Discrete Dynamics in Nature and Society

Theoretical and Numerical Results for Fractional Difference and Differential Equations

Lead Guest Editor: Qasem M. Al-Mdallal
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Editorial Board

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Editorial

Theoretical and Numerical Results for Fractional Difference and Differential Equations

Qasem M. Al-Mdallal,[1] Thabet Abdeljawad,[2] and Mohamed A. Hajji[1]

[1] *Department of Mathematical Sciences, United Arab Emirates University, P.O. Box 17551, Al Ain, Abu Dhabi, UAE*
[2] *Department of Mathematics and Physical Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*

Correspondence should be addressed to Qasem M. Al-Mdallal; q.almdallal@uaeu.ac.ae

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The fractional calculus is relatively an old topic dating back to more than 300 years. In 1695, L'Hopital wrote to Leibniz asking him about the notation of the nth derivative ($D^n f/dx^n$) which was mentioned for the first time in his publications. L'Hopital asked Leibniz about the result if $n = 1/2$. Leibniz's response was as follows: An apparent paradox, from which one day useful consequences will be drawn. This communication was what created the so-called fractional calculus. However, the gate to the fractional calculus subject was opened in 1832 by Liouville in a series of papers from 1832 to 1837. Later, the intensive investigations on the results of Liouville by Riemann led to the construction of the Riemann-Liouville fractional integral operator, given by

$$J^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)\, dt, \qquad (1)$$

where $y \in L_1(a,b)$ and $\alpha \in \mathbb{R}^+$. Notice that this definition for fractional integral operator had been a valuable cornerstone in fractional calculus subject and had led to the left Riemann-Liouville fractional derivative, defined by

$$\begin{aligned} D_R^{(\alpha)} y(x) &= \frac{d^n}{dx^n} J^{n-\alpha} y(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} y(t)\, dt, \end{aligned} \qquad (2)$$

where $y \in C^n[a,b]$ and $n = [\alpha]$ is the smallest integer greater than or equal to α.

Since then, many mathematicians were involved in the development of fractional calculus subject up to the middle of the twentieth century such as P. S. Laplace, J. B. J. Fourier, N. H. Abel, H. Holmgren, A. K. Grunwald, A. V. Letnikov, H. Laurent, P. A. Nekrassov, A. Krug, I. Hadamard, O. Heaviside, S. Pincherle, G. H. Hardy and I. E. Littlewood, H. Weyl, P. Levy, A. Marchaud, H. T. Davis, E. L. Post, A. Zygmund, E. R. Love, A. Erdelyi, H. Kober, D. V. Widder, and M. Riesz and W. Feller; for intensive historical background the reader is referred to the novel work on fractional calculus by Ross [1].

An alternative and novel definition of fractional derivative was introduced by Caputo in 1967, given by

$$\begin{aligned} D_C^{(\alpha)} y(x) &= J^{n-\alpha} \frac{d^n}{dx^n} y(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} y^{(n)}(t)\, dt, \end{aligned} \qquad (3)$$

where $y \in C^n[a,b]$ and $n = [\alpha]$.

Notice that this definition produces different properties than Riemann-Liouville fractional derivative. It should be pointed out that although the literature reveals several types of fractional derivative (such as Grunwald-Letnikov derivative, Sonin-Letnikov derivative, Hadamard derivative, Marchaud derivative, Riesz derivative, Riesz-Miller derivative, Miller-Ross derivative, Weyl derivative, and Erdelyi-Kober derivative), still the Riemann-Liouville and Caputo fractional derivatives are the most commonly used ones.

It is well-known that there is still a lack of geometric and physical interpretation of fractional integration and differentiation. However, the fractional derivatives have been proved to be a powerful technique for solving integral and differential equations resulting from several physical models. For instance, linear viscoelasticity is the field of the most applications of fractional calculus since it models hereditary phenomena with long memory. In fact, we may refer to other applications like finance, stochastic processes, signal processing, automatic control, electromagnetic theory, elasticity, diffusion and advection phenomena, and so on as applications of fractional calculus (the reader is referred to the survey paper [2]).

In this special issue, we focus on the recent theoretical and numerical studies on fractional ordinary differential equations, fractional partial differential equations, fractional stochastic differential equations, fractional delay differential equations, fractional difference equations, and fractional difference dynamical systems. We hope that the papers of this special issue will enrich our readers and stimulate researchers to extend, generalize, and apply the established results.

Qasem M. Al-Mdallal
Thabet Abdeljawad
Mohamed A. Hajji

References

[1] B. Ross, "A brief history and exposition of the fundamental theory of fractional calculus," in *Fractional Calculus and Its Applications*, vol. 457 of *Lecture Notes in Mathematics*, pp. 1–36, Springer Berlin Heidelberg, Berlin, Germany, 1975.

[2] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

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Research Article

Nontrivial Solutions for Time Fractional Nonlinear Schrödinger-Kirchhoff Type Equations

N. Nyamoradi,[1] Y. Zhou,[2,3] E. Tayyebi,[1] B. Ahmad,[3] and A. Alsaedi[3]

[1] *Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran*
[2] *Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, China*
[3] *Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Y. Zhou; yzhou@xtu.edu.cn

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We study the existence of solutions for time fractional Schrödinger-Kirchhoff type equation involving left and right Liouville-Weyl fractional derivatives via variational methods.

1. Introduction

In recent years, there has been a great interest in studying problems involving fractional Schrödinger equations [1–5], Kirchhoff type equations [6–8], fractional Navier-Stokes equations [9, 10], and fractional ordinary differential equations and Hamiltonian systems [11–17], and so forth. For further details and applications, we refer the reader to [18, 19] and the references cited therein.

On the other hand, the integer-order Schrödinger-Kirchhoff type equations have also been investigated by many authors; for example, see [20–23]. In fact, Schrödinger-Kirchhoff type equations play an important role in modelling several physical and biological systems. However, to the best of our knowledge, the existence of solutions to the time fractional Schrödinger-Kirchhoff type equations has yet to be addressed.

The objective of the present paper is to study time fractional Schrödinger-Kirchhoff type equation of the form

$$\begin{gathered}\left(a+b\int_{\mathbb{R}}\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2dt\right)^{\theta-1}{}_tD_{\infty}^{\alpha}\left({}_{-\infty}D_t^{\alpha}u(t)\right)\\ +\mu V(t)u=f(t,u),\quad t\in\mathbb{R},\ u\in H^{\alpha}(\mathbb{R}),\end{gathered}\tag{1}$$

where $\alpha\in(1/2,1]$, ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$, respectively, denote left and right Liouville-Weyl fractional derivatives of order α on $\mathbb{R}$, $a,b>0$ are constants, $\mu>0$ is parameter, $\theta>1$, $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $V:\mathbb{R}\to\mathbb{R}^+$ is a potential function.

The rest of the paper is organized as follows. Section 2 contains preliminary concepts of fractional calculus and fractional Sobolev space, while some important lemmas, which are needed in the proof of main results, are obtained in Section 3. We present our main results in Section 4.

2. Preliminaries

In this section, we recall important definitions and concepts of fractional calculus and then prove certain results about fractional Sobolev space $H^{\alpha}(\mathbb{R})$ related to our study of the problem at hand.

Definition 1 (see [24]). The left and right Liouville-Weyl fractional integrals of order $\alpha\in(0,1)$ on $\mathbb{R}$ are defined by

$$\begin{aligned}{}_{-\infty}I_x^{\alpha}\phi(x)&=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\phi(\xi)\,d\xi,\\ {}_xI_{\infty}^{\alpha}\phi(x)&=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}\phi(\xi)\,d\xi,\end{aligned}\tag{2}$$

respectively, where $x\in\mathbb{R}$.

The left and right Liouville-Weyl fractional derivatives of order $\alpha \in (0,1)$ on $\mathbb{R}$ are defined by

$$ {}_{-\infty}D_x^{\alpha}\phi(x) = \frac{d}{dx}\,{}_{-\infty}I_x^{1-\alpha}\phi(x), \quad {}_xD_{\infty}^{\alpha}\phi(x) = -\frac{d}{dx}\,{}_xI_{\infty}^{1-\alpha}\phi(x), \tag{3} $$

respectively, where $x \in \mathbb{R}$.

The definitions (3) may be written in an alternative form as follows:

$$ {}_{-\infty}D_x^{\alpha}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\phi(x)-\phi(x-\xi)}{\xi^{\alpha+1}}d\xi, \quad {}_xD_{\infty}^{\alpha}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\phi(x)-\phi(x+\xi)}{\xi^{\alpha+1}}d\xi. \tag{4} $$

Also, we define the Fourier transform $\mathscr{F}(u)(\xi)$ of $u(x)$ as

$$ \mathscr{F}(u)(\xi) = \int_{-\infty}^{\infty} e^{-ix\cdot\xi}u(x)\,dx. \tag{5} $$

For any $\alpha > 0$, we define the seminorm and norm, respectively, as [16]

$$ |u|_{I_{-\infty}^{\alpha}} = \left\| {}_{-\infty}D_x^{\alpha}u \right\|_{L^2}, \quad \|u\|_{I_{-\infty}^{\alpha}} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2\right)^{1/2}, \tag{6} $$

and let the space $I_{-\infty}^{\alpha}(\mathbb{R})$ denote the completion of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|\cdot\|_{I_{-\infty}^{\alpha}}$.

Next, for $0 < \alpha < 1$, we give the relationship between classical fractional Sobolev space $H^{\alpha}(\mathbb{R})$ and $I_{-\infty}^{\alpha}(\mathbb{R})$, where $H^{\alpha}(\mathbb{R})$ is defined by

$$ H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}, \tag{7} $$

with the norm

$$ \|u\|_{\alpha} = \left(\|u\|_{L^2}^2 + |u|_{\alpha}^2\right)^{1/2}, \tag{8} $$

and seminorm

$$ |u|_{\alpha} = \left\| |\xi|^{\alpha}\mathscr{F}(u) \right\|_{L^2}. \tag{9} $$

Observe that the spaces $H^{\alpha}(\mathbb{R})$ and $I_{-\infty}^{\alpha}(\mathbb{R})$ are equal and have equivalent norms (see [16]).

Therefore, we define

$$ H^{\alpha}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) \mid |\xi|^{\alpha}\mathscr{F}(u) \in L^2(\mathbb{R})\right\}. \tag{10} $$

Let

$$ X^{\alpha} = \left\{u \in H^{\alpha}(\mathbb{R}) \mid \int_{\mathbb{R}}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2 + |u(t)|^2\right)dt < \infty\right\}. \tag{11} $$

The space X^{α} is a reflexive and separable Hilbert space with the inner product

$$ \langle u, v\rangle_{X^{\alpha}} = \int_{\mathbb{R}}\left({}_{-\infty}D_t^{\alpha}u(t)\cdot {}_{-\infty}D_t^{\alpha}v(t) + u(t)v(t)\right)dt \tag{12} $$

and the corresponding norm

$$ \|u\|_{X^{\alpha}}^2 = \langle u, u\rangle_{X^{\alpha}}. \tag{13} $$

Define the space

$$ X_{\mu}^{\alpha} = \left\{u \in X^{\alpha} : \int_{\mathbb{R}} \mu V(t)|u|^2 dt < +\infty\right\}, \tag{14} $$

with the norm

$$ \|u\|_{X_{\mu}^{\alpha}} = \left(\int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}. \tag{15} $$

Lemma 2. *$(X_{\mu}^{\alpha}, \|\cdot\|_{X_{\mu}^{\alpha}})$ is a uniformly convex Banach space.*

Proof. X_{μ}^{α} is obviously Banach space. Now, we can prove that $(X_{\mu}^{\alpha}, \|\cdot\|_{X_{\mu}^{\alpha}})$ is uniformly convex. To this end, let $0 < \varepsilon < 2$ and $u, v \in X_{\mu}^{\alpha}$ with $\|u\|_{X_{\lambda}^{\alpha}} = \|v\|_{X_{\mu}^{\alpha}} = 1$ and $\|u - v\|_{X_{\mu}^{\alpha}} \geq \varepsilon$. Using the following inequality:

$$ \left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2 \leq \frac{1}{2}\left(|a|^2 + |b|^2\right), \quad \forall a, b \in \mathbb{R}, \tag{16} $$

we get

$$ \begin{aligned} &\left\|\frac{u+v}{2}\right\|_{X_{\mu}^{\alpha}}^2 + \left\|\frac{u-v}{2}\right\|_{X_{\mu}^{\alpha}}^2 \\ &= \int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}\left(\frac{u+v}{2}\right)(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)\left|\frac{u+v}{2}\right|^2 dt \\ &+ \int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}\left(\frac{u-v}{2}\right)(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)\left|\frac{u-v}{2}\right|^2 dt \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2\right)dt + \int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}v(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)|u|^2 dt + \int_{\mathbb{R}}\mu V(t)|v|^2 dt\right) = \frac{1}{2}\left(\|u\|_{X_{\mu}^{\alpha}}^2 + \|v\|_{X_{\mu}^{\alpha}}^2\right) = 1, \end{aligned} \tag{17} $$

which implies that $\|(u+v)/2\|_{X_{\mu}^{\alpha}}^2 \leq 1 - \varepsilon/2$. Hence, taking $\delta = \delta(\varepsilon)$ such that $1-\varepsilon/2 = 1-\delta$, we have $\|(u+v)/2\|_{X_{\mu}^{\alpha}}^2 \leq 1-\delta$. Therefore, $(X_{\mu}^{\alpha}, \|\cdot\|_{X_{\mu}^{\alpha}})$ is uniformly convex. □

In the sequel, we need the following assumptions.

(V1) $V(t) \in C(\mathbb{R}, \mathbb{R})$, $V_0 := \inf_{t\in\mathbb{R}} V(t) > 0$;

(V2) there exists $r > 0$ such that, for any $M > 0$,

$$\operatorname{meas}\left(\{t \in (y-r, y+r) : V(t) \leq M\}\right) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty; \tag{18}$$

(V3) there exists $l_0 > 0$ such that $\int_{|t|\geq l_0} V(t)^{-1} dt < \infty$;

(F1) $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_0, c_1, \ldots, c_l > 0$ and $q_j \in (2, 2\theta)$ such that

$$|f(t,u)| \leq c_0 |u| + \sum_{j=1}^{l} c_j |u|^{q_j - 1}, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}; \tag{19}$$

(F2) $f(t,u) = o(|u|)$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}^N$;

(F3) there exist $\lambda \in (2\theta, \infty)$ such that

$$\lambda F(t,u) \leq f(t,u)u, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}; \tag{20}$$

(F4) $F(t,u)/|u|^{2\theta} \to +\infty$ as $|u| \to +\infty$ uniformly in $t \in \mathbb{R}$;

(F5) $f(t,-u) = -f(t,u)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}$;

(F6) $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists $1 < p < 2$ such that

$$|f(t,u)| \leq |u|^{p-1}, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}; \tag{21}$$

(F7) there exist $\sigma_1 > 0$, $0 < \sigma_2 < 1/8D_2^2$ (D_2 is defined in Remark 6), $1 \leq \gamma < 2$, and small constants $0 < r_0 < r_1$ such that

$$\sigma_1 |u|^{\gamma} < F(t,u) \leq \sigma_2 |u|^2, \quad r_0 \leq |u| \leq r_1, \text{ a.e. } t \in \mathbb{R}. \tag{22}$$

Lemma 3. *Assume that (V1) holds. Then the embeddings $X_\mu^\alpha \hookrightarrow X^\alpha \hookrightarrow L^2(\mathbb{R})$ are continuous. In particular, there exists a constant $C_2 > 0$ such that*

$$\|u\|_{L^2(\mathbb{R})} \leq C_2 \|u\|_{X_\mu^\alpha} \quad \forall u \in X_\mu^\alpha. \tag{23}$$

Moreover, if (V1) and (V2) hold, then the embedding $X_\mu^\alpha \hookrightarrow L^2(\mathbb{R})$ is compact.

Proof. Clearly, the chain of embeddings $X_\mu^\alpha \hookrightarrow X^\alpha \hookrightarrow L^2(\mathbb{R})$ is continuous and consequently one can obtain (23). Also in view of (V1), (V2), and following the method of proof similar to that of Lemma 2.2 in [15], the embedding $X_\mu^\alpha \hookrightarrow L^2(\mathbb{R})$ is compact. □

Lemma 4. *Let $\alpha > 1/2$. Then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there exists a constant $C = C_\alpha$ such that*

$$\sup_{x\in\mathbb{R}} |u(x)| \leq C \|u\|_{X_\mu^\alpha}. \tag{24}$$

Proof. The proof is similar to that of Theorem 2.1 in [16], so we omit it. □

Also by Lemma 4, there is a constant $C_\alpha > 0$ such that

$$\|u\|_\infty \leq C_\alpha \|u\|_{X_\mu^\alpha}. \tag{25}$$

Remark 5. If $u \in H^\alpha(\mathbb{R})$ with $1/2 < \alpha < 1$, then it follows by Lemma 4 that $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty)$ as

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_\infty^{q-2} \|u\|_{L^2(\mathbb{R})}^2. \tag{26}$$

Remark 6. From Remark 5 and Lemma 3, it is easy to verify that the imbedding of X_μ^α in $L^q(\mathbb{R})$ is also compact for $q \in (2, \infty)$. Hence, for all $2 \leq q < \infty$, the imbedding of X_μ^α in $L^q(\mathbb{R})$ is continuous and compact, which together with Lemma 4 implies that there exists $D_q > 0$ such that

$$\|u\|_{L^q(\mathbb{R})} \leq D_q \|u\|_{X_\mu^\alpha}. \tag{27}$$

Lemma 7. *Assume that (V1) and (V3) hold. Then the embedding $X_\mu^\alpha \hookrightarrow L^p(\mathbb{R})$ is continuous and compact for $p \in [1, +\infty)$.*

Proof. By (V3) and Hölder's inequality, we have

$$\begin{aligned} &\int_{|t|\geq l_0} |u(t)| dt \\ &\leq \left(\int_{|t|\geq l_0} V(t) |u(t)|^2 dt\right)^{1/2} \left(\int_{|t|\geq l_0} V(t)^{-1/2} dt\right)^{1/2} \\ &\leq c_1 \|u\|_{X_\mu^\alpha}, \end{aligned} \tag{28}$$

for some positive constant c_1. So Lemma 4 implies that

$$\begin{aligned} \|u\|_1 &= \int_{-l_0}^{l_0} |u(t)| dt + \int_{|t|\geq l_0} |u(t)| dt \\ &\leq 2l_0 \|u\|_\infty + c_1 \|u\|_{X_\mu^\alpha} \leq c_2 \|u\|_{X_\mu^\alpha}, \end{aligned} \tag{29}$$

for some positive constant c_2. Hence, by Remark 6, we can get continuous embeddings X_μ^α into $L^p(\mathbb{R})$ for $p \in [1, +\infty)$. Now, we will show that the embedding is compact for $p \in [1, +\infty)$. Let $\{u_n\} \subset X_\mu^\alpha$ such that $u_n \rightharpoonup 0$ and $M > 0$ such that $\|u\|_{X_\mu^\alpha} \leq M$. In view of (V3), given $\varepsilon > 0$, for $l > 0$ large enough, one can obtain

$$\int_{|t|\geq l_0} V(t)^{-1/2} dt < \left(\frac{\varepsilon}{2M}\right)^2. \tag{30}$$

Then,

$$\begin{aligned} &\int_{|t|\geq l} |u(t)| dt \\ &\leq \left(\int_{|t|\geq l} V(t) |u(t)|^2 dt\right)^{1/2} \left(\int_{|t|\geq l} V(t)^{-1/2} dt\right)^{1/2} \\ &\leq \frac{\varepsilon}{2M} \|u\|_{X_\mu^\alpha} \leq \frac{\varepsilon}{2}. \end{aligned} \tag{31}$$

On the other hand, by Sobolev's theorem (see, e.g., [25]) which implies that $u_n \to 0$ uniformly on $[-l, l]$, there is n_0 such that $\int_{-l}^{l} |u(t)| dt < \varepsilon/2$ for all $n \geq n_0$. Thus $u_n \to 0$ in $L^1(\mathbb{R})$. So, for $1 < p < \infty$, we have

$$\int_{\mathbb{R}} |u(x)|^p \, dx \leq \|u\|_{\infty}^{p-1} \int_{\mathbb{R}} |u(t)| \, dt \leq c_3 \|u\|_1 \longrightarrow 0, \quad (32)$$

and consequently, $u_n \to 0$ in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$. □

Definition 8. Let X be a Banach space, $I \in C^1(X, \mathbb{R})$. One says that I satisfies the Palais-Smale (PS) condition if any sequence $(u_n) \in X$ for which $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

In order to establish the main results, we need the following known Theorems.

Theorem 9 (see [26, Theorem 2.2]). *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies (PS) condition. Suppose $I(0) = 0$ and*

(i) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq 0$;*

(ii) *there is an $e \in X \setminus \overline{B_\rho(0)}$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)), \quad (33)$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \ \gamma(1) = e\}. \quad (34)$$

Theorem 10 (see [26, Theorem 9.12]). *Let X be an infinite dimensional Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying (PS) condition, and $I(0) = 0$. If $X = Y \oplus Z$, where Y is finite dimensional and I satisfies the following conditions:*

(I1) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \geq \alpha$;*

(I2) *for any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $I(u) \leq 0$ on $\widetilde{X} \setminus B_R$,*

then I possesses an unbounded sequence of critical values.

3. Some Lemmas

Recall that $u \in X_\mu^\alpha$ is said to be a weak solution of problem (1) if

$$\begin{aligned} &\left(a + b \int_{\mathbb{R}} \left| {}_{-\infty}D_t^\alpha u(t) \right|^2 dt\right)^{\theta-1} \\ &\quad \cdot \int_{\mathbb{R}} {}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha \varphi(t) \, dt \\ &\quad + \int_{\mathbb{R}} \mu V(t) u(t) \varphi(t) \, dt = \int_{\mathbb{R}} f(t, u(t)) \varphi(t) \, dt, \\ &\qquad \forall \varphi \in X_\mu^\alpha, \end{aligned} \quad (35)$$

and the energy functional $I_{\mu,\theta} : X_\mu^\alpha \to \mathbb{R}$ is given by the formula

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta} \left(a + b \int_{\mathbb{R}} \left| {}_{-\infty}D_t^\alpha u(t) \right|^2 dt\right)^\theta \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \mu V(x) |u(t)|^2 \, dt \\ &\quad - \int_{\mathbb{R}} F(t, u(t)) \, dt, \end{aligned} \quad (36)$$

where $F(x, u) = \int_0^u f(t, s) ds$.

In view of assumptions (V1) and (F1), the functional $I_{\mu,\theta}$ is of class $C^1(X_\mu^\alpha, \mathbb{R})$ and by similar method in Theorem 4.1 in [27] and the definition of Gâteaux derivative, one can get

$$\begin{aligned} \left\langle I'_{\mu,\theta}(u), \varphi \right\rangle &= \left(a + b \int_{\mathbb{R}} \left| {}_{-\infty}D_t^\alpha u(t) \right|^2 dt\right)^{\theta-1} \\ &\quad \cdot \int_{\mathbb{R}} {}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha \varphi(t) \, dt \\ &\quad + \int_{\mathbb{R}} \mu(t) u(t) \varphi(t) \, dt \\ &\quad - \int_{\mathbb{R}} f(t, u(t)) \varphi(t) \, dt, \\ &\qquad \forall u, \varphi \in X_\mu^\alpha. \end{aligned} \quad (37)$$

Lemma 11. *Assume that (V) and (F1)–(F3) hold. Then $I_{\mu,\theta}$ satisfies the (PS) condition.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset X_\mu^\alpha$ be a sequence such that $\{I_{\mu,\theta}(u_n)\}_{n \in \mathbb{N}}$ is bounded and $I'_{\mu,\theta}(u_n) \to 0$ as $n \to \infty$. Then there exits $D > 0$ such that $|\langle I'_{\mu,\theta}(u_n), u_n \rangle| \leq D \|u\|_{X_\mu^\alpha}$ and $|I_{\mu,\theta}(u_n)| \leq D$. So, by (F3), (23), and the fact that $\lambda > 2\theta > 1$, we get

$$\begin{aligned} \lambda D + D \|u\|_{X_\mu^\alpha} &\geq \lambda I_{\mu,\theta}(u_n) - \left\langle I'_{\mu,\theta}(u_n), u_n \right\rangle \\ &= \frac{\lambda}{2b\theta} \left(a + b \int_{\mathbb{R}} \left| {}_{-\infty}D_t^\alpha u_n(t) \right|^2 dt\right)^\theta + \frac{\lambda}{2} \\ &\quad \cdot \int_{\mathbb{R}} \mu V(t) |u_n(t)|^2 dt - \lambda \int_{\mathbb{R}} F(t, u_n(t)) \, dt \\ &\quad - \left(a + b \int_{\mathbb{R}} \left| {}_{-\infty}D_t^\alpha u_n(t) \right|^2 dt\right)^{\theta-1} \\ &\quad \cdot \int_{\mathbb{R}} \left| {}_{-\infty}D_t^\alpha u_n(t) \right|^2 dt - \int_{\mathbb{R}} \mu V(t) |u_n(t)|^2 dt \\ &\quad + \int_{\mathbb{R}} f(t, u_n(t)) u_n(t) \, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{a\lambda}{2b\theta}\left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt\right)^{\theta-1} \\
&+ \frac{\lambda - 2\theta}{2\theta}\left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt\right)^{\theta-1} \\
&\cdot \int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt + \frac{\lambda - 2}{2} \\
&\cdot \int_{\mathbb{R}} \mu V(t) \left|u_n(t)\right|^2 dt \\
&+ \int_{\mathbb{R}} \left(f(t, u_n(t)) u_n(t) - \lambda F(t, u_n(t))\right) dt \\
&\geq \frac{\lambda - 2\theta}{2\theta} a^{\theta-1} \int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt + \frac{\lambda - 2}{2} \\
&\cdot \int_{\mathbb{R}} \mu V(t) \left|u_n(t)\right|^2 dt \geq \frac{\lambda - 2\theta}{2\theta} \|u\|_{X_{\mu}^{\alpha}}^2 .
\end{aligned} \tag{38}
$$

Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_{μ}^{α}.

So, passing onto subsequence if necessary, thanks to Lemma 3, we have

$$
\begin{aligned}
&u_n \rightharpoonup u, \quad \text{weakly in } X_{\mu}^{\alpha}, \\
&u_n \longrightarrow u, \quad \text{strongly a.e. in } \mathbb{R}, \\
&u_n \longrightarrow u, \\
&\text{strongly a.e. in } L^s(\mathbb{R}^N),\ 2 \leq s < +\infty,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt \longrightarrow \rho_1 \geq 0, \\
&\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt \longrightarrow \rho_2 \geq 0.
\end{aligned} \tag{40}
$$

We will prove that

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 dt = \rho_1, \\
&\int_{\mathbb{R}} \mu V(t) \left|u\right|^2 dx = \rho_2.
\end{aligned} \tag{41}
$$

Let $\varphi \in X_{\mu}^{\alpha}$ be fixed and denote by B_{φ} the linear functional on X_{μ}^{α} defined by

$$
B_{\varphi}(v) := \int_{\mathbb{R}} {}_{-\infty}D_t^{\alpha} \varphi(t) \cdot {}_{-\infty}D_t^{\alpha} v(t)\, dt, \tag{42}
$$

and set

$$
\Delta_{\alpha}(u) := \int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 dt, \tag{43}
$$

for all $v \in X_{\mu}^{\alpha}$. In view of the Hölder inequality and definition of B_{φ}, we have

$$
\begin{aligned}
&\left\langle I'_{\mu,\theta}(u_n) - I'_{\mu,\theta}(u), u_n - u\right\rangle = \left(a + b\Delta_{\alpha}(u_n)\right)^{\theta-1} \\
&\cdot B_{u_n}(u_n - u) - \left(a + b\Delta_{\alpha}(u_n)\right)^{\theta-1} B_u(u_n - u) \\
&+ \int_{\mathbb{R}} \mu V(t)(u_n - u)(u_n - u)\, dt \\
&- \int_{\mathbb{R}} \left(f(t, u_n) - f(t, u)\right)(u_n - u)\, dt \geq (a \\
&+ b\Delta_{\alpha}(u_n))^{\theta-1} \Delta_{\alpha}(u_n) - \left(a + b\Delta_{\alpha}(u_n)\right)^{\theta-1} \\
&\cdot \left(\Delta_{\alpha}(u_n)\right)^{(\theta-1)/2} \left(\Delta_{\alpha}(u_n)\right)^{1/2} + \left(a + b\Delta_{\alpha}(u)\right)^{\theta-1} \\
&\cdot \Delta_{\alpha}(u) - \left(a + b\Delta_{\alpha}(u)\right)^{\theta-1} \left(\Delta_{\alpha}(u)\right)^{(\theta-1)/2} \\
&\cdot \left(\Delta_{\alpha}(u_n)\right)^{1/2} + \int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt \\
&- \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2} \\
&+ \int_{\mathbb{R}} \mu V(t) |u|^2 dt - \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2} \\
&\cdot \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2} \\
&- \int_{\mathbb{R}} \left(f(t, u_n) - f(t, u)\right)(u_n - u)\, dt = (a \\
&+ b\Delta_{\alpha}(u_n))^{\theta-1} \left(\Delta_{\alpha}(u_n)\right)^{(\theta-1)/2} \left[\left(\Delta_{\alpha}(u_n)\right)^{1/2}\right. \\
&\left.- \left(\Delta_{\alpha}(u)\right)^{1/2}\right] + \left(a + b\Delta_{\alpha}(u)\right)^{\theta-1} \left(\Delta_{\alpha}(u)\right)^{(\theta-1)/2} \\
&\cdot \left[\left(\Delta_{\alpha}(u)\right)^{1/2} - \left(\Delta_{\alpha}(u_n)\right)^{1/2}\right] \\
&+ \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2} \left[\left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2}\right. \\
&\left.- \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2}\right] + \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2} \\
&\cdot \left[\left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2}\right. \\
&\left.- \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2}\right] \\
&- \int_{\mathbb{R}} \left(f(t, u_n) - f(t, u)\right)(u_n - u)\, dt \\
&= \left[\left(\Delta_{\alpha}(u_n)\right)^{1/2} - \left(\Delta_{\alpha}(u)\right)^{1/2}\right] \\
&\cdot \left[\left(a + b\Delta_{\alpha}(u_n)\right)^{\theta-1} \left(\Delta_{\alpha}(u_n)\right)^{(\theta-1)/2}\right. \\
&\left.- \left(a + b\Delta_{\alpha}(u)\right)^{\theta-1} \left(\Delta_{\alpha}(u)\right)^{(\theta-1)/2}\right]
\end{aligned}
$$

$$+\left[\left(\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt\right)^{1/2} - \left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 - \int_{\mathbb{R}}\left(f\left(t,u_n\right)-f(t,u)\right)\left(u_n-u\right)dt. \tag{44}$$

Since $u_n \rightharpoonup u$ in X_μ^α and $I'_{\mu,\theta}(u_n) \to 0$ as $n \to \infty$ in $(X_\mu^\alpha)^*$, therefore $\langle I'_{\mu,\theta}(u_n) - I'_{\mu,\theta}(u), u_n - u\rangle \to 0$ as $n \to \infty$. Now, using (F1) and Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\left(f\left(t,u_n\right)-f(t,u)\right)\left(u_n-u\right)dt \\
&\le \int_{\mathbb{R}}\left|c_0\left(\left|u_n\right|+|u|\right)+\sum_{j=1}^{l}c_j\left(\left|u_n\right|^{q_j-1}+|u|^{q_j-1}\right)\right| \\
&\cdot\left|u_n-u\right|dx \le c_0\left(\left\|u_n\right\|_{L^2(\mathbb{R})}+\|u\|_{L^2(\mathbb{R})}\right)\|u_n \\
&-u\|_{L^2(\mathbb{R})}+\sum_{j=1}^{l}c_j\left(\left\|u_n\right\|_{L^{q_j}(\mathbb{R})}^{q_j-1}+\left\|u_n\right\|_{L^{q_j}(\mathbb{R})}^{q_j-1}\right) \\
&\cdot\left\|u_n-u\right\|_{L^{q_j}(\mathbb{R})},
\end{aligned} \tag{45}$$

which, in view of (39), yields

$$\lim_{n\to\infty}\int_{\mathbb{R}}\left(f\left(t,u_n\right)-f(t,u)\right)\left(u_n-u\right)dt = 0. \tag{46}$$

Since $u_n \to u$ a.e. in $\mathbb{R}$, it follows by Fatou's lemma that

$$\begin{aligned}
\Delta_\alpha(u) &\le \liminf_{n\to\infty}\Delta_\alpha\left(u_n\right) = \rho_1, \\
\int_{\mathbb{R}}\mu V(t)|u|^2 dt &\le \liminf_{n\to\infty}\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt = \rho_2.
\end{aligned} \tag{47}$$

Noting that $\Pi(s) = (a+bs)^{\theta-1}s^{(\theta-1)/2}$ is a nondecreasing function for $s \ge 0$, we get

$$\begin{aligned}
&\left[\left(\rho_1\right)^{1/2}-\left(\Delta_\alpha(u)\right)^{1/2}\right]\left[\left(a+b\rho_1\right)^{\theta-1}\left(\rho_1\right)^{(\theta-1)/2}\right. \\
&\left.-\left(a+b\Delta_\alpha(u)\right)^{\theta-1}\left(\Delta_\alpha(u)\right)^{(\theta-1)/2}\right], \\
&\left[\left(\rho_2\right)^{1/2}-\left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 \ge 0.
\end{aligned} \tag{48}$$

Now, in view of $\langle I'_{\mu,\theta}(u_n) - I'_{\mu,\theta}(u), u_n - u\rangle \to 0$ as $n \to \infty$, (46), and (47), one has

$$\begin{aligned}
0 &\ge \liminf_{n\to\infty}\left\{\left[\left(\Delta_\alpha\left(u_n\right)\right)^{1/2}-\left(\Delta_\alpha(u)\right)^{1/2}\right]\right. \\
&\cdot\left[\left(a+b\Delta_\alpha\left(u_n\right)\right)^{\theta-1}\left(\Delta_\alpha\left(u_n\right)\right)^{(\theta-1)/2}\right. \\
&\left.-\left(a+b\Delta_\alpha(u)\right)^{\theta-1}\left(\Delta_\alpha(u)\right)^{(\theta-1)/2}\right] \\
&+\left[\left(\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt\right)^{1/2}\right. \\
&\left.-\left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 \\
&\left.-\int_{\mathbb{R}}\left(f\left(t,u_n\right)-f(t,u)\right)\left(u_n-u\right)dt\right\} \\
&\ge \lim_{n\to\infty}\left\{\left[\left(\Delta_\alpha\left(u_n\right)\right)^{1/2}-\left(\Delta_\alpha(u)\right)^{1/2}\right]\right. \\
&\cdot\left[\left(a+b\Delta_\alpha\left(u_n\right)\right)^{\theta-1}\left(\Delta_\alpha\left(u_n\right)\right)^{(\theta-1)/2}\right. \\
&\left.\left.-\left(a+b\Delta_\alpha(u)\right)^{\theta-1}\left(\Delta_\alpha(u)\right)^{(\theta-1)/2}\right]\right\} \\
&+\lim_{n\to\infty}\left[\left(\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt\right)^{1/2}\right. \\
&\left.-\left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 \\
&-\lim_{n\to\infty}\left\{\int_{\mathbb{R}}\left(f\left(t,u_n\right)-f(t,u)\right)\left(u_n-u\right)dt\right\} \\
&\ge\left[\left(\rho_1\right)^{1/2}-\left(\Delta_\alpha(u)\right)^{1/2}\right]\left[\left(a+b\rho_1\right)^{\theta-1}\left(\rho_1\right)^{(\theta-1)/2}\right. \\
&\left.-\left(a+b\Delta_\alpha(u)\right)^{\theta-1}\left(\Delta_\alpha(u)\right)^{(\theta-1)/2}\right]+\left[\left(\rho_2\right)^{1/2}\right. \\
&\left.-\left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2.
\end{aligned} \tag{49}$$

Then, from (48)-(49), we get

$$\begin{aligned}
\Delta_\alpha(u) &= \int_{\mathbb{R}}\left|{}_{-\infty}D_t^\alpha u(t)\right|^2 dt = \rho_1, \\
\int_{\mathbb{R}}\mu V(t)|u|^2 dt &= \rho_2.
\end{aligned} \tag{50}$$

Hence, we obtain $\|u_n\|_{X_\mu^\alpha} \to \|u\|_{X_\mu^\alpha}$. As X_μ^α is a reflexive Banach space (see Lemma 2), it is isomorphic to a locally uniformly convex space. So the weak convergence and norm convergence imply strong convergence. This completes the proof. □

Let $\{e_j\}$ be a total orthonormal basis of $L^2(\mathbb{R})$ and define $X_j = \mathbb{R}e_j$, $j \in \mathbb{N}$,

$$\begin{gathered} Y_k = \oplus_{j=1}^{k} X_j, \\ Z_k = \oplus_{j=k+1}^{\infty} X_j, \\ k \in \mathbb{N}. \end{gathered} \tag{51}$$

Lemma 12. *Assume that (V1) holds. Then, for $2 < p < +\infty$,*

$$\beta_k := \sup_{u\in Z_k, \|u\|_{X_\mu^\alpha}=1} \|u\|_{L^p(\mathbb{R})} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{52}$$

Proof. The proof is similar to that of Lemma 3.8 in [28]. So it is omitted. □

In view of Lemma 12, we can choose an integer $k \geq 1$ such that

$$\begin{aligned} &\int_{\mathbb{R}} |u|^2 \, dt \\ &\leq \frac{1}{2c_0} \left(\int_{\mathbb{R}} a \left(\left| {}_{-\infty}D_t^\alpha u(t) \right|^2 \right) dt + \int_{\mathbb{R}} \mu V(t) |u|^2 \, dt \right) \\ &\qquad \forall u \in Z_m \cap X_\mu^\alpha, \end{aligned} \tag{53}$$

where c_1 is a constant given in condition (F1). Let

$$\mathfrak{R}(t) = \begin{cases} 1, & |t| > r, \\ 0, & |t| \leq r, \end{cases} \tag{54}$$

and set $Y = \{(1-\mathfrak{R})u : u \in X_\mu^\alpha,\ (1-\mathfrak{R})u \in Y_k\}$ and $Z = \{(1-\mathfrak{R})u : u \in X_\mu^\alpha,\ (1-\mathfrak{R})u \in Z_k\} + \{\mathfrak{R}v : v \in X_\mu^\alpha\}$. Hence Y and Z are subspaces of X_μ^α, and $X_\mu^\alpha = Y \oplus Z$.

Lemma 13. *Suppose that (V1), (V2), and (F1) are satisfied. Then there exist constants $\varrho, \beta > 0$ such that $I_{\mu,\theta}|_{\partial B_\varrho \cap Z} \geq \alpha$.*

Proof. In view of (V2), (53), and definition of the space Z, we have

$$\begin{aligned} \|u\|_{L^2(\mathbb{R})}^2 &= \int_{|t|<r} |u(t)|^2 \, dt + \int_{|t|\geq r} |u(t)|^2 \, dt \\ &\leq \frac{1}{2c_0} \|u\|_{X_\mu^\alpha}^2 \\ &\quad + \frac{1}{\mu\omega} \int_{\{t\in\mathbb{R}, V(t)>\omega\}} \mu V(t) |u(t)|^2 \, dt \\ &\leq \frac{1}{2c_0} \|u\|_{X_\mu^\alpha}^2 + \frac{1}{\mu\omega} \|u\|_{X_\mu^\alpha}^2 \quad \forall u \in Z. \end{aligned} \tag{55}$$

Therefore, from (23), (55), and (F1) and for large enough value of μ, we get

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta} \left(a + b\Delta_\alpha(u) \right)^\theta + \frac{1}{2} \int_{\mathbb{R}} \mu V(t) |u|^2 \, dt \\ &\quad - \int_{\mathbb{R}} F(t,u) \, dt \\ &\geq \frac{a^{\theta-1}}{2} \Delta_\alpha(u) + \frac{1}{2} \int_{\mathbb{R}} \mu V(t) |u|^2 \, dt \\ &\quad - \int_{\mathbb{R}} F(t,u) \, dt \\ &\geq \frac{1}{2} \|u\|_{X_\mu^\alpha}^2 - \frac{c_0}{2} \|u\|_{L^2(\mathbb{R})}^2 - \sum_{j=1}^{l} \frac{c_j}{q_j} \|u\|_{L^{q_j}(\mathbb{R})}^{q_j} \\ &\geq \frac{1}{4} \|u\|_{X_\mu^\alpha}^2 - \frac{c_0}{\mu\omega 2} \|u\|_{X_\mu^\alpha}^2 - \sum_{j=1}^{l} \frac{c_j D_{q_j}^{q_j}}{q_j} \|u\|_{X_\mu^\alpha}^{q_j} \\ &\geq \frac{1}{8} \|u\|_{X_\mu^\alpha}^2 - \sum_{j=1}^{l} \frac{c_j D_{q_j}^{q_j}}{q_j} \|u\|_{X_\mu^\alpha}^{q_j}. \end{aligned} \tag{56}$$

Since $2 < q_j$ $(j = 1, \ldots, l)$, there exist constants $\varrho, \beta > 0$ such that $I_{\mu,\theta}|_{\partial B_\varrho \cap Z} \geq \beta$. □

Lemma 14. *Assume that (F1) and (F4) are satisfied. Then, for any finite dimensional subspace $\widetilde{X}_\mu^\alpha \subset X_\mu^\alpha$, there is $R = R(\widetilde{X}_\mu^\alpha) > 0$ such that $I_{\mu,\theta}(u) \leq 0$ on $\widetilde{X}_\mu^\alpha \setminus B_R$.*

Proof. Since all the norms in the finite dimensional space are equivalent, there exists a constant Υ such that

$$\|u\|_{L^{2\theta}(\mathbb{R})} \geq \Upsilon \|u\|_{X_\mu^\alpha}, \quad \forall u \in \widetilde{X}_\mu^\alpha. \tag{57}$$

From (F1) and (F4), for any $L > b^{\theta-1}/2\theta\Upsilon^{2\theta} a^{\theta(\theta-1)}$, there exists a constant $C_L > 0$ such that

$$F(t,u) \geq L|u|^{2\theta} - C_L |u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}. \tag{58}$$

Thus

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta} \left(a + b\Delta_\alpha(u) \right)^\theta + \frac{1}{2} \int_{\mathbb{R}} \mu V(t) |u|^2 \, dt \\ &\quad - \int_{\mathbb{R}} F(t,u) \, dt \\ &\leq \frac{1}{2b\theta} \left(a + \frac{b}{a^{\theta-1}} \|u\|_{X_\mu^\alpha}^2 \right)^\theta + \frac{1}{2} \|u\|_{X_\mu^\alpha}^2 \\ &\quad + C_L \|u\|_{L^2(\mathbb{R})}^2 - L \|u\|_{L^{2\theta}(\mathbb{R})}^{2\theta} \\ &\leq \frac{1}{2b\theta} \left(a + \frac{b}{a^{\theta-1}} \|u\|_{X_\mu^\alpha}^2 \right)^\theta \\ &\quad + \left(\frac{1}{2} + C_L D_2^2 \right) \|u\|_{X_\mu^\alpha}^2 - L\Upsilon^{2\theta} \|u\|_{X_\mu^\alpha}^{2\theta} \end{aligned} \tag{59}$$

for all $u \in \widetilde{X}_\mu^\alpha$. Consequently, there is a large $R > 0$ such that $I_{\mu,\theta}(u) \leq 0$ on $\widetilde{X}_\mu^\alpha \setminus B_R$. Therefore, the proof is completed. □

4. Existence of Weak Solutions

In this section, we present our main results.

Theorem 15. *Assume that (V1), (V2), (F1), (F3), (F4), and (F5) hold. Then problem (1) has infinitely many nontrivial weak solutions whenever $\mu > 0$ is sufficiently large.*

Proof. We know that $I_{\mu,\theta}(0) = 0$, and it is even by (F5). Let $X = X_\mu^\alpha$ and Y and Z be as defined in Section 2. By Lemmas 11, 13, and 14, it follows that $I_{\mu,\theta}$ satisfies all the condition of the Theorem 10. Therefore, problem (1) has infinitely many nontrivial weak solutions whenever $\mu > 0$ is sufficiently large. □

Theorem 16. *Assume that (V1), (V2), (F1), (F2), (F3), and (F4) hold. Then problem (1) has at least one nontrivial weak solution when $\mu > 0$.*

Proof. We complete the proof in three steps.

Step 1. Clearly $I_{\mu,\theta}(0) = 0$ and $I_{\mu,\theta} \in C^1(X_\mu^\alpha, \mathbb{R})$ satisfies the (PS) condition by Lemma 11.

Step 2. It will be shown that there exist constants $\varrho, \beta > 0$ such that $I_{\mu,\theta}$ satisfies condition (i) of Theorem 9. For any $\varepsilon > 0$, by (F1) and (F2), there exists a constant $c_\varepsilon > 0$ such that

$$|F(t,u)| \le \frac{\epsilon}{2}|u|^2 + \sum_{j=1}^{l} \frac{c_j^\varepsilon}{q_j}|u|^{q_j}. \tag{60}$$

Thus, by (23) and (60), for small $\rho > 0$, we get

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta}\left(a + b\Delta_\alpha(u)\right)^\theta + \frac{1}{2}\int_{\mathbb{R}} \mu V(t)|u|^2\,dt \\ &\quad - \int_{\mathbb{R}} F(t,u)\,dt \\ &\ge \frac{a^{\theta-1}}{2}\Delta_\alpha(u) + \frac{1}{2}\int_{\mathbb{R}} \mu V(t)|u|^2\,dt \\ &\quad - \int_{\mathbb{R}} F(t,u)\,dt \\ &\ge \frac{1}{2}\left(\|u\|_{X_\mu^\alpha}^2 - \varepsilon D_2^2\|u\|_{X_\mu^\alpha}^2\right) - \sum_{j=1}^{l}\frac{c_j^\varepsilon}{q_j}D_{q_j}^{q_j}\|u\|_{X_\mu^\alpha}^{q_j} \\ &\ge \frac{1}{8}\left(1 - \varepsilon D_2^2\right)\varrho^2, \end{aligned} \tag{61}$$

for all $u \in \overline{B}_\varrho$, where $B_\varrho = \{u \in X_\mu^\alpha : \|u\|_{X_\mu^\alpha} < \varrho\}$. So it suffices to choose $\varepsilon = 1/2D_2^2$ so that

$$I_{\mu,\theta}\big|_{\partial B_\varrho} \ge \frac{1}{16}\varrho^2 := \beta > 0. \tag{62}$$

Step 3. It remains to prove that there exists an $e \in X_\mu^\alpha$ such that $\|u\|_{X_\mu^\alpha} > \varrho$ and $I_{\mu,\theta}(e) \le 0$, where ρ is defined in Step 2. Let us consider

$$\begin{aligned} I_{\mu,\theta}(\sigma u) &= \frac{1}{2b\theta}\left(a + b\sigma^2\Delta_\alpha(u)\right)^\theta \\ &\quad + \frac{\sigma^2}{2}\int_{\mathbb{R}} \mu V(t)|u|^2\,dt - \int_{\mathbb{R}} F(t,\sigma u)\,dt, \end{aligned} \tag{63}$$

for all $\sigma \in \mathbb{R}$. Take $0 \ne u \in X_\mu^\alpha$. By (F1) and (F4), for any $\kappa > b^{\theta-1}(\Delta_\alpha(u))^\theta/2\theta\int_{\mathbb{R}}|u|^{2\theta}dt$, there is a constant $C_\kappa > 0$ such that

$$F(t,u) \ge \kappa|u|^{2\theta} - C_\kappa|u|^2. \tag{64}$$

So we have

$$\begin{aligned} I_{\mu,\theta}(\sigma u) &\le \frac{1}{2b\theta}\left(a + b\sigma^2\Delta_\alpha(u)\right)^\theta \\ &\quad + \frac{\sigma^2}{2}\int_{\mathbb{R}} \mu V(t)|u|^2\,dt + C_\kappa\sigma^2\int_{\mathbb{R}}|u|^2\,dt \\ &\quad - \kappa\sigma^{2\theta}\int_{\mathbb{R}}|u|^{2\theta}\,dt \longrightarrow -\infty, \end{aligned} \tag{65}$$

as $\sigma \to +\infty$. Thus, there is a point $e \in X_\mu^\alpha \setminus \overline{B}\varrho$ such that $I_{\mu,\theta}(e) \le 0$. By Theorem 9, $I_{\mu,\theta}$ possesses a critical value $c \ge \alpha > 0$ given by

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I_{\mu,\theta}(\gamma(s)), \tag{66}$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0,\ \gamma(1) = e\}. \tag{67}$$

Hence there is $u \in X_\mu^\alpha$ such that $I_{\mu,\theta}(u) = c$ and $I'_{\mu,\theta}(u) = 0$; that is, problem (1) has a nontrivial weak solution in X_μ^α. □

Theorem 17. *Assume that (V1), (V3), (F5), (F6), and (F7) hold. Then problem (1) has infinitely many nontrivial weak solutions for $\mu > 0$.*

Proof. One can obtain the proof by employing the method of proof for Theorem 15 and using Lemma 7. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] S. Ş. Bayın, "Time fractional Schrödinger equation: fox's H-functions and the effective potential," *Journal of Mathematical Physics*, vol. 54, Article ID 012103, 2013.

[2] J. L. Dubbeldam, Z. Tomovski, and T. Sandev, "Space-time fractional Schrödinger equation with composite time fractional derivative," *Fractional Calculus and Applied Analysis*, vol. 18, no. 5, pp. 1179–1200, 2015.

[3] A. Iomin, "Fractional-time Schrödinger equation: fractional dynamics on a comb," *Chaos, Solitons & Fractals*, vol. 44, no. 4-5, pp. 348–352, 2011.

[4] N. Laskin, "Fractional Schrödinger equation," *Physical Review E. Statistical, Nonlinear, and Soft Matter Physics*, vol. 66, Article ID 056108, 2002.

