Advances in Mechanical Engineering

New Developments in Multibody System Dynamics and Its Applications

Guest Editors: Xiaoting Rui, Chuanzeng Zhang, and Caishan Liu

New Developments in Multibody System Dynamics and Its Applications

Advances in Mechanical Engineering

New Developments in Multibody System Dynamics and Its Applications

Guest Editors: Xiaoting Rui, Chuanzeng Zhang, and Caishan Liu

Copyright © 2014 Hindawi Publishing Corporation. All rights reserved.

This is a special issue published in "Advances in Mechanical Engineering." All articles are open access articles distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Editorial Board

Koshi Adachi, Japan
Mehdi Ahmadian, USA
Rehan Ahmed, UK
Muhammad T. Akhtar, Japan
Nacim Alilat, France
M. Affan Badar, USA
Luis Baeza, Spain
R. Balachandran, UK
Claude Bathias, France
Adib Becker, UK
Leonardo Bertini, Italy
Liam A. Blunt, UK
Noël Brunetière, France
Marco Ceccarelli, Italy
Fakher Chaari, Tunisia
Chin-Lung Chen, Taiwan
Hyung H. Cho, Republic of Korea
Seung-Bok Choi, Korea
Ahmet S. Dalkilic, Turkey
Kangyao Deng, China
Francisco D. Denia, Spain
T. S. Dhanasekaran, USA
Nihad Dukhan, USA
Farzad Ebrahimi, Iran
Ali El Wahed, UK
Bogdan I. Epureanu, USA
Mohammad R. Eslami, Iran
Ali Fatemi, USA
Mario L. Ferrari, Italy
Siegfried Fouvry, France
Ian Frigaard, Canada
Mergen H. Ghayesh, Canada
Luís Godinho, Portugal
Tian Han, China
Francisco J. Huera-Huarte, Spain
Davood Jalali-Vahid, Iran
Jiin Y. Jang, Taiwan
Zhongmin Jin, UK
Xiaodong Jing, China
S.-W. Kang, Republic of Korea
Xianwen Kong, UK
Michal Kuciej, Poland
Yaguo Lei, China
Zili Li, The Netherlands
Yangmin Li, Macau
Jaw-Ren Lin, Taiwan
Cheng-Xian Lin, USA
Seyed N. Mahmoodi, USA
Oronzio Manca, Italy
Ramiro Martins, Portugal
Aristide F. Massardo, Italy
Francesco Massi, Italy
T. H. New, Singapore
Kim C. Ng, Singapore
Cong T. Nguyen, Canada
Hirosi Noguchi, Japan
Hakan F. Oztop, Turkey
Duc T. Pham, UK
Jurij Prezelj, Slovenia
Xiaotun Qiu, USA
Pascal Ray, France
Robert L. Reuben, UK
Pedro A. R. Rosa, Portugal
Elsa de Sá Caetano, Portugal
David R. Salgado, Spain
Mohammad R. Salimpour, Iran
Sunetra Sarkar, India
Pietro Scandura, Italy
A. S. Sekhar, India
Liyuan Sheng, China
Xi Shi, China
Seiichi Shiga, Japan
Chow-Shing Shin, Taiwan
Ray W. Snidle, UK
Margaret M. Stack, UK
Neil Stephen, UK
Kumar K. Tamma, USA
Yaya Tan, China
Anand Thite, UK
Cho W. Solomon To, USA
Yoshihiro Tomita, Japan
Shan-Tung Tu, China
Sandra Velarde-Suárez, Spain
Moran Wang, China
Junwu Wang, China
Jia-Jang Wu, Taiwan
Fengfeng Xi, Canada
Gongnan Xie, China
Wei Mon Yan, Taiwan
Jianqiao Ye, UK
Byeng D. Youn, USA
Bo Yu, China
Jianbo Yu, China
Zhongrong Zhou, China

Contents

New Developments in Multibody System Dynamics and Its Applications, Xiaoting Rui, Chuanzeng Zhang, and Caishan Liu
Volume 2014, Article ID 671604, 1 page

Automatic Deduction Theorem of Overall Transfer Equation of Multibody System, Xiaoting Rui, Jianshu Zhang, and Qinbo Zhou
Volume 2014, Article ID 378047, 12 pages

Optimal Design of One-Folded Leaf Spring with High Fatigue Life Applied to Horizontally Vibrating Linear Actuator in Smart Phone, Ki Bum Lee, Chang Hyun Park, and Jin Ho Kim
Volume 2014, Article ID 545126, 7 pages

Adaptive Terminal Sliding Mode Control of Electromagnetic Spacecraft Formation Flying in Near-Earth Orbits, Jingrui Zhang, Changqin Yuan, Dongmei Jiang, and Dawei Jin
Volume 2014, Article ID 512583, 9 pages

Application of Krylov Reduction Technique for a Machine Tool Multibody Modelling, M. Sulitka, J. Šindler, J. Sušeň, and J. Smolík
Volume 2014, Article ID 592628, 9 pages

Study on Launch Dynamics of Self-Propelled Artillery Based on Transfer Matrix Method of Multibody System, Hailong Yu and Xiaoting Rui
Volume 2014, Article ID 308049, 9 pages

Higher Order Variational Integrators for Multibody System Dynamics with Constraints, Jieyu Ding and Zhenkuan Pan
Volume 2014, Article ID 383680, 8 pages

Free Vibration Characteristic of Multilevel Beam Based on Transfer Matrix Method of Linear Multibody Systems, Laith K. Abbas and Xiaoting Rui
Volume 2014, Article ID 792478, 16 pages

Case Study on Human Walking during Wearing a Powered Prosthetic Device: Effectiveness of the System "Human-Robot", Svetlana Grosu, Pierre Cherelle, Chris Verheul, Bram Vanderborght, and Dirk Lefeber
Volume 2014, Article ID 365265, 9 pages

Controller Parameters Tuning Based on Transfer Matrix Method for Multibody Systems, Hossam Hendy, Xiaoting Rui, Qinbo Zhou, and Mostafa Khalil
Volume 2014, Article ID 957684, 8 pages

All Terrain Vehicle Flexible Multibody Dynamic Simulation for Fatigue Prediction, Jia-Shiun Chen and Hsiu-Ying Hwang
Volume 2013, Article ID 725315, 6 pages

Projectile Impact Point Prediction Based on Self-Propelled Artillery Dynamics and Doppler Radar Measurements, Mostafa Khalil, Xiaoting Rui, Qicheng Zha, Hailong Yu, and Hossam Hendy
Volume 2013, Article ID 153913, 12 pages

Study on the Dynamics of Laser Gyro Strapdown Inertial Measurement Unit System Based on Transfer Matrix Method for Multibody System, Gangli Chen, Xiaoting Rui, Fufeng Yang, Jianshu Zhang, and Qinbo Zhou
Volume 2013, Article ID 854583, 9 pages

Recursive Formulation for Dynamic Modeling and Simulation of Multilink Spatial Flexible Robotic Manipulators, Zhenjie Qian, Dingguo Zhang, and Jun Liu
Volume 2013, Article ID 216014, 15 pages

Trivariate Isogeometric Analysis for Flexible Multibody Dynamics, Ting Pi
Volume 2013, Article ID 139498, 8 pages

Dynamic Control and Disturbance Estimation of 3D Path Following for the Observation Class Underwater Remotely Operated Vehicle, Hai Huang, Qirong Tang, Yueming Li, Lei Wan, and Yongjie Pang
Volume 2013, Article ID 604393, 16 pages

Application of Finite Volume Method to Structural Stochastic Dynamics, Weidong Chen, Yanchun Yu, Ping Jia, Xiande Wu, and Fengchao Zhang
Volume 2013, Article ID 391704, 10 pages

Numerical Simulation on Dense Packing of Granular Materials by Container Oscillation, Jun Liu and Dongxu You
Volume 2013, Article ID 284693, 7 pages

Research on Flexible Joint Stiffness Test and Trajectory Planning of Space Manipulator, Yongtai Yang, Jili Rong, Jian Li, and Ling Tang
Volume 2013, Article ID 280453, 7 pages

Case Study on Influence of Step Blast-Excavation on Support Systems of Existing Service Tunnel with Small Interval, Shaorui Sun, Ling Yue, Jimin Wu, Jin Liu, and Jihong Wei
Volume 2013, Article ID 257457, 13 pages

The Effect of First-Order Bending Resonance of Wheelset at High Speed on Wheel-Rail Contact Behavior, Shuoqiao Zhong, Xinbiao Xiao, Zefeng Wen, and Xuesong Jin
Volume 2013, Article ID 296106, 19 pages

Dynamic Modeling and Analysis of Hub-Tapered Beam System, Si-jia Chen and Ding-guo Zhang
Volume 2013, Article ID 313279, 12 pages

Analysis of Elastic-Plastic Responses of a New Single-Story Frame-Bent Structure during an Earthquake Based on the Transfer Matrix Method of Multibody System, Jianguo Ding, Yuwei Dai, and Zhi Qiao
Volume 2013, Article ID 784525, 10 pages

Cooperative Search by Combining Simulated and Real Robots in a Swarm under the View of Multibody System Dynamics, Qirong Tang and Peter Eberhard
Volume 2013, Article ID 284782, 11 pages

Global Impact Dynamic Modeling and Verification of a Flexible Beam with Large Overall Motion, Yuechen Duan, Dingguo Zhang, and Jiazhen Hong
Volume 2013, Article ID 362317, 8 pages

Evaluating Energy Flux in Vibrofluidized Granular Bed, N. A. Sheikh, S. Manzoor, and U. Ghani
Volume 2013, Article ID 327379, 7 pages

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 671604, 1 page
http://dx.doi.org/10.1155/2014/671604

Editorial

New Developments in Multibody System Dynamics and Its Applications

Xiaoting Rui,[1] Chuanzeng Zhang,[2] and Caishan Liu[3]

[1] *Institute of Launch Dynamics, Nanjing University of Science and Technology, Nanjing 210094, China*
[2] *Department of Civil Engineering, University of Siegen, 57068 Siegen, Germany*
[3] *College of Engineering, Peking University, Beijing 100871, China*

Correspondence should be addressed to Xiaoting Rui; ruixt@163.net

Received 6 January 2014; Accepted 6 January 2014; Published 12 March 2014

Copyright © 2014 Xiaoting Rui et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Multibody system dynamics is nowadays a widely used toolbox in the development of modern engineering science and technology. It provides powerful dynamic design theories, computational and analytical methods, and test technology for innovative engineering applications. In order to systematically show the important effects of multibody system dynamics on the development of modern science and technology and to strengthen the academic exchanges in the research field of multibody system dynamics and its applications, we organized this special issue containing twenty-five papers selected from the contributions coming from Belgium, Czech Republic, Egypt, Germany, Iraq, Netherlands, Pakistan, China, Republic of Korea, United Kingdom, Taiwan, and so forth.

The twenty-five papers selected for this Special Issue are devoted to the following topics: an automatic deduction theorem of the overall transfer equations of the transfer matrix method for multibody systems (MSTMM), the application of MSTMM in dynamics of a multilevel beam system, the application of MSTMM in dynamics of an elastic-plastic system subjected to earthquakes, the application of MSTMM in dynamics of controlled cars, the application of MSTMM in dynamics of a laser gyro strapdown inertial measurement unit system, the application of MSTMM in launch dynamics of a self-propelled artillery system and flight dynamics of projectile, the modeling of trivariate isogeometrics for flexible multibody dynamics, the modeling of hub-tapered beam system dynamics, the formulations and modeling approaches of spatial flexible robotic manipulators, the impact in flexible systems, the modeling and test methods for flexible joint stiffness of a space manipulator, the computational stability and precision, the model and modal reduction techniques for multibody system dynamics, the optimization and control of a swarm of robots, the optimal design of smart systems, the dynamics and control of underwater vehicle and spacecraft formation, the perturbation-finite volume method for stochastic multibody system dynamics, the blast-excavation dynamics, the granular system dynamics, the vehicle-track coupling dynamics, and the human-robot dynamics. Thus, the general fundamental theories and several key engineering applications are covered representing the state of the art of recent intensive and extensive research activities in the field of multibody system dynamics to a certain extent.

The guest editors hope that this special issue will provide some valuable information and guidance for scientists working on multibody system dynamics and its applications in modern engineering science and technology.

Acknowledgments

Many thanks are due to Professor Edwin Kreuzer from Hamburg University of Technology and Professor Dieter Bestle from Brandenburg University of Technology Cottbus-Senftenberg for their great unselfish help in organizing the special issue, to reviewers and authors, and to the reviewers for their voluntary works and constructive comments. All of these promoted greatly the projected publication of this special issue.

Xiaoting Rui
Chuanzeng Zhang
Caishan Liu

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 378047, 12 pages
http://dx.doi.org/10.1155/2014/378047

Research Article

Automatic Deduction Theorem of Overall Transfer Equation of Multibody System

Xiaoting Rui, Jianshu Zhang, and Qinbo Zhou

Institute of Launch Dynamics, Nanjing University of Science and Technology, Nanjing 210094, China

Correspondence should be addressed to Xiaoting Rui; ruixt@163.net

Received 25 August 2013; Accepted 30 December 2013; Published 10 March 2014

Academic Editor: Caishan Liu

Copyright © 2014 Xiaoting Rui et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Transfer matrix method for multibody System (MSTMM) is a new multibody dynamics method developed in recent 20 years. It has been widely used in both science research and engineering for its special features as follows: without global dynamics equations of the system, high programming, low order of system matrix, and high computational speed. Based on MSTMM and its above features, a theorem to deduce automatically the overall transfer equations of multibody systems by handwriting or by computer is proposed in this paper. The theorem is effective for multibody systems with various topological structures, including chain systems, closed-loop systems, tree systems, general systems composed of one tree subsystem, and some closed-loop subsystems. This theorem makes it possible to program large scale software of multibody system dynamics with much higher programming, and much higher computational speed because of the above features of MSTMM. Formulations of the proposed method as well as two examples are given to verify this method.

1. Introduction

Lots of methods dealing with multibody system dynamics (MSD) have been studied by many authors since 1960s [1–18]. They are widely used in many engineering fields such as aeronautics, astronautics, spacecraft, vehicle, robot, precision machinery, and biomechanics. It is well known that almost all the previous ordinary methods for MSD have the same characteristics as follows: it is necessary and very complicated to develop the global dynamics equations of the system; the order of system matrix depends on the number of degrees of freedom of the system and hence it is rather high for complex multibody system.

To avoid establishing the global dynamics equations of the system, simplify the study procedure and especially keep high computational efficiency independent of the number of degree of freedom of system in studying MSD, new analytical method for MSD, namely, transfer matrix method for multibody system (MSTMM), is presented by Rui and his co-workers [19–21] and constantly developed in recent 20 years [22–25]. Nowadays, MSTMM is widely applied in science research and engineering [26–29] for the features of this method as follows: study MSD without global dynamics equations of the system, keep low order of the system matrix so very high computational speed, and avoid the difficulties in computation caused by high-order matrices and high programming. It has been proved by lots of theories and experiments that MSTMM is effective for linear time-invariant multibody systems [22], nonlinear time-variant multibody systems, multi-rigid-body systems [19, 21], multi-rigid-flexible-body systems [22–24], and controlled multi-body systems [19, 29].

Generally speaking, various multibody systems may be considered as one of the following four cases in topology [24]: (1) chain system, (2) closed-loop system, (3) tree system, (4) general systems composed of one tree subsystem, and some closed-loop subsystems. A chain system can be considered as a special example of a tree system at the case with only two boundary ends. By "cutting" at one connection point of a system, a closed-loop system can be considered as a chain system, and a general system composed of one tree subsystem and some closed-loop subsystems can be dealt with a tree system [24]. Based on MSTMM and its above features, a theorem to deduce automatically the overall

transfer equations of various multibody systems mentioned above by handwriting and by computer is proposed in this paper.

2. General Theorems and Steps of MSTMM

2.1. Basic Idea of MSTMM. The basic idea of MSTMM [19] is to break up a multibody system into the elements containing bodies (including rigid bodies, flexible bodies, lumped masses, etc.) and hinges (including joints, ball-and-sockets, pins, springs, rotary springs, dampers and rotary dampers, etc.) whose dynamics properties can be readily expressed in matrix forms. These matrices of elements are considered as building blocks that provide the dynamics properties of the entire system when assembling them together according to the topology of the system. Particularly, the positions of bodies and hinges are considered equivalent in transfer equations and transfer matrices, which is totally different from ordinary methods for MSD [1–18] and results in the very low order of system matrix and very high computational speed in MSTMM.

2.2. State Vector, Transfer Equation, and Transfer Matrix of Element. The same coordinate systems and sign conventions as are described in [19, 21, 30] will be used. The state vector of the connection point between any rigid body and hinge moving in space is defined as

$$\mathbf{z} = \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{z} & \ddot{\theta}_x & \ddot{\theta}_y & \ddot{\theta}_z & m_x & m_y & m_z & q_x & q_y & q_z & 1 \end{bmatrix}^T, \quad (1)$$

or

$$\mathbf{z} = \begin{bmatrix} \ddot{\mathbf{r}}^T & \ddot{\boldsymbol{\theta}}^T & \mathbf{m}^T & \mathbf{q}^T & 1 \end{bmatrix}^T, \quad (2)$$

where

$$\ddot{\mathbf{r}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}, \qquad \ddot{\boldsymbol{\theta}} = \begin{bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix},$$
$$\mathbf{m} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}. \quad (3)$$

The state vector of connection point between flexible body and hinge moving in space is defined as

$$\mathbf{z} = \begin{bmatrix} \ddot{x}, \ddot{y}, \ddot{z}, \ddot{\theta}_x, \ddot{\theta}_y, \ddot{\theta}_z, m_x, m_y, m_z, \\ q_x, q_y, q_z, \ddot{q}^1, \ddot{q}^2, \ldots, \ddot{q}^n, 1 \end{bmatrix}^T, \quad (4)$$

where $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$ are the accelerations of the connection point with respect to the global inertial coordinate system $oxyz$; $\ddot{\theta}_x$, $\ddot{\theta}_y$, and $\ddot{\theta}_z$ are the second time derivatives of the orientation angles of the body using space-three-angles 1-2-3 θ_x, θ_y, and θ_z defined in [18]; m_x, m_y, m_z, q_x, q_y, and q_z are the corresponding internal torques and internal forces in the same reference system, respectively; $\ddot{q}^1, \ddot{q}^2, \ldots, \ddot{q}^n$ are the second time derivatives of the generalized coordinates describing the deformation of flexible bodies using modal method; the superscript n is the highest order of the modal considered, $\ddot{\mathbf{r}}$, $\ddot{\boldsymbol{\theta}}$, $\mathbf{m}$, and $\mathbf{q}$ are the column matrices of accelerations, second time derivatives of the space-three angles, internal torques, and internal forces in $oxyz$, respectively.

For body and hinge elements moving in a plane, a similar definition of the state vector can be introduced, which is a special example of spatial motion.

The transfer equations of the jth element can be obtained easily by rewriting its dynamics equations as follows [19, 21, 30]

$$\mathbf{z}_{j,j+1} = \mathbf{U}_j(t_i)\,\mathbf{z}_{j,j-1}, \quad (5)$$

where $\mathbf{U}_j(t_i)$ is the transfer matrix of the jth element, which expresses the relationship between the state vectors of its output end $\mathbf{z}_{j,j+1}$ and input end $\mathbf{z}_{j,j-1}$ and is already known at time instant t_i.

It should be pointed out that there are general linear relations among accelerations, angular accelerations, forces, and torques of any mechanics system in an inertial coordinate system $oxyz$, according to Newton motion law and Euler theorem of moment of momentum. It is to say that there are strict linear relations between the state vectors of output end and input end of any element and among all state vectors of a multibody system. Thus, the transfer equation (5) is a general equation and effective for any mechanics element in the inertial coordinate system.

2.3. Overall Transfer Equation and Overall Transfer Matrix of the System. The overall transfer equations of a chain system can be deduced automatically as [19, 24]

$$\mathbf{z}_{1,0} = \mathbf{U}_{n-1}\mathbf{z}_{n,0}, \quad (6)$$

where the overall transfer matrix of the system is

$$\mathbf{U}_{n-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\cdots\mathbf{U}_n. \quad (7)$$

From equations (6) and (7), the features of the overall transfer equation for a chain system can be clearly seen; overall transfer matrix of a chain system can be deduced automatically by successive premultiplication of transfer matrices of every element of the system along the transfer path from the one end to another end.

2.4. Solutions of the System Motion. Applying the boundary conditions of the system, $\mathbf{z}_{1,0}$ and $\mathbf{z}_{n,0}$, to the overall system transfer equation (6), the unknown state variables in the boundary state vectors can be computed. Then, the state vectors of each element at time t_i can be computed by the repeated use of corresponding transfer equations of element (5). Then, using the computed values of the last step as the initial conditions, the entire procedure can be repeated from time t_{i+1} until the time required for complete analysis. It can be seen clearly from equations (5)–(7) that MSTMM belongs to one of strict analytical methods in principle.

3. Topology Figure of a Multibody System and Sign Conventions

3.1. Topology Figure of a Multibody System. A dynamics model of any complex multibody system can be constructed with dynamics elements including bodies and hinges. In order to describe the transfer relationship among the state vectors of elements and the transfer directions in a system, the topology figure of the model will be very useful for deduction of overall transfer equation in MSTMM. And the topology figure of the dynamics model of a system, for example, a tree multibody system, as shown in Figure 1, can be got very easily and directly from its dynamics model if using the following sign conventions.

3.2. The Sign Conventions. Besides the sign conventions introduced in [19, 21], the sign conventions as follows are used in the paper.

(1) A circle ○ denotes a body element and the number inside this circle is the sequence number of the body element.

(2) An arrow → denotes a hinge element and the transfer direction of state vectors; the number beside the arrow is the sequence number of the hinge element.

(3) Each body element is dealt with single output end and single input end if the body has two connection ends with other elements; otherwise, it is dealt with single output end and multiple input ends if the body has more two connection ends.

(4) For a nonboundary end, the first and second subscripts, i and j ($i, j \neq 0$), in a state vector $\mathbf{z}_{i,j}$ of the end denote the sequence numbers of the adjacent body element and hinge element, respectively. For a boundary end, the second subscript $j = 0$ in the state vector $\mathbf{z}_{i,j}$; that is, the second subscript 0 means boundary end; then the first subscript i in the state vector $\mathbf{z}_{i,0}$ of the boundary end stands for the sequence number of the element involved.

(5) In a multibody system, only one boundary end is considered as the root; the state vector of root is noted as $\mathbf{z}_{i,0}$, where i is the sequence number of the root element; all of other boundary ends are considered as the tips; the state vectors of tips are denoted as $\mathbf{z}_{j,0}$, where j is the sequence number of the tip element. The transfer directions of a system are always from its tips to the root.

(6) The subscript i in transfer matrix $\mathbf{U}_i$ denotes the sequence number of element i. The subscript $(i - k)$ in the transfer matrix $\mathbf{U}_{i-k}$ and the partitioned matrix $\mathbf{u}_{i-k}$ means from element i to element k. $\mathbf{U}_{i-k}$ and $\mathbf{u}_{i-k}$ mean the successive premultiplication of the transfer matrices of all elements in the transfer path from the element i to element k of the system.

Figure 1: Topology figure of a dynamics model of tree multibody system.

4. Automatic Deduction of Overall Transfer Equations of System

4.1. Automatic Deduction of the Overall Transfer Equation of a Chain System. The topology figure of any chain system is shown in Figure 2.

It is clear that we can rewrite the overall transfer equation (6) of the chain system as

$$\mathbf{U}_{\text{all}}\left[\mathbf{z}_{1,0}^T \quad \mathbf{z}_{n,0}^T\right]^T = 0, \quad \mathbf{U}_{\text{all}} = \left[-\mathbf{I} \quad \mathbf{U}_{n-1}\right]. \tag{8}$$

From equations (6) or (7), it can be seen clearly that the overall transfer matrix of any chain system $\mathbf{U}_{\text{all}}$ can be deduced automatically by handwriting and by computer. In the overall transfer matrix, the coefficient matrix of the state vector of root is a minus unit matrix $\mathbf{I}$, while the coefficient matrix of the state vector of tip $\mathbf{U}_{n-1}$ is the successive premultiplication of the transfer matrices of all elements in the transfer path from this tip to the root as shown in equation (7).

The highest order of the overall transfer matrix is 13 for spatial chain multi-rigid-body system or $(13 + n)$ for chain multi-rigid-flexible-body system, where n is the highest order of the modal considered.

4.2. Automatic Deduction of the Overall Transfer Equations of a Closed-Loop System. For any closed-loop system, whose topology figure is shown in Figure 3, after "cutting" at the junction of any two adjacent elements such as body 1 and hinge n as shown in Figure 4, consider the couple of "cutting point" as the "boundary ends" with the same state vectors noted as $\mathbf{z}_{n,0}$ and $\mathbf{z}_{1,0}$. Then the original closed-loop system becomes a chain system with the same "boundary ends" state vectors, $\mathbf{z}_{n,0}$ and $\mathbf{z}_{1,0}$, as shown in Figure 4.

Figure 2: Topology figure of a chain system.

Figure 3: Topology figure of a closed-loop system.

Thus, the transfer equation of the closed-loop system can be deduced automatically as is achieved for the chain system

$$\mathbf{z}_{1,0} = \mathbf{U}_{n-1}\mathbf{z}_{n,0} \tag{9}$$

according to the proposed sign conventions, where

$$\mathbf{U}_{n-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\cdots\mathbf{U}_n. \tag{10}$$

Attention should be paid that the state vectors of a couple of "cutting points" are the same, namely

$$\mathbf{z}_{n,0} = \mathbf{z}_{1,0}. \tag{11}$$

Then, the transfer equation (9) of the closed-loop system can be deduced automatically by handwriting and by computer

$$\mathbf{U}_{\text{all}}\mathbf{z}_{1,0} = 0, \quad \mathbf{U}_{\text{all}} = \left(\mathbf{I} - \mathbf{U}_{n-1}\right). \tag{12}$$

4.3. Automatic Deduction of the Overall Transfer Equations of a Tree System. For each body element with more than two ends, as is mentioned in the convention, only one of the ends is considered as output end and all the other ends are input ends. Moreover, its transfer equations should cover the geometrical relationship between its *first* input end and output end and describe the mechanical principle for the forces and moments acting on the element. Thus, it can be verified later that the transfer equations of a rigid body j with L input ends can be written in the following form

$$\mathbf{z}_{j,O} = \mathbf{U}_j\mathbf{z}_{j,I_1} + \mathbf{U}_{j,I_2}\mathbf{z}_{j,I_2} + \cdots + \mathbf{U}_{j,I_L}\mathbf{z}_{j,I_L}, \tag{13}$$

where the subscript j is the sequence number of the body; O and $I_1, I_2, I_3 \cdots I_L$ denote the output end and input ends respectively, and the *first* input end I_1 is considered as the dominant input end; $\mathbf{z}_{j,O}$ and $\mathbf{z}_{j,I_k}$ $(k = 2,3,4,\ldots,L)$ are the state vectors of the output end and the kth input end of the body element respectively; $\mathbf{U}_j$ is just the transfer matrix of element j when point I_1 and point O are considered as the only input end and output end of this element; $\mathbf{U}_{j,I_k}$ is the corresponding extraction matrix that extracts the force variables (including internal force and internal moment) from the state vector when it premultiplies $\mathbf{z}_{j,I_k}$, acted on the same point.

However, in equation(13), the number of unknown variables is more than that of algebraic equations. Therefore, geometrical equations of the body, which describes the geometrical relationship between the *first* input end and kth $(k = 2,3,4,\ldots,L)$ input end of the body, should be introduced for body elements with single output end and multiple input ends. It is verified later that the geometrical equation can be written in the form of

$$\mathbf{H}_j\mathbf{z}_{j,I_1} = \mathbf{H}_{j,I_k}\mathbf{z}_{j,I_k} \quad (k = 2,3,4,\ldots,L), \tag{14}$$

where $\mathbf{H}_j$ is a constant matrix extracting displacement variables (including position coordinates and orientation angles) from a state vector when premultiplying $\mathbf{z}_{j,I_1}$; $\mathbf{H}_{j,I_k}$ is related to the relative position between the *first* input end and kth input end of body. The number of the geometrical equations is $(L-1)$.

A spatial motion rigid body with more than two ends is shown in Figure 5; the concrete form of the transfer equations and geometrical equations of body element will be exhibited.

The state vectors of inboard ends and outboard ends of the rigid body, as defined in Section 2.2, are

$$\mathbf{z}_{I_k} = \left[\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\theta}_x, \ddot{\theta}_y, \ddot{\theta}_z, m_x, m_y, m_z, q_x, q_y, q_z, 1\right]_{I_k}^T \quad (k = 1,2,\ldots,L) \tag{15}$$

$$\mathbf{z}_O = \left[\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\theta}_x, \ddot{\theta}_y, \ddot{\theta}_z, m_x, m_y, m_z, q_x, q_y, q_z, 1\right]_O^T.$$

As shown in Figure 5, I_k $(k = 1,2,\ldots,L)$, O, and C denote the inboard ends, outboard end, and mass center of the rigid body, respectively; the coordinate system with subscript 2 denotes the body-fixed coordinate system whose initial point o_2 is on the *first* inboard end I_1 of the rigid body. According to the properties of a rigid body, the geometrical relationship between the *first* inboard end and the outboard end of the rigid body can be obtained as

$$\begin{aligned} \boldsymbol{\theta}_O &= \boldsymbol{\theta}_{I_1}, \\ \mathbf{r}_O &= \mathbf{r}_{I_1} + \mathbf{A}\mathbf{l}_{I_1O}, \end{aligned} \tag{16}$$

where $\boldsymbol{\theta}_{I_1}$ and $\boldsymbol{\theta}_O$ are the column matrices of the orientation coordinates of the *first* inboard end I_1 and the outboard end O; $\mathbf{r}_O$ and $\mathbf{l}_{I_1O}$ are the column matrices of the position coordinates of point O with respect to the global inertial coordinate system $oxyz$ and the body-fixed coordinate system $o_2x_2y_2z_2$ respectively; $\mathbf{A}$ is the direction cosine matrix that defines the orientation of the body-fixed coordinate system $o_2x_2y_2z_2$ in the global inertial coordinate system $oxyz$.

Similarly, the geometrical equations between the *first* inboard end I_1 and the other inboard ends I_k $(k = 1,2,\ldots,L)$ can be obtained easily

$$\begin{aligned} \boldsymbol{\theta}_{I_k} &= \boldsymbol{\theta}_{I_1} \quad (k = 2,3,\ldots,L) \\ \mathbf{r}_{I_k} &= \mathbf{r}_{I_1} + \mathbf{A}\mathbf{l}_{I_1I_k} \quad (k = 2,3,\ldots,L). \end{aligned} \tag{17}$$

Figure 4: Topology figure of a closed-loop system after "cutting" the hinge n.

Figure 5: A spatial motion rigid body with more than two ends.

Using Newton's laws of motion and considering the sign conventions, the dynamics equations of the rigid body can be obtained in the global inertial reference frame as [30]

$$\begin{aligned} m\ddot{\mathbf{r}}_C &= \sum_{k=1}^{L}\mathbf{q}_{I_k} - \mathbf{q}_O + \mathbf{f}_C, \\ \dot{\mathbf{G}}_{I_1} &= -\sum_{k=1}^{L}\mathbf{m}_{I_k} + \sum_{k=2}^{L}\tilde{\mathbf{r}}_{I_1I_k}\mathbf{q}_{I_k} + \mathbf{m}_O - \tilde{\mathbf{r}}_{I_1O}\mathbf{q}_O \\ &\quad - \mathbf{m}\tilde{\mathbf{r}}_{I_1C}\ddot{\mathbf{r}}_{I_1} + \mathbf{m}_C + \tilde{\mathbf{r}}_{I_1C}\mathbf{f}_C, \end{aligned} \tag{18}$$

where m and $\ddot{\mathbf{r}}_C$ are the mass and the column matrix of mass center acceleration of the rigid body respectively; $\mathbf{q}_{I_k}$ ($k = 1, 2, \ldots, L$) and $\mathbf{q}_O$ are the column matrices of internal forces acting on the point I_k ($k = 1, 2, \ldots, L$) and O, respectively; $\mathbf{f}_C$ and $\mathbf{m}_C$ are the column matrices of external force and the external torque acting on the mass center of the rigid body; $\mathbf{G}_{I_1}$ is the column matrix of relative moment of momentum with respect to the *first* input end I_1 of the body element; $\mathbf{m}_{I_k}$ ($k = 1, 2, \ldots, L$) and $\mathbf{m}_O$ are the column matrices of internal torques acting on points I_k ($k = 1, 2, \ldots, L$) and O, respectively; $\mathbf{r}_{I_1P}$ represents the column matrix of position vectors from I_1 to point P with respect to the inertial coordinate system, where P represents I_k ($k = 2, 3, \ldots, L$), C, or O.

By equations (16), (18), and rewriting them in the form of equation (13), transfer matrices can be obtained as

$$\mathbf{U}_j = \begin{bmatrix} \mathbf{I}_3 & \mathbf{E}_1 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{E}_2 \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times1} \\ \mathbf{E}_6\mathbf{E}_3 + \mathbf{E}_7 & \mathbf{E}_6\mathbf{E}_4 + \mathbf{E}_8 & \mathbf{I}_3 & \mathbf{E}_6 & \mathbf{E}_6\mathbf{E}_5 + \mathbf{E}_9 \\ \mathbf{E}_3 & \mathbf{E}_4 & \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{E}_5 \\ \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & 1 \end{bmatrix}, \tag{19}$$

$$\mathbf{U}_{j,I_k} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times1} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times1} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_3 & \tilde{\mathbf{r}}_{I_1I_k} & \mathbf{O}_{3\times1} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times1} \\ \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & 1 \end{bmatrix} \tag{20}$$

$$(k = 2, 3, 4, \ldots, L),$$

$$\begin{gathered} \mathbf{E}_1 = -\mathbf{A}\tilde{\mathbf{l}}_{IO}\mathbf{H}, \qquad \mathbf{E}_2 = \mathbf{A}\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{l}_{IO} - \mathbf{A}\tilde{\mathbf{l}}_{IO}\dot{\mathbf{H}}\dot{\boldsymbol{\theta}}_I, \\ \mathbf{E}_3 = -m\mathbf{I}_3, \qquad \mathbf{E}_4 = m\mathbf{A}\tilde{\mathbf{l}}_{IC}\mathbf{H}, \\ \mathbf{E}_5 = \mathbf{f}_C - m\mathbf{A}\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{l}_{IC} + m\mathbf{A}\tilde{\mathbf{l}}_{IC}\dot{\mathbf{H}}\dot{\boldsymbol{\theta}}_I, \\ \mathbf{E}_6 = \tilde{\mathbf{r}}_{IO}, \qquad \mathbf{E}_7 = m\tilde{\mathbf{r}}_{IC}, \qquad \mathbf{E}_8 = \mathbf{A}\mathbf{J}_I\mathbf{H}, \\ \mathbf{E}_9 = -\mathbf{m}_C - \tilde{\mathbf{r}}_{IC}\mathbf{f}_C + \mathbf{A}\tilde{\boldsymbol{\omega}}\mathbf{J}\boldsymbol{\omega} + \mathbf{A}\mathbf{J}\dot{\mathbf{H}}\dot{\boldsymbol{\theta}}_I, \\ \tilde{\boldsymbol{\omega}} = \mathbf{A}^T\dot{\mathbf{A}}, \qquad \boldsymbol{\omega} = \mathbf{H}\dot{\boldsymbol{\theta}}, \end{gathered} \tag{21}$$

$$\mathbf{A} = \begin{bmatrix} c_yc_z & s_xs_yc_z - c_xs_z & c_xs_yc_z + s_xs_z \\ c_ys_z & s_xs_ys_z + c_xc_z & c_xs_ys_z - s_xc_z \\ -s_y & s_xc_y & c_xc_y \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -s_y \\ 0 & c_x & s_xc_y \\ 0 & -s_x & c_xc_y \end{bmatrix},$$

where $\mathbf{l}_{AB}$ is the column matrix of the position vector from point A to point B in the body-fixed coordinate whose original point is the *first* input end I_1; $\tilde{\mathbf{l}}_{AB}$ is the cross product matrix of $\mathbf{l}_{AB}$. The meanings of other signs are the same with those in [19, 30].

By geometrical equations (17) and rewriting them in the form of equation (14), the concrete form of matrices $\mathbf{H}_j$ and $\mathbf{H}_{j,I_k}$ can be acquired as

$$\mathbf{H}_j = [\mathbf{I}_6 \;\; \mathbf{O}_{6\times7}]$$

$$\mathbf{H}_{j,I_k} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{A}\tilde{\mathbf{l}}_{I_1I_k}\mathbf{H} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{A}\tilde{\mathbf{l}}_{I_1I_k}\dot{\mathbf{H}}\dot{\boldsymbol{\theta}} - \mathbf{A}\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{l}_{I_1I_k} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times1} \end{bmatrix}$$

$$(k = 2, 3, 4, \ldots, L). \tag{22}$$

It can be clearly seen from equation (19) that the matrix is exactly the same with the transfer matrix of rigid body with single input end and single output end, which in fact can be

regarded as a special case of rigid body with more than two ends (multiple input ends and single output end).

Based on the transfer equations and geometrical equations of elements, it is then easy to get the overall transfer equation of the system automatically.

According to the topology figure of the system shown in Figure 1, the relations among the state vectors and transfer equations of elements can been described more intuitively and directly using topology described by state vectors and transfer equations as shown in Figure 6.

Then the main transfer equations of the system in Figure 6 can be easily deduced, that is,

$$\begin{aligned}\mathbf{z}_{1,0} &= \mathbf{U}_1\mathbf{z}_{2,1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{z}_{4,3} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{z}_{4,5} \\ &\quad + \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{z}_{4,6} \\ &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{z}_{8,9} + \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{z}_{10,6} \\ &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9\mathbf{z}_{9,11} \\ &\quad + \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_{9,12}\mathbf{z}_{9,12} \\ &\quad + \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{U}_{10}\mathbf{z}_{10,13} \\ &\quad + \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{U}_{10,14}\mathbf{z}_{10,14} \\ &= \mathbf{T}_{15-1}\mathbf{z}_{15,0} + \mathbf{T}_{16-1}\mathbf{z}_{16,0} + \mathbf{T}_{17-1}\mathbf{z}_{17,0} + \mathbf{T}_{18-1}\mathbf{z}_{18,0},\end{aligned} \tag{23}$$

where

$$\begin{aligned}\mathbf{T}_{15-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9\mathbf{U}_{11}\mathbf{U}_{15}, \\ \mathbf{T}_{16-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16}, \\ \mathbf{T}_{17-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{U}_{10}\mathbf{U}_{13}\mathbf{U}_{17}, \\ \mathbf{T}_{18-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{U}_{10,14}\mathbf{U}_{14}\mathbf{U}_{18}.\end{aligned} \tag{24}$$

Further, the geometrical equation of the body element 9 is

$$\mathbf{H}_9\mathbf{z}_{9,11} = \mathbf{H}_{9,12}\mathbf{z}_{9,12}. \tag{25}$$

Applying system topology described by state vectors and transfer equations in Figure 6, the state vectors $\mathbf{z}_{9,11}$ and $\mathbf{z}_{9,12}$ can be expressed using state vectors of system boundaries as

$$\begin{aligned}\mathbf{z}_{9,11} &= \mathbf{U}_{11}\mathbf{U}_{15}\mathbf{z}_{15,0}, \\ \mathbf{z}_{9,12} &= \mathbf{U}_{12}\mathbf{U}_{16}\mathbf{z}_{16,0}.\end{aligned} \tag{26}$$

Substituting equation (26) into equation (25), one obtains

$$-\mathbf{H}_9\mathbf{U}_{11}\mathbf{U}_{15}\mathbf{z}_{15,0} + \mathbf{H}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16}\mathbf{z}_{16,0} = \mathbf{0} \tag{27}$$

which can be written as

$$\mathbf{G}_{15-9}\mathbf{z}_{15,0} + \mathbf{G}_{16-9}\mathbf{z}_{16,0} = \mathbf{0}, \tag{28}$$

where

$$\begin{aligned}\mathbf{G}_{15-9} &= -\mathbf{H}_9\mathbf{U}_{11}\mathbf{U}_{15}, \\ \mathbf{G}_{16-9} &= \mathbf{H}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16}.\end{aligned} \tag{29}$$

Similarly, the geometrical equations corresponding to body elements 10 and 4 can be deduced as

$$\begin{aligned}&\mathbf{G}_{17-10}\mathbf{z}_{17,0} + \mathbf{G}_{18-10}\mathbf{z}_{18,0} = \mathbf{0}, \\ &\mathbf{G}_{15-4}\mathbf{z}_{15,0} + \mathbf{G}_{16-4}\mathbf{z}_{16,0} + \mathbf{G}_{17-4}\mathbf{z}_{17,0} \\ &\quad + \mathbf{G}_{18-4}\mathbf{z}_{18,0} = \mathbf{0},\end{aligned} \tag{30}$$

where

$$\begin{aligned}\mathbf{G}_{17-10} &= -\mathbf{H}_{10}\mathbf{U}_{13}\mathbf{U}_{17}, \\ \mathbf{G}_{18-10} &= \mathbf{H}_{10,14}\mathbf{U}_{14}\mathbf{U}_{18}, \\ \mathbf{G}_{15-4} &= -\mathbf{H}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9\mathbf{U}_{11}\mathbf{U}_{15}, \\ \mathbf{G}_{16-4} &= -\mathbf{H}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16}, \\ \mathbf{G}_{17-4} &= \mathbf{H}_{4,6}\mathbf{U}_6\mathbf{U}_{10}\mathbf{U}_{13}\mathbf{U}_{17}, \\ \mathbf{G}_{18-4} &= \mathbf{H}_{4,6}\mathbf{U}_6\mathbf{U}_{10,14}\mathbf{U}_{14}\mathbf{U}_{18}.\end{aligned} \tag{31}$$

The overall transfer equation of the system can be obtained by combining the main transfer equation (23) and all the geometrical equations (28) and (30) of the system:

$$\mathbf{U}_{\text{all}}\mathbf{z}_{\text{all}} = \mathbf{0}, \tag{32}$$

where

$$\begin{aligned}\mathbf{U}_{\text{all}} &= \begin{bmatrix} -\mathbf{I} & \mathbf{T}_{15-1} & \mathbf{T}_{16-1} & \mathbf{T}_{17-1} & \mathbf{T}_{18-1} \\ \mathbf{O} & \mathbf{G}_{15-9} & \mathbf{G}_{16-9} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{G}_{17-10} & \mathbf{G}_{18-10} \\ \mathbf{O} & \mathbf{G}_{15-4} & \mathbf{G}_{16-4} & \mathbf{G}_{17-4} & \mathbf{G}_{18-4} \end{bmatrix}, \\ \mathbf{z}_{\text{all}} &= \begin{bmatrix} \mathbf{z}_{1,0} \\ \mathbf{z}_{15,0} \\ \mathbf{z}_{16,0} \\ \mathbf{z}_{17,0} \\ \mathbf{z}_{18,0} \end{bmatrix}.\end{aligned} \tag{33}$$

According to the sign conventions and equations (32) and (33), it can be seen clearly that the overall transfer equation of the tree system, such as Figure 1, can be deduced automatically by handwriting and by computer. For more details, see Section 5 please.

As a short conclusion, in the overall transfer equation, $\mathbf{z}_{\text{all}}$ consists of all the state vectors at the boundary ends of system. Besides the main transfer equation, there exists a geometrical equation at multiple input body elements that finally leads to a branch in a tree system. Moreover, the number of geometrical equations in the overall transfer equation of a system is equal to the number of tips of a tree minus one. For the system shown in Figure 1, the number of the tips is 4; thus the number of geometric equations of the system is 3.

It can be seen clearly that overall transfer equation (8) of a chain system is a special example of overall transfer equations (32) and (33) at the case of the tree system with only two boundary ends.

4.4. Automatic Deduction of the Overall Transfer Equations of a General System. As is shown in Figure 7, a general multibody system can be considered as a system which consists

Figure 6: System topology described by state vectors and transfer equations.

of one tree subsystem and some closed-loop subsystems. After "cutting" at the junction of any two adjacent elements body 16 and hinge 19 in a closed-loop subsystem, a couple of "new boundaries" noted as $\mathbf{z}_{16,0}$ and $\mathbf{z}_{19,0}$, which are the same in nature, will emerge at the "cutting point." Then, as shown in Figure 8, the original nontree system shown in Figure 7 will become a tree system with the same two "new boundaries" state vectors $\mathbf{z}_{16,0}$ and $\mathbf{z}_{19,0}$ at the "cutting points." And the overall transfer equation of the general system with closed-loop subsystems can be deduced automatically by handwriting and by computer according to the method proposed in Section 4.3.

It should be pointed out that either $\mathbf{z}_{16,0}$ or $\mathbf{z}_{19,0}$ is the input end state vector of elements 16 or 19 respectively. Thus, the internal force and internal moment in $\mathbf{z}_{16,0}$ and $\mathbf{z}_{19,0}$ are of the same quantities but with opposite direction due to the sign conventions, that is,

$$\mathbf{z}_{19,0} = \mathbf{C}\mathbf{z}_{16,0}, \tag{34}$$

where

$$\mathbf{C} = \begin{bmatrix} I_n & O_n & O_{n\times 1} \\ O_n & -I_n & O_{n\times 1} \\ O_{1\times n} & 0 & 1 \end{bmatrix} \tag{35}$$

n is equal to 3 for planar motion or 6 for spatial motion.

Considering the method proposed in Section 4.3 for a tree system, and regarding equation (34) concerned with the relationship between $\mathbf{z}_{16,0}$ and $\mathbf{z}_{19,0}$, the overall transfer equations of a general system with a closed-loop subsystem shown in Figure 8 can be obtained automatically by handwriting and by computer:

$$\mathbf{U}_{\text{all}}\mathbf{z}_{\text{all}} = \mathbf{0}, \tag{36}$$

where

$$\mathbf{U}_{\text{all}} = \begin{bmatrix} -\mathbf{I} & \mathbf{T}_{15-1} & \mathbf{T}_{16-1} & \mathbf{T}_{19-1} & \mathbf{T}_{18-1} \\ \mathbf{O} & \mathbf{G}_{15-9} & \mathbf{G}_{16-9} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{G}_{19-10} & \mathbf{G}_{18-10} \\ \mathbf{O} & \mathbf{G}_{15-4} & \mathbf{G}_{16-4} & \mathbf{G}_{19-4} & \mathbf{G}_{18-4} \\ \mathbf{O} & \mathbf{O} & \mathbf{C} & -\mathbf{I} & \mathbf{O} \end{bmatrix},$$
$$\mathbf{z}_{\text{all}} = \begin{bmatrix} \mathbf{z}_{1,0} \\ \mathbf{z}_{15,0} \\ \mathbf{z}_{16,0} \\ \mathbf{z}_{19,0} \\ \mathbf{z}_{18,0} \end{bmatrix}. \tag{37}$$

Considering that $\mathbf{z}_{19,0}$ can be explicitly expressed by $\mathbf{z}_{16,0}$ due to equation (34), the overall transfer equation, the overall transfer matrix, and the overall state vectors equation (37) of the general system can be rewritten in the following form

$$\mathbf{U}_{\text{all}}\mathbf{z}_{\text{all}} = \mathbf{0}$$
$$\mathbf{U}_{\text{all}} = \begin{bmatrix} -\mathbf{I} & \mathbf{T}_{15-1} & \mathbf{T}_{16-1} + \mathbf{T}_{19-1}\mathbf{C} & \mathbf{T}_{18-1} \\ \mathbf{O} & \mathbf{G}_{15-9} & \mathbf{G}_{16-9} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{G}_{19-10}\mathbf{C} & \mathbf{G}_{18-10} \\ \mathbf{O} & \mathbf{G}_{15-4} & \mathbf{G}_{16-4} + \mathbf{G}_{19-4}\mathbf{C} & \mathbf{G}_{18-4} \end{bmatrix},$$
$$\mathbf{z}_{\text{all}} = \begin{bmatrix} \mathbf{z}_{1,0} \\ \mathbf{z}_{15,0} \\ \mathbf{z}_{16,0} \\ \mathbf{z}_{18,0} \end{bmatrix}, \tag{38}$$

Figure 7: Topology figure of a nontree system.

Figure 8: Topology figure of a nontree system becoming a tree system after "cutting" the hinge 19.

where

$$\begin{aligned}
\mathbf{T}_{15-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9\mathbf{U}_{11}\mathbf{U}_{15},\\
\mathbf{T}_{16-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16},\\
\mathbf{T}_{19-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{U}_{10}\mathbf{U}_{13}\mathbf{U}_{17}\mathbf{U}_{19},\\
\mathbf{T}_{18-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_{4,6}\mathbf{U}_6\mathbf{U}_{10,14}\mathbf{U}_{14}\mathbf{U}_{18},\\
\mathbf{G}_{15-9} &= -\mathbf{H}_9\mathbf{U}_{11}\mathbf{U}_{15},\\
\mathbf{G}_{16-9} &= \mathbf{H}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16},\\
\mathbf{G}_{19-10} &= -\mathbf{H}_{10}\mathbf{U}_{13}\mathbf{U}_{17}\mathbf{U}_{19},\\
\mathbf{G}_{18-10} &= \mathbf{H}_{10,14}\mathbf{U}_{14}\mathbf{U}_{18},\\
\mathbf{G}_{15-4} &= -\mathbf{H}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9\mathbf{U}_{11}\mathbf{U}_{15},\\
\mathbf{G}_{16-4} &= -\mathbf{H}_4\mathbf{U}_5\mathbf{U}_7\mathbf{U}_8\mathbf{U}_{9,12}\mathbf{U}_{12}\mathbf{U}_{16},\\
\mathbf{G}_{19-4} &= \mathbf{H}_{4,6}\mathbf{U}_6\mathbf{U}_{10}\mathbf{U}_{13}\mathbf{U}_{17}\mathbf{U}_{19},\\
\mathbf{G}_{18-4} &= \mathbf{H}_{4,6}\mathbf{U}_6\mathbf{U}_{10,14}\mathbf{U}_{14}\mathbf{U}_{18}.
\end{aligned} \tag{39}$$

5. Automatic Deduction Theorem of Overall Transfer Equation

The following features of the overall transfer equation of a multibody system can be clearly concluded from equations (23), (24), (28), and (30). These features make up the theorem to deduce automatically the overall transfer equation as the following for tree system (1, 2), for chain system (1, 3), for closed-loop system (1, 3, 4), and for general system (1, 2, 5).

(1) The state vectors involved in an overall transfer equation are the column matrix comprising the state vectors of all boundary ends of the system.

(2) For a tree system, in the first line of the overall transfer matrix, the coefficient matrix of the state vector of root is a minus unit matrix, while each coefficient matrix of the state vector of a tip is the successive premultiplication of the transfer matrices of all elements in the transfer path from this tip to the root; besides the first line in the overall transfer matrix, all coefficient matrices of state vectors in the first column are zero matrices. Except the first line, in each row, each nonzero partitioned matrix corresponds to the coefficient matrix of the tip state vector, from which there is a transfer path to the input end of the element with multiple input ends. Each nonzero coefficient matrix of the state vector of a tip is the successive premultiplication of all transfer matrices of elements in the transfer path from this tip to the kth input end I_k of jth body element, which has more than two ends, then premultiplied by $-\mathbf{H}_j$ for $k = 1$ or premultiplied by $\mathbf{H}_{j,I_k}$ for $k = 2, 3, 4 \cdots L$.

(3) For a chain system, its overall transfer matrix is deduced automatically by successive premultiplication of the transfer matrices of all elements in the transfer path from the tip to the root of system. In fact, any chain system can be considered as a special example of the tree system in the case with only two boundary ends.

(4) For a closed-loop system, its overall transfer equation is deduced automatically as the chain system, after treating the original system as the chain system by "cutting" a junction of any two adjacent elements and letting the couple of "cutting points" as the tip and root with the same state vectors of the chain system.

(5) For a general system composed of one tree subsystem and some closed-loop subsystems, its overall transfer

Figure 9: Tree multi-rigid-flexible-body system.

Figure 10: Topology figure of the system.

equation is deduced automatically as the tree system, after treating the original system as the tree system by "cutting" the junctions of any two adjacent elements in every closed-loop subsystem and letting every couple of "cutting points" as new "boundary ends" with the same state vectors of the tree system.

The theorem above is effective for various multi-rigid-body systems, chain multi-rigid-flexible-body systems, and any closed-loop multi-rigid-flexible-body systems and is effective for various tree multi-rigid-flexible-body systems and general multi-rigid-flexible-body systems if the bodies with more than two ends are rigid bodies. For more general systems including the flexible bodies with more than two ends, the theorem to deduce automatically overall transfer equation is undergoing study and will be discussed in another paper.

6. Numerical Examples

By comparison with Newton-Euler method, the numerical examples here are carried out to validate the proposed method.

Example 1. A tree multi-rigid-flexible-body system moving in plane, as shown in Figure 9, consists of two fixed hinges, 2 and 4, two elastic hinges, 6 and 7, three rigid bodies, 1, 5, and 8, and one uniform beam element, 3, with three boundary ends. The simulation parameters are given as follows: $m_1 = m_5 = m_8 = 7.8\,\text{kg}$, $J_{C,1} = J_{C,5} = J_{C,8} = 0.013\,\text{kg}\cdot\text{m}^2$, $l_3 = 3\,\text{m}$, $EA_3 = 1000\,\text{N}$, $\overline{m}_3 = 0.78\,\text{kg/m}$, $EI_3 = 166.67\,\text{N}\cdot\text{m}^2$, $K_{x,6} = K_{x,7} = 1000\,\text{N/m}$, $K_{y,6} = K_{y,7} = 500\,\text{N/m}$, and $K_6' = K_7' = 150\,\text{N}\cdot\text{m/rad}$.

According to the theorem to deduce automatically the overall transfer equation, the topology figure of the system can be got as in Figure 10, and the overall transfer equation of above system is deduced automatically by handwriting and by computer as follows

$$\mathbf{U}_{\text{all}}\mathbf{z}_{\text{all}} = \mathbf{0}, \tag{40}$$

where the overall transfer matrix is

$$\mathbf{U}_{\text{all}} = \begin{bmatrix} -\mathbf{I} & \mathbf{T}_{6-1} & \mathbf{T}_{8-1} \\ \mathbf{O} & \mathbf{G}_{6-5} & \mathbf{G}_{8-5} \end{bmatrix}. \tag{41}$$

State vectors of all boundary ends are

$$\mathbf{z}_{\text{all}} = \begin{bmatrix} \mathbf{z}_{1,0} \\ \mathbf{z}_{6,0} \\ \mathbf{z}_{8,0} \end{bmatrix}. \tag{42}$$

According to the proposed sign conventions, we know that

$$\begin{aligned} \mathbf{T}_{6-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6, \\ \mathbf{T}_{8-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_{5,7}\mathbf{U}_7\mathbf{U}_8, \\ \mathbf{G}_{6-5} &= -\mathbf{H}_5\mathbf{U}_6, \\ \mathbf{G}_{8-5} &= \mathbf{H}_{5,7}\mathbf{U}_7\mathbf{U}_8, \end{aligned} \tag{43}$$

where $\mathbf{z}_{1,0}$ is state vector of root, $\mathbf{z}_{6,0}$ and $\mathbf{z}_{8,0}$ are the state vectors of tips, and $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$, $\mathbf{U}_4$, $\mathbf{U}_5$, $\mathbf{U}_{5,7}$, $\mathbf{U}_6$, $\mathbf{U}_7$, and $\mathbf{U}_8$ are the transfer matrices of body 1, fixed hinge 2, uniform beam 3, fixed hinge 4, body 5, elastic hinge 6, elastic hinge 7, and body 8. All of the transfer matrices can be found directly from the library of transfer matrices in [19, 30].

There are boundary conditions of the system

$$\begin{aligned} \mathbf{z}_{1,0} &= \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{\theta}_z & 0 & 0 & 0 & 1 \end{bmatrix}_{1,0}^T, \\ \mathbf{z}_{6,0} &= \begin{bmatrix} 0 & 0 & 0 & m_z & q_x & q_y & 1 \end{bmatrix}_{6,0}^T, \\ \mathbf{z}_{8,0} &= \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{\theta}_z & 0 & 0 & 0 & 1 \end{bmatrix}_{8,0}^T. \end{aligned} \tag{44}$$

The system experiences a step upward force at the mass center of body element 1 at time instant zero, while the initial displacement and velocity of the whole system are zero. The computational results of the system dynamics obtained by the proposal method and by Newton-Euler method are shown in Figure 11. It can be seen from Figure 11 that the computational results obtained by the above two methods have good agreements.

Example 2. A multi-rigid-body system containing a close loop moving in plane is shown in Figure 12. The planar rigid bodies (numbered 1, 3, 5, 7, 9, 11, and 13) are connected by pin hinges (numbered 2, 4, 6, 8, 10, 12, and 14), and rigid body 1 is connected with the ground by a smooth pin. Each rigid body has the identical dynamics parameter as $m = 1\,\text{kg}$ and $J_C = (1/6)\,\text{kg}\cdot\text{m}^2$.

Figure 11: Computational results of the angle of the right end of the beam.

By "cutting" at the junction of body 9 and hinge 14 and following the proposed theorem, topology figure of the system can be drawn in Figure 13. And the overall transfer equation of the system can be derived automatically as

$$\mathbf{U}_{\text{all}}\mathbf{z}_{\text{all}} = \mathbf{0}, \tag{45}$$

where the overall transfer matrix takes the form:

$$\mathbf{U}_{\text{all}} = \begin{bmatrix} -\mathbf{I} & \mathbf{T}_{9-1} & \mathbf{T}_{14-1} \\ \mathbf{O} & \mathbf{G}_{9-3} & \mathbf{G}_{14-3} \\ \mathbf{O} & \mathbf{C} & -\mathbf{I} \end{bmatrix}. \tag{52}$$

State vectors of all boundary ends are

$$\mathbf{z}_{\text{all}} = \begin{bmatrix} \mathbf{z}_{1,0} \\ \mathbf{z}_{9,0} \\ \mathbf{z}_{14,0} \end{bmatrix}. \tag{46}$$

According to the proposed sign conventions, one can acquire

$$\begin{aligned} \mathbf{T}_{9-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9, \\ \mathbf{T}_{14-1} &= \mathbf{U}_1\mathbf{U}_2\mathbf{U}_{3,10}\mathbf{U}_{10}\mathbf{U}_{11}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}, \\ \mathbf{G}_{9-3} &= -\mathbf{H}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9, \\ \mathbf{G}_{14-3} &= \mathbf{H}_{3,10}\mathbf{U}_{10}\mathbf{U}_{11}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}. \end{aligned} \tag{47}$$

$\mathbf{z}_{1,0}$ is the state vector of the root; $\mathbf{z}_{9,0}$ and $\mathbf{z}_{14,0}$ are the state vectors of tips emerging after the "cutting." $\mathbf{U}_1$, $\mathbf{U}_3$, $\mathbf{U}_{3,10}$, $\mathbf{U}_5$, $\mathbf{U}_7$, $\mathbf{U}_9$, $\mathbf{U}_{11}$, and $\mathbf{U}_{13}$ are the transfer matrices of the corresponding planar rigid bodies. $\mathbf{U}_2$, $\mathbf{U}_4$, $\mathbf{U}_6$, $\mathbf{U}_8$, $\mathbf{U}_{10}$, $\mathbf{U}_{12}$, and $\mathbf{U}_{14}$ are transfer matrices of the corresponding planar pin hinges.

There are boundary conditions of the system:

$$\mathbf{z}_{1,0} = \begin{bmatrix} 0 & 0 & \ddot{\theta}_z & 0 & q_x & q_y & 1 \end{bmatrix}_{1,0}^T. \tag{48}$$

Figure 12: A general system moving in plane.

Figure 13: Topology figure of the system.

The initial angle of rigid body 1 is $(\pi/6)$ rad and the relative angles of pin hinges (numbered 2, 4, 6, 8, 10, 12, and 14) are all zero. The system moves from the rest under the effect of gravity. The computational results of the system dynamics are obtained by the proposal method and by Newton-Euler method. The time history of rigid body 1's angle is exhibited in Figure 14, which shows that the computational results obtained by the above two methods have good agreements.

7. Conclusions

Based on MSTMM and the features of the overall transfer equations of multibody systems with various topological structures, including chain systems, closed-loop systems, tree systems, general systems composed of one tree subsystem, and some closed-loop subsystems, the theorem to deduce automatically the overall transfer equation by handwriting and by computer is presented. Formulations of the proposed theorem as well as two numerical examples are given to verify the theorem. This makes it possible to program large-scale

Figure 14: Computational results of the angle of rigid body 1.

software for MSD with much higher computational speed and much higher programming.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research was supported by Research Fund for the Doctoral Program of Higher Education of China (no. 20113219110025). The authors owe special thanks to Professor Dieter Bestle for his important discussion with them during his five months visit in Nanjing University of Science and Technology.

References

[1] W. Schiehlen, *Multibody Systems Handbook*, Springer, Berlin, Germany, 1990.

[2] A. A. Shabana, *Dynamics of Multibody Systems*, Cambridge University Press, New York, NY, USA, 2009.

[3] D. Bestle and P. Eberhard, "Analyzing and optimizing multibody systems," *Mechanics of Structures and Machines*, no. 20, pp. 67–92, 1992.

[4] K. Thoma, F. Schäfer, S. Hiermaier, and E. Schneider, "An approach to achieve progress in spacecraft shielding," *Advances in Space Research*, vol. 34, no. 5, pp. 1063–1075, 2004.

[5] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart, Germany, 1977.

[6] O. A. Bauchau, *Flexible Multibody Dynamics*, Springer, Dodrecht, The Netherlands, 2011.

[7] J. A. C. Ambrósio, M. A. Neto, and R. P. Leal, "Optimization of a complex flexible multibody systems with composite materials," *Multibody System Dynamics*, vol. 18, no. 2, pp. 117–144, 2007.

[8] W. J. Book, "Recursive Lagrangian dynamics of flexible manipulator arms," *The International Journal of Robotics Research*, vol. 3, no. 3, pp. 87–101, 1984.

[9] R. L. Huston, "Multi-body dynamics including the effects of flexibility and compliance," *Computers and Structures*, vol. 14, no. 5-6, pp. 443–451, 1981.

[10] Y. A. Khulief and A. A. Shabana, "Dynamic analysis of constrained system of rigid and flexible bodies with intermittent motion," *ASME Journal of Mechanisms, Transmissions and Automation in Design*, vol. 108, no. 1, pp. 38–45, 1986.

[11] R. A. Laskin, P. W. Likins, and R. W. Longman, "Dynamical equations of a free-free beam subject to large overall motions," *Journal of the Astronautical Sciences*, vol. 31, no. 4, pp. 507–527, 1983.

[12] M. Pascal, "Dynamical analysis of a flexible manipulator arm," *Acta Astronautica*, vol. 21, no. 3, pp. 161–169, 1990.

[13] J. Rauh and W. Schiehlen, "A unified approach for the modeling of flexible robot arms," in *Proceedings of the 6th CISM-IFToMM Symposium on Theory and Practice of Robots and Manipulators*, Cracow, Poland, September 1986.

[14] R. E. Roberson, "A form of the translational dynamical equations for relative motion in systems of many non-rigid bodies," *Acta Mechanica*, vol. 14, no. 4, pp. 297–308, 1972.

[15] O. A. Bauchau, "Computational schemes for flexible, nonlinear multi-body systems," *Multibody System Dynamics*, vol. 2, no. 2, pp. 169–225, 1998.

[16] S. C. Chu and K. C. Pan, "Dynamic response of a high speed slider crank mechanism with an elastic connecting rod," *ASME Journal of Engineering for Industry*, vol. 97, no. 2, pp. 542–550, 1975.

[17] E. Kreuzer and O. Kust, "Analysis of long torsional strings by proper orthogonal decomposition," *Archive of Applied Mechanics*, vol. 67, no. 1-2, pp. 68–80, 1996.

[18] T. R. Kane, P. W. Likins, and D. A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, New York, NY, USA, 1983.

[19] X. Rui, L. Yun, Y. Lu et al., *Transfer Matrix Method for Multibody System and Its Application*, Science Press, Beijing, China, 2008.

[20] X. Rui, K. Huang, and Y. Qin, "Transfer matrix method of multibody system and its application in spacecraft dynamics," in *Proceeding of the 3rd China-Russia-Ukraine Symposium on Astronautical Science and Technology*, pp. 933–935, Northwestern Polytechnical University Press, Xi'an, China, 1994.

[21] X. Rui, B. He, Y. Lu, W. Lu, and G. Wang, "Discrete time transfer matrix method for multibody system dynamics," *Multibody System Dynamics*, vol. 14, no. 3-4, pp. 317–344, 2005.

[22] X. Rui, G. Wang, Y. Lu, and L. Yun, "Transfer matrix method for linear multibody system," *Multibody System Dynamics*, vol. 19, no. 3, pp. 179–207, 2008.

[23] X. Rui, B. He, B. Rong, G. Wang, and Y. Lu, "Discrete time transfer matrix method for dynamics of a multi-rigid- flexible-body system moving in plane," *Proceedings of the Institution of Mechanical Engineers K*, vol. 223, no. 1, pp. 23–42, 2009.

[24] X. Rui and J. Zhang, "Automatical transfer matrix method of multibody system," in *Proceedings of the 2nd Joint International Conference on Multibody System Dynamics*, Stuttgart, Germany, May-June 2012.

[25] X. Rui and B. Rong, "Advance in transfer matrix method for multibody system dynamics," *Advance in Mechanics*, vol. 42, no. 1, pp. 4–17, 2012.

[26] X. Rui, B. Rong, G. Wang, and B. He, "Discrete time transfer matrix method for dynamics analysis of complex weapon

systems," *Science China Technological Sciences*, vol. 54, no. 5, pp. 1061–1071, 2011.

[27] X. Rui, "Launch dynamics of multibody system and its application," *Engineering Science*, vol. 13, no. 10, pp. 76–81, 2011.

[28] G. Wang, X. Rui, F. Yang, and H. Xu, "Dynamics analysis for wheeled and tracked multiple launch rocket system," *Acta Armamentarii*, vol. 33, no. 11, pp. 1286–1290, 2012.

[29] B. Rong, X. Rui, G. Wang, and F. Yang, "Discrete time transfer matrix method for dynamics of multibody system with real-time control," *Journal of Sound and Vibration*, vol. 329, no. 6, pp. 627–643, 2010.

[30] X. Rui, D. Bestle, and J. Zhang, "A new form of the transfer matrix method for multibody systems," in *Proceedings of the ECCOMAS Thematic Conference on Multibody Dynamics*, Zagreb, Croatia, 2013.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 545126, 7 pages
http://dx.doi.org/10.1155/2014/545126

Research Article

Optimal Design of One-Folded Leaf Spring with High Fatigue Life Applied to Horizontally Vibrating Linear Actuator in Smart Phone

Ki Bum Lee,[1] Chang Hyun Park,[2] and Jin Ho Kim[1]

[1] *Department of Mechanical Engineering, Yeungnam University, Dae-dong, Gyeongsan-si, Gyeongsangbuk-do 712-160, Republic of Korea*
[2] *PIDOTECH Inc., Republic of Korea*

Correspondence should be addressed to Jin Ho Kim; jinho@ynu.ac.kr

Received 4 July 2013; Revised 12 November 2013; Accepted 25 November 2013; Published 24 February 2014

Academic Editor: Xiaoting Rui

Copyright © 2014 Ki Bum Lee et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Horizontally vibrating linear actuator (HVLA) instead of VVLA has been under study in a few past years and recently HVLA with thickness of 2.5 mm was developed. The one-folded leaf spring to guide the moving part is newly designed and applied in HVLA, but unfortunately it makes HVLA be wider. Accordingly, this paper presents the optimal design of one-folded leaf spring, which results in reduction of HLVA width. The commercial design optimization tool "PIAnO" was utilized based on design of experiments (DOE), approximation techniques, and optimization algorithm. In addition, for the vibration modal analysis and harmonic response analysis to generate metamodeling, the software "ANSYS" is utilized. The optimal width of leaf spring was reduced by 46% compared to the initial one, while all the design constraints were satisfied, which clearly showed the validity of the proposed design approach.

1. Introduction

The slimming of a smart phone has become a competitive issue in the smart phone industry because the display panel is enlarged to show more information to the user. A smart phone is comprised of numerous components: the battery, the camera module, the vibrating actuator, and so forth. One of the thickest components is vibrating actuator which imposes the restrictions on the slimming of smart phones. In currently manufactured smart phone in the market, a vertical vibrating linear actuator (VVLA) is being widely used as a vibration motor, as shown in Figure 1.

However, it imposes the thickness problem on a smart phone, because it needs vibration space in the vertical direction [1]. Therefore, a horizontally vibrating linear actuator (HVLA) has been developed in recent years, but it has not been commercialized in the smart phone industry so far [2]. The primary reason is that the fatigue fracture of guide spring due to the cyclic compression and tension loads has not been resolved. The guide spring has the role to support the moving part of the actuator and enable the actuator to vibrate elastically. Accordingly, the various designs of guide spring had been presented to achieve the high fatigue life [3, 4] and among them the one-folded leaf spring introduced by Lee and Kim has the highest fatigue life [5]. Figures 2(a) and 2(b) show the one-folded spring and the schematic diagrams of the HVLA which has the one-folded spring, respectively. Table 1 shows the performance characteristics of the HVLA.

This one-folded leaf spring enables the HVLA to improve the vibration force and product life. Instead, the HVLA has been enlarged in width due to the wide structure of one-folded spring. Figure 3 shows the top view of the HVLA with one-folded leaf spring.

This one-folded leaf spring has occupied 45% of total width of HVLA. Therefore, the design of one-folded leaf spring needs be optimized to reduce its width. Several researches about the optimal design of the mechanical springs such as helical, coil, and leaf spring had been reported in past decades. These springs are usually designed to avoid resonance, but one-folded leaf spring of HVLA needs to

Figure 1: Schematic diagram of VVLA.

Table 1: Specification of the HVLA.

Requirements	Values
Natural frequency	183 Hz
Thickness of actuator	2.5 mm
Vibration acceleration	2.1 to 2.3 G
Fatigue life	Over 1,000,000 cycles

accomplish the best performance. Moreover, there had been no studies reported about the optimal design of one-folded leaf spring. Therefore, this paper presents an optimization of the one-folded leaf spring. For optimal design process, we used the commercial optimization software PIAnO (Process Integration, Automation, and Optimization) which provides the users with various tools for the efficient optimization design [6]. (Ver.3.5, PIDOTECH) The optimal design objective is to minimize the width of one-folded spring to satisfy the stiffness and maximum stress at cyclic loading condition. Five design variables were selected to derive the optimal design of one-folded leaf spring and an orthogonal array used. Modal and harmonic response analyses were performed according to the design of experiments to obtain the natural frequency of the first vibration mode due to stiffness and the maximum stress at resonance using the commercial structural analysis program ANSYS. (Version 14.5, ANSYS) And then, analyses results were saved as a text file and reimported into the PIAnO for generating metamodel. Metamodels were generated based on the results of structural analysis using a Kriging model with PIAnO. Finally, we derived a global optimal point using evolution algorithms.

2. Design Problem

2.1. Design Requirements

2.1.1. Minimization of the Width of Leaf Spring. The width of one-folded spring should be minimized in order to reduce the size of the vibration actuator. This procedure is represented by the following equation:

$$\text{Minimize width of leaf spring.} \tag{1}$$

2.1.2. Design Constraint on the Maximum Stress. A one-folded spring that is affected by cyclic compression and tension loads should have a yield stress of approximately 215 MPa in order to maintain its fatigue life [7]. The one-folded spring is represented as follows:

$$\sigma_{\max} < \sigma_{\text{yield}}. \tag{2}$$

2.1.3. Design Constraint on the Mechanical Natural Frequency. The mechanical natural frequency of HVLA ranges from 175 Hz to 185.5 Hz:

$$175 \le f \le 185.5. \tag{3}$$

2.2. Design Variables. We selected the width, length, thickness, height, and radius as design variables for the leaf spring.

Figure 4 shows the shape of the leaf spring, and Table 2 shows the initial, lower, and upper bound values of the design variables.

2.3. Design Problem Formulation. The design problem for determining the design variables that satisfy all the design requirements can be mathematically formulated as

$$\begin{aligned} &\text{Find} && x_1, x_2, x_3, x_4, x_5, \\ &\text{to minimize} && \text{Width of spring} \\ &\text{subject to} && \sigma_{\max} \le \sigma_{\text{yield}} \\ & && 175 \le f \le 185.5 \\ & && 1.79 \le W \le 3.6 \\ & && 6 \le L \le 15.8 \\ & && 0.15 \le T \le 0.35 \\ & && 0.5 \le H \le 1.5 \\ & && 0.5 \le R \le 0.75. \end{aligned} \tag{4}$$

3. Analysis Procedures and Optimal Design

3.1. Analysis Procedures. Figure 5 shows our analysis procedures. First, we generated a design of experiments (DOE) and then the modal and harmonic analyses were response performed according to sampling points. After, metamodels were generated using the Kriging algorithm provided by the PIAnO software. We used an optimization technique using an evolution algorithm (EA) to find the optimum solution.

3.2. Design of Experiments. After determining the experimental design using an orthogonal array, which is one of the tools provided by $L_{98}(7^{15})$, the leaf spring was designed

Figure 2: (a) The one-folded spring of HVLA and (b) schematic diagram of HVLA.

Table 2: Initial, lower, and upper bound values of the selected design variables.

Design variables		Lower bound	Initial	Upper bound
x_1	W (mm)	1.79	3.6	3.6
x_2	L (mm)	6	11.6	15.8
x_3	T (mm)	0.15	0.25	0.35
x_4	H (mm)	0.5	1	1.5
x_5	R (mm)	0.5	0.64	0.75

Figure 3: Top view of HVLA with the one-folded leaf spring.

Figure 4: Design variables of the one-folded leaf spring.

according to each design variable. $L_{98}(7^{15})$ was selected by considering the number of saturated points and levels according to the number of design variables. The number of saturated points is the minimum required number of sampling points to generate a full quadratic polynomial model. An orthogonal array with the main effect and interaction of each factor was represented by a table, which allows the design of experiments to be easily established [8]. An orthogonal array, regardless of the lack of theoretical background,

Figure 5: Diagram of analysis procedures.

Figure 6: 3D model of the meshed FE model and constraints.

Figure 7: Fatigue fracture of the one-folded spring at folded area.

has the advantage of being able to apply easily fractional replication, decomposition method, and so on. Also, the experiment can use many factors without enlarging the size of the experiment, and the interpretation of the experiment data is simple.

3.3. Modal Analysis and Harmonic Response Analysis. 98 leaf spring FE models are created according to sampling points in the orthogonal array. And then the modal analysis was performed to obtain the vibration natural frequency and the harmonic response analysis was performed to calculate the maximum mechanical stress at the resonance using ANSYS.

(1) Modal Analysis. Figure 6 shows a 3-dimensional finite element (FE) model of moving part with the one-folded spring and constraint. This moving part consists of permanent magnets, yoke, weight, housing, and one-folded spring. Table 3 lists the material properties used in FE model. The modal analysis was conducted to calculate the natural vibration frequency according to DOE [9].

(2) Harmonic Response Analysis. Figure 7 shows the fatigue fracture of leaf spring at folded area due to the cyclic loading. Cyclic loading is created by the magnetic force which is energized by sinusoidal electric current. It is expressed by [10, 11]

$$F_{\text{magnetic}} = NB_g i L_{\text{eff}}, \qquad i = i_0 \sin(2\pi f t), \tag{5}$$

where N is the number of coil windings, B_g is the magnetic flux density in the airgap, i is the input electric current, L_{eff} is the effective coil length, and f is the input frequency of the electric current.

Accordingly, the harmonic response analysis was performed to calculate the maximum stress near the folded area at resonance frequency. The maximum stress must be less than the yield stress. The harmonic responses were performed with 30 intervals between ±10 natural frequencies [12]. Figure 8 shows the result of harmonic response analysis.

3.4. Metamodeling. Metamodeling builds a metamodel that approximates the relationship between performance indices and design variables of a real model by using the analysis results at the sampling points specified by a DOE. Figure 9 shows the result of a parameter study of the nonlinearity of $x_1(W)$. The simulation was performed such that $x_1(W)$ for the upper and lower bounds was divided into five sections. Then, we modeled the one-folded spring according to a divided $x_1(W)$ after a fixed initial value of $x_2(L)$, $x_3(T)$, $x_4(H)$, and $x_5(R)$.

We found that $x_3(T)$ was a sensitive value. In a previous optimization of the leaf spring, we found a sensitive value of $x_3(T)$ because the first frequency and stress level were very responsive, depending on the thickness of the spring. For example, the stiffness of cantilever beams is given by

$$k = \frac{3EI}{L^3} = \frac{EHT^3}{4L^3}, \qquad \left(I = \frac{HT^3}{12}\right). \tag{6}$$

Table 3: Material properties of the moving part.

Component	Material	Density (g/cc)	Modulus of elasticity (Gpa)	Poisson's ratio
Leaf spring	SUS304	8.00	193–200	0.29
Weight	Tungsten	19.3	400	0.28
Housing	SUS304	8.00	193–200	0.29
Magnet	NdFe35	7.50–7.80	150–160	0.24
Steel yoke	Steel 1010	7.84–7.87	200–205	0.29

Figure 8: Harmonic response of HVLA.

Figure 9: Nonlinearity for $x_1(W)$.

The stiffness of a cantilever beam is directly proportional to the cube of the thickness. Therefore, the thickness of the leaf spring has a nonlinearity, so we selected the Kriging model, which is one of the metamodels provided in the PIAnO software that was selected. The Kriging model, which a type of interpolation model, was mathematically established by Metheron in 1963 based on research conducted by mining researcher Krige in 1951 [13, 14]. And it shows superb predictive performance under many design variables and is in strongly nonlinear systems, provides a statistical estimation [15]. Also, there are no parameters that depend on the experience and intuition of customers when choosing the design parameters because the Kriging model can optimize design parameters through maximum likelihood estimation (MLE). Therefore, a recent trend is an increase in the use of Kriging models in the field of engineering. Our first Kriging model was generated by using an orthogonal array $L_{36}(3^{13})$, which was based on the analytical results of the width of the one-folded spring, the maximum stress, and the natural frequency. However, an accuracy evaluation of the Kriging model using R^2_{pred}, which corresponds to the maximum stress, was underestimated 55% as accuracy. Consequently, we added 39 types of sampling points to improve the accuracy of the metamodel and then performed the modal and harmonic response analyses. Metamodels were regenerated with respect to the width of the one-folded spring, maximum stress, and natural frequency of the first mode. At this point, we added only 39 types of sampling points to satisfy the 15 multiples of the number of design variables corresponding to the sampling points (75). We used augmented Latin hypercube design (ALHD) that is not overlapped with the existing sampling points [16]. Also, it is widely known for excellent performance of the space filling. Nevertheless, the accuracy of R^2_{pred} was only about 80%. Therefore, 98 sampling points were reextracted in order to obtain higher accuracy over 90% using orthogonal array $L_{98}(7^{15})$ at a time. Finally, in the evaluation results of the metamodel, the accuracy of R^2_{pred} is over 90%.

3.5. Optimization Technique. We selected an evolution algorithm that is provided in PIAnO. The evolution algorithm (EA) was proposed by Holland in 1975 and is a global optimization technique [17]. This algorithm describes the evolution of the biological genetic trait, so it searches for a global optimum through a process of selection, recombination, and mutation. On the downside, it requires considerable computation time, depending on the analytical model, because more calculation functions are required compared to

(a) Maximum stress at bending area on leaf spring

(b) Natural frequency of first mode

Figure 10: Comparison of the design constraints.

Table 4: Initial and optimal design variable values.

Design variables		Lower bound	Initial	Optimal	Upper bound
x_1	W (mm)	1.79	3.60	1.9	3.60
x_2	L (mm)	6.00	11.64	12.45	15.80
x_3	T (mm)	0.15	0.25	0.26	0.25
x_4	H (mm)	0.50	1.00	1.04	1.24
x_5	R (mm)	0.28	0.64	0.63	0.88

other optimization algorithms [18]. But we made up for the disadvantage of the evolution algorithm because we used a metamodel with a short analysis time.

3.6. Results. In our results for the optimal design using the metamodel (Opt_meta), the optimal design satisfied the constraints, including the maximum stress and natural frequency of first mode, and the width of one-folded leaf spring decreased by 47% compared to the initial width of 1.9 mm. However, the optimal design results can be changed based on using the metamodel instead of the actual analytical model in this research. The accuracy of the optimization results should be verified by actual analysis using ANSYS. To do this, the Kriging model results (Opt_meta) of the optimal design variables and the analysis results from ANSYS (Opt_exact) were compared as shown in Figures 10(a) and 10(b).

The Kriging model results (Opt_meta) and the ANSYS model results (Opt_exact) were very similar; therefore, we confirmed the high accuracy of the Kriging model's prediction. The initial and optimal values of the design variables were compared as shown in Table 4. Figure 11 shows the compared widths of the leaf spring. Figure 12 shows a comparison of the existing actuator and the actuator with the optimal width of one-folded spring.

Figure 11: Comparison of the objective function.

Figure 12: Comparison of HLVA according to the initial design and optimal design.

4. Conclusion

We minimized the existing size of a HVLA (16.5 × 15 × 2.5) by decreasing the width of the leaf spring. The following conclusions were drawn from our results.

(1) We formulated the design problem in order to minimize the maximum stress in the bending area and to satisfy the required first mode natural frequency, based on the design requirements.

(2) We generated an orthogonal array using the commercial optimization software PIAnO. Then, each leaf spring designed by sampling points was analyzed by using the commercial structural analysis software ANSYS.

(3) The Kriging model, provided PIAnO, was generated based on simulation results according to the design of experiments $L_{98}(7^{15})$. Then, the optimal design was determined using the evolution algorithm of the global optimization technique. As a result, all constraints were satisfied, and the derived the optimal width of one-folded spring was decreased by 47% compared to the initial values.

Abbreviations

T: Thickness of leaf spring
H: Height of leaf spring
L: Length of leaf spring
W: Width of leaf spring
R: Radius of leaf spring.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors express their gratitude to PIDOTECH, Inc., for providing their PIAnO software as a PIDO tool.

References

[1] J.-H. Kim, H.-K. Il, J.-W. Lee, and J.-H. Yoo, *Horizontally Vibration Motor for Slimming Mobile Smart Phone*, vol. 6, Korean Society for Precision Engineering, 2011.

[2] J.-H. Kim and J.-H. Kim, "A horizontally vibration linear actuator for slimming of smart phone," in *Proceedings of the International Conference on Computer, Electrical and Systems Conference (ICCESSE '11)*, Phuket, Thailand, December 2011.

[3] JA HWA ELECTRONICS CO., LTD., Linear type vibration motor of horizontal vibration capable of generating vibration to horizontal direction by linearly reciprocating mover, KOEAR Patent, 1007201970000 (20070514).

[4] SAMSUNG ELECTRO-MECHANICS CO., LTD, Linear vibration motor capable of minimizing a friction area with adjacent components when a motor is driven, KOREA Patent, 1010595990000 (20110819).

[5] K.-B. Lee and J.-H. Kim, "Design of leaf spring with high fatigue life applied to horizontal linear vibrating actuator," *The Korea Academia-Industrial Cooperation Society*, vol. 13, no. 12, pp. 5684–5688, 1975.

[6] PIDOTECH Inc, *Process Intergration, Automation, and Optimization*, PIAnO User's Manual, 2010.

[7] http://www.matweb.com/.

[8] A. S. Hedayat, N. J. A. Sloane, and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer, New York, NY, USA, 1999.

[9] Y. Kang, G.-J. Sheen, M.-H. Tseng, S.-H. Tu, and H.-W. Chiang, "Modal analyses and experiments for engine crankshafts," *Journal of Sound and Vibration*, vol. 214, no. 3, pp. 413–430, 1998.

[10] K. I. Hwang and J. H. Kim, "Single-axis flat electro-magnetic actuator using shorted turn for fast initial response," *Journal of Korean Medical Science*, vol. 19, no. 6, pp. 222–226, 2009.

[11] S.-M. Yoon, J.-H. Kim, and J.-H. Kim, "Mathematical modeling and analysis of vibration characteristics of smart-phone," *International Journal of Precision Engineering and Manufacturing*, vol. 14, no. 3, pp. 505–508, 2013.

[12] Z. Wu, C. Xu, J. Zhang, D. Yu, and P. Feng, "Modal and harmonic reponse analysis and evaluation of machine tools," in *Proceedings of the International Conference on Digital Manufacturing and Automation (ICDMA '10)*, vol. 1, pp. 929–933, Changsha, China, December 2010.

[13] D. G. Krige, "A statistical approach to some basic mine valuation problems on the witwatersrand," *Journal of the Chemical, Metallurgical and Mining Society of South Africa*, vol. 52, no. 6, pp. 119–139, 1951.

[14] G. Metheron, "Principles of geostatistics, economic geology," *Economic Geology*, vol. 58, no. 8, pp. 1246–1266, 1963.

[15] T. W. Simpson, J. J. Korte, T. M. Mauery, and F. Mistree, "Comparisons of response surface and kriging models for multidisciplinary design optimization," in *Proceedings of the 7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis & Oprimization*, vol. 1, pp. 381–3391, 1998.

[16] M. Stein, "Large sample properties of simulations using latin hypercube sampling," *Technometrics*, vol. 29, no. 2, pp. 143–151, 1987.

[17] J. H. Holland, *Adaptation in Natural and Artificial Systems*, The University of Michinan Press, 1975.

[18] R. L. Haupt and S. E. Haupt, *Practical Evolutionary Algorithms*, Wiley, 1998.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 512583, 9 pages
http://dx.doi.org/10.1155/2014/512583

Research Article

Adaptive Terminal Sliding Mode Control of Electromagnetic Spacecraft Formation Flying in Near-Earth Orbits

Jingrui Zhang,[1] Changqin Yuan,[2] Dongmei Jiang,[2] and Dawei Jin[2]

[1] *School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China*
[2] *Department of Aircraft and Dynamics, Aviation University of Air Force, Changchun 130022, China*

Correspondence should be addressed to Jingrui Zhang; ruierchat@163.com

Received 25 August 2013; Revised 17 November 2013; Accepted 9 December 2013; Published 11 February 2014

Academic Editor: Xiaoting Rui

Copyright © 2014 Jingrui Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

An adaptive terminal sliding mode control for six-degree-of-freedom electromagnetic spacecraft formation flying (EMFF) in near-Earth orbits is presented. By using terminal sliding mode (TSM) technique, the output tracking error can converge to zero in finite time, and strong robustness with respect to disturbance forces can be guaranteed. Based on a rotated frame F^r and the adaptive TSM controller, the special magnetic moment of the steerable magnetic dipole is computed. The angular momentum management strategy (AMM) is implemented in a periodically switching fashion, by which the angular momentum buildup was limited. Illustrative simulations of EMFF are conducted to verify the effectiveness of the proposed controller.

1. Introduction

Spacecraft formation flying (SFF) represents the concept of distributing the functionality of large spacecraft among smaller, less-expensive, cooperative spacecraft [1, 2]. Specifically, NASA and the U.S. Air Force have identified spacecraft formation flying as an enabling technology for future missions. The practical implementation of the SFF concept relies on the accurate control of the relative positions and orientations between the participating spacecraft for formation configuration. The conventional thruster-based schemes may require continuous expenditure of fuel to maintain formation geometry that can contaminate the sensitive sensors on board and mission lifetime also becomes dependent on the fuel available [3, 4].

To alleviate these concerns, several propellant-free formation flying methods have been proposed in the literature. The propulsive conducting tethers and spin-stabilized tether systems have been proposed in place of on-board propulsion systems to form and maintain satellite formations [5, 6]. King et al. [7] have presented Coulomb force approaches to maintain a formation. The flux pinning technology has been applied to achieve passively stable configurations by HTS electromagnetics [8]. LaPointe [9] has presented the microwave scattering formation flight method. Radiation forces on the order of 10-9 N/W may be generated using electromagnetic gradient forces or scattering forces; microwave beam powers of 10 kW can thus produce restoring forces of approximately 10 μN, which are sufficient to correct a number of orbital perturbations. Miller et al. [10, 11] address the novel concept of electromagnetic formation flying (EMFF) in which high temperature superconducting (HTS) wire technology is used to create magnetic dipoles on each spacecraft that can be used to maintain and reconfigure the spacecraft formation. However, Since magnetic force on each spacecraft in the formation can be applied in any arbitrary direction which can be easily created by steerable magnetic dipoles, EMFF has advantages in terms of controllability [4].

A critical component of EMFF is an effective formation flying control. The dynamics and control problem associated with EMFF become highly challenging, due to the nonlinear nature of the magnetic forces. Ahsun and Miller [3] have presented a hybrid adaptive control scheme in which translation control is implemented in a centralized fashion with a decentralized attitude control. Elias et al. [12] designed a linear optimal controller based on the linearized dynamics. Kong et al. [13] addressed the use of electromagnetic dipoles for relative position and orientation maintenance as needed for the terrestrial planet finder (TPF). Reference [14] derived the dynamics of an N-spacecraft EMFF (in 2D) for deep

Figure 1: Schematic representation of the EMFF system.

space missions and discussed a nonlinear control law using potential functions.

The orientation of a dipole obviously depends on the orientation of the body axes in the inertial space, and changing the dipole on one spacecraft affects actuation on all other spacecraft in EMFF. Therefore, general asymptotical stability may not deliver fast enough convergence to meet EMFF control for high-precision situation. The recently developed terminal sliding mode (TSM) control enables convergence to the desired state in finite time [15–18]. This technique has been used successfully in some control designs, such as robotic manipulators [17] and mobile target tracking [18]. The physical interpretation of finite time convergence lies in the fact that the convergence rate of TSM grows exponentially when the state is near equilibrium.

In this Paper, we confine our attention to adaptive TSMC design for EMFF in near-Earth orbits. The remainder of this paper is organized as follows. Section 2 presents a detailed 6-DOF dynamic modeling for EMFF in near-Earth orbits. In Section 3, a position/attitude tracking controller based on TSM is developed for EMFF, which ensures error convergence in finite time and strong robustness of the bounded disturbances. Based on a rotated frame F^r and the adaptive TSM controller, the special magnetic moments of the Steerable Magnetic Dipole (SMD) are computed. The AMM strategy is implemented in a periodically switching fashion. Simulation results are presented in Section 4. Finally, Section 5 concludes this paper.

2. System Model

Spacecraft formation constitutes an N-body mechanics problem. Figure 1 shows a typical formation system including N spacecrafts orbiting a central body (Earth). The spacecraft 0's designated as the "leader" spacecraft. We make the following considerations. (1) The ECI frame F^I is defined with its origin at the center of Earth, its X-axis points toward vernal equinox, Z-axis points toward celestial north pole, and Y-axis completes a right-handed axis system. (2) The Hill frame F^H is used to visualize the relative motion of each follower spacecraft with respect to the leader spacecraft. The x-axis aligned in the radial (zenith) direction, the z-axis is perpendicular to the orbital plane and points in the direction of the angular momentum vector, and the y-axis completes the right-hand system. (3) The orbital frame F^{Oi} is used in the attitude control. The reference frame is rotating about the y-axis with respect to ECI frame at orbital rate. The roll axis x is in the fight direction, the pitch axis y is perpendicular to the orbital plane, and the yaw axis z points toward the Earth. (4) A body frame F^{Bi} attached to the body of ith spacecraft with the center of mass of the spacecraft is also defined to describe the orientation of each spacecraft in the inertial space. (5) $\mathbf{R}_i \in \mathbb{R}^3$ denotes the position vector from the origin of ECI frame to ith spacecraft. $\mathbf{r}_{ij} \in \mathbb{R}^3$ denotes the relative position vector from ith spacecraft to the follower jth spacecraft. $\boldsymbol{\rho}_i \triangleq [x_i \;\; y_i \;\; z_i]^T \in \mathbb{R}^3$ is the expression of the relative position vector $\mathbf{r}_{0i}$ from the origin of the leader spacecraft coordinates system to the follower spacecraft i in the Hill frame F^H. (6) The rotated frame F^{ri} is used in a simplified algebraic form of the magnetic force equation. The x-axis is aligned with vector $\mathbf{r}_{ij}$ [3].

2.1. Translational Dynamics. Designating spacecraft 0 as the leader of the formation, the relative dynamics of ith spacecraft in the Hill reference frame F^H can be represented as

$$\ddot{\boldsymbol{\rho}}_i + \mathbf{C}_{iT}(\omega_O, \dot{\boldsymbol{\rho}}_i) + \mathbf{N}_{iT}(\boldsymbol{\rho}_i, \omega_O, \mathbf{R}_0) = \widehat{\mathbf{F}_{di}} + \widehat{\mathbf{F}_{ci}}, \tag{1}$$

where ω_O is the orbital angular velocity of the leader. $\mathbf{C}_{iT}(\cdot) \in \mathbb{R}^3$ is a nonlinear term defined as

$$\mathbf{C}_{iT}(\omega_O, \dot{\boldsymbol{\rho}}_i) \triangleq 2\omega_O[-\dot{y}_i \;\; \dot{x}_i \;\; 0]^T. \tag{2}$$

$\mathbf{N}_{iT}(\cdot) \in \mathbb{R}^3$ is nonlinear term defined as

$$\mathbf{N}_{iT}(\boldsymbol{\rho}_i, \omega_O, \mathbf{R}_0) \triangleq \begin{bmatrix} \mu\left(\dfrac{x_i + \|\mathbf{R}_0\|}{\|\mathbf{R}_0 + \boldsymbol{\rho}_i\|^3} - \dfrac{1}{\|\mathbf{R}_0\|^2}\right) - \omega_O^2 x_i - \dot{\omega}_O y_i \\ \mu\dfrac{y_i}{\|\mathbf{R}_0 + \boldsymbol{\rho}_i\|^3} - \omega_O^2 y_i + \dot{\omega}_O x_i \\ \mu\dfrac{z_i}{\|\mathbf{R}_0 + \boldsymbol{\rho}_i\|^3} \end{bmatrix}, \tag{3}$$

where μ is gravitational constant of earth.

Note that $\widehat{\mathbf{F}_{di}}$ denotes the parametric uncertainty and external disturbances, such as differential $\mathbf{J}_2$ and higher-order terms, differential solar pressure, and differential drag. The specific relative magnetic force $\widehat{\mathbf{F}_{ci}}$ provided by the coils needs to be strong enough to cancel these disturbances and at the same time provide additional force for trajectory following. Consider the following:

$$\widehat{\mathbf{F}_{ci}} = \frac{1}{m_i}\mathbf{F}_{cmi} - \frac{1}{m_0}\mathbf{F}_{cm0}, \tag{4}$$

where

$$\mathbf{F}_{cmi} = \sum_{j=0,\, j\neq i}^{N-1} \mathbf{F}_{ij}^m. \tag{5}$$

By approximating the coils on each spacecraft to SMD, the magnetic field due to the jth spacecraft can be written as [3, 10]

$$\mathbf{B}_j\left(\mathbf{r}_{ij}\right) = \frac{\mu_0}{4\pi}\left(\frac{3\boldsymbol{\mu}_j\cdot\mathbf{r}_{ij}}{r_{ij}^5}\mathbf{r}_{ij} - \frac{\boldsymbol{\mu}_j}{r_{ij}^3}\right) \tag{6}$$

and the force between spacecrafts can be written as

$$\begin{aligned}\mathbf{F}_{ij}^m &\triangleq f\left(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j, \mathbf{r}_{ij}\right)\\ &= \frac{3\mu_0}{4\pi}\left(-\frac{\boldsymbol{\mu}_i\cdot\boldsymbol{\mu}_j}{r_{ij}^5}\mathbf{r}_{ij} - \frac{\boldsymbol{\mu}_i\cdot\mathbf{r}_{ij}}{r_{ij}^5}\boldsymbol{\mu}_j - \frac{\boldsymbol{\mu}_i\cdot\mathbf{r}_{ij}}{r_{ij}^5}\boldsymbol{\mu}_i\right.\\ &\qquad\left. + 5\frac{\left(\boldsymbol{\mu}_i\cdot\mathbf{r}_{ij}\right)\left(\boldsymbol{\mu}_j\cdot\mathbf{r}_{ij}\right)}{r_{ij}^7}\mathbf{r}_{ij}\right),\end{aligned} \tag{7}$$

where $\mu_0 = 4\pi\times10^{-7}$ T·m/A is the permeability constant. $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_j$ are the dipole strength of the ith satellite and the jth satellite, respectively. Note that (7) gives the force on dipole i (located on ith spacecraft) due to dipole j (located on jth spacecraft), which depends on the distance between the two dipoles and the orientation of both dipoles in the inertial space. It is the dependence on the orientation of the dipoles that gives rise to the complexity of the expression for the force, since the orientation of a dipole obviously depends on the orientation of the body axes in the inertial space [3].

2.2. Attitude Dynamics. To avoid singular points, the Euler parameter is chosen to describe the attitude of the spacecraft. Let $\mathbf{q}_i$ represent the Euler parameters corresponding to the attitude of the body frame relative to the orbital frame F^{Oi} [19],

$$\begin{gathered}\mathbf{q}_i = \left[\boldsymbol{\varepsilon}_i^T \quad \eta_i\right]^T \in \mathbb{R}^4,\\ \boldsymbol{\varepsilon}_i \equiv \left[\varepsilon_{i1} \quad \varepsilon_{i2} \quad \varepsilon_{i3}\right]^T \in \mathbb{R}^{3\times1}, \quad \eta_i \in \mathbb{R}^{1\times1}.\end{gathered} \tag{8}$$

The Euler parameters, which are equivalent to the coefficients of unit quaternion, have unit norm by definition; hence

$$\boldsymbol{\varepsilon}_i^T\boldsymbol{\varepsilon}_i + \eta_i^2 = 1. \tag{9}$$

Attitude kinematics and dynamics of the spacecraft are governed by [19]

$$\boldsymbol{\Omega}_i = 2\mathbf{T}_i^{-1}\dot{\boldsymbol{\varepsilon}}_i + \boldsymbol{\omega}_{Oi}, \tag{10}$$

$$\mathbf{J}_i\dot{\boldsymbol{\Omega}}_i + \boldsymbol{\Omega}_i^{\times}\left(\mathbf{J}_i\boldsymbol{\Omega}_i + \mathbf{h}\right) = \boldsymbol{\tau}_{mi} + \boldsymbol{\tau}_{cwi} + \boldsymbol{\tau}_{di}, \tag{11}$$

where $\mathbf{T}_i \equiv \eta_i\mathbf{I} + \boldsymbol{\varepsilon}_i^{\times} \in \mathbb{R}^{3\times3}$, the superscript "×" denotes skew-symmetric matrix of a vector, and $\boldsymbol{\Omega}_i = \boldsymbol{\omega}_{Oi} + \boldsymbol{\omega}_{bi}$, $\boldsymbol{\omega}_{bi}$ is angular velocity of the spacecraft relative to the orbital frame F^{Oi}. $\boldsymbol{\omega}_{Oi}$ is the orbital angular velocity of ith spacecraft. $\mathbf{J}_i$ is the moment of inertia of the spacecraft. $\mathbf{h}$ is the angular momentum of the flywheel. The magnetic torque term in (11) can be written as

$$\boldsymbol{\tau}_{mi} = \sum_{j=0, i\neq j}^{N-1}\boldsymbol{\tau}_{ij}^m, \tag{12}$$

where $\boldsymbol{\tau}_{ij}^m$ is the torque exerted on the SMD of ith spacecraft due to that of jth spacecraft [3, 10],

$$\boldsymbol{\tau}_{ij}^m = \frac{\mu_0\boldsymbol{\mu}_i}{4\pi}\times\left(\frac{3\mathbf{r}_{ij}\left(\boldsymbol{\mu}_j\cdot\mathbf{r}_{ij}\right)}{r_{ij}^5} - \frac{\boldsymbol{\mu}_j}{r_{ij}^3}\right) \tag{13}$$

$\boldsymbol{\tau}_{cwi}$ and $\boldsymbol{\tau}_{di}$ are the control and disturbance torques, respectively. $\boldsymbol{\tau}_{di}$ includes the gravity gradient torques $\boldsymbol{\tau}_{gi}$ and the Earth magnetic torques $\boldsymbol{\tau}_{ei}$. Consider the following:

$$\boldsymbol{\tau}_{di} = \boldsymbol{\tau}_{gi} + \boldsymbol{\tau}_{ei}. \tag{14}$$

A very important factor presented in the disturbance torque term is the torque that acts on the spacecraft due to the Earth's magnetic field. This torque can be written as [4]

$$\boldsymbol{\tau}_{ei} = \boldsymbol{\mu}_i\times\mathbf{B}_e\left(\mathbf{R}_i\right), \tag{15}$$

where $\mathbf{B}_e$ is Earth's magnetic field strength at the location of the spacecraft.

Substituting (10) into (11), the attitude dynamics can be written compactly as

$$\begin{aligned}&\mathbf{M}_{iA}\left(\mathbf{J}_i, \mathbf{q}_i\right)\ddot{\boldsymbol{\varepsilon}}_i + \mathbf{C}_{iA}\left(\mathbf{J}_i, \boldsymbol{\Omega}_i, \mathbf{h}\right) + \mathbf{N}_{iA}\left(\mathbf{J}_i, \mathbf{q}_i, \boldsymbol{\omega}_{bi}, \boldsymbol{\omega}_{Oi}, \dot{\boldsymbol{\omega}}_{Oi}\right)\\ &\quad = \boldsymbol{\tau}_{cwi} + \boldsymbol{\tau}_{mi} + \boldsymbol{\tau}_{di},\end{aligned} \tag{16}$$

where

$$\begin{gathered}\mathbf{M}_{iA}\left(\mathbf{J}_i, \mathbf{q}_i\right) = 2\mathbf{J}_i\mathbf{T}_i^{-1},\\ \mathbf{C}_{iA}\left(\mathbf{J}_i, \boldsymbol{\Omega}_i, \mathbf{h}\right) = \boldsymbol{\Omega}_i^{\times}\left(\mathbf{J}_i\boldsymbol{\Omega}_i + \mathbf{h}\right), \qquad \mathbf{X}_i = \left[\boldsymbol{\rho}_i^T, \boldsymbol{\varepsilon}_i^T\right]^T,\\ \mathbf{N}_{iA}\left(\mathbf{J}_i, \mathbf{q}_i, \boldsymbol{\omega}_{bi}, \boldsymbol{\omega}_{Oi}, \dot{\boldsymbol{\omega}}_{Oi}\right) = 2\mathbf{J}_i\mathbf{T}_i^{-1}f_i, \quad f_i = -\boldsymbol{\omega}_{bi}^{\times}\boldsymbol{\omega}_{Oi} + \dot{\boldsymbol{\omega}}_{Oi}.\end{gathered} \tag{17}$$

2.3. Combined Attitude and Translational Dynamics. Combining the attitude dynamics in (1) and the translational dynamics in (16), the following 6-DOF dynamics equation for formation flying is obtained:

$$\mathbf{M}_i\ddot{\mathbf{X}}_i + \mathbf{C}_i + \mathbf{N}_i = \mathbf{u}_i + \mathbf{D}_i, \tag{18}$$

where

$$\begin{gathered}\mathbf{M}_i = \begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{iA}\end{bmatrix} \in \mathbb{R}^{6\times6}, \qquad \mathbf{C}_i = \begin{bmatrix}\mathbf{C}_{iT}\\ \mathbf{C}_{iA}\end{bmatrix} \in \mathbb{R}^{6\times1},\\ \mathbf{N}_i = \begin{bmatrix}\mathbf{N}_{iT}\\ \mathbf{N}_{iA}\end{bmatrix} \in \mathbb{R}^{6\times1}, \quad \mathbf{u}_i = \begin{bmatrix}\widehat{\mathbf{F}_{ci}}\\ \boldsymbol{\tau}_{cwi}\end{bmatrix},\\ \mathbf{D}_i \triangleq \left[d_{1i} \quad d_{2i} \quad d_{3i} \quad d_{4i} \quad d_{5i} \quad d_{6i}\right]^T = \begin{bmatrix}\widehat{\mathbf{F}_{di}}\\ \boldsymbol{\tau}_{mi} + \boldsymbol{\tau}_{di}\end{bmatrix}.\end{gathered} \tag{19}$$

3. Adaptive Terminal Sliding Mode Control Design

In this section, an adaptive terminal sliding mode controller is designed for the follower spacecraft based on the dynamic model in (18) and the TSM technique. With this controller, the follower spacecraft can track the desired attitude and relative position trajectories simultaneously.

3.1. Error Dynamics Equation. The trajectory tracking errors of the follower spacecraft are defined as

$$\boldsymbol{\rho}_i^e \triangleq \boldsymbol{\rho}_i - \boldsymbol{\rho}_i^d, \qquad \dot{\boldsymbol{\rho}}_i^e \triangleq \dot{\boldsymbol{\rho}}_i - \dot{\boldsymbol{\rho}}_i^d, \tag{20}$$

where $\boldsymbol{\rho}_i^d, \dot{\boldsymbol{\rho}}_i^d \in \mathbb{R}^{3\times1}$ are the relative position/velocity of the desired trajectory with respect to the leader. Differentiating (20) and substituting it into (1), one can obtain that

$$\ddot{\boldsymbol{\rho}}_i^e + \mathbf{C}_{iT}(\omega_O, \dot{\boldsymbol{\rho}}_i) + \mathbf{N}_{iT}^e\left(\boldsymbol{\rho}_i, \omega_O, \mathbf{R}_0, \ddot{\boldsymbol{\rho}}_i^d\right) = \widehat{\mathbf{F}_{di}} + \widehat{\mathbf{F}_{ci}}, \tag{21}$$

where $\mathbf{N}_{iT}^e(\boldsymbol{\rho}_i, \omega_O, \mathbf{R}_0, \ddot{\boldsymbol{\rho}}_i^d) = \mathbf{N}_{iT} + \ddot{\boldsymbol{\rho}}_i^d$.

