrak{F}$, one associates the graded vector space ${}_bH_a^r$ to this collection of circles. Taking direct sums over all crossingless matches gives a graded vector space

$$H^r = \bigoplus_{a,b} {}_bH_a^r\{r\}, \tag{4.1}$$

where the degree i component of ${}_bH_a^r\{r\}$ is the degree $i - r$ component of ${}_bH_a^r$. This graded vector space obtains the structure of an associative algebra via $\mathfrak{F}$; compare, for example, [5].

Let T be a tangle from $2r$ points to $2s$ points. Let a be a crossingless match for $2s$ points and b a crossingless match for $2s$ points. Then let ${}_aT_b$ be the concatenation $Ra \circ T \circ b$ and ${}_a\mathfrak{F}(T)_b = \mathfrak{F}({}_aT_b)$. See Figure 3 for an example when T is the identity tangle.

To any tangle diagram T from $2r$ points to $2s$ points, there is an (H^s, H^r)-bimodule

$$\mathfrak{F}(T) = \bigoplus_{\substack{a\in\mathrm{CM}_r \\ b\in\mathrm{CM}_s}} \mathfrak{F}({}_aT_b)\{r\}. \tag{4.2}$$

To any cobordism C between tangles T_1 and T_2, there is a bimodule map $\mathfrak{F}(C) : \mathfrak{F}(T_1) \to \mathfrak{F}(T_2)$, of degree $-\chi(C) - r - s$, where $\chi(C)$ is the Euler characteristic of C; compare, for example, Proposition 5 of [5].

Lemma 4.1. *Consider the tangles I and U_i in Figure 4. Then there are saddle cobordisms $S_i : U_i \to I$ and $S^i : I \to U_i$.*

Let T_i and T^i be the tangles in Figure 5.

(1) *There exists an (H^{n-1}, H^n)-bimodule homomorphism $\mu_i : \mathfrak{F}(T_i) \to \mathfrak{F}(T_{i+1})$ of degree one.*

(2) *There exists an (H^n, H^{n-1})-bimodule homomorphism $\mu^i : \mathfrak{F}(T^i) \to \mathfrak{F}(T^{i+1})$ of degree one.*

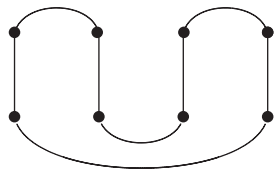

Figure 3: Concatenation ${}_aT_b$.

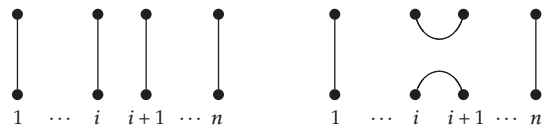

Figure 4: I and U_i.

Proof. There is a degree zero isomorphism of bimodules $\mathfrak{F}(T_i) \cong \mathfrak{F}(T_i)\bigotimes_{H^n}\mathfrak{F}(I)$. Then by [5] there is a bimodule map of degree one

$$1 \otimes \mathfrak{F}\left(S^{i+1}\right) : \mathfrak{F}(T_i)\bigotimes_{H^n}\mathfrak{F}(I) \longrightarrow \mathfrak{F}(T_i)\bigotimes_{H^n}\mathfrak{F}(U_{i+1}), \tag{4.3}$$

where 1 denotes the identity map. Finally note that $\mathfrak{F}(T_i)\bigotimes_{H^n}\mathfrak{F}(U_{i+1}) \cong \mathfrak{F}(T_{i+1})$. Then μ_i is the composition of these maps.

The construction of μ^i is similar. □

Remark 4.2. One may construct, in a similar way, maps of degree one: $\mathfrak{F}(T_i) \to \mathfrak{F}(T_{i-1})$ and $\mathfrak{F}(T^i) \to \mathfrak{F}(T^{i-1})$.

Lemma 4.3. *Let $a \in CM_n$ and $b \in CM_{n-1}$ be two crossingless matches. Let T^i be the tangle on the right side of Figure 5. Let U_i be the tangle in Figure 4. Consider the homomorphism induced by the cobordism $S^i, \mathfrak{F}(T^i) \to \mathfrak{F}(U_i)\bigotimes_{H^n}\mathfrak{F}(T^i) \cong \mathcal{A}\bigotimes_{\mathbb{C}}\mathfrak{F}(T^i)$. Let $\alpha \otimes \beta \in \mathfrak{F}({}_aT^i{}_b)$, where $\alpha \in \mathcal{A}$ corresponds to the circle passing through the point i on the top line and $\beta \in \mathcal{A}^{\otimes p}$ corresponds to the remaining circles. Then $\alpha \otimes \beta \mapsto \Delta(\alpha) \otimes \beta$.*

Proof. The map is induced by the cobordism S^i. On the set of circles, this cobordism is a union of identity cobordisms and a cobordism $\mathcal{C}_2$. The result now follows upon applying $\mathfrak{F}$. □

Lemma 4.4. *Let I be the identity tangle from $2r$ points to $2r$ points, T_i a tangle from $2(r+1)$ points to $2r$ points, and T^i a tangle from $2r$ points to $2(r+1)$ points. Let a and b be cup diagrams for $2r$ points ($a, b \in CM_r$). Consider the map*

$$\mathcal{A}\bigotimes_{\mathbb{C}}\mathfrak{F}(I) \longrightarrow \mathfrak{F}(T_i)\bigotimes_{H^{r+1}}\mathfrak{F}\left(T^i\right) \longrightarrow \mathfrak{F}(T_{i+1})\bigotimes_{H^{r+1}}\mathfrak{F}\left(T^i\right) \longrightarrow \mathfrak{F}(I), \tag{4.4}$$

where the first and last maps are isomorphisms and the middle map is $\mu_i \otimes 1$. Let $\beta \in \mathcal{A}$ correspond to the circle passing through point i of ${}_aI_b, \gamma \in \mathcal{A}^{\otimes r}$ correspond to the remaining circles, and $\alpha \in \mathcal{A}$. Then the map above sends $\alpha \otimes \beta \otimes \gamma \mapsto (\alpha\beta) \otimes \gamma$.

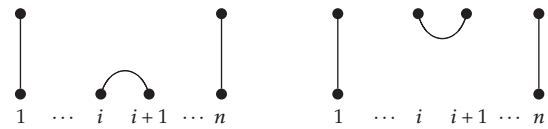

Figure 5: T_i and T^i.

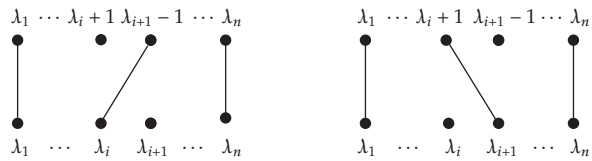

Figure 6: $D_{\lambda,i}$ and $D^{\lambda,i}$.

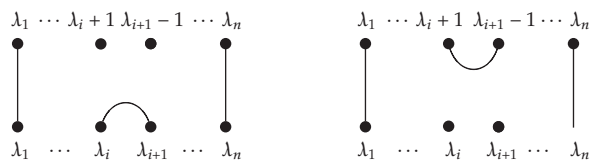

Figure 7: $T_{\lambda,i}$ and $T^{\lambda,i}$.

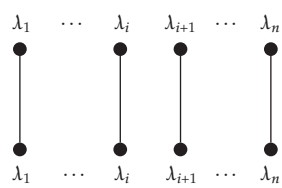

Figure 8: Identity tangle I_λ.

Proof. The map is induced by a cobordism S^{i+1}. On the set of circles, this cobordism is union of identity cobordisms and a cobordism $\mathcal{C}_1$. The result now follows upon applying $\mathfrak{F}$. □

4.2. The Huerfano-Khovanov Categorification

Let $\lambda \in P(2\omega_k)$. Recall that $\alpha_{-i} = -\alpha_i$. Hence, for $i \in I$, we have

$$\lambda + \alpha_i = (\lambda_1, \ldots, \lambda_i + \operatorname{sgn}(i), \lambda_{i+1} - \operatorname{sgn}(i), \ldots, \lambda_n). \tag{4.5}$$

Label n collinear points by the integers λ_i. Those points labeled by 0 or 2 will never be the boundaries of arcs but will rather just serve as place holders. Then define the algebra $H_\lambda = H^{\gamma(\lambda)}$ (as in Section 4.1), where $\gamma(\lambda) = (1/2) \mid \{\lambda_i \mid \lambda_i = 1\}\mid$. Let e_λ be the identity element.

Let $i \in I^+$. We define five special tangles $D_{\lambda,i}$, $D^{\lambda,i}$, $T_{\lambda,i}$, $T^{\lambda,i}$, I_λ in Figures 6, 7, and 8. If a point is labeled by zero or two, it will not be part of the boundary of any curve. Away from points $i, i+1$, the tangle is the identity.

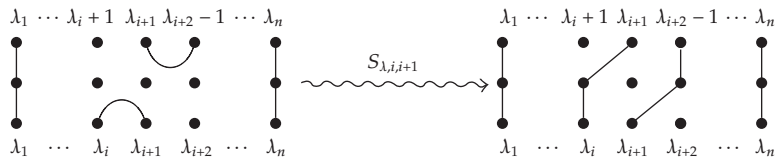

Figure 9: Cobordism $S_{\lambda,i,i+1}$.

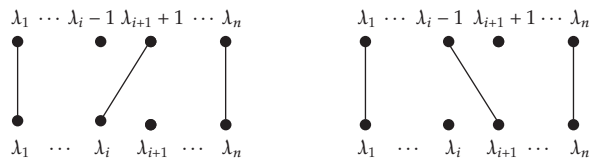

Figure 10: $D^{\lambda,-i}$ and $D_{\lambda,-i}$.

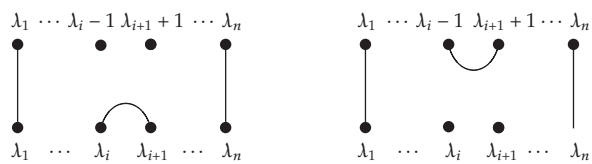

Figure 11: $T_{\lambda,-i}$ and $T^{\lambda,-i}$.

The cobordisms $S_{\lambda,i} : T^{\lambda+\alpha_i,i} \circ T_{\lambda,i} \to I_\lambda$ and $S_{\lambda,i,j} : T^{\lambda+\alpha_i,j} \circ T_{\lambda,i} \to D_{\lambda+\alpha_j,i} \circ D_{\lambda,j}$ are saddle cobordisms for $j = i \pm 1$. Similarly, the cobordisms $S^{\lambda,i}$, $S^{\lambda,i,j}$ are saddle cobordisms in the opposite direction. For example, the cobordism $S_{\lambda,i,i+1}$ is given in Figure 9.

Let $\mathcal{C}_\lambda$ be the category of finitely generated, graded H_λ-modules, and let $\mathbb{I}_\lambda : \mathcal{C}_\lambda \to \mathcal{C}_\lambda$ be the identity functor. For $\lambda, \lambda' \in P(2\omega_k)$, set $\mathbb{I}_{\lambda'}\mathbb{I}_\lambda = \delta_{\lambda,\lambda'}\mathbb{I}_\lambda$.

Let $i \in I^+$. To make future definitions more homogeneous, define $D_{\lambda,-i}$, $D^{\lambda,-i}$, $T_{\lambda,-i}$, $T^{\lambda,-i}$ as in Figures 10 and 11. Also, in what follows, interpret the pair $(\lambda_{-i}, \lambda_{-i+1})$ as $(\lambda_{i+1}, \lambda_i)$ and recall that $\alpha_{-i} = -\alpha_i$.

Let $i \in I$. Let $\mathbb{I}_\lambda : \mathcal{C}_\lambda \to \mathcal{C}_\lambda$ denote the identity functor which is tensoring with the (H_λ, H_λ)-bimodule H_λ. Let $\mathbb{E}_i\mathbb{I}_\lambda : \mathcal{C}_\lambda \to \mathcal{C}_{\lambda+\alpha_i}$ be the functor of tensoring with a bimodule defined as follows:

$$\mathbb{E}_i\mathbb{I}_\lambda = \begin{cases} \mathfrak{F}(D_{\lambda,i}) & \text{if } (\lambda_i, \lambda_{i+1}) = (1,2), \\ \mathfrak{F}(D^{\lambda,i}) & \text{if } (\lambda_i, \lambda_{i+1}) = (0,1), \\ \mathfrak{F}(T_{\lambda,i}) & \text{if } (\lambda_i, \lambda_{i+1}) = (1,1), \\ \mathfrak{F}(T^{\lambda,i}) & \text{if } (\lambda_i, \lambda_{i+1}) = (0,2), \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Evidently, $\mathbb{E}_i\mathbb{I}_\lambda = \mathbb{I}_{\lambda+\alpha_i}\mathbb{E}_i\mathbb{I}_\lambda$ for all $i \in I$, and $\mathbb{I}_\lambda = \mathfrak{F}(I_\lambda)$.

For $i \in I$, let $\mathbb{K}_i\mathbb{I}_\lambda : \mathcal{C}_\lambda \to \mathcal{C}_\lambda$ be the grading shift functor $\mathbb{K}_i\mathbb{I}_\lambda = \mathbb{I}_\lambda\{(\alpha_i, \lambda)\}$. Finally, set $\mathcal{C} = \bigoplus_{\lambda\in P(2\omega_k)}\mathcal{C}_\lambda$, $\mathbb{E}_i = \bigoplus_{\lambda\in P(2\omega_k)}\mathbb{E}_i\mathbb{I}_\lambda$, $\mathbb{K}_i = \bigoplus_{\lambda\in P(2\omega_k)}\mathbb{K}_i\mathbb{I}_\lambda$, and $\mathbb{I} = \bigoplus_{\lambda\in P(2\omega_k)}\mathbb{I}_\lambda$.

Propositions 2 and 3 of [4] are that these functors satisfy quantum $\mathfrak{sl}_n$ relations.

Proposition 4.5 (see [4, Propositions 2, 3]). *One has*

(1) $\mathbb{K}_i\mathbb{K}_{-i}\mathbb{I}_\lambda \cong \mathbb{I}_\lambda \cong \mathbb{K}_{-i}\mathbb{K}_i\mathbb{I}_\lambda$, *and* $\mathbb{K}_i\mathbb{K}_j\mathbb{I}_\lambda \cong \mathbb{K}_j\mathbb{K}_i\mathbb{I}_\lambda$ *for* $i, j \in I$,

(2) $\mathbb{K}_i\mathbb{E}_j\mathbb{I}_\lambda \cong \mathbb{E}_j\mathbb{K}_i\mathbb{I}_\lambda\{a_{ij}\}$, *for* $i, j \in I$,

(3) $\mathbb{E}_i\mathbb{E}_{-j}\mathbb{I}_\lambda \cong \mathbb{E}_{-j}\mathbb{E}_i\mathbb{I}_\lambda$ *if* $i, j \in I^+$, $i \neq j$,

(4) $\mathbb{E}_i\mathbb{E}_j\mathbb{I}_\lambda \cong \mathbb{E}_j\mathbb{E}_i\mathbb{I}_\lambda$ *if* $i, j \in I^\pm$, $|i-j| > 1$,

(5) $\mathbb{E}_i\mathbb{E}_i\mathbb{E}_j\mathbb{I}_\lambda \oplus \mathbb{E}_j\mathbb{E}_i\mathbb{E}_i\mathbb{I}_\lambda \cong \mathbb{E}_i\mathbb{E}_j\mathbb{E}_i\mathbb{I}_\lambda\{1\} \oplus \mathbb{E}_i\mathbb{E}_j\mathbb{E}_i\mathbb{I}_\lambda\{-1\}$ *if* $i, j \in I^\pm$, $|i-j| = 1$,

(6) *For* $i \in I$,

$$\mathbb{E}_i\mathbb{E}_{-i}\mathbb{I}_\lambda \cong \begin{cases} \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda \oplus \mathbb{I}_\lambda\{1\} \oplus \mathbb{I}_\lambda\{-1\} & \text{if } i \in I^+,\ (\lambda_i, \lambda_{i+1}) = (2,0), \\ \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda \oplus \mathbb{I}_\lambda\{1\} \oplus \mathbb{I}_\lambda\{-1\} & \text{if } i \in I^-,\ (\lambda_i, \lambda_{i+1}) = (0,2), \\ \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda \oplus \mathbb{I}_\lambda & \text{if } (\alpha_i, \lambda) = 1, \\ \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda & \text{if } (\alpha_i, \lambda) = 0. \end{cases} \tag{4.7}$$

Now we define the Huerfano-Khovanov 2-category $\mathcal{HK}_{k,n}$ over the field $\mathbb{k}$, $\mathrm{char}\mathbb{k} = 2$.

4.3. The Objects

The objects of $\mathcal{HK}_{k,n}$ are the categories $\mathcal{C}_\lambda$, $\lambda \in P(V_{2\omega_k})$.

4.4. The 1-Morphisms

For each $\lambda \in P(2\omega_k)$, $\mathbb{I}_\lambda \in \mathrm{End}_{\mathcal{HK}}(\lambda)$ is the identity morphism and, for $\lambda, \lambda' \in P$, set $\mathbb{I}_\lambda\mathbb{I}'_\lambda = \delta_{\lambda,\lambda'}\mathbb{I}_\lambda$ as above. For each $i \in I$, we have defined morphisms $\mathbb{E}_i\mathbb{I}_\lambda \in \mathrm{Hom}_{\mathcal{HK}}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\alpha_i})$. Evidently, we have $\mathbb{E}_i\mathbb{I}_\lambda = \mathbb{I}_{\lambda+\alpha_i}\mathbb{E}_i\mathbb{I}_\lambda$. For $\lambda, \lambda' \in P(2\omega_k)$, we have

$$\mathrm{Hom}_{\mathcal{HK}}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda'}) = \bigoplus_{\substack{\underline{i} \in \mathrm{Seq} \\ s \in \mathbb{Z}}} \mathbb{I}_{\lambda'}\mathbb{E}_{\underline{i}}\mathbb{I}_\lambda\{s\}, \tag{4.8}$$

where $\mathbb{E}_{\underline{i}} := \mathbb{E}_{i_1}\cdots\mathbb{E}_{i_r}\mathbb{I}_\lambda$ if $\underline{i} = (i_1, \ldots, i_r) \in I_\infty$, and s refers to a *grading shift*. Observe that $\mathbb{I}_{\lambda'}\mathbb{E}_{\underline{i}}\mathbb{I}_\lambda = 0$ unless $\mathrm{cont}(\underline{i}) = \lambda' - \lambda$, and $\mathbb{I}_{\lambda+\mathrm{cont}(\underline{i})}\mathbb{E}_{\underline{i}}\mathbb{I}_\lambda = \mathbb{E}_{\underline{i}}\mathbb{I}_\lambda$.

4.5. The 2-Morphisms

In this section we define natural transformations of functors. These maps were not explicitly defined in [4]. Note that the notation for these 2-morphisms is similar to the 2-morphisms in Section 3 since we will construct a 2-functor mapping one set of 2-morphisms to the other. Recall the convention $(\lambda_{-i}, \lambda_{-i+1}) = (\lambda_{i+1}, \lambda_i)$ for $i \in I^+$.

(1) The Maps $1_{i,\lambda}$, 1_λ

Let $i \in I$, and let $1_{i,\lambda} : \mathbb{E}_i\mathbb{I}_\lambda \to \mathbb{E}_i\mathbb{I}_\lambda$ and $1_\lambda : \mathbb{I}_\lambda \to \mathbb{I}_\lambda$ be the identity maps.

(2) The Maps $y_{i;\lambda}$

For $i \in I$ we define maps $y_{i;\lambda} : \mathbb{E}_i\mathbb{I}_\lambda \to \mathbb{E}_i\mathbb{I}_\lambda$ of degree 2. Let T be the tangle diagram for the functor $\mathbb{E}_i\mathbb{I}_\lambda$. It depends on the pair $(\lambda_i, \lambda_{i+1})$. Let a and b be crossingless matches such that $(Rb)Ta$ is a disjoint union of circles. Thus $\mathfrak{F}((Rb)Ta) = (\mathcal{A})^{\otimes p}$ for some natural number p. Define

$$y_{i;\lambda}\big((\beta_1 \otimes \cdots \otimes \beta_p)\big) = (\beta_1 \otimes \cdots \otimes x\beta_i \otimes \cdots \otimes \beta_p), \tag{4.9}$$

where

(a) if $(\lambda_i, \lambda_{i+1}) = (1,2)$, then the ith factor in $(\mathcal{A})^{\otimes p}$ corresponds to the circle passing through the ith point on the bottom set of dots for tangle $D_{\lambda,i}$ in Figure 6,

(b) if $(\lambda_i, \lambda_{i+1}) = (0,1)$, then the ith factor in $(\mathcal{A})^{\otimes p}$ corresponds to the circle passing through the ith point on the top set of dots for tangle $D^{\lambda,i}$ in Figure 6,

(c) if $(\lambda_i, \lambda_{i+1}) = (0,2)$, then the ith factor in $(\mathcal{A})^{\otimes p}$ corresponds to the circle passing through the ith point on the top set of dots for tangle $T^{\lambda,i}$ in Figure 7,

(d) if $(\lambda_i, \lambda_{i+1}) = (1,1)$, then the ith factor in $(\mathcal{A})^{\otimes p}$ corresponds to the circle passing through the ith point on the bottom set of dots for tangle $T_{\lambda,i}$ in Figure 7.