[5] M. Naber, "Time fractional Schrödinger equation," *Journal of Mathematical Physics*, vol. 45, no. 8, pp. 3339–3352, 2004.

[6] N. Nyamoradi and Y. Zhou, "Existence of solutions for a Kirchhoff type fractional differential equations via minimal principle and Morse theory," *Topological Methods in Nonlinear Analysis*, vol. 46, no. 2, pp. 617–630, 2015.

[7] M. Xiang and B. Zhang, "Degenerate Kirchhoff problems involving the fractional p-Laplacian without the (AR) condition," *Complex Variables and Elliptic Equations*, vol. 60, no. 9, pp. 1277–1287, 2015.

[8] M. Xiang, B. Zhang, and M. Ferrara, "Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 424, no. 2, pp. 1021–1041, 2015.

[9] Y. Zhou and L. Peng, "On the time-fractional Navier—Stokes equations," *Computers & Mathematics with Applications*, vol. 73, no. 6, pp. 874–891, 2017.

[10] Y. Zhou and L. Peng, "Weak solutions of the time-fractional Navier-Stokes equations and optimal control," *Computers & Mathematics with Applications*, vol. 73, no. 6, pp. 1016–1027, 2017.

[11] J. Wang, M. Fečkan, and Y. Zhou, "A survey on impulsive fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 19, no. 4, pp. 806–831, 2016.

[12] J. Wang, M. F**ě**ckan, and Y. Zhou, "Center stable manifold for planar fractional damped equations," *Applied Mathematics and Computation*, vol. 296, pp. 257–269, 2017.

[13] Y. Zhou, B. Ahmad, and A. Alsaedi, "Existence of nonoscillatory solutions for fractional neutral differential equations," *Applied Mathematics Letters*, vol. 72, pp. 70–74, 2017.

[14] Y. Zhou, V. Vijayakumar, and R. Murugesu, "Controllability for fractional evolution inclusions without compactness," *Evolution Equations and Control Theory*, vol. 4, no. 4, pp. 507–524, 2015.

[15] G. A. Mendez Cruz and C. E. Torres Ledesma, "Multiplicity of solutions for fractional Hamiltonian systems with Liouville-Weyl fractional derivatives," *Fractional Calculus and Applied Analysis*, vol. 18, no. 4, pp. 875–890, 2015.

[16] C. Torres, "Existence of solutions for fractional Hamiltonian systems," *Electronic Journal of Differential Equations*, vol. 2013, article 259, pp. 1–12, 2013.

[17] Y. Zhou and L. Zhang, "Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems," *Computers & Mathematics with Applications*, vol. 73, no. 6, pp. 1325–1345, 2017.

[18] A. Iomin, "Fractional-time quantum dynamics," *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, vol. 80, no. 2, Article ID 022103, 2009.

[19] N. Laskin, "Fractional quantum mechanics," *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, vol. 62, no. 3, pp. 3135–3145, 2000.

[20] F. Cammaroto and L. Vilasi, "On a Schrödinger-Kirchhoff-type equation involving the $p(x)$-Laplacian," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 81, pp. 42–53, 2013.

[21] Y. Guo and J. Nie, "Existence and multiplicity of nontrivial solutions for p-Laplacian Schrödinger-Kirchhoff-type equations," *Journal of Mathematical Analysis and Applications*, vol. 428, no. 2, pp. 1054–1069, 2015.

[22] J. Nie, "Existence and multiplicity of nontrivial solutions for a class of Schrödinger-Kirchhoff-type equations," *Journal of Mathematical Analysis and Applications*, vol. 417, no. 1, pp. 65–79, 2014.

[23] P. Pucci, M. Xiang, and B. Zhang, "Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$," *Calculus of Variations and Partial Differential Equations*, vol. 54, no. 3, pp. 2785–2806, 2015.

[24] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, the Netherlands, 2006.

[25] C. A. Stuart, *Bifurcation into Spectral Gaps*, Société Mathématique de Belgique, 1995.

[26] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65, American Mathematical Society, Providence, RI, USA, 1986.

[27] F. Jiao and Y. Zhou, "Existence of solutions for a class of fractional boundary value problems via critical point theory," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1181–1199, 2011.

[28] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, Mass, USA, 1996.

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Research Article

Asymptotic Solutions of Time-Space Fractional Coupled Systems by Residual Power Series Method

Wenjin Li[1] and Yanni Pang[2]

[1] *School of Applied Mathematics, Jilin University of Finance and Economics, Changchun, Jilin 130117, China*
[2] *School of Mathematics, Jilin University, Changchun, Jilin 130012, China*

Correspondence should be addressed to Yanni Pang; pangyn@jlu.edu.cn

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This paper focuses on the asymptotic solutions to time-space fractional coupled systems, where the fractional derivative and integral are described in the sense of Caputo derivative and Riemann-Liouville integral. We introduce the Residual Power Series (for short RPS) method to construct the desired asymptotic solutions. Furthermore, we apply this method to some time-space fractional coupled systems. The simplicity and efficiency of RPS method are shown by the application.

1. Introduction

Fractional derivative was mentioned in a letter from L'Hopital to Leibniz in 1695. In the letter, L'Hopital proposed a question "What is the result of $d^n y/dx^n$ if $n = 1/2$?" The answer of Leibniz was "$d^{1/2}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox, from which, one day useful consequences will be drawn" [1, 2]. Furthermore, the generalization of this framework indicates that it is more appropriate to talk of integration and differentiation of, such as fractional order, real number order and even complex number order just as the development of number system. However, there is a basic question: "What is fractional integral and derivative?" Or "How to define the fractional integral and derivative?" More and more mathematicians focused on this problem, such as J. L. Lagrange, P. S. Laplace, and Joseph B. J. Fourier. Some different definitions of fractional integrals and derivatives have been defined according to different needs, like Riemann-Liouville integral, Caputo derivative, Weyl derivative, and so on [2, 3]. But there is no uniform definition of fractional integral and derivative, and the frequently used definition is Riemann-Liouville integral and Caputo derivative.

Fractional differential equations, which involve fractional order derivatives, are applied in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so on. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. An essential topic is to construct the solutions to fractional differential equations. And there are some effective methods to obtain different kinds of solutions, like Sumudu transform and variational iteration method [4], fractional Taylor vector approximate method [5], iterative method [6–8], Residual Power Series (RPS) method [9–13], and so on [14–16]. On the other hand, the study of coupled systems which involve fractional differential equations is also important because fractional coupled systems occur in many fields [17–21]. In this paper, we generalize the RPS method to time-space fractional coupled systems and obtain the asymptotic series solutions.

The organization of this paper is as follows: In Section 2, some concepts and lemmas on fractional calculus are presented. In Section 3, we introduce the algorithm of RPS method for time-space fractional coupled system. In Section 4, asymptotic solutions of some examples are solved via RPS method. In Section 5, some concluding remarks are presented.

2. Preliminaries

In this section, some concepts and main lemmas we need in this paper are presented [2, 3, 22–24]. And more details about fractional calculus can be found in [2, 23, 24].

Definition 1. A real function $f(x)$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $\rho > \mu$ such that $f(x) = x^\rho f_1(x)$, where $f_1(x) \in C[0,\infty)$. And it is said to be in the space C_μ^n if $f^{(n)}(x) \in C_\mu,\ n \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu,\ \mu \geqslant -1$ is defined as

$$I_t^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, & \alpha > 0,\ t > \tau \geqslant 0, \\ f(t), & \alpha = 0, \end{cases} \quad (1)$$

where the symbol I_t^α represents the αth Riemann-Liouville fractional integral of f.

Definition 3. The Caputo fractional derivative of order $\alpha > 0$ of $f \in C_{-1}^n,\ n \in \mathbb{N}$ is defined as

$$D_t^\alpha f(t) := \begin{cases} I_t^{n-\alpha} f^{(n)}(t), & n-1 < \alpha < n,\ t > 0, \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n, \end{cases} \quad (2)$$

where the symbol $D_t^\alpha f(t)$ represents the αth Caputo fractional derivative of f.

Definition 4. The power series

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots \quad (3)$$

is called a fractional power series about $t = t_0$, where t is a variable and c_n $(n = 0,1,2\ldots)$ are the coefficients of the series, $\alpha \in \mathbb{R}^+$.

Remark 5. For convenience, we shall treat $t_0 = 0$. In fact, the transformation $\mathcal{T} : t' = t - t_0$ reduces the fractional power series about $t = t_0$ to the fractional power series about $t = 0$ and meanwhile the transformation $\mathcal{T}$ is reversible.

Definition 6. A function $f(t)$ is analytical at $t = 0$ if $f(t)$ can be written as a form of fractional power series.

Lemma 7. *Suppose that $f(t)$ is an analytic function at $t = 0$; then $f(t)$ can be written as follows:*

$$f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}, \quad 0 < \alpha \leq 1,\ |t| < R. \quad (4)$$

Furthermore, if $f(t) \in C(-R,R)$ and $D_t^{n\alpha} f(t) \in C(-R,R)$ for $n = 0,1,2\ldots$ then the coefficients c_n will take the form

$$c_n = \frac{D_t^{n\alpha} f(t)\big|_{t=0}}{\Gamma(n\alpha+1)}, \quad (5)$$

where $D_t^{n\alpha} = \underbrace{D_t^\alpha \cdot D_t^\alpha \cdots D_t^\alpha}_{n\text{-times}},\ n = 0,1,2\ldots.$

Proof. First of all, notice that if we put $t = 0$ into (5), it yields

$$c_0 = f(0) = \frac{D_t^{0\alpha} f(t)\big|_{t=0}}{\Gamma(0\alpha+1)}. \quad (6)$$

Applying the operator D_t^α one time on (4),

$$c_1 = \frac{D_t^{\alpha} f(t)\big|_{t=0}}{\Gamma(\alpha+1)}. \quad (7)$$

Again, by applying the operator D_t^α two times on (4),

$$c_2 = \frac{D_t^{2\alpha} f(t)\big|_{t=0}}{\Gamma(2\alpha+1)}. \quad (8)$$

Analogously

$$c_n = \frac{D_t^{n\alpha} f(t)\big|_{t=0}}{\Gamma(n\alpha+1)}, \quad n = 0,1,2,\ldots. \quad (9)$$

This completes the proof. □

3. Algorithm of RPS Method for Coupled Systems

In this section, we consider the following system:

$$\begin{aligned} D_t^{p\alpha} u + F(x,t) &= 0, \\ D_t^{q\alpha} v + G(x,t) &= 0, \\ D_t^{r\alpha} w + H(x,t) &= 0 \end{aligned} \quad (10)$$

with initial values

$$\begin{aligned} D_t^{i\alpha} u(x,t)\big|_{t=0} &= a_i(x), \quad i = 0,1,2,\ldots,p-1, \\ D_t^{j\alpha} v(x,t)\big|_{t=0} &= b_j(x), \quad j = 0,1,2,\ldots,q-1, \\ D_t^{k\alpha} w(x,t)\big|_{t=0} &= c_k(x), \quad k = 0,1,2,\ldots,r-1 \end{aligned} \quad (11)$$

as a generalized illustration for the main idea of RPS method, where the symbol $D_t^{(\cdot)\alpha}$ represents the $(\cdot)$th fractional derivative in the Caputo sense, $\max\{(p-1)/p, (q-1)/q, (r-1)/r\} < \alpha \leq 1$ $(p,q,r \in \mathbb{N})$, the functions u, v, w, F, G, H are analytic at $t = 0$, and the initial functions a_i, b_j, and c_k are infinitely many times differentiable for all $i = 0,1,2,\ldots,p$; $j = 0,1,2,\ldots,q$ and $k = 0,1,2,\ldots,r$.

Since u, v, w, F, G, H are analytic at $t = 0$, then they can be expanded in the form of fractional power series at $t = 0$ as follows:

$$\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
F(x,t) &= \sum_{i=0}^{\infty} \frac{f_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}; \\
v(x,t) &= \sum_{j=0}^{\infty} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}, \\
G(x,t) &= \sum_{j=0}^{\infty} \frac{g_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}; \\
w(x,t) &= \sum_{k=0}^{\infty} \frac{w_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha}, \\
H(x,t) &= \sum_{k=0}^{\infty} \frac{h_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha},
\end{aligned} \tag{12}$$

where $x \in \mathbb{R}$, $t \in (-R, R)$, R is the minimum convergence radius of functions $u(x,t), v(x,t), w(x,t), F(x,t), G(x,t), H(x,t)$, and

$$\begin{aligned}
a_i(x) &= D_t^{i\alpha} u(x,t)\big|_{t=0}, \\
f_i(x) &= D_t^{i\alpha} F(x,t)\big|_{t=0}, \\
&\qquad i = 0, 1, 2, \dots; \\
b_j(x) &= D_t^{j\alpha} u(x,t)\big|_{t=0}, \\
g_j(x) &= D_t^{j\alpha} G(x,t)\big|_{t=0}, \\
&\qquad j = 0, 1, 2, \dots; \\
c_k(x) &= D_t^{k\alpha} u(x,t)\big|_{t=0}, \\
h_k(x) &= D_t^{k\alpha} H(x,t)\big|_{t=0}, \\
&\qquad k = 0, 1, 2, \dots.
\end{aligned} \tag{13}$$

According to the initial conditions,

$$\begin{aligned}
u_i(x) &= a_i(x), \quad i = 0, 1, 2, \dots, p-1; \\
v_j(x) &= b_j(x), \quad j = 0, 1, 2, \dots, q-1; \\
w_k(x) &= c_k(x), \quad k = 0, 1, 2, \dots, r-1.
\end{aligned} \tag{14}$$

Thus the initial approximation of the solution u, v, w is as follows:

$$\begin{aligned}
u^{\text{init}}(x,t) &= \sum_{i=0}^{p-1} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
v^{\text{init}}(x,t) &= \sum_{j=0}^{q-1} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}, \\
w^{\text{init}}(x,t) &= \sum_{k=0}^{r-1} \frac{w_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha}.
\end{aligned} \tag{15}$$

Then we calculate the coefficients $u_i(x)$, $v_j(x)$, and $w_k(x)$ for $i = p, p+1, \dots$; $j = q, q+1, \dots$; $k = r, r+1, \dots$. Firstly, some symbols are given as follows:

$$\begin{aligned}
\text{Res}_u(x,t) &= D_t^{p\alpha} u + F(x,t), \\
\text{Res}_{u,l}(x,t) &= D_t^{p\alpha} u_l + F(x,t); \\
\text{Res}_v(x,t) &= D_t^{q\alpha} u + G(x,t), \\
\text{Res}_{v,m}(x,t) &= D_t^{q\alpha} v_m + F(x,t); \\
\text{Res}_w(x,t) &= D_t^{r\alpha} u + H(x,t), \\
\text{Res}_{w,n}(x,t) &= D_t^{r\alpha} u_n + F(x,t),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
u_l(x,t) &= \sum_{i=0}^{l} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} \\
&= u^{\text{init}}(x,t) + \sum_{i=p}^{l} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
&\qquad l = p, p+1, \dots; \\
v_m(x,t) &= \sum_{j=0}^{m} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha} \\
&= v^{\text{init}}(x,t) + \sum_{i=q}^{m} \frac{v_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
&\qquad m = q, q+1, \dots; \\
w_n(x,t) &= \sum_{k=0}^{n} \frac{w_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha} \\
&= w^{\text{init}}(x,t) + \sum_{i=r}^{n} \frac{w_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
&\qquad n = r, r+1, \dots.
\end{aligned} \tag{17}$$

Hence there are two facts:

$(\mathbf{F}_1)$

$$\lim_{l\to\infty} u_l(x,t) = u(x,t),$$
$$\lim_{m\to\infty} v_m(x,t) = u(x,t), \quad (18)$$
$$\lim_{n\to\infty} w_n(x,t) = w(x,t).$$

$(\mathbf{F}_2)$

$$\lim_{l\to\infty} \mathrm{Res}_{u,l}(x,t) = \mathrm{Res}_{u,\infty}(x,t) = \mathrm{Res}_u(x,t) \equiv 0;$$
$$\lim_{m\to\infty} \mathrm{Res}_{v,m}(x,t) = \mathrm{Res}_{v,\infty}(x,t) = \mathrm{Res}_v(x,t) \equiv 0; \quad (19)$$
$$\lim_{n\to\infty} \mathrm{Res}_{w,n}(x,t) = \mathrm{Res}_{w,\infty}(x,t) = \mathrm{Res}_w(x,t) \equiv 0.$$

Furthermore

$$\begin{aligned}
&0 = D_t^{(i-p)\alpha}\mathrm{Res}_{u,\infty}(x,t)\Big|_{t=0} = u_i(x) + D_t^{(i-p)\alpha}F(x,t), \quad i = p, p+1,\ldots \\
&\Longrightarrow u_i(x) = f_{i-p}(x) \triangleq a_i(x), \quad i = p, p+1,\ldots; \\
&0 = D_t^{(j-q)\alpha}\mathrm{Res}_{v,\infty}(x,t)\Big|_{t=0} = v_j(x) + D_t^{(j-q)\alpha}G(x,t), \quad j = q, q+1,\ldots \\
&\Longrightarrow v_j(x) = g_{j-q}(x) \triangleq b_j(x), \quad j = q, q+1,\ldots; \\
&0 = D_t^{(k-r)\alpha}\mathrm{Res}_{w,\infty}(x,t)\Big|_{t=0} = w_k(x) + D_t^{(k-q)\alpha}H(x,t), \quad k = r, r+1,\ldots \\
&\Longrightarrow w_k(x) = h_{k-p}(x) \triangleq c_k(x), \quad k = r, p+1,\ldots.
\end{aligned} \quad (20)$$

Thus the solutions of coupled system (10) are

$$u(x,t) = \sum_{i=0}^{\infty} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{n\alpha},$$
$$v(x,t) = \sum_{j=0}^{\infty} \frac{b_j(x)}{\Gamma(j\alpha+1)} t^{n\alpha}, \quad (21)$$
$$w(x,t) = \sum_{k=0}^{\infty} \frac{c_k(x)}{\Gamma(k\alpha+1)} t^{n\alpha}.$$

Remark 8. If

$$F(x,t) = f(x,t,u,v,w),$$
$$G(x,t) = g(x,t,u,v,w), \quad (22)$$
$$H(x,t) = h(x,t,u,v,w)$$

and even F, G, H include the term of space fractional derivative and the term of time-fractional derivative whose order is less than the order of the system, then RPS method is also effective in calculating the asymptotic solutions for coupled system (10). In fact, F, G, H could be expanded in the form of fractional power series about time variable t at the initial time t_0, and facts $(\mathbf{F}_1)$ and $(\mathbf{F}_2)$ are reasonable as well; thus the coefficients appearing in the asymptotic solutions could be obtained successfully.

4. Application of RPS Method to Time-Space Fractional Coupled Systems

4.1. The Time-Space Fractional Coupled KdV System. KdV equation plays an important role in nonlinear evolution equation for its wide application in physics and engineering. Coupled KdV system was introduced by Hirota and Satsuma [25] to describe the iterations of water waves and they claimed that the system exits a soliton solution. In [26], Fan and Zhang got several kinds of solutions by an improved homogeneous method. In [20], Bhrawy et al. reduced the time-fractional coupled KdV equations into a problem consisting of a system of algebraic equations that greatly simplifies the problem via the shifted Legendre polynomials. The time-fractional coupled KdV equation is a generalization of the classical coupled KdV equation and in this subsection we generalize time-fractional coupled KdV system to time-space fractional coupled system (23) and obtain the asymptotic solution using RPS method.

Consider the time-space fractional coupled KdV system:

$$D_t^{\alpha}u - aD_x^{3\beta}u - 6auD_x^{\gamma}u - 2bvD_x^{\delta}v = 0,$$
$$D_t^{\alpha}v + D_x^{3\lambda}v + 3uD_x^{\tau}v = 0 \quad (23)$$

with initial values

$$u(x,0) = a_0(x),$$
$$v(x,0) = b_0(x), \quad (24)$$

where $0 < \alpha, \gamma, \delta, \tau \le 1$, $2/3 < \beta, \lambda \le 1$, $u = u(x,t)$, $v = v(x,t)$, $(x,t) \in \mathbb{R}\times\mathbb{R}$.

If $u(x,t)$ and $v(x,t)$ are analytic at $t = 0$, then they can be expanded in the form of fractional power series

$$
\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
v(x,t) &= \sum_{j=0}^{\infty} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}, \\
F(x,t) &= \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \left\{ -aD_x^{3\beta} u_n(x) - \sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} \left(6au_s(x) D_x^{\gamma} u_{n-s}(x) + 2bv_s(x) D_x^{\delta} v_{n-s}(x)\right) \right\}, \\
G(x,t) &= \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \left\{ D_x^{3\lambda} v_n + \sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} \cdot 3u_s(x) D_x^{\tau} v_{i-1-s}(x) \right\}.
\end{aligned}
\tag{25}
$$

Under the initial conditions,

$$
\begin{aligned}
u_0(x) &= a_0(x), \\
v_0(x) &= b_0(x);
\end{aligned}
\tag{26}
$$

that is, the initial approximate solutions are

$$
\begin{aligned}
u^{\text{initial}}(x,t) &= a_0(x), \\
v^{\text{initial}}(x,t) &= b_0(x).
\end{aligned}
\tag{27}
$$

Set

$$
\begin{aligned}
\operatorname{Res}_u(x,t) &= D_t^{\alpha} u - aD_x^{3\beta} u - 6auD_x^{\gamma} u - 2bvD_x^{\delta} v, \\
\operatorname{Res}_v(x,t) &= D_t^{\alpha} v + D_x^{3\lambda} v + 3uD_x^{\tau} v, \\
\operatorname{Res}_{u,l}(x,t) &= D_t^{\alpha} u_l - aD_x^{3\beta} u - 6au_l D_x^{\gamma} u_l \\
&\quad - 2bv_m D_x^{\delta} v_m, \\
\operatorname{Res}_{v,m}(x,t) &= D_t^{\alpha} v_m + D_x^{3\lambda} v_m + 3u_l D_x^{\tau} v_m,
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
u_l(x,t) &= a_0(x) + \sum_{i=1}^{l} \frac{u_i(x)}{\Gamma(i\alpha)+1} t^{i\alpha}, \\
v_m(x,t) &= b_0(x) + \sum_{j=1}^{m} \frac{v_j(x)}{\Gamma(j\alpha)+1} t^{j\alpha}.
\end{aligned}
\tag{29}
$$

Using RPS method

$$
\begin{aligned}
&u_i(x) \\
&= aD_x^{3\beta} u_{i-1}(x) + \sum_{s=0}^{i-1} \frac{\Gamma((i-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((i-1-s)\alpha+1)} \\
&\quad \cdot \left\{ 6au_s(x) D_x^{\gamma} u_{i-1-s}(x) + 2bv_s(x) D_x^{\delta} v_{i-1-s}(x) \right\} \\
&\triangleq a_i(x), \quad i = 1, 2, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
&v_j(x) \\
&= -D_x^{3\lambda} v_{j-1}(x) \\
&\quad - \sum_{s=0}^{j-1} \frac{\Gamma((j-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((j-1-s)\alpha+1)} \\
&\quad \cdot 3u_s(x) D_x^{\tau} v_{j-1-s}(x) \triangleq b_j(x), \quad j = 1, 2, \ldots.
\end{aligned}
\tag{30}
$$

Thus the fractional power series solutions of coupled system (23) are

$$
\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
v(x,t) &= \sum_{j=0}^{\infty} \frac{b_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}.
\end{aligned}
\tag{31}
$$

4.2. The Time-Space Fractional Coupled KdV System of Generalized Hirota-Satsuma Type. In this subsection, we consider the time-space fractional coupled KdV system of generalized Hirota-Satsuma type

$$
\begin{aligned}
D_t^{\alpha} u - \frac{1}{2} D_x^{3\beta} u + 3uD_x^{\gamma} u - 3D_x^{\delta}(vw) &= 0, \\
D_t^{\alpha} v + D_x^{3\lambda} v - 3uD_x^{\tau} v &= 0, \\
D_t^{\alpha} w + D_x^{3\theta} w - 3uD_x^{\sigma} w &= 0, \\
u(x,0) &= a_0(x), \\
v(x,0) &= b_0(x), \\
w(x,0) &= c_0(x),
\end{aligned}
\tag{32}
$$

where $0 < \alpha, \gamma, \delta, \lambda, \tau \leq 1$, $2/3 < \beta, \sigma, \theta \leq 1$, $u = u(x,t)$, $v = v(x,t)$, $w = w(x,t)$, $(x,t) \in \mathbb{R}\times\mathbb{R}$. The equation describes an interaction of two long waves with different dispersion relations.

If u, v, w are analytic at $t = 0$, then u, v, w can be written as the form of fractional power series

$$
\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \\
v(x,t) &= \sum_{j=0}^{\infty} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}, \\
w(x,t) &= \sum_{k=0}^{\infty} \frac{w_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha}, \\
F(x,t) &= -\frac{1}{2} D_x^{3\beta} u + 3u D_x^{\gamma} u - 3D_x^{\delta}(vw) \\
&= \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \cdot \left\{ -\frac{1}{2} D_x^{3\beta} u_n(x) \right. \\
&+ 3\sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} \left(u_s(x) \right. \\
&\left. \left. \cdot D_x^{\gamma} u_{n-s}(x) - v_n(x) w_{n-s}(x)\right) \right\}, \qquad (33) \\
G(x,t) &= D_x^{3\lambda} v - 3u D_x^{\tau} v = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\
&\cdot \left\{ D_x^{3\lambda} v_n(x) - 3\sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} \right. \\
&\left. \cdot u_s(x) D_x^{\tau} v_{n-s}(x) \right\}, \\
H(x,t) &= D_x^{3\theta} w - 3u D_x^{\sigma} w = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\
&\cdot \left\{ D_x^{3\theta} w_n(x) - 3\sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} \right. \\
&\left. \cdot u_s(x) D_x^{\sigma} w_{n-s}(x) \right\}.
\end{aligned}
$$

With the initial values

$$
\begin{aligned}
u_0(x) &= a_0(x), \\
v_0(x) &= b_0(x), \\
w_0(x) &= c_0(x),
\end{aligned} \qquad (34)
$$

the initial approximate solutions are

$$
\begin{aligned}
u^{\text{initial}}(x,t) &= a_0(x), \\
v^{\text{initial}}(x,t) &= b_0(x), \\
w^{\text{initial}}(x,t) &= c_0(x).
\end{aligned} \qquad (35)
$$

Set

$$
\begin{aligned}
\operatorname{Res}_u(x,t) &= D_t^{\alpha} u - \frac{1}{2} D_x^{3\beta} u + 3u D_x^{\gamma} u - 3D_x^{\delta}(vw) \\
&= 0, \\
\operatorname{Res}_v(x,t) &= D_t^{\alpha} v + D_x^{3\lambda} v - 3u D_x^{\tau} v = 0, \\
\operatorname{Res}_w(x,t) &= D_t^{\alpha} w + D_x^{3\theta} w - 3u D_x^{\sigma} w = 0, \\
\operatorname{Res}_{u,l}(x,t) &= D_t^{\alpha} u_l - \frac{1}{2} D_x^{3\beta} u_l + 3u_l D_x^{\gamma} u_l \\
&\quad - 3D_x^{\delta}(v_m w_n), \\
\operatorname{Res}_{v,m}(x,t) &= D_t^{\alpha} v_m + D_x^{3\lambda} v - 3u_l D_x^{\tau} v_m, \\
\operatorname{Res}_{w,n}(x,t) &= D_t^{\alpha} w_n + D_x^{3\theta} w_n - 3u_l D_x^{\sigma} w_n,
\end{aligned} \qquad (36)
$$

where

$$
\begin{aligned}
u_l(x,t) &= a_0(x) + \sum_{i=1}^{l} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{\alpha}, \\
v_m(x,t) &= b_0(x) + \sum_{j=1}^{m} \frac{v_i(x)}{\Gamma(j\alpha+1)}, \\
w_n(x,t) &= c_0(x) + \sum_{k=1}^{n} \frac{w_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha},
\end{aligned} \qquad (37)
$$

with the results of RPS method:

$$
\begin{aligned}
u_i(x) &= \frac{1}{2} D_x^{3\beta} u_{i-1}(x) \\
&- 3\sum_{s=0}^{i-1} \frac{\Gamma((i-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((i-1-s)\alpha+1)} \\
&\cdot \left\{ u_s(x) D_x^{\gamma} u_{i-1-s}(x) - D_x^{\delta}\left(v_s(x) w_{i-1-s}(x)\right) \right\} \\
&\triangleq a_i(x), \quad i = 1, 2, \ldots, \\
v_j(x) &= -D_x^{3\sigma} v_{j-1}(x) \\
&+ 3\sum_{s=0}^{j-1} \frac{\Gamma((j-1)\alpha+1)}{\Gamma(j\alpha+1)\Gamma((j-1-s)\alpha+1)} u_s(x) \\
&\cdot D_x^{\lambda} v_{j-1-s}(x) \triangleq b_j(x), \quad j = 1, 2, \ldots, \\
w_k(x) &= -D_x^{3\theta} w_{k-1}(x) \\
&+ 3\sum_{s=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)}{\Gamma(j\alpha+1)\Gamma((k-1-s)\alpha+1)} u_s(x) \\
&\cdot D_x^{\tau} w_{k-1-j}(x) \triangleq c_k(x), \quad k = 1, 2, \ldots.
\end{aligned} \qquad (38)
$$

So the fractional power series solutions of coupled system (32) are

$$u(x,t) = \sum_{i=0}^{\infty} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha},$$
$$v(x,t) = \sum_{j=0}^{\infty} \frac{b_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}, \quad (39)$$
$$w(x,t) = \sum_{k=0}^{\infty} \frac{c_k(x)}{\Gamma(k\alpha+1)} t^{k\alpha}.$$

4.3. The Time-Space Fractional Coupled Whitham-Broer-Kaup (WBK) System. Whitham [27], Broer [28], and Kaup [29] obtained nonlinear WBK system. In [30], Rashidi et al. obtained the approximate traveling wave solutions of the coupled WBK system in shallow water using homotopy analysis method. In [31], Kadem and Baleanu applied the homotopy perturbation method to find an analytical approximate solution for the coupled WBK system. In this subsection, we consider the time-space fractional coupled WBK system and construct the approximate solution by RPS method.

Consider the time-space fractional coupled WBK system:

$$\begin{gathered} D_t^{\alpha} u + u D_x^{\beta} u + D_x^{\gamma} v + a D_x^{2\delta} u = 0, \\ D_t^{\alpha} v + D_x^{\lambda}(uv) - a D_x^{2\tau} v + b D_x^{3\theta} u = 0, \\ u(x,0) = a_0(x), \\ v(x,0) = b_0(x), \end{gathered} \quad (40)$$

where $0 < \alpha, \sigma, \tau, \lambda \le 1$, $1/2 < \delta, \eta \le 1$, $2/3 < \theta \le 1$, $(x,t) \in \mathbb{R} \times \mathbb{R}$, $a, b \in \mathbb{R}$ represent different dispersive power, $u = u(x,t)$ is the field of horizontal velocity, and $v = v(x,t)$ is the height deviating equilibrium position of liquid. And this is a very good model to describe dispersive wave.

If u, v are analytic at $t = 0$, then u, v can be expanded in the form of fractional power series

$$u(x,t) = \sum_{i=0}^{\infty} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha},$$
$$v(x,t) = \sum_{j=0}^{\infty} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha},$$
$$F(x,t) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \cdot \left\{ \sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} u_s(x) D_x^{\beta} u_{n-s}(x) + D_x^{\gamma} v_n(x) + a D_x^{2\delta} u_n(x) \right\},$$
$$G(x,t) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \left\{ \sum_{s=0}^{n} \frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)} \cdot D_x^{\lambda}(u_s(x) v_{n-s}(x)) + a D_x^{2\tau} v_n(x) + b D_x^{3\theta} u_n(x) \right\}. \quad (41)$$

Under the initial conditions

$$\begin{aligned} u_0(x) &= a_0(x), \\ v_0(x) &= b_0(x), \end{aligned} \quad (42)$$

and the initial approximate solutions are

$$\begin{aligned} u^{\text{initial}}(x,t) &= a_0(x), \\ v^{\text{initial}}(x,t) &= b_0(x,t). \end{aligned} \quad (43)$$

Set

$$\begin{aligned} \mathrm{Res}_u(x,t) &= D_t^{\alpha} u + u D_x^{\beta} u + D_x^{\gamma} v + a D_x^{2\delta} u, \\ \mathrm{Res}_v(x,t) &= D_t^{\alpha} v + D_x^{\lambda}(uv) - a D_x^{2\tau} v + b D_x^{3\theta} u, \\ \mathrm{Res}_{u,l}(x,t) &= D_t^{\alpha} u_l + u_l D_x^{\beta} u_l + D_x^{\gamma} v_m + a D_x^{2\delta} u_l, \\ \mathrm{Res}_{v,m}(x,t) &= D_t^{\alpha} v_m + D_x^{\lambda}(u_l v_m) - a D_x^{2\tau} v_m + b D_x^{3\theta} u_l, \end{aligned} \quad (44)$$

where

$$\begin{aligned} u_l(x,t) &= a_0(x) + \sum_{i=1}^{l} \frac{u_i(x)}{\Gamma(i\alpha+1)} t^{\alpha}, \\ v_m(x,t) &= b_0(x) + \sum_{j=1} \frac{v_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}. \end{aligned} \quad (45)$$

With RPS method

$$\begin{aligned} u_i(x) &= -\sum_{s=0}^{i-1} \frac{\Gamma((i-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((i-1-s)\alpha+1)} u_s(x) \cdot D_x^{\beta} u_{i-1-s}(x) + D_x^{\gamma} v_{i-1}(x) + a D_x^{2\delta} u_{i-1}(x) \\ &\triangleq a_i(x), \quad i = 1, 2, \ldots, \\ v_j(x) &= -\sum_{s=0}^{j-1} \frac{\Gamma((j-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((j-1-s)\alpha+1)} \cdot D_x^{\lambda}(u_s(x) v_{j-1-s}(x)) + a D_x^{2\tau} v_{j-1}(x) \\ &\quad - b D_x^{3\theta} u_{j-1}(x) \triangleq b_j(x), \quad j = 1, 2, \ldots. \end{aligned} \quad (46)$$

So the fractional power series solution of coupled system (40) is

$$u(x,t)=\sum_{i=0}^{\infty}\frac{a_i(x)}{\Gamma(i\alpha+1)}t^{i\alpha},$$
$$v(x,t)=\sum_{j=0}^{\infty}\frac{b_j(x)}{\Gamma(j\alpha+1)}t^{j\alpha}. \tag{47}$$

Remark 9. When $\alpha=\sigma=\tau=\delta=\lambda=\eta=\theta=1$, $\beta\neq 0$, $\gamma=0$, (40) reduces to the classical long-wave equations that describe the shallow water wave with diffusion.

Remark 10. When $\alpha=\sigma=\tau=\delta=\lambda=\eta=\theta=1$, $\beta=0$, $\gamma=1$, (40) reduces to the variant Boussinesq equation.

4.4. The Time-Space Fractional Coupled Shallow Water System. Shallow water systems are widely used in predicting hydrodynamics of surface flows such as water flows in rivers, channels, flood plains, and coastal regions. It is well known that the shallow water systems can accurately predict the hydraulic parameters under conditions of slow erosion and low sediment concentration of the time-space fractional coupled shallow water system [32]. In this subsection, consider the time-space fractional coupled shallow water system

$$D_t^{\alpha}u+uD_x^{\beta}u+D_x^{\gamma}v+aD_x^{2\delta}u=0,$$
$$D_t^{\alpha}v+vD_x^{\lambda}u+uD_x^{\tau}v-aD_x^{2\theta}v+bD_x^{3\sigma}u=0, \tag{48}$$

with initial values

$$u(x,0)=a_0(x),$$
$$v(x,0)=b_0(x), \tag{49}$$

where $0<\alpha,\beta,\gamma,\lambda,\tau\leq 1,\ 1/2<\delta,\theta\leq 1, 2/3<\sigma\leq 1,\ u=u(x,t),\ v=v(x,t),\ (x,t)\in\mathbb{R}\times\mathbb{R}$.

If u,v are analytic at $t=0$, then u,v can be written as the form of fractional power series:

$$u(x,t)=\sum_{i=0}^{\infty}\frac{u_i(x)}{\Gamma(i\alpha+1)}t^{i\alpha},$$

$$v(x,t)=\sum_{j=0}^{\infty}\frac{v_j(x)}{\Gamma(j\alpha+1)}t^{j\alpha},$$

$$F(x,t)=\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$
$$\cdot\left\{\sum_{s=0}^{n}\frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)}u_s(x)D_x^{\beta}u_{n-s}(x)\right.$$
$$\left.+D_x^{\gamma}v_n(x)+aD_x^{2\delta}u_n(x)\right\},$$

$$G(x,t)=\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$
$$\cdot\left\{\sum_{s=0}^{n}\frac{\Gamma(n\alpha+1)}{\Gamma(s\alpha+1)\Gamma((n-s)\alpha+1)}\right.$$
$$\cdot\left(v_s(x)D_x^{\lambda}u_{n-s}(x)+u_s(x)D_x^{\tau}v_{n-s}(x)\right)$$
$$\left.-aD_x^{2\theta}v_n(x)+bD_x^{3\sigma}u_n(x)\right\}. \tag{50}$$

With the initial conditions

$$u_0(x)=a_0(x),$$
$$v_0(x)=b_0(x), \tag{51}$$

and the initial approximate solutions are

$$u^{\text{initial}}(x,t)=a_0(x),$$
$$v^{\text{initial}}(x,t)=b_0(x). \tag{52}$$

Set

$$\mathrm{Res}_u(x,t)=D_t^{\alpha}u+uD_x^{\beta}u+D_x^{\gamma}v+aD_x^{2\delta}u,$$
$$\mathrm{Res}_v(x,t)\,D_t^{\alpha}v+vD_x^{\lambda}u+uD_x^{\tau}v-aD_x^{2\theta}v+bD_x^{3\sigma}u,$$
$$\mathrm{Res}_{u,l}(x,t)=D_t^{\alpha}u_l+u_lD_x^{\beta}u_l+D_x^{\gamma}v_m+aD_x^{2\delta}u_l, \tag{53}$$
$$\mathrm{Res}_{v,m}(x,t)=D_t^{\alpha}v_m+v_mD_x^{\lambda}u_l+u_lD_x^{\tau}v_m-aD_x^{2\theta}v_m$$
$$+bD_x^{3\sigma}u_l,$$

where

$$u_l(x,t)=a_0(x)+\sum_{i=1}^{l}\frac{u_i(x)}{\Gamma(i\alpha+1)}t^{i\alpha},$$
$$v_m(x,t)=b_0(x)+\sum_{j=1}^{m}\frac{v_j(x)}{\Gamma(j\alpha+1)}t^{j\alpha}. \tag{54}$$

Using the RPS method

$$u_i(x)=-\sum_{s=0}^{i-1}\frac{\Gamma((i-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((i-1-s)\alpha+1)}u_s(x)$$
$$\cdot D_x^{\beta}u_{i-1-s}(x)-D_x^{\gamma}v_{i-1}(x)+aD_x^{2\delta}u_{i-1}(x)$$
$$\triangleq a_i(x),\quad i=1,2,\ldots,$$
$$v_j(x)=-\sum_{s=0}^{i-1}\frac{\Gamma((i-1)\alpha+1)}{\Gamma(s\alpha+1)\Gamma((i-1-s)\alpha+1)}\left(v_s(x)\right.$$
$$\left.\cdot D_x^{\lambda}u_{j-1-s}+u_s(x)D_x^{\tau}v_{j-1-s}(x)\right)+aD_x^{2\theta}v_{j-1}(x)$$
$$-bD_x^{3\sigma}u_{i-1}(x)\triangleq b_j(x),\quad j=1,2,\ldots. \tag{55}$$

So the solutions of coupled system (48) are

$$u(x,t)=\sum_{n=0}^{\infty}\frac{a_n(x)}{\Gamma(n\alpha+1)}t^{n\alpha},$$
$$v(x,t)=\sum_{n=0}^{\infty}\frac{b_n(x)}{\Gamma(n\alpha+1)}t^{n\alpha}. \quad (56)$$

5. Concluding Remarks

This paper introduced a new analytical iterative technique to construct asymptotic solutions to time-space fractional coupled systems, which is based on the general Residual Power Series method. Furthermore, we apply this method to some specific time-space fractional coupled systems to obtain asymptotic solutions with respect to initial values, which shows that this method is efficient and does not require linearization or perturbation.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] G. Leibniz, “Letter from hanover, Germany, deptember 30, 1695 to ga lhospital,” *JLeibnizen Mathematische Schriften*, vol. 2, no. 301-302, 1849.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.

[3] K. Sakamoto and M. Yamamoto, “Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems,” *Journal of Mathematical Analysis and Applications*, vol. 382, no. 1, pp. 426–447, 2011.

[4] P. Goswami and R. T. Alqahtani, “Solutions of fractional differential equations by Sumudu transform and variational iteration method,” *Journal of Nonlinear Science and its Applications*, vol. 9, no. 4, pp. 1944–1951, 2016.

[5] V. K. Saraswathy, *The numerical solutions of fractional differential equations with fractional Taylor vector [Ph.D. thesis]*, ProQuest LLC, Mississippi State University, Ann Arbor, MI, USA, 2016.

[6] V. Daftardar-Gejji and H. Jafari, “An iterative method for solving nonlinear functional equations,” *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 753–763, 2006.

[7] C. D. Dhaigude and V. R. Nikam, “Solution of fractional partial differential equations using iterative method,” *Fractional Calculus and Applied Analysis*, vol. 15, no. 4, pp. 684–699, 2012.

[8] F. Xu, Y. Gao, and W. Zhang, “Construction of analytic solution for time-fractional Boussinesq equation using iterative method,” *Advances in Mathematical Physics*, vol. 2015, Article ID 506140, 7 pages, 2015.

[9] O. Abu Arqub, Z. Abo-Hammour, R. Al-Badarneh, and S. Momani, “A reliable analytical method for solving higher-order initial value problems,” *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 673829, 12 pages, 2013.

[10] O. A. Arqub, A. El-Ajou, Z. A. Zhour, and S. Momani, “Multiple solutions of nonlinear boundary value problems of fractional order: a new analytic iterative technique,” *Entropy*, vol. 16, no. 1, pp. 471–493, 2014.

[11] A. El-Ajou, O. Abu Arqub, and S. Momani, “Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: a new iterative algorithm,” *Journal of Computational Physics*, vol. 293, pp. 81–95, 2015.

[12] F. Xu, Y. Gao, X. Yang, and H. Zhang, “Construction of fractional power series solutions to fractional Boussinesq equations using residual power series method,” *Mathematical Problems in Engineering*, vol. 2016, Article ID 5492535, 15 pages, 2016.

[13] W. Yin, F. Xu, W. Zhang, and Y. Gao, “Asymptotic expansion of the solutions to time-space fractional Kuramoto-Sivashinsky equations,” *Advances in Mathematical Physics*, vol. 2016, 9 pages, 2016.

[14] S. S. Ezz-Eldien, “New quadrature approach based on operational matrix for solving a class of fractional variational problems,” *Journal of Computational Physics*, vol. 317, pp. 362–381, 2016.

[15] A. H. Bhrawy and S. S. Ezz-Eldien, “A new Legendre operational technique for delay fractional optimal control problems,” *Calcolo*, vol. 53, no. 4, pp. 521–543, 2016.

[16] S. S. Ezz-Eldien, A. H. Bhrawy, and A. A. El-Kalaawy, “Direct numerical method for isoperimetric fractional variational problems based on operational matrix,” *Journal of Vibration and Control*, article 107754631770034, 2017.

[17] C.-z. Bai and J.-x. Fang, “The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations,” *Applied Mathematics and Computation*, vol. 150, no. 3, pp. 611–621, 2004.

[18] Y. Chen and H.-L. An, “Numerical solutions of coupled Burgers equations with time- and space-fractional derivatives,” *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 87–95, 2008.

[19] A. Yldrm and A. Kelleci, “Homotopy perturbation method for numerical solutions of coupled Burgers equations with time- and space-fractional derivatives,” *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 20, no. 8, pp. 897–909, 2010.

[20] A. H. Bhrawy, E. H. Doha, S. S. Ezz-Eldien, and M. A. Abdelkawy, “A numerical technique based on the shifted Legendre polynomials for solving the time-fractional coupled KdV equations,” *Calcolo*, vol. 53, no. 1, pp. 1–17, 2016.

[21] S. S. Ray, “Soliton solutions for time fractional coupled modified KdV equations using new coupled fractional reduced differential transform method,” *Journal of Mathematical Chemistry*, vol. 51, no. 8, pp. 2214–2229, 2013.

[22] A. El-Ajou, O. Abu Arqub, Z. Al Zhour, and S. Momani, “New results on fractional power series: theories and applications,” *Entropy*, vol. 15, no. 12, pp. 5305–5323, 2013.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, Netherlands, 2006.

[24] B. Ross, “A brief history and exposition of the fundamental theory of fractional calculus,” in *Fractional Calculus and Its Applications*, vol. 457 of *Lecture Notes in Mathematics*, pp. 1–36, Springer Berlin Heidelberg, Berlin, Germany, 1975.

[25] R. Hirota and J. Satsuma, “Soliton solutions of a coupled Korteweg-de Vries equation,” *Physics Letters. A*, vol. 85, no. 8-9, pp. 407-408, 1981.

[26] E. Fan and H. Zhang, “New exact solutions to a solutions to a system of coupled KdV equations,” *Physics Letters. A*, vol. 245, no. 5, pp. 389–392, 1998.

[27] G. B. Whitham, “Variational methods and applications to water waves,” in *Hyperbolic Equations and Waves (Rencontres, Battelle RES. Inst., Seattle, Wash., 1968)*, pp. 153–172, Springer, Berlin, Germany, 1970.

[28] L. J. Broer, “Approximate equations for long water waves,” *Applied Scientific Research*, vol. 31, no. 5, pp. 377–395, 1975.

[29] D. J. Kaup, “A higher-order water-wave equation and the method for solving it,” *Progress of Theoretical Physics*, vol. 54, no. 2, pp. 396–408, 1975.

[30] M. M. Rashidi, D. D. Ganji, and S. Dinarvand, “Approximate traveling wave solutions of coupled Whitham-Broer-Kaup shallow water equations by homotopy analysis method,” *Differential Equations and Nonlinear Mechanics*, vol. 2008, Article ID 243459, 8 pages, 2008.

[31] A. Kadem and D. Baleanu, “On fractional coupled Whitham-Broer-Kaup equations,” *Romanian Journal of Physics*, vol. 56, no. 5-6, pp. 629–635, 2011.

[32] X. Liu, A. Mohammadian, A. Kurganov, and J. A. Infante Sedano, “Well-balanced central-upwind scheme for a fully coupled shallow water system modeling flows over erodible bed,” *Journal of Computational Physics*, vol. 300, pp. 202–218, 2015.

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Research Article

Arbitrary Order Fractional Difference Operators with Discrete Exponential Kernels and Applications

Thabet Abdeljawad,[1] Qasem M. Al-Mdallal,[2] and Mohamed A. Hajji[2]

[1]*Department of Mathematics and Physical Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*
[2]*Department of Mathematical Sciences, United Arab Emirates University, P.O. Box 17551, Al Ain, Abu Dhabi, UAE*

Correspondence should be addressed to Thabet Abdeljawad; tabdeljawad@psu.edu.sa

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Recently, Abdeljawad and Baleanu have formulated and studied the discrete versions of the fractional operators of order $0 < \alpha \leq 1$ with exponential kernels initiated by Caputo-Fabrizio. In this paper, we extend the order of such fractional difference operators to arbitrary positive order. The extension is given to both left and right fractional differences and sums. Then, existence and uniqueness theorems for the Caputo (CFC) and Riemann (CFR) type initial difference value problems by using Banach contraction theorem are proved. Finally, a Lyapunov type inequality for the Riemann type fractional difference boundary value problems of order $2 < \alpha \leq 3$ is proved and the ordinary difference Lyapunov inequality then follows as α tends to 2 from right. Illustrative examples are discussed and an application about Sturm-Liouville eigenvalue problem in the sense of this new fractional difference calculus is given.

1. Introduction

In the last few decades, the continuous and discrete fractional differential equations have received considerable interest due to their importance in many scientific fields; see, by way of example not exhaustive enumeration, [1–7].

In [8], the authors introduced a fractional derivative with an exponential kernel which tends to the ordinary derivative as α tends to 1. More properties of this fractional derivative have been studied in [9], where the correspondent fractional integral operator was formulated. Then, the authors in [7] defined the left and right fractional derivatives with exponential kernel in the Riemann sense and formulated the right fractional derivatives in the sense of Caputo-Fabrizio with complete investigation to the correspondent fractional integrals and all the discrete versions with integration and summation by parts applied in the fractional and discrete fractional variational calculus. Then, very recently, the same authors proved an interesting monotonicity result in the sense of this new fractional difference calculus in [10].

In the same direction, for the purpose of providing more fractional derivatives with different nonsingular kernels, the authors in [11] defined a fractional operator with Mittag-Leffler kernel and in [12, 13] the complete details and discrete versions have been studied. The exponential kernel fractional derivatives and hence their discrete counterparts are quite different from the Mittag-Leffler kernel fractional operators. For example, the integral operator corresponding to exponential kernel fractional derivatives consists of a multiple of the function f added to a multiple of the integration of f, whereas the Mittag-Leffler kernel correspondent integral operator consists of a multiple of f and a Riemann-Liouville fractional integral of the same order. Also, the monotonicity coefficient of the CFR fractional difference operator of order $0 < \alpha \leq 1$ is α as shown in [10], whereas for the discrete Mittag-Leffler CFR operator is α^2 as proven in [14].

Motivated, by what we mentioned above, we extend the order of fractional difference type operators with discrete exponential kernels to arbitrary positive order, prove existence and uniqueness theorems for the fractional initial value difference problems, and finally prove a Lyapunov type inequality for the CFR fractional difference operators of order $2 < \alpha \leq 3$. The ordinary discrete Lyapunov inequality is then confirmed as α tends to 2 from the right not as in the case of the classical fractional difference as α tends to 2 from the left [15]. For various fractional Lyapunov extensions we refer, for example, to [16–29]. All the authors there were motivated by the following theorem on ordinary Lyapunov inequality.