The error Euler parameter is chosen to describe the attitude error of the spacecraft with respect to the reference attitude. The error Euler parameter is determined as follow [19]:

$$\boldsymbol{\varepsilon}_i^e = \mathbf{U}_i^{d^T}\mathbf{q}_i, \tag{22}$$

$$\eta_i^e = \mathbf{q}_i^{d^T}\mathbf{q}_i, \tag{23}$$

where $\mathbf{U}_i^d \equiv \left[\mathbf{T}_i^{d^T} \quad -\boldsymbol{\varepsilon}_i\right]^T \in \mathbb{R}^{4\times3}$, $\mathbf{T}_i^d \equiv \eta_i^d\mathbf{I} + \boldsymbol{\varepsilon}_i^{\times d} \in \mathbb{R}^{3\times3}$, and $\mathbf{q}_i^d \equiv \left[\boldsymbol{\varepsilon}_i^{d^T} \quad \eta_i^d\right]^T \in \mathbb{R}^{4\times1}$ is the desired attitude.

Differentiating (22) with respect to time and utilizing (10), the attitude kinematics equations with error Euler parameter are given by

$$\boldsymbol{\Omega}_{bi} = 2\mathbf{T}_i^{e-1}\dot{\boldsymbol{\varepsilon}}_i^e + \mathbf{A}_i^e\boldsymbol{\omega}_{bi}^d + \mathbf{A}_{bIi}\boldsymbol{\omega}_{Oi}, \tag{24}$$

where $\mathbf{A}_i^e$ is the transformation matrix from the reference attitude frame to the body frame.

Differentiating (24), one can obtain that

$$\dot{\boldsymbol{\Omega}}_{bi} = 2\mathbf{T}_i^{e^{-1}}\ddot{\boldsymbol{\varepsilon}}_i^e + \mathbf{f}_i^e, \tag{25}$$

where

$$\begin{aligned}\mathbf{f}_i^e = \mathbf{f}_i &+ \frac{\boldsymbol{\varepsilon}_i^e}{2\eta_i^{e2}}\left(\boldsymbol{\omega}_{bi} - \mathbf{A}_i^e\boldsymbol{\omega}_{bi}^d\right)\left(\boldsymbol{\omega}_{bi} - \mathbf{A}_i^e\boldsymbol{\omega}_{bi}^d\right)\\ &+ \mathbf{A}_i^e\dot{\boldsymbol{\omega}}_{bi}^d - \boldsymbol{\omega}_{bi}^{\times}\mathbf{A}_i^e\boldsymbol{\omega}_{bi}^d\end{aligned} \tag{26}$$

Substituting (24) and (25) into (11), the error attitude dynamics are derived as follows [19]:

$$\begin{aligned}&\mathbf{M}_{iA}^e(\mathbf{J}_i, \mathbf{q}_i)\ddot{\boldsymbol{\varepsilon}}_i + \mathbf{C}_{iA}(\mathbf{J}_i, \boldsymbol{\Omega}_i, \mathbf{h}) + \mathbf{N}_{iA}^e\left(\mathbf{J}_i, \mathbf{q}_i^e, \boldsymbol{\omega}_{bi}, \boldsymbol{\omega}_{bi}^d, \dot{\boldsymbol{\omega}}_{bi}^d\right)\\ &\quad = \boldsymbol{\tau}_{cwi} + \boldsymbol{\tau}_{mi} + \boldsymbol{\tau}_{di},\end{aligned} \tag{27}$$

where $\mathbf{N}_{iA}^e = 2\mathbf{J}_i\mathbf{T}_i^{e^{-1}}\mathbf{f}_i^e$, $\mathbf{M}_{iA}^e = 2\mathbf{J}_i\mathbf{T}_i^{e^{-1}}$.

Let $\mathbf{e}_i = \left[\boldsymbol{\rho}_i^{e^T} \quad \boldsymbol{\varepsilon}_i^{e^T}\right]^T$, and utilizing (18), (21), and (27), the error dynamics of the whole system can be obtained

$$\mathbf{M}_i^e\ddot{\mathbf{e}}_i + \mathbf{C}_i + \mathbf{N}_i^e = \mathbf{u}_i + \mathbf{D}_i, \tag{28}$$

where

$$\mathbf{M}_i^e = \begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{iA}^e\end{bmatrix} \in \mathbb{R}^{6\times6}, \qquad \mathbf{N}_i^e = \begin{bmatrix}\mathbf{N}_{iT}^e\\ \mathbf{N}_{iA}^e\end{bmatrix} \in \mathbb{R}^{6\times1}. \tag{29}$$

3.2. Controller Design. The terminal switching planes can be describe as [16–18]

$$\mathbf{S}_i = \dot{\mathbf{e}}_i + \boldsymbol{\alpha}_i\mathbf{e}_i + \boldsymbol{\beta}_i\mathbf{e}_i^{\gamma_i}, \tag{30}$$

where $\mathbf{S}_i \triangleq \left[s_{1i} \;\; s_{2i} \;\; s_{3i} \;\; s_{4i} \;\; s_{5i} \;\; s_{6i}\right]^T$, $\boldsymbol{\alpha}_i = \mathrm{diag}(a_{1i}, a_{2i}, \ldots, a_{6i})$, $\boldsymbol{\beta}_i = \mathrm{diag}(b_{1i}, b_{2i}, \ldots, b_{6i})$, $a_n > 0$, $b_n > 0$ $(n = 1, 2, \ldots, 6)$, and $0 < \gamma_i < 1$. The unknown disturbances $\mathbf{D}_i$ in (18) are bounded but unknown. σ_{ni}^* is the upper bound of d_{ni}. Here, it can be estimated by

$$\dot{\hat{\sigma}}_{ni} = \frac{1}{\vartheta_{ni}}\left|s_{ni}\right|, \quad (n = 1, 2, \ldots, 6), \tag{31}$$

where ϑ_{ni} is the adaptive gain.

Utilizing (28), (30), and (31), an adaptive terminal sliding mode control law is designed as follows:

$$\begin{aligned}\mathbf{u}_i \triangleq \begin{bmatrix}\mathbf{u}_{iT}\\ \mathbf{u}_{iA}\end{bmatrix} &= \mathbf{C}_i + \mathbf{N}_i^e\\ &+ \mathbf{M}_i^e\left(-\boldsymbol{\alpha}_i\dot{\mathbf{e}}_i - \gamma_i\boldsymbol{\beta}_i\,\mathrm{diag}\left(\mathbf{e}_i^{\gamma_i-1}\right)\dot{\mathbf{e}}_i - \mathrm{diag}(\hat{\boldsymbol{\sigma}}_i)\,\mathrm{sign}(\mathbf{S}_i)\right),\end{aligned} \tag{32}$$

where $\hat{\boldsymbol{\sigma}}_i \triangleq \left[\hat{\sigma}_{1i} \;\; \hat{\sigma}_{2i} \;\; \hat{\sigma}_{3i} \;\; \hat{\sigma}_{4i} \;\; \hat{\sigma}_{5i} \;\; \hat{\sigma}_{6i}\right]^T$ and the vector signum function $\mathrm{sign}(\mathbf{S}_i)$ is a column of signum functions

$$\mathrm{sign}(\mathbf{S}_i) \triangleq \left[\mathrm{sign}(s_{1i}) \;\; \mathrm{sign}(s_{2i}) \;\; \mathrm{sign}(s_{3i}) \;\; \mathrm{sign}(s_{4i}) \;\; \mathrm{sign}(s_{5i}) \;\; \mathrm{sign}(s_{6i})\right]^T. \tag{33}$$

Substituting (32) into (18) produces the closed-loop dynamics

$$\ddot{\mathbf{e}}_i = -\boldsymbol{\alpha}_i\dot{\mathbf{e}}_i - \gamma_i\boldsymbol{\beta}_i\,\mathrm{diag}\left(\mathbf{e}_i^{\gamma-1}\right)\dot{\mathbf{e}}_i - \mathrm{diag}(\hat{\boldsymbol{\sigma}}_i)\,\mathrm{sign}(\mathbf{S}_i) + \mathbf{D}_i. \tag{34}$$

Consider a Lyapunov function as follows:

$$V = \frac{1}{2}\mathbf{S}_i^T\mathbf{S}_i + \frac{1}{2}\tilde{\boldsymbol{\sigma}}_i^T\,\mathrm{diag}(\boldsymbol{\vartheta}_i)\,\tilde{\boldsymbol{\sigma}}_i, \tag{35}$$

where $\boldsymbol{\vartheta}_i \triangleq [\vartheta_{1i}\ \vartheta_{2i}\ \vartheta_{3i}\ \vartheta_{4i}\ \vartheta_{5i}\ \vartheta_{6i}]^T$. The estimation error is defined as $\tilde{\boldsymbol{\sigma}}_i = \hat{\boldsymbol{\sigma}}_i - \boldsymbol{\sigma}_i^*$, $\boldsymbol{\sigma}_i^* \triangleq [\sigma_{1i}^*,\ \sigma_{2i}^*,\ \sigma_{3i}^*,\ \sigma_{4i}^*,\ \sigma_{5i}^*,\ \sigma_{6i}^*]^T$.

The first derivative of (30) can be expressed as

$$\dot{\mathbf{S}}_i = \ddot{\mathbf{e}}_i + \boldsymbol{\alpha}_i\dot{\mathbf{e}}_i + \gamma_i\boldsymbol{\beta}_i \operatorname{diag}\left(\mathbf{e}_i^{\gamma-1}\right)\dot{\mathbf{e}}_i. \tag{36}$$

Differentiating (35) and utilizing (36), one can obtain that

$$\begin{aligned}\dot{V} &= \mathbf{S}_i^T\dot{\mathbf{S}}_i + \tilde{\boldsymbol{\sigma}}_i^T \operatorname{diag}(\boldsymbol{\vartheta}_i)\dot{\tilde{\boldsymbol{\sigma}}}_i \\ &= \mathbf{S}_i^T\left(\ddot{\mathbf{e}}_i + \boldsymbol{\alpha}_i\dot{\mathbf{e}}_i + \gamma_i\boldsymbol{\beta}_i \operatorname{diag}\left(\mathbf{e}_i^{\gamma-1}\right)\dot{\mathbf{e}}_i\right) + \tilde{\boldsymbol{\sigma}}_i^T\|\mathbf{S}_i\|,\end{aligned} \tag{37}$$

where $\|\mathbf{S}_i\| \triangleq [|s_{1i}|\ \ |s_{2i}|\ \ |s_{3i}|\ \ |s_{4i}|\ \ |s_{5i}|\ \ |s_{6i}|]^T$. Substituting (31) and (34) into (37) produces that

$$\begin{aligned}\dot{V} &= -\mathbf{S}_i^T\left(\operatorname{diag}(\hat{\boldsymbol{\sigma}}_i)\operatorname{sign}(\mathbf{S}_i) - \mathbf{D}_i\right) + \left(\hat{\boldsymbol{\sigma}}_i^T - \boldsymbol{\sigma}_i^{*T}\right)\|\mathbf{S}_i\| \\ &\le -\sum_{n=1}^{6}|s_{ni}|\left(\sigma_{ni}^* - |d_{ni}|\right) < 0 \quad \text{for } s_{ni} \ne 0.\end{aligned} \tag{38}$$

It means that the switching planes s_{ni} ($n = 1, 2, \ldots, 6$) converge to zero [20]. On the other hand, in the TSM $s_{ni} = 0$ ($n = 1, 2, \ldots, 6$), the system state will reach zero in finite time [17].

Remark 1. For the purpose of eliminating chattering, a common practice is to replace the signum function of (32) with a continuous saturation function

$$\operatorname{sat}(s_{ni}, \xi_{ni}) = \begin{cases} \dfrac{s_{ni}}{\xi_{ni}} & \text{if } |s_{ni}| < \xi_{ni} \\ \operatorname{sign}(s_{ni}) & \text{if } |s_{ni}| \ge \xi_{ni}, \end{cases} \quad n = 1, 2, \ldots, 6, \tag{39}$$

where $\xi_{ni} > 0$ is the width of the boundary layer. Equation (32) can be written as

$$\begin{aligned}\mathbf{u}_i &= \mathbf{C}_i + \mathbf{N}_i^e \\ &\quad + \mathbf{M}_i^e\left(-\boldsymbol{\alpha}_i\dot{\mathbf{e}}_i - \gamma_i\boldsymbol{\beta}_i \operatorname{diag}\left(\mathbf{e}_i^{\gamma_i-1}\right)\dot{\mathbf{e}}_i \right. \\ &\quad \left. - \operatorname{diag}(\hat{\boldsymbol{\sigma}}_i)\operatorname{sat}(\mathbf{S}_i, \boldsymbol{\xi}_i)\right),\end{aligned} \tag{40}$$

where

$$\operatorname{sat}(\mathbf{S}_i, \boldsymbol{\xi}_i) = [\operatorname{sat}(s_{1i}, \xi_{1i})\ \ \operatorname{sat}(s_{2i}, \xi_{2i})\ \ \operatorname{sat}(s_{3i}, \xi_{3i})\ \ \operatorname{sat}(s_{4i}, \xi_{4i})\ \ \operatorname{sat}(s_{5i}, \xi_{5i})\ \ \operatorname{sat}(s_{6i}, \xi_{6i})]^T. \tag{41}$$

Remark 2. There exists a possible singularity in sliding mode controller as $\mathbf{e}_i \to 0$. Since $\mathbf{e}_i = 0$ only approaches along a sliding mode, we observe that for a general choice of γ_i

$$\dot{\mathbf{e}}_i = -\boldsymbol{\alpha}_i\mathbf{e}_i - \boldsymbol{\beta}_i \operatorname{diag}\left(\mathbf{e}_i^{\gamma}\right) \tag{42}$$

while sliding and that the component in (32)

$$\gamma_i\boldsymbol{\beta}_i \operatorname{diag}\left(\mathbf{e}_i^{\gamma_i-1}\right)\dot{\mathbf{e}}_i \longrightarrow -\gamma_i\boldsymbol{\beta}_i\left(\boldsymbol{\alpha}_i \operatorname{diag}\left(\mathbf{e}_i^{\gamma_i}\right) + \boldsymbol{\beta}_i \operatorname{diag}\left(\mathbf{e}_i^{2\gamma_i-1}\right)\right). \tag{43}$$

Consequently there will be a singularity in (32) unless γ_i is chosen so that $2\gamma_i > 1$. To satisfy this requirement we set $\gamma_i = 3/5$ for the examples to follow.

3.3. Compute the Special Magnetic Moments of SMD. The adaptive sliding mode controller $\mathbf{u}_{iT}$ gives the desired special forces that can be used as input for a thruster-based system. However, for EMFF, the desired special forces are produced by SMD. The SMD of each spacecraft is a complicated function of current position and orientation [3, 10]. For EMFF, the control variables consist of the dipoles of individual spacecraft. Utilizing (32) and (4), one can obtain that

$$\mathbf{F}_{cm1} = \frac{m_0 m_1}{m_0 + m_1}\mathbf{u}_{iT}, \qquad \mathbf{F}_{cm0} = -\frac{m_0 m_1}{m_0 + m_1}\mathbf{u}_{iT}. \tag{44}$$

Substituting (5) and (7) into (44) produces

$$\sum_{j=0,\, j\ne i}^{N-1} f_{ij}\left(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j, \mathbf{r}_{ij}\right) = \frac{m_i m_j}{m_i + m_j}\mathbf{u}_{iT}. \tag{45}$$

Note that (7) gives the force on the ith SMD due to the jth SMD and it depends on the distance between the two SMDs and the orientation of both SMDs in the inertial space. It rises to the complexity of the expression for the magnetic. Here, we consider a two-spacecraft electromagnetic formation flying (EMFF) array. By defining a rotated frame F^r (see Section 2), a simplified algebraic form of (7) is obtained

$$f_{ij}^r\left(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j, \mathbf{r}_{ij}\right) = \frac{3\mu_0}{4\pi r_{ij}^4}\begin{bmatrix} 2\mu_{ix}\mu_{jx} - \mu_{iy}\mu_{jy} - \mu_{iz}\mu_{jz} \\ -\mu_{ix}\mu_{jy} - \mu_{iy}\mu_{jx} \\ -\mu_{ix}\mu_{jz} - \mu_{iz}\mu_{jx} \end{bmatrix}. \tag{46}$$

Utilizing (45) and (46) produces that

$$\begin{bmatrix} 2\mu_{ix}\mu_{jx} - \mu_{iy}\mu_{jy} - \mu_{iz}\mu_{jz} \\ -\mu_{ix}\mu_{jy} - \mu_{iy}\mu_{jx} \\ -\mu_{ix}\mu_{jz} - \mu_{iz}\mu_{jx} \end{bmatrix} = \frac{4\pi r_{ij}^4}{3\mu_0}\frac{m_i m_j}{m_i + m_j}\mathbf{A}_{rIi}\mathbf{A}_{Ibi}\mathbf{u}_{iT}, \tag{47}$$

where $\mathbf{A}_{rIi}$ is the transformation matrix from the inertial ECI reference frame F^I to the rotated frame F^r. $\mathbf{A}_{Ibi}$ is

Table 1: Orbital parameters.

Spacecraft	Semimajor axis	Inclination	Right ascension of ascending node	Argument of perigee	Eccentricity	Mean anomaly
LS	6878 km	1.106538745764 rad	0.523598775598 rad	0 rad	0.003	0 rad
FS	6878 km	1.106538745764 rad	0.523598071512 rad	0.000000315261 rad	0.002999636522	0 rad

the transformation matrix from the body frame F^{Bi} of ith spacecraft to ECI frame.

Let $\breve{\mathbf{u}}_{iT}^{r} \triangleq [\breve{u}_{iTx}^{r}, \breve{u}_{iTy}^{r}, \breve{u}_{iTz}^{r}]^T = (4\pi r_{12}^4/3\mu_0)(m_1 m_2/(m_1 + m_2))\mathbf{A}_{rIi}\mathbf{A}_{Ibi}\mathbf{u}_{iT}$, $\mu_i = -\mu_j$, and utilizing (47), one can obtain that

$$\begin{aligned}
\mu_{ix} &= \frac{1}{2}\sqrt{\sqrt{\breve{u}_{iTx}^{r2} + 2\breve{u}_{iTy}^{r2} + 2\breve{u}_{iTz}^{r2}} - \breve{u}_{iTx}^{r2}}, \\
\mu_{iy} &= \frac{\breve{u}_{iTy}^{r2}}{\sqrt{\sqrt{\breve{u}_{iTx}^{r2} + 2\breve{u}_{iTy}^{r2} + 2\breve{u}_{iTz}^{r2}} - \breve{u}_{iTx}^{r2}}}, \\
\mu_{iz} &= \frac{\breve{u}_{iTz}^{r2}}{\sqrt{\sqrt{\breve{u}_{iTx}^{r2} + 2\breve{u}_{iTy}^{r2} + 2\breve{u}_{iTz}^{r2}} - \breve{u}_{iTx}^{r2}}}.
\end{aligned} \tag{48}$$

The special magnetic moments of SMD can be computed using (48), by which the follower spacecraft can track the desired relative position trajectories.

3.4. Angular Momentum Management. For EMFF in near-Earth orbits, a constant disturbance torque may act on each spacecraft due to the Earth's magnetic field that causes the reaction wheels on each spacecraft to quickly become saturated. In order to avoid this situation, the angular momentum management strategy herein can be utilized, which is based on a unique nonlinearity of the magnetic torques [3, 4]. The force acting between a pair of dipoles depends on the product of the individual dipole values. By switching the polarity of all dipoles in the EMFF, the torque acting on the spacecraft due to Earth's magnetic field changes sign, but the torques and forces due to the other spacecraft in the system do not. As can be seen from (15), it results in a net cancellation of the effect of the Earth's magnetic field on the average sense.

4. Numerical Simulation

The adaptive terminal sliding mode controller equation (40) was simulated for a two-spacecraft formation flying control. Considering the nonlinear dynamics with disturbance $\mathbf{D}_1$, the effectiveness of the proposed controller was verified. The orbital parameters of leader and follower spacecraft in the simulations are listed in Table 1. The parameters of spacecraft are summarized in Table 2. The dynamics includes the second harmonic of the gravitational field as well as the Earth's magnetic field, of which the vector is calculated using the World Magnetic (WMM2005)Model block in the paper.

The follower was commanded to move around the leader in an elliptic orbit. The desired trajectory was generated by solving nonlinear equation (1) numerically (set $\widehat{\mathbf{F}_{d1}}$ and $\widehat{\mathbf{F}_{c1}}$ equal to zero) with the following initial condition:

$$\begin{aligned}
\boldsymbol{\rho}_{d1}(0) &= [2.499998640217,\ 0.000000000217,\ 4.317138211336]\,[\mathrm{m}], \\
\dot{\boldsymbol{\rho}}_{d1}(0) &= [0.000000000000,\ -0.000005559083863,\ -0.000000000758]\,[\mathrm{m/s}].
\end{aligned} \tag{49}$$

The relative position/velocity of the follower was initialized to

$$\begin{aligned}
\boldsymbol{\rho}_{1}(0) &= [2.599999999855,\ 0.722841614669,\ 4.503332099822]\,[\mathrm{m}], \\
\dot{\boldsymbol{\rho}}_{1}(0) &= [-0.000000000000,\ -0.005557083863,\ -0.000000000758]\,[\mathrm{m/s}].
\end{aligned} \tag{50}$$

The parameters of the adaptive sliding mode controller are then

$$\begin{aligned}
\alpha_1 &= \mathrm{diag}(2, 2, 3, 250, 80, 80) \times 10^{-4}, \\
\beta_1 &= \mathrm{diag}(10, 15, 15, 1200, 800, 800) \times 10^{-5}, \\
\gamma_1 &= \frac{3}{5}, \qquad \xi = [5, 5, 5, 1, 1, 1] \times 10^{-3}, \\
\vartheta_1 &= [15, 22, 22, 80, 125, 125].
\end{aligned} \tag{51}$$

Table 2: Parameter values of model.

Spacecraft	Mass	Moment of inertia
Ls	100 kg	Diag (19, 19, 32) kg·m^2
FS	100 kg	Diag (19, 19, 32) kg·m^2

Figure 2: Actual trajectory $\boldsymbol{\rho}_1(t)$ of the follower spacecraft ($*$ the leader spacecraft.)

Figure 3: Position tracking error.

Figure 4: Attitude tracking error.

Figure 5: Magnetic dipole strength in F^{B1} frame.

In this simulation, a switching time of 100 second was used for AMM. Figures 2–6 show the simulation results, which are obtained by simulating the adaptive terminal sliding mode controller. The phase portrait of the trajectory $\boldsymbol{\rho}_1(t)$ of the follower spacecraft relative to the leader spacecraft is illustrated in Figure 3, where $*$ represents the leader spacecraft at the origin. Figure 3 depicts the position tracking error of the follower spacecraft. Figure 4 gives attitude error of the follower spacecraft. Figure 5 shows the magnetic dipole strength of the follower spacecraft. Figure 6 gives the reaction wheel angular momentum buildup of the follower spacecraft, and it is clear from this figure that the angular momentum buildup was limited by AMM. In a stable tracking situation, simulation results show that $\|\boldsymbol{\rho}_1^e\|$ is smaller than 6×10^{-3} m and $\|\boldsymbol{\varepsilon}_1^e\|$ is smaller than 0.075° which meets the required tracking accuracy.

Figure 6: Reaction wheel angular momentum.

5. Conclusions

In this paper, we consider the control problem for six-degree-of-freedom electromagnetic spacecraft formation flying (EMFF) in near-Earth orbits. Firstly, the combined attitude and translational dynamics are presented. Secondly, we confine our attention mainly to the position/attitude tracking control using the terminal sliding mode and adaptive control technique. Based on a rotated frame F^r and the adaptive TSM controller, the special magnetic moments of SMD are computed. The AMM strategy is implemented in a periodically switching fashion. Through numerical simulation, it is clear that the output tracking error can converge to zero in finite time by using the proposed controller and that the angular momentum buildup was limited by AMM. At the same time, strong robustness with respect to bounded disturbances can be guaranteed (in the simulation, the J_2 disturbance, the gravity gradient torques $\boldsymbol{\tau}_{g1}$, the Earth magnetic torques $\boldsymbol{\tau}_{e1}$, and the magnetic torque term $\boldsymbol{\tau}_{m1}$ have been considered).

Nomenclature

EMFF: Electromagnetic spacecraft formation flying
TSM: Terminal sliding mode
AMM: Angular momentum management
SFF: Spacecraft formation flying
HTS: High-temperature superconducting
TSMC: Terminal sliding mode control
SMD: Steerable magnetic dipole
ECI: earth center inertial frame
TPF: terrestrial planet finder.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by the Natural Science Foundation of China under Grant Nos. 11372353 and 10902125.

References

[1] J. Bristow, D. Folta, and K. Hartman, "A formation flying technology vision," in *Proceedings of the AIAA Space Conference and Exposition*, Long Beach, Calif, USA, September 2000, AIAA-2000-5194.

[2] D. Folta and A. Hawkins, "Results of NASA's first autonomous formation flying experiment: earth observing-1 (EO-1)," in *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, Monterey, Calif, USA, August 2002, AIAA-2002-4743.

[3] U. Ahsun and D. W. Miller, *Dynamics and control of electromagnetic satellite formations [Ph.D. thesis]*, Massachusetts Institute of Technology, 2007.

[4] U. Ahsun, D. W. Miller, and J. L. Ramirez, "Control of electromagnetic satellite formations in near-Earth orbits," *Journal of Guidance, Control, and Dynamics*, vol. 33, no. 6, pp. 1883–1891, 2010.

[5] L. Johnson, B. Gilchrist, R. D. Estes, and E. Lorenzini, "Overview of future NASA tether applications," Tech. Rep. 1998-0237034, NASA, 1998.

[6] D. Quinn and D. Folta, "A tethered formation flying concept for the SPECS missions," Tech. Rep. 2000-0032752, NASA, 2000.

[7] L. B. King, G. G. Parker, S. Deshmukh, and J. Chong, "Spacecraft formation-flying using inter-vehicle coulomb forces," NIAC Phase I Final Report, 2002, http://www.niac.usra.edu/.

[8] J. P. Shoer and M. A. Peck, "A flux-pinned magnet-superconductor pair for close-proximity station keeping and self-assembly of spacecraft," in *Proceedings of the AIAA Guidance, Navigation, and Control Conference and Exhibit*, pp. 611–628, Hilton Head, SC, USA, August 2007, AIAA Paper 2007-6352.

[9] M. R. LaPointe, "Formation flight method," NIAC Phase I Final Report, 2001, http://www.niac.usra.edu/.

[10] D. W. Miller, R. J. Sedwick, D. Jiang, D. Jin et al., "Electromagnetic formation flight," NIAC Phase I Final Report, Massachusetts Institute of Technology, 2002.

[11] D. W. Miller, R. J. Sedwick, E. M. C. Kong, and S. Schweighart, "Electromagnetic formation flight for sparse aperture telescopes," in *Proceedings of the IEEE Aerospace Conference*, Big Sky, Mont, USA, March 2002.

[12] L. M. Elias, D. W. Kwon, R. J. Sedwick, and D. W. Miller, "Electromagnetic formation flight dynamics including reaction wheel gyroscopic stiffening effects," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 2, pp. 499–511, 2007.

[13] E. M. C. Kong, D. W. Kwon, S. A. Schweighart, L. M. Elias, R. J. Sedwick, and D. W. Miller, "Electromagnetic formation flight for multisatellite arrays," *Journal of Spacecraft and Rockets*, vol. 41, no. 4, pp. 659–666, 2004.

[14] U. Ahsun and D. W. Miller, "Dynamics and control of electromagnetic satellite formations," in *Proceedings of the American*

Control Conference, pp. 1730–1735, Minneapolis, Minn, USA, June 2006.

[15] Y. Feng, X. Yu, and Z. Man, "Non-singular terminal sliding mode control of rigid manipulators," *Automatica*, vol. 38, no. 12, pp. 2159–2167, 2002.

[16] X. Yu and Z. Man, "Variable structure systems with terminal sliding modes," in *Variable Structure Systems: Towards the 21st Century*, vol. 274 of *Lecture Notes in Control and Information Sciences*, pp. 109–128, Springer, Berlin, Germany, 2002.

[17] X. Yu, Y. Wu, and Z. Man, "On global stabilization of nonlinear dynamical systems," in *Variable Structure Systems, Sliding Mode and Nonlinear Control*, vol. 274 of *Lecture Notes in Control and Information Sciences*, pp. 109–122, Springer, Berlin, Germany, 1999.

[18] D. J. Stonier and R. J. Stonier, "Obstacle avoidance and finite-time tracking of mobile targets," in *Proceedings of the 2nd International Conference on Autonomous Robots and Agents*, pp. 58–63, Palmerston North, New Zealand, December 2004.

[19] C.-Q. Yuan, J.-F. Li, T.-S. Wang, and H.-X. Baoyin, "Robust attitude control for rapid multi-target tracking in spacecraft formation flying," *Applied Mathematics and Mechanics*, vol. 29, no. 2, pp. 185–198, 2008.

[20] D. R. Merkin, *Introduction to the Theory of Stability*, Spring, New York, NY, USA, 1996.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 592628, 9 pages
http://dx.doi.org/10.1155/2014/592628

Research Article

Application of Krylov Reduction Technique for a Machine Tool Multibody Modelling

M. Sulitka, J. Šindler, J. Sušeň, and J. Smolík

Research Center of Manufacturing Technology, Czech Technical University in Prague, Horská 3, 128 00 Prague, Czech Republic

Correspondence should be addressed to M. Sulitka; m.sulitka@rcmt.cvut.cz

Received 26 August 2013; Accepted 22 November 2013; Published 5 February 2014

Academic Editor: Xiaoting Rui

Copyright © 2014 M. Sulitka et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Quick calculation of machine tool dynamic response represents one of the major requirements for machine tool virtual modelling and virtual machining, aiming at simulating the machining process performance, quality, and precision of a workpiece. Enhanced time effectiveness in machine tool dynamic simulations may be achieved by employing model order reduction (MOR) techniques of the full finite element (FE) models. The paper provides a case study aimed at comparison of Krylov subspace base and mode truncation technique. Application of both of the reduction techniques for creating a machine tool multibody model is evaluated. The Krylov subspace reduction technique shows high quality in terms of both dynamic properties of the reduced multibody model and very low time demands at the same time.

1. Introduction

In recent years, with increasing computational power of computers, machine tool virtual modeling has become one of the intensively developed tools for simulating the machine tool behavior in interaction with machining process, aiming especially at predicting the machining performance, process quality, and stability [1–4]. A key element in machine tool dynamic behavior represents the compliant machine tool mechanical system, including both the structural parts and the components of feed drive systems as well. Excitation of natural eigenfrequencies of the machine tool mechanics stands in the foreground of problems and represents one of the limiting factors related to enhancing the machining performance, since it can lead to either deteriorated machined surface quality, workpiece precision, or even generation of instable machining process.

The modern machine tool, as a typical mechatronic system, features a number of interactions between the control and mechanics [5]. Achievable parameters of both the CNC control and feed drive control as well relate directly to mechanical eigenfrequencies and damping of the mechanical system. Commonly, the setting of the control parameters is defined by the most compliant configuration of the machine tool mechanics, and thus the potential of the structural dynamics with respect to control remains not fully utilized. In the case of ball screw driven motion axes, the above mentioned issue relates predominantly to active length of the ball screw. Next to it, structural modal properties of large machine tools may vary significantly also depending on the kinematical configuration and actual position of the motion axes.

Relevant capturing of the machine tool structural properties represents an important tasks in virtual modeling. Coupled modeling techniques employing the reduced FE models of the machine tool structure have been developed for describing the ball screw feed drive and machine tool interaction. Commonly the coupled models rely on one kinematic configuration and application of modal or CMS reduction techniques [6–9]. Importance of proper prediction of higher frequency range dynamics of feed drive mechanical structure with respect to feed drive control parameters shows [10]. Using the coupled models, machining process stability and performance may be predicted by the oriented directional frequency response functions (FRFs) at tool center point (TCP) [11].

For updating of machine tool structural modal properties, a technique based on continuous redistribution of contact force along the motion axis linear guideway has been

developed in [12]. Another approach employing an additionally created FE node moving between the end positions of the motion axes stroke has been described in [13].

The finite element method (FEM) is used as a common way to simulate the dynamic behavior of a machine tool. The solution of FE problems usually involves the solution of a large set of sparse algebraic equations. The number of unknowns in ordinary engineering FE model nowadays is commonly 10^6-10^7. The computational time needed to solve static analysis of such problems on current hardware is acceptably low (~hours). The issue arises when the harmonic or transient analysis is required. The solution then requires 10 s or 100 s of iterations and therefore requires days or even weeks to solve. Such long solution time effectively hinders this kind of simulations in machine tool virtual model simulations. The model order reduction (MOR) methods are one of possible ways to speed up the solution and make it feasible in practice. The idea behind MOR is to reduce the number of unknowns while producing sufficiently good approximation to the input/output behavior.

The following text gives a basic review of model order reduction techniques. The idea of reduction of the number of unknowns during the solution of the FE model is almost as old as the FE method itself. The first MOR method was static reduction proposed by Guyan [14] and Irons [15]. This method was introduced for structural mechanics problems but it is also valid for thermal analysis and other analyses regardless of the underlying physics. However, this method is of questionable quality when using it for dynamic analyses as was shown in [16, 17].

To remedy the insufficiencies in the static condensation method, the component mode synthesis (CMS, [18]) was proposed by Bampton and Craig. The CMS has become widely used by the engineering community. The CMS was used to efficiently conduct not only large-scale structural eigenanalysis [19] but also transient heat conduction analysis [20] and heat conduction/convection analysis [21]. Another field of application of the CMS is coupled physics simulations. The weakly coupled thermomechanical models were studied in [22]. There is still active research regarding improvement of the CMS [23].

Another method enhanced to approximate dynamic systems well is the improved reduced system (IRS) proposed by O'Callahan in [24]. Later, Friswell et al. developed an iterated version of IRS in [25]. The static condensation, CMS, and IRS can be viewed as engineering approaches to reduce the number of equations.

The global error bounds and the preservation of passivity and stability are important questions posed on the MOR methods in a more mathematical point of view. Two of the MOR techniques proposed in accordance with these questions are Krylov subspace reduction [26] and Balanced truncation [26]. Balanced truncation methods [27] have a great advantage because there exists a priori global error bound. But it also has a great disadvantage in that the Lyapunov equation [28] needs to be solved in order to reduce the system. Thus the usage of balanced truncation in reduction of large-scale systems is limited.

The Krylov subspace methods [26, 29–31] are very interesting because of their iterative nature which allows the reduction of large-scale problems. The computational efficiency of Krylov subspace based MOR has encouraged wide interest in the method, and therefore wide knowledge in different fields is available. We will mention the most important observations to date. The passivity and stability preservation has been achieved using the Krylov MOR methods in [26, 32]. A Krylov algorithm preserving structure of second order ordinary differential equations has been presented in [33]. Handling of nonlinear convection coefficient was studied in [34]. Reduction of coupled physics problems was studied in [35] for the case of a thermomechanical model of packages and in [36] for the case of structural-acoustic coupled models. Krylov subspace MOR was also successfully used in optimization of MEMS devices [37] and sensitivity analysis of structural frequency response [38]. One of the most important directions in development on the Krylov base reductions is a parametric model order reduction (PMOR). The PMOR allows preservation of parameters which the system depends on [39, 40]. The dependence of parameters may be either linear or nonlinear.

The comparison of different model order reduction methods has been discussed in [17], where the Krylov subspace MOR method was found to be one of the best methods. The comparison of Krylov, CMS, and balanced truncation can be found in [41].

Based on the findings presented in previous works the Krylov subspace based MOR is very robust and computationally efficient. The goal of this study is to show a new strategy of employing the Krylov MOR technique for creating the coupled models of compliant systems. The idea is to reduce each part of the machine tool structure separately and to produce the coupled model of the whole assembly. This way enables coupling of reduced components in any kinematic configuration. To assess the quality of Krylov MOR and to show its soundness the following comparison properties will be evaluated:

(1) low error in approximation,

(2) fast computation.

A low error in the approximation of a full FE model is required, as the objective is to replace the full FE model in simulations. Fast computations are required to meet the requirements in virtual machine tools simulations.

The proposed procedure will be tested on industrial scale (10^6-10^7 DOFs) FE model of machine tool. The harmonic and transient simulations on such large models take a lot of time (days, weeks), and therefore some kind of acceleration is usually used. Mode truncation [42] is frequently used for fast simulations and will be compared with Krylov based MOR. Both methods will be employed in reduction of individual parts of machine tool. The reduced part will be then coupled in any kinematic position. The purpose is to simulate flexible multibody systems accurately and very fast (with possible applications in real-time CNC control simulations). The proposed method of coupling the Krylov based reduced substructures has not yet been reported in the literature, to the best the authors' knowledge.

The paper is organized in the following way: Section 1 contains the introduction and the motivation of work; approach to substructuring is considered in Section 2; Section 3 contains the description of Krylov subspace reduction; Section 4 introduces the used machine tool model; and Section 5 presents the comparison of the results obtained by the methods. Conclusion suggestions on future work are given in Section 6.

2. Substructuring

One way to couple dynamic systems is to connect their mass $\mathbf{M}_s$, damping $\mathbf{C}_s$, and stiffness $\mathbf{K}_s$ matrices; external force vectors f_s, and internal force vectors g_s by coupling equations [43]; the subscript s denotes the systems being coupled. This approach is similar to the global matrix assembly in FE software. The equations of motion of n coupled subsystems can be written as

$$\mathbf{M}\ddot{x} + \mathbf{C}\dot{x} + \mathbf{K}x = f + g. \tag{1}$$

The matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are diagonal matrices containing the subsystem matrices. Vectors f and g are column vectors containing the subsystem internal and external force vectors, respectively.

Consider

$$\begin{gathered}\mathbf{M} = \operatorname{diag}\left(\mathbf{M}^{(1)}, \ldots, \mathbf{M}^{(n)}\right),\\ \mathbf{C} = \operatorname{diag}\left(\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(n)}\right),\\ \mathbf{K} = \operatorname{diag}\left(\mathbf{K}^{(1)}, \ldots, \mathbf{K}^{(n)}\right),\\ x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(n)} \end{bmatrix}, \quad f = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(n)} \end{bmatrix}, \quad g = \begin{bmatrix} g^{(1)} \\ \vdots \\ g^{(n)} \end{bmatrix}.\end{gathered} \tag{2}$$

The compatibility condition (coupling equations) is expressed as

$$\mathbf{B}x = 0, \tag{3}$$

$\mathbf{B}$ is a Boolean if the interface degrees of freedom match (interfaces are conforming). In this case, the coupling equations are very simple $x^{(k)} - x^{(l)} = 0$. If the connected degrees of freedom do not coincide, the relations among them are more complex and the matrix $\mathbf{B}$ is real.

Equilibrium condition is given by

$$\mathbf{L}^T g = 0, \tag{4}$$

where $\mathbf{L}$ is a Boolean matrix describing the relations among the interface forces.

Equations (1), (2), (3), and (4) can be now used to couple any number of subsystems. Before it can be done, however, the unknown interface forces have to be eliminated:

$$x = \mathbf{L}x_u. \tag{5}$$

Here x_u is a vector of unique DOFs. Because (5) indicates that the substructures DOF are obtained from the unique set x_u, the compatibility equation (3) is satisfied for any set x_u:

$$\mathbf{B}x = \mathbf{B}\mathbf{L}x_u = 0 \quad \forall x_u. \tag{6}$$

This means that $\mathbf{L}$ is null space of $\mathbf{B}$ and vice versa. And it is only necessary to define one of the matrices $\mathbf{L}$ and $\mathbf{B}$:

$$\begin{gathered}\mathbf{L} = \operatorname{null}(\mathbf{B}),\\ \mathbf{B}^T = \operatorname{null}\left(\mathbf{L}^T\right).\end{gathered} \tag{7}$$

Substituting (5) into (1), one gets equation of motion

$$\mathbf{M}\mathbf{L}\ddot{x}_u + \mathbf{C}\mathbf{L}\dot{x}_u + \mathbf{K}\mathbf{L}x_u = f + g. \tag{8}$$

Further multiplication of (8) from the left-hand side by $\mathbf{L}^T$ yields the final form of the equation of motion of the coupled system

$$\widetilde{\mathbf{M}}\ddot{x}_u + \widetilde{\mathbf{C}}\dot{x}_u + \widetilde{\mathbf{K}}x_u = \tilde{f}, \tag{9}$$

where matrices $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{C}}$, and $\widetilde{\mathbf{K}}$ and vector $\tilde{f}$ are expressed as

$$\begin{gathered}\widetilde{\mathbf{M}} \triangleq \mathbf{L}^T\mathbf{M}\mathbf{L},\\ \widetilde{\mathbf{C}} \triangleq \mathbf{L}^T\mathbf{C}\mathbf{L},\\ \widetilde{\mathbf{K}} \triangleq \mathbf{L}^T\mathbf{K}\mathbf{L},\\ \tilde{f} \triangleq \mathbf{L}^T f.\end{gathered} \tag{10}$$

Coupling of Reduced Components. Let us consider the case of n independent substructures, each reduced using projection matrix $\mathbf{R}^{(i)}$:

$$x \cong \mathbf{R}\eta, \quad \mathbf{R} \triangleq \operatorname{diag}\left(R^{(1)} \cdots R^{(n)}\right), \tag{11}$$

where x are physical coordinates of substructures and η are generalized coordinates of substructures. Substituting (11) into (1) we get

$$\mathbf{M}^{(s)}\mathbf{R}\ddot{\eta}^{(s)} + \mathbf{C}\mathbf{R}\dot{\eta}^{(s)} + \mathbf{K}^{(s)}\mathbf{R}\eta^{(s)} = f^{(s)} + g^{(s)} + r^{(s)}, \tag{12}$$

where $r^{(s)}$ are residual forces due to model order reduction. Following Galerkin method, we enforce $\mathbf{R}^{(s)^T} r^{(s)} = 0$. The equations of motion in generalized coordinates η have the form

$$\mathbf{M}_m\ddot{\eta} + \mathbf{C}_m\dot{\eta} + \mathbf{K}\eta_m = f_m + g_m, \tag{13}$$

where

$$\begin{gathered}\mathbf{M}_m \triangleq \mathbf{R}^T\mathbf{M}\mathbf{R},\\ \mathbf{C}_m \triangleq \mathbf{R}^T\mathbf{C}\mathbf{R},\\ \mathbf{K}_m \triangleq \mathbf{R}^T\mathbf{K}\mathbf{R},\\ f_m \triangleq \mathbf{R}^T f,\\ g_m \triangleq \mathbf{R}^T g.\end{gathered} \tag{14}$$

Then following, procedure for coupling full matrices (3)–(10), we get coupled reduced system. This procedure allows reducing substructures independently and then coupling of reduced systems. The coupling interface is defined using matrix **B** (3), and therefore the specification of the coupling interface is arbitrary.

3. Model Order Reduction Methods of Second Order Systems

The underlying physics of the problem is described by a system of linear second order ordinary differential equations with constant coefficients

$$\begin{gathered}\mathbf{M}\ddot{x}(t)+\mathbf{C}\dot{x}(t)+\mathbf{K}x(t)=\mathbf{F}u(t),\\ y(t)=\mathbf{L}^{T}x(t),\\ x(0)=x_{0},\qquad \dot{x}(0)=x_{1},\end{gathered}\tag{15}$$

where $x(t) \in R^N$ is displacement vector of state variables, $u(t) \in R^N$ is input function, and $y(t) \in R^m$ is output function. The matrices **M**, **C**, and $\mathbf{K} \in R^{N\times N}$ are mass matrix, damping matrix and stiffness matrix. $\mathbf{F} \in R^{N\times l}$, and $\mathbf{L} \in R^{N\times m}$ are input and output matrices. $x_0, x_1 \in R^N$ are initial conditions. The system matrices **M**, **C**, and **K** are constant therefore the system (15) is a linear time invariant system (LTI).

3.1. Mode Truncation. One of the less complex methods of model order reduction is modal truncation method [42]. The projection matrix **T** is in this case obtained from modal analysis performed on the full model. The diagram of model reduction by modal truncation can be seen in Figure 1.

Figure 1: Diagram of modal truncation [42].

The equation of motion of the full model is

$$\mathbf{M}\ddot{x}+\mathbf{B}_{v}\dot{x}+\mathbf{K}x=f,\tag{16}$$

where **M** is the mass matrix, **B** is damping matrix, **K** is stiffness matrix, f is force vector and x is the displacement vector.

The projection matrix **T** consists of the system's mode shapes; each column of **T** corresponds to one mode shape V_i and each row represents a degree of freedom:

$$\mathbf{T}=\left[V_{1}\cdots V_{n}\right].\tag{17}$$

The vector x can be transformed to vector of modal coordinates using

$$x=\mathbf{T}q.\tag{18}$$

This equation can be then substituted to (16) and after multiplying the result by $\mathbf{T}^T$ from the left-hand side, one obtains the equation of motion in the full system in the modal coordinates of

$$\mathbf{E}\ddot{q}+\mathbf{C}_{q}\dot{q}+\mathbf{\Lambda}q=\mathbf{T}^{T}f,\tag{19}$$

where **E** is identity matrix, $\mathbf{C}_q$ is damping ratio matrix, and $\mathbf{\Lambda}$ is matrix of eigenvalues. The equations in (19) are independent because the matrices **E**, $\mathbf{C}_q$, and $\mathbf{\Lambda}$ are diagonal. This means that the whole system of equations can be looked at as a sum of many one-degree-of-freedom systems (Mode superposition).

The model order is reduced by including only a small number of mode shapes and degrees of freedom in (19). The number of mode shapes included depends on the required accuracy of the reduced model. It is recommended that the natural frequency of the last included mode shape is at least twice as high as the highest working frequency of the resulting reduced model. The number of included DOFs does not affect the model accuracy so only the important ones are included.

One disadvantage of this method is the fact that the reduced model's FRF never fits the full model's FRF near 0 Hz. Another drawback is the relatively long time needed to create the reduced model (modal analysis with extraction of lots of modes has to be performed). The method is on the other hand simple and accurate in chosen frequency range.

3.2. Krylov MOR. In this section only the basics behind Krylov reductions will be described. The reader is encouraged to read an excellent mathematical description of Krylov based reductions in [26]. An overview of the reduction methods is given in [27]. Although optimal Krylov based reduction algorithms are available [44], a simpler and a possibly more computationally efficient method will be used in this work—a block rational Krylov method [45]. The structure of second order ODEs in (15) will be preserved using Bai's algorithm [33].

3.2.1. The Treatment of Initial Conditions. Let us consider using the following coordinate transformation in case of (15):

$$x(t) = \tilde{x}(t) + x_0,$$
$$\dot{x}(t) = \dot{\tilde{x}}(t) + x_1, \tag{20}$$
$$\ddot{x}(t) = \ddot{\tilde{x}}(t).$$

Substituting (20) into (15),

$$\mathbf{M}\ddot{\tilde{x}}(t) + \mathbf{C}\dot{\tilde{x}}(t) + \mathbf{K}\tilde{x}(t) = \mathbf{F}u(t) - \mathbf{C}x_1 - \mathbf{K}x_0. \tag{21}$$

Right-hand side of (21) is enriched with the constant term $-\mathbf{C}x_1 - \mathbf{K}x_0$, and we may consider this term as additional force. Then we may assume initial conditions (15) without loss of generality in form of

$$x(0) = 0, \qquad \dot{x}(0) = 0. \tag{22}$$

This kind of treatment of a nonzero IC was introduced in [46].

3.2.2. Krylov Subspace Based MOR. The Laplace transform of (15) has the form of

$$\mathbf{H}(s) = \mathbf{L}^T\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{F}. \tag{23}$$

And the McLaurin series of transfer function (23) has the form

$$\mathbf{H}(s) = \sum_{i=0}^{\infty} m_l s^l, \tag{24}$$

where m_l are the so-called moments of the transfer function

$$m_l = \mathbf{L}^T r_l,$$
$$r_0 = \mathbf{K}^{-1}\mathbf{F},$$
$$r_1 = -\mathbf{K}^{-1}\mathbf{D}r_0, \tag{25}$$
$$r_l = -\mathbf{K}^{-1}\left(\mathbf{D}r_{l-1} - \mathbf{M}r_{l-2}\right).$$

The first n vectors r_l span Krylov space

$$\mathbf{K}_n = \operatorname{span}\left(r_0, \ldots, r_{n-1}\right). \tag{26}$$

Letting $\mathbf{V}_n$ be the orthonormal basis of $\mathbf{K}_n$

$$\mathbf{K}_n = \operatorname{span}\left(\mathbf{V}_n\right), \quad \mathbf{V}_n^T\mathbf{V}_n = \mathbf{I}, \quad \mathbf{V}_n \in R^{N\times n}. \tag{27}$$

The projection of state coordinates x onto $\mathbf{K}_n$ using $\mathbf{V}_n$ is called generalized state coordinates $q \in R^n$

$$x = \mathbf{V}q + \epsilon. \tag{28}$$

The error $\epsilon \in R^n$ in the projection rises while performing projection of x onto $\mathbf{K}_n$ (Figure 2).

We obtain reduced system of (29) by substituting generalized coordinates q into (15) and using the Galerkin method. The reduced equations have the form of

$$\mathbf{M}_n\ddot{q}(t) + \mathbf{C}_n\dot{q}(t) + \mathbf{K}_n q(t) = \mathbf{F}_n u(t),$$
$$\tilde{y}(t) = \mathbf{L}_n^T q(t), \tag{29}$$

Figure 2: Projection onto $\mathbf{K}_n$.

where

$$\mathbf{M}_n = \mathbf{V}_n^T\mathbf{M}\mathbf{V}_n,$$
$$\mathbf{K}_n = \mathbf{V}_n^T\mathbf{K}\mathbf{V}_n,$$
$$\mathbf{C}_n = \mathbf{V}_n^T\mathbf{C}\mathbf{V}_n, \tag{30}$$
$$\mathbf{F}_n = \mathbf{V}_n^T\mathbf{F},$$

where $\mathbf{M}_n$, $\mathbf{C}_n$, and $\mathbf{K}_n \in R^{n\times n}$, $\mathbf{F} \in R^n$, and $\mathbf{L}_n \in R^{n\times m}$.

The transfer function of the reduced system (30) has the form

$$\mathbf{H}_n(s) = \mathbf{L}_n^T\left(s\mathbf{C}_n + \mathbf{K}_n\right)^{-1}\mathbf{Q}_n. \tag{31}$$

The above procedure assures that the first n moments of the transfer function (23) of the full system equal the first n moments of the transfer function (31) of the reduced system [26].

The error induced by the projection (28) in the output function $y(t)$ has the form

$$\epsilon = \max_{t>0}\left|y(t) - \tilde{y}(t)\right|. \tag{32}$$

An a priori expression for error norm (32) is not known although there exist algorithms minimizing the error [44, 47]. The algorithm used in this paper to produce the reduced order systems is the block Arnoldi algorithm [45].

There exist wide possibilities to improve the computational performance of Krylov methods. One of the most obvious options is parallelization [48]. Another is to use an iterative algorithm to solve the system [49]. The presented case is of medium size, and it is therefore suitable to use the direct sparse solver [50].

The procedure is easily extended to a multi-input/multi-output case where $\mathbf{Q}, \mathbf{F} \in R^{N\times l}$ and $\mathbf{L} \in R^{N\times m}$. The size of the reduced system is determined by the size of $\mathbf{Q}$ and $\mathbf{L}$. However, it is possible to use the superposition property [51] to keep the matrices small.

4. Machine Tool Model for the Case Study

The dynamic performance of a machine tool may be evaluated at the TCP using the oriented directional FRFs, which directly relate according to stability theory to achievable

Figure 3: Machine tool FE model.

Figure 4: Multibody system of the machine tool structure.

chip thickness [52–54]. FRF can be obtained by harmonic simulation of either a full FE model or reduced models.

The purpose of the study is to perform mathematical verification of the FRFs generated at the TCP using a full FE model, or reduced order models. Modal reduction technique and Krylov subspace method are considered. At the same time, the study aims at testing the suitability of reduction methods for creating a machine tool multibody model, assembled from reduced FE models of separate structural parts. In this way, a model allowing for quick update of machine tool dynamic properties according to actual kinematic configuration is created.

A model of a large portal milling machine tool is considered. The FE model (Figure 3) consists of the volume, shell, spring, matrix, and mass elements. The mesh consists of about $1{,}5 \cdot 10^6$ nodes; the total number of degrees of freedom number is almost $6 \cdot 10^6$. The model was built in ANSYS v 14.5. Linear guideways are represented by linear spring elements; similarly the rack and pinion feed drives are modeled using linear spring elements, which reflect the closed position control.

Figure 4 shows the decomposition of the FE model into separate structural parts with a description of the type of coupling chosen according to the character of connections between the parts. The parts are connected either through the faces—typically, for example, connection of the column with the X-slide (face coupling)—or by force interaction in the case of linear guides (symbols of springs in Figure 4).

Table 1: Comparison of computational times.

	MOR	FRF simulation	Total time
Full harmonic	—	111 hours	111 hours
Mode truncation	40 min	<1 s	40 min
Krylov MOR	6,9 min	<1 s	6,9 min

5. Results

The following case studies compare computational efficiency of full harmonic in ANSYS software package and model order reduction using mode truncation and Krylov MOR. To assess the quality of results obtained using MOR the error norm is evaluated. The study is performed in one kinematic configuration, and it is assumed that properties of MOR techniques will be similar in other kinematic configurations. The model used in the study is a multibody system.

5.1. The Harmonic Simulation of Machine Tool. The frequency response function at the tip of the tool has been computed using a full FEA model and MOR using mode truncation with first 100 modes for each reduced component and Krylov subspace MOR with first 100 moments. Additionally FRF computed using modal reduction is provided for comparison. The comparison is shown in Table 1. The error norm is concluded to be

$$\begin{aligned} \epsilon_{\text{Mode Truncation}} &= \max_{f \in (0,400)} \left| y(f) - \tilde{y}(f) \right| > 4e-5, \\ \epsilon_{\text{KRYLOV}} &= \max_{f \in (0,400)} \left| y(f) - \tilde{y}(f) \right| < 4e-6. \end{aligned} \tag{33}$$

The Table 1 shows comparison of methods in terms of time needed to produce the reduced order model, time needed to computed FRF, and total time. The time needed to produce coupled reduced components is added to the time required to compute FRF. The reduced models are computed only once and then it is possible to couple them in any kinematic configuration. The coupling is computationally very cheap.

Figures 5 and 6 show FRFs in axes x and y. The results clearly show the quality of approximation obtained using Krylov MOR as well as computational efficiency compared to the full solution and modal reduction. The Krylov MOR is about 1000x more computationally efficient than the full solution, and the approximation error is almost negligible. The mode truncation is computationally less efficient and also the approximation error on frequencies higher than 50 Hz is very high rendering this method almost unusable.

6. Conclusions

In the paper a novel application of Krylov subspace based model order reduction method for coupling of compliant components, modelled as FE bodies, has been proposed and successfully tested. To the best of the authors' knowledge,

Figure 5: Comparison of FRF, X-axis.

Figure 6: Comparison of FRF, Y-axis.

coupling of arbitrary substructures reduced using Krylov MOR has not yet been reported in the literature. The proposed method has been tested on a dynamic simulation of multibody system of the machine tool structure.

The case study has shown superior quality of the Krylov subspace based MOR in comparison to mode truncation technique for creating the machine tool multibody model, in which each of the structural parts is reduced separately and coupled to each other using either faced, or force coupling. The multibody model employing the Krylov subspace base MOR features almost identical match of the dynamic properties with the full FE model. At the same time, Krylov subspace method is very time efficient and proves to provide a very effective tool for quick model updates related to varying structural dynamics with respect to various kinematic configurations or testing of different design variants in the machine tool development. The Krylov subspace will be next used for creating a multibody model with movable motion axes. Precise capturing of the varying dynamic properties will improve the quality of tool oscillation simulation and will help increasing the accuracy of machined surface quality and workpiece precision in virtual machining. Low time requirements of the Krylov subspace reduction technique set also a good prerequisite for possible future real-time applications of machine tool dynamic virtual model as an observer in a real CNC control.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research has been supported by the Competence Center-Manufacturing Technology Project TE01020075 funded by the Technology Agency of the Czech Republic.

References

[1] Y. Altintas, C. Brecher, M. Weck, and S. Witt, "Virtual machine tool," *CIRP Annals—Manufacturing Technology*, vol. 54, no. 2, pp. 115–138, 2005.

[2] Y. Altintas, "Virtual machining system," in *Proceedings of the 4th International Conference and Exhibition on Design and Production of Machines and Dies/Molds*, Cesme, Turkey, 2007.

[3] M. F. Zaeh, M. Hennauer, and A. Poehler, "The machine tool of tomorrow: an optimised, complete system," in *Proceedings of the 6th International Conference on High Speed Machining*, San Sebastian, Spain, 2007.

[4] G. Pritschow and N. Croon, "Wege zur virtuellen werkzeugmaschine," *Wt Werkstattstechnik*, vol. 92, no. 5, pp. 194–199, 2002.

[5] J. Veselý, M. Sulitka, and J. Hornych, "Simulated impact of machine tool CNC system and feed drive control on workpiece quality," in *Proceedings of the 2nd International conference Process Machine Interactions*, Vancouver, Canada, 2010.

[6] M. Zatarain, E. Lejardi, and F. Egaña, "Modular synthesis of machine tools," *CIRP Annals—Manufacturing Technology*, vol. 47, no. 1, pp. 333–336, 1998.

[7] P. De Fonseca, D. Vandepitte, H. Van Brussel, and P. Sas, "Dynamic model reduction of a flexible three-axis milling machine," in *Proceedings of the International Seminar on Modal Analysis (ISMA '98)*, pp. 1197–1206, Leuven, Belgium, September 1998.

[8] R. Maj and G. Bianchi, "Mechatronic analysis of machine tools," in *Proceedings of the 9th SAMTECH Users Conference*, Paris, France, February 2005.

[9] M. Sulitka and P. Strakoš, "Complex model of a real machine tool feed drive axis with ball screw," in *Proceedings of the Conference VIDA*, Poznan, Poland, 2007.

[10] J. Vesely and M. Sulitka, "Machine tool virtual model," in *Proceedings of the International Conference on Machine Tools, Automation, Robotics and Technology (MATAR '08)*, Prague, Czech Republic, 2008.

[11] J. J. Zulaika, F. J. Campa, and L. N. Lopez de Lacalle, "An integrated processmachine approach for designing productive and lightweight milling machines," *International Journal of*

Machine Tools and Manufacture, vol. 51, no. 7-8, pp. 591–604, 2011.

[12] M. Zaeh and D. Siedl, "A new method for simulation of machining performance by integrating finite element and multi-body simulation for machine tools," *CIRP Annals—Manufacturing Technology*, vol. 56, no. 1, pp. 383–386, 2007.

[13] R. Neugebauer, M. Kolouch, M. Richter, V. Wittstock, and S. N. Saha, "Model updating of machine tools," in *Proceedings of the 9th International Conference on Machine Tools, Automation, Robotics and Technology (MATAR '12)*, Prague, Czech Republic, September 2012, (MM Science Journal 2012, Special Issue).

[14] R. J. Guyan, "Reduction of stiffness and mass matrices," *AIAA Journal*, vol. 3, p. 380, 1965.

[15] B. M. Irons, "Structural eigenvalue problems: elimination of unwanted variables," *AIAA Journal*, vol. 3, no. 5, pp. 961–962, 1965.

[16] L. B. Bushard, "On the value of Guyan reduction in dynamic thermal problems," *Computers & Structures*, vol. 13, no. 4, pp. 525–531, 1981.

[17] P. Koutsovasilis and M. Beitelschmidt, "Comparison of model reduction techniques for large mechanical systems," *Multibody System Dynamics*, vol. 20, no. 2, pp. 111–128, 2008.

[18] M. Bampton and J. R. Craig, "Coupling of substructures for dynamic analyses," *AIAA Journal*, vol. 6, no. 7, pp. 1313–1319, 1968.

[19] Y. Aoyama and G. Yagawa, "Component mode synthesis for large-scale structural eigenanalysis," *Computers & Structures*, vol. 79, no. 6, pp. 605–615, 2001.

[20] D. Botto, S. Zucca, M. Gola, E. Troncarelli, and G. Pasquero, "Component modes synthesis applied to a thermal transient analysis of a turbine disc," in *Proceedings of the 3rd Worldwide Aerospace Conference & Technology Showcase*, Toulouse, France, 2002.

[21] D. Botto, S. Zucca, and M. M. Gola, "Reduced-order models for the calculation of thermal transients of heat conduction/convection FE models," *Journal of Thermal Stresses*, vol. 30, no. 8, pp. 819–839, 2007.

[22] P. Nachtergaele, D. J. Rixen, and A. M. Steenhoek, "Efficient weakly coupled projection basis for the reduction of thermo-mechanical models," *Journal of Computational and Applied Mathematics*, vol. 234, no. 7, pp. 2272–2278, 2010.

[23] Z. Bai and B.-S. Liao, "Towards an optimal substructuring method for model reduction," in *Applied Parallel Computing. State of the Art in Scientific Computing*, J. Dongarra, K. Madsen, and J. Waśniewski, Eds., vol. 3732 of *Lecture Notes in Computer Science*, pp. 276–285, Springer, Berlin, Germany, 2006.

[24] J. O'Callahan, "A procedure for an improved reduced system (IRS) model," in *Proceedings of the 7th International Modal Analysis Conference (IMAC '89)*, Las Vegas, Nev, USA, 1989.

[25] M. I. Friswell, S. D. Garvey, and J. E. T. Penny, "Model reduction using dynamic and iterated IRS techniques," *Journal of Sound and Vibration*, vol. 186, no. 2, pp. 311–323, 1995.

[26] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, Advances in Design and Control, SIAM, Philadelphia, Pa, USA, 2005.

[27] A. C. Antoulas and D. C. Sorensen, "Approximation of large-scale dynamical systems: an overview," *International Journal of Applied Mathematics and Computer Science*, vol. 11, no. 5, pp. 1093–1121, 2001.

[28] T. Stykel, *Analysis and numerical solution of generalized Lyapunov equations [Ph.D. thesis]*, Technical University of Berlin, Berlin, Germany, 2002.

[29] E. Grimme, *Krylov projection methods for model reduction [Ph.D. thesis]*, ECE Department, University of Illinois, Urbana-Champaign, Ill, USA, 1997.

[30] E. Grimme and K. Gallivan, "Krylov projection methods for rational interpolation," 1997 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.54.5011&rep=rep1&type=pdf.

[31] Z. Bai, "Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems," *Applied Numerical Mathematics*, vol. 43, no. 1-2, pp. 9–44, 2002.

[32] A. Odabasioglu, M. Celik, and L. T. Pileggi, "PRIMA: passive reduced-order interconnect macromodeling algorithm," in *Proceedings of the IEEE/ACM International Conference on Computer-Aided Design (ICCAD '97)*, pp. 58–65, Washington, DC, USA, November 1997.

[33] Z. Bai, K. Meerbergen, and Y. Su, "Arnoldi methods for structure-preserving dimension reduction of second-order dynamical systems," in *Dimension Reduction of Large-Scale Systems*, P. Benner, D. Sorensen, and V. Mehrmann, Eds., vol. 45 of *Lecture Notes in Computational Science and Engineering*, pp. 173–189, Springer, Berlin, Germany, 2005.

[34] L. H. Feng, E. B. Rudnyi, J. G. Korvink, C. Bohm, and T. Hauck, "Compact electro-thermal model of semiconductor device with nonlinear convection coefficient," in *Proceedings of the 6th International Conference on Thermal, Mechanical and Multi-Physics Simulation and Experiments in Micro-Electronics and Micro-Systems (EuroSimE '05)*, pp. 372–375, Berlin, Germany, April 2005.

[35] E. Zukowski, J. Wilde, E.-B. Rudnyi, and J.-G. Korvink, "Model reduction for thermo-mechanical simulation of packages," in *Proceedings of the 11th International Workshop on Thermal Investigations of ICs and Systems (THERMINIC '05)*, pp. 134–138, Belgirate, Italy, 2006.

[36] R. S. Puri, D. Morrey, A. J. Bell, J. F. Durodola, E. B. Rudnyi, and J. G. Korvink, "Reduced order fully coupled structural-acoustic analysis via implicit moment matching," *Applied Mathematical Modelling*, vol. 33, no. 11, pp. 4097–4119, 2009.

[37] J. S. Han, E. B. Rudnyi, and J. G. Korvink, "Efficient optimization of transient dynamic problems in MEMS devices using model order reduction," *Journal of Micromechanics and Microengineering*, vol. 15, no. 4, pp. 822–832, 2005.

[38] J. S. Han, "Efficient frequency response and its direct sensitivity analyses for large-size finite element models using Krylov subspace-based model order reduction," *Journal of Mechanical Science and Technology*, vol. 26, no. 4, pp. 1115–1126, 2012.

[39] U. Baur, C. Beattie, P. Benner, and S. Gugercin, "Interpolatory projection methods for parameterized model reduction," *SIAM Journal on Scientific Computing*, vol. 33, no. 5, pp. 2489–2518, 2011.

[40] J. Šindler, A. Suleng, T. Jelstad Olsen, and P. Bárta, "Krylov model order reduction of a thermal Subsea model," *World Academy of Science, Engineering and Technology*, vol. 77, pp. 217–224, 2013.

[41] W. Witteveen, "Comparison of CMS, Krylov and balanced truncation based model reduction from a mechanical application engineer's perspective," in *Topics in Experimental Dynamics Substructuring and Wind Turbine Dynamics*, R. Mayes, D. Rixen, D. Griffith et al., Eds., vol. 2 of *Conference Proceedings of the Society for Experimental Mechanics Series*, pp. 319–331, Springer, New York, NY, USA, 2012.

[42] Z. Qu, *Model Order Reduction Techniques: With Applications in Finite Element Analysis*, Springer, New York, NY, USA, 2004.

[43] D. De Klerk, D. J. Rixen, and S. N. Voormeeren, "General framework for dynamic substructuring: history, review, and classification of techniques," *AIAA Journal*, vol. 46, no. 5, pp. 1169–1181, 2008.

[44] S. Gugercin, A. C. Antoulas, and C. Beattie, "H_2 model reduction for large-scale linear dynamical systems," *SIAM Journal on Matrix Analysis and Applications*, vol. 30, no. 2, pp. 609–638, 2008.

[45] I. M. Elfadel and D. D. Ling, "A block rational Arnoldi algorithm for multipoint passive model-order reduction of multiport RLC networks," in *Proceedings of the IEEE/ACM International Conference on Computer-Aided Design (ICCAD '97)*, pp. 66–71, Washington, DC, USA, November 1997.

[46] L. F. L. Feng, D. Koziol, E. Rudnyi, and J. Korvink, "Model order reduction for scanning electrochemical microscope: the treatment of nonzero initial condition," in *Proceedings of IEEE Sensors*, vol. 3, pp. 1236–1239, 2004.

[47] G. Flagg, C. Beattie, and S. Gugercin, "Convergence of the iterative rational Krylov algorithm," *Systems & Control Letters*, vol. 61, no. 6, pp. 688–691, 2012.

[48] E. F. Yetkin and H. Dağ, "Parallel implementation of iterative rational Krylov methods for model order reduction," in *Proceedings of the 5th International Conference on Soft Computing, Computing with Words and Perceptions in System Analysis, Decision and Control (ICSCCW '09)*, Famagusta, Cyprus, September 2009.

[49] C. A. Beattie and S. Gugercin, "Inexact solves in Krylov-based model reduction," in *Proceedings of the 45th IEEE Conference on Decision and Control (CDC '06)*, pp. 3405–3411, San Diego, Calif, USA, December 2006.

[50] T. Davis, *Direct Methods for Sparse Linear Systems*, Fundamentals of Algorithms, SIAM, Philadelphia, Pa, USA, 2006.

[51] P. Benner, L. Feng, and E. B. Rudnyi, "Using the superposition property for model reduction of linear systems with a large number of inputs," in *Proceedings of the 18th International Symposium on Mathematical Theory of Networks and Systems (MTNS '08)*, pp. 1–12, Blacksburg, Va, USA, 2008.

[52] Y. Altintas, *Manufacturing Automation: Metal Cutting Mechanics, Machine Tool Vibrations, and CNC Design*, Cambridge University Press, New York, NY, USA, 2000.

[53] Y. Altintas and E. Budak, "Analytical prediction of stability lobes in milling," *CIRP Annals—Manufacturing Technology*, vol. 44, no. 1, pp. 357–362, 1995.

[54] S. D. Merdol and Y. Altintas, "Multi frequency solution of chatter stability for low immersion milling," *Journal of Manufacturing Science and Engineering*, vol. 126, no. 3, pp. 459–466, 2004.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 308049, 9 pages
http://dx.doi.org/10.1155/2014/308049

Research Article

Study on Launch Dynamics of Self-Propelled Artillery Based on Transfer Matrix Method of Multibody System

Hailong Yu and Xiaoting Rui

Institute of Launch Dynamics, Nanjing University of Science and Technology, Nanjing 210094, China

Correspondence should be addressed to Xiaoting Rui; ruixt@163.net

Received 26 August 2013; Accepted 18 December 2013; Published 30 January 2014

Academic Editor: Ch Zhang

Copyright © 2014 H. Yu and X. Rui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Due to the importance of increasing the firing precision of self-propelled artillery system corresponding to the projectile impact point, the launch dynamics of self-propelled artillery based on the transfer matrix method for multibody system (MSTMM) is studied in this paper. By using MSTMM to study launch dynamics of self-propelled artillery, there is no need of global dynamical equation of the system, the order of the system matrix is low, the computational speed is high, and it meets the requirements of dynamics of self-propelled artillery multibody system. The dynamical model of multibody system and its topological figure, the overall transfer equation, overall transfer matrix, characteristics equation and generalized coordinates equation, and launch dynamical equation of self-propelled artillery are developed. The vibration characteristics, initial disturbance of projectile, dynamical response, and firing dispersion for self-propelled artillery are simulated. The results of eigenfrequency and the time history of system dynamics got by simulations and experiments have good agreements. The study results can be used to improve the firing precision and design of self-propelled artillery.

1. Introduction

The research results of launch dynamics provided the theoretical basis and technical means for design and test of various weapon systems [1–6]. The movements of projectile and artillery are very complicated because of the complex mechanical structure of self-propelled artillery and the severe mechanical environments, such as high temperature, high pressure, high speed, instantaneous state, multibody, and mutation in launch process. Because of their great influence on firing precision of self-propelled artillery, the vibration characteristics and dynamical response of the system are paid great attention in the study of launch dynamics. To increase firing precision of self-propelled artillery, it is necessary to compute accurately the frequencies distribution and establish the quantitative relationship among the global structure parameters of the system and its vibration characteristics, dynamical response, and firing precision. By adjusting the structure parameters to design system vibration frequency distribution, it is then able to improve the firing precision for self-propelled artillery system.

At present, the methods to study vibration characteristics of mechanical system are mainly finite element method, modal analysis method, and the structural modal synthesis method. The multibody system dynamics methods, such as Wittenburg method, Schiehlen method, and Kane method have been widely accepted and applied by engineer of mechanical system dynamics [7–9]. Finite element method and multibody system dynamics method have been an important basis of weapon dynamics. However, when calculating natural vibration characteristics of weapons for complex multirigid-flexible body system containing rigid and flexible bodies using these methods, it faces the difficulties that computational scale is large and it is easy to result in computation singularity caused by computation ill-condition. To meet the requirement of accurate dynamics modeling of launch dynamics and fast calculation, Rui et al. presented transfer matrix method for multibody systems [2, 10–20]. Due to the important feature that order of the overall transfer matrix is very low, MSTMM avoids eigenvalue computation ill-condition for complex multi-rigid-flexible body system, which significantly improves the computational speed of

Figure 1: Self-propelled artillery.

vibration characteristics and is successfully applied to many engineering design and test of various types of vehicles, airborne, and ship-borne weapons [1, 2, 4].

In this paper, by using MSTMM and its automatic deduction method [2, 20], the dynamical model of multibody system for self-propelled artillery is developed; the overall transfer equation, overall transfer matrix, and characteristics equation are deduced. Generalized coordinates equation of self-propelled artillery is obtained by using the orthogonality of augmented eigenvector and body dynamical equation. Combining with launch dynamical equation of projectile, launch dynamical equation of self-propelled artillery system is developed. The vibration characteristics, initial disturbance of projectile, dynamical response, and firing dispersion are simulated, and the factors of different ground conditions and connection stiffness among various components, which influence the firing precision and vibration characteristics, are analyzed. It provides a theoretical basis and simulation tool for improving the firing precision by adjusting structural parameters of the system to change the vibration characteristics of the self-propelled artillery.

2. Dynamical Model of Self-Propelled Artillery Multibody System

Main components of self-propelled artillery are muzzle brake, barrel, gun breech, recoil and counter-recoil mechanisms, cradle, elevating mechanism, equilibrator, turret, traversing mechanism, chassis, torsion bar, balance elbow, shock absorber, track chain, road wheel, and so on. According to the motion state of each component, the firepower system of self-propelled artillery can be divided into recoil part, elevating part, revolving part, suspension part, walking part, and so on. The recoil part contains muzzle brake, barrel, gun breech, and recoil and counter-recoil mechanisms. The elevating part contains total recoil part, cradle, and the components moving with cradle, which includes elevating mechanism, equilibrator, and so on. The revolving part contains total elevating part, turret, and the elements moving with turret that include traversing mechanism and so on. The walking part is used to support the weight of the self-propelled artillery and drive self-propelled artillery to run placidly, which contains track chain and road wheels. The suspension part is used to connect chassis to walking part, which contains torsion bar, balance elbow, and shock absorber. The self-propelled artillery is shown in Figure 1.

Taking self-propelled artillery as an example, according to its system structure, from bottom to top in sequence, it can be divided into road wheels, hull, and revolving part which does not contain elevating part, elevating part which does not contain recoil part, gun breech, and muzzle brake. Each component can be regarded as rigid body and elastic beam according to its natural attribute. Rigid body and elastic beam are called "body," the connection between "body" and "body" is called the "hinge," which can be seen in [2].

The ground that supports the self-propelled artillery is regarded as an infinity rigid body, whose sequence number is 0. The elastic and damping effect of each road wheel and the interaction between ground and each road wheel are, respectively, modeled as springs, rotary springs, and the accompanying dampers connected in parallel, which can represent relative linear motion and relative angular motion in 3 directions at the same time; their sequence numbers are $42, 43, \ldots, 53$. Each road wheel can be regarded as a rigid body, whose sequence number is $30, 31, \ldots, 41$. The interaction between each road wheel and hull are, respectively, modeled as springs, rotary springs, and the accompanying dampers connected in parallel, which can represent relative linear motion and relative angular motion in 3 directions at the same time; their sequence numbers are $18, 19, \ldots, 29$. Hull, revolving part, elevating part, gun breech, and muzzle brake can be regarded as a rigid body, respectively, whose sequence numbers are 17, 15, 13, 10, and 1. The effect of traversing mechanism associating the elastic, and damping effects of hull, the effect of elevating mechanism and equilibrator associating the elastic and damping effects between revolving part and elevating part are, respectively, modeled as springs and rotary springs accompanying dampers which can represent relative linear motion and relative angular motion in 3 directions at the same time, whose sequence numbers are 16, 14. The interaction between barrel and elevating par, is, respectively, modeled as springs and rotary springs accompanying dampers which can represent relative linear motion and relative angular motion in 3 directions at the same time, whose sequence numbers are 11, 12. According to its structure characteristics, the barrel is divided into 6 segments, each segment can be regarded as a beam with equal sectional area, whose sequence numbers are 2, 3, 4, 5, 7, and 9. The connection points among joints and beam can be regarded as massless rigid body, whose sequence numbers are 6, 8. The dynamical model of self-propelled artillery multi-rigid-flexible system is composed of 19 rigid bodies, 6 beams,

and 28 joints, as shown in Figure 2. The topology figure of dynamical model of the self-propelled system is shown in Figure 3.

3. Overall Transfer Equation of Self-Propelled Artillery

3.1. The State Vector of Self-Propelled Artillery. According to the dynamical model and its topology figure, the state vectors at boundary points for self-propelled artillery are defined as follows

$$\mathbf{Z}_{1,0} = \left[X, Y, Z, \Theta_x, \Theta_y, \Theta_z, M_x, M_y, M_z, Q_x, Q_y, Q_z\right]_{1,0}^T, \tag{1}$$

where, 1 is the sequence number of body and 0 denotes the boundary.

The form of $\mathbf{Z}_{10,0}$, $\mathbf{Z}_{11,13}$, $\mathbf{Z}_{13,0}$, $\mathbf{Z}_{42,0}$, $\mathbf{Z}_{43,0}$, $\mathbf{Z}_{44,0}$, $\mathbf{Z}_{45,0}$, $\mathbf{Z}_{46,0}$, $\mathbf{Z}_{47,0}$, $\mathbf{Z}_{48,0}$, $\mathbf{Z}_{49,0}$, $\mathbf{Z}_{50,0}$, $\mathbf{Z}_{51,0}$, $\mathbf{Z}_{52,0}$, and $\mathbf{Z}_{53,0}$ is similar to $\mathbf{Z}_{1,0}$.

3.2. Overall Transfer Equation of Self-Propelled Artillery System. According to MSTMM and the topology figure of the dynamical model, the overall transfer equation of self-propelled artillery system is automatically deduced as follows:

$$\mathbf{U}_{\text{all}}\mathbf{Z}_{\text{all}} = \mathbf{0}, \tag{2}$$

where overall transfer matrix

$$\mathbf{U}_{\text{all}} = \begin{bmatrix}
-\mathbf{I}_{12} & \mathbf{T}_{10-1} & \mathbf{T}_{42-1} & \mathbf{T}_{43-1} & \mathbf{T}_{44-1} & \mathbf{T}_{45-1} & \mathbf{T}_{46-1} & \mathbf{T}_{47-1} & \mathbf{T}_{48-1} & \mathbf{T}_{49-1} & \mathbf{T}_{50-1} & \mathbf{T}_{51-1} & \mathbf{T}_{52-1} & \mathbf{T}_{53-1} & \mathbf{T}_{13-1}+\mathbf{T}_{11-1} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{G}_{43-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{G}_{44-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{45-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{46-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{47-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{48-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{49-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{50-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{51-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{52-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-17} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{53-17} & \mathbf{0}_{6\times12} \\
\mathbf{0}_{6\times12} & \mathbf{0}_{6\times12} & \mathbf{G}_{42-13} & \mathbf{G}_{43-13} & \mathbf{G}_{44-13} & \mathbf{G}_{45-13} & \mathbf{G}_{46-13} & \mathbf{G}_{47-13} & \mathbf{G}_{48-13} & \mathbf{G}_{49-13} & \mathbf{G}_{50-13} & \mathbf{G}_{51-13} & \mathbf{G}_{52-13} & \mathbf{G}_{53-13} & \mathbf{G}_{13,I_2} \\
\mathbf{0}_{6\times12} & \mathbf{G}_{10-8} & \mathbf{G}_{42-8} & \mathbf{G}_{43-8} & \mathbf{G}_{44-8} & \mathbf{G}_{45-8} & \mathbf{G}_{46-8} & \mathbf{G}_{47-8} & \mathbf{G}_{48-8} & \mathbf{G}_{49-8} & \mathbf{G}_{50-8} & \mathbf{G}_{51-8} & \mathbf{G}_{8-52,b12} & \mathbf{G}_{53-8} & \mathbf{G}_{13-8} \\
\mathbf{0}_{6\times12} & \mathbf{G}_{10-6} & \mathbf{G}_{42-6} & \mathbf{G}_{43-6} & \mathbf{G}_{44-6} & \mathbf{G}_{45-6} & \mathbf{G}_{46-6} & \mathbf{G}_{47-6} & \mathbf{G}_{48-6} & \mathbf{G}_{49-6} & \mathbf{G}_{50-6} & \mathbf{G}_{51-6} & \mathbf{G}_{52-6} & \mathbf{G}_{53-6} & \mathbf{G}_{13-6}+\mathbf{G}_{11-6}
\end{bmatrix}. \tag{3}$$

The overall state vector of all boundary points is given by

$$\mathbf{Z}_{\text{all}} = \left[\mathbf{Z}_{1,0}, \mathbf{Z}_{10,0}, \mathbf{Z}_{42,0}, \mathbf{Z}_{43,0}, \mathbf{Z}_{44,0}, \mathbf{Z}_{45,0}, \mathbf{Z}_{46,0}, \mathbf{Z}_{47,0}, \mathbf{Z}_{48,0}, \mathbf{Z}_{49,0}, \mathbf{Z}_{50,0}, \mathbf{Z}_{51,0}, \mathbf{Z}_{52,0}, \mathbf{Z}_{53,0}, \mathbf{Z}_{13,0}\right]. \tag{4}$$

The elements of the overall transfer matrix $\mathbf{U}_{\text{all}}$ are determined in appendix.

4. Characteristics Equation of Self-Propelled Artillery System

In (4), $\mathbf{Z}_{\text{all}}$ is composed of the state vectors $\mathbf{Z}_{1,0}$, $\mathbf{Z}_{10,0}$, $\mathbf{Z}_{13,0}$, $\mathbf{Z}_{42,0}$, $\mathbf{Z}_{43,0}$, $\mathbf{Z}_{44,0}$, $\mathbf{Z}_{45,0}$, $\mathbf{Z}_{46,0}$, $\mathbf{Z}_{47,0}$, $\mathbf{Z}_{48,0}$, $\mathbf{Z}_{49,0}$, $\mathbf{Z}_{50,0}$, $\mathbf{Z}_{51,0}$, $\mathbf{Z}_{52,0}$, and $\mathbf{Z}_{53,0}$ at the system boundary points. For the state vectors at the boundary points, half of its elements have been identified using boundary conditions. For self-propelled artillery system, each state vector $\mathbf{Z}_{i,0}$ ($i = 42, 43, \ldots, 53$) between 12 road wheels and the ground contains 12 variables, including displacements, angle displacements, forces, and moments in 3 directions respectively, in which the 6 variables representing the displacements and angle displacements are always equal to 0; therefore, note symbol $\overline{\mathbf{Z}}_{i,0}$ ($i = 42, 43, \ldots, 53$) as the state vector that only includes the remnant variables after these 0 elements are removed from $\mathbf{Z}_{i,0}$ ($i = 42, 43, \ldots, 53$), which contains 6 variables. $\mathbf{Z}_{10,0}$ and $\mathbf{Z}_{1,0}$ are the state vectors at the boundary points of gun breech and muzzle. Each of them contains 12 variables, including displacements, angle displacements, forces, and moments in 3 directions, respectively. Both gun breech and muzzle are free boundary, therefore, the 6 variables representing the forces and moments are always equal to 0. In the same way, let the symbol $\overline{\mathbf{Z}}_{10,0}$ and $\overline{\mathbf{Z}}_{1,0}$ represent the remnant after these zeros are removed from $\mathbf{Z}_{10,0}$ and $\mathbf{Z}_{1,0}$, respectively, and each of them has 6 variables. In this case, there are 84 variables always equal to 0 in $\mathbf{Z}_{\text{all}}$; eliminating them from $\mathbf{Z}_{\text{all}}$, a new state vector $\overline{\mathbf{Z}}_{\text{all}}$ can be obtained, which has 96 variables, given by

$$\overline{\mathbf{Z}}_{\text{all}} = \left[\overline{\mathbf{Z}}_{1,0}, \overline{\mathbf{Z}}_{10,0}, \overline{\mathbf{Z}}_{42,0}, \overline{\mathbf{Z}}_{43,0}, \overline{\mathbf{Z}}_{44,0}, \overline{\mathbf{Z}}_{45,0}, \overline{\mathbf{Z}}_{46,0}, \overline{\mathbf{Z}}_{47,0}, \overline{\mathbf{Z}}_{48,0}, \overline{\mathbf{Z}}_{49,0}, \overline{\mathbf{Z}}_{50,0}, \overline{\mathbf{Z}}_{51,0}, \overline{\mathbf{Z}}_{52,0}, \overline{\mathbf{Z}}_{53,0}, \mathbf{Z}_{13,0}\right]. \tag{5}$$

Deleting the columns 7~12, 19~24, 25~30, 37~42, 49~54, 61~66, 73~78, 85~90, 97~102, 109~114, 121~126, 133~138, 145~150, 157~162 in $\mathbf{U}_{\text{all}}$, a new matrix $\overline{\mathbf{U}}_{\text{all}}$ with 96 orders can be obtained; (2) can be written as

$$\underset{96\times96}{\overline{\mathbf{U}}_{\text{all}}}\ \underset{96\times1}{\mathbf{Z}_{\text{all}}} = 0. \tag{6}$$

The components of $\mathbf{U}_{\text{all}}$ or $\overline{\mathbf{U}}_{\text{all}}$ are only decided by parameters and eigenfrequency of the system. When the system parameters are determined, (6) should have nonzero solution for any

Figure 2: Dynamical model of the self-propelled artillery multibody system.

Figure 3: Topology figure of the self-propelled system dynamical model.

eigenvalue of the system. Hence, the system characteristics equation is obtained by

$$\det\left(\overline{\mathbf{U}}_{\text{all}}\right) = 0. \tag{7}$$

By solving (7), the eigenfrequency of self-propelled artillery, ω_k $(k = 1, 2, \ldots)$, can be obtained. Solving (6), the state vector $\overline{\mathbf{Z}}_{\text{all}}^{k}$ corresponding to ω_k can be obtained, in other words, $\mathbf{Z}_{\text{all}}^{k}$ is obtained. That is to say, the state vectors $\mathbf{Z}_{1,0}$, $\mathbf{Z}_{10,0}$, $\mathbf{Z}_{13,0}$, $\mathbf{Z}_{42,0}$, $\mathbf{Z}_{43,0}$, $\mathbf{Z}_{44,0}$, $\mathbf{Z}_{45,0}$, $\mathbf{Z}_{46,0}$, $\mathbf{Z}_{47,0}$, $\mathbf{Z}_{48,0}$, $\mathbf{Z}_{49,0}$, $\mathbf{Z}_{50,0}$, $\mathbf{Z}_{51,0}$, $\mathbf{Z}_{52,0}$, and $\mathbf{Z}_{53,0}$ corresponding to ω_k are obtained. Based on these state vectors at the system boundaries, through using the transfer equation of each element one by one, all the state vectors in system including each connection points and an arbitrary point on beam (barrel) can be obtained. Thus, the vibration characteristics of self-propelled artillery are obtained, which include the eigenfrequency ω_k $(k = 1, 2, \ldots)$ and the mode shape corresponding to each ω_k, and the datum of mode shape is included in the variables of all state vectors of system, so that the mode shape can be got after the corresponding variables selected them from the state vectors.

It can be seen from (6) that the order of the overall transfer matrix of the self-propelled artillery system is only 96 and is much lower than that for other multibody system dynamics methods so computational speed is high and the computational ill-condition caused by high matrix order and large stiffness gradient is avoided.

5. Launch Dynamical Equation of Self-Propelled Artillery

5.1. Body Dynamical Equation of Self-Propelled Artillery. Body dynamical equation of self-propelled artillery can be written as

$$\mathbf{M}_j\mathbf{v}_{j,tt} + \mathbf{C}_j\mathbf{v}_{j,t} + \mathbf{K}_j\mathbf{v}_j = \mathbf{f}_j \quad (j = 1, 2, 3, 4, 5, 7, 9, 10, 13, 15, 17, 30, 31, \ldots, 41), \tag{8}$$

where j is the sequence number of all body elements of the self-propelled artillery and $\mathbf{M}_j$, $\mathbf{C}_j$, $\mathbf{K}_j$, $\mathbf{v}_j$, and $\mathbf{f}_j$ are, respectively, named as parameter matrix of mass, parameter matrix of damper, parameter matrix of stiffness, displacement matrix, and external force matrix, which can be seen in [2].

5.2. Dynamic Response of Self-Propelled Artillery System. The dynamic response in the physical coordinate system may be expanded using augmented eigenvectors $\mathbf{V}^{k_1}$ up to a selected modal order n,

$$\mathbf{v}_j = \sum_{k_1=1}^{+\infty} \mathbf{V}_j^{k_1} q^{k_1}(t). \tag{9}$$

Substituting (9) into (8), thus we obtain

$$\sum_{k_1=1}^{+\infty} \left(\mathbf{M}_j\mathbf{V}_j^{k_1}\right)\ddot{q}^{k_1}(t) + \sum_{k_1=1}^{+\infty} \left(\mathbf{C}_j\mathbf{V}_j^{k_1}\right)\dot{q}^{k_1}(t) + \sum_{k_1=1}^{+\infty} \left(\mathbf{K}_j\mathbf{V}_j^{k_1}\right) q^{k_1}(t) = \mathbf{f}_j. \tag{10}$$

Taking the inner product of both sides of (10) with $\mathbf{V}_j^k$ ($k = 1, 2, \ldots, n$), thus we obtain

$$\ddot{q}^k(t) + \frac{\sum_j \left\langle \sum_{k_1=1}^{+\infty} \left(\mathbf{C}_j\mathbf{V}_j^{k_1}\right)\dot{q}^{k_1}(t), \mathbf{V}_j^k \right\rangle}{d^k} + \omega_k^2 q^k(t) = \frac{\sum_j \left\langle \mathbf{f}_j, \mathbf{V}_j^k \right\rangle}{d^k}. \tag{11}$$

For proportional damping,

$$\mathbf{C}_j = \alpha\mathbf{M}_j + \beta\mathbf{K}_j. \tag{12}$$

There are

$$\sum_j \left\langle \sum_{k_1=1}^{+\infty} \left(\mathbf{C}_j\mathbf{V}_j^{k_1}\right)\dot{q}^{k_1}(t), \mathbf{V}_j^k \right\rangle = \left(\alpha + \beta\omega_k^2\right)\dot{q}^k(t)\, d^k. \tag{13}$$

Generalized coordinate equation of self-propelled artillery is obtained,

$$\ddot{q}^k(t) + \left(\alpha + \beta\omega_k^2\right)\dot{q}^k(t) + \omega_k^2 q^k(t) = p^k(t), \tag{14}$$

where

$$p^k(t) = \frac{\sum_j \left\langle \mathbf{f}_j, \mathbf{V}_j^k \right\rangle}{d^k} \quad (j = 1, 2, 3, 4, 5, 7, 9, 10, 13, 15, 17, 30, 31, \ldots, 41). \tag{15}$$

5.3. Launch Dynamical Equation of the Projectile. Launch dynamical equation of a projectile is given by [1, 5]

$$\begin{aligned}
a_p &= \frac{P_b S_b}{m\varphi_3} - \frac{\partial^2 x'}{\partial t^2} \\
\ddot{y}'_{oc} &= -g\cos\theta_1 - \frac{K}{m}\left[y'_{oo} - \mu z'_{oo}\sin\alpha\right] + \frac{F_y^{sf}}{m} - \ddot{y}'_o, \\
\ddot{z}'_{oc} &= \frac{-K}{m}\left[z'_{oo} + \mu y'_{oo}\sin\alpha\right] + \frac{F_z^{sf}}{m} - \ddot{z}'_o, \\
\ddot{\delta}_1^I &= -\frac{C}{A}\dot{\gamma}\left(\dot{\psi}_2^I + \dot{\delta}_2^I\right) + \left(1 - \frac{C}{A}\right)\left(\dot{\gamma}^2\beta_{D_\eta} + \ddot{\gamma}\beta_{D_\xi}\right) \\
&\quad - \frac{C\ddot{\gamma}}{A}\delta_2^I + \frac{P_b S_b}{A\varphi_3}y'_{oc} \\
&\quad - \frac{Kh^2}{12A}\left(\delta_1^I - \delta_2^I\mu\sin\alpha\right) + \frac{K\left(l_R + r_b\mu\right)}{A}y'_{oo} \\
&\quad - \frac{Kl_R\mu\sin\alpha}{A}z'_{oo} + \frac{l_1}{A}F_y^{sf} - \ddot{\psi}_1^I, \\
\ddot{\delta}_2^I &= \frac{C}{A}\dot{\gamma}\left(\dot{\psi}_1^I + \dot{\delta}_1^I\right) + \left(1 - \frac{C}{A}\right)\left(\dot{\gamma}^2\beta_{D_\xi} - \ddot{\gamma}\beta_{D_\eta}\right) \\
&\quad + \frac{C\ddot{\gamma}}{A}\delta_1^I + \frac{P_b S_b}{A\varphi_3}z'_{oc} \\
&\quad - \frac{Kh^2}{12A}\left(\delta_2^I + \delta_1^I\mu\sin\alpha\right) + \frac{K\left(l_R + r_b\mu\right)}{A}z'_{oo} \\
&\quad + \frac{K}{A}l_R\mu y'_{oo}\sin\alpha + \frac{l_1}{A}F_z^{sf} - \ddot{\psi}_2^I, \\
\gamma &= \begin{cases} \dfrac{2tg\alpha_0}{d_0}x_q + \dfrac{k_\alpha}{d_0}x_q^2 & \left(x_q < l_\alpha\right), \\ \dfrac{2tg\alpha_g}{d_0}x_q - \dfrac{k_\alpha}{d_0}l_\alpha^2 & \left(x_q \geq l_\alpha\right), \end{cases}
\end{aligned} \tag{16}$$

where the specific meaning of all symbols is no longer given, which can be seen in [1, 5].

6. Numerical Simulation and Experimental Validation

Launch dynamics of self-propelled artillery can be computed by combining (14) and (16). Eigenfrequency, vertical target dispersion, the time history of barrel recoil displacement, and the time history of muzzle displacement in the plumb direction in cement ground for a self-propelled artillery got by simulation and test are shown, respectively, in Tables 1 and 2 and Figures 4 and 5. The simulation parameters can be seen in [1].

It can be seen from Tables 1 and 2 and Figures 4 and 5 that results got by simulation and test have good agreements,

Table 1: Eigenfrequencies of self-propelled artillery got by simulation and test (rad/s).

Mode order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Simulation	3.0	15.3	18.5	29.1	44.0	46.8	64.7	75.8	87.5	94.2	101.5	—	141.0	214.2	225.9	243.7
Test	—	15.5	18.7	29.9	42.2	49.8	—	—	82.1	—	105.5	128.4	136.3	202.9	229.2	244.4
Error (%)		−1.3	−1.0	−2.7	4.3	6.0			6.2		−3.8		3.4	5.6	−1.4	−0.3

Figure 4: The time history of barrel recoil displacement got by simulation and test.

Figure 5: The time history of muzzle displacement in plumb direction got by simulation and test.

Table 2: Vertical target dispersion of self-propelled artillery got by simulation and test.

	Vertical target dispersion	
	Vertical dispersion	Direction dispersion
Test	0.400	0.380
Simulation	0.391	0.404
Error (%)	−2.3	6.3

which shows that the study on launch dynamics of self-propelled artillery by using MSTMM is very effective.

7. Analysis of Factors Influenced Firing Precision of Self-Propelled Artillery

Many shooting tests have shown that the difference of firing dispersion is big due to different ground conditions and different connection stiffness among the various components. In fact, different ground conditions mean that connection stiffness between road wheel and ground is different; the different ground hardness can be simulated by changing connection stiffness between road wheel and ground. The vibration characteristics and firing dispersion on the different ground are simulated for self-propelled artillery. Eigenfrequencies and firing dispersion of self-propelled artillery got by simulation are shown, respectively, in Tables 3 and 4.

It can be seen from Tables 3 and 4 that the effect of different ground conditions on the natural vibration characteristics is significantly great and then affects the firing precision of self-propelled artillery. The simulation results explained this phenomenon that firing precision of self-propelled artillery is different due to different ground conditions.

For self-propelled artillery, the vibration characteristics and firing dispersion on the cement ground are simulated by changing the connection stiffness between hull and revolving part; eigenfrequencies and firing dispersion got by simulation are shown, respectively, in Tables 5 and 6.

For a self-propelled artillery, the vibration characteristic and firing dispersion on the cement ground are simulated by changing the connection stiffness between revolving part and elevating part; eigenfrequencies and firing dispersion got by simulation are shown, respectively, in Tables 7 and 8.

It can be seen from Tables 5–8 that the connection stiffness between hull and revolving part, revolving part, and elevating part has great effect on vibration characteristic and firing dispersion of self-propelled artillery. The factors, such as different ground hardness, connection stiffness among various components, have a great influence on vibration characteristics, and then greatly influence firing precision of self-propelled artillery.

8. Conclusion

In this paper, the launch dynamics of self-propelled artillery is studied using MSTMM and the automatic deduction method

Table 3: Eigenfrequencies of self-propelled artillery on the different ground got by simulation.

Different ground conditions	Natural vibration frequency/(rad/s)								
Cement	Mode order	1	2	3	4	5	6	7	8
	Eigenfrequencies	3.0	15.3	18.5	29.1	44.0	46.8	64.7	75.8
Soil	Mode order	1	2	3	4	5	6	7	8
	Eigenfrequencies	3.0	7.7	9.2	12.0	15.2	21.1	30.2	31.2

Table 4: Firing dispersion of self-propelled artillery on the different ground got by simulation.

Different ground conditions	Vertical target dispersion	
	Vertical dispersion/m	Direction dispersion/m
Cement	0.411	0.367
Soil	0.283	0.393

of the overall transfer equation. Dynamical model of self-propelled artillery multibody system and its topology figure are developed. According to MSTMM, dynamical model of self-propelled artillery and its topology figure, overall transfer equation, characteristics equation, and launch dynamical equations of self-propelled artillery are developed. By solving launch dynamical equations of self-propelled artillery, eigenfrequency, firing dispersion, and dynamical response of self-propelled artillery are obtained. Results got by simulation and experiment have good agreements, which show that MSTMM and the automatic deduction method of overall transfer equation are useful to study the launch dynamics of self-propelled artillery.

Different ground conditions and connection stiffness among various components greatly influences vibration characteristics and firing precision of self-propelled artillery. Decreasing connection stiffness between hull and revolving part, revolving part, and elevating part makes firing dispersion of self-propelled change significantly. Firing precision of self-propelled artillery can be significantly improved by reasonable adjustment of the system stiffness or connection stiffness among various components.

Appendix

Overall Transfer Matrix Elements for Self-Propelled Artillery

All elements for overall transfer matrix mentioned in (3) are expressed as

$$\mathbf{T}_{10-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7\mathbf{U}_8\mathbf{U}_9\mathbf{U}_{10},$$

$$\mathbf{T}_{42-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17}\mathbf{U}_{18}\mathbf{U}_{30}\mathbf{U}_{42},$$

$$\mathbf{T}_{43-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,19}\mathbf{U}_{19}\mathbf{U}_{31}\mathbf{U}_{43},$$

$$\mathbf{T}_{44-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,20}\mathbf{U}_{20}\mathbf{U}_{32}\mathbf{U}_{44},$$

$$\mathbf{T}_{45-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,21}\mathbf{U}_{21}\mathbf{U}_{33}\mathbf{U}_{45},$$

$$\mathbf{T}_{46-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,22}\mathbf{U}_{22}\mathbf{U}_{34}\mathbf{U}_{46},$$

$$\mathbf{T}_{47-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,23}\mathbf{U}_{23}\mathbf{U}_{35}\mathbf{U}_{47},$$

$$\mathbf{T}_{48-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,24}\mathbf{U}_{24}\mathbf{U}_{36}\mathbf{U}_{48},$$

$$\mathbf{T}_{49-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7\mathbf{U}_{8,12} \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,25}\mathbf{U}_{25}\mathbf{U}_{37}\mathbf{U}_{49},$$

$$\mathbf{T}_{50-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,26}\mathbf{U}_{26}\mathbf{U}_{38}\mathbf{U}_{50},$$

$$\mathbf{T}_{51-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,27}\mathbf{U}_{27}\mathbf{U}_{39}\mathbf{U}_{51},$$

$$\mathbf{T}_{52-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,28}\mathbf{U}_{28}\mathbf{U}_{40}\mathbf{U}_{52},$$

$$\mathbf{T}_{53-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7 \times \mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,29}\mathbf{U}_{29}\mathbf{U}_{41}\mathbf{U}_{53},$$

$$\mathbf{T}_{13-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_6\mathbf{U}_7\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13,0},$$

$$\mathbf{T}_{11-1} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\mathbf{U}_{6,11}\mathbf{U}_{11}\mathrm{C},$$

$$\mathbf{G}_{42-17} = -\mathbf{H}_{17}\mathbf{U}_{18}\mathbf{U}_{30}\mathbf{U}_{42},$$

$$\mathbf{G}_{43-17} = \mathbf{H}_{17,19}\mathbf{U}_{19}\mathbf{U}_{31}\mathbf{U}_{43},$$

$$\mathbf{G}_{44-17} = \mathbf{H}_{17,20}\mathbf{U}_{20}\mathbf{U}_{32}\mathbf{U}_{44},$$

$$\mathbf{G}_{45-17} = \mathbf{H}_{17,21}\mathbf{U}_{21}\mathbf{U}_{33}\mathbf{U}_{45},$$

$$\mathbf{G}_{46-17} = \mathbf{H}_{17,22}\mathbf{U}_{22}\mathbf{U}_{34}\mathbf{U}_{46},$$

$$\mathbf{G}_{47-17} = \mathbf{H}_{17,23}\mathbf{U}_{23}\mathbf{U}_{35}\mathbf{U}_{47},$$

$$\mathbf{G}_{48-17} = \mathbf{H}_{17,24}\mathbf{U}_{24}\mathbf{U}_{36}\mathbf{U}_{48},$$

$$\mathbf{G}_{49-17} = \mathbf{H}_{17,25}\mathbf{U}_{25}\mathbf{U}_{37}\mathbf{U}_{49},$$

Table 5: Eigenfrequencies of a self-propelled artillery got by simulation on the cement ground.

Different connection stiffness	Eigenfrequencies/(rad/s)								
$(2.2\times10^7, 2.2\times10^7, 1.1\times10^8)$	Mode order	1	2	3	4	5	6	7	8
$(5.5\times10^7, 1.3\times10^7, 1.1\times10^7)$	Eigenfrequencies	3.0	15.3	18.5	29.1	44.0	46.8	64.7	75.8
$(2.2\times10^6, 2.2\times10^6, 1.1\times10^7)$	Mode order	1	2	3	4	5	6	7	8
$(5.5\times10^6, 1.3\times10^6, 1.1\times10^6)$	Eigenfrequencies	1.6	3.0	10.9	19.7	25.6	39.1	45.1	52.0

Table 6: Firing dispersion of a self-propelled artillery got by simulation on the cement ground.