(3) The Map $\cup_{i;\lambda}$

We define a map $\cup_{i;\lambda} : \mathbb{I}_\lambda \to \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda$. There are four nontrivial cases for $(\lambda_i, \lambda_{i+1})$ to consider.

(a) $(\lambda_i, \lambda_{i+1}) = (1,2)$. The identity functor is induced from the identity tangle I_λ. The functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(D^{\lambda+\alpha_i,i} \circ D_{\lambda,i})$ which is equal to $\mathfrak{F}(I_\lambda)$. Thus in this case $\cup_{i;\lambda}$ is given by the identity map.

(b) $(\lambda_i, \lambda_{i+1}) = (1,1)$. Then the functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(T^{\lambda+\alpha_i,i} \circ T_{\lambda,i})$. Then $\cup_{i;\lambda}$ is $\mathfrak{F}(S^{\lambda,i})$.

(c) $(\lambda_i, \lambda_{i+1}) = (0,2)$. Then the functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(T_{\lambda+\alpha_i,i} \circ T^{\lambda,i}) = \mathfrak{F}(I_\lambda) \otimes \mathcal{A}$. Then the bimodule map is given by $1_\lambda \otimes \iota$.

(d) $(\lambda_i, \lambda_{i+1}) = (0,1)$. The functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(D_{\lambda+\alpha_i,i} \circ D^{\lambda,i})$. As in case 1, this tangle is isotopic to the identity so the map between the functors is the identity map.

(4) The Map $\cap_{i;\lambda}$.

We define a map $\cap_{i;\lambda} : \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda \to \mathbb{I}_\lambda$. There are four non-trivial cases for $(\lambda_i, \lambda_{i+1})$ to consider.

(a) $(\lambda_i, \lambda_{i+1}) = (1,2)$. The functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(D^{\lambda+\alpha_i,i} \circ D_{\lambda,i})$ which is equal to $\mathfrak{F}(I_\lambda)$. Thus in this case $\cap_{i;\lambda}$ is given by the identity map.

(b) $(\lambda_i, \lambda_{i+1}) = (1,1)$. Then the functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(T^{\lambda+\alpha_i,i} \circ T_{\lambda,i})$. Then the homomorphism is $\mathfrak{F}(S_{\lambda,i})$.

(c) $(\lambda_i, \lambda_{i+1}) = (0,2)$. Then the functor $\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(T_{\lambda+\alpha_i,i} \circ T^{\lambda,i}) = \mathfrak{F}(I_\lambda) \otimes \mathcal{A}$. Then the bimodule map is given by $1_\lambda \otimes \mathrm{Tr}$.

(d) $(\lambda_i, \lambda_{i+1}) = (0,1)$. The functor $\mathbb{E}_{-i}\mathbb{E}_i$ is given by tensoring with the bimodule $\mathfrak{F}(D_{\lambda+\alpha_i,i} \circ D^{\lambda,i})$. As in case 1, this tangle is isotopic to the identity so the map between the functors is the identity map.

(5) The Maps $\varphi_{i,j;\lambda}$

We define a map $\varphi_{i,j;\lambda} : \mathbb{E}_i\mathbb{E}_j\mathbb{I}_\lambda \to \mathbb{E}_j\mathbb{E}_i\mathbb{I}_\lambda$ for $i, j \in I^{\pm}$.

There are four cases for i and j to consider and then subcases for λ.

(a) $i = j$. In this case, the functors are non-trivial only if $\lambda_i = 0$ and $\lambda_{i+1} = 2$. The bimodule for $\mathbb{E}_i\mathbb{E}_i$ is isomorphic to tensoring with the bimodule $\mathfrak{F}(T_{\lambda+\alpha_i,i} \circ T^{\lambda,i}) = \mathfrak{F}(I_\lambda) \otimes \mathcal{A}$. Then $\varphi_{i,i} = 1_\lambda \otimes \kappa$.

(b) $|i-j| > 1$. In this case, the functors $\mathbb{E}_i\mathbb{E}_j$ and $\mathbb{E}_j\mathbb{E}_i$ are isomorphic via an isomorphism induced from a cobordism isotopic to the identity so set $\varphi_{i,j}$ to the identity map.

(c) $\varphi_{i,i+1} : \mathbb{E}_i\mathbb{E}_{i+1} \to \mathbb{E}_{i+1}\mathbb{E}_i$. There are four non-trivial subcases to consider.

(i) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,1,2)$. The bimodule for $\mathbb{E}_i\mathbb{E}_{i+1}$ is $\mathfrak{F}(D_{\lambda+\alpha_{i+1},i} \circ D_{\lambda,i+1})$. The bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$ is $\mathfrak{F}(T_{\lambda+\alpha_i,i+1} \circ T_{\lambda,i})$. In this case we define the bimodule map to be $\mathfrak{F}(S^{\lambda,i,i+1})$.

(ii) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,1,1)$. The functor $\mathbb{E}_i\mathbb{E}_{i+1}$ is given by tensoring with a bimodule isomorphic to

$$\mathfrak{F}(D_{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1}) \cong \mathfrak{F}(D_{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1}) \bigotimes_{H_\lambda} \mathfrak{F}(I_\lambda). \tag{4.10}$$

The bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$ is isomorphic to $\mathfrak{F}(D^{\lambda+\alpha_i,i+1} \circ T_{\lambda,i})$. Then define $\varphi_{i,j}$ to be $1_\lambda \bigotimes_{H_\lambda} \mathfrak{F}(S^{\lambda,i})$ since

$$\mathfrak{F}(D_{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1}) \bigotimes_{H_\lambda} \mathfrak{F}\left(T^{\lambda+\alpha_i,-i} \circ T_{\lambda,i}\right) \cong \mathfrak{F}\left(D^{\lambda+\alpha_i,i+1} \circ T_{\lambda,i}\right). \tag{4.11}$$

(iii) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,2)$. The bimodule for $\mathbb{E}_i\mathbb{E}_{i+1}$ is isomorphic to

$$\mathfrak{F}\left(T^{\lambda+\alpha_{i+1},i} \circ D_{\lambda,i+1}\right) \cong \mathfrak{F}(\mathbb{I}_{\lambda+\alpha_i+\alpha_{i+1}}) \bigotimes_{H_{\lambda+\alpha_i+\alpha_{i+1}}} \mathfrak{F}\left(T^{\lambda+\alpha_{i+1},i} \circ D_{\lambda,i+1}\right). \tag{4.12}$$

The bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$ is isomorphic to $\mathfrak{F}(T^{\lambda+\alpha_i,i+1} \circ D^{\lambda,i})$. Then define $\varphi_{i,j}$ to be $\mathfrak{F}(S^{\lambda+\alpha_i+\alpha_{i+1},i}) \bigotimes_{H_\lambda} 1_\lambda$ since

$$\begin{aligned} &\mathfrak{F}\left(T^{\lambda+2\alpha_i+\alpha_{i+1},-(i+1)} \circ T_{\lambda+\alpha_i+\alpha_{i+1},i+1}\right) \bigotimes_{H_{\lambda+\alpha_i+\alpha_{i+1}}} \mathfrak{F}\left(T^{\lambda+\alpha_{i+1},i} \circ D_{\lambda,i+1}\right) \\ &\quad \cong \mathfrak{F}\left(T^{\lambda+\alpha_i,i+1} \circ D^{\lambda,i}\right). \end{aligned} \tag{4.13}$$

(iv) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,1)$. The bimodule for $\mathbb{E}_i\mathbb{E}_{i+1}$ is $\mathfrak{F}(T^{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1})$. The bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$ is $\mathfrak{F}(D^{\lambda+\alpha_i,i+1} \circ D^{\lambda,i})$. Then set $\varphi_{i,j} = \mathfrak{F}(S_{\lambda,i+1,i})$.

(d) $\varphi_{i+1,i} : \mathbb{E}_{i+1}\mathbb{E}_i \to \mathbb{E}_i\mathbb{E}_{i+1}$. We essentially just have to read the maps in cases (c)(i)–(iv) above backwards.

(i) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1, 1, 2)$. The functors are just as in case (c)(i). Now the map is $\mathfrak{F}(S_{\lambda,i,i+1})$.

(ii) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1, 1, 1)$. The bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$ is isomorphic to

$$\mathfrak{F}\left(D^{\lambda+\alpha_i,i+1} \circ T_{\lambda,i}\right) \cong \mathfrak{F}\left(D^{\lambda+\alpha_i,i+1} \circ T_{\lambda,i}\right) \bigotimes_{H_\lambda} \mathfrak{F}(I_\lambda). \tag{4.14}$$

Then define $\varphi_{i+1,i} = 1_\lambda \bigotimes_{H_\lambda} \mathfrak{F}(S^{\lambda,i+1})$.

(iii) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0, 1, 2)$. The bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$ is isomorphic to

$$\mathfrak{F}\left(T^{\lambda+\alpha_i,i+1} \circ D^{\lambda,i}\right) \cong \mathfrak{F}(I_{\lambda+\alpha_i+\alpha_{i+1}}) \bigotimes_{H_{\lambda+\alpha_i+\alpha_{i+1}}} \mathfrak{F}\left(T^{\lambda+\alpha_i,i+1} \circ D^{\lambda,i}\right). \tag{4.15}$$

Then define $\varphi_{i+1,i} = \mathfrak{F}(S^{\lambda+\alpha_i+\alpha_{i+1},i}) \bigotimes_{H_\lambda} 1_\lambda$.

(iv) $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0, 1, 1)$. The functors are just as in case (c)(iv). Now the map is $\mathfrak{F}(S^{\lambda,i+1,i})$.

Proposition 4.6. *For all $i, j \in I$, and $\lambda \in P(V_{2\omega_k})$, the maps $y_{i;\lambda}$, $\varphi_{i,j,\lambda}$, $\cup_{i,\lambda}$, $\cap_{i,\lambda}$ are bimodule homomorphisms.*

For convenience of notation, we define the following 2-morphisms. If $\theta \in \mathrm{End}(\mathbb{E}_{\underline{i}})$, let $\theta^{[j]} = \underbrace{\theta \circ \cdots \circ \theta}_{j}$. For each $i \in I$, define the *bubble*

$$\overset{\bullet N}{\underset{i;\lambda}{\bigcirc}} = \cap_{i;\lambda} \circ \left(1_{-i;\lambda+\alpha_i} y_{i;\lambda}\right)^{[N]} \circ \cup_{i;\lambda}, \tag{4.16}$$

and define *fake bubbles* inductively by the formula

$$\left(\sum_{n\geq 0} \overset{\bullet(\alpha_{-i},\lambda)-1+n}{\underset{i;\lambda}{\bigcirc}} t^n\right)\left(\sum_{n\geq 0} \overset{\bullet(\alpha_{-i},\lambda)-1+n}{\underset{-i;\lambda}{\bigcirc}} t^n\right) = 1 \tag{4.17}$$

and $\bigcirc_{i;\lambda}^{\bullet -1} = 1$ whenever $(\alpha_i, \lambda) = 0$. Also, define *half-bubbles*

$$\overset{\bullet N}{\underset{i;\lambda}{\bigcup}} = \left(1_{-i;\lambda+\alpha_i} y_{i;\lambda}\right)^{[N]} \circ \cup_{i;\lambda}, \qquad \overset{\bullet N}{\underset{i;\lambda}{\bigcap}} = \cap_{i;\lambda} \circ \left(y_{i;\lambda+\alpha_i} 1_{i,\lambda}\right)^{[N]}. \tag{4.18}$$

Finally, for $i, j \in I^{\pm}$, define

$$\varphi_{i,-j} = \left(1_{-j} 1_i \cap_{-j}\right) \circ \left(1_{-j} \varphi_{j,i} 1_{-j}\right) \circ \left(\cup_j 1_i 1_{-j}\right). \tag{4.19}$$

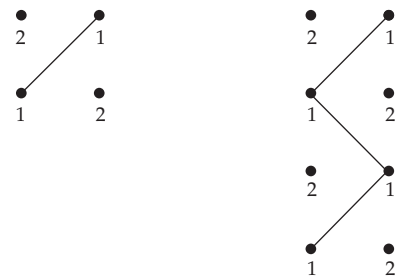

Figure 12: Tangles for $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$, $(\lambda_i, \lambda_{i+1}) = (1, 2)$.

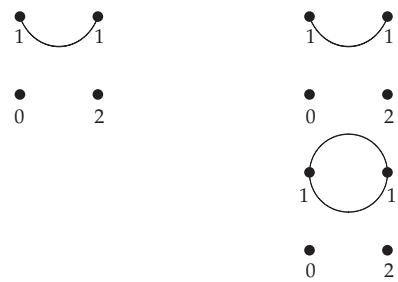

Figure 13: Tangles for $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$, $(\lambda_i, \lambda_{i+1}) = (0, 2)$.

4.6. The 2-Morphism Relations

In this section we prove certain relations between the 2-morphisms defined in Section 4.5. This will allow us to define a 2-functor from the Khovanov-Lauda 2-category to the Huerfano-Khovanov 2-category. Again, we will often omit the argument λ when it is clear from context.

4.6.1. $\mathfrak{sl}_2$ Relations

Proposition 4.7. *For all $i \in I$, $(\cap_{-i}1_i) \circ (1_i\cup_i) = 1_i = (1_i\cap_i) \circ (\cup_{-i}1_i)$.*

Proof. The second equality is similar to the first equality. The case $i \in I^-$ is similar to the case $i \in I^+$ so we just compute the map $(\cap_i 1_i) \circ (1_i\cup_i)$ on the bimodule for the functor $\mathbb{E}_i$ for $i \in I^+$. There are four cases to consider.

Suppose that $(\lambda_i, \lambda_{i+1}) = (1, 2)$. Then the tangle diagrams for the functors $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ are $D_{\lambda,i}$ and $D_{\lambda,i} \circ D^{\lambda+\alpha_i} \circ D_{\lambda,i}$ and can be found in Figure 12.

The cobordism between the tangles is isotopic to the identity map so in this case the composition is equal to the identity map.

The case $(\lambda_i, \lambda_{i+1}) = (0, 1)$ is similar to the $(1, 2)$ case.

Now let $(\lambda_i, \lambda_{i+1}) = (0, 2)$. Then the tangle diagrams for the functors $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ can be found in Figure 13.

Let B be the bimodule for the functor $\mathbb{E}_i$. Then the bimodule for $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ is isomorphic to $\mathcal{A} \otimes B$. The map $\mathbb{E}_i \to \mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ is given by the unit map which sends an element $b \in B$ to $1 \otimes b$. The map $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i \to \mathbb{E}_i$ is obtained from the cobordism joining the circle to the upper cup which induces the multiplication map. This maps $1\otimes b$ to b. Thus the composition is equal to the identity.

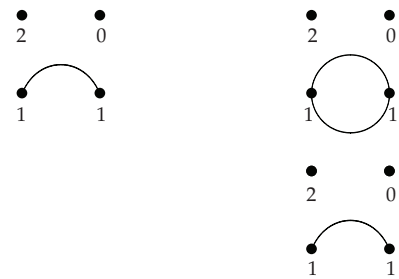

Figure 14: Tangles for $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$, $(\lambda_i, \lambda_{i+1}) = (1,1)$.

Finally consider the case $(\lambda_i, \lambda_{i+1}) = (1,1)$. The tangle diagrams for the functors $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ can be found in Figure 14.

Let B be the bimodule giving rise to the functor $\mathbb{E}_i$ and let $\mathcal{A}\otimes B$ be the bimodule giving rise to the functor $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$. Let $\alpha\otimes\beta \in B$, where α is in the tensor factor corresponding to the circle passing through point i on the bottom row of the left side of Figure 14 and β belongs to the remaining tensor factors.

The cobordism between the two tangle diagrams is a saddle which, on the level of bimodule maps, sends $\alpha\otimes\beta \mapsto \Delta(\alpha)\otimes\beta$. Then the map from $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ to $\mathbb{E}_i$ is given by $\mathrm{Tr}\otimes 1_\lambda$ so $\Delta(\alpha)\otimes\beta \mapsto \alpha\otimes\beta$ by considering the two cases $\alpha = 1$ or x. Thus the composition is equal to the identity map. □

Proposition 4.8. *One has*

$$y_i = (\cap_{-i}1_i)\circ(1_i y_{-i} 1_i)\circ(1_i\cup_i) = (1_i\cap_i)\circ(1_i y_{-i} 1_i)\circ(\cup_{-i}1_i). \tag{4.20}$$

Proof. We prove only the first equality as the second is similar. There are four cases to consider for which the functor $\mathbb{E}_i$ is nonzero.

Suppose that $(\lambda_i, \lambda_{i+1}) = (1,2)$. Then the tangle diagrams for the functors $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ can be found in Figure 12.

Note that the bimodules for $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ are the same. Denote this bimodule by B. Let $\alpha\otimes\beta \in B$, where α is an element in the tensor factor corresponding to a circle passing through point i in the bottom row of Figure 12. Then the first map $1_i\cup_i$ is given by the identity cobordism and is thus the identity. The second map is multiplication by x on all tensor components corresponding to circles passing through the point $i+1$ in the second row of the right side of Figure 12. The final map $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i \to \mathbb{E}_i$ is also given by the identity cobordism. Thus the composition maps $\alpha\otimes\beta \mapsto \alpha\otimes\beta \mapsto x\alpha\otimes\beta \mapsto \alpha\otimes\beta$. On the other hand, $y_i(\alpha\otimes\beta) = x\alpha\otimes\beta$.

The case $(\lambda_i, \lambda_{i+1}) = (0,1)$ is similar to the previous case.

Suppose that $(\lambda_i, \lambda_{i+1}) = (0,2)$. Then the bimodule for the functor $\mathbb{E}_i$ is $B = \mathfrak{F}(T^{\lambda,i})$ and the tangle diagram for $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ is $\mathfrak{F}(T^{\lambda,i}\circ T_{\lambda-\alpha_i,i}\circ T^{\lambda,i}) \cong \mathcal{A}\otimes B$. Let $\alpha\otimes\beta \in B$, where α is an element of the tensor factor corresponding to the circle passing through the point i in the top row of the tangle $T^{\lambda,i}$ and β is an element in the remaining tensor factors. Then the composition of maps sends $\alpha\otimes\beta \mapsto 1\otimes\alpha\otimes\beta \mapsto x\otimes\alpha\otimes\beta \mapsto x\alpha\otimes\beta$. This is equal to $y_i(\alpha\otimes\beta)$.

Suppose that $(\lambda_i, \lambda_{i+1}) = (1,1)$. Then the tangle diagrams for the functors $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$ can be found in Figure 14.

Let B be the bimodule for the functor $\mathbb{E}_{-i}$ and let $\mathcal{A} \otimes B$ be the bimodule for $\mathbb{E}_i\mathbb{E}_{-i}\mathbb{E}_i$. Let $\alpha \otimes \beta \in B$, where α is an element in the tensor factor corresponding to the circle passing through point i on the bottom row of Figure 14 and β is an element in the remaining tensor factors. First let $\alpha = 1$. Then

$$1 \otimes \beta \longmapsto x \otimes 1 \otimes \beta + 1 \otimes x \otimes \beta \longmapsto x \otimes x \otimes \beta \longmapsto x \otimes \beta = y_i(1 \otimes \beta), \tag{4.21}$$

where the last map is $\mathrm{Tr} \otimes 1$. If $\alpha = x$, then

$$x \otimes \beta \longmapsto x \otimes x \otimes \beta \longmapsto 0 = y_i(x \otimes \beta). \tag{4.22}$$

□

Proposition 4.9. *Suppose $i \in I$ and $(-\alpha_i, \lambda) > r + 1$, then $\bigcirc_{i;\lambda}^{\bullet r} = 0$.*

Proof. In order that $r \geq 0$, it must be the case that $(-\alpha_i, \lambda) \geq 2$. Thus the only possibility is $(\lambda_i, \lambda_{i+1}) = (0, 2)$ and $r = 0$. Then the bimodule for $\mathbb{E}_{-i}\mathbb{E}_i$ is $\mathcal{A} \otimes \mathfrak{F}(\mathbb{I}_\lambda)$. Thus the map $1 \to \mathbb{E}_{-i}\mathbb{E}_i$ is given by the unit map. The map $\mathbb{E}_{-i}\mathbb{E}_i \to 1$ is given by the trace map. Thus the composition of the maps in the proposition sends an element $\beta \mapsto 1 \otimes \beta \mapsto \mathrm{Tr}(1) \otimes b = 0$. □

Proposition 4.10. *If $(\alpha_i, \lambda) \leq -1$, then $\bigcirc_{i;\lambda}^{\bullet(-\alpha_i,\lambda)-1} = 1$.*

Proof. The only cases to consider are $(\lambda_i, \lambda_{i+1}) = (0, 2), (1, 2), (0, 1)$.

Consider the case $(0, 2)$. Let $B = \mathfrak{F}(\mathbb{I}_\lambda)$. Then the bimodule corresponding to $\mathbb{E}_{-i}\mathbb{E}_i$ is $\mathcal{A} \otimes B$. Let $\beta \in B$. Then $\cup_i(\beta) = 1 \otimes \beta$, $y_i(1 \otimes \beta) = x \otimes \beta$, and $\cap_i(x \otimes \beta) = \mathrm{Tr}(x)\beta = \beta$. Thus in this case, the composition is the identity map.