Theorem 1 (see [30]). *If the boundary value problem*

$$y''(t) + q(t)\,y(t) = 0, \quad t \in (a,b), \; y(a) = y(b) = 0, \quad (1)$$

has a nontrivial solution, where q is a real continuous function; then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \quad (2)$$

Notice that inequality (2) is known as the classical Lyapunov inequality. It is worth mentioning that Cheng [31] had pointed out that Lyapunov neither stated nor proved Theorem 1 but he only stated the following result.

Theorem 2 (see [26]). *Let $q(t)$ be a nontrivial, continuous, and nonnegative function with period ω and let*

$$\int_0^\omega q(s)\,ds \le \frac{4}{\omega}. \quad (3)$$

Then the roots of the characteristic equation corresponding to Hills equation

$$x''(t) + q(t)\,x(t) = 0, \quad -\infty < t < \infty, \quad (4)$$

are purely imaginary with modulus one.

For the classical fractional calculus which is behind many extensions, we refer the reader to [32–35] and for the sake of comparison with the classical discrete fractional case we refer to [36] and the references therein. In addition, for the discrete fractional operators and their duality we refer to [37–39].

The article will be organised as follows: In the remaining part of this section we shall give some basics about the discrete CFC and CFR fractional differences and their correspondent sums as used in [7, 10]. In Section 2, we extend the order of CFC and CFR fractional differences and their correspondent sums to arbitrary positive order and give some illustrative examples. In Section 3, we prove some existence and uniqueness theorems by means of Banach fixed point theorem and give some illustrative examples. In Section 4, we prove a Lyapunov type inequality for a fractional CFR difference boundary value problem of order $2 < \alpha \le 3$ and give an application to the fractional difference Sturm-Liouville Eigenvalue problem (SLEP) to enrich the applicability of our proven Lyapunov inequality in the frame of fractional difference operators with discrete exponential kernels.

2. Preliminaries

Definition 3 (see [36]). For $\alpha > 0$, $a \in \mathbb{R}$, $\rho(s) = s-1$, and f a real-valued function defined on $\mathbb{N}_a = \{a, a+1, \ldots\}$, the left Riemann-Liouville fractional sum of order $\alpha > 0$ is defined by

$$\left({}_a\nabla^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t-\rho(s)\right)^{\overline{\alpha-1}} f(s). \quad (5)$$

This is fractionalising of the n-iterated nabla sum

$$\left({}_a\nabla^{-n} f\right)(t) = \frac{1}{(n-1)!} \sum_{a+1}^{t} \left(t-\rho(s)\right)^{\overline{n-1}} f(s). \quad (6)$$

The right fractional integral ending at b, where usually we assume that $a \equiv b \pmod 1$, is defined by

$$\left(\nabla_b^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \left(s-\rho(t)\right)^{\overline{\alpha-1}} f(s), \quad (7)$$

where $t^{\overline{\alpha}} = \Gamma(t+\alpha)/\Gamma(t)$ and $\Gamma(t)$ is the well-known gamma special function of t.

Definition 4 (see [7, 10]). For $\alpha \in (0,1)$ and f defined on $\mathbb{N}_a$, or ${}_b\mathbb{N} = \{b, b-1, \ldots\}$ in right case, we have the following definitions:

(i) The left (nabla) CFC fractional difference is given by

$$\begin{aligned}\left({}^{\mathrm{CFC}}_{\;\;a}\nabla^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{t} \left(\nabla_s f\right)(s)\,(1-\alpha)^{t-\rho(s)} \\ &= B(\alpha) \sum_{s=a+1}^{t} \left(\nabla_s f\right)(s)\,(1-\alpha)^{t-s}.\end{aligned} \quad (8)$$

(ii) The right (nabla) CFC fractional difference has the following form:

$$\begin{aligned}\left({}^{\mathrm{CFC}}\nabla_b^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=t}^{b-1} \left(-\Delta_s f\right)(s)\,(1-\alpha)^{s-\rho(t)} \\ &= B(\alpha) \sum_{s=t}^{b-1} \left(-\Delta_s f\right)(s)\,(1-\alpha)^{s-t}.\end{aligned} \quad (9)$$

(iii) The left (nabla) CFR fractional difference is written as

$$\begin{aligned}\left({}^{\mathrm{CFR}}_{\;\;a}\nabla^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \nabla_t \sum_{s=a+1}^{t} f(s)\,(1-\alpha)^{t-\rho(s)} \\ &= B(\alpha)\,\nabla_t \sum_{s=a+1}^{t} f(s)\,(1-\alpha)^{t-s}.\end{aligned} \quad (10)$$

(iv) The right (nabla) CFR fractional difference is given by

$$\begin{aligned}\left({}^{\mathrm{CFR}}\nabla_b^{\alpha} f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \left(-\Delta_t\right) \sum_{s=t}^{b-1} f(s)\,(1-\alpha)^{s-\rho(t)} \\ &= B(\alpha) \left(-\Delta_t\right) \sum_{s=t}^{b-1} f(s)\,(1-\alpha)^{s-t},\end{aligned} \quad (11)$$

where $B(\alpha)$ is a normalization positive constant depending on α satisfying $B(0) = B(1) = 1$, $(\nabla g)(t) = g(t) - g(t-1)$, and $(\Delta g)(t) = g(t+1) - g(t)$.

In [7, 8], it was verified that $({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}{}^{\mathrm{CFR}}_{a}\nabla^{\alpha} f)(t) = f(t)$ and $({}^{\mathrm{CFR}}_{a}\nabla^{\alpha}{}^{\mathrm{CF}}_{a}\nabla^{-\alpha} f)(t) = f(t)$. Also, in the right case $({}^{\mathrm{CF}}I_{b}^{\alpha}{}^{\mathrm{CFR}}\nabla_{b}^{\alpha} f)(t) = f(t)$ and $({}^{\mathrm{CFR}}\nabla_{b}^{\alpha}{}^{\mathrm{CF}}\nabla_{b}^{-\alpha} f)(t) = f(t)$. From [7, 8] we recall the relation between the CFC and CFR fractional differences as

$$\left({}^{\mathrm{CFC}}_{a}\nabla^{\alpha} f\right)(t) = \left({}^{\mathrm{CFR}}_{a}\nabla^{\alpha} f\right)(t) - \frac{B(\alpha)}{1-\alpha} f(a)(1-\alpha)^{t-a}, \tag{12}$$

and for the right case by

$$\left({}^{\mathrm{CFC}}\nabla_{b}^{\alpha} f\right)(t) = \left({}^{\mathrm{CFR}}\nabla_{b}^{\alpha} f\right)(t) - \frac{B(\alpha)}{1-\alpha} f(b)(1-\alpha)^{b-t}. \tag{13}$$

Notice that we extend Definition 4 to arbitrary $\alpha > 0$ in the next section.

Lemma 5 (see [7]). *For $0 < \alpha < 1$, we have*

$$\begin{aligned} \left({}^{CF}_{a}\nabla^{-\alpha}{}^{CFC}_{a}\nabla^{\alpha} f\right)(t) &= f(t) - f(a), \\ \left({}^{CF}\nabla_{b}^{-\alpha}{}^{CFC}\nabla_{b}^{\alpha} f\right)(t) &= f(t) - f(b). \end{aligned} \tag{14}$$

Notation. For a positive integer n, we have

(i) $(\nabla^{n} f)(t) = (\underbrace{\nabla\nabla\cdots\nabla}_{n \text{ times}} f)(t)$.

(ii) $(\Delta^{n} f)(t) = (\underbrace{\Delta\Delta\cdots\Delta}_{n \text{ times}} f)(t)$.

(iii) $({}_{\ominus}\Delta^{n} f)(t) = (-1)^{n}(\Delta^{n} f)(t)$.

3. Higher Order Fractional Differences and Sums

Definition 6. Let $n < \alpha \leq n+1$ and f be defined on $\mathbb{N}_{a} \cap {}_{b}\mathbb{N}$. Set $\beta = \alpha - n$ and define

$$\begin{aligned} \left({}^{\mathrm{CFC}}_{a}\nabla^{\alpha} f\right)(t) &= \left({}^{\mathrm{CFC}}_{a}\nabla^{\beta}\nabla^{n} f\right)(t), \\ \left({}^{\mathrm{CFR}}_{a}\nabla^{\alpha} f\right)(t) &= \left({}^{\mathrm{CFR}}_{a}\nabla^{\beta}\nabla^{n} f\right)(t). \end{aligned} \tag{15}$$

The associated fractional sum is given by

$$\left({}^{\mathrm{CF}}_{a}\nabla^{-\alpha} f\right)(t) = \left({}_{a}\nabla^{-n}\,{}^{\mathrm{CF}}_{a}\nabla^{-\beta} f\right)(t). \tag{16}$$

Note that if we use the convention that $({}_{a}\nabla^{-0} f)(t) = f(t)$, then for the case $0 < \alpha \leq 1$ we have $\beta = \alpha$ and hence we obtain Definition 4. Also, the convention $(\nabla^{0} f)(t) = f(t)$ leads to $({}^{\mathrm{CFR}}_{a}\nabla^{\alpha} f)(t)$ and $({}^{\mathrm{CFC}}_{a}\nabla^{\alpha} f)(t)$ as in Definition 4 for $0 < \alpha \leq 1$.

Remark 7. In Definition 6, if we let $\alpha = n+1$ then $\beta = 1$ and hence $({}^{\mathrm{CFR}}_{a}\nabla^{\alpha} f)(t) = ({}^{\mathrm{CFR}}_{a}\nabla^{1}\nabla^{n} f)(t) = (\nabla^{n+1} f)(t)$. Also, by noting that $({}^{\mathrm{CF}}_{a}\nabla^{-1} f)(t) = ({}_{a}\nabla^{-1} f)(t)$, we see that for $\alpha = n+1$ we have $({}^{\mathrm{CF}}_{a}\nabla^{-\alpha} f)(t) = ({}_{a}\nabla^{-(n+1)} f)(t)$. Also, for $0 < \alpha \leq 1$ we reobtain the concepts defined in Definition 4. Therefore, our generalization to higher order case is confirmed.

Analogously, in the right case we have the following extension.

Definition 8. Let $n < \alpha \leq n+1$ and f be defined on $\mathbb{N}_{a} \cap {}_{b}\mathbb{N}$. Set $\beta = \alpha - n$. Then $\beta \in (0,1]$ and we define

$$\begin{aligned} \left({}^{\mathrm{CFC}}\nabla_{b}^{\alpha} f\right)(t) &= {}^{\mathrm{CFC}}\nabla_{b}^{\beta}\left({}_{\ominus}\Delta^{n} f\right)(t), \\ \left({}^{\mathrm{CFR}}\nabla_{b}^{\alpha} f\right)(t) &= {}^{\mathrm{CFR}}\nabla_{b}^{\beta}\left({}_{\ominus}\Delta^{n} f\right)(t). \end{aligned} \tag{17}$$

The associated fractional integral is given by

$$\left({}^{\mathrm{CF}}\nabla_{b}^{-\alpha} f\right)(t) = \left(\nabla_{b}^{-n}\,{}^{\mathrm{CF}}\nabla_{b}^{-\beta} f\right)(t). \tag{18}$$

An immediate extension of (12) and (13) by using Definition 6 is the following.

Proposition 9. *For f defined on $\mathbb{N}_{a} \cap {}_{b}\mathbb{N}$ and $n < \alpha \leq n+1$, we have*

$$\left({}^{CFC}_{a}\nabla^{\alpha} f\right)(t) = \left({}^{CFR}_{a}\nabla^{\alpha} f\right)(t) - \frac{B(\alpha)}{1-\alpha}(\nabla^{n} f)(a)(1-\alpha)^{t-a}, \tag{19}$$

and for the right case

$$\left({}^{CFC}\nabla_{b}^{\alpha} f\right)(t) = \left({}^{CFR}\nabla_{b}^{\alpha} f\right)(t) - \frac{B(\alpha)}{1-\alpha}\left({}_{\ominus}\Delta^{n} f\right)(b)(1-\alpha)^{b-t}. \tag{20}$$

Next proposition explains the action of the arbitrary order sum operator ${}^{\mathrm{CF}}_{a}\nabla^{-\alpha}$ on the arbitrary order CFR and CFC differences (and vice versa) and the action of the CFR difference on the CF correspondent sum operator.

Proposition 10. *For $u(t)$ defined on $\mathbb{N}_{a} \cap {}_{b}\mathbb{N}$ and for some $n \in \mathbb{N}_{0}$ with $n < \alpha \leq n+1$, we have*

(i) $({}^{CFR}_{a}\nabla^{\alpha}{}^{CF}_{a}\nabla^{-\alpha} u)(t) = u(t)$.

(ii) $({}^{CF}_{a}\nabla^{-\alpha}{}^{CFR}_{a}\nabla^{\alpha} u)(t) = u(t) - \sum_{k=0}^{n-1}((\nabla^{k} u)(a)/k!)(t-a)^{\overline{k}}$.

(iii) $({}^{CF}_{a}\nabla^{-\alpha}{}^{CFC}_{a}\nabla^{\alpha} u)(t) = u(t) - \sum_{k=0}^{n}((\nabla^{k} u)(a)/k!)(t-a)^{\overline{k}}$.

Proof. (i) By Definition 6 and the statement after Definition 4, we have

$$\begin{aligned} \left({}^{\mathrm{CFR}}_{a}\nabla^{\alpha}{}^{\mathrm{CF}}_{a}\nabla^{-\alpha} u\right)(t) &= {}^{\mathrm{CFR}}_{a}\nabla^{\beta}\nabla^{n}{}_{a}\nabla^{-n}\,{}^{\mathrm{CF}}_{a}\nabla^{-\beta} u(t) \\ &= {}^{\mathrm{CFR}}_{a}\nabla^{\beta}{}^{\mathrm{CF}}_{a}\nabla^{-\beta} u(t) = u(t). \end{aligned} \tag{21}$$

(ii) By Definition 6 and the statement after Definition 4, we have

$$\begin{aligned}\left({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}{}^{\mathrm{CFR}}_{a}\nabla^{\alpha}u\right)(t) &= \left({}_{a}\nabla^{-n}{}^{\mathrm{CF}}_{a}\nabla^{-\beta}{}^{\mathrm{CFR}}_{a}\nabla^{\beta}\nabla^{n}u\right)(t)\\ &= {}_{a}\nabla^{-n}\nabla^{n}u(t)\\ &= u(t)-\sum_{k=0}^{n-1}\frac{\left(\nabla^{k}u\right)(a)}{k!}(t-a)^{\overline{k}},\end{aligned} \tag{22}$$

where $\beta = \alpha - n$.

(iii) By Lemma 5 applied to $f(t) = (\nabla^n u)(t)$, we have

$$\begin{aligned}\left({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}{}^{\mathrm{CFC}}_{a}\nabla^{\alpha}u\right)(t) &= {}_{a}\nabla^{-n}{}_{a}\nabla^{-\beta}{}^{\mathrm{CFC}}_{a}\nabla^{\beta}\nabla^{n}u(t)\\ &= {}_{a}\nabla^{-n}\left[\nabla^{n}u(t)-\left(\nabla^{n}u\right)(a)\right]\\ &= u(t)-\sum_{k=0}^{n-1}\frac{\left(\nabla^{k}u\right)(a)}{k!}(t-a)^{\overline{k}}\\ &\quad-\left(\nabla^{n}u\right)(a)\frac{(t-a)^{\overline{n}}}{n!}\\ &= u(t)-\sum_{k=0}^{n}\frac{\left(\nabla^{k}u\right)(a)}{k!}(t-a)^{\overline{k}}.\end{aligned} \tag{23}$$

□

Using the facts that ${}_{\ominus}\Delta^{n}\nabla_{b}^{-n}g(t) = g(t)$,

$$\nabla_{b}^{n}{}_{\ominus}\Delta^{n}g(t) = g(t)-\sum_{k=0}^{n-1}\frac{\left({}_{\ominus}\Delta^{k}g\right)(b)}{k!}(b-t)^{\overline{k}}, \tag{24}$$

and making use of Lemma 5 and the statement after Definition 4, we can state, for the right case, the following.

Proposition 11. *For $u(t)$ defined on $\mathbb{N}_{a,b}$ and $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}_0$, we have*

(i) $({}^{CFR}\nabla_{b}^{\alpha}{}^{CF}\nabla_{b}^{-\alpha}u)(t) = u(t)$.

(ii) $({}^{CF}\nabla_{b}^{-\alpha}{}^{CFR}\nabla_{b}^{\alpha}u)(t) = u(t)-\sum_{k=0}^{n-1}(({}_{\ominus}\Delta^{k}u)(b)/k!)(b-t)^{\overline{k}}$.

(iii) $({}^{CF}\nabla_{b}^{-\alpha}{}^{CFC}\nabla_{b}^{\alpha}u)(t) = u(t)-\sum_{k=0}^{n}(({}_{\ominus}\Delta^{k}u)(b)/k!)(b-t)^{\overline{k}}$.

Example 12. Consider the initial value problem:

$$\left({}^{\mathrm{CFC}}_{a}\nabla^{\alpha}y\right)(t) = F(t), \tag{25}$$

where $F(t)$ is defined on $\mathbb{N}_{a,b} = \mathbb{N}_a \cap {}_{b}\mathbb{N}$. Let us consider two cases depending on the order $\alpha > 0$:

(i) Assume $0 < \alpha \leq 1$, $y(a) = c$, and $F(a) = 0$. By applying ${}^{\mathrm{CF}}_{a}\nabla^{-\alpha}$ and making use of Proposition 10, we get the solution

$$y(t) = c+\frac{1-\alpha}{B(\alpha)}F(t)+\frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}F(s). \tag{26}$$

Notice that the condition $F(a) = 0$ verifies the initial condition $y(a) = c$. In addition, when $\alpha \to 1$ we obtain the solution of the ordinary difference initial value problem $(\nabla y)(t) = F(t)$, $y(a) = c$.

(ii) Assume $1 < \alpha \leq 2$, $F(a) = 0$, $y(a) = c_1$, and $(\nabla y)(a) = c_2$. By applying ${}^{\mathrm{CF}}_{a}\nabla^{-\alpha}$ and making use of Proposition 10 and Definition 6 with $\beta = \alpha - 1$, we obtain the solution

$$\begin{aligned}y(t) &= c_1+c_2(t-a)+\frac{2-\alpha}{B(\alpha-1)}\sum_{s=a+1}^{t}K(s)\\ &\quad+\frac{\alpha-1}{B(\alpha-1)}\sum_{s=a+1}^{t}(t-\rho(s))F(s).\end{aligned} \tag{27}$$

Notice that the solution $y(t)$ verifies $y(a) = c_1$ without the use of $F(a) = 0$. However, it verifies $(\nabla y)(a) = c_2$ under the assumption $F(a) = 0$. Also, note that when $\alpha \to 2$ we recover the solution of the second-order ordinary difference initial value problem $(\nabla^2 y)(t) = F(t)$, $y(a) = c_1$, and $(\nabla y)(a) = c_2$.

In the next section, we prove existence and uniqueness theorems for some types of CFC and CFR initial value difference problems.

Example 13. Consider the CFC difference boundary value problem

$$\begin{aligned}&\left({}^{\mathrm{CFC}}_{a}\nabla^{\alpha}y\right)(t)+q(t)y(t) = 0,\\ &\quad 1<\alpha\leq 2,\ t\in\mathbb{N}_{a,b},\ y(a) = y(b) = 0.\end{aligned} \tag{28}$$

Then $\beta = \alpha - 1$ and by Proposition 10 if we apply the operator ${}^{\mathrm{CF}}_{a}\nabla^{-\alpha}$, we obtain the solution

$$y(t) = c_1+c_2(t-a)-\left({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}q(\cdot)y(\cdot)\right)(t). \tag{29}$$

But

$$\begin{aligned}\left({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}q(\cdot)y(\cdot)\right)(t) &= \frac{1-\beta}{B(\beta)}\sum_{s=a+1}^{t}q(s)y(s)\\ &\quad+\frac{\beta}{B(\beta)}{}_{a}\nabla^{-2}q(t)y(t).\end{aligned} \tag{30}$$

Hence, the solution has the form

$$\begin{aligned}y(t) &= c_1+c_2(t-a)-\frac{2-\alpha}{B(\alpha-1)}\sum_{s=a+1}^{t}q(s)y(s)\\ &\quad-\frac{\alpha-1}{B(\alpha-1)}\sum_{s=a+1}^{t}(t-\rho(s))q(s)y(s).\end{aligned} \tag{31}$$

The boundary conditions imply that $c_1 = 0$ and

$$\begin{aligned}c_2 &= \frac{2-\alpha}{(b-a)B(\alpha-1)}\sum_{s=a+1}^{b}q(s)y(s)\\ &\quad+\frac{\alpha-1}{(b-a)B(\alpha-1)}\sum_{s=a+1}^{b}(b-\rho(s))q(s)y(s).\end{aligned} \tag{32}$$

Hence,

$$y(t)=\frac{(2-\alpha)(t-a)}{(b-a)B(\alpha-1)}\sum_{s=a+1}^{b}q(s)y(s)-\frac{(\alpha-1)(t-a)}{(b-a)B(\alpha-1)}\sum_{s=a+1}^{b}(b-\rho(s))q(s)y(s)-\frac{2-\alpha}{B(\alpha-1)}\sum_{s=a+1}^{t}q(s)y(s)-\frac{\alpha-1}{B(\alpha-1)}\sum_{s=a+1}^{t}(t-\rho(s))q(s)y(s). \quad (33)$$

4. Existence and Uniqueness Theorems for the Initial Value Problem Types

In this section we prove existence and uniqueness theorems for CFC and CFR type initial value problems.

Theorem 14. *Consider the system*

$$\left({}^{CFC}_{a}\nabla^{\alpha}y\right)(t)=f(t,y(t)),\quad t\in\mathbb{N}_{a,b},\ 0<\alpha\leq1,\ y(a)=c, \quad (34)$$

such that $f(a,y(a))=0$, $A((1-\alpha)/B(\alpha)+\alpha(b-a)/B(\alpha))<1$, *and* $|f(t,y_1)-f(t,y_2)|\leq A|y_1-y_2|$, $A>0$, *where* $f:\mathbb{N}_{a,b}\times\mathbb{R}\to\mathbb{R}$ *and* $y:\mathbb{N}_{a,b}\to\mathbb{R}$. *Then, system (34) has a unique solution of the form*

$$y(t)=c+{}^{CF}_{a}\nabla^{-\alpha}f(t,y(t)). \quad (35)$$

Proof. First, by the help of Proposition 10, (12), and taking into account the fact that $f(a,y(a))=0$, it is straight forward to prove that $y(t)$ satisfies system (34) if and only if it satisfies (35).

Let $X=\{x:\max_{t\in\mathbb{N}_{a,b}}|x(t)|<\infty\}$ be the Banach space endowed with the norm $\|x\|=\max_{t\in\mathbb{N}_{a,b}}|x(t)|$. On X define the linear operator

$$(Tx)(t)=c+{}^{CF}_{a}\nabla^{-\alpha}f(t,x(t)). \quad (36)$$

Then, for arbitrary $x_1,x_2\in X$ and $t\in\mathbb{N}_{a,b}$, we have by assumption that

$$\begin{aligned}\left|(Tx_1)(t)-(Tx_2)(t)\right| &=\left|{}^{CF}_{a}\nabla^{-\alpha}\left[f(t,x_1(t))-f(t,x_2(t))\right]\right|\\ &\leq A\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha(b-a)}{B(\alpha)}\right)\|x_1-x_2\|,\end{aligned} \quad (37)$$

and hence T is a contraction. By Banach fixed point theorem, there exists a unique $x\in X$ such that $Tx=x$ and hence the proof is complete. □

Remark 15. Similar existence and uniqueness theorems can be proved for system (34) with higher order by making use of Proposition 10. The condition $f(a,y(a))=0$ always can not be avoided as we have seen in Example 12 with $f(t,y(t))=F(t)$. As a result of Theorem 14, we conclude that the fractional difference linear initial value problem

$$\left({}^{CFC}_{a}\nabla^{\alpha}y\right)(t)=ry(t),\quad r\in\mathbb{R},\ t\in\mathbb{N}_{a,b},\ 0<\alpha\leq1,\ y(a)=c, \quad (38)$$

can have only the trivial solution unless $\alpha=1$. Indeed, the solution satisfies $y(t)=c+r((1-\alpha)/B(\alpha))y(t)+(\alpha r/B(\alpha))\sum_{s=a+1}^{t}y(s)$. This solution is only verified at a if $(1-\alpha)y(a)=0$.

Theorem 16. *Consider the system*

$$\left({}^{CFR}_{a}\nabla^{\alpha}y\right)(t)=f(t,y(t)),\quad t\in\mathbb{N}_{a,b},\ 1<\alpha\leq2,\ y(a)=c, \quad (39)$$

such that $(A/B(\alpha-1))((2-\alpha)(b-a)+(\alpha-1)(b-a)^{\overline{2}}/2)<1$, *and* $|f(t,y_1)-f(t,y_2)|\leq A|y_1-y_2|$, $A>0$, *where* $f:\mathbb{N}_{a,b}\times\mathbb{R}\to\mathbb{R}$ *and* $y:\mathbb{N}_{a,b}\to\mathbb{R}$. *Then, system (34) has a unique solution of the form*

$$\begin{aligned}y(t)&=c+{}^{CF}_{a}\nabla^{-\alpha}f(t,y(t))\\&=c+\frac{2-\alpha}{B(\alpha-1)}\sum_{s=a+1}^{t}f(s,y(s))\\&\quad+\frac{\alpha-1}{B(\alpha-1)}\left({}_{a}\nabla^{-2}f(\cdot,y(\cdot))(t)\right).\end{aligned} \quad (40)$$

Proof. If we apply ${}^{CF}_{a}\nabla^{-\alpha}$ to system (39) and make use of Proposition 10 with $\beta=\alpha-1$ then we reach at the representation (40). Conversely, if we apply ${}^{CFR}_{a}\nabla^{\alpha}$, make use of Proposition 10 and by noting that

$${}^{CFR}_{a}\nabla^{\alpha}={}^{CFR}_{a}\nabla^{\beta}\nabla_{t}c=0, \quad (41)$$

we obtain system (39). Hence, $y(t)$ satisfies system (39) if and only if it satisfies (40).

Let $X=\{x:\max_{t\in\mathbb{N}_{a,b}}|x(t)|<\infty\}$ be the Banach space endowed with the norm $\|x\|=\max_{t\in\mathbb{N}_{a,b}}|x(t)|$. On X define the linear operator

$$(Tx)(t)=c+{}^{CF}_{a}\nabla^{-\alpha}f(t,x(t)). \quad (42)$$

Then, for arbitrary $x_1,x_2\in X$ and $t\in\mathbb{N}_{a,b}$, we have by assumption that

$$\begin{aligned}&\left|(Tx_1)(t)-(Tx_2)(t)\right|\\&=\left|{}^{CF}_{a}\nabla^{-\alpha}\left[f(t,x_1(t))-f(t,x_2(t))\right]\right|\\&\leq\frac{A}{B(\alpha-1)}\left((2-\alpha)(b-a)+\frac{(\alpha-1)(b-a)^{\overline{2}}}{2}\right)\\&\quad\cdot\|x_1-x_2\|,\end{aligned} \quad (43)$$

and hence T is a contraction. By Banach Contraction Principle, there exists a unique $x\in X$ such that $Tx=x$ and hence the proof is complete. □

5. The Lyapunov Inequality for the CFR Difference Boundary Value Problem

In this section, we prove a Lyapunov inequality for a CFR boundary value difference problem of order $2 < \alpha \leq 3$.

Consider the boundary value problem

$$\begin{gathered}\left({}^{\mathrm{CFR}}_{\ \ a}\nabla^{\alpha} y\right)(t) + q(t)\, y^{\rho}(t) = 0, \\ 2 < \alpha \leq 3,\ t \in \mathbb{N}_{a+1,b-1},\ y(a) = y(b) = 0,\end{gathered} \tag{44}$$

where, $y^{\rho}(t) = y(\rho(t)) = y(t-1)$.

Lemma 17. *$y(t)$ is a solution of the boundary value problem (44) if and only if it satisfies the equation*

$$y(t) = \sum_{s=a+1}^{b} G(t,s)\, T(s, y(s)), \tag{45}$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-a)(b-\rho(s))}{b-a}, & t+1 \leq s,\ t,s \in \mathbb{N}_{a,b}, \\ \left(\dfrac{(t-a)(b-\rho(s))}{b-a} - (t-\rho(s))\right), & s-1 \leq t,\ t,s \in \mathbb{N}_{a,b}, \end{cases} \tag{46}$$

$$\begin{aligned} T(t, y(t)) &= {}^{\mathrm{CF}}_{\ a}\nabla^{\beta}\left(q(\cdot)\, y^{\rho}(\cdot)\right)(t) \\ &= \frac{1-\beta}{B(\beta)} q(t)\, y^{\rho}(t) + \frac{\beta}{B(\beta)}\left({}_{a}\nabla^{-1} q(\cdot)\, y^{\rho}(\cdot)\right)(t), \quad \beta = \alpha - 2. \end{aligned}$$

Proof. Apply the integral ${}^{\mathrm{CF}}_{\ a}\nabla^{-\alpha}$ to (44) and make use of Definition 6 and Proposition 10 with $n = 2$ and $\beta = \alpha - 2$ to reach

$$\begin{aligned} y(t) &= c_1 + c_2(t-a) - \left({}_{a}\nabla^{-2} T(\cdot, y(\cdot))\right)(t) \\ &= c_1 + c_2(t-a) - \sum_{s=a+1}^{t} (t-\rho(s))\, T(s, y(s)). \end{aligned} \tag{47}$$

The condition $y(a) = 0$ implies that $c_1 = 0$ and the condition $y(b) = 0$ implies that $c_2 = (1/(b-a)) \sum_{s=a+1}^{b} (b - \rho(s)) T(s, y(s))$, and hence

$$\begin{aligned} y(t) = {} & \frac{t-a}{b-a} \sum_{s=a+1}^{b} (b-\rho(s))\, T(s, y(s)) \\ & - \sum_{s=a+1}^{t} (t-\rho(s))\, q(s)\, T(s, y(s))\, ds. \end{aligned} \tag{48}$$

Then, the result follows by splitting the summation

$$\begin{aligned} \sum_{s=a+1}^{b} (b-\rho(s))\, T(s, y(s)) &= \sum_{s=a+1}^{t} (b-\rho(s))\, T(s, y(s)) \\ &\quad + \sum_{s=t+1}^{b} (b-\rho(s))\, T(s, y(s)). \end{aligned} \tag{49}$$

□

Lemma 18. *Given that $b \equiv a \pmod 1$, the Green function $G(t,s)$ defined in Lemma 17 has the following properties:*

(1) *$G(t,s) \geq 0$ for all $t, s \in \mathbb{N}_{a,b}$.*

(2) *$\max_{t \in \mathbb{N}_{a,b}} G(t,s) = G(\rho(s), s)$ for $s \in \mathbb{N}_{a+1,b-1}$.*

(3) *$f(s) = G(\rho(s), s) = (\rho(s) - a)(b - \rho(s))/(b-a)$ has a unique maximum, given by*

$$\max_{s \in \mathbb{N}_{a,b}} f(s) = \begin{cases} f\left(\dfrac{(a+b+2)}{2}\right) = \dfrac{b-a}{4} & \text{if } b-a \text{ is even} \\ f\left(\dfrac{(a+b+3)}{2}\right) = \dfrac{(b-a)^2 - 1}{4(b-a)} & \text{if } b-a \text{ is odd.} \end{cases} \tag{50}$$

Hence, in either cases $\max_{s \in \mathbb{N}_{a,b}} f(s) \leq (b-a)/4$.

Proof. (1) It is clear that $g_1(t,s) = (t-a)(b-\rho(s))/(b-a) \geq 0$. Regarding the part $g_2(t,s) = ((t-a)(b-\rho(s))/(b-a) - (t-\rho(s)))$ we see that $(t - \rho(s)) = ((t-a)/(b-a))(b - (a + (\rho(s)-a)(b-a)/(t-a)))$ and that $a + (\rho(s)-a)(b-a)/(t-a) \geq \rho(s)$ if and only if $\rho(s)(b-t) + a(t-b) \geq 0$ if and only if $s \geq a+1$. Hence, we conclude that $g_2(t,s) \geq 0$ as well.

(2) Clearly, $g_1(t,s)$ is an increasing function in t. Applying ∇ to g_2 with respect to t for every fixed $s \geq a+1$ we see that $\nabla_t g(t,s) = (b-a)^{-1}(a - \rho(s))$ and hence g_2 is a decreasing function in t.

(3) Let $f(s) = G(\rho(s), s) = (\rho(s) - a)(b - \rho(s))/(b-a)$. Then,

$$(\nabla f)(s) = \frac{a+b-2s+3}{b-a} = 0, \tag{51}$$

if $s = (a+b+3)/2$ and hence for $a, b \in \mathbb{N}$, f attains its maximum at $s_1 = (a+b+3)/2$ if $a+b$ (or $b-a$) is odd and at $(a+b+2)/2$ if $a+b$ (or $b-a$) is even. More generally, if a and b are such that $b \equiv a \pmod 1$, we see that $s_1 \equiv a \pmod 1$ if $b-a$ is odd and $s_2 \equiv a \pmod 1$ if $b-a$ is even. Finally, $f(s_1) = ((b-a)^2 - 1)/4(b-a)$ and $f(s_2) = (b-a)/4$. □

In next lemma, we estimate $T(t, y(t))$ for a function $y \in B[\mathbb{N}_{a,b}]$, the Banach space of all Banach-valued finite sequences on $\mathbb{N}_{a,b}$ with $\|y\| = \max_{t \in \mathbb{N}_{a,b}} |y(t)|$, where $|\cdot|$ is the norm in the Banach space.

Lemma 19. *For $y \in B[\mathbb{N}_{a,b}]$ and $2 < \alpha \leq 3$, $\beta = \alpha - 2$, we have, for any $t \in \mathbb{N}_{a,b}$,*

$$\left|T(t, y(t))\right| \leq R(t)\, \|y\|, \tag{52}$$

where

$$R(t) = \left[\frac{3-\alpha}{B(\alpha-2)} \left|q(t)\right| + \frac{\alpha-2}{B(\alpha-2)} \sum_{s=a+1}^{t} \left|q(s)\right|\right]. \tag{53}$$

Theorem 20 (the CFR fractional difference Lyapunov inequality). *If the boundary value problem (44) has a nontrivial solution, where $q(t)$ is a real-valued bounded function on $\mathbb{N}_{a,b}$, then*

$$\sum_{s=a+1}^{b} R(s) > \frac{4}{b-a}. \tag{54}$$

Proof. Assume $y \in Y = B[\mathbb{N}_{a,b}]$ is a nontrivial solution of the boundary value problem (44). By Lemma 17, y must satisfy

$$y(t) = \sum_{s=a+1}^{b} G(t,s) T(s, y(s)). \quad (55)$$

Then, by using the properties of the Green function $G(t,s)$ proved in Lemmas 18 and 19, we come to the conclusion that

$$\|y\| < \frac{b-a}{4} \sum_{s=a+1}^{b} R(s) \|y\| \quad (56)$$

from which (54) follows. □

Remark 21. Note that if $\alpha \to 2^+$, then $R(t)$ tends to $|q(t)|$ and hence we obtain the classical nabla discrete version of the Lyapunov inequality (2). For the sake of more comparisons of Lyapunov inequalities on time scales we refer to [40].

Example 22. Consider the following CFR Sturm-Liouville difference eigenvalue problem (SLDEP) of order $2 < \alpha \le 3$

$$\begin{gathered}\left({}_{0}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) + \lambda y^{\rho}(t) = 0, \\ t \in \mathbb{N}_{1,b-1}, \ b \in \mathbb{N}, \ y(0) = y(b) = 0.\end{gathered} \quad (57)$$

If λ is an eigenvalue of (57), then by Theorem 20 with $q(t) = \lambda$, we have

$$\begin{aligned} T(t) &= \left[\frac{3-\alpha}{B(\alpha-2)}|\lambda| + \frac{\alpha-2}{B(\alpha-2)}\left({}_{0}\nabla^{-1}|\lambda|\right)(t)\right] \\ &= |\lambda|\left[\frac{3-\alpha}{B(\alpha-2)} + \frac{\alpha-2}{B(\alpha-2)}t\right]. \end{aligned} \quad (58)$$

Hence, we must have

$$\sum_{s=1}^{b} T(s) = |\lambda|\left[\frac{b(3-\alpha)}{B(\alpha-2)} + \frac{b^{\overline{2}}(\alpha-2)}{2B(\alpha-2)}\right] > \frac{4}{b}. \quad (59)$$

Notice that the limiting case $\alpha \to 2^+$ implies that $|\lambda| > 4/b^2$ which is the lower bound for the eigenvalues of the ordinary difference eigenvalue problem:

$$\begin{gathered}\left(\nabla^2 y\right)(t) + \lambda y^{\rho}(t) = 0, \\ t \in \mathbb{N}_{1,b-1}, \ y(0) = y(b) = 0.\end{gathered} \quad (60)$$

6. Conclusions

Fractional differences and their correspondent fractional sum operators are of importance in discrete modeling of various problems in science. We extended the fractional difference calculus whose difference operators depend on a discrete exponential function kernel to arbitrary positive order. The correspondent arbitrary order fractional sum operators have been defined as well and applied to solve fractional initial and boundary value difference problems. The extension for right fractional differences and sums is also achieved. To set up the basic concepts, we proved existence and uniqueness theorems by means of Banach fixed point theorem for initial value problems in the frame of CFC and CFR fractional differences. We have come to the conclusion that the condition $f(a, y(a)) = 0$ is necessary to guarantee the existence of solution and hence fractional linear difference initial value problem with constant coefficients results in the trivial solution unless the order is positive integer. We used our extension to arbitrary order to prove a Lyapunov type inequality for a CFR boundary value problem of order $2 < \alpha \le 3$ and then obtain the classical ordinary case when α tends to 2 from right. This proves different behavior from the classical fractional difference case, where the Lyapunov inequality was proved for a fractional difference boundary problem of order $1 < \alpha \le 2$ and the classical ordinary case was then recovered when α tends to 2 from left.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] M. A. Hajji, Q. M. Al-Mdallal, and F. M. Allan, "An efficient algorithm for solving higher-order fractional Sturm-Liouville eigenvalue problems," *Journal of Computational Physics*, vol. 272, pp. 550–558, 2014.

[2] Q. M. Al-Mdallal and M. A. Hajji, "A convergent algorithm for solving higher-order nonlinear fractional boundary value problems," *Fractional Calculus and Applied Analysis*, vol. 18, no. 6, pp. 1423–1440, 2015.

[3] Q. M. Al-Mdallal, "On the numerical solution of fractional Sturm-Liouville problems," *International Journal of Computer Mathematics*, vol. 87, no. 12, pp. 2837–2845, 2010.

[4] A. H. Bhrawy, T. M. Taha, and J. A. Machado, "A review of operational matrices and spectral techniques for fractional calculus," *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 81, no. 3, pp. 1023–1052, 2015.

[5] F. Wu and J. Liu, "Discrete fractional creep model of salt rock," *J. Comput. Complex. Appl*, vol. 2, pp. 1–6, 2016.

[6] R. Abu-Saris and Q. Al-Mdallal, "On the asymptotic stability of linear system of fractional-order difference equations," *Fractional Calculus and Applied Analysis. An International Journal for Theory and Applications*, vol. 16, no. 3, pp. 613–629, 2013.

[7] T. Abdeljawad and D. Baleanu, "On fractional derivatives with exponential kernel and their discrete versions," *Journal of Reports in Mathematical Physics*, https://arxiv.org/abs/1606.07958.

[8] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernal," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 73–85, 2015.

[9] J. Losada and J. J. Nieto, "Properties of a new fractional derivative without singular kernal," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 87–92, 2015.

[10] T. Abdeljawad and D. Baleanu, "Monotonicity results for fractional difference operators with discrete exponential kernels," *Advances in Difference Equations*, vol. 2017, Article ID 78, 2017.

[11] A. Atangana and D. Baleanu, "New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[12] T. Abdeljawad and D. Baleanu, "Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 03, pp. 1098–1107, 2017.

[13] T. Abdeljawad and D. Baleanu, "Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels," *Advances in Difference Equations*, vol. 2016, Article ID 232, 2016.

[14] T. Abdeljawad and D. Baleanu, "Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel," *Chaos, Solitons & Fractals*, 2017.

[15] R. A. Ferreira, "Some discrete fractional Lyapunov-type inequalities," *Fractional Differential Calculus*, vol. 5, no. 1, pp. 87–92, 2015.

[16] R. A. Ferreira, "A Lyapunov-type inequality for a fractional boundary value problem," *Fractional Calculus and Applied Analysis. An International Journal for Theory and Applications*, vol. 16, no. 4, pp. 978–984, 2013.

[17] A. Chidouh and D. F. Torres, "A generalized Lyapunov's inequality for a fractional boundary value problem," *Journal of Computational and Applied Mathematics*, vol. 312, pp. 192–197, 2017.

[18] M. Jleli and B. Samet, "Lyapunov-type inequalities for fractional boundary-value problems," *Electronic Journal of Differential Equations*, vol. 2015, no. 88, 11 pages, 2015.

[19] D. O'Regan and B. Samet, "Lyapunov-type inequalities for a class of fractional differential equations," *Journal of Inequalities and Applications*, vol. 2015, no. 247, 10 pages, 2015.

[20] J. Rong and C. Bai, "Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions," *Advances in Difference Equations*, vol. 2015, no. 82, 10 pages, 2015.

[21] M. Jleli and B. Samet, "A Lyapunov-type inequality for a fractional q-difference boundary value problem," *Journal of Nonlinear Science and its Applications. JNSA*, vol. 9, no. 5, pp. 1965–1976, 2016.

[22] H. Esser, "Stability inequalities for discrete nonlinear two-point boundary value problems," *Applicable Analysis. An International Journal*, vol. 10, no. 2, pp. 137–162, 1980.

[23] N. G. Abuj and D. B. Pachpatte, "Lyapunov type inequality for fractional differential equation with k-Prabhakar derivative," https://arxiv.org/abs/1702.01562.

[24] D. B. Pachpatte, N. G. Abuj, and A. D. Khandagale, "Lyapunov type inequality for hybrid fractional differential equation with Prabhakar Derivative," https://arxiv.org/abs/1609.03027.

[25] X. Yang, "On Liapunov-type inequality for certain higher-order differential equations," *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 307–317, 2003.

[26] A. n. Tiryaki, "Recent developments of Lyapunov-type inequalities," *Advances in Dynamical Systems and Applications*, vol. 5, no. 2, pp. 231–248, 2010.

[27] N. Parhi and S. Panigrahi, "Liapunov-type inequality for higher order differential equations," *Mathematica Slovaca*, vol. 52, no. 1, pp. 31–46, 2002.

[28] J. X. Cui, X. Huang, and M. C. Hou, "Lyapunov-type inequality for fractional order difference equations," *Global Journal of Science Frontier Research*, vol. 16, no. 1, 2016.

[29] T. Abdeljawad, "A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel," *Journal of Inequalities and Applications*, 2017:130 pages, 2017.

[30] A. M. Lyapunov, "Probleme général de la stabilité du mouvment," *Annales de la Faculté des Sciences de Toulouse*, vol. 2, pp. 227–247, 1907, Reprinted in: Ann. Math. Studies, no. 17, Princeton, 1947.

[31] S. S. Cheng, "A discrete analogue of the inequality of Lyapunov," *Hokkaido Mathematical Journal*, vol. 12, no. 1, pp. 105–112, 1983.

[32] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[33] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[34] A. Kilbas, M. H. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, vol. 204 of *North Holland Mathematics Studies*, 2006.

[35] F. Bozkurt, T. Abdeljawad, and M. A. Hajji, "Stability analysis of a fractional order differential equatin model of a brain tumor growth depending on the density," *Applied and Computational Mathematics. An International Journal*, vol. 14, no. 1, pp. 50–62, 2015.

[36] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer International, New York, NY, USA, 2015.

[37] T. Abdeljawad, "On delta and nabla Caputo fractional differences and dual identities," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 406910, 12 pages, 2013.

[38] T. Abdeljawad, "Dual identities in fractional difference calculus within Riemann," *Advances in Difference Equations*, vol. 2013, no. 36, 2013.

[39] T. Abdeljawad and D. F. M. Torres, "Symmetric duality for left and right Riemann-Liouville and Caputo fractional differences," *Arab Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 157–172, 2017.

[40] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Springer, Berlin, Germany, 2001.

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Research Article

Fractional Stochastic Differential Equations with Hilfer Fractional Derivative: Poisson Jumps and Optimal Control

Fathalla A. Rihan,[1] Chinnathambi Rajivganthi,[1] and Palanisamy Muthukumar[2]

[1] *Department of Mathematical Sciences, College of Science, UAE University, Al-Ain 15551, UAE*
[2] *Department of Mathematics, Gandhigram Rural Institute-Deemed University, Gandhigram, Tamil Nadu 624 302, India*

Correspondence should be addressed to Fathalla A. Rihan; frihan@uaeu.ac.ae

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In this work, we consider a class of fractional stochastic differential system with Hilfer fractional derivative and Poisson jumps in Hilbert space. We study the existence and uniqueness of mild solutions of such a class of fractional stochastic system, using successive approximation theory, stochastic analysis techniques, and fractional calculus. Further, we study the existence of optimal control pairs for the system, using general mild conditions of cost functional. Finally, we provide an example to illustrate the obtained results.

1. Introduction

The subject of fractional calculus has gained importance and attractiveness due to its applications in widespread fields of engineering and science. Fractional calculus is successful in describing systems which have long-time memory and long-range interaction [1–3]. *Fractional-Order Differential Equations* (FODEs) models have been successfully applied in biology systems [3, 4], physics [5, 6], chemistry and biochemistry [7], hydrology [8], engineering [9, 10], medicine [11], finance [12], and control problems [13, 14]. In most cases, the models of FODEs seem to be more regular with the real events compared with integer-order models, because fractional integrals and derivatives allow the explanation of the hereditary and memory properties inherent in various processes and materials [15, 16]. Many authors described the fractional-order models with the most common definitions of fractional derivatives defined by Caputo and Riemann-Liouville sense [17].

Hilfer [5] proposed a general operator for fractional derivative, called "Hilfer fractional derivative," which combines Caputo and Riemann-Liouville fractional derivatives. Hilfer fractional derivative is performed, for example, in the theoretical simulation of dielectric relaxation in glass forming materials. Sandev et al. [18] derived the existence results of fractional diffusion equation with Hilfer fractional derivative which attained in terms of Mittag Leffler functions. Mahmudov and McKibben [19] studied the controllability of fractional dynamical equations with generalized Riemann-Liouville fractional derivative by using Schauder fixed point theorem and fractional calculus. Recently, Gu and Trujillo [20] reported the existence results of fractional differential equations with Hilfer derivative based on noncompact measure method. The set of two parameters in "Hilfer fractional derivative" $D_{a^+}^{\nu,\mu}$ of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ permits one to connect between the Caputo and Riemann-Liouville derivatives [17, 21, 22]. This set of parameters gives an extra degree of freedom on the initial conditions and produces more types of stationary states. Models with Hilfer fractional derivatives are discussed in [23, 24].

The deterministic models often fluctuate due to noise. Naturally, the extension of such models is necessary to consider stochastic models, where the related parameters are considered as appropriate Brownian motion and stochastic processes. The modeling of most problems in real situations is described by stochastic differential equations rather than deterministic equations. Thus, it is of great importance to design stochastic effects in the study of fractional-order dynamical systems. Chen and Li [25] reported the existence results of fractional stochastic integrodifferential equations with nonlocal initial conditions in Hilbert space. Wang [26] investigated the existence results of fractional stochastic differential equations by using Picard type approximation. Lu and Liu [27] studied, recently, the controllability of fractional stochastic hemivariational inequalities based on multivalued maps and fixed point theorem. The above-mentioned research papers discussed the detail of stochastic differential equations (SDEs) with Brownian motion, Although Brownian motion cannot be used to define the stochastic disturbances in some real systems such as the fluctuations in the financial markets and price dynamics of financial instruments with jumps (see [28]). The authors in [29] studied the existence results of jumps in stock markets and the foreign exchange markets which are based on SDEs with Poisson jumps. Ren et al. [30] reported the existence and stability results of time-dependent stochastic delayed differential equations with Poisson jumps. Recently, Rajivganthi and Muthukumar [31] studied the properties of almost automorphic solutions of fractional stochastic evolution equations with Poisson jumps with the help of solution operator.

To the best of our knowledge, the existence and uniqueness of mild solutions for fractional stochastic differential equations with Hilfer fractional derivative are an untreated topic in the present literature. Herein, we convert the deterministic fractional differential equations into a stochastic fractional differential equation with Hilfer fractional derivative. We then study the existence and uniqueness of mild solutions by using successive approximation. We study the existence and uniqueness of mild solutions by using successive approximation theory. This theory possesses some advantages of linearization for the nonlinear functional with respect to the state variables. We then study the existence of optimal control pairs for the system, using general mild conditions of cost functional.

Consider the fractional stochastic differential equations with Hilfer fractional derivative and Poisson jumps of the form

$$\begin{aligned} D_{0^+}^{\nu,\mu} x(t) &= Ax(t) + f(t, x(t)) \\ &\quad + \int_0^t g(s, x(s))\, dW(s) \\ &\quad + \int_Z h(t, x(t), \eta)\, \widetilde{N}(dt, d\eta), \qquad (1) \\ &\qquad\qquad t \in J' := (0, b], \\ I_{0^+}^{(1-\nu)(1-\mu)} x(0) &= x_0. \end{aligned}$$

Here, $D_{0^+}^{\nu,\mu}$ is the Hilfer fractional derivative: $0 \le \nu \le 1, 0 < \mu < 1$ and $J := [0, b]$. A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\ge0}$ in Hilbert space H. The state variable $x(\cdot)$ is considered in H with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let K be another separable Hilbert space and $\{W(t)\}_{t\ge0}$ is a given K-valued Wiener process or Brownian motion with a finite trace nuclear covariance operator $Q \ge 0$. Let $q(\cdot)$ be a Poisson point process in a measurable space $(Z, \mathscr{B}(Z))$ and induced compensating martingale measure $\widetilde{N}(dt, d\eta)$ described on a complete probability space $(\Omega, \mathfrak{F}, P)$. $f : J \times H \to H$, $g : J \times H \to L_Q(K, H)$ and $h : J \times H \times Z \to H$ are appropriate functions and $L_Q(K, H)$ defines the space of all $Q-$ Hilbert Schmidt operators from K into H.