Different connection stiffness	Vertical target dispersion	
	Vertical dispersion/m	Direction dispersion/m
$(2.2\times10^7, 2.2\times10^7, 1.1\times10^8)$ $(5.5\times10^7, 1.3\times10^7, 1.1\times10^7)$	0.411	0.367
$(2.2\times10^6, 2.2\times10^6, 1.1\times10^7)$ $(5.5\times10^6, 1.3\times10^6, 1.1\times10^6)$	0.543	0.380

$$
\begin{aligned}
\mathbf{G}_{50-17} &= \mathbf{H}_{17,26}\mathbf{U}_{26}\mathbf{U}_{38}\mathbf{U}_{50},\\
\mathbf{G}_{51-17} &= \mathbf{H}_{17,27}\mathbf{U}_{27}\mathbf{U}_{39}\mathbf{U}_{51},\\
\mathbf{G}_{52-17} &= \mathbf{H}_{17,28}\mathbf{U}_{28}\mathbf{U}_{40}\mathbf{U}_{52},\\
\mathbf{G}_{53-17} &= \mathbf{H}_{17,29}\mathbf{U}_{29}\mathbf{U}_{41}\mathbf{U}_{53},\\
\mathbf{G}_{42-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17}\mathbf{U}_{18}\mathbf{U}_{30}\mathbf{U}_{42},\\
\mathbf{G}_{43-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,19}\mathbf{U}_{19}\mathbf{U}_{31}\mathbf{U}_{43},\\
\mathbf{G}_{44-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,20}\mathbf{U}_{20}\mathbf{U}_{32}\mathbf{U}_{44},\\
\mathbf{G}_{45-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,21}\mathbf{U}_{21}\mathbf{U}_{33}\mathbf{U}_{45},\\
\mathbf{G}_{46-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,22}\mathbf{U}_{22}\mathbf{U}_{34}\mathbf{U}_{46},\\
\mathbf{G}_{47-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,23}\mathbf{U}_{23}\mathbf{U}_{35}\mathbf{U}_{47},\\
\mathbf{G}_{48-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,24}\mathbf{U}_{24}\mathbf{U}_{36}\mathbf{U}_{48},\\
\mathbf{G}_{49-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,25}\mathbf{U}_{25}\mathbf{U}_{37}\mathbf{U}_{49},\\
\mathbf{G}_{50-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,26}\mathbf{U}_{26}\mathbf{U}_{38}\mathbf{U}_{50},\\
\mathbf{G}_{51-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,27}\mathbf{U}_{27}\mathbf{U}_{39}\mathbf{U}_{51},\\
\mathbf{G}_{52-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,28}\mathbf{U}_{28}\mathbf{U}_{40}\mathbf{U}_{52},\\
\mathbf{G}_{53-13} &= \mathbf{H}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,29}\mathbf{U}_{29}\mathbf{U}_{41}\mathbf{U}_{53},\\
\mathbf{G}_{13,\mathrm{I}_2} &= -\mathbf{H}_{13,0},\\
\mathbf{G}_{10-8} &= -\mathbf{H}_{8}\mathbf{U}_{9}\mathbf{U}_{10},\\
\mathbf{G}_{42-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17}\mathbf{U}_{18}\mathbf{U}_{30}\mathbf{U}_{42},\\
\mathbf{G}_{43-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,19}\mathbf{U}_{19}\mathbf{U}_{31}\mathbf{U}_{43},\\
\mathbf{G}_{44-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,20}\mathbf{U}_{20}\mathbf{U}_{32}\mathbf{U}_{44},\\
\mathbf{G}_{45-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,21}\mathbf{U}_{21}\mathbf{U}_{33}\mathbf{U}_{45},\\
\mathbf{G}_{46-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,22}\mathbf{U}_{22}\mathbf{U}_{34}\mathbf{U}_{46},\\
\mathbf{G}_{47-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,23}\mathbf{U}_{23}\mathbf{U}_{35}\mathbf{U}_{47},\\
\mathbf{G}_{48-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,24}\mathbf{U}_{24}\mathbf{U}_{36}\mathbf{U}_{48},\\
\mathbf{G}_{49-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,25}\mathbf{U}_{25}\mathbf{U}_{37}\mathbf{U}_{49},\\
\mathbf{G}_{50-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,26}\mathbf{U}_{26}\mathbf{U}_{38}\mathbf{U}_{50},\\
\mathbf{G}_{51-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,27}\mathbf{U}_{27}\mathbf{U}_{39}\mathbf{U}_{51},\\
\mathbf{G}_{52-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,28}\mathbf{U}_{28}\mathbf{U}_{40}\mathbf{U}_{52},\\
\mathbf{G}_{53-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,29}\mathbf{U}_{29}\mathbf{U}_{41}\mathbf{U}_{53},\\
\mathbf{G}_{13-8} &= \mathbf{H}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13,0},\\
\mathbf{G}_{42-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17}\mathbf{U}_{18}\mathbf{U}_{30}\mathbf{U}_{42},\\
\mathbf{G}_{43-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,19}\mathbf{U}_{19}\mathbf{U}_{31}\mathbf{U}_{43},\\
\mathbf{G}_{44-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,20}\mathbf{U}_{20}\mathbf{U}_{32}\mathbf{U}_{44},\\
\mathbf{G}_{45-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,21}\mathbf{U}_{21}\mathbf{U}_{33}\mathbf{U}_{45},\\
\mathbf{G}_{46-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,22}\mathbf{U}_{22}\mathbf{U}_{34}\mathbf{U}_{46},\\
\mathbf{G}_{47-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,23}\mathbf{U}_{23}\mathbf{U}_{35}\mathbf{U}_{47},\\
\mathbf{G}_{48-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,24}\mathbf{U}_{24}\mathbf{U}_{36}\mathbf{U}_{48},\\
\mathbf{G}_{49-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,25}\mathbf{U}_{25}\mathbf{U}_{37}\mathbf{U}_{49},\\
\mathbf{G}_{50-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,26}\mathbf{U}_{26}\mathbf{U}_{38}\mathbf{U}_{50},\\
\mathbf{G}_{51-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,27}\mathbf{U}_{27}\mathbf{U}_{39}\mathbf{U}_{51},\\
\mathbf{G}_{52-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,28}\mathbf{U}_{28}\mathbf{U}_{40}\mathbf{U}_{52},\\
\mathbf{G}_{53-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13}\mathbf{U}_{14}\mathbf{U}_{15}\mathbf{U}_{16}\mathbf{U}_{17,29}\mathbf{U}_{29}\mathbf{U}_{41}\mathbf{U}_{53},\\
\mathbf{G}_{13-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8,12}\mathbf{U}_{12}\mathbf{U}_{13,0},\\
\mathbf{G}_{11-6} &= -\mathbf{H}_{6,11}\mathbf{U}_{11}\mathrm{C},\\
\mathbf{G}_{10-6} &= \mathbf{H}_{6}\mathbf{U}_{7}\mathbf{U}_{8}\mathbf{U}_{9}\mathbf{U}_{10},\\
\mathrm{C} &= \begin{bmatrix} \mathrm{I}_6 & \mathrm{O}_{6\times6} \\ \mathrm{O}_{6\times6} & -\mathrm{I}_6 \end{bmatrix}.
\end{aligned}
\tag{A.1}
$$

All matrices is no longer given in detail, which can be seen in [1, 2, 20].

Table 7: Eigenfrequencies of self-propelled artillery got by simulation on the cement ground.

Different connection stiffness	Natural vibration frequency/(rad/s)								
$(2.2 \times 10^7, 2.1 \times 10^7, 5.3 \times 10^7)$	Mode order	1	2	3	4	5	6	7	8
$(5.5 \times 10^7, 1.1 \times 10^7, 1.1 \times 10^8)$	Eigenfrequencies	3.0	15.3	18.5	29.1	44.0	46.8	64.7	75.8
$(2.2 \times 10^6, 2.1 \times 10^6, 5.3 \times 10^6)$	Mode order	1	2	3	4	5	6	7	8
$(5.5 \times 10^6, 1.1 \times 10^6, 1.1 \times 10^7)$	Eigenfrequencies	2.1	2.9	7.9	16.6	27.9	36.5	42.9	50.1

Table 8: Firing dispersion of self-propelled artillery got by simulation on the cement ground.

Different connection stiffness	Vertical target dispersion	
	Vertical dispersion/m	Direction dispersion/m
$(2.2 \times 10^7, 2.1 \times 10^7, 5.3 \times 10^7)$ $(5.5 \times 10^7, 1.1 \times 10^7, 1.1 \times 10^8)$	0.411	0.367
$(2.2 \times 10^6, 2.1 \times 10^6, 5.3 \times 10^6)$ $(5.5 \times 10^6, 1.1 \times 10^6, 1.1 \times 10^7)$	0.458	0.371

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research was supported by the Research Fund for the Doctoral Program of Higher Education of China (20113219110025), the Natural Science Foundation of China Government (11102089), and the Program for New Century Excellent Talents in University (NCET-10-0075).

References

[1] X. Rui, Y. Liu, and H. Yu, *Launch Dynamics of Tank and Self-Propelled Artillery*, Science Press, Beijing, China, 2008.

[2] X. Rui, L. Yun, Y. Lu et al., *Transfer Matrix Method for Multibody System and its Application*, Science Press, Beijing, China, 2008.

[3] X. Rui, L. Yun, G. Wang et al., *Direction to Launch Safety of Ammunition*, National Defense Industry Press, Beijing, China, 2009.

[4] X. Rui, Y. Lu, G. Wang et al., *Simulation and Test Methods of Launch Dynamics of Multiple Launch Rocket System*, National Defense Industry Press, Beijing, China, 2003.

[5] X. Rui, *Launch Dynamics of Multibody System*, National Defense Industry Press, Beijing, China, 1995.

[6] X. Rui and Q. Yang, *Launch Process of Projectile*, South East University Press, Nanjing, China, 1991.

[7] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart, Germany, 1977.

[8] W. Schiehlen, *Multibody Systems Handbook*, Springer, Berlin, Germany, 1990.

[9] T. R. Kane, P. W. Likins, and D. A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, New York, NY, USA, 1983.

[10] X. Rui, D. Bestle, and J. Zhang, "A new form of the transfer matrix method for multibody systems," in *Proceedings of the ECCOMAS Thematic Conference on Multibody Dynamics*, Zagreb, Croatia, 2013.

[11] X. Rui and W. Schiehlen, "Multibody system dyanamics: preface," *Multibody System Dynamics*, vol. 18, no. 4, pp. 485–492, 2007.

[12] X. Rui, L. Yun, G. Wang et al., "Augmented eigenvector and its orthogonality of linear multi-rigid-flexible-body system," *Journal of China Ordnance*, vol. 4, no. 2, pp. 100–105, 2008.

[13] X. Rui, B. He, B. Rong et al., "Discrete time transfer matrix method for dynamics of multi-rigid-flexible-body system moving in plane," *Journal of Multi-Body Dynamics*, vol. 23, no. 1, pp. 23–42, 2009.

[14] X. Rui, G. Wang, Y. Lu, and L. Yun, "Transfer matrix method for linear multibody system," *Multibody System Dynamics*, vol. 19, no. 3, pp. 179–207, 2008.

[15] X. Rui, E. Kreuzer, B. Rong, and B. He, "Discrete time transfer matrix method for dynamics of multibody system with flexible beams moving in space," *Acta Mechanica Sinica*, vol. 28, pp. 490–504, 2012.

[16] X. Rui, B. Rong et al., "Some new developments in discrete time transfer matrix method of multibody systems," in *Proceedings of the International Conference on Dynamics Vibration and Control*, Hangzhou, China, 2010.

[17] X. Rui, B. Rong, G. Wang, and B. He, "Discrete time transfer matrix method for dynamics analysis of complex weapon systems," *Science China Technological Sciences*, vol. 54, no. 5, pp. 1061–1071, 2011.

[18] X. Rui, B. He, Y. Lu, W. Lu, and G. Wang, "Discrete time transfer matrix method for multibody system dynamics," *Multibody System Dynamics*, vol. 14, no. 3-4, pp. 317–344, 2005.

[19] B. He, X. Rui, and G. Wang, "Riccati discrete time transfer matrix method for elastic beam undergoing large overall motion," *Multibody System Dynamics*, vol. 18, no. 4, pp. 579–598, 2007.

[20] X. Rui and J. Zhang, "Automatical transfer matrix method of multibody system," in *Proceedings of the 2nd Joint International Conference on Multibody System Dynamics*, Stuttgart, Germany, May-June 2012.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 383680, 8 pages
http://dx.doi.org/10.1155/2014/383680

Research Article

Higher Order Variational Integrators for Multibody System Dynamics with Constraints

Jieyu Ding and Zhenkuan Pan

College of Information Engineering, Qingdao University, Qingdao 266071, China

Correspondence should be addressed to Jieyu Ding; djy@qdu.edu.cn

Received 25 August 2013; Revised 16 November 2013; Accepted 25 November 2013; Published 23 January 2014

Academic Editor: Xiaoting Rui

Copyright © 2014 J. Ding and Z. Pan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The continuous and discrete Euler-Lagrangian equations with holonomic constraints are presented based on continuous and discrete Hamiltonian Principle. Using Lagrangian polynomial to interpolate state variables and Gauss quadrature formula to approximate Hamiltonian action integral, the higher order variational Galerkin integrators for multibody system dynamics with constraints and the computation procedure are given. Numerical examples are provided to show the long-time behavior of the methods proposed in this paper via comparisons with traditional Runge-Kutta methods.

1. Introduction

Dynamics of multibody systems [1] are usually described by a set of differential-algebraic equations (DAEs) [2, 3], which are solved traditionally using numerical methods of ordinary differential equations (ODEs) and combining some constraint stabilization techniques. In the community of numerical analysis, numerical methods for ODEs are always designed in a small area with local stability analysis, which lead to a lot of difficulties for long-time simulation. Geometric numerical integration methods [3] originated from long-term simulation of molecular dynamics and planet dynamics in solar system with global numerical stability due to the property of structure-preserving provide alternatives to overcome these problems. These numerical methods are referred to as the ones preserving the invariants in their corresponding continuous systems, such as preserving symplectic, energy, momentum, symmetry, and orthogonality, in the original systems, thus revolutionizing the traditional concepts of numerical stability. It has been widely accepted that the more invariants conserved in a numerical method, the more stable it is.

Symplectic algorithms [4] and energy conservation methods [5] are two frequently used methods for numerical integration of constrained Hamiltonian systems, but they are designed for conservative systems and it is difficult to design an algorithm that is symplectic and energy preserving at the same time [6]. The variational integrators [7, 8] based on discrete variational principle [9] provides an excellent framework for design of geometric numerical integrators with many merits: it preserves momentum naturally with good behavior of approximate energy conservation, even preserves strictly symplectic-momentum-energy [10]; it can be easily extended to a large class of problems [11, 12], such as the construction of geometric structure-preserving numerical integrators for PDEs [13], nonsmooth collisions [14, 15], stochastic systems [16, 17], nonholonomic systems [18, 19], and constrained systems [20], dissipative systems [21], optimal control [22, 23], parameter optimization [24]; it can generate easily a large class of higher order methods based on polynomial interpolation and numerical quadrature formulas systematically [25–29] and some Lie group variational integrators [30–32] combining Lie group methods [33].

For the general conservative systems without constraints, [25, 26] propose the same framework of design higher order variational Galerkin integrators using numerical interpolation and quadrature techniques for state variables and action integral, respectively, during one integration step based on discrete variational principle. Motivated by these works, we will investigate higher order variational Galerkin integrators for multibody system dynamics with constraints for the long-term simulation purpose. Due to the space limit, we will

focus on the cases with holonomic constraints, but the results can be easily extended to the cases with nonholonomic constraints and the cases with nonconservative forces.

The paper is organized as follows. In Section 2, the continuous and discrete Euler-Lagrangian equations with holonomic constraints are presented based on continuous and discrete Hamiltonian Principle, respectively; in Section 3, we derive the higher order variational Galerkin integrators for multibody system dynamics with constraints using Lagrangian polynomial to interpolate state variables and using Gauss quadrature formula to approximate Hamiltonian action integral; the computation procedure is given in Section 4; numerical examples are provided in Section 5 to show the long-time behavior of the methods proposed in this paper via comparisons with traditional methods. The last one is concluding remarks including summary and future works.

2. Discrete Euler-Lagrangian Equations with Constraints

The Hamilton Principle of a multibody system with holonomic constraints and conservative forces can be stated as

$$\delta S = \delta \int_0^{t_f} \left(L(\mathbf{q}, \dot{\mathbf{q}}, t) - \boldsymbol{\lambda}^T \boldsymbol{\Phi}(\mathbf{q}, t) \right) dt = 0, \tag{1}$$

where S is Hamilton action integral, L is Lagrangian, $\mathbf{q}$, $\dot{\mathbf{q}}$ are generalized displacement and velocity, respectively, and $\boldsymbol{\lambda}$ is Lagrange multiplier vector corresponding to the constraint

$$\boldsymbol{\Phi}(\mathbf{q}, t) = \mathbf{0}. \tag{2}$$

Via variational method, the following Euler-Lagrange equations of a constrained mechanical system can be got:

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}} + \boldsymbol{\Phi}^T(\mathbf{q}, t)\boldsymbol{\lambda} = \mathbf{0}, \\ \boldsymbol{\Phi}(\mathbf{q}, t) = \mathbf{0}. \end{gathered} \tag{3}$$

It is a typical differential-algebraic equation which can be solved using different traditional numerical methods of DAEs [2, 3].

Following the variational integrators based on discrete variational principle, the time interval $t \in [0, t_f]$ is divided into N subintervals with time step $h = t_f/N$ equally along with $N+1$ grids, $0 = t_0, t_1, t_2, \ldots, t_{N-1}, t_N = t_f$, $t_i = ih$, $i = 0, 1, 2, \ldots, N$. The discrete generalized displacement and velocity and Lagrangian multiplier can be denoted as $\mathbf{q}_i = \mathbf{q}(t_i)$, $\dot{\mathbf{q}}_i = \dot{\mathbf{q}}(t_i)$, $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}(t_i)$, $i = 0, 1, 2, \ldots, N$. Equation (1) can be rewritten as the following discrete counterpart:

$$\begin{aligned} &\delta S_d(\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_{N-1}, \mathbf{q}_N, \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{N-2}, \boldsymbol{\lambda}_{N-1}) \\ &\quad = \delta \sum_{i=0}^{N-1} L_d(\mathbf{q}_i, \mathbf{q}_{i+1}, t_i) \\ &\qquad - \delta \sum_{i=0}^{N-1} f_d(\mathbf{q}_i, \mathbf{q}_{i+1}, \boldsymbol{\lambda}_i, t_i) \\ &\quad = 0, \end{aligned} \tag{4}$$

where $L_d(\mathbf{q}_i, \mathbf{q}_{i+1}, h)$ and $f_d(\mathbf{q}_i, \mathbf{q}_{i+1}, \boldsymbol{\lambda}_i, t_i)$ are defined as the discrete approximation of the integration of $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and $\boldsymbol{\lambda}^T \boldsymbol{\Phi}(\mathbf{q}, t)$ on the interval $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} L_d(\mathbf{q}_i, \mathbf{q}_{i+1}, t_i) &= \int_{t_i}^{t_{i+1}} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt, \\ f_d(\mathbf{q}_i, \mathbf{q}_{i+1}, \boldsymbol{\lambda}_i, t_i) &= \int_{t_i}^{t_{i+1}} \boldsymbol{\lambda}^T \boldsymbol{\Phi}(\mathbf{q}, t)\, dt. \end{aligned} \tag{5}$$

Using standard variational method, the discrete Euler-Lagrangian (DEL) equations are derived as

$$\begin{gathered} D_1 L_d(\mathbf{q}_i, \mathbf{q}_{i+1}, t_i) + D_2 L_d(\mathbf{q}_{i-1}, \mathbf{q}_i, t_{i-1}) \\ - h\boldsymbol{\Phi}_{\mathbf{q}}^T(\mathbf{q}_i, t_i)\boldsymbol{\lambda}_i = \mathbf{0}, \\ \boldsymbol{\Phi}(\mathbf{q}_{i+1}, t_{i+1}) = \mathbf{0}, \end{gathered} \tag{6}$$

where $D_j L_d$ $(j = 1, 2)$ is the partial derivative of L_d with respect to the jth variable.

3. Higher Order Variational Galerkin Integrators of Multibody Systems with Constraints

The key technique to improve the accuracy of variational integrators is the approximation of the discrete Hamiltonian action integral, which can be realized through general Galerkin methods [24, 25]. Using the same methods in this section, Hamiltonian action integral is approximated in a subinterval $[t_i, t_{i+1}]$ with an introduced local time parameter $\tau \in [0, 1]$, where

$$\tau = \frac{t - t_i}{t_{i+1} - t_i}. \tag{7}$$

Given $s+1$ control points $0 = \tau_0 < \tau_1 < \cdots < \tau_s = 1$, for $k = 0, 1, \ldots, s$, Lagrange interpolation polynomial $l_k(\tau) : [0, 1] \to R$ is defined as

$$l_k(\tau) = \prod_{0 \le j \le s, j \ne k} \frac{\tau - \tau_j}{\tau_k - \tau_j}. \tag{8}$$

The state variables $\mathbf{q}(t)$ on $[t_i, t_{i+1}]$ can be interpolated as follows:

$$\mathbf{q}_d(\tau) = \sum_{k=0}^{s} l_k(\tau)\, \mathbf{q}_{i,k}, \tag{9}$$

where $\mathbf{q}_{i,k} = \mathbf{q}(t_i + k \cdot (h/s))$. The derivative of $\mathbf{q}_d(\tau)$ with respect to t provides an approximation of $\dot{\mathbf{q}}(t)$ on $[t_i, t_{i+1}]$ as

$$\dot{\mathbf{q}}_d(\tau) = \frac{1}{h} \sum_{k=0}^{s} l_k'(\tau)\, \mathbf{q}_{i,k}, \tag{10}$$

where $l_k'(\tau) = (\partial/\partial\tau) l_k(\tau)$. Then, Hamiltonian action integral can be approximated on $[t_i, t_{i+1}]$ as follows:

$$\begin{aligned} &\int_{t_i}^{t_{i+1}} \left(L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) - \boldsymbol{\lambda}^T \boldsymbol{\Phi}(\mathbf{q}, t) \right) dt \\ &\quad = \int_0^1 h \Big(L(\mathbf{q}_d(\tau), \dot{\mathbf{q}}_d(\tau), t_i + \tau h) \\ &\qquad - \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}(\mathbf{q}_d(\tau), t_i + \tau h) \Big)\, d\tau. \end{aligned} \tag{11}$$

Table 1: Lagrange interpolation polynomials and comparable derivatives with respect to τ.

s	$l_k(\tau)$	$l'_k(\tau)$
1	$l_0(\tau) = 1 - \tau$	$l'_0(\tau) = -1$
	$l_1(\tau) = \tau$	$l'_1(\tau) = 1$
2	$l_0(\tau) = (2\tau - 1)(\tau - 1)$	$l'_0(\tau) = 4\tau - 3$
	$l_1(\tau) = -4\tau(\tau - 1)$	$l'_1(\tau) = -4(2\tau - 1)$
	$l_2(\tau) = \tau(2\tau - 1)$	$l'_2(\tau) = 4\tau - 1$

Table 2: Quadrature points and weights of the Gauss quadrature formula with respect to the interval $[-1, 1]$.

n	$\tilde{\tau}_r$	A_r
2	$\pm\sqrt{3}/3$	1
3	$\pm\sqrt{15}/5$	5/9
	0	8/9

Lagrange interpolation polynomials in (9) and comparable derivatives with respect to τ in (10) are listed in Table 1.

Using Gauss quadrature formula, which has maximal degree of accuracy $2n - 1$ for a fixed number n of quadrature points, we have

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \left(L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) - \boldsymbol{\lambda}^T \boldsymbol{\Phi}(\mathbf{q}, t)\right) dt \\
&\quad = \sum_{r=1}^{n} A_r h \left(L(\mathbf{q}_d(\tilde{\tau}_r), \dot{\mathbf{q}}_d(\tilde{\tau}_r), t_i + \tilde{\tau}_r h)\right. \\
&\qquad\qquad \left. - \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}(\mathbf{q}_d(\tilde{\tau}_r), t_i + \tilde{\tau}_r h)\right) \\
&\quad = L_d(\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}, t_i) \\
&\qquad - f_d(\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}, \boldsymbol{\lambda}_i, t_i),
\end{aligned} \tag{12}$$

where A_r is the weights and $\tilde{\tau}_r$ is the quadrature point. Quadrature points and weights of the Gauss quadrature formula with respect to the interval $[-1, 1]$ are listed in Table 2.

Apply discrete Hamilton's principle, the derivatives of the action with respect to $\mathbf{q}_{i,k}$ have to vanish for all $i = 0, 1, \ldots, N - 1$ and $k = 0, 1, \ldots, s$. That leads to the discrete Euler-Lagrange equations

$$\begin{aligned}
&D_1 L_d(\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}, t_i) - D_1 f_d(\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}, \boldsymbol{\lambda}_i, t_i) \\
&\quad + D_{s+1} L_d(\mathbf{q}_{i-1,0}, \mathbf{q}_{i-1,1}, \ldots, \mathbf{q}_{i-1,s}, t_{i-1}) \\
&\quad - D_{s+1} f_d(\mathbf{q}_{i-1,0}, \mathbf{q}_{i-1,1}, \ldots, \mathbf{q}_{i-1,s}, \boldsymbol{\lambda}_{i-1}, t_{i-1}) = \mathbf{0} \\
&D_k L_d(\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}, t_i) \\
&\quad - D_k f_d(\mathbf{q}_{i,0}, \mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}, \boldsymbol{\lambda}_i, t_i) = \mathbf{0}, \quad k = 1, \ldots, s \\
&\qquad \boldsymbol{\Phi}(\mathbf{q}_{i+1}, t_{i+1}) = \mathbf{0}.
\end{aligned} \tag{13}$$

4. Computation Procedure

The following computation procedure can be used to get the solution of the discrete Euler-Lagrange equations (13).

Figure 1: Double pendulum.

Step 1. Given the order of Lagrange interpolation polynomial s, the number of Gauss quadrature points n, fixed time step h, initial state variables $\mathbf{q}_0$, and velocity variables $\dot{\mathbf{q}}_0$.

Step 2. Get $\mathbf{q}_1$, $\dot{\mathbf{q}}_1$, and $\boldsymbol{\lambda}_1$ by Euler method or Runge-Kutta method. Use $\mathbf{q}_{0,0} = \mathbf{q}_0$, and $\mathbf{q}_{0,s} = \mathbf{q}_1$ to calculate $\mathbf{q}_{0,k}$, $k = 1, \ldots, s - 1$.

Step 3. For $i = 2$ to $N - 1$, solve (13) by Newton iteration method to get $\mathbf{q}_{i,1}, \ldots, \mathbf{q}_{i,s}$ and $\boldsymbol{\lambda}_i$, thus $\mathbf{q}_{i+1} = \mathbf{q}_{i,s}$, $\dot{\mathbf{q}}_{i+1} = \dot{\mathbf{q}}_d(1) = (1/h)\sum_{k=0}^{s} l'_k(1)\mathbf{q}_{i,k}$.

Step 4. Use the solution $\mathbf{q}_i$, $\dot{\mathbf{q}}_{i+1}$, $i = 0, 1, \ldots, N - 1$ to get the potential energy T, kinetic energy V, total energy of the system $H = T + V$, and constraints $\boldsymbol{\Phi}$.

5. Numerical Example

Consider a simple double pendulum system shown in Figure 1 the state variables are chosen as $\mathbf{q} = [x_1 \;\; y_1 \;\; x_2 \;\; y_2]^T$, then the potential energy, kinetic energy of the system are

$$\begin{aligned}
T &= \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right), \\
V &= m_1 g y_1 + m_2 g y_2.
\end{aligned} \tag{14}$$

The mass matrix is a constant matrix as follows:

$$\mathbf{M} = \operatorname{diag}\left[m_1 \;\; m_1 \;\; m_2 \;\; m_2\right]. \tag{15}$$

The constraint equations are

$$\boldsymbol{\Phi}(\mathbf{q}) = \begin{pmatrix} x_1^2 + y_1^2 - l_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2 \end{pmatrix} = \mathbf{0}. \tag{16}$$

Given $m_1 = m_2 = 1$ kg, $l_1 = l_2 = 1$ m, with time step $h = 0.01$ s, the results of the method presented in this paper are

Table 3: Comparison of the methods mentioned with time step $h = 0.01$.

Numerical method	Run time (s)	$\max\lvert\varepsilon(H)\rvert$	$\max\lvert\varepsilon(\mathbf{\Phi})\rvert$
Runge-Kutta	15.0385	3.0045	0.2345
P1L-Q2G	16.0993	0.4049	1.9984×10^{-15}
P2L-Q2G	24.8510	0.2270	8.8818×10^{-16}
P2L-Q3G	31.2938	0.2273	1.3323×10^{-15}

Table 4: Comparison of the methods mentioned with time step $h = 0.001$.

Numerical method	Run time (s)	$\max\lvert\varepsilon(H)\rvert$	$\max\lvert\varepsilon(\mathbf{\Phi})\rvert$
Runge-Kutta	99.7938	4.7378×10^{-4}	3.6633×10^{-5}
P1L-Q2G	121.4468	0.0405	9.6040×10^{-11}
P2L-Q2G	212.9882	0.0023	1.2912×10^{-9}
P2L-Q3G	265.9505	0.0021	1.2912×10^{-9}

Figure 2: Total energy of double pendulum by different method, $h = 0.01$.

Figure 3: A rotary rod slider system.

Figure 4: Energies of rotary rod slider system by method P1L-Q2G, $h = 0.01$.

compared with the Runge-Kutta method in Table 3 for the terminal time 100 s. Here, P1L-Q2G means that the order of Lagrange interpolation polynomial is 1 and the number of Gauss quadrature points is 2, and the same as P2L-Q2G and P2L-Q3G. $\varepsilon(\mathrm{H})$, $\varepsilon(\mathbf{\Phi})$ are the maximum errors of the total energy H and constraints $\mathbf{\Phi}$.

Using the traditional method such as Runge-Kutta method, to keep the errors of the total energy H and constraints $\mathbf{\Phi}$ are all smaller, the time step should be chosen as about 0.001 s in this example, and the total computation time is 99.7938 s. But using the discrete variational methods presented in this paper, the time step can be chosen longer as 0.01 s. Table 3 shows that, with this longer time step, the discrete variational methods can keep the total energy and constraints in higher accuracy than Runge-Kutta method, the cost instead is the longer computation time because of the iterations during the nonlinear equations solving process, but it is still shorter than the computation time of Runge-Kutta method with smaller time step to reach the same accuracy of the results.

It is also shown in Tables 3 and 4 that when the order of Lagrange interpolation polynomial is higher, the total energy

Figure 5: Constraints of rotary rod slider system by method P1L-Q2G, $h = 0.01$.

and constraints can be kept better, but there are no big differences between 2 or 3 points in Gauss quadrature formula.

Figure 2 shows the energies of the previously mentioned methods. With the time step $h = 0.01$ s, the total energy is up and down around zero with little errors during the long-time simulation by the methods presented in this paper, while increases quickly during the simulation by Runge-Kutta method. The results of high order variational Galerkin integrators are better than low order integrators.

In the case of nonconstant mass matrix, the methods of higher order variational Galerkin integrators are also applicable. For another example, Figure 3 shows a rotary rod slider system. OA is a rigid rod with uniform mass m_A and length l, which rotates round O in the plane OXY. B is a slider with the mass m_B, and the stiffness of the spring on it is k; the mass of the spring is ignored. Suppose only gravity in the plane OXY is considered for the system.

The state variables are chosen as $\mathbf{q} = [R_{x^i} \;\; R_{y^i} \;\; R_{x^j} \;\; R_{y^j} \theta \;\; x_i]^T$. Then, the mass matrix is a nonconstant matrix

$$\mathbf{M} = \begin{bmatrix} m_B & 0 & 0 & 0 & -m_B x_i \sin\theta & m_B \cos\theta \\ 0 & m_B & 0 & 0 & m_B x_i \cos\theta & m_B \sin\theta \\ 0 & 0 & m_A & 0 & 0 & 0 \\ 0 & 0 & 0 & m_A & 0 & 0 \\ -m_B x_i \sin\theta & m_B x_i \cos\theta & 0 & 0 & \frac{1}{3} m_A l^2 + m_B x_i^2 & 0 \\ m_B \cos\theta & m_B \sin\theta & 0 & 0 & 0 & m_B \end{bmatrix}. \tag{17}$$

The constraint equations are

$$\mathbf{\Phi} = \begin{pmatrix} R_{x^j} - \frac{l}{2}\cos\theta \\ R_{y^j} - \frac{l}{2}\sin\theta \\ \left(R_{x^j} - R_{x^i}\right)\cos\theta + \left(R_{y^j} - R_{y^i}\right)\sin\theta - x_i \\ \left(R_{x^j} - R_{x^i}\right)\sin\theta - \left(R_{y^j} - R_{y^i}\right)\cos\theta \end{pmatrix} = \mathbf{0}. \tag{18}$$

Given $l_1 = 2m$, $m_A = 2\,\text{kg}$, $m_B = 5\,\text{kg}$, $k = 1000\,\text{N/m}$. With the initial step length $h = 0.01\,\text{s}$ and the terminal time 100 s, using the method P1L-Q2G, where the order of Lagrange interpolation polynomial is 1 and the number of Gauss quadrature points is 2, the results are also satisfying. The total computer time is 86.0034 s; the maximum errors of the total energy H and constraints $\mathbf{\Phi}$ are 0.2611 and 3.8858×10^{-16}, which is illustrated in Figures 4 and 5.

6. Conclusions

Based on Lagrange interpolation and Gauss quadrature, higher order variational Galerkin integrators have been developed for multibody systems with constraints. Numerical results validate the accuracy of the method for longer time step and long-time simulation. Nevertheless, as the computation is more complex, the computation cost is higher. Further study is needed to have more efficient integrators.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grants nos. 11272166, 11002075, 10972110).

References

[1] A. A. Shabana, *Dynamics of Multibody Systems*, Cambridge University Press, 3rd edition, 2005.

[2] A. Laulusa and O. A. Bauchau, "Review of classical approaches for constraint enforcement in multibody systems," *Journal of Computational and Nonlinear Dynamics*, vol. 3, no. 1, Article ID 011004, 2008.

[3] O. A. Bauchau and A. Laulusa, "Review of contemporary approaches for constraint enforcement in multibody systems," *Journal of Computational and Nonlinear Dynamics*, vol. 3, no. 1, Article ID 011005, 2008.

[4] L. Jay, "Symplectic partitioned Runge-Kutta methods for constrained Hamiltonian systems," *SIAM Journal on Numerical Analysis*, vol. 33, no. 1, pp. 368–387, 1996.

[5] O. Gonzalez, "Mechanical systems subject to holonomic constraints: differential-algebraic formulations and conservative integration," *Physica D*, vol. 132, no. 1-2, pp. 165–174, 1999.

[6] Z. Ge and J. E. Marsden, "Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators," *Physics Letters A*, vol. 133, no. 3, pp. 134–139, 1988.

[7] J. M. Wendlandt and J. E. Marsden, "Mechanical integrators derived from a discrete variational principle," *Physica D*, vol. 106, no. 3-4, pp. 223–246, 1997.

[8] J. E. Marsden and M. West, "Discrete Mechanics and Variational Integrators," *Acta Numerica*, vol. 10, pp. 357–514, 2001.

[9] J. Moser and A. P. Veselov, "Discrete versions of some classical integrable systems and factorization of matrix polynomials," *Communications in Mathematical Physics*, vol. 139, no. 2, pp. 217–243, 1991.

[10] C. Kane, J. E. Marsden, and M. Ortiz, "Symplectic-energy-momentum preserving variational integrators," *Journal of Mathematical Physics*, vol. 40, no. 7, pp. 3353–3371, 1999.

[11] A. Lew, J. E. Marsden, M. Ortiz, and M. West, "Variational time integrators," *International Journal for Numerical Methods in Engineering*, vol. 60, no. 1, pp. 153–212, 2004.

[12] A. Lew, J. E. Marsden, M. Ortiz, and M. West, "An overview of variational integrators," in *Finite Element Methods: 1970s and Beyond*, pp. 85–146, CIMNE, Barcelona, Spain, 2003.

[13] A. Lew, J. E. Marsden, M. Ortiz, and M. West, "Asynchronous variational integrators," *Archive for Rational Mechanics and Analysis*, vol. 167, no. 2, pp. 85–146, 2003.

[14] A. Pandolfi, C. Kane, J. E. Marsden, and M. Ortiz, "Time-discretized variational formulation of non-smooth frictional contact," *International Journal for Numerical Methods in Engineering*, vol. 53, no. 8, pp. 1801–1829, 2002.

[15] R. C. Fetecau, J. E. Marsden, M. Ortiz, and M. West, "Nonsmooth lagrangian mechanics and variational collision integrators," *SIAM Journal on Applied Dynamical Systems*, vol. 2, no. 3, pp. 381–416, 2003.

[16] N. Bou-Rabee and H. Owhadi, "Stochastic variational integrators," *IMA Journal of Numerical Analysis*, vol. 29, no. 2, pp. 421–443, 2009.

[17] W. Hu, Z. Deng et al., "Generalized multi-symplectic integrators for a class of Hamiltonian nonlinear wave PDEs," *Journal of Computational Physics*, vol. 235, pp. 394–406, 2013.

[18] J. Cortés and S. Martínez, "Non-holonomic integrators," *Nonlinearity*, vol. 14, no. 5, pp. 1365–1392, 2001.

[19] R. McLachlan and M. Perlmutter, "Integrators for nonholonomic mechanical systems," *Journal of Nonlinear Science*, vol. 16, no. 4, pp. 283–328, 2006.

[20] S. Leyendecker, J. E. Marsden, and M. Ortiz, "Variational integrators for constrained dynamical systems," *ZAMM Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 88, no. 9, pp. 677–708, 2008.

[21] C. Kane, J. E. Marsden, M. Ortiz, and M. West, "Variational integrators and the newmark algorithm for conservative and dissipative mechanical systems," *International Journal for Numerical Methods in Engineering*, vol. 49, no. 10, pp. 1295–1325, 2000.

[22] S. Leyendecker, S. Ober-Blöbaum, J. E. Marsden, and M. Ortiz, "Discrete mechanics and optimal control for constrained systems," *Optimal Control Applications and Methods*, vol. 31, no. 6, pp. 505–528, 2010.

[23] S. Ober-Blöbaum, O. Junge, and J. E. Marsden, "Discrete mechanics and optimal control: an analysis," *Control, Optimisation and Calculus of Variations*, vol. 17, no. 2, pp. 322–352, 2011.

[24] K. Witting, S. Ober-Blöbaum, and M. Dellnitz, "A variational approach to define robustness for parametric multi-objective

optimization problems," *Journal of Global Optimization*, vol. 57, no. 2, pp. 331–345, 2013.

[25] M. Leok and J. Zhang, "Discrete Hamiltonian variational integrators," *IMA Journal of Numerical Analysis*, vol. 31, no. 4, pp. 1497–1532, 2011.

[26] S. Ober-Blöbaum and N. Saake, "Construction and analysis of higher order Galerkin variational integrators," *Numerical Analysis*. In press. http://arxiv.org/abs/1304.1398 .

[27] M. Leok, T. Shingel, and T. P, "rolongation-collocation variational integrators," *IMA Journal of Numerical Analysis*, vol. 32, no. 3, pp. 1194–1216, 2012.

[28] S. Leyendecker and S. Ober-Blöbaum, "A variational approach to multirate integration for constrained systems," in *Multibody Dynamics*, J.-C. Samin and P. Fisette, Eds., vol. 28 of *Computational Methods in Applied Sciences*, pp. 97–121, Springer, 2013.

[29] C. M. Campos, O. Junge, and S. Ober-Blöbaum, "Higher order variational time discretization of optimal control problems," in *Proceedings of the 20th International Symposium on Mathematical Theory of Networks and Systems,*, pp. 9–13, Melbourne, Australia, July 2012.

[30] M. Leok and T. Shingel, "General techniques for constructing variational integrators," *Frontiers of Mathematics in China*, vol. 7, no. 2, pp. 273–303, 2012.

[31] N. Bou-Rabee and J. E. Marsden, "Hamilton-Pontryagin integrators on lie groups part I: introduction and structure-preserving properties," *Foundations of Computational Mathematics*, vol. 9, no. 2, pp. 197–219, 2009.

[32] T. Lee, M. Leok, and N. H. McClamroch, "Lie group variational integrators for the full body problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 29-30, pp. 2907–2924, 2007.

[33] A. Iserles, H. Z. Munthe-Kaas, S. P. Nørsett, and A. Zanna, "Lie-Group methods," *Acta Numerica*, vol. 9, pp. 215–365, 2000.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 792478, 16 pages
http://dx.doi.org/10.1155/2014/792478

Research Article

Free Vibration Characteristic of Multilevel Beam Based on Transfer Matrix Method of Linear Multibody Systems

Laith K. Abbas and Xiaoting Rui

Institute of Launch Dynamics, Nanjing University of Science and Technology, Nanjing 210094, China

Correspondence should be addressed to Laith K. Abbas; laithabbass@yahoo.com

Received 18 August 2013; Accepted 28 November 2013; Published 20 January 2014

Academic Editor: Caishan Liu

Copyright © 2014 L. K. Abbas and X. Rui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, an approach based on transfer matrix method of linear multibody systems (MS-TMM) is developed to analyze the free vibration of a multilevel beam, coupled by spring/dashpot systems attached to them in-span. The Euler-Bernoulli model is used for the transverse vibration of the beams, and the spring/dashpot system represents a simplified model of a viscoelastic material. MS-TMM reduces the dynamic problem to an overall transfer equation which only involves boundary state vectors. The state vectors at the boundaries are composed of displacements, rotation angles, bending moments, and shear forces, which are partly known and partly unknown, and end up with reduced overall transfer matrix. Nontrivial solution requires the coefficient matrix to be singular to yield the required natural frequencies. This paper implements two novel algorithms based on the methodology by reducing the zero search of the reduced overall transfer matrix's determinate to a minimization problem and demonstrates a simple and robust algorithm being much more efficient than direct enumeration. The proposal method is easy to formulate, systematic to apply, and simple to code and can be extended to complex structures with any boundary conditions. Numerical results are presented to show the validity of the proposal method against the published literature.

1. Introduction

The vibration problem of beam-type structures is of particular urgent issue in many branches of modern aerospace, mechanical, and civil engineering. Natural vibration frequencies and modes are one of the most important dynamic characteristics of these kinds of systems. For example, the precision in manufacturing can be highly influenced by vibrations. If the vibration characteristics cannot be solved or preestimated exactly when designing a mechanism system, it is often hard to obtain a good dynamic performance of the mechanism system and consequently hard to control its vibration.

There are different types of beam models. One of the well-known models is the Euler-Bernoulli beam theory that works well for slender beams. According to the Euler-Bernoulli beam theory, the length of each beam section is much greater than the height of each section and the shear and rotary inertia effects are ignored.

The vibration theory of single-beam systems is well developed and studied in detail in hundreds of contributions. The vibration of systems composed of uniform double-beam coupled by translational springs or elastic layers have been studied extensively in the literature. Inceoğlu and Gürgöze [1] studied the bending vibrations of a combined system consisting of two clamped-free beams carrying tip masses to which several double spring-mass systems are attached across the span. Using Green's function method, the frequency equation of the system is established. Kukla [2] solved the problem of free vibration of two axially loaded beams which are connected by many translational springs. The solution contains possible combinations of the classical boundary conditions. The technique of the solution consists of developing a Green function. In [3], the vibrations of uniform beams connected by homogeneous elastic layer are devoted. Oniszczuk [4] discussed the free transverse vibrations of two parallel simply supported beams continuously joined by a Winkler elastic layer. The motion of the system was described by a homogeneous set of two partial differential equations, which was solved by using the classical Bernoulli-Fourier method. Oniszczuk [5] is devoted to analyze the undamped forced transverse vibrations of an elastically

connected complex double-beam system in the case of simply supported beams. The classical modal expansion method was applied to determine the dynamic responses of the beams due to arbitrarily distributed continuous loads. Vu et al. [6] presented an exact method for solving the vibration problem of a damped double-beam system subjected to harmonic excitation. The double-beam system consists of two identical beams with the same boundary conditions on both sides. The beams are connected by a viscoelastic layer. Natural frequencies and mode shapes of vibration of the system are determined and the forced vibrations are investigated. Gürgöze and Erol [7] determined the natural frequencies of a clamped-free double-beam system carrying tip masses to which several spring mass systems are attached across the span. However, there are only few contributions dealing with the vibration of multibeam systems. That is, probably, the general vibration analyses of an elastically connected multibeam system are complicated and laborious in view of a large variety of possible combinations of boundary conditions, and thus, the solution of the governing coupled partial differential equations is difficult [8].

Multibody system dynamics (MSD) has become an important theoretical tool for wide engineering problems analysis in the world. Lots of methods of MSD have been studied by many authors on theory and computational method [9–15]. Professor Rui Xiaoting and his students have been enlightened by the method of letting state vectors (SVs) be transferred into classical transfer matrix method and built up a new multibody dynamics method called "Transfer Matrix Method of Linear Multibody Systems-MS-TMM" [16]. Using MS-TMM, the eigenvalue of linear multi-rigid-flexible-body system is computed easily, the computational ill-condition is overcome, and the computational efficiency is increased. Over 20 years, MS-TMM has been developed and used widely in engineering applications.

Motivated by the interesting study by Kukla [2] which was published on the problem of the natural longitudinal vibrations of two rods coupled by many translational springs and by the two novel algorithms based on the new recursive scanning approach presented by Bestle et al. [17], this paper presents a unique yet simple scenario of obtaining the exact free vibration characteristics of undamped/damped multilevel beam coupled elastically. The scenario developed in this paper is based on MS-TMM and Euler-Bernoulli beam theory.

The text is organized as follows. The problem statement is presented in Section 2. In Sections 3 and 4, the general theorem brief of MS-TMM and problem solution scenario are shown. In Section 5, some results calculated by MS-TMM and the other method are given which can validate the proposed method. The conclusions are presented in Section 6.

2. Problem Statement

The transversely vibrating system in a plane under consideration consists of multi-level parallel, elastic, and homogeneous Euler-Bernoulli beam with general boundary conditions. Beam bending stiffness is EI, where E is the elastic modulus and I is the area moment of inertia. Beam mass per unit length $\overline{m} = \rho A$, where ρ is the material density and A is the beam cross-section area. Of course, any different materials, different dimensions, or different boundary conditions could be considered in MS-TMM. The beams have the same length L and are joined by the j-system of spring/dashpot located at different positions (e.g., x_1 and x_j, where j is the number of spring/dashpot systems between two levels). k_y and c are linear translational spring constant and damping coefficient, respectively, as shown in Figure 1.

3. MS-TMM Strategy in the Context of Free Vibration Characteristics

3.1. MS Topologies. According to the natural attribute of bodies, a complicated MS can be represented by various bodies (e.g., rigid bodies, elastic bodies, lumped masses, etc.) interconnected by hinges (e.g., spherical joints, sliding joints, cylindrical joints, dampers, springs, etc.). In MS-TMM, there are different topologies based on a certain set of modeling variables to formulate the dynamic equations of MS. Such topologies are chain, tree, closed loop (as illustrated in Figure 2 for reference), network, and so forth.

3.2. State Vector, State Variables, and Transfer Direction. The state vector (SV) at a connection point of MS is a column vector denoting the mechanics state of this point. It includes the displacements of the point (including angular displacements) and the corresponding internal forces (including internal moments). Therefore, the SV is given by kinematics (displacements) and kinetics (internal forces) quantities, called *state variables*. For convention in this paper, $\mathbf{z}$ with bold lowercase represents the SV in the physical coordinates and $\mathbf{Z}$ with bold capital represents the SV in the modal coordinates. Vibrations in space are described by displacement coordinates x, y, z along the Cartesian axes and angular rotations $\theta_x, \theta_y, \theta_z$ about Cartesian axes. Cutting forces and moments are given by q_x, q_y, q_z and m_x, m_y, m_z, respectively. Positive directions at input points are shown in Figure 3(a). Positive directions of forces and moments at output points (Figure 3(b)) are opposite due to the principle of action equals reaction. In 3D case with $n_s = 12$ (n_s is the number of state variables in the SV), the SVs in physical and modal coordinates at the connection point $p_{i,k}$ (where the first subscript i is the serial number of element for boundary end and the second subscript k is the serial number of the hinge element and $k = 0$ for boundary end) are summarized in a vector, receptively:

$$\begin{aligned} &\mathbf{z}_{i,k}\big|_{\text{physical coordinates}} \\ &\quad = \left[x, y, z, \theta_x, \theta_y, \theta_z, m_x, m_y, m_z, q_x, q_y, q_z\right]_{i,k}^T, \\ &\mathbf{Z}_{i,k}\big|_{\text{modal coordinates}} \\ &\quad = \left[X, Y, Z, \Theta_x, \Theta_y, \Theta_z, M_x, M_y, M_z, Q_x, Q_y, Q_z\right]_{i,k}^T. \end{aligned} \tag{1}$$

Figure 1: A system of a multi-level beam with a distributed spring/dashpot system(s).

Figure 2: Multibody systems (a) chain, (b) tree, and (c) closed loop topology.

In case of 1D or 2D applications, the SV will be reduced as shown later. Defining a boundary point of the MS as the transfer end, the direction from all other boundary points to the transfer end is called *transfer direction*. Along the transfer direction, the nodes entering into elements are called inputs denoted by I and the nodes leaving from elements are called outputs O.

3.3. Transfer Equation, Transfer Matrix, Overall System Transfer Matrix, and Overall System State Vector. A vibrating MS comprised of n-components, see Figure 4, is used as an example to show how to deduce the overall transfer equation and overall transfer matrix of the system. In order to describe conveniently the idea, the chain topology is considered in the following. This vibrating system is comprised of n components and $n+1$ connection points. The SVs of the boundary right extremity and other boundary left extremity of the system are expressed as $\mathbf{z}_{n,0}$ and $\mathbf{z}_{1,0}$, respectively. Transfer direction of the system is always from another boundary end to may call it the *root*. Following the transfer direction shown in Figure 3, the *transfer equation* between the component input and output is

$$\begin{aligned}
\mathbf{z}_{n,0} &= \mathbf{U}_n \mathbf{z}_{n-1,n} \\
\mathbf{z}_{n-1,n} &= \mathbf{U}_{n-1} \mathbf{z}_{n-2,n-1} \\
\mathbf{z}_{n-2,n-1} &= \mathbf{U}_{n-2} \mathbf{z}_{n-3,n-2} \\
&\vdots \\
\mathbf{z}_{1,2} &= \mathbf{U}_1 \mathbf{z}_{1,0}.
\end{aligned} \tag{2}$$

The constant matrix $\mathbf{U}_k$ is the *transfer matrix* of the kth component. Transfer matrices of basic components are considered as building blocks, which can be assembled together to provide the transfer matrix of the whole system according to the chain MS-TMM topology as follows:

$$\mathbf{z}_{n,0} = \mathbf{T}\mathbf{z}_{1,0}, \quad \text{where } \mathbf{T} = \prod_{k=0}^{n-1} \mathbf{U}_{n-k}. \tag{3}$$

Figure 3: Sign convention at (a) input; (b) output.

Figure 4: TMM-MS in the sense of chain topology. A vibrating system comprised of n-components with $n+1$ connections.

Rewrite (2) as

$$\mathbf{U}_{\text{all}}\big|_{n_s\times(2\times n_s)}\,\mathbf{z}_{\text{all}}\big|_{(2\times n_s)\times 1} = \mathbf{0}, \tag{4a}$$

where

$$\mathbf{z}_{\text{all}}^T = \left[\mathbf{z}_{1,0}^T \quad \mathbf{z}_{n,0}^T\right]^T, \tag{4b}$$

$$\mathbf{U}_{\text{all}} = \left[\mathbf{T} \quad -\mathbf{I}_{n_s}\right]. \tag{4c}$$

Herein, $\mathbf{U}_{\text{all}}$ is the *overall system transfer matrix* and $\mathbf{z}_{\text{all}}$ is the *overall system state vector*.

3.4. Eigenfrequency Equation of the Whole MS. The overall transfer equation (4a) only involves the boundary SVs, and the SVs at all other connection points do not appear. The SVs at the boundary are composed of displacements, rotation angles, moments, and shears, which are partly known and partly unknown. For common boundary conditions, half of state variables of $\mathbf{z}_{\text{all}}$ (4b) are zeros due to known constraints. Thus, (4a) reduces to $\overline{\mathbf{U}}_{\text{all}}\overline{\mathbf{z}}_{\text{all}} = \mathbf{0}$, where $\overline{\mathbf{z}}_{\text{all}}$ is composed of the unknown state variables and $\overline{\mathbf{U}}_{\text{all}}$ is a square matrix resulting from elimination of all columns of $\mathbf{U}_{\text{all}}$ (4c) associated zeros in $\mathbf{z}_{\text{all}}$. For harmonic vibrations, solutions maybe written as $\mathbf{z}_{i,j} = \mathbf{Z}_{i,j}e^{\lambda t}$ where $\lambda = -\lambda^r \pm i\lambda^i$, $\lambda^r, \lambda^i \in \mathbb{R}$ are the eigenvalues. The real part $(-\lambda^r)$ is related to the magnitude of damping, where the imaginary part (λ^i) is related to the vibration frequency of the damped system. For undamped systems, $\lambda^r = 0$ and $\lambda^i = \omega$. Finally, $\overline{\mathbf{U}}_{\text{all}}$ is only a function of the unknown λ_i of the system. For nontrivial solutions, the Eigenfrequency equation

$$\Delta(\lambda) = \det \overline{\mathbf{U}}_{\text{all}} \overset{!}{=} 0 \tag{5}$$

has to be fulfilled. The natural frequencies of the system can now be computed.

3.5. Beam Transfer Matrix. The full derivation of transfer matrices for the Timoshenko and Euler-Bernoulli beams vibrating in a plane (with kinematics and kinetics' SV defined as $\mathbf{z} = [y, \theta_z, m_z, q_y]^T$) may be found in [18], which is an open access article and the reader may download it from the Internet. However, for completeness, only the transfer matrix

for the Euler-Bernoulli beam will be presented, Figure 5. The differential equation of a Euler-Bernoulli beam is

$$EI\frac{\partial^4 y}{\partial x^4}+\overline{m}\frac{\partial^2 y}{\partial t^2}=0 \xrightarrow{y(x,t)=Y(x)e^{\lambda t}} \frac{\partial^4 Y(x)}{\partial x^4}+\frac{\overline{m}\lambda^2}{EI}Y(x)=0,$$

$$\text{where } Y(x)=A_1\cosh\beta x+A_2\sinh\beta x+A_3\cos\beta x+A_4\sin\beta x, \tag{6}$$

A_1, A_2, A_3, and A_4 are arbitrary constants, and $\beta=\sqrt[4]{-\overline{m}\lambda^2/(EI)}$. For the Euler-Bernoulli beam, the linearized relations in modal coordinates $\Theta_z=Y'$, $M_z=EIY''$, and $Q_y=M_z'$ maybe added to end up with the transfer relation:

$$\mathbf{Z}(x)=\mathbf{B}(x)\,\mathbf{a} \quad \text{or} \quad \begin{bmatrix} Y \\ \Theta_z \\ M_z \\ Q_y \end{bmatrix}_x = \begin{bmatrix} \cosh\beta x & \sinh\beta x & \cos\beta x & \sin\beta x \\ \beta\sinh\beta x & \beta\cosh\beta x & -\beta\sin\beta x & \lambda\cos\beta x \\ EI\beta^2\cosh\beta x & EI\beta^2\sinh\beta x & -EI\beta^2\cos\beta x & -EI\beta^2\sin\beta x \\ EI\beta^3\sinh\beta x & EI\beta^3\cosh\beta x & EI\beta^3\sin\beta x & -EI\beta^3\cos\beta x \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}. \tag{7}$$

The coefficient vector $\mathbf{a}=[A_1,A_2,A_3,A_4]^T$ summarizes the unknown constants to be adopted to boundary conditions. At input end $\mathbf{Z}_I$ $(x=0)$, we get $\mathbf{Z}_I=[\mathbf{B}(0)]\mathbf{a}$. Thus, the coefficient vector can be expressed as $\mathbf{a}=[\mathbf{B}(0)]^{-1}\mathbf{Z}_O$ and substituting it into (7) for the beam output end at $x=l$, one gets

$$\mathbf{Z}_O=[\mathbf{B}(l)]\,\mathbf{a}=[\mathbf{B}(l)]\,[\mathbf{B}(0)]^{-1}\mathbf{Z}_I=\mathbf{U}\mathbf{Z}_I, \tag{8a}$$

where

$$\mathbf{U}=\mathbf{B}(l)\,\mathbf{B}^{-1}(0)=\begin{bmatrix} S & \frac{T}{\beta} & \frac{U}{EI\beta^2} & \frac{V}{EI\beta^3} \\ \beta V & S & \frac{T}{EI\beta} & \frac{U}{EI\beta^2} \\ EI\beta^2 U & EI\beta V & S & \frac{T}{\beta} \\ EI\beta^3 T & EI\beta^2 U & \beta V & S \end{bmatrix},$$

$$S=\frac{ch+c}{2},\quad T=\frac{sh+s}{2},$$

$$U=\frac{ch-c}{2},\quad V=\frac{sh-s}{2},$$

$$ch=\cosh(\beta l),\quad sh=\sinh(\beta l),$$

$$c=\cos(\beta l),\quad s=\sin(\beta l) \tag{8b}$$

is the transfer matrix of the Euler-Bernoulli beam component.

3.6. Summary. In the context of free vibration characteristic, the general strategy of linear MS-TMM, in summary, is as follows.

(1) Break up the complicated MS into components with simple dynamic properties, which can be expressed in matrix form and SVs (for each component, it is possible to obtain the close form expression of the transfer matrix giving the displacements and the forces applied to one extremity to the displacements and forces applied to the other extremity). In other words, on the component level, the governing partial and ordinary differential or algebraic equations are transformed to algebraic transfer equations, where the output state results from a product of the input state and an element specific transfer matrix. These component matrices are considered as building blocks. In fact, the transfer matrix of such components needs not to be rededuced but may be taken directly from a transfer matrix library.

(2) Following the transfer direction that has been designed already by the analyst and according to the topology of the MS, these component transfer matrices are then assembled and end up with a system of linear algebraic equations called the overall transfer equation.

(3) Substitute the boundary conditions into the overall transfer equation to construct the eigenfrequency equation. Consequently, the vibration characteristics such as frequencies can be deduced as the roots of a transcendental equation. Due to narrow couples of natural frequencies, the classical zero search method is likely to fail. However, a new recursive scanning approach for minima of the absolute values of the determinant shows much more efficiency and reliability than direct enumeration.

4. Problem Solution Scenario

Figure 6 illustrates the suggested scenario for the solution of the problem statement and as follows. There are $(1:m)$ multi-level beam. Each beam level is divided into $(1:n)$ components, which have $(0:n+1)$ connection points and coupled with another beam level by a system (say $1:j$) of a viscoelastic material (modeled as a spring/dashpot system). The massless dummy body as shown in Figure 6

Figure 5: The direction of the state vector of Euler-Bernoulli beam component.

Figure 6: Structure of a multi-level beam with the distribution of spring/dashpot.

(e.g., A, B, C, and so on) is a connection point between the two beam segment components and spring/dashpot system(s). The main key of MS-TMM is transferring the SV from one component to another following the general transfer equation (2). For the beam segment component, the transfer matrix is available (8b), while it is not for spring/dashpot system at the connection point between two or multi-level beam in this paper. However, it is based on the kinematics and kinetics of the spring/dashpot to formulate the transfer matrix and it needs two steps. First, as an example, let us consider a connection point j between two beam segment components and system of spring/dashpot may be grounded or connected to another connection point. Figure 7(a) sketches the first step. According to the continuity, j has identical displacements, angles, and moments at input and output:

$$Y_O = Y_I, \qquad \Theta_{z_O} = \Theta_{z_I}, \qquad M_{z_O} = M_{z_I}. \tag{9a}$$

And from the force analysis shown in Figure 7(b), the spring/dashpot force F changes the shear forces:

$$Q_{y_O} = Q_{y_I} + F. \tag{9b}$$

Equations (9a) and (9b) in the view of the SV as a matrix form

$$\mathbf{Z}_O = \mathbf{Z}_I + \mathbf{e}_4 F. \tag{9c}$$

FIGURE 7: (a) Two beam segments and spring/dashpot system connected at massless dummy body, (b) state vectors (SV) at the massless dummy body j-connection point, (c) force analysis of a spring/dashpot system, (d) a system 1,1 connected two levels 1-2 through two connection points A and B shown in Figure 6, and (e) two systems $j, 1$ and $1, m-1$ connected with multi-level (1-2-3) through three connection points G, C, and V shown in Figure 6.

$\mathbf{e}_4 = [0\ 0\ 0\ 1]^T$ is a unit vector assigning F to the transfer equation of the massless connection body. The spring compression and damping forces are given as (see Figure 7(c))

$$F_{\text{spring}} = k_y\left(Y_I - Y_O\right) = k_y \Delta Y = k_y \mathbf{e}_1^T \Delta \mathbf{Z}$$
$$F_{\text{dashpot}} = c\left(\dot{y}_I - \dot{y}_O\right) \xrightarrow{y = Ye^{\lambda t}} = c\lambda\left(Y_I - Y_O\right) = c\lambda \mathbf{e}_1^T \Delta \mathbf{Z}$$
$$\longrightarrow F_{\text{sys}} = \left(k_y + c\lambda\right)\mathbf{e}_1^T \Delta \mathbf{Z}. \tag{10}$$

Herein, $\mathbf{e}_1^T = [1\ 0\ 0\ 0]$ is a unit vector, which extracts Y from the transfer SV. Substitute (10) into (9c) and knowing that $F = \pm F_{\text{sys}}$, we ended the first step with

$$\mathbf{Z}_O = \mathbf{Z}_I \pm \mathbf{D}\Delta\mathbf{Z},$$
$$\text{where } \mathbf{D} = \left(k_y + c\lambda\right)\mathbf{e}_4\mathbf{e}_1^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_y + c\lambda & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

For system 1,1 connected two levels 1-2 through two connection points A and B shown in Figure 6, the second step is how to deduce the transfer equation between the input and output related to these two levels. For the connection point A (see Figure 7(d)) and using (11) with $F = F_{\text{sys}}$,

$$\begin{aligned} \mathbf{Z}_{1,O} &= \mathbf{Z}_{1,I} + \mathbf{D}_{1,1}\left(\mathbf{Z}_{2,I} - \mathbf{Z}_{1,I}\right) \\ &= \underbrace{(\mathbf{I} - \mathbf{D}_{1,1})}_{\mathbf{U}_{1,1}^d}\mathbf{Z}_{1,I} + \underbrace{\mathbf{D}_{1,1}}_{\mathbf{U}_{1,1}^c}\mathbf{Z}_{2,I} \\ \mathbf{Z}_{1,O} &= \mathbf{U}_{1,1}^d\mathbf{Z}_{1,I} + \mathbf{U}_{1,1}^c\mathbf{Z}_{2,I}. \end{aligned} \tag{12a}$$

Similarly, for connection point B and using (11) with $F = -F_{\text{sys}}$,

$$\begin{aligned} \mathbf{Z}_{2,O} &= \mathbf{Z}_{2,I} - \mathbf{D}_{1,1}\left(\mathbf{Z}_{2,I} - \mathbf{Z}_{1,I}\right), \\ \mathbf{Z}_{2,O} &= \mathbf{U}_{1,1}^c\mathbf{Z}_{1,I} + \mathbf{U}_{1,1}^d\mathbf{Z}_{2,I}. \end{aligned} \tag{12b}$$

Combining (12a) and (12b) in one matrix form, we ended up the second step by writing the transfer equation and transfer matrix of a system located between two levels as

$$\begin{Bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{Bmatrix}_O = \underbrace{\begin{bmatrix} \mathbf{U}_{1,1}^d & \mathbf{U}_{1,1}^c \\ \mathbf{U}_{1,1}^c & \mathbf{U}_{1,1}^d \end{bmatrix}}_{\text{transfer matrix}} \begin{Bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{Bmatrix}_I. \tag{12c}$$

The transfer equation for two systems j, 1 and 1, $m-1$ connected with multi-level (1-2-3) through three connection points G, C, and V shown in Figures 6 and 7(e) can be obtained similarly:

$$\begin{Bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}_{j,1}^d & \mathbf{U}_{j,1}^c & \mathbf{0} \\ \mathbf{U}_{j,1}^c & \mathbf{U}_{j,1/1,m-1}^d & \mathbf{U}_{1,m-1}^c \\ \mathbf{0} & \mathbf{U}_{1,m-1}^c & \mathbf{U}_{1,m-1}^d \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \end{Bmatrix}_I,$$
$$\text{where } \mathbf{U}_{j,1}^d \equiv \mathbf{I} - \mathbf{D}_{j,1}, \quad \mathbf{U}_{j,1}^c \equiv \mathbf{D}_{j,1} \tag{12d}$$
$$\mathbf{U}_{1,m-1}^d \equiv \mathbf{I} - \mathbf{D}_{1,m-1}, \quad \mathbf{U}_{1,m-1}^c \equiv \mathbf{D}_{1,m-1}$$
$$\mathbf{U}_{j,1/1,m-1}^d \equiv \mathbf{I} - \mathbf{D}_{j,1/1,m-1},$$
$$\mathbf{D}_{j,1/1,m-1} = \left(\left(k_y + c\lambda\right)_{j,1} + \left(k_y + c\lambda\right)_{1,m-1}\right)\mathbf{e}_4\mathbf{e}_1^T.$$

Now we are in the position to end the solution scenario. From the MS-TMM topology point of view, the system shown in Figure 1 or equivalently Figure 6 can be modeled as a multichain, following the similar procedures mentioned above in Section 3.3 to determine the overall transfer equation and ending up with determinant solution of the reduced overall transfer matrix.

5. Numerical Examples

Before the free vibration analysis of a multi-level elastic beam coupled by a spring/dashpot system(s) is performed, the reliability of the suggested scenario and the Matlab computer program developed for this paper are confirmed by comparing the present results with those obtained from the existing literature.

5.1. Reliability of the Suggested Scenario. Figure 8(a) consists of two free-free and clamped-clamped undamped beams connected to each other with two symmetrically distributed linear translational springs. The hybrid system is vibrating transversely in the $x - y$ plane. The beams are supposed to have the same length, material, and geometrical data, that is, ($L = 1$ m, $\overline{m}_1 = \overline{m}_2 = \overline{m}$ and $(EI)_1 = (EI)_2 = EI$). It is worth mentioning here that the proposal scenario can be extended to a system consisting of any number of uniform/nonuniform beams coupled with any number of spring/dashpot systems. However, the two springs $k_{y_{1,1}}$ and $k_{y_{2,1}}$ are located at distances x_1 and $x_2 = L - x_1$, respectively, and are moving as a function of x_1 from 0 to 0.5 m as shown in Figure 8(a). This example is presented by Kukla [2], which studied the dynamics of the longitudinal vibrations of two rods coupled by several translational springs using Green's functional method. Figure 8(b) shows a vibrating two-level beam comprised of 10 components with 4 boundaries, namely, $\mathbf{Z}_{1,0}$, $\mathbf{Z}_{6,0}$ on the system left side and $\mathbf{Z}_{5,0}$, $\mathbf{Z}_{10,0}$ on the right side. Herein, $\mathbf{Z}_{i,k} =$

(a) (b)

Figure 8: (a) Free-free and clamped-clamped beams connected with two linear translational springs, (b) chain TMM-MS: State vectors and transfer direction of the suggested solution scenario.

$[Y, \Theta_z, M_z, Q_y]^T_{i,k}$. Chain MS-TMM topology is systematic to apply. Following the transfer direction from the left to right and applying (2) for two-level, the transfer equation can be written as

$$\begin{Bmatrix} \mathbf{Z}_{5,0} \\ \mathbf{Z}_{8,0} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_8 \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{4,5} \\ \mathbf{Z}_{4,8} \end{Bmatrix}_I$$

$$\downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{4,5} \\ \mathbf{Z}_{4,8} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}^{4,d}_{2,1} & \mathbf{U}^{4,c}_{2,1} \\ \mathbf{U}^{4,c}_{2,1} & \mathbf{U}^{4,d}_{2,1} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{3,4} \\ \mathbf{Z}_{7,4} \end{Bmatrix}_I$$

$$\downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{3,4} \\ \mathbf{Z}_{7,4} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_7 \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{2,3} \\ \mathbf{Z}_{2,7} \end{Bmatrix}_I$$

$$\downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{2,3} \\ \mathbf{Z}_{2,7} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}^{2,d}_{1,1} & \mathbf{U}^{2,c}_{1,1} \\ \mathbf{U}^{c}_{1,1} & \mathbf{U}^{2,d}_{1,1} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{1,2} \\ \mathbf{Z}_{6,2} \end{Bmatrix}_I$$

$$\downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{1,2} \\ \mathbf{Z}_{6,2} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_6 \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{1,0} \\ \mathbf{Z}_{6,0} \end{Bmatrix}_I$$

$$\Downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{5,0} \\ \mathbf{Z}_{8,0} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{U}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_8 \end{bmatrix} \begin{bmatrix} \mathbf{U}^{4,d}_{2,1} & \mathbf{U}^{4,c}_{2,1} \\ \mathbf{U}^{4,c}_{2,1} & \mathbf{U}^{4,d}_{2,1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_7 \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{U}^{2,d}_{1,1} & \mathbf{U}^{2,c}_{1,1} \\ \mathbf{U}^{c}_{1,1} & \mathbf{U}^{2,d}_{1,1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_6 \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{1,0} \\ \mathbf{Z}_{6,0} \end{Bmatrix}_I$$

$$\downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{5,0} \\ \mathbf{Z}_{8,0} \end{Bmatrix}_O = \begin{bmatrix} \begin{pmatrix} \mathbf{U}_5\mathbf{U}^{4,d}_{2,1}\mathbf{U}_3\mathbf{U}^{2,d}_{1,1}\mathbf{U}_1 \\ +\mathbf{U}_5\mathbf{U}^{4,c}_{2,1}\mathbf{U}_7\mathbf{U}^{2,c}_{1,1}\mathbf{U}_1 \end{pmatrix} & \begin{pmatrix} \mathbf{U}_5\mathbf{U}^{4,d}_{2,1}\mathbf{U}_3\mathbf{U}^{2,c}_{1,1}\mathbf{U}_6 \\ +\mathbf{U}_5\mathbf{U}^{4,c}_{2,1}\mathbf{U}_7\mathbf{U}^{2,d}_{1,1}\mathbf{U}_6 \end{pmatrix} \\ \begin{pmatrix} \mathbf{U}_8\mathbf{U}^{4,c}_{2,1}\mathbf{U}_3\mathbf{U}^{2,d}_{1,1}\mathbf{U}_1 \\ +\mathbf{U}_8\mathbf{U}^{4,d}_{2,1}\mathbf{U}_7\mathbf{U}^{2,c}_{1,1}\mathbf{U}_1 \end{pmatrix} & \begin{pmatrix} \mathbf{U}_8\mathbf{U}^{4,c}_{2,1}\mathbf{U}_3\mathbf{U}^{2,c}_{1,1}\mathbf{U}_6 \\ +\mathbf{U}_8\mathbf{U}^{4,d}_{2,1}\mathbf{U}_7\mathbf{U}^{2,d}_{1,1}\mathbf{U}_6 \end{pmatrix} \end{bmatrix}$$

$$\times \begin{Bmatrix} \mathbf{Z}_{1,0} \\ \mathbf{Z}_{6,0} \end{Bmatrix}_I$$

$$\downarrow$$

$$\begin{Bmatrix} \mathbf{Z}_{5,0} \\ \mathbf{Z}_{8,0} \end{Bmatrix}_O = \begin{bmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,1} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_{1,0} \\ \mathbf{Z}_{6,0} \end{Bmatrix}_I. \tag{13}$$

$\mathbf{U}_1$, $\mathbf{U}_3$, $\mathbf{U}_5$, $\mathbf{U}_6$, $\mathbf{U}_7$, and $\mathbf{U}_8$ represent the beam segment transfer matrices (8b). $\mathbf{U}^{2,d}_{1,1}$, $\mathbf{U}^{2,c}_{1,1}$, and $\mathbf{U}^{4,d}_{2,1}$, $\mathbf{U}^{4,c}_{2,1}$ are the transfer matrices of a system between two connection points located between two beams (12c). The overall transfer equation, overall SV, and overall transfer matrix are, respectively,

$$\mathbf{U}_{\text{all}}\big|_{8\times 16}\ \mathbf{Z}_{\text{all}}\big|_{16\times 1} = \mathbf{0}, \tag{14a}$$

$$\text{where } \mathbf{Z}^T_{\text{all}} = \left[\underbrace{\mathbf{Z}_{1,0}}_{4\times 1}{}^T\ \underbrace{\mathbf{Z}_{6,0}}_{4\times 1}{}^T\ \underbrace{\mathbf{Z}_{5,0}}_{4\times 1}{}^T\ \underbrace{\mathbf{Z}_{8,0}}_{4\times 1}{}^T \right]^T, \tag{14b}$$

$$\mathbf{U}_{\text{all}} = \begin{bmatrix} \mathbf{T}_{1,1}\big|_{4\times 4} & \mathbf{T}_{1,2}\big|_{4\times 4} & -\mathbf{I}_{4\times 4} & \mathbf{O}_{4\times 4} \\ \mathbf{T}_{2,1}\big|_{4\times 4} & \mathbf{T}_{2,2}\big|_{4\times 4} & \mathbf{O}_{4\times 4} & -\mathbf{I}_{4\times 4} \end{bmatrix}. \tag{14c}$$

Applying the boundary conditions listed in Table 1, half of state variables of $\mathbf{Z}_{\text{all}}$ (14b) are zeros due to known constraints. Thus, the overall transfer equation reduced to $\overline{\mathbf{U}}_{\text{all}}|_{8\times 8}\overline{\mathbf{Z}}_{\text{all}}|_{8\times 1} = \mathbf{0}$ that is ready for eigenproblem (5).

Table 1: Common boundary conditions for a beam vibrating in a plane.

Support type	Zero terms	Nonzero terms
Fixed	Y, Θ_z	Q_y, M_z
Pinned	Y, M_z	Q_y, Θ_z
Free	Q_y, M_z	Y, Θ_z
Guided	Q_y, Θ_z	Y, M_z

The natural frequencies of the system can now be computed by zero search of the determinant, which is based on sign change of $\Delta(\omega)$ (in case of real numbers) or $\Delta(\lambda)$ (in case of complex numbers) during a scanning of an interesting frequency range. This procedure can be cumbersome for several reasons. Therefore, reliable and efficient algorithms called recursive scanning approach are applied (see [17] for more details of the proposed algorithm) by switching from zero search for Δ to minimization of the absolute value $|\Delta|$ of the determinant, which is equally well applicable to both the real and the complex cases. The algorithm general idea is as follows. In a first iteration step, it divides an interesting band of frequencies into a number of sample points and searches for lower peaks. Each region having a lower peak is then divided again into small intervals to find more narrow regions of lower peaks as second iteration step. The algorithm proceeds until the required precision of roots is achieved. MS-TMM natural frequency results are obtained using fMin1D algorithm [17] (frequency ranges $\underbrace{[1, 3000]}_{\lambda^i=\omega}$ rad/sec, scanning grids $N_{x0} = 500$, absolute precision tolerance $\varepsilon = 10^{-6}$). Figure 9 shows $\log_{10}|\Delta|$ obtained from the fMin1D algorithm versus the first 10 dimensionless frequency values ($\overline{\omega} = \sqrt[4]{(\lambda^i \equiv \omega)^2 \overline{m} L^4/EI}$) for $k_{y_{1,1}} = k_{y_{2,1}} = k_y = 100(EI/L^3)$ and $x_1 = 0.3$ m. $\overline{\omega}$ values are plotted versus the location of the two springs in Figures 10(a) and 10(b) for $k_y = 100(EI/L^3)$ and $k_y = 1000(EI/L^3)$, respectively. The first and second springs are moving opposite to each other from $x_1 = 0$ at the beams support ends and coincide together at the middle of the beams ($x_1 = 0.5$ m). Reference [2] does not provide the results as tables or specific data but only showed figures. The reader may download the reference from the library or from the Internet because the authors of the present paper cannot include the figures to maintain the journal publication rights. However, the MS-TMM results are in very good agreement with Figures 2(a) and 2(b) [2, page 131].

5.2. Free Vibration of Damped Elastically Coupled Triple Beams. The physical model of the transversely vibrating system under consideration is composed of three parallel uniform rectangular Euler-Bernoulli beams of homogenous properties. Each two-level beam is joined (connected) together by two spring/dashpot systems located at x_1 and x_2, respectively (see Figure 1). The beams have the same length and are pinned at their ends. The small damped vibrations of the system are considered. In the sense of the chain MS-TMM (see Figure 11), the overall transfer equation, overall transfer matrix, and overall SV are, respectively,

$$\mathbf{U}_{\text{all}}\mathbf{Z}_{\text{all}} = \mathbf{0}, \tag{15a}$$

$$\text{where } \mathbf{U}_{\text{all}} = \begin{bmatrix} \mathbf{T}_{1,1}\big|_{4\times4} & \mathbf{T}_{1,2}\big|_{4\times4} & \mathbf{T}_{1,3}\big|_{4\times4} & -\mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{T}_{2,1}\big|_{4\times4} & \mathbf{T}_{2,2}\big|_{4\times4} & \mathbf{T}_{2,3}\big|_{4\times4} & \mathbf{O}_{4\times4} & -\mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{T}_{3,1}\big|_{4\times4} & \mathbf{T}_{3,2}\big|_{4\times4} & \mathbf{T}_{3,3}\big|_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & -\mathbf{I}_{4\times4} \end{bmatrix}_{12\times24}, \tag{15b}$$

$$\mathbf{Z}_{\text{all}}^T = \left[\mathbf{Z}_{1,0}{}^T \;\; \mathbf{Z}_{6,0}{}^T \;\; \mathbf{Z}_{11,0}{}^T \;\; \mathbf{Z}_{5,0}{}^T \;\; \mathbf{Z}_{10,0}{}^T \;\; \mathbf{Z}_{13,0}{}^T\right]_{24\times1}^T, \tag{15c}$$

$$\begin{aligned}
\mathbf{T}_{1,1} &= \mathbf{U}_5\mathbf{U}_{2,1}^{4,d}\mathbf{U}_3\mathbf{U}_{1,1}^{2,d}\mathbf{U}_1 + \mathbf{U}_5\mathbf{U}_{2,1}^{4,c}\mathbf{U}_8\mathbf{U}_{1,1}^{2,c}\mathbf{U}_1, \\
\mathbf{T}_{1,2} &= \mathbf{U}_5\mathbf{U}_{2,1}^{4,d}\mathbf{U}_3\mathbf{U}_{1,1}^{2,c}\mathbf{U}_6 + \mathbf{U}_5\mathbf{U}_{2,1}^{4,c}\mathbf{U}_8\mathbf{U}_{1,1/1,2}^{2/7,d}\mathbf{U}_6, \\
\mathbf{T}_{1,3} &= \mathbf{U}_5\mathbf{U}_{2,1}^{4,c}\mathbf{U}_8\mathbf{U}_{1,2}^{7,c}\mathbf{U}_{11}, \\
\mathbf{T}_{2,1} &= \mathbf{U}_{10}\mathbf{U}_{2,1}^{4,c}\mathbf{U}_3\mathbf{U}_{1,1}^{2,d}\mathbf{U}_1 + \mathbf{U}_{10}\mathbf{U}_{2,1/2,2}^{4/9,d}\mathbf{U}_8\mathbf{U}_{1,1}^{2,c}\mathbf{U}_1, \\
\mathbf{T}_{2,2} &= \mathbf{U}_{10}\mathbf{U}_{2,1}^{4,c}\mathbf{U}_3\mathbf{U}_{1,1}^{2,c}\mathbf{U}_6 + \mathbf{U}_{10}\mathbf{U}_{2,1/2,2}^{4/9,d}\mathbf{U}_8\mathbf{U}_{1,1/1,2}^{2/7,d}\mathbf{U}_6 + \mathbf{U}_{10}\mathbf{U}_{2,2}^{9,c}\mathbf{U}_{12}\mathbf{U}_{1,2}^{7,c}\mathbf{U}_6, \\
\mathbf{T}_{2,3} &= \mathbf{U}_{10}\mathbf{U}_{2,1/2,2}^{4/9,d}\mathbf{U}_8\mathbf{U}_{1,2}^{7,c}\mathbf{U}_{11} + \mathbf{U}_{10}\mathbf{U}_{2,2}^{9,c}\mathbf{U}_{12}\mathbf{U}_{1,2}^{7,d}\mathbf{U}_{11}, \\
\mathbf{T}_{3,1} &= \mathbf{U}_{13}\mathbf{U}_{2,2}^{9,c}\mathbf{U}_8\mathbf{U}_{1,1}^{2,c}\mathbf{U}_1, \\
\mathbf{T}_{3,2} &= \mathbf{U}_{13}\mathbf{U}_{2,2}^{9,c}\mathbf{U}_8\mathbf{U}_{1,1/1,2}^{2/7,d}\mathbf{U}_6 + \mathbf{U}_{13}\mathbf{U}_{2,2}^{9,d}\mathbf{U}_{12}\mathbf{U}_{1,2}^{7,c}\mathbf{U}_6, \\
\mathbf{T}_{3,3} &= \mathbf{U}_{13}\mathbf{U}_{2,2}^{9,c}\mathbf{U}_8\mathbf{U}_{1,2}^{7,c}\mathbf{U}_{11} + \mathbf{U}_{13}\mathbf{U}_{2,2}^{9,d}\mathbf{U}_{12}\mathbf{U}_{1,2}^{7,d}\mathbf{U}_{11}.
\end{aligned} \tag{15d}$$

Figure 9: fMin1D function determinant of a system consisting of free-free and clamped-clamped beams coupled with two springs. $k_y = 100(EI/L^3)$ and $x_1 = 0.3$ m.

Herein, $\mathbf{U}_1$, $\mathbf{U}_3$, $\mathbf{U}_5$, $\mathbf{U}_6$, $\mathbf{U}_8$, $\mathbf{U}_{10}$, $\mathbf{U}_{11}$, $\mathbf{U}_{12}$, and $\mathbf{U}_{13}$ represent the beam segment transfer matrices; $\mathbf{U}_{1,1}^{2,d}, \mathbf{U}_{1,1}^{2,c}, \mathbf{U}_{2,1}^{4,d}, \mathbf{U}_{2,1}^{4,c}, \mathbf{U}_{1,2}^{7,d}, \mathbf{U}_{1,2}^{7,c}, \mathbf{U}_{2,2}^{9,d}$, and $\mathbf{U}_{2,2}^{9,c}$ are the transfer matrices of a system between two connection points located between two beams (12c). $\mathbf{U}_{1,1/1,2}^{2/7,d}$ and $\mathbf{U}_{2,1/2,2}^{4/9,d}$ are the transfer matrices for two systems connected with three-level through three connection points (12d). Applying the boundary conditions, that is, $\mathbf{Z}_{1,0} = [0, \Theta_z, 0, Q_y]_{1,0}^T$; $\mathbf{Z}_{6,0} = [0, \Theta_z, 0, Q_y]_{6,0}^T$; $\mathbf{Z}_{11,0} = [0, \Theta_z, 0, Q_y]_{11,0}^T$, $\mathbf{Z}_{5,0} = [0, \Theta_z, 0, Q_y]_{5,0}^T$, $\mathbf{Z}_{10,0} = [0, \Theta_z, 0, Q_y]_{10,0}^T$, and $\mathbf{Z}_{13,0} = [0, \Theta_z, 0, Q_y]_{13,0}^T$, the overall transfer equation (15a) reduced to $\underbrace{\overline{\mathbf{U}}_{\text{all}}}_{12\times 12}\underbrace{\overline{\mathbf{Z}}_{\text{all}}}_{12\times 1} = \mathbf{0}$. The geometric and material properties (steel) of the multi-level beam system are given as follows: $L = 1$ m, $b_1 = b_2 = b_3 = 0.02$ m, $h_1 = h_2 = h_3 = 0.01$ m, $E_1 = E_2 = E_3 = E = 2.069 \times 10^{11}$ N/m^2, and $\rho_1 = \rho_2 = \rho_3 = \rho = 7850$ kg/m^3. Here, b_i and h_i represent the width and height of the beam, respectively. The cross-sectional area and the moment of inertial of the beam cross-section are $A_i = b_i \times h_i$ and $I_i = b_i \times h_i^3/12$, respectively. For comparison, the lowest three "exact" dimensionless frequency values $\overline{\omega}$ for the single pinned-pinned beam that might be found in structural or vibration text books

Figure 10: Dimensionless frequency parameter $\overline{\omega}$ values as a function of the spring locations ($x_1 = 0 \rightarrow 0.5$ m) for a system consisting of free-fee and clamped-clamped beams coupled by two springs. (a) $k_y = 100(EI/L^3)$; (b) $k_y = 1000(EI/L^3)$.

Figure 11: Chain TMM-MS: state vectors and transfer direction of pinned-pinned multi-level beam connected by spring/dashpot systems.

Figure 12: fMin1D function determinant to evaluate the lowest three $\overline{\omega}$ for undamped, uncoupled pinned-pinned three-beam.

FIGURE 13: (a) The lowest three $\overline{\omega}$ values as a function of the spring systems locations ($x_1 = 0 \rightarrow 0.5$ m) for undamped, coupled pinned-pinned three-beam; (b) fMin1D function determinant to evaluate the dimensionless frequency parameter $\overline{\omega}$ for $k_y = 25(EI/L^3)$ and $x_1 = 0.25$ m.

FIGURE 14: fMin2D first step: damping ($\lambda^r = \delta$) range scanning of a specific frequency band ($\lambda^i = \omega \leq 1500$ rad/sec) for damped, coupled pinned-pinned three-beam, (a) $[-2000, 2000]$, (b) $[-1000, 1000]$, (c) $[-500, 500]$, and (d) $[-100, 600]$.

Figure 15: fMin2D second step: function determinant to evaluate the eigenvalues ($\lambda = -\lambda^r \pm i\lambda^i$) for a damped, coupled pinned-pinned three-beam.

are 3.1416, 6.2832 and 9.4248. MS-TMM results for undamped, uncoupled pinned-pinned three-beam are 3.141592, 6.28318, and 9.424777 as shown in Figure 12 using fMin1D algorithm. For undamped elastically coupled three-beam, Figure 13 shows that $\overline{\omega}$ values were plotted versus the location movement of spring systems. Systems 2 and 4 ($k_{y_{1,1}} = k_{y_{1,2}} = k_y = 25\ (EI/L^3)$) and systems 7 and 9 ($k_{y_{1,2}} = k_{y_{2,2}} = k_y = 25\ (EI/L^3)$) are moving opposite to each other from ($x_1 = 0 \rightarrow 0.5\,\text{m}$). The system shows symmetric and antisymmetric vibrations. For symmetric vibration, the frequency parameters $\overline{\omega}_{1,1}$, $\overline{\omega}_{1,2}$, and $\overline{\omega}_{1,3}$ (where l and p are the beam number and vibration mode, resp.) are constant throughout the spring systems movement as seen in Figure 13(a). The antisymmetric of the first mode vibration $\overline{\omega}_{2,1}$ and $\overline{\omega}_{3,1}$ started with equally values as $\overline{\omega}_{1,1}$ at $x_1 = 0$ (spring systems are rigidly mounted at the support ends) and increased significantly at $x_1 = 0.5\,\text{m}$ (both spring systems are in the same position). However, it is evident that $\overline{\omega}_{2,1}$ and $\overline{\omega}_{3,1}$ are sensitive to the variation of spring positions. The results presented of the other antisymmetric vibration modes, $\overline{\omega}_{2,2}$, $\overline{\omega}_{3,2}$, $\overline{\omega}_{2,3}$, and $\overline{\omega}_{3,3}$, indicate that as the order of the frequency increases, the shape of the frequency vibration versus the spring systems location resembles a half or full sine wave with insignificant variation in the frequency ranges. The strong ability of fMin1D algorithm is shown in Figure 13(b) by capturing the very narrow regions of lower peaks.

The effect of the dashpot is included in the system making the model more complicated. The two spring/dashpot systems coupled with three-beam are located at position $x_1 = 0.25\,\text{m}$. The spring/dashpot system in parallel is similar to viscoelastic material model. The given values of damping are $c_{1,1} = c_{2,1} = c_{1,2} = c_{2,2} = c = 5\sqrt{EI\ \overline{m}/L^2}$. Next, it is the turn to implement fMin2D algorithm [17] for complex eigenproblem solution. For undamped given system as it is computed above, $\overline{\omega}$ values do not exceed more than 10, which is equivalent to ($\lambda^i = \omega \leq 1500\,\text{rad/sec}$). However, the damping ($\lambda^r = \delta$) is unknown in range. Therefore, within the required system frequency band, a couple scanning grids are demanded to understand and explore the damping ranges of the system. Figure 14 shows different scanning grids for damping ranges as follows: (a) $[-2000, 2000]$, (b) $[-1000, 1000]$, (c) $[-500, 500]$, and (d) $[-100, 600]$. It is obviously seen that there are 9 promising regions distributed randomly on the surface. These regions represent the complex eigenvalues of the system. In the following searching complex roots, the damping range (Figure 14(d)) is considered. MS-TMM for damped, coupled three-beam results are shown in Figure 15 and Table 2 after using fMin2D algorithm coded under Matlab environment with the following input data: ($\underbrace{[-1, 1500]}_{\lambda^i}$, $\underbrace{[-100, 600]}_{\lambda^r}$, $N_{x0} = 35$, $N_{y0} = 20$, $\varepsilon_x = 10^{-8}$, $\varepsilon_y = 10^{-8}$).

Chain MS-TMM for undamped/damped coupled multi-level beam with the suggested scenario and the two novel recursive scanning algorithms provide a closed-form solution, not only presents the principles of the vibration problem but also shed light on practical applications. Since the solution is almost exact, it allows a complete understanding of a problem.

6. Conclusions

Starting from the principle of mechanics and the elementary formulations for the flexible beam, the free vibration analysis of laterally vibrating system made up of a multi-level Euler-Bernoulli beam to which spring/dashpot systems are attached across the span is performed using one of the Transfer Matrix Method of Linear Multibody Systems (MS-TMM) scenarios. Although the number of coupling springs or spring/dashpot systems considered in the examples given was limited to three, there is no inherent difficulty in extending the current method to solve the problems of vibration of systems consisting of any number of uniform/nonuniform beams with different boundary conditions and coupled with any number of spring/dashpot systems. The numerical results obtained to reveal that the eigenfrequencies calculated by this method are in very good agreement with those obtained by the published literature. Moreover, MS-TMM is encouraging for further

Table 2: Chain MS-TMM eigenvalues results of damped, coupled pinned-pinned three-beam.

$\lambda = -\lambda^r \pm i\lambda^i$	Chain MS-TMM results using fMin2D λ^i (rad/sec)
λ_1	$-2.463512 \times 10^2 + 4.010226 \times 10^1 i$
λ_2	$-2.074369 \times 10^{-9} + 1.462698 \times 10^2 i$
λ_3	$-7.409510 \times 10^1 + 1.652322 \times 10^2 i$
λ_4	$-4.577982 \times 10^2 + 4.615502 \times 10^2 i$
λ_5	$-2.074369 \times 10^{-9} + 5.850790 \times 10^2 i$
λ_6	$-1.484833 \times 10^2 + 5.861790 \times 10^2 i$
λ_7	$-2.159398 \times 10^2 + 1.243120 \times 10^3 i$
λ_8	$-7.448404 \times 10^1 + 1.311750 \times 10^3 i$
λ_9	$-2.074369 \times 10^{-9} + 1.316428 \times 10^3 i$

investigations of more complex multibody systems of this type with rigid bodies due to simplicity in the formulation of the transfer equation, being systematic to apply, and being easy to program.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research was supported by the Research Fund for the Doctoral Program of Higher Education of China (20113219110025), the Natural Science Foundation of China Government (11102089), and the Program for New Century Excellent Talents in University (NCET-10-0075).

References

[1] S. Inceoğlu and M. Gürgöze, "Bending vibrations of beams coupled by several double spring-mass systems," *Journal of Sound and Vibration*, vol. 243, no. 2, pp. 370–379, 2001.

[2] S. Kukla, "Free vibration of the system of two beams connected by many translational springs," *Journal of Sound and Vibration*, vol. 172, no. 1, pp. 130–135, 1994.

[3] S. Kukla and B. Skalmierski, "Free vibration of a system composed of two beams separated by an elastic layer," *Journal of Theoretical and Applied Mechanics*, vol. 32, no. 3, pp. 581–590, 1994.

[4] Z. Oniszczuk, "Free transverse vibrations of elastically connected simply supported double-beam complex system," *Journal of Sound and Vibration*, vol. 232, no. 2, pp. 387–403, 2000.

[5] Z. Oniszczuk, "Forced transverse vibrations of an elastically connected complex simply supported double-beam system," *Journal of Sound and Vibration*, vol. 264, no. 2, pp. 273–286, 2003.

[6] H. V. Vu, A. M. Ordóñez, and B. H. Karnopp, "Vibration of a double-beam system," *Journal of Sound and Vibration*, vol. 229, no. 4, pp. 807–822, 2000.

[7] M. Gürgöze and H. Erol, "On laterally vibrating beams carrying tip masses, coupled by several double spring-mass systems," *Journal of Sound and Vibration*, vol. 269, no. 1-2, pp. 431–438, 2004.

[8] M. Abu-Hilal and N. Beithou, "Free transverse vibrations of a triple-beam system," *Journal of Mechanical Engineering*, vol. 58, no. 1, pp. 30–50, 2007.

[9] W. Schiehlen, *Multibody Systems Handbook*, Springer, Berlin, Germany, 1990.

[10] W. Schiehlen, "Multibody system dynamics: roots and perspectives," *Multibody System Dynamics*, vol. 1, no. 2, pp. 149–188, 1997.

[11] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, Edited by: B. G. Teubner, Stuttgart, Germany, 1977.

[12] J. Wittenburg, *Dynamics of Multibody Systems*, Springer, Berlin, Germany, 2nd edition, 2008.

[13] A. A. Shabana, *Dynamics of Multibody Systems*, Cambridge University Press, New York, NY, USA, 3rd edition, 2010.

[14] A. A. Shabana, *Computational Dynamics*, John Wiley & Sons, New York, NY, USA, 3rd edition, 2010.

[15] A. A. Shabana, "Flexible multibody dynamics: review of past and recent developments," *Multibody System Dynamics*, vol. 1, no. 2, pp. 189–222, 1997.

[16] X. Rui, L. Yun, Y. Lu, B. He, and G. Wang, *Transfer Matrix Method of Multibody System and Its Application*, Science Press, Beijing, China, 2008, (Chinese).

[17] D. Bestle, L. K. Abbas, and X. Rui, "Recursive eigenvalue search algorithm for transfer matrix method of linear flexible multibody systems," *Multibody System Dynamics*, 2013.

[18] L. K. Abbas, M. J. Li, and X. Rui, "Transfer matrix method for the determination of the natural vibration characteristics of realistic thrusting launch vehicle—part I," *Mathematical Problems in Engineering*, vol. 2013, Article ID 764673, 16 pages, 2013.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 365265, 9 pages
http://dx.doi.org/10.1155/2014/365265

Research Article

Case Study on Human Walking during Wearing a Powered Prosthetic Device: Effectiveness of the System "Human-Robot"

Svetlana Grosu,[1] Pierre Cherelle,[1] Chris Verheul,[2] Bram Vanderborght,[1] and Dirk Lefeber[1]

[1] *Department of Mechanical Engineering, Vrije Universiteit Brussel, 1050 Brussels, Belgium*
[2] *SayField International, Broeksloot 10, 3474 HP Zegveld, The Netherlands*

Correspondence should be addressed to Svetlana Grosu; sgrosu@vub.ac.be

Received 12 September 2013; Accepted 16 November 2013; Published 9 January 2014

Academic Editor: Xiaoting Rui

Copyright © 2014 Svetlana Grosu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It is known that force exchanges between a robotic assistive device and the end-user have a direct impact on the quality and performance of a particular movement task. This knowledge finds a special reflective importance in prosthetic industry due to the close human-robot collaboration. Although lower-extremity prostheses are currently better able to provide assistance as their upper-extremity counterparts, specific locomotion problems still remain. In a framework of this contribution the authors introduce the multibody dynamic modelling approach of the transtibial prosthesis wearing on a human body model. The obtained results are based on multibody dynamic simulations against the real experimental data using AMP-Foot 2.0, an energy efficient powered transtibial prosthesis for actively assisted walking of amputees.

1. Introduction

A definition of the functionalities/duties between a human and a robotic device, also the organization of their interaction, basically, includes a number of different criteria that influence the effectiveness of the "human-robot" system. The hierarchy of criteria importance depends on a general approach in a certain domain application. Generally, the requirements in a robotic device design should assure the maximum economical effectiveness of the system in combination with a personal security of the end-user.

Robots for physical assistance to humans are meant to reduce fatigue and stress, increase human capabilities in terms of force, speed, and precision, and improve in general the quality of life. In other words, the crucial goal of a robot for physical human-robot interactions (pHRI) is a generation of supplementary forces to overcome human physical limits. Moreover, the human can bring experience, global knowledge, and understanding for a correct execution of movements [1]. In case of assistive devices, an improved analysis of the problems related to the physical interaction with robots becomes mandatory. Also, in a special perspective for the interaction with humans should be considered the design of the mechanism, sensors selection, actuators, and control architecture [2].

Compared with healthy persons, walking amputees require 10–60% more metabolic energy depending on walking speed, physical individual properties, cause of amputation, amputation level, and prosthetic intervention characteristics. Furthermore, amputees walk at 11–40% slower self-selected gait speed than do persons with intact limbs [3, 4]. To date, commercially available prostheses comprise spring structures that store and release elastic energy throughout each walking stance period [5]. Due to their passive nature, such prostheses are unable to generate more mechanical energy than what is stored during each walking step. Also, experiences in clinical environment indicate that transtibial (TT) amputees suffer from a nonsymmetrical gait while wearing a completely passive prosthesis [6]. In distinction, the human ankle performs positive net work and has a greater peak power over the stance period, especially at moderate to fast walking speeds [7, 8].

In literature several engineering challenges still slowing down the further development of a powered ankle-foot prosthesis [9, 10] are discussed. In the field of prosthetic leg design, a critical objective is to progress a powered ankle

prosthesis capable of mimicking the dynamics of the human ankle. A study of TT prosthetics research contributions shows that none of the commercially available devices are skillful of mimicking an human ankle-foot complex. With current actuator technology, it is challenging to build an ankle-foot prosthesis that matches the size and weight of the human ankle-foot structure but still provides sufficient stance-period work and instantaneous power output to drive an amputee. In 1998, Klute and colleagues [11] were the first to build a powered ankle-foot prosthesis efficient in performing net positive work. Their device employed a pneumatic actuation strategy with an off-board power supply. More recent work has focused on the design of energetically autonomous powered systems [12–14]. In the growing field of rehabilitation robotics, the use of compliant actuators is becoming a standard where accurate trajectory tracking is not required. The advantages of such actuators are represented by safely interaction with the patient and large forces absorption during gait. In the particular case of trans-tibial prostheses, compliance of the actuation provides even more advantages. Besides shock absorption in case of collision with objects during walking, energy provided by the actuator (e.g., electric motor) can be stored into its elastic element (e.g., spring in series). The current state-of-the-art in powered ankle-foot prostheses with focus on devices using compliant actuators has been presented in [15].

In the next sections, the authors are focused on the influences on normal human gait of the forces that are generated by a motor and forces that are stored in and then released by springs, also the reaction of the mechanism as a whole in interaction with user. Then, the comparison between the real "human-robot" setup and virtual model is discussed.

2. The Ankle Mimicking Prosthetic- (AMP-) Foot 2.0: Background

This section is dedicated to a summary description of the prosthesis used in the study. The section includes presentation of the AMP-Foot 2.0 mechanical design and validation part in the framework of real experiments, with an amputee subject participation.

2.1. Design of the AMP-Foot 2.0. The AMP-Foot 2.0, see Figure 1, claims a new energy concept, based on a principle of optimal power distribution which is elaborated in [16]. Basically, the working principle is similar to existing powered assistive devices, except that the actuator is working during the complete stance phase. Gravitational potential energy is gradually stored into a series elastic element, in time which the drive still has to provide the same torque and power output. But there is approximately 3 times more time available to generate the power by the electric motor. As a consequence, the overall power rating of the actuator can be divided by approximately the same amount, reducing the weight and size of the drive power considerably.

The functional principle of the AMP-Foot 2.0 uses a "plantar flexion (PF)" spring, which stores energy from the controlled dorsiflexion phase of stance. An electric actuator is loading a "push-off (PO)" spring during the complete stance phase. The prosthesis includes a locking mechanism which provides the energy implementation into the PO spring and can be delayed and released at push-off. This way, full torque and power required for locomotion can be obtained with less power from the actuator.

Figure 1: Design representation of AMP-Foot 2.0.

Structurally, the device consists of three bodies: a leg, a foot, and a lever arm, pivoting around the ankle axis; see Figure 2. As mentioned before, the system comprises 2 spring sets: a PF and a PO spring set. The PF spring is placed between a fixed point p on the foot and a cable that runs over a pulley a to the lever arm at point b and is attached to the lever arm at point c, while the PO spring is placed between the motor-ballscrew assembly and a fixed point d on the lever arm. A critical part of this device is the locking mechanism, that is meant to bear high forces while being as compact and lightweight as possible and is represented by a four bar linkage moving in and out of its singular position. The working principle of such a system has been proposed in [17].

The locking mechanism which provides a rigid connection between the leg and the lever arm when energy is injected into the system is not represented in Figure 2.

In order to illustrate the behaviour of the AMP-Foot 2.0 prosthesis, the gait cycle is divided in 5 phases starting with a controlled plantar flexion from *heel strike* (HS) to *foot flat* (FF); see Figure 3. A step is initiated by touching the ground with the heel. During this phase the foot rotates with respect to the leg, until θ (= ϕ) reaches approximately -5°. This is followed by a controlled dorsiflexion phase ending in push-off at *heel off* (HO), during which a generation of propulsive forces by Soleus and Gastrocnemius muscle groups will have place. In the late stance phase, the torque produced by the ankle decreases until the leg enters the swing phase at *toe off* (TO). Once the leg is engaged in the *swing phase*, the foot resets the locking mechanism.

Figure 2: Schematic representation of AMP-Foot 2.0.

The more detailed description of AMP-Foot 2.0 dynamic behaviour during a complete gait cycle can be found under contribution [16].

2.2. Validation of the AMP-Foot 2.0 Device. The experiments with participation of one disabled patient were effectuated by the research group of Vrije Universiteit Brussel [18] and, in present paper, the authors are referring to results obtained from those experiments. Briefly, in the performed experiments one transfemoral amputee subject of 75 kg was considered and the experiment was divided in three trials: (1) walking at a self-selected speed on a treadmill; (2) walking speed raised to a faster cadence; (3) an overground walking at self-selected speed; see Figure 4 [19].

Conform analysed data acquired during the experiments, and compared to existing powered prosthetic devices, it was observed that the AMP-Foot 2.0 prosthesis can improve an amputee gait, in conditions when little power is required for the actuation. Since the present contribution is focused on modelling and simulation approach of human-robot system, all additional details regarding the real experiment design, prosthesis behaviour, power consumption, and torque characteristics are given in [18].

3. Modelling and Simulations in Virtual Environment

In this section, the authors present modelling methods of the AMP-Foot 2.0 device. Then, modelling and simulation results of an amputee walking using the assistive device are described. Further, the comparison of a normal walking (healthy person) and the same person wearing the AMP-Foot 2.0 is effectuated. The section is concluded with discussion on obtained results from simulations and the real experiments, referred in Section 2.2.

Computational dynamics has grown in last year's along the need to develop simulations and analysis for mechanical systems that consist of interconnected bodies. Simulation is meant to describe and analyze the behaviour of various system configurations, ask what-if questions about the real system, and optimize the structural design.

Due to the high complexity of modern robotic systems, almost any research conducted in the area of robotics can benefit from a simulation of the system behaviour before experiments on a real platform take place [20]. The computational modelling including anatomic, physiologic, and engineering analyses serve to study various activities in a normal and pathological condition of humans. Computer simulations represent an effective, faster, and cheapest approach than experiments, which necessarily consume physical resources. Computer modelling is considered the most effective, when employed in combination with real experiments.

3.1. Modelling Approach for AMP-Foot 2.0. A 3D model was developed in conformity with the real mechanical properties of AMP-Foot 2.0 device. In other words, all material and structural characteristics are preserved relative to the existing prototype. In order to model and simulate the AMP-Foot 2.0 prosthesis, MSC Adams 3D multibody dynamics software was used [21]. The geometries of the AMP-Foot 2.0 component parts were exported from Autodesk Inventor software in MSC Adams environment and were converted into a set of Adams/View geometry elements. This importing approach reduces the need to recreate geometry primitives within Adams software and enhances the ability to realistically view the behavior of complicated mechanical systems. After importing, the co-rrelated constraints between all geometry parts were defined and applied to the model. Two spring sets (PF and PO), which play a critical role in AMP-Foot 2.0 dynamic behaviour, have been modelled according to the real design stiffness and damping characteristics. The stiffness of the PF spring is about 300 N/mm and for every PO spring the value of stiffness is modeled as 60 N/mm. Damping values were determined experimentally, and were found to be 10 Ns/mm for PF spring set and 1.2 Ns/mm for each PO spring respectively. The FF phase is considered as the initial position of the model. In the beginning stage of this study the model dynamic behaviour, see Figure 5, was analysed without human body model and was elaborated in [22]. The AMP-Foot 2.0 model complexity can be appreciated and visualised in Figure 5, where moving parts and constraints of the model are presented. There are 11 degrees of freedom (DoF) in the AMP-Foot 2.0 system.

The model includes 2 actuation forces: one represents the motor actuator, which is constantly pulling up the PO springs from HS to TO period of the gait cycle; the second force is responsible for triggering the locking mechanism. The control of the force actuation period is based on timing approach, implemented in the model in form of a STEP function.

The STEP function approximates the Heaviside step function with a cubic polynomial and has the following format:

Figure 3: Working principle of AMP-Foot 2.0 during a complete gait cycle.

(a) (b)

Figure 4: Overground walking at self-selected speed (~2.9 km/h).

Figure 5: The AMP-Foot 2.0 model composition and DoF.

STEP (x, x_0, h_0, x_1, h_1). It has continuous first derivatives. Its second derivatives are discontinuous at $x = x_0$ and $x = x_1$, where x is the independent variable (time, in present model), x_0 is a real variable that specifies the x value at which the STEP function begins, x_1 is a real variable that specifies the x value at which the STEP function ends, h_0 is the initial value of the step, and h_1 is the final value of the step.

The arrows, that can be observed in Figure 6, are representing the actuation and tension forces during the simulation.

Due to cables nonlinear geometric properties and a complex static and dynamic behaviour, the challenge was to model the cable segments which play an important role in a force transmission system of the AMP-Foot 2.0. The cable transmission system was modelled using TKC toolbox feature and the tension forces inside of the system were obtained [22].

The cable properties, such as density, Young's modulus, axial stiffness, and strain value, are defined with respect to the ones used in real mechanical system and were determined experimentally. These properties are presented in Figure 6(c) and can be easily modified for further experimental purposes within the framework of Graphical User Interface (GUI), developed in MSC Adams software; see Figure 6(c); the mass of the cable elements is assumed to be negligible.

3.2. Modelling and Simulation of Human Walking during Wearing the AMP-Foot 2.0 Prosthesis. There are specialized commercially available simulation tools that can be used for analysis of human walking, such as AnyBody [23], OpenSim [24], SIMM [25], and LifeMod [26]. However, these tools cannot be used for analyzing human-robot interactions, which is becoming an essential requirement for modeling and simulation of robotic systems as their physical interaction with humans becomes more complex. In general, the modeling and simulation phase of the development of such systems is becoming demanding. Since the robot and robot-human interactions increase in complexity, the simulation with a single simulation tool is not effective anymore. Therefore, in this research work the combination of MSC Adams with LifeMode plug-in was used as the most appropriate.