For the case $(1, 2)$, $(-\alpha_i, \lambda) - 1 = 0$. The cobordism between the tangle diagrams for the identity functor and $\mathbb{E}_{-i}\mathbb{E}_i$ is isotopic to the identity cobordism. Similarly, the cobordism between the tangle diagrams for the functors $\mathbb{E}_{-i}\mathbb{E}_i$ and the identity functor is isotopic to the identity cobordism. Thus the bimodule map is equal to the identity.

The case $(0, 1)$ is the same as the case $(1, 2)$. □

Proposition 4.11. *Let $i \in I$. If $(\alpha_i, \lambda) \geq 1$, then*

$$1_{i;\lambda-\alpha_i} 1_{-i;\lambda} = \varphi_{-i,i;\lambda} \circ \varphi_{i,-i;\lambda} + \sum_{f=0}^{(\alpha_i,\lambda)-1} \sum_{g=0}^{f} \bigcup_{-i;\lambda}^{\bullet(\alpha_i,\lambda)-f-1} \circ \bigcirc_{i;\lambda}^{\bullet-(\alpha_i,\lambda)-1+g} \circ \bigcap_{-i;\lambda}^{\bullet f-g}. \tag{4.23}$$

Proof. There are three cases to consider: $(\lambda_i, \lambda_{i+1}) = (1, 0), (2, 1), (2, 0)$.

For the case $(1, 0)$, the first term on the right-hand side is zero since that map passes through the functor $\mathbb{E}_i\mathbb{E}_i\mathbb{E}_{-i}$ which is zero for this λ. The summation on the right-hand side reduces to

$$\bigcup_{-i;\lambda}^{\bullet 0} \circ \bigcirc_{i;\lambda}^{\bullet -2} \circ \bigcap_{i;\lambda}^{\bullet 0} = \cup_{-i;\lambda} \circ \cap_{-i;\lambda} \tag{4.24}$$

by definition (4.17) of the fake bubbles. This map is a composition $\mathbb{E}_i\mathbb{E}_{-i} \to 1 \to \mathbb{E}_i\mathbb{E}_{-i}$. This composition of maps is the identity.

The case $(2, 1)$ is similar to the $(1, 0)$ case.

For the case $(2,0)$, the first term on the right-hand side is zero as in the previous two cases. The summation on the right-hand side consists of three terms, which simplifies by (4.17) to

$$\overset{\bullet 1}{\bigcup_{-i;\lambda}} \circ \cap_{-i;\lambda} + \cup_{-i;\lambda} \circ \overset{\bullet 1}{\bigcup_{-i;\lambda}} + \cup_{-i;\lambda} \circ \overset{\bullet 2}{\bigcirc_{i;\lambda}} \circ \cap_{-i;\lambda}. \tag{4.25}$$

Let $B = \mathfrak{F}(\mathbb{I}_\lambda)$. Then the bimodule for $\mathbb{E}_i\mathbb{E}_{-i}$ is $\mathcal{A} \otimes B$. Then

$$\overset{\bullet 1}{\bigcup_{-i;\lambda}} \circ \cap_{-i;\lambda} : \mathbb{E}_i\mathbb{E}_{-i} \longrightarrow \mathbb{I} \rightarrow \mathbb{E}_i\mathbb{E}_{-i} \longrightarrow \mathbb{E}_i\mathbb{E}_{-i}. \tag{4.26}$$

Under this composition of maps, $1 \otimes b$ maps to zero since the first map is given by a trace map on the first component. The element $x \otimes b$ gets mapped to $x \otimes b$ as follows:

$$x \otimes b \longmapsto b \longmapsto 1 \otimes b \longmapsto x \otimes b, \tag{4.27}$$

where the first map is the trace map, the second map is the unit map, and the third map is multiplication by x. Similarly,

$$\cup_{-i;\lambda} \circ \overset{\bullet 1}{\bigcup_{-i;\lambda}} : \mathbb{E}_i\mathbb{E}_{-i} \longrightarrow \mathbb{E}_i\mathbb{E}_{-i} \longrightarrow \mathbb{I} \longrightarrow \mathbb{E}_i\mathbb{E}_{-i}. \tag{4.28}$$

Under this composition, $1 \otimes b \mapsto 1 \otimes b$ and $x \otimes b \mapsto 0$. Finally, the map

$$\cup_{-i;\lambda} \circ \overset{\bullet 2}{\bigcirc_{i;\lambda}} \circ \cap_{-i;\lambda} \tag{4.29}$$

is zero because the middle term is zero. Thus the right-hand side is the identity as well. □

Proposition 4.12. *Let $i \in I^+$.*

(1) *If $(\alpha_i, \lambda) \leq 0$, then*

$$(1_i \cap_{-i;\lambda}) \circ \left(\varphi_{i,i;\lambda-\alpha_i} 1_{-i}\right) \circ (1_i \cup_{-i;\lambda}) = \sum_{f=0}^{-(\alpha_i,\lambda)} y_i^{-(\alpha_i,\lambda)-f} \overset{\bullet(\alpha_i,\lambda)-1+f}{\bigcirc_{-i;\lambda}}. \tag{4.30}$$

(2) *If $(\alpha_i, \lambda) \geq -2$, then*

$$(\cap_{i;\lambda+\alpha_i} 1_i) \circ \left(1_i \varphi_{i,i;\lambda}\right) \circ (\cup_{i;\lambda+\alpha_i} 1_i) = \sum_{g=0}^{(\alpha_i,\lambda)+2} \overset{\bullet -(\alpha_i,\lambda)-3+g}{\bigcirc_{i,\lambda}} y_i^{(\alpha_i,\lambda)-g+2}. \tag{4.31}$$

Proof. We prove (1), the proof of (2) being similar. Since the maps on both sides pass through the functor $\mathbb{E}_i\mathbb{E}_i\mathbb{E}_{-i}$, the maps on both sides are zero unless $(\lambda_i, \lambda_{i+1}) = (1,1)$. The functors for $\mathbb{E}_i$ and $\mathbb{E}_i\mathbb{E}_i\mathbb{E}_{-i}$ are given by tangles in Figure 14.

Let B be the bimodule for the functor $\mathbb{E}_i$ so $\mathcal{A} \otimes B$ is the bimodule for the functor $\mathbb{E}_i\mathbb{E}_i\mathbb{E}_{-i}$. Let $\alpha \otimes \beta \in B$, where α is an element in the tensor factor corresponding to a circle passing through point i in the bottom row of the left side of Figure 14 and β is an element in the other tensor factors. Consider first $\alpha = 1$. The left-hand side maps an element $\alpha \otimes \beta$ as follows:

$$1 \otimes \beta \longmapsto x \otimes 1 \otimes \beta + 1 \otimes x \otimes \beta \longmapsto 1 \otimes 1 \otimes \beta \longmapsto 1 \otimes \beta, \tag{4.32}$$

where the first map is $\Delta \otimes 1$, the second map is $\kappa \otimes 1 \otimes 1$, and the third map is $m \otimes 1$. If $\alpha = x$, the left-hand side maps $\alpha \otimes \beta$ as follows:

$$x \otimes \beta \longmapsto x \otimes x \otimes \beta \longmapsto 1 \otimes x \otimes \beta \longmapsto x \otimes \beta. \tag{4.33}$$

The right-hand side is 1 by convention. □

4.6.2. nil-Hecke Relations.

Proposition 4.13. *For $i \in I^+, \psi_{i,i}^{[2]} = 0$.*

Proof. Since $\mathbb{E}_i\mathbb{E}_i$ is identically zero unless $(\lambda_i, \lambda_{i+1}) = (0,2)$, we need only to consider this case. Let $B = \mathfrak{F}(\mathbb{I}_\lambda)$. Then the bimodule for $\mathbb{E}_i\mathbb{E}_i$ is isomorphic to $\mathfrak{F}(T_{\lambda,i} \circ T^{\lambda,i}) = \mathcal{A} \otimes B$.

Then $\psi_{i,i} \circ \psi_{i,i} : \mathcal{A} \otimes B \to \mathcal{A} \otimes B \to \mathcal{A} \otimes B$. This map sends $1 \otimes b \mapsto 0$ and $x \otimes b \mapsto 1 \otimes b \mapsto 0$. □

Proposition 4.14. *Let $i \in I^+$. Then, $(\psi_{i,i}1_i) \circ (1_i\psi_{i,i}) \circ (\psi_{i,i}1_i) = (1_i\psi_{i,i}) \circ (\psi_{i,i}1_i) \circ (1_i\psi_{i,i})$.*

Proof. Both sides are natural transformations of the functor $\mathbb{E}_i\mathbb{E}_i\mathbb{E}_i$. However, by definition this composition is zero. □

Proposition 4.15. *For $i \in I^+, (1_i1_i) = (\psi_{i,i}) \circ (y_i1_i) - (1_iy_i) \circ (\psi_{i,i}) = (y_i1_i) \circ (\psi_{i,i}) - (\psi_{i,i}) \circ (1_iy_i)$.*

Proof. The only case to check is $(\lambda_i, \lambda_{i+1}) = (0,2)$ since otherwise $\mathbb{E}_i\mathbb{E}_i = 0$. Let $B = \mathfrak{F}(\mathbb{I}_\lambda)$. Then the bimodule for $\mathbb{E}_i\mathbb{E}_i$ is isomorphic to $\mathcal{A} \otimes B$. Then

$$(\psi_{i,i}) \circ (y_i1_i) : \mathcal{A} \otimes B \longrightarrow \mathcal{A} \otimes B. \tag{4.34}$$

Under this map, $1 \otimes b \mapsto x \otimes b \mapsto 1 \otimes b$ and $x \otimes b \mapsto 0$. For the map $(1_iy_i) \circ (\psi_{i,i}), 1 \otimes b \mapsto 0$, and $x \otimes b \mapsto 1 \otimes b \mapsto x \otimes b$. This gives the first equality since our field has characteristic two.

For the second equality, $(y_i1_i) \circ (\psi_{i,i}) : 1 \otimes b \mapsto 0, (y_i1_i) \circ (\psi_{i,i}) : x \otimes b \mapsto 1 \otimes b \mapsto x \otimes b$. Similarly, $(\psi_{i,i}) \circ (1_iy_i) : 1 \otimes b \mapsto x \otimes b \mapsto 1 \otimes b$ and $(\psi_{i,i}) \circ (1_iy_i) : x \otimes b \mapsto 0$. □

Proposition 4.16. *For $i, j \in I^-$,*

$$\begin{aligned}\psi_{j,i} &= (\cap_{-j}1_i1_j) \circ (1_j\cap_{-i}1_{-j}1_i1_j) \circ (1_j1_i\psi_{-j,-i}1_i1_j) \circ (1_j1_i1_{-j}\cup_i1_j) \circ (1_j1_i\cup_j) \\ &= (1_i1_j\cap_i) \circ (1_i1_j1_{-i}\cap_j1_i) \circ (1_i1_j\psi_{-j,-i}1_j1_i) \circ (1_i\cup_{-j}1_{-i}1_j1_i) \circ (\cup_{-i}1_j1_i).\end{aligned} \tag{4.35}$$

Proof. Let $i, j \in I^-$. We prove only the first equality. If $|i-j| > 1$, the proposition is easy because then $\psi_{\pm i,\pm j}$ are identity morphisms. Therefore, we take $i = j+1$, the case $i = j-1$ being similar. The natural transformation on the right side of the proposition is a composition of natural transformations:

$$\begin{aligned}\mathbb{E}_j\mathbb{E}_{j+1} &\longrightarrow \mathbb{E}_j\mathbb{E}_{j+1}\mathbb{E}_{-j}\mathbb{E}_j \longrightarrow \mathbb{E}_j\mathbb{E}_{j+1}\mathbb{E}_{-j}\mathbb{E}_{-j-1}\mathbb{E}_{j+1}\mathbb{E}_j \\ &\longrightarrow \mathbb{E}_j\mathbb{E}_{j+1}\mathbb{E}_{-j-1}\mathbb{E}_{-j}\mathbb{E}_{j+1}\mathbb{E}_j \longrightarrow \mathbb{E}_j\mathbb{E}_{-j}\mathbb{E}_{j+1}\mathbb{E}_j \longrightarrow \mathbb{E}_{j+1}\mathbb{E}_j.\end{aligned} \tag{4.36}$$

There are four nontrivial cases for λ. We prove the case $(\lambda_j, \lambda_{j+1}, \lambda_{j+2}) = (2,1,1)$. The proofs of the remaining cases $(2,1,0)$, $(1,1,0)$, and $(1,1,1)$ are similar.

Let B be the bimodule representing the functor $\mathbb{E}_j\mathbb{E}_{j+1}$ and B' the bimodule representing the functor $\mathbb{E}_{j+1}\mathbb{E}_j$. Then the morphism is the composition $B \to B \to B \to \mathcal{A} \otimes B \to B \to B'$ induced by the tangle cobordisms in Figure 15. The first and second maps are the identity maps. The third map is comultiplication. The fourth map is the trace map and the last map is $\psi_{j,j+1}$. Computing this composition on elements as in previous propositions easily gives that it is equal to $\psi_{j,j+1}$. □

4.6.3. $R(\nu)$ Relations

Proposition 4.17. *For $i, j \in I^{\pm}$, $i \neq j$,*

$$\psi_{-j,i} \circ \psi_{i,-j} = 1_i1_{-j}. \tag{4.37}$$

Proof. Note that, for $|i-j| > 1$, the left-hand side is easily seen to be the identity so let $j = i+1$. The case $j = i-1$ is similar. Thus the left-hand side is

$$\begin{aligned}\psi_{-j,i} \circ \psi_{i,-j} : \mathbb{E}_i\mathbb{E}_{-i-1} &\longrightarrow \mathbb{E}_{-i-1}\mathbb{E}_{i+1}\mathbb{E}_i\mathbb{E}_{-i-1} \longrightarrow \mathbb{E}_{-i-1}\mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_{-i-1} \longrightarrow \mathbb{E}_{-i-1}\mathbb{E}_i \\ &\longrightarrow \mathbb{E}_{-i-1}\mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_{-i-1} \longrightarrow \mathbb{E}_{-i-1}\mathbb{E}_{i+1}\mathbb{E}_i\mathbb{E}_{-i-1} \longrightarrow \mathbb{E}_i\mathbb{E}_{-i-1}.\end{aligned} \tag{4.38}$$

There are four non-trivial cases for λ.

Case 1 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,2,1))$. Let B be the bimodule representing the functor $\mathbb{E}_i\mathbb{E}_{-i-1}$. Then

$$\psi_{-j,i} \circ \psi_{i,-j} : B \longrightarrow \mathcal{A} \otimes B \longrightarrow B \longrightarrow B \longrightarrow B \longrightarrow \mathcal{A} \otimes B \longrightarrow B. \tag{4.39}$$

The first map is $\iota \otimes 1_\lambda$. The second map is multiplication m. The third and fourth maps are the identity. The fifth map is comultiplication Δ. The last map is $\mathrm{Tr} \otimes 1$. It is easy to check on elements that this is the identity map.

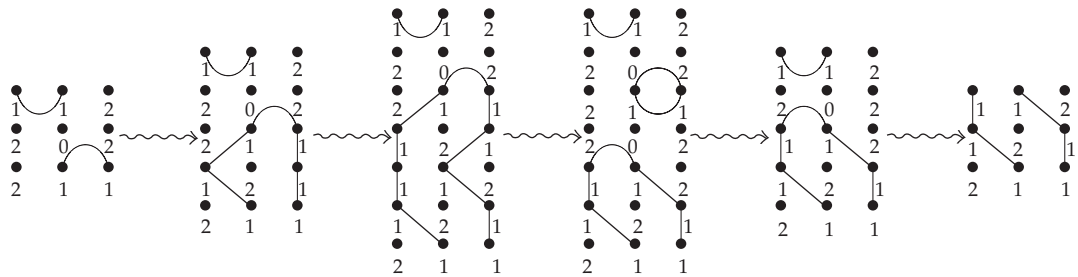

Figure 15: Tangles for compositions of natural transformations in the $(2,1,1)$ case.

Case 2 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,2,0))$. Let B be the bimodule representing the functor $\mathbb{E}_i\mathbb{E}_{-i-1}$. Then

$$\psi_{-j,i} \circ \psi_{i,-j} : B \longrightarrow B \longrightarrow \mathcal{A} \otimes B \longrightarrow B \longrightarrow \mathcal{A} \otimes B \longrightarrow B \longrightarrow B. \tag{4.40}$$

The first map is the identity. The second map is Δ by Lemma 4.3. The third map is $\mathrm{Tr} \otimes 1$ where the trace map is applied to the tensor factor arising from the new circle component. The fourth map is $\iota \otimes 1$. The fifth map is multiplication by Lemma 4.4. The last map is the identity. It is easy to check that this composition is the identity on all elements.

Case 3 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,2,1))$. This is similar to Case 2.

Case 4 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,2,0))$. This is similar to Case 1.

□

Proposition 4.18. *If $i, j \in I^+$ and $|i-j| > 1$, then $\psi_{j,i} \circ \psi_{i,j} = 1_i 1_j$.*

Proof. The tangle diagrams for the bimodules for $\mathbb{E}_i\mathbb{E}_j$ and $\mathbb{E}_j\mathbb{E}_i$ are the same up to isotopy. The maps in the proposition are obtained from cobordisms isotopic to the identity so they are identity maps. □

Proposition 4.19. *If $i, j \in I^+$ and $|i-j| = 1$, then $\psi_{j,i} \circ \psi_{i,j} = (y_i 1_j + 1_i y_j)$.*

Proof. Assume that $j = i+1$. The case $j = i-1$ is similar. There are eight cases for λ such that $\mathbb{E}_i\mathbb{E}_{i+1}$ is non-zero. In all cases let a and b be cup diagrams. Let B be the bimodule for $\mathbb{E}_i\mathbb{E}_{i+1}$ and B' the bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$.

Case 1. $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,0,1)$. Since $\mathbb{E}_{i+1}\mathbb{E}_i = 0$, the map $\psi_{i+1,i} \circ \psi_{i,i+1} = 0$. The bimodule representing the functor $\mathbb{E}_i\mathbb{E}_{i+1}$ is isomorphic to $\mathfrak{F}(D^{\lambda+\alpha_{i+1},i} \circ D^{\lambda,i+1})$. Since the circle passing through point i on the bottom row of $D^{\lambda+\alpha_{i+1},i} \circ D^{\lambda,i+1}$ is the same as the circle passing through point $i+1$ in the middle row, the map on the right side of the proposition is zero as well.

Case 2 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,0,1))$. This is similar to Case 1.

Case 3 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,0,2))$. This is similar to Case 1.

Case 4 $((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,0,2))$. This is similar to Case 1.

Case 5 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,1)$). In this case $B \cong \mathfrak{F}(T^{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1})$ and $B' \cong \mathfrak{F}(D^{\lambda+\alpha_i,i+1} \circ D^{\lambda,i})$. Let a and b be crossingless matches.

(i) Suppose that the circle passing through point $i+1$ on the bottom row of ${}_a(T^{\lambda+\alpha_{i+1},i}) \circ T_{\lambda,i+1})_b$ is the same as the circle passing through point i of the top row. Then ${}_aB_b = \mathcal{A} \otimes R$ and ${}_aB'_b = \mathcal{A} \otimes \mathcal{A} \otimes R$, where R is a tensor product of $\mathcal{A}$ corresponding to the remaining circles. Then the map on the left side of the proposition is $(m \otimes 1) \circ (\Delta \otimes 1)$. Thus it maps an element $1 \otimes r$ to $2x \otimes r$. On the other hand, $y_i(1 \otimes r) = x \otimes r$. Also, $y_{i+1}(1 \otimes r) = x \otimes r$. Thus both sides are the same.

(ii) Suppose that the circle passing through point $i+1$ on the bottom is different from the circle passing through point i on the top. Then ${}_aB_b = \mathcal{A} \otimes \mathcal{A} \otimes R$ and ${}_aB'_b = \mathcal{A} \otimes R$. Then the map on the left side of the proposition is $(\Delta \otimes 1_\lambda) \circ (m \otimes 1_\lambda)$. Thus it maps an element $1 \otimes 1 \otimes r$ to $x \otimes 1 \otimes r + 1 \otimes x \otimes r$. On the other hand, $y_i(1 \otimes 1 \otimes r) = x \otimes 1 \otimes r$. Also, $y_{i+1}(1 \otimes r) = 1 \otimes x \otimes r$. Thus both sides are the same

Case 6 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,1,1)$). In this case, $B \cong \mathfrak{F}(D_{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1})$ and $B' \cong \mathfrak{F}(D^{\lambda+\alpha_i,i+1} \circ T_{\lambda,i})$. Let a and b be crossingless matches.

(i) Suppose that the circle passing through point $i+1$ on the bottom row of $D_{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1}$ is the same as the circle passing through point i on the bottom row. Then ${}_aB_b = \mathcal{A} \otimes R$ and ${}_aB'_b = \mathcal{A} \otimes \mathcal{A} \otimes R$. Then the map on the left side of the proposition is $(m \otimes 1) \circ (\Delta \otimes 1)$. Thus it maps an element $1 \otimes r$ to $2x \otimes r$. On the other hand, $y_i(1 \otimes r) = x \otimes r$. Also, $y_{i+1}(1 \otimes r) = x \otimes r$. Thus both sides are the same.