Frequently, the optimal control problems stand up in system engineering. The main goal of optimal control is to find, in an open-loop control, the optimal values of the control variables for the dynamic system which maximize or minimize a given performance index. The determination of optimal control is a difficult task and is open-ended due to the nonlinear nature of dynamic systems. If the FODEs are described in conjunction with a set of initial conditions and performance index, they become *Fractional Optimal Control Problems* (FOCPs). The FOCP refers to optimization of the performance index subject to dynamical constraints on the control and state which have fractional-order models. There has been some work done in the area of deterministic FOCPs in finite dimensional spaces [32, 33] and infinite dimensional cases [34, 35]. Ren and Wu [36] discussed the optimal control problem associated with multivalued SDEs with Levy jumps by using Yosida approximation theory. Ahmed [37] studied the existence and optimal control of stochastic initial boundary value problems subject to boundary noise. Rajivganthi et al. [38] investigated the optimal control results of fractional stochastic neutral differential equations in Hilbert space. Motivated by the work done by the authors [20, 35, 38], in this paper, we study additionally the sufficient conditions that guarantee the optimal control results for the fractional stochastic system (1).

This paper is prepared as follows. In Section 2, we provide some remarks, definitions, and lemmas which are useful to prove the main results. Suitable sufficient conditions for existence and uniqueness of (1) are studied in Section 3. Optimal control results are discussed in Section 4. An example is given in Section 5 to verify the theoretical results. We then conclude the paper in the last Section.

2. Preliminaries

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub-σ-algebras $\{\mathfrak{F}_t, t \in J\}$ satisfying $\mathfrak{F}_t \subset \mathfrak{F}$. We assume that $\mathfrak{F}_t = \sigma(W(s) : 0 \le s \le t)$ is the σ-algebra generated by W and $\mathfrak{F}_t = \mathfrak{F}$. Let $\varphi \in L(K, H)$ and define $\|\varphi\|_Q^2 = \mathrm{Tr}(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi\zeta_n\|^2$. If $\|\varphi\|_Q < \infty$, then φ is called a Q-Hilbert Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert Schmidt operators $\varphi : K \to H$. The collection of all strongly measurable, square integrable H-valued random variables, denoted by $L_2(\Omega, \mathfrak{F}, P; H) = L_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot; w)\|_H^2)^{1/2}$, where the expectation E is defined by $E(h_1) = \int_\Omega h_1(w)dP$. Let $C(J, L_2(\Omega; H))$ be the Banach space of all continuous maps from J into $L_2(\Omega; H)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$. Suppose that $\{q(t) : t \in J\}$ is the Poisson point process, taking its value in a measurable space $(Z, \mathscr{B}(Z))$ with a σ-finite intensity measure $\lambda(d\eta)$. The compensating martingale measure and Poisson counting measure are defined by $\widetilde{N}(ds, d\eta) = N(ds, d\eta) - \lambda(d\eta)ds$ and $N(ds, d\eta)$. Let us assume that the filtration $\mathfrak{F}_t = \sigma\{N((0,s], A);\ s \le t,\ A \in \mathscr{B}(Z)\} \vee \mathscr{N},\ t \in J$, produced by $q(\cdot)$ Poisson point process and is augmented, where $\mathscr{N}$ is the class of P-null sets.

Define $C^{\nu,\mu}(J, L_2(\Omega; H)) = \{x \in C((0,b], L_2(\Omega; H));\ \lim_{t\to 0^+} t^{(1-\nu)(1-\mu)}x(t)$ exists and its finite$\}$ and let

$$\|x\|^2_{\nu,\mu} = \sup_{0<t\le b} \left\| t^{(1-\nu)(1-\mu)} x(t) \right\|^2. \tag{2}$$

Obviously, $C^{\nu,\mu}(J, L_2(\Omega; H))$ is a Banach space.

Definition 1. The fractional integral of order $\alpha > 0$ with the lower limit a for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a,\ \alpha > 0, \tag{3}$$

provided that the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the Gamma function.

Definition 2 (see [5]). The Hilfer fractional derivative of order $0 \le \nu \le 1$ and $0 < \mu < 1$ with lower limit a is defined as

$$D^{\nu,\mu}_{a^+} f(t) = I^{\nu(1-\mu)}_{a^+} \frac{d}{dt} I^{(1-\nu)(1-\mu)}_{a^+} f(t) \tag{4}$$

for functions such that the expression on the right-hand side exists.

For more details about the Caputo and Riemann-Liouville fractional derivatives, the reader may refer to [22].

Remark 3. When $\nu = 0$, $0 < \mu < 1$, the Hilfer fractional derivative coincides with classical Riemann-Liouville fractional derivative:

$$D^{0,\mu}_{a^+} f(t) = \frac{d}{dt} I^{1-\mu}_{a^+} f(t) = {}^{L}D^{\mu}_{a^+} f(t). \tag{5}$$

When $\nu = 1, 0 < \mu < 1$, the Hilfer fractional derivative coincides with classical Caputo fractional derivative:

$$D^{1,\mu}_{a^+} f(t) = I^{1-\mu}_{a^+} \frac{d}{dt} f(t) = {}^{C}D^{\mu}_{a^+} f(t). \tag{6}$$

For $x \in H$, let us define the operators $\{S_{\nu,\mu}(t) : t \ge 0\}$ and $\{P_{\mu}(t) : t \ge 0\}$ by

$$\begin{aligned} S_{\nu,\mu}(t) &= I^{\nu(1-\mu)}_{0^+} P_{\mu}(t), \\ P_{\mu}(t) &= t^{\mu-1} T_{\mu}(t), \\ T_{\mu}(t) &= \int_0^{\infty} \mu\theta\Psi_{\mu}(\theta) T(t^{\mu}\theta)\, d\theta, \end{aligned} \tag{7}$$

where $\Psi_{\mu}(\theta) = \sum_{n=1}^{\infty}((-\theta)^{n-1}/(n-1)!\Gamma(1-n\mu))\sin(n\pi\alpha), \theta \in (0,\infty)$, is a function of Wright-type defined on $(0, \infty)$ and verifies $\Psi_{\alpha}(\theta) \ge 0$, $\int_0^{\infty}\Psi_{\alpha}(\theta)d\theta = 1$, $\int_0^{\infty}\theta^{\xi}\Psi_{\mu}(\theta)d\theta = \Gamma(1+\xi)/\Gamma(1+\mu\xi)$, $\xi \in (-1, \infty)$, and $\|T(t)\|^2 \le M$.

Lemma 4 (see [20]). *The operators $S_{\nu,\mu}$ and P_{μ} have the following properties:*

(i) *For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_{\mu}(t)$ are bounded and linear operators, and $\|P_{\mu}(t)x\|^2 \le (Mt^{2(\mu-1)}/(\Gamma(\mu))^2)\|x\|^2$ and $\|S_{\nu,\mu}(t)x\|^2 \le (Mt^{2(\nu-1)(1-\mu)}/(\Gamma(\nu(1-\mu)+\mu))^2)\|x\|^2$.*

(ii) *$\{P_{\mu}(t) : t > 0\}$ is compact, if $\{T(t) : t > 0\}$ is compact.*

Definition 5 (see [19, 20]). An H-valued stochastic process $\{x(t) \in C(J', L_2(\Omega; H))\}$ is a mild solution of system (1) if the process x satisfies the following integral equation:

$$\begin{aligned} x(t) &= S_{\nu,\mu}(t)x_0 + \int_0^t P_{\mu}(t-s) \\ &\quad \cdot \left[f(s, x(s)) + \int_0^s g(\tau, x(\tau))\, dW(\tau) \right] ds \\ &\quad + \int_0^t \int_Z P_{\mu}(t-s) h(s, x(s), \eta) \widetilde{N}(ds, d\eta), \\ &\qquad \forall t \in J'. \end{aligned} \tag{8}$$

Remark 6. (i) $D^{\nu(1-\mu)}_{0^+} S_{\nu,\mu}(t) = P_{\mu}(t),\ t \in J'$.

(ii) When $\nu = 1$, the fractional stochastic equation (1) simplifies to the classical Caputo fractional equation which has been discussed by Chen and Li [25]. In this case, $S_{1,\mu}(t) = S_{\mu}(t),\ 0 \le t \le b$, where $S_{\mu}(t)$ is defined in [25].

We impose the following assumptions to show the main results:

(H_1) The maps $f : J \times H \to H$, $g : J \times H \to L_Q(K, H)$, and $h : J \times H \times Z \to H$ satisfy, for all $t \in J$ and $x_1, x_2 \in H$,

$$\begin{aligned} &\|f(t, x_1) - f(t, x_2)\|^2 \le \mathscr{K}\left(\|x_1 - x_2\|^2\right), \\ &\int_0^s \|g(\tau, x_1) - g(\tau, x_2)\|^2 d\tau \le \mathscr{K}\left(\|x_1 - x_2\|^2\right), \\ &\int_Z \|h(s, x_1, \eta) - h(s, x_2, \eta)\|^2 \lambda(d\eta) \\ &\quad \vee \left(\int_Z \|h(s, x_1, \eta) - h(t, x_2, \eta)\|^4 \lambda(d\eta) \right)^{1/2} \\ &\quad \le \mathscr{K}\left(\|x_1 - x_2\|^2\right), \\ &\left(\int_Z \|h(s, x_1, \eta)\|^4 \lambda(d\eta) \right)^{1/2} \le \mathscr{K}\left(\|x_1\|^2\right), \end{aligned} \tag{9}$$

where $\mathscr{K}(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\mathscr{K}(0) = 0$, $\mathscr{K}(\vartheta) > 0$ for $\vartheta > 0$ and $\int_{0^+}(d\vartheta/\mathscr{K}(\vartheta)) = +\infty$.

(H_2) For all $t \in J$, there exists a constant $M_0 > 0$ such that

$$\begin{aligned} &\|f(t, 0)\|^2 \vee \int_0^s \|g(\tau, 0)\|^2 d\tau \vee \int_Z \|h(t, 0, \eta)\|^2 \lambda(d\eta) \\ &\le M_0. \end{aligned} \tag{10}$$

The reader may refer to Remark 2.3 and Lemmas 2.4 and 2.5 in [30], which are useful to prove the main results.

Let the solution $x(t) \in C^{\nu,\mu}(J, L_2(\Omega; H))$ of (1) be defined as follows:

$$\lim_{t\to 0^+} t^{(1-\nu)(1-\mu)} S_{\nu,\mu}(t)\, x_0 = \frac{1}{\Gamma(\nu(1-\mu))\,\Gamma(\mu)} \int_0^1 (1-s)^{\nu(1-\mu)-1} s^{\mu-1} x_0 ds = \frac{x_0}{\Gamma((\nu(1-\mu)+\mu))},$$
$$t^{(1-\nu)(1-\mu)} x(t) = \begin{cases} \dfrac{x_0}{\Gamma((\nu(1-\mu)+\mu))}, & \text{for } t = 0, \\ t^{(1-\nu)(1-\mu)} S_{\nu,\mu}(t)\, x_0 + t^{(1-\nu)(1-\mu)} \displaystyle\int_0^t P_\mu(t-s) \left[f(s, x(s)) + \int_0^s g(\tau, x(\tau))\, dW(\tau) \right] ds + t^{(1-\nu)(1-\mu)} \int_0^t \int_Z P_\mu(t-s)\, h(s, x(s), \eta)\, \widetilde{N}(ds, d\eta), & \text{for } 0 < t \le b. \end{cases} \tag{11}$$

We refer to [25, 38, 39] for further discussion of stochastic concepts.

3. Existence and Uniqueness of Mild Solutions

In order to prove the existence of mild solution for system (1), let us consider the sequence of successive approximations defined as follows:

$$\begin{aligned} x^0(t) &= t^{(1-\nu)(1-\mu)} S_{\nu,\mu}(t)\, x_0, \quad 0 < t \le b, \\ x^n(t) &= \frac{x_0}{\Gamma((\nu(1-\mu)+\mu))}, \quad t = 0,\ n = 1, 2, \ldots, \\ x^n(t) &= t^{(1-\nu)(1-\mu)} S_{\nu,\mu}(t)\, x_0 + t^{(1-\nu)(1-\mu)} \int_0^t P_\mu(t-s) \\ &\quad \cdot \left[f\left(s, x^{n-1}(s)\right) + \int_0^s g\left(\tau, x^{n-1}(\tau)\right) dW(\tau) \right] ds \\ &\quad + t^{(1-\nu)(1-\mu)} \int_0^t \int_Z P_\mu(t-s)\, h\left(s, x^{n-1}(s), \eta\right) \\ &\quad \cdot \widetilde{N}(ds, d\eta), \quad 0 < t \le b,\ n = 1, 2, \ldots. \end{aligned} \tag{12}$$

Theorem 7. *If the assumptions (H_1)-(H_2) are satisfied, then system (1) has a unique mild solution in the space $C^{\nu,\mu}(J, L_2(\Omega; H))$, provided that $(3M/(\Gamma(\mu))^2) t^{2(1-\nu)(1-\mu)}(b + \mathrm{Tr}(Q) + 2C) < 1$, with $1/2 < \mu < 1$ and $t \in J$.*

Proof. For better readability, we break the proof into a sequence of steps.

Step 1. For all $t \in J$, the sequence $x^n(t)$, $n \ge 1$, is bounded.

It is obvious that $x^0(t) \in C^{\nu,\mu}(J, L_2(\Omega; H))$. Let x^0 be a fixed initial approximation to (12). Let us use the assumptions (H_1) and (H_2), Holder inequality, Doob Martingale inequality and Burkholder-Davis-Gundy inequality for pure jump stochastic integral in H ([30]). We have

$$\begin{aligned} E\left\|x^n(t)\right\|^2 &\le \frac{4M\left\|x_0\right\|^2}{(\Gamma(\nu(1-\mu)+\mu))^2} + \frac{4Mbt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2} \\ &\quad \cdot \int_0^t (t-s)^{2(\mu-1)} E\left\|f\left(s, x^{n-1}(s)\right)\right\|^2 ds \\ &\quad + \frac{4M\mathrm{Tr}(Q)\, t^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2} \int_0^t (t-s)^{2(\mu-1)} \\ &\quad \cdot \left(\int_0^s E\left\|g\left(\tau, x^{n-1}(\tau)\right)\right\|^2 d\tau \right) ds \\ &\quad + \frac{4MCt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2} \left[\int_0^t (t-s)^{2(\mu-1)} \right. \\ &\quad \cdot \int_Z E\left\|h\left(s, x^{n-1}(s), \eta\right)\right\|^2 \lambda(d\eta)\, ds + \int_0^t (t \\ &\quad - s)^{2(\mu-1)} \\ &\quad \left. \cdot \left(\int_Z E\left\|h\left(s, x^{n-1}(s), \eta\right)\right\|^4 \lambda(d\eta) \right)^{1/2} ds \right] \\ &\le \frac{4M\left\|x_0\right\|^2}{(\Gamma(\nu(1-\mu)+\mu))^2} + \frac{4Mbt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2} \int_0^t (t \\ &\quad - s)^{2(\mu-1)} \left[E\left\|f\left(s, x^{n-1}(s)\right) - f(s,0)\right\|^2 \right. \\ &\quad \left. + E\left\|f(s,0)\right\|^2 \right] ds + \frac{4M\mathrm{Tr}(Q)\, t^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2} \\ &\quad \cdot \int_0^t (t-s)^{2(\mu-1)} \\ &\quad \cdot \left(\int_0^s E\left\|g\left(\tau, x^{n-1}(\tau)\right) - g(\tau,0)\right\|^2 d\tau \right. \\ &\quad \left. + \int_0^s E\left\|g(\tau,0)\right\|^2 d\tau \right) ds \\ &\quad + \frac{4MCt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2} \left[\int_0^t (t-s)^{2(\mu-1)} \right. \\ &\quad \cdot \left(\int_Z E\left\|h\left(s, x^{n-1}(s), \eta\right) - h(s,0,\eta)\right\|^2 \right. \\ &\quad \left. + E\left\|h(s,0,\eta)\right\|^2 \right) \lambda(d\eta)\, ds + \int_0^t (t-s)^{2(\mu-1)} \\ &\quad \left. \cdot \left(\int_Z E\left\|h\left(s, x^{n-1}(s), \eta\right)\right\|^4 \lambda(d\eta) \right)^{1/2} ds \right] \end{aligned}$$

$$\leq \frac{4M\|x_0\|^2}{(\Gamma(\nu(1-\mu)+\mu))^2} + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + C) M_0 \frac{b^{2\mu-1}}{2\mu-1} + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) \int_0^t (t-s)^{2(\mu-1)} \cdot \mathscr{K}\left(E\left\|x^{n-1}(s)\right\|^2\right) ds, \tag{13}$$

where $M_1 = 4M\|x_0\|^2/(\Gamma(\nu(1-\mu)+\mu))^2 + (4Mt^{2(1-\nu)(1-\mu)}/(\Gamma(\mu))^2)(b + \operatorname{Tr}(Q) + C)M_0(b^{2\mu-1}/(2\mu-1))$ and $C > 0$ is constant. Here, $\mathscr{K}(\cdot)$ is concave and $\mathscr{K}(0) = 0$, and one can find a pair of positive constants a_1 and a_2 such that $\mathscr{K}(t) \leq a_1 + a_2 t$, for $t \geq 0$. Then

$$E\left\|x^n(t)\right\|^2 \leq M_1 + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) \cdot a_1 \frac{b^{2\mu-1}}{2\mu-1} + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) \cdot a_2 \int_0^t (t-s)^{2(\mu-1)} E\left\|x^{n-1}(s)\right\|^2 ds \leq M_2 + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) \cdot a_2 \int_0^t (t-s)^{2(\mu-1)} E\left\|x^{n-1}(s)\right\|^2 ds, \tag{14}$$

where

$$M_2 = M_1 + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) a_1 \frac{b^{2\mu-1}}{2\mu-1}. \tag{15}$$

For any $k \geq 1$,

$$\max_{1\leq n\leq k} E \sup_{0\leq s\leq t}\left\|x^{n-1}(s)\right\|^2 \leq E\left\|x^0(s)\right\|^2 + \max_{1\leq n\leq k} E \cdot \sup_{0\leq s\leq t}\left\|x^n(s)\right\|^2,$$
$$\max_{1\leq n\leq k} E \sup_{0\leq s\leq t}\left\|x^n(s)\right\|^2 \leq M_2 + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) \cdot a_2 \int_0^t (t-s)^{2(\mu-1)} E\left\|x^0(s)\right\|^2 ds + \frac{4Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^2}(b + \operatorname{Tr}(Q) + 2C) \cdot a_2 \int_0^t (t-s)^{2(\mu-1)} \max_{1\leq n\leq k} E \sup_{0\leq r\leq s}\left\|x^n(r)\right\|^2 ds \leq M_3 + M_4 \int_0^t (t-s)^{2(\mu-1)} E\left\|x^n(s)\right\|^2 ds \leq M_3 \cdot \exp\left(\frac{M_4 b^{2\mu-1}}{2\mu-1}\right), \tag{16}$$

where $M_3 = M_2 + (4Mt^{2(1-\nu)(1-\mu)}/(\Gamma(\mu))^2)(b + \operatorname{Tr}(Q) + 2C)a_2 \int_0^t (t-s)^{2(\mu-1)} E\|x^0(s)\|^2 ds < \infty$ and $M_4 = (4Mt^{2(1-\nu)(1-\mu)}/(\Gamma(\mu))^2)(b + \operatorname{Tr}(Q) + 2C)a_2$. Thus $E\|x^n(t)\|^2 < \infty$, for $n \geq 1$, $t \in J$, which shows that the sequence $x^n(t)$, $n \geq 1$, is bounded in $C^{\nu,\mu}(J, L_2(\Omega; H))$.

Step 2. Sequence $x^n(t)$, $n \geq 1$, is a Cauchy sequence.

From (12), for all $n \geq 1$ and $0 \leq t \leq b$,

$$E\left\|x^{n+1}(t) - x^n(t)\right\|^2 = E\left\|t^{(1-\nu)(1-\mu)} \int_0^t P_\mu(t-s) \cdot \left[\left(f\left(s, x^n(s)\right) - f\left(s, x^{n-1}(s)\right)\right) + \left(\int_0^s g\left(\tau, x^n(\tau)\right) dW(\tau) - \int_0^s g\left(\tau, x^{n-1}(\tau)\right) dW(\tau)\right)\right] ds + \int_0^t \int_Z P_\mu(t-s)\left(h\left(s, x^n(s), \eta\right) - h\left(s, x^{n-1}(s), \eta\right)\right) \cdot \widetilde{N}(ds, d\eta)\right\|^2 \leq \frac{3M}{(\Gamma(\mu))^2} t^{2(1-\nu)(1-\mu)}(b + \operatorname{Tr}(Q) + 2C) \int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(E\left\|x^n(s) - x^{n-1}(s)\right\|^2\right) ds. \tag{17}$$

Let $\Phi_n(t) = \sup_{t\in[0,b]} E\|x^{n+1}(t) - x^n(t)\|^2$. Thus, we have in the above inequality that

$$\Phi_n(t) \leq \frac{3M}{(\Gamma(\mu))^2} t^{2(1-\nu)(1-\mu)}(b + \operatorname{Tr}(Q) + 2C) \cdot \int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(E\left\|x^n(s) - x^{n-1}(s)\right\|^2\right) ds,$$
$$\Phi_n(t) \leq \frac{3M}{(\Gamma(\mu))^2} t^{2(1-\nu)(1-\mu)}(b + \operatorname{Tr}(Q) + 2C) \cdot \int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(\Phi_{n-1}(s)\right) ds, \quad 0 \leq t \leq b. \tag{18}$$

Choose $b_1 \in [0, b]$ such that

$$M_5 \int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(\Phi_{n-1}(s)\right) ds \leq M_5 \int_0^t (t-s)^{2(\mu-1)} \Phi_{n-1}(s) ds, \quad 0 \leq t \leq b_1, \; n \geq 1. \tag{19}$$

Moreover,

$$E\left\|x^{1}(t)-x^{0}(t)\right\|^{2} \leq \frac{3M}{(\Gamma(\mu))^{2}} \cdot t^{2(1-\nu)(1-\mu)}(b+\operatorname{Tr}(Q)+2C) \cdot \int_{0}^{t}(t-s)^{2(\mu-1)} \mathscr{K}\left(E\left\|x^{0}(s)\right\|^{2}\right) ds. \quad (20)$$

We take the supreme over t and use Φ_n:

$$\Phi_{0}(t)=\sup_{t\in[0,b]} E\left\|x^{1}(t)-x^{0}(t)\right\|^{2} \leq \frac{3M}{(\Gamma(\mu))^{2}} \cdot t^{2(1-\nu)(1-\mu)}(b+\operatorname{Tr}(Q)+2C) \cdot \int_{0}^{t}(t-s)^{2(\mu-1)} \mathscr{K}\left(E\left\|x^{0}(s)\right\|^{2}\right) ds=C_{1}. \quad (21)$$

Now, for $n = 1$ in (18), we have

$$\begin{aligned}\Phi_{1}(t) &\leq \frac{3M}{(\Gamma(\mu))^{2}} t^{2(1-\nu)(1-\mu)}(b+\operatorname{Tr}(Q)+2C) \cdot \int_{0}^{t}(t-s)^{2(\mu-1)} \mathscr{K}\left(\Phi_{0}(s)\right) ds \\ &\leq M_{5} \int_{0}^{t}(t-s)^{2(\mu-1)} \mathscr{K}\left(\Phi_{0}(s)\right) ds \\ &\leq M_{5} \int_{0}^{t}(t-s)^{2(\mu-1)} \Phi_{0}(s) ds \\ &\leq M_{5} \int_{0}^{t}(t-s)^{2(\mu-1)} C_{1} ds \leq M_{5} C_{1} \frac{b^{2(\mu-1)+1}}{2(\mu-1)+1}.\end{aligned} \quad (22)$$

And, for $n = 2$ in (18), we have

$$\begin{aligned}\Phi_{2}(t) &\leq M_{5} \int_{0}^{t}(t-s)^{2(\mu-1)} \mathscr{K}\left(\Phi_{1}(s)\right) ds \\ &\leq M_{5} \int_{0}^{t}(t-s)^{2(\mu-1)} \Phi_{1}(s) ds \\ &\leq M_{5} \int_{0}^{t} M_{5} C_{1} \frac{b^{2(\mu-1)+1}}{2(\mu-1)+1} ds \\ &\leq C_{1} \frac{b^{2(\mu-1)}}{2(\mu-1)+1}\left(M_{5}\right)^{2} \frac{b^{2}}{2!}.\end{aligned} \quad (23)$$

By applying mathematical induction in (18) and with the above work, we have

$$\Phi_{n}(t) \leq C_{1} \frac{b^{2(\mu-1)}}{2(\mu-1)+1}\left(M_{5}\right)^{n} \frac{b^{n}}{n!}, \quad n \geq 1,\ t \in\left[0, b_{1}\right]. \quad (24)$$

So, for any $m \geq n \geq 0$,

$$\begin{aligned}&\sup_{t\in[0,b_1]} E\left\|x^{m}(t)-x^{n}(t)\right\|^{2} \\ &\leq \sum_{r=n}^{+\infty} \sup_{t\in[0,b_1]} E\left\|x^{r+1}(t)-x^{r}(t)\right\|^{2} \\ &\leq \sum_{r=n}^{+\infty} C_{1} \frac{b^{2(\mu-1)}}{2(\mu-1)+1}\left(M_{5}\right)^{r} \frac{b^{r}}{r!} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \quad (25)$$

Step 3. The existence and uniqueness of solution for system (1) are tackled as follows.

The Borel-Cantelli Lemma says that $x^n(t) \to x(t)$, as $n \to \infty$ uniformly for $0 \leq t \leq b$. Thus, for all $t \in J$, taking limits on both sides of (12), one can obtain that $x(t)$ is a solution to (1). Next, to show the uniqueness, let $x_1, x_2 \in C^{\nu,\mu}(J, L_2(\Omega; H))$ be two solutions on $t \in J$. For $t \in J$,

$$\begin{aligned}&E\left\|x_{1}(t)-x_{2}(t)\right\|^{2} \\ &\leq \frac{3Mt^{2(1-\nu)(1-\mu)}}{(\Gamma(\mu))^{2}}(b+\operatorname{Tr}(Q)+2C) \cdot \int_{0}^{t}(t-s)^{2(\mu-1)} \mathscr{K}\left(E\left\|x_{1}(s)-x_{2}(s)\right\|^{2}\right) ds.\end{aligned} \quad (26)$$

Thus, from Bihari inequality, it yields that

$$\sup_{t\in J} E\left\|x_{1}(t)-x_{2}(t)\right\|^{2}=0. \quad (27)$$

Therefore, $x_1(t) = x_2(t)$, for all $t \in J$. □

4. Optimal Control Results

Let Y be reflexive Banach space in which controls u take values. Let us denote a class of nonempty convex and closed subsets of Y by $2^Y \setminus \{\emptyset\}$. The multivalued function $\nu : J \to 2^Y \setminus \{\emptyset\}$ is measurable and $\nu(\cdot) \subset \mathscr{E}$, where $\mathscr{E}$ is a bounded set of Y. The admissible control set $U_{\text{ad}} = \{u(\cdot) \in L_2(\mathscr{E}) \mid u(t) \in \nu(t).\text{a.e}\}$. Then $U_{\text{ad}} \neq \emptyset$ and $U_{\text{ad}} \subset L_2(J, Y)$ is bounded, closed, and convex [35]. The fractional stochastic control problem with Hilfer fractional derivative is of the form

$$\begin{aligned}D_{0^{+}}^{\nu,\mu} x(t) &= Ax(t)+B(t)u(t)+f(t,x(t)) + \int_{0}^{t} g(s,x(s)) dW(s) + \int_{Z} h(t,x(t),\eta) \widetilde{N}(dt,d\eta), \quad t \in J', \\ I_{0^{+}}^{(1-\nu)(1-\mu)} x(0) &= x_{0}.\end{aligned} \quad (28)$$

By using Definition 5 for every $u \in U_{\text{ad}}$, there exists $b > 0$, and the solution of the control problem (28) is defined as

$$\begin{aligned} x(t) = {} & S_{\nu,\mu}(t)\, x_0 + \int_0^t P_\mu(t-s) \Big[B(s)\, u(s) \\ & + f(s, x(s)) + \int_0^s g(\tau, x(\tau))\, dW(\tau) \Big] ds \\ & + \int_0^t \int_Z P_\mu(t-s)\, h(s, x(s), \eta)\, \widetilde{N}(ds, d\eta), \\ & \forall t \in J. \end{aligned} \tag{29}$$

(H_3) The operator $B \in L_2(J, L(Y,H))$; $\|B\|_2$ denotes the norm of operator B in Banach space $L_2(J, L(Y,H))$. Obliviously, $Bu \in L_2(J,H)$ for every $u \in U_{\text{ad}}$.

Lemma 8. *Let (H_1)–(H_3) hold. If system (28) is mildly solvable on J with respect to $u \in U_{ad}$ and $1/2 < \mu < 1$, then there exists a constant $\rho > 0$ such that $E\|x(t)\|^2 \le \rho$ for all $t \in J$.*

Proof. If x is a mild solution of (28) with respect to $u \in U_{\text{ad}}$, then x satisfies equation (29). Using hypotheses (H_1)–(H_3), as well as Burkholder-Davis-Gundy inequality ([30]), we obtain

$$\begin{aligned} E\|x(t)\|^2 \le {} & 5E\left\|S_{\nu,\mu}(t)\, x_0\right\|^2 + 5b\int_0^t \left\|P_\mu(t-s)\right\|^2 \\ & \cdot \|B\|^2 \|u(s)\|^2 ds + 5b\int_0^t \left\|P_\mu(t-s)\right\|^2 \\ & \cdot E\left\|f(s,x(s))\right\|^2 ds + 5\operatorname{Tr}(Q)\int_0^t \left\|P_\mu(t-s)\right\|^2 \\ & \cdot \int_0^s E\left\|g(\tau, x(\tau))\right\|^2 d\tau\, ds + 5C\int_0^t \Big\|P_\mu(t \\ & - s)\Big\|^2 \left(\int_Z E\left\|h(s,x(s),\eta)\right\|^2 \lambda d(\eta)\, ds \right. \\ & \left. + \left(\int_Z E\left\|h(s,x(s),\eta)\right\|^4 \lambda d(\eta) \right)^{1/2} ds \right) \\ \le {} & \frac{5Mt^{2(\nu-1)(1-\mu)}}{(\Gamma(\nu(1-\mu)+\mu))^2}\left\|x_0\right\|^2 + \frac{5Mb\|B\|^2}{(\Gamma(\mu))^2}\int_0^t (t \\ & - s)^{2(\mu-1)}\|u(s)\|^2 ds + \frac{5Mb}{(\Gamma(\mu))^2}\int_0^t (t-s)^{2(\mu-1)} \\ & \cdot \left(E\left\|f(s,x(s)) - f(s,0)\right\|^2 + E\left\|f(s,0)\right\|^2 \right) ds \\ & + \frac{5M\operatorname{Tr}(Q)}{(\Gamma(\mu))^2}\int_0^t (t-s)^{2(\mu-1)} \\ & \cdot \left(\int_0^\tau E\left\|g(\tau,x(\tau)) - g(\tau,0)\right\|^2 d\tau \right. \\ & \left. + \int_0^\tau E\left\|g(\tau,0)\right\|^2 d\tau \right) ds + \frac{5MC}{(\Gamma(\mu))^2}\left\{ \int_0^t (t \right. \\ & - s)^{2(\mu-1)}\left(\int_Z E\left\|h(s,x(s),\eta) - h(s,0,\eta)\right\|^2 \right. \\ & \left.\left. + E\left\|h(s,0,\eta)\right\|^2 \right) \lambda d(\eta)\, ds \right\} + \frac{5MC}{(\Gamma(\mu))^2}\int_0^t (t \\ & - s)^{2(\mu-1)}\left(\int_Z E\left\|h(s,x(s),\eta)\right\|^4 \lambda d(\eta) \right)^{1/2} ds \\ \le {} & \frac{5Mb^{2(\nu-1)(1-\mu)}}{(\Gamma(\nu(1-\mu)+\mu))^2}\left\|x_0\right\|^2 + \frac{5Mb\|B\|^2}{(\Gamma(\mu))^2}\left(\int_0^t (t \right. \\ & \left. - s)^{2(\mu-1)(p/(p-1))} ds \right)^{(p-1)/p} \\ & \cdot \left(\int_0^t \|u(s)\|^{2p} ds \right)^{1/p} + \frac{5MM_0(b + \operatorname{Tr}(Q) + C)}{(\Gamma(\mu))^2} \\ & \cdot \frac{b^{2\mu-1}}{2\mu-1} + \frac{5M(b + \operatorname{Tr}(Q) + 2C)}{(\Gamma(\mu))^2}\int_0^t (t-s)^{2(\mu-1)} \\ & \cdot \mathscr{K}\left(E\|x(s)\|^2 \right) ds, \end{aligned} \tag{30}$$

where $M_6 = (5Mb^{2(\nu-1)(1-\mu)}/(\Gamma(\nu(1-\mu)+\mu))^2)\|x_0\|^2 + (5Mb\|B\|^2/(\Gamma(\mu))^2)(b^{(2\mu p-p-1)/(p-1)}/((2\mu p - p - 1)/(p-1)))^{(p-1)/p}\|u\|^2_{L_p(J,Y)} + (5MM_0(b+\operatorname{Tr}(Q)+C)/(\Gamma(\mu))^2)(b^{2\mu-1}/(2\mu-1))$ and $M_7 = 5M(b + \operatorname{Tr}(Q) + 2C)/(\Gamma(\mu))^2$. Given that $\mathscr{K}(\cdot)$ is concave and $\mathscr{K}(0) = 0$, one can find a pair of positive constants a_1 and a_2 such that $\mathscr{K}(t) \le a_1 + a_2 t$, for $t \ge 0$. Then

$$\begin{aligned} E\|x(t)\|^2 \le {} & M_6 + M_7 a_1 \frac{b^{2\mu-1}}{2\mu-1} \\ & + M_7 a_2 \int_0^t (t-s)^{2(\mu-1)} E\|x(s)\|^2 ds. \end{aligned} \tag{31}$$

By using Gronwall's inequality,

$$\begin{aligned} & E\|x(t)\|^2 \\ & \le \left(M_6 + M_7 a_1 \frac{b^{2\mu-1}}{2\mu-1} \right) \exp\left(M_7 a_2 \frac{b^{2\mu-1}}{2\mu-1} \right) = \rho \\ & < \infty. \end{aligned} \tag{32}$$

□

Theorem 9 (see [35]). *Under hypotheses (H_1)–(H_3) and for each $u \in U_{ad}$, system (28) is mildly solvable on J and the solution is unique.*

Minimize a performance index of the following form:

$$\mathfrak{J}(x,u) = \int_0^b \mathscr{L}(t, x^u(t), u(t))\, dt, \tag{33}$$

among all the admissible state control pairs of system (28); that is, find an admissible state control pair $(x^0, u^0) \in C(J, L_2(\Omega; H)) \times U_{ad}$ such that

$$\mathfrak{J}\left(x^0, u^0\right) \leq \mathfrak{J}(x, u) \quad \forall u \in U_{ad}; \tag{34}$$

here $x^u(t)$ defines the mild solution of (28) corresponding to $u \in U_{ad}$. Assume that

(H_4) the cost functional $\mathscr{L} : J \times H \times Y \to \mathbb{R} \cup \{+\infty\}$ is such that

(i) $(t, x, u) \to \mathscr{L}(t, x, u)$ is measurable,

(ii) $\mathscr{L}(t, \cdot, \cdot)$ is lower semicontinuous on $H \times Y$ for almost all $t \in J$,

(iii) $\mathscr{L}(t, x, \cdot)$ is convex on Y for all $x \in H$ and almost all $t \in J$,

(iv) there exist constants $d \geq 0$, $j > 0$, $\rho_1 \geq 0$, and $\rho_1 \in L_1(J, \mathbb{R})$ such that

$$\mathscr{L}(t, x(t), u(t)) \geq \rho_1(t) + d\|x\|_H + j\|u\|_Y^2. \tag{35}$$

Theorem 10. *If B is strongly continuous operator, hypotheses (H_1)–(H_4) and Theorem 9 hold. Then the stochastic control problem (28) permits at least one optimal control pair.*

Proof. The main aim is to minimize the value of $\mathfrak{J}(x, u)$. If $\inf_{(x,u)\in\mathscr{A}_{ad}} \mathfrak{J}(x, u) = +\infty$, ($\mathscr{A}_{ad} = \{(x, u)$ such that x is a mild solution of (28) with $u \in U_{ad}\}$); then there is nothing to prove. Assume that $\inf_{(x,u)\in\mathscr{A}_{ad}} \mathfrak{J}(x, u) = \epsilon < \infty$. Using (H_4), we have $\epsilon > -\infty$. By definition of infimum, there exists a minimizing sequence feasible pair $\{(x_n, u_n)\}_{n\geq 1} \subset \mathscr{A}_{ad}$, such that $\mathfrak{J}(x_n, u_n) \to \epsilon$ as $n \to +\infty$. Since $u_n \in U_{ad}$, $\{u_n\}_{n\geq 1} \subset L_2(J, Y)$ is bounded. Thus, there exists $\widehat{u} \in L_2(J, Y)$ and a subsequence extracted from (u_n) (still called (u_n)) such that $u_n \rightharpoonup \widehat{u}$ weakly in $L_2(J, Y)$. Moreover, from the convexity and closeness of U_{ad} and Mazur's Theorem, we know that $\widehat{u} \in U_{ad}$. Suppose that x_n and $\widehat{x}$ are the mild solutions of (28) corresponding to u_n and $\widehat{u}$, respectively. x_n and $\widehat{x}$ satisfy the following equations, respectively:

$$\begin{aligned} x_n(t) &= S_{\nu,\mu}(t) x_0 + \int_0^t P_\mu(t-s)\left[B(s) u_n(s)\right. \\ &\quad \left. + f(s, x_n(s)) + \int_0^s g(\tau, x_n(\tau))\, dW(\tau)\right] ds \\ &\quad + \int_0^t \int_Z P_\mu(t-s) h(s, x_n(s), \eta) \widetilde{N}(ds, d\eta), \end{aligned} \tag{36}$$

$$\begin{aligned} \widehat{x}(t) &= S_{\nu,\mu}(t) x_0 + \int_0^t P_\mu(t-s)\left[B(s) \widehat{u}(s)\right. \\ &\quad \left. + f(s, \widehat{x}(s)) + \int_0^s g(\tau, \widehat{x}(\tau))\, dW(\tau)\right] ds \\ &\quad + \int_0^t \int_Z P_\mu(t-s) h(s, \widehat{x}(s), \eta) \widetilde{N}(ds, d\eta). \end{aligned} \tag{37}$$

From the boundedness of u_n and $\widehat{u}$, Lemma 8, one can verify that there exists a positive number ρ such that $\|x_n\|, \|\widehat{x}\| \leq \rho$. Then, for $t \in J$, $(p+1)/2p < \mu < 1$.

$$\begin{aligned} E\|x_n(t) - \widehat{x}(t)\|^2 &= E\Bigg\|\int_0^t P_\mu(t-s) \\ &\quad \cdot \Big[\left(f(s, x_n(s)) - f(s, \widehat{x}(s))\right) + \left(B(s) u_n(s) - B(s) \widehat{u}(s)\right) \\ &\quad + \left(\int_0^s g(\tau, x_n(\tau))\, dW(\tau) - \int_0^s g(\tau, \widehat{x}(\tau))\, dW(\tau)\right)\Big] ds \\ &\quad + \int_0^t \int_Z P_\mu(t-s)\left(h(s, x_n(s), \eta) - h(s, \widehat{x}(s), \eta)\right) \\ &\quad \cdot \widetilde{N}(ds, d\eta)\Bigg\|^2 \leq \frac{4M}{(\Gamma(\mu))^2}(b + \mathrm{Tr}(Q) + 2C)\int_0^t (t-s)^{2(\mu-1)} \\ &\quad \cdot \mathscr{K}\left(E\|x_n(s) - \widehat{x}(s)\|^2\right) ds + \frac{4Mb}{(\Gamma(\mu))^2}\int_0^t (t-s)^{2(\mu-1)} \\ &\quad \cdot E\|B(s) u_n(s) - B(s) \widehat{u}(s)\|^2 ds \leq \frac{4M}{(\Gamma(\mu))^2}(b \\ &\quad + \mathrm{Tr}(Q) + 2C)\int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(E\|x_n(s) - \widehat{x}(s)\|^2\right) ds \\ &\quad + \frac{4Mb}{(\Gamma(\mu))^2}\left(\int_0^t (t-s)^{2(\mu-1)(p/(p-1))} ds\right)^{(p-1)/p}\left(\int_0^t E\|B(s)\right. \\ &\quad \left. \cdot u_n(s) - B(s) \widehat{u}(s)\|^{2p} ds\right)^{1/p} \leq \frac{4M}{(\Gamma(\mu))^2}(b + \mathrm{Tr}(Q) \\ &\quad + 2C)\int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(E\|x_n(s) - \widehat{x}(s)\|^2\right) ds \\ &\quad + \frac{4Mb}{(\Gamma(\mu))^2}\left((p-1)\frac{b^{(2\mu p-p-1)/(p-1)}}{2\mu p - p - 1}\right)^{(p-1)/p}\left(\int_0^b E\|B(s)\right. \\ &\quad \left. \cdot u_n(s) - B(s) \widehat{u}(s)\|^{2p} ds\right)^{1/p}. \end{aligned} \tag{38}$$

Using the continuous operator B and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\frac{4M}{(\Gamma(\mu))^2}(b + \mathrm{Tr}(Q) + 2C) \\ &\quad \cdot \int_0^t (t-s)^{2(\mu-1)} \mathscr{K}\left(E\|x_n(s) - \widehat{x}(s)\|^2\right) ds \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty, \\ &\frac{4Mb}{(\Gamma(\mu))^2}\left((p-1)\frac{b^{(2\mu p-p-1)/(p-1)}}{2\mu p - p - 1}\right)^{(p-1)/p} \\ &\quad \cdot \left(\int_0^b E\|B(s) u_n(s) - B(s) \widehat{u}(s)\|^{2p} ds\right)^{1/p} \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty. \end{aligned} \tag{39}$$

Consequently, $E\|x_n(t)-\hat{x}(t)\|^2 \to 0$ as $n\to\infty$. Thus, by (H_4) and Balder's theorem [40], we obtain

$$\mathfrak{J}(\hat{x},\hat{u}) \le \lim_{n\to\infty}\mathfrak{J}(x_n,u_n) = m. \tag{40}$$

This shows that $\mathfrak{J}$ attains its minimum at $\hat{u}\in U_{\mathrm{ad}}$. □

5. Example

In this section, we provide an example to verify the theoretical results. Consider the control problem

$$\begin{aligned}
&D_{0^+}^{\nu,3/4}y(t,x)\\
&\quad=\frac{\partial^2}{\partial x^2}y(t,x)+B(t)u(t)x\\
&\qquad+\frac{e^{-t}y(t,x)}{(1+e^t)(1+y(t,x))}+\int_0^t\frac{\sin y(t,x)}{t^{1/3}}d\beta(s)\\
&\qquad+\int_Z\left(1+e^{-t}\right)\cos y(t,x)\,\eta\widetilde{N}(dt,d\eta),\\
&\qquad\qquad 0\le x\le\pi,\ u\in U_{\mathrm{ad}},\\
&y(t,0)=y(t,\pi)=0,\quad t>0,\\
&I_{0^+}^{(1-\nu)(1/4)}y(0)=y_0,\quad 0<x<\pi,\ 0\le t\le b.
\end{aligned}\tag{41}$$

Here, $D_{0^+}^{\nu,3/4}$ is the Hilfer fractional derivative, $0\le\nu\le1$, $\mu=3/4$, $b>0$. Let $\beta(t)$ denote a standard one-dimensional Wiener process in $H=L_2([0,\pi])$ defined on $(\Omega,\mathfrak{F},P)$. The operator $A: H\to H$ is defined by $Ay=y''$ with the domain $D(A)=\{y\in H: y, y'$ absolutely continuous, $y''\in H$, $y(0)=y(\pi)=0\}$. Then $Ay=\sum_{n=1}^{\infty}-n^2\langle y,y_n\rangle y_n$, $y\in D(A)$, where $y_n(x)=\sqrt{2/\pi}\sin(nx)$, $n\in\mathbb{N}$, is the orthogonal set of eigenvectors of A. It is well known that A generates a compact semigroup $(T(t))_{t\ge0}$ in H and is given by $T(t)y=\sum_{n=1}^{\infty}e^{-n^2t}\langle y,y_n\rangle y_n$, for $y\in H$. Moreover, for any $y\in H$, we have

$$\begin{aligned}
&\mathscr{T}_{3/4}(t)=\frac{3}{4}\int_0^\infty\theta\Psi_{3/4}(\theta)T\left(t^{3/4}\theta\right)d\theta,\\
&\mathscr{T}_{3/4}(t)y\\
&\quad=\frac{3}{4}\sum_{n=1}^{\infty}\int_0^\infty\theta\Psi_{3/4}(\theta)\exp\left(-n^2t^{3/4}\theta\right)d\theta\,\langle y,y_n\rangle\,y_n.
\end{aligned}\tag{42}$$

The Poisson point process $\{q(t); t\in J\}$ induced the Poisson counting measure $N(ds,d\eta)$ and the compensating martingale measure defined as

$$\widetilde{N}(ds,d\eta)=N(ds,d\eta)-\lambda(d\eta)\,ds. \tag{43}$$

The nonlinear functions $f: J\times H\to H$, $g: J\times H\to L_Q(H)$, and $h: J\times H\to H$ are defined by

$$\begin{aligned}
f(y)(x)&=\frac{e^{-t}y(t,x)}{(1+e^t)(1+y(t,x))},\\
g(y)(x)&=\frac{\sin y(t,x)}{t^{1/3}},\\
h(y)(x)&=\left(1+e^{-t}\right)\cos y(t,x)
\end{aligned}\tag{44}$$

and assuming that $\int_Z\eta^2\lambda(d\eta)<\infty$, $\int_Z\eta^4\lambda(d\eta)<\infty$. Clearly, the functions f, g, and h satisfy the assumptions (H_1)-(H_2). If $B=0$, then problem (41) can be written as the form of (1). All the conditions stated in Theorem 7 are satisfied for system (41) and can be applied to ensure the existence and uniqueness of the mild solution of (41). The controls are the functions $u: Ty([0,\pi])\to\mathbb{R}$, such that $u\in L_2(Ty([0,\pi]))$. It means that $t\to u(\cdot,t)$ going from J into Y is measurable. Set $U(t)=\{u\in Y: \|u\|_Y\le\tau_1\}$, where $\tau_1\in L_2(J,\mathbb{R}^+)$. We restrict the admissible controls U_{ad} to be all $u\in L_2(Ty([0,\pi]))$ such that $\|u(\cdot,t)\|^2\le\tau_1(t)$ almost everywhere.

Let us define $B(t)u(t)x=\int_{[0,\pi]}k_1(x,\gamma)u(\gamma,t)d\gamma$ and make the following assumptions:

(i) k_1 is continuous.

(ii) $u\in L_2(J\times[0,\pi])$ and $\mathscr{L}: J\times H\times Y\to\mathbb{R}\cup\{+\infty\}$ is defined by

$$\begin{aligned}
&\mathscr{L}(t,y^u(t),u(t))\\
&\quad=\int_{[0,\pi]}\left(\|y(t)(x)\|^2+\|u(t)(x)\|^2\right)dx.
\end{aligned}\tag{45}$$

Then, system (41) can be written as in the form of (28). All the conditions stated in Theorem 10 are verified. Therefore, there exists an admissible control pair (y,u) such that the associated cost functional

$$\mathfrak{J}(y,u)=\int_0^b\mathscr{L}(t,y^u(t),u(t))\,dt \tag{46}$$

attains its minimum.

6. Concluding Remarks

In this paper, we studied the existence of solutions and optimal control results of fractional stochastic differential equations with Hilfer fractional derivative and Poisson jumps. The existence and uniqueness of mild solutions for the system have been obtained by using the successive approximation theory and stochastic analysis techniques. New sufficient conditions for optimal control results of fractional stochastic control system have been deduced. Throughout an example, the effectiveness of the obtained results is proven, under suitable conditions, for fractional stochastic partial differential equations with Poisson jumps.

The optimal control analysis for fractional stochastic differential inclusions with distributed delays, time varying delays, and impulsive effects will be our future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] A. Dzieliński, D. Sierociuk, and G. Sarwas, “Some applications of fractional order calculus,” *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 58, no. 4, pp. 583–592, 2010.

[2] F. A. Rihan, “Numerical modeling of fractional-order biological systems,” *Abstract and Applied Analysis*, vol. 2013, Article ID 816803, 2013.

[3] F. A. Rihan, S. Lakshmanan, A. H. Hashish, R. Rakkiyappan, and E. Ahmed, “Fractional-order delayed predator-prey systems with Holling type-II functional response,” *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 80, no. 1-2, pp. 777–789, 2015.

[4] A. M. El-Sayed, A. E. El-Mesiry, and H. A. El-Saka, “On the fractional-order logistic equation,” *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 20, no. 7, pp. 817–823, 2007.

[5] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[6] G. M. Zaslavsky, “Chaos, fractional kinetics, and anomalous transport,” *Physics Reports. A Review Section of Physics Letters*, vol. 371, no. 6, pp. 461–580, 2002.