3.2.1. Methods. The creation of human models begins by generating a base human segment set, followed by joints, soft tissues, and contact elements between the model and the

Figure 6: (a, b) Simulation of AMP-Foot 2.0 behaviour. (c) Cable properties GUI.

Figure 7: Joints representation and definition.

environment. In order to study the influences of AMP-Foot 2.0 device on normal overground walking the mechanical model of the human body was built using the Lifemodeler plug-in in framework of MSC Adams environment. This combination of tools supports the analysis of the "human-robot system" effectiveness and mutual interaction. The considered model, see Figure 8, includes a model of human's lower body (two legs and pelvis) which is rigidly attached to the right extremity AMP-Foot 2.0 prosthesis. Once the segments of the lower body are established, joints are created between the segments. Along an inverse-dynamics simulation, joints learn angulation patterns while the model is being driven by the motion capture data. The nominal joints stiffness for both legs at the hip, knee, and the ankle is

Figure 8: Model of human lower limbs cosimulated with AMP-Foot 2.0.

Figure 9: Ankle angle during normal walking of the same subject: red graph with AMP-Foot 2.0; blue line using the healthy foot.

assumed to be 1e5 Ns/mm and the damping to be 1e4 Ns/mm. The joint consists of a triaxis hinge and passive or active forces acting on each of the three degrees of freedom. They are implemented as an assembly of two "virtual" bodies of negligible mass and inertia and three revolute joints; see Figure 7. For every joint there are three axes: sagittal, transverse, and frontal. The Lifemode software offers the possibility to define every axis in a way suitable to certain application. The axes of lower limb joints in present human model are defined as in the table in Figure 7.

The settings specified for every joint axis in the model are interpreted as follows.

Driven. Kinematically driven using data from a driving spline.

Passive. A torsional spring force with user-specified stiffness, damping, angular limits, and limit stiffness values. These joints are used in an inverse dynamics analysis to record the joint angulations while the model is being manipulated with motion agents.

Hybrid III. The Hybrid III strength model is created for the individual joint axis with a user-specified scale value. The Hybrid III strength model is based on physical measurements of an actual crash dummy. The strength model consists of nonlinear stiffness, damping, and frictional values and also includes joint limit stop stiffness with hysteresis.

Servo. This selection creates a trained PD-servo type controller on the joint axis. The joint is commanded to track an angular history spline with a user-specified gain on the error between the actual angle and the commanded error. A user-specified derivative gain is specified to control the derivative of the error.

It is important to notice that the mass (75 kg) of the whole human body is considered in simulation, although just the lower part is represented. The human body model generated by the LifeMod plug-in stores the joint motion trajectories in MSC Adams. These trajectories are used during simulation as reference inputs for the joints' PD controllers which are also implemented in MSC Adams.

For simulation, driving of the human body is based on captured motion data obtained through marker-based motion capture system. Motion capture (MOCAP) systems track the trajectories of markers attached at various locations on the body. The marker trajectories are then used to train the human model. During the training, the response of the body is recorded and later used for a forward dynamics simulation. Marker trajectory data is used to drive elements called "motion agents," which are massless parts fixed to the body segments using spring elements.

The right foot part was removed and replaced by the AMP-Foot 2.0 model and connected to the right lower leg by a fixed joint; see Figure 8. Then, contact forces between prosthesis toe, heel parts, and the ground were defined as an IMPACT function model. The IMPACT function represents a simple model for contacts. It evaluates a force that turns on when a distance falls below a nominal free length (when two parts collide). The force has two components: a spring or stiffness component and a damping or viscous component. The stiffness component opposes the penetration. The damping component of the force is a function of the speed of penetration and opposes the direction of relative motion.

3.2.2. Results. The obtained simulation data were filtered by low pass filter and processed accordingly. In Figure 9 the ankle angle data of the human body model with the same physical properties as in real experiment are illustrated (weight 75 kg, height 175 cm, male) during normal overground walking. In the first simulation the human was walking without AMP-Foot 2.0 prosthesis, using a normal human foot model (see Figure 9, blue dot-line).

Afterwards, the normal foot was removed and replaced by the AMP-Foot 2.0 model and, therefore, we can follow from the graph (see red line representation) the consequences for this change in gait kinematics. Please notice that in the first slot of time the AMP-Foot 2.0 prosthesis is passive; there is no force acting on the motor part and pulling the spring sets. The single actuation is coming from the human leg during

Figure 10: Torque values for right limb.

Figure 11: Knee angle: red line, with AMP-Foot 2.0 model; grey line, with human foot model.

walking only. In the grey zone, see Figure 9, the AMP-Foot 2.0 becomes actuated and, as a result, we can observe an increase of the ankle angle, due to the provided push off propulsion force of the prosthetic device.

In Figure 10 are illustrated the obtained torque values of the right lower limb with the AMP-Foot 2.0 prosthesis attached.

Also, a slight difference was observed at the knee level joint. In case the AMP-Foot 2.0 is attached to the human leg model, the authors noticed a small knee angle increase; see Figure 11. This occurrence can be explained by the lower compliancy in the TO moment. Then the locking mechanism opens, providing the additional propulsion force to the leg.

As was discussed before, the authors were interested to make a comparison between data from simulations with the data, based on real experiments; see Figure 12.

Conform Figure 12, the time-based data on level ground walking, one can observe the similarities between the angle ankle data of the human model wearing the AMP-Foot 2.0 device and the walking pattern of real amputee subject with attached prosthesis. It can be noticed for the human model with the AMP-Foot 2.0 that there is an increase of the ankle angle around HO moment. This fact can be the result of the influence of contact forces between the heel and the ground part. In other words, when the heel is touching the ground the

Figure 12: Ankle angle comparison: red line: simulation of human model during wearing the AMP-Foot 2.0; blue line: simulation of human model without AMP-Foot 2.0; green line: real experiment data of subject during wearing the AMP-Foot 2.0 device.

reaction forces are pushing up the AMP-Foot 2.0 prosthesis, by this way creating an additional torque motion.

4. Discussion

It is crucial for design and control developing of assistive devices to have a model of the robotic system interacting with the human before the actual physical system is ready for the use. Sometimes, performing the real experiments is too exhausting, expensive, or almost impossible and in this case modeling and simulation approach is considered very effective. In the framework of this paper the authors have researched in virtual environment the differences between human walking during wearing an assistive device and normal walking of the healthy person. In both simulation and real-life testing the same human model was considered. This experiment is impossible to be repeated in real life. It was expected to have some differences in the ankle angle data, but the main conclusion is that the human walking is not affected in negative way while wearing the AMP-Foot 2.0. The powered device is providing the human leg with additional propulsion force that helps locomotion.

If we will refer to real experiments, performed with an amputee person with comparison to results obtained from simulations with human model during wearing the AMP-Foot 2.0, we can notice many common similarities in walking pattern. Even, if the gait kinematics of people is individual, (in simulations a similar subject with the same physical characteristics was considered) the obtained results remain valid and can be interpreted. If one will do the comparison between the normal walking simulation and both experiments which uses the AMP-Foot 2.0 prosthesis (Figure 12), one will observe some difference in TO moment. In case of normal walking without the device the transaction of the foot from TO to a swing moment is more compliant. In case of simulations there can be other external factors that can slightly influence obtained data results, such as the simplified

contact definition (which is not so complex as in case of real foot-ground contact).

5. Conclusions

The main focus in the framework of the presented study was to inspect how the attached trans-tibial prosthesis will influence normal healthy human walking kinematics and how close the AMP-Foot 2.0 working behaviour comes to a real human foot subsystem. The effectiveness of human-robot interaction was analyzed during the tests with a real subject and in virtual environment. The authors admit that it is difficult to compare individuals having a different walking pattern, even if for simulation a subject with the same physical parameters as the one tested in real experiment was considered. Moreover, the simulation comparison (subject walking with and without AMP-Foot 2.0) is almost impossible to arrange in real life. This fact that the simulation results more interesting and significant.

Thanks to the built human-prosthesis model further improvements on prototype design parameters can be achieved and the prototype can be adapted to certain subject's individual properties. The result data demonstrate that AMP-Foot 2.0 prosthesis can successfully replace the human ankle-foot subsystem, provided some individual adjustments are made.

Conflict of Interests

The authors, Grosu Svetlana, Pierre Cherelle, Chris Verheul, Bram Vanderborght, and Dirk Lefeber, declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work has been partially supported by the European Commission 7th Framework Program as part of the CORBYS (Cognitive Control Framework for Robotic Systems) project under Grant Agreement no. FP7 ICT-270219.

References

[1] O. Khatib, K. Yokoi, O. Brock, K. Chang, and A. Casal, "Robots in human environments: basic autonomous capabilities," *International Journal of Robotics Research*, vol. 18, no. 7, pp. 684–696, 1999.

[2] A. de Santis, B. Siciliano, A. de Luca, and A. Bicchi, "An atlas of physical human-robot interaction," *Mechanism and Machine Theory*, vol. 43, no. 3, pp. 253–270, 2008.

[3] M.-J. Hsu, D. H. Nielsen, S.-J. Lin-Chan, and D. Shurr, "The effects of prosthetic foot design on physiologic measurements, self-selected walking velocity, and physical activity in people with transtibial amputation," *Archives of Physical Medicine and Rehabilitation*, vol. 87, no. 1, pp. 123–129, 2006.

[4] A. Esquenazi and R. DiGiacomo, "Rehabilitation after amputation," *Journal of the American Podiatric Medical Association*, vol. 91, no. 1, pp. 13–22, 2001.

[5] S. Ron, *Prosthetics and Orthotics: Lower Limb and Spinal*, Lippincott Williams & Wilkins, Baltimore, Md, USA, 2002.

[6] H. Bateni and S. J. Olney, "Kinematic and kinetic variations of below-knee amputee gait," *Journal of Prosthetics and Orthotics*, vol. 14, no. 1, pp. 2–10, 2002.

[7] D. A. Winter, "Biomechanical motor patterns in normal walking," *Journal of Motor Behavior*, vol. 15, no. 4, pp. 302–330, 1983.

[8] A. H. Hansen, D. S. Childress, S. C. Miff, S. A. Gard, and K. P. Mesplay, "The human ankle during walking: implications for design of biomimetic ankle prostheses," *Journal of Biomechanics*, vol. 37, no. 10, pp. 1467–1474, 2004.

[9] S. K. Au, P. Dilworth, and H. Herr, "An ankle-foot emulation system for the study of human walking biomechanics," in *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA '06)*, pp. 2939–2945, Orlando, Fla, USA, May 2006.

[10] D. A. Winter and S. E. Sienko, "Biomechanics of below-knee amputee gait," *Journal of Biomechanics*, vol. 21, no. 5, pp. 361–367, 1988.

[11] G. K. Klute, J. Czerniecki, and B. Hannaford, "Development of powered prosthetic lower limb," in *Proceedings of the 1st National Meeting, Veterans Affairs Rehabilitation R&D Service*, Washington, DC, USA, October 1998.

[12] S. K. Au, *Powered ankle-foot prosthesis for the improvement of amputee walking economy [Ph.D. thesis]*, MIT Department of Mechanical Engineering, Cambridge, Mass, USA, 2007.

[13] J. K. Hitt, R. Bellman, M. Holgate, T. G. Sugar, and K. W. Hollander, "The SPARKy (spring ankle with regenerative kinetics) project: design and analysis of a robotic transtibial prosthesis with regenerative kinetics," in *Proceedings of the 2007 ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE '07)*, pp. 1587–1596, Las Vegas, Nev, USA, September 2007.

[14] R. Versluys, G. Lenaerts, M. Van Damme et al., "Successful preliminary walking experiments on a transtibial amputee fitted with a powered prosthesis," *Prosthetics and Orthotics International*, vol. 33, no. 4, pp. 368–377, 2009.

[15] R. Versluys, P. Beyl, M. Van Damme, A. Desomer, R. Van Ham, and D. Lefeber, "Prosthetic feet: state-of-the-art review and the importance of mimicking human anklefoot biomechanics," *Disability and Rehabilitation: Assistive Technology*, vol. 4, no. 2, pp. 65–75, 2009.

[16] P. Cherelle, A. Matthys, V. Grosu, B. Vanderborght, and D. Lefeber, "The AMP-Foot 2.0: Mimicking intact ankle behavior with a powered transtibial prosthesis," in *Proceedings of the IEEE International Conference on Biomedical Robotics and Biomechatronics*, pp. 544–549, June 2012.

[17] G. van Oort, R. Carloni, D. J. Borgerink, and S. Stramigioli, "An energy efficient knee locking mechanism for a dynamically walking robot," in *Proceedings of the IEEE International Conference on Robotics and Automation*, pp. 2003–2008, May 2011.

[18] P. Cherelle, V. Grosu, A. Matthys, B. Vanderborght, and D. Lefeber, "Design and validation of the ankle mimicking prosthetic (AMP-) foot 2.0," *Transaction on Neural Systems and Rehabilitation Engineering*, 2013.

[19] http://www.youtube.com/watch?v=tXLQq9a5kyI.

[20] J. G. de Jalón and E. Bayo, *Kinematic and Dynamic Simulation of Multibody Systems: The Real-Time Challenge*, Springer, New York, NY, USA, 1994.

[21] http://www.mscsoftware.com/.

[22] S. Grosu, P. Cherelle, C. Verheul, B. Vanderborght, and D. Lefeber, "AMP-Foot 2.0 prosthesis dynamic behavior, preliminary computational multibody dynamics simulation results,"

in *Proceedings of the ECCOMAS Thematic Conference on Multibody Dynamics*, Zagreb, Croatia, July 2013.

[23] AnyBody, http://www.anybodytech.com.

[24] OpenSim, https://simtk.org.

[25] SIMM, http://www.musculographics.com.

[26] LifeMod, http://www.lifemodeler.com.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2014, Article ID 957684, 8 pages
http://dx.doi.org/10.1155/2014/957684

Research Article

Controller Parameters Tuning Based on Transfer Matrix Method for Multibody Systems

Hossam Hendy, Xiaoting Rui, Qinbo Zhou, and Mostafa Khalil

Institute of Launch Dynamics, Nanjing University of Science and Technology, Nanjing 210094, China

Correspondence should be addressed to Hossam Hendy; hosam_hendi@yahoo.com

Received 25 August 2013; Accepted 9 December 2013; Published 5 January 2014

Academic Editor: Caishan Liu

Copyright © 2014 Hossam Hendy et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Transfer matrix method for multibody systems (MS-TMM) is a rife method to multi-rigid-flexible-body systems dynamics model deduction due to that there are no needs to establish the global dynamics equations of the system. Its basic idea is transferring a state vector between the body input(s) and output(s); this idea is close to the linear theories in control analysis and design. In this paper, three controllers' parameters tuning techniques for the proposed system model using MS-TMM are utilized; one technique is applied to get the stability regions via the frequency response of MS-TMM derived model. Another technique considers a classical PID controller design through the analysis of step input response of the system, and the last technique can be applied in both time and frequency domains if the model has a known mathematical model. A car suspension system is considered to represent modeling and tuning problems. In-depth study of MS-TMM with control techniques and defining the controllers' parameters stability regions provide an opportunity to formulate a relationship between MS-TMM and control design for novel control applications due to the powerful strength of MS-TMM dealing with more complex problems of the controlled multibody systems.

1. Introduction

With the increments of complexity of multibody systems and the development of their design and control methods, the need for more elegant formulations of the equations of motion becomes an issue of paramount importance. Many methods and theories for developing the model of the multibody system dynamics and control are presented for such reasons. In transfer matrix method for multibody systems (MS-TMM) there are two cases to deal with control element, one is to express the control force with state of system of previous time such as the delay control, and the second is that the control force is relative to present state of system, such as real-time control systems. Rui et al. have trials to develop new controlled systems depending on deriving the dynamics equations using MS-TMM [1, 2]. Bestle et al. reformulated the car suspension system using MS-TMM as similar as classical control theory's structural diagram [3].

Proportional-Integration-Differentiation (PID) controller is prevalent in industry applications. Reference [4] indicated that more than 90% of feedback control loops are based on PID control and [5] indicated that more than 97% of regulatory controllers utilize the PID algorithm. But the tuning of the controller gains is a problem because many industrial models suffer some burdens such as nonlinearities, higher order, and time delay [6]. The basic idea of MS-TMM is transferring the state vector between the body input(s) and output(s); this idea is close to linear theories in control analysis and design. The control and feedback variables could be considered in the transfer equation of MS-TMM. The tuning of the controller parameters is a great issue to get a stabilized system. However Ziegler-Nichols technique is one of the simplest techniques of PID tuning; there are several disadvantages, such as many trials are needed to find the stability limits and also not all tuning values work efficiently for all situations [7–10]. Shamsuzzoha and Skogestad (2010) developed a new procedure for PI/PID tuning method in closed-loop mode, only one closed-loop step test is required to deduce PI controller values. In this method also it is simple to obtain the PID tuning parameters in one step [11, 12]. This paper is organized as follows: the proposed system is modeled by MS-TMM including the control law in Section 2. The

Figure 1: Dynamic model of controlled car.

Figure 2: The structural diagram of the controlled system [3].

design of PI controller values stability region that achieves a specified gain and phase margins, in addition to tuning the controller parameters using different techniques, is presented in Section 3. Simulation and results of car active suspension system model with the designed controllers are provided in Section 4. Conclusions are offered in Section 5.

2. Problem Formulation

A two-degree-of-freedom car dynamic model is established to simulate the system as shown in Figure 1. It does not only simplify the system design and analysis, but also represents most of full car features [13]. In Figure 1, m_1 is the sprung mass that represents the car body, m_3 is the unsprung mass which stands for the wheel, C_2 is the damping coefficient, and K_2 and K_4 are the springs stiffness. In this section, this model is deduced using MS-TMM. According to MS-TMM, the system is portioned into many elements which are numbered as 1 to 4 from up to down, where body 1 and 3 are lumped masses, element 4 is a linear spring, while the controlled element 2 includes spring and damper in parallel connections. The boundaries at two ends are noted as 0. The positive direction of x axis is also shown in the figure. It should be mentioned that the controlled force f_c serves as an internal force of the system.

The transfer direction is stipulated as the positive direction of x axis. The state vector of an arbitrary connection point of the system is defined as $\mathbf{Z}_{i,j} = [X, Q_x]^T_{i,j}$, where i and j stands for the sequence number of the body element and hinge element, respectively. Thereby, the transfer equation of each element is discussed as follows:

Considering the rise and fall of the road, the boundary end $x_{4,0}$ is assumed to be in the form of $x_{4,0} = Ae^{i\Omega t}$, where A is the complex amplitude and Ω is the frequency of the movement. Thus, in a steady-state sense, the whole system experiences the forced vibration in the sine form.

According to MS-TMM for the lumped mass 1 and 3, their transfer equations are

$$\mathbf{Z}_{j,O} = \mathbf{U}_j \mathbf{Z}_{j,I}, \quad (j = 1, 3), \tag{1}$$

where $\mathbf{U}_j = \begin{bmatrix} 1 & 0 \\ m_j\Omega^2 & 1 \end{bmatrix}$ and Ω is the external excitation frequency.

For the linear spring 4, the transfer equation is

$$\mathbf{Z}_{4,O} = \mathbf{U}_4 \mathbf{Z}_{4,I} \tag{2}$$

with transfer matrix $\mathbf{U}_4 = \begin{bmatrix} 1 & -1/K_4 \\ 0 & 1 \end{bmatrix}$.

For the controlled element 2, the transfer equation should be

$$\mathbf{Z}_{2,O} = \mathbf{U}_2 \mathbf{Z}_{2,I} + \mathbf{E}_c F_c, \tag{3}$$

where $\mathbf{U}_2 = \begin{bmatrix} 1 & -(1/(K_2+iC_2\Omega)) \\ 0 & 1 \end{bmatrix}$, $\mathbf{E}_c = \begin{bmatrix} 1/(K_2+iC_2\Omega) \\ 0 \end{bmatrix}$, and F_c is the control force.

Considering the negative PID feedback control and regarding $x_{1,0}$ as the measurement signal, the control force takes the form

$$f_c = -\left(k_P x_{1,0} + k_I \int_0^t x_{1,0} dt + k_D \frac{dx_{1,0}}{dt}\right). \tag{4}$$

By transformation $x = Xe^{i\Omega t}$, $f_c = F_c e^{i\Omega t}$, it yields

$$F_c = -\left(k_P + \frac{k_I}{i\Omega} + i\Omega k_D\right) X_{1,0} =: -C(i\Omega) X_{1,0}. \tag{5}$$

Meanwhile there should be

$$X_{1,0} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Q_x \end{bmatrix}_{1,0} =: \mathbf{E}_m \mathbf{Z}_{1,0}. \tag{6}$$

Regarding the transfer direction and (1)~(3) and (5)~(6), the structure diagram of the controlled system can be drawn in Figure 2.

From Figure 2, the overall transfer equation of the controlled system can be written as

$$\begin{aligned} \mathbf{Z}_{1,0} &= \mathbf{U}_1 \mathbf{Z}_{2,1} \\ &= \mathbf{U}_1 \left(\mathbf{U}_2 \mathbf{U}_3 \mathbf{U}_4 \mathbf{Z}_{4,0} + \mathbf{E}_c F_c\right) \\ &= \mathbf{U}_1 \left(\mathbf{U}_2 \mathbf{U}_3 \mathbf{U}_4 \mathbf{Z}_{4,0} - \mathbf{E}_c C(i\Omega) \mathbf{E}_m \mathbf{Z}_{1,0}\right). \end{aligned} \tag{7}$$

Table 1: Elements values of controlled car.

Name	Value	Company
m_3	50	kg
m_1	250	kg
K_2	16	kN/m
K_4	160	kN/m
C_2	1500	N·s/m

which finally ends up with

$$\mathbf{Z}_{4,0} = \left(\mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\right)^{-1}\left[\mathbf{I}_2 + C\left(i\Omega\right)\mathbf{U}_1\mathbf{E}_c\mathbf{E}_m\right]\mathbf{Z}_{1,0}. \quad (8)$$

This can be rewritten considering the boundary condition of the system $\mathbf{Z}_{1,0} = \left[\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right]_{1,0}$, $\mathbf{Z}_{4,0} = \left[\begin{smallmatrix} A \\ Q_x \end{smallmatrix}\right]_{4,0}$ as

$$\begin{bmatrix} A \\ Q_x \end{bmatrix}_{4,0} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} X \\ 0 \end{bmatrix}_{1,0}. \quad (9)$$

Thus the frequency response function from the disturbance to the output can be read as

$$\frac{X_{1,0}}{A} = \frac{1}{U_{11}}. \quad (10)$$

Setting $k_P = k_I = k_D = 0$, one can acquire the formula for the uncontrolled system.

From the classical approaches for modeling the transfer function of a passive suspension system using ordinary differential equations, the equations which describe the system can be found in [14], and the transfer equation is

$$\begin{aligned} G(s) = {} & \frac{K_4C_2}{m_3m_1}\left(s + \frac{K_2}{C_2}\right) \\ & \times\left(s^4 + \left(\frac{C_2}{m_3} + \frac{C_2}{m_1}\right)s^3 \right. \\ & \left. + \left(\frac{K_2}{m_3} + \frac{K_2}{m_1} + \frac{K_4}{m_3}\right)s^2 + \frac{K_4C_2}{m_3m_1}s + \frac{K_4K_2}{m_3m_1}\right)^{-1}. \end{aligned} \quad (11)$$

The described system elements values are given in Table 1 [15].

For the uncontrolled case, Bode plots for both classical method and MS-TMM results are shown in Figure 3, considering the elements values given in Table 1 and substituting into (10) after eliminating the controller gains, the transfer equation of MS-TMM system as a function of the excitation frequency Ω is

$$\begin{aligned} G(j\Omega) = {} & \left(K_4K_2 + (j\Omega)C_2K_4\right) \\ & \times\left(m_3m_1\Omega^4 - j\Omega^3C_2\left(m_3 + m_1\right)\right. \\ & - \Omega^2\left(K_2m_3 + K_2m_1 + K_4m_3\right) \\ & \left. + (j\Omega)K_4C_2 + K_4K_2\right)^{-1}. \end{aligned} \quad (12)$$

This equation represents the open loop uncontrolled part of the system enclosed in the solid blue line in Figure 2; results from Figure 3 indicate that both techniques are identical.

Figure 3: Bode plots for MS-TMM and classical method.

3. Controller Parameters Tuning

This section illustrates different approaches for designing both PI and PID controllers and some methods for tuning their parameters. Considering the control scheme shown in Figure 2, at the beginning a PI controller is designed; from (5) the designed PI controller will be at the form of

$$C(s) = k_P + \frac{k_I}{s} = \frac{k_Ps + k_I}{s}. \quad (13)$$

From the previous literature reviews, instability is the disadvantage of feedback control system. There is a risk that the closed-loop system becomes unstable when using feedback. Thus, analyzing the closed-loop stability of system is essential requirement for feedback control system. The mean obstacle is to indicate the controller parameters or gains stabilizing region [16].

Decomposing (12) numerator and denominator to their even and odd parts, the equation can be rewritten as

$$G(j\Omega) = \frac{N_e\left(-\Omega^2\right) + j\Omega N_o\left(-\Omega^2\right)}{D_e\left(-\Omega^2\right) + j\Omega D_o\left(-\Omega^2\right)}. \quad (14)$$

From the control scheme diagram shown in Figure 2, and considering the control law in (13), the closed-loop characteristic polynomial derived in (10) can be separated and solved to zero; then the control parameters can be written as [17]

$$k_P = \frac{X(\Omega)U(\Omega) - Y(\Omega)R(\Omega)}{Q(\Omega)U(\Omega) - R(\Omega)S(\Omega)}, \quad (15)$$

$$k_I = \frac{Y(\Omega)Q(\Omega) - X(\Omega)S(\Omega)}{Q(\Omega)U(\Omega) - R(\Omega)S(\Omega)}, \quad (16)$$

where

$$\begin{aligned} Y(\Omega) &= -\Omega D_e\left(-\Omega^2\right), & Q(\Omega) &= -\Omega^2N_0\left(-\Omega^2\right), \\ X(\Omega) &= \Omega^2D_0\left(-\Omega^2\right), & S(\Omega) &= \Omega N_e\left(-\Omega^2\right), \\ U(\Omega) &= \Omega N_o\left(-\Omega^2\right), & R(\Omega) &= N_e\left(-\Omega^2\right). \end{aligned} \quad (17)$$

FIGURE 4: Stability region for frequency range $\Omega \rightarrow [0, 56.7]$.

To determine the missing expressions in (15), (16), and (17), the transfer function in (12) can be divided into odd and even portions for both numerator and denominator, and after the comparison with (14), the following variables can be substituted as

$$\begin{aligned} N_e\left(-\Omega^2\right) &= K_1K_3, \qquad N_o\left(-\Omega^2\right) = K_1C_3, \\ D_e\left(-\Omega^2\right) &= m_2m_4\Omega^4 \\ &\quad - \left(K_1m_4 + K_3m_2 + K_3m_4\right)\Omega^2 + K_1K_3, \\ D_o\left(-\Omega^2\right) &= -C_3\left(m_2 + m_4\right)\Omega^2 + K_1C_3. \end{aligned} \tag{18}$$

For the controller design problems which based on gain and phase margins design, it is inevitable to estimate the stabilizing region of the controller gains. The stability boundary locus in proportional and integral gains plane can be obtained from (15) and (16) after substituting the missing terms from (17) and (18), then verifying the presented model using the table of contents in Table 1.

The region of stability of the proportional gain k_P and the integral gain k_I that cause the closed-loop polynomial to satisfy Hurwitz stability test is shown in Figure 4, where line $k_I = 0$ divides the controller gains plane into stable and unstable regions.

For designing of controller to meet a certain demanded gain and phase margins which are important frequency domain requirements for user specifications, consider a control function:

$$G_c\left(j\Omega\right) = Me^{-j\varphi}, \tag{19}$$

where M and φ are gain and phase margins, respectively.

FIGURE 5: Stability region for two cases.

FIGURE 6: Bode plot for the controlled and uncontrolled system.

Substituting (17) and (18) into (15) and (16) and considering (19) yields to

$$\begin{aligned} K_P &= \frac{N_A\cos\left[\varphi\right] + N_B\sin\left[\varphi\right]}{-M\left(N_e^2 + \Omega^2N_o^2\right)}, \\ K_I &= \frac{\Omega N_B\cos\left[\varphi\right] - \Omega N_A\sin\left[\varphi\right]}{-M\left(N_e^2 + \Omega^2N_o^2\right)}, \end{aligned} \tag{20}$$

where

$$N_A = \left(\Omega^2N_oD_o + N_eD_e\right), \quad N_B = \Omega\left(N_oD_e - N_eD_o\right). \tag{21}$$

Using these formulae makes it easy to satisfy certain conditions and find the stabilization region for the controller parameters. Assuming two cases, one as the phase margin is greater than 45° and gain margin is greater than 1, another as the phase margin is greater than 60° and gain margin is greater than 1.

Figure 5 shows two regions of stabilities for two different conditions. The blue circles locus shows the stability region for a demanded conditions gain margin =1 and phase margin 45°, the maximum frequency for this region is 41.5 rad/sec,

FIGURE 7: Phase margin versus damping ratio.

the red solid locus shows another design conditions with gain margin =1 and phase margin 60°, and the maximum frequency for this region is 31.5 rad/sec.

It is an applicable method to achieve user specified gain and phase margins to define the values of controller parameters with the ability to justify the maximum frequency without the need of sweeping over the parameters and also without using programming for solving inequality problems. This method can be expanded to estimate the stabilizing region of PID controller gains as shown in [17].

From Figure 6, the plotted line in red solid color indicates the output response of the uncontrolled system and also shows the stability region of frequencies shown in red points in the range $\Omega \in [10.2, 15.6]$ rad/sec, while the blue dashed locus indicates the system response after applying the PI controller which is chosen from the stability region shown in Figure 6 as $K_P = 1$ and $K_I = 2.5$; it is obvious that the stability region of frequencies shown in blue squares is expanded in the range $\Omega \in [0, 15.5]$ rad/sec.

The relationship between the phase margin Φ_M, the overshoot (OS), and the damping ratio ζ can be expressed as:

$$\Phi_M = 90^\circ - \tan^{-1} \frac{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}{2\zeta}, \tag{22}$$

where

$$\mathrm{OS}\% = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100\%. \tag{23}$$

The relation between Φ_M and ζ is plotted in Figure 7.

So it is noticeable to get different overshoot values due to the changes of the phase margins with the controller design parameters; Figure 9 shows the effect of changing these values as follow.

In Figure 8 the red dash-dot line indicates the uncontrolled system step response; the overshoot is about 46%, and choosing a test point from Figure 3 with values of $K_P = 1$ and $K_I = 25$, it is clear that it locates outside the regions of stabilities from the used technique, so it is clear from the figure that the green solid line response overshoot value is about 55%, and unlike the line in blue dot color it is clear that the overshoot value is improved to be 7% when the test point is chosen as $K_P = 1$ and $K_I = 7.5$.

FIGURE 8: Step response of the system.

Returning back to (10) and substituting $j\Omega = s$, another technique is applied in this section; a developed methodology by Shamsuzzoha and Skogestad requires only one closed-loop step test to obtain PI controller setting. In this method it is also simple to obtain the PID tuning parameters [11, 12].

Considering an ideal PID controller in the form, as shown in (5)

$$c(s) = K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right) \Longleftrightarrow c(s) = \left(K_P + \frac{K_I}{s} + K_D s\right). \tag{24}$$

K_c is the controller gain and τ_I, τ_D are the controller integral and derivative time, respectively.

The defined parameters can be calculated using the following basic formulae:

$$K_c = \frac{2\tau + \theta}{3k\theta}, \qquad \tau_I = \min\left\{\left(r + \frac{\theta}{2}\right), 8\theta\right\}, \qquad \tau_D = \frac{r\theta}{2\tau + \theta}, \tag{25}$$

where k is the gain, τ is the time lag constant, and θ is the time delay. The controller gain is only depending on the overshoot, while the integral and derivative times are function of the system peak time. Applying this approach, the tuning results simulation for the designed problem assuming different values for τ/θ is indicated in Table 2.

An improved definition of (25) parameters were concluded in Shamsuzzoha et al. [12], a closed-loop test with P-controller was applied to adjust overshoot value of 0.3021 for P-only control then assigned the other parameters which leaded to gain $K_c = 2.5$ with $\tau_I = 0.2040$ and peak time 0.153 for the controller gains.

In spite of Zeigler-Nichols methods' drawbacks which are mentioned in previous sections, it is still one of the easiest methods for determining the proportional gain K_P, integral time T_i, and derivative time T_d based on the transient response of a given system. Zeigler-Nichols suggested the

Table 2: Controller tuning parameters.

τ/θ	Controller parameters		
	K_P	K_I	K_D
0.1	0.4000	0.6667	0.0333
0.5	0.6667	0.6667	0.1667
2.5	2.0000	0.6667	0.8333
10	7.0000	0.8750	3.3333
20	13.6667	1.7083	6.6667
50	33.6667	4.2083	16.6667
100	67.0000	8.3750	33.3333

Table 3: Zeigler-Nichols tuning rules.

Type of controller	K_P	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$0.83\,P_{cr}$	0
PID	$0.6K_{cr}$	$0.5\,P_{cr}$	$0.125\,P_{cr}$

Table 4: Zeigler-Nichols tuning values.

Type of controller	K_P	T_i	T_d
P	2.55	∞	0
PI	2.295	0.091976	0
PID	3.06	0.036568	0.013852

values of the controller parameters according to the shown formulae in Table 3.

For the systems which have a mathematical model, root-locus method can be used to find the critical gain K_{cr} and the sustained oscillation frequency ω_{cr}; then get $P_{cr} = 2\pi/\omega_{cr}$. Also the systems' open loop system (passive suspension) and closed-loop system (Active suspension) can be analyzed in frequency domain to find the bandwidth, gain, and phase margins, and then the design parameters can be found easily.

Considering the previous controlled car problem, the results are found as follows: the critical gain is 5.1, and the sustained oscillation frequency is 56.7 rad/sec. It is the same result shown in Figure 4, where the stability frequency for the case of assuming gain margin 1 and phase margin 0 is $\Omega \rightarrow [0, 56.7]$.

From the previous values, the P, PI, and PID Zeigler-Nichols controller values can be tabulated in Table 4.

Applying the controller parameters obtained from the different tuning techniques, the system response is plotted in Figure 9.

From Figure 9 and related to the different tuning methods, the systems' time of the first peak, the peak values, and the overshoot values can be tabulated in Table 5.

4. Simulations and Results

After defining the controller parameters, it is necessary to test the system performance for some situations. The suspension

Figure 9: Step response of the system.

Table 5: System response analysis parameters.

	Peak time T_p	Peak value M_p	Overshoot % O.S%	Marker
Passive suspension	0.336 sec	1.45	45%	
Z-N PID	0.0789	1.33	33%	
Shamsuzzoha PID	0.158	1.08	8%	
PI	0.23	1.08	8%	

response for a vehicle coming out of a pothole can be simulated as a step input with amplitude of 0.08 as [18]:

$$\text{Amp} = \begin{cases} 0 \Longrightarrow t \leq 0, \\ 0.08 \Longrightarrow t \geq 0. \end{cases} \tag{26}$$

The system output displacement is shown in Figure 10.

Figure 10 shows that the peak value of the passive suspension system in red solid line is nearly 0.116 m although the input step amplitude is 0.08 m; that is, the overshoot is 45%, and after using the estimated values from the described techniques, the overshoot and response of the system improved although the transient time is increased for some of them but the displacement action is smoothed; Zeigler-Nichols results are plotted in dash-dot line, Shamsuzzoha results are plotted in dash lines, and PI controller is plotted in x marker line.

Another simulation model for random roughness road is applied where the road model is expressed as a differential equation with the formula [19]:

$$\ddot{Z}(t) + avZ(t) = W(t), \tag{27}$$

where $\ddot{Z}(t)$ is the road random incentive and the constant "a" depends on road grade; the values of "a" are shown in Table 6.

$W(t)$ is the white noise, with zero mean value and amplitude equals 1, and v is the vehicle speed.

Focusing for more details can be shown in Figure 12.

The desired problem is validated using the results of road simulation of grade B and assuming the velocity is

Figure 10: The output displacement from the system.

Figure 11: Output displacement of the system.

Table 6: "a" value under different road grades.

Road grade	A	B	C	D	E
a	0.132	0.1303	0.12	1.1007	0.09

Figure 12: Output displacement zoomed in.

50 km/hr; the results show the effectiveness of the used controllers to reduce the displacement motion as shown in Figure 11 and zoomed in Figure 12, also results illustrated that different control techniques can be applied to MS-TMM, results of the passive suspension response are plotted in blue solid line, Zeigler-Nichols result is plotted in blue dot plot, Shamsuzzohas' result is plotted in green dash-dot plot, and finally the PI controls' result is plotted in red dash plot.

5. Conclusion and Future Work

As the increments of complexity of multibody systems and the development of their design and control methods, the need for more elegant formulations of the equations of motion becomes an issue of paramount importance. The results showed identical values for MS-TMM results with the ordinary classical methods. Thus, ongoing researches, MS-TMM strategies, and its related topologies are highly recommended for analyzing and solving the controlled systems due to easy formulations, being systematic to apply, and simple program coding with less computational time. The results also showed that different controller design methods can be impeded with MS-TMM which leads to more availability for other controllers applications; one depends on the frequency response analysis, another considers the step input response of the system, and the last technique can be applied for both time and frequency domains if the model has a known mathematical model. All controllers' parameters values are applied to a model of controlled car system with brief analysis for the obtained results. The future work will concern with deeply impeding different control techniques with MS-TMM to develop new techniques for the more complex multibody controlled systems considering nonstationary inputs by taking into consideration the system's equations of motion with studying practical systems in frequency domain due to its advantage of indicating inputs and outputs easily such as power spectral density measurements for such kinds of active vibration isolation techniques.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

Acknowledgments

The research was supported by the Research Fund for the Doctoral Program of Higher Education of China (20113219110025), the Natural Science Foundation of China Government (11102089), and the Program for New Century Excellent Talents in University (NCET-10-0075).

References

[1] X.-T. Rui, B. He, B. Rong, G.-P. Wang, and Y.-Q. Lu, "Discrete time transfer matrix method for dynamics of a multi-rigid-flexible-body system moving in plane," *Proceedings of the*

Institution of Mechanical Engineers K, vol. 223, no. 1, pp. 23–42, 2009.

[2] B. Rong, X. Rui, G. Wang, and F. Yang, "Discrete time transfer matrix method for dynamics of multibody system with real-time control," *Journal of Sound and Vibration*, vol. 329, no. 6, pp. 627–643, 2010.

[3] D. Bestle, "Application of the transfer matrix method to control problems," in *Proceedings of the Thematic Conference, Multibody Dynamics*, Zagreb, Croatia, 2013.

[4] S. Yamato and I. Hashimoto, "Present status and future needs the view from Japanese industry," in *Proceedings of the 4th International Conference on Chemical Process Control*, Y. Arkun and W. H. Ray, Eds., Padre Island, Tex, USA, 1991.

[5] C. C. Yu, *Auto Tuning of PID Controllers*, Springer, 2nd edition, 2006.

[6] G. J. Suva, A. Datta, and S. P. Bhattacharyya, "New results on the synthesis of PID controllers," *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 241–252, 2002.

[7] J. G. Ziegler and N. B. Nichols, "Optimum settings for automatic controllers," *Journal of Dynamic Systems, Measurement, and Control*, vol. 115, no. 2, pp. 220–222, 1993.

[8] T. S. Schei, "A method for closed loop automatic tuning of PID controllers," *Automatica*, vol. 28, no. 3, pp. 587–591, 1992.

[9] B. D. Tyreus and W. L. Luyben, "Tuning PI controllers for integrator/dead time processes," *Industrial and Engineering Chemistry Research*, vol. 31, no. 11, pp. 2625–2628, 1992.

[10] K. J. Astrom and T. Hagglund, *PID Controllers: Theory, Design and Tuning*, Instrument Society of America, Research Triangle Park, NC, USA, 2nd edition, 1995.

[11] M. Shamsuzzoha and S. Skogestad, "The setpoint overshoot method: a simple and fast closed-loop approach for PID tuning," *Journal of Process Control*, vol. 20, no. 10, pp. 1220–1234, 2010.

[12] M. Shamsuzzoha, M. Lee, and H. Seki, "Closed-loop PI/PID controller tuning for stable and unstable processes," in *Proceedings of the American Control Conference*, pp. 2368–2373, Montreal, Canada, 2012.

[13] A. J. Barr and J. L. Ray, "Control of an active suspension using fuzzy logic," in *Proceedings of the 5th IEEE International Conference on Fuzzy Systems*, pp. 42–48, September 1996.

[14] M. Cai, H. Liu, P. Du, J. Guo, and H. Chang, "Suspension dynamics simulation based on simulink," in *Proceedings of the International Conference on Electric Information and Control Engineering*, pp. 2715–2718, April 2011.

[15] N. Changizi and M. Rouhani, "Comparing PID and fuzzy logic control a quarter-car suspension system," *The Journal of Mathematics and Computer Science*, vol. 2, no. 3, pp. 559–564, 2011.

[16] K. J. Åström and T. Hägglund, "The future of PID control," *Control Engineering Practice*, vol. 9, no. 11, pp. 1163–1175, 2001.

[17] N. Tan, I. Kaya, C. Yeroglu, and D. P. Atherton, "Computation of stabilizing PI and PID controllers using the stability boundary locus," *Energy Conversion and Management*, vol. 47, no. 18-19, pp. 3045–3058, 2006.

[18] J. Y. Wong, *Theory of Ground Vehicles*, John Wiley & Sons, New York, NY, USA, 1988.

[19] J. Zhihong, E. Zhang, L. Zhonghua, J. Jing, and Z. Fan, "Simulation and experiment research on human riding comfort for vehicle vibration in dynamic environment," in *Proceedings of the 9th International Conference on Computer-Aided Industrial Design and Conceptual Design: Multicultural Creation and Design*, pp. 175–181, November 2008.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 725315, 6 pages
http://dx.doi.org/10.1155/2013/725315

Research Article

All Terrain Vehicle Flexible Multibody Dynamic Simulation for Fatigue Prediction

Jia-Shiun Chen and Hsiu-Ying Hwang

Department of Vehicle Engineering, National Taipei University of Technology, No. 1, Section 3, Zhong-Xiao E. Road, Taipei 10608, Taiwan

Correspondence should be addressed to Jia-Shiun Chen; chenjs@mail.ntut.edu.tw

Received 26 August 2013; Accepted 12 December 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 J.-S. Chen and H.-Y. Hwang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This study presents the flexible-body dynamic analysis and simulated stress recovery of vehicle components to predict their lifetime when maneuvering on an uneven random road. The subject of this study was an all-terrain vehicle (ATV). The body frame and suspension system were modeled as flexible elements for multibody dynamic simulation. The simulations in this study revealed the stress from the flexible elements and predicted the component fatigue life using the retrieved stresses. This approach considers the interaction between dynamic forces and structure deformation and achieves more accurate structure stress prediction and fatigue life prediction.

1. Introduction

Fatigue is one of the major concerns in automotive engineering. The structure and components of a vehicle are constantly under cyclic loading, and especially on rough roads. This could cause the fatigue failure of the structure or components. Ohchida [1] showed that 60% of machine equipment failures are caused by component fatigue. Therefore, a reliable method for predicting the potential fatigue failures of vehicle components is highly desirable. In the design cycle of automotive vehicles, the majority of automotive manufacturers use empirical methods or dynamic simulations to predict the component and structure loadings [2]. The FEA model was designed to predict the stress of components and fatigue failure [3]. However, the FEA analysis process does not consider the effects of the dynamic loads caused by structural deformations. The dynamic loading that occurs when the vehicle is running can cause structural deformation, which in turn may affect load conditions and further change the stress of components. Thus, dynamic analysis with a rigid body system is insufficient. The dynamic loading changes caused by structural deformation should be considered in structure and component stress analysis. Cuadrado et al. [4] presented flexible-body dynamic simulations considering the effects of structural deformation and can therefore predict more accurate loads and stresses of the vehicle components than the general rigid-body simulation can. The good correlation between simulation and experimental results was presented as well in their study. Some studies have focused on vehicle components by modeling them as flexible elements in multibody dynamic simulation. Moon et al. [5] presented a taper leaf spring with hysteretic characteristics. This study developed a flexible multi-body dynamic simulation to interface the leaf spring finite-element model and computation model and correlated simulation and experimental results. Shabana and Sany [6] presented a rolling contact theory with multi-body dynamics to simulate the effects because of the structural flexibility of the vehicle component and track. Zhu et al. [7] predicted the fatigue life of a truck frame. They modeled the frame as a flexible element and the vehicle system as rigid multi-body dynamic model. Yang et al. [8] predicted the fatigue life of a wheel by simulating the EMU wheel as a flexible body in multi-body simulations. Rathod et al. [9] presented a multi-body railroad vehicle system that accounted for the dynamic coupling between a 3D wheel and the rail structure flexibility. However, these studies did not include the deformation of the entire vehicle structure.

This study uses ADAMS to perform flexible-body dynamic simulation and models all structural components as flexible elements. The stress history of the structure was then retrieved from the dynamic simulation for fatigue calculation.

The remaining sections are as follows: the theoretical background of the flexible-body dynamics and fatigue analysis, the simulation model, the simulation results, and the conclusion.

2. Theory Background

2.1. Equations of Flexible-Body Dynamics. In equations of motion, the linear deformation of a structure can be represented by the combination of mode shapes and mode coordinates as follows:

$$u = \sum_{i=1}^{M} \phi_i q_i, \tag{1}$$

where ϕ is the mode shape and q is the mode coordinate.

With the Cartesian coordinate (x, y, z), Euler's angle (ψ, θ, ϕ), mode coordinate q, and the generalized coordinates of the flexible element can be expressed as follows:

$$\xi = \begin{pmatrix} x \\ \psi \\ q \end{pmatrix} = \begin{Bmatrix} x \\ y \\ z \\ \psi \\ \theta \\ \phi \\ q_i \end{Bmatrix}, \quad (i = 1, 2, 3, \ldots, M). \tag{2}$$

The position on the deformed body is written as

$$r_i = x + A\left(S_i + \phi_i q\right), \tag{3}$$

where A is the transformation matrix between global coordinates and the body local coordinates. The term S_i represents the position in the body before deformation, and ϕ_i is the mode shape matrix.

From the position, the velocity of the deformed body is

$$\begin{aligned} v_i &= \frac{dr_i}{dt} = \frac{dx}{dt} + \frac{dA}{dt}\left(s_i + \phi_i q\right) + A\frac{d\left(s_i + \phi_i q\right)}{dt} \\ &= \dot{x} - A\left(\tilde{s}_i + \tilde{\phi}_i q\right)\omega + A\phi_i \dot{q} \\ &= \left|F - A\left(\tilde{s}_i + \tilde{\phi}_i q\right)B + A\phi_i\right|\dot{\xi}. \end{aligned} \tag{4}$$

The term ω is the angular velocity vector of the body coordinate, and F is the transfer matrix between the time derivative of Euler's angle and the angular velocity.

The kinematic energy of the body is

$$\begin{aligned} T &= \frac{1}{2}\int \rho v^T v dV \\ &\approx \frac{1}{2}\sum_{i=1}^{N}\left(m_i v_i^T v_i + \omega_i^T I_i \omega_i\right) \\ &= \frac{1}{2}\dot{\xi}^T M(\xi)\dot{\xi}, \end{aligned} \tag{5}$$

where m_i and I_i represent the modal mass and modal inertia of moving body, respectively.

The mass matrix of flexible body, M, can be expressed as

$$M(\xi) = \begin{bmatrix} M_{tt} & M_{tr} & M_{tm} \\ M_{tr}^T & M_{rr} & M_{rm} \\ M_{tm} & M_{rm}^T & M_{mm} \end{bmatrix}, \tag{6}$$

where t, r, and m represent the translational, rotational, and modal degrees of freedom, respectively.

The stiffness matrix of the flexible body, K, is relatively simple because there is no rigid body contribution:

$$K(\xi) = \begin{bmatrix} K_{tt} & K_{tr} & K_{tm} \\ K_{tr}^T & K_{rr} & K_{rm} \\ K_{tm}^T & K_{rm}^T & K_{mm} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{mm} \end{bmatrix}. \tag{7}$$

The equations of motion with a Lagrange multiplier can be written as follows:

$$M\ddot{\xi} + \dot{M}\dot{\xi} - \frac{1}{2}\left[\frac{\partial M}{\partial \xi}\dot{\xi}\right]^T \dot{\xi} + K\xi + f_g + D\dot{\xi} + \left[\frac{\partial \psi}{\partial \xi}\right]^T \lambda = Q. \tag{8}$$

The term D is the modal damping matrix, and $K\xi$ and $D\dot{\xi}$, respectively, represent the structural internal force caused by the elastic deformation and velocity. The term λ is the Lagrange multiplier, f_g is the structure weight, and Q is the external force.

The strain and stress can be retrieved using the mode coordinate [10]. This study calculates the stress in the structure using the modal stress recovery technique:

$$\begin{aligned} \{\varepsilon\} &= [H]\{x\}, \\ \{\sigma\} &= [E]\{\varepsilon\}. \end{aligned} \tag{9}$$

The term $\{\sigma\}$ is the stress vector, $\{\varepsilon\}$ is the strain vector, and $[H]$ is the finite-element geometric deformation-to-strain matrix. The term $[E]$ is the material property matrix.

2.2. Fatigue Prediction. Metal material under cycling stress can fail even when the stress is under the ultimate stress limit. The relationship between fatigue stress and life cycle is usually represented by a stress-cycles curve (S-N diagram). For steel, a cycle life greater than 10^6 is treated as a lasting life, and the stress corresponding to a 10^6 cycle is the endurance limit S_e [11].

The S-N diagram is plotted in log scale, and the formula of S-N curve can be simplified as

$$\log S_f = \log a + b \log N \Longrightarrow S_f = aN^b, \tag{10}$$

where b is the slop of curve and N is the life cycle.

During vehicle maneuvering, the structural stress is fluctuating. The amplitude of stress variation varies with the driving and road conditions. The rain-flow counting technique

Figure 1: Vehicle structure and suspension frames.

can be applied to measure different stress variations during driving. The Palmgrem-Miner cycle-ratio summation rule can then be used to accumulate the damages of various loads and cycles:

$$\sum_{i=1}^{i=N} \frac{n_i}{N_{fi}} = \text{Damage}, \tag{11}$$

where n_i is the number of cycles at the change of stress amplitude ($\Delta\sigma_i$) and N_{ft} is the number of cycles until failure at the change of stress amplitude ($\Delta\sigma_i$). The "Damage" parameter usually ranges from 0.7 to 2.2. When the accumulated damage is larger than the "Damage" parameter, the component fails [12].

3. Vehicle Model

The object of research in this study is a dune buggy. Figure 1 shows the geometric model based on the measurement on the vehicle structure. For each component of vehicle structure, MSC Nastran was used to perform modal analysis, and MSC Patran was used to process finite-element models and modal analysis data to create flexible elements for dynamic simulation [13]. These elements were imported to the vehicle model to assemble the whole vehicle. Then, we used ADAMS to perform the flexible-body dynamic simulations [14] and determine the stress generated by random road maneuvers.

3.1. Components. The structure of vehicle model included three primary parts: the body frame, the front suspension, and the rear suspension (Figure 2). The powertrain and passenger weights were attached to the structure, and the total weight of vehicle was 423 kg. All the structure components were modeled as a flexible body. The tires were simulated as force elements in dynamic simulation [14]. Figure 2 shows the connecting joint and forces applied between components [14, 15], and Table 1 shows material properties of the frame [16].

In addition to the weight of the vehicle structure and the driveline, two dummies weighing 80 kg each were modeled and seated on the vehicle structure. The dummies were modeled as two parts (the upper torso; pelvis and legs)

Figure 2: Assembled vehicle and connecting joints and forces.

Table 1: Material property of vehicle structure.

Property	Steel 1015	Steel 1018
Elastic modulus (MPa)	2.07E05	2.07E05
Poisson ratio	0.29	0.29
Density (ton/mm^3)	7.9E − 09	7.9E − 09
Ultimate strength (MPa)	414	455

[10], which were connected with a revolute joint. The upper torso, including the seat back, and the pelvis, including the seat, were attached to the vehicle structure by springs and dampers. Table 2 lists the spring stiffness and damping coefficient. The driveline was seated on the engine mounts, and the engine mounts were modeled as rubber bushings. Table 2 lists the parameters.

3.2. Tire and Road Models. The simulations in this study modeled tires as force elements, and these tire models included the tire mass and inertia. The simulations in this study use the Fiala tire model, which was designed for vertical and longitudinal motion. Table 3 presents the tire parameters.

For simulation road condition, this study applies a random 2D surface (Figure 3).

4. Simulation and Result

In the simulation, the vehicle maneuvered in straight runs over a random uneven surface. The vehicle accelerated from standing still, and the data for analysis was collected when it reached a constant speed of 50 km/h. The duration of data collecting was 10 s. Several thicknesses of the structure frame tube were applied in the simulation: 1 mm, 1.5 mm, 2 mm, and 2.5 mm.

4.1. Dynamic Simulation. In flexible-body dynamic simulation, ADAMS can retrieve the structural stress which could be applied to failure or fatigue analysis. Figure 4 shows the highest stress locations during the dynamic simulation. Table 4

Table 2: Properties of dummies and driveline.

	Mass (kg)	Spring stiffness (N/mm)	Damping coefficient (N-s/mm)
Upper torso (including seat back)	45	10	0.4
Hips and leg (including seat)	35	15	0.2
Powertrain	40	52	0.12

Table 3: Tire parameters.

Property	Front tire	Rear tire
Tire model	Fiala tire type	Fiala tire type
Mass (kg)	10	11
Moment of inertia (kg-mm^2)	Ixx, Iyy: 8E5; Izz: 16E5	Ixx, Iyy: 8E5; Izz: 16E5
Tire radius (mm)	254	280
Tread width (mm)	200	200

Figure 3: 2D random road profile.

Figure 4: Highest stress locations at random road simulation.

shows the maximum von Misses stress of these locations in uneven random road simulations.

To compare the differences in dynamic load running on an uneven road surface, this study created both rigid-body and flexible-body models. Because the rigid-body dynamic simulation cannot determine the structural stress, the maximum rear suspension joint forces were used to present the difference between these two types of simulation (Table 5). Figure 5 shows the joint locations. The frame tube thickness of the model was 2.0 mm in this simulation.

Figure 5: Joint locations.

In addition to the suspension and tire elements, the flexible element in the vehicle structure also absorbed part of dynamic energy during the uneven road simulation. Thus, the joint force of the flexible-body model was smaller than that of the rigid-body model during the simulation.

4.2. Fatigue Analysis. The FEM model in this study adopts several thicknesses of the frame tube. The flexible elements were built based on these models, and flexible-body dynamic simulations were run with different tube thicknesses. The tube thicknesses were 1 mm, 1.5 mm, 2 mm, and 2.5 mm. The maximum stress was retrieved and plotted in a stress-time curve. Figure 6 shows the stress history of the structure with a 2.5 mm tube thickness in 10 s simulation at a constant speed of 50 km/h.

The rainflow counting method was applied to the stress history curve, and the stress cycle number of different stress amplitudes was used for fatigue analysis. Figure 7 shows the stress range and cumulative cycle number of different frame tube thicknesses.

Based on the stress cycle and material S-N diagram, the structural life can be predicted using the Miner rule. Using a 10 s simulation cycle, Table 6 presents the structural life of different frame tube thicknesses. This duration time and the maximum stress position listed in Table 4 show that the structure connecting points of two or more structure members are the most vulnerable to fatigue damage.

Table 4: Structure stress retrieved from flexible body dynamic simulation.

Frame tube thickness (mm)	Road surface	Max. stress grid point	Max. stress (MPa)
1	Flat	582887	215.36
	Random	585076	498.30
1.5	Flat	582887	163.08
	Random	586151	430.28
2	Flat	582887	137.87
	Random	586151	361.84
2.5	Flat	582887	105.33
	Random	585079	347.21

Table 5: Joint force comparison between flexible-body and rigid-body simulation.

Frame tube thickness (mm)	Right and left joints average force at rear suspension (N)	Difference
Flexible-body model	623.01	—
Rigid-body model	758.09	21.7%

Figure 6: Stress history of the max. stress grid point (585079).

Figure 7: Cumulative cycle number of different frame tube thicknesses.

Table 6: Fatigue analysis of different tube thicknesses.

Tube thickness	1 mm	1.5 mm	2 mm	2.5 mm
Cycle number	Less than 1	$N_f = 218$	$N_f = 1265$	$N_f = 2780$

5. Conclusions

Conventional analysis methods model the vehicle body as a rigid body in dynamic simulations. In this case, the dynamic energy passing from the tire running over an uneven road can only be absorbed and damped by tires, springs, and dampers in suspension, and bushings. After this rigid body dynamic simulation, the component load was applied to a finite-element model to determine the structural stress of the component for further analysis. The approach used in this paper is different from the conventional one. The dynamic model in this study uses flexible bodies. The advantage of using flexible-body dynamic simulation is that it simulates structural stresses in dynamic analysis and can be retrieved through the simulation. The dynamic response considers the effect of structural deformation. This provides more realistic and better prediction, especially for not-so-rigid vehicle structures. The structure can also absorb part of the energy. Thus, the loads at joints were smaller compared with the loads obtained in rigid-body simulation.

The stress received from flexible dynamic analysis can be used in fatigue analysis. The structure connecting points where two or more structure members were welded together appeared to be the most vulnerable to fatigue damage. Therefore, these points should be reinforced during structure design and assembly to increase the expected lifetime.

Vehicle fatigue testing is costly and time consuming. Flexible-body dynamic simulations can assist in the life prediction of vehicle structures and components. The proposed analysis tool can shorten the design cycle and can reduce vehicle development cost and time.

Notations

A: Transformation matrix between global coordinates and the body local coordinates
D: Modal damping matrix
E: Material property matrix
F: Transfer matrix between the time derivative of Euler's angle and angular velocity
H: Finite-element geometric deformation-to-strain matrix
I_i: Modal inertia
K: Stiffness matrix of the flexible body
M: Mass matrix of flexible body
Q: External force
S_i: Position before deformation
f_g: Structure weight
m_i: Modal mass
q: Mode coordinate
ε: Strain vector
λ: Lagrange multiplier
ξ: The generalized coordinate of flexible elements
σ: Stress vector
ϕ: Mode shape
ϕ_i: Mode shape matrix
ω: Angular velocity vector.

References

[1] H. Ohchida, "Analysis of service failures of Hitachi products 1970–1975," Hitachi Company Report, 1979.

[2] S. Lu, Y. Li, and Y. Dong, "Dynamic modeling and virtual test of a new all terrain off-road vehicle," *Applied Mechanics and Materials*, vol. 86, pp. 125–128, 2011.

[3] B. R. Miao, W. H. Zhang, L. M. Zhang, T. Zhu, and H. T. Yin, "Research of modern railway vehicle structure fatigue design method based on multibody dynamic simulation," *Advanced Materials Research*, vol. 118–120, pp. 327–331, 2010.

[4] J. Cuadrado, R. Gutiérrez, M. A. Naya, and M. González, "Experimental validation of a flexible MBS dynamic formulation through comparison between measured and calculated stresses on a prototype car," *Multibody System Dynamics*, vol. 11, no. 2, pp. 147–166, 2004.

[5] I.-D. Moon, H.-S. Yoon, and C.-Y. Oh, "A flexible multi-body dynamic model for analyzing the hysteretic characteristics and the dynamic stress of a taper leaf spring," *Journal of Mechanical Science and Technology*, vol. 20, no. 10, pp. 1638–1645, 2006.

[6] A. A. Shabana and J. R. Sany, "A survey of rail vehicle track simulations and flexible multibody dynamics," *Nonlinear Dynamics*, vol. 26, no. 2, pp. 179–210, 2001.

[7] S. Zhu, Z. Xiao, and X. Li, "Vehicle frame fatigue life prediction based on finite element and multi-body dynamic," *Applied Mechanics and Materials*, vol. 141, no. 1, pp. 578–585, 2012.

[8] J. Yang, Q. Shi, G. Zhang, and J. Zhang, "The fatigue life simulation of the wheel of CHR3 EMU in random loading," *Advanced Materials Research*, vol. 430–432, pp. 1424–1427, 2012.

[9] C. Rathod, R. Chamorro, J. L. Escalona, M. El-Sibaie, and A. A. Shabana, "Validation of three-dimensional multi-body system approach for modelling track flexibility," *Proceedings of the Institution of Mechanical Engineers K*, vol. 223, no. 4, pp. 269–282, 2010.

[10] T. M. Wasfy and A. K. Noor, "Computational strategies for flexible multibody systems," *Applied Mechanics Reviews*, vol. 56, no. 6, pp. 553–613, 2003.

[11] W. Yao, *Structure Fatigue Analysis*, Defence Inductry Publisher, 2003.

[12] J. E. Shigley and C. R. Mischke, *Mechanical Engineering Design*, McGraw Hill, 2010.

[13] MSC Software, *Nastran User's Manual*, MSC Software, Newport Beach, Calif, USA, 2010.

[14] MSC Software, *MD_Adams. ADAMS User's Manual*, MSC Software, Newport Beach, Calif, USA, 2010.

[15] M. Azadi, S. Behzadipour, and G. Faulkner, "Performance analysis of a semi-active mount made by a new variable stiffness spring," *Journal of Sound and Vibration*, vol. 330, no. 12, pp. 2733–2746, 2011.

[16] R. C. Hibbeler, *Mechanics of Materials*, Prentice Hall, 2010.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 153913, 12 pages
http://dx.doi.org/10.1155/2013/153913

Research Article

Projectile Impact Point Prediction Based on Self-Propelled Artillery Dynamics and Doppler Radar Measurements

Mostafa Khalil, Xiaoting Rui, Qicheng Zha, Hailong Yu, and Hossam Hendy

Institute of Launch Dynamics, Nanjing University of Science and Technology, 200 Xiao Ling Wei, Nanjing 210094, China

Correspondence should be addressed to Mostafa Khalil; mostafakhalil80@gmail.com

Received 24 August 2013; Accepted 26 November 2013

Academic Editor: Ch Zhang

Copyright © 2013 Mostafa Khalil et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Any trajectory calculation method has three primary sources of errors, which are model error, parameter error, and initial state error. In this paper, based on initial projectile flight trajectory data measured using Doppler radar system; a new iterative method is developed to estimate the projectile attitude and the corresponding impact point to improve the second shot hit probability. In order to estimate the projectile initial state, the launch dynamics model of practical 155 mm self-propelled artillery is defined, and hence, the vibration characteristics of the self-propelled artillery is obtained using the transfer matrix method of linear multibody system MSTMM. A discrete time transfer matrix DTTM-4DOF is developed using the modified point mass equations of motion to compute the projectile trajectory and set a direct algebraic relation between any two successive radar data. During iterations, adjustments to the repose angle are made until an agreement with acceptable tolerance occurs between the Doppler radar measurements and the estimated values. Simulated Doppler radar measurements are generated using the nonlinear six-degree-of-freedom trajectory model using the resulted initial disturbance. Results demonstrate that the data estimated using the proposed algorithm agrees well with the simulated Doppler radar data obtained numerically using the nonlinear six-degree-of-freedom model.

1. Introduction

There are different computation methods to estimate the projectile ballistic trajectory and subsequent impact point. These methods have three primary sources of errors, which are model error, parameter error, and initial state error. The model error depends on how much the projectile equations of motion are simplified. Increasing model complexity means decreasing model error, but the parameter and initial state errors will increase [1]. In 1964, the projectile six-degree-of-freedom (6-DOF) equations of motion in terms of direction cosines [2] was developed by the ballistic research laboratory, which is the most accurate model that can simulate the rigid body flight dynamics under the condition that all the aerodynamic forces and moments, and the initial state, are known to a high degree of accuracy. Using this system of equations, a discrete time transfer matrix method DTTM-6DOF [3] has been developed to accurately compute the projectile ballistic trajectory and minimize the corresponding computational time.

During testing new weapons, an accurate instrumentation is needed to measure the in-flight projectile parameters. Projectiles do not have enough space to install on-board sensors to acquire these parameters. Further, considering the severe environment faced by the projectile during flight such as velocity and spinning rate, a low cost and accurate measurement method is required. The most promising method is using a Doppler radar system [4] due to its robustness, portability, accuracy, and providing nearly continuous velocity-time information on the projectile during the flight down range. A number of authors presented different algorithms based on Doppler radar measurements. In 1958, Shapiro [5] proposed three different estimation methods to compute the six parameters that define the orbital parameters describing the ballistic missile trajectory using single radar measurements which include range, range-rate, and the azimuth and elevation angles. These methods are the maximum likelihood, the weighted least squares, and a deterministic method. In 1985, a new iterative method had been developed [6] and applied to a modified version of 6-DOF model to extract

actual thrust made during the flight of rocket-assisted projectiles using Doppler radar measurements. In 2008, an iterative algorithm had been developed [7] using maximum likelihood method to predict the drag characteristics of projectiles in motion by processing data acquired by Doppler radar system, where the point mass model was used due to the projectile small angle of attack along the trajectory.

The movements of artillery and projectile are very complex in launch process because of the complex structure of self-propelled artillery and the severe environments, such as, high temperature, high pressure, high speed, instantaneous state, multibody, and mutation, in launch process [8]. Therefore, gun launch dynamics has a primary concern in designing embedded navigation and guidance systems in case of guided projectiles and evaluating the projectile initial disturbance in case of unguided projectiles. In 2000, the launch dynamics of self-propelled artillery [9] based on transfer matrix method of multibody system MSTMM was developed for studying the vibration characteristics and dynamics of self-propelled artillery and the projectile's initial disturbance. In 2001, a finite element analysis [10] had been done for the breech closure for the 155 mm Cannon M199, which is normally mounted on the Towed Howitzer M198. The analysis is for a 9 body problem with 13 contact surfaces and was solved for both static and dynamic load cases. In 2004, a simplified model [11] had been done to predict joint forces and accelerations along the length of a simplified 155 mm projectile using modal superposition. These predicted loads used to locate all guidance equipment, sensors, joints, and computers that must operate reliably after exiting the gun. In 2005, a sophisticated 3D-finite-element model [12] had been developed to investigate the survivability of embedded electrical systems by simulating the launch dynamics of a surrogate Excalibur projectile. This study determined free mounting locations trouble for sensitive components and parametric investigation in identifying sensitive factors affecting the muzzle-exit motion of projectile substructures. In 2008, a new method [13] named as transfer matrix method of linear multibody system MSTMM for linear hybrid multibody system dynamics was developed. This method had been applied to a Multiple Rocket System as a linear multi-rigid-flexible-body system. In 2011, the dynamics problem of a shipboard gun system had been solved [14] using the discrete time transfer matrix method of multibody system MSDTTMM [15].

In this paper, an iterative method is developed to determine the projectile kinematics including attitude and angular motion using Doppler radar measurements. Doppler radar measurements are only available during the first portion of projectile trajectory including the range, the range-rate, and the azimuth and elevation angles (position and velocity vectors), in order not to be recognized by the enemy counterattack systems. Simulated Doppler radar measurements are generated during flight time using 6-DOF trajectory model including the projectile initial disturbance problem. The projectile initial disturbance is computed by solving the self-propelled artillery and projectile dynamics as a multi-rigid-flexible system using Transfer Matrix Method of Multibody System MSTMM [8], where the overall transfer equation of self-propelled artillery was obtained to compute the corresponding vibration characteristics. In order to iteratively compute the projectile attitude, a discrete time transfer matrix DTTM-4DOF is developed based on linearization of the nonlinear modified point mass equations of motion using second-order Taylor series [16]. The system state vector, which defines the projectile trajectory parameters during Doppler radar observations, is defined by 13 state variables, which are the projectile position, velocity, spin rate, repose angle, and wind velocity. During iterations, adjustments to the repose angle are made until agreement with a certain tolerance occurs between the Doppler radar velocity measurements and the value predicted using DTTM-4DOF. Finally, projectile impact point can be predicted precisely depending on the length of radar data available, where the data estimated at the end of radar observations is used as an initial state to the 6-DOF trajectory model.

2. Launch Dynamics Model for Self-Propelled Artillery System

To accurately estimate the projectile initial state, launch process has to be decided precisely by studying the artillery and projectile dynamics from the moment of firing to the state of muzzle point. Four interactions have to be considered, the interaction between projectile and artillery, the interaction between the artillery transverse and longitudinal motion, the interaction between rigid and flexible body, and, finally, the interaction between space coordinate and time coordinate for describing motion parameters.

The launch dynamics model of self-propelled artillery is a multi-rigid-flexible system which has been studied in [8]. This system is composed of 51 elements (23 bodies and 28 joints) as shown in Figure 1. It has 3 boundary ends including ground, gun breech, and muzzle. The ground (0) is regarded as an infinity rigid body and the first input point, the gun breech (44) is regarded as rigid body and the second input point, and the muzzle (52) is the output point. The road wheels (13–24) are regarded as lumped masses. The hull (37), revolving part (39), elevating part (41), and muzzle brake (51) are regarded as rigid bodies; the barrel is divided into 6 segments (45–50) and each is regarded as a beam with equal sectional area. The interactions (1–12) between ground and road wheel and the connections (25–36) between road wheel and the hull are, respectively, modeled with springs and dampers in 3 directions. Connections (38, 40, 42, and 43) are modeled with springs and rotary springs accompanying dampers to represent relative linear motion and relative angular motion in 3 directions at the same time. All connections between barrel and gun breech, barrel and muzzle brake, and each segment of barrel are regarded as fixed. The masses of traversing mechanism, elevating mechanism, and equilibrator fall into elevating part, revolving parts, and hull, respectively. According to the launch dynamics model of self-propelled artillery and the sequence number of each element, there are 49 connection points which are $P_{i,i-12}$, $P_{i,i+12}$ (i = 13–24), $P_{37,i}$ (i = 25–36), $P_{37,38}$, $P_{39,38}$, $P_{39,40}$, $P_{41,40}$, $P_{41,42}$, $P_{41,43}$, $P_{44,45}$, $P_{45,46}$, $P_{46,47}$, $P_{47,48}$, $P_{48,49}$, $P_{49,50}$, and $P_{51,50}$ and 14 boundary points which are $P_{0,i}$ (i = 1–12),

Figure 1: Launch dynamic model of self-propelled artillery [8].

$P_{44,0}$, $P_{51,52}$, respectively. The overall transfer equation of the practical 155 mm self-propelled artillery shown in Figure 1 was obtained using the transfer matrix method of linear multibody system MSTMM [8].

During launch process, all forces and moments acting between barrel and elevating part, elevating part and revolving part, revolving part and the hull, the hull and road wheels, and road wheels and the ground are regarded as internal forces. The forces that acted on the barrel by projectile [8, 17] and the forces that acted on the barrel and muzzle brake by propellant gas [8] are regarded as external forces of the artillery. The recoil resistance between the barrel and elevating part produced by recoil and counter-recoil mechanism is decomposed into external force and internal force, and the internal force is proportional to the relative displacement between barrel and elevating part.

3. Projectile Flight Trajectory Model

Due to the complexity of projectile initial disturbance determination, a modified point mass 4-DOF model for an unguided rigid projectile was stated in [18, 19]. These equations assume that the epicyclic pitching and yawing motion of the projectile are small everywhere along the trajectory; therefore, the yaw and pitch moments considered in 6-DOF trajectory model [19] are neglected.

The projectile modified point mass equations of motion with respect to earth fixed coordinate system (X_1, X_2, X_3) as shown in Figure 2 are given by [19]

$$\dot{\mathbf{X}} = \mathbf{V} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}^T, \tag{1}$$

$$\dot{\mathbf{V}} = -\frac{\rho v S_{\text{ref}} C_D}{2m}\mathbf{V_t} + \frac{\rho S_{\text{ref}} C_L^{\alpha}}{2m} v^2 \boldsymbol{\alpha}_R + \frac{\rho S_{\text{ref}} d C_N^{p\alpha}}{2m} p \left[\mathbf{V_t} \times \boldsymbol{\alpha}_R\right] + \mathbf{g} + \boldsymbol{\Lambda}, \tag{2}$$

$$\dot{p} = \frac{\rho S_{\text{ref}} d^2 v}{2I_x} p C_l^p, \tag{3}$$

Figure 2: Coordinate system and directions of ballistic target.

where C_D, C_L^{α}, $C_N^{p\alpha}$, and C_l^p are the aerodynamic drag force, lift force, Magnus force, and spin damping moment coefficients, respectively; $\mathbf{V}$ is the projectile velocity vector in earth fixed coordinate; $\mathbf{V_t}$ is the projectile total velocity vector with respect to air; v is the projectile total aerodynamic speed; ρ is the air density; m, I_x, I_y, d, and S_{ref} are the projectile mass, axial and transverse moments of inertia, reference diameter, and reference area, respectively; and $\mathbf{g}$ and $\boldsymbol{\Lambda}$ are the earth's gravitational and Coriolis accelerations vectors.

The projectile repose angle $\boldsymbol{\alpha}_R$ is defined by [19]

$$\boldsymbol{\alpha}_R = \mathbf{x} - \left(\cos\left|\alpha_t\right|\right)\mathbf{i}, \tag{4}$$

where $\mathbf{x}$ is the projectile's axis of symmetry unit vector and $\mathbf{i}$ is the unit vector in the direction of $\mathbf{V_t}$.

And the total aerodynamic angle of attack is [3, 18, 19]

$$\alpha_t = \cos^{-1}\frac{\mathbf{V_t}\cdot\mathbf{x}}{v}. \tag{5}$$

The simplified form of the repose angle is given by [19]

$$\boldsymbol{\alpha}_R = \frac{2I_x p}{\rho S_{\text{ref}} d v^4 C_m^{\alpha}}\left[\dot{\mathbf{V}} \times \mathbf{V_t}\right], \tag{6}$$

where C_m^{α} is the aerodynamic pitching moment coefficient.

For discrete systems, a good approximation was done for the projectile repose angle [20] as

$$\boldsymbol{\alpha}_R = \frac{2I_x p}{\rho S_{\text{ref}} d v^4 C_m^{\alpha}} \left[\mathbf{g} \times \mathbf{V_t}\right]. \tag{7}$$

The Coriolis acceleration due to earth's rotation and gravitational acceleration are given by [3, 19]

$$\boldsymbol{\Lambda} = 2\Omega \begin{bmatrix} 0 & -\cos\mu\sin\lambda & -\sin\mu \\ \cos\mu\sin\lambda & 0 & \cos\mu\cos\lambda \\ \sin\mu & -\cos\mu\cos\lambda & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

$$\mathbf{g} = -g \cdot \begin{bmatrix} \dfrac{X_1}{R_e} & 1 - \dfrac{2X_2}{R_e} & 0 \end{bmatrix}^T, \tag{8}$$

where $g = g_o \cdot [1 - 0.0026\cos(2\mu)]$, $g_o = 9.80665\ \text{m/s}^2$; Ω is the earth's angular velocity; μ, λ are the corresponding latitude and longitude of the firing sight; and R_e is the average radius of the earth (=6356766 m).

Due to spherical earth approximation, the instantaneous projectile altitude H is defined as [19]

$$H = X_2 + \frac{X_1^2}{2 \cdot R_e}. \tag{9}$$

4. Discrete Time Transfer Matrix for Modified Point Mass Equations of Motion

To develop a discrete time transfer matrix DTTM-4DOF using the modified point mass equations of motion mentioned above, a system state vector $\mathbf{S}$ is defined by 13 state variables to define the projectile trajectory during flight time. These state variables are the projectile position, velocity, spin rate, repose angle, and wind velocity, and are as follows:

$$\mathbf{S}_{13\times1} = \left[X_1 \ \ X_2 \ \ X_3 \ \ V_1 \ \ V_2 \ \ V_3 \ \ p \ \ \alpha_1 \ \ \alpha_2 \ \ \alpha_3 \ \ W_1 \ \ W_2 \ \ W_3\right]^T. \tag{10}$$

DTTM-4DOF is the relation between any two successive state vectors, where

$$\mathbf{S}_{10\times1}^{i+1} = \text{DTTM-4DOF}_{10\times13} \cdot \mathbf{S}_{13\times1}^{i}. \tag{11}$$

The 2nd-order Taylor series is given by [16]

$$\mathbf{S}_{10\times1}^{i+1} = \mathbf{S}_{10\times1}^{i} + \dot{\mathbf{S}}_{10\times1}^{i} \cdot \Delta t + \ddot{\mathbf{S}}_{10\times1}^{i} \cdot \frac{\Delta t^2}{2}, \tag{12}$$

where

$$\dot{\mathbf{S}}_{10\times1} \cdot \Delta t = \mathbf{I}_{10\times13}\mathbf{A}_{13\times13} \cdot \mathbf{S}_{13\times1}, \tag{13a}$$

$$\ddot{\mathbf{S}}_{10\times1}^{i} \cdot \frac{\Delta t^2}{2} = \left[\frac{\partial \dot{\mathbf{S}}^i}{\partial t} \cdot \frac{\Delta t^2}{2}\right] = \left[\left(\frac{\partial \dot{\mathbf{S}}}{\partial \mathbf{S}}\right)_{10\times13}^{i} \cdot \dot{\mathbf{S}}_{13\times1}^{i} \cdot \frac{\Delta t^2}{2}\right] = \left[\mathbf{T}_{10\times10}\mathbf{J}_{10\times13}\mathbf{A}_{13\times13}\mathbf{S}_{13\times1}\right]. \tag{13b}$$

The equations of motion matrix $\mathbf{A}$, summarizing (1)–(9) multiplied by the time step Δt, are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{G}_{3\times3}\dfrac{\Delta t^2}{2} & \mathbf{I}_3 \cdot \Delta t & \mathbf{O}_{3\times1} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{G}_{3\times3} \cdot \Delta t & \dot{\mathbf{V}}_{\mathbf{V}} \cdot \Delta t & \dot{\mathbf{V}}_p \cdot \Delta t & \dot{\mathbf{V}}_{\boldsymbol{\alpha}} \cdot \Delta t & \dot{\mathbf{V}}_{\mathbf{W}} \cdot \Delta t \\ \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & \dot{p}_p \cdot \Delta t & \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} \\ \boldsymbol{\alpha}_{\dot{\mathbf{V}}}\mathbf{G}_{3\times3} & \boldsymbol{\alpha}_{\dot{\mathbf{V}}}\dot{\mathbf{V}}_{\mathbf{V}} & \boldsymbol{\alpha}_{\dot{\mathbf{V}}}\dot{\mathbf{V}}_p & \boldsymbol{\alpha}_{\dot{\mathbf{V}}}\dot{\mathbf{V}}_{\boldsymbol{\alpha}} & \boldsymbol{\alpha}_{\dot{\mathbf{V}}}\dot{\mathbf{V}}_{\mathbf{W}} \\ \mathbf{O}_{3\times3} & \dot{\mathbf{W}}_{\mathbf{V}} \cdot \Delta t & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \end{bmatrix}, \tag{14}$$

where $\mathbf{I_3}$ is the 3×3 identity matrix.

All elements of matrix $\mathbf{A}$ in (14) are stated as follows:

$$\mathbf{G}_{3\times3} = \begin{bmatrix} -\overline{g} & 0 & 0 \\ 0 & 2\overline{g} - g_{X_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

where $g_{X_2} = g/X_2$, $\overline{g} = g/R_e$,

$$\dot{\mathbf{V}}_{\mathbf{V}} = \boldsymbol{\Lambda}_{\mathbf{V}} - \underbrace{\rho a_1 C_D v \mathbf{I}_3}_{\dot{\mathbf{V}}_{\mathbf{W}}}, \tag{16}$$

where $\boldsymbol{\Lambda}_{\mathbf{V}} = 2 \cdot \Omega \cdot \begin{bmatrix} 0 & -\cos\mu\cdot\sin\lambda & -\sin\mu \\ \cos\mu\cdot\sin\lambda & 0 & \cos\mu\cdot\cos\lambda \\ \sin\mu & -\cos\mu\cdot\cos\lambda & 0 \end{bmatrix}$,

$$\begin{aligned} \dot{\mathbf{V}}_p &= \rho a_2 C_N^{p\alpha} \mathbf{B}_1, \\ \dot{\mathbf{V}}_{\boldsymbol{\alpha}} &= \rho a_1 C_L^{\alpha} v^2 \mathbf{I}_3, \\ \dot{p}_p &= \rho b_1 v C_l^p, \\ \boldsymbol{\alpha}_{\dot{\mathbf{V}}} &= \frac{-b_2 p}{\rho v^4 C_m^{\alpha}} \left[\mathbf{V_t}\right]_{\times}, \end{aligned} \tag{17}$$

where $a_1 = (S_{\text{ref}}/2m)$, $a_2 = ((S_{\text{ref}}d)/2m)$, $b_1 = ((S_{\text{ref}} \cdot d^2)/(2 \cdot I_x))$, $b_2 = (d/b_1)$, and $\mathbf{B}_1 = \mathbf{V_t} \times \boldsymbol{\alpha}_R$.

The change in wind velocity vector with the projectile position $\dot{\mathbf{W}}_{\mathbf{V}}$ is given by

$$\dot{\mathbf{W}}_{\mathbf{V}} = \frac{\partial \mathbf{W}/\partial t}{\partial \mathbf{V}} = \frac{\partial \mathbf{W}/\partial t}{\partial \mathbf{X}/\partial t} = \frac{\partial \mathbf{W}}{\partial \mathbf{X}}. \tag{18}$$

The Jacobian matrix $\mathbf{J}$ of the system, mentioned in (13b), which represents discretization errors in matrix $\mathbf{A}$, is given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times1} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{j}_{22} & \dot{\mathbf{V}}_p & \mathbf{O}_{3\times3} & \mathbf{j}_{25} \\ \mathbf{O}_{1\times3} & \mathbf{j}_{32} & \dot{p}_p & \mathbf{O}_{1\times3} & -\mathbf{j}_{32} \\ \mathbf{O}_{3\times3} & \mathbf{j}_{42} & \mathbf{j}_{43} & \mathbf{O}_{3\times3} & \mathbf{j}_{45} \end{bmatrix}. \tag{19}$$

The partial derivatives of the projectile acceleration in (2) with respect to wind velocity and projectile velocity are given by

$$\mathbf{j}_{25} = \rho a_1 C_D v \mathbf{I}_3 + \left[\rho a_1 \left(\left(C_D + C_{D_M} M \right) \frac{\mathbf{V_t}}{v} - \left(2C_L^{\alpha} + C_{L_M}^{\alpha} M \right) \boldsymbol{\alpha}_R \right) - \frac{\rho a_2 p C_{N_M}^{p\alpha} M}{v^2} \mathbf{B}_1 \right] \mathbf{V}_{\mathbf{t}}^T + \rho a_2 p C_N^{p\alpha} [\boldsymbol{\alpha}_R]_{\times}, \tag{20}$$

where $C_{i_M} = \partial C_i / \partial M$ is the rate of change of any aerodynamic coefficient with respect to Mach number

$$\mathbf{j}_{22} = \boldsymbol{\Lambda}_{\mathbf{V}} - \mathbf{j}_{25}. \tag{21}$$

Similarly, using (3), the rate of change of projectile spin rate with respect to its velocity is given by

$$\mathbf{j}_{32} = \frac{\rho b_1 p}{v} \left(C_l^p + C_{l_M}^p M \right) \mathbf{V_t}. \tag{22}$$

The partial derivatives of the projectile yaw of repose in (7) with respect to projectile velocity and spin rate are as follows:

$$\mathbf{j}_{42} = \frac{b_2 p}{\rho v^4 C_m^{\alpha}} [\mathbf{g}]_{\times} + \frac{b_2 p \left(4C_m^{\alpha} + C_{m_M}^{\alpha} M \right)}{\rho v^6 \left(C_m^{\alpha} \right)^2} \mathbf{B}_2 \mathbf{V}_{\mathbf{t}}^T, \qquad \mathbf{j}_{43} = \frac{b_2}{\rho v^4 C_m^{\alpha}} \mathbf{B}_2, \tag{23}$$

where $\mathbf{B}_2 = \mathbf{g} \times \mathbf{V_t}$.

The time matrix $\mathbf{T}$ mentioned in (13b) is given by

$$\mathbf{T}_{10\times10} = \begin{bmatrix} \mathbf{I}_3 \frac{\Delta t}{2}. & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times1} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 . \frac{\Delta t}{2} & \mathbf{O}_{3\times1} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{1\times3} & \mathbf{O}_{1\times3} & \frac{\Delta t}{2} & \mathbf{O}_{1\times3} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times1} & \mathbf{I}_3 \end{bmatrix}. \tag{24}$$

By Applying (14), (19), and (25) into (12), the overall discrete time transfer matrix is given by

$$\mathbf{S}_{10\times1}^{i+1} = \left[\mathbf{M}_{10\times13} + \left[\mathbf{I}_{10\times13} + \mathbf{T}_{10\times10} \cdot \mathbf{J}_{10\times13} \right] \cdot \mathbf{A}_{13\times13} \right] \cdot \mathbf{S}_{13\times1}^{i}, \tag{25}$$

where, $\mathbf{M}_{10\times13} = \begin{bmatrix} \mathbf{I}_{7\times7} & \mathbf{O}_{7\times6} \\ \mathbf{O}_{3\times7} & \mathbf{O}_{3\times6} \end{bmatrix}$,

$$\text{DTTM-4DOF} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{14} & \mathbf{m}_{15} \\ \mathbf{m}_{21} & \mathbf{m}_{22} & \mathbf{m}_{23} & \mathbf{m}_{24} & \mathbf{m}_{25} \\ \mathbf{m}_{31} & \mathbf{m}_{32} & \mathbf{m}_{33} & \mathbf{m}_{34} & \mathbf{m}_{35} \\ \mathbf{m}_{41} & \mathbf{m}_{42} & \mathbf{m}_{43} & \mathbf{m}_{44} & \mathbf{m}_{45} \end{bmatrix}, \tag{26}$$

$$\mathbf{S}_{10\times1}^{i+1} = [\text{DTTM-4DOF}] \cdot \mathbf{S}_{13\times1}^{i}. \tag{27}$$

All elements $\mathbf{m}_{ij}$ of the matrix DTTM-4DOF are listed in the appendix.

5. Projectile Attitude Determination Algorithm

The algorithm presented here is an iterative method to compute (1) the projectile muzzle velocity and line of sight by using the first projectile displacement observed by radar, and (2) the projectile flight attitude and the corresponding angular momentum by using the Doppler velocity observed by radar. At each time step, the difference between the data measured by radar and the estimated parameters by using DTTM-4DOF are linearly related as follows:

$$\begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{V} \end{bmatrix}_{i+1} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} & \mathbf{m}_{13} & \mathbf{m}_{14} \\ \mathbf{m}_{21} & \mathbf{m}_{22} & \mathbf{m}_{23} & \mathbf{m}_{24} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{V} \\ \Delta p \\ \Delta\boldsymbol{\alpha}_R \end{bmatrix}_i. \tag{28}$$

At the beginning of flight time, the initial guess of the projectile velocity $\mathbf{V}_o$ is iteratively corrected by adding $\Delta\mathbf{V}_o$ using the first radar data available ($i = 1$) until the position difference $\Delta\mathbf{X}_1$ between the estimated and observed value is small enough (less than a given tolerance) as follows:

$$\Delta\mathbf{V}_o = \left[\mathbf{m}_{12} \right]^{-1} \Delta\mathbf{X}_{(i=1)}. \tag{29a}$$

The projectile muzzle spin rate is corrected by adding Δp_o using the velocity difference $\Delta\mathbf{V}_o$ as

$$\Delta p_o = \frac{2\pi \cdot |\mathbf{V}_o| - |\mathbf{V}_o + \Delta\mathbf{V}_o|}{n \cdot d}, \tag{29b}$$

where n is the gun twist rate.

At the beginning of each iteration cycle, the DTTM-4DOF matrix is computed and hence the projectile states $\mathbf{S}_{10\times1}^{i+1}$ in (27) to obtain the projectile velocity difference $\Delta\mathbf{V}_{i+1}$ which will be reduced until reaching a predefined tolerance by adding $\Delta\boldsymbol{\alpha}_{Ri}$ to the initial state as

$$\Delta\boldsymbol{\alpha}_{Ri} = \left[\mathbf{m_{24}} \right]^{-1} \Delta\mathbf{V}_{i+1}, \tag{30a}$$

$$\boldsymbol{\alpha}_{Ri} = \boldsymbol{\alpha}_{Ri} + \Delta\boldsymbol{\alpha}_{Ri}. \tag{30b}$$

Then, the estimated parameters are used as initials for the next time step.

As shown in Figure 3, during each time step, the projectile attitude and the corresponding angular momentum can be estimated using the following procedure.

(1) The projectile total angle of attack can be computed by

$$\alpha_t = |\boldsymbol{\alpha}_R|. \tag{31}$$

(2) The projectile's axis of symmetry unit vector $\mathbf{x}$ is computed by [19]

$$\mathbf{x} = \boldsymbol{\alpha}_R + (\cos\alpha_t)\, \mathbf{i}, \tag{32}$$

where $\mathbf{i} = \mathbf{V_t} / |\mathbf{V_t}|$.

FIGURE 3: Projectile attitude and impact point determination flow chart.

(3) The rate of change of the projectile total velocity unit vector is given by [19]

$$\dot{\mathbf{i}} = \frac{\rho S_{\text{ref}} v C_L^{\alpha}}{2m} \boldsymbol{\alpha}_R + \frac{1}{v} \left[\mathbf{i} \times (\mathbf{g} \times \mathbf{i})\right]. \tag{33}$$

(4) Due to the small variation in projectile repose angle during flight [19], its rate of change is approximated to be linear during each time step as

$$\dot{\boldsymbol{\alpha}}_R \approx \frac{\boldsymbol{\alpha}_{R_{i+1}} - \boldsymbol{\alpha}_{R_i}}{\Delta t}. \tag{34}$$

(5) The rate of change of projectile's axis of symmetry unit vector $\dot{\mathbf{x}}$ is computed by [19]

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\alpha}}_R + (\cos\alpha_t)\,\dot{\mathbf{i}} - (\sin\alpha_t)\,\mathbf{i}. \tag{35}$$

(6) The projectile angular momentum divided by I_y in earth fixed coordinate is given by [19]

$$\mathbf{h} = \left(\frac{I_x \cdot p}{I_y}\right)\mathbf{x} + [\mathbf{x} \times \dot{\mathbf{x}}]. \tag{36}$$

6. Simulated Trajectory Generation

In order to validate the proposed algorithm, a simulated noiseless ballistic trajectory for 155 mm high explosive HE spin stabilized projectile is generated to simulate the Doppler radar velocity measurements data. The projectile mass, length, center of mass measured from the nose, and the axial and transverse mass moments of inertia are 46.5 kg, 0.902 m, 0.593 m, 0.1585 kg/m^2, and 1.8816 kg/m^2, respectively. This trajectory is generated using the nonlinear 6-DOF trajectory model [2, 19] including projectile initial disturbance problem with 52° firing elevating angle.

TABLE 1: Corresponding relation between the input point and the sequence number.

Input point	Sequence no.
1	Ground
2–13	$P_{13,1}, P_{14,2}, \ldots, P_{24,12}$
14	$P_{37,25}$
15	$P_{37,38}$
16	$P_{39,38}$
17	$P_{39,40}$
18	$P_{41,40}$
19	$P_{41,42}$
20	$P_{44,0}$
21	$P_{44,45}$
22–29	45 (beam 1)
30	$P_{45,46}$
31–68	46 (beam 2)
69	$P_{46,47}$
70–104	47 (beam 3)
105–126	48 (beam 4)
127–148	49 (beam 5)
149–229	50 (beam 6)
230	$P_{51,52}$

6.1. Natural Vibration Characteristics of the Self-Propelled Artillery. To compute the vibration characteristics for 155 mm self-propelled artillery, the input points corresponding to sequence numbers of connection points and elements shown in Figure 1 are identified in Table 1.

All simulation and test results of natural frequencies for the first sixteen ranks are listed in [8], where the simulation results have a good agreement with that obtained by experiments. Figures 4 and 5 show part of the resulted system mode shapes, where the 1st mode mainly represents the recoil

FIGURE 4: The 1st and 2nd mode shapes in X- and Z-direction, respectively.

FIGURE 5: The 3rd and 16th mode shape in Y-direction.

FIGURE 6: Projectile longitudinal displacement.

FIGURE 7: Projectile longitudinal velocity and acceleration.

FIGURE 8: Projectile lateral displacement.

FIGURE 9: Projectile angular displacement during launch.

motion of barrel, the 2nd mode represents the horizontal revolving motions of turret, cradle, and barrel, the 3rd mode mainly represents the elevating motions of cradle and barrel, and the 16th mode mainly represents the transverse motion of barrel and complex motions of other elements. The low rank mode shapes do not represent the transverse elastic deformation of the barrel that is only represented in high rank [8].

6.2. Computation of Projectile Initial Disturbance. The resulted projectile launch dynamics parameters with respect to gun tube coordinate system are shown in Figures 6–10. The gun tube coordinate system $O_3x_Oy_Oz_O$ is defined as follows: the origin O_3 is the intersection point between axis of gun tube and the vertical plane of projectile's axis of symmetry, the x_O-axis is along gun tube axis of symmetry and points to muzzle, the y_O-axis (plumb direction) is vertical

FIGURE 10: Projectile angular velocity during launch.

FIGURE 11: Projectile velocity vector versus flight time.

FIGURE 12: Projectile total angle of attack vector versus flight time.

TABLE 2: Simulation results of initial disturbance of projectile.

Initial disturbance	Plumb	Sidewise
Deflection angle/(10^{-3} rad)	0.71131	1.4600
Swing angle/(10^{-3} rad)	2.11265	−0.06758
Swing angle velocity/(rad/s)	−2.92724	−3.31593

to x_O-axis and points up, the direction of z_O-axis (swing direction) is determined by the right-hand role. The projectile longitudinal displacement, velocity, and acceleration with respect to gun tube axis are shown in Figures 6 and 7, where L_b is the total barrel length. Figure 8 shows the projectile center of gravity plumb and sidewise displacement with respect to gun tube axis as function of nondimensionalized longitudinal displacement.

The projectile axes orientation with respect to line of sight LOS, which includes the projectile roll, plumb and sidewise swing components, angular displacement, and velocity, are illustrated in Figures 9 and 10, respectively. The corresponding projectile angular displacement and velocity components at muzzle are shown in Table 2, where the deflection angle is the angle between the projectile velocity vector and LOS

Table 3: Simulation results of the projectile parameters at muzzle.

V_1 (m/s)	V_2 (m/s)	V_3 (m/s)	α (10^{-3} rad)	β (10^{-3} rad)	p (rad/s)	q (rad/s)	r (rad/s)
429.3847	551.9871	1.021021	1.40134	−1.52758	1279.72	−2.92724	−3.15983

Figure 13: The projectile's axis of symmetry unit vector.

Figure 14: Rate of change of the projectile AOA.

while projectile is just flying off from muzzle, and the swing angle is the angle between projectile axes and LOS at the same moment.

The projectile initial flight conditions corresponding to the self-propelled artillery launch dynamics, and hence, the projectile initial disturbance calculations proposed before are listed in Table 3, which include the projectile initial velocity V_1, V_2, V_3, initial roll p, pitch q and yaw r rates, initial angle of attack α, and side-slip angle β.

7. Estimation Results

A computer program has been developed by using the nonlinear 6-DOF dynamic equations stated in [19]. All simulations were computed with fixed time step size of 0.001 s and integrated using Runge-Kutta-Gill method.

Firstly, a simulated noiseless Doppler radar velocity measurements data 6DOF-R are generated, including the projectile's position and velocity vectors, using the initial flight conditions listed in Table 3. The Doppler radar output data frequency is assumed to be 100 Hz. The 6DOF-R data is assumed to be the real projectile trajectory data. The corresponding impact point range, drift, maximum altitude, velocity, and flight time are 19689.2 m, 1027.85 m, 8180.9 m, 338.76 m/s, and 81.932 s, respectively.

Secondly, a nondisturbed trajectory simulation has been performed as nominal trajectory 6DOF-N, where the initial flight conditions used are 700 m/s, 52°, and 1279.7 rad/s for muzzle velocity, elevating angle, and spin rate, respectively. The corresponding impact point range, drift, maximum altitude, velocity, and flight time are 19729 m, 995 m, 8162 m, 338.66 m/s, and 81.829 s, respectively.

Finally, the iterative algorithm shown in Figure 3 has been applied to estimate the projectile attitude, using the generated Doppler radar data 6DOF-R, where the initial flight conditions are as the data used to simulate the projectile nominal

FIGURE 15: Rate of change of the projectile's axis of symmetry unit vector.

FIGURE 16: Projectile angular momentum per unit I_y.

trajectory 6DOF-N. Figures 11, 12, and 13 show that the estimated projectile velocity, AOA, and attitude, respectively, are matching very well with the simulated Doppler radar trajectory 6DOF-R, although the nominal (non-disturbed) trajectory results 6DOF-N are mismatching the 6DOF-R. Figures 14 and 15 show the rate of change of the projectile AOA and attitude, respectively, during flight time, where the estimated values are matched very well with 6DOF-R except at some peaks/dips data points due to the large difference between the data frequency obtained from radar and the simulation speed of the real trajectory data (1 : 10). Therefore, the final estimated value shown in Figure 16, which represents the projectile angular momentum per-unit I_y has mismatched with 6DOF-R data during these peaks/dips data points.

For nominal trajectory data 6DOF-N, the impact range ε_R and drift ε_D errors relative to 6DOF-R are 0.202% and −3.2%, respectively. All impact point parameters inaccuracies for the proposed algorithm as function of simulated Doppler radar data availability are shown in Figure 17, where these errors, ε_R and ε_D, can be reduced by approximately 8 and 7 times, respectively, as the radar data is available for the first 10% of total flight time and approximately by 10 and 70 times to be −0.0205% and −0.0448%, respectively, as the radar data availability is 15% of total flight time.

8. Conclusion

The main objective of this paper is determining the effect of self-propelled artillery dynamics on the projectile impact point prediction accuracy, and hence, an iterative projectile attitude determination algorithm was developed to accurately compute the projectile ballistic trajectory and the corresponding impact point parameters by using a Doppler radar data. The projectile initial disturbance was computed by solving the interaction between the self-propelled artillery and the projectile dynamics during launch process. A discrete time transfer matrix DTTM-4DOF was developed by linearizing the projectile nonlinear modified point mass equations using second-order Taylor series. The main advantages of using DTTM-4DOF are (1) linearly relating any two successive radar data, and (2) increasing the computation speed due to solving a set of algebraic equations instead of using traditional numerical methods. An iterative projectile attitude determination algorithm was developed to accurately compute the projectile ballistic trajectory and predict the

Figure 17: Impact range and drift error versus Doppler Radar data availability.

impact point parameters by using a Doppler radar data. In order to validate the proposed algorithm, a simulated noiseless trajectory data was generated as Doppler radar velocity measurements data 6DOF-R for 155 mm HE spin stabilized projectile using the nonlinear six-degree-of-freedom trajectory model including projectile initial disturbance problem. A nominal ballistic trajectory 6DOF-N was generated with 52° firing elevating angle and without including the effect of launch dynamics to show the importance of the projectile initial disturbance. Based on the results obtained, the proposed algorithm estimated the projectile velocity and attitude very well compared to the 6DOF-R trajectory data. The impact range and drift inaccuracies were reduced from 0.202% and −3.2% in case of nominal trajectory 6DOF-N to be −0.0205% and −0.0448% by applying the proposed algorithm as the radar data availability is 15% of total flight time. As the projectile dynamic stability increased, the radar data availability can be decreased. Finally, the proposed algorithm can estimate the projectile impact point accurately without computing the projectile initial disturbance, where different projectiles have different initial disturbances based on many factors, such as, gun barrel wear and erosion, projectile initial position and attitude inside barrel, and the propellant charge amount, composition, and homogeneity.

Appendix

Modified Point Mass Discrete Time Transfer Matrix Elements

All DTTM-4DOF elements mentioned in (26) are expressed as follows:

$$\begin{gathered}
\mathbf{m}_{11} = \mathbf{I}_3 + \mathbf{G}\Delta t^2, \\
\mathbf{m}_{12} = \mathbf{I}_3 \cdot \Delta t + \dot{\mathbf{V}}_{\mathbf{V}} \frac{\Delta t^2}{2}, \\
\left[\mathbf{m}_{13} \;\; \mathbf{m}_{14} \;\; \mathbf{m}_{15}\right] = \left[\dot{\mathbf{V}}_p \;\; \dot{\mathbf{V}}_{\boldsymbol{\alpha}} \;\; \dot{\mathbf{V}}_{\mathbf{W}}\right] \frac{\Delta t^2}{2}, \\
\mathbf{m}_{21} = \left(\mathbf{I}_3 + \mathbf{j}_{22} \frac{\Delta t}{2}\right) \mathbf{G}\Delta t, \\
\mathbf{m}_{22} = \mathbf{I}_3 + \dot{\mathbf{V}}_{\mathbf{V}}\Delta t + \left(\mathbf{j}_{22}\dot{\mathbf{V}}_{\mathbf{V}} + \mathbf{j}_{25}\dot{\mathbf{W}}_{\mathbf{V}}\right) \frac{\Delta t^2}{2}, \\
\mathbf{m}_{23} = \dot{\mathbf{V}}_p \Delta t + \left[\mathbf{j}_{22}\dot{\mathbf{V}}_p + \dot{\mathbf{V}}_p \dot{p}_p\right] \frac{\Delta t^2}{2}, \\
\mathbf{m}_{24} = \dot{\mathbf{V}}_{\boldsymbol{\alpha}}\Delta t + \mathbf{j}_{22}\dot{\mathbf{V}}_{\boldsymbol{\alpha}} \frac{\Delta t^2}{2}, \\
\mathbf{m}_{25} = \dot{\mathbf{V}}_{\mathbf{W}}\Delta t + \mathbf{j}_{22}\dot{\mathbf{V}}_{\mathbf{W}} \frac{\Delta t^2}{2}, \\
\mathbf{m}_{31} = \mathbf{j}_{32}\mathbf{G} \frac{\Delta t^2}{2}, \\
\mathbf{m}_{32} = \mathbf{j}_{32}\left[\dot{\mathbf{V}}_{\mathbf{V}} - \dot{\mathbf{W}}_{\mathbf{V}}\right] \frac{\Delta t^2}{2}, \\
\mathbf{m}_{33} = 1 + \dot{p}_p \Delta t + \left[\mathbf{j}_{32}\dot{\mathbf{V}}_p + \dot{p}_p^2\right] \frac{\Delta t^2}{2}, \\
\left[\mathbf{m}_{34} \;\; \mathbf{m}_{35}\right] = \mathbf{j}_{32}\left[\dot{\mathbf{V}}_{\boldsymbol{\alpha}} \;\; \dot{\mathbf{V}}_{\mathbf{W}}\right] \frac{\Delta t^2}{2}, \\
\mathbf{m}_{41} = \left[\boldsymbol{\alpha}_{\dot{\mathbf{V}}} + \mathbf{j}_{42}\Delta t\right] \mathbf{G}, \\
\mathbf{m}_{42} = \left[\boldsymbol{\alpha}_{\dot{\mathbf{V}}} + \mathbf{j}_{42}\Delta t\right] \dot{\mathbf{V}}_{\mathbf{V}} + \mathbf{j}_{45}\dot{\mathbf{W}}_{\mathbf{V}}\Delta t, \\
\mathbf{m}_{43} = \left[\boldsymbol{\alpha}_{\dot{\mathbf{V}}} + \mathbf{j}_{42}\Delta t\right] \dot{\mathbf{V}}_p + \mathbf{j}_{43}\dot{p}_p \Delta t, \\
\mathbf{m}_{44} = \left[\boldsymbol{\alpha}_{\dot{\mathbf{V}}} + \mathbf{j}_{42}\Delta t\right] \dot{\mathbf{V}}_{\boldsymbol{\alpha}}, \\
\mathbf{m}_{45} = \left[\boldsymbol{\alpha}_{\dot{\mathbf{V}}} + \mathbf{j}_{42}\Delta t\right] \dot{\mathbf{V}}_{\mathbf{W}}.
\end{gathered} \tag{A.1}$$

Acknowledgments

The research was supported by the Research Fund for the Doctoral Program of Higher Education of China (20113219110025), the Natural Science Foundation of China Government (11102089), and the Program for New Century Excellent Talents in University (NCET-10-0075).

References

[1] F. Fresconi, G. Cooper, and M. Costello, "Practical assessment of real-time impact point estimators for smart weapons," *Journal of Aerospace Engineering*, vol. 24, no. 1, pp. 1–11, 2011.

[2] R. F. Lieske and R. L. McCoy, "Equations of Motion of a Rigid Projectile," Report 1244 (AD, 441598), U.S. Army Ballistic Research Laboratory, Aberdeen Proving Ground, Md, USA, 1964.

[3] M. Khalil, X. Rui, and H. Hendy, "Discrete time transfer matrix method for projectile trajectory prediction," *Journal of Aerospace Engineering*, 2013.

[4] J. D. Pinezich, J. Heller, and T. Lu, "Ballistic projectile tracking using CW doppler radar," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 46, no. 3, pp. 1302–1311, 2010.

[5] I. I. Shapiro, *The Prediction of Ballistic Missile Trajectories From Radar Observations*, McGraw-Hill, New York, NY, USA, 1958.

[6] E. M. Friedman and M. J. Amoruso, "Determining Thrust or Drag of a Rocket-Assisted Projectile From Doppler Trajectory Data," Tech. Rep. ARAED-TR-85008, Army Armament Research Development and Engineering Center, New Jersey, NJ, USA, 1985.