(ii) Suppose that the circle passing through point $i+1$ on the bottom row of $D_{\lambda+\alpha_{i+1},i} \circ T_{\lambda,i+1}$ is different from the circle passing through point i on the bottom row. Then ${}_aB_b = \mathcal{A} \otimes \mathcal{A} \otimes R$ and ${}_aB'_b = \mathcal{A} \otimes R$. Then the map on the left side of the proposition is $(\Delta \otimes 1) \circ (m \otimes 1)$. Thus it maps an element $1 \otimes 1 \otimes r$ to $x \otimes 1 \otimes r + 1 \otimes x \otimes r$. On the other hand, $y_i(1 \otimes 1 \otimes r) = x \otimes 1 \otimes r$. Also, $y_{i+1}(1 \otimes r) = 1 \otimes x \otimes r$. Thus both sides are the same.

Case 7 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,1,2)$). This is similar to Case 5.

Case 8 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,2)$). This is similar to Case 6. □

Proposition 4.20. *Let $i, j \in I^+$. If $i \neq j$, then*

(1) $(1_j y_i) \circ \varphi_{i,j} = \varphi_{i,j} \circ (y_i 1_j)$,

(2) $(y_j 1_i) \circ \varphi_{i,j} = \varphi_{i,j} \circ (1_i y_j)$.

Proof. We prove only the first statement. Assume further that $j = i+1$, the case $j = i-1$ being similar. The case for $|j-i| > 1$ is easy because the bimodules for $\mathbb{E}_i\mathbb{E}_j$ and $\mathbb{E}_j\mathbb{E}_i$ are equal.

There are four non-trivial cases for $(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$. Let a and b be crossingless matches. Let B be the bimodule for $\mathbb{E}_i\mathbb{E}_{i+1}$ and let B' be the bimodule for $\mathbb{E}_{i+1}\mathbb{E}_i$.

Case 1 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,1,2)$). (i) Suppose that the circle passing through point i on the bottom row of the tangle for $\mathbb{E}_i\mathbb{E}_{i+1}$ is the same as the circle passing through point $i+1$ on the bottom row. Then ${}_aB_b = \mathcal{A} \otimes R$ and ${}_aB'_b = \mathcal{A} \otimes \mathcal{A} \otimes R$, where R denotes a tensor product of $\mathcal{A}$ corresponding to the remaining circles. Then $\varphi_{i,i+1}$ is given by $\Delta \otimes 1$. Then $\varphi_{i,i+1} y_i(1 \otimes r) = \varphi_{i,i+1}(x \otimes r) = x \otimes x \otimes r$. Then $y_i \varphi_{i,i+1}(1 \otimes r) = y_i(x \otimes 1 \otimes r + 1 \otimes x \otimes r) = x \otimes x \otimes r$.

(ii) Suppose that the circle passing through point i on the bottom row of the tangle for $\mathbb{E}_i\mathbb{E}_{i+1}$ is different from the circle passing through point $i+1$ on the bottom row. Then ${}_aB_b = \mathcal{A}\otimes\mathcal{A}\otimes R$ and ${}_aB'_b = \mathcal{A}\otimes R$. Then $\psi_{i,i+1} = m\otimes 1$. Then it is easy to verify that $\psi_{i,i+1}y_i(1\otimes 1\otimes r) = y_i\psi_{i,i+1}(1\otimes 1\otimes r) = x\otimes r$.

Case 2. $(\lambda_i,\lambda_{i+1},\lambda_{i+2}) = (0,1,1)$. This is similar to Case 1.

Case 3. $(\lambda_i,\lambda_{i+1},\lambda_{i+2}) = (1,1,1)$.

(i) Suppose that the circle passing through point i on the bottom row of the tangle is the same as the circle passing through point $i+1$ on the bottom row. Then ${}_aB_b = \mathcal{A}\otimes R$ and ${}_aB'_b = \mathcal{A}\otimes\mathcal{A}\otimes R$. Then $\psi_{i,i+1}$ is given by $\Delta\otimes 1$. This then follows as in Case 1.

(ii) Suppose that the circle passing through point i on the bottom row of the tangle is different from the circle passing through the point $i+1$ on the bottom row. Then ${}_aB_b = \mathcal{A}\otimes\mathcal{A}\otimes R$ and ${}_aB'_b = \mathcal{A}\otimes R$. Then $\psi_{i,i+1} = m\otimes 1$. This then follows as in Case 1.

Case 4 $((\lambda_i,\lambda_{i+1},\lambda_{i+2}) = (0,1,2))$. This is similar to Case 3. □

Proposition 4.21. *For $i,j,k\in I^+$,*

$$(\psi_{j,k}1_i)\circ(1_j\psi_{i,k})\circ(\psi_{i,j}1_k)+(1_k\psi_{i,j})\circ(\psi_{i,k}1_j)\circ(1_i\psi_{j,k}) = \begin{cases} 0, & i\neq k \text{ or } |i-j|\neq 1,\\ 1_i1_j1_i, & i=k \text{ and } |i-j|=1.\end{cases} \tag{4.41}$$

Proof. The proof of the first part consists of verifying the equality in many different cases, each of which is similar to the second part. We only prove the second part in the case $j=i+1$ as the case $j=i-1$ is similar. There are four cases for $(\lambda_i,\lambda_{i+1},\lambda_{i+2})$ for which $\mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i$ is non-zero.

Case 1 $((\lambda_i,\lambda_{i+1},\lambda_{i+2}) = (0,1,1))$. In this case, $(\psi_{j,i}1_i)\circ(1_j\psi_{i,i})\circ(\psi_{i,j}1_i) = 0$ because it passes through the functor $\mathbb{E}_{i+1}\mathbb{E}_i\mathbb{E}_i$ which is zero on the category corresponding to this λ. On the other hand,

$$(1_i\psi_{i,j})\circ(\psi_{i,i}1_j)\circ(1_i\psi_{j,i}) : \mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i \longrightarrow \mathbb{E}_i\mathbb{E}_i\mathbb{E}_{i+1} \longrightarrow \mathbb{E}_i\mathbb{E}_i\mathbb{E}_{i+1} \longrightarrow \mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i. \tag{4.42}$$

Let B be the bimodule for the functor $\mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i$. Then this is a sequence of maps

$$B \longrightarrow \mathcal{A}\otimes B \longrightarrow \mathcal{A}\otimes B \longrightarrow B, \tag{4.43}$$

where the first map is given by comultiplication, the middle map is given by the map $1\otimes\kappa$, and the last map is multiplication. This sequence of maps acts on $1\otimes\alpha\in B$ as follows:

$$1\otimes\alpha \longmapsto x\otimes 1\otimes\alpha + 1\otimes x\otimes\alpha \longmapsto 1\otimes 1\otimes\alpha \longmapsto 1\otimes\alpha. \tag{4.44}$$

Clearly, $(\psi_{j,i}1_i)\circ(1_j\psi_{i,i})\circ(\psi_{i,j}1_i)(1\otimes\alpha) = 0$. Similarly, $x\otimes\alpha\mapsto x\otimes\alpha$.

Case 2 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,2,2)$). This is similar to Case 1 except that now $(1_i\psi_{i,j}) \circ (\psi_{i,i}1_j) \circ (1_i\psi_{j,i}) = 0$ and $(\psi_{j,i}1_i) \circ (1_j\psi_{i,i}) \circ (\psi_{i,j}1_i) = 1_i1_j1_i$.

Case 3. $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,2)$. In this case, $(\psi_{j,i}1_i) \circ (1_j\psi_{i,i}) \circ (\psi_{i,j}1_i) = 0$ since this map passes through the functor $\mathbb{E}_{i+1}\mathbb{E}_i\mathbb{E}_i$ which is zero on the category corresponding to λ.

On the other hand,

$$(1_i\psi_{i,j}) \circ (\psi_{i,i}1_j) \circ (1_i\psi_{j,i}) : \mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i \longrightarrow \mathbb{E}_i\mathbb{E}_i\mathbb{E}_{i+1} \longrightarrow \mathbb{E}_i\mathbb{E}_i\mathbb{E}_{i+1} \longrightarrow \mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i. \tag{4.45}$$

Let B be the bimodule for the functor $\mathbb{E}_i\mathbb{E}_{i+1}\mathbb{E}_i$. Then this is a sequence of maps

$$B \longrightarrow \mathcal{A} \otimes B \longrightarrow \mathcal{A} \otimes B \longrightarrow B, \tag{4.46}$$

where the first and third maps are given by Lemmas 4.3 and 4.4, respectively, and the middle map is given in Section 4.5. This sequence of maps acts on $1 \otimes \alpha,\ x \otimes \alpha \in B$ as follows:

$$\begin{gathered} 1 \otimes \alpha \longmapsto x \otimes 1 \otimes \alpha + 1 \otimes x \otimes \alpha \longmapsto 1 \otimes 1 \otimes \alpha \longmapsto 1 \otimes \alpha, \\ x \otimes \alpha \longmapsto x \otimes x \otimes \alpha \longmapsto x \otimes 1 \otimes \alpha \longmapsto x \otimes \alpha. \end{gathered} \tag{4.47}$$

Case 4 ($(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,2,1)$). This is similar to Case 1 except that now $(1_i\psi_{i,j}) \circ (\psi_{i,i}1_j) \circ (1_i\psi_{j,i}) = 0$ and $(\psi_{j,i}1_i) \circ (1_j\psi_{i,i}) \circ (\psi_{i,j}1_i)(\beta \otimes \alpha) = \beta \otimes \alpha$. □

The relations of the 2-morphisms proven in this section give the following.

Theorem 4.22. *There is a 2-functor $\Omega_{k,n} : \mathcal{KL} \to \mathcal{HK}_{k,n}$ such that, for all $i, j \in I$,*

(1) $\Omega_{k,n}(\lambda) = \mathcal{C}_\lambda$,

(2) $\Omega_{k,n}(\mathcal{O}_\lambda) = \mathbb{I}_\lambda$,

(3) $\Omega_{k,n}(\mathcal{E}_i\mathcal{O}_\lambda) = \mathbb{E}_i\mathbb{I}_\lambda$,

(4) $\Omega_{k,n}(\Upsilon_{i;\lambda}) = y_{i;\lambda}$,

(5) $\Omega_{k,n}(\Psi_{i,j;\lambda}) = \psi_{i,j;\lambda}$,

(6) $\Omega_{k,n}(\bigcup_{i;\lambda}) = \cup_{i;\lambda}$,

(7) $\Omega_{k,n}(\bigcap_{i;\lambda}) = \cap_{i;\lambda}$,

(8) $\Omega_{k,n}(\mathbf{1}_{i;\lambda}) = 1_{i;\lambda}$.

5. The 2-Category $\mathcal{P}_{k,n}$

5.1. Graded Category ${}_{\mathbb{Z}}\mathcal{O}$

Let $\mathfrak{g} = \mathfrak{gl}_{2k}$ be the Lie algebra of $2k \times 2k$-matrices, let $\mathfrak{d}$ denote the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices, and let $\mathfrak{p}$ be the Borel subalgebra of upper triangular matrices. For $i = 1, \ldots, 2k$, let e_{ij} denote the (i,j)-matrix unit, and let $\varepsilon_i \in \mathfrak{d}^*$ be the coordinate functional

$\varepsilon_i(e_{jj}) = \delta_{ij}$. Let $\mathcal{O}$ be the category of finitely generated $\mathfrak{g}$-modules which are diagonalizable with respect to $\mathfrak{d}$ and locally finite with respect to $\mathfrak{p}$. Let

$$X = \bigoplus_{i=1}^{2k} \mathbb{Z}\varepsilon_i, \qquad Y = \bigoplus_{i=1}^{2k-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}) \subset X \tag{5.1}$$

denote the weight lattice and root lattice of $\mathfrak{gl}_{2k}$, respectively. The dominant weights are given by the set $X^+ = \{\ \mu = \mu_1\varepsilon_1 + \cdots + \mu_{2k}\varepsilon_{2k} \in X \mid \mu_1 \geq \cdots \geq \mu_{2k}\ \}$. Denote half the sum of the positive roots by ρ. Let $\mu \in X^+$, and let $\mathcal{O}_\mu$ be the block of $\mathcal{O}$ consisting of modules that have a generalized central character corresponding to μ under the Harish-Chandra homomorphism. Let $\mathcal{O}_\mu^{(k,k)}$ be the full subcategory $\mathcal{O}$ consisting of modules which are locally finite with respect to the parabolic subalgebra whose reductive part is $\mathfrak{gl}_k \oplus \mathfrak{gl}_k$. Finally, let $\mathcal{P}_\mu^{(k,k)}$ be the full subcategory of $\mathcal{O}_\mu^{(k,k)}$ whose objects have projective presentations by projective-injective modules.

Let μ and μ' be integral dominant weights of $\mathfrak{g}$, and let $\mathrm{Stab}(\mu)$ denote the stabilizer of μ under the ρ-shifted action of the symmetric group $\mathbb{S}_{2k}$. Suppose that $\mu' - \mu$ is an integral dominant weight. Then, let $\theta_\mu^{\mu'} : \mathcal{O}_\mu^{(k,k)} \to \mathcal{O}_{\mu'}^{(k,k)}$ be the translation functor of tensoring with the finite-dimensional irreducible representation of highest weight $\mu' - \mu$ composed with projecting onto the μ'-block, and let $\theta_{\mu'}^{\mu}$ be its adjoint.

Let P_μ be a minimal projective generator of $\mathcal{O}_\mu$. It was shown that $A_\mu = \mathrm{End}_{\mathfrak{g}}(P_\mu)$ has the structure of a graded algebra [11]. Since $\mathcal{O}_\mu$ is Morita equivalent to A_μ-mod, we consider the category of graded A_μ-modules which we denote by ${}_{\mathbb{Z}}\mathcal{O}_\mu$. Let the graded lift of $\mathcal{O}_\mu^{(k,k)}$ and $\mathcal{P}_\mu^{(k,k)}$ be ${}_{\mathbb{Z}}\mathcal{O}_\mu^{(k,k)}$ and ${}_{\mathbb{Z}}\mathcal{P}_\mu^{(k,k)}$, respectively. It is known that if $\mathrm{Stab}(\mu) \subset \mathrm{Stab}(\mu')$, there is a graded lift of the translation functors, compare, for example, [14], which by abuse of notation we denote again by $\widetilde{\theta}_{\mu'}^{\mu}$ and $\widetilde{\theta}_{\mu}^{\mu'}$.

The key tool in the construction of graded category $\mathcal{O}$ is the Soergel functor. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a composition of $2k$ and $\mathbb{S}_\lambda = \mathbb{S}_{\lambda_1} \times \cdots \times \mathbb{S}_{\lambda_n}$. Denote the longest coset representative in $\mathbb{S}_{2k}/\mathbb{S}_\mu$ by w_0^μ. Let $P(w_0^\mu \cdot \mu)$ be the unique up to isomorphism, indecomposable projective-injective object of $\mathcal{O}_\mu$. Let $C = S(\mathfrak{h})/S(\mathfrak{h})_+^{\mathbb{S}_{2k}}$ be the coinvariant algebra of the symmetric algebra for the Cartan subalgebra with respect to the action of the symmetric group. Let $\{x_1, \ldots, x_{2k}\}$ be a basis of $S(\mathfrak{h})$ and by abuse of notation also let x_i denote its image in C. Let C^λ be the subalgebra of elements invariant under the action of $\mathbb{S}_\lambda$. Soergel proved in [15] the following.

Proposition 5.1. *One has* $\mathrm{End}_{\mathfrak{g}}(P(w_0^\mu \cdot \mu)) \cong C^{\mathrm{Stab}(\mu)}$.

Define the Soergel functor $\mathbb{V}_\mu : \mathcal{O}_\mu \to C^{\mathrm{Stab}(\mu)}$-mod to be $\mathrm{Hom}_{\mathfrak{g}}(P(w_0 \cdot \mu), \bullet)$.

Proposition 5.2. *Let P be a projective object. Then there is a natural isomorphism* $\mathrm{Hom}_{C^{\mathrm{Stab}(\mu)}}(\mathbb{V}_\mu P, \mathbb{V}_\mu M) \cong \mathrm{Hom}_{\mathfrak{g}}(P, M)$.

Proof. This is the Structure Theorem of [15]. □

Proposition 5.3. *Let $\mu, \mu' \in X^+$ be integral dominant weights such that there is a containment of stabilizers:* $\mathrm{Stab}(\mu) \subset \mathrm{Stab}(\mu')$. *Then there are isomorphisms of functors*

(1) $\mathbb{V}_{\mu'}\theta_{\mu}^{\mu'} \cong \mathrm{Res}_{C^{\mathrm{Stab}(\mu)}}^{C^{\mathrm{Stab}(\mu')}}\mathbb{V}_{\mu}$,

(2) $\mathbb{V}_{\mu}\theta_{\mu'}^{\mu} \cong C^{\mathrm{Stab}(\mu)}\bigotimes_{C^{\mathrm{Stab}(\mu')}}\mathbb{V}_{\mu'}$.

Proof. These are Theorem 12 and Proposition 6 of [16]. □

5.2. The Objects of $\mathcal{P}_{k,n}$

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition of $2k$ with $\lambda_i \in \{0, 1, 2\}$ for all i. To each such λ, we associate an integral dominant weight

$$\overline{\lambda} = \sum_{j=1}^{r}\sum_{i=1}^{\lambda_j}(r-j+1)\varepsilon_{\lambda_1+\cdots+\lambda_{j-1}+i} - \rho \tag{5.2}$$

of $\mathfrak{gl}_{2k}$, where $\lambda_0 = 0$. Note that the stabilizer of this weight under the action of $\mathbb{S}_{2k}$ is $\mathbb{S}_{\lambda_1} \times \cdots \times \mathbb{S}_{\lambda_n}$.

The set of objects of $\mathcal{P}_{k,n}$ are the categories ${}_{\mathbb{Z}}\mathcal{P}_{\overline{\lambda}}^{(k,k)}$, $\lambda \in P(V_{2\omega_k})$.

5.3. The 1-Morphisms of $\mathcal{P}_{k,n}$

Let $\lambda \in P(\mathrm{V}_{2\omega_k})$, and let $\mathbb{I}_\lambda \in \mathrm{End}_{\mathfrak{g}}({}_{\mathbb{Z}}\mathcal{P}_{\overline{\lambda}}^{(k,k)})$ be the identity functor.

For each $i \in I$, we define functors $\mathbb{E}_i\mathbb{I}_\lambda$ and $\mathbb{K}_i\mathbb{I}_\lambda$. To this end, let λ be a weight of $V_{2\omega_k}$ and $i \in I^+$. Then we have compositions of $2k$ into $n+1$ parts:

$$\lambda(i) = (\lambda_1, \ldots, \lambda_i, 1, \lambda_{i+1} - 1, \ldots, \lambda_n), \qquad \lambda(-i) = (\lambda_1, \ldots, \lambda_i - 1, 1, \lambda_{i+1}, \ldots, \lambda_n) \tag{5.3}$$

Also, if $\lambda = \sum_i a_i\omega_i \in P$, set $r_{i,\lambda} = 1 + a_1 + \cdots + a_{i-1} + a_{i+1}$ and $s_{i,\lambda} = 2 - a_i - a_{i+1}$.

Let $i \in I$. Suppose that $(\lambda_i, \lambda_{i+1}) \in \{(0,1), (0,2), (1,1), (1,2)\}$. Then we define, as in [17], $\mathbb{E}_i\mathbb{I}_\lambda : {}_{\mathbb{Z}}\mathcal{P}_{\overline{\lambda}}^{(k,k)} \to {}_{\mathbb{Z}}\mathcal{P}_{\overline{\lambda+\alpha_i}}^{(k,k)}$ which is given by tensoring with the following bimodule:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}\left(P_{\overline{\lambda+\alpha_i}}, \theta_{\overline{\lambda(i)}}^{\overline{\lambda+\alpha_i}}\theta_{\overline{\lambda}}^{\overline{\lambda(i)}}P_{\overline{\lambda}}\{r_{i,\lambda}\}\right) &\cong \mathrm{Hom}_{C^{\lambda+\alpha_i}}\left(\mathbb{V}_{\overline{\lambda+\alpha_i}}P_{\overline{\lambda+\alpha_i}}, \mathbb{V}_{\overline{\lambda+\alpha_i}}\theta_{\overline{\lambda(i)}}^{\overline{\lambda+\alpha_i}}\theta_{\overline{\lambda}}^{\overline{\lambda(i)}}P_{\overline{\lambda}}\{r_{i,\lambda}\}\right)\\ &\cong \mathrm{Hom}_{C^{\lambda+\alpha_i}}\left(\mathbb{V}_{\overline{\lambda+\alpha_i}}P_{\overline{\lambda+\alpha_i}}, C^{\lambda+\alpha_i}\bigotimes_{C^{\lambda(i)}}\mathrm{Res}_{C^{\lambda}}^{C^{\lambda(i)}}\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\{r_{i,\lambda}\}\right).\end{aligned} \tag{5.4}$$

For all other values of $(\lambda_i, \lambda_{i+1})$, set $\mathbb{E}_i\mathbb{I}_\lambda = 0$. Let $\mathbb{K}_i\mathbb{I}_\lambda : {}_{\mathbb{Z}}\mathcal{P}_{\overline{\lambda}}^{(k,k)} \to {}_{\mathbb{Z}}\mathcal{P}_{\overline{\lambda}}^{(k,k)}$ be the grading shift functor $\mathbb{K}_i\mathbb{I}_\lambda = \mathbb{I}_\lambda\{(\alpha_i, \lambda)\}$.