[7] S. B. Yuste, L. Acedo, and K. Lindenberg, “Reaction front in an $A + \vec{B}C$ reaction-subdiffusion process,” *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 69, no. 3, Article ID 036126, 2004.

[8] H. Sheng, Y. Q. Chen, and T. S. Qiu, *Fractional Processes and Fractional-Order Signal Processing*, Springer, New York, NY, USA, 2012.

[9] J. A. Tenreiro Machado, “Entropy analysis of integer and fractional dynamical systems,” *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 62, no. 1-2, pp. 371–378, 2010.

[10] J. A. Tenreiro Machado and A. M. Galhano, “Fractional order inductive phenomena based on the skin effect,” *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 68, no. 1-2, pp. 107–115, 2012.

[11] Y. Ferdi, “Some applications of fractional order calculus to design digital filters for biomedical signal processing,” *Journal of Mechanics in Medicine and Biology*, vol. 12, no. 2, Article ID 12400088, 13 pages, 2012.

[12] W. C. Chen, “Nonlinear dynamics and chaos in a fractional-order financial system,” *Chaos, Solitons and Fractals*, vol. 36, no. 5, pp. 1305–1314, 2008.

[13] R. Caponetto, G. Dongola, and L. Fortuna, *Fractional Order Systems: Modeling and Control Applications*, World Scientific, London, UK, 2010.

[14] J. A. Tenreiro Machado, “Analysis and design of fractional-order digital control systems,” *Systems Analysis Modelling Simulation*, vol. 27, pp. 107–122, 1997.

[15] R. Hilfer, “Experimental evidence for fractional time evolution in glass forming materials,” *Chemical Physics*, vol. 284, no. 1-2, pp. 399–408, 2002.

[16] D. Sierociuk, A. Dzielinski, G. Sarwas, I. Petras, I. Podlubny, and T. Skovranek, “Modelling heat transfer in heterogeneous media using fractional calculus,” *Philosophical Transactions of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, vol. 371, no. 1990, Article ID 20120146, 20120146, 10 pages, 2013.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, “Preface,” *North-Holland Mathematics Studies*, vol. 204, no. C, pp. vii–x, 2006.

[18] T. Sandev, R. Metzler, and Z. Tomovski, “Fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative,” *Journal of Physics. A. Mathematical and Theoretical*, vol. 44, no. 25, Article ID 255203, 255203, 21 pages, 2011.

[19] N. I. Mahmudov and M. A. McKibben, “On the approximate controllability of fractional evolution equations with generalized Riemann-Liouville fractional derivative,” *Journal of Function Spaces*, Article ID 263823, Art. ID 263823, 9 pages, 2015.

[20] H. Gu and J. J. Trujillo, “Existence of mild solution for evolution equation with Hilfer fractional derivative,” *Applied Mathematics and Computation*, vol. 257, pp. 344–354, 2015.

[21] K. M. Furati, M. D. Kassim, and N. e. Tatar, “Existence and uniqueness for a problem involving Hilfer fractional derivative,” *Computers & Mathematics with Applications. An International Journal*, vol. 64, no. 6, pp. 1616–1626, 2012.

[22] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, UK, 2009.

[23] E. Gerolymatou, I. Vardoulakis, and R. Hilfer, “Modelling infiltration by means of a nonlinear fractional diffusion model,” *Journal of Physics D: Applied Physics*, vol. 39, no. 18, article no. 022, pp. 4104–4110, 2006.

[24] R. Hilfer, Y. Luchko, and Z. Tomovski, “Operational method for the solution of the fractional differential equations with generalized Riemann-Liouville fractional derivatives,” *Fractional Calculus and Applied Analysis*, vol. 12, pp. 299–318, 2009.

[25] P. Chen and Y. Li, “Nonlocal Cauchy problem for fractional stochastic evolution equations in Hilbert spaces,” *Collectanea Mathematica*, vol. 66, no. 1, pp. 63–76, 2015.

[26] J. Wang, “Approximate mild solutions of fractional stochastic evolution equations in Hilbert spaces,” *Applied Mathematics and Computation*, vol. 256, pp. 315–323, 2015.

[27] L. Lu and Z. Liu, “Existence and controllability results for stochastic fractional evolution hemivariational inequalities,” *Applied Mathematics and Computation*, vol. 268, pp. 1164–1176, 2015.

[28] R. C. Merton, “Option pricing when underlying stock returns are discontinuous,” *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 125–144, 1976.

[29] P. Jorion, “On jump processes in the foreign exchange and stock markets,” *Review of Financial Studies*, vol. 1, pp. 427–445, 1988.

[30] Y. Ren, Q. Zhou, and L. Chen, “Existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay,” *Journal of Optimization Theory and Applications*, vol. 149, no. 2, pp. 315–331, 2011.

[31] C. Rajivganthi and P. Muthukumar, “Almost automorphic solutions for fractional stochastic differential equations and its optimal control,” *Optimal Control Applications & Methods*, vol. 37, no. 4, pp. 663–681, 2016.

[32] S. Hosseinpour and A. Nazemi, "Solving fractional optimal control problems with fixed or free final states by Haar wavelet collocation method," *IMA Journal of Mathematical Control and Information*, vol. 33, no. 2, pp. 543–561, 2016.

[33] R. Kamocki and M. Majewski, "Fractional linear control systems with Caputo derivative and their optimization," *Optimal Control Applications & Methods*, vol. 36, no. 6, pp. 953–967, 2015.

[34] Z. Fan and G. Mophou, "Existence of optimal controls for a semilinear composite fractional relaxation equation," *Reports on Mathematical Physics*, vol. 73, no. 3, pp. 311–323, 2014.

[35] J. Wang, Y. Zhou, and M. Medved, "On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay," *Journal of Optimization Theory and Applications*, vol. 152, no. 1, pp. 31–50, 2012.

[36] J. Ren and J. Wu, "The optimal control problem associated with multi-valued stochastic differential equations with jumps," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 86, pp. 30–51, 2013.

[37] N. U. Ahmed, "Stochastic initial boundary value problems subject to distributed and boundary noise and their optimal control," *Journal of Mathematical Analysis and Applications*, vol. 421, no. 1, pp. 157–179, 2015.

[38] C. Rajivganthi, P. Muthukumar, and B. Ganesh Priya, "Successive approximation and optimal controls on fractional neutral stochastic differential equations with Poisson jumps," *Optimal Control Applications & Methods*, vol. 37, no. 4, pp. 627–640, 2016.

[39] Y. Ren, X. Cheng, and R. Sakthivel, "Impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm," *Applied Mathematics and Computation*, vol. 247, pp. 205–212, 2014.

[40] E. J. Balder, "Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functionals," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 11, no. 12, pp. 1399–1404, 1987.

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Research Article

Bifurcations and Synchronization of the Fractional-Order Bloch System

Xiaojun Liu,[1,2] Ling Hong,[2] Honggang Dang,[1] and Lixin Yang[1]

[1] *School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China*
[2] *State Key Laboratory for Strength and Vibration of Mechanical Structures, Xi'an Jiaotong University, Shaanxi, Xi'an 710049, China*

Correspondence should be addressed to Xiaojun Liu; flybett3952@126.com

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In this paper, bifurcations and synchronization of a fractional-order Bloch system are studied. Firstly, the bifurcations with the variation of every order and the system parameter for the system are discussed. The rich dynamics in the fractional-order Bloch system including chaos, period, limit cycles, period-doubling, and tangent bifurcations are found. Furthermore, based on the stability theory of fractional-order systems, the adaptive synchronization for the system with unknown parameters is realized by designing appropriate controllers. Numerical simulations are carried out to demonstrate the effectiveness and flexibility of the controllers.

1. Introduction

Nowadays, fractional calculus is a hot topic in the research field. It is well known that fractional calculus has an equally long history with classical calculus. It did not attract enough attention for the absence of geometrical interpretation and applications at the initial stage of development. As the development of technology and science continues, fractional calculus has been applied in many fields, such as control theory, dynamics, mathematics, mechanics, and physics [1–5].

As the research of fractional calculus moves along, many nonlinear systems with fractional orders are proposed and investigated. The chaos and bifurcations which are observed in integer-order systems are also found in fractional-order ones, such as fractional versions of Duffing system, Lorenz system, and Chen system [6–11]. It is well known that the Bloch system is very important for interpretation of the underlying physical process of nuclear magnetic resonance. Recently, the fractional-order Bloch equations with and without delay were studied [12–14]. Meanwhile, physical interest in the fractional-order Bloch equation has been growing [15, 16] with the goal of improving the modeling of relaxation, diffusion, and perfusion in biological tissues. In [17], for the fractional-order Bloch system, the chaotic dynamics including the chaotic attractors in different system parameters sets, bifurcations with the derivative order in commensurate-order case, were analyzed. Rich dynamics such as period-doubling and subharmonic cascade routes to chaos were found for the system in the commensurate-order case. Based on these results, we want to know the bifurcations of the fractional-order Bloch system with the variation of every order in incommensurate-order case as well as every system parameter.

Motivated by the above discussed, in this paper, the bifurcations with the derivative order in incommensurate-order case and system parameters are studied firstly. A series of period-doubling bifurcations and tangent bifurcations are obtained by numerical simulations. Meanwhile, different chaotic and periodic attractors are also observed. Furthermore, based on the stability theory of fractional-order systems, the adaptive synchronization of the fraction-order systems with uncertain parameters is realized by designing appropriate controllers. Numerical simulations are carried out to demonstrate the effectiveness and flexibility of the controllers.

The paper is organized as follows. In Section 2, the definitions for the fractional calculus and numerical algorithms for fractional differential equations are given. The bifurcations of the fractional-order Bloch system are investigated in detail in Section 3. In Section 4, the adaptive synchronization of the system is investigated. Numerical simulations are used to demonstrate the effectiveness of the controllers. Finally, we summarize the results in Section 5.

2. Fractional Derivatives

There are many definitions for the general fractional derivative. The three most frequently used ones are the Grunwald-Letnikov definition and the Riemann-Liouville and the Caputo definitions. It is well known that the initial conditions for the fractional differential equations with Caputo derivatives take on the same form as those for the integer-order ones, which is very suitable for practical problems. Therefore, we will use the Caputo definition for the fractional derivatives in this paper.

The Caputo fractional derivative is defined as follows:

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f^{(n)}(\tau)\, d\tau, \qquad n-1 < q < n. \tag{1}$$

As the initial conditions for the fractional differential equations with Caputo derivatives take on the same form as those for the integer-order ones, we will use the Caputo definition for the fractional derivatives in this paper.

In the following, we will give the definitions of commensurate-order and incommensurate-order fractional-order systems [18].

Definition 1. For a fractional-order system, which can be described by $D^{\mathbf{q}}\mathbf{x} = f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the state vector, $\mathbf{q} = (q_1, q_2, \ldots, q_n)^{\mathrm{T}}$ is the fractional derivative orders vector, and $q_i > 0$. The fractional-order system is commensurate-order when all the derivative orders satisfy $q_1 = q_2 = \cdots = q_n$; otherwise it is an incommensurate-order system.

Compared with the numerical algorithm for solving an ordinary differential equation, the numerical solution of a fractional differential equation is not easy to obtain. There are two approximation methods which can frequently be used for numerical computation on chaos and bifurcations with fractional differential equations. One is an improved version of Adams-Bashforth-Moulton algorithm based on the predictor-correctors scheme [19–21], which is a time domain approach. The other is a method, known as frequency domain approximation [22], based on numerical analysis of fractional-order systems in the frequency domain.

Simulation of fractional-order systems using the time domain methods is complicated and, due to long memory characteristics of these systems, requires a very long simulation time but on the other hand, it is more accurate [23]. Therefore, we employ the improved predictor-corrector algorithm for fractional-order differential equations in this paper.

3. A Fractional-Order Bloch System

The Bloch system is usually used to describe an ensemble of spins. The integer-order and fractional-order Bloch systems were studied in [17]. In this section, the bifurcations of the fractional-order Bloch system with the variation of different system parameters and derivative orders will be investigated.

The fractional-order Bloch system can be described as follows:

$$\begin{aligned} D^{q_1}x &= \delta y + \gamma z\left(x\sin(c) - y\cos(c)\right) - \frac{x}{\Gamma_2} \\ D^{q_2}y &= -\delta x - z + \gamma z\left(x\cos(c) + y\sin(c)\right) - \frac{y}{\Gamma_2} \\ D^{q_3}z &= y - \gamma\sin(c)\left(x^2 + y^2\right) - \frac{z-1}{\Gamma_1}, \end{aligned} \tag{2}$$

where x, y, z are the state variables, q_1, q_2, q_3 the derivative orders, and $\delta, \gamma, c, \Gamma_1, \Gamma_2$ the system parameters. When the orders $q_1 = q_2 = q_3 = q = 0.99$, system (2) has a chaotic attractor with system parameters $\delta = 1.26$, $\gamma = 10$, $c = 0.7764$, $\Gamma_1 = 0.5$, $\Gamma_2 = 0.25$, which is plotted in Figure 1. The initial conditions for the numerical simulation are $x(0) = 0.1$, $y(0) = 0.1$, $z(0) = 0.1$ and also used in the rest of the paper.

3.1. Bifurcations with the Variation of the Orders. When the system parameters are fixed as $\delta = 1.26$, $\gamma = 10$, $c = 0.7764$, $\Gamma_1 = 0.5$, $\Gamma_2 = 0.25$, and the orders $q_2 = 1$, $q_3 = 1$, the bifurcation of system (2) as the order q_1 is varied is depicted in Figure 2(a). From this, it is clear that the system has a long period-1 window when q_1 is slightly less than 0.72. As the order q_1 further increases, an evolution procedure of system (2) through period-doubling route to chaos is obtained. Meanwhile, to verify the chaotic behaviors for the system, the corresponding largest Lyapunov exponent (LLE for short) diagram by the algorithm of small data sets is shown in Figure 2(b). Phase portraits and the corresponding time series diagrams are shown in Figures 3(a)–3(h), from which we can see that the system has period-1, period-2, period-4, and chaos for different values of the order q_1. The route out of chaos for system (2) is tangent bifurcation when $q_1 = 0.833$. There are three limit cycles coexisting until the secondary period-doubling bifurcations occur.

When the system parameters are fixed as the same values, and the orders $q_1 = 1$, $q_3 = 1$, the affection of the variation of order $q_2 \in [0.5, 1]$ for the system is showed in Figure 4(a). It is clear that a series of period-doubling and tangent bifurcations can also be seen from the figure. Meanwhile, the corresponding LLE diagram is shown in Figure 4(b). As the order q_3 is varied, the bifurcations of the system are similar to that of q_2, so more details are not displayed in here.

3.2. Bifurcations with the Variation of the System Parameters. When the other parameters are taken as $\gamma = 10$, $c = 0.7764$,

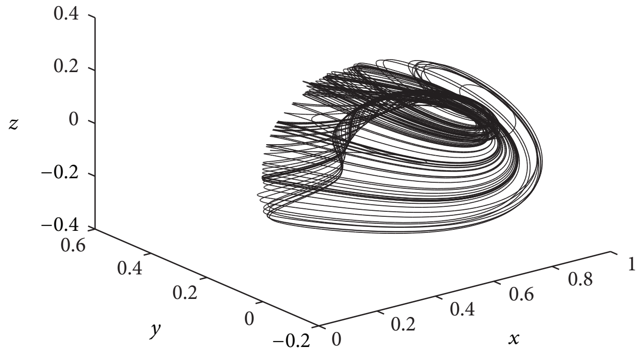
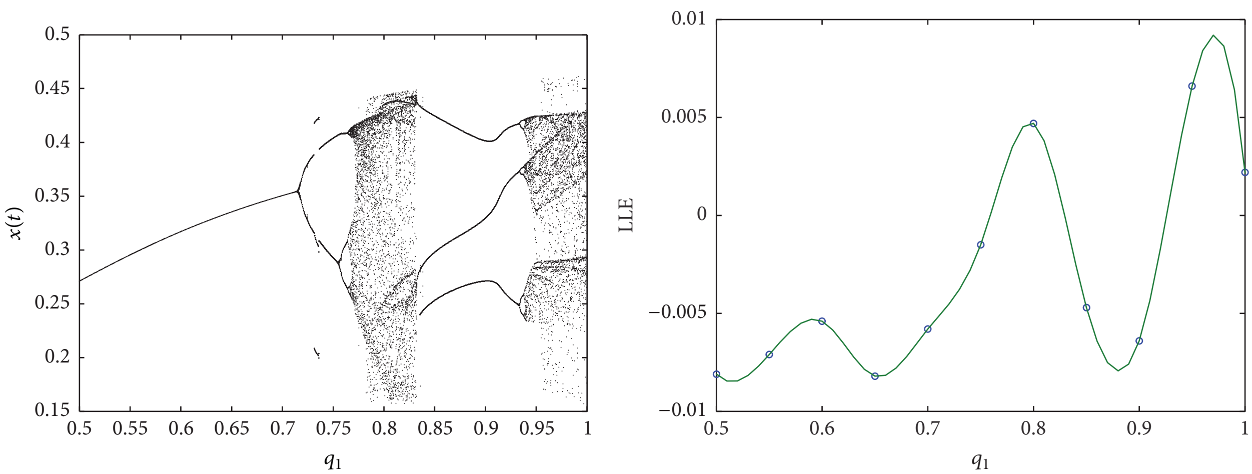


FIGURE 1: The chaotic attractor of system (2).



(a) The bifurcation diagram

(b) The corresponding LLE

FIGURE 2: The bifurcation and the corresponding LLE diagrams of system (2) with the order $q_1 \in [0.5, 1]$.

$\Gamma_1 = 0.5$, $\Gamma_2 = 0.25$, and the orders $q_1 = q_2 = q_3 = 0.98$, the bifurcation and the corresponding LLE diagrams of system (2) varying the parameter δ are depicted in Figure 5. With the increase of the parameter δ from -1.26, the system is period-1 firstly when $\delta \in [-1.26, -0.38]$. Then system (2) is period-2 motion until a series of period-doubling bifurcations occur. The phase portraits and the corresponding time series diagrams of system (2) with different values of the parameter δ are showed in Figure 6, from which it is clear that the system has only one limit cycle for $\delta = -1.26$, two cycles for $\delta = -0.7$, four cycles for $\delta = 0.7$, and six cycles for $\delta = 0.8$.

When $\delta = 1.26$, the bifurcation and the corresponding LLE diagrams of the system with the variation of γ are depicted in Figure 7. From this, we can see that a series of period-doubling and tangent bifurcations can be observed. Meanwhile, the bifurcation and the corresponding LLE diagrams of system (2) with the system parameters c, Γ_1, and Γ_2 are also obtained by numerical simulations, respectively (see Figure 8). It is clear that the bifurcations of the system with c and Γ_2 are similar to that of the parameter γ. The system undergoes a series of period-doubling bifurcations when the parameter Γ_1 is varied and the alternation between the period and chaos.

4. Adaptive Synchronization

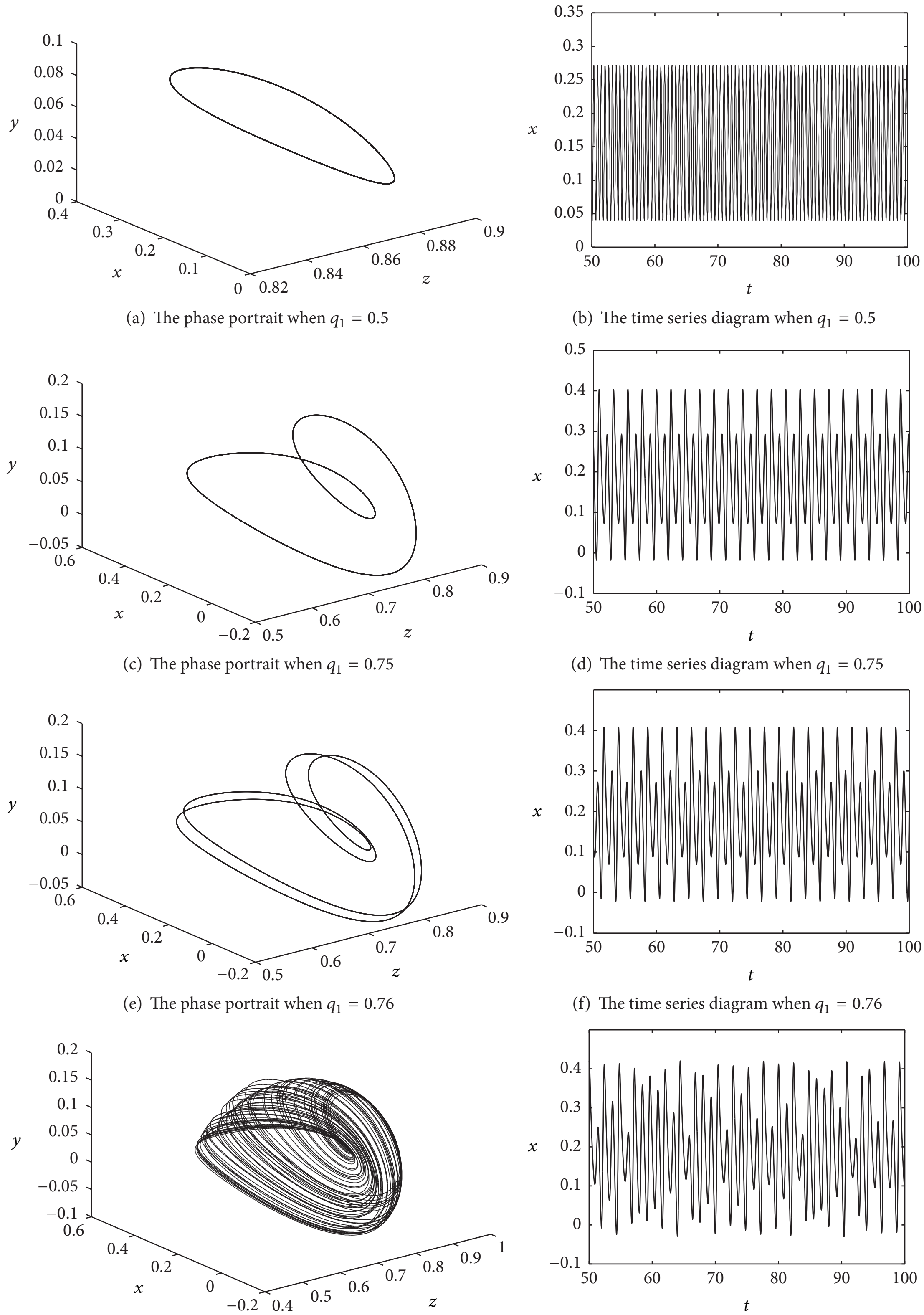
In this section, the adaptive synchronization for system (2) with uncertain parameters will be investigated.

For simplicity, system (2) in commensurate-order case is taken as the drive system and can be rewritten as follows:

$$
\begin{aligned}
D^q x_1 &= \delta y_1 + \gamma z_1 \left(x_1 \sin(c) - y_1 \cos(c)\right) - \frac{x_1}{\Gamma_2} \\
D^q y_1 &= -\delta x_1 - z_1 + \gamma z_1 \left(x_1 \cos(c) + y_1 \sin(c)\right) - \frac{y_1}{\Gamma_2} \\
D^q z_1 &= y_1 - \gamma \sin(c) \left(x_1^2 + y_1^2\right) - \frac{z_1 - 1}{\Gamma_1},
\end{aligned}
\tag{3}
$$

where δ and γ are uncertain parameters which are needed to be identified. And the response system is described by the following differential equations:

$$
\begin{aligned}
D^q x_2 &= \tilde{\delta} y_2 + \tilde{\gamma} z_2 \left(x_2 \sin(c) - y_2 \cos(c)\right) - \frac{x_2}{\Gamma_2} + u_1 \\
D^q y_2 &= -\tilde{\delta} x_2 - z_2 + \tilde{\gamma} z_2 \left(x_2 \cos(c) + y_2 \sin(c)\right) - \frac{y_2}{\Gamma_2} \\
&\quad + u_2
\end{aligned}
$$



(a) The phase portrait when $q_1 = 0.5$

(b) The time series diagram when $q_1 = 0.5$

(c) The phase portrait when $q_1 = 0.75$

(d) The time series diagram when $q_1 = 0.75$

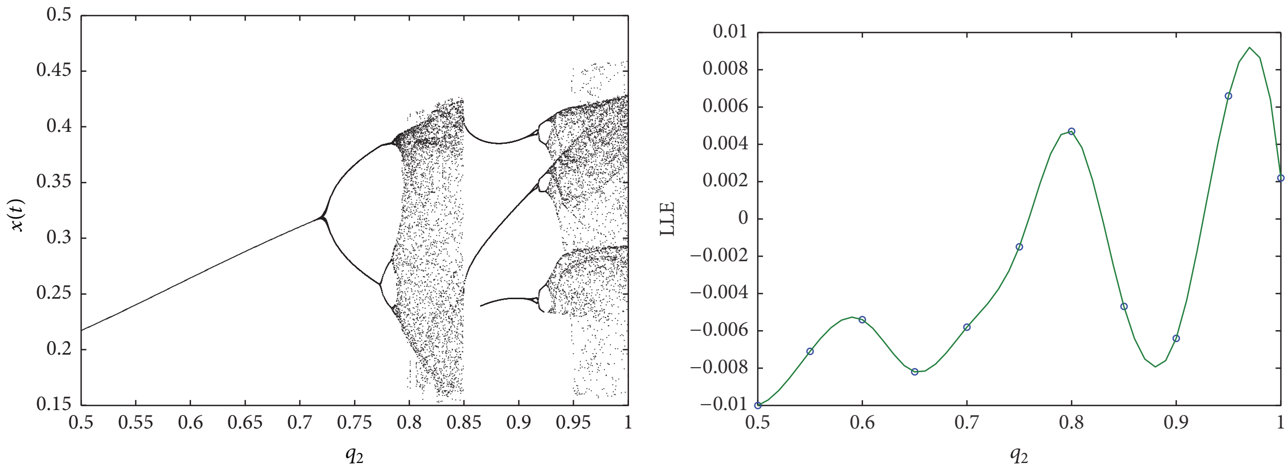
(e) The phase portrait when $q_1 = 0.76$

(f) The time series diagram when $q_1 = 0.76$

(g) The phase portrait when $q_1 = 0.78$

(h) The time series diagram when $q_1 = 0.78$

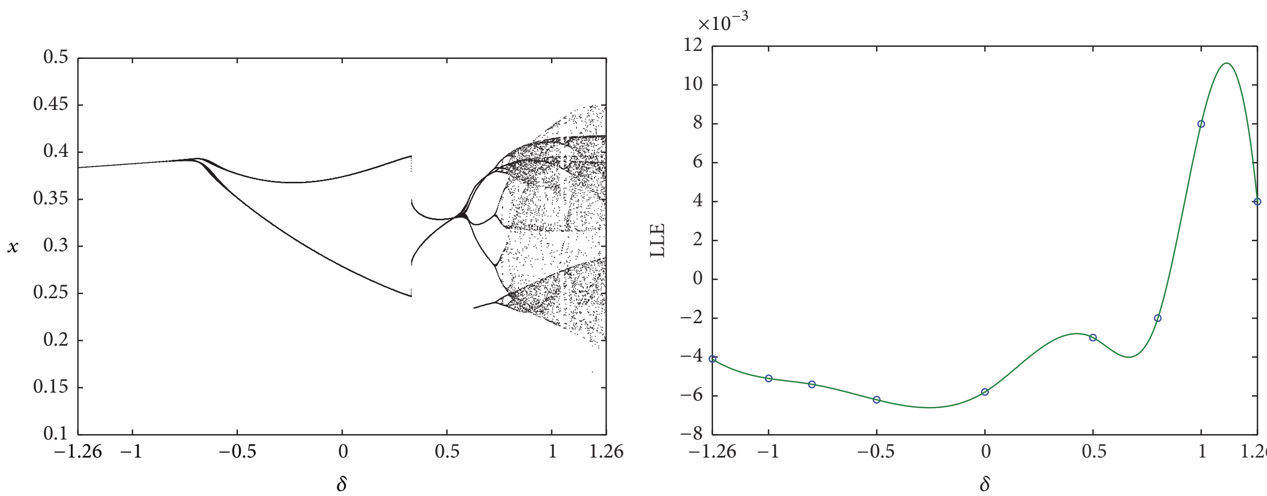
FIGURE 3: The phase portraits and time series diagrams of system (2) with different values of the order q_1.



(a) The bifurcation diagram

(b) The corresponding LLE

FIGURE 4: The bifurcation and the corresponding LLE diagrams of system (2) with the order $q_2 \in [0.5, 1]$.



(a) The bifurcation diagram

(b) The corresponding LLE

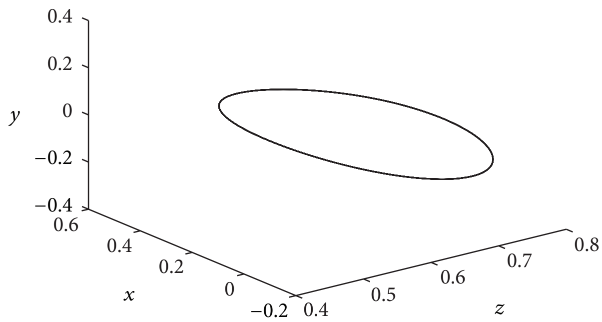
FIGURE 5: The bifurcation and the corresponding LLE diagrams of system (2) with the parameter $\delta \in [-1.26, 1.26]$.

$$D^q z_2 = y_2 - \tilde{\gamma}\sin(c)\left(x_2^2 + y_2^2\right) - \frac{z_2 - 1}{\Gamma_1} + u_3, \tag{4}$$

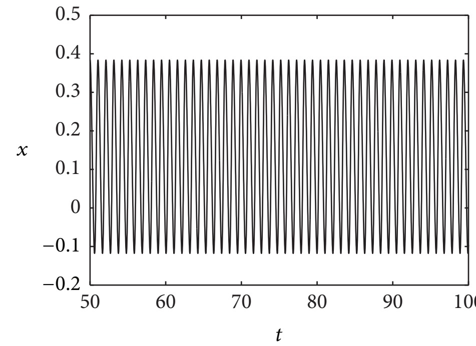
where u_1, u_2, and u_3 are the synchronization controllers and $\tilde{\delta}$, $\tilde{\gamma}$ are the estimations of unknown parameters. Then, the synchronization error variables are defined as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$, and estimation errors of uncertain parameters $e_\delta = \tilde{\delta} - \delta$, $e_\gamma = \tilde{\gamma} - \gamma$. By subtracting system (3) from (4), the error dynamical system is obtained, which is given as follows:

$$\begin{aligned}
D^q e_1 &= \tilde{\delta} e_2 + y_1 e_\delta + \tilde{\gamma} e_3 \left(x_2 \sin(c) - y_2 \cos(c)\right) \\
&\quad + \tilde{\gamma} z_1 \left(e_1 \sin(c) - e_2 \cos(c)\right) \\
&\quad + e_\gamma z_1 \left(x_1 \sin(c) - y_1 \cos(c)\right) - \frac{e_1}{\Gamma_2} + u_1 \\
D^q e_2 &= -\tilde{\delta} e_1 - x_1 e_\delta - e_3 + \tilde{\gamma} e_3 \left(x_2 \cos(c) + y_2 \sin(c)\right) \\
&\quad + \tilde{\gamma} z_1 \left(e_1 \cos(c) + e_2 \sin(c)\right) \\
&\quad + e_\gamma z_1 \left(x_1 \cos(c) + y_1 \sin(c)\right) - \frac{e_2}{\Gamma_2} + u_2 \\
D^q e_3 &= e_2 - \tilde{\gamma}\left(e_1 \left(x_1 + x_2\right) + e_2 \left(y_1 + y_2\right)\right) \sin(c) \\
&\quad - e_\gamma \left(x_1^2 + y_1^2\right) \sin(c) - \frac{e_3}{\Gamma_1} + u_3.
\end{aligned} \tag{5}$$

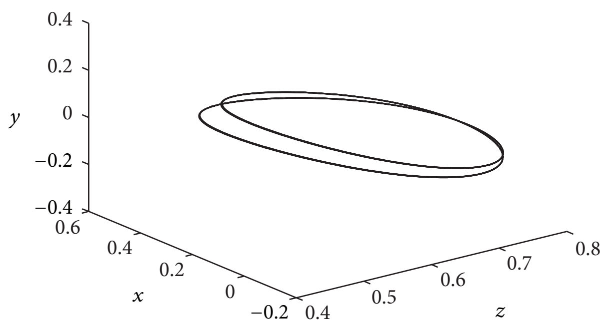
In order to realize the synchronization of the drive and response systems, the controllers should be designed properly. Therefore, the following criterion is presented to ensure system (3) effectively synchronizes system (4).



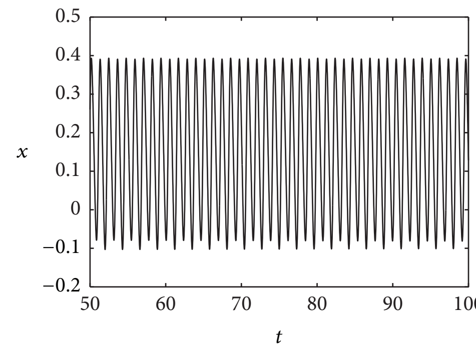
(a) The phase portrait when $\delta = -1.26$



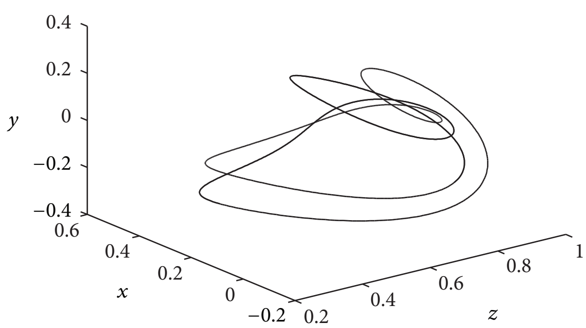
(b) The time series diagram when $\delta = -1.26$



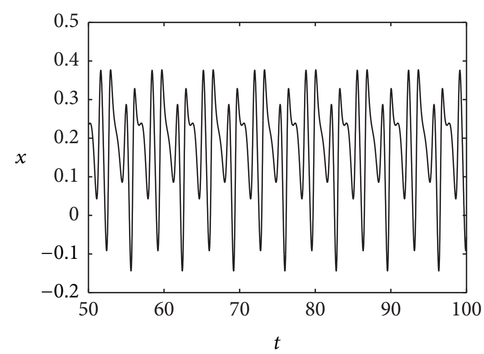
(c) The phase portrait when $\delta = -0.7$



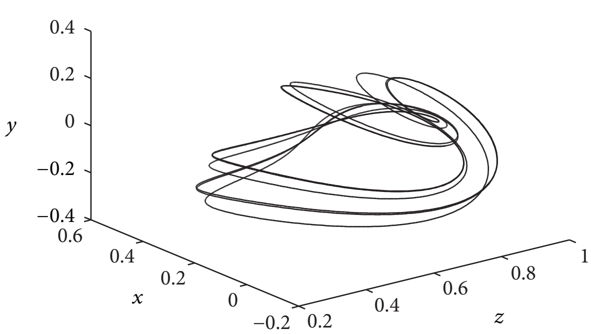
(d) The time series diagram when $\delta = -0.7$



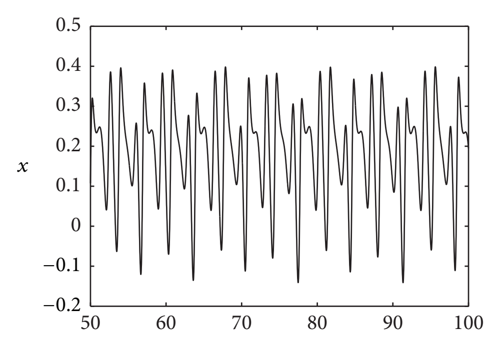
(e) The phase portrait when $\delta = 0.7$



(f) The time series diagram when $\delta = 0.7$

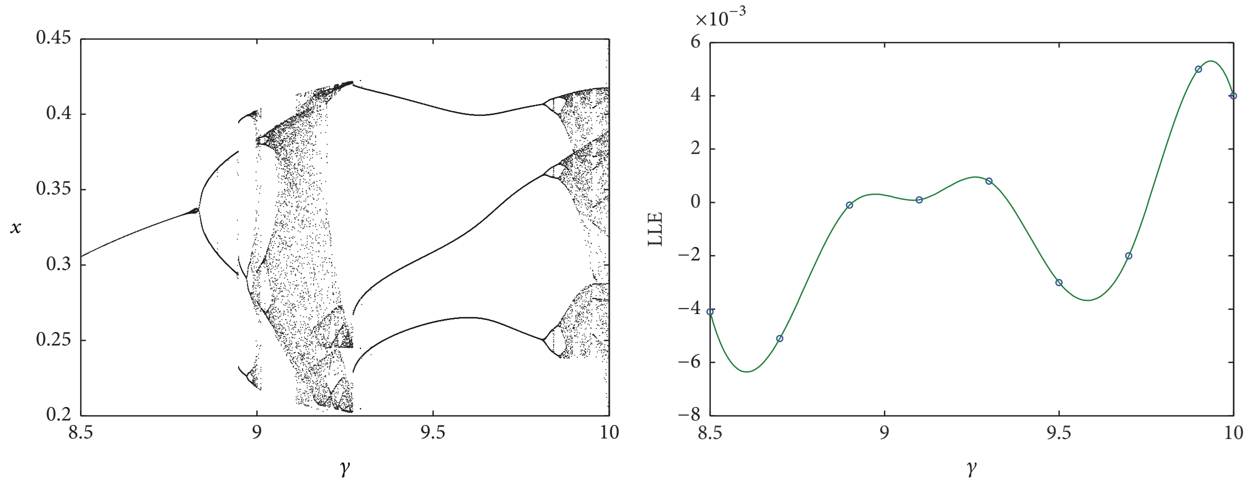


(g) The phase portrait when $\delta = 0.8$



(h) The time series diagram when $\delta = 0.8$

Figure 6: The phase portraits and time series diagrams of system (2) with different values of the parameter δ.



(a) The bifurcation diagram

(b) The corresponding LLE

FIGURE 7: The bifurcation and the corresponding LLE diagrams of system (2) with the parameter $\gamma \in [8.5, 10]$.

Theorem 2. *Adaptive synchronization between systems (3) and (4) is realized when the controllers and laws of the uncertain parameter are designed as follows:*

$$\begin{aligned} u_1 &= -\tilde{\delta} e_2 - \tilde{\gamma} e_3 \left(x_2 \sin(c) - y_2 \cos(c)\right) \\ &\quad - \tilde{\gamma} z_1 \left(e_1 \sin(c) - e_2 \cos(c)\right) + \left(\frac{1}{\Gamma_2} - 1\right) e_1 \\ u_2 &= \tilde{\delta} e_1 + e_3 - \tilde{\gamma} e_3 \left(x_2 \cos(c) + y_2 \sin(c)\right) \\ &\quad - \tilde{\gamma} z_1 \left(e_1 \cos(c) + e_2 \sin(c)\right) + \left(\frac{1}{\Gamma_2} - 1\right) e_2 \\ u_3 &= -e_2 + \tilde{\gamma} \left(e_1 \left(x_1 + x_2\right) + e_2 \left(y_1 + y_2\right)\right) \sin(c) \\ &\quad + \left(\frac{1}{\Gamma_1} - 1\right) e_3, \end{aligned} \tag{6}$$

$$\begin{aligned} D^q \tilde{\delta} &= -y_1 e_1 + x_1 e_2 \\ D^q \tilde{\gamma} &= -z_1 \left(x_1 \sin(c) - y_1 \cos(c)\right) e_1 \\ &\quad - z_1 \left(x_1 \cos(c) + y_1 \sin(c)\right) e_2 \\ &\quad + \left(x_1^2 + y_1^2\right) \sin(c)\, e_3. \end{aligned} \tag{7}$$

Proof. Firstly, controllers (6) are substituted into system (5), and then the error dynamical system is expressed as

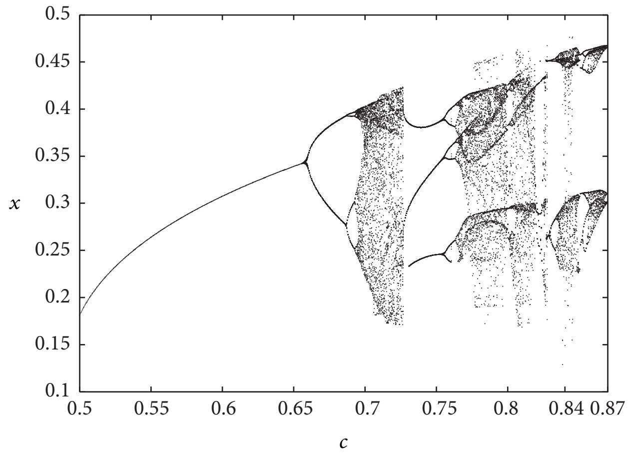
$$\begin{aligned} D^{q_1} e_1 &= y_1 e_\delta + e_\gamma z_1 \left(x_1 \sin(c) - y_1 \cos(c)\right) - e_1 \\ D^{q_1} e_2 &= -x_1 e_\delta + e_\gamma z_1 \left(x_1 \cos(c) + y_1 \sin(c)\right) - e_2 \\ D^{q_1} e_3 &= -e_\gamma \left(x_1^2 + y_1^2\right) \sin(c) - e_3. \end{aligned} \tag{8}$$

By combining systems (8) and (7), we can get

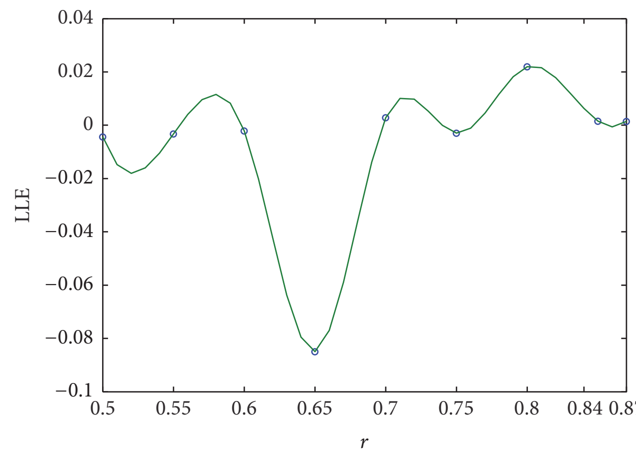
$$\begin{aligned} &\left(D^q e_1, D^q e_2, D^q e_3, D^q e_\delta, D^q e_\gamma\right)^{\mathrm{T}} \\ &\quad = A \left(e_1, e_2, e_3, e_\delta, e_\gamma\right)^{\mathrm{T}}, \end{aligned} \tag{9}$$

where

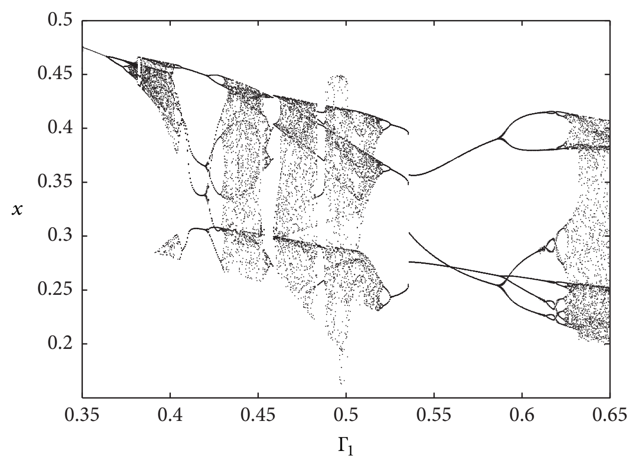
$$A = \begin{pmatrix} -1 & 0 & 0 & y_1 & z_1 \left(x_1 \sin(c) - y_1 \cos(c)\right) \\ 0 & -1 & 0 & -x_1 & z_1 \left(x_1 \cos(c) + y_1 \sin(c)\right) \\ 0 & 0 & -1 & 0 & -\left(x_1^2 + y_1^2\right) \sin(c) \\ -y_1 & x_1 & 0 & 0 & 0 \\ -z_1 \left(x_1 \sin(c) - y_1 \cos(c)\right) & -z_1 \left(x_1 \cos(c) + y_1 \sin(c)\right) & \left(x_1^2 + y_1^2\right) \sin(c) & 0 & 0 \end{pmatrix}. \tag{10}$$



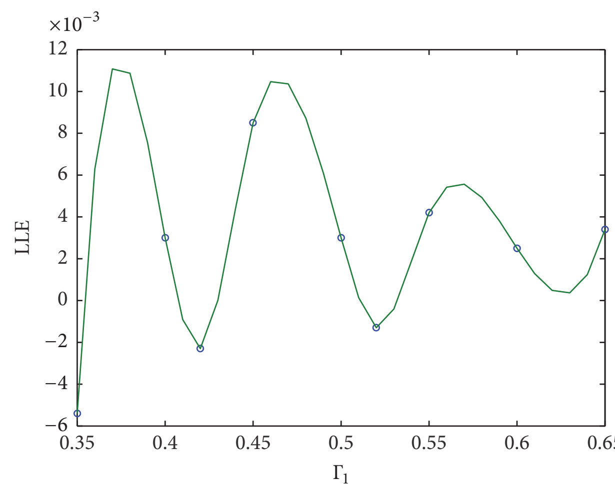
(a) The bifurcation diagram for $c \in [0.5, 0.87]$



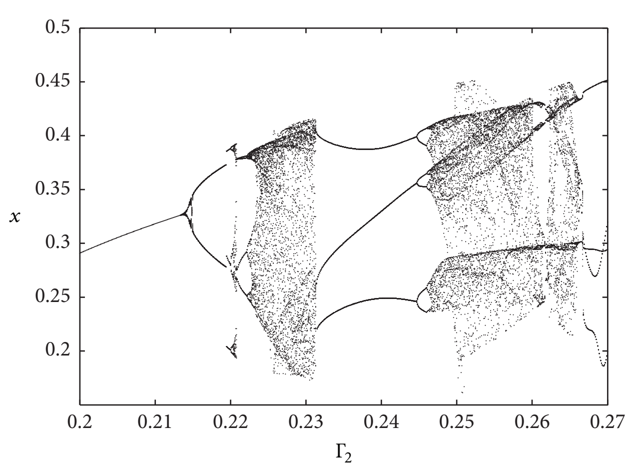
(b) The corresponding LLE for $c \in [0.5, 0.87]$



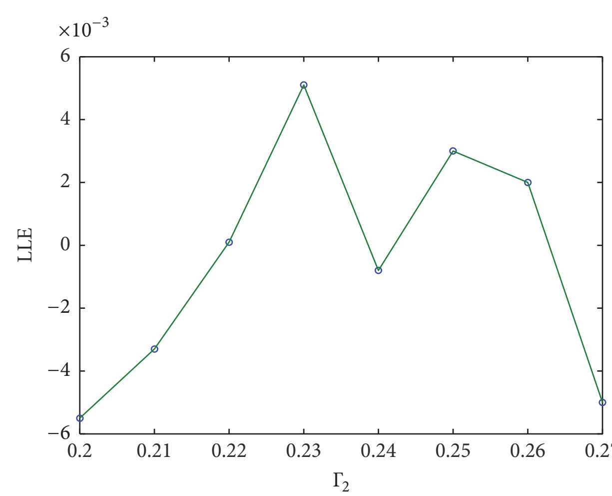
(c) The bifurcation diagram for $\Gamma_1 \in [0.35, 0.65]$



(d) The corresponding LLE for $\Gamma_1 \in [0.35, 0.65]$

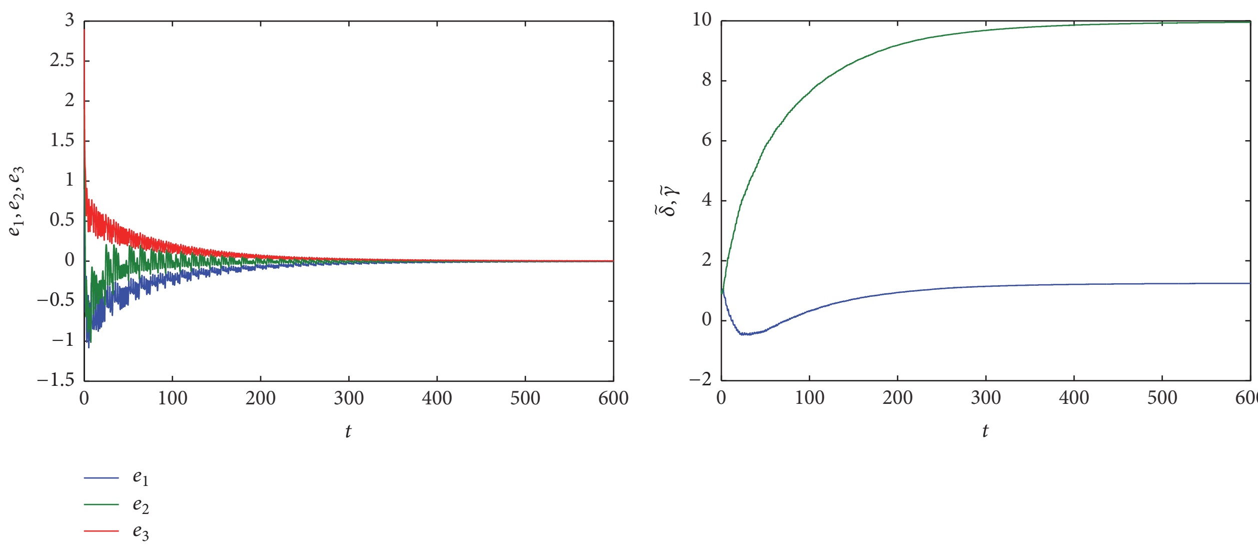


(e) The bifurcation diagram for $\Gamma_2 \in [0.2, 0.27]$



(f) The corresponding LLE for $\Gamma_2 \in [0.2, 0.27]$

FIGURE 8: The bifurcation and the corresponding LLE diagrams of system (2) with different parameters.