[7] G. G. Dutta, A. Singhal, A. Kushari, and A. K. Ghosh, "Estimation of drag coefficient from radar-tracked flight data of a cargo shell," *Defence Science Journal*, vol. 58, no. 3, pp. 377–389, 2008.

[8] X. Rui, L. Yun, Y. Lu, B. He, and G. Wang, *Transfer Matrix Method of Multibody System and Its Application*, Science Press, Beijing, China, 2008.

[9] X. Rui, Y. Lu, W. Lu, J. Qian, and G. Wang, "A study on the launch dynamics of self-propelled artillery," *Acta Armamentarii*, vol. 21, supplement 1, pp. 38–40, 2000.

[10] G. P. O'Hara, "Dynamic analysis of a 155 mm cannon breech," *Shock and Vibration*, vol. 8, no. 3-4, pp. 215–221, 2001.

[11] J. A. Cordes, D. Carlucci, and R. Jafar, "Dynamics of a simplified 155-MM projectile," in *Proceedings of the 21st International Symposium on Ballistics*, vol. 2, pp. 1164–1170, Adelaide, Australia, April 2004.

[12] M. Chowdhury, A. M. R. Frydman, J. Cordes, L. Reinhardt, and D. Carlucci, "3-D finite-element gun launch simulation of a surrogate Excalibur 155 mm guided artillery projectile-modeling capabilities and its implications," in *Proceedings of the 22nd International Symposium on Ballistics*, pp. 259–267, Vancouver, Canada, November 2005.

[13] X. Rui, G. Wang, Y. Lu, and L. Yun, "Transfer matrix method for linear multibody system," *Multibody System Dynamics*, vol. 19, no. 3, pp. 179–207, 2008.

[14] X. Rui, B. Rong, G. Wang, and B. He, "Discrete time transfer matrix method for dynamics analysis of complex weapon systems," *Science China Technological Sciences*, vol. 54, no. 5, pp. 1061–1071, 2011.

[15] X. Rui, B. He, Y. Lu, W. Lu, and G. Wang, "Discrete time transfer matrix method for multibody system dynamics," *Multibody System Dynamics*, vol. 14, no. 3-4, pp. 317–344, 2005.

[16] E. Miletics and G. Molnárka, "Taylor series method with numerical derivatives for initial value problems," *Journal of Computational Methods in Sciences and Engineering*, vol. 4, no. 1, pp. 105–114, 2004.

[17] B. Rong, X. Rui, and L. Tao, "Discrete time transfer matrix method for launch dynamics modeling and cosimulation of self-propelled artillery system," *Journal of Applied Mechanics*, vol. 80, no. 1, 2013.

[18] R. F. Lieske and R. L. McCoy, "Equations of Motion for a Modified Point Mass Trajectory," Report R1314 (AD, 485869), U.S. Army Ballistic Research Laboratory, Aberdeen Proving Ground, Md, USA, 1966.

[19] R. L. McCoy, *Modern Exterior Ballistics: the Launch and Flight Dynamics of Symmetric Projectiles*, Schiffer Publishing, 1999.

[20] J. W. Bradley, "An Alternative form of the Modified Point-Mass Equation of Motion," Report BRL-MR-3875 (AD A229514), U.S. Army Ballistic Research Laboratory, Aberdeen Proving Ground, Md, USA, 1990.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 854583, 9 pages
http://dx.doi.org/10.1155/2013/854583

Research Article

Study on the Dynamics of Laser Gyro Strapdown Inertial Measurement Unit System Based on Transfer Matrix Method for Multibody System

Gangli Chen, Xiaoting Rui, Fufeng Yang, Jianshu Zhang, and Qinbo Zhou

Institute of Launch Dynamics, Nanjing University of Science and Technology, Nanjing 210094, China

Correspondence should be addressed to Xiaoting Rui; ruixt@163.net

Received 26 August 2013; Accepted 1 December 2013

Academic Editor: Caishan Liu

Copyright © 2013 Gangli Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The dynamic test precision of the strapdown inertial measurement unit (SIMU) is the basis of estimating accurate motion of various vehicles such as warships, airplanes, spacecrafts, and missiles. So, it is paid great attention in the above fields to increase the dynamic precision of SIMU by decreasing the vibration of the vehicles acting on the SIMU. In this paper, based on the transfer matrix method for multibody system (MSTMM), the multibody system dynamics model of laser gyro strapdown inertial measurement unit (LGSIMU) is developed; the overall transfer equation of the system is deduced automatically. The computational results show that the frequency response function of the LGSIMU got by the proposed method and Newton-Euler method have good agreements. Further, the vibration reduction performance and the attitude error responses under harmonic and random excitations are analyzed. The proposed method provides a powerful technique for studying dynamics of LGSIMU because of using MSTMM and its following features: without the global dynamics equations of the system, high programming, low order of system matrix, and high computational speed.

1. Introduction

As there is no need to use external information for measurements, LGSIMU is widely used in tactical missiles, chariots, torpedoes, warships, airplanes, and so on [1–4]. Inertial measurement unit (IMU) usually operates in bad dynamic environment such as vibration, shock, and overload. For instance, broadband random excitations are produced by missiles in powered phase and reentry phase flights as a result of its engine thrust, jet noise, and turbulent boundary layer pressure. LGSIMU is fixed on the vehicle and undergoes the dynamic environments directly, which makes its operational environment very bad. Thus, its dynamic accuracy is declined.

In order to improve the dynamic accuracy, passive control method is usually adopted to restrain the dynamics response of LGSIMU system in the dynamic environment. The methods such as optimizing the bracket structure, matching the stiffness of isolator, and improving the sustaining mode are the main ways of passive control for the moment [5, 6]. Dynamics analysis is the basis of LGSIMU system design and error compensation. The simplified model which has two degrees of freedom is often used for the dynamics analysis of LGSIMU [7, 8]. Due to the excessive simplification, the model can not accurately reflect the influence of some realistic factors on the system dynamics performance. To analyze the dynamics performance of LGSIMU, attention is gradually paid to more realistic dynamics model, which is established based on the theory of multibody system dynamics. Liu et al. [9] used the theory of multibody system dynamics to establish the dynamics model of LGSIMU and analyze its dynamic characteristics. The ordinary dynamics methods are complicated due to the necessity to deduce the global dynamics equation and inconvenience for engineering applications. Rui et al. [10] put forward the transfer matrix method for multibody system (MSTMM) which does not need to establish the global dynamics equation and is highly stylized. MSTMM has been widely used in the fields of multiple launch rocket system, shipboard gun system, spacecraft, underwater towed system, and so on [11–15]. Yang et al. [16] used

MSTMM to establish the dynamics model of LGSIMU and analyzed the effect of shock on the output precision. Ma et al. [17] used MSTMM to simulate the output power spectral density (PSD) of LGSIMU undergoing random excitation. Their works are both based on the body dynamics equations in MSTMM and modal superposition.

As a result of the bad dynamic environment, the shifting of gyroscope centroid and the drift produced by structure deformation greatly reduce the accuracy of strapdown IMU. However, due to a lack of the error analysis approach, ordinary inertial instruments can hardly compensate the error caused by dynamic environment. Savage [18] established the technique for compensating the attitude, velocity, and position errors under sinusoidal and random vibrations. Savage also presented the simplified dynamics model of strapdown IMU system which had two degrees of freedom including linear and angular motions. It provides an effective approach for error analysis and compensation, but the simplified model cannot reflect the reality perfectly. So it has great engineering significance to establish more realistic dynamics model for error analysis of strapdown IMU.

In this paper, MSTMM is used to establish more realistic dynamics model of LGSIMU system, elements transfer equations, and transfer matrices. The overall transfer equation and overall transfer matrix of the system are derived based on the MSTMM and its automatic deduction method [19]. The method for computing frequency response functions of LGSIMU system based on MSTMM is presented. The responses of LGSIMU system undergoing various sinusoidal and random excitations are obtained. Then the dynamics performance is analyzed. Combining the formulae of errors caused by vibration, the attitude errors of LGSIMU under sinusoidal and random vibrations are discussed. The methodology presented in this paper provides an effective technique for improving the dynamics performance of LGSIMU under dynamic environment.

2. Introduction to MSTMM: A Simple Example

In order to describe the proposed method presented in the following sections conveniently, the simple system shown in Figure 1 is taken as an example. The system consists of lumped mass 1, spring-and-damper hinge 2, and base 3. The free boundary is numbered as 0. The mass of the lumped mass is m, and the stiffness and viscous coefficients of hinge 2 are k and d, respectively. The input and output of the system are the displacements of base 3 and lamped mass 1, respectively. The input of the system is assumed as harmonic excitation $w = We^{i\Omega t}$. In the following of this section, the frequency response function of the system will be derived by using MSTMM.

Using the sign conventions in [10], the positive directions of displacements and forces are shown in the free body diagrams of Figures 2 and 3. For the lumped mass, the displacements of output end and input end are equal, namely,

$$x_{1,0} = x_{1,2}. \tag{1}$$

Figure 1: One degree of freedom system used to introduce MSTMM.

Figure 2: Free body diagram of mass 1.

Applying Newton's second law of motion to the lumped mass yields

$$m\ddot{x}_{1,2} = f_{1,2} - f_{1,0}. \tag{2}$$

For the system undergoing harmonic excitation $w = We^{i\Omega t}$, its steady state responses can be written as $x_{1,0} = X_{1,0}e^{i\Omega t}$, $x_{1,2} = X_{1,2}e^{i\Omega t}$, $f_{1,2} = F_{1,2}e^{i\Omega t}$, $f_{1,0} = F_{1,0}e^{i\Omega t}$ [20]. Substituting these equations into (1) and (2), the transfer equation of lumped mass 1 can be found as

$$\mathbf{Z}_{1,0} = \mathbf{U}_1\mathbf{Z}_{1,2}, \tag{3}$$

where $\mathbf{Z} = [X \quad F]^{\mathrm{T}}$ is the state vector which consists of the complex amplitudes of displacement and internal force. The transfer matrix is

$$\mathbf{U}_1 = \begin{bmatrix} 1 & 0 \\ m\Omega^2 & 1 \end{bmatrix}. \tag{4}$$

For hinge 2, considering its constitutive relation, one can obtain

$$f_{1,2} = f_{3,2} = -k\left(x_{1,2} - x_{3,2}\right) - d\left(\dot{x}_{1,2} - \dot{x}_{3,2}\right). \tag{5}$$

By substituting the steady state responses into (5), the transfer equation of hinge 2 can be acquired as

$$\mathbf{Z}_{1,2} = \mathbf{U}_2\mathbf{Z}_{3,2}, \tag{6}$$

where the state vector also has the form $\mathbf{Z} = [X \quad F]^{\mathrm{T}}$ and the transfer matrix is

$$\mathbf{U}_2 = \begin{bmatrix} 1 & -\dfrac{1}{k + i\Omega d} \\ 0 & 1 \end{bmatrix}. \tag{7}$$

Combining (3) and (6), the overall transfer equation of the system can be obtained as

$$\mathbf{U}_{\mathrm{all}}\mathbf{Z}_{\mathrm{all}} = \mathbf{0}, \tag{8}$$

Figure 3: Free body diagram of spring-and-damper hinge 2.

where

$$\mathbf{U}_{\text{all}} = [-\mathbf{I} \quad \mathbf{T}], \qquad \mathbf{Z}_{\text{all}} = \left[\mathbf{Z}_{1,0}^{\text{T}} \quad \mathbf{Z}_{3,2}^{\text{T}}\right]^{\text{T}},$$
$$\mathbf{T} = \mathbf{U}_1\mathbf{U}_2 = \begin{bmatrix} 1 & -\dfrac{1}{k+\text{i}\Omega d} \\ m\Omega^2 & 1-\dfrac{m\Omega^2}{k+\text{i}\Omega d} \end{bmatrix}. \tag{9}$$

The boundary conditions of the system are

$$\mathbf{Z}_{1,0} = [X_{1,0} \quad 0]^{\text{T}}, \qquad \mathbf{Z}_{3,2} = [W \quad F_{3,2}]^{\text{T}}. \tag{10}$$

Substituting the boundary conditions into (8) yields

$$\overline{\mathbf{U}}_{\text{all}}\overline{\mathbf{Z}}_{\text{all}} = -\mathbf{B}W, \tag{11}$$

where

$$\overline{\mathbf{Z}}_{\text{all}} = [X_{1,0} \quad F_{3,2}]^{\text{T}}, \qquad \overline{\mathbf{U}}_{\text{all}} = \begin{bmatrix} -1 & -\dfrac{1}{k+\text{i}\Omega d} \\ 0 & 1-\dfrac{m\Omega^2}{k+\text{i}\Omega d} \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 1 \\ m\Omega^2 \end{bmatrix}. \tag{12}$$

Solving (11), the unknown elements in the state vectors of boundary points can be obtained as

$$\overline{\mathbf{Z}}_{\text{all}} = -\overline{\mathbf{U}}_{\text{all}}^{-1}\mathbf{B}W = \frac{W(k+\text{i}\Omega d)}{-m\Omega^2+k+\text{i}\Omega d}\begin{bmatrix} 1 \\ -m\Omega^2 \end{bmatrix}. \tag{13}$$

As a result, the frequency response function of the system is

$$H(\Omega) = \frac{X_{1,0}}{W} = \frac{k+\text{i}\Omega d}{-m\Omega^2+k+\text{i}\Omega d}. \tag{14}$$

Equation (14) can easily be verified using classical frequency response function analysis (i.e., Newton's second law of motion and Fourier transform).

3. Dynamics Model of LGSIMU System

According to the elements dynamics properties of LGSIMU system, the dynamics model shown in Figure 4 is established as a multibody system. The system can be divided into elements which can be modeled as follows: three gyros 1~3, platform 7, and foundation 12 are considered as rigid bodies, respectively; since the accelerometers are fixed on the platform and their masses are relatively small, they are included in the platform; the connections 4~6 between platform and gyros are considered as spatial spring-and-damper hinges; meanwhile, platform and foundation are linked by isolators 8~11, which are regarded as spring-and-damper hinges longitudinally vibrating in space. The motion of the vehicle is the input of LGSIMU system. The isolators 13~16 between foundation and platform are considered as spring-and-damper hinges longitudinally vibrating in space.

Figure 4: Dynamics model of LGSIMU system.

The inertial coordinate system shown in Figure 4 is used to describe the motion of elements. The state vectors of the input and output points of every element and the boundary points take the form

$$\mathbf{Z} = \left[X, Y, Z, \Theta_x, \Theta_y, \Theta_z, M_x, M_y, M_z, Q_x, Q_y, Q_z\right]^{\text{T}}. \tag{15}$$

The state variables in the state vector are complex amplitudes of displacements, angular displacements, internal forces, and internal torques when the system undergoes harmonic excitation, respectively.

4. Topology Figure of the Dynamics Model of LGSIMU System

The dynamics model of LGSIMU system is constructed with dynamics elements including bodies and hinges. In order to describe the transfer relationship among the state vectors of elements in the system, the topology figure of the model will be used for deducing the overall transfer equation of LGSIMU system. Isolators 8~11 and 13~16 can be treated as one equivalent hinge element, respectively, which will be shown in Section 5.3 ; thus the dynamics model of LGSIMU system can be dealt with a tree system. Its topology figure can be got readily as illustrated in Figure 5.

Besides the sign conventions introduced in [10], the sign conventions used in this paper are introduced as follows.

(1) A circle ○ denotes a body element and the number inside it is the sequence number of the body element.

Figure 5: Topology figure of LGSIMU system.

(2) An arrow $\rightarrow$ denotes a hinge element and the transfer direction of state vectors, and the number beside it is the sequence number of the hinge element.

(3) Body element 7 is dealt with three input ends and one output end and the other body elements are dealt with single input end and single output end.

(4) For a nonboundary end, the first and second subscript i and j $(i, j \neq 0)$ in the state vector $\mathbf{Z}_{i,j}$ of the end denote the sequence numbers of the adjacent body element and hinge element, respectively. For a boundary end, the second subscript $j = 0$ in the state vector $\mathbf{Z}_{i,j}$, that is, the second subscript means boundary end, and the first subscript i stands for the sequence number of the element involved.

(5) The boundary end corresponding to element 1 is considered as the root, its state vector is denoted as $\mathbf{Z}_{1,0}$, other boundary ends are considered as the tips, and their state vectors are denoted as $\mathbf{Z}_{2,0}$, $\mathbf{Z}_{3,0}$, and $\mathbf{Z}_{13\sim16,0}$, respectively. The transfer directions are from the tips to the root.

5. Transfer Equations and Transfer Matrices of LGSIMU Elements

5.1. Transfer Equations and Transfer Matrices of Gyros 1~3 and Foundation 12. The gyros 1~3 and foundation 12 are rigid bodies with single input end and single output end and their transfer equations are

$$\mathbf{Z}_{i,\mathrm{O}} = \mathbf{U}_i \mathbf{Z}_{i,\mathrm{I}} \quad (i = 1, 2, 3, 12), \tag{16}$$

where $\mathbf{Z}_{i,\mathrm{I}}$ and $\mathbf{Z}_{i,\mathrm{O}}$ denote the state vectors of input and output ends of element i, respectively. The transfer matrices are

$$\mathbf{U}_i = \begin{bmatrix} \mathbf{I}_3 & -\widetilde{\mathbf{l}}_{\mathrm{IO}} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ m\Omega^2\widetilde{\mathbf{l}}_{\mathrm{CO}} & -\Omega^2\left(m\widetilde{\mathbf{l}}_{\mathrm{IO}}\widetilde{\mathbf{l}}_{\mathrm{IC}} + \mathbf{J}_{\mathrm{I}}\right) & \mathbf{I}_3 & \widetilde{\mathbf{l}}_{\mathrm{IO}} \\ m\Omega^2\mathbf{I}_3 & -m\Omega^2\widetilde{\mathbf{l}}_{\mathrm{IC}} & \mathbf{O}_{3\times3} & \mathbf{I}_3 \end{bmatrix}, \quad (i = 1, 2, 3, 12), \tag{17}$$

where C denotes the mass center and m is the mass of rigid body, $\mathbf{J}_{\mathrm{I}}$ is the inertia matrix relative to the input point I, $\widetilde{\mathbf{l}}_{\mathrm{IO}}$ and $\widetilde{\mathbf{l}}_{\mathrm{IC}}$ are the cross product matrices from the input point I to the output point O and mass center C, respectively, and $\widetilde{\mathbf{l}}_{\mathrm{CO}}$ is the cross product matrix from the mass center C to output point O.

5.2. Transfer Equations and Transfer Matrices of Spatial Spring-and-Damper Hinges 4~6. The transfer equations of elements 4~6 are

$$\mathbf{Z}_{i,\mathrm{O}} = \mathbf{U}_i \mathbf{Z}_{i,\mathrm{I}} \quad (i = 4, 5, 6). \tag{18}$$

The transfer matrices are

$$\mathbf{U}_i = \begin{bmatrix} \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{U}_{14} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{U}_{23} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_3 \end{bmatrix} \quad (i = 4, 5, 6), \tag{19}$$

where

$$\mathbf{U}_{14} = \begin{bmatrix} -\dfrac{1}{k_x + \mathrm{i}\Omega d_x} & 0 & 0 \\ 0 & -\dfrac{1}{k_y + \mathrm{i}\Omega d_y} & 0 \\ 0 & 0 & -\dfrac{1}{k_z + \mathrm{i}\Omega d_z} \end{bmatrix},$$
$$\mathbf{U}_{23} = \begin{bmatrix} \dfrac{1}{k'_x + \mathrm{i}\Omega d'_x} & 0 & 0 \\ 0 & \dfrac{1}{k'_y + \mathrm{i}\Omega d'_y} & 0 \\ 0 & 0 & \dfrac{1}{k'_z + \mathrm{i}\Omega d'_z} \end{bmatrix}. \tag{20}$$

k_x, k_y, k_z represent the stiffness coefficients of linear spring, k'_x, k'_y, k'_z denote the stiffness coefficients of rotary spring, d_x, d_y, d_z stand for the damping coefficients of linear damper, and d'_x, d'_y, d'_z denote the damping coefficients of rotary damper, respectively.

5.3. Transfer Equations and Transfer Matrices of Isolators 8~11 and 13~16. As four isolators are sandwiched between two rigid bodies, they can be treated as one new element, namely, an equivalent spring-and-damper hinge with single input end and single output end. The input and output points of the new element can be selected at any positions on the two rigid

Figure 6: Equivalent model of isolators.

bodies, respectively. Hereby, isolators 13~16 are taken as an example to deduce the transfer equation and transfer matrix of the new element. The input and output points of isolator 13 are selected as those of the new element, respectively. Two coordinate systems O_1xyz and O_2xyz shown in Figure 6 are introduced for deduction, whose origins are at the equilibrium positions of the input and output points of the new element. And their directions are the same with the coordinate system shown in Figure 4. The coordinates of the input points of isolators 13~16 in O_1xyz are noted as (a_i, b_i, c_i) $(i = 13 \sim 16)$. The same goes for their output points in O_2xyz. The stiffness and damping coefficients of isolators 13~16 are (k_{xi}, k_{yi}, k_{zi}) and (d_{xi}, d_{yi}, d_{zi}) $(i = 13 \sim 16)$, respectively. Simplify the system of forces acting on the input ends of isolators 13~16 into a system of forces only acting on the input point of the new element, and simplify the system of forces acting on the output ends of isolators 13~16 into a system of forces only acting on the output point of the new element. Further, since the principle vectors and principle moments of the two new systems of forces should be equal, the transfer equation of the equivalent spring-and-damper hinge can be obtained as

$$\mathbf{Z}_{13\sim16,\mathrm{O}} = \mathbf{U}_{13\sim16}\mathbf{Z}_{13\sim16,\mathrm{I}}. \tag{21}$$

The transfer matrix is

$$\mathbf{U}_{13\sim16} = \begin{bmatrix} \mathbf{I}_6 & \left(\mathbf{K}_{13\sim16} + \mathrm{i}\Omega\mathbf{D}_{13\sim16}\right)^{-1} \\ \mathbf{O}_{6\times6} & \mathbf{I}_6 \end{bmatrix}, \tag{22}$$

where Ω is the frequency of harmonic excitation,

$$\mathbf{K}_{13\sim16} = \begin{bmatrix} 0 & -\sum_{i=13}^{16} k_{yi}c_i & -\sum_{i=13}^{16} k_{zi}b_i & \sum_{i=13}^{16}\left(k_{yi}c_i^2 + k_{zi}b_i^2\right) & -\sum_{i=13}^{16} k_{zi}a_ib_i & -\sum_{i=13}^{16} k_{yi}a_ic_i \\ \sum_{i=13}^{16} k_{xi}c_i & 0 & -\sum_{i=13}^{16} k_{zi}a_i & -\sum_{i=13}^{16} k_{zi}b_ia_i & \sum_{i=13}^{16}\left(k_{xi}c_i^2 + k_{zi}a_i^2\right) & -\sum_{i=13}^{16} k_{xi}b_ic_i \\ -\sum_{i=13}^{16} k_{xi}b_i & \sum_{i=13}^{16} k_{yi}a_i & 0 & -\sum_{i=13}^{16} k_{yi}c_ia_i & -\sum_{i=13}^{16} k_{xi}c_ib_i & \sum_{i=13}^{16}\left(k_{xi}b_i^2 + k_{yi}a_i^2\right) \\ -\sum_{i=13}^{16} k_{xi} & 0 & 0 & 0 & -\sum_{i=13}^{16} k_{xi}c_{\mathrm{i}} & \sum_{i=13}^{16} k_{xi}b_{\mathrm{i}} \\ 0 & -\sum_{i=13}^{16} k_{yi} & 0 & \sum_{i=13}^{16} k_{yi}c_{\mathrm{i}} & 0 & -\sum_{i=13}^{16} k_{yi}a_{\mathrm{i}} \\ 0 & 0 & -\sum_{i=13}^{16} k_{zi} & -\sum_{i=13}^{16} k_{zi}b_{\mathrm{i}} & \sum_{i=13}^{16} k_{zi}a_{\mathrm{i}} & 0 \end{bmatrix}. \tag{23}$$

Replacing the stiffness coefficients (k_{xi}, k_{yi}, k_{zi}) $(i = 13 \sim 16)$ in $\mathbf{K}_{13\sim16}$ by damping coefficients (d_{xi}, d_{yi}, d_{zi}) $(i = 13 \sim 16)$, one can immediately achieve $\mathbf{D}_{13\sim16}$. The transfer equation and transfer matrix of the equivalent spring-and-damper hinge of elements 8~11 have the same forms as (21) and (22).

5.4. Transfer Equations and Transfer Matrices of Platform 7. Platform 7 is considered as a rigid body with three input ends and single output end. The first input end is the connection point between platform 7 and isolator 8 and the second and third input points are the connection points between platform and gyros 2~3, respectively. The output point is the connection point between the platform and gyro 1. The transfer equation of platform 7 is

$$\mathbf{Z}_{7,\mathrm{O}} = \mathbf{U}_7\mathbf{Z}_{7,\mathrm{I}_1} + \mathbf{U}_{7,\mathrm{I}_2}\mathbf{Z}_{7,\mathrm{I}_2} + \mathbf{U}_{7,\mathrm{I}_3}\mathbf{Z}_{7,\mathrm{I}_3}. \tag{24}$$

The transfer matrices are

$$\mathbf{U}_7 = \begin{bmatrix} \mathbf{I}_3 & -\widetilde{\mathbf{l}}_{\mathrm{I}_1\mathrm{O}} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ m\Omega^2\widetilde{\mathbf{l}}_{\mathrm{CO}} & -\Omega^2\left(m\widetilde{\mathbf{l}}_{\mathrm{I}_1\mathrm{O}}\widetilde{\mathbf{l}}_{\mathrm{I}_1\mathrm{C}} + \mathbf{J}_{\mathrm{I}_1}\right) & \mathbf{I}_3 & \widetilde{\mathbf{l}}_{\mathrm{I}_1\mathrm{O}} \\ m\Omega^2\mathbf{I}_3 & -m\Omega^2\widetilde{\mathbf{l}}_{\mathrm{I}_1\mathrm{C}} & \mathbf{O}_{3\times3} & \mathbf{I}_3 \end{bmatrix},$$

$$\mathbf{U}_{7,\mathrm{I}_r} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_3 & \widetilde{\mathbf{l}}_{\mathrm{I}_r\mathrm{O}} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_3 \end{bmatrix} \quad (r = 2, 3). \tag{25}$$

The geometrical equations of platform 7 are

$$\mathbf{H}_7\mathbf{Z}_{7,\mathrm{I}_1} = \mathbf{H}_{7,\mathrm{I}_r}\mathbf{Z}_{7,\mathrm{I}_r}, \quad r = 2, 3 \tag{26}$$

where

$$\mathbf{H}_7 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \end{bmatrix},$$
$$\mathbf{H}_{7,\mathrm{I}_r} = \begin{bmatrix} \mathbf{I}_3 & \widetilde{\mathbf{l}}_{\mathrm{I}_1\mathrm{I}_r} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{I}_3 & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \end{bmatrix} \quad (r = 2, 3). \tag{27}$$

The notations in these matrices are similar to those in Section 5.1.

6. Automatic Deduction of the Overall Transfer Equation of LGSIMU System

According to the topology figure and the automatic deduction method of MSTMM, one can obtain the overall transfer equation of LGSIMU system:

$$\mathbf{U}_{\mathrm{all}}\mathbf{Z}_{\mathrm{all}} = \mathbf{0}. \tag{28}$$

The overall transfer matrix is

$$\mathbf{U}_{\mathrm{all}} = \begin{bmatrix} -\mathbf{I}_{12} & \mathbf{T}_{13\sim16-1} & \mathbf{T}_{2-1} & \mathbf{T}_{3-1} \\ \mathbf{O}_{6\times12} & \mathbf{G}_{13\sim16-7} & \mathbf{G}_{2-7} & \mathbf{O}_{6\times12} \\ \mathbf{O}_{6\times12} & \mathbf{G}_{13\sim16-7} & \mathbf{O}_{6\times12} & \mathbf{G}_{3-7} \end{bmatrix}, \tag{29}$$

where

$$\begin{aligned} \mathbf{T}_{13\sim16-1} &= \mathbf{U}_1\mathbf{U}_4\mathbf{U}_7\mathbf{U}_{8\sim11}\mathbf{U}_{12}\mathbf{U}_{13\sim16}, \\ \mathbf{T}_{2-1} &= \mathbf{U}_1\mathbf{U}_4\mathbf{U}_{7,5}\mathbf{U}_5\mathbf{U}_2, \\ \mathbf{T}_{3-1} &= \mathbf{U}_1\mathbf{U}_4\mathbf{U}_{7,6}\mathbf{U}_6\mathbf{U}_3, \\ \mathbf{G}_{13\sim16-7} &= -\mathbf{H}_7\mathbf{U}_{8\sim11}\mathbf{U}_{12}\mathbf{U}_{13\sim16}, \\ \mathbf{G}_{2-7} &= \mathbf{H}_{7,5}\mathbf{U}_5\mathbf{U}_2, \\ \mathbf{G}_{3-7} &= \mathbf{H}_{7,6}\mathbf{U}_6\mathbf{U}_3. \end{aligned} \tag{30}$$

$\mathbf{Z}_{\mathrm{all}}$ is a column matrix consisting of the state vectors of system boundary points, that is

$$\mathbf{Z}_{\mathrm{all}} = \left[\mathbf{Z}_{1,0}^{\mathrm{T}} \quad \mathbf{Z}_{13\sim16,0}^{\mathrm{T}} \quad \mathbf{Z}_{2,0}^{\mathrm{T}} \quad \mathbf{Z}_{3,0}^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{31}$$

7. Dynamic Responses of LGSIMU System

In order to introduce the method based on MSTMM to calculate the frequency response, the input and output are assumed as the displacements of the vehicle and the mass center of platform along y-axis, respectively. Setting the input to be a harmonic excitation $w = \mathrm{e}^{\mathrm{i}\Omega t}$ and according to the theory of vibration [20], the steady state response can be expressed as $y_{7,\mathrm{C}} = Y_{7,\mathrm{C}}\mathrm{e}^{\mathrm{i}\Omega t}$ and the complex amplitude $Y_{7,\mathrm{C}}$ is the frequency response of LGSIMU system.

The boundary conditions of LGSIMU system are

$$\begin{aligned} \mathbf{Z}_{1,0} &= \left[X, Y, Z, \Theta_x, \Theta_y, \Theta_z, 0, 0, 0, 0, 0, 0\right]_{1,0}^{\mathrm{T}}, \\ \mathbf{Z}_{13\sim16,0} &= \left[0, 1, 0, 0, 0, 0, M_x, M_y, M_z, Q_x, Q_y, Q_z\right]_{13\sim16,0}^{\mathrm{T}}, \\ \mathbf{Z}_{2,0} &= \left[X, Y, Z, \Theta_x, \Theta_y, \Theta_z, 0, 0, 0, 0, 0, 0\right]_{2,0}^{\mathrm{T}}, \\ \mathbf{Z}_{3,0} &= \left[X, Y, Z, \Theta_x, \Theta_y, \Theta_z, 0, 0, 0, 0, 0, 0\right]_{3,0}^{\mathrm{T}}. \end{aligned} \tag{32}$$

Substituting these boundary conditions into (28) yields

$$\overline{\mathbf{U}}_{\mathrm{all}}\overline{\mathbf{Z}}_{\mathrm{all}} = -\mathbf{B}, \tag{33}$$

where $\overline{\mathbf{U}}_{\mathrm{all}}$ is a 24×24 square matrix composed of the 1st~6th, 19th~30th, and 37th~42nd columns of $\mathbf{U}_{\mathrm{all}}$, $\overline{\mathbf{Z}}_{\mathrm{all}}$ is a column matrix consisting of the unknown elements in $\mathbf{Z}_{\mathrm{all}}$, and $\mathbf{B}$ is the 14th column of $\mathbf{U}_{\mathrm{all}}$. For a damped system, $\overline{\mathbf{U}}_{\mathrm{all}}$ is a nonsingular square matrix. Solving (33), the unknown elements in the state vectors of boundary points can be acquired as

$$\overline{\mathbf{Z}}_{\mathrm{all}} = -\overline{\mathbf{U}}_{\mathrm{all}}^{-1}\mathbf{B}. \tag{34}$$

Using the transfer equations of elements, one can easily obtain the frequency response function:

$$H(\Omega) = Y_{7,\mathrm{C}} = \mathbf{E}_{7,\mathrm{I}_1\mathrm{C}}\mathbf{U}_{8\sim11}\mathbf{U}_{12}\mathbf{U}_{13\sim16}\mathbf{Z}_{13\sim16,0}, \tag{35}$$

where $\mathbf{E}_{7,\mathrm{I}_1\mathrm{C}} = [0, 1, 0, -c_{\mathrm{C}}, 0, a_{\mathrm{C}}, 0, 0, 0, 0, 0, 0]$, and $(a_{\mathrm{C}}, b_{\mathrm{C}}, c_{\mathrm{C}})$ denotes the coordinate of the platform's mass center in the body fixed coordinate system.

Similarly, the frequency response functions of other points can be got. Using the frequency response function, one can easily obtain the responses under sinusoidal and random excitations.

If the input is $w = W\sin(\Omega t + \alpha)$, the steady state response is

$$y_{7,\mathrm{C}} = W\left|H(\Omega)\right|\sin\left(\Omega t + \alpha + \varphi(\Omega)\right), \tag{36}$$

where $|H(\Omega)|$ is the amplitude of $H(\Omega)$ and $\varphi(\Omega)$ is the phase angle of $H(\Omega)$.

If the input is a stationary random excitation $w(t)$ with PSD $S_w(\Omega)$, the response is also a stationary random process and its PSD can be expressed as

$$S_{y_{7,\mathrm{C}}}(\Omega) = \left|H(\Omega)\right|^2 S_w(\Omega). \tag{37}$$

For the case that the input is a stationary acceleration random excitation $\ddot{w}(t)$ with PSD $S_{\ddot{w}}(\Omega)$ and the output is the acceleration of platform's mass center $\ddot{y}_{7,\mathrm{C}}(t)$ whose PSD is denoted as $S_{\ddot{y}_{7,\mathrm{C}}}(\Omega)$, regarding the relationship between

acceleration PSD and displacement PSD $S_{\ddot{w}}(\Omega) = \Omega^4 S_w(\Omega)$, $S_{\ddot{y}_{7,C}}(\Omega) = \Omega^4 S_{y_{7,C}}(\Omega)$, and (37), one can obtain

$$S_{\ddot{y}_{7,C}}(\Omega) = |H(\Omega)|^2 S_{\ddot{w}}(\Omega). \tag{38}$$

According to the Wiener-Khintchine theorem, the autocorrelation function of the response can be obtained as

$$R_{\ddot{y}_{7,C}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\ddot{y}_{7,C}}(\Omega)\, e^{i\Omega\tau} d\Omega. \tag{39}$$

Substituting (38) into (39) and setting $\tau = 0$, the response mean square value can be written as

$$\phi^2_{\ddot{y}_{7,C}} = R_{\ddot{y}_{7,C}}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\Omega)|^2 S_{\ddot{w}}(\Omega)\, d\Omega. \tag{40}$$

Similarly, the response PSDs and mean square values of other points can be readily achieved. Then one can analyze the dynamics performance and errors of LGSIMU system.

8. Error Analysis of LGSIMU

In order to improve the precision of LGSIMU in the dynamic environment, the errors must be compensated. According to the error analysis method presented in [18], the attitude errors are taken as examples to discuss the errors under sinusoidal and random vibrations in this section.

Assuming that LGSIMU system is excited by a sinusoidal excitation $W\sin(\Omega t + \alpha)$, the angular motion response of the platform can be written as

$$\boldsymbol{\theta}(t) = \mathbf{e}_x\theta_x(t) + \mathbf{e}_y\theta_y(t) + \mathbf{e}_z\theta_z(t), \tag{41}$$

where

$$\begin{aligned} \theta_x(t) &= \Theta_x \sin(\Omega t + \phi_x), \\ \theta_y(t) &= \Theta_y \sin(\Omega t + \phi_y), \\ \theta_z(t) &= \Theta_z \sin(\Omega t + \phi_z). \end{aligned} \tag{42}$$

$\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors of the body-fixed coordinate frame of platform. Θ_x, Θ_y, Θ_z and ϕ_x, ϕ_y, ϕ_z are the amplitudes and phase angles of the angular motion steady state responses, respectively, which can be computed using MSTMM as presented above.

The error caused by sinusoidal vibration is

$$\begin{aligned} \delta\dot{\boldsymbol{\Phi}}_{\text{Algo}} &= \frac{1}{2}\mathbf{e}_z\Omega\Theta_x\Theta_y \sin(\phi_x - \phi_y) \\ &\quad \times \left\{\left[1 + \frac{1}{3}(1 - \cos\Omega T_l)\right]\frac{\sin\Omega T_l}{\Omega T_l} - 1\right\}, \end{aligned} \tag{43}$$

where $\delta\dot{\boldsymbol{\Phi}}_{\text{Algo}}$ is the average attitude error rate, l is high speed algorithm computation cycle index, and T_l is time interval between l cycles.

If the response PSD of LGSIMU system undergoing random excitation is $S_O(\Omega)$, the mean value of the attitude error caused by random vibration is

$$\begin{aligned} &E\left(\delta\dot{\boldsymbol{\Phi}}_{\text{Algo}}\right) \\ &= \int_0^{\infty} \Omega \left|\left[1 + \frac{1}{3}(1 - \cos\Omega T_l)\right]\frac{\sin\Omega T_l}{\Omega T_l} - 1\right| S_O(\Omega)\, d\Omega. \end{aligned} \tag{44}$$

9. Numerical Example

For the LGSIMU system shown in Figure 4, the mass and inertia matrix of foundation 12 are given as

$$\begin{aligned} m_{12} &= 1.2\,\text{kg}, \\ \mathbf{J}_{12,C} &= \begin{bmatrix} 0.0045 & 0 & 0 \\ 0 & 0.0045 & 0 \\ 0 & 0 & 0.0045 \end{bmatrix} \text{kg}\cdot\text{m}^2. \end{aligned} \tag{45}$$

The mass and inertia matrix of platform 7 are

$$\begin{aligned} m_7 &= 2\,\text{kg}, \\ \mathbf{J}_{7,C} &= \begin{bmatrix} 0.0075 & 0 & 0 \\ 0 & 0.0075 & 0 \\ 0 & 0 & 0.0075 \end{bmatrix} \text{kg}\cdot\text{m}^2. \end{aligned} \tag{46}$$

The masses and inertia matrices of gyros 1~3 are

$$\begin{aligned} m_i &= 0.2\,\text{kg}, \\ \mathbf{J}_{i,C} &= \begin{bmatrix} 0.0006 & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.0006 \end{bmatrix} \text{kg}\cdot\text{m}^2 \quad (i = 1 \sim 3). \end{aligned} \tag{47}$$

The amplitude and phase frequency responses of the LGSIMU system got by the proposed method and Newton-Euler method are shown in Figures 7 and 8, respectively. It can be seen that the results got by the two methods have good agreements, which verifies the effectiveness of the proposed method. The attitude error rate under sinusoidal vibration is demonstrated in Figure 9, where the excitation frequency varies from 10 to 2000 Hz, the time interval is 20 ms, and the amplitude of excitation is 0.2 mm. From this figure, it can be concluded that when the excitation frequency is close to the natural frequencies of the LGSIMU system (25 Hz, 46 Hz, and 85 Hz), attitude error rate reaches maximum values. So the natural frequencies cannot be close to excitation frequency.

Currently, the vibration reduction system of LGSIMU is in the development phase. Typically, its vibration reduction index is the root mean square value of acceleration response should be less than 2 g, and the PSD curve comes down rapidly after 200 Hz [21]. In the following, the dynamics response and the attitude error response of the LGSIMU system under random excitation is computed and analyzed. The PSD of the excitation is shown in Figure 10 and its root mean square value is 6.03 g. The PSD of the centroid

Figure 7: Amplitude frequency response.

Figure 8: Phase frequency response.

Figure 9: Attitude error rate under sinusoidal vibration.

Figure 10: Input PSD.

acceleration of platform along y-axis is shown in Figure 11 and its root mean square value is 1.55 g. Since the root mean square value is less than 2 g and the PSD curve comes down rapidly after 200 Hz, the vibration reduction index is achieved. The isolators 8~11 are usually identical. The mean value of the attitude error rate versus the damping coefficient of isolators 8~11 is shown in Figure 12. The mean value of the attitude error rate decreases with the increasing of the damping coefficient. Consequently, within the index of vibration reduction, the damping coefficient of isolators should be as large as possible.

10. Conclusion

Improving the precision of LGSIMU in a dynamic environment is an important problem over a long period of time. In this paper, the method for calculating the frequency response function of LGSIMU system is established by using MSTMM. Based on this method, the responses of LGSIMU system are obtained and the dynamics performance is discussed. Furthermore, the attitude errors under sinusoidal and random vibrations are calculated and the effect of damping coefficient on the errors is analyzed. The proposed method can also be used to analyze the velocity and position errors caused by vibration. The simulation results verify the effectiveness of the proposed method. Compared with the ordinary method, the proposed method does not need the global dynamics equation. It is also highly stylized, flexible for modeling, and easy to program and provides a powerful technique for studying the dynamics of LGSIMU.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Figure 11: Output PSD of platform along y-axis.

Figure 12: The mean values of attitude error rates.

Acknowledgments

The research was supported by Research Fund for the Doctoral Program of Higher Education of China (no. 20113219110025), Natural Science Foundation of China Government (no. 11102089), and Program for New Century Excellent Talents in University (NCET-10-0075).

References

[1] J. M. Oelschlaeger and L. O. Thielman, "GG1308 ring laser gyro inertial measurement systems: Honeywell's low-cost solution for tactical applications," in *Proceedings of the Position Location and Navigation Symposium*, pp. 528–536, March 1990.

[2] A. T. Daniel, G. John, and R. E. Mark, "Ring laser gyroscope dither drive system and method," United States Patent, No. 4981359, January 1, 1991.

[3] R. D. Curby, G. H. McCammon, and S. Vallev, "Dither controller for ring laser angular rotation sensor," United States Patent, No. 4740084, April 26, 1988.

[4] J. I. Lahham and J. R. Brazell, "Acoustic noise reduction in the MK49 ship's inertial navigation system (SINS)," in *Proceedings of the IEEE Position Location and Navigation Symposium*, pp. 32–39, March 1992.

[5] J. I. Lahham, D. J. Wigent, and A. L. Coleman, "Tuned support structure for structure-borne noise reduction of inertial navigator with dithered ring laser gyros (RLG)," in *Proceedings of the IEEE 2000 Position, Location and Navigation Symposium*, pp. 419–428, March 2000.

[6] J. F. Li, J. J. LIU, and X. L. Tang, "Aseismatic control of system structure for laser strapdown IMU," *Journal of Chinese Inertial Technology*, vol. 14, no. 1, pp. 21–26, 2006.

[7] B. Wu, H. F. Wang, T. Qu et al., "Design of mounting plate for a missile-bone laser gyro SINS," *Journal of Chinese Inertial Technology*, vol. 15, no. 2, pp. 132–135, 2007.

[8] Y. Xie, Y. J. Lei, J. P. Zhou et al., "Optimum parameter of INS system absorber," *Structure and Environment Engineering*, vol. 32, no. 4, pp. 39–45, 2005.

[9] P. Liu, H. P. Liu, and C. Zhang, "Research on dynamical characteristics of laser gyro strapdown IMU," *Optics and Optoelectronic Technology*, vol. 9, no. 5, pp. 70–74, 2011.

[10] X. T. Rui, L. F. Yun, Y. Q. Lu et al., *Transfer Matrix Method for Multibody System and Its Applications*, Science Press, Beijing, China, 2008.

[11] X. Rui, G. Wang, Y. Lu, and L. Yun, "Transfer matrix method for linear multibody system," *Multibody System Dynamics*, vol. 19, no. 3, pp. 179–207, 2008.

[12] X. Rui, B. Rong, G. Wang, and B. He, "Discrete time transfer matrix method for dynamics analysis of complex weapon systems," *Science China Technological Sciences*, vol. 54, no. 5, pp. 1061–1071, 2011.

[13] X. Rui, B. He, Y. Lu, W. Lu, and G. Wang, "Discrete time transfer matrix method for multibody system dynamics," *Multibody System Dynamics*, vol. 14, no. 4, pp. 317–344, 2005.

[14] B. Rong, X. Rui, G. Wang, and H. Yu, "Discrete time transfer matrix method for dynamic modeling of complex spacecraft with flexible appendages," *Journal of Computational and Nonlinear Dynamics*, vol. 6, no. 1, Article ID 011013, 2011.

[15] G. P. Wang, B. Rong, L. Tao et al., "Riccati discrete time transfer matrix method for dynamic modeling and simulation of an underwater towed system," *Journal of Applied Mechanics*, vol. 79, no. 1, Article ID 041014, 2012.

[16] F. F. Yang, X. T. Rui, and L. Ma, "Dynamical model and simulation of laser gyro strapdown inertial navigation system," *Journal of Chinese Inertial Technology*, vol. 16, no. 3, pp. 301–305, 2008.

[17] L. Ma, X. Rui, F. Yang, and B. Rong, "Power spectrum analysis for laser gyro damping system of strapdown inertial navigation," *Journal of Vibration Engineering*, vol. 22, no. 6, pp. 603–607, 2009.

[18] P. G. Savage, *Strapdown Analytics*, Strapdown Associates, Maple Plain, Minn, USA, 2000.

[19] X. T. Rui and J. S. Zhang, "Automatical transfer matrix method of multibody system," in *Proceedings of the 2nd Joint International Conference on Multibody System Dynamics*, Stuttgart, Germany, 2012.

[20] W. T. Thomson and M. D. Dahleh, *Theory of Vibration with Applications*, Prentice Hall, Englewood Cliffs, NJ, USA, 5th edition, 1993.

[21] H. F. Wang, B. Wu, M. Zhang et al., "Research on technique of aseismatic control for laser strapdown IMU," *Aerospace Control*, vol. 25, no. 6, pp. 81–85, 2007.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 216014, 15 pages
http://dx.doi.org/10.1155/2013/216014

Research Article

Recursive Formulation for Dynamic Modeling and Simulation of Multilink Spatial Flexible Robotic Manipulators

Zhenjie Qian, Dingguo Zhang, and Jun Liu

School of Sciences, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China

Correspondence should be addressed to Dingguo Zhang; zhangdg419@mail.njust.edu.cn

Received 25 August 2013; Accepted 11 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Zhenjie Qian et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The dynamics for spatial manipulator arms consisting of n flexible links and n flexible joints is presented. All the transversal, longitudinal, and torsional deformation of flexible links are considered. Within the total longitudinal deformation, the nonlinear coupling term, also known as the longitudinal shortening caused by transversal deformation, also is considered here. Each flexible joint is modeled as a linearly elastic torsional spring, and the mass of joint is considered. Lagrange's equations are adopted to derive the governing equations of motion of the system. The algorithmic procedure is based on recursive formulation using 4×4 homogenous transformation matrices where all the kinematical expressions as well as the final equations of motion are suited for computation. A corresponding general-purpose C++ software package for dynamic simulation is developed. Several examples are simulated to illustrate the performance of the algorithm.

1. Introduction

Flexible multibody dynamics has become a key methodology for various engineering fields, such as robotic manipulator arms, large radar antennas, solar panels, transportation vehicles, and manufacturing equipment as well as flexible ligament in human musculoskeletal system [1], and so forth.

In particular, the flexible robotic manipulators are typically precise and complex operated at high speed. They are designed to be light with low inertia in order to achieve cost-reduction, energy-saving, and high-performance. Therefore, the dynamic analysis of flexible manipulator arms is complicated due to the coupling between the large rigid body motion and deformation.

A large number of the literature related to the so-called dynamic stiffening effects have been published, which is first proposed by Kane et al. [2] due to link's high-order coupling flexibility. It was observed that an industrial link under high-speed rotational motion would exhibit instability problems when its angular velocity exceeds a certain limit. The incorrect simulation solutions are attributed to neglecting the high-order coupling deformation terms in the dynamic equations. Wu and Haug [3] modeled a flexible multibody system by means of substructure synthesis formulation to account for structural geometric non-linear effects. Xi and Fenton [4] studied a manipulator consisting of one flexible link and one flexible joint based on the assumption that the link is constrained to move only in a horizontal plane, whereas the gravity and some coupling terms between the equations of motions have been dropped. It has been demonstrated by Wallrapp and Schwertassek [5], that the so-called "geometric stiffening" problem can be solved by keeping second order terms in the expression of the deformations of the material, and the amplitude of the flexible motions must remain small in those formalisms which are generally based on kinematic restrictions as regards the flexibility.

Low [6] presented vibration analysis of a rotating beam carrying a tip mass at its end by using Hamilton's principle and the associated boundary conditions. Yoo et al. [7] used a non-Cartesian variable along with two Cartesian variables to describe the elastic deformation and investigate the dynamic stiffness effect. Ryu et al. [8] put forward a criterion on inclusion of stiffening effects such that it clarifies the limit of the validity of the linear modeling method. El-Absy and Shabana [9] introduced the effect of longitudinal deformation due to bending and studied the influence of geometric stiffness on instability problem of nonlinear elastic model. Al-Bedoor and Hamdan [10], based on the condition of inextensibility to

relate the axial and transverse deformations of the material point, studied a rotating flexible arm deformation undergoing large planar motion. Hong et al. [11, 12] established higher-order rigid-flexible coupling dynamic model of rotating flexible beam undergoing large overall motion. When the deformation of flexible beam or the deformation rate is larger, the first-order or zero-order approximation coupling model will appear with divergent results, nevertheless the complete high-order coupling model's results are still converged. Schwertassek et al. [13] presented the fundamental shape functions choice in floating frame of reference formulation, by separating the flexible body motion into a reference motion and deformation.

Book [14] used the 4×4 homogenous transformation matrices and assumed modes to describe the kinematics of the rigid-joint and flexible-link robots, whereas the dynamic model cannot deal with the torsional deformation of links. A recursive formulation for the spatial kinematic and dynamic analysis of open chain mechanical systems containing interconnected deformable bodies is given by Changizi and Shabana [15], and Kim and Haug [16]. Jain and Rodriguez [17] developed new spatially recursive dynamics algorithms for flexible multibody systems by using spatial operators, which is based on Newton-Euler factorization and innovations factorization of the system mass matrix. Hwang [18] developed a recursive formulation for the flexible dynamic manufacturing analysis of open-loop robotic systems with the generalized Newton-Euler equations. Zhang and Zhou [19, 20] did a further work based on Book's work in [14]. Both the bending and torsional flexibility of links were taken into account. Dynamic simulation of a spatial flexible manipulator arm was given as an example to validate the algorithm. However, the dynamic stiffening effect was not considered yet.

When the system's operating speed becomes high, neglecting the flexibility of joints is quite devastating, which will usually give rise to the errors of position precision. A typical dynamic model for taking into account the joint flexibility was presented by Spong [21]. In his work, the joint flexibility is modeled as a torsional spring with more emphasis on simplifying the equations of motion for control purposes. Other dynamic analyses of robots with joint flexibilities can be seen in the work of Wasfy and Noor [22], Dwivedy and Eberhard [23], and Na and Kim [24].

As mentioned above, there are a lot of research work in the dynamic modeling and simulation. But it is still very difficult for us to deal with the dynamics of the complex multibody systems, such as the spatial flexible-link and flexible-joint robotic manipulators with consideration of the axial, bending, and torsional deformation for links, and the flexibility and mass effects for joints, and the so-called "dynamic stiffening" effects. In this paper, we will present the dynamic modeling methodology to include all such terms. In the following section, the kinematics of the system are presented, in which coordinate frames are established, and 4×4 homogeneous transformation matrices are used to describe the kinematics of flexible links and flexible joints. The approach of assumed modes is employed to describe the deformation of the flexible links. Section 3 firstly deals with the description of the kinetic and elastic potential energy

Figure 1: Flexible robotic arms.

as well as the gravitational potential energy of system, and then focuses on the derivation of the recursive rigid-flexible coupling dynamic equations of the system. In the modeling, the high-order coupling terms related to the non-linear geometry deformation are retained and the recursive strategy for kinematics is adopted. In Section 4, several examples are simulated to illustrate the performance of the algorithm proposed in the paper. Finally, Section 5 summarizes the results and draws conclusions from them.

2. Kinematics of Flexible Robots

As a point of departure, the system considered here is an assembly of n flexible links connected by n rotary joints, as shown in Figure 1.

2.1. Simplified Model of Flexible Joint. Figure 2 shows the flexible joint model. We can model the flexibility of joint i as a linear torsional spring with stiffness $Kt_i \cdot J_{ri}$ is the moment of inertia of rotor i about its spinning axis. And τ_i is the torque exerted at joint i. For simplicity, we neglect friction or damping in the flexible joint. Let q_{1i} be the theoretical rotational angle of link i, q_{2i} be the real rotational angle of link i, ε_i be the torsional angle of joint i, φ_i be the angular displacement of rotor i, and ϕ_i be the gear ratio, respectively. The relationships among them are as follows:

$$\varepsilon_i = q_{2i} - q_{1i}, \tag{1}$$

$$\varphi_i = \phi_i q_{1i}. \tag{2}$$

2.2. Simplified Model of Flexible Link. Assume that the links are slender beams. Analysis here is based on the Euler-Bernoulli beam theory in the elastic small displacements field.

2.3. Coordinate Systems and Transformation Matrices. To express the transformation between different coordinate systems clearly, we establish four coordinate systems for link i. Fix the coordinate system $(X_b Y_b Z_b)_i$ at the proximal end of link i (oriented so that the X_b coincides with the neutral

Figure 2: Flexible joint model.

axis of link i in undeformed shape). This will be referred to as the base reference frame of link i. Fix the coordinate system $(X_dY_dZ_d)_i$ at the distal end of link i. This is the distal frame of link i. When link i is in its undeformed state, the distal frame can be located by a pure translation of the base reference frame $(X_bY_bZ_b)_i$ along the length L_i of link i. Let $(H_xH_yH_z)_i'$ and $(H_xH_yH_z)_i$ be two Denavit-Hartenberg (D-H) frames fixed at the proximal end (at joint i) and the distal end (at joint $i+1$) of link i, respectively. When joint i is motionless, $(H_xH_yH_z)_i'$ is coincident with $(H_xH_yH_z)_{i-1}$, and matrix $\mathbf{HH}_i^{i-1}$, that is, the transformation matrix between them, is the function of q_{2i}. Matrix $\mathbf{H}_i$, the 4×4 homogeneous transformation matrix between frames $(X_dY_dZ_d)_i$ and $(H_xH_yH_z)_i$, is a constant matrix. The transformation matrix $\mathbf{Hb}_i$ of $(H_xH_yH_z)_i'$ and $(X_bY_bZ_b)_i$ is also a constant matrix. Define the joint-transformation matrix $\mathbf{A}_i$ of joint i to be the transformation matrix from $(X_dY_dZ_d)_{i-1}$ to $(X_bY_bZ_b)_i$. Then,

$$\mathbf{A}_i = \mathbf{dH}_{i-1}\mathbf{HH}_i^{i-1}\mathbf{Hb}_i. \tag{3}$$

Obviously, $\mathbf{A}_i$ is a function of q_{2i}. Define $\mathbf{E}_i$ to be the link-transformation matrix of link i from $(X_b\ Y_b\ Z_b)_i$ to $(X_dY_dZ_d)_i$. According to the assumption of small deformation of the links, the small deformable angles can be added vectorially. $\mathbf{E}_i$ can be written as

$$\mathbf{E}_i = \mathbf{H}_i + \sum_{j=1}^{N_i}\delta_{ij}\mathbf{M}_{ij}, \tag{4}$$

in which

$$\mathbf{H}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

$$\mathbf{M}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_{ij} & 1 & -\theta_{zij} & \theta_{yij} \\ y_{ij} & \theta_{zij} & 0 & -\theta_{xij} \\ z_{ij} & -\theta_{yij} & \theta_{xij} & 0 \end{bmatrix}. \tag{6}$$

Here, x_{ij}, y_{ij}, and z_{ij} are the x_{bi}, y_{bi}, and z_{bi} components of the elastic linear displacement mode j of link i at the origin of the coordinate $(X_d\ Y_d\ Z_d)_i$, respectively. θ_{xij}, θ_{yij}, and θ_{zij} are the x_{bi}, y_{bi}, and z_{bi} rotation components of the elastic angular displacement mode j of link i at the origin of the coordinate $(X_d\ Y_d\ Z_d)_i$, respectively. δ_{ij} is the time-varying amplitude of mode j of link i, and N_i is the number of modes used to describe the deformation of link i.

Define ${}^0\mathbf{W}_i$ or $\mathbf{W}_i$ to be the 4×4 homogeneous transformation matrix from the base coordinate frame $(X_b\ Y_b\ Z_b)_0$ to $(X_b\ Y_b\ Z_b)_i$. Then, we have

$${}^0\mathbf{W}_i = \mathbf{W}_i = \mathbf{W}_{i-1}\mathbf{E}_{i-1}\mathbf{A}_i = \widehat{\mathbf{W}}_{i-1}\mathbf{A}_i, \tag{7}$$

where $\widehat{\mathbf{W}}_{i-1}$ is the 4×4 homogeneous transformation matrix from the base coordinate frame $(X_bY_bZ_b)_0$ to the distal coordinate system $(X_dY_dZ_d)_{i-1}$ of link $i-1$.

2.4. Velocity of a Point of Link i. Let ${}^i\mathbf{h}_i(\eta)$ be the homogeneous coordinates in the system $(X_bY_bZ_b)_i$ of a point of the deformed link i at position η with the link under an undeformed condition from the origin of $(X_bY_bZ_b)_i$. Then, ${}^i\mathbf{h}_i(\eta)$ can be approximated as

$$\begin{aligned} {}^i\mathbf{h}_i(\eta) &= \left[1\ \ \eta\ \ 0\ \ 0\right]^{\mathrm{T}} + \sum_{j=1}^{N_i}\delta_{ij}\left[0\ \ x_{ij}(\eta)\ \ y_{ij}(\eta)\ \ z_{ij}(\eta)\right]^{\mathrm{T}} \\ &\quad - \frac{1}{2}\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\delta_{ij}\delta_{ik}\left[\ 0\ \ \underline{x}_{ijk}(\eta)\ \ 0\ \ 0\ \right]^{\mathrm{T}}. \end{aligned} \tag{8}$$

Here, $\underline{x}_{ijk}$ is the nonlinear strain coupling term, also known as the axial shortening due to the bending deformations of the link. When the flexible links are undergoing a high speed, this term will bring the so-called dynamic stiffening effect which will have a great influence on the dynamic behavior of flexible arms. In the dynamic modeling presented here, the high-order terms related to the non-linear coupling term are retained, which are ignored in the first-order and zero-order approximation coupling modeling.

In terms of the fixed inertial coordinates of the base $(X_bY_bZ_b)_0$, the position ${}^0\mathbf{h}_i$ or $\mathbf{h}_i$ of the point is given as

$$\mathbf{h}_i = \mathbf{W}_i\,{}^i\mathbf{h}_i\,. \tag{9}$$

Taking the time derivative of the position $\mathbf{h}_i$, we have the velocity of the point as

$$\frac{d\mathbf{h}_i}{dt} = \dot{\mathbf{h}}_i = \dot{\mathbf{W}}_i\,{}^i\mathbf{h}_i + \mathbf{W}_i\,{}^i\dot{\mathbf{h}}_i\,. \tag{10}$$

To accelerate the computation of the matrices $\dot{\mathbf{W}}_i$ or $\ddot{\mathbf{W}}_i$, we use the recursive kinematics method. By differentiating (7), one obtains

$$\begin{aligned} \dot{\mathbf{W}}_i &= \dot{\widehat{\mathbf{W}}}_{i-1}\mathbf{A}_i + \widehat{\mathbf{W}}_{i-1}\dot{\mathbf{A}}_i, \\ \ddot{\mathbf{W}}_i &= \ddot{\widehat{\mathbf{W}}}_{i-1}\mathbf{A}_i + 2\dot{\widehat{\mathbf{W}}}_{i-1}\dot{\mathbf{A}}_i + \widehat{\mathbf{W}}_{i-1}\ddot{\mathbf{A}}_i, \end{aligned} \tag{11}$$

where

$$\dot{\mathbf{A}}_i = \mathbf{U}_i \dot{q}_{2i},$$
$$\ddot{\mathbf{A}}_i = \mathbf{U}_{2i}\dot{q}_{2i}^2 + \mathbf{U}_i \ddot{q}_{2i}. \tag{12}$$

Here, $\mathbf{U}_i \triangleq \partial \mathbf{A}_i / \partial q_{2i}$, $\mathbf{U}_{2i} \triangleq \partial^2 \mathbf{A}_i / \partial q_{2i}^2$, and q_{2i} is the joint variable of joint i. Thus, $\dot{\mathbf{W}}_i$ and $\ddot{\mathbf{W}}_i$ can be computed recursively from $\widehat{\mathbf{W}}_{i-1}$ and its derivatives. Here, one additionally needs $\widehat{\mathbf{W}}_{i-1}$ and its derivatives. These can be computed recursively from $\mathbf{W}_{i-1}$ and its derivatives as follows:

$$\widehat{\mathbf{W}}_i = \mathbf{W}_i \mathbf{E}_i,$$
$$\dot{\widehat{\mathbf{W}}}_i = \dot{\mathbf{W}}_i \mathbf{E}_i + \mathbf{W}_i \dot{\mathbf{E}}_i, \tag{13}$$
$$\ddot{\widehat{\mathbf{W}}}_i = \ddot{\mathbf{W}}_i \mathbf{E}_i + 2\dot{\mathbf{W}}_i \dot{\mathbf{E}}_i + \mathbf{W}_i \ddot{\mathbf{E}}_i,$$

where

$$\dot{\mathbf{E}}_i = \sum_{j=1}^{N_i} \dot{\delta}_{ij} \mathbf{M}_{ij},$$
$$\ddot{\mathbf{E}}_i = \sum_{j=1}^{N_i} \ddot{\delta}_{ij} \mathbf{M}_{ij}. \tag{14}$$

3. Dynamics of Flexible Robots

To use Lagrange's equations, we need the kinetic and potential energy of the system.

3.1. The System Kinetic Energy. The system kinetic energy K contains two parts: the links kinetic energy K^b and the joints kinetic energy K^r as follows:

$$K = K^b + K^r. \tag{15}$$

Assume that the links are slender beams, so the rotary inertia and shear effects can be neglected. Therefore, the present analysis is based on the Euler-Bernoulli beam theory. Also assume that the links can undergo a large overall rigid motion, however the elastic displacements are small. The kinetic energy of the ith link is

$$K_i^b = \frac{1}{2}\int_0^{L_i} \mu(\eta)\, \mathrm{Tr}\left\{\dot{\mathbf{h}}_i \dot{\mathbf{h}}_i^{\mathrm{T}}\right\} d\eta + \frac{1}{2}\int_0^{L_i} J_{xi}(\eta) \left(\frac{\partial \theta_{xi}(\eta, t)}{\partial t}\right)^2 d\eta, \tag{16}$$

where $\mathrm{Tr}\{\cdot\}$ is the trace operator; $\mu(\eta)$ and J_{xi} are the mass per unit length and the polar moment of inertia per unit length of the link about the neutral axis x, respectively. For slender beams with uniform cross section area along x axis, $\mu(\eta) = \mu_i$. The first term in (16) is the kinetic energy of link i accounting for the rigid-body motion and the lateral and longitudinal deformation due to flexibility, whereas the second term is the kinetic energy accounting for the torsional deformation. Note that the torsional angle of link i, θ_{xi}, is expressed as

$$\theta_{xi} = \sum_{j=1}^{N_i} \delta_{ij} \theta_{xij}, \tag{17}$$

where δ_{ij} and θ_{xij} are mentioned in (4) and (6), respectively. δ_{ij} is the function of time, whereas θ_{xij} is the function of position η.

Substituting (10) and (17) into (16), expanding it, and then summing over all n links, one finds the links' kinetic energy to be

$$K^b = \sum_{i=1}^{n} K_i^b = \sum_{i=1}^{n} \mathrm{Tr}\left\{\dot{\mathbf{W}}_i \mathbf{B}_{3i} \mathbf{W}_i^{\mathrm{T}} + 2\dot{\mathbf{W}}_i \mathbf{B}_{2i} \mathbf{W}_i^{\mathrm{T}} + \mathbf{W}_i \mathbf{B}_{1i} \mathbf{W}_i^{\mathrm{T}}\right\} + \sum_{i=1}^{n}\sum_{j=1}^{N_i}\sum_{k=1}^{N_i} \dot{\delta}_{ij}\dot{\delta}_{ik} T_{ijk}, \tag{18}$$

where

$$T_{ijk} = \frac{1}{2}\int_0^{L_i} J_{xi}\theta_{xij}\theta_{xik}\, d\eta,$$
$$\mathbf{B}_{1i} = \frac{1}{2}\int_0^{L_i} \mu_i\, {}^i\dot{\mathbf{h}}_i\, {}^i\dot{\mathbf{h}}_i^{\mathrm{T}}\, d\eta,$$
$$\mathbf{B}_{2i} = \frac{1}{2}\int_0^{L_i} \mu_i\, {}^i\mathbf{h}_i\, {}^i\dot{\mathbf{h}}_i^{\mathrm{T}}\, d\eta, \tag{19}$$
$$\mathbf{B}_{3i} = \frac{1}{2}\int_0^{L_i} \mu_i\, {}^i\mathbf{h}_i\, {}^i\mathbf{h}_i^{\mathrm{T}}\, d\eta.$$

With the consideration of ${}^i\mathbf{h}_i$ expressed in (8) and its derivative, consider

$${}^i\dot{\mathbf{h}}_i(\eta) = \sum_{j=1}^{N_i} \dot{\delta}_{ij}\left[0 \;\; x_{ij}(\eta) \;\; y_{ij}(\eta) \;\; z_{ij}(\eta)\right]^{\mathrm{T}} - \sum_{j=1}^{N_i}\sum_{k=1}^{N_i} \dot{\delta}_{ij}\delta_{ik}\left[0 \;\; \underline{x}_{ijk}(\eta) \;\; 0 \;\; 0\right]^{\mathrm{T}}, \tag{20}$$

the matrices $\mathbf{B}_{1i}$, $\mathbf{B}_{2i}$, and $\mathbf{B}_{3i}$ can be written as

$$\mathbf{B}_{1i} = \sum_{j=1}^{N_i}\sum_{k=1}^{N_i} \dot{\delta}_{ij}\dot{\delta}_{ik}\mathbf{D}_{ijk} + \underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i} \dot{\delta}_{ij}\dot{\delta}_{ik}\delta_{il}\left(-\mathbf{F}_{ijkl} - \mathbf{F}_{ijkl}^{\mathrm{T}}\right)} + \underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\sum_{s=1}^{N_i} \dot{\delta}_{ij}\delta_{ik}\dot{\delta}_{il}\delta_{is}\mathbf{E}_{ijkls}},$$
$$\mathbf{B}_{2i} = \sum_{j=1}^{N_i} \dot{\delta}_{ij}\mathbf{D}_{ij} + \sum_{j=1}^{N_i}\sum_{k=1}^{N_i} \dot{\delta}_{ij}\delta_{ik}\mathbf{D}_{ijk}^{\mathrm{T}} + \underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i} \dot{\delta}_{ij}\delta_{ik}\left(-\mathbf{E}_{ijk}\right)} \tag{21}$$

$$+\underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\dot{\delta}_{ij}\delta_{ik}\delta_{il}\left(-\frac{1}{2}\mathbf{F}_{ijkl}^{\mathrm{T}}-\mathbf{F}_{iljk}\right)}$$
$$+\underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\sum_{s=1}^{N_i}\dot{\delta}_{ij}\delta_{ik}\delta_{il}\delta_{is}\left(\frac{1}{2}\mathbf{E}_{ijlks}^{\mathrm{T}}\right)}, \tag{22}$$

$$\begin{aligned}\mathbf{B}_{3i}&=\mathbf{D}_i+\sum_{j=1}^{N_i}\delta_{ij}\left(\mathbf{D}_{ij}+\mathbf{D}_{ij}^{\mathrm{T}}\right)+\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\delta_{ij}\delta_{ik}\mathbf{D}_{ijk}\\&+\underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\delta_{ij}\delta_{ik}\left(-\frac{1}{2}\mathbf{E}_{ijk}-\frac{1}{2}\mathbf{E}_{ijk}^{\mathrm{T}}\right)}\\&+\underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\delta_{ij}\delta_{ik}\delta_{il}\left(-\frac{1}{2}\mathbf{F}_{ijkl}-\frac{1}{2}\mathbf{F}_{ijkl}^{\mathrm{T}}\right)}\\&+\underline{\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\sum_{s=1}^{N_i}\delta_{ij}\delta_{ik}\delta_{il}\delta_{is}\left(\frac{1}{4}\mathbf{E}_{ijlks}\right)},\end{aligned} \tag{23}$$

where

$$\begin{aligned}\mathbf{D}_i&=\frac{1}{2}\int_0^{L_i}\mu_i\mathbf{L}_i\mathbf{L}_i^{\mathrm{T}}d\eta,\\\mathbf{D}_{ij}&=\frac{1}{2}\int_0^{L_i}\mu_i\mathbf{L}_i\mathbf{U}_{ij}^{\mathrm{T}}d\eta,\\\mathbf{D}_{ijk}&=\frac{1}{2}\int_0^{L_i}\mu_i\mathbf{U}_{ij}\mathbf{U}_{ik}^{\mathrm{T}}d\eta,\\\mathbf{E}_{ijk}&=\frac{1}{2}\int_0^{L_i}\mu_i\mathbf{L}_i\mathbf{S}_{ijk}^{\mathrm{T}}d\eta,\\\mathbf{F}_{ijkl}&=\frac{1}{2}\int_0^{L_i}\mu_i\mathbf{U}_{ij}\mathbf{S}_{ikl}^{\mathrm{T}}d\eta,\\\mathbf{E}_{ijkls}&=\frac{1}{2}\int_0^{L_i}\mu_i\mathbf{S}_{ijk}\ \mathbf{S}_{ils}^{\mathrm{T}}d\eta,\end{aligned} \tag{24}$$

in which

$$\begin{aligned}\mathbf{L}_i&=\begin{bmatrix}1&\eta&0&0\end{bmatrix}^{\mathrm{T}},\\\mathbf{U}_{ij}&=\begin{bmatrix}0&x_{ij}(\eta)&y_{ij}(\eta)&z_{ij}(\eta)\end{bmatrix}^{\mathrm{T}},\\\mathbf{S}_{ijk}&=\begin{bmatrix}0&\underline{x}_{ijk}(\eta)&0&0\end{bmatrix}^{\mathrm{T}}.\end{aligned} \tag{25}$$

In (21)–(23), the terms with underline remained in high-order approximation coupling model.

It should be noted that the link shape mentioned above is restricted to be the slender beam type. In fact, the link shape can further be extended to the other cases.

Case 1. Link i is the rigid-body with irregular shape. In this case, $N_i = 0$, $\mathbf{B}_{1i} = \mathbf{B}_{2i} = 0$, and $\mathbf{B}_{3i} = \mathbf{D}_i$. The term $\mathbf{D}_i$ is the equivalent of the inertia moment of the rigid-link. It is actually the pseu-matrix of the inertia moment in [25] with its more complex form compared with that of (23). Thus, for rigid-link, the link shape can be arbitrary, and one should only input the corresponding matrix $\mathbf{D}_i$.

Case 2. Link i consists of a flexible beam with the concentrated mass m_i at its proximal end. To account for the contribution of the concentrated mass to the kinetic energy of link i, the extra term $\mathbf{D}_i'$ should be added to the matrix $\mathbf{D}_i$, where

$$\mathbf{D}_i'=\frac{1}{2}m_i\begin{bmatrix}1&0&0&0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}1&0&0&0\end{bmatrix}. \tag{26}$$

Should the concentrated mass m_i locate at the position (x_0, y_0, z_0) near the proximal end, the extra matrix $\mathbf{D}_i'$ is modified to be

$$\mathbf{D}_i'=\frac{1}{2}m_i\begin{bmatrix}1&x_0&y_0&z_0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}1&x_0&y_0&z_0\end{bmatrix}. \tag{27}$$

Case 3. Link i consists of a flexible beam with the concentrated mass m_i at its distal end. Considering the contribution of the concentrated mass to the kinetic energy of link i, the extra terms $\mathbf{D}_i''$, $\mathbf{D}_{ij}''$, $\mathbf{D}_{ijk}''$, $\mathbf{E}_{ijk}''$, $\mathbf{F}_{ijkl}''$, and $\mathbf{E}_{ijkls}''$ should be added to the corresponding matrix $\mathbf{D}_i$, $\mathbf{D}_{ij}$, $\mathbf{D}_{ijk}$, $\mathbf{E}_{ijk}$, $\mathbf{F}_{ijkl}$, and $\mathbf{E}_{ijkls}$, respectively. Here,

$$\begin{aligned}\mathbf{D}_i''&=\frac{1}{2}m_i\begin{bmatrix}0&L_i&0&0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}0&L_i&0&0\end{bmatrix},\\\mathbf{D}_{ij}''&=\frac{1}{2}m_i\begin{bmatrix}0&L_i&0&0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}0&x_{ij}(L_i)&y_{ij}(L_i)&z_{ij}(L_i)\end{bmatrix},\\\mathbf{D}_{ijk}''&=\frac{1}{2}m_i\begin{bmatrix}0&x_{ij}(L_i)&y_{ij}(L_i)&z_{ij}(L_i)\end{bmatrix}^{\mathrm{T}}\\&\quad\times\begin{bmatrix}0&x_{ik}(L_i)&y_{ik}(L_i)&z_{ik}(L_i)\end{bmatrix},\\\mathbf{E}_{ijk}''&=\frac{1}{2}m_i\begin{bmatrix}0&L_i&0&0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}0&\underline{x}_{ijk}(L_i)&0&0\end{bmatrix},\\\mathbf{F}_{ijkl}''&=\frac{1}{2}m_i\begin{bmatrix}0&x_{ij}(L_i)&y_{ij}(L_i)&z_{ij}(L_i)\end{bmatrix}^{\mathrm{T}}\\&\quad\times\begin{bmatrix}0&\underline{x}_{ikl}(L_i)&0&0\end{bmatrix},\\\mathbf{E}_{ijkls}''&=\frac{1}{2}m_i\begin{bmatrix}0&\underline{x}_{ijk}(L_i)&0&0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}0&\underline{x}_{ils}(L_i)&0&0\end{bmatrix}.\end{aligned} \tag{28}$$

For the calculation of the kinetic energy of the joint i, we can lump its mass to link $i-1$ in accordance with the assumptions made in [21] for simplicity. Thus, the kinetic energy needs to be included only in the part accounting for spinning kinetic energy of the rotor of joint i as follows:

$$K_i^r=\frac{1}{2}J_{ri}\dot{\phi}_i^2. \tag{29}$$

Considering the relationship in (2). We can obtain the kinetic energy of the joints as follows:

$$K^r=\sum_{i=1}^{n}K_i^r=\sum_{i=1}^{n}\left(\frac{1}{2}J_{ri}\dot{\phi}_i^2\dot{q}_{1i}^2\right). \tag{30}$$

3.2. The System Potential Energy. The potential energy of the flexible manipulators V mainly includes the elastic potential energy of flexible joints V^r, the elastic potential energy of the

flexible links V^b, and their gravitational potential energy V_g. Therefore, the whole system potential energy is

$$V = V^r + V^b + V_g. \tag{31}$$

Firstly, the elastic potential energy of joint i is

$$V_i^r = \frac{1}{2}Kt_i(q_{2i} - q_{1i})^2. \tag{32}$$

Secondly, the elastic potential of the flexible links considered here have three parts: one by bending about the transverse y_i and z_i axes, one by compressing about the longitudinal x_i axis, and one by twisting about the longitudinal x_i axis. Along an incremental length $d\eta$, the elastic potential energy is

$$dV_i^b = \frac{1}{2}\left\{E_i\left[I_{zi}\left(\frac{\partial\theta_{zi}}{\partial\eta}\right)^2 + I_{yi}\left(\frac{\partial\theta_{yi}}{\partial\eta}\right)^2 + A_{xi}\left(\frac{\partial x_i}{\partial\eta}\right)^2\right] + G_iI_{xi}\left(\frac{\partial\theta_{xi}}{\partial\eta}\right)^2\right\}d\eta. \tag{33}$$

Here, θ_{xi}, θ_{yi}, and θ_{zi} are the ith link's rotations of the neutral axis at the point η in the x_i, y_i, and z_i directions, respectively; E_i is Young's modulus of material of link i; G_i is the shear modulus of material of link i; A_{xi} is the ith link's area of cross section about the x_i axis; I_{yi} and I_{zi} are the area moment of inertia of ith link's cross section about the y_i and z_i axes, respectively; and I_{xi} is the polar area moment of inertia of the ith link's cross section about the neutral axis. Similar to the torsional angle θ_{xi} of (17), θ_{yi} and θ_{zi} can be expressed as

$$\theta_{yi} = \sum_{k=1}^{N_i}\delta_{ik}\theta_{yik}, \qquad \theta_{zi} = \sum_{k=1}^{N_i}\delta_{ik}\theta_{zik}, \tag{34}$$

where θ_{yik} and θ_{zik} are mentioned in (6). By integrating dV_i^b of (33) over the link, and summing over all n links, one can obtain V^b as

$$V^b = \sum_{i=1}^{n}\left(\int_0^{L_i} dV_i^b\right) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{N_i}\sum_{k=1}^{N_i}\delta_{ij}\delta_{ik}K_{ijk}, \tag{35}$$

where

$$\begin{aligned} K_{ijk} &= K_{xijk} + K_{yijk} + K_{zijk} + K_{tijk}, \\ K_{xijk} &= \int_0^{L_i} E_iA_{xi}(\eta)\frac{\partial x_{ij}}{\partial\eta}\frac{\partial x_{ik}}{\partial\eta}d\eta, \\ K_{yijk} &= \int_0^{L_i} E_iI_{yi}(\eta)\frac{\partial\theta_{yij}}{\partial\eta}\frac{\partial\theta_{yik}}{\partial\eta}d\eta, \\ K_{zijk} &= \int_0^{L_i} E_iI_{zi}(\eta)\frac{\partial\theta_{zij}}{\partial\eta}\frac{\partial\theta_{zik}}{\partial\eta}d\eta, \\ K_{tijk} &= \int_0^{L_i} G_iI_{xi}(\eta)\frac{\partial\theta_{xij}}{\partial\eta}\frac{\partial\theta_{xik}}{\partial\eta}d\eta. \end{aligned} \tag{36}$$

Finally, by using the same procedure given by Book [14], the gravity potential of the system is

$$V_g = -\mathbf{g}^{\mathrm{T}}\sum_{i=1}^{n}\mathbf{W}_i\mathbf{r}_i, \tag{37}$$

in which $\mathbf{g}$ is the gravity vector with respect to the inertial base, and it has the following form:

$$\begin{aligned} \mathbf{g}^{\mathrm{T}} &= [0 \;\; g_x \;\; g_y \;\; g_z]; \\ \mathbf{r}_i &= M_i\mathbf{r}_{ri} + \sum_{k=1}^{N_i}\delta_{ik}\boldsymbol{\varepsilon}_{ik} - \frac{1}{2}\sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\delta_{ik}\delta_{il}\boldsymbol{\kappa}_{ikl}. \end{aligned} \tag{38}$$

Here, M_i is the total mass of link i and $\mathbf{r}_{ri}$ is the homogenous coordinates of the gravity center of link i (undeformed) in the frame $(X_bY_bZ_b)_i$,

$$\begin{aligned} \mathbf{r}_{ri} &= [1 \;\; r_{xi} \;\; 0 \;\; 0]^{\mathrm{T}}, \\ \boldsymbol{\varepsilon}_{ik} &= \int_0^{L_i}\mu_i[0 \;\; x_{ik} \;\; y_{ik} \;\; z_{ik}]^{\mathrm{T}}d\eta, \\ \boldsymbol{\kappa}_{ikl} &= \int_0^{L_i}\mu_i[0 \;\; \underline{x}_{ikl} \;\; 0 \;\; 0]^{\mathrm{T}}d\eta. \end{aligned} \tag{39}$$

Note that $\boldsymbol{\varepsilon}_{ik}$ and $\boldsymbol{\kappa}_{ikl}$ can be found in the top row of $\mathbf{D}_{ik}$ and $\mathbf{E}_{ijk}$, respectively.

3.3. Dynamic Equations of the System. We use the Lagrange method to derive the dynamics of the system and accord with the method of [14]. Then, the form of Lagrange's equations will be as follows.

For the joint variable q_{1j}, the following simply joint variable equation:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{q}_{1j}}\right) - \frac{\partial K}{\partial q_{1j}} + \frac{\partial V}{\partial q_{1j}} = \tau_j \quad (j = 1, 2, \ldots, n). \tag{40}$$

For the joint variable q_{2j}, the following simply joint variable equation:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{q}_{2j}}\right) - \frac{\partial K}{\partial q_{2j}} + \frac{\partial V}{\partial q_{2j}} = 0 \quad (j = 1, 2, \ldots, n). \tag{41}$$

For the deformation variable δ_{jf}, the following simply deformation variable equation:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{\delta}_{jf}}\right) - \frac{\partial K}{\partial\delta_{jf}} + \frac{\partial V}{\partial\delta_{jf}} = 0 \quad (j = 1, 2, \ldots, n;\; f = 1, 2, \ldots, N_j). \tag{42}$$

Upon the substitution of the system kinetic energy of (15), the system elastic potential energy of (31), and the gravity energy of (37) into (40), (41), and (42), thus becoming as follows.

For the joint variable q_{1j},

$$J_{rj}n_i^2\ddot{q}_{1j} = -Kt_jq_{1j} + Kt_jq_{2j} + \tau_j \quad (j = 1, 2, \ldots, n). \tag{43}$$

For the joint variables q_{2j} $(j = 1, 2, \ldots, n)$,

$$2\sum_{i=j}^{n} \mathrm{Tr}\left\{\frac{\partial \mathbf{W}_i}{\partial q_{2j}}\left[\mathbf{B}_{3i}\ddot{\mathbf{W}}_i^{\mathrm{T}} + \sum_{k=1}^{N_i}\mathbf{H}_{ik}\mathbf{W}_i^{\mathrm{T}}\ddot{\delta}_{ik} + \sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\mathbf{L}_{ikl}\mathbf{W}_i^{\mathrm{T}}\dot{\delta}_{ik}\dot{\delta}_{il} + 2\sum_{k=1}^{N_i}\mathbf{H}_{ik}\dot{\mathbf{W}}_i^{\mathrm{T}}\dot{\delta}_{ik}\right]\right\} = \mathbf{g}^{\mathrm{T}}\sum_{i=j}^{n}\frac{\partial \mathbf{W}_i}{\partial q_{2j}}\mathbf{r}_i + Kt_j q_{1j} - Kt_j q_{2j}. \tag{44}$$

For the deformation variables δ_{jf} $(j = 1, 2, \ldots, n;\ f = 1, 2, \ldots, N_j)$,

$$\begin{aligned} &2\sum_{i=j+1}^{n} \mathrm{Tr}\left\{\frac{\partial \mathbf{W}_i}{\partial \delta_{jf}}\left[\mathbf{B}_{3i}\ddot{\mathbf{W}}_i^{\mathrm{T}} + \sum_{k=1}^{N_i}\mathbf{H}_{ik}\mathbf{W}_i^{\mathrm{T}}\ddot{\delta}_{ik} + \sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\mathbf{L}_{ikl}\mathbf{W}_i^{\mathrm{T}}\dot{\delta}_{ik}\dot{\delta}_{il} + 2\sum_{k=1}^{N_i}\mathbf{H}_{ik}\dot{\mathbf{W}}_i^{\mathrm{T}}\dot{\delta}_{ik}\right]\right\} \\ &+ 2\,\mathrm{Tr}\left\{\left[\ddot{\mathbf{W}}_j\mathbf{H}_{jf} + \sum_{k=1}^{N_j}\mathbf{W}_j\mathbf{N}_{jfk}\ddot{\delta}_{jk} + \sum_{k=1}^{N_j}\sum_{l=1}^{N_j}\mathbf{W}_j\mathbf{P}_{jfkl}\dot{\delta}_{jk}\dot{\delta}_{jl} + 2\sum_{k=1}^{N_j}\dot{\mathbf{W}}_j\mathbf{N}_{jfk}\dot{\delta}_{jk}\right]\mathbf{W}_j^{\mathrm{T}}\right\} + 2\sum_{k=1}^{N_j}\ddot{\delta}_{jk}T_{jfk} \\ &= \mathbf{g}^{\mathrm{T}}\sum_{i=j+1}^{n}\left(\frac{\partial \mathbf{W}_i}{\partial \delta_{jf}}\mathbf{r}_i\right) + \mathbf{g}^{\mathrm{T}}\mathbf{W}_j\boldsymbol{\varepsilon}_{jf} - \mathbf{g}^{\mathrm{T}}\mathbf{W}_j\sum_{k=1}^{N_j}\delta_{jk}\boldsymbol{\kappa}_{jkl} - \sum_{k=1}^{N_j}\delta_{jk}K_{jfk}. \end{aligned} \tag{45}$$

Here,

$$\begin{aligned} \mathbf{H}_{ij} &= \mathbf{D}_{ij} + \sum_{k=1}^{N_i}\delta_{ik}\mathbf{D}_{ijk}^{\mathrm{T}} + \sum_{k=1}^{N_i}\delta_{ik}\left(-\mathbf{E}_{ijk}\right) + \sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\delta_{ik}\delta_{il}\left(-\frac{1}{2}\mathbf{F}_{ijkl}^{\mathrm{T}} - \mathbf{F}_{iljk}\right) + \sum_{k=1}^{N_i}\sum_{l=1}^{N_i}\sum_{s=1}^{N_i}\delta_{ik}\delta_{il}\delta_{is}\left(\frac{1}{2}\mathbf{E}_{ijlks}^{\mathrm{T}}\right), \\ \mathbf{L}_{ijk} &= -\mathbf{E}_{ijk} + \sum_{l=1}^{N_i}\delta_{il}\left(-\mathbf{F}_{iljk}\right) + \sum_{l=1}^{N_i}\sum_{s=1}^{N_i}\delta_{il}\delta_{is}\left(\frac{1}{2}\mathbf{E}_{ijkls}^{\mathrm{T}} + \mathbf{E}_{ijlks}^{\mathrm{T}} - \mathbf{E}_{ijlks}\right), \\ \mathbf{N}_{ijk} &= \mathbf{D}_{ijk}^{\mathrm{T}} + \sum_{l=1}^{N_i}\delta_{il}\left(-\mathbf{F}_{ijkl}^{\mathrm{T}} - \mathbf{F}_{ikjl}\right) + \sum_{l=1}^{N_i}\sum_{s=1}^{N_i}\delta_{il}\delta_{is}\mathbf{E}_{ijlks}^{\mathrm{T}}, \\ \mathbf{P}_{ijkl} &= -\mathbf{F}_{ijkl}^{\mathrm{T}} + \sum_{s=1}^{N_i}\delta_{is}\mathbf{E}_{ijskl}^{\mathrm{T}}. \end{aligned} \tag{46}$$

Finally, by complicated derivation and assembling, and using the recursive scheme to reduce the number of calculation, the dynamic equations of the system are obtained in the following formulation:

$$\mathbf{J}\ddot{\mathbf{z}} = \mathbf{R}^{I}, \tag{47}$$

where the generalized coordinates column $\mathbf{z}$ is defined as

$$\mathbf{z} = \left[q_{11}\ \ q_{21}\ \ \delta_{11}\ \ \delta_{12}\ \ \cdots\ \ \delta_{1N_1}\ \ q_{12}\ \ q_{22}\ \ \delta_{21}\ \ \cdots\ \ \delta_{2N_2}\ \ \cdots\ \ q_{1n}\ \ q_{2n}\ \ \delta_{n1}\ \ \cdots\ \ \delta_{nN_n}\right]^{\mathrm{T}}, \tag{48}$$

where $\mathbf{J}$ is the inertia matrix of the system composed of coefficients of the generalized acceleration $\ddot{\mathbf{z}}$ in the system dynamic equations. It is positive-definite and symmetric. The elements of $\mathbf{J}$ are arranged in the order of the generalized coordinate column $\mathbf{z}$, the $\mathbf{J}_i$ are diagonal elements, the ${}^{j}\mathbf{J}_i$ are off-diagonal elements, and these elements are given as follows:

$$\mathbf{J}_i = \begin{pmatrix} J_{ri}n_i^2 & 0 & 0 & \cdots & 0 \\ 0 & J_{ii} & J_{ii1} & \cdots & J_{iiN_i} \\ 0 & I_{i1i} & I_{i1i1} & \cdots & I_{i1iN_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & I_{iN_ii} & I_{iN_ii1} & \cdots & I_{iN_iiN_i} \end{pmatrix}, \tag{49}$$

$${}^{j}\mathbf{J}_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & J_{ij} & J_{ij1} & \cdots & J_{ijN_j} \\ 0 & I_{i1j} & I_{i1j1} & \cdots & I_{i1jN_j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & I_{iN_ij} & I_{iN_ij1} & \cdots & I_{iN_ijN_j} \end{pmatrix}. \tag{50}$$

(I) In the joint variable equation j $(j = 1, 2, \ldots, n)$, the inertia coefficient J_{jh} of the second derivative $\ddot{q}_{2h}$ of the joint variable q_{2h} is

$$J_{jh} = 2\,\mathrm{Tr}\left\{\left(\widehat{\mathbf{W}}_{j-1}\mathbf{U}_j\right)^{j}\ \widetilde{\mathbf{F}}_h\mathbf{U}_h^{\mathrm{T}}\widehat{\mathbf{W}}_{h-1}^{\mathrm{T}}\right\}, \tag{51}$$

where

$$ {}^{j}\widetilde{\mathbf{F}}_h = \sum_{i=\max(h,j)}^{n} {}^{j}\widetilde{\mathbf{W}}_i \mathbf{B}_{3i} {}^{h}\widetilde{\mathbf{W}}_i^{\mathrm{T}}, \tag{52} $$

$$ {}^{h}\widetilde{\mathbf{W}}_i = \mathbf{E}_h \mathbf{A}_{h+1} \cdots \mathbf{E}_{i-1} \mathbf{A}_i. \tag{53} $$

Here, ${}^{j}\widetilde{\mathbf{W}}_i$ is the 4×4 homogeneous transformation matrix from the proximal reference frame $(X_b Y_b Z_b)_j$ of link j to the proximal reference frame $(X_b Y_b Z_b)_i$ of link i. From (52), we have $J_{jh} = J_{hj}$ (symmetric).

(II) In the joint variable equation j ($j = 1, 2, \ldots, n$), the inertia coefficient J_{jhk} of the second derivative $\ddot{\delta}_{hk}$ of the joint variable δ_{hk} (for $1 \le k \le N_k$) is expressed as follows:

for $h = n$; $j = 1, \ldots, n$,

$$ J_{jhk} = 2\,\mathrm{Tr}\left\{\left(\widehat{\mathbf{W}}_{j-1}\mathbf{U}_j\right) {}^{j}\widetilde{\mathbf{W}}_n \mathbf{H}_{nk} \mathbf{W}_n^{\mathrm{T}}\right\}; \tag{54} $$

for $h = j, \ldots, n-1$; $j = 1, \ldots, n-1$,

$$ J_{jhk} = 2\,\mathrm{Tr}\left\{\widehat{\mathbf{W}}_{j-1}\mathbf{U}_j \left[{}^{j}\mathbf{F}_h \mathbf{M}_{hk}^{\mathrm{T}} + {}^{j}\widetilde{\mathbf{W}}_h \mathbf{H}_{hk}\right] \mathbf{W}_h^{\mathrm{T}}\right\}; \tag{55} $$

for $h = 1, \ldots, j-1$; $j = 2, \ldots, n$,

$$ J_{jhk} = 2\,\mathrm{Tr}\left\{\left(\widehat{\mathbf{W}}_{j-1}\mathbf{U}_j\right) {}^{j}\mathbf{F}_h \mathbf{M}_{hk}^{\mathrm{T}} \mathbf{W}_h^{\mathrm{T}}\right\}. \tag{56} $$

Here,

$$ \begin{aligned} {}^{j}\mathbf{F}_h &= \sum_{i=\max(h+1,j)}^{n} {}^{j}\widetilde{\mathbf{W}}_i \mathbf{B}_{3i} {}^{h}\mathbf{W}_i^{\mathrm{T}}, \\ {}^{h}\mathbf{W}_i &= \mathbf{A}_{h+1}\mathbf{E}_{h+1} \cdots \mathbf{E}_{i-1}\mathbf{A}_i, \end{aligned} \tag{57} $$

where ${}^{j}\mathbf{W}_i$ is the 4×4 homogeneous transformation matrix from the distal coordinate system $(X_d Y_d Z_d)_j$ of link j to the proximal coordinate system $(X_b Y_b Z_b)_i$ of link i. Note that the inertia coefficient for the deformation variable δ_{hk} in the joint equation j is the same as that for the joint variable q_{2h} in the deformation equation h, k. This further extends the symmetry of the inertia matrix and reduces the computation necessary.

(III) In the joint variable equation j ($j = 1, 2, \ldots, n$), the inertia coefficients I_{jfh} of the second derivative $\ddot{q}_{2h}$ of the joint variable q_{2h} are expressed as follows:

for $j = n; h = 1, \ldots, n$,

$$ I_{jfn} = 2\,\mathrm{Tr}\left\{\left(\widehat{\mathbf{W}}_{h-1}\mathbf{U}_h\right) {}^{h}\widetilde{\mathbf{W}}_n \mathbf{H}_{nf} \mathbf{W}_n^{\mathrm{T}}\right\}; \tag{58} $$

for $j = 1, \ldots, n-1$; $h = 1, \ldots, j$,

$$ I_{jfh} = 2\,\mathrm{Tr}\left\{\widehat{\mathbf{W}}_{h-1}\mathbf{U}_h \left[{}^{j}\widetilde{\phi}_h^{\mathrm{T}} \mathbf{M}_{jf}^{\mathrm{T}} + {}^{h}\widetilde{\mathbf{W}}_j \mathbf{H}_{jf}\right] \mathbf{W}_j^{\mathrm{T}}\right\}; \tag{59} $$

for $j = 1, \ldots, n-1$; $h = j+1, \ldots, n-1$,

$$ I_{jfh} = 2\,\mathrm{Tr}\left\{\left(\mathbf{W}_j \mathbf{M}_{jf}\right) {}^{j}\widetilde{\phi}_h \mathbf{U}_h^{\mathrm{T}} \widehat{\mathbf{W}}_{h-1}^{\mathrm{T}}\right\}. \tag{60} $$

(IV) In the deformation variable equation j, f, the inertia coefficients I_{jfhk} of the second derivative $\ddot{\delta}_{hk}$ of the deformation variable δ_{hk} ($1 \le f \le N_j$, $1 \le k \le N_h$) are expressed as follows:

for $j = h = n$,

$$ I_{nfnk} = 2\,\mathrm{Tr}\left\{\mathbf{N}_{nfk}\right\} + 2T_{nfk}; \tag{61} $$

for $j = n$; $h = 1, \ldots, n-1$,

$$ I_{nfhk} = 2\,\mathrm{Tr}\left\{\left(\mathbf{W}_h \mathbf{M}_{hk}\right) {}^{h}\mathbf{W}_n \mathbf{H}_{nf} \mathbf{W}_n^{\mathrm{T}}\right\}; \tag{62} $$

for $j = 1, \ldots, n-1$; $h = n$,

$$ I_{jfnk} = 2\,\mathrm{Tr}\left\{\left(\mathbf{W}_j \mathbf{M}_{jf}\right) {}^{j}\mathbf{W}_n \mathbf{H}_{nk} \mathbf{W}_n^{\mathrm{T}}\right\}; \tag{63} $$

for $j = h$; $h = 1, \ldots, n-1$,

$$ I_{jfhk} = 2\,\mathrm{Tr}\left\{\mathbf{M}_{jf} {}^{j}\phi_h \mathbf{M}_{jk}^{\mathrm{T}} + \mathbf{N}_{jfk}\right\} + 2T_{jfk}; \tag{64} $$

for $j = 2, \ldots, n-1$; $h = 1, \ldots, j-1$,

$$ I_{jfhk} = 2\,\mathrm{Tr}\left\{\mathbf{W}_j \left[\mathbf{M}_{jf} {}^{j}\phi_h + \mathbf{H}_{jf}^{\mathrm{T}} {}^{h}\mathbf{W}_j^{\mathrm{T}}\right] \mathbf{M}_{hk}^{\mathrm{T}} \mathbf{W}_h^{\mathrm{T}}\right\}; \tag{65} $$

for $j = 1, \ldots, n-2$; $h = j+1, \ldots, n-1$,

$$ I_{jfhk} = 2\,\mathrm{Tr}\left\{\mathbf{W}_j \mathbf{M}_{jf} \left[{}^{j}\phi_h \mathbf{M}_{hk}^{\mathrm{T}} + {}^{j}\mathbf{W}_h \mathbf{H}_{hk}\right] \mathbf{W}_h^{\mathrm{T}}\right\}. \tag{66} $$

Here,

$$ \begin{aligned} {}^{j}\widetilde{\phi}_h &= \sum_{i=\max(j+1,h)}^{n} {}^{j}\mathbf{W}_i \mathbf{B}_{3i} {}^{h}\widetilde{\mathbf{W}}_i^{\mathrm{T}}, \\ {}^{j}\phi_h &= \sum_{i=\max(j+1,h+1)}^{n} {}^{j}\mathbf{W}_i \mathbf{B}_{3i} {}^{h}\mathbf{W}_i^{\mathrm{T}}. \end{aligned} \tag{67} $$

For the symmetry of the coefficients, we have $I_{jfhk} = I_{hkjf}$.

(V) $\mathbf{R}^I$ is generalized force, containing the remaining dynamics and external forcing terms except impact force as follows:

$$ \mathbf{R}^I = \left[R_{r1}^I \;\; R_{J1}^I \;\; R_{11}^I \;\; \cdots \;\; R_{1N_1}^I \;\; R_{r2}^I \;\; R_{J2}^I \;\; R_{21}^I \;\; \cdots \;\; R_{2N_2}^I \;\; \cdots \;\; R_{rn}^I \;\; R_{Jn}^I \;\; R_{n1}^I \;\; \cdots \;\; R_{nN_n}^I\right]^{\mathrm{T}}. \tag{68} $$

The R_{rj}^I is the remains of (43) with the second derivatives removed as follows:

$$ R_{rj}^I = -Kt_j q_{1j} + Kt_j q_{2j} + \tau_j \quad (j = 1, \ldots, n). \tag{69} $$

The R_{Jj}^I is the remains of (44) with the second derivatives removed as follows: for $j = 1$,

$$ R_{J1}^I = -2\,\mathrm{Tr}\left\{\mathbf{U}_1 \mathbf{Q}_1\right\} + \mathbf{g}^{\mathrm{T}} \mathbf{U}_1 \mathbf{P}_1 + Kt_1 q_{11} - Kt_1 q_{21}; \tag{70} $$

for $j = 2, \ldots, n$,

$$R_{Jj}^{I} = -2\,\mathrm{Tr}\left\{\widehat{\mathbf{W}}_{j-1}\mathbf{U}_j\mathbf{Q}_j\right\} + \mathbf{g}^{\mathrm{T}}\widehat{\mathbf{W}}_{j-1}\mathbf{U}_j\mathbf{P}_j + Kt_j q_{1j} - Kt_j q_{2j}. \tag{71}$$

The R_{jf}^{I} is the remains of (45) with the second derivatives removed as follows.

For $j = 1, \ldots, n-1$,

$$\begin{aligned} R_{Jj}^{I} = -2\,\mathrm{Tr}\Big\{&\mathbf{W}_j\mathbf{M}_{jf}\mathbf{A}_{j+1}\mathbf{Q}_{j+1} \\ &+ \left[\ddot{\mathbf{W}}_{\mathbf{v}j}\mathbf{H}_{jf} + 2\dot{\mathbf{W}}_j\left(\sum_{k=1}^{N_j}\mathbf{N}_{jfk}\dot{\delta}_{jk}\right)\right. \\ &\left. + \mathbf{W}_j\left(\sum_{k=1}^{N_j}\sum_{l=1}^{N_j}\mathbf{P}_{jfkl}\dot{\delta}_{jk}\dot{\delta}_{jl}\right)\right] \\ &\times\mathbf{W}_j^{\mathrm{T}}\Big\} \\ &- \sum_{k=1}^{N_j}\delta_{jk}K_{jfk} + \mathbf{g}^{\mathrm{T}}\mathbf{W}_j\mathbf{M}_{jf}\mathbf{A}_{j+1}\mathbf{P}_{j+1} \\ &+ \mathbf{g}^{\mathrm{T}}\mathbf{W}_j\varepsilon_{jf} - \mathbf{g}^{\mathrm{T}}\mathbf{W}_j\sum_{k=1}^{N_j}\delta_{jk}\boldsymbol{\kappa}_{jfk}; \end{aligned} \tag{72}$$

for $j = n$,

$$\begin{aligned} R_{nf}^{\mathrm{I}} = -2\,\mathrm{Tr}&\left[\ddot{\mathbf{W}}_{\mathbf{v}n}\mathbf{H}_{nf} + 2\dot{\mathbf{W}}_n\left(\sum_{k=1}^{N_n}\mathbf{N}_{nfk}\dot{\delta}_{nk}\right)\right. \\ &\left. + \mathbf{W}_j\left(\sum_{k=1}^{N_n}\sum_{l=1}^{N_n}\mathbf{P}_{nfkl}\dot{\delta}_{nk}\dot{\delta}_{nl}\right)\mathbf{W}_n^{\mathrm{T}}\right] \\ &- \sum_{k=1}^{N_n}\delta_{nk}K_{nfk} + \mathbf{g}^{\mathrm{T}}\mathbf{W}_n\boldsymbol{\varepsilon}_{nf} \\ &- \mathbf{g}^{\mathrm{T}}\mathbf{W}_n\sum_{k=1}^{N_n}\delta_{nk}\boldsymbol{\kappa}_{nfk}. \end{aligned} \tag{73}$$

Here,

$$\begin{aligned} \mathbf{Q}_j = \sum_{i=j}^{n}\Bigg({}^{j}\widetilde{\mathbf{W}}_i\mathbf{B}_{3i}\ddot{\mathbf{W}}_{\mathbf{v}i} + 2\sum_{k=1}^{N_i}{}^{j}\widetilde{\mathbf{W}}_i\mathbf{H}_{ik}\dot{\mathbf{W}}_i^{\mathrm{T}}\dot{\delta}_{ik} \\ + \sum_{k=1}^{N_i}\sum_{l=1}^{N_i}{}^{j}\widetilde{\mathbf{W}}_i\,\mathbf{L}_{ikl}\mathbf{W}_i^{\mathrm{T}}\dot{\delta}_{ik}\dot{\delta}_{il}\Bigg), \\ \mathbf{P}_j = \sum_{i=j}^{n}{}^{j}\widetilde{\mathbf{W}}_i\,\mathbf{r}_i, \end{aligned} \tag{74}$$

where the value of $\ddot{\mathbf{W}}_{\mathbf{v}i}$ is the remainder of $\ddot{\mathbf{W}}_i$ by only eliminating the terms involving $\ddot{q}_{2h}$ and $\ddot{\delta}_{hk}$, and it can be calculated recursively by

$$\ddot{\mathbf{W}}_i = \sum_{h=1}^{i}\widehat{\mathbf{W}}_{h-1}\mathbf{U}_h{}^{h}\overline{\mathbf{W}}_i\ddot{q}_{2h} + \sum_{h=1}^{i-1}\sum_{k=1}^{N_h}\mathbf{W}_h\mathbf{M}_{hk}{}^{h}\mathbf{W}_i\ddot{\delta}_{hk} + \ddot{\mathbf{W}}_{\mathbf{v}i}. \tag{75}$$

The value of $\ddot{\widehat{\mathbf{W}}}_i$ can be obtained similarly as follows:

$$\ddot{\widehat{\mathbf{W}}}_i = \sum_{h=1}^{i}\left(\widehat{\mathbf{W}}_{h-1}\mathbf{U}_h{}^{h}\overline{\mathbf{W}}_i\ddot{q}_{2h} + \sum_{k=1}^{N_h}\mathbf{W}_h\mathbf{M}_{hk}{}^{h}\widehat{\mathbf{W}}_i\ddot{\delta}_{hk}\right) + \ddot{\widehat{\mathbf{W}}}_{\mathbf{v}i}, \tag{76}$$

where

$$\begin{aligned} {}^{h}\overline{\mathbf{W}}_i &= \mathbf{E}_h\mathbf{A}_{h+1}\cdots\mathbf{A}_i\mathbf{E}_i, \\ {}^{h}\widehat{\mathbf{W}}_i &= \mathbf{A}_{h+1}\mathbf{E}_{h+1}\cdots\mathbf{A}_i\mathbf{E}_i. \end{aligned} \tag{77}$$

For accelerating the computation of the generalized inertia matrix of the system, the terms $\ddot{\mathbf{W}}_{\mathbf{v}i}$, $\ddot{\widehat{\mathbf{W}}}_{\mathbf{v}i}$, $\mathbf{Q}_j$, $\mathbf{P}_j$, ${}^{j}\widetilde{\mathbf{F}}_h$, ${}^{j}\widetilde{\phi}_h$, ${}^{h}\mathbf{F}_j$, and ${}^{j}\phi_h$ defined in (52)–(76), respectively, can be calculated recursively as follows:

$$\begin{aligned} \ddot{\mathbf{W}}_{\mathbf{v}1} &= \mathbf{U}_{21}\dot{q}_{21}^2, \\ \ddot{\mathbf{W}}_{\mathbf{v}i} &= \ddot{\widehat{\mathbf{W}}}_{\mathbf{v}i-1}\mathbf{A}_i + 2\dot{\widehat{\mathbf{W}}}_{i-1}\mathbf{U}_i\dot{q}_{2i} + \widehat{\mathbf{W}}_{i-1}\mathbf{U}_{2i}\dot{q}_{2i}^2, \\ \ddot{\widehat{\mathbf{W}}}_{\mathbf{v}i} &= \ddot{\mathbf{W}}_{\mathbf{v}i}\mathbf{E}_i + 2\dot{\mathbf{W}}_i\dot{\mathbf{E}}_i, \\ \mathbf{Q}_n &= \mathbf{B}_{3n}\ddot{\mathbf{W}}_{\mathbf{v}n}^{\mathrm{T}} + 2\left(\sum_{k=1}^{N_n}\mathbf{H}_{nk}\dot{\delta}_{nk}\right)\dot{\mathbf{W}}_n^{\mathrm{T}} \\ &\quad + \left(\sum_{k=1}^{N_n}\sum_{l=1}^{N_n}\mathbf{L}_{nkl}\dot{\delta}_{nk}\dot{\delta}_{nl}\right)\mathbf{W}_n^{\mathrm{T}}, \\ \mathbf{Q}_j &= \mathbf{B}_{3i}\ddot{\mathbf{W}}_{\mathbf{v}i}^{\mathrm{T}} + 2\left(\sum_{k=1}^{N_j}\mathbf{H}_{jk}\dot{\delta}_{jk}\right)\dot{\mathbf{W}}_j^{\mathrm{T}} \\ &\quad + \left(\sum_{k=1}^{N_j}\sum_{l=1}^{N_j}\mathbf{L}_{jkl}\dot{\delta}_{jk}\dot{\delta}_{jl}\right)\mathbf{W}_j^{\mathrm{T}} + {}^{j}\widetilde{\mathbf{W}}_{j+1}\,\mathbf{Q}_{j+1}, \end{aligned} \tag{78}$$

$$\mathbf{P}_n = \mathbf{r}_n = \mathbf{M}_n\mathbf{r}_{rn} + \sum_{k=1}^{N_n}\delta_{nk}\varepsilon_{nk} - \frac{1}{2}\sum_{k=1}^{N_n}\sum_{l=1}^{N_n}\delta_{nk}\delta_{nl}\boldsymbol{\kappa}_{nkl}, \tag{79}$$

$$\mathbf{P}_j = \mathbf{r}_j + {}^{j}\widetilde{\mathbf{W}}_{j+1}\,\mathbf{P}_{j+1}. \tag{80}$$

As for ${}^{j}\widetilde{\mathbf{F}}_h$, for $j = h = n$,

$${}^{n}\widetilde{\mathbf{F}}_n = \mathbf{B}_{3n}, \tag{81}$$

for $j = n;\ h = 1, \ldots, n-1$,

$${}^{n}\widetilde{\mathbf{F}}_h = \mathbf{B}_{3i}{}^{h}\widetilde{\mathbf{W}}_n^{\mathrm{T}}, \tag{82}$$

for $j = h;\ h = 1, \ldots, n-1$,

$${}^{h}\widetilde{\mathbf{F}}_h = \mathbf{B}_{3h} + \mathbf{E}_h\mathbf{A}_{h+1}{}^{h+1}\widetilde{\mathbf{F}}_h. \tag{83}$$

And for $j \neq h \neq n$,

$$\begin{aligned} {}^{j}\widetilde{\mathbf{F}}_h &= \mathbf{B}_{3j}\cdot{}^{h}\widetilde{\mathbf{W}}_j^{\mathrm{T}} + \left(\mathbf{E}_i\mathbf{A}_{j+1}\right)\cdot{}^{j+1}\widetilde{\mathbf{F}}_h, \\ {}^{j}\widetilde{\mathbf{F}}_h &= \left({}^{h}\widetilde{\mathbf{F}}_j\right)^{\mathrm{T}}. \end{aligned} \tag{84}$$

FIGURE 3: $\omega_0 < 3$ rad/s.