Let ${}_{\mathbb{Z}}p^{(k,k)}_{\underline{\lambda}}$ and ${}_{\mathbb{Z}}p^{(k,k)}_{\underline{\lambda}'}$ be two objects. Then

$$\operatorname{Hom}\left({}_{\mathbb{Z}}p^{(k,k)}_{\underline{\lambda}}, {}_{\mathbb{Z}}p^{(k,k)}_{\underline{\lambda}'}\right) = \bigoplus_{\substack{\underline{i}\in \mathrm{Seq} \\ s\in\mathbb{Z}}} \mathbb{I}_{\lambda'}\mathbb{E}_{\underline{i}}\mathbb{I}_{\lambda}\{s\}, \tag{5.5}$$

where $\mathbb{E}_{\underline{i}} := \mathbb{E}_{i_1}\cdots\mathbb{E}_{i_r}\mathbb{I}_{\lambda}$ if $\underline{i} = (i_1,\ldots,i_r) \in I_{\infty}$, and s refers to a *grading shift*.

5.4. Bimodule Categories over the Cohomology of Flag Varieties

A review of certain bimodules and bimodule maps over the cohomology of flag varieties developed in [1, 2, 18] is given here. Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be a composition of $2k$ into n parts. Let $x(\lambda)_{j,r} = x_{\lambda_1+\cdots+\lambda_{j-1}+r}$. There is an isomorphism of algebras:

$$C^{\lambda} \cong \frac{\bigotimes_{1\le j\le n}\mathbb{C}\left[x(\lambda)_{j,1}, x(\lambda)_{j,2},\ldots,x(\lambda)_{j,\lambda_j}\right]}{J_{\lambda,n}}, \tag{5.6}$$

where $J_{\lambda,n}$ is the ideal generated by the homogeneous terms in the equation

$$\prod_{1\le j\le n}\left(1 + x(\lambda)_{j,1}t + x(\lambda)_{j,2}t^2 + \cdots + x(\lambda)_{j,\lambda_j}t^{\lambda_j}\right) = 1. \tag{5.7}$$

Let $\widehat{x}(\lambda)_{i,k}$ be the homogenous term of degree $2k$ in the product

$$\prod_{\substack{1\le j\le n \\ j\ne i}}\left(1 + x(\lambda)_{j,1}t + x(\lambda)_{j,2}t^2 + \cdots + x(\lambda)_{j,\lambda_j}t^{\lambda_j}\right). \tag{5.8}$$

Then, using (5.7), we see that

$$\sum_{j=1}^{k} x(\lambda)_{i,j}\widehat{x}(\lambda)_{i,k-j} = \delta_{k,0}, \tag{5.9}$$

compare, for example, [1, Section 5.1] for details.

We must also consider $C^{\lambda(i)}$. There is an isomorphism of algebras:

$$\begin{aligned} C^{\lambda(i)} \cong & \bigotimes_{\substack{1\le j\le n, \\ j\ne i+1}} \mathbb{C}\left[x(\lambda)_{j,1}, x(\lambda)_{j,2},\ldots,x(\lambda)_{j,\lambda_j}\right] \otimes \mathbb{C}[\zeta_i] \\ & \otimes \frac{\mathbb{C}\left[x(\lambda)_{i+1,1}, x(\lambda)_{i+1,2},\ldots,x(\lambda)_{i+1,\lambda_{i+1}-1}\right]}{J_{\lambda(i),n}}, \end{aligned} \tag{5.10}$$

where $J_{\lambda(i),n}$ is the ideal generated by the homogeneous terms in the equation

$$\prod_{\substack{1\le j\le n,\\ j\ne i+1}}(1+\zeta_i t)\sum_{r=0}^{\lambda_{i+1}-1}x(\lambda)_{i+1,r}t^r\sum_{s=0}^{\lambda_j}x(\lambda)_{j,s}t^s=1. \tag{5.11}$$

There is also an isomorphism of algebras:

$$C^{\lambda(-i)}\cong\bigotimes_{\substack{1\le j\le n,\\ j\ne i}}\mathbb{C}\left[x(\lambda)_{j,1},x(\lambda)_{j,2},\dots,x(\lambda)_{j,\lambda_j}\right]\otimes\mathbb{C}\left[x(\lambda)_{i,1},x(\lambda)_{i,2},\dots,x(\lambda)_{i,\lambda_i-1}\right]\otimes\mathbb{C}[\zeta_i]/J_{\lambda(-i),n}, \tag{5.12}$$

where $J_{\lambda(-i),n}$ is the ideal generated by the homogeneous terms in the equation

$$\prod_{\substack{1\le j\le n,\\ j\ne i}}(1+\zeta_i t)\sum_{r=0}^{\lambda_i-1}x(\lambda)_{i,r}t^r\sum_{s=0}^{\lambda_j}x(\lambda)_{j,s}t^s=1. \tag{5.13}$$

5.5. The 2-Morphisms

In light of Propositions 5.2 and 5.3, we may define the 2-morphisms on the algebras C^λ, $\lambda\in P(V_{2\omega_k})$ in order to define natural transformations of functors.

The Maps $\overline{y}_{i;\lambda}$

Let $i\in I$. Define $\overline{y}_{i;\lambda}:C^{\lambda(i)}\to C^{\lambda(i)}$ which is a map of $(C^{\lambda+\alpha_i},C^\lambda)$-bimodules by $\overline{y}_{i;\lambda}((\zeta_i)^r)=(\zeta_i)^{r+1}$.

The Maps $\overline{\cup}_{i;\lambda},\overline{\cap}_{i;\lambda}$

Let $i\in I^+$. Define a map of (C^λ,C^λ)-bimodules

$$\overline{\cup}_{i;\lambda}:C^\lambda\longrightarrow C^{\lambda(i)}\bigotimes_{C^{\lambda+\alpha_i}}C^{\lambda(i)}\{1-\lambda_i-\lambda_{i+1}\} \tag{5.14}$$

by

$$\overline{\cup}_{i;\lambda}(1)=\sum_{f=0}^{\lambda_i}(-1)^{\lambda_i-f}\zeta_i^f\otimes x(\lambda)_{i,\lambda_i-f}. \tag{5.15}$$

Next define a map of (C^λ,C^λ)-bimodules

$$\overline{\cup}_{-i;\lambda}:C^\lambda\longrightarrow C^{\lambda(-i)}\bigotimes_{C^{\lambda-\alpha_i}}C^{\lambda(-i)}\{1-\lambda_i-\lambda_{i+1}\} \tag{5.16}$$

by

$$\overline{\cup}_{-i;\lambda}(1) = \sum_{f=0}^{\lambda_{i+1}} (-1)^{\lambda_{i+1}-f} \zeta_i^f \otimes x(\lambda)_{i+1,\lambda_{i+1}-f}. \tag{5.17}$$

Next define a map of (C^λ, C^λ)-bimodules

$$\overline{\cap}_{i;\lambda} : C^{\lambda(i)} \bigotimes_{C^{\lambda+\alpha_i}} C^{\lambda(i)}\{1-\lambda_i-\lambda_{i+1}\} \longrightarrow C^\lambda \tag{5.18}$$

by

$$\overline{\cap}_{i;\lambda}\left(\zeta_i^{r_1} \otimes \zeta_i^{r_2}\right) = (-1)^{r_1+r_2+1-\lambda_{i+1}} \widehat{x}(\lambda)_{i+1,r_1+r_2+1-\lambda_{i+1}}. \tag{5.19}$$

Next define a map of (C^λ, C^λ)-bimodules

$$\overline{\cap}_{-i;\lambda} : C^{\lambda(-i)} \bigotimes_{C^{\lambda-\alpha_i}} C^{\lambda(-i)}\{1-\lambda_i-\lambda_{i+1}\} \longrightarrow C^\lambda \tag{5.20}$$

by

$$\overline{\cap}_{-i;\lambda}\left(\zeta_i^{r_1} \otimes \zeta_i^{r_2}\right) = (-1)^{r_1+r_2+1-\lambda_i} \widehat{x}(\lambda)_{i,r_1+r_2+1-\lambda_i}. \tag{5.21}$$

The Maps $\overline{\varphi}_{i,j;\lambda}$

Let $i, j \in I^+$. Define a map of $(C^{\lambda+\alpha_i+\alpha_j}, C^\lambda)$-bimodules

$$\overline{\varphi}_{i,j;\lambda} : C^{(\lambda+\alpha_j)(i)} \bigotimes_{C^{\lambda+\alpha_j}} C^{\lambda(j)} \longrightarrow C^{(\lambda+\alpha_i)(j)} \bigotimes_{C^{\lambda+\alpha_i}} C^{\lambda(i)} \tag{5.22}$$

by

$$\overline{\varphi}_{i,j;\lambda}\left(\zeta_i^{r_1} \otimes \zeta_j^{r_2}\right) = \begin{cases} \zeta_j^{r_2} \otimes \zeta_i^{r_1} & \text{if } |i-j| > 1, \\ \displaystyle\sum_{f=0}^{r_1-1} \zeta_i^{r_1+r_2-1-f} \otimes \zeta_i^f - \sum_{g=0}^{r_2-1} \zeta_i^{r_1+r_2-1-g} \otimes \zeta_i^g & \text{if } j = i, \\ \left(\zeta_j^{r_2} \otimes \zeta_i^{r_1+1} - \zeta_j^{r_2+1} \otimes \zeta_i^{r_1}\right)\{-1\} & \text{if } i = j+1, \\ \left(\zeta_j^{r_2} \otimes \zeta_i^{r_1}\right)\{1\} & \text{if } j = i+1. \end{cases} \tag{5.23}$$

Define a map of $(C^{\lambda-\alpha_i-\alpha_j}, C^\lambda)$-bimodules

$$\overline{\varphi}_{-i,-j;\lambda} : C^{(\lambda-\alpha_j)(-i)} \bigotimes_{C^{\lambda-\alpha_j}} C^{\lambda(-j)} \longrightarrow C^{(\lambda-\alpha_i)(-j)} \bigotimes_{C^{\lambda-\alpha_i}} C^{\lambda(-i)} \tag{5.24}$$

by

$$\overline{\varphi}_{-i,-j}\left(\zeta_i^{r_1}\otimes\zeta_j^{r_2}\right)=\begin{cases}\zeta_j^{r_2}\otimes\zeta_i^{r_1} & \text{if } |i-j|>1,\\ \displaystyle\sum_{f=0}^{r_2-1}\zeta_i^{r_1+r_2-1-f}\otimes\zeta_i^{f}-\sum_{g=0}^{r_1-1}\zeta_i^{r_1+r_2-1-g}\otimes\zeta_i^{g} & \text{if } j=i,\\ \left(\zeta_j^{r_2}\otimes\zeta_i^{r_1+1}\right)\{-1\} & \text{if } i=j+1,\\ \left(\zeta_j^{r_2+1}\otimes\zeta_i^{r_1}-\zeta_j^{r_2}\otimes\zeta_i^{r_1+1}\right)\{1\} & \text{if } j=i+1.\end{cases}\tag{5.25}$$

5.6. The 2-Morphisms of $\rho_{k,n}$

Let $i,j\in I^+$.

The Maps $1_{i;\lambda}$

Let $1_{i;\lambda}:\mathbb{E}_i\mathbb{I}_\lambda\to\mathbb{E}_i\mathbb{I}_\lambda$ and $1_{-i;\lambda}:\mathbb{E}_{-i}\mathbb{I}_\lambda\to\mathbb{E}_{-i}\mathbb{I}_\lambda$ be the identity morphisms.

The Maps $y_{i;\lambda}$

Next we define a morphism of degree 2, $y_{i;\lambda}:\mathbb{E}_i\mathbb{I}_\lambda\to\mathbb{E}_i\mathbb{I}_\lambda$. Recall that

$$\mathbb{E}_i\mathbb{I}_\lambda\cong\mathrm{Hom}_{C^{\lambda+\alpha_i}}\left(\mathbb{V}_{\overline{\lambda+\alpha_i}}P_{\overline{\lambda+\alpha_i}},C^{\lambda(i)}\bigotimes_{C^\lambda}\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\{r_{i,\lambda}\}\right).\tag{5.26}$$

Let f be such a homomorphism. Suppose that $f(m)=\gamma\otimes n$. Then set $(y_{i;\lambda}\cdot f)(m)=\overline{y}_i(\gamma)\otimes n$. Similarly,

$$\mathbb{E}_{-i}\mathbb{I}_\lambda\cong\mathrm{Hom}_{C^{\lambda-\alpha_i}}\left(\mathbb{V}_{\overline{\lambda-\alpha_i}}P_{\overline{\lambda-\alpha_i}},C^{\lambda(-i)}\bigotimes_{C^\lambda}\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\{s_{i,\lambda}\}\right).\tag{5.27}$$

Let f be such a homomorphism. Suppose that $f(m)=\gamma\otimes n$. Then set $(y_{-i;\lambda}\cdot f)(m)=\overline{y}_{-i;\lambda}(\gamma)\otimes n$.

The Maps $\cup_{i;\lambda},\cap_{i;\lambda}$

Note that

$$\begin{gathered}\mathbb{I}_\lambda\cong J=\mathrm{Hom}_{C^\lambda}\left(\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}},\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\right),\\ \mathbb{E}_{-i}\circ\mathbb{E}_i\mathbb{I}_\lambda\cong K=\mathrm{Hom}_{C^\lambda}\left(\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}},C^{\lambda+\alpha_i(-i)}\bigotimes_{C^{\lambda+\alpha_i}}C^{\lambda(i)}\bigotimes_{C^\lambda}\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\{r_{\lambda,i}+s_{\lambda+\alpha_i,i}\}\right),\\ \mathbb{E}_i\circ\mathbb{E}_{-i}\mathbb{I}_\lambda\cong L=\mathrm{Hom}_{C^\lambda}\left(\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}},C^{\lambda-\alpha_i(i)}\bigotimes_{C^{\lambda-\alpha_i}}C^{\lambda(-i)}\bigotimes_{C^\lambda}\mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\{s_{\lambda,i}+r_{\lambda-\alpha_i,i}\}\right).\end{gathered}\tag{5.28}$$

Let $f \in J$. Then define $\cup_{i;\lambda} : \mathbb{I}_\lambda \to \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda$ by

$$\cup_{i;\lambda}(f)(m) = \overline{\cup}_{i;\lambda}(1) \otimes f(m) \tag{5.29}$$

and $\cup_{-i;\lambda} : \mathbb{I}_\lambda \to \mathbb{E}_i\mathbb{E}_{-i}\mathbb{I}_\lambda$ by

$$\cup_{-i;\lambda}(f)(m) = \overline{\cup}_{-i;\lambda}(1) \otimes f(m). \tag{5.30}$$

Now define $\cap_{i;\lambda} : \mathbb{E}_{-i}\mathbb{E}_i\mathbb{I}_\lambda \to \mathbb{I}_\lambda$. Suppose that $f \in K$ such that $f(m) = \gamma \otimes n$. Then set $\cap_{i;\lambda}(f)(m) = \overline{\cap}_{i;\lambda}(\gamma) \otimes n$.

Next define $\cap_{-i;\lambda} : \mathbb{E}_i\mathbb{E}_{-i}\mathbb{I}_\lambda \to \mathbb{I}_\lambda$. Suppose that $f \in L$ such that $f(m) = \gamma \otimes n$. Then set $\cap_{-i;\lambda}(f)(m) = \overline{\cap}_{-i;\lambda}(\gamma) \otimes n$.

The Maps $\psi_{i,j;\lambda}$

First we define a map $\psi_{i,j;\lambda} : \mathbb{E}_i\mathbb{E}_j\mathbb{I}_\lambda \to \mathbb{E}_j\mathbb{E}_i\mathbb{I}_\lambda$.

Set

$$J^+_{i,j} = \mathbb{E}_i\mathbb{E}_j\mathbb{I}_\lambda \cong \mathrm{Hom}_{C^{\lambda+\alpha_i+\alpha_j}}\left(\mathbb{V}_{\overline{\lambda+\alpha_i+\alpha_j}}P_{\overline{\lambda+\alpha_i+\alpha_j}}, C^{(\lambda+\alpha_j)(i)} \bigotimes_{C^{\lambda+\alpha_j}} C^{\lambda(j)} \bigotimes_{C^\lambda} \mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\left\{r_{\lambda,j} + r_{\lambda+\alpha_j,i}\right\}\right),$$

$$K^+_{i,j} = \mathbb{E}_j\mathbb{E}_i\mathbb{I}_\lambda \cong \mathrm{Hom}_{C^{\lambda+\alpha_j+\alpha_i}}\left(\mathbb{V}_{\overline{\lambda+\alpha_j+\alpha_i}}P_{\overline{\lambda+\alpha_j+\alpha_i}}, C^{(\lambda+\alpha_i)(j)} \bigotimes_{C^{\lambda+\alpha_i}} C^{\lambda(i)} \bigotimes_{C^\lambda} \mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\left\{r_{\lambda,i} + r_{\lambda+\alpha_i,j}\right\}\right). \tag{5.31}$$

Let $f \in J^+_{i,j}$ and suppose that $f(m) = \gamma_1 \otimes \gamma_2 \otimes n$. Then define $\psi_{i,j;\lambda}f(m) = \overline{\psi}_{i,j;\lambda}(\gamma_1 \otimes \gamma_2) \otimes n$.

Set

$$J^-_{i,j} = \mathbb{E}_{-i}\mathbb{E}_{-j}\mathbb{I}_\lambda \cong \mathrm{Hom}_{C^{\lambda-\alpha_i-\alpha_j}}\left(\mathbb{V}_{\overline{\lambda-\alpha_i-\alpha_j}}P_{\overline{\lambda-\alpha_i-\alpha_j}}, C^{(\lambda-\alpha_j)(-i)} \bigotimes_{C^{\lambda-\alpha_j}} C^{\lambda(-j)} \bigotimes_{C^\lambda} \mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\left\{s_{\lambda,j} + s_{\lambda-\alpha_j,i}\right\}\right),$$

$$K^-_{i,j} = \mathbb{E}_{-j}\mathbb{E}_{-i}\mathbb{I}_\lambda \cong \mathrm{Hom}_{C^{\lambda-\alpha_j-\alpha_i}}\left(\mathbb{V}_{\overline{\lambda-\alpha_j-\alpha_i}}P_{\overline{\lambda-\alpha_j-\alpha_i}}, C^{(\lambda-\alpha_i)(-j)} \bigotimes_{C^{\lambda-\alpha_i}} C^{\lambda(-i)} \bigotimes_{C^\lambda} \mathbb{V}_{\overline{\lambda}}P_{\overline{\lambda}}\left\{s_{\lambda,i} + s_{\lambda-\alpha_i,j}\right\}\right). \tag{5.32}$$

Let $f \in J^-_{i,j}$ and suppose that $f(m) = \gamma_1 \otimes \gamma_2 \otimes n$. Then define $\psi_{-i,-j;\lambda}f(m) = \overline{\psi}_{-i,-j;\lambda}(\gamma_1 \otimes \gamma_2) \otimes n$.

Theorem 5.4. *There is a 2-functor $\Pi_{k,n} : \mathcal{KL} \to \mathcal{P}_{k,n}$ such that, for all $i, j \in I$,*

(1) $\Pi_{k,n}(\lambda) = {}_{\mathbb{Z}}\mathcal{P}^{(k,k)}_{\overline{\lambda}}$,

(2) $\Pi_{k,n}(\mathcal{O}_\lambda) = \mathbb{I}_\lambda$,

(3) $\Pi_{k,n}(\mathcal{E}_i\mathcal{O}_\lambda) = \mathbb{E}_i\mathbb{I}_\lambda$,

(4) $\Pi_{k,n}(Y_{i;\lambda}) = y_{i;\lambda}$,

(5) $\Pi_{k,n}(\Psi_{i,j;\lambda}) = \psi_{i,j;\lambda}$,

(6) $\Pi_{k,n}(\bigcup_{i;\lambda}) = \cup_{i;\lambda}$,

(7) $\Pi_{k,n}(\bigcap_{i;\lambda}) = \cap_{i;\lambda}$,

(8) $\Pi_{k,n}(\mathbf{1}_{i;\lambda}) = 1_{i;\lambda}$.

Proof. This now follows from the computations in [1, Section 6.2] for bimodules over the cohomology of flag varieties using the naturality of the isomorphism in Proposition 5.2. □

Finally we show that the category $\mathcal{P}_{k,n}$ is a categorification of the module $V_{2\omega_k}$. Denote the Grothendieck group of $\mathcal{P}_{k,n}$ by $[\mathcal{P}_{k,n}]$, and let $[\mathcal{P}_{k,n}]_{\mathbb{Q}(q)} = \mathbb{C}(q)\otimes_{\mathbb{Z}[q,q^{-1}]}[\mathcal{P}_{k,n}]$.

Proposition 5.5. *There is an isomorphism of $\mathcal{U}_q(\mathfrak{sl}_n)$-modules $[\mathcal{P}_{k,n}]_{\mathbb{Q}(q)} \cong V_{2\omega_k}$.*

Proof. Since projective functors map projective-injective modules to projective-injective modules, it follows from Theorem 5.4 and [1] that $[\mathcal{P}_{k,n}]_{\mathbb{Q}(q)}$ is a $\mathcal{U}_q(\mathfrak{sl}_n)$-module. By construction, it contains a highest weight vector of weight $2\omega_k$ so it suffices to compute the dimension of its weight spaces.