(a) The synchronization error curves

(b) The identification curves of the unknown parameters

FIGURE 9: The synchronization results of the numerical simulation.

Assume that λ is one of the eigenvalues of matrix $\mathbf{A}$, and then the corresponding nonzero eigenvectors are $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^{\mathrm{T}}$; then we have

$$\mathbf{A}\zeta = \lambda\zeta, \tag{11}$$

and the following relation can be easily obtained when conjugate transpose H is taken on both sides of (11)

$$\overline{(A\zeta)^{\mathrm{T}}} = \bar{\lambda}\zeta^{\mathrm{H}}. \tag{12}$$

Using (11) multiplied left by $(1/2)\zeta^{\mathrm{H}}$ plus (12) multiplied right by $(1/2)\zeta$, we get

$$\zeta^{\mathrm{H}}\left(\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^{\mathrm{H}}\right)\zeta = \frac{1}{2}\left(\lambda + \bar{\lambda}\right)\zeta^{\mathrm{H}}\zeta. \tag{13}$$

The above formula can be rewritten as the following form:

$$\frac{1}{2}\left(\lambda + \bar{\lambda}\right) = \frac{\zeta^{\mathrm{H}}\left((1/2)\,\mathbf{A} + (1/2)\,\mathbf{A}^{\mathrm{H}}\right)\zeta}{\zeta^{\mathrm{H}}\zeta}. \tag{14}$$

By substituting the matrix $\mathbf{A}$ into (14), then we can have

$$\frac{1}{2}\left(\lambda + \bar{\lambda}\right) = \frac{1}{\zeta^{\mathrm{H}}\zeta}\zeta^{\mathrm{H}}\mathbf{B}\zeta, \tag{15}$$

where $\mathbf{B} = \mathrm{diag}(-1, -1, -1, 0, 0)$. Since $\lambda + \bar{\lambda} \le 0$, then all the eigenvalues of matrix $\mathbf{A}$ satisfy the following relationship:

$$\left|\arg(\lambda)\right| \ge \frac{\pi}{2} > \frac{\pi}{2}q, \quad \left(0 < q < 1\right). \tag{16}$$

According to the stability theory of fractional-order systems, then the error dynamical system (5) is asymptotically stable. Therefore

$$\lim_{t\to\infty} \|e(t)\| = 0, \tag{17}$$

which means that the synchronization between the drive and response systems is realized. The proof is completed. □

In numerical simulations, the real values of the uncertain parameters are $\delta = 1.26$, $\gamma = 10$ when $c = 0.7764$, $\Gamma_1 = 0.5$, $\Gamma_2 = 0.25$, $q = 0.98$. The initial conditions of the drive and response systems are $(0.1, 0.1, 0.1)$ and $(1, 2, 3)$, respectively. The synchronization results of the numerical simulation are depicted in Figure 9. From this we can see that the error variables tend to 0, and the estimations of unknown parameters converge to their real values. These results demonstrate the effectiveness of the synchronization controllers and laws of unknown parameters.

5. Conclusions

In this paper, bifurcations and synchronization of a fractional-order Bloch system have been studied. Firstly, bifurcations of the system as every order is varied are obtained by the numerical simulations. Period-doubling and tangent bifurcations can be observed in the system. Meanwhile, bifurcations of the system with the variation of every system parameter are also determined by numerical computation. Besides the period-doubling and tangent bifurcations, limit cycles coexisting are also found for the fractional-order Bloch system. From these results, it can be seen clearly that the derivative orders are also the important parameters which affect the dynamics of the fractional-order Bloch system. The bifurcations of the system in such a parameter set are demonstrated in detail. Finally, based on the stability theory of fractional-order systems, the adaptive synchronization for the system with unknown parameters is realized by designing appropriate controllers. Numerical simulations are carried out to demonstrate the effectiveness and flexibility of the controllers.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

[1] M. M. Asheghan, S. S. Delshad, M. T. Hamidi Beheshti, and M. S. Tavazoei, "Non-fragile control and synchronization of a new fractional order chaotic system," *Applied Mathematics and Computation*, vol. 222, pp. 712–721, 2013.

[2] S. Dadras, H. R. Momeni, G. Qi, and Z.-l. Wang, "Four-wing hyperchaotic attractor generated from a new 4D system with one equilibrium and its fractional-order form," *Nonlinear Dynamics*, vol. 67, no. 2, pp. 1161–1173, 2012.

[3] C. Zeng, Q. Yang, and J. Wang, "Chaos and mixed synchronization of a new fractional-order system with one saddle and two stable node-foci," *Nonlinear Dynamics*, vol. 65, no. 4, pp. 457–466, 2011.

[4] K. Diethelm and N. J. Ford, "Multi-order fractional differential equations and their numerical solution," *Applied Mathematics and Computation*, vol. 154, no. 3, pp. 621–640, 2004.

[5] K. Diethelm, N. J. Ford, A. D. Freed, and Y. Luchko, "Algorithms for the fractional calculus: a selection of numerical methods," *Computer Methods in Applied Mechanics and Engineering*, vol. 194, no. 6–8, pp. 743–773, 2005.

[6] T. T. Hartley, C. F. Lorenzo, and H. K. Qammer, "Chaos in a fractional order Chua's System," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 485–490, 1995.

[7] Z.-M. Ge and C.-Y. Ou, "Chaos in a fractional order modified Duffing system," *Chaos, Solitons and Fractals*, vol. 34, no. 2, pp. 262–291, 2007.

[8] I. Grigorenko and E. Grigorenko, "Chaotic dynamics of the fractional Lorenz system," *Physical Review Letters*, vol. 91, no. 3, Article ID 034101, 2003.

[9] J. G. Lu, "Chaotic dynamics of the fractional-order Ikeda delay system and it synchronization," *Physics Letters A*, vol. 59, no. 2, pp. 301–305, 2006.

[10] W. Zhang, S. Zhou, H. Li, and H. Zhu, "Chaos in a fractional-order Rössler system," *Chaos, Solitons and Fractals*, vol. 42, no. 3, pp. 1684–1691, 2009.

[11] V. Daftardar-Gejji and S. Bhalekar, "Chaos in fractional ordered Liu system," *Computers and Mathematics with Applications*, vol. 59, no. 3, pp. 1117–1127, 2010.

[12] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu, and R. L. Magin, "Generalized fractional order bloch equation with extended delay," *International Journal of Bifurcation and Chaos*, vol. 22, no. 4, pp. 91–116, 2012.

[13] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu, and R. Magin, "Fractional Bloch equation with delay," *Computers and Mathematics with Applications*, vol. 61, no. 5, pp. 1355–1365, 2011.

[14] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu, and R. Magin, "Transient chaos in fractional Bloch equations," *Computers and Mathematics with Applications*, vol. 64, no. 10, pp. 3367–3376, 2012.

[15] R. L. Magin, O. Abdullah, D. Baleanu, and X. J. Zhou, "Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation," *Journal of Magnetic Resonance*, vol. 190, no. 2, pp. 255–270, 2008.

[16] R. Magin, X. Feng, and D. Baleanu, "Solving the fractional order Bloch equation," *Concepts in Magnetic Resonance Part A: Bridging Education and Research*, vol. 34, no. 1, pp. 16–23, 2009.

[17] H. Nasr-eddine and H. Tarek, "Chaotic dynamics of the fractional order nonlinear Bloch system," *Electronic Journal of Theoretical Physics*, vol. 8, no. 25, pp. 233–244, 2011.

[18] I. Petráš, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Higher Education Press, Beijing, China, 2011.

[19] E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, vol. 8 of *Springer Series in Computational Mathematics*, Springer, Berlin, Germany, 2nd edition, 1993.

[20] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential- Algebraic Problems*, vol. 14, Springer, Berlin, Germany, 1991.

[21] K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dynamics*, vol. 29, no. 1–4, pp. 3–22, 2002.

[22] A. Charef, H. H. Sun, Y.-Y. Tsao, and B. Onaral, "Fractal system as represented by singularity function," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, pp. 1465–1470, 1992.

[23] M. S. Tavazoei and M. Haeri, "Limitations of frequency domain approximation for detecting chaos in fractional order systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 4, pp. 1299–1320, 2008.

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Research Article

An Efficient Series Solution for Nonlinear Multiterm Fractional Differential Equations

Moh'd Khier Al-Srihin and Mohammed Al-Refai

Department of Mathematical Sciences, United Arab Emirates University, Al-Ain, UAE

Correspondence should be addressed to Mohammed Al-Refai; m_alrefai@uaeu.ac.ae

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In this paper, we introduce an efficient series solution for a class of nonlinear multiterm fractional differential equations of Caputo type. The approach is a generalization to our recent work for single fractional differential equations. We extend the idea of the Taylor series expansion method to multiterm fractional differential equations, where we overcome the difficulty of computing iterated fractional derivatives, which are difficult to be computed in general. The terms of the series are obtained sequentially using a closed formula, where only integer derivatives have to be computed. Several examples are presented to illustrate the efficiency of the new approach and comparison with the Adomian decomposition method is performed.

1. Introduction

Fractional differential equations (FDEs) are generalization to differential equations (DEs) for noninteger orders. In recent years, FDEs caught the attention of many researchers because of their appearance in modeling several phenomenon in the physical sciences. As many FDEs do not possess exact solutions on closed forms, analytical and numerical techniques have been implemented to study these equations. Iterative methods, such as the variational iteration method (VIM) in [1], the homotopy analysis method (HAM) in [2, 3], the Adomian decomposition method (ADM) in [4–9], and the fractional differential transform method in [10], have been implemented to solve various types of FDEs. These methods produce a solution in a series form whose terms are determined sequentially. We refer the reader to [11, 12] for a comprehensive study of series solutions of fractional differential equations. Recently, we have introduced a new series solution for single fractional differential equations [13]. The new approach is a modified form of the well-known Taylor series expansion, where we overcome the difficulty of computing iterative fractional derivatives. The efficiency of the new approach has been illustrated through several examples. In this paper we extend the idea to multiterm fractional differential equations. The presented work is a part of the Master thesis [14]. We consider the left Caputo fractional derivative $D_{0^+}^{\alpha}$, defined by [15, 16]

$$D_{0^+}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, & \text{if } n-1<\alpha<n, \\ f^{(n)}(t), & \text{if } \alpha = n \in \mathbb{N} \end{cases} \quad (1)$$

provided the integral exists, where Γ is the well-known Gamma function. The left Riemann-Liouville fractional integral, $I_{0^+}^{\alpha}$, of order $\alpha \in \mathbb{R}^+$, is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t>0,\ \alpha>0. \quad (2)$$

The left Caputo derivative is related to the left Riemann-Liouville fractional integral by

$$D_{0^+}^{\alpha} f(t) = I_{0^+}^{n-\alpha} f^{(n)}(t), \quad n-1<\alpha<n. \quad (3)$$

It is known that

$$\left(\left(I_{0^+}^{\alpha} D_{0^+}^{\alpha}\right) f\right)(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k,$$
$$\left(\left(D_{0^+}^{\alpha} I_{0^+}^{\alpha}\right) f\right)(t) = f(t). \tag{4}$$

Also, for $n-1 < \alpha < n, n \in \mathbb{N}, t > 0$, it holds that

$$D_{0^+}^{\alpha}\left(t^{\mu}\right) = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, & \text{if } \mu > n-1, \\ 0, & \text{if } \mu \in \{0, 1, \ldots, n-1\}. \end{cases} \tag{5}$$

This paper is organized as follows. In Section 2, we present the series solution of nonlinear two-term fractional differential equations. We illustrate the efficiency of the presented technique through several examples. We also compare our results with the ones obtained by the Adomian decomposition method. In Section 3, we present and illustrate the efficiency of the new series solution for three-term fractional differential equations of several types. Finally, we conclude with some remarks in Section 4.

2. Two-Term Fractional Differential Equations

We start with the nonlinear two-term fractional initial value problems of the form

$$c_1 D_{0^+}^{\alpha_1} u(t) + c_2 D_{0^+}^{\alpha_2} u(t) = f(t, u(t)), \quad t > 0 \tag{6}$$

with

$$u(0) = b, \tag{7}$$

where $0 < \alpha_2 \le \alpha_1 < 1$, and c_1 and c_2 are nonzero constants. We assume that $f(t, u(t))$ is continuous and smooth with respect to $u(t)$. We also assume that α_1 and α_2 are rational numbers with $\alpha_1 = p_1/q_1$ and $\alpha_2 = p_2/q_2$, $p_1, p_2, q_1, q_2 \in \mathbb{N}$.

2.1. The Expansion Procedure. Let $q = \operatorname{lcm}(q_1, q_2)$; we have $q = sq_1 = rq_2$ for some $s, r \in \mathbb{N}$.

In the following we expand the solution of problem (6)-(7) in an infinite series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^{n/q}, \tag{8}$$

where the coefficients a_n: $n \geqslant 0$ have to be determined sequentially in the following manner: From the initial condition (7) we have $u(0) = b = a_0$. Since $D_{0^+}^{\alpha}(c) = 0$, for c being constant, we have

$$D_{0^+}^{\alpha_1} u(t) = \sum_{n=1}^{\infty} a_n s_n t^{n/q - p_1/q_1} = \sum_{n=1}^{\infty} a_n s_n t^{(n - sp_1)/q},$$
$$D_{0^+}^{\alpha_2} u(t) = \sum_{n=1}^{\infty} a_n r_n t^{n/q - p_2/q_2} = \sum_{n=1}^{\infty} a_n r_n t^{(n - rp_2)/q}, \tag{9}$$

where

$$s_n = \frac{\Gamma(n/q + 1)}{\Gamma(n/q - \alpha_1 + 1)},$$
$$r_n = \frac{\Gamma(n/q + 1)}{\Gamma(n/q - \alpha_2 + 1)}. \tag{10}$$

By substituting (9) in (6) we have

$$c_1 \sum_{n=1}^{\infty} a_n s_n t^{(n - sp_1)/q} + c_2 \sum_{n=1}^{\infty} a_n r_n t^{(n - rp_2)/q} = f\left(t, \sum_{n=0}^{\infty} a_n t^{n/q}\right). \tag{11}$$

Applying the well-known Taylor series method to compute the coefficients $\{a_n; n \ge 1\}$ will lead to computing iterated fractional derivatives, which are not easily computed in general. To avoid this difficulty, let $t = w^q$; we have

$$c_1 \sum_{n=1}^{\infty} a_n s_n w^{n - sp_1} + c_2 \sum_{n=1}^{\infty} a_n r_n w^{n - rp_2} = f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right). \tag{12}$$

Shifting the index to zero yields

$$c_1 \sum_{n=0}^{\infty} a_{n+1} s_{n+1} w^{n - sp_1 + 1} + c_2 \sum_{n=0}^{\infty} a_{n+1} r_{n+1} w^{n - rp_2 + 1} = f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right). \tag{13}$$

To avoid the singularity at $w = 0$, we multiply (13) by $w^{sp_1 - 1}$; we have

$$c_1 \sum_{n=0}^{\infty} a_{n+1} s_{n+1} w^{n} + c_2 \sum_{n=0}^{\infty} a_{n+1} r_{n+1} w^{n - rp_2 + sp_1} = w^{sp_1 - 1} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right). \tag{14}$$

Now, since $\alpha_1 = p_1/q_1 = sp_1/q > rp_2/q = p_2/q_2 = \alpha_2$, thus $sp_1 - rp_2 > 0$, and (14) has no singularity at $w = 0$.

Let $k = sp_1 - rp_2 - 1 \ge 0$, and $g(w) = f(w^q, \sum_{n=0}^{\infty} a_n w^n)$; then (14) can be written as

$$c_1 \sum_{n=0}^{k} a_{n+1} s_{n+1} w^n + \sum_{n=k+1}^{\infty} \left(c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k}\right) w^n = w^{k + rp_2} g(w). \tag{15}$$

We first determine the coefficients a_n for $n \le k$. By performing the nth derivativeof (15) with respect to w and substituting $w = 0$, we have

$$c_1 n! a_{n+1} s_{n+1} = \frac{d^n}{dw^n}\left(w^{k+rp_2} g(w)\right)\Big|_{w=0} \tag{16}$$

which yields

$$a_{n+1} = \frac{1}{c_1 n! s_{n+1}} \frac{d^n}{dw^n}\left(w^{k+rp_2} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right)\right)\Big|_{w=0}. \tag{17}$$

Since $k + rp_2 \geq n + 1$, and f is smooth, then for $n \leq k$, we have

$$\frac{d^n}{dw^n}\left(w^{k+rp_2} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right)\right)\Big|_{w=0} = 0, \tag{18}$$

and hence $a_{n+1} = 0$, for $n \leq k$.

We now determine a_n, for $n \geq k+1$. By performing the nth derivative of (15) with respect to w and substituting $w = 0$, we have

$$\begin{aligned} n!\left(c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k}\right) &= \frac{d^n}{dw^n}\left(w^{k+rp_2} g(w)\right)\Big|_{w=0} \\ &= \frac{d^n}{dw^n}\left(w^{k+rp_2} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right)\right)\Big|_{w=0}. \end{aligned} \tag{19}$$

Using the well-known Leibniz rule for differentiating the products, we have

$$\begin{aligned} &\frac{d^n}{dw^n}\left(w^{k+rp_2} g(w)\right) \\ &\quad = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{dw^j}\left(w^{k+rp_2}\right) \frac{d^{n-j}}{dw^{n-j}}\left(g(w)\right). \end{aligned} \tag{20}$$

Since

$$\frac{d^j}{dw^j}\left(w^{k+rp_2}\right)\Big|_{w=0} = \begin{cases} 0 & \text{if } j \neq k + rp_2, \\ j! & \text{if } j = k + rp_2, \end{cases} \tag{21}$$

we have

$$\begin{aligned} &c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k} \\ &\quad = \frac{1}{(n-j)!}\left(\frac{d^{n-j}}{dw^{n-j}} g(w)\right)\Big|_{w=0}, \\ &\qquad\qquad \text{where } j = k + rp_2. \end{aligned} \tag{22}$$

From the last equation we determine $a_n : n \geq k + 1$ and thus the solution

$$u(t) = \sum_{n=0}^{\infty} a_n t^{n/q} = a_0 + \sum_{n=k+1}^{\infty} a_n t^{n/q} \tag{23}$$

is obtained.

Remark 1. In (6), assuming $\alpha_1 = \alpha_2 = 1$, then $s_n = r_n = n, k = -1$ and $k + rp_2 = 0$. Thus (17) is reduced to

$$a_{n+1} = \frac{1}{c_1 (n+1)!} \frac{d^n}{dw^n}\left[f\left(w, \sum_{n=0}^{\infty} a_n w^n\right)\right]_{w=0}, \tag{24}$$

which coincides with the coefficients obtained by the Taylor series expansion method. Same comment applies for the coefficient a_{n+1} in (22).

Remark 2. The algorithm can be generalized for the two-term fractional differential equation (6) with $0 < \alpha_2 \leq \alpha_1 < N$, for arbitrary integer N. But we have to take care of the existence of the fractional derivative $D_{0^+}^{\alpha_1}(t^{n/q})$ (see Eq. (5)), by choosing the coefficient $a_n = 0$, for $n = 1, \ldots, qN - 1$. This case is discussed in Section 3.

2.2. Numerical Results

Example 3. Consider the nonlinear two-term fractional initial value problem

$$\begin{aligned} &2D_{0^+}^{1/2} u(t) + 2\Gamma\left(\frac{13}{10}\right) D_{0^+}^{1/5} u(t) \\ &\quad = \Gamma\left(\frac{1}{2}\right)\left(u^2 + t^{3/10} - t + 1\right), \quad t > 0, \end{aligned} \tag{25}$$

with

$$u(0) = 0. \tag{26}$$

The exact solution of the problem is $u(t) = t^{1/2}$.

Applying the current algorithm, we have $\alpha_1 = 1/2 = p_1/q_1$, $\alpha_2 = 1/5 = p_2/q_2$, $q = \mathrm{l.c.m}(q_1, q_2) = 10$, $s = 5$, and $r = 2$. We expand the solution in an infinite series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^{n/10}. \tag{27}$$

The initial condition in (26) yields $a_0 = 0$. We have $t = w^q = w^{10}$ and

$$\begin{aligned} g(w) &= f\left(w^{10}, u(w)\right) \\ &= \Gamma\left(\frac{1}{2}\right)\left(\left(\sum_{n=0}^{\infty} a_n w^n\right)^2 + w^3 - w^{10} + 1\right). \end{aligned} \tag{28}$$

Since $g(w)$ is continuous and smooth with respect to w, we have

$$a_{n+1} = 0, \quad \text{for } n \leq k = sp_1 - rp_2 - 1 = 2. \tag{29}$$

Thus $a_1 = a_2 = a_3 = 0$. The function $g(w)$ satisfies the assumption of the proposed algorithm, and it holds that

$$\frac{d^m}{dw^m}(g(w))\bigg|_{w=0} = \Gamma\left(\frac{1}{2}\right)\begin{cases} 1 & m=0 \\ 3! & m=3 \\ 6!a_3^2 & m=6 \\ 2\times 7!a_3a_4 & m=7 \\ 8!a_4^2 + 2\times 8!a_3a_5 & m=8 \\ 2!\left(a_4a_5 + a_3a_3\right) & m=9 \\ -10! + 10!a_5^2 + 2\times 10!\left(a_4a_6 + a_3a_7\right) & m=10 \\ \vdots & \vdots \end{cases} \tag{30}$$

The computation above is made using the software Mathematica version 9. For $n \geq 3$; substituting (30) in (22) yields

$$\begin{aligned} &2a_{n+1}s_{n+1} + 2\Gamma\left(\frac{13}{10}\right)a_{n-2}r_{n-2} \\ &\quad = \frac{1}{(n-4)!}\frac{d^{n-4}}{dw^{n-4}}(g(w))\bigg|_{w=0}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} s_{n+1} &= \frac{\Gamma((n+1)/10+1)}{\Gamma((n+1)/10+1/2)}, \\ r_{n-2} &= \frac{\Gamma((n-2)/10+1)}{\Gamma((n-2)/10+4/5)}. \end{aligned} \tag{32}$$

Applying (31) together with $a_1 = a_2 = a_3 = 0$, we have

$$\begin{aligned} a_4 &= 0, \\ a_5 &= 1, \\ a_n &= 0, \\ n &\geq 6. \end{aligned} \tag{33}$$

Thus,

$$u(t) = a_5 t^{5/10} = t^{1/2}, \tag{34}$$

and the exact solution of problem (25)-(26) is obtained.

In the following we compare our results with the Adomian decomposition method (ADM). Assume that the nonlinear function $f(u) = u^2$ and the solution $u(t)$ can be expressed in the following series form:

$$\begin{aligned} u(t) &= \sum_{n=0}^{\infty} u_n(t), \\ f(u) &= \sum_{n=0}^{\infty} A_n, \end{aligned} \tag{35}$$

where $A_n, n = 0, 1, 2, \ldots$ are the well-known Adomian polynomials that can be derived from [17]

$$A_n = \frac{1}{n!}\frac{d^n}{d\beta^n}\left[f\left(\sum_{j=0}^{\infty}\beta^j u_j\right)\right]_{\beta=0}, \quad j \geq 0. \tag{36}$$

For recent advances in the Adomian decomposition method we refer the reader to [18, 19]. For $f(u) = u^2$, we have

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\ &\vdots \end{aligned} \tag{37}$$

Applying the Riemann-Liouville fractional integral operator $I_{0^+}^{1/2}$ to (25) and substituting

$$I_{0^+}^{1/2} D_{0^+}^{1/5} u(t) = I_{0^+}^{3/10} u(t), \tag{38}$$

we have

$$\begin{aligned} u(t) &= \frac{\Gamma(1/2)}{2} I_{0^+}^{1/2}\left(1 + t^{3/10} - t\right) - \Gamma\left(\frac{13}{10}\right) I_{0^+}^{3/10} u(t) \\ &\quad + \frac{\Gamma(1/2)}{2} I_{0^+}^{1/2}\left(u^2(t)\right) \\ &= t^{1/2} + 0.853958t^{4/5} - 0.666667t^{3/2} \\ &\quad - \Gamma\left(\frac{13}{10}\right) I_{0^+}^{3/10}\left(\sum_{n=0}^{\infty} u_n(t)\right) \\ &\quad + \frac{\Gamma(1/2)}{2} I_{0^+}^{1/2}\left(\sum_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} u_n(t). \end{aligned} \tag{39}$$

Set

$$u_0 = t^{1/2} + 0.853958t^{4/5} - 0.666667t^{3/2} \quad (40)$$

and balancing (39) yields

$$u_{n+1} = -\Gamma\left(\frac{13}{10}\right) I_{0^+}^{3/10}(u_n) + \frac{\Gamma(1/2)}{2} I_{0^+}^{1/2}(A_n), \quad n = 0, 1, 2, \ldots \quad (41)$$

Evaluating the first 5 term of the power series solution, we obtain the approximate solution $u^{(4)}(t) = \sum_{n=0}^{4} u_n(t)$, where

$$\begin{aligned} u^{(4)}(t) = {} & t^{0.5} + 0.257998t^2 - 2.798359t^{2.7} \\ & - 1.454522t^3 + 8.855605t^{3.4} + 9.799202t^{3.7} \\ & + 2.317936t^4 - 11.599906t^{4.1} \\ & - 22.429449t^{4.4} - 11.273123t^{4.7} \\ & + 6.685081t^{4.8} - 1.388180t^5 \\ & + 23.109057t^{5.1} + 20.278401t^{5.4} \\ & - 1.404129t^{5.5} + 5.305238t^{5.7} \\ & - 11.070905t^{5.8} + 0.278605t^6 \\ & - 17.334122t^{6.1} - 7.918245t^{6.4} \\ & + 2.012699t^{6.5} - 0.885674t^{6.7} \\ & + 7.156236t^{6.8} + 5.827855t^{7.1} \\ & + 1.140180t^{7.4} - 1.153180t^{7.5} \\ & - 2.127835t^{7.8} - 0.742611t^{8.1} \\ & + 0.309488t^{8.5} + 0.244589t^{8.8} \\ & - 0.032578t^{9.5}, \end{aligned} \quad (42)$$

which is not the exact solution. Solving the problem by the ADM with $u_0 = t^{0.5}$ yields $U^{(4)}(t) = \sum_{n=0}^{4} U_n(t)$, where

$$\begin{aligned} U^{(4)}(t) = {} & t^{0.5} - 0.853958t^{0.8} + 0.682106t^{1.1} \\ & - 0.515733t^{1.4} + 0.666667t^{1.5} \\ & + 0.0372208t^{1.7} - 1.527774t^{1.8} \\ & + 2.25291t^{2.1} - 2.68363t^{2.4} + 0.711111t^{2.5} \\ & - 2.576649t^{2.8} + 5.473545t^{3.1} \\ & + 0.8533333t^{3.5} - 4.19714t^{3.8} \\ & + 1.078789t^{4.5}, \end{aligned} \quad (43)$$

which is again not the exact solution. Figure 1 depicts the exact solution $u(t)$ obtained by the current algorithm and the approximate solutions $u^{(4)}(t)$ and $U^{(4)}(t)$ obtained by the Adomian decomposition method.

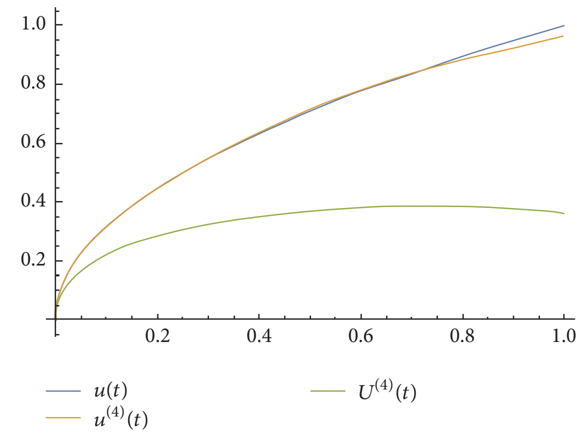

Figure 1: Comparison between the exact and approximate solutions obtained by the ADM for problem ((25)-(26)).

Example 4. Consider the nonlinear two-term fractional initial value problem

$$D_{0^+}^{4/5} u(t) + D_{0^+}^{1/2} u(t) = \frac{u(t)\sin(t)}{u^2(t) + 1}, \quad t > 0, \quad (44)$$

with

$$u(0) = 1. \quad (45)$$

This example has been discussed in [20], where the problem is transformed to a fractional integral equation, and then the Adams-Bashforth-Moulton method is used with step size $h = 1/50$ to approximate the solution.

Applying the current algorithm, we have $p_1 = 4, q_1 = 5, p_2 = 1, q_2 = 2, q = \text{l.c.m}(5,2) = 10, s = 2, r = 5, k = 2$, and

$$a_{n+1}s_{n+1} + a_{n-2}r_{n-2} = \frac{1}{(n-7)!} \left(\frac{d^{n-7}}{dw^{n-7}} f\left(w^{10}, \sum_{n=0}^{\infty} a_n w^n\right)\right)\Bigg|_{w=0}, \quad (46)$$

where

$$f\left(w^{10}, \sum_{n=0}^{\infty} a_n w^n\right) = \frac{\sin(w^{10}) \sum_{n=0}^{\infty} a_n w^n}{\left(\sum_{n=0}^{\infty} a_n w^n\right)^2 + 1}. \quad (47)$$

Table 1: The error of Example 4 for $N = 35$ and different values of t.

t_i	$E_{35}(t_i)$
0.1	1.16844×10^{-10}
0.2	8.36081×10^{-9}
0.3	9.93967×10^{-8}
0.4	5.28261×10^{-7}
0.5	1.93074×10^{-6}
0.6	5.47664×10^{-6}
0.7	0.0000130833
0.8	0.0000276624
0.9	0.0000534896
1.0	0.0000967893

Table 2: The error of Example 4 at $t = 1$ and for different values of N.

N	$E_N(1)$
5	0.0681689
10	0.0681689
15	0.0681689
20	0.067155
25	0.0248829
30	0.00882027
35	0.0000967893

The initial condition $u(0) = 1$ yields $a_0 = 1$. We apply (46) to compute the first 13 nonzero terms of a_n, and we have

$$\begin{aligned} u(t) = {} & 1 + 0.298242t^{1.8} - 0.227519t^{2.1} \\ & + 0.167717t^{2.4} - 0.119885t^{2.7} + 0.0833333t^{3} \\ & - 0.0564631t^{3.3} + 0.0373656t^{3.6} \\ & - 0.0280303t^{3.8} - 0.0241927t^{3.9} \\ & + 0.0179008t^{4.1} + 0.0153477t^{4.2} \\ & - 0.0112111t^{4.4}. \end{aligned} \tag{48}$$

Since the exact solution of problem (44)-(45) is not available in a closed form, we define the error $E_N(t)$ by

$$E_N(t_i) = \int_0^{t_i} (\mathscr{P}u_N(t))^2 \, dt, \quad t \in [0, t_i], \tag{49}$$

where

$$\begin{aligned} \mathscr{P}u(t) &= D_{0^+}^{4/5}u(t) + D_{0^+}^{1/2}u(t) - \frac{u(t)\sin(t)}{u^2(t)+1}, \\ u_N(t) &= \sum_{k=0}^{N} a_k t^{k/10}. \end{aligned} \tag{50}$$

Tables 1 and 2 present the error E_N for different values of N. One can see that the error decreases with N and more accuracy can be achieved by considering more terms. Also, the error increases with t, as any other series solutions.

3. Three-Term Fractional Differential Equations

We consider the nonlinear three-term fractional initial value problem of the form

$$c_1 D_{0^+}^{\alpha_1}u(t) + c_2 D_{0^+}^{\alpha_2}u(t) + c_3 D_{0^+}^{\alpha_3}u(t) = f(t, y(t)), \quad t > 0, \tag{51}$$

with

$$u^{(i)}(0) = b_i, \quad i = 0, 1, \ldots, N_1 - 1, \tag{52}$$

where

$$\begin{gathered} 0 < \alpha_3 \le \alpha_2 \le \alpha_1 < N_1, \\ N_1 - 1 < \alpha_1 < N_1, \\ N_2 - 1 < \alpha_2 \le N_2 \le N_1, \\ 0 < N_3 - 1 < \alpha_3 \le N_3 \le N_2, \end{gathered} \tag{53}$$

and c_1, c_2, and c_3 are nonzero constants. We assume that $f(t, y(t))$ is continuous and smooth with respect to $u(t)$. We also assume that α_1, α_2, and α_3 are rational numbers with $\alpha_1 = p_1/q_1, \alpha_2 = p_2/q_2$, and $\alpha_3 = p_3/q_3$, where $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{N}$. Let $q = \operatorname{lcm}(q_1, q_2, q_3)$; we have $q = sq_1 = rq_2 = vq_3$ for s, r and $v \in \mathbb{N}$.

Applying analogous steps for the case of the two-term fractional equations, we seek a solution of the form

$$\begin{aligned} u(t) &= \sum_{n=0}^{\infty} a_n t^{n/q} = \sum_{n=0}^{(N_1-1)q} a_n t^{n/q} + \sum_{n=(N_1-1)q+1}^{\infty} a_n t^{n/q} \\ &= \sum_{n=0}^{N_1-1} a_{nq} t^{n} + \sum_{n=(N_1-1)q+1}^{\infty} a_n t^{n/q}. \end{aligned} \tag{54}$$

From the initial condition (52) we have $u^{(i)}(0) = b_i = i! a_{iq}$, and thus, $a_{iq} = b_i/i!$, $i = 0, 1, \ldots, N_1 - 1$.

Using the fact that $D_{0^+}^{\alpha}(t^n) = 0$, for $\alpha > n \in \mathbb{N}$, we have

$$\begin{aligned} D_{0^+}^{\alpha_1}u(t) &= \sum_{n=(N_1-1)q+1}^{\infty} a_n s_n t^{(n-sp_1)/q}, \\ D_{0^+}^{\alpha_2}u(t) &= \sum_{n=N_2}^{N_1-1} a_{nq} \rho_{nq} t^{n-\alpha_2} + \sum_{n=(N_1-1)q+1}^{\infty} a_n r_n t^{(n-rp_2)/q}, \\ D_{0^+}^{\alpha_3}u(t) &= \sum_{n=N_3}^{N_1-1} a_{nq} \nu_{nq} t^{n-\alpha_3} + \sum_{n=(N_1-1)q+1}^{\infty} a_n v_n t^{(n-vp_3)/q}, \end{aligned} \tag{55}$$

where $s_n = \Gamma(n/q+1)/\Gamma(n/q-\alpha_1+1)$, $r_n = \Gamma(n/q+1)/\Gamma(n/q-\alpha_2+1)$, $\rho_n = \Gamma(n+1)/\Gamma(n+1-\alpha_2)$, $v_n = \Gamma(n/q+1)/\Gamma(n/q-\alpha_3+1)$, and $\nu_n = \Gamma(n+1)/\Gamma(n+1-\alpha_3)$.

Substituting (55) in (51), we have

$$\begin{aligned} &c_1 \sum_{n=(N_1-1)q+1}^{\infty} a_n s_n t^{(n-sp_1)/q} + c_2 \sum_{n=N_2}^{N_1-1} a_{nq}\rho_{nq} t^{n-\alpha_2} \\ &\quad + c_2 \sum_{n=(N_1-1)q+1}^{\infty} a_n r_n t^{(n-rp_2)/q} + c_3 \sum_{n=N_3}^{N_1-1} a_{nq}\nu_{nq} t^{n-\alpha_3} \\ &\quad + c_3 \sum_{n=(N_1-1)q+1}^{\infty} a_n \nu_n t^{(n-\nu p_3)/q} = f\left(t, \sum_{n=0}^{\infty} a_n t^{n/q}\right). \end{aligned} \tag{56}$$

Since the coefficients $a_{nq}, n = 0, \ldots, N_1 - 1$, are known, we write the last equation in the form

$$\begin{aligned} &c_1 \sum_{n=(N_1-1)q+1}^{\infty} a_n s_n t^{(n-sp_1)/q} + c_2 \sum_{n=(N_1-1)q+1}^{\infty} a_n r_n t^{(n-rp_2)/q} \\ &\quad + c_3 \sum_{n=(N_1-1)q+1}^{\infty} a_n \nu_n t^{(n-\nu p_3)/q} \\ &\quad = f\left(t, \sum_{n=0}^{\infty} a_n t^{n/q}\right) - c_2 \sum_{n=N_2}^{N_1-1} a_{nq}\rho_{nq} t^{n-\alpha_2} \\ &\quad - c_3 \sum_{n=N_3}^{N_1-1} a_{nq}\nu_{nq} t^{n-\alpha_3}. \end{aligned} \tag{57}$$

Let $w = t^{1/q}$; then (57) reduces to

$$\begin{aligned} &\sum_{n=(N_1-1)q+1}^{\infty} c_1 a_n s_n w^{n-sp_1} + \sum_{n=(N_1-1)q+1}^{\infty} c_2 a_n r_n w^{n-rp_2} \\ &\quad + \sum_{n=(N_1-1)q+1}^{\infty} c_3 a_n \nu_n w^{n-\nu p_3} \\ &\quad = f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right) - \sum_{n=N_2}^{N_1-1} c_2 a_{nq}\rho_{nq} w^{q(n-\alpha_2)} \\ &\quad - \sum_{n=N_3}^{N_1-1} c_3 a_{nq}\nu_{nq} w^{q(n-\alpha_3)}. \end{aligned} \tag{58}$$

Substituting $z = (N_1 - 1)q + 1$ and

$$\begin{aligned} g(w) &= f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right) - \sum_{n=N_2}^{N_1-1} c_2 a_{nq}\rho_{nq} w^{q(n-\alpha_2)} \\ &\quad - \sum_{n=N_3}^{N_1-1} c_3 a_{nq}\nu_{nq} w^{q(n-\alpha_3)} \end{aligned} \tag{59}$$

and shifting the indices to zero, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} c_1 a_{n+z} s_{n+z} w^{n-sp_1+z} + \sum_{n=0}^{\infty} c_2 a_{n+z} r_{n+z} w^{n-rp_2+z} \\ &\quad + \sum_{n=0}^{\infty} c_3 a_{n+z} \nu_{n+z} w^{n-\nu p_3+z} = g(w). \end{aligned} \tag{60}$$

To avoid singularity we multiply both sides in (60) by w^{sp_1-z}; we have

$$\begin{aligned} &\sum_{n=0}^{\infty} c_1 a_{n+z} s_{n+z} w^{n} + \sum_{n=0}^{\infty} c_2 a_{n+z} r_{n+z} w^{n-rp_2+sp_1} \\ &\quad + \sum_{n=0}^{\infty} c_3 a_{n+z} \nu_{n+z} w^{n-\nu p_3+sp_1} = w^{sp_1-z} g(w). \end{aligned} \tag{61}$$

Equation (61) can be written as

$$\begin{aligned} &\sum_{n=0}^{k_1-1} c_1 a_{n+z} s_{n+z} w^n + \sum_{n=k_1}^{k_2-1} \left(c_1 a_{n+z} s_{n+z} + c_2 a_{n+z-k_1} r_{n+z-k_1}\right) \\ &\quad \cdot w^n + \sum_{n=k_2}^{\infty} \left(c_1 a_{n+z} s_{n+z} + c_2 a_{n+z-k_1} r_{n+z-k_1}\right. \\ &\quad \left. + c_3 a_{n+z-k_2} \nu_{n+z-k_2}\right) w^n = w^{sp_1-z} g(w), \end{aligned} \tag{62}$$

where $k_1 = sp_1 - rp_2$ and $k_2 = sp_1 - \nu p_3$.

Applying analogs steps for deriving (22), we have

$$\begin{aligned} &c_1 a_{n+z} s_{n+z} + c_2 a_{n+z-k_1} r_{n+z-k_1} + c_3 a_{n+z-k_2} \nu_{n+z-k_2} \\ &\quad = \frac{1}{(n-j)!} \left[\frac{d^{n-j}}{dw^{n-j}} g(w)\right]_{w=0} \quad \text{for } n \geq k_2, \end{aligned} \tag{63}$$

where $j = sp_1 - z$. From the last recursion we determine $a_{n+z}, n \geq k_2$.

Remark 5. Our derivation is based on the facts that the nonlinear function is smooth and the fractional differential equations is of constant coefficients. In case if one of these conditions does not hold, a modified treatment will be considered as we will see in Example 7.

3.1. Numerical Results

Example 6. Consider the Bagely-Torvik initial value problem

$$D_{0^+}^{5/2} u(t) + D_{0^+}^{2} u(t) - 2\sqrt{\pi} D_{0^+}^{1/2} u(t) + 4u(t) = g(t), \quad t \in [0, 1], \tag{64}$$

$$u(0) = u'(0) = u''(0) = 0, \tag{65}$$

where

$$g(t) = 4t^9 - \frac{131072}{12155} t^{17/2} + 72t^7 + \frac{49152}{143\sqrt{\pi}} t^{13/2}. \tag{66}$$

This example has been discussed in [21] using a Chebyshev spectral method, where the solution has been approximated by the shifted Chebyshev polynomials with different degrees. Then the exact solution $u(t) = t^9$ was obtained by considering the shifted Chebyshev polynomial of degree 9. We mention here that there are simple typos in presenting the example in (64) and in (66) and we correct them here. Applying the current algorithm we have

$$\begin{aligned} p_1 &= 5, \\ p_2 &= 4, \\ p_3 &= 1, \\ q_1 &= q_2 = q_3 = 2, \\ q &= \operatorname{lcm}(q_1, q_2, q_3) = 2, \\ s &= 1, \\ r &= 1, \\ v &= 1, \\ k_1 &= 1, \\ k_2 &= 4. \end{aligned} \tag{67}$$

We expand the solution in infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{n/2}$. The initial condition in (65) yields $a_0 = a_2 = a_4 = 0$. Let $t = w^2$; applying (59) we have

$$\begin{aligned} g(w) = {} & 4w^{18} - \frac{131072}{12155} w^{17} + 72w^{14} + \frac{49152}{143\sqrt{\pi}} w^{13} \\ & - 4\left(\sum_{n=0}^{\infty} a_n w^n\right), \end{aligned} \tag{68}$$

and it holds that

$$\left.\frac{d^m}{dw^m} g(w)\right|_{w=0} = \begin{cases} -4a_1 & \text{if } m = 1 \\ -4 \times 2! a_2 & \text{if } m = 2 \\ \vdots & \vdots \\ 13!\left(\dfrac{49152}{143\sqrt{\pi}} - 4a_{13}\right) & \text{if } m = 13 \\ 14!\,(72 - 4a_{14}) & \text{if } m = 14 \\ -4 \times 15! a_{15} & \text{if } m = 15 \\ -4 \times 16! a_{16} & \text{if } m = 16 \\ 17!\left(-\dfrac{131072}{12155} - 4a_{17}\right) & \text{if } m = 17 \\ 18!\,(4 - 4a_{18}) & \text{if } m = 18 \\ -4! a_{19} & \text{if } m = 19 \\ \vdots & \vdots \end{cases} \tag{69}$$

Using (63) and (69), we have

$$\begin{aligned} & a_{n+5} s_{n+5} + a_{n+4} r_{n+4} - 2\sqrt{\pi} a_{n+1} v_{n+1} \\ & \quad = \frac{1}{n!}\left.\left(\frac{d^n}{dw^n} g(w)\right)\right|_{w=0}, \quad n \geq 4, \end{aligned} \tag{70}$$

where

$$\begin{aligned} s_{n+5} &= \frac{\Gamma((n+5)/2+1)}{\Gamma((n+5)/2-3/2)}, \\ r_{n+4} &= \frac{(n+4)(n+2)}{4}, \\ v_{n+1} &= \frac{\Gamma((n+1)/2+1)}{\Gamma((n+1)/2+1/2)}. \end{aligned} \tag{71}$$

Since $g(w)$ is smooth, then $a_{n+5} = 0$, for $n < 4$. We now apply the last recursion together with $a_0 = \cdots = a_8 = 0$, to compute a_{n+5}, for $n \geq 4$. For $n = 4$, we have

$$a_9 s_9 + a_8 r_8 - 2\sqrt{\pi} a_5 v_5 = g^{(4)}(0) = 0, \tag{72}$$

and thus $a_9 = 0$.

Applying analogous arguments yields $a_{10} = \cdots = a_{17} = 0$. For $n = 13$, we have

$$a_{18} s_{18} + a_{17} r_{17} - 2\sqrt{\pi} a_{14} v_{14} = \frac{49152}{143\sqrt{\pi}} - 4a_{13}, \tag{73}$$

which, together with $a_{17} = a_{14} = a_{13} = 0$, implies that $a_{18} = 1$. For $n = 14$, we have

$$a_{19} s_{19} + a_{18} r_{18} - 2\sqrt{\pi} a_{15} v_{15} = 72 - 4a_{14}, \tag{74}$$

but $a_{18} = 1$ and $a_{15} = a_{14} = 0$; thus $a_{19} = 0$. Applying the same steps yields $a_{20} = a_{21} = 0$. For $n = 17$, we have

$$a_{22} s_{22} + a_{21} r_{21} - 2\sqrt{\pi} a_{18} v_{18} = -\frac{131072}{12155} - 4a_{17}, \tag{75}$$

and thus $a_{22} = 0$.

Proceeding in the same manner, we have $a_{n+1} = 0$ for $n \geq 18$. Thus,

$$u(t) = a_{18} t^{18/2} = t^9, \tag{76}$$

and the exact solution of problem (64)-(65) is obtained.

In the following example we show that the current algorithm can be applied to more general multiterm fractional differential equations which are not necessary of the form in (51).

Example 7. Consider the nonlinear three-term fractional initial value problem

$$\begin{aligned} & t D_{0^+}^{4/3} u(t) + D_{0^+}^{1/3} u(t) + t^{19/6} D_{0^+}^{1/6} u(t) \\ & \quad = \frac{33}{4\Gamma(8/3)} t^{8/3} + \frac{1296}{935\Gamma(5/6)} u^2(t), \end{aligned} \tag{77}$$

with

$$u(0) = u'(0) = 0. \tag{78}$$

The exact solution for this problem is $u(t) = t^3$.

Applying the current algorithm we have $q = 6$. We then expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{n/6}$. From the initial condition (78), we have $a_0 = a_6 = 0$. To guarantee the existence of the fractional derivative $D_{0^+}^{4/3} u(t)$ we choose $a_1 = \cdots = a_5 = 0$. Let $w = t^{1/6}$; then (77) reduces to

$$\begin{aligned} a_1 r_1 &+ \sum_{n=0}^{18} \left(a_{n+2} s_{n+2} + a_{n+2} r_{n+2}\right) w^n \\ &+ \sum_{n=19}^{\infty} \left(a_{n+2} s_{n+2} + a_{n+2} r_{n+2} + a_{n-18} v_{n-18}\right) w^n \\ &= \frac{33 w^{16}}{4\Gamma(8/3)} + \frac{1296}{935\Gamma(5/6)} \left(\sum_{n=0}^{\infty} a_n w^n\right)^2, \end{aligned} \tag{79}$$

where $s_n = \Gamma(n/6+1)/\Gamma(n/6-1/3)$, $r_n = \Gamma(n/6+1)/\Gamma(n/6+2/3)$, and $v_n = \Gamma(n/6+1)/\Gamma(n/6+5/6)$.

Performing the nth derivatives one can see that $a_n = 0$ for $n < 16$. For $n = 16$, we have

$$a_{18} s_{18} + a_{18} r_{18} = \frac{33}{4\Gamma(8/3)}, \tag{80}$$

which yields $a_{18} = 1$.

Also, $a_{19} = \cdots = a_{35} = 0$.

For $n = 36$, we have

$$a_{38} s_{38} + a_{38} r_{38} + a_{18} v_{18} = \frac{1296}{935\Gamma(5/6)}, \tag{81}$$

which yields $a_{38} = 0$.

Following analogous steps, we have $a_n = 0$ for $n > 36$. Thus

$$u(t) = a_{18} t^{18/6} = t^3, \tag{82}$$

which is the exact solution.

4. Conclusion

For fractional differential equations of order $n-1 < \alpha < n$, it is common to obtain a series solution in the form $\sum_{n=0}^{\infty} a_n t^{\alpha(n)}$. The question is how to obtain the coefficients $a_n, n = 0, 1, \ldots$. Naturally, if the problem is of fractional order, the differentiation is also of fractional order. In this paper, we presented a new algorithm for obtaining a series solution for nonlinear multiterm fractional differential equations of Caputo type, where we overcome the use of fractional differentiation. We employed a transformation that allows us to use ordinary differentiation rather than fractional differentiation to recursively compute the coefficient of the series expansion. Then the terms of the series, a_n, are computed sequentially using a closed form formula. We applied the new algorithm to several types of multiterm fractional differential equations, where accurate solutions as well as exact solutions in closed forms have been obtained. For one example it is noted that the current algorithm is more efficient than the ADM as it is more easier to apply and it produces the exact solution while the ADM does not. We have developed the new algorithm for two- and three-term fractional equations, while the idea can be extended to multiterm fractional equations of arbitrary order; obtaining a general formula in this case is not an easy task. The current algorithm can be modified to deal with the fractional multiterm time-diffusion equations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] M. Inc, "The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 476–484, 2008.

[2] L. Song and H. Zhang, "Application of homotopy analysis method to fractional KdV-Burgers-KURamoto equation," *Physics Letters A*, vol. 367, no. 1-2, pp. 88–94, 2007.

[3] N. H. Sweilam, M. M. Khader, and R. F. Al-Bar, "Numerical studies for a multi-order fractional differential equation," *Physics Letters A*, vol. 371, no. 1-2, pp. 26–33, 2007.

[4] V. Daftardar-Gejji and S. Bhalekar, "Solving multi-term linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 113–120, 2008.

[5] W. He, "Adomian decomposition method for fractional differential equations of Caputo Hadamard type," *Journal of Computational Complexity and Applications*, vol. 2, no. 4, pp. 160–162, 2016.

[6] H. Jafari and V. Daftardar-Gejji, "Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 488–497, 2006.

[7] S. Momani and K. Al-Khaled, "Numerical solutions for systems of fractional differential equations by the decomposition method," *Applied Mathematics and Computation*, vol. 162, no. 3, pp. 1351–1365, 2005.