FIGURE 4: $\omega_0 = 3$ rad/s.

As for ${}^{j}\mathbf{F}_h$, only if $N_h > 0$, calculate

$$ {}^{j}\mathbf{F}_h = {}^{j}\widetilde{\mathbf{F}}_{h+1} \cdot \mathbf{A}_{h+1}^{\mathrm{T}} \quad \left(j = 1, \ldots, n;\ h = 1, \ldots, n-1\right). \tag{85} $$

As for ${}^{j}\widetilde{\phi}_h$, it can be obtained from ${}^{h}\mathbf{F}_j$ easily as follows:

$$ {}^{j}\widetilde{\phi}_h = \left({}^{h}\mathbf{F}_j\right)^{\mathrm{T}}. \tag{86} $$

As for ${}^{j}\phi_h$, only if $N_j > 0$ and $N_h > 0$, calculate

$$ {}^{j}\phi_h = {}^{j}\widetilde{\phi}_{h+1} \cdot \mathbf{A}_{h+1}^{\mathrm{T}} \quad \left(j = 1, \ldots, n-1;\ h = j, \ldots, n-1\right). \tag{87} $$

4. Example of Dynamic Simulation

A general-purpose software package for dynamic simulation of flexible-links and flexible-joints robotic manipulators based on the said algorithm is written in C++. The data structure is pointers and linked lists, so that the data length can be changed based on different initial conditions of the system. The first step is setting up a given interface to input system parameters via reading files. Then, the basic module matrix can be generated automatically. With that, reducing a second order differential equation to a first order linear ordinary differential equation. Eventually, a process based on Adams predictor-corrector method for solving ordinary differential equation calculates till the end time, with an output file per step. It is universal for usage, computationally efficient, and numerically stable.

4.1. Flexible Single-Link Undergoing Low/High Rotational Speed. Ignoring the gravity and the flexible effect of the joint, here, consider the flexible single-link undergoing low/high rotational speed in order to discover the applicability of the high-order geometric nonlinearity coupling model. The properties of the flexible beam are the same as those in [3, 8], and they are given as follows. The length $L_1 = 8$ m, the cross-section area $A_{x1} = 7.2968 \times 10^{-5}$ m^2, the area moment of inertia $I = 8.2189 \times 10^{-9}$ m^4, the mass density $\rho = 2.7667 \times 10^3$ kg/m^3, and the elasticity modulus $E_1 = 6.8952 \times 10^{10}$ N/m^2. For the flexible cantilever beam without large overall motion, the fundamental frequency is 0.464 Hz. The large rotating motion law of the system adopted $\dot{\theta}$ in [3, 8] is given by

$$ \dot{\theta} = \begin{cases} \dfrac{\omega_0}{T} t - \dfrac{\omega_0}{2\pi} \sin\left(\dfrac{2\pi}{T} t\right), & 0 \le t \le T, \\ \omega_0, & t > T, \end{cases} \tag{88} $$

where $T = 15$ s, and ω_0 is chosen here to be 0.6, 2, 3, 8, and 10 rad/s, respectively, in the following numerical simulations. The tip deformations of link u_y are shown from Figures 3, 4, and 5.

It can be observed that, undergoing a low rotating speed (0.6 rad/s or 2 rad/s), there is an obvious difference between zero-order and high-order coupling model, and the difference becomes larger and larger with the increasing of the rotating speed. When the high-order coupling model is at the rotating speed 3 rad/s (≈0.477 Hz), closing to the fundamental frequency (0.464 Hz), the vibrating amplitude of the link tip is aroused much higher, but it is convergent. Whereas, the results form zero-order coupling model turn to be divergente. When reaching a high rotating speed, 8 rad/s (≈1.272 Hz) or 10 rad/s (≈1.59 Hz), the zero-order coupling model will fail to get the convergent results. These phenomena are coincided with the numerical result in [2, 11].

4.2. Spatial Flexible-Link and Flexible-Joint Robotic Arm. In order to validate the algorithm and package presented in this paper, the dynamic simulation of Canadarm2 (the Space Station remote manipulator system serving on the international space station) is given as an example.

FIGURE 5: $\omega_0 > 3$ rad/s.

FIGURE 6: The space robot arm Canadarm2.

FIGURE 7: A spatial flexible robot arms.

FIGURE 8: The flexible robot arms motion.

FIGURE 9: The torsional angle of 1st joint, ε_1.

As shown in Figure 6, Canadarm2 consists of four parts: shoulder, upper arm boom, lower arm boom, and wrist, in which the shoulder and the wrist both have 3 rotational degrees of freedom (DOF), and the elbow connecting the upper arm and the lower arm has 1 rotational DOF. Hence, the total system can be regarded as a 7-DOF manipulator with seven links and seven revolute joints. The links making up the shoulder and the wrist are short and thick in shape; therefore, they can be regarded as rigid links, whereas the upper and lower arms are assumed to be flexible due to their slender beam type. Therefore, Canadarm2 system is

Figure 10: u_x of 1st link tip.

Figure 11: v_y of 1st link tip.

Figure 12: w_z of 1st link tip.

Figure 13: θ_x of 1st link tip.

simplified to contain two flexible links and three rigid links which are connected through flexible joints, as shown in Figure 7. In this example, there is a distance $d = 0.475$ m between the 2nd link and 1st link along the Z-axis, which consequently arouse the rotation of the flexible links. And the longitudinal stretching (along X-axis), transversal bending (along Y-axis and Z-axis), and torsion (around X-axis) are considered here. Correspondingly, a total of 12 eigenfunctions (the first three preorientation of deformation) of a clamped-free beam are utilized to represent the mode shapes of each link deformation. The links parameters are shown in Table 1. The material of the links is aluminum, cross-section parameter $E_iI_{yi} = E_iI_{zi} = 3.8\times10^6$ Nm^2, $E_iA_i = 2.84\times10^7$ N, $G_i = 0.27 \times 10^5$ MPa, and $J_{xi} = 15.41$ kg $\times$ m^2. Initial joint angle is $q_{11} = 0.4\pi$ rad, $q_{21} = 0.4\pi$ rad, $q_{12} = -0.4\pi$ rad, and $q_{22} = -0.4\pi$ rad, respectively. Initial joint angle velocity is zero.

4.2.1. Dropping Motion in the Gravity Field. Figure 8 describes the motion of the high-order coupling flexible five-links system dropping in the gravity field.

Table 1: The parameters of space robot system.

Link	Mass (kg)	Centre (m)	Transverse parameters (kg × m^{-2})	Length (m)
1	139.35	(3.1883, 0.0, 0.0282)	$I_{xx} = 3.225$, $I_{yy} = 998.630$, $I_{zz} = 997.180$, $I_{yx} = -1.928$, $I_{zx} = 9.185$, $I_{yz} = -0.0302$	(6.377, 0, 0.1524)
2	87.42	(3.53, 0.0, −0.0066)	$I_{xx} = 1.59$, $I_{yy} = 632.9$, $I_{zz} = 632.51$, $I_{yx} = 0.8021$, $I_{zx} = -3.786$, $I_{yz} = -0.08$	(7.06, 0, 0.1524)
3	8.48	(0.2286, 0.0, 0.0)	$I_{xx} = 0.0893$, $I_{yy} = 0.402$, $I_{zz} = 0.402$, $I_{yx} = 0.0$, $I_{zx} = 0.0$, $I_{yz} = 0.0$	(0.4572, 0, 0)
4	45.94	(0.381, 0.0, 0.0)	$I_{xx} = 0.408$, $I_{yy} = 4.86$, $I_{zz} = 4.86$, $I_{yx} = 0.0$, $I_{zx} = 0.0$, $I_{yz} = 0.0$	(0.762, 0, 0)
5	45.21	(0.332, 0.0, 0.0)	$I_{xx} = 1.903$, $I_{yy} = 3.816$, $I_{zz} = 2.823$, $I_{yx} = 0.385$, $I_{zx} = 0.966$, $I_{yz} = 0.353$	(0.6604, 0, 0)

Figure 14: The 1st joint, q_{21}.

Figure 15: The 2nd joint, q_{22}.

Figure 16: The 3rd joint, q_{23}.

4.2.2. Effects of Joint Flexibility on the System. Figure 9 shows the torsional angle of the 1st joint ε_1 in the high-order coupling model with different mass $J_{ri}n_i^2$, 5, 10, or 15, respectively. Figures 10, 11, 12, and 13 show the X, Y, and Z deformation u_x, v_y, w_z, and torsional angle θ_x of 1st link tip with different joint mass. The larger joint's mass is, the larger joint angles of motion the system has. The same phenomena can be seen form Figures 10 to 13.

The real rotational angle q_{2i} of five joint in the high-order coupling model is shown in Figures 14, 15, 16, 17, and 18 with different stiffness. The joint stiffness Kt_i is chosen here to be 0.60, 1.33, 2.66, and 5.00×10^6 N/mm, respectively.

5. Conclusions

This paper aims at building a comprehensive model of spatial manipulator arms consisting of n flexible links and n flexible joints. The deformations of stretching, bending, and torsion of the links are all considered. And the flexibility and the mass of the joints are also included. In the modeling, the

Figure 17: The 4th joint, q_{24}.

Figure 18: The 5th joint, q_{25}.

so-called dynamic stiffening effects are considered via the adoption of nonlinear deformation description. Based on the presented method, a general-purpose software package of multilink manipulator arms is developed. The results demonstrate that dynamic stiffening effects cannot be neglected. The coupling effects of flexible links and flexible joints on dynamic characteristics are significant to the system's accurate dynamic response predictions, performance evaluation, and implementation of procedure control.

Acknowledgments

This work is supported by the National Natural Science Foundations of China (11272155, 11132007, and 10772085), 333 Project of Jiangsu Province (BRA2011172), and the Fundamental Research Funds for Central Universities (30920130112009).

References

[1] A. A. Shabana, F. M. Gantoi, and M. A. Brown, "Integration of finite element and multibody system algorithms for the analysis of human body motion," in *Proceedings of the IUTAM Symposium on Human Body Dynamics: From Multibody Systems to Biomechanics*, vol. 2, pp. 233–240, June 2011.

[2] T. R. Kane, R. R. Ryan, and A. K. Banerjee, "Dynamics of a cantilever beam attached to a moving base," *Journal of Guidance, Control, and Dynamics*, vol. 10, no. 2, pp. 139–151, 1987.

[3] S.-C. Wu and E. J. Haug, "Geometric non-linear substructuring for dynamics of flexible mechanical systems," *International Journal for Numerical Methods in Engineering*, vol. 26, no. 10, pp. 2211–2226, 1988.

[4] F. Xi and R. G. Fenton, "Coupling effect of a flexible link and a flexible joint," *International Journal of Robotics Research*, vol. 13, no. 5, pp. 443–453, 1994.

[5] O. Wallrapp and R. Schwertassek, "Representation of geometric stiffening in multibody system simulation," *International Journal for Numerical Methods in Engineering*, vol. 32, no. 8, pp. 1833–1850, 1991.

[6] K. H. Low, "Vibration analysis of a tip-loaded beam attached to a rotating joint," *Computers and Structures*, vol. 52, no. 5, pp. 955–968, 1994.

[7] H. H. Yoo, R. R. Ryan, and R. A. Scott, "Dynamics of flexible beams undergoing overall motions," *Journal of Sound and Vibration*, vol. 181, no. 2, pp. 261–278, 1995.

[8] J. Ryu, S. S. Kim, and S.-S. Kim, "A criterion on inclusion of stress stiffening effects in flexible multibody dynamic system simulation," *Computers and Structures*, vol. 62, no. 6, pp. 1035–1048, 1997.

[9] H. El-Absy and A. A. Shabana, "Geometric stiffness and stability of rigid body modes," *Journal of Sound and Vibration*, vol. 207, no. 4, pp. 465–496, 1997.

[10] B. O. Al-Bedoor and M. N. Hamdan, "Geometrically non-linear dynamic model of a rotating flexible arm," *Journal of Sound and Vibration*, vol. 240, no. 1, pp. 59–72, 2001.

[11] J. Y. Liu and J. Z. Hong, "Geometric stiffening effect on rigid-flexible coupling dynamics of an elastic beam," *Journal of Sound and Vibration*, vol. 278, no. 4-5, pp. 1147–1162, 2004.

[12] Z.-Y. Liu, J.-Z. Hong, and J.-Y. Liu, "Complete geometric nonlinear formulation for rigid-flexible coupling dynamics," *Journal of Central South University of Technology*, vol. 16, no. 1, pp. 119–124, 2009.

[13] R. Schwertassek, O. Wallrapp, and A. A. Shabana, "Flexible multibody simulation and choice of shape functions," *Nonlinear Dynamics*, vol. 20, no. 4, pp. 361–380, 1999.

[14] W. J. Book, "Recursive Lagrangian dynamics of flexible manipulator arms," *International Journal of Robotics Research*, vol. 3, no. 3, pp. 87–101, 1984.

[15] K. Changizi and A. A. Shabana, "Recursive formulation for the dynamic analysis of open loop deformable multibody systems," *Journal of Applied Mechanics, Transactions ASME*, vol. 55, no. 3, pp. 687–693, 1988.

[16] S.-S. Kim and E. J. Haug, "A recursive formulation for flexible multibody dynamics—part I: open-loop systems," *Computer*

Methods in Applied Mechanics and Engineering, vol. 71, no. 3, pp. 293–314, 1988.

[17] A. Jain and G. Rodriguez, "Recursive flexible multibody system dynamics using spatial operators," *Journal of Guidance, Control, and Dynamics*, vol. 15, no. 6, pp. 1453–1466, 1992.

[18] Y.-L. Hwang, "Recursive Newton-Euler formulation for flexible dynamic manufacturing analysis of open-loop robotic systems," *International Journal of Advanced Manufacturing Technology*, vol. 29, no. 5, pp. 598–604, 2006.

[19] D.-G. Zhang and S.-F. Zhou, "Dynamic analysis of flexible-link and flexible-joint robots," *Applied Mathematics and Mechanics*, vol. 27, no. 5, pp. 695–704, 2006.

[20] D.-G. Zhang, "Recursive Lagrangian dynamic modeling and simulation of multi-link spatial flexible manipulator arms," *Applied Mathematics and Mechanics*, vol. 30, no. 10, pp. 1283–1294, 2009.

[21] M. W. Spong, "Modeling and control of elastic joint robots," *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME*, vol. 109, no. 4, pp. 310–319, 1987.

[22] T. M. Wasfy and A. K. Noor, "Computational strategies for flexible multibody systems," *Applied Mechanics Reviews*, vol. 56, no. 6, pp. 553–613, 2003.

[23] S. K. Dwivedy and P. Eberhard, "Dynamic analysis of flexible manipulators, a literature review," *Mechanism and Machine Theory*, vol. 41, no. 7, pp. 749–777, 2006.

[24] K.-S. Na and J.-H. Kim, "Deployment of a multi-link flexible structure," *Journal of Sound and Vibration*, vol. 294, no. 1-2, pp. 298–313, 2006.

[25] K. S. Fu, R. C. Gonzalez, and C. S. G. Lee, *Robotics: Control, Sensing, Vision, and Intelligence*, McGraw-Hill, New York, NY, USA, 1987.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 139498, 8 pages
http://dx.doi.org/10.1155/2013/139498

Research Article

Trivariate Isogeometric Analysis for Flexible Multibody Dynamics

Ting Pi

Wuhan Second Ship Design & Research Institute, Wuhan 430064, China

Correspondence should be addressed to Ting Pi; piting007@gmail.com

Received 25 July 2013; Accepted 5 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Ting Pi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Isogeometric analysis (IGA) has been a fundamental step forward in the computational mechanics for the past few years, which maintains the accuracy of the description of computational domain geometry throughout the analysis process. However, the research on IGA in the area of flexible multibody dynamics is little and mainly concentrates on the univariate or bivariate NURBS geometry. This paper applies the trivariate IGA to the flexible multibody dynamics and proposes a continuum mechanics-based method to construct the system dynamic equations within the framework of IGA. A significant feature of this method is that it only employs the position coordinates of the control points as the system variables. To solve the large rotation and deformation coupled problems without introducing any rotation angles, the Green-Lagrange strain tensor is adopted. The evaluation of the elastic force and its Jacobian is easy and accurate by exploiting the appropriate mathematical transformation. In addition, the mass matrix and the generalized gravity force remain constant, and the centrifugal and Coriolis inertia forces equal zero. A numerical experiment is conducted using a thin-plate pendulum, which proves the feasibility and effectiveness of this method.

1. Introduction

The kinematic description of flexible bodies that undergo large rotation and deformation is a research hotspot in the area of flexible multibody dynamics. Various methods have been proposed over the years [1], among which the floating frame of reference (FFRF) is the most widely used way. FFRF uses two sets of coordinates to describe the configuration of the deformable bodies; one set describes the position and orientation of a body-fixed coordinate system, and the other describes the deformation of the body with respect to its body-fixed coordinate system. As a consequence, the stiffness matrix used to obtain elastic forces remains the same. However, the mass matrix, centrifugal and Coriolis inertia forces, and generalized gravity forces are highly nonlinear. Moreover, the small deformation assumption limits the use of FFRF in large deformation problems.

In order to overcome the limitation of FFRF, the incremental finite-element approach introduces infinitesimal rotation angles as nodal variables. However, this approach cannot exactly describe the rigid-body motion, which is a fundamental requirement in flexible multibody dynamics.

To solve the drawback of the incremental finite-element approach, the large rotation vector formulation replaces the infinitesimal rotation angles with finite ones. However, this method has not been widely accepted due to the redundancy of representing the large rotation of the cross-section which may lead to singularity problems and unrealistic shear forces.

Absolute nodal coordinate formulation (ANCF) introduces absolute displacements and global slopes as nodal coordinates, which prevents the system equations from being highly nonlinear because the mass matrix and generalized force remain constant and centrifugal and Coriolis inertia forces equal zero. ANCF can exactly describe the rigid body modes and solve the large rotation and deformation problems, while FFRF is just effective in small deformation situations.

Different from the above methods, isogeometric analysis (IGA) [2, 3], which has been considered as a fundamental step forward in computational mechanics, aims to maintain

the same exact description of the computational domain geometry throughout the analysis process. The basic idea is the use of the same NURBS-based mathematical model, which is the predominant technology used by CAD, for both modeling and FEA analysis. This makes it possible to exactly represent certain geometries that can only be approximated by polynomial functions, such as conic and circular sections. Moreover, it uses the shape functions with higher regularity than the traditional FEM. IGA exhibits many advantages and has been successfully applied in many problems, for example, contact problems [4], optimization problems [5], shell and plate problems [6–10], fluid-structure interaction problems [11], and so on. In flexible multibody dynamics, [12, 13] compare IGA and ANCF. Reference [14] analyzes the Kirchhoff-Love shells within the framework of IGA. However, most of the prior research uses the univariate or bivariate NURBS geometries to describe the deformable bodies. This paper applies the trivariate IGA to flexible multibody dynamics and proposes a continuum mechanics-based systematic method to solve the large rotation and deformation coupled problems.

The rest of the paper is organized as follows. Section 2 makes a short review of the geometric modeling and computational domain discretization techniques used in IGA. Section 3 proposes a systematic method to construct the dynamic models within the framework of IGA. Section 4 describes the computational strategies used to solve the system equations. To verify the feasibility and effect of this method, numerical tests are performed and the results are presented in Section 5. A conclusion is made at last in Section 6.

2. Geometric Modeling and Space Discretization

IGA forms the computational domain in a way much the same as that of the traditional FEMs. In these methods, the position of any point in an element is evaluated by interpolation of element nodes, and the interpolation weights are determined by the element shape functions. IGA employs NURBS as the geometric description method which is based on the knot vectors, and control points. Meshing and element shape functions are formed by knot vectors and the control points play the same role as that of the element nodes.

A knot vector, $\Omega = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, is constituted by nondecreasing real numbers, where p is the order of the B-Spline and n is the number of basis functions (and control points) necessary to describe it. A knot vector is said to be uniform if its knots are uniformly spaced and nonuniform otherwise. Moreover, a knot vector is said to be open if its first and last knots are repeated $p+1$ times. In this paper, the open knot vectors are adopted. In addition, ξ_1 and ξ_{n+p+1} are set to 0 and 1, respectively.

Given a knot vector, Ω, B-Spline basis functions are defined recursively starting with $p = 0$ (piecewise constants) as follows:

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi \in [\xi_i, \xi_{i+1}), \\ 0, & \xi \notin [\xi_i, \xi_{i+1}). \end{cases} \tag{1}$$

For $p > 1$,

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi). \tag{2}$$

If the internal knots are not repeated, B-Spline basis functions are C^{p-1}-continuous. If a knot has multiplicity k, the basis is C^{p-k}-continuous at that knot. In particular, when a knot has multiplicity p, the basis is C^0-continuous and interpolatory at that location.

With the knot vectors and the control points, NURBS can form curves, surfaces, and solids exactly. In this paper, we concentrate on the trivariate NURBS solid, on which any point $\mathbf{C}$ is expressed as follows:

$$\mathbf{C}(\xi_X, \xi_Y, \xi_Z) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} S_{i,j,k}(\xi_X, \xi_Y, \xi_Z)\, \mathbf{e}_{i,j,k}, \tag{3}$$

where ξ_X, ξ_Y, and ξ_Z are the three parameters located in the knot vectors Ω_X, Ω_Y, and Ω_Z, respectively; $\mathbf{e}_{i,j,k}$ are the control points; $S_{i,j,k}(\xi_X, \xi_Y, \xi_Z)$ are the NURBS basis functions as follows:

$$S_{i,j,k}(\xi_X, \xi_Y, \xi_Z) = \frac{N_{i,u}(\xi_X) N_{j,v}(\xi_Y) N_{k,w}(\xi_Z) \omega_{i,j,k}}{\sum_{\hat{i}=1}^{n} \sum_{\hat{j}=1}^{m} \sum_{\hat{k}=1}^{l} N_{\hat{i},u}(\xi_X) N_{\hat{j},v}(\xi_Y) N_{\hat{k},w}(\xi_Z) \omega_{\hat{i},\hat{j},\hat{k}}}, \tag{4}$$

where $\omega_{i,j,k}$ are weights; $\{N_{i,u}(\xi_X)\}$, $N_{j,v}(\xi_Y)$, and $N_{k,w}(\xi_Z)$ are the B-Spline basis functions defined on the knot vectors Ω_X, Ω_Y, and Ω_Z, respectively. It can be easily deduced that the number of NURBS basis functions is $N_e = n \times m \times l$.

The most distinguished feature of IGA is the discretization of the computational domain. To directly perform the meshing on the same exact NURBS geometry, knot vectors $\widehat{\Omega}_X$, $\widehat{\Omega}_Y$, and $\widehat{\Omega}_Z$ are generated by deleting the repeated knot values in knot vectors Ω_X, Ω_Y, and Ω_Z, separately. Taking $\widehat{\Omega}_X$ as an example, every knot span $[\widehat{\xi}_i, \widehat{\xi}_{i+1})$ is regarded as an element in the parametric space. Then, the product of knot vectors $\widehat{\Omega}_X \times \widehat{\Omega}_Y \times \widehat{\Omega}_Z$ forms a mesh of the three-dimensional parametric space. Finally, the corresponding mesh of the NURBS solids is formed by projecting these parametric elements into the physical space. Because of this "inherent" meshing scheme, IGA can not only perform FEM typical refinement strategy by means of knot insertion and degree elevation, but it can also offer an effective high-regularity refinement strategy [2].

3. Dynamic Equations

The large rotation and deformation problems of the flexible bodies belong to the highly nonlinear problems. To extend IGA to these problems, a continuum mechanics-based method is proposed to describe the elastic forces.

Suppose that the position vector $\mathbf{r}$ of an arbitrary point on a flexible body can be expressed in the global coordinate system as

$$\mathbf{r} = \mathbf{S}\mathbf{e}, \tag{5}$$

where $\mathbf{S}$ is the shape function given as follows:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{3D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{3D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_{3D} \end{bmatrix}^T, \tag{6}$$

where $\mathbf{S}_{3D} = [S_1, S_2, \ldots, S_{N_e}]^T$, an arrangement of all of the NURBS bases in (4). $\mathbf{e}$ is the corresponding arrangement of all of the control points $\mathbf{e}_{i,j,k}$:

$$\mathbf{e} = \begin{bmatrix} \mathbf{x}_\mathbf{e}^T & \mathbf{y}_\mathbf{e}^T & \mathbf{z}_\mathbf{e}^T \end{bmatrix}^T, \tag{7}$$

where $\mathbf{x}_\mathbf{e}$, $\mathbf{y}_\mathbf{e}$, and $\mathbf{z}_\mathbf{e}$ are the coordinates of the control points in three directions.

Given the initial configuration $\mathbf{r}_o$ and current configuration $\mathbf{r}$, the position vector gradients $\mathbf{J}$ are defined as

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_o} = \frac{\partial \mathbf{r}}{\partial \boldsymbol{\xi}} \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{r}_o} = \frac{\partial \mathbf{r}}{\partial \boldsymbol{\xi}} \left(\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}} \right)^{-1}. \tag{8}$$

To solve the large rotation and deformation problems within the framework of IGA, the Green-Lagrange strain tensor and the corresponding second Piola-Kirchhoff stress tensor are employed to guarantee that the work of elastic forces and strain energy are not affected when the continuum undergoes pure rigid-body motion.

The Green-Lagrange strain tensor is defined as

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left(\mathbf{J}^T \mathbf{J} - \mathbf{I}_{3\times 3} \right). \tag{9}$$

With this strain tensor, the weak form of dynamic equilibrium equations of the continuous body in the reference configuration can be written as

$$\begin{aligned} &\int_V \rho \mathbf{a} \cdot \delta \mathbf{r} dV + \int_V |\mathbf{J}| \, \mathbf{J}^{-1} \boldsymbol{\sigma} \mathbf{J}^{-T} : \delta \boldsymbol{\varepsilon} dV \\ &\quad - \int_A \frac{|\mathbf{J}|}{\sqrt{\mathbf{n}^T \mathbf{J} \mathbf{J}^T \mathbf{n}}} \boldsymbol{\sigma}_n \cdot \delta \mathbf{r} dA - \int_V \mathbf{f}_b \cdot \delta \mathbf{r} dV = 0. \end{aligned} \tag{10}$$

The first integral in (10) represents the virtual work of inertia forces, where ρ is the density of material, $\mathbf{a}$ is the acceleration vector, dV is the volume element, and the symbol $\cdot$ means the scalar product of two vectors. The second integral is the virtual work of the internal elastic forces, where $|\mathbf{J}|$ is the determinant of position vector gradients, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and the symbol : means the scalar product of two second-order tensors. The third integral is the virtual work of surface forces, where dA is the area element, $\mathbf{n}$ is a unit vector directed along the outward normal to the surface, and $\boldsymbol{\sigma}_n$ is the mean surface traction. The fourth integral is the virtual work of the body forces, where $\mathbf{f}_b$ is the vector of body forces.

3.1. Inertia Forces. The virtual work of inertia forces can be written as

$$\begin{aligned} \int_V \rho \mathbf{a} \cdot \delta \mathbf{r} dV &= \int_V \rho \mathbf{a}^T \delta \mathbf{r} dV = \ddot{\mathbf{e}}^T \int_V \rho \mathbf{S}^T \mathbf{S} dV \delta \mathbf{e} \\ &= \left(\int_V \rho \mathbf{S}^T \mathbf{S} dV \ddot{\mathbf{e}} \right)^T \delta \mathbf{e}. \end{aligned} \tag{11}$$

It can be inferred from (11) that the vector of inertia forces is

$$\mathbf{Q}_i = \int_V \rho \mathbf{S}^T \mathbf{S} dV \ddot{\mathbf{e}}, \tag{12}$$

and the mass matrix is

$$\mathbf{M} = \int_V \rho \mathbf{S}^T \mathbf{S} dV. \tag{13}$$

It can be deduced that the mass matrix $\mathbf{M}$ is a constant symmetric matrix, which leads to zero centrifugal and Coriolis forces when the body experiences an arbitrary deformation and rotation.

3.2. Elastic Forces. As shown in (10), the Green-Lagrange strain tensor is used to measure the degree of deformation in this paper. In order to deduce the expression of the elastic forces, the detailed formula of the Green-Lagrange strain tensor is needed. Suppose $\mathbf{S}_{i,\xi_1}$, $\mathbf{S}_{i,\xi_2}$, and $\mathbf{S}_{i,\xi_3}$, $i = 1, 2, 3$, are the partial derivatives of the ith row of shape functions $\mathbf{S}$ with respect to parameters ξ_1, ξ_2, and ξ_3, respectively, and

$$\left(\mathbf{S}_{,j} \right)_i \triangleq \left(\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}} \right)_j^{-T} \begin{bmatrix} \mathbf{S}_{i,\xi_1}^T & \mathbf{S}_{i,\xi_2}^T & \mathbf{S}_{i,\xi_3}^T \end{bmatrix}^T, \quad i, j = 1, 2, 3. \tag{14}$$

Substituting (14) into (8), the position vector gradients can be written as

$$\mathbf{J} = \begin{bmatrix} \mathbf{S}_{,1}\mathbf{e} & \mathbf{S}_{,2}\mathbf{e} & \mathbf{S}_{,3}\mathbf{e} \end{bmatrix}. \tag{15}$$

Substituting (15) into (9), the elements of the Green-Lagrange strain tensor can be written as

$$\begin{aligned} \boldsymbol{\varepsilon}_{11} &= \frac{1}{2} \left(\mathbf{e}^T \mathbf{S}_{,1}^T \mathbf{S}_{,1} \mathbf{e} - 1 \right), & \boldsymbol{\varepsilon}_{22} &= \frac{1}{2} \left(\mathbf{e}^T \mathbf{S}_{,2}^T \mathbf{S}_{,2} \mathbf{e} - 1 \right), \\ \boldsymbol{\varepsilon}_{33} &= \frac{1}{2} \left(\mathbf{e}^T \mathbf{S}_{,3}^T \mathbf{S}_{,3} \mathbf{e} - 1 \right), & \boldsymbol{\varepsilon}_{12} &= \boldsymbol{\varepsilon}_{21} = \frac{1}{2} \left(\mathbf{e}^T \mathbf{S}_{,1}^T \mathbf{S}_{,2} \mathbf{e} \right), \\ \boldsymbol{\varepsilon}_{13} &= \boldsymbol{\varepsilon}_{31} = \frac{1}{2} \left(\mathbf{e}^T \mathbf{S}_{,1}^T \mathbf{S}_{,3} \mathbf{e} \right), & \boldsymbol{\varepsilon}_{23} &= \boldsymbol{\varepsilon}_{32} = \frac{1}{2} \left(\mathbf{e}^T \mathbf{S}_{,2}^T \mathbf{S}_{,3} \mathbf{e} \right). \end{aligned} \tag{16}$$

For isotropic elastic materials, the virtual work of elastic forces, U_s, can be written as

$$\begin{aligned} U_s &= \int_V \frac{\lambda + 2G}{2}\left(\boldsymbol{\varepsilon}_{11}^2 + \boldsymbol{\varepsilon}_{22}^2 + \boldsymbol{\varepsilon}_{33}^2\right) \\ &\quad + \lambda\left(\boldsymbol{\varepsilon}_{11}\boldsymbol{\varepsilon}_{22} + \boldsymbol{\varepsilon}_{11}\boldsymbol{\varepsilon}_{33} + \boldsymbol{\varepsilon}_{22}\boldsymbol{\varepsilon}_{33}\right) \\ &\quad + 2G\left(\boldsymbol{\varepsilon}_{12}^2 + \boldsymbol{\varepsilon}_{13}^2 + \boldsymbol{\varepsilon}_{23}^2\right) dV \\ &= \int_\Omega \left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right| \left[\frac{\lambda + 2G}{2}\left(\boldsymbol{\varepsilon}_{11}^2 + \boldsymbol{\varepsilon}_{22}^2 + \boldsymbol{\varepsilon}_{33}^2\right)\right. \\ &\quad + \lambda\left(\boldsymbol{\varepsilon}_{11}\boldsymbol{\varepsilon}_{22} + \boldsymbol{\varepsilon}_{11}\boldsymbol{\varepsilon}_{33} + \boldsymbol{\varepsilon}_{22}\boldsymbol{\varepsilon}_{33}\right) \\ &\quad \left. + 2G\left(\boldsymbol{\varepsilon}_{12}^2 + \boldsymbol{\varepsilon}_{13}^2 + \boldsymbol{\varepsilon}_{23}^2\right)\right] d\Omega, \end{aligned} \tag{17}$$

where λ is Lame's constant and G is the shear modulus. The vector of elastic forces is defined as the derivative of U_s with respect to generalized coordinates $\mathbf{e}$:

$$\begin{aligned} \mathbf{Q}_s &= \frac{\partial U_s}{\partial \mathbf{e}} \\ &= \int_V \frac{\lambda + 2G}{2}\left(2\boldsymbol{\varepsilon}_{11}\frac{\partial \boldsymbol{\varepsilon}_{11}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{22}\frac{\partial \boldsymbol{\varepsilon}_{22}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{33}\frac{\partial \boldsymbol{\varepsilon}_{33}}{\partial \mathbf{e}}\right) \\ &\quad + \lambda\left(\boldsymbol{\varepsilon}_{11}\frac{\partial \boldsymbol{\varepsilon}_{22}}{\partial \mathbf{e}} + \frac{\partial \boldsymbol{\varepsilon}_{11}}{\partial \mathbf{e}}\boldsymbol{\varepsilon}_{22} + \boldsymbol{\varepsilon}_{11}\frac{\partial \boldsymbol{\varepsilon}_{33}}{\partial \mathbf{e}} + \frac{\partial \boldsymbol{\varepsilon}_{11}}{\partial \mathbf{e}}\boldsymbol{\varepsilon}_{33}\right. \\ &\quad \left. + \boldsymbol{\varepsilon}_{22}\frac{\partial \boldsymbol{\varepsilon}_{33}}{\partial \mathbf{e}} + \frac{\partial \boldsymbol{\varepsilon}_{22}}{\partial \mathbf{e}}\boldsymbol{\varepsilon}_{33}\right) \\ &\quad + 2G\left(2\boldsymbol{\varepsilon}_{12}\frac{\partial \boldsymbol{\varepsilon}_{12}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{13}\frac{\partial \boldsymbol{\varepsilon}_{13}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{23}\frac{\partial \boldsymbol{\varepsilon}_{23}}{\partial \mathbf{e}}\right) dV \\ &= \int_\Omega \left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right| \left[\frac{\lambda + 2G}{2}\left(2\boldsymbol{\varepsilon}_{11}\frac{\partial \boldsymbol{\varepsilon}_{11}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{22}\frac{\partial \boldsymbol{\varepsilon}_{22}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{33}\frac{\partial \boldsymbol{\varepsilon}_{33}}{\partial \mathbf{e}}\right)\right. \\ &\quad + \lambda\left(\boldsymbol{\varepsilon}_{11}\frac{\partial \boldsymbol{\varepsilon}_{22}}{\partial \mathbf{e}} + \frac{\partial \boldsymbol{\varepsilon}_{11}}{\partial \mathbf{e}}\boldsymbol{\varepsilon}_{22} + \boldsymbol{\varepsilon}_{11}\frac{\partial \boldsymbol{\varepsilon}_{33}}{\partial \mathbf{e}}\right. \\ &\quad \left. + \frac{\partial \boldsymbol{\varepsilon}_{11}}{\partial \mathbf{e}}\boldsymbol{\varepsilon}_{33} + \boldsymbol{\varepsilon}_{22}\frac{\partial \boldsymbol{\varepsilon}_{33}}{\partial \mathbf{e}} + \frac{\partial \boldsymbol{\varepsilon}_{22}}{\partial \mathbf{e}}\boldsymbol{\varepsilon}_{33}\right) \\ &\quad + 2G\left(2\boldsymbol{\varepsilon}_{12}\frac{\partial \boldsymbol{\varepsilon}_{12}}{\partial \mathbf{e}} + 2\boldsymbol{\varepsilon}_{13}\frac{\partial \boldsymbol{\varepsilon}_{13}}{\partial \mathbf{e}}\right. \\ &\quad \left.\left. + 2\boldsymbol{\varepsilon}_{23}\frac{\partial \boldsymbol{\varepsilon}_{23}}{\partial \mathbf{e}}\right)\right] d\Omega, \end{aligned} \tag{18}$$

where

$$\frac{\partial \varepsilon_{ij}}{\partial \mathbf{e}} = \frac{1}{2}\left(\mathbf{S}_{,i}^T\mathbf{S}_{,j} + \mathbf{S}_{,j}^T\mathbf{S}_{,i}\right)\mathbf{e}, \quad i, j = 1, 2, 3, \quad j \geq i. \tag{19}$$

It can be seen from (14), (16), (18), and (19) that $\mathbf{S}_{i,\xi_j}$ is the key to evaluate the elastic forces. $\mathbf{S}_{i,\xi_j}$ is the collection of all of the derivatives of NURBS bases with respect to their parameters, which has been fully studied in computer graphics [15]. In the numerical integration methods, the derivatives of elastic forces with respect to generalized coordinates are necessary for the calculation of the Jacobian. Reference [16] gives an elaborately deduced form of the Jacobian:

$$\begin{aligned} \frac{\partial \mathbf{Q}_s}{\partial \mathbf{e}} &= \sum_{\alpha=1}^{3} \frac{\lambda + 2G}{2}\int_V \left(\left(\mathbf{e}^T\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\mathbf{e}\right)\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right. \\ &\qquad \left. + 2\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\mathbf{e}\mathbf{e}^T\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha} - \mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right) dV \\ &\quad + \sum_{\alpha=1}^{3}\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{3} \frac{\lambda}{2}\int_V \left(\left(\mathbf{e}^T\mathbf{S}_{,\beta}^T\mathbf{S}_{,\beta}\mathbf{e}\right)\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right. \\ &\qquad \left. + 2\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\mathbf{e}\mathbf{e}^T\mathbf{S}_{,\beta}^T\mathbf{S}_{,\beta} - \mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right) dV \\ &\quad + \sum_{\alpha=1}^{3}\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{3} G\int_V \left(\left(\mathbf{e}^T\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta}\mathbf{e}\right)\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta}\right. \\ &\qquad \left. + \mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta}\mathbf{e}\mathbf{e}^T\left(\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta} + \mathbf{S}_{,\beta}^T\mathbf{S}_{,\alpha}\right)\right) dV \\ &= \sum_{\alpha=1}^{3} \frac{\lambda + 2G}{2}\left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right|\int_\Omega \left(\left(\mathbf{e}^T\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\mathbf{e}\right)\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right. \\ &\qquad \left. + 2\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\mathbf{e}\mathbf{e}^T\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha} - \mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right) d\Omega \\ &\quad + \sum_{\alpha=1}^{3}\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{3} \frac{\lambda}{2}\left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right|\int_\Omega \left(\left(\mathbf{e}^T\mathbf{S}_{,\beta}^T\mathbf{S}_{,\beta}\mathbf{e}\right)\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right. \\ &\qquad \left. + 2\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\mathbf{e}\mathbf{e}^T\mathbf{S}_{,\beta}^T\mathbf{S}_{,\beta} - \mathbf{S}_{,\alpha}^T\mathbf{S}_{,\alpha}\right) d\Omega \\ &\quad + \sum_{\alpha=1}^{3}\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{3} G\left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right|\int_\Omega \left(\left(\mathbf{e}^T\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta}\mathbf{e}\right)\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta}\right. \\ &\qquad \left. + \mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta}\mathbf{e}\mathbf{e}^T\left(\mathbf{S}_{,\alpha}^T\mathbf{S}_{,\beta} + \mathbf{S}_{,\beta}^T\mathbf{S}_{,\alpha}\right)\right) d\Omega. \end{aligned} \tag{20}$$

With (18) and (20), the elastic forces and the Jacobian can be effectively evaluated. In engineering problems, mechanical properties of isotropic elastic materials are usually given in terms of Young's modulus, E_Y, and Poisson's ratio, μ. The relationships between these parameters and Lame's constants are the following:

$$\begin{aligned} \lambda &= \frac{\mu E_Y}{(1+\mu)(1-2\mu)}, \\ G &= \frac{E_Y}{2(1+\mu)}. \end{aligned} \tag{21}$$

3.3. Body Forces. Taking gravity forces as an example, the virtual work of body forces can be written as

$$\begin{aligned} \int_V \mathbf{f}_b \cdot \delta\mathbf{r} dV &= \int_V \begin{bmatrix} 0 & 0 & -\rho g \end{bmatrix} \mathbf{S} dV \delta\mathbf{e} \\ &= \begin{bmatrix} 0 & 0 & -\rho g \end{bmatrix} \left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right| \int_\Omega \mathbf{S} d\Omega \delta\mathbf{e}. \end{aligned} \tag{22}$$

(1) $\mathbf{q}_{n+1} = \mathbf{q}_n + h\dot{\mathbf{q}}_n + h^2(0.5-\beta)\mathbf{a}$;
(2) $\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h(1-\gamma)\mathbf{a}$;
(3) $\boldsymbol{\lambda}_{n+1} = \mathbf{0}$;
(4) $\mathbf{a} = 1/(1-\alpha_m)(\alpha_f\ddot{\mathbf{q}}_n - \alpha_m\mathbf{a})$;
(5) $\mathbf{q}_{n+1} = \mathbf{q}_{n+1} + h^2\beta\mathbf{a}$;
(6) $\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_{n+1} + h\gamma\mathbf{a}$;
(7) $\ddot{\mathbf{q}}_{n+1} = \mathbf{0}$;
(8) **for** $i = 1 : i_{\max}$ **do**
(9) compute system residual error $\boldsymbol{\Upsilon}$;
(10) **if** $\|\boldsymbol{\Upsilon}\| \le \epsilon$ **then**
(11) break;
(12) **end**
(13) $\begin{bmatrix}\Delta\ddot{\mathbf{q}}\\ \Delta\boldsymbol{\lambda}\end{bmatrix} = \mathbf{J}^{-1}_{\text{Gen-}\alpha}\boldsymbol{\Upsilon}$;
(14) $\begin{bmatrix}\ddot{\mathbf{q}}_{n+1}\\ \boldsymbol{\lambda}_{n+1}\end{bmatrix} = \begin{bmatrix}\ddot{\mathbf{q}}_{n+1}\\ \boldsymbol{\lambda}_{n+1}\end{bmatrix} - \begin{bmatrix}\Delta\ddot{\mathbf{q}}\\ \Delta\boldsymbol{\lambda}\end{bmatrix}$;
(15) $\mathbf{q}_{n+1} = \mathbf{q}_{n+1} - (1-\alpha_f)/(1-\alpha_m)\beta h^2\Delta\ddot{\mathbf{q}}$;
(16) $\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_{n+1} - (1-\alpha_f)/(1-\alpha_m)\gamma h\Delta\ddot{\mathbf{q}}$;
(17) **end**
(18) $\mathbf{a} = \mathbf{a} + (1-\alpha_f)/(1-\alpha_m)\ddot{\mathbf{q}}_{n+1}$;

Algorithm 1: Generalized-α scheme.

It can be inferred that the vector of gravity forces is

$$\mathbf{Q}_g = \left|\frac{\partial \mathbf{r}_o}{\partial \boldsymbol{\xi}}\right| \int_\Omega \mathbf{S}^T d\Omega \begin{bmatrix} 0 \\ 0 \\ -\rho g \end{bmatrix}. \tag{23}$$

4. Computational Strategies

Multibody systems usually contain various forms of constraints. Different kinds of system equations for the constrained multibody systems have been proposed over the years [17, 18]. This paper adopts the frequently used DAE system of index-3 with holonomic constraints which is usually written as

$$\begin{gathered}\mathbf{M}(\mathbf{e})\ddot{\mathbf{e}} + \boldsymbol{\Phi}_{\mathbf{e}}^T(\mathbf{e},t)\boldsymbol{\lambda} - \mathbf{Q}(t,\mathbf{e},\dot{\mathbf{e}}) = \mathbf{0},\\ \boldsymbol{\Phi}(\mathbf{e},t) = \mathbf{0},\end{gathered} \tag{24}$$

where $\mathbf{M}$ is the system mass matrix, $\mathbf{e}$, $\dot{\mathbf{e}}$, and $\ddot{\mathbf{e}}$ are the position coordinates, velocities, and accelerations of control points in the global coordinate system, t is the time variable, $\boldsymbol{\Phi}$ are the system constraint equations, $\boldsymbol{\Phi}_{\mathbf{e}}$ is the Jacobian of constraints $\boldsymbol{\Phi}$, $\boldsymbol{\lambda}$ is the vector of the Lagrange multipliers, and $\mathbf{Q}$ is the vector of generalized forces, including body forces and elastic forces.

In general, DAE-solvers can be classified into two categories [17, 18]: algebraic elimination and constraint violation stabilization. The latter is paid continuously increasing attention due to the simplicity and efficiency in implementation, among which the commonly used methods include Newmark [19], BDF [20], HHT [21], generalized-α [22, 23]. The Newmark method has only order-1 convergence of errors, while the others possess order-2 convergence. In addition, the generalized-α method has adjustable control of numerical dissipation; which leads to an optimal combination of high-frequency and low-frequency dissipation, that is, for a desired level of high-frequency dissipation, the low-frequency dissipation is minimized. Based on these considerations, the generalized-α method is employed to

Figure 1: The configuration of the thin plate at different moments (36 elements).

Figure 2: Relationship between the element and the corresponding control points.

(a) 0.25 s (b) 0.75 s

(c) 1.25 s (d) 2 s

Figure 3: The configuration of the plate and related control points at different moments (1 element).

solve the system equation (24). With this method, system equations of time step $n+1$ are discretized as

$$\begin{gathered}\mathbf{M}\left(\mathbf{e}_{n+1}\right)\ddot{\mathbf{e}}_{n+1}+\left(\mathbf{\Phi}_{\mathbf{e}}^{T}\boldsymbol{\lambda}\right)_{n+1}-\mathbf{Q}_{n+1}=\mathbf{0},\\ \frac{1-\alpha_m}{\left(1-\alpha_f\right)\beta h^2}\mathbf{\Phi}\left(\mathbf{e}_{n+1},t_{n+1}\right)=\mathbf{0},\end{gathered}\tag{25}$$

where h is the time step, $\alpha_m=(2\rho_\infty-1)/(\rho_\infty+1)$, $\alpha_f=(\rho_\infty)/(\rho_\infty+1)$, $\rho_\infty\in[0,1]$, and $\beta=(1+\alpha_f-\alpha_m)^2/4$. ρ_∞ is a parameter controlling numerical dissipation ($\rho_\infty=0$ for maximal dissipation). The position and velocity variables of time step $n+1$ are discretized as

$$\begin{gathered}\mathbf{e}_{n+1}=\mathbf{e}_n+h\dot{\mathbf{e}}_n+\frac{h^2}{2}\left[\left(1-2\beta\right)\mathbf{q}_n+2\beta\mathbf{q}_{n+1}\right],\\ \dot{\mathbf{e}}_{n+1}=\dot{\mathbf{e}}_n+h\left[\left(1-\gamma\right)\mathbf{q}_n+\gamma\mathbf{q}_{n+1}\right],\end{gathered}\tag{26}$$

where $\gamma=1/2+\alpha_f-\alpha_m$; $\mathbf{q}_n$ is constituted by the acceleration-like auxiliary variables, which require that

$$\begin{gathered}\left(1-\alpha_m\right)\mathbf{q}_{n+1}+\alpha_m\mathbf{q}_n=\left(1-\alpha_f\right)\ddot{\mathbf{e}}_{n+1}+\alpha_f\ddot{\mathbf{e}}_n,\\ \mathbf{q}_0=\ddot{\mathbf{e}}_0.\end{gathered}\tag{27}$$

Algorithm 1 sums the process of generalized-α scheme to solve dynamic equilibrium at time t_{n+1}.

The Jacobian, $\mathbf{J}_{\text{Gen-}\alpha}$ of system equations (25) in Algorithm 1 is

$$\mathbf{J}_{\text{Gen-}\alpha}=\begin{bmatrix}\mathbf{M}+\beta h^2\dfrac{1-\alpha_f}{1-\alpha_m}\left(\mathbf{M}\left(\mathbf{e}\right)\ddot{\mathbf{e}}+\mathbf{\Phi}_{\mathbf{e}}^{T}\boldsymbol{\lambda}-\mathbf{Q}\right)_{\mathbf{e}}-\gamma h\dfrac{1-\alpha_f}{1-\alpha_m}\mathbf{Q}_{\dot{\mathbf{e}}} & \mathbf{\Phi}_{\mathbf{e}}^{T}\\ \mathbf{\Phi}_{\mathbf{e}} & \mathbf{0}\end{bmatrix}.\tag{28}$$

5. Numerical Tests

In this section, a numerical experiment with a thin-plate pendulum is performed to verify the proposed continuum mechanics-based trivariate isogeometric analysis for the large rotation and deformation coupled problems.

As shown in Figure 1, the plate is fixed in one corner without any deformation at the initial time. Then, it falls down freely under the action of gravity. Both the length and width of the plate are set to 3 m, and its thickness is chosen to be 0.1 m. The density, Young's modulus, and Poisson ratio of the material are assumed to be 5000 kg/cm^3, 1.0×10^7 N/m^2, and 0.3, respectively.

Figure 4: Comparison of plate deformation of different meshing plans (1 element: yellow; 36 elements: cyan; 100 elements: red).

The employed plate element adopts NURBS bases of order 3 in the direction of both length and width. In order not to make a distraction, these NURBS bases are assumed to be C^2-continuous at all of the internal knot values, which are uniformly distributed between 0 and 1. The polynomial order in the thickness direction is chosen to be 1. Therefore, the number of control points in a single element is $4 \times 4 \times 2 = 32$.

To better illustrate the NURBS element, Figure 2 presents the relationship between the element and its related control points when the whole plate is regarded as a single element, that is, $\Omega_x = \Omega_y = \{0, 0, 0, 0, 1, 1, 1, 1\}$, and $\Omega_z = \{0, 0, 1, 1\}$. As shown in the figure, control points are marked with circles and connected with dashed line to form the control net. It should be noted that this is a special case that NURBS basis functions are interpolatory at all of the corners of the element. In most cases, control points do not need to lie on the geometry.

To see the influence of refinement on the plate's deformation, other 36-elements ($\Omega_x = \Omega_y = \{0, 0, 0, 0, 1/6, 1/3, 1/2, 2/3, 5/6, 1, 1, 1, 1\}$, $\Omega_z = \{0, 0, 1, 1\}$) and 100-element ($\Omega_x = \Omega_y = \{0, 0, 0, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1\}$, $\Omega_z = \{0, 0, 1, 1\}$) meshing plans are performed separately. A 2 s simulation with 2^{-12} s step size is performed for each of the above three meshing plans. Figure 1 shows the configurations of the moving thin plate at different moments for the 36-element meshing plan. Figure 3 shows the position relationship between the control points and the thin plate configuration for the single-element meshing plan at different moments. Figure 4 compares the plate deformation of single-element, 36-element and 100-element meshing plans at different moments. It is obvious that the plate configuration changes significantly when the number of elements increases from 1 to 36, while it differs not that much from 36 to 100. But the ratio of CPU time used in the three cases is 1 : 22 : 66. Therefore, it can be concluded that, using this method, a balance between the accuracy and efficiency can be achieved with a relatively coarse meshing.

6. Conclusion

This paper applies the trivariate IGA to the flexible multibody system dynamics and proposes a continuum mechanics-based method to construct the system dynamic equations within the framework of IGA. A significant feature of this method is that it only employs the position coordinates of the control points as the system variables. To solve the large rotation and deformation coupled problems without introducing any rotation angles, the Green-Lagrange strain tensor is employed. Elastic force and its Jacobian are easily and accurately evaluated by using appropriate mathematical transformations. In addition, the mass matrix and generalized gravity force remain constant, and the centrifugal and Coriolis inertia forces equal zero. The feasibility and effectiveness of this method are verified through a thin-plate pendulum experiment. In addition, the trivariate IGA has a good symmetric form, which facilitates the computer implementation. The element types and the comparison with other methods will be studied in the future.

References

[1] A. A. Shabana, "Flexible multibody dynamics: review of past and recent developments," *Multibody System Dynamics*, vol. 1, no. 2, pp. 189–222, 1997.

[2] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs, "Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement," *Computer Methods in Applied Mechanics and Engineering*, vol. 194, no. 39–41, pp. 4135–4195, 2005.

[3] J. A. Cottrell, T. J. Hughes, and Y. Bazilevs, *Isogeometric Analysis: toward Integration of CAD and FEA*, John Wiley & Sons, New York, NY, USA, 2009.

[4] T. Temizer, P. Wriggers, and T. J. R. Hughes, "Contact treatment in isogeometric analysis with NURBS," *Computer Methods in Applied Mechanics and Engineering*, vol. 200, no. 9–12, pp. 1100–1112, 2011.

[5] W. A. Wall, M. A. Frenzel, and C. Cyron, "Isogeometric structural shape optimization," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 33–40, pp. 2976–2988, 2008.

[6] D. J. Benson, Y. Bazilevs, M.-C. Hsu, and T. J. Hughes, "A large deformation, rotation-free, isogeometric shell," *Computer Methods in Applied Mechanics and Engineering*, vol. 200, no. 13–16, pp. 1367–1378, 2011.

[7] D. J. Benson, Y. Bazilevs, M. C. Hsu, and T. J. R. Hughes, "Isogeometric shell analysis: the Reissner-Mindlin shell," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 5–8, pp. 276–289, 2010.

[8] J. Kiendl, K.-U. Bletzinger, J. Linhard, and R. Wüchner, "Isogeometric shell analysis with Kirchhoff-Love elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 198, no. 49–52, pp. 3902–3914, 2009.

[9] L. Beirão da Veiga, A. Buffa, C. Lovadina, M. Martinelli, and G. Sangalli, "An isogeometric method for the Reissner-Mindlin plate bending problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 209-212, pp. 45–53, 2012.

[10] R. Echter, B. Oesterle, and M. Bischoff, "A hierarchic family of isogeometric shell finite elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 254, pp. 170–180, 2013.

[11] Y. Bazilevs, V. M. Calo, T. J. R. Hughes, and Y. Zhang, "Isogeometric fluid-structure interaction: theory, algorithms, and computations," *Computational Mechanics*, vol. 43, no. 1, pp. 3–37, 2008.

[12] G. G. Sanborn and A. A. Shabana, "On the integration of computer aided design and analysis using the finite element absolute nodal coordinate formulation," *Multibody System Dynamics*, vol. 22, no. 2, pp. 181–197, 2009.

[13] P. Lan and A. A. Shabana, "Integration of B-spline geometry and ANCF finite element analysis," *Nonlinear Dynamics*, vol. 61, no. 1-2, pp. 193–206, 2010.

[14] A. Goyal, M. R. Dörfel, B. Simeon, and A. V. Vuong, "Isogeometric shell discretizations for flexible multibody dynamics," *Multibody System Dynamics*, vol. 30, no. 2, pp. 139–151, 2013.

[15] L. A. Piegl and W. Tiller, *The NURBS Book*, Springer, New York, NY, USA, 1997.

[16] D. García-Vallejo, J. Mayo, J. L. Escalona, and J. Domínguez, "Efficient evaluation of the elastic forces and the jacobian in the absolute nodal coordinate formulation," *Nonlinear Dynamics*, vol. 35, no. 4, pp. 313–329, 2004.

[17] A. Laulusa and O. A. Bauchau, "Review of classical approaches for constraint enforcement in multibody systems," *Journal of Computational and Nonlinear Dynamics*, vol. 3, no. 1, Article ID 011004, 2008.

[18] O. A. Bauchau and A. Laulusa, "Review of contemporary approaches for constraint enforcement in multibody systems," *Journal of Computational and Nonlinear Dynamics*, vol. 3, no. 1, Article ID 011 005, 2008.

[19] N. M. Newmark, "A method of computation for structural dynamics," *Journal of the Engineering Mechanics Division*, vol. 85, pp. 67–94, 1959.

[20] C. W. Gear, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice Hall; PTR, New York, NY, USA, 1971.

[21] H. M. Hilber, T. J. Hughes, and R. L. Taylor, "Improved numerical dissipation for time integration algorithms in structural dynamics," *Earthquake Engineering & Structural Dynamics*, vol. 5, no. 3, pp. 283–292, 1977.

[22] J. Chung and G. M. Hulbert, "Time integration algorithm for structural dynamics with improved numerical dissipation: the generalized- α method," *Journal of Applied Mechanics, Transactions ASME*, vol. 60, no. 2, pp. 371–375, 1993.

[23] O. L. Jay and D. Negrut, "A second order extension of the generalized-α method for constrained systems in mechanics," *Multibody Dynamics*, pp. 143–158, 2008.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 604393, 16 pages
http://dx.doi.org/10.1155/2013/604393

Research Article

Dynamic Control and Disturbance Estimation of 3D Path Following for the Observation Class Underwater Remotely Operated Vehicle

Hai Huang,[1] Qirong Tang,[2] Yueming Li,[1] Lei Wan,[1] and Yongjie Pang[1]

[1] *National Key Laboratory of Science and Technology on Underwater Vehicle, Harbin Engineering University, Harbin 150001, China*
[2] *University of Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany*

Correspondence should be addressed to Qirong Tang; qirong.tang@outlook.com

Received 25 August 2013; Accepted 23 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Hai Huang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This study addresses the question of 3D path following for the observation class underwater remotely operated vehicle. The dynamic model of the investigated remote operated vehicle is taken as a coupled multibody system composing of a flexible body and a rigid body. For precise control, the tether cable disturbance has been investigated as well via a dynamic model. Each element of the tethered cable even has been taken as an elastic body, and the waves and current disturbances have been taken into consideration. Based on the multibody system model, an adaptive backstepping sliding mode controller has been designed. To improve the controller's systematic robustness against disturbances, the sliding mode surface and adaptive control rule have been designed, too. Experiments have been performed in a tank, including the 3D path following controls of depth, heading, advance, sideway, polygon line, and spiral line. With current and wave disturbances having been taken into consideration, the tether effect has been analyzed, the efficacy and superiority of adaptive backstepping sliding mode control have been verified. It is further confirmed from the comparisons that the investigated method outperforms those S surface based controllers.

1. Introduction

Observation class underwater remotely operated vehicles (ROVs) have been used widely in many areas such as hull and harbor inspection, underwater salvage, and pollution surveillance. High accuracy positioning and path following are demanded for underwater inspections and manipulations. The 3D path following is extremely important for accomplishing missions executed by subsea remotely operated vehicles. Observation class ROV is not so big; thus its motion is easy to be affected by the tethered cable, and also by current, wave, and wind. Therefore, the robust 3D path-following control for observation used ROV has been recognized as one of the most challenging issues [1, 2].

In order to meet the requirements of path following (sometimes refer to trajectory tracking when it is under a temporal law) for underwater vehicles, various types of control approaches have been studied. Kokegei et al. proposed a sliding mode controller for underwater vehicle to realize coupled 6 DoF trajectory control [3]. Bian et al. presented a nonlinear feedback controller based on input state for path following [4]. But those control techniques assume that the unknown dynamic model can be represented by a linear one with unknown parameters and a regression matrix is specified for the ROVs [5]. Fuzzy linguistic rules are more flexible for dealing with nonlinear control problems. Thus, Liu et al. in [6] and Wang et al. in [7] both provided an S surface (plane) controller in combination with fuzzy control for underwater vehicles navigation. A trained neural-fuzzy system can approximate the linear or nonlinear mapping of disturbance, but the objectives for training are usually difficult to be obtained; see in [8, 9]. Marzbanrad et al. modified sliding mode with robust adaptive fuzzy control algorithm for ROVs path following control. In that case, fuzzy algorithm was used for online estimation of external disturbances as well as unknown nonlinear terms of the dynamic model of the ROV. A robust control rule is employed to compensate the estimation errors [2]. Zhang et al. also

derived an output feedback controller using nonlinear control theory [10]. They proved that a nonlinear output feedback integral controller provides exponential stability. In consideration of disturbance caused by wave-induced hydrodynamic forces, Patompak and Nilkhamhang provided an adaptive backstepping sliding mode controller for station keeping and path following of underwater vehicle based on its dynamic model [11]. Lapierre and Jouvencel extended a kinematic controller with backstepping and Lyapunov-based techniques so as to cope with parametric uncertainties and external disturbances [12]. The controllers shown in [13–15] confirmed that the model-based backstepping controller outperforms the conventional linear controller for the wide range of velocities.

A teleoperated ROV can be taken as a mass-spring system, and periodic waves accompanied with strong current may excite the motion of the vehicle. Tethered cable tensions may cause forces and position nonlinear oscillations on ROV as stated in [16, 17]. Thus a quasistatic or dynamic model reflecting the umbilical cable behavior is very necessary, however, being quite complicated, for precise trajectory control [18–20]. Prabhakar and Buckham developed a computational tethered cable model by applying finite element technique with decoupled PD controller [21]. Researchers de Souza and Maruyama estimated tethered cable dynamics from a lumped mass tethered cable model and issued a PI feed forward control strategy for ROV position control [22]. Soylu et al. considered the tethered cable as a series of lumped point masses connected by linear, massless, and visco-elastic springs and proposed a model-based sliding mode controller to improve the ROV performances [23]. Montano et al. modeled the tethered cable dynamics in a quasi-stationary state and presented adaptive control scheme for tethered cable perturbations [24]. Unfortunately, based on our knowledge, there are very few of previous studies that consider the effects from tethered cable and the disturbances from current, wave, and wind all together. However, none of these effects can be neglected during the practical applications. This study exactly focuses on this issue and contributes an adaptive backstepping sliding mode controller which takes into consideration all the mentioned aspects for the high precision 3D path following of one of observation class ROVs.

The rest of this study is organized as follows. Section 2 sets up the dynamic model of the ROV multibody system especially including the considered umbilical cable. A controller which is for the 3D path following is designed in Section 3, while Section 4 presents the experimental results. Conclusions are given in Section 5 to close this paper.

2. Dynamic Model of the ROV System

2.1. Modeling the Whole ROV Multibody System. The ROV is usually operating with the support vessel (mother ship) in the ocean; see Figure 1. The tethered cable not only provides power and communication media but also brings nonlinear drag forces upon ROV. In this study, the considered system is a multibody system including a flexible body, that is, the tethered cable, and a rigid body, that is, the ROV itself.

Figure 1: Multibody system of the considered ROV with tethered cable.

The dynamic equation of the whole ROV multibody system according to [5] can be represented by

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}(\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \Delta\mathbf{d} = \mathbf{T}_{\text{tru}} + \boldsymbol{\tau}_t, \quad (1)$$

where $\mathbf{M}(\mathbf{q})$ is the 6×6 mass matrix of the ROV, vector $\mathbf{q} = [i, j, k, \varphi, \theta, \psi]^T$ is a 6×1 pose vector of the ROV, $\mathbf{C}(\dot{\mathbf{q}})$ is a 6×6 matrix of centrifugal and Coriolis terms, $\mathbf{D}(\dot{\mathbf{q}})$ is the damping matrix, $\mathbf{G}(\mathbf{q})$ is the vector of gravitational forces and moments, $\Delta\mathbf{d}$ is a disturbance vector, and $\mathbf{T}_{\text{tru}}$ contains the forces and torques from thrusts, while $\boldsymbol{\tau}_t$ is the drag effects from the tethered cable. For observation class ROV, the drag force from tethered cable is one of the most important nonlinear disturbances particularly when the current or the wave is strong. It is one of the most important factors, which restricts the path-following accuracy. The following subsection focuses on the investigation of drag forces from the tethered cable.

2.2. Tethered Cable Drag Forces and Boundary Conditions. The tethered cable is a long flexible cylinder which can only sustain tensile loads and its length varies over the time. The tension force $\mathbf{T}$ at any point of a tethered cable in fluid according to [5] can be described by

$$\mathbf{W} + \mathbf{F} + \frac{\partial \mathbf{T}}{\partial s} = m\mathbf{a}, \quad (2)$$

where m is the effective mass per unit length, $\mathbf{a}$ is the inertial acceleration, s is the unstretched Lagrangian coordinate, $\mathbf{F}$ is the fluid dynamic force per unit length, and $\mathbf{W}$ is the cable unit length weight which can be written as W since its direction is fixed. We define ρ_t as the density of the tethered cable, A as the cross-sectional area of the tether, and $A = \pi D^2/4$ with D for the profile diameter of the cable. If we take EA as the cable stiffness and p as the stretched Lagrangian coordinate of the tethered cable profile which has

$$dp = \sqrt{dx_t^2 + dz_t^2}, \quad (3)$$

Figure 2: Model of tethered cable of ROV system in the water.

Figure 3: The ith microunit length of tethered cable in the air and its forces analysis.

Figure 4: The jth microunit length of tethered cable in the water and its forces analysis.

then the tension force in (2) exerted at s with its value is calculated by

$$T_s = \mathrm{EA}\left(\frac{dp}{ds} - 1\right) \tag{4}$$

due to Hooke's law. In Figure 2, h_1 is the diving depth of the ROV, and H_0 and V_0 are the horizontal and vertical drag forces, respectively, at the ship end of the tethered cable, that is, at point O_c.

In order to analyze the motion of the tethered cable and its effect on the vehicle, three coordinate systems have been established which are the fixed frame $[x, y, z]$, the local frame $[\zeta, \eta, \xi]$ on one of the cable ends at the mother ship side, and the ROV frame $[i, j, k]$. We define ϕ as the angle formed by the horizontal and local tangential direction at the considered point, $\cos\phi = d\zeta/dp$, $\sin\phi = d\xi/dp$. Therefore,

$$\left(\frac{d\zeta}{dt}\right)^2 + \left(\frac{d\xi}{dt}\right)^2 = \left|\mathbf{v}_t\right|^2, \tag{5}$$

where $\mathbf{v}_t$ is the velocity of any point of the tethered cable.

For the tethered cable in the air, the first item at the left side of (2) is $W = W_{\text{air}} = A\rho_t g$ in which g is the value of gravitational acceleration. Here only the vertical direction is considered since it is a gravitational force. If we set $\mathbf{v}_{\text{wind}}$ as the wind velocity relative to the cable, we have

$$\mathbf{F}_{\text{wind}} = \frac{1}{2} d\xi C_{\text{wind}} \rho_{\text{air}} A \left|\mathbf{v}_{\text{wind}}\right| \mathbf{v}_{\text{wind}} \tag{6}$$

as the wind effect on the $d\xi$ length tethered cable, where ρ_{air} is the air density, the strength of local wind velocity above the sea surface can be expressed by

$$\left|\mathbf{v}_{\text{wind}}\right| = v_{10}\left(\frac{d_1 - \xi_w}{10}\right)^{1/7} \tag{7}$$

in which v_{10} is the wind speed at 10 meters above the sea surface, d_1 is the cable height out of the water counting from O_c to the water plane, and ξ_w is only the cable height out of the water counting from the considered point to the water plane; see Figure 2. Here C_{wind} is the drag coefficient in the air. If we focus on the ith microunit length of tethered cable in the air; (see Figure 3), one has

$$\begin{aligned} T_{i+1}\frac{d\zeta_i}{dp} - T_i\frac{d\zeta_i}{dp} + F_{i,\text{wind}} &= m\frac{d\mathbf{v}_t}{dt}\frac{d\zeta_i}{dp}, \\ T_i\frac{d\xi_i}{dp} - T_{i+1}\frac{d\xi_i}{dp} - W_{i,\text{air}} &= m\frac{d\mathbf{v}_t}{dt}\frac{d\xi_i}{dp}. \end{aligned} \tag{8}$$

Here we consider as usual that the wind is horizontal only, and in this case in right horizontal, T_i and T_{i+1} are the two tension forces at the two ends of the ith microunit length cable, $W_{i,\text{air}} = A\rho_t g$.

Similarly, as shown in Figure 4, the tethered cable is in the water.

Its dynamics in the water hold

$$\begin{aligned} T_{j+1}\frac{d\zeta_j}{dp} - T_j\frac{d\zeta_j}{dp} + F_{j,f} &= m\frac{d\mathbf{v}_t}{dt}\frac{d\zeta_j}{dp}, \\ T_j\frac{d\xi_j}{dp} - T_{j+1}\frac{d\xi_j}{dp} - W_{j,\text{water}} &= m\frac{d\mathbf{v}_t}{dt}\frac{d\xi_j}{dp}, \end{aligned} \tag{9}$$

where $W_{j,\text{water}} = (\rho_t - \rho_w)Ag$, and ρ_w is the density of water. Here T_j and T_{j+1} are the two tension forces at the two ends of the jth microunit length cable under water, and $F_{j,f}$ is the force due to the wave and current disturbances which usually

(a) SY-II ROV (b) Control diagram

Figure 5: SY-II open-frame underwater vehicle.

(a) Water tank with wave generator (b) Local current generation device

Figure 6: Water tank with wave generator and local current generation device.

only considers for the horizontal direction. If $\mathbf{v}_{wc}$ is assumed to be the resultant velocity of wave and current velocities, then the relative velocity between the cable and this resultant velocity is

$$\mathbf{v}_{\text{rel}} = \mathbf{v}_t - \mathbf{v}_{wc}. \tag{10}$$

Thus, if the drag coefficient in the water is represented by C_f, then

$$\mathbf{F}_f = \frac{1}{2} d\xi C_f \rho_w A \left|\mathbf{v}_{\text{rel}}\right| \mathbf{v}_{\text{rel}}. \tag{11}$$

In this study we consider that the wave and current forces are in horizontal direction, for example, in Figures 2 and 4, and they are in the right horizontal direction.

Considering the boundary condition at $s = 0$, that is, at point O_c, the cable tension force is

$$\left|\mathbf{T}_{O_c}\right| = \sqrt{H_0^2 + V_0^2}. \tag{12}$$

At the point R on ROV, the cable tension force can be calculated by

$$\begin{aligned} H_R + F_{\text{wind}} + F_f - H_0 &= \int_{O_c}^{\text{sur}} m\mathbf{a}_{i,H} + \int_{\text{sur}}^{R} m\mathbf{a}_{j,H}, \\ V_0 - W - V_R &= \int_{O_c}^{\text{sur}} m\mathbf{a}_{i,V} + \int_{\text{sur}}^{R} m\mathbf{a}_{j,V} \end{aligned} \tag{13}$$

Figure 7: The 6-meter depth control experiments.

assuming that H_R is in right horizontal direction and V_R is in down vertical direction. Here $H_R = -H_R^{(r)}$, $V_R = -V_R^{(r)}$ in which $H_R^{(r)}$ and $V_R^{(r)}$ are the reactive forces in horizontal and vertical directions, respectively, at point R on ROV. At the right side of (13), $\mathbf{a}_{i,H}$, $\mathbf{a}_{j,H}$, $\mathbf{a}_{i,V}$, and $\mathbf{a}_{j,V}$ are the corresponding microunits accelerations. The cable tension force at point R is $|\mathbf{T}_R| = \sqrt{H_R^2 + V_R^2}$.

In (13), other corresponding terms can be governed by

$$\begin{aligned} F_{\text{wind}} &= \left|\mathbf{F}_{\text{wind}}\right| = \frac{1}{2} d_1 C_{\text{wind}} \rho_{\text{air}} A \left|\mathbf{v}_{\text{wind}}\right|^2, \\ F_f &= \left|\mathbf{F}_f\right| = \frac{1}{2} h_1 C_f \rho_w A \left|\mathbf{v}_{\text{rel}}\right|^2, \\ W &= A L_s \rho_t g - A\left(L_s - L_a\right) \rho_w g, \end{aligned} \tag{14}$$

where L_s is the total length of the cable, that is, from O_c to R, and L_a is the length of the cable out of water, that is, from O_c to the water surface.

In the following, we analyse the cable tension force at the water surface. This boundary condition can be considered either from the cable part in the air or the part in the water. Considering the cable in the air, one has

$$\begin{aligned} F_{\text{wind}} - H_0 + H_{\text{sur}}^{\text{air}} &= \int_{O_c}^{\text{sur}} m \mathbf{a}_{i,H}, \\ V_0 - W_{L_a} - V_{\text{sur}}^{\text{air}} &= \int_{O_c}^{\text{sur}} m \mathbf{a}_{i,V}, \end{aligned} \tag{15}$$

where $H_{\text{sur}}^{\text{air}}$ and $V_{\text{sur}}^{\text{air}}$ are the cable tension forces at horizontal and vertical directions, respectively. The actual directions of

Figure 8: The 8-meter depth control experiments.

these cable tension forces are determined via the positive or negative characteristics of the force values. If it is positive, its actual direction is the same as assumed, otherwise in the opposite of the assumed. Here we assume that $H_{\text{sur}}^{\text{air}}$ is in right horizontal direction and $V_{\text{sur}}^{\text{air}}$ is in down vertical direction. In (15), $W_{L_a} = AL_a\rho_t g$. If we consider the cable part in the water, one further has

$$F_f + H_R - H_{\text{sur}}^{\text{water}} = \int_{\text{sur}}^{R} m\mathbf{a}_{j,H},$$
$$-V_R - W_{L_s-L_a,\rho_t-\rho_w} + V_{\text{sur}}^{\text{water}} = \int_{\text{sur}}^{R} m\mathbf{a}_{j,V}, \tag{16}$$

where $H_{\text{sur}}^{\text{water}}$ and $V_{\text{sur}}^{\text{water}}$ are the cable tension forces at horizontal and vertical directions, respectively, on the water surface and considered at the water side. In addition, the following equations should be met which are

$$W_{L_s-L_a,\rho_t-\rho_w} = A\left(L_s - L_a\right)\left(\rho_t - \rho_w\right)g,$$
$$H_{\text{sur}}^{\text{air}} = -H_{\text{sur}}^{\text{water}},$$
$$V_{\text{sur}}^{\text{air}} = -V_{\text{sur}}^{\text{water}}, \tag{17}$$
$$\left|\mathbf{T}_{\text{sur}}\right| = \sqrt{\left(H_{\text{sur}}^{\text{air}}\right)^2 + \left(V_{\text{sur}}^{\text{air}}\right)^2} = \sqrt{\left(H_{\text{sur}}^{\text{water}}\right)^2 + \left(V_{\text{sur}}^{\text{water}}\right)^2}.$$

Figure 9: The −30° heading control experiments.

The moment at R is governed by

$$\begin{aligned} &-H_0\left(d_1+h_1\right)+V_0 l_1-\frac{1}{2} l_1 A L_s \rho_t g \\ &+\frac{1}{2}\left(l_1-a_1\right) A\left(L_s-L_a\right) \rho_w g \\ &+\left(\frac{1}{2} d_1+h_1\right) F_{\text{wind}}+\frac{1}{2} h_1 F_f=0 \end{aligned} \tag{18}$$

when it is in the equilibrium state. Here a_1 is the horizontal length of the cable in the air, l_1 is the horizontal length of the total cable.

3. Controller Design for Path Following

The ROV dynamic model described in state space can be expressed as

$$\begin{gathered} \dot{\mathbf{x}}_1=\mathbf{x}_2, \\ \dot{\mathbf{x}}_2=\mathbf{M}^{-1}(\mathbf{q})\left(\mathbf{T}_{\text{tru}}+\boldsymbol{\tau}_t-\mathbf{C}(\dot{\mathbf{q}}) \mathbf{x}_2-\mathbf{D}(\dot{\mathbf{q}}) \mathbf{x}_2-\mathbf{G}(\mathbf{q})-\Delta \mathbf{d}\right), \\ \mathbf{Y}=\mathbf{x}_1, \end{gathered} \tag{19}$$

where $\mathbf{x}_1=\mathbf{q} \in R^{6\times 1}$ is the state vector and $\mathbf{Y}$ is the observation variable. Set $\mathbf{Y}_d$ to be the desired trajectory of ROV and it is twice differentiable. The main steps of the

Figure 10: The 40° heading control experiments.

adaptive backstepping sliding mode control can be described as follows.

Step 1. According to the path following objective, $\mathbf{e}_1$ is selected as the trajectory tracking error and defined as

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{Y} - \mathbf{Y}_d, \\ \dot{\mathbf{e}}_1 &= \dot{\mathbf{Y}} - \dot{\mathbf{Y}}_d = \mathbf{x}_2 - \dot{\mathbf{Y}}_d. \end{aligned} \tag{20}$$

Then define virtual control coefficient

$$\boldsymbol{\alpha}_1 = k_1 \mathbf{e}_1, \tag{21}$$

where $k_1 > 0$. The velocity error of the ROV is defined as

$$\mathbf{e}_2 = \dot{\mathbf{e}}_1 + \boldsymbol{\alpha}_1 = \mathbf{x}_2 - \dot{\mathbf{Y}}_d + \boldsymbol{\alpha}_1 = \mathbf{x}_2 - \dot{\mathbf{Y}}_d + k_1 \mathbf{e}_1. \tag{22}$$

The corresponding Lyapunov function is selected as

$$V_1 = \frac{1}{2} \mathbf{e}_1^T \mathbf{e}_1. \tag{23}$$

Thus,

$$\begin{aligned} \dot{V}_1 &= \frac{1}{2} \dot{\mathbf{e}}_1^T \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_1^T \dot{\mathbf{e}}_1 \\ &= \dot{\mathbf{e}}_1^T \mathbf{e}_1 = \mathbf{e}_1^T \dot{\mathbf{e}}_1 \\ &= \mathbf{e}_1^T \left(\mathbf{e}_2 - \boldsymbol{\alpha}_1 \right) \\ &= \mathbf{e}_1^T \mathbf{e}_2 - \mathbf{e}_1^T k_1 \mathbf{e}_1. \end{aligned} \tag{24}$$

Figure 11: Advance toward north with two main thrusters, $V_{\text{curr}} = 0$.

It is possible to adjust $\mathbf{e}_2$ and let it satisfy $\mathbf{e}_2 = 0$; then $\dot{V}_1 \leq 0$ is satisfied which means the first subsystem of our whole ROV system is stable.

Step 2. To improve the controller's systematic robustness against the disturbances, the following switching function of the terminal sliding mode surface is defined which is

$$\mathbf{S} = c_1\mathbf{e}_1 + \mathbf{e}_2, \tag{25}$$

where $\mathbf{S}$ is the switching function of the terminal sliding mode, $c_1 \geq 0$. Then the Lyapunov function is chosen as

$$V_2 = V_1 + \frac{1}{2}\mathbf{S}^T\mathbf{S}. \tag{26}$$

Therefore,

$$\begin{aligned}\dot{V}_2 = {} & \mathbf{e}_1^T\mathbf{e}_2 - \mathbf{e}_1^T k_1\mathbf{e}_1 \\ & + \mathbf{S}^T\Big[c_1\left(-k_1\mathbf{e}_1 + \mathbf{e}_2\right) \\ & \quad + \mathbf{M}^{-1}(\mathbf{q}) \\ & \quad \cdot\Big(\mathbf{T}_{\text{tru}} + \boldsymbol{\tau}_t - \left(\mathbf{C}(\dot{\mathbf{q}}) + \mathbf{D}(\dot{\mathbf{q}})\right)\left(\mathbf{e}_2 + \dot{\mathbf{Y}}_d - \boldsymbol{\alpha}_1\right) \\ & \quad - \mathbf{G}(\mathbf{q}) - \Delta\mathbf{d}\Big) - \ddot{\mathbf{Y}}_d + \dot{\boldsymbol{\alpha}}_1\Big].\end{aligned} \tag{27}$$

To satisfy the Lyapunov stability theory and the reaching law of the controller as in [25], the derivative $\dot{\mathbf{S}}$ is defined as

$$\dot{\mathbf{S}} = -h\beta\,\mathrm{sgn}(\mathbf{S}) - h\mathrm{S}, \tag{28}$$

FIGURE 12: Advance toward north with two main thrusters, $V_{\text{curr}} = 0.4$ m/s normal eastward.

where h and β are positive constants. We design the backstepping sliding mode controller as

$$\begin{aligned}\mathbf{u} &= \mathbf{T}_{\text{tru}} \\ &= \mathbf{M}(\mathbf{q})\left[-c_1\left(-k_1\mathbf{e}_1+\mathbf{e}_2\right)+\ddot{\mathbf{Y}}_d-\dot{\boldsymbol{\alpha}}_1-h\beta\,\text{sgn}(\mathbf{S})-h\mathbf{S}\right] \\ &\quad -\boldsymbol{\tau}_t+\left(\mathbf{C}(\dot{\mathbf{q}})+\mathbf{D}(\dot{\mathbf{q}})\right)\left(\mathbf{e}_2+\dot{\mathbf{Y}}_d-\boldsymbol{\alpha}_1\right) \\ &\quad +\mathbf{G}(\mathbf{q})+\Delta\tilde{\mathbf{d}}\,\text{sgn}(\mathbf{S}),\end{aligned} \tag{29}$$

where $\Delta\tilde{\mathbf{d}}$ is the estimation error of the disturbance $\Delta\mathbf{d}$; that is, $\Delta\tilde{\mathbf{d}} = \Delta\mathbf{d} - \Delta\hat{\mathbf{d}}$, in which $\Delta\hat{\mathbf{d}}$ is the estimation of $\Delta\mathbf{d}$. Here $\mathbf{u}$ is the control input.

Step 3. In order to estimate further disturbances $\Delta\mathbf{d}$ and avoid the undetermined upper bounds, we define Lyapunov function:

$$V_3 = V_2 + \frac{1}{2\gamma}\Delta\tilde{\mathbf{d}}^T\Delta\tilde{\mathbf{d}}, \tag{30}$$

where γ is a positive constant. If we assume the disturbance $\Delta\mathbf{d}$ being time invariant in a short period, thus

$$\begin{aligned}\dot{V}_3 &= \mathbf{e}_1^T\mathbf{e}_2-\mathbf{e}_1^Tk_1\mathbf{e}_1-\frac{1}{\gamma}\Delta\tilde{\mathbf{d}}^T\left(\Delta\dot{\hat{\mathbf{d}}}-\gamma\mathbf{M}^{-1}(\mathbf{q})\mathbf{S}\right) \\ &\quad +\mathbf{S}^T\left[c_1\left(-k_1\mathbf{e}_1+\mathbf{e}_2\right)\right. \\ &\quad +\mathbf{M}^{-1}(\mathbf{q})\end{aligned}$$

Figure 13: Lateral motion toward east with two side thrusters, $V_{\text{curr}} = 0$.

$$\cdot \left(\mathbf{T}_{\text{tru}} + \boldsymbol{\tau}_t - (\mathbf{C}(\dot{\mathbf{q}}) + \mathbf{D}(\dot{\mathbf{q}}))\left(\mathbf{e}_2 + \dot{\mathbf{Y}}_d - \boldsymbol{\alpha}_1\right) - \mathbf{G}(\mathbf{q}) - \Delta\mathbf{d}\right) - \ddot{\mathbf{Y}}_d + \dot{\boldsymbol{\alpha}}_1\Big]. \tag{31}$$

The adaptive control rule can be selected as

$$\Delta\dot{\hat{\mathbf{d}}} = \gamma\mathbf{M}^{-1}(\mathbf{q})\,\mathbf{S}. \tag{32}$$

Further we can design the backstepping sliding mode controller as same as in (29); that is,

$$\begin{aligned}\mathbf{u} &= \mathbf{T}_{\text{tru}} \\ &= \mathbf{M}(\mathbf{q})\left[-c_1\left(-k_1\mathbf{e}_1 + \mathbf{e}_2\right) + \ddot{\mathbf{Y}}_d - \dot{\boldsymbol{\alpha}}_1 - h\beta\,\text{sgn}(\mathbf{S}) - h\mathbf{S}\right] \\ &\quad - \boldsymbol{\tau}_t + (\mathbf{C}(\dot{\mathbf{q}}) + \mathbf{D}(\dot{\mathbf{q}}))\left(\mathbf{e}_2 + \dot{\mathbf{Y}}_d - \boldsymbol{\alpha}_1\right) \\ &\quad + \mathbf{G}(\mathbf{q}) + \Delta\tilde{\mathbf{d}}\,\text{sgn}(\mathbf{S}).\end{aligned} \tag{33}$$

With (31)–(33), one can obtain

$$\dot{V}_3 = \mathbf{e}_1^T\mathbf{e}_2 - \mathbf{e}_1^T k_1\mathbf{e}_1 - h\beta\mathbf{S}^T\,\text{sgn}(\mathbf{S}) - h\mathbf{S}^T\mathbf{S}. \tag{34}$$

If we set

$$\mathbf{Q} = \begin{pmatrix} k_1 + hc_1^2 & hc_1 - \dfrac{1}{2} \\ hc_1 - \dfrac{1}{2} & h \end{pmatrix}, \tag{35}$$

it yields

$$\mathbf{e}^T\mathbf{Q}\mathbf{e} = k_1\mathbf{e}_1^T\mathbf{e}_1 - \mathbf{e}_1^T\mathbf{e}_2 + h\mathbf{S}^T\mathbf{S}. \tag{36}$$

Figure 14: Lateral motion toward east with two side thrusters, $V_{\text{curr}} = 0.4$ m/s normal northward.

By selecting proper values of h, c_i, and k_i to satisfy $\|\mathbf{Q}\| \geq 0$; that is, the transformation matrix $\mathbf{Q}$ is regarded as the positive definite matrix; then

$$\dot{V}_3 = -\mathbf{e}^T \mathbf{Q} \mathbf{e} - h\beta \mathbf{S}^T \operatorname{sgn}(\mathbf{S}) \leq 0, \tag{37}$$

where $\mathbf{e} = [\mathbf{e}_1, \mathbf{e}_2]^T$. This means the third subsystem is also stable which is sufficient for our ROV system.

4. Experiments

4.1. Experimental Setup. In order to verify and analysis the investigated tethered cable dynamic model and the backstepping controller, experiments have been performed based on an SY-II remote operated vehicle; see Figure 5. The SY-II ROV is an open frame underwater vehicle, which is purposed for ship hull inspection. Its power supply and communication cable are a neutrally buoyant tether which is attached to the vehicle at the tail. The control commands are sent through network communication between surface computer and PC104 embedded processor. It is equipped with depth gauge, ultrasonic Doppler velocity meter (DVL), magnetic compass as motion sensors, underwater CCD, image sonar, ultrasonic thickness gauge as environmental sensors, and 6 thrusters including 2 main thrusters, 2 side thrusters, and 2 vertical thrusters. Its tethered cable and hydrodynamic and inertial parameters are illustrated in Tables 1, 2, and 3, respectively. The experiments have been performed in a 50 m × 30 m × 10 m tank at the Key Laboratory of Science and Technology on Underwater Vehicle in Harbin Engineering University. Comparisons have been made between the S surface controller investigated in [6] and our adaptive backstepping sliding mode controller. Current and waves are generated from local current generation device (Figure 6(a))

FIGURE 15: 3D path-following experiments, polygon path, $V_{curr} = 0.2$ m/s, $V_{wave} = 0.2$ m/s normal northward.

and multidirectional waves generator (Figure 6(b)), respectively, in the experiments to simulate real oceanic conditions.

4.2. Experiments and Results Analysis. In the depth control experiments of Figures 7 and 8, the tension force from tethered cable upon SY-II ROV changes little mainly because its neutral buoyancy, and the peak values of tethered cable moment are decided by diving speed. The tethered cable makes little effect upon depth control when diving is not very fast. Adaptive backstepping sliding mode controller has improved the stability and speed constringency in depth control in comparing with S surface controller since it provides dynamic compensation during the experiments.

In the heading control experiments of Figures 9 and 10, the desired heading is 40° and −30° relative to the normal north of earth's magnetic field. The tension force and moment from tethered cable upon SY-II ROV change a lot at first during revolution process. The tethered cable and ROV's

TABLE 1: Tethered cable parameters.

Item	ρ_t (kg/m^3)	C_f (m^{-1})	E (10^5 MPa)	C_{wind} (m^{-1})	D (m)	L_{total} (m)
Value	1000	1.2	0.02	0.4	10^{-2}	300

TABLE 2: SY-II hydrodynamic parameters.

Dimensionless coefficient	x	y	z	r	q	p
First-order terms	27.65	47.26	44.86	62.63	57.55	141.33

Figure 16: 3D path-following experiments, spiral path, $V_{curr} = 0.2$ m/s, $V_{wave} = 0.2$ m/s, normal northward.

Table 3: SY-II inertial parameters.

Mass (kg)	I_x (N·m^2)	I_y (N·m^2)	I_z (N·m^2)	I_{xy} (N·m^2)	I_{xz} (N·m^2)	I_{xz} (N·m^2)
111.9	97.3	26.1	56.8	0	0	0

rectangle shape have made nonlinear effects to the ROV manipulation. As a result, the S surface controller cannot realize precise heading control, while adaptive backstepping sliding mode controller qualifies, since the controller delivers command according to its dynamic manipulation and character.

During the northward advancing process (see Figures 11 and 12), the SY-II ROV that is advancing mainly depends on two main thrusters. Two conditions, that is, $V_{curr} = 0$ and $V_{curr} = 0.4$ m/s normal east (V_{curr} is the currents speed), are compared. Here the current has made great influence upon tethered cable and generates tension force and moment. Nonlinear disturbance from current and tether has produced much greater errors for the S surface controller than the adaptive backstepping sliding mode controller.

During the eastward lateral motion process (see Figures 13 and 14), the SY-II ROV that is laterally moving mainly depends on two lateral thrusters. Two conditions, that is, $V_{curr} = 0$ and $V_{curr} = 0.4$ m/s normal north, are compared. At first SY-II ROV has been blown a few meters away, and then the control started. The current has generated great tension force and moment effect upon SY-II ROV. Adaptive backstepping sliding mode controller has realized more accuracy result compared to the S surface controller.

During the 3D path-following experiments in the tank (see Figures 15 and 16), the disturbances are current and waves. The current speed is $V_{\text{curr}} = 0.4\,\text{m/s}$, the wave is a sine wave with speed $V_{\text{wave}} = 0.2\,\text{m/s}$, and height $H_{\text{wave}} = 0.2\,\text{m}$, respectively. Polygon path and spiral path are followed, respectively. As the experiments we made above, environmental disturbances exerted on the tethered cable have contributed nonlinear effect to the ROV. The S surface controller manages to follow the polygon path but have to adjust continuously and heavily to follow the spiral path. Comparatively, adaptive backstepping sliding mode controller can compensate more disturbances and can follow the desired path more precisely. Deduced from dynamic model, adaptive backstepping sliding mode controller is established to calculate and compensate nonlinear disturbances caused by current and tether. Therefore, it outperforms S surface controller in control accuracy and robustness.

For the experiments to be performed successfully, we have also run some preliminary simulations to get the initial value of some uncertainties. The simulation and experimental results agree with each other well. Details of simulation results can be found in [26, 27].

5. Conclusions

The main contributions of this paper are summarized as follows. At first, the dynamic tether model has been established based on the lumped mass cable model. Each element of the cable is taken as an elastic body and takes into consideration waves and current disturbances. Based on the dynamic model, an adaptive backstepping sliding mode controller has been designed. Secondly, the controller has taken dynamic model into consideration at the first and second steps (backstepping steps) based on Lyapunov functions. To improve the controller's systematic robustness and against disturbances, the sliding mode surface has been defined. An adaptive control rule has been chosen to further resist environmental disturbance which is the most difficult part for ROV 3D path following.

Experiments have been performed in the tank, including depth control, heading control, advance control, sideway control, polygon line, and spiral line 3D path-following control. With current and waves disturbances being taken into consideration, tethered cable effect has been analyzed, and the efficacy and superiority of our designed controller have been verified. The comparisons to the S surface controller have shown better performances of the investigated adaptive backstepping sliding mode controller.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This project is supported by National Science Foundation of China under the Grants of no. 51209050, no. 51179035, no. 51279221, and no. 61100006, the Doctoral Fund of Ministry of Education for Young Scholar with no. 20122304120003, State Key Laboratory of Ocean Engineering of Shanghai Jiao Tong University no. 1102, State Key Laboratory of Robotics and Systems of Harbin Institute of Technology no. SKLRS-2012-ZD-03, and the Fundamental Research Funds for the Central Universities with no. HEUCFR1101.

References

[1] K. Zhu and L. Gu, "A MIMO nonlinear robust controller for work-class ROVs positioning and trajectory tracking control," in *Proceedings of the Chinese Control and Decision Conference (CCDC '11)*, pp. 2565–2570, Mianyang, China, May 2011.

[2] A. R. Marzbanrad, M. Eghtesad, and R. Kamali, "A robust adaptive fuzzy sliding mode controller for trajectory tracking of ROVs," in *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pp. 2863–2870, Orlando, Fla, USA, December 2011.

[3] M. Kokegei, F. He, and K. Sammut, "Fully coupled 6 degree-of-freedom control of an over-actuated autonomous underwater vehicle," in *Autonomous Underwater Vehicles*, N. A. Cruz, Ed., pp. 147–170, InTech, Rijeka, Croatia, 2011.

[4] X. Bian, Y. Qu, Z. Yan, and W. Zhang, "Nonlinear feedback control for trajectory tracking of an unmanned underwater vehicle," in *Proceedings of the IEEE International Conference on Information and Automation (ICIA '10)*, pp. 1387–1392, Harbin, China, June 2010.

[5] A. Bagheri, T. Karimi, and N. Amanifard, "Tracking performance control of a cable communicated underwater vehicle using adaptive neural network controllers," *Applied Soft Computing Journal*, vol. 10, no. 3, pp. 908–918, 2010.

[6] J. Liu, H. Yu, and Y. Xu, "Improved S plane control algorithm for underwater vehicles," *Journal of Harbin Engineering University*, vol. 23, no. 1, pp. 33–36, 2002 (Chinese).

[7] J. Wang, G. Wu, L. Wan, Y. Sun, and L. Wang, "Controller design of underwater robots based on generalized S-plane," *Electric Machines and Control*, vol. 13, supplement 1, pp. 144–148, 2009 (Chinese).

[8] P. W. J. van de Ven, C. Flanagan, and D. Toal, "Neural network control of underwater vehicles," *Engineering Applications of Artificial Intelligence*, vol. 18, no. 5, pp. 533–547, 2005.

[9] W. Luo and Z. Zou, "Neural network based robust controller for trajectory tracking of underwater vehicles," *China Ocean Engineering*, vol. 21, no. 2, pp. 281–292, 2007.

[10] L. Zhang, X. Qi, and Y. Pang, "Adaptive output feedback control based on DRFNN for AUV," *Ocean Engineering*, vol. 36, no. 9-10, pp. 716–722, 2009.

[11] P. Patompak and I. Nilkhamhang, "Adaptive backstepping sliding-mode controller with bound estimation for underwater robotics vehicles," in *Proceedings of the 9th International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology*, pp. 1–4, Phetchaburi, Thailand, May 2012.

[12] L. Lapierre and B. Jouvencel, "Robust nonlinear path-following control of an AUV," *IEEE Journal of Oceanic Engineering*, vol. 33, no. 2, pp. 89–102, 2012.

[13] G. Antonelli, F. Caccavale, S. Chiaverini, and G. Fusco, "A novel adaptive control law for underwater vehicles," *IEEE Transactions on Control Systems Technology*, vol. 11, no. 2, pp. 221–232, 2003.

[14] J. Wang and C. S. G. Lee, "Self-adaptive recurrent neuro-fuzzy control of an autonomous underwater vehicle," *IEEE Transactions on Robotics and Automation*, vol. 19, no. 2, pp. 283–295, 2003.

[15] M. Xiao, "Variable structure control of a catastrophic course in a high-speed underwater vehicle launched out of the water," *Advances in Mechanical Engineering*, vol. 2013, Article ID 398707, 8 pages, 2013.

[16] F. Muttin, "Umbilical deployment modeling for tethered UAV detecting oil pollution from ship," *Applied Ocean Research*, vol. 33, no. 4, pp. 332–343, 2011.

[17] K. Zhu, H. Zhu, Y. Zhang, and J. Gao, "A multi-body space-coupled motion simulation for a deep-sea tethered remotely operated vehicle," *Journal of Hydrodynamics*, vol. 20, no. 2, pp. 210–215, 2008.

[18] Z. Feng and R. Allen, "Evaluation of the effects of the communication cable on the dynamics of an underwater flight vehicle," *Ocean Engineering*, vol. 31, no. 8-9, pp. 1019–1035, 2004.

[19] F. R. Driscoll, R. G. Lueck, and M. Nahon, "Development and validation of a lumped-mass dynamics model of a deep-sea ROV system," *Applied Ocean Research*, vol. 22, no. 3, pp. 169–182, 2000.

[20] J. Wu, J. Ye, C. Yang, Y. Chen, H. Tian, and X. Xiong, "Experimental study on a controllable underwater towed system," *Ocean Engineering*, vol. 32, no. 14-15, pp. 1803–1817, 2005.

[21] S. Prabhakar and B. Buckham, "Dynamics modeling and control of a variable length remotely operated vehicle tether," in *Proceedings of the MTS/IEEE OCEANS*, pp. 1255–1262, Washington, DC, USA, September 2005.

[22] E. C. de Souza and N. Maruyama, "Intelligent UUVs: some issues on ROV dynamic positioning," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 43, no. 1, pp. 214–226, 2007.

[23] S. Soylu, B. J. Buckham, and R. P. Podhorodeski, "Dynamics and control of tethered underwater-manipulator systems," in *Proceedings of the MTS/IEEE OCEANS*, pp. 1–8, Seattle, Wash, USA, September 2010.

[24] A. Montano, M. Restelli, and R. Sacco, "Numerical simulation of tethered buoy dynamics using mixed finite elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 41–44, pp. 4117–4129, 2007.

[25] N. Chen, F. Song, G. Li, X. Sun, and C. Ai, "An adaptive sliding mode backstepping control for the mobile manipulator with nonholonomic constraints," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 10, pp. 2885–2899, 2013.

[26] H. Huang, L. Wan, Y. Li, and Y. Pang, "The fault tolerable control system structure of SY-II remote operated vehicle," *Advanced Materials Research*, vol. 308–310, pp. 1483–1491, 2011.

[27] H. Huang, S. Jiang, L. Wan, and Y. Pang, "Type-2 fuzzy logic neural network control and target following for remote operated vehicles," in *Proceedings of the IEEE International Conference on Mechatronics and Automation*, pp. 516–521, Chengdu, China, August 2012.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 391704, 10 pages
http://dx.doi.org/10.1155/2013/391704

Research Article

Application of Finite Volume Method to Structural Stochastic Dynamics

Weidong Chen, Yanchun Yu, Ping Jia, Xiande Wu, and Fengchao Zhang

Harbin Engineering University, 145 Nantong Street, Harbin 150001, China

Correspondence should be addressed to Yanchun Yu; yuyanchun@hrbeu.edu.cn

Received 24 July 2013; Accepted 2 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Weidong Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The stochastic dynamic problems were becoming more difficult after considering the influences of stochastic factors and the complexity of the dynamic problems. To this background, the finite volume method combined with Perturbation Method was proposed for the stochastic dynamic analysis. The equations of perturbation-finite volume method were derived; the explicit expressions between random response and basic random variables were given; the method of stochastic dynamic analysis was discussed; and the examples were presented to verify the perturbation-finite volume method. The results of perturbation-finite volume method were compared with the results of Monte Carlo Method, which proved that the proposed method was correct and accurate. Because the proposed method was simple and clear, the equations were easy to establish and the efficiency was improved. Meanwhile, the proposed method was successfully applied to the stochastic dynamic analysis of linear multibody system, which was verified through the example in this paper.

1. Introduction

In recent years, it has been gradually accepted by workers and engineers that various random factors should be considered in structural analysis and design. In this context, stochastic structural analysis has become a hot topic in the field [1–12]. The problems of static stochastic structural analysis can be solved very well, and it has been applied to engineering practice successfully. However, the study of the stochastic dynamic analysis was still in the primary stage [13–19]. The main reasons were that the issues of stochastic dynamic analysis were difficult; there were many problems and even some of the basic problems are still to be studied. For example, the stochastic dynamic response is difficult to solve; the explicit expressions between random response and basic random variables are difficult to give and so on.

Currently, the effective solution methods for stochastic dynamic problems were mostly stochastic simulations based on dynamics analysis. The finite element method was applied mostly in the dynamic problems, and the calculation time was longer. The stochastic dynamic analysis in the time and space domain was random; the stochastic simulation based on the finite element method will lead to the low-solving efficiency. When using semianalytical methods to solve the problem of stochastic dynamic analysis, the main difficulty was that the methods of dynamic analysis such as the finite element method were mostly based on implicit solution. It was difficult to give the explicit expressions between random response and basic random variables, and it was difficult to obtain the sensitivity information. Therefore, it was difficult to carry out the stochastic dynamic analysis.

To solve these problems, Finite volume method combined with perturbation method was proposed for the stochastic dynamic analysis, and the method was successfully applied to the stochastic dynamic analysis of linear multibody system. The finite volume method was originally used in computational fluid mechanics; the application in the field of solid mechanics has only been considered in recent years, and it has been successfully applied in many engineering practical problems [20, 21]. However, the finite volume method is mostly studied on the problems of solid mechanics for deterministic variables. For stochastic problems, it has been rarely reported

in the field of solid mechanics. The advantage of finite volume method was that the explicit expressions between random response and basic random variables could be given, which was essential for the stochastic dynamic analysis. Therefore, finite volume method combined with perturbation method was proposed for the stochastic dynamic analysis.

2. The Equations of Finite Volume Method

The process of solution for finite volume method was that the solution domain was partitioned into finite continuous grids, to form a nonoverlapping control volume which surrounds each grid point according to certain rules; the governing equations were integrated in the control volume to form the discrete equations. Explicit central difference scheme was used in time field. First, the acceleration of every control volume was calculated by governing equations. The velocity and displacement of every control volume were calculated by central difference method, and then the grid stress was calculated. Finally, the stress was taken back to the governing equations for the next time step.