By [19, Theorem 4.8], the number of projective-injective objects in $\mathcal{O}_{\bar{\lambda}}^{(k,k)}(\mathfrak{gl}_{2k})$ is equal to the number of column decreasing and row nondecreasing tableau for a diagram with k rows and 2 columns with entries from the set

$$\left\{\underbrace{n,\ldots,n}_{\lambda_1},\ldots,\underbrace{1,\ldots,1}_{\lambda_n}\right\}. \tag{5.33}$$

Call the set of such tableau T.

Let $S = \{i \in I^+ \mid \lambda_i = 1\}$. Denote by $|S|$ the cardinality of this set. Consider a Young diagram with $|S|/2$ rows and 2 columns. Let T' denote the set of tableau on such a column with entries from S such that the rows and columns are decreasing. It is well known that the cardinality of the set T' is the Catalan number $\binom{2|S|}{|S|}/(|S|+1)$. There is a bijection between T and T'. For any tableaux $t' \in T'$, one constructs a tableaux $t \in T$ by inserting a new box with the entry i in each column for each $i \in I^+$ such that $\lambda_i = 2$. The inverse is given by box removal.

Finally, the Weyl character formula gives that the dimension of the λ weight space of $V_{2\omega_k}$ is $\binom{2|S|}{|S|}/(|S|+1)$. □

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Research Article

Universal Verma Modules and the Misra-Miwa Fock Space

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The Misra-Miwa v-deformed Fock space is a representation of the quantized affine algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$. It has a standard basis indexed by partitions, and the nonzero matrix entries of the action of the Chevalley generators with respect to this basis are powers of v. Partitions also index the polynomial Weyl modules for $U_q(\mathfrak{gl}_N)$ as N tends to infinity. We explain how the powers of v which appear in the Misra-Miwa Fock space also appear naturally in the context of Weyl modules. The main tool we use is the Shapovalov determinant for a universal Verma module.

1. Introduction

Fock space is an infinite dimensional vector space which is a representation of several important algebras, as described in, for example, [1, Chapter 14]. Here we consider the charge zero part of Fock space, which we denote by $\mathbf{F}$, and its v-deformation $\mathbf{F}_v$. The space $\mathbf{F}$ has a standard $\mathbb{Q}$-basis $\{|\mu\rangle \mid \lambda \text{ is a partition}\}$ and $\mathbf{F}_v := \mathbf{F}\otimes_{\mathbb{Q}}\mathbb{Q}(v)$. Following Hayashi [2], Misra and Miwa [3] define an action of the quantized universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$. The only nonzero matrix elements $\langle\mu|F_{\bar{i}}|\lambda\rangle$ of the Chevalley generators $F_{\bar{i}}$ in terms of the standard basis occur when μ is obtained by adding a single $\bar{i}$-colored box to λ, and these are powers of v.

We show that these powers of v also appear naturally in the following context: partitions with at most N parts index polynomial Weyl modules $\Delta(\lambda)$ for the integral quantum group $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. Let V be the standard N dimensional representation of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. If the matrix element $\langle\mu|F_{\bar{i}}|\lambda\rangle$ is nonzero then, for sufficiently large N, $(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ contains the highest weight vector of weight μ. There is a unique such highest weight vector v_μ which satisfies a certain triangularity condition with respect to an integral basis of $\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V$. We show that the matrix element $\langle\mu|F_{\bar{i}}|\lambda\rangle$ is equal to $v^{\mathrm{val}_{\phi_{2\ell}}(v_\mu,v_\mu)}$, where $(\cdot,\cdot)$ is the Shapovalov form and $\mathrm{val}_{\phi_{2\ell}}$ is the valuation at the cyclotomic polynomial $\phi_{2\ell}$.

Our proof is computational, making use of the Shapovalov determinant [4–6]. This is a formula for the determinant of the Shapovalov form on a weight space of a Verma module. The necessary computation is most easily done in terms of the universal Verma modules introduced in the classical case by Kashiwara [7] and studied in the quantum case by Kamita [8]. The statement for Weyl modules is then a straightforward consequence.

Before beginning, let us discuss some related work. In [9], Kleshchev carefully analyzed the $\mathfrak{gl}_{N-1}$ highest weight vectors in a Weyl module for $\mathfrak{gl}_N$ and used this information to give modular branching rules for symmetric group representations. Brundan and Kleshchev [10] have explained that highest weight vectors in the restriction of a Weyl module to $\mathfrak{gl}_{N-1}$ give information about highest weight vectors in a tensor product $\Delta(\lambda)\otimes V$ of a Weyl module with the standard N-dimensional representation of $\mathfrak{gl}_N$. Our computations put a new twist on the analysis of the highest weight vectors in $\Delta(\lambda)\otimes V$, as we study them in their "universal" versions and by the use of the Shapovalov determinant. Our techniques can be viewed as an application of the theory of Jantzen [11] as extended to the quantum case by Wiesner [12].

Brundan [13] generalized Kleshchev's [9] techniques and used this information to give modular branching rules for Hecke algebras. As discussed in [14, 15], these branching rules are reflected in the fundamental representation of $\widehat{\mathfrak{sl}}_p$ and its crystal graph, recovering much of the structure of the Misra-Miwa Fock space. Using Hecke algebras at a root of unity, Ryom-Hansen [16] recovered the full $U_v(\widehat{\mathfrak{sl}}_\ell)$ action on Fock space. To complete the picture, one should construct a graded category, where multiplication by v in the $\widehat{\mathfrak{sl}}_\ell$ representation corresponds to a grading shift. Recent work of Brundan-Kleshchev [17] and Ariki [18] explains that one solution to this problem is through the representation theory of Khovanov-Lauda-Rouquier algebras [19, 20]. It would be interesting to explicitly describe the relationship between their category and the present work. Another related construction due to Brundan-Stroppel considers the case when the Fock space is replaced by $\wedge^m V\otimes\wedge^n V$, where V is the natural $\mathfrak{gl}_\infty$ module and m,n are fixed natural numbers.

We would also like to mention very recent work of Peng Shan [21] which independently develops a similar story to the one presented here, but using representations of a quantum Schur algebra where we use representations of $U_\varepsilon(\mathfrak{gl}_N)$. The approach taken there is somewhat different and in particular relies on localization techniques of Beĭlinson and Bernstein [22].

This paper is arranged as follows. Sections 2 and 3 are background on the quantum group $U_q(\mathfrak{gl}_N)$ and the Fock space $\mathbf{F}_v$. Sections 4 and 5 explain universal Verma modules and the Shapovalov determinant. Section 6 contains the statement and proof of our main result relating Fock space and Weyl modules.

2. The Quantum Group $U_q(\mathfrak{gl}_N)$ and Its Integral Form $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$

This is a very brief review, intended mainly to fix notation. With slight modifications, the construction in this section works in the generality of symmetrizable Kac-Moody algebras. See [23, Chapters 6 and 9] for details.

2.1. The Rational Quantum Group

$U_q(\mathfrak{gl}_N)$ is the associative algebra over the field of rational functions $\mathbb{Q}(q)$ generated by

$$X_1,\dots,X_{N-1},\qquad Y_1,\dots,Y_{N-1},\qquad L_1^{\pm1},\dots,L_N^{\pm1}, \tag{2.1}$$

with relations

$$L_iL_j = L_jL_i, \qquad L_iL_i^{-1} = L_i^{-1}L_i = 1, \qquad X_iY_j - Y_jX_i = \delta_{i,j}\frac{L_iL_{i+1}^{-1} - L_{i+1}L_i^{-1}}{q - q^{-1}},$$

$$L_iX_jL_i^{-1} = \begin{cases} qX_j, & \text{if } i = j, \\ q^{-1}X_j, & \text{if } i = j+1, \\ X_j, & \text{otherwise}, \end{cases} \qquad L_iY_jL_i^{-1} = \begin{cases} q^{-1}Y_j, & \text{if } i = j, \\ qY_j, & \text{if } i = j+1, \\ Y_j, & \text{otherwise}, \end{cases} \tag{2.2}$$

$$X_iX_j = X_jX_i, \qquad Y_iY_j = Y_jY_i, \quad \text{if } |i-j| \geq 2,$$

$$X_i^2X_j - \left(q + q^{-1}\right)X_iX_jX_i + X_jX_i^2 = Y_i^2Y_j - \left(q + q^{-1}\right)Y_iY_jY_i + Y_jY_i^2 = 0, \quad \text{if } |i-j| = 1.$$

The algebra $U_q(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct and antipode given by

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, & S(L_i) &= L_i^{-1}, \\ \Delta(X_i) &= X_i \otimes L_iL_{i+1}^{-1} + 1 \otimes X_i, & S(X_i) &= -X_iL_i^{-1}L_{i+1}, \\ \Delta(Y_i) &= Y_i \otimes 1 + L_i^{-1}L_{i+1} \otimes Y_i, & S(Y_i) &= -L_iL_{i+1}^{-1}Y_i, \end{aligned} \tag{2.3}$$

respectively, (see [23, Section 9.1]).

As a $\mathbb{Q}(q)$-vector space, $U_q(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q(g\mathfrak{l}_N) \cong U_q(g\mathfrak{l}_N)^{<0} \otimes U_q(g\mathfrak{l}_N)^{0} \otimes U_q(g\mathfrak{l}_N)^{>0}, \tag{2.4}$$

where the inverse isomorphism is given by multiplication (see [23, Proposition 9.1.3]). Here $U_q(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the Y_i for $i = 1, \ldots, N-1$, $U_q(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the X_i for $i = 1, \ldots, N-1$, and $U_q(\mathfrak{gl}_N)^{0}$ is the subalgebra generated by the $L_i^{\pm 1}$ for $i = 1, \ldots, N$.

2.2. The Integral Quantum Group

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. For $n, k \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}$, let

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad x^{(k)} := \frac{x^k}{[k][k-1]\cdots[2][1]}, \qquad \begin{bmatrix} x; c \\ k \end{bmatrix} := \prod_{s=1}^{k} \frac{xq^{c+1-s} - x^{-1}q^{s-1-c}}{q^s - q^{-s}}, \tag{2.5}$$

in $\mathbb{Q}(q,x)$. The *restricted integral form* $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ is the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{gl}_N)$ generated by $X_i^{(k)}, Y_i^{(k)}, L_i^{\pm 1}$ and $\left[\begin{smallmatrix} L_i;c \\ k \end{smallmatrix}\right]$ for $1 \le i \le N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$. As discussed in [24, Section 6], this is an integral form in the sense that

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{Q}(q) = U_q(\mathfrak{gl}_N). \tag{2.6}$$

As with $U_q(\mathfrak{gl}_N)$, the algebra $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0} \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}, \tag{2.7}$$

where the isomorphism is given by multiplication (see [23, Proposition 9.3.3]). In this case, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the $Y_i^{(k)}$, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the $X_i^{(k)}$, and $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0}$ is generated by $L_i^{\pm 1}$ and $\left[\begin{smallmatrix} L_i;c \\ k \end{smallmatrix}\right]$ for $1 \le i \le N$, $c \in \mathbb{Z}$, and $k \in \mathbb{Z}_{>0}$.

2.3. Rational Representations

The Lie algebra $\mathfrak{gl}_N = M_N(\mathbb{C})$ of $N \times N$ matrices has standard basis $\{E_{ij} \mid 1 \le i,j \le N\}$, where E_{ij} is the matrix with 1 in position (i,j) and 0 everywhere else. Let $\mathfrak{h} = \operatorname{span}\{E_{11}, E_{22}, \ldots, E_{NN}\}$. Let $\varepsilon_i \in \mathfrak{h}^*$ be the weight of $\mathfrak{gl}_N$ given by $\varepsilon_i(E_{jj}) = \delta_{i,j}$. Define

$$\begin{aligned}
\mathfrak{h}_{\mathbb{Z}}^* &:= \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \mathfrak{h}^* \mid \lambda_1, \ldots, \lambda_N \in Z\},\\
(\mathfrak{h}_{\mathbb{Z}}^*)^+ &:= \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N\},\\
P^+ &:= \left\{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in (\mathfrak{h}_{\mathbb{Z}}^*)^+ \mid \lambda_N \ge 0\right\},\\
R^+ &:= \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le N\},\\
Q &:= \operatorname{span}_{\mathbb{Z}}(R^+), \qquad Q^+ := \operatorname{span}_{\mathbb{Z}_{\ge 0}}(R^+), \qquad Q^- := \operatorname{span}_{\mathbb{Z}_{\le 0}}(R^+)
\end{aligned} \tag{2.8}$$

to be the set of *integral weights*, the set of *dominant integral weights*, the set of *dominant polynomial weights*, the set of *positive roots*, the *root lattice*, the *positive part of the root lattice*, and the *negative part of the root lattice*, respectively.

For an integral weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_N\varepsilon_N$, the *Verma module* $M(\lambda)$ for $U_q(\mathfrak{gl}_N)$ of the highest weight λ is

$$M(\lambda) := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\ge 0}} \mathbb{Q}(q)_\lambda, \tag{2.9}$$

where $\mathbb{Q}(q)_\lambda = \operatorname{span}_{\mathbb{Q}(q)}\{v_\lambda\}$ is the one dimensional vector space over $\mathbb{Q}(q)$ with $U_q(\mathfrak{gl}_N)^{\ge 0}$ action given by

$$X_i \cdot v_\lambda = 0, \qquad L_j \cdot v_\lambda = q^{\lambda_j} v_\lambda, \quad \text{for } 1 \le i \le N-1,\ 1 \le j \le N. \tag{2.10}$$

Theorem 2.1 (see [23, Chapter 10.1]). *If $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})^+$ then $M(\lambda)$ has a unique finite dimensional quotient $\Delta(\lambda)$ and the map $\lambda \mapsto \Delta(\lambda)$ is a bijection between $(\mathfrak{h}^*_{\mathbb{Z}})^+$, and the set of irreducible finite dimensional $U_q(\mathfrak{gl}_N)$-modules.*

A *singular vector* in a representation of $U_q(\mathfrak{gl}_N)$ is a vector v such that $X_i \cdot v = 0$ for all i.

2.4. Integral Representations

The *integral Verma module* $M^{\mathcal{A}}(\lambda)$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of $M(\lambda)$ generated by v_λ. The *integral Weyl module* $\Delta^{\mathcal{A}}(\lambda)$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of $\Delta(\lambda)$ generated by v_λ. Using (2.6) and (2.4),

$$M^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}\mathbb{Q}(q) = M(\lambda), \qquad \Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}\mathbb{Q}(q) = \Delta(\lambda). \tag{2.11}$$

In general, $\Delta^{\mathcal{A}}(\lambda)$ is not irreducible as a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module.

3. Partitions and Fock Space

We now describe the v-deformed Fock space representation of $U_v(\widehat{\mathfrak{sl}}_\ell)$ constructed by Misra and Miwa [3] following work of Hayashi [2]. Our presentation largely follows [25, Chapter 10].

3.1. Partitions

A partition λ is a finite length nonincreasing sequence of positive integers. Associated to a partition is its Ferrers diagram. We draw these diagrams as in Figure 1 so that, if $\lambda = (\lambda_1, \ldots, \lambda_N)$, then λ_i is the number of boxes in row i (rows run southeast to northwest $\nwarrow$). Say that λ is contained in μ if the diagram for λ fits inside the diagram for μ and let μ/λ be the collection of boxes of μ that are not in λ. For each box $b \in \lambda$, the *content* $c(b)$ is the horizontal position of b and the *color* $\overline{c}(b)$ is the residue of $c(b)$ modulo ℓ. In Figure 1, the numbers $c(b)$ are listed below the diagram. The *size* $|\lambda|$ of a partition λ is the total number of boxes in its Ferrers diagram.

The set P^+ of dominant polynomial weights from Section 2.3 is naturally identified with partitions with at most N parts. If $\lambda \in P^+$, then

$$\Delta(\lambda) \otimes \Delta(\varepsilon_1) \cong \bigoplus_{\substack{1\le k\le N \\ \lambda+\varepsilon_k\in P^+}} \Delta(\lambda + \varepsilon_k) \tag{3.1}$$

as $U_q(\mathfrak{gl}_N)$-modules. The diagram of $\lambda+\varepsilon_k$ is obtained from the diagram of λ by adding a box on row k, and $\Delta(\lambda + \varepsilon_k)$ appears in the sum on the right side of (3.1) if and only if $\lambda + \varepsilon_k$ is a partition. See, for example, [26, Section 6.1, Formula 6.8] for the classical statement and [23, Proposition 10.1.16] for the quantum case.

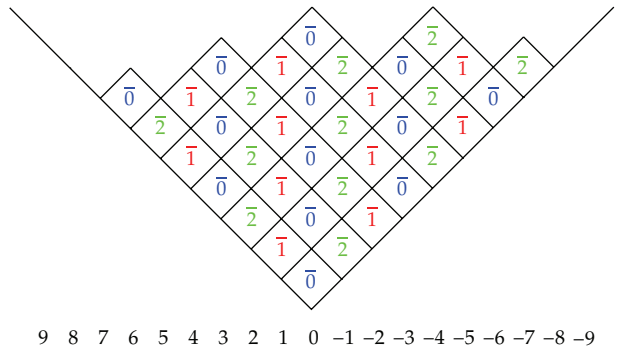
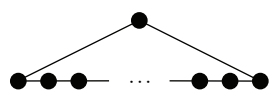


Figure 1: The partition $(7,6,6,5,5,3,3,1)$ with each box containing its color for $\ell = 3$. The content $c(b)$ of a box b is the horizontal position of b reading right to left. The contents of boxes are listed beneath the diagram so that $c(b)$ is aligned with all boxes b of that content.

3.2. The Quantum Affine Algebra

Let $U_v'(\widehat{\mathfrak{sl}}_\ell)$ be the quantized universal enveloping algebra corresponding to the ℓ-node Dynkin diagram



More precisely, $U_v'(\widehat{\mathfrak{sl}}_\ell)$ is the algebra generated by $E_{\bar{i}}, F_{\bar{i}}, K_{\bar{i}}^{\pm 1}$, for $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$, with relations

$$K_{\bar{i}}K_{\bar{j}} = K_{\bar{j}}K_{\bar{i}}, \qquad K_{\bar{i}}K_{\bar{i}}^{-1} = K_{\bar{i}}^{-1}K_{\bar{i}} = 1, \qquad E_{\bar{i}}F_{\bar{j}} - F_{\bar{j}}E_{\bar{i}} = \delta_{\bar{i},\bar{j}}\frac{K_{\bar{i}} - K_{\bar{i}}^{-1}}{v - v^{-1}},$$

$$K_{\bar{i}}E_{\bar{j}}K_{\bar{i}}^{-1} = \begin{cases} v^2 E_{\bar{j}}, & \text{if } \bar{i} = \bar{j}, \\ v^{-1}E_{\bar{j}}, & \text{if } \bar{i} = \bar{j} \pm 1, \\ E_{\bar{j}}, & \text{otherwise}, \end{cases} \qquad K_{\bar{i}}F_{\bar{j}}K_{\bar{i}}^{-1} = \begin{cases} v^{-2}F_{\bar{j}}, & \text{if } \bar{i} = \bar{j}, \\ vF_{\bar{j}}, & \text{if } \bar{i} = \bar{j} \pm 1, \\ F_{\bar{j}}, & \text{otherwise}, \end{cases} \tag{3.2}$$

$$E_{\bar{i}}E_{\bar{j}} = E_{\bar{j}}E_{\bar{i}}, \qquad F_{\bar{i}}F_{\bar{j}} = F_{\bar{j}}F_{\bar{i}}, \quad \text{if } \left|\bar{i} - \bar{j}\right| \geq 2,$$

$$E_{\bar{i}}^2E_{\bar{j}} - \left(v + v^{-1}\right)E_{\bar{i}}E_{\bar{j}}E_{\bar{i}} + E_{\bar{j}}E_{\bar{i}}^2 = F_{\bar{i}}^2F_{\bar{j}} - \left(v + v^{-1}\right)F_{\bar{i}}F_{\bar{j}}F_{\bar{i}} + F_{\bar{j}}F_{\bar{i}}^2 = 0, \quad \text{if } \left|\bar{i} - \bar{j}\right| = 1.$$

See [23, Definition Proposition 9.1.1]. The algebra $U_v'(\widehat{\mathfrak{sl}}_\ell)$ is the quantum group corresponding to the nontrivial central extension $\widehat{\mathfrak{sl}}_\ell' = \mathfrak{sl}_\ell[t, t^{-1}] \oplus \mathbb{C}c$ of the algebra of polynomial loops in $\mathfrak{sl}_\ell$.