[8] G. C. Wu, D. Baleanu, and H. P. Xie, "Riesz Riemann-Liouville difference on discrete domains," *Chaos*, vol. 26, no. 8, Article ID 084308, 2016.

[9] C. Yang and J. Hou, "An approximate solution of nonlinear fractional differential equation by laplace transform and adomian polynomials," *Journal of Information and Computational Science*, vol. 10, no. 1, pp. 213–222, 2013.

[10] A. Elsaid, M. S. Abdel Latif, and F. M. Kamal, "Series solution for fractional Riccati differential equation and its convergence," *Journal of Fractional Calculus and Applications*, vol. 6, no. 2, pp. 186–196, 2015.

[11] H. M. Srivastava, J. A. Tenreiro Machado, and X.-J. Yang, "Approximate methods for local fractional differential equations," in *Fractional Dynamics*, C. Cattani, H. M. Srivastava, and X.-J. Yang, Eds., pp. 244–258, Emerging Science Publishers (De Gruyter Open), Berlin, Germany, 2015.

[12] X.-J. Yang, D. Baleanu, and H. M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press (Elsevier Science Publishers), Amsterdam, The Netherlands, 2016.

[13] M. Al-Refai, M. A. Hajji, and M. Syam, "An efficient series solution for fractional differential equations," *Abstract and Applied Analysis*, vol. 2014, Article ID 891837, 7 pages, 2014.

[14] M. K. Al-Srihin, *Series solutions of multi-term fractional differential equations [M.S. thesis]*, United Arab Emirates University, Al Ain, United Arab Emirates, 2015.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematical Studies*, Elsevier Science Publishers, Amsterdam, The Netherlands, 2006.

[16] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[17] G. Adomian, "A review of the decomposition method and some recent resuluts for nonlinear equations," *Mathematical and Computer Modelling*, vol. 13, no. 7, pp. 17–43, 1990.

[18] J.-S. Duan, "Convenient analytic recurrence algorithms for the Adomian polynomials," *Applied Mathematics and Computation*, vol. 217, no. 13, pp. 6337–6348, 2011.

[19] L.-L. Huang, G.-C. Wu, M. M. Rashidi, and W.-H. Luo, "Chaos analysis of the nonlinear duffing oscillators based on the new Adomian polynomials," *Journal of Nonlinear Science and its Applications. JNSA*, vol. 9, no. 4, pp. 1877–1881, 2016.

[20] M. Blasik, "Numerical scheme for a two-term sequential fractional differential equation," *Scientific Research of the Institute of Mathematics and Computer Science*, vol. 2, no. 10, pp. 17–29, 2011.

[21] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, "A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order," *Computers & Mathematics with Applications*, vol. 62, no. 5, pp. 2364–2373, 2011.

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Research Article

Hartman-Wintner-Type Inequality for a Fractional Boundary Value Problem via a Fractional Derivative with respect to Another Function

Mohamed Jleli,[1] Mokhtar Kirane,[2] and Bessem Samet[1]

[1] *Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *LaSIE, Pôle Sciences et Technologies, Université de La Rochelle, avenue M. Crépeau, 17042 La Rochelle Cedex, France*

Correspondence should be addressed to Bessem Samet; bsamet@ksu.edu.sa

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We consider a fractional boundary value problem involving a fractional derivative with respect to a certain function g. A Hartman-Wintner-type inequality is obtained for such problem. Next, several Lyapunov-type inequalities are deduced for different choices of the function g. Moreover, some applications to eigenvalue problems are presented.

1. Introduction

In this work, we are concerned with the following fractional boundary value problem:

$$\begin{gathered}\left(D_{a,g}^{\alpha}u\right)(t)+q(t)\,u(t)=0,\quad a<t<b,\\ u(a)=u(b)=0,\end{gathered}\tag{1}$$

where $(a,b)\in\mathbb{R}^2$, $a<b$, $\alpha\in(1,2)$, $q:[a,b]\to\mathbb{R}$ is a continuous function, and $D_{a,g}^{\alpha}$ is the fractional derivative operator of order α with respect to a certain nondecreasing function $g\in C^1([a,b];\mathbb{R})$ with $g'(x)>0$, for all $x\in(a,b)$. A Hartman-Wintner-type inequality is derived for problem (1). As a consequence, several Lyapunov-type inequalities are deduced for different types of fractional derivatives. Next, we end the paper with some applications to eigenvalue problems.

Let us start by describing some historical backgrounds about Lyapunov inequality and some related works. In the late XIX century, the mathematician A. M. Lyapunov established the following result (see [1]).

Theorem 1. *If the boundary value problem*

$$\begin{gathered}u''(t)+q(t)\,u(t)=0,\quad a<t<b,\\ u(a)=u(b)=0\end{gathered}\tag{2}$$

has a nontrivial solution, where $q:[a,b]\to\mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\,ds>\frac{4}{b-a}.\tag{3}$$

Inequality (3) is known as Lyapunov inequality. It is proved to be very useful in various problems in connection with differential equations, including oscillation theory, asymptotic theory, eigenvalue problems, and disconjugacy. For more details, we refer the reader to [2–12] and references therein.

In [8], Hartman and Wintner proved that if boundary value problem (2) has a nontrivial solution, then

$$\int_a^b (s-a)(b-s)\,q^+(s)\,ds>b-a,\tag{4}$$

where

$$q^+(s)=\max\{q(s),0\},\quad s\in[a,b].\tag{5}$$

Using the fact that

$$\max_{a\le s\le b}(s-a)(b-s)=\frac{(b-a)^2}{4},\tag{6}$$

Lyapunov inequality (3) follows immediately from inequality (4). Many other generalizations and extensions of inequality (3) exist in the literature; see, for instance, [7, 13–22] and references therein.

Due to the positive impact of fractional calculus on several applied sciences (see, for instance, [23]), several authors investigated Lyapunov-type inequalities for various classes of fractional boundary value problems. The first work in this direction is due to Ferreira [24], where he considered the fractional boundary value problem

$$\begin{gathered}\left({}^{\mathrm{RL}}D_a^{\alpha}u\right)(t)+q(t)\,u(t)=0,\quad a<t<b,\\ u(a)=u(b)=0,\end{gathered}\tag{7}$$

where $(a,b)\in\mathbb{R}^2$, $a<b$, $\alpha\in(1,2)$, $q:[a,b]\to\mathbb{R}$ is a continuous function, and ${}^{\mathrm{RL}}D_a^{\alpha}$ is the Riemann-Liouville fractional derivative operator of order α. The main result obtained in [24] is the following fractional version of Theorem 1.

Theorem 2. *If fractional boundary value problem (7) has a nontrivial solution, then*

$$\int_a^b |q(s)|\,ds>\Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1},\tag{8}$$

where Γ is the Gamma function.

Observe that (3) can be deduced from Theorem 2 by passing to the limit as $\alpha\to 2$ in (8). For other related works, we refer the reader to Ferreira [25, 26], Jleli and Samet [27, 28], Jleli et al. [29, 30], O'Regan and Samet [31], Al Arifi et al. [32], Rong and Bai [33], Chidouh and Torres [34], Agarwal and Özbekler [35], Ma [36], and the references therein.

Very recently, Ma et al. [37] investigated the fractional boundary value problem

$$\begin{gathered}\left({}^{H}D_1^{\alpha}u\right)(t)+q(t)\,u(t)=0,\quad 1<t<e,\\ u(1)=u(e)=0,\end{gathered}\tag{9}$$

where $\alpha\in(1,2)$, $q:[1,e]\to\mathbb{R}$ is a continuous function, and ${}^{H}D_1^{\alpha}$ is the Hadamard fractional derivative operator of order α. The main result in [37] is the following.

Theorem 3. *If fractional boundary value problem (9) has a nontrivial solution, then*

$$\int_1^e |q(s)|\,ds>\Gamma(\alpha)\,\lambda^{1-\alpha}(1-\lambda)^{1-\alpha}e^{\lambda},\tag{10}$$

where $\lambda=(2\alpha-1-\sqrt{(2\alpha-2)^2+1})/2$.

In the same paper [37], the authors formulated the following question: How to get the Lyapunov inequality for the following Hadamard fractional boundary value problem:

$$\begin{gathered}\left({}^{H}D_a^{\alpha}u\right)(t)+q(t)\,u(t)=0,\quad a<t<b,\\ u(a)=u(b)=0,\end{gathered}\tag{11}$$

where $(a,b)\in\mathbb{R}^2$, $1\le a<b$, $\alpha\in(1,2)$, and $q:[a,b]\to\mathbb{R}$ is a continuous function. Note that one of our obtained results is an answer to the above question.

2. Preliminaries

Before stating and proving the main results in this work, some preliminaries are needed.

Let $I=[a,b]$ be a certain interval in $\mathbb{R}$, where $(a,b)\in\mathbb{R}^2$, $a<b$. We denote by $\mathrm{AC}(I;\mathbb{R})$ the space of real valued and absolutely continuous functions on I. For $n=1,2,\ldots$, we denote by $\mathrm{AC}^n(I;\mathbb{R})$ the space of real valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on I with $f^{(n-1)}\in\mathrm{AC}(I;\mathbb{R})$; that is,

$$\begin{aligned}\mathrm{AC}^n(I;\mathbb{R})=\Big\{&f:I\to\mathbb{R}\text{ such that }D^{n-1}f\\ &\in\mathrm{AC}(I;\mathbb{R})\left(D=\frac{d}{dx}\right)\Big\}.\end{aligned}\tag{12}$$

Clearly, we have $\mathrm{AC}^1(I;\mathbb{R})=\mathrm{AC}(I;\mathbb{R})$.

Definition 4 (see [23]). Let $f\in L^1((a,b);\mathbb{R})$. The Riemann-Liouville fractional integral of order $\alpha>0$ of f is defined by

$$\left({}^{\mathrm{RL}}I_a^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\quad \text{a.e } t\in[a,b].\tag{13}$$

Definition 5 (see [23]). Let $\alpha>0$ and n be the smallest integer greater than or equal to α. Let $f:[a,b]\to\mathbb{R}$ be a function such that ${}^{\mathrm{RL}}I_a^{n-\alpha}f\in\mathrm{AC}^n([a,b];\mathbb{R})$. Then the Riemann-Liouville fractional derivative of order α of a function f is defined by

$$\begin{aligned}\left({}^{\mathrm{RL}}D_a^{\alpha}f\right)(t)&=\left(\frac{d}{dt}\right)^n {}^{\mathrm{RL}}I_a^{n-\alpha}f(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t\frac{f(s)}{(t-s)^{\alpha-n+1}}ds,\end{aligned}\tag{14}$$

for a.e. $t\in[a,b]$.

Let $\alpha>0$ and n be the smallest integer greater than or equal to α. By $\mathrm{AC}^{\alpha}([a,b];\mathbb{R})$ (see [38]), one denotes the set of all functions $f:[a,b]\to\mathbb{R}$ that have the representation:

$$\begin{aligned}f(t)=\sum_{i=0}^{n-1}\frac{c_i}{\Gamma(\alpha-n+1+i)}(t-a)^{\alpha-n+i}+{}^{\mathrm{RL}}I_a^{\alpha}\varphi(t),\\ \text{a.e } t\in[a,b],\end{aligned}\tag{15}$$

where $c_0,c_1,\ldots,c_{n-1}\in\mathbb{R}$ and $\varphi\in L^1((a,b);\mathbb{R})$.

Lemma 6 (see [38]). *Let $\alpha > 0$, n be the smallest integer greater than or equal to α, and $f \in L^1((a,b);\mathbb{R})$. Then ${}^{\mathrm{RL}}D_a^{\alpha} f(t)$ exists almost everywhere on $[a,b]$ if and only if $f \in \mathrm{AC}^{\alpha}([a,b];\mathbb{R})$; that is, f has representation (15). In such a case, one has*

$$\left({}^{\mathrm{RL}}D_a^{\alpha} f\right)(t) = \varphi(t), \quad a.e\ t \in [a,b]. \tag{16}$$

Let $g \in C^1([a,b];\mathbb{R})$ be a nondecreasing function with $g'(x) > 0$, for all $x \in (a,b)$.

Definition 7 (see [23]). Let $f \in L^1((a,b);\mathbb{R})$. The fractional integral of order $\alpha > 0$ of f with respect to the function g is defined by

$$\left(I_{a,g}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s) f(s)}{\left(g(t) - g(s)\right)^{1-\alpha}} ds, \quad \text{a.e } t \in [a,b]. \tag{17}$$

Definition 8 (see [23]). Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Let $f : [a,b] \to \mathbb{R}$ be a function such that $((1/g'(t))(d/dt))^n I_{a,g}^{n-\alpha} f$ exists almost everywhere on $[a,b]$. In this case, the fractional derivative of order α of f with respect to the function g is defined by

$$\begin{aligned} D_{a,g}^{\alpha} f(t) &= \left(\frac{1}{g'(t)} \frac{d}{dt}\right)^n I_{a,g}^{n-\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(t)} \frac{d}{dt}\right)^n \int_a^t \frac{g'(s) f(s)}{\left(g(t) - g(s)\right)^{\alpha-n+1}} ds, \end{aligned} \tag{18}$$

for a.e. $t \in [a,b]$.

The following lemma is crucial for the proof of our main result.

Lemma 9. *Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Suppose that the function $y \mapsto f(g^{-1}(y))$ belongs to the space $\mathrm{AC}^{\alpha}([g(a), g(b)];\mathbb{R})$. Then*

$$D_{a,g}^{\alpha} f\left(g^{-1}(y)\right) = {}^{\mathrm{RL}}D_{g(a)}^{\alpha} \left(f \circ g^{-1}\right)(y), \quad a.e\ y \in \left[g(a), g(b)\right]. \tag{19}$$

Proof. At first, observe that, from Lemma 6, ${}^{\mathrm{RL}}D_{g(a)}^{\alpha}(f \circ g^{-1})(y)$ exists for a.e. $y \in [g(a), g(b)]$. Now, using the change of variable $x = g^{-1}(y)$, $y \in (g(a), g(b))$, the chain rule yields

$$\frac{d}{dy} = \frac{d}{dx}\frac{dx}{dy} = \frac{d}{dx}\frac{1}{g'\left(g^{-1}(y)\right)} = \frac{1}{g'(x)}\frac{d}{dx}. \tag{20}$$

Therefore, we obtain

$$\begin{aligned} &{}^{\mathrm{RL}}D_{g(a)}^{\alpha}\left(f \circ g^{-1}\right)(y) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dy}\right)^n \int_{g(a)}^{y} \frac{f\left(g^{-1}(s)\right)}{\left(y - s\right)^{\alpha-n+1}} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(x)}\frac{d}{dx}\right)^n \int_{g(a)}^{g(x)} \frac{f\left(g^{-1}(s)\right)}{\left(g(x) - s\right)^{\alpha-n+1}} ds. \end{aligned} \tag{21}$$

Next, using the change of variable $t = g^{-1}(s)$, we obtain

$$\begin{aligned} &{}^{\mathrm{RL}}D_{g(a)}^{\alpha}\left(f \circ g^{-1}\right)(y) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(x)}\frac{d}{dx}\right)^n \int_a^x \frac{g'(t) f(t)}{\left(g(x) - g(t)\right)^{\alpha-n+1}} dt \\ &= D_{a,g}^{\alpha} f(x) = D_{a,g}^{\alpha} f\left(g^{-1}(y)\right), \end{aligned} \tag{22}$$

which proves the desired result. □

In the sequel, we denote by $\Xi_g([a,b];\mathbb{R})$ the functional space defined by

$$\Xi_g([a,b];\mathbb{R}) = \left\{f : [a,b] \to \mathbb{R} : f \circ g^{-1} \in \mathrm{AC}^{\alpha}\left(\left[g(a), g(b)\right];\mathbb{R}\right)\right\}. \tag{23}$$

Definition 10 (see [23]). Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Let $g(x) = \ln x$, where $x \in [a,b]$ and $0 < a < b$. The Hadamard fractional derivative of order α of $f \in \Xi_g([a,b];\mathbb{R})$ is defined by

$${}^{H}D_a^{\alpha} f(t) = D_{a,g}^{\alpha} f(t), \quad \text{a.e } t \in [a,b]. \tag{24}$$

We refer the reader to Ferreira [24] for the proofs of the following results.

Lemma 11. *Let $h \in C([A,B];\mathbb{R})$, $(A,B) \in \mathbb{R}^2$, $A < B$, and $1 < \alpha < 2$. Then $F \in \mathrm{AC}^{\alpha}([a,b];\mathbb{R}) \cap C([a,b];\mathbb{R})$ is a solution of the boundary value problem*

$$\begin{aligned} \left({}^{\mathrm{RL}}D_A^{\alpha} F\right)(t) + h(t) = 0, &\quad A < t < B, \\ F(A) = F(B) = 0 & \end{aligned} \tag{25}$$

if, and only if, F satisfies the integral equation

$$F(t) = \int_A^B G(t,s) h(s) ds, \quad A \le t \le B, \tag{26}$$

where

$$\Gamma(\alpha) G(t,s) = \begin{cases} \dfrac{(t-A)^{\alpha-1}}{(B-A)^{\alpha-1}}(B-s)^{\alpha-1} - (t-s)^{\alpha-1}, & A \le s \le t \le B, \\ \dfrac{(t-A)^{\alpha-1}}{(B-A)^{\alpha-1}}(B-s)^{\alpha-1}, & A \le t \le s \le B. \end{cases} \quad (27)$$

Lemma 12. *The Green function G defined by (27) satisfies the following properties:*

(i) $G(t,s) \ge 0$ *for all* $A \le t$, $s \le B$.

(ii) *For all* $s \in [A,B]$, *one has*

$$\max_{t\in[A,B]} G(t,s) = G(s,s). \quad (28)$$

3. A Hartman-Wintner-Type Inequality for Boundary Value Problem (1)

In this section, a Hartman-Wintner-type inequality is established for fractional boundary value problem (1).

Problem (1) is investigated under the following assumptions:

(A1) $\alpha \in (1,2)$.

(A2) $q \in C([a,b];\mathbb{R})$.

(A3) $g \in C^1([a,b];\mathbb{R})$.

(A4) g is a nondecreasing function with $g'(x) > 0$, for all $x \in (a,b)$.

We have the following result.

Theorem 13. *Under assumptions (A1)–(A4), if fractional boundary value problem (1) has a nontrivial solution $u \in \Xi_g([a,b];\mathbb{R}) \cap C([a,b];\mathbb{R})$, then*

$$\int_a^b [(g(s)-g(a))(g(b)-g(s))]^{\alpha-1} g'(s)|q(s)|\,ds \ge \Gamma(\alpha)(g(b)-g(a))^{\alpha-1}. \quad (29)$$

Proof. Suppose that u is a nontrivial solution of (1). Let us define the function $v : [g(a), g(b)] \to \mathbb{R}$ by

$$v(y) = u(g^{-1}(y)), \quad y \in [g(a), g(b)]. \quad (30)$$

Using Lemma 9, for all $y \in (g(a), g(b))$, we have

$$\begin{aligned} D^{\alpha}_{a,g} u(g^{-1}(y)) &= {}^{\mathrm{RL}}D^{\alpha}_{g(a)}(u \circ g^{-1})(y) \\ &= {}^{\mathrm{RL}}D^{\alpha}_{g(a)} v(y). \end{aligned} \quad (31)$$

On the other hand, since u is a solution of (1), we have

$$\begin{aligned} D^{\alpha}_{a,g} u(g^{-1}(y)) &= -q(g^{-1}(y))\,u(g^{-1}(y)), \\ &\qquad y \in (g(a), g(b)), \\ u(g^{-1}(g(a))) &= u(g^{-1}(g(b))) = 0. \end{aligned} \quad (32)$$

Therefore, v is a nontrivial solution of the Riemann-Liouville fractional boundary value problem

$$\begin{gathered} ({}^{\mathrm{RL}}D^{\alpha}_{A} v)(y) + Q(y)v(y) = 0, \quad A < y < B, \\ v(A) = v(B) = 0, \end{gathered} \quad (33)$$

where $A = g(a)$, $B = g(b)$, and $Q : [A,B] \to \mathbb{R}$ is the function defined by

$$Q(y) = q(g^{-1}(y)), \quad y \in [A,B]. \quad (34)$$

Now, by Lemma 11, we obtain

$$v(y) = \int_A^B G(y,s)Q(s)v(s)\,ds, \quad A \le y \le B, \quad (35)$$

where G is the Green function defined by (27). Next, let us consider the Banach space $C([A,B];\mathbb{R})$ equipped with the standard norm

$$\|v\|_\infty = \max\{|v(y)| : A \le y \le B\}. \quad (36)$$

Clearly, since v is nontrivial, we have $\|v\|_\infty > 0$. Further, using (35) and Lemma 12, we have

$$|v(y)| \le \|v\|_\infty \int_A^B G(s,s)|Q(s)|\,ds, \quad y \in [A,B], \quad (37)$$

which yields

$$\|v\|_\infty \le \|v\|_\infty \int_A^B G(s,s)|Q(s)|\,ds. \quad (38)$$

Therefore, we obtain

$$\int_A^B G(s,s)|Q(s)|\,ds \ge 1; \quad (39)$$

that is,

$$\int_{g(a)}^{g(b)} G(s,s)\left|q(g^{-1}(s))\right|\,ds \ge 1. \quad (40)$$

Using the change of variable $s = g(t)$, we get

$$\int_a^b G(g(t),g(t))|q(t)|\,g'(t)\,dt \ge 1. \quad (41)$$

Note that by (27) we have

$$G(g(t),g(t)) = \frac{(g(t)-g(a))^{\alpha-1}(g(b)-g(t))^{\alpha-1}}{\Gamma(\alpha)(g(b)-g(a))^{\alpha-1}}, \quad t \in [a,b]. \quad (42)$$

Therefore,

$$\int_a^b \left[(g(t)-g(a))(g(b)-g(t))\right]^{\alpha-1} g'(t)\,|q(t)|\,dt \geq \Gamma(\alpha)\,(g(b)-g(a))^{\alpha-1}, \tag{43}$$

which is desired inequality (29). □

4. Lyapunov-Type Inequalities for Different Choices of the Function g

In this section, using Theorem 13, several Lyapunov-type inequalities are deduced for different choices of the function g.

4.1. The Case $g(x)=x^\beta$, $\beta>0$. Taking $g(x)=x^\beta$, $\beta>0$, in Theorem 13, we deduce the following Hartman-Wintner-type inequality.

Corollary 14. *If fractional boundary value problem (1) has a nontrivial solution $u\in \Xi_g([a,b];\mathbb{R})\cap C([a,b];\mathbb{R})$, where $g(x)=x^\beta$, $x\in[a,b]$, $0<a<b$, then*

$$\int_a^b \left[\left(s^\beta-a^\beta\right)\left(b^\beta-s^\beta\right)\right]^{\alpha-1} s^{\beta-1}\,|q(s)|\,ds \geq \frac{\Gamma(\alpha)\left(b^\beta-a^\beta\right)^{\alpha-1}}{\beta}. \tag{44}$$

Next, let us define the function $\varphi_{\alpha,\beta}:[a,b]\to[0,\infty)$ by

$$\varphi_{\alpha,\beta}(s)=\left[\left(s^\beta-a^\beta\right)\left(b^\beta-s^\beta\right)\right]^{\alpha-1} s^{\beta-1},\quad s\in[a,b]. \tag{45}$$

Since $\varphi_{\alpha,\beta}$ is continuous on $[a,b]$ and $\varphi_{\alpha,\beta}(a)=\varphi_{\alpha,\beta}(b)=0$, there exists some $s^*(\alpha,\beta)\in(a,b)$ such that

$$\varphi_{\alpha,\beta}\left(s^*(\alpha,\beta)\right)=\max\left\{\varphi_{\alpha,\beta}(s): s\in[a,b]\right\}>0. \tag{46}$$

Therefore, from inequality (44), we obtain the following Lyapunov-type inequality.

Corollary 15. *If fractional boundary value problem (1) has a nontrivial solution $u\in \Xi_g([a,b];\mathbb{R})\cap C([a,b];\mathbb{R})$, where $g(x)=x^\beta$, $x\in[a,b]$, $0<a<b$, then*

$$\int_a^b |q(s)|\,ds \geq \frac{\Gamma(\alpha)\left(b^\beta-a^\beta\right)^{\alpha-1}}{\beta\varphi_{\alpha,\beta}\left(s^*(\alpha,\beta)\right)}. \tag{47}$$

In order to compute the value of $s^*(\alpha,\beta)$ for $\alpha\in(1,2)$ and $\beta>0$, we have to study the variations of the function $\varphi_{\alpha,\beta}$ defined by (45). Observe that

$$\varphi_{\alpha,\beta}(s)=\phi_{\alpha,\beta}\left(s^\beta\right),\quad s\in[a,b], \tag{48}$$

where $\phi_{\alpha,\beta}:[M,N]\to[0,\infty)$ is the function defined by

$$\phi_{\alpha,\beta}(x)=\left[(x-M)(N-x)\right]^{\alpha-1} x^{(\beta-1)/\beta},\quad x\in[M,N] \tag{49}$$

with $M=a^\beta$ and $N=b^\beta$. A simple computation yields

$$\phi'_{\alpha,\beta}(x)=\frac{\phi_{\alpha,\beta}(x)}{x(x-M)(N-x)}\Big((\gamma-2\delta)x^2 + (M+N)(\delta-\gamma)x+\gamma MN\Big), \tag{50}$$

for all $x\in(M,N)$, where $\gamma=(1-\beta)/\beta$ and $\delta=\alpha-1$. Next, we put

$$P(x)=(\gamma-2\delta)x^2+(M+N)(\delta-\gamma)x+\gamma MN,\quad x\in[M,N]. \tag{51}$$

We consider three cases.

Case 1 (if $\beta=1/(2\alpha-1)$). In this case, we have $\gamma=2\delta$ and $P(x)=0$ if and only if $x=2MN/(M+N)$. Moreover, we have $P(x)\geq0$ for $x\in[M,2MN/(M+N)]$ and $P(x)\leq0$ for $x\in[2MN/(M+N),N]$. Therefore,

$$\begin{aligned}\phi_{\alpha,\beta}\left(\frac{2MN}{M+N}\right)&=\max\left\{\phi_{\alpha,\beta}(x): x\in[M,N]\right\},\\ s^*(\alpha,\beta)&=\left(\frac{2MN}{M+N}\right)^{1/\beta}=\left(\frac{2}{a^\beta+b^\beta}\right)^{1/\beta}ab.\end{aligned} \tag{52}$$

Thus, in this case we obtain

$$\varphi_{\alpha,\beta}\left(s^*(\alpha,\beta)\right)=\left[\frac{\left(b^\beta-a^\beta\right)^2}{4(ab)^\beta}\right]^{\alpha-1}. \tag{53}$$

Next, using (53), we deduce from Corollary 15 the following Lyapunov-type inequality in the case $\beta(2\alpha-1)=1$.

Corollary 16 (the case $\beta(2\alpha-1)=1$). *If fractional boundary value problem (1) has a nontrivial solution $u\in \Xi_g([a,b];\mathbb{R})\cap C([a,b];\mathbb{R})$, where $g(x)=x^\beta$, $x\in[a,b]$, $0<a<b$, then*

$$\int_a^b |q(s)|\,ds \geq \frac{\Gamma(\alpha)}{\beta}\left[\frac{4(ab)^\beta}{b^\beta-a^\beta}\right]^{\alpha-1}. \tag{54}$$

Case 2 (if $0<\beta<1/(2\alpha-1)$). In this case, we have $\gamma>2\delta>0$ and $P(x)$ has two distinct zeros at

$$\begin{aligned}x_1&=\frac{(\gamma-\delta)(M+N)-\sqrt{\Delta}}{2(\gamma-2\delta)},\\ x_2&=\frac{(\gamma-\delta)(M+N)+\sqrt{\Delta}}{2(\gamma-2\delta)},\end{aligned} \tag{55}$$

where

$$\Delta=(M-N)^2(\delta-\gamma)^2+4MN\delta^2. \tag{56}$$

It can be easily seen that

$$0<M<x_1<N<x_2. \tag{57}$$

Moreover, we have $P(x) \geq 0$ for $x \in [M, x_1]$ and $P(x) \leq 0$ for $x \in [x_1, N]$. Therefore,

$$\phi_{\alpha,\beta}(x_1) = \max\left\{\phi_{\alpha,\beta}(x) : x \in [M, N]\right\},$$
$$s^*(\alpha,\beta) = x_1^{1/\beta} = \left(\frac{(1-\alpha\beta)\left(a^\beta + b^\beta\right) - \sqrt{\left(a^\beta - b^\beta\right)^2(1-\alpha\beta)^2 + 4a^\beta b^\beta \beta^2 (1-\alpha)^2}}{2\left[\beta(1-2\alpha)+1\right]}\right)^{1/\beta}. \tag{58}$$

Case 3 (if $\beta > 1/(2\alpha - 1)$). In this case, we have $\gamma < 2\delta$ and $P(x)$ has two distinct zeros at x_1 and x_2. It can be easily seen that

$$x_2 < M < x_1 < N. \tag{59}$$

Moreover, we have $P(x) \geq 0$ for $x \in [M, x_1]$ and $P(x) \leq 0$ for $x \in [x_1, N]$. Therefore,

$$\phi_{\alpha,\beta}(x_1) = \max\left\{\phi_{\alpha,\beta}(x) : x \in [M, N]\right\},$$
$$s^*(\alpha,\beta) = x_1^{1/\beta} = \left(\frac{(1-\alpha\beta)\left(a^\beta + b^\beta\right) - \sqrt{\left(a^\beta - b^\beta\right)^2(1-\alpha\beta)^2 + 4a^\beta b^\beta \beta^2 (1-\alpha)^2}}{2\left[\beta(1-2\alpha)+1\right]}\right)^{1/\beta}. \tag{60}$$

Observe that, for $\beta = 1$ ($g(x) = x$), problem (1) is equivalent to problem (7). Moreover, in this case we have

$$s^*(\alpha, 1) = x_1 = \frac{a+b}{2},$$
$$\varphi_{\alpha,1}\left(s^*(\alpha,1)\right) = \varphi_{\alpha,1}\left(\frac{a+b}{2}\right) = \left[\frac{(b-a)^2}{4}\right]^{\alpha-1}. \tag{61}$$

Therefore, using (61) and Corollary 15, we obtain inequality (8), which is due to Ferreira [24].

4.2. A Lyapunov-Type Inequality via Hadamard Fractional Derivative. Taking $g(x) = \ln x$ in Theorem 13, we deduce the following Hartman-Wintner-type inequality for the following Hadamard fractional boundary value problem:

$$\left({}^H D_a^\alpha u\right)(t) + q(t)\,u(t) = 0, \quad a < t < b,$$
$$u(a) = u(b) = 0, \tag{62}$$

where $(a, b) \in \mathbb{R}^2$, $0 < a < b$, $\alpha \in (1, 2)$, and $q : [a, b] \to \mathbb{R}$ is a continuous function.

Corollary 17. *If fractional boundary value problem (62) has a nontrivial solution $u \in \Xi_g([a,b];\mathbb{R}) \cap C([a,b];\mathbb{R})$, where $g(x) = \ln x$, $x \in [a, b]$, then*

$$\int_a^b \left[\left(\ln\frac{s}{a}\right)\left(\ln\frac{b}{s}\right)\right]^{\alpha-1} \frac{|q(s)|}{s} ds \geq \Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}. \tag{63}$$

Now, define the function $\psi : [a, b] \to [0, \infty)$ by

$$\psi(s) = \left[\left(\ln\frac{s}{a}\right)\left(\ln\frac{b}{s}\right)\right]^{\alpha-1} s^{-1}, \quad s \in [a, b]. \tag{64}$$

Observe that

$$\psi(s) = \mu(\ln s), \quad s \in [a, b], \tag{65}$$

where $\mu : [A, B] \to [0, \infty)$ is the function defined by

$$\mu(x) = \left[(x - A)(B - x)\right]^{\alpha-1} e^{-x}, \quad x \in [A, B] \tag{66}$$

with $A = \ln a$ and $B = \ln b$. A simple computation yields

$$\mu'(x) = \left[(x - A)(B - x)\right]^{\alpha-2} e^{-x} R(x), \quad x \in (A, B), \tag{67}$$

where

$$R(x) = x^2 - (2(\alpha - 1) + A + B)\,x + (\alpha - 1)(A + B) + AB, \quad x \in [A, B]. \tag{68}$$

Observe that $R(x)$ has two distinct zeros at

$$x_1 = \frac{2(\alpha-1) + A + B - \sqrt{4(\alpha-1)^2 + (A-B)^2}}{2} := \lambda(a, b),$$
$$x_2 = \frac{2(\alpha-1) + A + B + \sqrt{4(\alpha-1)^2 + (A-B)^2}}{2}. \tag{69}$$

It can be easily seen that

$$A < x_1 < B < x_2. \tag{70}$$

Moreover, we have $R(x) \geq 0$ for $x \in [A, x_1]$ and $R(x) \leq 0$ for $x \in [x_1, B]$. Therefore, we deduce that

$$\mu(x_1) = \max\{\mu(x) : x \in [A, B]\}, \tag{71}$$

$$\begin{aligned} \psi(e^{x_1}) &= \max\{\psi(s) : s \in [a, b]\} \\ &= [(\lambda(a,b) - \ln a)(\ln b - \lambda(a,b))]^{\alpha-1} e^{-\lambda(a,b)}. \end{aligned} \tag{72}$$

Next, combining (63) with (72), we obtain the following Lyapunov-type inequality for fractional boundary value problem (62).

Corollary 18. *If fractional boundary value problem (62) has a nontrivial solution* $u \in \Xi_g([a,b];\mathbb{R}) \cap C([a,b];\mathbb{R})$, *where* $g(x) = \ln x$, $x \in [a,b]$, *then*

$$\begin{aligned} &\int_a^b |q(s)|\, ds \\ &\geq \Gamma(\alpha) \left[\frac{\ln b - \ln a}{(\lambda(a,b) - \ln a)(\ln b - \lambda(a,b))} \right]^{\alpha-1} e^{\lambda(a,b)}, \end{aligned} \tag{73}$$

where

$$\begin{aligned} &\lambda(a,b) \\ &= \frac{2(\alpha-1) + \ln a + \ln b - \sqrt{4(\alpha-1)^2 + (\ln a - \ln b)^2}}{2}. \end{aligned} \tag{74}$$

Observe that, in the particular case $(a,b) = (1,e)$, inequality (73) reduces to inequality (10) which is due to Ma et al. [37].

Remark 19. Corollary 18 is an answer to the open problem proposed in [37].

5. Applications to Eigenvalue Problems

Now, we present an application of the Hartman-Wintner-type inequality given by Theorem 13 to eigenvalue problems.

We say that the scalar λ is an eigenvalue of the fractional boundary value problem

$$\begin{gathered} \left(D_{a,g}^{\alpha} u\right)(t) + \lambda u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{gathered} \tag{75}$$

where $(a,b) \in \mathbb{R}^2$, $a < b$, $\alpha \in (1,2)$, and $g \in C^1([a,b];\mathbb{R})$ with $g'(x) > 0$, for all $x \in (a,b)$, if problem (75) has at least a nontrivial solution $u_\lambda \in \Xi_g([a,b];\mathbb{R}) \cap C([a,b];\mathbb{R})$.

We have the following result which provides a lower bound of the eigenvalues of problem (75).

Corollary 20. *If* λ *is an eigenvalue of problem (75), then*

$$|\lambda| \geq \frac{\Gamma(\alpha)(B-A)^{\alpha-1}}{\int_A^B (x-A)^{\alpha-1}(B-x)^{\alpha-1}\, dx}, \tag{76}$$

where $A = g(a)$ *and* $B = g(b)$.

Proof. Suppose that λ is an eigenvalue of problem (75). Then problem (75) admits a nontrivial solution. Applying Theorem 13 with $q \equiv \lambda$, we obtain

$$\begin{aligned} &|\lambda| \int_a^b (g(s) - g(a))^{\alpha-1} (g(b) - g(s))^{\alpha-1} g'(s)\, ds \\ &\quad \geq \Gamma(\alpha)(g(b) - g(a))^{\alpha-1}. \end{aligned} \tag{77}$$

Using the change of variable $x = g(s)$, we obtain

$$|\lambda| \int_A^B (x-A)^{\alpha-1}(B-x)^{\alpha-1}\, dx \geq \Gamma(\alpha)(B-A)^{\alpha-1}, \tag{78}$$

which proves the desired inequality. □

Taking $g(x) = x^\beta$, $\beta > 0$, in Corollary 20, we obtain the following result.

Corollary 21. *If* λ *is an eigenvalue of problem (75) with* $g(x) = x^\beta$, $\beta > 0$, $x \in [a,b]$, $0 < a < b$, *then*

$$|\lambda| \geq \frac{\Gamma(\alpha)\left(b^\beta - a^\beta\right)^{\alpha-1}}{\int_{a^\beta}^{b^\beta} \left(x - a^\beta\right)^{\alpha-1} \left(b^\beta - x\right)^{\alpha-1} dx}. \tag{79}$$

Taking $g(x) = \ln x$ in Corollary 20, we obtain the following result.

Corollary 22. *If* λ *is an eigenvalue of the Hadamard fractional boundary value problem*

$$\begin{gathered} \left({}^{H}D_a^{\alpha} u\right)(t) + \lambda u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{gathered} \tag{80}$$

where $(a,b) \in \mathbb{R}^2$, $0 < a < b$, *and* $\alpha \in (1,2)$, *then*

$$|\lambda| \geq \frac{\Gamma(\alpha)(\ln b - \ln a)^{\alpha-1}}{\int_{\ln a}^{\ln b} (x - \ln a)^{\alpha-1} (\ln b - x)^{\alpha-1}\, dx}. \tag{81}$$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] A. Liapounoff, "Problème général de la stabilité du mouvement," *Annales de la faculté des sciences de Toulouse Mathématiques*, vol. 9, pp. 203–474, 1907.

[2] A. Beurling, "Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel," *Acta Mathematica*, vol. 77, pp. 127–136, 1945.

[3] G. Borg, “On a Liapounoff criterion of stability,” *American Journal of Mathematics*, vol. 71, pp. 67–70, 1949.

[4] R. C. Brown and D. B. Hinton, “Opial's inequality and oscillation of 2nd order equations,” *Proceedings of the American Mathematical Society*, vol. 125, no. 4, pp. 1123–1129, 1997.

[5] R. S. Dahiya and B. Singh, “A Lyapunov inequality and nonoscillation theorem for a second order non-linear differential-difference equation,” *Journal of Mathematical and Physical Sciences*, vol. 7, pp. 163–170, 1973.

[6] G. S. Guseinov and A. Zafer, “Stability criteria for linear periodic impulsive Hamiltonian systems,” *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1195–1206, 2007.

[7] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 1964, Birkhuser, Boston, Mass, USA 1982.

[8] P. Hartman and A. Wintner, “On an oscillation criterion of Liapounoff,” *American Journal of Mathematics*, vol. 73, pp. 885–890, 1951.

[9] B. G. Pachpatte, “Inequalities related to the zeros of solutions of certain second order differential equations,” *Facta Universitatis, Series: Mathematics and Informatics*, vol. 16, pp. 35–44, 2001.

[10] W. T. Reid, “A matrix equation related to a non-oscillation criterion and Liapunov stability,” *Quarterly of Applied Mathematics*, vol. 23, pp. 83–87, 1965.

[11] B. Singh, “Forced oscillation in general ordinary differential equations,” *Tamkang Journal of Mathematics*, vol. 6, pp. 5–11, 1975.

[12] A. Wintner, “On the non-existence of conjugate points,” *American Journal of Mathematics*, vol. 73, pp. 368–380, 1951.

[13] D. Çakmak, “Lyapunov-type integral inequalities for certain higher order differential equations,” *Applied Mathematics and Computation*, vol. 216, no. 2, pp. 368–373, 2010.

[14] S. B. Eliason, “Lyapunov type inequalities for certain second order functional differential equations,” *SIAM Journal on Applied Mathematics*, vol. 27, no. 1, pp. 180–199, 1974.

[15] L. Jiang and Z. Zhou, “Lyapunov inequality for linear Hamiltonian systems on time scales,” *Journal of Mathematical Analysis and Applications*, vol. 310, no. 2, pp. 579–593, 2005.

[16] C. Lee, C. Yeh, C. Hong, and R. P. Agarwal, “Lyapunov and Wirtinger inequalities,” *Applied Mathematics Letters*, vol. 17, no. 7, pp. 847–853, 2004.

[17] B. G. Pachpatte, “Lyapunov type integral inequalities for certain differential equations,” *Georgian Mathematical Journal*, vol. 4, no. 2, pp. 139–148, 1997.

[18] S. Panigrahi, “Liapunov-type integral inequalities for certain higher-order differential equations,” *Electronic Journal of Differential Equations*, vol. 28, pp. 1–14, 2009.

[19] N. Parhi and S. Panigrahi, “Liapunov-type inequality for higher order differential equations,” *Mathematica Slovaca*, vol. 52, no. 1, pp. 31–46, 2002.

[20] A. n. Tiryaki, “Recent developments of Lyapunov-type inequalities,” *Advances in Dynamical Systems and Applications*, vol. 5, no. 2, pp. 231–248, 2010.

[21] X. Yang, “On Liapunov-type inequality for certain higher-order differential equations,” *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 307–317, 2003.

[22] X. Yang and K. Lo, “Lyapunov-type inequality for a class of even-order differential equations,” *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3884–3890, 2010.

[23] A. A. Kilbas, H. M. Srivastava, and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[24] R. A. C. Ferreira, “A Lyapunov-type inequality for a fractional boundary value problem,” *Fractional Calculus and Applied Analysis*, vol. 16, no. 4, pp. 978–984, 2013.

[25] R. A. C. Ferreira, “On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function,” *Journal of Mathematical Analysis and Applications*, vol. 412, no. 2, pp. 1058–1063, 2014.

[26] R. A. C. Ferreira, “Lyapunov-type inequalities for some sequential fractional boundary value problems,” *Advances in Dynamical Systems and Applications*, vol. 11, no. 1, pp. 33–43, 2016.

[27] M. Jleli and B. Samet, “Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions,” *Mathematical Inequalities & Applications*, vol. 18, no. 2, pp. 443–451, 2015.

[28] M. Jleli and B. Samet, “Lyapunov-type inequalities for fractional boundary value problems,” *Electronic Journal of Differential Equations*, vol. 88, pp. 1–11, 2015.

[29] M. Jleli, M. Kirane, and B. Samet, “Lyapunov-type inequalities for fractional partial differential equations,” *Applied Mathematics Letters*, vol. 66, pp. 30–39, 2017.

[30] M. Jleli, L. Ragoub, and B. Samet, “A Lyapunov-type inequality for a fractional differential equation under a Robin boundary condition,” *Journal of Function Spaces*, vol. 2015, Article ID 468536, 5 pages, 2015.

[31] D. O'Regan and B. Samet, “Lyapunov-type inequalities for a class of fractional differential equations,” *Journal of Inequalities and Applications*, vol. 2015, article 247, 10 pages, 2015.

[32] N. Al Arifi, I. Altun, M. Jleli, A. Lashin, and B. Samet, “Lyapunov-type inequalities for a fractional p-Laplacian equation,” *Journal of Inequalities and Applications*, vol. 2016, article 189, 2016.

[33] J. Rong and C. Bai, “Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions,” *Advances in Difference Equations*, vol. 2015, article 82, 2015.

[34] A. Chidouh and D. F. Torres, “A generalized Lyapunov's inequality for a fractional boundary value problem,” *Journal of Computational and Applied Mathematics*, vol. 312, pp. 192–197, 2017.

[35] R. P. Agarwal and A. Özbekler, “Lyapunov type inequalities for mixed nonlinear Riemann-Liouville fractional differential equations with a forcing term,” *Journal of Computational and Applied Mathematics*, vol. 314, pp. 69–78, 2017.

[36] D. Ma, “A generalized Lyapunov inequality for a higher-order fractional boundary value problem,” *Journal of Inequalities and Applications*, vol. 2016, article no. 261, 2016.

[37] Q. Ma, C. Ma, and J. Wang, “A Lyapunov-type inequality for a fractional differential equation with Hadamard derivative,” *Journal of Mathematical Inequalities*, vol. 11, no. 1, pp. 135–141, 2007.

[38] D. Idczak and S. Walczak, “Fractional sobolev spaces via Riemann-Liouville derivatives,” *Journal of Function Spaces and Applications*, vol. 2013, Article ID 128043, 15 pages, 2013.

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Research Article

Centralized Data-Sampling Approach for Global $O(t^{-\alpha})$ Synchronization of Fractional-Order Neural Networks with Time Delays

Jin-E Zhang

Hubei Normal University, Hubei 435002, China

Correspondence should be addressed to Jin-E Zhang; zhang86021205@163.com

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In this paper, the global $O(t^{-\alpha})$ synchronization problem is investigated for a class of fractional-order neural networks with time delays. Taking into account both better control performance and energy saving, we make the first attempt to introduce centralized data-sampling approach to characterize the $O(t^{-\alpha})$ synchronization design strategy. A sufficient criterion is given under which the drive-response-based coupled neural networks can achieve global $O(t^{-\alpha})$ synchronization. It is worth noting that, by using centralized data-sampling principle, fractional-order Lyapunov-like technique, and fractional-order Leibniz rule, the designed controller performs very well. Two numerical examples are presented to illustrate the efficiency of the proposed centralized data-sampling scheme.

1. Introduction

Fractional-order calculus has gained an increasing attention in physical systems and engineering systems. Fractional-order dynamic systems containing fractional derivatives and integrals have been investigated in the field of control systems [1–19]. Investigating analytical skills in fractional-order dynamic systems is one important theme. However, it is considered that many control schemes for fractional differential equation are only a mathematical concept. A lot of control methods in integer-order dynamic systems cannot be introduced to fractional-order dynamic systems [2, 3].

As a kind of the important dynamic systems, the concept of neurodynamic systems can be traced back to the early 1940s. Neurodynamic systems behave like a synthesizer evaluating the performance of system itself via the topology structure. In recent years, modeling fractional phenomenon to neurodynamic systems has been developed in an effort to improve the neurodynamic processes [2, 3, 14, 16]. Appealing feature of fractional-order neurodynamic systems is that the infinite memory property can take the past inherited information into account, which well suits describing complex dynamic processes. For control system applications, fractional-order neurodynamic systems have greatly expanded systems of conventional difference equations, which provide an adaptive control system for standard application.

Data-sampling control of systems has been studied in a number of publications. Actually, as stated in [20–30], for complex or multivariable control systems, it is unrealistic or even impossible to sample all real-time physical signals at one single rate. In such situation, one is forced to use multirate data-sampling control. Multirate data-sampling control conditions have been derived there which are less conservative [31–35]. Multirate data-sampling control can achieve what single-rate data-sampling control cannot, for instance, gain margin improvement, centralized control, and decentralized control.

Many control schemes have been established for complex control systems, such as adaptive control [36, 37] and sliding mode control [38]. In view of the feature of fractional-order systems, compared with other control strategies, centralized data-sampling scheme is more applicable for implementation in fractional-order systems. For one thing, centralized data-sampling scheme itself is relatively cheaper and simpler to operate. Unlike a lot of data-sampling designs, these schemes are usually designed for continuous sampling, and the control cost is very high. For another thing, considering that the system structures of fractional-order systems are complex and ever changing, which may have unpredictable nonlinear effects, it is more reasonable and implementable for centralized data-sampling only carried out at part of timing nodes.

In this paper, a centralized data-sampling architecture enabled by low-bandwidth communication is proposed. The centralized data-sampling approach is developed to globally $O(t^{-\alpha})$ synchronize the drive-response-based coupled fractional-order neural networks with time delays. And then we present an intelligent control method for designing synchronization scheme based on centralized data-sampling principle, fractional-order Lyapunov-like technique, and fractional-order Leibniz rule. The obtained results provide novel and higher performance extension for the designed controller. The use of centralized data-sampling approach facilitates utilizing low-bandwidth communication to transmit harmonic signals. The operation principle and numerical examples based on computer simulations are also presented.

2. Preliminaries and Problem Formulation

2.1. Notation. For n-dimensional vector $v = (v_1, v_2, \ldots, v_n)^T$, the norm of vector v is recorded as $\|v\| = \sum_{i=1}^{n} |v_i|$. Denote $\mathscr{C}^{\iota}([t_0, +\infty), \mathfrak{R}^n)$ as the space of ι-order continuous and differentiable functions from $[t_0, +\infty)$ into $\mathfrak{R}^n$. $\mathscr{C}_\tau := \mathscr{C}([-\tau, 0], \mathfrak{R})$ is a Banach space of all continuous functions $\varphi : [-\tau, 0] \to \mathfrak{R}$. For any $\varphi \in \mathscr{C}_\tau$, let $\|\varphi\|_{\mathscr{C}} = \sup_{s \in [-\tau, 0]} \|\varphi(s)\|$.

2.2. Preliminaries. In order to facilitate understanding, we first introduce some concepts of fractional calculation.

Fractional integral with order $\alpha > 0$ of function $\mathscr{F}(t)$ is characterized as

$$ {}^{\mathrm{RL}}_{t_0}D^{-\alpha}_{t} \mathscr{F}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathscr{F}(s)\, \mathrm{d}s, \tag{1} $$

where $t \geq t_0$, $\Gamma(\cdot)$ is the Gamma function, which is defined as

$$ \Gamma(\alpha) = \int_{t_0}^{+\infty} s^{\alpha-1} \exp\{-s\}\, \mathrm{d}s. \tag{2} $$

Riemann-Liouville derivative with order $\alpha > 0$ of function $\mathscr{F}(t)$ is characterized as

$$ {}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} \mathscr{F}(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{\mathscr{F}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \tag{3} $$

where $t \geq t_0$, $n-1 < \alpha < n$, and n is a positive integer.

Caputo derivative with order $\alpha > 0$ of function $\mathscr{F}(t) \in \mathscr{C}^{n+1}([t_0, +\infty), \mathfrak{R})$ is characterized as

$$ {}^{C}_{t_0}D^{\alpha}_{t} \mathscr{F}(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{\mathscr{F}^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \tag{4} $$

where $t \geq t_0$, $n-1 < \alpha < n$, and n is a positive integer.