The governing equations of finite volume method were the mass, momentum, and energy conservation equations. Discretization was mainly focused on the momentum conservation equations; the governing equations were integrated in the control volume. For three-dimensional solid structure, three-dimensional unstructured grid was used as shown in Figure 1; connecting the gravity of the tetrahedron, the center of each side, and the midpoint of each edge, which comprised six planes. The tetrahedron was divided into four parts by the six planes, and each part was the part of the control volume.

The governing equations were integrated in the control volume:

$$\int_V \rho \ddot{u}_i dV = \int_V \left(\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right) dV. \tag{1}$$

Here, u_i $(i = 1, 2, 3)$ was the displacement component of the control volume and ρ was the material density. When all of the control volumes satisfied (1), the solution domain naturally satisfied the momentum conservation. If the body force was ignored, the volume integral was converted into surface integral through the Gauss formula. Equation (1) could be written as:

$$\rho \ddot{u}_i \int_V dV = \oint_S \sigma_{ji} n_j dS, \tag{2}$$

where n_j $(j = 1, 2, 3)$ was the unit vector component on the outward normal direction for the control volume. For node 1 in Figure 1, if it was connected with m tetrahedral, the left integral of (2) could be expressed as

$$\rho \ddot{u}_i \int_V dV = \frac{\ddot{u}_i}{4} \sum_{n=1}^{m} M_n, \tag{3}$$

where M_n was the quality of the n tetrahedral.

Supposing that the stress of tetrahedral was constant, then the stress on the boundary of the control volume was constant. Because the surface of the control volume was formed

Figure 1: Three-dimensional unstructured grid.

by some planes, the unit vector components on the outward normal direction for the control volume were constant. For the node 1 in Figure 1, it is connected with m tetrahedral and accumulated the integrals of m tetrahedral; the final form could be expressed as

$$\frac{\ddot{u}_i}{4} \sum_{n=1}^{m} M_n = \sum_{n=1}^{m} (a_1 \sigma_{1i})_n + \sum_{n=1}^{m} (a_2 \sigma_{2i})_n + \sum_{n=1}^{m} (a_3 \sigma_{3i})_n. \tag{4}$$

Here, a_1, a_2, and a_3 were the coefficients of the n tetrahedral which was connected with node 1. Its value was related to the node coordinates of the tetrahedral. For other nodes, it had the same expressions. We have

$$\begin{aligned} a_1 &= -\frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ y_4 & y_3 & y_2 \\ z_4 & z_3 & z_2 \end{vmatrix}, \\ a_2 &= -\frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ z_4 & z_3 & z_2 \\ x_4 & x_3 & x_2 \end{vmatrix}, \\ a_3 &= -\frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ x_4 & x_3 & x_2 \\ y_4 & y_3 & y_2 \end{vmatrix}. \end{aligned} \tag{5}$$

Equation (4) in matrix form is as follows:

$$\ddot{\mathbf{u}} = \frac{1}{M} \sum_{n=1}^{m} \mathbf{A}_n \boldsymbol{\sigma}_n, \tag{6}$$

where ρ was the material density M was the quality of the control volume. We have

$$\begin{aligned} \ddot{\mathbf{u}} &= \left[\ddot{u}_1 \;\; \ddot{u}_2 \;\; \ddot{u}_3 \right]^T, \\ \boldsymbol{\sigma}_n &= \left[\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{33} \;\; \sigma_{12} \;\; \sigma_{23} \;\; \sigma_{31} \right]^T, \\ \mathbf{A}_n &= \frac{1}{6} \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & a_3 \\ 0 & a_2 & 0 & a_1 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_2 & a_1 \end{bmatrix}. \end{aligned} \tag{7}$$

The boundary conditions and the initial conditions were given as follows.

Boundary conditions are as follows:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \overline{\mathbf{X}}, \qquad \mathbf{u} = \overline{\mathbf{u}}, \tag{8}$$

where $\overline{\mathbf{X}}$ was the outside force of boundary and $\overline{\mathbf{u}}$ was the displacement of boundary.

Initial conditions are as follows:

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0, \qquad \dot{\mathbf{u}}\big|_{t=0} = \dot{\mathbf{u}}_0. \tag{9}$$

Explicit central difference scheme was used in time domain [22], in the form of

$$\dot{u}^{n+1/2} = \dot{u}^{n-1/2} + \Delta t^n \ddot{u}^n, \qquad u^{n+1} = u^n + \Delta t^{n+1/2} \dot{u}^{n+1/2}, \tag{10}$$

where $\Delta t^{n+1/2} = t^{n+1} - t^n$ and $\Delta t^n = t^{n+1/2} - t^{n-1/2} = (\Delta t^{n-1/2} + \Delta t^{n+1/2})/2$. In order to ensure the stability of the numerical results, the value of the time step should satisfy $\Delta t_1 = l/c$; here, c was the speed of sound.

3. The Equations of Perturbation-Finite Volume Method

There was a stochastic field $\{Y\}$, where

$$\{Y\} = \left[Y_1 \;\; Y_2 \;\; \cdots \;\; Y_n\right]^T. \tag{11}$$

When the stochastic field was not dependent on time, it could be expressed as

$$\{Y\} = \left\{\overline{Y}\right\} + \{\alpha\}, \tag{12}$$

where $\{\overline{Y}\}$ was the mean value of the stochastic field,

$$\left\{\overline{Y}\right\} = \left[\overline{Y}_1 \;\; \overline{Y}_2 \;\; \cdots \;\; \overline{Y}_n\right]^T. \tag{13}$$

$\{\alpha\}$ was the small perturbation on the mean value point $\{\overline{Y}\}$, where

$$\{\overline{\alpha}\} = \left[\overline{\alpha}_1 \;\; \overline{\alpha}_2 \;\; \cdots \;\; \overline{\alpha}_n\right]^T = \{0\}. \tag{14}$$

If the stochastic field was dependent on time, then, for any instant, (12) was correct.

Since the displacement field $\{u\}$ depended on stochastic field, $\{u\}$ could be expanded by second-order perturbation as follows:

$$\{u\} = \{\overline{u}\} + \sum_{i=1}^{n} \alpha_i \left\{u_i^{(1)}\right\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \left\{u_{ij}^{(2)}\right\} \tag{15}$$

where $\{\overline{u}\}$ was the mean value of displacement field $\{u\}$ at point $\{\overline{Y}\}$. $\{u_i^{(1)}\}$ was the first-order partial derivatives of displacement field $\{u\}$ on Y_i at point $\{\overline{Y}\}$. $\{u_{ij}^{(2)}\}$ was the second-order partial derivatives of displacement field $\{u\}$ on Y_i and Y_j at point $\{\overline{Y}\}$.

For the governing equation (6), due to the randomness of load or structure, the displacement and stress were random. Therefore, the displacement and stress were expressed as second-order perturbation expressions. The second-order perturbation expressions of displacement were shown in (15). The first moment and second moment of $\{u\}$ were as follows:

$$\begin{aligned}
E[\{u\}] &= \{\overline{u}\} + \sum_{i=1}^{n} E\left[\alpha_i\right]\left\{u_i^{(1)}\right\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\alpha_i\alpha_j\right]\left\{u_{ij}^{(2)}\right\} \\
&= \{\overline{u}\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\alpha_i\alpha_j\right]\left\{u_{ij}^{(2)}\right\}, \\
D[\{u\}] &= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\alpha_i\alpha_j\right]\left\{u_i^{(1)}\right\}\left\{u_j^{(1)}\right\}^T \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left\{u_i^{(1)}\right\}\left\{u_{jk}^{(2)}\right\}^T \\
&\quad \cdot \left(E\left[\alpha_i\alpha_j\right] - E\left[\alpha_i\right]E\left[\alpha_j\alpha_k\right]\right) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{u_{ij}^{(2)}\right\}\left\{u_{kl}^{(2)}\right\}^T \\
&\quad \times \left(E\left[\alpha_i\alpha_j\alpha_k\alpha_l\right] - E\left[\alpha_i\alpha_j\right]E\left[\alpha_k\alpha_l\right]\right),
\end{aligned} \tag{16}$$

where

$$E\left[\alpha_i\alpha_j\right] = \mathrm{Cov}\left(Y_i, Y_j\right). \tag{17}$$

Generally higher moments of the stochastic field were difficult to obtain and the error was large, so the first item was used as follows:

$$D[\{u\}] = \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\alpha_i\alpha_j\right]\left\{u_i^{(1)}\right\}\left\{u_j^{(1)}\right\}^T. \tag{18}$$

The second-order perturbation expression of stress was

$$\{\sigma\} = \{\overline{\sigma}\} + \sum_{i=1}^{n}\alpha_i\left\{\sigma_i^{(1)}\right\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\left\{\sigma_{ij}^{(2)}\right\}, \tag{19}$$

where $\{\overline{\sigma}\}$ was the mean value of stress field $\{\sigma\}$ at point $\{\overline{Y}\}$. $\{\sigma_i^{(1)}\}$ was the first-order partial derivatives of stress field $\{\sigma\}$ on Y_i at point $\{\overline{Y}\}$. $\{\sigma_{ij}^{(2)}\}$ was the second-order partial derivatives of stress field $\{\sigma\}$ on Y_i and Y_j at point $\{\overline{Y}\}$.

The relationship between stress matrix and strain matrix was

$$\{\sigma\} = [D]\{\varepsilon\}. \tag{20}$$

It could be obtained by the perturbation method as follows:

$$\begin{aligned}\{\overline{\sigma}\} &= \left[\overline{D}\right]\{\overline{\varepsilon}\},\\ \left\{\sigma_i^{(1)}\right\} &= \left[D_i^{(1)}\right]\{\overline{\varepsilon}\} + \left[\overline{D}\right]\left\{\varepsilon_i^{(1)}\right\},\\ \left\{\sigma_{ij}^{(2)}\right\} &= \left[\overline{D}\right]\left\{\varepsilon_{ij}^{(2)}\right\} + \left[D_i^{(1)}\right]\left\{\varepsilon_j^{(1)}\right\}\\ &\quad + \left[D_j^{(1)}\right]\left\{\varepsilon_i^{(1)}\right\} + \left[D_{ij}^{(2)}\right]\{\overline{\varepsilon}\}.\end{aligned} \tag{21}$$

According to (21), the mean value and variance of stress were

$$\begin{aligned}E[\{\sigma\}] &= \{\overline{\sigma}\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\alpha_i\alpha_j\right]\left\{\sigma_{ij}^{(2)}\right\}\\ &= \left[\overline{D}\right][B]\{\overline{u}\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\alpha_i\alpha_j\right]\left[\overline{D}\right][B]\left\{u_{ij}^{(2)}\right\}\\ &\quad + \left[D_i^{(1)}\right][B]\left\{u_j^{(1)}\right\} + \left[D_j^{(1)}\right][B]\left\{u_i^{(1)}\right\}\\ &\quad + \left[D_{ij}^{(2)}\right][B]\{\overline{u}\},\\ D[\{\sigma\}] &= \sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\alpha_i\alpha_j\right]\left\{\sigma_i^{(1)}\right\}\left\{\sigma_j^{(1)}\right\}^T\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\alpha_i\alpha_j\right]\left(\left[D_i^{(1)}\right][B]\{\overline{u}\} + \left[\overline{D}\right][B]\left\{u_i^{(1)}\right\}\right)\\ &\quad \cdot\left(\left[D_j^{(1)}\right][B]\{\overline{u}\} + \left[\overline{D}\right][B]\left\{u_j^{(1)}\right\}\right)^T\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\alpha_i\alpha_j\right]\left(\left[S_i^{(1)}\right]\left[R_{\overline{a}\overline{a}}\right]\left[\overline{S}\right]^T\right.\\ &\quad + \left[\overline{S}\right]\left[R_{a_i\overline{a}}\right]\left[S_i^{(1)}\right]^T\\ &\quad \left. + \left[\overline{S}\right]\left[R_{a_ia_j}\right]\left[\overline{S}\right]^T\right),\end{aligned} \tag{22}$$

where **D** was elastic matrix and **B** was strain matrix. We have

$$\begin{aligned}\left[S_i^{(1)}\right] &= \left[D_i^{(1)}\right][B], \qquad \left[\overline{S}\right] = \left[\overline{D}\right][B],\\ \left[R_{\overline{a}\overline{a}}\right] &= \{\overline{u}\}\{\overline{u}\}^T,\\ \left[R_{a_i\overline{a}}\right] &= \left[R_{\overline{a}a_i}\right]^T = \{\overline{u}\}\left\{u_i^{(1)}\right\}^T,\\ \left[R_{a_ia_j}\right] &= \left\{u_i^{(1)}\right\}\left\{u_j^{(1)}\right\}.\end{aligned} \tag{23}$$

It was seen from the derivation that, in order to get the statistical properties of the response, it was necessary to obtain the first-order and second-order partial derivatives of the response to random variables. Explicit algorithm was used in the finite volume method so that the partial derivatives could be solved directly, which was the advantage of the finite volume method for solving stochastic dynamic problems. The partial derivative of (6) to random variable was

$$\begin{aligned}\ddot{\mathbf{u}}_i^{(1)} &= \frac{1}{M}\sum_{n=1}^{m}\mathbf{A}_n\left(\boldsymbol{\sigma}_n\right)_i^{(1)},\\ \ddot{\mathbf{u}}_{ij}^{(2)} &= \frac{1}{M}\sum_{n=1}^{m}\mathbf{A}_n\left(\boldsymbol{\sigma}_n\right)_{ij}^{(2)}.\end{aligned} \tag{24}$$

If the physical quantities at time t were known, the partial derivatives of displacement at time $t + \Delta t$ could be obtained by explicit central difference as follows:

$$\left(\dot{\mathbf{u}}^{t+\Delta t/2}\right)_i^{(1)} = \left(\dot{\mathbf{u}}^{t-\Delta t/2}\right)_i^{(1)} + \left(\ddot{\mathbf{u}}^{t}\right)_i^{(1)}\Delta t, \tag{25}$$

$$\left(\mathbf{u}^{t+\Delta t}\right)_i^{(1)} = \left(\mathbf{u}^{t}\right)_i^{(1)} + \left(\dot{\mathbf{u}}^{t+\Delta t/2}\right)_i^{(1)}\Delta t, \tag{26}$$

$$\left(\dot{\mathbf{u}}^{t+\Delta t/2}\right)_{ij}^{(2)} = \left(\dot{\mathbf{u}}^{t-\Delta t/2}\right)_{ij}^{(2)} + \left(\ddot{\mathbf{u}}^{t}\right)_{ij}^{(2)}\Delta t, \tag{27}$$

$$\left(\mathbf{u}^{t+\Delta t}\right)_{ij}^{(2)} = \left(\mathbf{u}^{t}\right)_{ij}^{(2)} + \left(\dot{\mathbf{u}}^{t+\Delta t/2}\right)_{ij}^{(2)}\Delta t. \tag{28}$$

Figure 2: The calculation model of cantilever beam.

Figure 3: Mean value of displacement on node (0.03, 0.06, 0.1) m on direction y.

Figure 4: Variance of displacement on node (0.03, 0.06, 0.1) m on direction y.

Figure 5: Mean value of equivalent stress on node (0.03, 0.06, 0.0) m.

Figure 6: Variance of equivalent stress on node (0.03, 0.06, 0.0) m.

The partial derivatives of stress could be given according to (26) and (28):

$$\left(\boldsymbol{\sigma}^{t+\Delta t}\right)_i^{(1)} = D_i^{(1)}\mathbf{B}\mathbf{u}^{t+\Delta t} + \mathbf{D}\mathbf{B}\left(\mathbf{u}^{t+\Delta t}\right)_i^{(1)},$$
$$\left(\boldsymbol{\sigma}^{t+\Delta t}\right)_{ij}^{(2)} = D_{ij}^{(2)}\mathbf{B}\mathbf{u}^{t+\Delta t} + D_i^{(1)}\mathbf{B}\left(\mathbf{u}^{t+\Delta t}\right)_j^{(1)} + D_j^{(1)}\mathbf{B}\left(\mathbf{u}^{t+\Delta t}\right)_i^{(1)} + \mathbf{D}\mathbf{B}\left(\mathbf{u}^{t+\Delta t}\right)_{ij}^{(2)}. \quad (29)$$

For the perturbation-finite volume method, the boundary conditions were shown in (30) and the initial conditions were shown in (31) as follows.

Boundary conditions are as follows:

$$\frac{\partial \boldsymbol{\sigma}}{\partial x_i}\cdot\mathbf{n} = \frac{\overline{\mathbf{X}}}{\partial x_i},$$
$$\frac{\partial \mathbf{u}}{\partial x_i} = \frac{\partial \overline{\mathbf{u}}}{\partial x_i},$$
$$\frac{\partial^2 \boldsymbol{\sigma}}{\partial x_i \partial x_j}\cdot\mathbf{n} = \frac{\partial^2 \overline{\mathbf{X}}}{\partial x_i \partial x_j},$$
$$\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j} = \frac{\partial^2 \overline{\mathbf{u}}}{\partial x_i \partial x_j}. \quad (30)$$

Initial conditions are as follows:

$$\left.\frac{\partial \mathbf{u}}{\partial x_i}\right|_{t=0} = \mathbf{u}_0,$$
$$\left.\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j}\right|_{t=0} = \mathbf{u}_0,$$
$$\left.\frac{\partial \dot{\mathbf{u}}}{\partial x_i}\right|_{t=0} = \dot{\mathbf{u}}_0,$$
$$\left.\frac{\partial^2 \dot{\mathbf{u}}}{\partial x_i \partial x_j}\right|_{t=0} = \dot{\mathbf{u}}_0. \quad (31)$$

Figure 7: The calculation model of clamped beam.

Figure 8: Mean value of displacement on node (0.03, 0.06, 0.1) m on direction y.

For the boundary conditions of load, the partial derivatives of load to random variables were added to (24). For the boundary conditions of displacement, if a degree of freedom of a node was constrained, the partial derivative of the acceleration was forced to zero at each time step.

4. Examples Analysis

The stochastic dynamic problems for cantilever beam and clamped beam subjected stochastic dynamic load were solved by perturbation-finite volume method. The results were compared with that produced by monte carlo method, which proved the proposed method was correct. To ensure the accuracy of Monte Carlo, according to the law of large numbers, a large number of samples were required to ensure the credibility of the results. Through the results of perturbation-finite volume method, the structural failure probability was estimated, and the value was 44%. The literature [23] recommended that the number of samples should satisfy certain conditions:

$$N \geq \frac{100}{P_f}. \tag{32}$$

Figure 9: Variance of displacement on node (0.03, 0.06, 0.1) m on direction y.

Figure 10: Mean value of equivalent stress on node (0.03, 0.06, 0.0) m.

Therefore, 10000 samples were used in Monte Carlo Method to satisfy the accuracy.

(A) Cantilever Beam. The calculation model of cantilever beam was shown in Figure 2; the parameters of material were shown in Table 1.

The expression of load was

$$P(t) = P_m \exp\left(-\frac{t}{\theta}\right), \tag{33}$$

Figure 11: Variance of equivalent stress on node (0.03, 0.06, 0.0) m.

Figure 12: Mean value of displacement on node (0.03, 0.06, 0.1) m on direction y.

where

$$P_m = 52.4\left(\frac{w^{1/3}}{R}\right)^{1.13} \text{(MPa)}, \quad (34)$$

$$\left(3.4\,\text{MPa} < P_m < 138\,\text{MPa}\right),$$

$$\theta = 0.048w^{1/3}\left(\frac{w^{1/3}}{R}\right)^{-0.23} \text{(ms)}. \quad (35)$$

Here, w and R were the independent random variables. The statistical characteristics of random variables were shown in Table 2.

Figure 13: Variance of displacement on node (0.03, 0.06, 0.1) m on direction y.

Figure 14: Mean value of equivalent stress on node (0.03, 0.06, 0.0) m.

Table 1: The parameters of material.

Density (kg/m^3)	Modulus of elasticity (GPa)	Poisson Ratio
7800	210	0.3

The comparison charts of the two methods were shown in Figures 3, 4, 5, and 6.

(B) Clamped Beam. The basic parameters of loads and materials were the same as example A (Figure 7), and then the comparison charts of the two methods were shown in Figures 8, 9, 10, and 11.

From the results of the two examples, it showed that the results of two methods were basically the same, and the calculation accuracy was verified. It has be seen from the comparison of the two methods that the mean value of the displacement and stress fitted better, while the variance

Table 2: Statistical characteristics of random variables.

Random variables	Distribution	Mean value	Coefficient of variation
w (kg)	Normal distribution	200.0	0.20
R (m)	Normal distribution	50.0	0.20

Figure 15: Variance of equivalent stress on node (0.03, 0.06, 0.0) m.

of displacement and stress produced a small error in a certain period of time. It was because the mean value of the results was second-order accuracy, the variance was first-order accuracy. It was difficult to obtain the higher moments of stochastic field and the error was large, so only the first term was taken, which leads to a lower calculation accuracy. It needed to be improved in the future. Finally, the correctness of perturbation-finite volume method has been proved by the examples.

(C) The Randomness of Structure and Load. The randomness of structure and load was considered in this example. The calculation model was shown in Figure 2. The density of material was 7800 kg/m^3, and the Poisson Ratio was 0.3. The expression of load was the same as (33)–(35), but R was constant and its value was 50 m in this example. Here w and E, structural Modulus of elasticity, were the independent random variables. The statistical characteristics of random variables were shown in Table 3.

The mean value and variance of the responses solved by the Perturbation-Finite Volume Method were shown in Figures 12, 13, 14, and 15.

(D) The Stochastic Dynamic Analysis of Linear Multibody System. The perturbation-finite volume method was successfully applied in the stochastic dynamic analysis of linear multibody system, which was verified by the next example.

Table 3: Statistical characteristics of random variables.

Random variables	Distribution	Mean value	Coefficient of variation
w (kg)	Normal distribution	200.0	0.20
E (GPa)	Normal distribution	200.0	0.002

Table 4: The parameters of material.

Materials	Density (kg/m^3)	Modulus of elasticity (GPa)	Poisson Ratio
Copper	8930	121	0.30
Aluminum	2785	72	0.30
Steel	7800	210	0.30

The calculation model of linear multibody system was shown in Figure 16. The sections of copper and steel were the same.

The parameters of materials were shown in Table 4.

The load and the statistical characteristics of random variables were the same as example A. The mean value and variance of the responses solved by the perturbation-finite volume method were shown in Figures 17, 18, 19, and 20.

5. Conclusions

The examples were presented to verify the perturbation-finite volume method. The results of perturbation-finite volume method were compared with the results of Monte Carlo method, which proved that the proposed method was correct and accurate. Therefore, the proposed method was reasonable and effective. This method, theoretically, can be applied to problems under any forms of random dynamic loads. Meanwhile, the perturbation-finite volume Method was successfully applied to the stochastic dynamic analysis of linear multibody system, which was verified by the example D. A new method has been provided for the stochastic dynamic analysis, and it has laid the foundation for the reliability analysis of dynamics.

The benefit of the finite volume method was that the explicit expressions between random response and basic random variables could be given, which greatly reduced the difficulty of stochastic dynamic analysis. It provided some theoretical references for engineering applications. However, this method has something to be improved yet. For example, in the process of calculation, the grid stress was assumed to be constant, which required a small grid to satisfy the accuracy; it restricted the size of components. So, the development of high-precision grid needed to be studied strongly, and the accuracy of the variance of the response should be improved in the future because of the low accuracy.

Figure 16: The calculation model of linear multibody system.

Figure 17: Mean value of displacement on node (0.0, 0.06, 0.35) m on direction y.

Figure 19: Mean value of equivalent stress on node (0.0, 0.058, 0.213) m.

Figure 18: Variance of displacement on node (0.0, 0.06, 0.35) m on direction y.

Figure 20: Variance of equivalent stress on node (0.0, 0.058, 0.213) m.

Acknowledgment

This work was supported by the Science and Technology on Combustion and Explosion Laboratory Foundation (9140C350406150C35126).

References

[1] Q. Chen and X. Liu, *Stochastic Finite Element Method and Engineering Application*, Southwest Jiaotong University Press, Chengdu, China, 1993.

[2] W. An, W. Zhu, and X. Yan, "Application of stochastic finite element method to analysis of uncertainties," *Journal of Harbin Engineering University*, vol. 23, no. 1, pp. 132–135, 2002.

[3] H. Cui and W. Wen, "Stochastic FEM and its applications in engineering," *Journal of Nanjing University of Aeronautics & Astronautics*, vol. 32, no. 1, pp. 91–98, 2000.

[4] X.-C. Yan and Y. Hua, "Structural system reliability analysis based on stochastic finite element method," *Journal of Wuhan University of Technology*, vol. 32, no. 9, pp. 69–75, 2010.

[5] Z. Rong and S. Lin, "Reliability analysis for connecting-rod based on SFEM," *Ship Sciences and Technology*, vol. 33, no. 9, pp. 68–70, 2011.

[6] L. Yang and Z. Li, "Vectorial cooperative response surface method for stochastic analysis of structures," *Engineering Mechanics*, vol. 29, no. 11, pp. 58–64, 2012.

[7] P. Jing and S. Huang, "Analysis on structure reliability collocation based stochastic finite element method," *Sichuan Building Science*, vol. 38, no. 2, pp. 39–41, 2012.

[8] X. Xu, Z. Lv, and P. Wei, "Probability and compound uncertainty importance measures of structure stochastic analysis and their computational method," *Chinese Journal of Computational Mechanics*, no. 2, pp. 64–71, 2013.

[9] T. Xu, T. Xiang, C. Yang, and R. Zhao, "Uncertainty analysis of long-term deformation for prestress concrete beam," *Journal of Civil Architectural & Environmental Engineering*, vol. 35, no. 1, pp. 111–116, 2013.

[10] A. Saltelli, "Sensitivity analysis for importance assessment," *Risk Analysis*, vol. 22, no. 3, pp. 579–590, 2002.

[11] H. C. Frey and S. R. Patil, "Identification and review of sensitivity analysis methods," *Risk Analysis*, vol. 22, no. 3, pp. 553–578, 2002.

[12] A. Saltelli, S. Tarantola, and F. Campolongo, "Sensitivity analysis as an ingredient of modeling," *Statistical Science*, vol. 15, no. 4, pp. 377–395, 2000.

[13] W.-P. Mu, "An extended precise integration method for response of a structure subjected to evolutionary random exciation," *Journal of Vibration and Shock*, vol. 28, no. 7, pp. 131–142, 2009.

[14] X. Lv, Y. Zhao, X. Y. Huang, G. H. Xia, and X. H. Su, "A matrix-free implicit unstructured multigrid finite volume method for simulating structural dynamics and fluid-structure interaction," *Journal of Computational Physics*, vol. 225, no. 1, pp. 120–144, 2007.

[15] J.-B. Chen and J. Li, "Dynamic response and reliability analysis of non-linear stochastic structures," *Probabilistic Engineering Mechanics*, vol. 20, no. 1, pp. 33–44, 2005.

[16] C. S. Putcha, R. Patev, and M. A. Leggett, "Time-dependent reliability analysis of steel miter gates," *International Journal of Modelling and Simulation*, vol. 23, no. 1, pp. 1–12, 2003.

[17] M. Xiao, C. Gao, and L. Ye, "Dynamic reliability analysis for structure seismic response based on wavelet," *World Earthquake Engineering*, vol. 18, no. 1, pp. 27–33, 2002.

[18] H. Cao, M. Lai, and S. Bai, "Estimations of the reliability of structures based on wavelet transform," *World Information on Earthquake Engineering*, vol. 16, no. 4, pp. 70–77, 2000.

[19] H. Qiao and Z. Lv, "Dynamic reliability analysis of stochastic structures under non-stationary random excitation," *Chinese Journal of Solid Mechanics*, vol. 29, no. 1, pp. 36–40, 2008.

[20] N. Fallah, "A cell vertex and cell centred finite volume method for plate bending analysis," *Computer Methods in Applied Mechanics and Engineering*, vol. 193, no. 33–35, pp. 3457–3470, 2004.

[21] G. Xia and C.-L. Lin, "An unstructured finite volume approach for structural dynamics in response to fluid motions," *Computers and Structures*, vol. 86, no. 7-8, pp. 684–701, 2008.

[22] C. Yan, *The Method of Computational Fluid Dynamics and Its Application*, Beijing University of Aeronautics and Astronautics Press, Beijing, China, 2003.

[23] W. An, Y. Cai, and W. Chen, *Reliability Analysis and Optimization Design of Stochastic Structural System*, Harbin Engineering University Press, Harbin, China, 2005.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 284693, 7 pages
http://dx.doi.org/10.1155/2013/284693

Research Article

Numerical Simulation on Dense Packing of Granular Materials by Container Oscillation

Jun Liu[1,2] and Dongxu You[2]

[1] *Key Laboratory of Ministry of Education for Geomechanics and Embankment Engineering, Hohai University, 1 Xikang Road, Nanjing, Jiangsu 210098, China*
[2] *Institute of Engineering Safety and Disaster Prevention, Hohai University, 1 Xikang Road, Nanjing, Jiangsu 210098, China*

Correspondence should be addressed to Jun Liu; ljun8@263.net

Received 26 August 2013; Accepted 11 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 J. Liu and D. You. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The packing of granular materials is a basic and important problem in geomechanics. An approach, which generates dense packing of spheres confined in cylindrical and cuboidal containers in three steps, is introduced in this work. A loose packing structure is first generated by means of a reference lattice method. Then a dense packing structure is obtained in a container by simulating dropping of particles under gravitational forces. Furthermore, a scheme that makes the bottom boundary fluctuate up and down was applied to obtain more denser packing. The discrete element method (DEM) was employed to simulate the interactions between particle-particle and particle-boundary during the particles' motions. Finally, two cases were presented to indicate the validity of the method proposed in this work.

1. Introduction

The compaction of granular materials is a current subject of keen interest, which is widely used in sciences and engineering; for example, a compaction structure of granular materials can be used to model the structure of granular media, liquids, living cells, glasses, and random media. The spheres packing problem has its roots in geometry and number theory (it is part of Hilbert's 18th problem) [1]. Finding sphere packing with maximal density ais important to most unsolved problems with many applications.

The bulk density of a packed assembly is characterized by the packing fraction, which is the ratio of the particles' volume to the total occupied volume of the system [2]. A collection of uniformly sized balls in Euclidean 3-space is called a packing if no two balls have a common interior point. The packing fraction of equal spheres' packing in three dimensions cannot exceed that of the face-centered cubic packing, $\pi/\sqrt{18}$ (it can exceed this value in the case of sphere packing with varying sizes) and this is the famous Kepler conjecture. Recently, Hales has proved it by a program [3].

At present, many methods have been developed to obtain higher density of particle packing. These methods can be divided into four types: geometric method (GM), discrete element method (DEM), combined geometry and particle motion (CGPM), and Monte Carlo method (MC). In the GM, the particles' motions are not considered and a dense packing was formed by a purely geometrical computation; for example, Place and Mora, Delarue and Jeulin [4, 5] randomly placed a given number of particles within 3D space and a dense packing was obtained by filling the remaining voids with particles. In the DEM, the interaction forces between particles in contact and between particles and boundaries were calculated and the Newtonian second law was employed to describe particles' motions; for example, Siiriä and Yliruusi used DEM to simulate the particles' packing formation [6–8]. In the CGPM, the geometrical method was not only used, but also the particles' motions were considered, for example, Lubachevsky and Stillinger's compression algorithm [9–11], and the spheres grow in size during the process of the simulation at a certain expansion rate until a final state with diverging collision rate is reached. In the Monte Carlo method, a loose particle packing structure is first generated. Then a slight random displacement is exerted on each particle. Furthermore, the new configuration can be

Figure 1: The sketch map of reference cuboid for cylindrical (a) and cuboidal (b) container, respectively.

accepted or not by identifying overlap and variety of potential energy. Finally, a dense packing can be reached by repeating the above course many times [12]. These above mentioned methods mainly focus on reaching the highest packing density instead of describing real mechanical characteristics during the course of forming dense packing.

In this work, the objective is to generate a dense packing structure of particles to model practical dense packing, for instance, structure of concrete, packing of grain, soil, and sand, and so forth, instead of only obtaining the highest packing density. Therefore, the mechanical characteristics of forming dense packing are considered in our study. The Discrete Element Method (DEM) is employed to simulate particles' packing confined in container (cylindrical container and cuboidal container). A loose structure was first generated. Then the dropping of particles onto the bottom of the container due to gravity is simulated. Furthermore, the bottom boundary of the container was fluctuated up and down to reach a more dense packing.

2. Initial Loose Packing

A loose packing structure must be generated in order to simulate the process of particle packing under gravitational forces. Loose packing means that there is no overlap existing between particle and its neighbors or between particle and boundaries. The objective is to avoid interactions among particles and between particles and boundaries at initial state. A simple scheme has been applied to generate loose packing. First, a reference cuboid whose projection in xoy plane is a square has been constructed within container. The symmetrical axis of reference cuboid is the same as that of container. For a cylindrical container, the reference cuboid has the maximal size when it is inscribed to the container (see Figure 1(a)), but for cuboidal container, the maximal size of reference cuboid is the same as that of the container within the range of the container height (see Figure 1(b)). The size of the reference cuboid does not have a determinate value for a certain container because the length of its section must be integral times of the size of lattice. The height of the reference cuboid is determined by the number of particles. Secondly, the reference cuboid is partitioned using cubic lattice. The cubic lattices are regarded as reference lattice for generating coordinates of spheres' centers. The size of lattice is determined by the sizes of particles and container. Finally, a random point is selected within each cubic lattice and the point is regarded as a particle's center. It should be noted that the random point within a cubic lattice must meet the request of no existing intersection between particle's surface and lattice's faces.

Let N_p represent the number of particles; $r_{\max}$ and $r_{\min}$ denote the maximal and minimal radius of the particles; R_{cylinder} and H_{cylinder} are the radius and height of cylinder; the size of the cuboidal container is $L_{\text{cuboid}} \times L_{\text{cuboid}} \times H_{\text{cuboid}}$; and the size of the reference cuboid is $L_r \times L_r \times H_r$. For the cylindrical container, the size of reference lattice is limited by the following equation:

$$2R_{\max} \le L_{\text{lattice}} \le \sqrt{2}R_{\text{cylinder}}. \tag{1}$$

The size of reference cuboid for cylindrical container is

$$L_r = \operatorname{mod}\left(\frac{\sqrt{2}R_{\text{cylinder}}}{L_{\text{lattice}}}\right)L_{\text{lattice}},$$
$$H_r = \operatorname{mod}\left\{\frac{N_p}{\left[\operatorname{mod}\left(\sqrt{2}R_{\text{cylinder}}/L_{\text{lattice}}\right)\right]^2 + 0.5}\right\}L_{\text{lattice}}, \tag{2}$$

where mod means remaining integral part of the result.

For the cuboidal container, the size of reference lattice is

$$2R_{\max} \le L_{\text{lattice}} \le L_{\text{cuboid}}. \tag{3}$$

The size of reference cuboid for cuboidal container is

$$L_r = \operatorname{mod}\left(\frac{L_{\text{cuboid}}}{L_{\text{lattice}}}\right)L_{\text{lattice}},$$
$$H_r = \operatorname{mod}\left\{\frac{N_p}{\left[\operatorname{mod}\left(L_{\text{cuboid}}/L_{\text{lattice}}\right)\right]^2 + 0.5}\right\}L_{\text{lattice}}. \tag{4}$$

3. Mechanical Model

In the current study, the Discrete Element Method (DEM) [13] has been employed to simulate the interactions among

particles and between particles and boundaries. DEM is a finite difference scheme. It is used to study assemblies of individual particles. By monitoring the interaction between particles, the behavior of the material is known.

3.1. Contact Forces. The sketch map of contact model is shown in Figure 2.

The normal contact force F_N and shear contact force F_T can be expressed by

$$F_N = k_d \delta_{ij} + \eta_d \mathbf{V}_{ij} \cdot \mathbf{n}_{ij}, \tag{5}$$

$$F_T = k_\tau \tau_{ij} + \eta_\tau \mathbf{V}_{ij} \cdot \mathbf{t}_{ij}, \tag{6}$$

where k_d, k_τ represent the normal and shear spring constant; η_d, η_τ are the normal and shear damping coefficient due to the dissipative force; $\mathbf{n}_{ij}$ and $\mathbf{t}_{ij}$ are the unit vector in normal direction and tangential direction, respectively, which can be specified by

$$\begin{gathered} \mathbf{n}_{ij} = \frac{\mathbf{X}_i - \mathbf{X}_j}{\left|\mathbf{X}_i - \mathbf{X}_j\right|}, \\ \mathbf{t}_{ij} = \frac{\mathbf{V}_{ij} - \left(\mathbf{V}_{ij} \cdot \mathbf{n}_{ij}\right) \mathbf{n}_{ij}}{\left|\mathbf{V}_{ij} - \left(\mathbf{V}_{ij} \cdot \mathbf{n}_{ij}\right) \mathbf{n}_{ij}\right|}, \end{gathered} \tag{7}$$

where $\mathbf{X}_i$ and $\mathbf{X}_j$ are the positions of particle i and j, respectively; $\mathbf{V}_{ij}$ is the relative velocity of particle i and j at contact points when forming contact, which can be expressed by

$$\mathbf{V}_{ij} = \dot{\mathbf{X}}_i - \dot{\mathbf{X}}_j + \dot{\boldsymbol{\theta}}_i \times \left(r_i \mathbf{n}_{ij}\right) - \dot{\boldsymbol{\theta}}_j \times \left(r_j \mathbf{n}_{ij}\right), \tag{8}$$

where $\dot{\mathbf{X}}_i$, $\dot{\mathbf{X}}_j$, $\dot{\boldsymbol{\theta}}_i$, $\dot{\boldsymbol{\theta}}_j$ are the velocities and angular velocities of the particles i and j, respectively; δ_{ij} and τ_{ij} represent the normal and tangential contact depth. The forces are only applied to the particles in the case of $\delta_{ij} > 0$, which can be expressed as follows:

$$\begin{gathered} \delta_{ij} = r_i + r_j - \left(\mathbf{X}_i - \mathbf{X}_j\right) \cdot \mathbf{n}_{ij}, \\ \tau_{ij} = \left(\mathbf{V}_{ij} \cdot \mathbf{t}_{ij}\right) \Delta t, \end{gathered} \tag{9}$$

where Δt is the time step.

The shear force should be modified because of frictional effects:

$$\left|F_T\right| = \min\left(\left|F_T\right|, \mu \left|F_N\right|\right). \tag{10}$$

In (10), the sign obtained from (6) is preserved. In (10), μ is the coefficient according to Coulomb's frictional theory.

3.2. Particles Motions. The particles' motions are simulated using Newton's second law; that is,

$$\begin{gathered} m_l \ddot{\mathbf{X}}_l = \sum \mathbf{F}_l, \\ I_l \ddot{\boldsymbol{\theta}}_l = \sum \mathbf{M}_l, \end{gathered} \tag{11}$$

where m_l, I_l represent the mass and moment of inertia of particle l, $l = 1, 2, \ldots, N_p$, $\ddot{\mathbf{X}}_l$, $\ddot{\boldsymbol{\theta}}_l$ represent acceleration and angular acceleration of particle l, $\mathbf{F}_l$, $\mathbf{M}_l$ are the resultant force and resultant moment of particle l, and $\sum \mathbf{M}_l = \sum \mathbf{F}_T r_l$. The equations of motion (11) are solved by means of Verlet integrator. See [10] for details.

3.3. Oscillations of Bottom Boundary. To improve the volume fraction, the oscillations of bottom boundary are applied after a dense packing formation under gravity forces. A sinusoidal function has been used to describe increment of containers' bottom boundaries displacements; see the following equation:

$$\Delta U(t) = A \sin\left(\frac{2\pi}{N\Delta t} t\right), \tag{12}$$

where $\Delta U(t)$ is the increment of bottom boundary displacement, A is the amplitude of increment, $N\Delta t$ is the period of oscillation, N is the number of time step within a period, and t is the current time. The increment of displacement must be small enough in each time step to keep stability of calculation. The central coordinates of the bottom of containers vary with calculation time; that is,

$$\mathbf{P}(t) = \mathbf{P}_0 + \Delta U(t) \mathbf{n}_d, \tag{13}$$

where $\mathbf{P}(t)$ is the central position vector of the bottom boundary of container, $\mathbf{n}_d$ is a normal vector of the bottom boundary, and $\mathbf{P}_0$ is the initial value of $\mathbf{P}(t)$ (when $t = 0$).

4. Two Cases

A computer program has been developed using the above-mentioned algorithm. To verify validity of the method proposed in this work, two cases are introduced for cylindrical and cuboidal containers, respectively.

4.1. Initial Loose Packing. The parameters used in initial loose packing simulation are listed in Table 1. The configurations of loose packing structures are shown in Figure 3.

4.2. Dense Packing under Gravity Forces. The dense packing under gravity forces process is simulated using the methods described in Section 3. Parameters used in the simulation are listed in Table 2. The simulation results are shown in Figures 4 and 5.

4.3. Dense Packing Induced by Oscillations of Containers. The packing fraction has been calculated in the oscillating process. The sum of particles volume is a constant when oscillating containers, but the volume occupied by particles within the container varies with the container oscillating. So, the packing fraction is calculated approximately for the cuboidal container as follows:

$$f_{p-v} = \frac{V_p}{\left(a^2\left(z_{p\max} - \Delta U(t) + r\right)\right)}, \tag{14}$$

Figure 2: The sketch map of calculating contact force ((a) two particles in contact; (b) the mechanical model of contact. N-normal direction, T-tangential direction).

Figure 3: Initial loose packing configuration for cylindrical (a) and cuboidal (b) container, respectively.

where V_p is the sum of particles volume, a is the side length of the bottom boundary, $z_{p\,\max}$ is the maximal z coordinate of particles during particles motions, and r is the corresponding radius of the particle. For cylindrical container, replace a^2 by $\pi {R_{\text{cylinder}}}^2$.

Figures 6(a), 6(b), 6(c), and 6(d) are the simulation results before and after container oscillations for cylindrical and cuboidal container, respectively. In addition, the dense packings of particles with four kinds of grain size distribution were also simulated for cylindrical and cuboidal container, respectively, as shown in Figure 7. Figure 7 shows that the particles with smaller sizes can reach a larger volume fraction both for cylindrical and cuboidal containers. Furthermore, the effects of amplitude of oscillation on volume in the case of keeping other factors the same were analyzed. An optimal oscillation amplitude exists as shown in Figure 8. The frequency of oscillation was also analyzed in this work. In some values of oscillation frequency, the dense packing structure can reach a higher volume fraction. However, there is no definite relationship between oscillation frequency and the resulted volume fraction, as shown in Figure 9.

5. Conclusions and Discussions

The simulating results implicate that oscillation of container can significantly affect the packing density. However, different oscillation amplitudes and frequencies have different effects on packing density. For a given particles system, there exists an optimal amplitude of oscillation corresponding to the maximum volume fraction. But there is no definite relationship between the frequency of oscillation and the volume fraction.

The packing density from our simulation does not exceed that from the method in the view of pure geometry. The main reason lies that the mechanical behaviors of interactions between particles and between particles and boundaries

Figure 4: The dropping process of particles under gravity forces within a cylindrical container ((a), (b), (c) and (d) are the configuration at 0.2 s, 0.35 s, 0.65 s, and 1 s, resp.).

Figure 5: The dropping process of particles under gravity forces within a cuboidal container ((a), (b), (c), and (d) are the configuration at 0.2 s, 0.35 s, 0.65 s, and 1 s, resp.).

Figure 6: The dense packing of particles in cylindrical and cuboidal container before and after container oscillation ((a), (c) before container oscillation; (b), (d) after container oscillation).

Table 1: Parameters used in the simulation of initial loose packing.

	N_p	$R_{cylinder}$ (m)	$H_{cylinder}$ (m)	L_{cuboid} (m)	H_{cuboid} (m)	L_r (m)	H_r (m)	$L_{lattice}$ (m)	r_{min} (m)	r_{max} (m)
Cylindrical container	800	0.03	0.15	—	—	0.04	0.50	0.01	0.002	0.005
Cuboidal container	900	—	—	0.06	0.16	0.06	0.25	0.01	0.002	0.005

Table 2: Parameters used in dense packing simulation.

Density (Kg/m^3)	Elastic modulus (N/m)	Time length (s)	Time step (s)	Contact damping
2700	1.5×10^8	1	10^{-7}	0.27

Figure 7: The relationship between volume fraction and average size of particles.

Figure 8: The relationship between volume fraction and oscillation amplitude.

Figure 9: The relationship between volume fraction and oscillation frequency.

was simulated such as motions, collisions, and frictions in condition of container oscillation.

It should be noted that there exists slight overlap in the final packing structure since the gravitational forces of particles have been considered in the simulation. However, the penetration depth is small enough (10^{-6} m in the simulation). Therefore, the effects of small penetration depth in particles on the packing structure can be ignored in the simulation.

Acknowledgment

This work was supported by the Grant from the National Natural Science Foundation of China (no. 51174076). This grant is gratefully acknowledged.

References

[1] D. J. Daley, "Packings and approximate packings of spheres," Tech. Rep. 104, National Institute of Statistical Sciences, Research Triangle Park, NC, USA, http://www.niss.org.

[2] J. A. Elliott, A. Kelly, and A. H. Windle, "Recursive packing of dense particle mixtures," *Journal of Materials Science Letters*, vol. 21, no. 16, pp. 1249–1251, 2002.

[3] T. C. Hales, "A computer verification of the Kepler conjecture," *Proceedings of the International Congress of Mathematicians*, vol. 3, no. 1–3, pp. 1–10, 2002.

[4] D. Place and P. Mora, "A random lattice solid model for simulation of fault zone dynamics and fracture processes," in *Bifurcation and Localisation Theory for Soils and Rocks*, H.-B. Muhlhaus, A. V. Dyskin, and E. Pasternak, Eds., A.A. Balkema, Rotterdam, The Netherlands, 1999.

[5] A. Delarue and D. Jeulin, "Multi-scale simulation of spherical aggregates," *Image Analysis & Stereology*, vol. 20, pp. 181–186, 2001.

[6] S. Siiriä and J. Yliruusi, "Particle packing simulations based on Newtonian mechanics," *Powder Technology*, vol. 174, no. 3, pp. 82–92, 2007.

[7] Y. Shi and Y. Zhang, "Simulation of random packing of spherical particles with different size distributions," *Applied Physics A*, vol. 92, no. 3, pp. 621–626, 2008.

[8] X. L. Deng and R. N. Davé, "Dynamic simulation of particle packing influenced by size, aspect ratio and surface energy," *Granular Matter*, vol. 15, pp. 401–415, 2013.

[9] B. D. Lubachevsky and F. H. Stillinger, "Geometric properties of random disk packings," *Journal of Statistical Physics*, vol. 60, no. 5-6, pp. 561–583, 1990.

[10] A. R. Kansal, S. Torquato, and F. H. Stillinger, "Computer generation of dense polydisperse sphere packings," *Journal of Chemical Physics*, vol. 117, no. 18, pp. 8212–8218, 2002.

[11] A. Donev, S. Torquato, F. H. Stillinger, and R. Connelly, "Jamming in hard sphere and disk packings," *Journal of Applied Physics*, vol. 95, no. 3, pp. 989–999, 2004.

[12] V. A. Luchnikov, M. L. Gavrilova, N. N. Medvedev, and V. P. Voloshin, "The Voronoi-Delaunay approach for the free volume analysis of a packing of balls in a cylindrical container," *Future Generation Computer Systems*, vol. 18, no. 5, pp. 673–679, 2002.

[13] P. A. Cundall and O. D. L. Strack, "A discrete numerical model for granular assemblies," *Geotechnique*, vol. 29, no. 1, pp. 47–65, 1979.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 280453, 7 pages
http://dx.doi.org/10.1155/2013/280453

Research Article

Research on Flexible Joint Stiffness Test and Trajectory Planning of Space Manipulator

Yongtai Yang,[1] Jili Rong,[1] Jian Li,[2] and Ling Tang[3]

[1] *Department of Mechanics, School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China*
[2] *Automotive and Transportation Engineering Institute, Guangxi University of Science and Technology, Liuzhou 545006, China*
[3] *China Academy of Space Technology, Beijing 100094, China*

Correspondence should be addressed to Jili Rong; rongjili@bit.edu.cn

Received 23 August 2013; Revised 15 October 2013; Accepted 23 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Yongtai Yang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Both Natural Coordinate Formulation describing rigid bodies and Absolute Nodal Coordinate Formulation describing flexible bodies are used to model a flexible manipulator with flexible joint and flexible link. The torsional stiffness of flexible joint is tested using a specialized stiffness test equipment, and then the nonlinear torsional stiffness is determined by fitting the experimental data. A new trajectory planning function called the cosine-based function is proposed to design the joint trajectory, which is smoother than the fifth-polynomial and cycloidal motion functions. Finally, a one-link manipulator with flexible joint and flexible link is used to compare the performance of the three trajectory planning functions. Results show that residual vibration can be remarkably reduced by the proposed cosine-based function, which exhibits a significantly better performance than the fifth-polynomial and cycloidal motion functions.

1. Introduction

A space manipulator undertakes tasks of cabin translocation, transfer, and installation of equipment, as well as serves as an astronaut auxiliary in a station system. It serves an important function in space engineering, with the advantages of lighter structure, lower launching cost, higher operational speed, greater payload-to-manipulator-weight ratio, smaller actuators, lower energy consumption, and better maneuverability. However, the greatest disadvantage of the space manipulator is the vibration problem due to the flexibility of the joint and link. For instance, it took many cumulative hours in order to damp down the residual vibration in the remote manipulator system in a Space Station-assembly Shuttle flight, which occupied 20% to 30% of the total time [1]. The residual vibration significantly reduces the end-point accuracy of the manipulator. Therefore, the best scheme is in which the flexible manipulator completes the required move with minimal residual vibration.

Joint flexibility and link flexibility are basic reasons of manipulator vibration, which cannot be neglected in the dynamic modeling. Spong [2] first modeled the flexible joint as a linear spring, and this modeling technique was adopted by most researchers [3, 4]. However, simple linear spring is not capable of describing the flexible joint precisely. In the dynamic modeling of flexible link, the assumed mode method is typically used by most researchers. Martins et al. [5] and Tso et al. [6] studied a flexible manipulator using the Lagrangian equation and the assumed mode method. In assumed mode model formulation, the link flexibility is usually represented by a truncated finite modal series in terms of spatial mode eigenfunctions and time-varying mode amplitudes. The main drawback of this method is that there are several ways to choose link boundary conditions and mode eigenfunctions. Moreover, the assumed mode method is only capable of describing small deformation, not large deformation. The previous works solely focused on either joint flexibility or link flexibility.

Several different approaches have been suggested to reduce residual vibration, which can be categorized as active control, passive control, and trajectory planning methods. Different trajectories of joint space, significantly affect the vibration of a flexible manipulator. The fifth-polynomial function is widely used to plan the joint trajectory, but it

Figure 1: One-link manipulator with flexile joint and flexible link.

Figure 2: Space rigid body described by NCF.

contains the undesirable higher harmonics that excite the system resonances. Cycloidal motion [7] is another trajectory planning function of joint space, whose performance is not clear.

Due to the above deficiencies in the previous work, both Natural Coordinate Formulation (NCF) [8] describing rigid bodies and Absolute Nodal Coordinate Formulation (ANCF) [9, 10] especially suitable for flexible bodies with large deformation are applied to model the flexible manipulator in this paper. The flexibilities of joint and link are simultaneously considered in the dynamic model. The torsional stiffness of flexible joint is tested by a specialized stiffness test equipment, and the nonlinear torsional stiffness is finally determined by fitting the experiment data, which is more precise than the constant torsional stiffness in describing the flexible joint. To suppress the residual vibration, a new planning function called the cosine-based function is proposed to design the joint trajectory. Finally, a one-link manipulator with flexible joint and flexible link is used to verify the effectiveness of the proposed cosine-based function.

This paper is organized as follows. Section 2 is dedicated to the dynamic modeling of a flexible manipulator by NCF and ANCF, with flexible joint and flexible link considered. Section 3 is devoted to the torsional stiffness test for the flexible joint. In Section 4, two trajectory planning functions of the joint are introduced first and then a new planning function called the cosine-based function is proposed. Section 5 gives simulation results and analyses. Finally, the related conclusions are drawn in Section 6.

2. Dynamic Modeling of a Flexible Manipulator

The one-link manipulator with flexible joint and flexible link is shown in Figure 1. The motor rotor and the joint shell are rigid bodies modeled by NCF. They are connected by a torsional spring, whose stiffness is determined from the test experiment introduced in Section 3. The link is a flexible body modeled by ANCF.

NCF is adopted to describe rigid bodies, as shown in Figure 2. The position vectors of two fixed points and two unit vectors on the rigid body are set as generalized coordinates expressed as

$$\mathbf{q} = \left[\mathbf{r}_i^{\mathrm{T}} \quad \mathbf{r}_j^{\mathrm{T}} \quad \mathbf{u}^{\mathrm{T}} \quad \mathbf{v}^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{1}$$

The global position vector of an arbitrary point on the rigid body can be written as

$$\mathbf{r} = \mathbf{C}\mathbf{q}, \tag{2}$$

where $\mathbf{C}$ is the element shape function expressed as

$$\mathbf{C} = \left[(1 - c_1)\,\mathbf{I}_3 \quad c_1\mathbf{I}_3 \quad c_2\mathbf{I}_3 \quad c_3\mathbf{I}_3\right], \tag{3}$$

where $\mathbf{I}_3$ is the unit matrix and c_1, c_2, and c_3 are constant components of the local coordinate.

According to the principle of virtual work, the constant mass matrix can be determined as

$$\mathbf{M} = \int_V \rho \mathbf{C}^{\mathrm{T}}\mathbf{C}\, dV. \tag{4}$$

The three-dimensional beam element of two nodes described by ANCF, as shown in Figure 3, as first proposed by Shabana and Yakoub [9, 10], is used to describe the flexible link. The generalized coordinates of the beam element are written as

$$\mathbf{e} = \left[\mathbf{r}_i^{\mathrm{T}}, \mathbf{r}_{i,x}^{\mathrm{T}}, \mathbf{r}_{i,y}^{\mathrm{T}}, \mathbf{r}_{i,z}^{\mathrm{T}}, \mathbf{r}_j^{\mathrm{T}}, \mathbf{r}_{j,x}^{\mathrm{T}}, \mathbf{r}_{j,y}^{\mathrm{T}}, \mathbf{r}_{j,z}^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{5}$$

The global position vector of an arbitrary point on the body can be written as

$$\mathbf{r} = \mathbf{S}\mathbf{e}, \tag{6}$$

where $\mathbf{S}$ is the element shape function expressed as

$$\mathbf{S} = \left[s_1\mathbf{I}_3, ls_2\mathbf{I}_3, ls_3\mathbf{I}_3, ls_4\mathbf{I}_3, s_5\mathbf{I}_3, ls_6\mathbf{I}_3, ls_7\mathbf{I}_3, ls_8\mathbf{I}_3\right], \tag{7}$$

where $\mathbf{I}_3$ is the unit matrix. $s_1 = 1 - 3\xi^2 + 2\xi^3$, $s_2 = \xi - 2\xi^2 + \xi^3$, $s_3 = \eta - \xi\eta$, $s_4 = \zeta - \xi\zeta$, $s_5 = 3\xi^2 - 2\xi^3$, $s_6 = -\xi^2 + \xi^3$, $s_7 = \xi\eta$, $s_8 = \xi\zeta$; $\xi = x/l$, $\eta = y/l$, and $\zeta = z/l$. x, y, and z are the local

Figure 3: Three-dimensional beam element of two nodes described by ANCF.

coordinates of the element. $x \in [\,0 \;\; l\,]$, and l is the initial length of the element.

By using the first class of Lagrange's equation, the equations of the motions for constrained rigid-flexible multibody systems can be expressed in a compact form as a set of differential and algebraic equations written as [11]

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} &= \mathbf{Q}(\mathbf{q}) - \mathbf{F}(\mathbf{q}), \\ \mathbf{C}(\mathbf{q}, t) &= \mathbf{0}, \end{aligned} \tag{8}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{q}$ denotes the generalized coordinates, $\mathbf{C}$ stands for the constraint equations, $\mathbf{C}_{\mathbf{q}}$ is the Jacobi of constraint equations, $\boldsymbol{\lambda}$ is the Lagrangian multiplier, $\mathbf{Q}$ is the generalized external force, and $\mathbf{F}$ is the generalized elastic force. The calculation of the elastic force and its Jacobi can be referred to in [12, 13].

Several algorithms can be used to solve differential-algebraic equations of multibody system numerically. These algorithms include the Baumgarte, Newmark, generalized-a, and augmented Lagrangian methods. In this paper, the iteration strategy of the generalized-a method proposed by Arnold and Brüls [14] and the scaling technology proposed by Bottasso et al. [15] are adopted for numerically solving the differential-algebraic equations efficiently and precisely.

3. Torsional Stiffness Test for the Flexible Joint

The specialized stiffness test equipment developed by our research group is used to test the torsional stiffness of the flexible joint. The optical schematic of double angle is shown in Figure 4. The laser beam emitted by the laser locates at point A_0 on the screen after being reflected by the mirror. According to the basic optical principle, if the mirror rotates by α degree, the reflected laser beam rotates by 2α degree, and the laser spot on the screen moves from point A_0 to point A_1. Let the moving distance of the laser spot be Δ_A, and let the distance from the mirror to the screen be S_A. S_A is generally significantly greater than Δ_A; that is, $S_A \gg \Delta_A$. Thus, α has a small value. According to the triangle geometry, the rotation angle of the mirror can be obtained as

Figure 4: Optical schematic of double angle.

Figure 5: Schematic of torsional stiffness test.

$$\alpha = \frac{\Delta_A}{2S_A}. \tag{9}$$

The stiffness test schematic is shown in Figure 5. Let A be the measuring point and let B be the reference point. Two mirrors are attached to points A and B separately. The two lasers emit laser beams, and the laser beams, respectively, locate at points A_0 and B_0 on the screen after being reflected by the mirrors. When torque τ is applied to the joint, the two mirrors rotate by α_A and α_B. According to the basic optical principle, the reflected laser beams rotate by $2\alpha_A$ and $2\alpha_B$, and the laser spots on the screen move from points A_0 and B_0 to points A_1 and B_1, respectively. The moving distances of laser spots are Δ_A and Δ_B. Then, the rotational angle of the measuring point A relative to the rotational angle of the datum point B can be obtained as

$$\alpha = \alpha_A - \alpha_B = \frac{\Delta_A}{2S_A} - \frac{\Delta_B}{2S_B}. \tag{10}$$

A mechanical testing machine is used to apply torque τ_i to the joint, after which the relative angle α_i is measured. The typical τ-α curve is shown in Figure 6.

Figure 6: Torque-angle curve.

Figure 7: Nonlinear stiffness of the joint.

The torque-angle curve has two segments: nonlinear and linear. The quadratic polynomial is adopted to fit the nonlinear segment, whereas the least square method is adopted to fit the linear segment. Finally, the relationship between torque and angle can be expressed by the following formula:

$$\tau = \begin{cases} \tau_1 \left(\dfrac{\alpha}{\alpha_1}\right)^2, & |\alpha| < \alpha_1, \\ \tau_1 + K_1 (\alpha - \alpha_1), & |\alpha| \geq \alpha_1, \end{cases} \tag{11}$$

where (α_1, τ_1) is the turning point between nonlinear segment and linear segment and K_1 is the stiffness of linear segment.

The nonlinear stiffness of the joint, as shown in Figure 7, can be obtained by differentiating the above equation.

4. Trajectory Planning of the Joint

In engineering, smooth and continuous functions are typically adopted to plan the joint trajectory, the first and second derivatives of which are also smooth and continuous, such as the fifth-polynomial and cycloidal motion functions. The six boundary conditions of angle, angular velocity, and angular acceleration of the joint are as follows:

$$\begin{aligned} &\tilde{\theta}(0) = \theta_0, \qquad \dot{\tilde{\theta}}(0) = 0, \qquad \ddot{\tilde{\theta}}(0) = 0, \\ &\tilde{\theta}(t_f) = \theta_f, \qquad \dot{\tilde{\theta}}(t_f) = 0, \qquad \ddot{\tilde{\theta}}(t_f) = 0, \end{aligned} \tag{12}$$

where t_f is the planning time, θ_0 is the initial angle of joint, and θ_f is the final angle of joint.

(1) The fifth-polynomial is adopted to plan the joint trajectory. The polynomial coefficients are obtained by the six boundary conditions described by (12). Finally, the angle, angular velocity, and angular acceleration of the joint are as follows:

$$\begin{aligned} \tilde{\theta} &= (\theta_f - \theta_0)\left[6\left(\frac{t}{t_f}\right)^5 - 15\left(\frac{t}{t_f}\right)^4 + 10\left(\frac{t}{t_f}\right)^3\right] + \theta_0, \\ \dot{\tilde{\theta}} &= \frac{(\theta_f - \theta_0)}{t_f}\left[30\left(\frac{t}{t_f}\right)^4 - 60\left(\frac{t}{t_f}\right)^3 + 30\left(\frac{t}{t_f}\right)^2\right], \\ \ddot{\tilde{\theta}} &= \frac{(\theta_f - \theta_0)}{t_f^2}\left[120\left(\frac{t}{t_f}\right)^3 - 180\left(\frac{t}{t_f}\right)^2 + 60\left(\frac{t}{t_f}\right)\right]. \end{aligned} \tag{13}$$

(2) Cycloidal motion is adopted to plan the joint trajectory expressed as

$$\tilde{\theta}(t) = (\theta_f - \theta_0)\left[\frac{t}{t_f} - \frac{1}{2\pi}\sin\left(\frac{2\pi t}{t_f}\right)\right] + \theta_0. \tag{14}$$

The angular velocity and angular acceleration of the joint can be obtained by differentiating the above equation and can be determined as

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= \frac{(\theta_f - \theta_0)}{t_f}\left[1 - \cos\left(\frac{2\pi t}{t_f}\right)\right], \\ \ddot{\tilde{\theta}}(t) &= \frac{(\theta_f - \theta_0)}{t_f^2}\left[2\pi \sin\left(\frac{2\pi t}{t_f}\right)\right]. \end{aligned} \tag{15}$$

Cycloidal motion can evidently satisfy all boundary conditions described by (12).

(3) In this paper, a cosine-based function is proposed to plan the joint trajectory, and it can be written as

$$\tilde{\theta}(t) = (\theta_f - \theta_0)\left[a_0 + a_1\cos\left(\frac{\pi t}{t_f}\right) + a_3\cos\left(\frac{3\pi t}{t_f}\right)\right] + \theta_0. \tag{16}$$

According to the six boundary conditions described by (12), the three unknown coefficients a_0, a_1, and a_3 can be determined. Finally, the angle, angular velocity, and angular acceleration of the joint are as follows:

$$\begin{aligned} \tilde{\theta}(t) &= (\theta_f - \theta_0)\left[\frac{1}{2} - \frac{9}{16}\cos\left(\frac{\pi t}{t_f}\right) + \frac{1}{16}\cos\left(\frac{3\pi t}{t_f}\right)\right] + \theta_0, \\ \dot{\tilde{\theta}}(t) &= \frac{(\theta_f - \theta_0)}{t_f}\left[\frac{9}{16}\pi\sin\left(\frac{\pi t}{t_f}\right) - \frac{3}{16}\pi\sin\left(\frac{3\pi t}{t_f}\right)\right], \\ \ddot{\tilde{\theta}}(t) &= \frac{(\theta_f - \theta_0)}{t_f^2}\left[\frac{9}{16}\pi^2\cos\left(\frac{\pi t}{t_f}\right) - \frac{9}{16}\pi^2\cos\left(\frac{3\pi t}{t_f}\right)\right]. \end{aligned} \tag{17}$$

Table 1: Maximal angular velocity and angular acceleration of the three planning functions.

	Fifth-polynomial function	Cycloidal motion function	Cosine-based function
$\dot{\tilde{\theta}}_{\max}$	$\frac{15}{8}\left(\frac{\theta_f-\theta_0}{t_f}\right)$	$2\left(\frac{\theta_f-\theta_0}{t_f}\right)$	$\frac{3}{4}\pi\left(\frac{\theta_f-\theta_0}{t_f}\right)$
$\ddot{\tilde{\theta}}_{\max}$	$\frac{10}{\sqrt{3}}\left(\frac{\theta_f-\theta_0}{t_f^2}\right)$	$2\pi\left(\frac{\theta_f-\theta_0}{t_f^2}\right)$	$\frac{\sqrt{3}}{2}\pi^2\left(\frac{\theta_f-\theta_0}{t_f^2}\right)$

Table 2: Parameters of the flexible manipulator link.

Length	Inner radius	Outer radius	Young's modulus	Poisson's ratio
3.5 (m)	6 (cm)	8 (cm)	90 (GPa)	0.3

The maximal angular velocity and angular acceleration of the fifth-polynomial, cycloidal motion, and cosine-based functions are shown in Table 1. The ratio of the maximal angular velocity of the three functions is 1 : 1.07 : 1.26, and the ratio of the maximal angular acceleration of the three functions is 1 : 1.09 : 1.48.

The planning time t_f is set to be 1 [s], the initial angle θ_0 is set to be 0°, and the final angle θ_f is set to be 5°.

Figures 8, 9, and 10 show the angle, angular velocity, and angular acceleration of the three planning functions, respectively. The angle, angular velocity, and angular acceleration of the cosine-based function are smoother than those of the fifth-polynomial and cycloidal motion functions at the initial and final times, whereas the maximal angular velocity and the maximal angular acceleration of the cosine-based function are larger than those of the fifth-polynomial and cycloidal motion functions.

5. Simulation Studies

The planning time is set to be 1 [s], the simulation time is set to be 2 [s], the initial angle of the joint is set to be 0°, and the final angle of the joint is set to be 5°.

The geometrical parameters and mechanical parameters of the flexible link are shown in Table 2. Also, six ANCF three-dimensional beam elements are used to discretized the flexible link.

The above three planning functions are used to plan the joint trajectory. The motor is driven by the speed mode; that is, the motor rotor is assumed to be capable of absolutely tracking the given trajectory using the three planning functions, respectively. To determine the effects of the three planning functions on manipulator vibration, two cases are studied: (1) with joint flexibility considered and (2) with joint flexibility and link flexibility considered.

Figure 11 shows the angle errors of the three planning functions with joint flexibility considered. In the course of motion ($0\,\mathrm{s} \le t \le 1\,\mathrm{s}$), the maximal angle errors of the fifth-polynomial, cycloidal motion, and cosine-based functions are, respectively, 0.056°, 0.052°, and 0.056°, whereas after motion ($1\,\mathrm{s} \le t \le 2\,\mathrm{s}$), the residual angle errors are, respectively, 0.028°, 0.019°, and 0.002°. The tracking errors caused by the three functions are evidently similar in the course of motion, but the residual vibration caused by the proposed cosine-based function is far less than that caused by the fifth-polynomial and cycloidal motion functions after motion.

Figure 12 shows the angle errors of the joint of the three planning functions with joint flexibility and link flexibility considered. Figure 13 shows the vibration of the endpoint of the three planning functions with joint flexibility and link flexibility considered. In the course of motion ($0\,\mathrm{s} \le t \le 1\,\mathrm{s}$), the maximal angle errors of the fifth-polynomial, cycloidal motion, and cosine-based functions are, respectively, 0.069°,

Figure 8: Joint angle of the three planning functions.

Figure 9: Joint angular velocity of the three planning functions.

Figure 10: Joint angular acceleration of the three planning functions.

Figure 11: Angle errors of the joint (with joint flexibility considered).

Figure 12: Angle errors of the joint (with joint flexibility and link flexibility considered).

Figure 13: Vibration of the endpoint (with joint flexibility and link flexibility considered).

0.070°, and 0.095°, whereas the maximal vibration amplitudes of the endpoint are, respectively, 1.84 cm, 1.87 cm, and 2.38 cm. After motion ($1\,\mathrm{s} \le t \le 2\,\mathrm{s}$), the maximal residual angle errors are, respectively, 0.060°, 0.040°, and 0.012°, whereas the maximal residual vibration amplitudes of the endpoint are, respectively, 1.43 cm, 1.03 cm, and 0.10 cm. The tracking errors caused by the three functions are evidently similar in the course of motion, but the error caused by the cosine-based function is a lightly higher. However, the residual vibration caused by the cosine-based function is significantly less than that caused by the fifth-polynomial and cycloidal motion functions after motion.

The above analysis shows that, whether with joint flexibility considered or with both joint flexibility and link flexibility considered, the tracking errors caused by the three planning functions are similar in the course of motion, but the residual vibration caused by the cosine-based function proposed in this paper is significantly less than that caused by the fifth-polynomial and cycloidal motion functions after motion. We can thus conclude that the cosine-based planning function proposed in this paper is more effective in weakening the residual vibration than the fifth-polynomial planning and cycloidal motion planning functions.

6. Conclusions

The flexible manipulator is modeled by NCF and ANCF, considering the flexibility of joint and link. The nonlinear torsional stiffness is finally determined by fitting the experimental data. Moreover, a cosine-based function is proposed to suppress the residual vibration. The maximal angular velocity ratio and angular acceleration ratio of the fifth-polynomial, cycloidal motion, and cosine-based functions are, respectively, $1:1.07:1.26$ and $1:1.09:1.48$. A one-link manipulator with flexible joint and flexible link is used to evaluate the proposed trajectory planning function. Results show that the residual vibration by the proposed cosine-based function is reduced to 7.1% of that by the fifth-polynomial function and to 9.7% of that by the cycloidal motion function.

Notably, the effectiveness of the proposed cosine-based function in weakening the residual vibration is only verified by numerical simulations. The corresponding experimental validation is left for our future work.

Acknowledgments

This work is supported by Grant 10972033 of the Natural Science Foundation of China. This work is also supported by Grant 20100141107 of the China Academy of Space Technology.

References

[1] S. Dubowsky, "Dealing with vibrations in the deployment structures of space robotic systems," in *Proceedings of the 5th International Conference on Adaptive Structures*, pp. 5–7, Sendai International Center, Sendai, Japan, 1994.

[2] M. W. Spong, "Modeling and control of elastic joint robots," *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME*, vol. 109, no. 1, pp. 310–319, 1987.

[3] S. K. Ider and M. K. Özgören, "Trajectory tracking control of flexible-joint robots," *Computers and Structures*, vol. 76, no. 6, pp. 757–763, 2000.

[4] G. Magnani, P. Rocco, and A. Rusconi, "Modeling and position control of a joint prototype of DEXARM," in *Proceedings of the 10th International Workshop on Advanced Motion Control (AMC' 08)*, pp. 562–567, March 2008.

[5] J. Martins, M. Ayala Botto, and J. S. Da Costa, "Modeling of flexible beams for robotic manipulators," *Multibody System Dynamics*, vol. 7, no. 1, pp. 79–100, 2002.

[6] S. K. Tso, T. W. Yang, W. L. Xu, and Z. Q. Sun, "Vibration control for a flexible-link robot arm with deflection feedback," *International Journal of Non-Linear Mechanics*, vol. 38, no. 1, pp. 51–62, 2003.

[7] A. Abe, "Trajectory planning for residual vibration suppression of a two-link rigid-flexible manipulator considering large deformation," *Mechanism and Machine Theory*, vol. 44, no. 9, pp. 1627–1639, 2009.

[8] D. J. García and E. Bayo, *Kinematic and Dynamic Simulation of Multibody Systems: the Real-Time Challenge*, Springer, New York, NY, USA, 1994.

[9] A. A. Shabana and R. Y. Yakoub, "Three dimensional absolute nodal coordinate formulation for beam elements: theory," *Journal of Mechanical Design*, vol. 123, no. 4, pp. 606–613, 2001.

[10] R. Y. Yakoub and A. A. Shabana, "Three dimensional absolute nodal coordinate formulation for beam elements: implementation and applications," *Journal of Mechanical Design*, vol. 123, no. 4, pp. 614–621, 2001.

[11] A. A. Shabana, *Computational Dynamics*, John Wiley & Sons, New York, NY, USA, 3rd edition, 2010.

[12] D. García-Vallejo, J. Mayo, J. L. Escalona, and J. Domínguez, "Efficient evaluation of the elastic forces and the jacobian in the absolute nodal coordinate formulation," *Nonlinear Dynamics*, vol. 35, no. 4, pp. 313–329, 2004.

[13] C. Liu, Q. Tian, and H. Hu, "Efficient computational method for dynamics of flexible multibody systems based on absolute nodal coordinate," *Journal of Theoretical and Applied Mechanics*, vol. 42, no. 6, pp. 1197–1205, 2010 (Chinese).

[14] M. Arnold and O. Brüls, "Convergence of the generalized-α scheme for constrained mechanical systems," *Multibody System Dynamics*, vol. 18, no. 2, pp. 185–202, 2007.

[15] C. L. Bottasso, D. Dopico, and L. Trainelli, "On the optimal scaling of index three DAEs in multibody dynamics," *Multibody System Dynamics*, vol. 19, no. 1-2, pp. 3–20, 2008.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 257457, 13 pages
http://dx.doi.org/10.1155/2013/257457

Research Article

Case Study on Influence of Step Blast-Excavation on Support Systems of Existing Service Tunnel with Small Interval

Shaorui Sun, Ling Yue, Jimin Wu, Jin Liu, and Jihong Wei

College of Earth Science and Engineering, Hohai University, Nanjing 210098, China

Correspondence should be addressed to Shaorui Sun; ssrfish@hhu.edu.cn

Received 19 August 2013; Accepted 8 October 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Shaorui Sun et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

During the construction of newly built tunnel (NBT) adjacent to the existing service tunnel (EST), stability of the EST with small interval is affected by vibration waves which are caused by blasting load. The support structures of the EST will be cracked and damaged, while the unreasonable blast-excavation methods are adopted. Presently, the studies on behavior of support structure in the EST under blasting load are not totally clear, especially for the bolts system. Besides, the responses of support structure on blasting load are lacking comprehensive research. In this paper, New Zuofang tunnel is taken as a study case to study the influence of step blast-excavation in NBT on support structures of the EST through field experiment and numerical simulation. Some data, such as blasting vibration velocity (BVV) and frequency of support structures, are obtained through field measurement. Based on these data, the formula of BVVs is obtained. Research on stability of tunnel support structures affected by step blast-excavation is conducted using numerical simulation method. The dynamic-plastic constitutive model is adopted in the software ABAQUS to assess safety of support structures. The range and degree of damage for the support structures are obtained. In addition, change laws of axial force and stress with time for the bolts are analyzed.

1. Introduction

The structural types of tunnel with small interval (less than 1.5 times of tunnel width) exist widely in the tunnelling projects. The major risks resulted from vibration failure induced by blasting load in NBT when they are excavated by the drill and blast method. The blasting load in NBT will be easy to cause the adverse influence of blast-induced vibrations in the EST lining. How to protect the support structures of the EST during blast-excavation in NBT has become to a significant and valuable research work. It is necessary to carry out both experiments and numerical analysis to research the impact of BVVs on the support structures of the EST.

The damage of concrete lining in the EST under blasting load has been studied by lots of researchers. For example, Hisatake et al. [1] have researched a dynamic method with respect to effects of adjacent blast operation on vibration behavior of the EST. Krajeinovie [2] has introduced the damage acoustic wave of surrounding rock and obtained an important result that damage cracking of surrounding rock can be expressed by change rate of acoustic wave in rock masses. Kuszmaul [3] has obtained the "KUS damage" constitutive model for fragmentation of rock under the dynamic loading. Oriard [4] and Law et al. [5] have researched the accumulative damage of surrounding rock due to multitimes repetitive blasting loads by monitoring changes of peak BVVs and velocity of acoustic wave. Preece and Thlone [6] have studied the detonation time and fragmentation using PRONTO-3D dynamic finite-element program and obtained a modified "KUS damage" constitutive model. Doucet et al. [7], Villaescusa et al. [8], and Ramulu et al. [9] have studied the damage degree and damage range of surrounding rock under multi-times blasting by use of sliding micrometer and imaging well log technique. Singh [10] has introduced the blasting vibration damage to underground coal mines from adjacent to open-pit blasting. Toraño et al. [11] have researched FEM models including randomness and its application to the blasting vibrations prediction. Nateghi et al. [12] have researched a negative effects method of blasting waves on concrete by analyzing parameters of underground vibration. The control method of blasting waves has been proposed based on the peak BVVs of underground structure.

From the aforementioned research results, it can be seen that most works on blasting vibration response focus on the theoretical study which is very little combined with engineering cases, especially for the research on the blast-excavation effect on the EST. The vibration wave gathering method is mostly realized through the natural vibration method. However, there are usually many differences between vibration waves caused by natural earthquake and those caused by blasting vibration. So it is important to adopt a reasonablility natural earthquake analysis theory. Generally, the numerical simulation methods are used to analyze damage cracking of support structures in the EST, although the results based on different calculation methods might vary greatly. Currently, the wave function expansion method is a main way to analyze BVVs of the EST. However, the surrounding rock is regarded as a homogeneous elastic medium in this method. In addition, the shape of tunnel and structure of rock masses are complicated actually, and the discontinuities exist widely in surrounding rock. Therefore, only some conventional views can be obtained in the past researches.

About the damage of the EST lining under dynamic load, the analysis on field experimental data and numerical simulation have been focused on in the recent decades. For instance, Wang et al. [13] have found that the BVVs and tensile stress of side-walls are bigger than the sections of the EST. The most dangerous part in the EST is the side-walls. Li et al. [14] and Xia et al. [15] have researched the damage properties of rock masses under blasting load in nuclear power station project. Feng and Wen [16] have simulated the dynamic response process of the EST under blasting load and found that blasting in adjacent distance is a key factor for the safety of concrete lining. Zhang et al. [17] have researched the vibration properties and change laws of middle wall and tunnel face in tunnel with small interval by vibration effect experiment of surrounding rock under blasting load. Shen et al. [18] have found that the attenuation of BVV is very obvious and amplitude of particles will likewise decrease with the distance increasing between monitoring point and blasting center. Chen [19] has studied the vibration response and accumulative damage of rock pillar in the EST with small interval. The attenuation formula of blasting wave in the rock pillar under blasting load has been obtained. Yang et al. [20] have researched the degree and range of rock masses damage in the EST under blast-excavation by using of two-side-wall pilot tunnel method. Although there are many successful cases in which the adverse influences of blast-induced vibrations on EST lining have been avoided or minimized, the studies on response of support structures of the EST under the blast-excavation are mostly only to analyze the damage cracking of concrete lining, while ignoring the influences on other support structures, such as bolts system and shotcrete. Therefore, it is important to study the vibration response of bolts support system placed in the surrounding rock of arch section and shotcrete support placed between the secondary lining and surrounding rock. At present, it is not clear that the deformation and failure laws of shotcrete and bolts under blasting loads are consistent with secondary lining. In addition, the dynamic response laws of bolts installed in surrounding rock at the arch section under dynamic loads are not understood uniformly. In this paper, the field experiment and numerical simulation are used to study the response laws of support structures in the EST under blasting load. The responses of blasting load on bolts and shotcrete support are analyzed based on stress wave theories and vibration mechanics theories.

Figure 1: Location and layout of New Zuofang Tunnel.

2. Site Description and Experimental Methods

2.1. General and Geological Information. New Zuofang Tunnel in Lanzhou-Chongqing railway is located in Hechuan District of Chongqing, next to Fu River, China. As shown in Figure 1, the NBT (New Zuofang Tunnel) is located on the left side of the EST (Old Zuofang Tunnel) with length of 487 m (DK888+156–DK888+643). The minimum distance between the two side-walls is less than 10 m (less than 1.5 times tunnel width with 33 m). The NBT built three railway lines is approximately parallel to the EST, where the Suining-Chongqing railway is a single railway line and Lanzhou-Chongqing railway is double lines. The tunnel becomes double-arch tunnel from single-arch tunnel at DK888+600. The largest span of single-arch tunnel is 22 m, and the maximum burial depth is 75 m. There is a single railway line in the EST built in 2005 and used for eight years on this unique single railway line. The support system of EST is consisted of bolts in surrounding rock, shotcrete support (its thickness is 25 cm) and secondary lining (its thickness is 80 cm). The arch section of the EST is supported by the grouted rockbolt each 1.0 m. The elevation of NBT's bottom boundary is higher than that of the EST (see Figure 2), which will seriously affect the stability of the support structure of arch section in the EST.

The NBT and the EST are located in a region of the denuded hills, where the lithology is Jurassic fine sandstone and mudstone with weak strength. The thickness of mudstone intercalations is about 1.0-2.0 m. It tends to mud and is easy to expand when it is mixed with water. The occurrence of whole strata varies little with small dip angle. Some intercalary strata of impurities develops little in surrounding rock

Table 1: Properties of surrounding rock with different grades.

Milepost	DK888+177–250	DK888+250–290	DK888+290–510	DK888+510–550	DK888+550–626
Grade	V	IV	III	IV	V
Properties	Mudstone intercalation (horizontal bedding) and fragment rock	Mudstone intercalation (horizontal bedding)	Intact sandstone	Mudstone intercalation (horizontal bedding)	Fragment rock (fracture development)

Figure 2: Excavation steps and monitoring points.

with grade IV and V. The rock masses classified based on Chinese Code for Design of Railway Tunnel (TB10003-2005) which considers many influence factors, such as strength of intactness rock, geometrical characteristics and mechanical properties of discontinuity, intactness of rock mass, groundwater, ground stress, and so forth. The properties of different surrounding rocks are shown in Table 1.

2.2. Field Experimental Method

(1) Method of Blast-Excavation. The NBT is step-excavated by use the two side-wall pilot tunnel method, which is a kind of drill and blast method, and the sequence of excavation is shown in Figure 2. The parallel excavation is applied to side-wall pilot tunnel on both sides, and the distance of excavation face is more than 100 m to decrease influence on each other. The measures of smooth blasting and microdifference blasting in deep borehole are applied to decrease the influence on the surrounding rock and the EST lining, and the depth in horizontal direction of each round excavation is about 2.0 m. The method of the decoupling charge with 2# rock emulsion explosive is applied in blasting of periphery borehole. The depth of blast boreholes is about 2.0 m and is divided into 9–11 segments to be filled with emulsion explosive. The excavation cross-section of NBT is divided into seven steps (see Figure 2). Because of the minor effects on the fifth step and the sixth step of the EST, the stress and deformation of the EST affected by these two steps would not be considered in this paper.

(2) Measure Method. The vibration transducers are fixed on the side-wall of the EST to obtain BVVs under blasting load at the same position (see Figure 3). There are 4 monitoring points numbered one, two, three, and four on the side-walls. Two vibration transducers with horizontal (along tunnel axis) and vertical directions are fixed one very monitoring point. After blasting occurred, the vibration signal of particles at the concrete lining subsurface caused by the blasting vibration waves is transformed as the electrical signal and inputted into memory instrument through the vibration transducer, the memory instrument connected with PC. The vibration signal is transferred into BVVs and frequency to analyze the response spectrum. This vibration experiment is carried out for every excavation step to obtain corresponding data.

3. Field Experimental Results and Analysis

3.1. Statistical Analysis of Field Experimental Data. The variation of peak BVVs with distance to the blasting center is shown in Figure 4. As seen, the vertical BVVs of particles caused by longitudinal and Rayleigh waves are different from the horizontal BVVs caused by shear and Love waves. Most of the BVV values are small. The maximum vertical BVV is 8.668 cm/s which is close to critical safety state of concrete lining. The maximum horizontal BVV is 3.788 cm/s, which is less than vertical BVV. The energy increases with the increasing of BVV, and the BVV of vertical is usually larger than the one of the horizontal, so the vertical BVV is often taken as a safety control indicator of concrete lining. The maximum BVV value occurs at the 4th step for the same monitoring point and then reduces from the 7th step, the 3rd step, and the 2nd step to the 1st step. The maximum BVV is at the monitoring point 1 for the same blast-excavation step. The monitoring numbers of vertical BVV from the maximum value to the minimum value are point 1, point 2, point 3, and point 4. Therefore, the BVVs attenuate rapidly with the increasing of distance to the blasting center. The energy density and strength decrease rapidly with the expansion of spread range of vibration waves and influence of rock fracture during the spread process of stress wave. The pre-reinforcement distance of NBT is designed to be 20.0 m because of the very small value of BVV at point 4.

Presently, based on the principle of dimensional analysis, the attenuation formula (1) of BVV put forward by Sadov (1978) is widely used:

$$V = k\left(\frac{Q^{1/3}}{R}\right)^{\alpha}, \tag{1}$$

where R is the distance from monitoring point to blasting center (m); V is BVV of particle at the monitoring point (cm/s); Q is a single-stage initiation charge weight (kg) and Q is equal to 3 in here; and k and α are the coefficients related to media properties, blasting method, and spread path.

Based on the measurement data from field experiment, the coefficients of k and α of BVVs by fitting Sadov's formula

(a) Monitoring points in the EST (b) Monitoring points in the field

Figure 3: Setup and installation of monitoring instruments.

(a) Vertical BVVs (b) Horizontal BVVs

Figure 4: Relationship between horizontal and vertical peak BVVs and distance to the blasting center.

are 1.185 and 3.544 for the vertical BVV and 2.443 and 2.427 for the horizontal BVV, respectively.

3.2. Application of Sadov's Formula. The BVVs of concrete lining at different positions and excavation steps are predicted by using Sadov's formula and shown in Figure 5. As seen, the BVV values of the 4th and 7th steps are bigger at point 1, and these values are close to the limited safety velocity controlled by "Blasting Safety Code" published by Chinese State Administration of Work Safety. In addition, the cracks occur at the hance of the EST at the 4th step and the 7th step; this phenomenon is consistent with the observation in the field.

3.3. Analysis on Dominant Vibration Frequency. The dominant vibration frequency distribution of BVVs measured at the different monitoring points is shown in Figure 6. As seen, BVVs less than 5 cm/s have a high frequency and their dominant vibration frequencies are distributed at the range of 50–150 Hz. This indicates that the vibration waves caused by blasting load are low-frequency waves, and its frequency is much larger than the natural frequency of concrete lining. The dominant vibration frequency is less than 50 Hz when the BVV is more than 6 cm/s. The vibration frequency at the position of the greater BVV is mostly low frequency. In general, the frequency decreases with distance increasing in the inhomogeneous rock masses. The existing joints and fractures in intermediate strata have a great effect on the stress wave spread. High BVV and low frequency intensify the damage of concrete lining, because the low vibration frequency is close to the natural frequency of the EST lining (about 10–15 Hz) and causes resonance of concrete lining. Therefore, the concrete lining is not safe at the position with the maximum BVV. The high BVV occurs mostly at the 4th step, which leads to the greater possibility of damage at the position of hance and spandrel facing the blasting side.

3.4. Time-Domain Analysis of BVV. The attenuation laws of particle BVV are obtained by the time-domain analysis of BVV. The curve of BVV of point 2 at the 4th step is shown in Figure 7. As seen, the time-dominant curve has lots of wave peaks and troughs with continuous intervals. This is caused by the millisecond blasting in each borehole and vibration

(a) Vertical BVVs (b) Horizontal BVVs

Figure 5: Vertical and horizontal BVVs forecasted by Sadov's formula.

Figure 6: Distribution of dominant vibration frequency.

reflection at the interface. Meanwhile, there exist certain time intervals among periphery boreholes, satellite boreholes, cutting boreholes, and so on during blast-excavation. Due to the adoption of presplitting blasting, the first maximum BVV peaks and troughs are caused by the blasting in periphery boreholes. The blasting in periphery holeh as the greatest impact on the support structures of the EST. The serious oscillation stage and stationary oscillation stage are shown on the BVV curve. The serious oscillation stage is the main phase caused by superposition of body wave (longitudinal wave on the vertical BVV curve and transverse wave on the horizontal BVV curve) and surface wave. The stationary oscillation stage is called after shock phase caused by complementary waves from surface wave.

The complex vibration process will be arised at the measured point, and every round blasting corresponds to a peak BVV, which lasts for 0.55 s. The massive stress waves generate in the process of blasting, and the properties of stress waves analyzed by the principle of superposition are the same except for amplitude. The damage possibility of concrete lining is caused by the longer duration of vibration. The vibrations processes of particles at point 3 and point 4 are quicker finished than those at point 1 and point 2, because the spread of vibration waves attenuates before arriving at measured points due to far spread path. The blasting vibration wave is triangle waveform along with severe vibration which is different from the natural vibration wave. The total time of blasting is 0.55 s when BVV reduces to zero. After blasting, there are still small amplitude oscillations, and the horizontal residual vibration lasts a little longer. This is because the spread direction of transverse wave is horizontal and the decay speed is less than that of longitudinal wave. The discontinuous BVV shown in Figure 7 can be divided into several different regions with interval every 0.1 s by means of millisecond delay blasting. The vibration superposition can be avoided by using interval blasting in different position and stage blasting at the same position.

4. Theoretical Verifications of Field Experimental Results

In order to understand the response on tunnel support structures under blast-excavation load specifically, the numerical simulation method is used to analyze the surrounding rock with grade III (DK888+290–DK888+510). Firstly, the calculation results of vertical peak BVVs are compared with the field monitoring results to validate the reasonable of numerical simulation. Secondly, the variation laws of stress and displacement in the support structures are analyzed

(a) Vertical BVVs (b) Horizontal BVVs

Figure 7: Time-history of horizontal and vertical BVV for the point 2.

Table 2: Physical-mechanical parameters of surrounding rock and bolts.

Materials	Weight/kN/m^3	Elastic modulus/GPa	Poisson's ratio	Internal friction angle/°	Dilation angle/°	Cohesion/kPa
Surrounding rock	25.50	2.06	0.3	30.6	30.6	20
Bolts	78.50	210	0.2	/	/	/

under the different blast-excavation steps. Finally, the plastic damage in the concrete lining and the laws of response on bolts are studied by introducing dynamic-plastic constitutive model of concrete.

4.1. Physical-Mechanical Parameters of Rock Masses and Concrete

(1) Physical-Mechanical Parameters of Surrounding Rock. The plastic state under blasting load might appear in the surrounding rock, so the Drucker-Prager model is used to simulate damage of concrete lining, and the elastic model is used to simulate bolts. According to the reports of engineering geological investigation and the results of laboratorial experiments, the physical-mechanical parameters are listed as Table 2.

(2) Physical-Mechanical Parameters of Concrete Lining. Damage-plastic constitutive model is defined as the constitutive model of shotcrete and secondary lining. The physical-mechanical parameters of concrete lining are shown in Table 3.

4.2. Rational Verification of Numerical Calculation. The average values of vertical BVVs at point 1 by the numerical simulation based on the software ABAQUS and the field experiment are shown in Table 4. The BVVs obtained by numerical simulation are larger than those of field experiment. This is because there are a large number of discontinuities in rock masses in field experiment to make the stress waves decay faster than those in the rock masses regarded as a homogeneous material in numerical simulation. In addition, the calculation formula of blasting-borehole pressure is set up in an ideal condition. The blasting energy is totally translated into the force to throw the rock into the air. The allowed error range of BVVs larger than actual values is about 10%, so the calculation results obtained by numerical calculation are reliable to analyze the law of stress and strain of support structures under different blasting excavation steps.

4.3. Analysis on Damage Degree of the EST Lining

4.3.1. Damage Mechanism of Concrete Lining under Cyclic Load. The representation of damage variable proposed by Kachanov is widely used to describe the damage variable. The formula is listed as follows:

$$D = 1 - \frac{A^*}{A} = \frac{A - A^*}{A} = \frac{\Delta A}{A}, \tag{2}$$

where D is the damage variable value, $0 \le D \le 1$. It indicates no damage when D is equal to zero. It indicates complete damage when D is equal to 1.0; A is initial sectional area of damage zone; A^* is the effective bearing area after the structure is damaged.

The process of damage in the concrete lining can be regarded as a cyclic loading process with high loading rate, because of the influence of repeated vibration caused by multi-times blast-excavation and a high loading rate being 5 m/s with a longer time of blasting vibration. This process is also regarded as an accumulative process of multi-times dynamic damages. The distribution of damage in the concrete lining and degree of damage in different points can be obtained by the damage-plastic constitutive model of concrete in the process of numerical analysis.

4.3.2. Comparison Analysis on Damage under Different Blast-Excavation Steps. The damage process of the EST lining is irreversible because of the sublevel millisecond blasting. The fatigue damages of concrete lining occur on the support structures of the EST under a longtime dynamic load. The damage process of concrete lining caused by blasting load at different excavation steps is shown in Figure 8.

As seen in Figure 8, there is no damage appearing in concrete lining at the 1st step and the 2nd step, which means that the stress wave is not strong enough to produce damage

Table 3: Physical-mechanical parameters of concrete lining.

Concrete lining	γ/kN/m^3	E_0/GPa	μ	Ψ/°	ξ	$\sigma_{b_0}/\sigma_{c_0}$	K	ω_t	ω_c
Secondary lining	25	30	0.2	15	0.1	1.16	0.6667	0	1
Shotcrete	24	23	0.2	/	0.1	1.16	0.6667	0	1

Notes: γ is weight; E_0 is initial elastic modulus; μ is Poisson's ratio; Ψ is dilation angle; ξ is eccentricity ratio; σ_{b_0} is initial yield compressive stress of equal axial; σ_{c_0} is initial yield compressive stress of unequal axial; K is yield constant; ω_t is tensile restore factor; ω_c is compressive restore factor.

Table 4: Comparison between vertical BVVs of field experiment and that of numerical calculation.

Excavation steps	The 1st step	The 2nd step	The 3rd step	The 4th step	The 7th step
Field experiment/cm/s	2.642	3.622	4.740	8.616	8.668
Numerical simulation/cm/s	3.123	4.342	6.346	10.125	10.084
Error	18.2%	19.9%	33.9%	17.5%	16.3%

(a) The 1st step and the 2nd step (C) (b) The 3rd step (C) (c) The 4th step (C)

(d) The 7th step (C) (e) The 1st step and the 2nd step (T) (f) The 3rd step (T)

(g) The 4th step (T) (h) The 7th step (T)

Figure 8: Damage of concrete lining under different excavation steps (C: compression; T: tensile).

(a) Compression (b) Tensile

Figure 9: Damage range and zone under different excavation steps.

because the wave intensity attenuates while spreading to the concrete lining. Damage of concrete lining begins at the 3rd step. The range and area of damaged zone of secondary lining are almost the same to those of shotcrete. The compressive damage zone focuses on secondary lining and arch foot facing the blasting side. The cyclic-dynamic stress concentration takes place in the aforementioned positions due to their acute angle types. This position can be damaged easily under compressive stress. The maximum damage value less than 1.0 under the compressive stress is smaller than the limited value, so only a few of microcracks appear on the subsurface of concrete lining. The buckling failure of concrete lining and shotcrete is not produced at the 3rd step. The macrocracks begin to emerge when the maximum damage value caused by tensile stress in the concrete lining is 0.8213. However, these cracks without connection fail to result in the failure of concrete lining. The maximum damage value in the shotcrete is 0.6974. The damage degree of shotcrete is smaller than that of secondary lining, but the damage range of shotcrete is greater than that of secondary lining. The minor damage at the back facing the blasting side of concrete lining can be seen from the damage graph. This phenomenon shows that the intensity diffracted here is still very strong when the stress waves spread to this position. At the 4th step, the minor compressed damage in shotcrete and small range of compressive damage in secondary lining appear at the same position with the 3rd step. The maximum damage value is 0.6123, and the degree of damage is larger than that at the 3rd step. The maximum damage value in tensile stress zone is 0.8720, and the macrocracks begin to emerge but not posing failure of concrete lining. Larger damage zone exists at arch foot, which is similar to the 3rd step, and the maximum damage value is 0.8882. Comparing with the 3rd step, there is no damage emergeing at the back facing the blasting side of concrete lining. Damage generation concentrates mainly on the inside of invert at the 7th step. The range and degree of damage are small under the compressive stress and become bigger under the tensile stress. The punctuate distribution of damage zones appears in the concrete lining at the 7th step, and the damage degree in shotcrete is far less than that in secondary lining under the tensile stress.

According to the aforementioned analysis, the distance from blasting center to monitoring point has great impact on the range and degree of damage. Damage appears at the 3rd step, the 4th step and the 7th step, and the largest one is at the 4th step. Therefore, the blasting parameters should be controlled mainly at the 4th step and the fifth step. The maximum damage value is between 0.8 and 0.9 with some microcracks appearing at the damage zone. The total damage in the concrete lining after every round blast-excavation is shown in Figure 9. As seen, the macrocracks appear at the hance facing the blasting side of concrete lining, arch foot, and arch spandrels, and at spandrel back facing the blasting side of concrete lining.

It can be seen from Figure 9(b) that the damage degree of concrete lining at back facing the blasting side is higher than that of facing the blasting side. This phenomenon is consistent with the distribution of peak principle stress. The simulation results can be proved by field observation. Some reinforcement measures of concrete lining are taken by railway administrative department. Two main reasons lead to the cracks in the concrete lining. Firstly, because of the closeness of EST to NBT, the charge is too excessive to produce a large blasting load. Secondly, lots of initial defects exist in the concrete lining because of concrete aging. Comparing Figure 9(a) with Figure 9(b), it can be found that the range and degree of damage due to tensile stress are larger than those of damage due to compressive stress. This is because the cyclic principle tensile stress in the concrete lining is big and it is close to tensile strength of concrete. Besides, the duration of blasting load with 65 ms promotes the development of damage zone.

4.3.3. Development Process of Damage in the Concrete Lining. The expansion of damage range and deepening of damage degree at the 4th step are taken as example to analyze the development of damage in the concrete lining under the successive blasting load.

(1) Expansion Process of Damage Zone. The range of damage zone in the concrete lining becomes larger with the excavation steps firstly and then tends to be stable. The distribution range of damage at different time is shown in Figure 10. As seen, the damage caused by tensile stress occurs firstly near subsurface facing the blasting side at 0.012 s, and the damage value is 0.01463. The damage zone enlarges to two sides with

Figure 10: Damage of concrete lining at different time.

the lasting function of stress wave, and the degree of damage deepens continually. The damage zone of hance facing the blasting side enlarges obviously at 0.057 s, and the maximum value is up to 0.1480. The damage zone emerges at arch foot facing the blasting side, and two aforementioned sections are damaged seriously at 0.077 s. The maximum value reaches 0.8313 at 0.207 s. Macrocracks begin to generate here, and cracks extend with the deepening of damage zone. Damaged zone begins to expand to the side near shotcrete along the deepening direction, and the range of damaged zone reaches the maximum value at 0.293 s. Damage value of outside (near subsurface) is larger than that of inside. This phenomenon shows that the macrocracks do not enlarge to inside of concrete lining (there are microcracks at inside) and just extend along profile of concrete lining at outside. Therefore, the fracture of concrete lining along the deepening direction is avoided.

(2) Deepening Process of Damage Degree. The damage emergence of concrete lining in the EST is caused by the blast-excavation. Under the cyclic loading, the mechanical parameters of concrete lining began to decrease, and damage degree enlarged gradually. Meanwhile, the damage value D began to increase from zero to a stable value.

The damage process of concrete lining with time at the 4th step is shown in Figure 11. As seen, the damage of concrete lining does not appear immediately but later after blast-excavation. This may be caused by two main reasons. Firstly, it requires enough time to make stress wave spread from blasting center to the concrete lining. Secondly, the concrete lining has inertia effect under the dynamic cyclic loading.

Figure 11: Development process of damage value at the 4th step.

The increase of strain and displacement does not coincide with stress under high strain rate load. This means that it does not have enough time to make deformation increase with increase of stress under the inertia function. This is the theory of damage effect lag that the cracks lag to appear in the concrete lining. The theory based on the deformation of concrete lining is the direct reason of damage generation.

It needs some time to make the damage of concrete lining reach the maximum value once the concrete lining emerges

Figure 12: The arch section of the EST.

Figure 13: Arrangement of bolts at the arch section.

damage. According to the time-history curve of stress, the stress superposition is in the form of triangle. Stress will decrease and cracks will stop extending when stress value reaches the critical value of producing cracks. Subsequently, stress will accumulate continuously, and cracks will continue to extend. Damage of concrete lining will stop development when stress wave doesn't make cracks further extend. As seen in Figure 11, the increase tendency of damage value is coincident with logarithm curve. So the express formula is listed as follows:

$$D_t = \begin{cases} 0, & t < t_0, \\ A \ln t + B, & t_0 \le t \le t_1, \\ D_{\max}, & t > t_1, \end{cases} \tag{3}$$

where D_t is damage value and A and B are two coefficients which are related to materials of concrete lining and can be fitted with least square method. It is noticed that A and B are different under compressive damage and tensile damage even under the same engineering condition. $D_{\max}$ is a final damage value; t_0 is the initial time of damage; t_1 is the final time of damage; t is one time of damage.

According to (3), the damage development speed of concrete lining will become slower with time lasting. This is because the reflection and diffraction of stress wave occur at the position of concrete cracks to make the stress and energy releases. The development speed of damage in the concrete lining decreases gradually with strength decrease of stress wave.

4.4. Dynamic Response of Bolts on Blasting Load under Different Steps

4.4.1. Support Effect of Bolts under Blasting Load. In order to understand the dynamic response of bolts on blasting load in the surrounding rock at the arch section of the EST under blasting load, the 3rd step is taken as an example to analyze the distribution of stress and displacement along the arch section of tunnel. The arch section of the EST and the arrangement of bolts at the arch section are shown in Figures 12 and 13, respectively.

The distribution laws of stress and displacement of different arch section are shown in Figure 14. As seen, the horizontal stress at the arch section is negative at the 3rd step. This proves that the stress in surrounding rock at the arch section is a compressive one along horizontal direction. From the distribution of horizontal stress, it can be found that the pressure at the arch section facing the blasting side is larger than that of other parts, and the stress distribution of surrounding rock with bolts is similar to that without bolts. The stress of spandrel facing the blasting side with bolts is smaller than that without bolts, while it is the opposite at other sections. The maximum vertical stress happens at the spandrel facing the blasting side. The stress decreases rapidly firstly and then increases from the spandrel to the center of arch section. After entering into the side-wall back facing the blasting side, it decreases gradually and even becomes negative so that the rock mass here is subjected to compressive stress. Comparing the stress with bolts and that without bolts is shown in Figure 14. It can be found that the stress with bolts is generally larger than that without bolts in the arch section. The maximum compressive stress being 0.222 MPa appears at the arch section where it is smaller than that of other sections. Horizontal displacement is smaller, and the displacement direction of particle moves from right to left at the side-walls facing the blasting side. The displacement increases at first and then decreases from the side-walls facing the blasting side, the arch section, to the side-walls back facing the blasting side. The maximum displacement being 23.0 mm happens at the arch section of the EST. It can be found that the displacements with bolts are similar to those without bolts. The vertical displacement at the 3rd step follows the similar change law with horizontal displacement. The maximum value of vertical displacement at the arch section is 16.0 mm without bolts and 2.3 mm with bolts.

It can be seen from Figure 14 that the displacement of surrounding rock without bolts is much larger than that with bolts at the arch section. The difference at the center of arch section is larger than that at two sides of arch section. The largest difference of displacement between them is 14.0 mm. The dynamic stress in the surrounding rock with bolts is larger than that without bolts. Simultaneously, the displacement of the former is smaller than that of the latter. The reason is that the reinforcement role of bolts at the arch section makes the rock masses close to each other. The attenuation speed of stress wave decreases in rock masses. The stress wave in the surrounding rock with bolts is stronger than that without bolts at the same section. The displacement of rock masses reinforced by bolts becomes smaller than

(a) Stress (b) Displacement

Figure 14: Stress and displacement at the arch section (H: horizontal direction; V: vertical direction; NA: no bolt; A: bolt).

Table 5: Peak axial forces on the middle of bolts under different excavation steps (units: kN).

No. of bolts	No. 1	No. 3	No. 5	No. 8	No. 10
The 1st step	4.134	6.35	2.459	1.775	1.235
The 2nd step	1.523	2.432	1.166	0.809	0.809
The 3rd step	5.814	8.384	7.053	4.773	3.591
The 4th step	35.553	45.904	28.033	8.432	6.483
The 7th step	9.606	22.017	6.365	2.409	3.804

that of without bolts. The arrangement of bolts enhances equivalently the stiffness of surrounding rock at the arch section. Comparing with surrounding rock without bolts, the capacity of bearing dynamic load of surrounding rock with bolts is enhanced obviously.

4.4.2. Axial Force Analysis of Bolts at the Arch Section under Different Steps. It can be seen from Figure 13 the bolts under different blasting loads. They also suffer different axial forces with different angles even at the same excavation step. The peak axial force on the middle of No. 1, No. 3, No. 5, No. 8, and No. 10 bolt is shown in Table 5.

As seen in Table 5, the peak axial forces of bolts at the arch section facing the blasting side are generally larger than those of bolts at other section. This is because the bolts in this position are influenced by stress waves firstly, and the stress waves attenuate when they spread to the other side. About the axial forces of bolts at the same step, the sequence from big to small is No. 3, No. 1, No. 5, No. 8, and No. 10 bolts, respectively. It indicates that the angle of bolts has great effect on the axial force of bolts. Dynamic load has the greatest impact on inclined bolts, horizontal bolts secondly, and vertical bolts at last. About the axial force of the same bolt, the sequence from big to small is the 4th step, the 7th step, the 3rd step, the 1st step, and the 2nd step, respectively. Stress waves generating at the 4th step have greatest impact on bolts, and the smallest influence appears at the 2nd step. This is because the small distance between monitoring point and blasting center has great effect on axial forces of bolts and the change of blasting center and angles of bolts have significant impact on axial forces of bolts.