3.3. Fock Space

Define *v-deformed Fock space* to be the $\mathbb{Q}(v)$ vector space $\mathbf{F}_v$ with basis $\{|\mu\rangle \mid \lambda \text{ is a partition}\}$. Our $\mathbf{F}_v$ is only the charge 0 part of Fock space described in [27]. Fix $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$ and partitions $\lambda \subseteq \mu$ such that μ/λ is a single box. Define

$$\begin{aligned}
A_{\bar{i}}(\lambda) &:= \left\{\text{boxes } b \mid b \notin \lambda,\ b \text{ has color } \bar{i} \text{ and } \lambda \cup b \text{ is a partition}\right\},\\
R_{\bar{i}}(\lambda) &:= \left\{\text{boxes } b \mid b \in \lambda,\ b \text{ has color } \bar{i} \text{ and } \lambda \setminus b \text{ is a partition}\right\},\\
N_{\bar{i}}^{l}(\mu/\lambda) &:= \left|\{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}\right| - \left|\{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}\right|,\\
N_{\bar{i}}^{r}(\mu/\lambda) &:= \left|\{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the right of } \mu/\lambda\}\right| - \left|\{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the right of } \mu/\lambda\}\right|,
\end{aligned} \tag{3.3}$$

to be the set of *addable boxes of color $\bar{i}$*, the set of *removable boxes of color $\bar{i}$*, the *left removable-addable difference*, and the *right removable-addable difference*, respectively.

Theorem 3.1 (see [25, Theorem 10.6]). *There is an action of $U_v'(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$ determined by*

$$E_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\lambda/\mu)=\bar{i}} v^{-N_{\bar{i}}^{r}(\lambda/\mu)}|\mu\rangle, \qquad F_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\mu/\lambda)=\bar{i}} v^{N_{\bar{i}}^{l}(\mu/\lambda)}|\mu\rangle, \tag{3.4}$$

where $\bar{c}(\lambda/\mu)$ denotes the color of λ/μ and the sum is over partitions μ which differ from λ by removing (resp. adding) a single $\bar{i}$-colored box.

As a $U_v'(\widehat{\mathfrak{sl}}_\ell)$-module, $\mathbf{F}_v$ is isomorphic to an infinite direct sum of copies of the basic representation $V(\Lambda_0)$. Using the grading of $\mathbf{F}_v$ where $|\lambda\rangle$ has degree $|\lambda|$, the highest weight vectors in $\mathbf{F}_v$ occur in degrees divisible by ℓ, and the number of the highest weight vectors in degree ℓk is the number of partitions of k. Then, $\mathbf{F}_v \cong V(\Lambda_0) \otimes \mathbb{C}[x_1, x_2, \ldots]$, where x_k has degree ℓk, and $U_v'(\widehat{\mathfrak{sl}}_\ell)$ acts trivially on the second factor (see [27, Proposition 2.3]). Note that we are working with the "derived" quantum group $U_v'(\widehat{\mathfrak{sl}}_\ell)$, not the "full" quantum group $U_v(\widehat{\mathfrak{sl}}_\ell)$, which is why there are no δ-shifts in the summands of $\mathbf{F}_v$.

Comment 1. Comparing with [25, Chapter 10], our $N_{\bar{i}}^{l}(\mu/\lambda)$ is equal to Ariki's $-N_{\bar{i}}^{a}(\mu/\lambda)$ and our $N_{\bar{i}}^{r}(\mu/\lambda)$ is equal to Ariki's $-N_{\bar{i}}^{b}(\mu/\lambda)$. However, these numbers play a slightly different role in Ariki's work, which is explained by a different choice of conventions.

4. Universal Verma Modules

The purpose of this section is to construct a family of representations which are universal Verma modules in the sense that each can be "evaluated" to obtain any given Verma module. This notion was defined by Kashiwara [7] in the classical case and was studied in the quantum case by Kamita [8].

4.1. Rational Universal Verma Modules

Let $\mathbb{K} := \mathbb{Q}(q, z_1, z_2, \ldots, z_N)$. This field is isomorphic to the field of fractions of $U_q(\mathfrak{gl}_N)^0$ via the map

$$\psi : U_q(\mathfrak{gl}_N)^0 \longrightarrow \mathbb{K}, \quad \text{defined by } \psi\left(L_i^{\pm 1}\right) = z_i^{\pm 1}. \tag{4.1}$$

For each $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, define a $\mathbb{Q}(q)$-linear automorphism $\sigma_\mu : \mathbb{K} \to \mathbb{K}$ by

$$\sigma_\mu(z_i) := q^{(\mu,\varepsilon_i)} z_i, \quad \text{for } 1 \le i \le N, \tag{4.2}$$

where $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}_{\mathbb{Z}}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. Let $\mathbb{K}_\mu = \operatorname{span}_{\mathbb{K}}\{v_{\mu+}\}$ be the one-dimensional vector space over $\mathbb{K}$ with basis vector v_μ^+ and $U_q(\mathfrak{gl}_N)^{\ge 0}$ action given by

$$X_i \cdot v_{\mu+} = 0, \quad \text{for } 1 \le i \le N-1, \quad a \cdot v_{\mu+} = \sigma_\mu(\psi(a)) v_{\mu+}, \ \text{for } a \in U_q(\mathfrak{gl}_N)^0. \tag{4.3}$$

The *μ-shifted rational universal Verma module* ${}^\mu\widetilde{M}$ is the $U_q(\mathfrak{gl}_N)$-module

$${}^\mu\widetilde{M} := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\ge 0}} \mathbb{K}_\mu. \tag{4.4}$$

The universal Verma module ${}^\mu\widetilde{M}$ is actually a module over $U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^0} \widetilde{U}_q(\mathfrak{gl}_N)^0$, where $\widetilde{U}_q(\mathfrak{gl}_N)^0$ is the field of fractions of $U_q(\mathfrak{gl}_N)^0$. However, if we identify $\widetilde{U}_q(\mathfrak{gl}_N)^0$ with $\mathbb{K}$ using the map ψ, the action of $\widetilde{U}_q(\mathfrak{gl}_N)^0$ on ${}^\mu\widetilde{M}$ is not by multiplication, but rather is twisted by the automorphism σ_μ. It is to keep track of the difference between the action of $U_q(\mathfrak{gl}_N)^0$ and multiplication that we use different notation for the generators of $\mathbb{K}$ and $U_q(\mathfrak{gl}_N)^0$ (i.e., z_i versus L_i).

4.2. Integral Universal Verma Modules

The field $\mathbb{K}$ contains an $\mathcal{A}$-subalgebra

$$\mathcal{R} \text{ generated by } z_i^{\pm 1}, \qquad \begin{bmatrix} z_i; c \\ k \end{bmatrix}, \quad (1 \le i \le N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}), \tag{4.5}$$

which is isomorphic to $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0$ via the restriction of the map ψ in (4.1). The *integral universal Verma module* ${}^\mu\widetilde{M}^{\mathcal{R}}$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of ${}^\mu\widetilde{M}$ generated by $v_{\mu+}$. By restricting (4.4),

$${}^\mu\widetilde{M}^{\mathcal{R}} = U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\ge 0}} \mathcal{R}_\mu, \tag{4.6}$$

where $\mathcal{R}_\mu$ is the $\mathcal{R}$-submodule of $\mathbb{K}_\mu$ spanned by $v_{\mu+}$. In particular, ${}^\mu\widetilde{M}^{\mathcal{R}}$ is a free $\mathcal{R}$-module.

4.3. Evaluation

Let $\mathrm{ev}_\lambda^{\mathcal{R}} : \mathcal{R} \to \mathcal{A}$ be the map defined by

$$\mathrm{ev}_\lambda^{\mathcal{R}}(z_i) = q^{(\lambda,\varepsilon_i)}, \qquad \mathrm{ev}_\lambda^{\mathcal{R}}\begin{bmatrix} z_i; c \\ n \end{bmatrix} = \begin{bmatrix} q^{(\lambda,\varepsilon_i)}; c \\ n \end{bmatrix}, \tag{4.7}$$

where $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. There is a surjective $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-module homomorphism "evaluation at λ"

$$\mathrm{ev}_\lambda : {}^{\mu}\widetilde{M}^{\mathcal{R}} \longrightarrow M^{\mathcal{A}}(\mu+\lambda) \text{ defined by } \mathrm{ev}_\lambda(a \cdot v_{\mu+}) := a \cdot v_{\mu+\lambda}, \quad \forall a \in U_q^{\mathcal{A}}(\mathfrak{gl}_N). \tag{4.8}$$

For fixed λ, the maps $\mathrm{ev}_\lambda^{\mathcal{R}}$ and ev_λ extend to a map from the subspace of $\mathbb{K}$ and ${}^{\mu}\widetilde{M} = {}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{R}} \mathbb{K}$, respectively, where no denominators evaluate to 0. Where it is clear we denote both these extended maps by ev_λ.

Example 4.1. Computing the action of L_i on $v_{\mu+}$ and $v_{\mu+\lambda}$,

$$L_i \cdot v_{\mu+} = q^{(\mu,\varepsilon_i)} z_i v_{\mu+}, \qquad L_i \cdot v_{\mu+\lambda} = \mathrm{ev}_\lambda\left(q^{(\mu,\varepsilon_i)} z_i\right) v_{\mu+\lambda} = q^{(\mu,\varepsilon_i)} q^{(\lambda,\varepsilon_i)} v_{\mu+\lambda} = q^{(\mu+\lambda,\varepsilon_i)} v_{\mu+\lambda}. \tag{4.9}$$

4.4. Weight Decompositions

Let $\widetilde{V}$ be a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module. For each $\nu \in \mathfrak{h}_{\mathbb{Z}}^*$, we define the *$\nu$-weight space* of $\widetilde{V}$ to be

$$\widetilde{V}_\nu := \left\{ v \in \widetilde{V} : L_i \cdot v = q^{(\nu,\varepsilon_i)} z_i v \right\}. \tag{4.10}$$

The universal Verma module ${}^{\mu}\widetilde{M}^{\mathcal{R}}$ is a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module, where the second factor acts as multiplication. The weight space ${}^{\mu}\widetilde{M}_\eta \neq 0$ if and only if $\eta = \mu - \nu$ with ν in the positive part Q^+ of the root lattice. These nonzero weight spaces and the weight decomposition of ${}^{\mu}\widetilde{M}$ can be described explicitly by

$${}^{\mu}\widetilde{M}_{\mu-\nu}^{\mathcal{R}} = U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{-\nu}^{<0} \cdot \mathcal{R}_\mu, \qquad {}^{\mu}\widetilde{M}^{\mathcal{R}} = \bigoplus_{\nu \in Q^+} {}^{\mu}\widetilde{M}_{\mu-\nu}^{\mathcal{R}}. \tag{4.11}$$

Here, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{-\nu}^{<0}$ is defined using the grading of $U_q(\mathfrak{gl}_N)^{<0}$ with $F_i \in U_q(\mathfrak{gl}_N)_{-(\varepsilon_i - \varepsilon_{i+1})}^{<0}$.

4.5. Tensor Products

Let $\widetilde{V}$ be a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module and W a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-module. The tensor product $\widetilde{V} \otimes_{\mathcal{A}} W$ is a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module, where the first factor acts via the usual coproduct and the second

factor acts by multiplication on $\widetilde{V}$. In the case when $\widetilde{V}$ and W both have weight space decompositions, the weight spaces of $\widetilde{V}\otimes_{\mathcal{A}} W$ are

$$\left(\widetilde{V}\otimes_{\mathcal{A}} W\right)_{\nu} = \bigoplus_{\gamma+\eta=\nu} \widetilde{V}_{\gamma}\otimes_{\mathcal{A}} W_{\eta}. \tag{4.12}$$

We also need the following.

Proposition 4.2. *The tensor product of a universal Verma module with a Weyl module satisfies*

$$\left({}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} \Delta^{\mathcal{A}}(\nu)\right)\otimes_{\mathcal{R}}\mathbb{K} \cong \left(\bigoplus_{\gamma} ({}^{\mu+\gamma}\widetilde{M}^{\mathcal{R}})^{\oplus \dim \Delta^{\mathcal{A}}(\nu)_{\gamma}}\right)\otimes_{\mathcal{R}}\mathbb{K}. \tag{4.13}$$

Proof. Fix $\nu \in P^{+}$. In general, $M(\lambda+\mu)\otimes\Delta(\nu)$ has a Verma filtration (see, e.g., [28, Theorem 2.2]) and if $\lambda+\mu+\gamma$ is dominant for all γ such that $\Delta(\nu)_{\gamma}\neq 0$ then

$$M(\lambda+\mu)\otimes\Delta(\nu) \cong \bigoplus_{\gamma} M(\lambda+\mu+\gamma)^{\oplus \dim \Delta(\nu)_{\gamma}}, \tag{4.14}$$

which can be seen by, for instance, taking central characters. The proposition follows since this is true for a Zariski dense set of weights λ. □

5. The Shapovalov Form and the Shapovalov Determinant

5.1. The Shapovalov Form

The *Cartan involution* $\omega : U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)$ is the $\mathbb{Q}(q)$-algebra anti-involution of $U_q(\mathfrak{gl}_N)$ defined by

$$\omega\left(L_i^{\pm 1}\right) = L_i^{\pm 1}, \qquad \omega(X_i) = Y_i L_i L_{i+1}^{-1}, \qquad \omega(Y_i) = L_i^{-1} L_{i+1} X_i. \tag{5.1}$$

The map ω is also a coalgebra involution. An *ω-contravariant* form on a $U_q(\mathfrak{gl}_N)$-module V is a symmetric bilinear form $(\cdot,\cdot)$ such that

$$(u, a\cdot v) = (\omega(a)\cdot u, v), \quad \text{for } u, v \in V,\ a \in U_q(\mathfrak{gl}_N). \tag{5.2}$$

It follows by the same argument used in the classical case [4] that there is an ω-contravariant form (the Shapovalov form) on each Verma module $M(\lambda)$ and this is unique up to rescaling. The radical of $(\cdot,\cdot)$ is the maximal proper submodule of $M(\lambda)$, so $\Delta(\lambda) = M(\lambda)/\mathrm{Rad}(\cdot,\cdot)$ for all $\lambda \in P^{+}$. In particular, $(\cdot,\cdot)$ descends to an ω-contravariant form on $\Delta(\lambda)$.

Since ω fixes $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \subseteq U_q(\mathfrak{gl}_N)$, there is a well-defined notion of an ω-contravariant form on a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module. In particular, the restriction of the Shapovalov form on $\Delta(\lambda)$ to $\Delta^{\mathcal{A}}(\lambda)$ is ω-contravariant.

5.2. Universal Shapovalov Forms

There are surjective maps of $\mathcal{A}$-algebras $p_- : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \to \mathbb{Q}(q)$ and $p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \to \mathbb{Q}(q)$ defined by $p_-(F_i) = 0$ and $p_+(E_i) = 0$, for $1 \le i \le N$. Using the triangular decomposition (2.7), there is an $\mathcal{A}$-linear surjection

$$\pi_0 := p_- \otimes \mathrm{Id} \otimes p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0 \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \longrightarrow U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0. \tag{5.3}$$

The *standard universal Shapovalov form* is the $\mathcal{R}$-bilinear form $(\cdot,\cdot)_{{}^\mu\widetilde{M}^{\mathcal{R}}} : {}^\mu\widetilde{M}^{\mathcal{R}} \otimes {}^\mu\widetilde{M}^{\mathcal{R}} \to \mathcal{R}$ defined by

$$\left(a_1 \cdot v_{\mu+}, a_2 \cdot v_{\mu+}\right)_{{}^\mu\widetilde{M}^{\mathcal{R}}} = \left(\sigma_\mu \circ \psi \circ \pi_0\right)\left(\omega(a_2)a_1\right) \tag{5.4}$$

for all $a_1, a_2 \in U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}$. Here, ψ and σ_μ are as in (4.1) and (4.2). Since

$$\left(a_1 a_2 \cdot v_{\mu+}, a_3 \cdot v_{\mu+}\right)_{{}^\mu\widetilde{M}^{\mathcal{R}}} = \left(\sigma_\mu \circ \psi \circ \pi_0\right)\left(\omega(a_2)\omega(a_1)a_3\right) = \left(a_2 \cdot v_{\mu+}, \omega(a_1)a_3 \cdot v_{\mu+}\right)_{{}^\mu\widetilde{M}^{\mathcal{R}}} \tag{5.5}$$

for $a_1, a_2, a_3 \in U_q(\mathfrak{gl}_N)$, the form $(\cdot,\cdot)_{{}^\mu\widetilde{M}^{\mathcal{R}}}$ is ω-contravariant. As with the usual Shapovalov form, distinct weight spaces are orthogonal, where weight spaces are defined as in Section 4.4.

Evaluation at λ gives an $\mathcal{A}$-valued ω-contravariant form $(\cdot,\cdot)_{M^{\mathcal{A}}(\mu+\lambda)}$ on $M^{\mathcal{A}}(\mu+\lambda)$ by

$$\left(\mathrm{ev}_\lambda(u_1), \mathrm{ev}_\lambda(u_2)\right)_{M^{\mathcal{A}}(\mu+\lambda)} = \mathrm{ev}_\lambda\left((u_1, u_2)_{{}^\mu\widetilde{M}^{\mathcal{R}}}\right) \quad \text{for } u_1, u_2 \in {}^\mu\widetilde{M}^{\mathcal{R}}. \tag{5.6}$$

The form $(\cdot,\cdot)_{{}^\mu\widetilde{M}^{\mathcal{R}}}$ can be extended by linearity to an ω-contravariant form $(\cdot,\cdot)_{{}^\mu\widetilde{M}}$ on ${}^\mu\widetilde{M}$.

5.3. The Shapovalov Determinant

Let $\widetilde{V}$ be a $(U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R})$-module with a chosen ω-contravariant form. Let B_η be an $\mathcal{R}$ basis for the η-weight space $\widetilde{V}_\eta$ of $\widetilde{V}$. Let $\det \widetilde{V}_{B_\eta}$ be the determinant of the form evaluated on the basis B_η. Changing the basis B_η changes the determinant by a unit in $\mathcal{R}$, and we sometimes write $\det \widetilde{V}_\eta$ to mean the determinant calculated on an unspecified basis ($\det \widetilde{V}_\eta$ which is only defined up to multiplication by unit in $\mathcal{R}$). The *Shapovalov determinant* is

$$\det \widetilde{M}_\eta^{\mathcal{R}} := \det\left(\left(b_i, b_j\right)_{\widetilde{M}^{\mathcal{R}}}\right)_{b_i, b_j \in B_\eta}. \tag{5.7}$$

Define the *partition function* $p : \mathfrak{h}^* \to \mathbb{Z}_{\ge 0}$ by

$$p(\gamma) := \dim M(0)_\gamma. \tag{5.8}$$

Then, $p(\gamma) = \dim M(\lambda)_{\gamma+\lambda}$ for any λ, and $\eta \notin Q^-$ implies that $p(\eta) = 0$ and $\det \widetilde{M}_\eta^{\mathcal{R}} = 1$.

Theorem 5.1 (see [5, Proposition 1.9A], [6, Theorem 3.4], [4]). *For any weight η,*

$$\det \widetilde{M}_{\eta}^{\mathcal{R}} = c_{\eta} \prod_{\substack{1\le i<j\le N \\ m>0}} \left(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j \right)^{p(\eta+m\varepsilon_i-m\varepsilon_j)}, \tag{5.9}$$

where c_η is a unit in $\mathcal{R}\otimes_{\mathcal{A}} \mathbb{Q}(q) = Q(q)[z_1^{\pm 1},\dots,z_N^{\pm 1}]$.

Proposition 5.2. *Fix $\mu,\eta \in \mathfrak{h}_{\mathbb{Z}}^*$ with $\eta-\mu \in Q^-$. Choose an $\mathcal{A}$-basis $B_{\eta-\mu}$ for $U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{\eta-\mu}$. Consider the $\mathcal{R}$-bases $\widetilde{B}_{\eta-\mu} := \{b\cdot v_+ \mid b\in B_{\eta-\mu}\}$ for $\widetilde{M}_{\eta-\mu}^{\mathcal{R}}$ and ${}^{\mu}\widetilde{B}_{\eta} := \{b\cdot v_{\mu+} \mid b\in B_{\eta-\mu}\}$ for ${}^{\mu}\widetilde{M}_{\eta}^{\mathcal{R}}$. Then $\det {}^{\mu}\widetilde{M}_{({}^{\mu}\widetilde{B}_{\eta})}^{\mathcal{R}} = \sigma_\mu(\det \widetilde{M}_{\widetilde{B}_{\eta-\mu}}^{\mathcal{R}})$.*

Proof. For $b,b'\in B_{\eta-\mu}$,

$$(b\cdot v_{\mu+}, b'\cdot v_{\mu+})_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} = \sigma_\mu\circ\psi\circ\pi_0(\omega(b')b) = \sigma_\mu\left((b\cdot v_{0+}, b'\cdot v_{0+})_{\widetilde{M}^{\mathcal{R}}}\right). \tag{5.10}$$

The result follows by taking determinants. □

5.4. Contravariant Forms on Tensor Products

If V and W are $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-modules with ω-contravariant forms $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, define an $\mathcal{A}$-bilinear form $(\cdot,\cdot)_{W\otimes V}$ by $(w_1\otimes v_1, w_2\otimes v_2)_{W\otimes V} = (w_1,w_2)_W(v_1,v_2)_V$. Similarly, for a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)\otimes_{\mathcal{A}}\mathcal{R}$ module $\widetilde{W}$ with $\mathcal{R}$-bilinear ω-contravariant form $(\cdot,\cdot)_{\widetilde{W}}$, define a $\mathcal{R}$-bilinear form $(\cdot,\cdot)_{\widetilde{W}\otimes_{\mathbb{Q}(q)}V}$ on $\widetilde{W}\otimes_{\mathbb{Q}(q)}V$ by

$$(u_1\otimes v_1, u_2\otimes v_2)_{\widetilde{W}\otimes_{\mathbb{Q}(q)}V} = (u_1,u_2)_{\widetilde{W}}(v_1,v_2)_V. \tag{5.11}$$

Since ω is a coalgebra involution (i.e., $\Delta(\omega(a)) = (\omega\otimes\omega)\Delta(a)$, for $a\in U_q(\mathfrak{gl}_N)$), the forms $(\cdot,\cdot)_{V\otimes W}$ and $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}\otimes_{Q(q)}V}$ are ω-contravariant.