In this paper, consider a class of fractional-order neural networks with time delays governed by

$$ \begin{aligned} {}^{C}_{t_0}D^{\alpha}_{t} x(t) &= -Dx(t) + Ag(x(t)) + Bf(x(t-\tau(t))) \\ &\quad + I, \\ y(t) &= Cx(t), \end{aligned} \tag{5} $$

where $0 < \alpha < 1$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathfrak{R}^n$ is the state vector, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is a positive diagonal matrix, $A = (a_{ij})_{n\times n} \in \mathfrak{R}^{n\times n}$ and $B = (b_{ij})_{n\times n} \in \mathfrak{R}^{n\times n}$ denote the weight matrix and the delayed weight matrix, respectively, $\tau(t)$ represents the transmission delay satisfying $0 \leq \tau(t) \leq \tau$, $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T \in \mathfrak{R}^n$ and $f(x(t-\tau(t))) = (f_1(x_1(t-\tau(t))), f_2(x_2(t-\tau(t))), \ldots, f_n(x_n(t-\tau(t))))^T \in \mathfrak{R}^n$ denote the activation functions at times t and $t-\tau(t)$, respectively, $I = (I_1, I_2, \ldots, I_n)^T \in \mathfrak{R}^n$ is the bias, $C = (\mathfrak{C}_{ij})_{m\times n} \in \mathfrak{R}^{m\times n}$ is a constant matrix, and $y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T \in \mathfrak{R}^m$ is the output vector.

Obviously, system (5) is a more general model. In [2, 16], the model is a fractional-order system without time-delay. By comparing the system models, system (5) contains some existing fractional-order neural networks.

The initial condition of system (5) is $x(t) = \chi(t) \in \mathscr{C}_\tau$.

In addition, in (5), the activation functions $g(\cdot)$ and $f(\cdot)$ are global Lipschitz; that is, for $i = 1, 2, \ldots, n$, there exist positive constants $G_i > 0$ and $F_i > 0$ such that

$$ \begin{aligned} |g_i(u) - g_i(v)| &\leq G_i |u - v|, \quad \text{for any } u \in \mathfrak{R},\ v \in \mathfrak{R}, \\ |f_i(u) - f_i(v)| &\leq F_i |u - v|, \quad \text{for any } u \in \mathfrak{R},\ v \in \mathfrak{R}. \end{aligned} \tag{6} $$

2.3. Problem Formulation. In this paper, consider system (5) as the master/drive system, and then the slave/response system is described as

$$ \begin{aligned} {}^{C}_{t_0}D^{\alpha}_{t} \widehat{x}(t) &= -D\widehat{x}(t) + Ag(\widehat{x}(t)) + Bf(\widehat{x}(t-\tau(t))) \\ &\quad + I + u(t), \\ \widehat{y}(t) &= C\widehat{x}(t), \end{aligned} \tag{7} $$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathfrak{R}^n$ is the controller to be designed.

The initial condition of system (7) is $\widehat{x}(t) = \widehat{\chi}(t) \in \mathscr{C}_\tau$.

Define $e(t) = \widehat{x}(t) - x(t) \in \mathfrak{R}^n$, from (5) and (7); then we can obtain the error dynamics system

$$ \begin{aligned} {}^{C}_{t_0}D^{\alpha}_{t} e(t) &= -De(t) + Ag(e(t)) + Bf(e(t-\tau(t))) \\ &\quad + u(t), \end{aligned} \tag{8} $$

where

$$ \begin{aligned} g(e(t)) &= (g_1(e_1(t)), g_2(e_2(t)), \ldots, g_n(e_n(t)))^T \\ &= g(\widehat{x}(t)) - g(x(t)), \\ f(e(t-\tau(t))) &= (f_1(e_1(t-\tau(t))), f_2(e_2(t-\tau(t))), \end{aligned} $$

$$\ldots, f_n\left(e_n\left(t-\tau\left(t\right)\right)\right)\right)^T = f\left(\hat{x}\left(t-\tau\left(t\right)\right)\right) - f\left(x\left(t-\tau\left(t\right)\right)\right). \tag{9}$$

The initial condition of system (8) is $e(t) = \psi(t) := \hat{\chi}(t) - \chi(t) \in \mathscr{C}_\tau$.

In our control design, the structure-dependent centralized data-sampling is used. Moreover, the measured output of error dynamics system is sampled and then the data-sampling information is sent to the controller of the response system as

$$z\left(t\right) = \hat{y}\left(t_k\right) - y\left(t_k\right) = Ce\left(t_k\right), \quad t \in \left[t_k, t_{k+1}\right), \ k = 0, 1, 2, \ldots, \tag{10}$$

where $z(t) \in \mathfrak{R}^m$ represents the output of (8) and t_k denotes the sampling instant satisfying $\lim_{k\to+\infty} t_k = +\infty$.

To study global $O(t^{-\alpha})$ synchronization, next, we will introduce some related definitions.

Definition 1. For dynamic system

$${}_{t_0}^{C}D_t^{\alpha}\mathscr{X}\left(t\right) = \mathscr{F}\left(t, \mathscr{X}_t\right), \tag{11}$$

where $\mathscr{X}_t(\theta) = \mathscr{X}(t+\theta)$ is defined on $-\tau \le \theta \le 0$, the initial condition of system (11) is $\mathscr{X}(t) = \varphi(t) \in \mathscr{C}_\tau$. System (11) is said to be globally $O(t^{-\alpha})$ stable if there exists a positive constant $M > 0$ such that $\|\mathscr{X}(t, t_0, \varphi)\| \le M\|\varphi\|_{\mathscr{C}} O(t^{-\alpha})$ for any $\varphi \in \mathscr{C}_\tau$ and $t \ge t_0$.

Remark 2. Convergence of fractional-order systems is totally different from conventional exponential convergence or absolute convergence, which possesses abnormal convergence behavior. In addition, according to Definition 1, global $O(t^{-\alpha})$ stability and global Mittag-Leffler stability are "essentially the same." On Mittag-Leffler stability, please see some publications [2, 16].

Definition 3. For the master/drive system

$${}_{t_0}^{C}D_t^{\alpha}\mathscr{X}\left(t\right) = \mathscr{F}\left(t, \mathscr{X}_t\right) \tag{12}$$

and the slave/response system

$${}_{t_0}^{C}D_t^{\alpha}\mathscr{Y}\left(t\right) = \mathscr{F}\left(t, \mathscr{Y}_t, u\left(t\right)\right), \tag{13}$$

where $\mathscr{X}_t(\theta) = \mathscr{X}(t+\theta)$ and $\mathscr{Y}_t(\theta) = \mathscr{Y}(t+\theta)$ are defined on $-\tau \le \theta \le 0$, the initial conditions for systems (12) and (13) are $\mathscr{X}(t) = \phi(t) \in \mathscr{C}_\tau$ and $\mathscr{Y}(t) = \hat{\phi}(t) \in \mathscr{C}_\tau$, respectively. The coupled systems (12) and (13) are said to be globally $O(t^{-\alpha})$ synchronized if the zero solution of the error dynamics system

$${}_{t_0}^{C}D_t^{\alpha}\mathscr{E}\left(t\right) = \mathscr{F}\left(t, \mathscr{Y}_t, u\left(t\right)\right) - \mathscr{F}\left(t, \mathscr{X}_t\right) \tag{14}$$

is globally $O(t^{-\alpha})$ stable, where $\mathscr{E}(t) = \mathscr{Y}(t) - \mathscr{X}(t)$.

3. Main Results

Based on the discussion in preceding section, then the data-sampling controller can be designed as

$$u\left(t\right) = Kz\left(t\right) = KCe\left(t_k\right), \quad t \in \left[t_k, t_{k+1}\right), \ k = 0, 1, 2, \ldots, \tag{15}$$

where $K = (\mathfrak{K}_{ij})_{m\times m} \in \mathfrak{R}^{m\times m}$ is the constant gain matrix to be determined. Therefore, the error dynamics system can be transformed into the following form:

$${}_{t_0}^{C}D_t^{\alpha}e\left(t\right) = -De\left(t\right) + Ag\left(e\left(t\right)\right) + Bf\left(e\left(t-\tau\left(t\right)\right)\right) + KCe\left(t_k\right), \quad t \in \left[t_k, t_{k+1}\right). \tag{16}$$

For technical convenience, we also give mathematical expression for (16) represented by components: for $i = 1, 2, \ldots, n$,

$${}_{t_0}^{C}D_t^{\alpha}e_i\left(t\right) = -d_ie_i\left(t\right) + \sum_{j=1}^{n} a_{ij}g_j\left(e_j\left(t\right)\right) + \sum_{j=1}^{n} b_{ij}f_j\left(e_j\left(t-\tau\left(t\right)\right)\right) + \sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\mathfrak{K}_{\ell\hbar}\mathfrak{C}_{\hbar i}e_i\left(t_k\right), \quad t \in \left[t_k, t_{k+1}\right). \tag{17}$$

Before we began to develop theoretical criterion, we first state an important lemma, which will be used in the proving process.

Lemma 4 (see [3]). *If $\mathscr{F} \in \mathscr{C}^1([0, +\infty), \mathfrak{R})$, then the following properties hold:*

(1) ${}_{t_0}^{C}D_t^{\alpha}\mathscr{F}(t) = {}_{t_0}^{\mathrm{RL}}D_t^{\alpha}\mathscr{F}(t) - (\mathscr{F}(t_0)/\Gamma(1-\alpha))(t-t_0)^{-\alpha}$.

(2) *If $\mathscr{F}(t)$ and $\vartheta(t)$ and their all derivatives are continuous in $[t_0, t]$, then*

$${}_{t_0}^{\mathrm{RL}}D_t^{\alpha}\left(\vartheta\left(t\right)\mathscr{F}\left(t\right)\right) = \sum_{\kappa=0}^{n}\frac{\mathrm{d}^{\kappa}\vartheta\left(t\right)}{\mathrm{d}t^{\kappa}}\binom{\alpha}{\kappa}{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-\kappa}\mathscr{F}\left(t\right) - R_n^{\alpha}\left(t\right), \tag{18}$$

where $0 < \alpha < 1$, $n \ge \alpha$,

$$\begin{aligned} R_n^{\alpha}\left(t\right) &= \frac{\left(-1\right)^n\left(t-\alpha\right)^{n-\alpha+1}}{n!\Gamma\left(-\alpha\right)}\int_0^1\int_0^1\mathscr{F}_{\alpha}\left(t,\xi,\eta\right)\mathrm{d}\xi\,\mathrm{d}\eta, \\ \mathscr{F}_{\alpha}\left(t,\xi,\eta\right) &= \mathscr{F}\left(t_0+\eta\left(t-t_0\right)\right)\vartheta^{n+1}\left(t_0+\left(t-t_0\right)\left(\xi+\eta-\xi\eta\right)\right), \\ \binom{\alpha}{\kappa} &= \frac{\Gamma\left(\alpha+1\right)}{\kappa!\Gamma\left(\alpha-\kappa+1\right)}. \end{aligned} \tag{19}$$

Theorem 5. *If there exist positive constants $r > \tau \geq 0$ and $\beta_i > 0$ $(i = 1, 2, \ldots, n)$ such that*

$$d_i - \frac{1}{\beta_i}\sum_{j=1}^{n}\beta_j\left|a_{ij}\right|G_j - \sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar i}\right| - \left(\frac{r}{r-\tau}\right)^{\alpha}\sum_{j=1}^{n}\frac{\beta_j}{\beta_i}\left|b_{ij}\right|F_j - \frac{1+\alpha}{r^{\alpha}\Gamma(2-\alpha)} \geq 0, \tag{20}$$

for $i = 1, 2, \ldots, n$, set t_{k+1} as the sampling time point such that

$$t_{k+1} = \sup_{\delta \geq t_k}\left\{\delta : \left|e_i(t_k)\right| \leq \beta_i \max_{1\leq\sigma\leq n}\left[\frac{\left|e_\sigma(t)\right|}{\beta_\sigma}\right],\ \forall t \in [t_k, \delta),\ i = 1, 2, \ldots, n\right\} \tag{21}$$

for all $k = 0, 1, 2, \ldots$; then system (17) is globally $O(t^{-\alpha})$ stable. That is, the drive-response-based coupled systems (5) and (7) can reach global $O(t^{-\alpha})$ synchronization.

Proof. Consider the Lyapunov functions candidate as

$$\begin{aligned}\mathscr{W}(t) &= \max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\},\\ \mathscr{V}(t) &= (t - t_0 + r)^{\alpha}\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\end{aligned} \tag{22}$$

and set

$$\begin{aligned}\overline{\mathscr{W}}(t) &= \sup_{-\tau\leq\theta\leq t}\mathscr{W}(\theta),\\ \overline{\mathscr{V}}(t) &= \sup_{-\tau\leq\theta\leq t}\mathscr{V}(\theta),\end{aligned} \tag{23}$$

for $t \geq t_0$. □

Using Lemma 4, we can obtain

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}\mathscr{V}(t) &= {}_{t_0}^{\mathrm{RL}}D_t^{\alpha}\mathscr{V}(t) - \frac{\mathscr{V}(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}\\
&= {}_{t_0}^{\mathrm{RL}}D_t^{\alpha}\left((t-t_0+r)^{\alpha}\cdot\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\right)\\
&\quad - \frac{r^{\alpha}\max\left\{\left|e_i(t_0)\right|/\beta_i,\ i = 1, 2, \ldots, n\right\}}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}\\
&\leq (t-t_0+r)^{\alpha}\,{}_{t_0}^{C}D_t^{\alpha}\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\\
&\quad + (t-t_0+r)^{\alpha}\frac{\max\left\{\left|e_i(t_0)\right|/\beta_i,\ i = 1, 2, \ldots, n\right\}}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha} + \alpha^2(t-t_0+r)^{\alpha-1}\,{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1}\\
&\quad\cdot\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\} + \frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2}\times{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-2}\\
&\quad\cdot\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\\
&\quad - \frac{r^{\alpha}\max\left\{\left|e_i(t_0)\right|/\beta_i,\ i = 1, 2, \ldots, n\right\}}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}\\
&\leq (t-t_0+r)^{\alpha}\,{}_{t_0}^{C}D_t^{\alpha}\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\\
&\quad + \left[\left(\frac{t-t_0+r}{t-t_0}\right)^{\alpha} - \left(\frac{r}{t-t_0}\right)^{\alpha}\right]\cdot\frac{\max\left\{\left|e_i(t_0)\right|/\beta_i,\ i = 1, 2, \ldots, n\right\}}{\Gamma(1-\alpha)} + \alpha^2(t-t_0\\
&\quad + r)^{\alpha-1}\,{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1}\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\\
&\quad + \frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2}\times{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-2}\\
&\quad\cdot\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\} \leq (t-t_0+r)^{\alpha}\\
&\quad\cdot{}_{t_0}^{C}D_t^{\alpha}\mathscr{W}(t) + (1+2\alpha)\frac{\overline{\mathscr{V}}(t)}{r^{\alpha}\Gamma(1-\alpha)} + \alpha^2(t-t_0\\
&\quad + r)^{\alpha-1}\,{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1}\mathscr{W}(t) + \frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2}\\
&\quad\cdot{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-2}\mathscr{W}(t),\end{aligned} \tag{24}$$

for $t \geq t_0$.

On the other hand, it is not difficult to follow

$$\begin{aligned}\alpha^2(t-t_0+r)^{\alpha-1}\,{}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1}\mathscr{W}(t) &= \alpha^2(t-t_0+r)^{-(1-\alpha)}\\
&\quad\cdot{}_{t_0}^{\mathrm{RL}}D_t^{-(1-\alpha)}\max\left\{\frac{\left|e_i(t)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\\
&\leq \alpha^2(t-t_0+r)^{-(1-\alpha)}\frac{1}{\Gamma(1-\alpha)}\\
&\quad\times\sup_{-\tau\leq\theta\leq t}\left[\max\left\{\frac{\left|e_i(\theta)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\right]\int_{t_0}^{t}(t-s)^{-\alpha}\,ds\\
&\leq \alpha^2(t-t_0+r)^{-(1-\alpha)}\frac{(t-t_0)^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\\
&\quad\times\sup_{-\tau\leq\theta\leq t}\left[\max\left\{\frac{\left|e_i(\theta)\right|}{\beta_i},\ i = 1, 2, \ldots, n\right\}\right]\end{aligned}$$

$$\leq \alpha^2 \left(\frac{t-t_0}{t-t_0+r} \right)^{1-\alpha} \frac{1}{\Gamma(2-\alpha)} \times \sup_{-\tau\leq\theta\leq t} \left[\max \left\{ \frac{|e_i(\theta)|}{\beta_i},\ i=1,2,\ldots,n \right\} \right] \leq \frac{\alpha^2}{\Gamma(2-\alpha)} \cdot \mathscr{W}\left(\tilde{\theta}\right) \leq \frac{\alpha^2}{r^{\alpha}\Gamma(2-\alpha)} \overline{\mathscr{V}}(t),$$

$$\frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2}\, {}^{\mathrm{RL}}_{t_0}D^{\alpha-2}_{t}\mathscr{W}(t) \leq \frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2}\frac{1}{\Gamma(2-\alpha)} \times \int_{t_0}^{t}(t-s)^{1-\alpha}\max\left\{\frac{|e_i(s)|}{\beta_i},\ i=1,2,\ldots,n\right\}\mathrm{d}s \leq \frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2} \times \frac{\sup_{-\tau\leq\theta\leq t}\left[\max\left\{|e_i(\theta)|/\beta_i,\ i=1,2,\ldots,n\right\}\right]}{\Gamma(2-\alpha)} \cdot \int_{t_0}^{t}(t-s)^{1-\alpha}\mathrm{d}s \leq \frac{\alpha^2(\alpha-1)^2}{2}(t-t_0+r)^{\alpha-2} \times \frac{(t-t_0)^{2-\alpha}\sup_{-\tau\leq\theta\leq t}\left[\max\left\{|e_i(\theta)|/\beta_i,\ i=1,2,\ldots,n\right\}\right]}{(2-\alpha)\Gamma(2-\alpha)} \leq \frac{\alpha^2(\alpha-1)^2}{2}\left(\frac{t-t_0}{t-t_0+r}\right)^{2-\alpha} \times \frac{\sup_{-\tau\leq\theta\leq t}\left[\max\left\{|e_i(\theta)|/\beta_i,\ i=1,2,\ldots,n\right\}\right]}{(2-\alpha)\Gamma(2-\alpha)} \leq \frac{\alpha^2(\alpha-1)^2}{2} \cdot \frac{\sup_{-\tau\leq\theta\leq t}\left[\max\left\{|e_i(\theta)|/\beta_i,\ i=1,2,\ldots,n\right\}\right]}{(2-\alpha)\Gamma(2-\alpha)} \leq \frac{\alpha^2(\alpha-1)^2}{2r^{\alpha}(2-\alpha)\Gamma(2-\alpha)}\overline{\mathscr{V}}(t) \leq \frac{\alpha^2}{r^{\alpha}\Gamma(2-\alpha)}\overline{\mathscr{V}}(t), \tag{25}$$

for $t \geq t_0$, $\overline{\mathscr{W}}(t) = \mathscr{W}(\tilde{\theta})$, where $\tilde{\theta} \in [-\tau, t]$.

Substituting (25) into (24), it is obvious to derive that

$${}^{C}_{t_0}D^{\alpha}_{t}\mathscr{V}(t) \leq (t-t_0+r)^{\alpha}\, {}^{C}_{t_0}D^{\alpha}_{t}\max\left\{\frac{|e_i(t)|}{\beta_i},\ i=1,2,\ldots,n\right\} + \frac{\Delta}{r^{\alpha}\Gamma(2-\alpha)} + \frac{\Theta}{r^{\alpha}\Gamma(1-\alpha)} = (t-t_0+r)^{\alpha}\, {}^{C}_{t_0}D^{\alpha}_{t}\mathscr{W}(t) + \frac{1+\alpha}{r^{\alpha}\Gamma(2-\alpha)}\overline{\mathscr{V}}(t), \tag{26}$$

where

$$\Delta = 2\alpha^2 \sup_{-\tau\leq\theta\leq t}\left[(\theta-t_0+r)^{\alpha} \cdot \max\left\{\frac{|e_i(\theta)|}{\beta_i},\ i=1,2,\ldots,n\right\}\right], \quad \Theta = (1+2\alpha)\sup_{-\tau\leq\theta\leq t}\left[(\theta-t_0+r)^{\alpha}\right] \cdot \max\left\{\frac{|e_i(\theta)|}{\beta_i},\ i=1,2,\ldots,n\right\}. \tag{27}$$

Utilizing (22), it is obvious that there exists a $\hat{k} \in \{1,2,\ldots,n\}$ such that

$$\mathscr{W}(t) = \frac{|e_{\hat{k}}(t)|}{\beta_{\hat{k}}}, \tag{28}$$

for $t \geq t_0$.

From (17), together with the data-sampling principle (21), we compute

$${}^{C}_{t_0}D^{\alpha}_{t}\mathscr{W}(t) = \frac{1}{\beta_{\hat{k}}}\, {}^{C}_{t_0}D^{\alpha}_{t}|e_{\hat{k}}(t)| \leq \frac{1}{\beta_{\hat{k}}} \cdot \operatorname{sgn}(e_{\hat{k}}(t))\, {}^{C}_{t_0}D^{\alpha}_{t}e_{\hat{k}}(t) \leq \frac{1}{\beta_{\hat{k}}}\operatorname{sgn}(e_{\hat{k}}(t))\left[-d_{\hat{k}}e_{\hat{k}}(t) + \sum_{j=1}^{n}a_{\hat{k}j}g_j(e_j(t)) + \sum_{j=1}^{n}b_{\hat{k}j}f_j(e_j(t-\tau(t))) + \sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\mathfrak{K}_{\ell\hbar}\mathfrak{C}_{\hbar\hat{k}}e_{\hat{k}}(t_k)\right] \leq -\frac{d_{\hat{k}}|e_{\hat{k}}(t)|}{\beta_{\hat{k}}} + \frac{1}{\beta_{\hat{k}}} \cdot \sum_{j=1}^{n}\beta_j|a_{\hat{k}j}|G_j\frac{|e_j(t)|}{\beta_j} + \frac{1}{\beta_{\hat{k}}} \cdot \sum_{j=1}^{n}\beta_j|b_{\hat{k}j}|F_j\frac{|e_j(t-\tau(t))|}{\beta_j} + \frac{1}{\beta_{\hat{k}}} \cdot \sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}|\mathfrak{K}_{\ell\hbar}||\mathfrak{C}_{\hbar\hat{k}}||e_{\hat{k}}(t_k)| \leq -\left[d_{\hat{k}} - \frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j|a_{\hat{k}j}|G_j - \sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}|\mathfrak{K}_{\ell\hbar}||\mathfrak{C}_{\hbar\hat{k}}|\right]\mathscr{W}(t) + \frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j|b_{\hat{k}j}|F_j\overline{\mathscr{W}}(t-\tau(t)). \tag{29}$$

Then

$$
\begin{aligned}
&(t-t_0+r)^{\alpha}\,{}_{t_0}^{C}D_t^{\alpha}\mathscr{W}(t) \le -(t-t_0+r)^{\alpha}\Bigg[d_{\hat{k}} \\
&-\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|a_{\hat{k}j}\right|G_j-\sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar\hat{k}}\right|\Bigg]\mathscr{W}(t)+(t \\
&-t_0+r)^{\alpha}\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|b_{\hat{k}j}\right|F_j\overline{\mathscr{W}}(t-\tau(t)) \le -\Bigg[d_{\hat{k}} \\
&-\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|a_{\hat{k}j}\right|G_j-\sum_{\ell=1}^{\mathrm{m}}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar\hat{k}}\right|\Bigg]\mathscr{V}(t) \\
&+\frac{(t-t_0+r)^{\alpha}}{\beta_{\hat{k}}\left(t-t_0+r+\overline{\theta}\right)^{\alpha}}\sum_{j=1}^{n}\beta_j\left|b_{\hat{k}j}\right|F_j\overline{\mathscr{V}}(t) \le -\Bigg[d_{\hat{k}} \\
&-\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|a_{\hat{k}j}\right|G_j-\sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar\hat{k}}\right| \\
&-\left(\frac{r}{r+\overline{\theta}}\right)^{\alpha}\sum_{j=1}^{n}\frac{\beta_j}{\beta_{\hat{k}}}\left|b_{\hat{k}j}\right|F_j\Bigg]\mathscr{V}(t) \le -\Bigg[d_{\hat{k}} \\
&-\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|a_{\hat{k}j}\right|G_j-\sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar\hat{k}}\right| \\
&-\left(\frac{r}{r-\tau}\right)^{\alpha}\sum_{j=1}^{n}\frac{\beta_j}{\beta_{\hat{k}}}\left|b_{\hat{k}j}\right|F_j\Bigg]\mathscr{V}(t),
\end{aligned}
\tag{30}
$$

where $\overline{\theta}\in[-\tau,t]$ such that $\overline{\mathscr{V}}(t)=(t-t_0+\overline{\theta}+r)^{\alpha}\overline{\mathscr{W}}(t)$, when $\overline{\mathscr{V}}(t)=\mathscr{V}(t)$.

Substituting (30) into (26), we have

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha}\mathscr{V}(t) \le -\Bigg[d_{\hat{k}}-\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|a_{\hat{k}j}\right|G_j \\
&-\sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar\hat{k}}\right|-\left(\frac{r}{r-\tau}\right)^{\alpha}\sum_{j=1}^{n}\frac{\beta_j}{\beta_{\hat{k}}}\left|b_{\hat{k}j}\right|F_j\Bigg] \\
&\cdot\mathscr{V}(t)+\frac{1+\alpha}{r^{\alpha}\Gamma(2-\alpha)}\overline{\mathscr{V}}(t) \le -\Bigg[d_{\hat{k}} \\
&-\frac{1}{\beta_{\hat{k}}}\sum_{j=1}^{n}\beta_j\left|a_{\hat{k}j}\right|G_j-\sum_{\ell=1}^{m}\sum_{\hbar=1}^{m}\left|\mathfrak{K}_{\ell\hbar}\right|\left|\mathfrak{C}_{\hbar\hat{k}}\right| \\
&-\left(\frac{r}{r-\tau}\right)^{\alpha}\sum_{j=1}^{n}\frac{\beta_j}{\beta_{\hat{k}}}\left|b_{\hat{k}j}\right|F_j-\frac{1+\alpha}{r^{\alpha}\Gamma(2-\alpha)}\Bigg]\mathscr{V}(t),
\end{aligned}
\tag{31}
$$

when $\overline{\mathscr{V}}(t)=\mathscr{V}(t)$.

According to (20) and (31),

$$
{}_{t_0}^{C}D_t^{\alpha}\overline{\mathscr{V}}(t) \le 0, \tag{32}
$$

for all $t \ge t_0$.

By the definition of Caputo derivative, from (32), we can obtain

$$
\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\frac{\overline{\mathscr{V}}(s)}{(t-s)^{\alpha}}\mathrm{d}s \le \frac{\overline{\mathscr{V}}(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}; \tag{33}
$$

then

$$
\overline{\mathscr{V}}(t) \le \overline{\mathscr{V}}(t_0), \tag{34}
$$

for $t \ge t_0$. Thus, for $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
\left|e_i(t)\right| &\le \beta_i\mathscr{W}(t)=\beta_i\frac{\mathscr{V}(t)}{(t-t_0+r)^{\alpha}} \le \beta_i\frac{\overline{\mathscr{V}}(t)}{(t-t_0+r)^{\alpha}} \\
&\le \beta_i\frac{\mathscr{V}(t_0)}{(t-t_0+r)^{\alpha}}=\beta_i\frac{r^{\alpha}\overline{\mathscr{W}}(t_0)}{(t-t_0+r)^{\alpha}} \\
&\le \frac{\beta_i r^{\alpha}\left\|\psi\right\|_{\mathscr{C}}}{\beta_{\min}(t-t_0+r)^{\alpha}},
\end{aligned}
\tag{35}
$$

for $t \ge t_0$, where $\beta_{\min} = \min\{\beta_i,\ i = 1, 2, \ldots, n\}$; then

$$
\|e(t)\| \le \frac{\Lambda r^{\alpha}\left\|\psi\right\|_{\mathscr{C}}}{(t-t_0+r)^{\alpha}}, \tag{36}
$$

where $\Lambda = (1/\beta_{\min})\sum_{i=1}^{n}\beta_i$. Therefore, system (17) is globally $O(t^{-\alpha})$ stable, which implies that the drive-response-based coupled systems (5) and (7) can reach global $O(t^{-\alpha})$ synchronization.

Remark 6. In Theorem 5, (20) is an algebraic condition, which relies on only system parameters, free-weighting parameters r and β_i ($i = 1, 2, \ldots, n$). Besides, (21) is used to characterize the sampling time point, which can be determined by trigger mechanism. Generally, the discrimination conditions in Theorem 5 can be easily rectified. In addition, although the results established are based on $0 < \alpha < 1$, clearly, such results can be also applied to $\alpha = 1$.

Remark 7. Aiming at the realistic environment under the limited bandwidth of communication channel, it is very critical to reduce the data transmission rate for networked systems. As (21), the centralized data-sampling mechanism for the sampling time point is an effective way, which does not waste the bandwidth of network to be with needless signals, and then reduces the data transmission and power consumption.

4. Two Illustrative Examples

In this section, two illustrative examples are given to demonstrate the effectiveness of theoretical criterion.

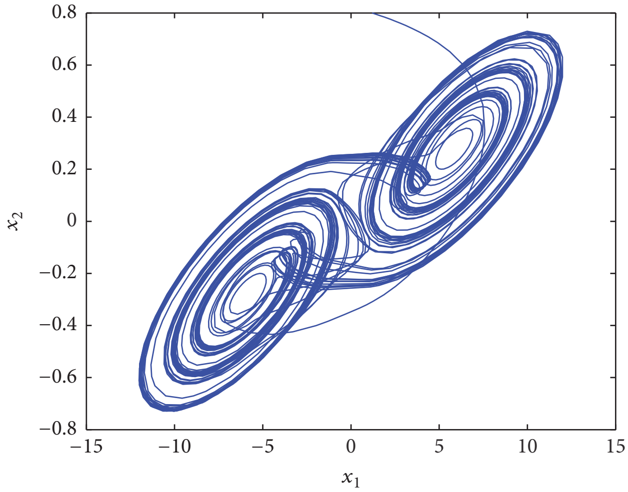

Figure 1: Chaotic behavior.

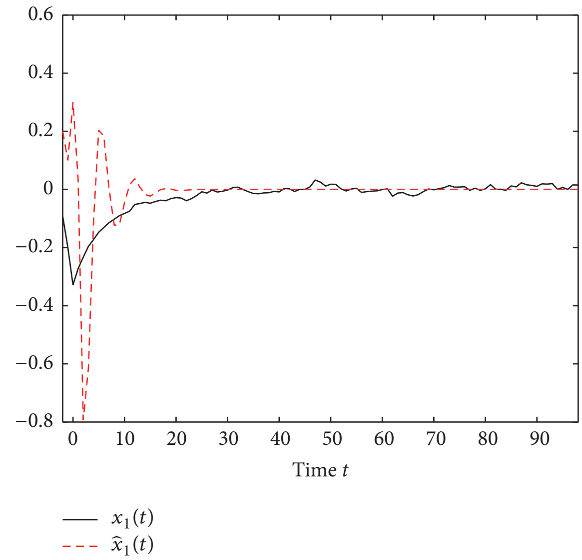

Figure 2: Evolutive behavior of $x_1(t)$ and $\hat{x}_1(t)$.

Example 1. Consider a two-dimensional neural network model (5) described by

$$\begin{gathered}\alpha = 0.5,\\ D = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix},\\ A = \begin{pmatrix} 0.2 & 0.2 \\ 0.5 & 0.5 \end{pmatrix},\\ B = \begin{pmatrix} 0.2 & 0.8 \\ 0.2 & 0.8 \end{pmatrix},\\ \tau(t) = |\sin(t)|,\\ I = (0.5, 0.5)^T,\\ C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\\ g(\Upsilon) = f(\Upsilon) = \tanh(\Upsilon).\end{gathered} \tag{37}$$

Figure 1 shows simulation result of the above neural network model, which can exhibit chaotic behavior.

Obviously, we can obtain $G_1 = G_2 = 1$, $F_1 = F_2 = 1$, and $\tau = 1$; to apply Theorem 5, it requires

$$\begin{aligned}&6 - \frac{1}{\beta_1}\sum_{j=1}^{2}\beta_j\left|a_{1j}\right| - \sum_{\ell=1}^{2}\sum_{\hbar=1}^{2}\left|\mathfrak{K}_{\ell\hbar}\right|\\ &\quad - \left(\frac{r}{r-1}\right)^{0.5}\sum_{j=1}^{2}\frac{\beta_j}{\beta_1}\left|b_{1j}\right| - \frac{1.5}{r^{0.5}\Gamma(1.5)} \geq 0,\\ &6 - \frac{1}{\beta_2}\sum_{j=1}^{2}\beta_j\left|a_{2j}\right| - \sum_{\ell=1}^{2}\sum_{\hbar=1}^{2}\left|\mathfrak{K}_{\ell\hbar}\right|\\ &\quad - \left(\frac{r}{r-1}\right)^{0.5}\sum_{j=1}^{2}\frac{\beta_j}{\beta_2}\left|b_{2j}\right| - \frac{1.5}{r^{0.5}\Gamma(1.5)} \geq 0.\end{aligned} \tag{38}$$

Select $r = 4$, $\beta_1 = 1$, $\beta_2 = 2$, and

$$K = \begin{pmatrix} \mathfrak{K}_{11} & \mathfrak{K}_{12} \\ \mathfrak{K}_{21} & \mathfrak{K}_{22} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.6 \\ -0.6 & -0.6 \end{pmatrix}; \tag{39}$$

then (38) are satisfied, so the designed controller is

$$u(t) = KCe(t_k) = \begin{pmatrix} 1.2 & 1.2 \\ -1.2 & -1.2 \end{pmatrix} e(t_k), \quad t \in [t_k, t_{k+1}). \tag{40}$$

Figure 2 is utilized to show the simulation result for the master/drive system state $x_1(t)$ and the slave/response system state $\hat{x}_1(t)$ in Example 1. Figure 3 is utilized to show the simulation result for the master/drive system state $x_2(t)$ and the slave/response system state $\hat{x}_2(t)$ in Example 1. Clearly, the drive-response-based coupled systems in Example 1 can reach global $O(t^{-\alpha})$ synchronization. Figure 4 shows the centralized data-sampling release instants and the corresponding release intervals. These results via computer simulations nicely demonstrate that the designed controller performs very well.

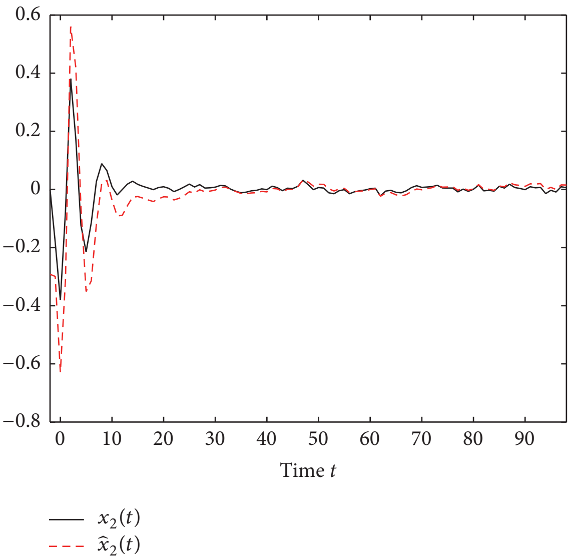

FIGURE 3: Evolutive behavior of $x_2(t)$ and $\hat{x}_2(t)$.

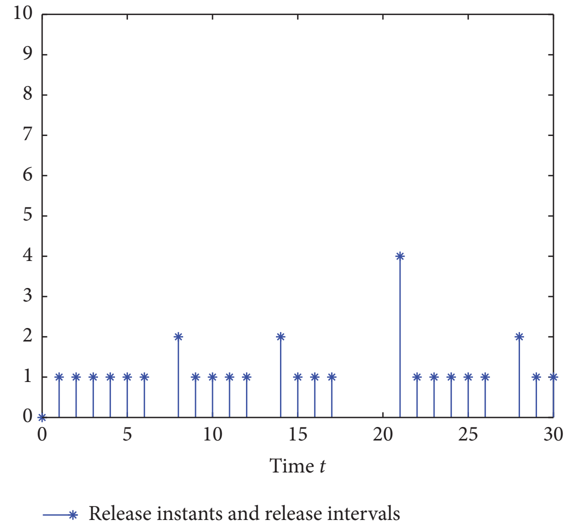

FIGURE 4: Release instants and release intervals.

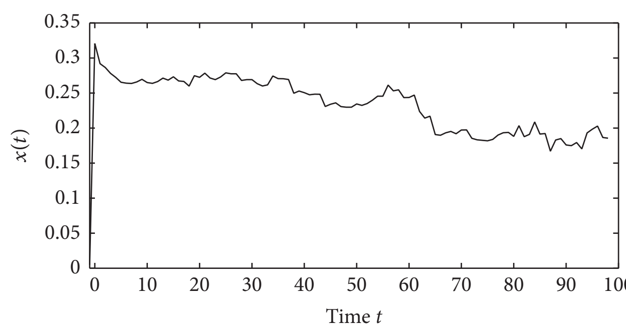

FIGURE 5: Disorganized behavior.

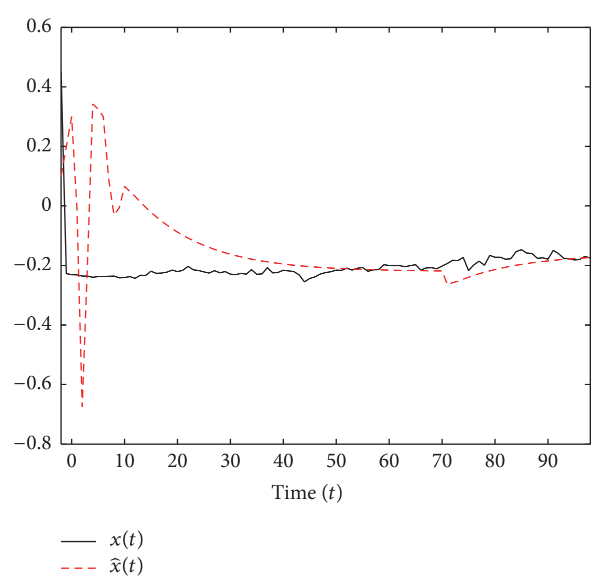

FIGURE 6: Evolutive behavior of $x(t)$ and $\hat{x}(t)$.

Example 2. Consider a one-dimensional neural network model (5) described by

$$\begin{aligned} \alpha &= 0.5, \\ D &= 6, \\ A &= 1, \\ B &= -3.5, \\ \tau(t) &= |\sin(t)|, \\ I &= 1, \\ C &= 1, \\ g(\Upsilon) &= f(\Upsilon) = \tanh(\Upsilon). \end{aligned} \tag{41}$$

Figure 5 shows simulation result of the above neural network model without any external control, which generates disorganized behavior.

Obviously, we can obtain $G = 1$, $F = 1$, and $\tau = 1$; to apply Theorem 5, it requires

$$D - |A| - |\mathfrak{K}| - \left(\frac{r}{r-1}\right)^{0.5} |B| - \frac{1.5}{r^{0.5}\Gamma(1.5)} \geq 0. \tag{42}$$

Select $r = 4$ and

$$K = \mathfrak{K} = 0.1; \tag{43}$$

then (42) is satisfied, so the designed controller is

$$u(t) = KCe(t_k) = 0.1e(t_k), \quad t \in [t_k, t_{k+1}). \tag{44}$$

Figure 6 is utilized to show the simulation result for the master/drive system state $x(t)$ and the slave/response system state $\hat{x}(t)$ in Example 2. Figure 7 shows the centralized data-sampling release instants and the corresponding release intervals. Such results via computer simulations also indicate that the designed controller performs very well.

5. Conclusion

In this paper, the centralized data-sampling approach for global $O(t^{-\alpha})$ synchronization of fractional-order neural net-

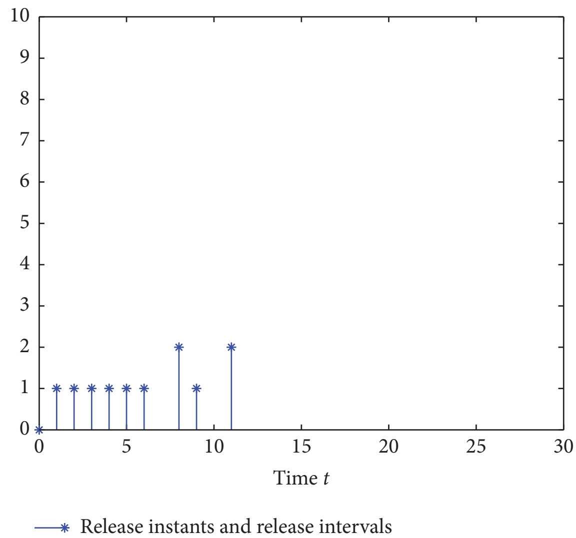

Figure 7: Release instants and release intervals.

works with time delays is investigated. Some sufficient conditions for centralized data-sampling principle are derived and proved to guarantee the global $O(t^{-\alpha})$ synchronization for drive-response-based coupled neural networks. The proposed theoretical results are sufficient conditions for global $O(t^{-\alpha})$ synchronization and contain a lot of space for further improvement. Future research can be extended to (1) more economical and efficient event-triggered mechanism and (2) more realistic networked systems involving stochastic effect or incomplete measurement.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] I. Ameen and P. Novati, "The solution of fractional order epidemic model by implicit Adams methods," *Applied Mathematical Modelling. Simulation and Computation for Engineering and Environmental Systems*, vol. 43, pp. 78–84, 2017.

[2] A. Wu and Z. Zeng, "Global Mittag–leffler stabilization of fractional-order memristive neural networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 1, pp. 206–217, 2017.

[3] B. S. Chen and J. J. Chen, "Global O($t^{-\alpha}$) stability and global asymptotical periodicity for a non-autonomous fractional-order neural networks with time-varying delays," *Neural Networks*, vol. 73, pp. 47–57, 2016.

[4] J. Palanivel, K. Suresh, S. Sabarathinam, and K. Thamilmaran, "Chaos in a low dimensional fractional order nonautonomous nonlinear oscillator," *Chaos, Solitons & Fractals*, vol. 95, pp. 33–41, 2017.

[5] N. Miljković, N. Popović, O. Djordjević, L. Konstantinović, and T. B. Šekara, "ECG artifact cancellation in surface EMG signals by fractional order calculus application," *Computer Methods and Programs in Biomedicine*, vol. 140, pp. 259–264, 2017.

[6] S. Saha Ray and S. Sahoo, "Two efficient reliable methods for solving fractional fifth order modified Sawada–Kotera equation appearing in mathematical physics," *Journal of Ocean Engineering and Science*, vol. 1, no. 3, pp. 219–225, 2016.

[7] K. Amoura, R. Mansouri, M. Bettayeb, and U. M. Al-Saggaf, "Closed-loop step response for tuning PID-fractional-order-filter controllers," *ISA Transactions*, vol. 64, pp. 247–257, 2016.

[8] R. Rakkiyappan, G. Velmurugan, and J. Cao, "Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays," *Nonlinear Dynamics*, vol. 78, no. 4, pp. 2823–2836, 2014.

[9] R. Rakkiyappan, G. Velmurugan, and J. Cao, "Stability analysis of memristor-based fractional-order neural networks with different memductance functions," *Cognitive Neurodynamics*, vol. 9, no. 2, pp. 145–177, 2015.

[10] R. Rakkiyappan, G. Velmurugan, and J. Cao, "Stability analysis of fractional-order complex-valued neural networks with time delays," *Chaos, Solitons and Fractals*, vol. 78, pp. 297–316, 2015.

[11] D. K. Raju, B. S. Umre, A. Junghare, and B. C. Babu, "Mitigation of subsynchronous resonance with fractional-order PI based UPFC controller," *Mechanical Systems and Signal Processing*, vol. 85, pp. 698–715, 2017.

[12] F. Wang, Y. Yang, M. Hu, and X. Xu, "State estimation for fractional-order neural networks," *Optik*, vol. 126, no. 23, pp. 4083–4086, 2015.

[13] Y. Arya and N. Kumar, "BFOA-scaled fractional order fuzzy PID controller applied to AGC of multi-area multi-source electric power generating systems," *Swarm and Evolutionary Computation*, vol. 32, pp. 202–218, 2017.

[14] A. Wu and Z. Zeng, "Boundedness, Mittag-Leffler stability and asymptotical ω-periodicity of fractional-order fuzzy neural networks," *Neural Networks*, vol. 74, pp. 73–84, 2016.

[15] G. M. Mahmoud, M. E. Ahmed, and T. M. Abed-Elhameed, "On fractional-order hyperchaotic complex systems and their generalized function projective combination synchronization," *Optik*, vol. 130, pp. 398–406, 2017.

[16] A. Wu, Z. Zeng, and X. Song, "Global Mittag-Leffler stabilization of fractional-order bidirectional associative memory neural networks," *Neurocomputing*, vol. 177, pp. 489–496, 2016.

[17] R. Sharma, P. Gaur, and A. P. Mittal, "Design of two-layered fractional order fuzzy logic controllers applied to robotic manipulator with variable payload," *Applied Soft Computing*, vol. 47, pp. 565–576, 2016.

[18] J. A. Gallegos and M. A. Duarte-Mermoud, "On the Lyapunov theory for fractional order systems," *Applied Mathematics and Computation*, vol. 287- 288, pp. 161–170, 2016.

[19] S. Zhang, Y. Yu, and H. Wang, "Mittag-Leffler stability of fractional-order Hopfield neural networks," *Nonlinear Analysis. Hybrid Systems*, vol. 16, pp. 104–121, 2015.

[20] Q.-L. Han, "A discrete delay decomposition approach to stability of linear retarded and neutral systems," *Automatica*, vol. 45, no. 2, pp. 517–524, 2009.

[21] Y. He, J. Zeng, M. Wu, and C.-K. Zhang, "Robust stabilization and H_∞ controllers design for stochastic genetic regulatory networks with time-varying delays and structured uncertainties," *Mathematical Biosciences*, vol. 236, no. 1, pp. 53–63, 2012.

[22] H. Huang, T. Huang, and X. Chen, "Global exponential estimates of delayed stochastic neural networks with Markovian switching," *Neural Networks*, vol. 36, pp. 136–145, 2012.

[23] J. Liang, Z. Wang, Y. Liu, and X. Liu, "State estimation for two-dimensional complex networks with randomly occurring nonlinearities and randomly varying sensor delays," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 1, pp. 18–38, 2014.

[24] D. Liu and Q. Wei, "Finite-approximation-error-based optimal control approach for discrete-time nonlinear systems," *IEEE Transactions on Cybernetics*, vol. 43, no. 2, pp. 779–789, 2013.

[25] J. Lu, D. W. Ho, J. Cao, and J. Kurths, "Single impulsive controller for globally exponential synchronization of dynamical networks," *Nonlinear Analysis. Real World Applications*, vol. 14, no. 1, pp. 581–593, 2013.

[26] A. Wu and Z. Zeng, "Output convergence of fuzzy neurodynamic system with piecewise constant argument of generalized type and time-varying input," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 46, no. 12, pp. 1689–1702, 2016.

[27] R. Wang, P. Shi, Z.-G. Wu, and Y.-T. Sun, "Stabilization of switched delay systems with polytopic uncertainties under asynchronous switching," *Journal of the Franklin Institute*, vol. 350, no. 8, pp. 2028–2043, 2013.

[28] Z. Wang, B. Shen, H. Shu, and G. Wei, "Quantized H_∞ control for nonlinear stochastic time-delay systems with missing measurements," *Institute of Electrical and Electronics Engineers. Transactions on Automatic Control*, vol. 57, no. 6, pp. 1431–1444, 2012.

[29] Z. Wang, H. Zhang, and B. Jiang, "LMI-based approach for global asymptotic stability analysis of recurrent neural networks with various delays and structures," *IEEE Transactions on Neural Networks*, vol. 22, no. 7, pp. 1032–1045, 2011.

[30] H. G. Zhang, Z. S. Wang, and D. Liu, "A comprehensive review of stability analysis of continuous-time recurrent neural networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 25, no. 7, pp. 1229–1262, 2014.

[31] H. Li, "Event-triggered state estimation for a class of delayed recurrent neural networks with sampled-data information," *Abstract and Applied Analysis*, vol. 2012, Article ID 731453, 21 pages, 2012.

[32] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nesic, "Robust event-triggered output feedback controllers for nonlinear systems," *Automatica*, vol. 75, pp. 96–108, 2017.

[33] A. Tanwani, C. Prieur, and M. Fiacchini, "Observer-based feedback stabilization of linear systems with event-triggered sampling and dynamic quantization," *Systems and Control Letters*, vol. 94, pp. 46–56, 2016.

[34] A. Seuret, C. Prieur, S. Tarbouriech, and L. Zaccarian, "Event-triggered control via reset control systems framework," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 170–175, 2016.

[35] N. K. Sharma and T. V. Sreenivas, "Event-triggered sampling using signal extrema for instantaneous amplitude and instantaneous frequency estimation," *Signal Processing*, vol. 116, pp. 43–54, 2015.

[36] S. Dashkovskiy and S. Pavlichkov, "Constructive design of adaptive controllers for nonlinear MIMO systems with arbitrary switchings," *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 2001–2007, 2016.

[37] A. A. Pyrkin and A. A. Bobtsov, "Adaptive controller for linear system with input delay and output disturbance," in *Proceedings of the 52nd IEEE Conference on Decision and Control (CDC '13)*, vol. 61, pp. 4229–4234, IEEE, Florence, Italy, December 2013.

[38] J. E. Azzaro and R. A. Veiga, "Sliding mode controller with neural network identification," *IEEE Latin America Transactions*, vol. 13, no. 12, pp. 3754–3757, 2015.