In the case when $\widetilde{W}={}^{\mu}\widetilde{M}^{\mathcal{R}}$, evaluation of the ω-contravariant form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}\otimes_{\mathcal{A}}V}$ at λ gives an ω-contravariant form $(\cdot,\cdot)_{M^{\mathcal{A}}(\mu+\lambda)\otimes_{\mathcal{A}}V}$:

$$\begin{aligned}(u_1\otimes v_1, u_2\otimes v_2)_{M^{\mathcal{A}}(\mu+\lambda)\otimes_{\mathcal{A}}V} &= \mathrm{ev}_\lambda\left((u_1\otimes v_1, u_2\otimes v_2)_{{}^{\mu}\widetilde{M}\otimes_{\mathcal{A}}V}\right)\\ &= (\mathrm{ev}_\lambda(u_1)\otimes v_1, \mathrm{ev}_\lambda(u_2)\otimes v_2)_{M(\mu+\lambda)\otimes_{\mathcal{A}}V},\end{aligned} \tag{5.12}$$

for $u_1,u_2\in{}^{\mu}\widetilde{M}$ and $v_1,v_2\in V$. As in Section 4.3, this form can be extended to the $\mathcal{A}$-submodule of the rational module where no denominators evaluate to zero.

6. The Misra-Miwa Formula for $F_{\bar{i}}$ from $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ Representation Theory

Let us prepare the setting for our main result (Theorem 6.1). Fix $\ell \geq 2$ and a partition λ. Let N be a positive integer greater than the number of parts of λ. All calculations below are in terms of representations of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$.

(1) Let $V = \Delta^{\mathcal{A}}(\varepsilon_1)$ be the standard N-dimensional module. Since $\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}\mathbb{Q}(q) = \Delta(\lambda)$, (3.1) implies

$$\left(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V\right)\otimes_{\mathcal{A}}\mathbb{Q}(q) \simeq \bigoplus \Delta^{\mathcal{A}}\left(\lambda+\varepsilon_{k_j}\right)\otimes_{\mathcal{A}}\mathbb{Q}(q), \tag{6.1}$$

where the sum is over those indices $1 = k_1 < k_2 < \cdots < k_{m_\lambda} \leq N$ for which $\lambda + \varepsilon_{k_j}$ is a partition. For ease of notation, let $\mu^{(j)} = \lambda + \varepsilon_{k_j}$.

(2) Fix an $\mathcal{A}$-basis $\{v_1, \ldots, v_N\}$ of V where v_k has weight ε_k and $Y_i(v_k) = \delta_{i,k}v_{k+1}$. Recursively, define singular weight vectors $v_{\mu^{(j)}}$ in $(\Delta^{\mathcal{A}}(\lambda)\otimes V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ by

(i) $v_{\mu^{(1)}} = v_\lambda \otimes v_1$

(ii) for each k, the submodule W_k of $(\Delta(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ generated by $\{v_\lambda \otimes v_i \mid 1 \leq i \leq k\}$ contains all weight vectors of $(\Delta(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ of weight greater than or equal to $\lambda + \varepsilon_k$. Thus, using (6.1), for each $1 \leq j \leq m_\lambda$ there is a one-dimensional space of singular vectors of weight $\mu^{(j)}$ in W_{k_j}, and this is not contained in $W_{k_{j-1}}$ (since $k_j > k_{j-1}$). This implies that there unique singular vector $v_{\mu^{(j)}}$ of weight $\mu^{(j)}$ in

$$v_\lambda \otimes v_{k_j} + \bigoplus_{1\leq i<j} U_q(\mathfrak{gl}_N)v_{\mu^{(i)}} \subseteq \left(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V\right)\otimes_{\mathcal{A}}\mathbb{Q}(q), \tag{6.2}$$

where we recall that $U_q(\mathfrak{gl}_N) = U_q^{\mathcal{A}}(\mathfrak{gl}_N)\otimes_{\mathcal{A}}\mathbb{Q}(q)$.

(3) There is a unique ω-contravariant form on $\Delta^{\mathcal{A}}(\lambda)$ normalized so that $(v_\lambda, v_\lambda) = 1$ and a unique ω-contravariant form on V normalized so that $(v_1, v_1) = 1$. As in Section 5.4, define a ω-contravariant form on $(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ by $(u_1\otimes w_1, u_2\otimes w_2) = (u_1, u_2)(w_1, w_2)$. For each $1 \leq j \leq m_\lambda$, define an element $r_j(\lambda) \in \mathbb{Q}(q)$ by

$$r_j(\lambda) := \left(v_{\mu^{(j)}}, v_{\mu^{(j)}}\right). \tag{6.3}$$

Theorem 6.1. *The Misra-Miwa operators $F_{\bar{i}}$ from Section 3.3 satisfy*

$$F_{\bar{i}}|\lambda\rangle = \sum_{\bar{c}\left(b^{(j)}\right)=\bar{i}} v^{val_{\phi_{2\ell}} r_j(\lambda)}\left|\mu^{(j)}\right\rangle, \tag{6.4}$$

where $b^{(j)}$ is the box $\mu^{(j)}/\lambda$, $\overline{c}(b^{(j)})$ is the color of box $b^{(j)}$ as in Figure 1, $\phi_{2\ell}$ is the 2ℓth cyclotomic polynomial in q, and $\mathrm{val}_{\phi_{2\ell}} r$ is the number of factors of $\phi_{2\ell}$ in the numerator of r minus the number of factors of $\phi_{2\ell}$ in the denominator of r.

The proof of Theorem 6.1 will occupy the rest of this section. We will first prove a similar statement, Proposition 6.6, where the role of the Weyl modules is played by the universal Verma modules from Section 4. For ease of notation, let $\widetilde{M}^{\mathcal{R}}$ denote the module ${}^{0}\widetilde{M}^{\mathcal{R}}$ from Section 4.2.

Definition 6.2. Recursively define singular weight vectors $v_{\varepsilon_k+} \in (\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V) \otimes_{\mathcal{R}} \mathbb{K}$ and elements $s_k \in \mathbb{K}$ for $1 \le k \le N$ by

(i) $v_{\varepsilon_1+} = v_+ \otimes v_1$,

(ii) since $\{v_+ \otimes v_j \mid 1 \le j \le N\}$ generates $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ as a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\le 0}$ module, Proposition 4.2 implies that, for each $1 \le k \le N$, there is a unique singular vector v_{ε_k+} in $v_+ \otimes v_k + \oplus_{1 \le j < k} U_q^{\mathbb{K}}(\mathfrak{gl}_N) v_{\varepsilon_j+} \subseteq (\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V) \otimes_{\mathcal{R}} \mathbb{K}$, where $U_q^{\mathbb{K}}(\mathfrak{gl}_N) := U_q(\mathfrak{gl}_N) \otimes_{(q)} \mathbb{K}$ and the factor of $\mathbb{K}$ acts by multiplication on $\widetilde{M}^{\mathcal{R}}$.

Let $s_k = (v_{\varepsilon_k+}, v_{\varepsilon_k+})$.

The s_k are quantized versions of the Jantzen numbers first calculated in [11, Section 5] and quantized in [12]. It follows immediately from the definition that $s_1 = 1$.

Lemma 6.3. *For any weight η, up to multiplication by a power of q,*

$$\prod_{1 \le k \le N} s_k^{p(\eta - \varepsilon_k)} = \prod_{1 \le k \le N} \frac{\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}{\sigma_{\varepsilon_k} \det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}, \tag{6.5}$$

where, as in Section 5.3, $\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}$ is the determinant of the Shapovalov form evaluated on an $\mathcal{R}$-basis for the $\eta - \varepsilon_k$ weight space of $\widetilde{M}^{\mathcal{R}}$.

Comment 2. In order for Lemma 6.3 to hold as stated, for each $1 \le k \le N$, one must calculate the $\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}$ in the numerator and denominator with respect to the same $\mathcal{R}$-basis. The power of q which appears depends on this choice of $\mathcal{R}$-bases.

Proof of Lemma 6.3. For each $\gamma \in \mathrm{span}_{\mathbb{Z}_{\le 0}}(R^+)$ fix an $\mathcal{R}$-basis B_γ for $U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}_{\gamma}$. Consider the following three $\mathbb{K}$-bases for $((\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta) \otimes_{\mathcal{R}} \mathbb{K}$:

$$\begin{aligned} A_\eta &:= \{(b \cdot v_+) \otimes v_k \mid b \in B_{\eta - \varepsilon_k}, 1 \le k \le N\}, \\ C_\eta &:= \{b \cdot (v_+ \otimes v_k) \mid b \in B_{\eta - \varepsilon_k}, 1 \le k \le N\}, \\ D_\eta &:= \{b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta - \varepsilon_k}, 1 \le k \le N\}. \end{aligned} \tag{6.6}$$

Let $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_B$ denote the determinant of $(\cdot,\cdot)_{(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta}$ calculated on B, where B is one of A_η, C_η, or D_η. Let $\det {}^{\nu}\widetilde{M}^{\mathcal{R}}_{B_{\eta - \nu}}$ denote $\det {}^{\nu}\widetilde{M}^{\mathcal{R}}_{\eta}$ calculated with respect to the basis $B_{\eta - \nu} \cdot v_{\nu+}$.

By the definition of the ω-contravariant form on $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ (see Section 4.5),

$$\det\left(\widetilde{M}^{\mathcal{R}} \otimes V\right)_{A_\eta} = \prod_{k=1}^{N}\left(\det \widetilde{M}^{\mathcal{R}}_{B_{\eta-\varepsilon_k}}\right)^{\dim V_{\varepsilon_k}} (\det V_{\varepsilon_k})^{\dim \widetilde{M}^{\mathcal{R}}_{\eta-\varepsilon_k}}. \tag{6.7}$$

For $1 \le k \le N$, V_{ε_k} is one dimensional and $\det V_{\varepsilon_k}$ is a power of q. Hence, up to multiplication by a power of q, (6.7) simplifies to

$$\det\left(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V\right)_{A_\eta} = \prod_{k=1}^{N} \det \widetilde{M}^{\mathcal{R}}_{B_{\eta-\varepsilon_k}}. \tag{6.8}$$

Notice that $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \cdot v_{\varepsilon_k+}$ is isomorphic to ${}^{\varepsilon_k}\widetilde{M}$, and D_η is the union of $\mathcal{R}$-bases for each of these submodules. For each $1 \le k \le N$, and each $\eta \in \mathfrak{h}_Z^*$ define an $\mathcal{R}$ basis of ${}^{\varepsilon_k}\widetilde{M}_\eta$ by

$${}^{\varepsilon_k}\widetilde{B}_\eta := \left\{ b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta-\varepsilon_k} \right\}. \tag{6.9}$$

Using $(v_{\varepsilon_k+}, v_{\varepsilon_k+}) = s_k$,

$$\det\left(\widetilde{M}^{\mathcal{R}} \otimes V\right)_{D_\eta} = \prod_{k=1}^{N} s_k^{\dim({}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_\eta)} \det {}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_{({}^{\varepsilon_k}\widetilde{B}_\eta)} = \prod_{k=1}^{N} s_k^{p(\eta-\varepsilon_k)} \sigma_{\varepsilon_k}\left(\det \widetilde{M}^{\mathcal{R}}_{\widetilde{B}_{\eta-\varepsilon_k}}\right), \tag{6.10}$$

where the last equality uses Proposition 5.2. Here, as in Section 5.3, $\det {}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_{({}^{\varepsilon_k}\widetilde{B}_\eta)}$ is the Shapovalov determinant calculated with respect to the basis ${}^{\varepsilon_k}\widetilde{B}_\eta$.

The change of basis from A_η to C_η is unitriangular and the change of basis from C_η to D_η is unitriangular. Thus, $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{A_\eta} = \det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{D_\eta}$, and so the right sides of (6.8) and (6.10) are equal. The lemma follows from this equality by rearranging. □

Lemma 6.4. *Up to multiplication by a power of q,*

$$s_k = \prod_{1 \le j < k}\left(\frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j}\left(z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k\right)}\right). \tag{6.11}$$

Proof. Fix $1 \le k \le N$. Setting $\eta = \varepsilon_k$ in Lemma 6.3 and applying Theorem 5.1 we see that, up to multiplication by a power of q,

$$\prod_{1\le x\le N} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1\le x\le N} \frac{\det \widetilde{M}^{\mathcal{R}}_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x} \det \widetilde{M}^{\mathcal{R}}_{\varepsilon_k-\varepsilon_x}}$$
$$= \prod_{1\le x\le N} \prod_{\substack{1\le i<j\le N \\ m>0}} \left(\frac{c_{\varepsilon_k-\varepsilon_x}\left(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j\right)}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})\sigma_{\varepsilon_x}\left(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_x+m\varepsilon_i-m\varepsilon_j)}, \tag{6.12}$$

where, for each $1 \le x \le N$, $c_{\varepsilon_k-\varepsilon_x}$ is a unit in $\mathbb{Q}(q)[z_1^{\pm1}, \ldots, z_N^{\pm1}]$. The value $p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)$ is 0 unless $m = 1$ and $x \le i < j \le k$. If $i > x$, then σ_{ε_x} acts as the identity on $z_i z_j^{-1} - q^{2+2i-2j} z_i^{-1} z_j$, so the corresponding factors in the numerator and denominator cancel. Hence, we need only consider factors on the right hand side where $m = 1$, $i = x$, and $x < j \le k$. If $x > k$, then $\varepsilon_k - \varepsilon_x \notin Q^-$, and hence $p(\varepsilon_k - \varepsilon_x) = 0$, so on the left hand since we only need to consider those factors where $1 \le x \le k$. Up to multiplication by a power of q, the expression reduces to

$$\prod_{1\le x\le k} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1\le x<k} \left(\frac{c_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})} \right)^{p(\varepsilon_k-\varepsilon_j)} \prod_{x<j\le k} \left(\frac{z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x}\left(z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_j)}$$
$$= \prod_{1<j\le k} \left(\prod_{1\le x<j} \frac{z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x}\left(z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_j)}. \tag{6.13}$$

The last two expressions are equal because they are each a product over pairs (x, j) with $1 \le x < j \le k$, and the factors of $c_{\varepsilon_k-\varepsilon_x}/(\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x}))$ have been dropped because they are powers of q. Using the fact that $s_1 = 1$ and making the change of variables $j \to x$ and $x \to j$ on the right side, (6.13) becomes

$$\prod_{1<x\le k} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1<x\le k} \left(\prod_{1\le j<x} \frac{z_j z_x^{-1} - q^{2+2j-2x} z_j^{-1} z_x}{\sigma_{\varepsilon_j}\left(z_j z_x^{-1} - q^{2+2j-2x} z_j^{-1} z_x\right)} \right)^{p(\varepsilon_k-\varepsilon_x)}. \tag{6.14}$$

For $k \ge 2$, the lemma now follows by induction. For $k = 1$, the result simply says that $s_1 = 1$, which we already know. □

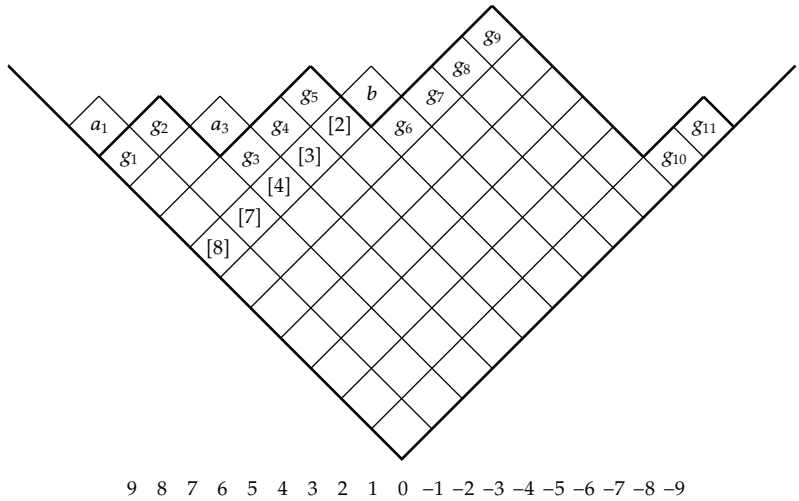


Figure 2: The partition enclosed by the thick lines is $\lambda = (10,10,8,8,8,6,6,6,6,1,1)$. If $k = 6$ then $A(\lambda,<6) = \{a_1,a_3\}$, $R(\lambda,<6) = \{g_2,g_5\}$, and $\mathrm{ev}_\lambda(s_6) = ([2]/[3])([3]/[4])([4]/[5])([7]/[8])([8]/[9]) = ([2]/[5])([7]/[9]) = ([c(g_5)-c(b)][c(g_2)-c(b)])/([c(a_3)-c(b)][c(a_1)-c(b)])$. The factors in the numerator of the first expression are displayed. These are the q-integers corresponding to the hook lengths of the boxes in the same column as the addable box b in row 6.

Proposition 6.5. *Let λ be a partition. Let $A(\lambda,<k)$ (resp. $R(\lambda,<k)$) be the set of boxes which can be added to (resp. removed from) λ on rows λ_j with $j<k$ such that the result is still a partition. Let $b = (\lambda+\varepsilon_k)/\lambda$ and let $c(\cdot)$ be as in Figure 1. Then, up to multiplication by a power of q,*

$$ev_\lambda(s_k) = \begin{cases} \dfrac{\prod_{r\in R(\lambda,<k)}[c(r)-c(b)]}{\prod_{a\in A(\lambda,<k)}[c(a)-c(b)]}, & \text{if } \lambda+\varepsilon_k \text{ is a partition,} \\ 0, & \text{if } \lambda+\varepsilon_k \text{ is not a partition.} \end{cases} \tag{6.15}$$

Proof. For $1\le j\le N$, let g_j be the last box in row j of λ. By Lemma 6.4, up to multiplication by a power of q,

$$\mathrm{ev}_\lambda(s_k) = \mathrm{ev}_\lambda\left(\prod_{1\le j<k}\frac{z_jz_k^{-1}-q^{2+2j-2k}z_j^{-1}z_k}{\sigma_{\varepsilon_j}\left(z_jz_k^{-1}-q^{2+2j-2k}z_j^{-1}z_k\right)}\right) = \prod_{1\le j<k}\frac{[c(g_j)-c(b)]}{[c(g_j)-c(b)+1]}, \tag{6.16}$$

where the last equality is a simple calculation from definitions. The denominator on the right side is never zero, and the numerator is zero exactly when $\lambda_k = \lambda_{k-1}$, so that $\lambda+\varepsilon_k$ is no longer a partition. If $\lambda_j = \lambda_{j+1}$ for any $j<k$, then there is cancellation, giving (6.15). See Figure 2. □

Proposition 6.6. *Let $N^l_{\bar{j}}(\mu/\lambda)$ be as in Section 3.3. For any partition λ,*

$$\begin{aligned} val_{\phi_{2\ell}}ev_\lambda(s_k) &= N^l_{\bar{i}}(\mu/\lambda), \quad \text{if } \mu=\lambda+\varepsilon_k \text{ is a partition, and } \mu/\lambda \text{ is an } \bar{i} \text{ colored box,} \\ ev_\lambda(s_k) &= 0, \quad \text{otherwise.} \end{aligned} \tag{6.17}$$

Proof. By Proposition 6.5, $\mathrm{ev}_\lambda(s_k) = 0$ if $\lambda + \varepsilon_k$ is not a partition. If $\lambda + \varepsilon_k$ is a partition, then

$$\begin{aligned} \{b \in A(\lambda, < k) : \overline{c}(b) = \overline{c}(\mu/\lambda)\} &= \{b \in A_{\overline{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}, \\ \{b \in R(\lambda, < k) : \overline{c}(b) = \overline{c}(\mu/\lambda)\} &= \{b \in R_{\overline{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}, \end{aligned} \tag{6.18}$$

where the notation is as in Section 3.3. Since

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = q^{-x}\left(q - q^{-1}\right)^{-1} \prod_{d|2x} \phi_d, \tag{6.19}$$

$[x]$ is divisible by $\phi_{2\ell}$ if and only if x is divisible by ℓ, and $[x]$ is never divisible by $\phi_{2\ell}^2$. The result now follows from Proposition 6.5. □

Proof of Theorem 6.1. Fix λ and $1 \le k \le m_\lambda$. From definitions, $(\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+} = v_{\mu^{(j)}}$. Thus, using (5.12),

$$r_j(\lambda) = \left(v_{\mu^{(j)}}, v_{\mu^{(j)}}\right) = \left((\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+}, (\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+}\right) = \mathrm{ev}_\lambda\left(v_{\varepsilon_{k_j}+}, v_{\varepsilon_{k_j}+}\right) = \mathrm{ev}_\lambda\left(s_{k_j}\right). \tag{6.20}$$

The result now follows from Proposition 6.6. □

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