4.4.3. Time-History Analysis on Axial Stress of Bolts. The strength of bolts is enhanced obviously after lasting stress waves from surrounding rock. The axial stress of bolts vibrates intensely with time. The time-history analysis on axial stress on the middle of No. 3, No. 5, and No. 8 bolts is shown in Figure 15.

As seen in Figure 15, bolts with different angles have different change trends of axial stress. It indicates that the stress waves with different angles have the different spread laws. Just as the time-history of axial stress in the concrete lining, the stresses in bolts vibrate intensely and last for 0.1 s, which is almost consistent with the function time of blasting load, and then concuss slightly because of inertia function. They do not decrease to zero during the calculation time while vibrating appreciably around a certain value. The permanent compressive stress and tensile stress in bolts caused by the vibration function of blasting load, rotation, or extrusion of rocks in the arch section. The tensile stress or compressive stress of bolts prevents the rotation or extrusion of rock masses. These stresses are put together with static stress of

(a) No. 3 bolt

(b) No. 5 bolt

(c) No. 8 bolt

Figure 15: Time-history analysis on different bolts.

bolts caused by ground stress. Bolts force will increase if two kinds of stress are in the same direction. On the contrary, bolts force will decrease. The concussion process of stress in bolts with different angles exists significant differences, for example, the superposition phenomenon of axial stress appears in bolts (as No. 3 bolt) facing the blasting side under blasting load. However, it does not appear in bolts back facing the blasting side (as No. 5 and No. 8 bolts), which are vibrated by tensile stress and compressive stress circularly. As a result, the yield strength of bolts decreases obviously under cyclic load.

5. Conclusions

(1) At the same monitoring point, BVVs caused by different blast-excavation steps decrease with the distance to blasting center increasing. After analysis on particle BVVs from different positions to blasting center, Sodev's formula which is suitable for similar projects is obtained. The particle BVVs in the concrete lining subsurface are forecasted by this formula.

(2) Under different blast-excavation steps, the peak principle stress appears at spandrel at the 1st step and the 3rd step and appears at hance at the 2nd step and the 4th step. Because the largest peak principle stress appears at the 4th step and the 7th step, the damage zones of the concrete lining are mainly concentrated in hance and arch foot facing the blasting side. Stress in support structures concusses with many peak values. The concussion lasts for very long time which is far longer than that of stress in borehole. It is shown on the stress wave curve that the first peak of stress is the largest one, and then peaks attenuate gradually. Stress waves which spread in the concrete lining present superposition phenomenon at the 1st step and the 2nd step.

(3) Damage of support structures happens at the 3rd step, the 4th step, and the 7th step. It is mainly concentrated on the concrete lining facing the blasting side. The most serious damage happens at the 4th step. This shows that the damage degree is related to the distance from blasting center in terms of the same charge. The cracks at hance and spandrel facing the blasting side are caused by the cumulative damage of these three steps. The area of damage zone, as well as distribution law of peak damage value, begins to extend from one point on the subsurface of concrete lining to any direction. At the same monitoring point, damage does not happen immediately after blast-excavation occurs, but it experiences a delay time before enlarging rapidly.

(4) It can be seen from the distribution of damage zone that the range and degree of damage in the surrounding rock at the arch section are very small. This proves that bolts improve the capacity of bearing dynamic load for the surrounding rock efficiently. In addition, bolts with different angles suffer different stresses. The inclined bolts suffer larger stress than that of horizontal bolts and vertical bolts. This shows that stress wave with oblique incident is stronger than that with forward incident. After blast-excavation is finished, the fact that the residual stress is left in bolts proves that the interaction dislocation happens in the surrounding rocks at the arch section.

Acknowledgments

This paper is financially supported by the National Natural Science Foundation of China (Grants nos. 41002089 and 41102162) and Jiangsu Overseas Research & Training Program for University Prominent Young and Middle-Aged Teachers and Presidents. The authors gratefully acknowledge M. S. Chuanlei Zhang at the School of Earth Sciences and Engineering, Hohai University, China, for his contribution to the field experiment. The authors would also like to acknowledge the editors and reviewers of this paper for their very helpful comments and valuable remarks.

References

[1] M. Hisatake, S. Sakurai, and T. Ito, "Effects of adjacent blast operation on vibration behavior of existing tunnel," *Proceedings of the Japan Society of Civil Engineers*, vol. 1983, no. 332, pp. 67–74, 1983.

[2] D. Krajeinovie, "Continuum damage mechanics," *Applied Mechanics Reviews*, vol. 37, no. 1, pp. 1–32, 1984.

[3] J. S. Kuszmaul, "New constitution model for fragmentation of rock under dynamic loading," in *Proceedings of the 2nd International Symposium on Rock Fragmentation by Blasting*, pp. 412–423, 1987.

[4] L. L. Oriad, "The scale of effects in evaluating vibration damage potential," in *Proceedings of the 15th Annual Conference on Explode Sives and Blasting Research*, pp. 161–176, International Society of Explosives Engineers, New Orleans, La, USA, 1989.

[5] T. M. Law, J. May, A. T. Spathis, A. T. du Plessis, and A. M. Palmer, "Blast damage and blast dilution control: the application of bulk emulsion systems at the WMC St Ives junction mine," *Fragblast*, vol. 5, no. 1-2, pp. 1–20, 2001.

[6] D. S. Preece and B. J. Thlone, "A study of detonation time and fragmentation using 3-D finite element techniques and a damage constitutive model," in *Proceedings of the 5nd International Symposium on Rock Fragmentation by Blasting*, pp. 147–156, 1996.

[7] C. Doucet, A. Cameron, and Y. Lizotte, "Effects of rock mass characteristics on fragmentation in controlled blasting experiments in small development headings," in *Proceedings of the 22nd ISEE's Annual Conference on Explosives and Blasting Technique*, pp. 1–10, Orlando, Fla, USA, February 1996.

[8] E. Villaescusa, I. Onederra, and C. Scott, "Blast induced damage and dynamic behaviour of hangingwalls in bench stoping," *Fragblast*, vol. 8, no. 1, pp. 23–40, 2004.

[9] M. Ramulu, A. K. Chakraborty, and T. G. Sitharam, "Damage assessment of basaltic rock mass due to repeated blasting in a railway tunnelling project—a case study," *Tunnelling and Underground Space Technology*, vol. 24, no. 2, pp. 208–221, 2009.

[10] P. K. Singh, "Blast vibration damage to underground coal mines from adjacent open-pit blasting," *International Journal of Rock Mechanics and Mining Sciences*, vol. 39, no. 8, pp. 959–973, 2002.

[11] J. Toraño, R. Rodríguez, I. Diego, J. M. Rivas, and M. D. Casal, "FEM models including randomness and its application to the blasting vibrations prediction," *Computers and Geotechnics*, vol. 33, no. 1, pp. 15–28, 2006.

[12] R. Nateghi, M. Kiany, and O. Gholipouri, "Control negative effects of blasting waves on concrete of the structures by analyzing of parameters of ground vibration," *Tunnelling and Underground Space Technology*, vol. 24, no. 6, pp. 608–616, 2009.

[13] M.-N. Wang, X.-M. Pan, C.-M. Zhang, X.-D. Wen, and K.-K. Wang, "Study of blasting vibration influence on close-spaced tunnel," *Rock and Soil Mechanics*, vol. 25, no. 3, pp. 412–414, 2004.

[14] J.-R. Li, X. Xia, H.-B. Li, X.-W. Wang, and Q.-C. Zhou, "Study on blast-induced bedrock damage extension for a nuclear power station project," *Chinese Journal of Rock Mechanics and Engineering*, vol. 23, no. A01, pp. 4674–4678, 2005.

[15] X. Xia, J. Li, H. Li et al., "Study on damage characteristics of rock mass under blasting load in Ling'ao Nuclear Power Station, Guangdong Province," *Chinese Journal of Rock Mechanics and Engineering*, vol. 26, no. 12, pp. 2510–2516, 2007.

[16] Z. R. Feng and X. Wen, "Numerical analysis on existing service tunnel under blasting load in new tunnel," *Blasting*, vol. 25, no. 4, pp. 20–23, 2008.

[17] Q.-S. Zhang, L.-P. Li, S.-C. Li, W.-T. Ding, and W.-L. Hong, "Experimental study of blasting dynamic vibration of closely adjacent tunnels," *Rock and Soil Mechanics*, vol. 29, no. 10, pp. 2655–2666, 2008.

[18] Y. S. Shen, B. Gao, and Z. J. Wang, "Effect of blasting in double line tunnel on existing tunnel," *Chinese Journal of Underground Space and Engineering*, vol. 5, no. 5, pp. 980–984, 2009.

[19] L. J. Chen, "Experimental study on blasting vibration influence on nearby tunnels with small interval," *Blasting*, vol. 27, no. 2, pp. 36–39, 2010.

[20] Y.-B. Yang, M.-G. Liu, G.-H. Zhang, and Q. Li, "Analysis of construction parameter optimization for new large cross-section tunnel next to existing tunnels," *Rock and Soil Mechanics*, vol. 31, no. 4, pp. 1217–1226, 2010.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 296106, 19 pages
http://dx.doi.org/10.1155/2013/296106

Research Article

The Effect of First-Order Bending Resonance of Wheelset at High Speed on Wheel-Rail Contact Behavior

Shuoqiao Zhong, Xinbiao Xiao, Zefeng Wen, and Xuesong Jin

State Key Laboratory of Traction Power, Southwest Jiaotong University, Chengdu 610031, China

Correspondence should be addressed to Xuesong Jin; xsjin@home.swjtu.edu.cn

Received 29 June 2013; Accepted 3 September 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Shuoqiao Zhong et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The first-order bending deformation of wheelset is considered in the modeling vehicle/track coupling dynamic system to investigate its effect on wheel/rail contact behavior. In considering the effect of the first-order bending resonance on the rolling contact of wheel/rail, a new wheel/rail contact model is derived in detail in the modeling vehicle/track coupling dynamic system, in which the many intermediate coordinate systems and complex coordinate system transformations are used. The bending mode shape and its corresponding frequency of the wheelset are obtained through the modal analysis by using commercial software ANSYS. The modal superposition method is used to solve the differential equations of wheelset motion considering its flexible deformation due to the first-order bending resonance. In order to verify the present model and clarify the influence of the first-order bending deformation of wheelset on wheel/track contact behavior, a harmonic track irregularity with a fixed wavelength and a white-noise roughness are, respectively used as the excitations in the two models of vehicle-rail coupling dynamic system, one considers the effect of wheelset bending deformation, and the other does not. The numerical results indicate that the wheelset first-order bending deformation has an influence on wheel/rail rolling contact behavior and is easily excited under wheel/rail roughness excitation.

1. Introduction

The development of high-speed railway is associated with dynamic problems, for example, the wear of their wheels/rails. When a train with polygonal wheels due to wear operates on a track at high speed, the train and the track vibrate fiercely in an extensive frequency range. The wheel polygonal wear excites the strong dynamic behavior of the vehicle/track structures in mid- and high-frequency ranges. In the extensive experiments conducted on China metro lines by Jin et al. [1], the test results showed that all of the wheels of one type of metro trains in service present the 9th-order polygonal wear, and the excitation frequency of polygonal wheel was close to the first-order bending resonance frequency of the wheelset. The root of the 9th order polygonal wear was attributed to the first-order bending resonance occurring of wheelset in operation, as indicated in Figure 1. Now, the problem of wheel polygonal wear is very common on the existing high-speed trains in China. The mechanism of this phenomenon has not been explained perfectly until now, and it also has become one of the most difficult problems of the wheel-rail system to be solved. The wheel polygon due to wear usually takes a long time and so far cannot be reproduced through the numerical simulation due to the existing insufficiently advanced theoretical models for railway vehicle coupling with track.

The dynamics analysis of railway vehicle by using the existing rigid multibody dynamics models for vehicle/track coupling system is limited in the low-frequency domain of 0–20 Hz [2]. An effective way to broaden the analysis frequency range is to take the structural flexibility of vehicle/track into consideration in modeling the vehicle/track system [3–6]. And the key part in modeling structural flexibility of the vehicle/track is to consider the effect of the flexibility of wheelset on the rolling contact of wheelset/rail.

Many scholars have carried out a lot of researches on the vehicle/track dynamic system modeling that considers the effect of flexible wheelset. In [5], the wheelset was treated

Figure 1: The root cause of harmonic wear of metro wheels [1].

as a lumped and discrete structure by Popp et al. In [4, 6–8], the existing models of the flexible wheelset were continuous models. A vehicle/track dynamic model developed by Meinders considering wheelset flexibility was based on the cosimulation of finite element analysis software ANSYS and multibody dynamics simulation software NEWEUL [9, 10]. Baeza et al. [11–13] proposed a numerical dynamic system model originating from a rotating elastic cylinder model, taking inertial effect and effect of Coriolis force into account.

In terms of wheel/rail rolling contact modeling, the effect of wheelset deformation in mid- and high-frequency ranges should not be neglected [14]. Though many literatures presented different methods to model flexible wheelsets, the detailed derivations of rolling contact geometry of a flexible wheelset and a pair of flexible rails and the detailed calculations and discussions on the effect of the high-frequency bending of wheelset on the rolling contact behavior were not given. This is a very difficult research. The influence exerted by the wheelset bending formation is much concerned with wheelset polygonal wear. In [15], the effect of wheelset flexible deformation on the creepages and the creep forces in static state was indicated to be significant. The flexible deformation of a wheelset in service always contains mid- and high-frequency components. The rolling contact behavior of the wheel/rail caused by wheelset deformation would be in mid- and high-frequency ranges, which brings a great difficulty into the on-line calculation of wheel/rail contact geometry. And the existing models for the wheel/rail contact geometry are not suitable for the model with flexible wheelsets and need to be further improved.

In this paper, a new wheel/rail contact model is developed to consider the effect of the wheelset first-order bending deformation on the wheel/rail rolling contact behavior. The main object is to analyze the effect of wheelset flexibility on the wheel/rail contact behavior mainly in mid-frequency range. The mid-frequency components of the excitation from the wheel/rail interaction system are filtered by the suspension system, and the wheel/rail mid-frequency excitation cannot easily excite the flexibility of the car-body and bogies. Hence, the influence of the structural flexibility of car-body and bogies is ignorable in this present paper. In Section 2, the model of the flexible wheelset and the wheel/rail contact model are introduced. In Section 3, a harmonic track irregularity with a fixed wavelength and white-noise roughness of the rail are adopted as the excitations to analyze the effect of the wheelset flexibility.

2. Modeling

2.1. Vehicle/Track Model. Conventional vehicle system models consist of one carriage, two bogies, and four wheelsets, which are considered as rigid bodies or lumped masses connected by spring-damping systems. A ballasted track is usually modeled as a triple-layer model of discrete elastic support. The rail is modeled as a Timoshenko beam on elastic point supporting foundation. The two ends of the calculation rail are simply supported. The vertical and lateral bending deformations and twisting of the rails are taken into account. The sleepers are modeled as rigid bodies, and the ballasts are regarded as discrete equivalent bodies. The equivalent spring-damping systems are used as the connections between the rails and the sleepers, the sleepers and the equivalent ballast bodies, and the ballast bodies and the subgrade. Figure 2 illustrates such a model of the vehicle and the track.

Figure 2: Vehicle/track coupling model (elevation).

Its detailed description and the derivation of motion equations refer to the published papers, for example, [16–18]. The important parameters of the vehicle and the track used in this paper calculation are given in Table 1.

2.2. Flexible Wheelset Model. The flexible wheelset modeling mainly considers the contribution of the first-order bending deformations of the wheelset axle in the two planes, perpendicular and parallel to the track level plane, respectively, and neglects their crossing effect. The wheelset modes are calculated by using the finite element software ANSYS. The corresponding resonance frequency of the first-bending, shown in Figure 3, is about 117 Hz. This mode function will be used in the analytical solution to the partial differential equations of the wheelset deformation based on the modal superposition method [16].

Figure 3 indicates the force system on the flexible wheelset. O_{fL} and O_{fR} are, respectively, the left and right points of the application of the primary suspension forces on the axle. O_{CL} and O_{CR} are, respectively, the left and right contact points of wheel/rail. O indicates the origin of the coordinate system O-XYZ that is a coordinate system with a translational motion along with a tangent track central line at the vehicle speed. If the vehicle speed is a constant, this coordinate system is an inertial coordinate system, therefore, regarded as an absolute coordinate system (geodetic coordinate system).

Based on the above assumptions that the wheels are regarded as the rigid bodies and perpendicular to the axle central lines at their centers, their spatial poses are determined by the positions of nominal rolling circle centers and the directions of the normal lines through nominal rolling circle centers on the deformed axle centerline. The wheelset axle is modeled as an Euler-Bernoulli beam. So according to the force analysis diagram of the wheelset given in Figure 3, the left and right wheel/rail forces are, respectively, translated to the nominal circle centers, O_L and O_R. Such a force translation introduces the additional moments applied to the axle. The additional moments are the products of the wheel/rail forces and the distance vector between the wheel/rail contact points and the rolling circle centers. Then, the force analysis diagrams in the two planes of the wheelset axle could be obtained, as shown in Figures 4(a) and 4(b), based on which the differential equations of the axle can be established.

The partial differential equations of the wheelset in bending deformation in the two planes, (1) and (2), give the equations for the perpendicular and the parallel planes to the track level plane, respectively, as follows

$$EI_z \frac{\partial^4 x(y,t)}{\partial y^4} + \rho A \frac{\partial^2 x(y,t)}{\partial t^2} = -Q_x(y,t) - \frac{\partial M_z(y,t)}{\partial y}, \tag{1}$$

$$EI_x \frac{\partial^4 z(y,t)}{\partial y^4} + \rho A \frac{\partial^2 z(y,t)}{\partial t^2} = Q_z(y,t) - \frac{\partial M_x(y,t)}{\partial y}. \tag{2}$$

In (1) and (2), $Q_x(y,t)$ and $Q_z(y,t)$ are, respectively, given as

$$\begin{aligned} -Q_x(y,t) &= (N_{Lx} + F_{Lx})\,\delta(y - y_{L3}) + F_{xfL}\delta(y - y_{L4}) \\ &\quad + (N_{Rx} + F_{Rx})\,\delta(y - y_{R3}) + F_{xfR}\delta(y - y_{R4}), \\ Q_z(y,t) &= -(N_{Lz} + F_{Lz})\,\delta(y - y_{L3}) + F_{zfL}\delta(y - y_{L4}) \\ &\quad - (N_{Rz} + F_{Rz})\,\delta(y - y_{R3}) + F_{zfR}\delta(y - y_{R4}) \\ &\quad + G\delta(y - 0). \end{aligned} \tag{3}$$

$M_z(y,t)$ and $M_x(y,t)$ are, respectively, given as

$$\begin{aligned} M_z(y,t) &= \left(-(N_{Lx} + F_{Lx})\,y_{L3} + \left(N_{Ly} + F_{Ly}\right)x_{L3} + M_{Lz}\right) \\ &\quad \times \delta(y - y_{L3}) \\ &\quad + \left(-(N_{Rx} + F_{Rx})y_{R3} + \left(N_{Ry} + F_{Ry}\right)x_{R3} + M_{Rz}\right) \\ &\quad \times \delta(y - y_{R3}) \\ &\quad + \left(-F_{xfL}y_{L4} + F_{yfL}x_{L4}\right)\delta(y - y_{L4}) \\ &\quad + \left(-F_{xfR}y_{R4} + F_{yfR}x_{R4}\right)\delta(y - y_{R4}), \end{aligned} \tag{4}$$

$$\begin{aligned} M_x(y,t) &= \left(-\left(N_{Ly} + F_{Ly}\right)z_{L3} + (N_{Lz} + F_{Lz})\,y_{L3} + M_{Lx}\right) \\ &\quad \times \delta(y - y_{L3}) \\ &\quad + \left(-\left(N_{Ry} + F_{Ry}\right)z_{R3} + (N_{Rz} + F_{Rz})\,y_{R3} + M_{Rx}\right) \\ &\quad \times \delta(y - y_{R3}) \end{aligned}$$

Figure 3: Force analysis diagram of the flaxible wheelset.

Figure 4: Force analysis diagrams (a) in the X-Y plane parallel to the track level plane and (b) in the Z-Y plane parallel to the track level plane.

$$+\left(-F_{yfL}z_{L4}+F_{zfL}y_{L4}\right)\delta\left(y-y_{L4}\right)$$
$$+\left(-F_{yfR}z_{R4}+F_{zfR}y_{R4}\right)\delta\left(y-y_{R4}\right). \tag{5}$$

The function $\delta(\cdot)$ in (3)~(5) reads as

$$\delta\left(y-y_i\right)=\begin{cases}1, & y=y_i,\\ 0, & y\neq y_i,\end{cases}\quad i=L3,L4,R3,R4. \tag{6}$$

The boundary conditions of (1) and (2) are given as

$$\begin{aligned}x''\left(-l_{\text{axle}},t\right)&=x'''\left(-l_{\text{axle}},t\right)=0,\\ x''\left(l_{\text{axle}},t\right)&=x'''\left(l_{\text{axle}},t\right)=0,\\ z''\left(-l_{\text{axle}},t\right)&=z'''\left(-l_{\text{axle}},t\right)=0,\\ z''\left(l_{\text{axle}},t\right)&=z'''\left(l_{\text{axle}},t\right)=0,\end{aligned} \tag{7}$$

Table 1

Physical parameter	Value	Unit	Notation
M_c	33.766×10^3	kg	Car-body mass
M_{bi}	2.4×10^3	kg	The ith bogie mass
M_{wi}	1.85×10^3	kg	The ith wheelset mass
C_{ty}	20×10^3	Ns/m	The equivalent lateral damping of secondary suspension (considering damping of lateral shock absorber joint)
K_{ty}	18.13×10^6	N/m	The equivalent lateral stiffness of secondary suspension (considering stiffness of lateral shock absorber joint and lateral stiffness of air spring)
C_{tz}	40×10^3	Ns/m	The equivalent vertical damping of secondary suspension (considering vertical damping of air spring)
K_{tz}	299×10^3	N/m	The equivalent vertical stiffness of secondary suspension (considering vertical stiffness of air spring)
C_{fy}	0	Ns/m	The equivalent lateral damping of primary suspension
K_{fy}	6.47×10^6	N/m	The equivalent lateral stiffness of primary suspension (considering the lateral stiffness locating node of the axle-box rotary arm)
C_{fz}	15×10^3	Ns/m	The equivalent vertical damping of primary suspension (considering damping of vertical shock absorber joint)
K_{fz}	6.076×10^6	N/m	The equivalent vertical stiffness of primary suspension (considering stiffness of vertical shock absorber joint and steel spring)
M_r	60.64	kg/m	Rail mass per unit length
M_s	349	kg	Mass of sleeper
M_b	466	kg	Mass of ballast element
L_s	0.6	m	Sleeper bay
E	2.06×10^{11}	N/m^2	Young's modulus
K_{pLi}	20×10^6	N/m	Lateral stiffness of the ith pad
C_{pLi}	5×10^4	N/m	Lateral damping of the ith pad
K_{pVi}	40×10^6	N/m	Vertical stiffness of the ith pad
C_{pVi}	5×10^4	N/m	Vertical damping of the ith pad
$K_{bv(L,R)i}$	80×10^6	N/m	Vertical stiffness between sleeper and the ith ballast element
$C_{bv(L,R)i}$	1×10^5	Ns/m	Vertical damping between sleeper and the ith ballast element
K_w	78×10^6	N/m	Vertical stiffness between sleeper and the ith ballast element
C_w	8×10^4	Ns/m	Vertical damping between sleeper and the ith ballast element
$K_{fv(L,R)i}$	65×10^6	N/m	Vertical stiffness between road bed and the ith ballast element
$C_{fv(L,R)i}$	31000	N/m	Vertical damping between road bed and the ith ballast element

where "$'$" means one-order partial derivative with respect to y coordinate.

Based on the modal superposition principle, the solutions to (1) and (2) are, respectively, given as

$$x(y,t) = \sum_{k=1}^{\mathrm{NM}} X_k(y)\, q_{xk}(t),$$
$$z(y,t) = \sum_{k=1}^{\mathrm{NM}} Z_k(y)\, q_{zk}(t). \tag{8}$$

Using the normalized modal functions which are mutually orthogonal, we have that

$$\ddot{q}_{xk} + \lambda_{xk}^2 q_{xk} = P_{xk}, \qquad \ddot{q}_{zk} + \lambda_{zk}^2 q_{zk} = P_{zk}. \tag{9}$$

P_{xk} and P_{zk} are, respectively, given as

$$\begin{aligned} P_{xk} = {} & (N_{Lx} + F_{Lx}) X_k(y_{L3}) + F_{xfL} X_k(y_{L4}) \\ & + (N_{Rx} + F_{Rx}) X_k(y_{R3}) + F_{xfR} X_k(y_{R4}) \\ & + \left(-(N_{Lx} + F_{Lx}) y_{L3} + (N_{Ly} + F_{Ly}) x_{L3} + M_{Lz}\right) \\ & \times X_k'(y_{L3}) \\ & + \left(-(N_{Rx} + F_{Rx}) y_{R3} + (N_{Ry} + F_{Ry}) x_{R3} + M_{Rz}\right) \\ & \times X_k'(y_{R3}) \\ & + \left(-F_{xfL} y_{L4} + F_{yfL} x_{L4}\right) X_k'(y_{L4}) \\ & + \left(-F_{xfR} y_{R4} + F_{yfR} x_{R4}\right) X_k'(y_{R4}), \end{aligned}$$

$$\begin{aligned}P_{zk} = &-\left(N_{Lz}+F_{Lz}\right)Z_k\left(y_{L3}\right)+F_{zfL}Z_k\left(y_{L4}\right)\\&-\left(N_{Rz}+F_{Rz}\right)Z_k\left(y_{R3}\right)+F_{zfR}Z_k\left(y_{R4}\right)+GZ_k\left(0\right)\\&\times\left(-\left(N_{Ly}+F_{Ly}\right)z_{L3}+\left(N_{Lz}+F_{Lz}\right)y_{L3}+M_{Lx}\right)\\&\times Z_k'\left(y_{L3}\right)\\&+\left(-\left(N_{Ry}+F_{Ry}\right)z_{R3}+\left(N_{Rz}+F_{Rz}\right)y_{R3}+M_{Rx}\right)\\&\times Z_k'\left(y_{R3}\right)\\&+\left(-F_{yfL}z_{L4}+F_{zfL}y_{L4}\right)Z_k'\left(y_{L4}\right)\\&+\left(-F_{yfR}z_{R4}+F_{zfR}y_{R4}\right)Z_k'\left(y_{R4}\right).\end{aligned} \tag{10}$$

In addition, the contribution of the rigid modes to the wheelset motion is also taken into account. The total displacements of the wheelset in the two planes, which are, respectively, parallel and perpendicular to the track level plane, read as

$$\begin{aligned}x\left(y,t\right) &= \sum_{k=1}^{\mathrm{NME}} X_{Ek}\left(y\right)q_{Exk}\left(t\right)+\sum_{k=1}^{\mathrm{NMR}} X_{Rk}\left(y\right)q_{Rxk}\left(t\right),\\ z\left(y,t\right) &= \sum_{k=1}^{\mathrm{NME}} Z_{Ek}\left(y\right)q_{Ezk}\left(t\right)+\sum_{k=1}^{\mathrm{NMR}} Z_{Rk}\left(y\right)q_{Rzk}\left(t\right).\end{aligned} \tag{11}$$

The elastic mode number is given as NME = NM = 1, and the rigid model number is given as NMR = 2. The rigid modal functions read as

$$\begin{aligned}X_{Rk}\left(y\right) &= -Z_{Rk}\left(y\right)\\ &= \begin{cases}-\sqrt{\dfrac{1}{\rho A}}, & k=1,\\ -\sqrt{\dfrac{3}{\rho A}}\left(1-\dfrac{y}{l_{\mathrm{axle}}}\right), & k=2.\end{cases}\end{aligned} \tag{12}$$

The wheelset bending deformation in the two planes has an influence on the primary suspension forces in the directions of X- and Z-axes, which read, respectively, as

$$\begin{aligned}&F_{xf(L,R)i}\\&\quad= K_{fx}\left(\pm l_b\psi_b\left(t\right)+H_{bw}\beta_b\left(t\right)-x_i\left(\mp l_w,t\right)\right)\\&\qquad+C_{fx}\left(\pm l_b\dot{\psi}_b\left(t\right)+H_{bw}\dot{\beta}_b\left(t\right)-\dot{x}_i\left(\mp l_w,t\right)\right),\\&F_{zf(L,R)i}\\&\quad= K_{fz}\left(Z_b\left(t\right)-z_i\left(\mp l_w,t\right)+\left(-1\right)^i l_b\beta_b\left(t\right)\mp l_b\phi_b\left(t\right)\right)\\&\qquad+C_{fz}\left(\dot{Z}_b\left(t\right)-\dot{z}_i\left(\mp l_w,t\right)+\left(-1\right)^i l_b\dot{\beta}_b\left(t\right)\mp l_b\dot{\phi}_b\left(t\right)\right).\end{aligned} \tag{13}$$

The lateral primary suspension forces on the four wheelsets read as

$$\begin{aligned}&F_{yf(L,R)i}\\&\quad= K_{fy}\left(u_{yi}\left(\mp l_w,t\right)-Y_b\left(t\right)+H_{bw}\phi_b\left(t\right)+\left(-1\right)^i l_b\psi_b\left(t\right)\right)\\&\qquad+C_{fy}\left(\dot{u}_{yi}\left(\mp l_w,t\right)-\dot{Y}_b\left(t\right)+H_{bw}\dot{\phi}_b\left(t\right)+\left(-1\right)^i l_b\dot{\psi}_b\left(t\right)\right).\end{aligned} \tag{14}$$

The effect of the wheelset bending deformation on the lateral primary suspension forces is neglected. They are mainly influenced by the wheelset rigid motion in the Y-direction. Substituting (11) into (13), the components of the primary suspension forces in the directions of X- and Z-axes are written as follows:

$$\begin{aligned}&F_{xf(L,R)i}\\&\quad= K_{fx}\left(-\left(\sum_{k=1}^{2}X_{Ek}\left(\mp l_w\right)q_{Exki}\left(t\right)+\sum_{k=1}^{\mathrm{NMR}}X_{Rk}\left(\mp l_w\right)q_{Rxki}\left(t\right)\right)\right.\\&\qquad\left.\pm l_b\psi_b\left(t\right)+H_{bw}\beta_b\left(t\right)\right)\\&\qquad+C_{fx}\left(-\left(\sum_{k=1}^{2}X_{Ek}\left(\mp l_w\right)\dot{q}_{Exki}\left(t\right)+\sum_{k=1}^{\mathrm{NMR}}X_{Rk}\left(\mp l_w\right)\dot{q}_{Rxki}\left(t\right)\right)\right.\\&\qquad\left.\pm l_b\dot{\psi}_b\left(t\right)+H_{bw}\dot{\beta}_b\left(t\right)\right),\\&F_{zf(L,R)i}\\&\quad= K_{fz}\left(Z_b\left(t\right)-\left(\sum_{k=1}^{2}Z_{Ek}\left(\mp l_w\right)q_{Ezki}\left(t\right)+\sum_{k=1}^{\mathrm{NMR}}Z_{Rk}\left(\mp l_w\right)q_{Rzki}\left(t\right)\right)\right.\\&\qquad\left.+\left(-1\right)^i l_b\beta_b\left(t\right)\mp l_b\phi_b\left(t\right)\right)\\&\qquad+C_{fz}\left(\dot{Z}_b\left(t\right)-\left(\sum_{k=1}^{2}Z_{Ek}\left(\mp l_w\right)\dot{q}_{Ezki}\left(t\right)+\sum_{k=1}^{\mathrm{NMR}}Z_{Rk}\left(\mp l_w\right)\dot{q}_{Rzki}\left(t\right)\right)\right.\\&\qquad\left.+\left(-1\right)^i l_b\dot{\beta}_b\left(t\right)\mp l_b\dot{\phi}_b\left(t\right)\right).\end{aligned} \tag{15}$$

The flexible wheelset model is successfully coupled with the bogie through (14) and (15). The calculation formulas of the wheel/rail contact forces are to be discussed below.

2.3. Wheel/Rail Rolling Contact Model

2.3.1. Description of Coordinate Systems for Wheel/Rail Rolling Contact Calculation. It is complex and tedious to carry out the rolling contact behavior calculation of a flexible wheelset running on a pair of rails in service. The calculation is carried out with the aid of the transformations of multicoordinate systems. A wheel/rail rolling contact model considering flexible wheelset effect is much different from that without considering flexible wheelset effect.

Figure 5 indicates a detailed contact geometry description of a flexible wheelset in rolling contact with a pair of rails, the global coordinate system, and the local coordinate systems used in the rolling contact calculation. In Figure 4, the rolling angles, $\phi_{w(L,R)}$, are, actually, the projections of the rotation angles of the normal line of the axle cross-section at the wheel nominal rolling circle centers in the O-ZY plane, but the yawing angles $\psi_{w(L,R)}$ of the wheels, which are the projections in the O-XY plane, are not shown in the figure and can be described in the top view of Figure 5 that is omitted here.

The left and right contact points are regarded as, respectively, the origins O_{L1} and O_{R1} of the local coordinates $O_{(L,R)1}$-$X_{(L,R)1}Y_{(L,R)1}Z_{(L,R)1}$. o'_L and o'_R are the centers of the left and right instant rolling circles, respectively. The two auxiliary lines are created, and they are, respectively, perpendicular to the left and right instant circles and pass through the centers o'_L and o'_R. These two auxiliary lines, respectively, intersect with the common planes of the wheel/rail contact surfaces, and points A_L and A_R are, respectively, their intersection points, as indicated in Figure 5. Making a line connects point A_L to the left contact point O_{L1} and the other line connects point A_R to the left contact point O_{R1}. The included angle between line A_LO_{L1} and line $A_Lo'_L$ and the one between line A_RO_{R1}, and line $A_Ro'_R$ are exactly the contact angles δ_L and δ_R, respectively. Each coordinate system in Figures 2 and 4 has the following detailed description and requirements.

(1) $O - XYZ$: this coordinate system is illustrated in Section 2.2. Its base vector is that $\mathbf{e} = [\mathbf{i}, \mathbf{j}, \mathbf{k}]$. Its unit vectors, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are, respectively, in the longitudinal, vertical and lateral directions of the tangent track.

(2) O_1-$X_1Y_1Z_1$: it is a translating coordinate system that is always parallel to the coordinate system O-XYZ, as indicated in Figure 5. It translates with the wheelset mass center. Its base vector is that $\mathbf{e}_1 = [\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1]$.

(3) $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$: they are the local coordinate systems attached to the two rigid wheels, as indicated in Figure 5. Their origins O_L and O_R are at the centers of the nominal rolling circles of the left and right wheels, respectively, which means that their poses are determined by the wheel poses and their $Y_{(L,R)}$ axes are always perpendicular to the corresponding nominal rolling circle planes. The nominal rolling circles of the left and right wheels coincide with the planes O_L-X_LZ_L and O_R-X_RZ_R, respectively. But the rolling motions of the left and right wheels are around the axes X_L and X_R, respectively, and their yawing motions are around the axes Z_L and Z_R, respectively. So the axes X_L and X_R are always parallel to the horizontal plane, and the axes Z_L and Z_R are always parallel to the vertical plane. Their base vectors are that $\mathbf{e}_{(L,R)} = [\mathbf{i}_{(L,R)}, \mathbf{j}_{(L,R)}, \mathbf{k}_{(L,R)}]$. The unit vectors $\mathbf{i}_{(\cdot)}$, $\mathbf{j}_{(\cdot)}$, and $\mathbf{k}_{(\cdot)}$ indicate the directions of the axes $X_{(\cdot)}$, $Y_{(\cdot)}$, and $Z_{(\cdot)}$, respectively.

(4) $O_{(L,R)1}$-$X_{(L,R)1}Y_{(L,R)1}Z_{(L,R)1}$: their axes Y_{L1} and Y_{R1}, respectively, coincide with the lines A_LO_{L1} and A_RO_{R1}, as indicated in Figure 5. The planes O_{L1}-$X_{L1}Y_{L1}$ and O_{R1}-$X_{R1}Y_{R1}$ coincide with the common tangent planes of the contact surfaces of the left and right wheels/rails, respectively. The axes Z_{L1} and Z_{R1} coincide with the common normal lines of the contact surfaces of the left and right wheels/rails, respectively. Their base vectors are that $\mathbf{e}_{(L,R)1} = [\mathbf{i}_{(L,R)1}, \mathbf{j}_{(L,R)1}, \mathbf{k}_{(L,R)1}]$.

(5) $O_{r(L,R)}$-$X_{r(L,R)}Y_{r(L,R)}Z_{r(L,R)}$: these two coordinate systems are used to describe the configurations and motions of the left and right rail cross-sections which are assumed to be rigid. Their origins are fixed at the left and right rail cross-session centers, O_{rL} and O_{rR}, and their rotation and translation are determined by those of the rail cross-sections. Their base vectors are, that, $\mathbf{e}_{r(L,R)} = [\mathbf{i}_{r(L,R)}, \mathbf{j}_{r(L,R)}, \mathbf{k}_{r(L,R)}]$.

The transformations from the base vector of coordinate system O_1-$X_1Y_1Z_1$ to those of the coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$ read as

$$\begin{Bmatrix} \mathbf{i}_{(L,R)} \\ \mathbf{j}_{(L,R)} \\ \mathbf{k}_{(L,R)} \end{Bmatrix} = \begin{bmatrix} \cos\psi_{w(L,R)} & \sin\psi_{w(L,R)} & 0 \\ -\cos\phi_{w(L,R)}\sin\psi_{w(L,R)} & \cos\phi_{w(L,R)}\cos\psi_{w(L,R)} & \sin\phi_{w(L,R)} \\ \sin\phi_{w(L,R)}\sin\psi_{w(L,R)} & -\sin\phi_{w(L,R)}\cos\psi_{w(L,R)} & \cos\phi_{w(L,R)} \end{bmatrix} \times \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix} = \mathbf{A}_{w(L,R)} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix}. \quad (16)$$

The transformation from the wheel coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$ to the coordinate system O-XYZ is expressed in (17), in which $[x_{(L,R)}, y_{(L,R)}, z_{(L,R)}]$ are the coordinates of any point described in the coordinate system O_L-$X_LY_LZ_L$ or O_R-$X_RY_RZ_R$, and $[x, y, z]$ is the coordinates of the same point described in the absolute coordinate system O-XYZ. $\overrightarrow{O_1O}$ indicates the vector form the origin

Figure 5: A flexible wheelset in rolling contact with a pair of rail.

O_1 of O_1-$X_1Y_1Z_1$ to the point O. $[x_{O(L,R)}, y_{O(L,R)}, z_{O(L,R)}]$ are the coordinates of the nominal rolling circle centers $O_{(L,R)}$ in the coordinate system O-XYZ. It is noted that in (17) the bases $[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ are the same as $[\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1]$, as indicated in Figure 4. The transformation in (17) includes two times of translations and one rotation of the coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$. Consider the following:

$$\begin{aligned}
&\begin{bmatrix} x_{(L,R)} \\ y_{(L,R)} \\ z_{(L,R)} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i}_{(L,R)} \\ \mathbf{j}_{(L,R)} \\ \mathbf{k}_{(L,R)} \end{Bmatrix} \\
&\quad = \overrightarrow{O_{(L,R)}O_1} + \begin{bmatrix} x_{(L,R)} \\ y_{(L,R)} \\ z_{(L,R)} \end{bmatrix}^T \mathbf{A}_{w(L,R)} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix} + \overrightarrow{O_1O} \\
&\quad = \overrightarrow{O_{(L,R)}O} + \begin{bmatrix} x_{(L,R)} \\ y_{(L,R)} \\ z_{(L,R)} \end{bmatrix}^T \mathbf{A}_{w(L,R)} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix} \\
&\quad = -\begin{bmatrix} x_{O(L,R)} \\ y_{O(L,R)} \\ z_{O(L,R)} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} + \begin{bmatrix} x_{(L,R)} \\ y_{(L,R)} \\ z_{(L,R)} \end{bmatrix}^T \mathbf{A}_{w(L,R)} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} \\
&\quad = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}.
\end{aligned} \tag{17}$$

The transformations from the base vectors of the local coordinate systems $O_{(L,R)1}$-$X_{(L,R)1}Y_{(L,R)1}Z_{(L,R)1}$ at the contact points to the absolute coordinate system read as

$$\begin{Bmatrix} \mathbf{i}_{(L,R)1} \\ \mathbf{j}_{(L,R)1} \\ \mathbf{k}_{(L,R)1} \end{Bmatrix} = \mathbf{A}_{(L,R)1} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}, \tag{18}$$

where

$$\mathbf{A}_{(L,R)1} = \begin{bmatrix} \cos\psi_{w(L,R)} & \sin\psi_{w(L,R)} & 0 \\ -\cos\left(\delta_{(L,R)} + \phi_{w(L,R)}\right)\sin\psi_{w(L,R)} & \cos\left(\delta_{(L,R)} + \phi_{w(L,R)}\right)\cos\psi_{w(L,R)} & \sin\left(\delta_{(L,R)} + \phi_{wL}\right) \\ \sin\left(\delta_{(L,R)} + \phi_{w(L,R)}\right)\sin\psi_{w(L,R)} & -\sin\left(\delta_{(L,R)} + \phi_{w(L,R)}\right)\cos\psi_{w(L,R)} & \cos\left(\delta_{(L,R)} + \phi_{wL}\right) \end{bmatrix}. \tag{19}$$

The transformations from the bases of the local coordinate systems O_{rL}-$X_{rL}Y_{rL}Z_{rL}$ and O_{rR}-$X_{rR}Y_{rR}Z_{rR}$ at the rail cross-section centers to those of the coordinate system O_1-$X_1Y_1Z_1$ (or O-XYZ) are given as

$$\begin{Bmatrix} \mathbf{i}_{r(L,R)} \\ \mathbf{j}_{r(L,R)} \\ \mathbf{k}_{r(L,R)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_{r(L,R)} + \phi_0) & \sin(\phi_{r(L,R)} + \phi_0) \\ 0 & -\sin(\phi_{r(L,R)} + \phi_0) & \cos(\phi_{r(L,R)} + \phi_0) \end{bmatrix} \times \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix} = \mathbf{A}_{r(L,R)} \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{Bmatrix} = \mathbf{A}_{r(L,R)} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}. \quad (20)$$

The wheel/rail rolling contact calculation consists of three parts, wheel/rail contact points, wheel/rail normal forces, and wheel/rail tangential forces. They are elaborated as follows.

2.3.2. Wheel/Rail Contact Point Calculation. Before the calculation of the wheel/rail contact points, the given discrete data points of the profiles of the wheel treads are usually described in the local coordinate system associated with the wheel tread, and the rail top profile described in the local coordinate system associated with the rail [19]. In the calculation, they have to be transformed into the expression in the absolute coordinate system O-XYZ through a series of the coordinate system transformations which include rotating and translating the coordinate system.

The prescribed profile of the wheel is usually described in the local coordinate system associated with the wheel tread. The local coordinate system associated with the wheel tread is, actually, parallel to the rigid wheel coordinate system, O_L-$X_LY_LZ_L$, or O_R-$X_RY_RZ_R$, and their origins are at the lowest points of the nominal rolling circles. In order to obtain the coordinates of the wheel tread profiles described in the absolute coordinate system O-XYZ, firstly, the coordinates of the profiles of the wheels described in the local coordinate systems associated with the wheel treads have a translation that equals the radius of the nominal rolling circles, and can be described in $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$. Then, they are transformed into the description in the absolute coordinate system O-XYZ by using formula (17) and with the help of the wheelset coordinate system O_1-$X_1Y_1Z_1$.

The transformation from $O_{r(L,R)}$-$X_{r(L,R)}Y_{r(L,R)}Z_{r(L,R)}$ to O-XYZ is through translating and rotating the coordinates of the rail top profiles described in $O_{r(L,R)}$-$X_{r(L,R)}Y_{r(L,R)}Z_{r(L,R)}$. The translation is just done along the vector $\overrightarrow{O_{(L,R)1}O}$ or $\mathbf{R}_{(L,R)}$, and (20) is the transformation from the base vector of the coordinate systems $O_{r(L,R)}$-$X_{r(L,R)}Y_{r(L,R)}Z_{r(L,R)}$ to that of the coordinate system O-XYZ.

The existing models for the calculation of rigid wheel/rail contact geometrical relationship need to be further improved to be applicable to the contact geometry calculation of a flexible wheelset in contact with a pair of rails. Figure 6 illustrates the spatial contact geometrical relationship of a wheel of a rigid wheelset in contact with a rail [19], in which the contact state on the right side is shown. In Figure 6, C_R is the contact point of the right wheel/rail, O_3 is the center of the rigid wheelset, the left and right wheels have the same rolling angle and yawing angle, the wheelset axle centerline O_3O_{3R} is perpendicular to the instant rolling circle plane,

Figure 6: Schematic diagram of wheel/rail contact spatial geometry.

O_{3R} is the intersection point of the wheelset axle centerline and the common tangent line of the contact surfaces, d_B is the distance between the point O_3 and the rolling circle, B and BC_R are, respectively, the center and radius of the rolling circle, A is the intersection point of the common normal line of the contact surfaces and the wheelset axle centerline, O_3-$X_3Y_3Z_3$ is the coordinate system of the rigid wheelset, and $O_3'A'B'C_R'$ are, respectively, the projections of O_3ABC_R on the O-XY plane, so the line $O_3'B'$ is the projection of the axle centerline on the O-XY plane.

The wheel contact point trace line, on which the contact point C_R is, in the absolute coordinate system O-XYZ can be calculated with (21) according to [18] as follows:

$$\begin{aligned} x &= x_B + l_x R_R \tan\delta_R, \\ y &= y_B - \frac{R_R\left(l_x^2 l_y \tan\delta_R + l_z m\right)}{\left(1 - l_x^2\right)}, \\ z &= z_B - \frac{R_R\left(l_x^2 l_z \tan\delta_R - l_y m\right)}{\left(1 - l_x^2\right)}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} l_x &= -\cos\phi\sin\psi, \\ l_y &= \cos\phi\cos\psi, \\ l_z &= \sin\phi, \\ m &= \sqrt{1 - l_x^2\left(1 + \tan^2\delta_R\right)}. \end{aligned} \quad (22)$$

(x_B, y_B, z_B) are the coordinates of the instant rolling circle center B in the absolute coordinate system, and they are given

as follows:

$$x_B = X_{ow}^{\text{rigid}} + d_B l_x,$$
$$y_B = Y_{ow}^{\text{rigid}} + d_B l_y, \quad (23)$$
$$z_B = Z_{ow}^{\text{rigid}} + d_B l_z,$$

where $(X_{ow}^{\text{rigid}}, Y_{ow}^{\text{rigid}}, Z_{ow}^{\text{rigid}})$ are described in the coordinate system O_3-$X_3Y_3Z_3$ defined as a rigid wheelset coordinate system. So the series of the wheel rolling circles can be determined with d_B.

But, for the contact between a flexible wheelset and a pair of rails, due to the axle deformation, as indicated in Figure 5, strictly speaking, d_B is not constant, and the directioncosine components, (l_x, l_y, l_z), of the central lines of the left and right wheels should be modified. (l_x, l_y, l_z) in (22) and (23) should be replaced by the direction cosine components, $(l_{(L,R)x}, l_{(L,R)y}, l_{(L,R)z})$, of the normal lines at the instant rolling circle centers in O-XYZ; namely, in the calculation of $(l_{(L,R)x}, l_{(L,R)y}, l_{(L,R)z})$, the rolling and yawing angles (ϕ, ψ) of the wheelset should be replaced by $(\phi_{w(L,R)}, \psi_{w(L,R)})$ of the left and right wheels of the flexible wheelset. It should be noted that the normal lines through the instant rolling circle centers are, actually, those through the centers of the nominal rolling circles since the wheels are assumed to be rigid. For the flexible wheelset, the coordinates of the centers of the left and right instant rolling circles should be written as, based on formula (23) and Figure 6,

$$x_{o'_{(L,R)}} = X_{ow}^{\text{rigid}} + \Delta x + d_{O_1 o'_{(L,R)}} l_{(L,R)x},$$
$$y_{o'_{(L,R)}} = Y_{ow}^{\text{rigid}} + \Delta y + d_{O_1 o'_{(L,R)}} l_{(L,R)y}, \quad (24)$$
$$z_{o'_{(L,R)}} = Z_{ow}^{\text{rigid}} + \Delta z + d_{O_1 o'_{(L,R)}} l_{(L,R)z},$$

where $d_{O_1 o'_{(L,R)}}$ are the distances between the center O_1 of the flexible wheelset and the centers $o'_{(L,R)}$ of the left and right instant rolling circles, respectively. And Δx, Δy, and Δz are the increments of the wheelset center translation in the X-, Y-, and Z-directions, respectively. They are caused by the bending deformation of the flexible wheelset. Comparing (23) with (24), for the flexible wheelset, the calculation of the coordinates of the wheelset center O_3 and the instant rolling circle center (x_B, y_B, z_B) considers the combined effect of the rigid motion and flexible deformation of the wheelset. But its stretching and compressing deformations caused by the axle bending deformation in the axle direction are very small, compared with the axle bending deformation amplitudes. So Δy is neglected. Since the analyzed wheelset is not a powered wheelset, the component along the track of the wheel/rail force is quite small, compared with the vertical component, the bending deformation of the wheelset is very small in a horizontal plane of the track. So Δx caused by the wheelset bending deformation can also be neglected. It should be noted that $(X_{ow}^{\text{rigid}}, Y_{ow}^{\text{rigid}}, Z_{ow}^{\text{rigid}})$ are obtained through the dynamic analysis using the traditional multibody dynamic model of the vehicle/track, which is introduced in the present paper.

Once the contact circles and the rail top profiles are described in the absolute coordinate system O-XYZ, it will be easy to calculate the minimum distances $\Delta Z_{\min L}(t)$ and $\Delta Z_{\min R}(t)$ between the wheel treads and rail top surfaces, as well as the coordinates of the wheel/rail contact points. The wheel/rail contact angles can be subsequently calculated with the known coordinates of the wheel/rail contact points and the given wheel/rail profiles.

2.3.3. Wheel/Rail Normal Forces. The vertical instant compressions of wheel/rail at the contact points, $\delta Z_L(t)$ and $\delta Z_R(t)$, are written as

$$\delta Z_L(t) = Z_{owL}(t) - \left(\Delta Z_{\min L}(t) - \Delta Z_L(0)\right) - Z_{\text{irr}L}(t),$$
$$\delta Z_R(t) = Z_{owL}(t) - \left(\Delta Z_{\min R}(t) - \Delta Z_R(0)\right) - Z_{\text{irr}R}(t). \quad (25)$$

The instant normal compressions, $\delta Z_{Lc}(t)$ and $\delta Z_{Rc}(t)$, are calculated according to (26) and the contact geometrical relationship shown in Figure 6, and read they as

$$\delta Z_{Lc}(t) = \frac{\delta Z_L(t)}{\cos\left(\delta_L - \phi_{wL}\right)},$$
$$\delta Z_{Rc}(t) = \frac{\delta Z_R(t)}{\cos\left(\delta_R + \phi_{wR}\right)}. \quad (26)$$

The left and right wheel/rail normal forces $F_{wrn(L,R)}$ are calculated by

$$F_{wrn(L,R)}(t) = \begin{cases} \left\{\frac{1}{G}\left[\delta Z_{(L,R)c}(t) - \delta Z_{(L,R)c}(0)\right]\right\}^{3/2}, & \delta Z_{(L,R)c}(t) - \delta Z_{(L,R)c}(0) > 0, \\ 0, & \delta Z_{(L,R)c}(t) - \delta Z_{(L,R)c}(0) \le 0. \end{cases} \quad (27)$$

Then, the normal force should be decomposed into three components in the X-, Y-, and Z-directions in establishing the motion equations of the wheelset and the rails.

2.3.4. Wheel/Rail Creep Forces and Spin Moments. The wheel/rail creep forces are calculated by using the model developed by Shen-Hedrick-Elkins, as described in [18]. In the calculation of each time step, the wheel/rail creepages ξ_x, ξ_y, and ξ_z should be calculated in advance. The existing creepage calculation formulas need to be improved and can be used to calculate the creepages of a flexible wheelset in rolling contact with a pair of rails. The creepages represent the ratios of the relative velocities of the contact surfaces at the contact points

to the forward velocities, $v_{(L,R)}$, of the instant rolling circles of the wheels, as shown in (28). Consider the following:

$$\xi_{x(L,R)} = \frac{\Delta v_{(L,R)x}}{v_{(L,R)}},$$
$$\xi_{y(L,R)} = \frac{\Delta v_{(L,R)y}}{v_{(L,R)}}, \tag{28}$$
$$\xi_{z(L,R)} = \frac{\Delta \omega_{(L,R)z}}{v_{(L,R)}},$$

where $\Delta v_{(L,R)x}$, $\Delta v_{(L,R)y}$, and $\Delta \omega_{(L,R)z}$ are, respectively, the relative longitudinal velocity, the relative lateral velocity, and angular velocities. The relative velocities are firstly calculated in the absolute coordinate system and then transformed into the expressions in the local coordinate systems associated with the contact surfaces, which will be discussed in the following.

$v_{(L,R)}$ can be calculated in two ways. The first way is to make a summation of the rigid motion and the flexible deformation of the wheelset, and the other way is to take the derivatives of the coordinates of the centers of the instant rolling circles with respect to time directly. But, actually, the difference between $v_{(L,R)}$ and v is quite small due to the very small effect of the rigid yaw, motion and the bending deformation of the wheelset on the forward speeds, $v_{(L,R)}$, of the instant rolling circles. So, by using formulas (28), for simplicity, $v_{(L,R)}$ is replaced with v. v is the speed of the vehicle or the nominal forward speed of the wheelset center.

Based on the composite motion theorem of rigid body motion, the absolute velocity $\mathbf{v}_a$ of a particle is the resultant vector of the relative velocity $\mathbf{v}_r$ and the transport velocity $\mathbf{v}_e$, given in (29). In other words, $\mathbf{v}_r$ is the velocity of the moving point, described in the moving coordinate system, and $\mathbf{v}_e$ is the velocity of the moving coordinate system relative to the absolute coordinate system. Consider the following:

$$\mathbf{v}_a = \mathbf{v}_e + \mathbf{v}_r. \tag{29}$$

The wheel coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$, as shown in Figure 5, can be regarded as the moving coordinate systems. So the absolute velocities of the left and right contact points $O_{(L,R)1}$, respectively, on the left and right wheels, have the following expressions:

$$\mathbf{v}_{wa(L,R)} = \mathbf{v}_{we(L,R)} + \mathbf{v}_{wr(L,R)}, \tag{30}$$

where $\mathbf{v}_{we(L,R)}$ and $\mathbf{v}_{wr(L,R)}$ are, respectively, the transport velocities and the relative velocities. The relative velocities are the velocities of the wheel contact points $O_{(L,R)1}$ relative to the origins $O_{(L,R)}$ of the moving coordinate systems. The transport velocities $\mathbf{v}_{we(L,R)}$ are the velocities of two points in $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$, and these two points coincide with the contact points $O_{(L,R)1}$, respectively. So $\mathbf{v}_{we(L,R)}$ read as

$$\mathbf{v}_{we(L,R)} = \boldsymbol{\omega}_{(L,R)} \times \mathbf{R}_{O_{(L,R)}O_{(L,R)1}} + \dot{\mathbf{R}}_{OO_{(L,R)}}, \tag{31}$$

where $\mathbf{R}_{O_{(L,R)}O_{(L,R)1}}$ indicate the vectors $\overrightarrow{O_{(L,R)}O_{(L,R)1}}$. $\dot{\mathbf{R}}_{OO_{(L,R)}}$ indicate the derivatives of the vectors $\mathbf{R}_{OO_{(L,R)}}$ with respect to time. $\mathbf{R}_{OO_{(L,R)}}$ are the vectors $\overrightarrow{OO_{(L,R)}}$. $\dot{\mathbf{R}}_{OO_{(L,R)}}$ represent the translational velocities of the centers of the nominal rolling circles of the left and right wheels, described in the absolute coordinate system. $\boldsymbol{\omega}_{(L,R)}$ are determined by the angular velocity vectors of the left and right wheels, $\dot{\psi}_{w(L,R)}$ and $\dot{\phi}_{w(L,R)}$, as indicated in (32). It is noted that $\boldsymbol{\omega}_{(L,R)}$ is just the rotation of the transport coordinate systems, $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$, relative to the absolute coordinate system O-XYZ, as shown in Figure 5. Consider the following:

$$\boldsymbol{\omega}_{(L,R)} = \begin{bmatrix} \dot{\phi}_{w(L,R)} \\ 0 \\ \dot{\psi}_{w(L,R)} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix},$$
$$\mathbf{R}_{O_{(L,R)}O_{(L,R)1}} = \begin{bmatrix} \dot{R}'_{(L,R)x} \\ \dot{R}'_{(L,R)y} \\ \dot{R}'_{(L,R)z} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i}_{(L,R)} \\ \mathbf{j}_{(L,R)} \\ \mathbf{k}_{(L,R)} \end{Bmatrix}. \tag{32}$$

It should also be noted that $\boldsymbol{\omega}_{(L,R)}$ do not include the rolling (ω_0) of the rigid wheelset over the track and neglect the rolling angular velocities of the wheels due to the twisting deformation of the wheelset, which produce the longitudinal creepages. The relative velocities $\mathbf{v}_{wr(L,R)}$ are the velocities of the left and right contact points described in the local coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$, due to the rolling of the wheelset over the two rails. They are written as

$$\mathbf{v}_{wr(L,R)} = \boldsymbol{\omega}'_{(L,R)} \times \mathbf{R}_{O_{(L,R)}O_{(L,R)1}}, \tag{33}$$

where

$$\boldsymbol{\omega}'_{(L,R)} = \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix}^T \begin{Bmatrix} \mathbf{i}_{(L,R)} \\ \mathbf{j}_{(L,R)} \\ \mathbf{k}_{(L,R)} \end{Bmatrix}. \tag{34}$$

Equation (34) is expressed in the local coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$.

Substituting (31) and (33) into (30), the equation could be deduced as follows:

$$\begin{aligned} \mathbf{v}_{wa(L,R)} &= \mathbf{v}_{we(L,R)} + \mathbf{v}_{wr(L,R)} \\ &= \dot{\mathbf{R}}_{OO_{(L,R)}} + \boldsymbol{\omega}_{(L,R)} \times \mathbf{R}_{O_{(L,R)}O_{(L,R)1}} \\ &\quad + \boldsymbol{\omega}'_{(L,R)} \times \mathbf{R}_{O_{(L,R)}O_{(L,R)1}}. \end{aligned} \tag{35}$$

It should be noted that $\mathbf{v}_{wr(L,R)}$ should be transformed into the expression in absolute coordinate system O-XYZ through using transformation matrices $\mathbf{A}_{w(L,R)}$, as indicated in (16), before being added to the calculation.

The absolute velocities of the contact points on the left and right rails are written as

$$\mathbf{v}_{\text{rail}(L,R)} = \begin{bmatrix} \dot{X}_{r(L,R)} \\ \dot{Y}_{r(L,R)} \\ \dot{Z}_{r(L,R)} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} + \begin{bmatrix} \delta\dot{X}_{r(L,R)} \\ \delta\dot{Y}_{r(L,R)} \\ \delta\dot{Z}_{r(L,R)} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}, \tag{36}$$

where $\dot{Y}_{r(L,R)}$ and $\dot{Z}_{r(L,R)}$ are, respectively, the lateral and vertical velocities of the left and right rail cross-sections and

$\delta\dot{Y}_{r(L,R)}$ and $\delta\dot{Z}_{r(L,R)}$ are, respectively, the rates of changes of the lateral and vertical irregularities on the left and right railheads.

Using (35) and (36), the velocity differences $\Delta\mathbf{v}_{(L,R)}$ between the contact surfaces of the wheels and the rails can be written as

$$\Delta\mathbf{v}_{(L,R)} = \mathbf{v}_{wa(L,R)} - \mathbf{v}_{\mathrm{rail}(L,R)} = \begin{bmatrix} \Delta v_{x(L,R)} \\ \Delta v_{y(L,R)} \\ \Delta v_{z(L,R)} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}. \quad (37)$$

The velocity differences $\Delta\mathbf{v}_{(L,R)}$, are expressed in the absolute coordinate system, shown in (37), and then need to be transformed into the expressions in the local coordinate systems $O_{(L,R)1}$-$X_{(L,R)1}Y_{(L,R)1}Z_{(L,R)1}$ through (18).

Then, plugging (18) into (37), the velocity differences are

$$\begin{aligned} \Delta\mathbf{v}_{(L,R)} &= \begin{bmatrix} \Delta v_{x(L,R)} \\ \Delta v_{y(L,R)} \\ \Delta v_{z(L,R)} \end{bmatrix}^T \mathbf{A}_{(L,R)1}{}^{-1} \begin{Bmatrix} \mathbf{i}_{(L,R)1} \\ \mathbf{j}_{(L,R)1} \\ \mathbf{k}_{(L,R)1} \end{Bmatrix} \\ &= \begin{bmatrix} \Delta v'_{(L,R)x} \\ \Delta v'_{(L,R)y} \\ \Delta v'_{(L,R)z} \end{bmatrix}^T \begin{Bmatrix} \mathbf{i}_{(L,R)1} \\ \mathbf{j}_{(L,R)1} \\ \mathbf{k}_{(L,R)1} \end{Bmatrix}. \end{aligned} \quad (38)$$

Equation (38) is plugged into (28) to calculate the lateral and longitudinal creepages. The calculation of the spin creepages by using the third formula in (28) neglects the effect of the wheelset axle bending since its effect on the spin creepages is quite small.

3. Results and Discussions

In order to verify the present model considering the effect of the first-order bending modes of the wheelset in the two planes mentioned above on the rolling contact behavior, a harmonic track irregularity with fixed wavelength of 1.18 m is used. The excitation frequency of the irregularity is 117 Hz at the speed of 500 km/h, which is exactly equal to the resonance frequency of the wheelset first-order bending mode. The fluctuating amplitude of the irregularity is 0.05 mm. The calculation cases include the speeds of 200 km/h, 300 km/h, 400 km/h, and 500 km/h. The parameters used in the calculation are listed in Table 1 [20].

In Figure 7, the right vertical axis denotes the difference (DIFF) between the normal forces calculated by the models of the rigid (RW) and flexible (EW) wheelsets. The lines of DIFF in Figure 7 are obtained by subtracting the lines of RW from the corresponding lines of EW. The numerical computation benefits from fast double precision. Figure 7 shows that the amplitudes of the total vertical wheel/rail forces increase with the speed increase, as indicated by the red and blue broken lines. The amplitude of the normal force difference denoted by DIFF, that is, the normal force variation caused by the first-order wheelset bending deformation, is about 0.2 N when the speed is less than 500 km/h, as shown in Figures 7(a), 7(b), and 7(c). In the case of 500 km/h, as shown in Figure 7(d), the maximum amplitude of the difference reaches about 1.2 N, which is 6 times larger than those of the cases of Figures 7(a), 7(b), and 7(c). This is caused by the excited resonance of first-order bending mode. An increase in the wheel/rail normal force caused by the first-order bending resonance is quite small, compared with the static wheel/rail forces. But such a small difference of the force applied on wheels at a fixed frequency could cause a polygonal wear on high-speed wheels in a long-time service. For instance, now the high-speed vehicles in service widely use the wheelsets with diameters in the range of 840 mm–920 mm, and the operating speed of the high-speed trains is 300 km/h. Large amounts of wheel tread roughness measured from the sites show that the trilateral wear on the wheels is very common due to the wheelset first-order bending resonance excited probably.

Figure 8 indicates the PSDs (power spectral densities) of the vertical forces, corresponding to those analyzed in time domain in Figure 7. The frequencies of the dominant vibration components (at peaks) in Figure 8 are the excitation frequencies of the harmonic track irregularity, which are, respectively, 46 Hz, 69 Hz, 92 Hz and 117 Hz at the speeds of 200 km/h, 300 km/h, 400 km/h, and 500 km/h. This indicates that track irregularity plays a dominant role in the influence on wheel/rail vertical force. In other words, the influence of the wheelset flexibility is not significant if the resonance frequency of the first-order bending mode is far away from the passing irregularity frequency. The lines of DIFF illustrate the PSDs of the differences between the vertical forces applied on the rigid and flexible wheelsets. In Figures 7(a), 7(b), and 7(c), three frequency components dominate the PSDs of the force differences. The first is that $f_1 = 31$ Hz, which is the natural frequency of the wheel-rail contact system independent of the operating speed. The second is the passing harmonic track irregularity frequency, f_2 that is equal to, respectively, 46 Hz, 69 Hz, 92 Hz and 117 Hz, corresponding to the speeds of 200 km/h, 300 km/h, 400 km/h and 500 km/h. The third is the first-order bending resonance frequency, f_3. At the speed of 500 km/h, the peak value, as seen in Figure 8(d), is about two orders of magnitude larger than the values at the frequency of 117 Hz at the other three speeds. So the difference in the dynamic behavior of the rigid and flexible wheelsets has been clearly understood.

The differences of the lateral wheel/rail forces and their PSDs have almost the same features. So the discussion on them is omitted in this paper.

The excited wheelset bending mode makes a contribution to the wheel/rail interaction to some extent, as known from the above analysis on the wheel/rail forces. The first-order bending resonance causes the oscillations of the wheel/rail contact points and the relative velocity of the wheel/rail contact surfaces. Figure 9 shows the changes of contact points on the rigid and flexible wheelsets with the travelling distance. In Figure 8, the vertical axis represents the coordinates of the wheel contact points in the direction of $Y_{(L,R)}$ axis of the local coordinate systems $O_{(L,R)}$-$X_{(L,R)}Y_{(L,R)}Z_{(L,R)}$. The black heavy lines indicate the traces of the contact points on the rigid wheelset and the fine red lines denote those on the flexile wheelset, which are, respectively, indicated by RW and EW. The contact points on the rigid wheelset are closer to the nominal rolling circles, and the contact points on the flexible wheelset move toward their corresponding field sides due to

Figure 7: Wheel/rail vertical forces at the speeds of (a) 200 km/h, (b) 300 km/h, (c) 400 km/h, and (d) 500 km/h. RW and EW denote the cases of the rigid wheelset and the elastic wheelset, respectively, and DIFF illustrates the difference between the analysis results of the rigid and the elastic wheelsets.

the bending deformation. The first-order bending resonance of the wheelset in the plane perpendicular to the track level causes the opposite rolling angles of the left and right wheels, which are nearly symmetrical about the wheelset centre. This mode shape is similar to the downward bending deformation of the wheelset axis, as shown in Figure 3. With the increase of the speed the passing irregularity frequency gradually approaches the wheelset resonance frequency of the first-order bending mode, so that the trace of the contact points oscillates more and more seriously. When the vehicle speeds up to 500 km/h, as displayed in Figure 9(d), the first-order bending resonance of the wheelset occurs and the left and right contact points on the elastic wheelset fiercely fluctuate around the traces of the contact points on the rigid wheelset. The fluctuating amplitude reaches 0.1 μm, and is five times larger than those at the speeds of 200 km/h 300 km/h and 400 km/h.

It is very interesting that the contact point traces of the rigid wheelset do not present the obvious oscillation at the passing frequency of the harmonic regularity excitation, as indicated by the black lines of RW. However, the RW lines show the position oscillations of the contact points on the rigid wheelset at a low-frequency about 22 Hz. The frequency is probably the natural frequency of the bogie in the lateral direction. The resonance of the bogie is caused by the harmonic track irregularity. When the speed is 400 km/h, the low frequency oscillation reaches the largest, as indicated in Figure 9(c), and when the speed is 500 km/h, the oscillation disappears, as indicated by the coarse black lines in Figure 9(d). The further discussion on dynamic behavior of the bogie does not belong to the task of this paper.

The above analysis shows the evident impact of the wheelset bending deformation on the contact points. The wheel/rail contact forces rely on not only the contact point

Figure 8: The PSDs of the vertical wheel/rail forces at the speeds of (a) 200 km/h, (b) 300 km/h, (c) 400 km/h, and (d) 500 km/h: $f_1 = 31$ Hz, $f_2 = 46$ Hz in (a), $f_2 = 69$ Hz in (b), $f_2 = 92$ Hz in (c), $f_2 = 117$ Hz in (d), and $f_3 = 117$ Hz.

positions but also the relative speeds of the contact surfaces represented by the creepages. There is a small difference between the creepages calculated with the models of the elastic and rigid wheelsets, as indicated in Figure 10.

As well known, rail corrugation and wheel polygonization are mainly related to the sensitive resonances of the wheel/rail system [1, 21, 22]. In order to verify that the first-order bending resonance of the wheelset is active and find the sensitive resonance frequencies of the wheel/rail systems, the present model of the high-speed vehicle coupled with the track is used to analyze the dynamic behaviour of the wheel/rail systems under a white-noise excitation and at different speeds. The analyzed frequency range is from 0 to 500 Hz and the operation speed is from 200 km/h to 400 km/h. Figure 11 shows the wheel/rail force responses analyzed in frequency-domain. Figures 11(a) and 11(b) illustrate the linear spectra of the lateral and vertical wheel/rail forces of the rigid wheelset, respectively, and Figures 11(c) and 11(d) indicate the corresponding results of the flexible wheelset. In Figure 11(b), f_0 indicates the resonance frequency of the wheel/rail system, as mentioned above. It is about 31 Hz, and independent of the operation speed. In Figures 11(a) and 11(b), f_s is the passing frequency of the sleepers and $2f_s$ is the twice passing frequency of the sleepers or called harmonic frequency due to the excitation of the multisleepers with an equal pitch between the neighboring sleepers. In Figures 11(c) and 11(d), f_{m1} denotes the bending resonance frequency of the wheelset. f_{m1} equals 117 Hz, and it is evident that the resonance frequency is independent of the vehicle speed. In all the figures of Figure 11, f_t always appears at about 380 Hz. It is one of the resonance frequencies of the system consisting of the track and the bogie of the vehicle. It will be further analyzed in the following.

Carefully making a comparison of Figures 11(a), 11(b), 11(c), and 11(d) shows the interesting results, in which one should carefully pay attention to the vertical coordinate scales of the figures. Under the present excitation of the white-noise, the wheel/rail force response has high peaks at the passing frequencies of the sleepers, or the wheel/rail force response to the excitation of the periodically discrete sleepers is clearly characterized by using the existing model of the coupling vehicle/track without considering the effect of the first-order

Figure 9: Lateral coordinates of contact points on wheel treads in associated coordinate systems at the speeds and of (a) 200 km/h, (b) 300 km/h, (c) 400 km/h, and (d) 500 km/h.

Figure 10: Lateral creepage at the speed of 500 km/h.

bending resonance of wheelset, as indicated in Figures 11(a) and 11(b). But the wheel/rail force response has very low peaks at the passing frequencies of the sleepers in Figures 11(c) and 11(d) in which the results are calculated by using the model considering the effect of the flexible wheelset. Actually, the wheel/rail force response to the excitation of the periodically discrete sleepers is overwhelmed by the responses at the other resonance frequencies. So the first-order bending resonance of the wheelset plays a dominant role in the wheel/rail force response under the present excitation of the white-noise and in the analyzed speed range.

The peaks at $f_t \approx 380$ Hz are always remarkable in the results obtained by the two models, or the vibration at about 380 Hz always makes a great contribution to the wheel/rail force response. This is because f_t is the resonance frequency of the track including the effect of the sprung mass and the bogie frame of the vehicle, and in the employed two models the track modelling is the same. In order to find the mode shape of the track resonance, a finite element model

(a) Rigid wheelset (b) Rigid wheelset

(c) Flexible wheelset (d) Flexible wheelset

Figure 11: Wheel/rail force amplitudes in frequency domain at different speeds and under a white-noise roughness excitation: (a) the lateral forces on the rigid wheelset; (b) the vertical forces on the rigid wheelset; (c) the lateral forces on the flexible wheelset; and (d) the vertical forces on the flexible wheelset.

Figure 12: Track mode shapes with resonance frequencies near 380 Hz.

for the track with a bogie is developed and used to calculate the resonance frequencies and the corresponding mode shapes of them. In Figure 12, the two figures indicate the mode shapes of the track with a bogie, the corresponding frequencies of which are, respectively, 374 Hz and 390 Hz. They are close to 380 Hz.

4. Conclusions

This paper presents a new vehicle/track coupling dynamics model considering the effect of the first-order bending resonance of the flexible wheelset. A new wheel/rail contact geometry model is derived in detail and considers

the interaction of a flexible wheelset and a pair of rails. So the existing coupling dynamics model of railway vehicle/track is further improved. In order to verify the new model and clarify the influence of the first-order bending resonance of the flexible wheelset on the wheel/rail contact behavior, the new and existing models are used to analyze the difference of the wheel/rail contact behavior under the excitation of a harmonic track irregularity with a fixed wavelength and a white-noise at different operating speeds. The results calculated by the two models are compared to understand the effect of the wheelset first-order bending on the rolling contact behavior of the wheel/rail system. The conclusions are given as follows.

(1) In the analysis cases that indicate the different operating speeds and a harmonic irregularity and white-noise excitations, the first-order bending mode of the wheelset in service can be excited easily. When the passing irregularity frequency equals the natural frequency of the first-order bending mode of wheelset, the resonance of the wheel/rail systems occurs. In such a situation, the wheelset bending has a great influence on the wheel/rail rolling contact behavior.

(2) When the first-order bending mode in the vertical plane is excited at the speed of 500 km/h, the wheel/rail forces and contact points oscillate more seriously than those at the other speeds at the first-order bending resonance frequency. Besides, the resonance easily occurs between the flexibility wheelset and the rails when the passing frequency of the track irregularity excitation is equal or close to the wheelset bending resonance frequency at any speed. Though the numeric differences between the dynamic responses of the rigid and flexible wheelsets are small, they could contribute to the development of the polygonal wear on the wheel treads after a long-time service of the wheelset.

(3) Developing the coupling dynamics model of railway vehicle and track considering the effect of structural flexibility of the key parts is a complex subject. The present work is just at a preliminary stage. The next step will be to consider modeling the high-order bending resonances of wheelset, wheel flexibility and powered wheelset flexibility in the wheel/rail rolling contact model.

Nomenclature

i (=1–4):	The number of wheelsets
j (=1-2):	The number of bogies
M_{Lx}, M_{Lz}:	The components in X- and Z-directions of the spin moment on the left wheel
M_{Rx}, M_{Rz}:	The components in X- and Z-directions of the spin moment on the right wheel
N_{Lx}, N_{Ly}, N_{Lz}:	Components in X-, Y-, and Z-directions of wheel/rail normal force on left wheel
N_{Rx}, N_{Ry}, N_{Rz}:	The components in X-, Y-, and Z-directions of wheel/rail normal force on the right wheel
F_{Lx}, F_{Ly}, F_{Lz}:	The components in X-, Y-, and Z-directions of wheel/rail tangent force on the left wheel
F_{Rx}, F_{Ry}, F_{Rz}:	The components in X-, Y-, and Z-directions of wheel/rail tangent force on the right wheel
$F_{xfL}, F_{yfL}, F_{zfL}$:	The components in X-, Y-, and Z-directions of primary suspension forces on the left sides of a wheelset
$F_{xfR}, F_{yfR}, F_{zfR}$:	The components in X-, Y-, and Z-directions of primary suspension forces on the right sides of a wheelset
t:	Time
x:	The displacement of the bending deformation in the directions of X-axis
z:	The displacement of the bending deformation in the directions of Z-axis
E:	Young's modulus of the wheelset material
ρ:	The wheelset density
A:	The equivalent cross-sectional area of the axle
G:	The gravity of the wheelset
I_z, I_x:	The inertia moments of the cross-section with respect to the Z-axis and the X-axis, respectively
Q_x, Q_z:	The concentrated forces on the wheelset axle in the X- and Z-directions
M_z, M_x:	The concentrated moments on the wheelset axle, respectively, in the negative Z-axis direction and the X-axis direction
$(x_{(L,R)3}, y_{(L,R)3}, z_{(L,R)3})$:	The coordinates describing the distances from the wheelset axle to the contact points of the left and right wheels/rails, respectively
$(x_{(L,R)4}, y_{(L,R)4}, z_{(L,R)4})$:	The coordinates describing the distances from the axle to the left and right primary suspensions, respectively
X_k, Z_k:	The kth normalized modal functions describing the bending deformations of the axle in the planes parallel and perpendicular to the track level plane, respectively
q_{xk}, q_{zk}:	The kth generalized coordinates of the bending deformations in the directions of the X-axis and the Z-axis, respectively
$\ddot{q}_{xk}, \ddot{q}_{zk}$:	The second-order derivatives of $q_{xk}(t)$ and $q_{zk}(t)$ with respect to time t

$q_{Exk}(t), q_{Ezk}(t)$: The generalized coordinates corresponding to the flexible modes
$q_{Rxk}(t), q_{Rzk}(t)$: The generalized coordinates corresponding to the rigid modes
$\lambda_{xk}, \lambda_{zk}$: The kth circular frequencies
$X_{Ek}(y), Z_{Ek}(y)$: The flexible modal functions normalized in the directions of the X-axis and the Z-axis, respectively
$X_{Rk}(y), Z_{Rk}(y)$: The rigid modal functions normalized in the directions of the X-axis and the Z-axis, respectively
NM: The total number of the modes in the calculation
NMR: The total number of the rigid modal functions
NME: The total number of the flexible modal functions
l_w: The half distance between the left and right action points of the primary suspension forces on the wheelset
l_b: The half distance between the left and right action points of the primary suspension forces on the bogie
l_{axle}: The half of the length of the wheelset axle
ψ_b: The yawing angle of the bogie
ϕ_b: The pitching angles of the bogie
β_b: The rolling angles of the bogie
K_{fx}, K_{fy}, K_{fz}: The primary suspension stiffness in X-, Y-, and Z-directions, respectively
C_{fx}, C_{fy}, C_{fz}: The primary suspension damping in X-, Y-, and Z-directions, respectively
H_{bw}: The vertical distance between the mass centers of the bogie and the wheelset
$\delta_{(L,R)}$: The left and right wheel/rail contact angles
$\phi_{w(L,R)}$: The rolling angles of the left and right wheels
$\psi_{w(L,R)}$: The yawing angles of the left and right wheels
ϕ_0: The rail cant
$\phi_{r(L,R)}$: The left and right rail rolling angles
d_B: The abscissa ordinate of the instant rolling circle center
l_x, l_y, l_z: The cosine components of the axle centerline in the X-, Y-, and Z-directions, respectively, of a rigid wheelset
$(l_{(L,R)x}, l_{(L,R)y}, l_{(L,R)z})$: The cosine components of the axle centerline in the X-, Y-, and Z-directions, respectively, of left and right wheels of a flexible wheelset
ϕ: The rolling angle of a rigid wheelset
ψ: The yawing angle of a rigid wheelset
$(X_{ow}^{\text{rigid}}, Y_{ow}^{\text{rigid}}, Z_{ow}^{\text{rigid}})$: The coordinates of a rigid wheelset center O_3
$\delta Z_{(L,R)}$: The relative vertical displacements of the left and right wheels of the flexible wheelset, respectively
$\triangle Z_{(L,R)}(0)$: The initial vertical compressions of the left and right wheels at $t = 0$
$Z_{ow(L,R)}$: The vertical displacements of the left and right wheels caused by the wheelset bending deformation occurred in the transverse section of the track
$Z_{\text{irr}(L,R)}$: The vertical irregularities of the left and right rails
$\delta Z_{(L,R)c}$: The left and right wheel/rail normal compressions
$F_{\text{wrn}(L,R)}$: The left and right wheel/rail normal forces
ξ_x, ξ_y, ξ_z: The longitudinal, lateral, and spin creepages
$v_{(L,R)}$: Forward velocities for left and right wheels
$\Delta v_{(L,R)x}, \Delta v_{(L,R)y}$: The relative longitudinal and lateral velocities
$\Delta\omega_{(L,R)z}$: The relative angular velocities
$\mathbf{v}_a$: The absolute velocity
$\mathbf{v}_r$: A relative velocity
$\mathbf{v}_e$: A transport velocity
$\mathbf{v}_{we(L,R)}$: The transport velocity used in creepage calculation of the flexible wheelset
$\mathbf{v}_{wa(L,R)}$: The relative velocities used in creepage calculation of the flexible wheelset
$\boldsymbol{\omega}_{(L,R)}$: The angular velocity vectors of the left and right wheels
$\Delta\mathbf{v}_{(L,R)}$: The velocity differences of the contact points on left and right rail and wheel, respectively.

Acknowledgments

The present work is supported by the National Natural Science Foundation of China (U1134202), the National Basic Research Program of China (2011CB711103), and the Program for Changjiang Scholars and Innovative Research Team in university (IRT1178 and SWJTU12ZT01).

References

[1] X. S. Jin, L. Wu, J. Y. Fang, S. Q. Zhong, and L. Ling, "An investigation into the mechanism of the polygonal wear of metro train wheels and its effect on the dynamic behavior of a wheel/ rail system," *Vehicle System Dynamics*, vol. 50, no. 12, pp. 1817–1834, 2012.

[2] K. L. Knothe and S. L. Grassie, "Modelling of railway track and vehicle/track interaction at high frequencies," *Vehicle System Dynamics*, vol. 22, no. 3-4, pp. 209–262, 1993.

[3] T. B. Meinders, P. Meinke, and W. O. Schiehlen, "Wear estimation in flexible multibody systems with application to railroads," in *Proceedings of the ECCOMAS Thematic Conference on Multi-Body Dynamics*, pp. 21–24, Madrid, Spain, June 2005.

[4] T. Szolc, "Medium frequency dynamic investigation of the railway wheelset-track system using a discrete-continuous model," *Archive of Applied Mechanics*, vol. 68, no. 1, pp. 30–45, 1998.

[5] K. Popp, H. Kruse, and I. Kaiser, "Vehicle-track dynamics in the mid-frequency range," *Vehicle System Dynamics*, vol. 31, no. 5-6, pp. 423–464, 1999.

[6] N. Chaar, *Wheelset structural flexibility and track flexibility in vehicle/track dynamic interaction [Ph.D. thesis]*, Royal Institute of Technology, Stockholm, Sweden, 2007.

[7] M. Meywerk, "Polygonalization of railway wheels," *Archive of Applied Mechanics*, vol. 69, no. 2, pp. 105–120, 1999.

[8] T. Szolc, "Medium frequency dynamic analysis of various designs of the high-speed railway bogies," in *Proceedings of the 7th Mini Conference on Vehicle System Dynamics, Identification and Anomalies*, pp. 255–262, November 2000.

[9] T. Meinders, "Modeling of a railway wheelset as a rotating elastic multibody system," *Machine Dynamics Problems*, vol. 20, pp. 209–219, 1998.

[10] T. Meinders and P. Meinker, "Rotor dynamics and irregular wear of elastic wheelsets," in *System Dynamics and Long-Term Behaviour of Railway Vehicles, Track and Subgrade*, vol. 6 of *Lecture Notes in Applied Mechanics*, pp. 133–152, 2002.

[11] J. Fayos, L. Baeza, F. D. Denia, and J. E. Tarancón, "An Eulerian coordinate-based method for analysing the structural vibrations of a solid of revolution rotating about its main axis," *Journal of Sound and Vibration*, vol. 306, no. 3–5, pp. 618–635, 2007.

[12] L. Baeza, P. Vila, A. Roda, and J. Fayos, "Prediction of corrugation in rails using a non-stationary wheel-rail contact model," *Wear*, vol. 265, no. 9-10, pp. 1156–1162, 2008.

[13] L. Baeza, P. Vila, G. Xie, and S. D. Iwnicki, "Prediction of rail corrugation using a rotating flexible wheelset coupled with a flexible track model and a non-Hertzian/non-steady contact model," *Journal of Sound and Vibration*, vol. 330, no. 18-19, pp. 4493–4507, 2011.

[14] A. A. Shabana, "Flexible multibody dynamics: review of past and recent developments," *Multibody System Dynamics*, vol. 1, no. 2, pp. 189–222, 1997.

[15] X. Jin, P. Wu, and Z. Wen, "Effects of structure elastic deformations of wheelset and track on creep forces of wheel/rail in rolling contact," *Wear*, vol. 253, no. 1-2, pp. 247–256, 2002.

[16] X. Xiao, X. Jin, Z. Wen, M. Zhu, and W. Zhang, "Effect of tangent track buckle on vehicle derailment," *Multibody System Dynamics*, vol. 25, no. 1, pp. 1–41, 2011.

[17] X. Xiao, X. Jin, and Z. Wen, "Effect of disabled fastening systems and ballast on vehicle derailment," *Journal of Vibration and Acoustics*, vol. 129, no. 2, pp. 217–229, 2007.

[18] X. Xiao, X. Jin, Y. Deng, and Z. Zhou, "Effect of curved track support failure on vehicle derailment," *Vehicle System Dynamics*, vol. 46, no. 11, pp. 1029–1059, 2008.

[19] W. M. Zhai, *Vehicle/Track Coupling Dynamics*, Chinese Science Press, 3rd edition, 2007.

[20] S. G. Zhang, *Gaosu Lieche Sheji Fangfa Yanjiu*, Study on Design Method of High-Speed Trains, China Railway Publishing House, Beijing, China, 2009.

[21] W. Li, X. Li, L. Wu et al., "Investigation into mechanism of metro rail corrugation using experimental and theoretical methods," in *Proceedings of the 22th Symposium of the International Association of Vehicle System Dynamics (IAVSD '11)*, Manchester, UK, 2011.

[22] X. S. Jin, B. Q. Wu, L. Wu, J. Y. Fang, and S. Q. Zhong, "An investigation into the mechanism of the polygonal wear of metro train wheels and its effect on the dynamical behavior of wheel/rail system," in *Proceedings of the 22th Symposium of the International Association of Vehicle System Dynamics (IAVSD '11)*, Manchester, UK, 2011.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 313279, 12 pages
http://dx.doi.org/10.1155/2013/313279

Research Article

Dynamic Modeling and Analysis of Hub-Tapered Beam System

Si-jia Chen[1,2] and Ding-guo Zhang[2]

[1] *Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, China*
[2] *School of Sciences, Nanjing University of Science and Technology, Nanjing 210094, China*

Correspondence should be addressed to Ding-guo Zhang; zhangdg419@njust.edu.cn

Received 9 July 2013; Revised 21 August 2013; Accepted 4 September 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 S.-j. Chen and D.-g. Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The dynamic modeling of a hub-tapered flexible beam system which is rotating in a plane is investigated. In the modeling, the high-order terms related with the nonlinear coupling term are retained, which are ignored in the first-order approximation coupling modeling. So, this model can also be called a high-order approximation coupling model, which cannot only be used in the case of small deformation but also in the case of large deformation. Solving for this mathematical model, we can obtain the dynamics of the system. Examples are given to indicate the differences between the high-order approximation coupling model and the first-order approximation coupling model. Also, the natural frequencies are studied through the transversal bending vibration analysis. The width ratio and the height ratio of the tapered beam are shown to have great influences on the transversal bending natural frequencies.

1. Introduction

Dynamics of the flexible structures can be affected by many factors, such as the shape and the motion. It is found that the dynamics of the flexible structure with large overall motion has an essential difference from the dynamics of that on an immobile base. In the past three decades, the dynamic modeling of a rotating flexible beam has received extensive research efforts [1–18]. Kane et al. [1] investigated a rotating flexible cantilever beam by using the traditional coupling model, which showed that this model fails to describe the dynamic behavior of the beam when it is at high rotation speed; and "Dynamic Stiffening" [1] was first pointed out. Then, many methodologies were developed to capture the dynamic stiffening term. Yoo and Shin [6] derived the motion equations of a rotating cantilever beam based on a new dynamic modeling method. The derived equations (governing stretching and bending motions) were all linear, so they could be directly used for the vibration analysis including the coupling effect, which could not be considered in the conventional modeling method. References [7–10] presented the first-order approximation coupling (FOAC) model, which is based on the theory of continuum medium mechanics and the theory of analysis dynamics. The FOAC model considered the second-order coupling term of longitudinal displacement caused by transversal deformation, while the traditional zeroth-order approximation coupling (ZOAC) model assumed small deformation in structural dynamics, where the longitudinal and transversal deformations are uncoupled. However, the FOAC model can only be used in the case of small deformation. Liu and Hong [11] presented a high-order approximation coupling (HOAC) model and showed that the HOAC model can be approximated to the FOAC model in the case of small deformation. However, the distinctions between HOAC model and FOAC model were not indicated.

Great progress has been made in the research on rotating flexible beam, but all the work mentioned above focused on the uniform beam. In some realistic engineering examples, such as turbine blades, the huge flagelliform antennas on the spacecraft, the cross sections of these structures are always variational along the length. We will get a more accurate dynamic analysis if we use the tapered beam model instead of the uniform beam model. Reference [14] investigated the free bending vibration of rotating tapered beams by using the dynamic stiffness method. The range of problems considered included beams for which the height and/or width of the cross

section vary linearly along the length. A parametric study was carried out to demonstrate the effects of rotational speed, taper ratio, and hub radius on the natural frequencies and mode shapes of the tapered beams. Reference [15] investigated the intrinsic characteristics and the flexible motion of the tapered Euler-Bernoulli beam with tip mass and a rotating hub. Dynamic equations were derived from the extended Hamilton principle and the FOAC model. By comparing the uniform and the tapered system in the same condition, the obtained results indicated that a little difference in section can cause a significant difference in natural frequencies and dynamics response.

Since the FOAC model neglected some high-order terms in the dynamic equations, it could not be used in the large deformation problems [16]. So an exact dynamic model is needed to solve the large deformation problems. Shabana [17, 18] presented an absolute nodal coordinate method for the large rotation and deformation problems, and some examples were given to demonstrate the use of the absolute nodal coordinate formulation in the large rotation and deformation analysis of flexible bodies. The absolute nodal coordinate method can solve the large deformation problem of the flexible body very well. However, it cannot distinguish the rigid body motion from the elastic deformation of the flexible body. Small deformation problems also should be treated as large deformation problems.

In this paper, we will study the dynamics of a hub-tapered beam system by using a high-order approximation coupling (HOAC) model, which cannot only be used in the small deformation problems but also large deformation problems. Both the transversal deformation and the longitudinal deformation of the flexible beam are considered. And in the total longitudinal deformation, the nonlinear coupling term, also known as the longitudinal shortening caused by transversal deformation, is considered. The assumed mode method (AMM) is used to describe the deformation of the flexible beam. The rigid-flexible coupling dynamic equations are established via employing the second kind of Lagrange's equation, retaining the high-order terms synchronously. Solving for this mathematical model, we can obtain the dynamics of the system.

2. Dynamic Equations

The system considered here is a flexible tapered beam driven by a hub rotor, as shown in Figure 1, which we call hub-beam system. The hub is assumed to be rigid with radius a and rotary inertia J_{oh}. Applied at the hub, there is a torque τ. The flexible beam is a tapered Euler-Bernoulli beam (Figure 2) with length L and mass density ρ. The section area of the beam is $S(x)$, and the section moment of inertia about $\mathbf{k}'$ axis is $I(x)$. Then, the bending stiffness of the beam is $EI(x)$ and the compressive stiffness is $ES(x)$, where $S(x)$ and $I(x)$ are formed as

$$\begin{aligned}\mathbf{S}(x) &= \left(\frac{x}{L}b_1 + \frac{L-x}{L}b_2\right)\left(\frac{x}{L}h_1 + \frac{L-x}{L}h_2\right),\\ \mathbf{I}(x) &= \frac{1}{12}\left(\frac{x}{L}b_1 + \frac{L-x}{L}b_2\right)^3\left(\frac{x}{L}h_1 + \frac{L-x}{L}h_2\right).\end{aligned} \tag{1}$$

Figure 1: Schematic diagram of the system.

Figure 2: Schematic diagram of the tapered beam.

The coordinate systems used in developing the model are shown in Figure 3. Wherein $o - \mathbf{ij}$ is the inertial reference frame, $o' - \mathbf{i}'\mathbf{j}'$ is a rotating frame attached to the hub such that its $\mathbf{i}'$ axis is directed along the undeformed configuration of the beam. The $\mathbf{i}'$ axis makes an angle of θ with the $\mathbf{i}$ axis. The beam deformation is expressed with respect to the rotating frame. There is an assumption that the beam is constrained in the horizontal plane of $\mathbf{ij}$.

2.1. Kinetic Energy. To a generic point $P(x)$ on the beam, the deformed position vector to point o can be expressed as

$$\mathbf{r} = \left(a + x + w_1(x,t) + w_{\mathrm{c}}(x,t)\right)\mathbf{i}' + w_2(x,t)\,\mathbf{j}', \tag{2}$$

where $w_1(x,t)$ is the axial extension quantity, $w_2(x,t)$ is the transversal deformation, and $w_c(x,t)$ is the axial shorten caused by transversal deformation. $w_c(x,t)$ is also called the coupling deformation. In the ZOAC model, we usually used the small deformation assumption in structure dynamics but ignored the coupling deformation $w_c(x,t)$. When the large overall motion is at high speed, the coupling deformation $w_c(x,t)$ will have a significant influence on the dynamic characteristic of the system. $w_c(x,t)$ can be expressed as

$$w_{\mathrm{c}}(x,t) = -\frac{1}{2}\int_0^x \left(\frac{\partial w_2(\xi,t)}{\partial \xi}\right)^2 d\xi. \tag{3}$$

Figure 3: The deformation of the beam.

The velocity vector of the point $P(x)$ can be obtained by differentiating Equation (2) as follows:

$$\begin{aligned}\dot{\mathbf{r}} &= \left(\dot{w}_1(x,t) + \dot{w}_c(x,t) - w_2(x,t)\dot{\theta}\right)\mathbf{i}' \\ &\quad + \left[\left(a + x + w_1(x,t) + w_c(x,t)\right)\dot{\theta} + \dot{w}_2(x,t)\right]\mathbf{j}';\end{aligned} \tag{4}$$

then, the kinetic energy of the system can be expressed as

$$K = \frac{1}{2}J_{\text{oh}}\dot{\theta}^2 + \frac{1}{2}\int_0^L \rho S(x)\,\dot{\mathbf{r}}^T\dot{\mathbf{r}}dx, \tag{5}$$

where the first part is the kinetic energy of the hub and the second part is the kinetic energy of the beam.

2.2. Potential Energy. As the beam is constrained in the horizontal plane, the gravitational potential energy can be ignored here. So, the potential energy of the system can be expressed as

$$\begin{aligned}V &= \frac{1}{2}\int_0^L ES(x)\left(\frac{\partial w_1(x,t)}{\partial x}\right)^2 dx \\ &\quad + \frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 w_2(x,t)}{\partial x^2}\right)^2 dx,\end{aligned} \tag{6}$$

where the first part is the stretching energy of the beam and the second part is the bending energy of the beam.

2.3. Dynamic Equations. The assumed mode method (AMM) is used to describe the deformation of the flexible beam. The axial deformation $w_1(x,t)$ and the transversal deformation $w_2(x,t)$ can be expressed as

$$\begin{aligned}w_1(x,t) &= \mathbf{\Phi}_x(x)\,\mathbf{A}(t), \\ w_2(x,t) &= \mathbf{\Phi}_y(x)\,\mathbf{B}(t),\end{aligned} \tag{7}$$

where $\mathbf{\Phi}_x(x) \in \mathbf{R}^{1\times N}$ and $\mathbf{\Phi}_y(x) \in \mathbf{R}^{1\times N}$ are the mode functions of axial and transverse vibrations of the beam, respectively. Any compact set of functions, which satisfy the boundary conditions of the cantilever beam, can be used, while $\mathbf{A}(t) \in \mathbf{R}^{N\times 1}$ and $\mathbf{B}(t) \in \mathbf{R}^{N\times 1}$ are the mode coordinates of axial and transverse vibrations of the beam, respectively. So, there is

$$w_c(x,t) = -\frac{1}{2}\,\mathbf{B}^T\mathbf{H}(x)\,\mathbf{B}, \tag{8}$$

where $\mathbf{H}(x) \in \mathbf{R}^{N\times N}$ is the coupling shape function, as follows:

$$\mathbf{H}(x) = \int_0^x \mathbf{\Phi}_y'^{\,T}(\xi)\,\mathbf{\Phi}_y'(\xi)\,d\xi. \tag{9}$$

Take the generalized coordinates $\mathbf{q} = (\theta, \mathbf{A}^T, \mathbf{B}^T)^T$ and substitute the kinetic and potential energy expressions (Equations (5) and (6)) into the second kind of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = -\frac{\partial V}{\partial q} + F_q. \tag{10}$$

Then, we will get the dynamic equation of the system written in compact form as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}, \tag{11}$$

where $\mathbf{M}$ is called generalized mass matrix and $\mathbf{Q}$ is called generalized force matrix. $\mathbf{M}$ and $\mathbf{Q}$ can be formed as

$$\mathbf{M} = \begin{bmatrix} M_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} Q_\theta \\ \mathbf{Q}_A \\ \mathbf{Q}_B \end{bmatrix} + \begin{bmatrix} \tau \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned}M_{11} &= J_{\text{oh}} + J_{\text{ob}} + 2\mathbf{S}_x\mathbf{A} + \mathbf{A}^T\mathbf{M}_x\mathbf{A} + \mathbf{B}^T\mathbf{M}_y\mathbf{B} - \underline{\mathbf{B}^T\mathbf{C}\mathbf{B}} \\ &\quad + \underline{\underline{\frac{1}{4}\int_0^L \rho S(x)\left(\mathbf{B}^T\mathbf{H}\mathbf{B}\mathbf{B}^T\mathbf{H}\mathbf{B}\right)dx}} \\ &\quad - \underline{\underline{\int_0^L \rho S(x)\left(\mathbf{A}^T\mathbf{\Phi}_x{}^T\mathbf{B}^T\mathbf{H}\mathbf{B}\right)dx}} \\ \mathbf{M}_{22} &= \mathbf{M}_x \\ \mathbf{M}_{33} &= \mathbf{M}_y + \underline{\underline{\int_o^L \rho S(x)\left(\mathbf{H}\mathbf{B}\mathbf{B}^T\mathbf{H}\right)dx}} \\ \mathbf{M}_{21} &= \mathbf{M}_{12}{}^T = -\mathbf{M}_{xy}\mathbf{B}\end{aligned}$$

$$
\begin{aligned}
\mathbf{M}_{31} &= \mathbf{M}_{13}{}^{T} = \mathbf{S}_{y}{}^{T} + \mathbf{M}_{xy}{}^{T}\mathbf{A} \\
&\quad + \underline{\underline{\int_{0}^{L} \rho S(x)\left(\mathbf{HB\Phi}_{y}\mathbf{B} - \frac{1}{2}\mathbf{\Phi}_{y}{}^{T}\mathbf{B}^{T}\mathbf{HB}\right)dx}} \\
\mathbf{M}_{32} &= \mathbf{M}_{23}{}^{T} = -\underline{\underline{\int_{0}^{L} \rho S(x)\left(\mathbf{HB\Phi}_{x}\right)dx}} \\
Q_{\theta} &= -2\dot{\theta}\left[\mathbf{S}_{x}\dot{\mathbf{A}} + \mathbf{A}^{T}\mathbf{M}_{x}\dot{\mathbf{A}} + \mathbf{B}^{T}\mathbf{M}_{y}\dot{\mathbf{B}} - \underline{\mathbf{B}^{T}\mathbf{C}\dot{\mathbf{B}}}\right] \\
&\quad - \underline{\underline{\int_{0}^{L} \rho S(x)\left(\mathbf{B}^{T}\mathbf{\Phi}_{y}{}^{T}\dot{\mathbf{B}}^{T}\mathbf{H}\dot{\mathbf{B}}\right)dx}} \\
&\quad + \underline{\underline{\dot{\theta}\int_{0}^{L} \rho S(x)\left(\dot{\mathbf{A}}^{T}\mathbf{\Phi}_{x}{}^{T}\mathbf{B}^{T}\mathbf{HB} + 2\mathbf{A}^{T}\mathbf{\Phi}_{x}{}^{T}\mathbf{B}^{T}\mathbf{H}\dot{\mathbf{B}}\right.}} \\
&\qquad \underline{\underline{\left. -\mathbf{B}^{T}\mathbf{HBB}^{T}\mathbf{H}\dot{\mathbf{B}}\right)dx}} \\
\mathbf{Q}_{A} &= \dot{\theta}^{2}\left(\mathbf{S}_{x}^{T} + \mathbf{M}_{x}\mathbf{A}\right) + 2\dot{\theta}\mathbf{M}_{xy}\dot{\mathbf{B}} - \mathbf{K}_{1}\mathbf{A} \\
&\quad - \underline{\underline{\frac{1}{2}\dot{\theta}^{2}\int_{0}^{L} \rho S(x)\left(\mathbf{\Phi}_{x}{}^{T}\mathbf{B}^{T}\mathbf{HB}\right)dx}} \\
&\quad + \underline{\underline{\int_{0}^{L} \rho S(x)\left(\mathbf{\Phi}_{x}{}^{T}\dot{\mathbf{B}}^{T}\mathbf{H}\dot{\mathbf{B}}\right)dx}} \\
\mathbf{Q}_{B} &= \dot{\theta}^{2}\left(\mathbf{M}_{y} - \underline{\mathbf{C}}\right)\mathbf{B} - 2\dot{\theta}\mathbf{M}_{xy}{}^{T}\dot{\mathbf{A}} - \mathbf{K}_{2}\mathbf{B} \\
&\quad + \underline{\underline{\dot{\theta}^{2}\int_{0}^{L} \rho S(x)\left(\frac{1}{2}\mathbf{HBB}^{T}\mathbf{HB} - \mathbf{HB\Phi}_{x}\mathbf{A}\right)dx}} \\
&\quad - \underline{\underline{2\dot{\theta}\int_{0}^{L} \rho S(x)\left(\mathbf{HB\Phi}_{y}\dot{\mathbf{B}} - \mathbf{\Phi}_{y}{}^{T}\mathbf{B}^{T}\mathbf{H}\dot{\mathbf{B}}\right)dx}} \\
&\quad - \underline{\underline{\int_{0}^{L} \rho S(x)\left(\mathbf{HB}\dot{\mathbf{B}}^{T}\mathbf{H}\dot{\mathbf{B}}\right)dx.}}
\end{aligned}
\tag{13}
$$

Equation (11) is the rigid-flexible coupling dynamic equation of the system. The parts with single underline in (13) are called the one-order coupling terms. The parts with double underlines in (13) are called the high-order coupling terms. If without the large overall motion, (11) can be degenerated as the structure dynamic equation. Actually, (11) contains four different models: the high-order approximation coupling (HOAC) model with large overall motion, the first-order approximation coupling (FOAC) model with large overall motion, the zero-order approximation coupling (ZOAC) model with large overall motion, and the structure dynamic model without large overall motion. The constant matrices in (13) can be expressed as

$$
\begin{aligned}
\mathbf{S}_{x} &= \int_{0}^{L} \rho S(x)(a+x)\mathbf{\Phi}_{x}(x)\,dx \\
\mathbf{S}_{y} &= \int_{0}^{L} \rho S(x)(a+x)\mathbf{\Phi}_{y}(x)\,dx \\
\mathbf{M}_{x} &= \int_{0}^{L} \rho S(x)\mathbf{\Phi}_{x}{}^{T}(x)\mathbf{\Phi}_{x}(x)\,dx \\
\mathbf{M}_{y} &= \int_{0}^{L} \rho S(x)\mathbf{\Phi}_{y}{}^{T}(x)\mathbf{\Phi}_{y}(x)\,dx \\
\mathbf{M}_{xy} &= \int_{0}^{L} \rho S(x)\mathbf{\Phi}_{x}^{T}(x)\mathbf{\Phi}_{y}(x)\,dx \\
\mathbf{C} &= \int_{0}^{L} \rho S(x)(a+x)\mathbf{H}(x)\,dx \\
\mathbf{K}_{1} &= \int_{0}^{L} ES(x)\mathbf{\Phi}_{x}'^{T}(x)\mathbf{\Phi}_{x}'(x)\,dx \\
\mathbf{K}_{2} &= \int_{0}^{L} EI(x)\mathbf{\Phi}_{y}''^{T}(x)\mathbf{\Phi}_{y}''(x)\,dx \\
J_{\mathrm{ob}} &= \int_{0}^{L} \rho S(x)(a+x)^{2}dx.
\end{aligned}
\tag{14}
$$

3. First-Order Approximation Coupling (FOAC) Model

When evolving the kinetic energy expression, since the coupling deformation $w_c(x,t)$ is the second-order amount of the transversal deformation $w_2(x,t)$, some high-order amount respected to $w_c(x,t)$ can be ignored. Thus, the parts with double underlines in (13) can be ignored. So, (11) can be transferred as another form

$$
\begin{aligned}
&\begin{bmatrix} M_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{0} \\ \mathbf{M}_{31} & \mathbf{0} & \mathbf{M}_{33} \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\mathbf{A}} \\ \ddot{\mathbf{B}} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{23} \\ \mathbf{0} & \mathbf{G}_{32} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\mathbf{A}} \\ \dot{\mathbf{B}} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{33} \end{bmatrix}\begin{bmatrix} \theta \\ \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} Q_{\theta} \\ \mathbf{Q}_{1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},
\end{aligned}
\tag{15}
$$

where

$$M_{11} = J_{\mathrm{oh}} + J_{\mathrm{ob}} + 2\mathbf{S}_{x}\mathbf{A} + \mathbf{A}^{T}\mathbf{M}_{x}\mathbf{A} + \mathbf{B}^{T}\mathbf{M}_{y}\mathbf{B} - \underline{\mathbf{B}^{T}\mathbf{CB}} \tag{16}$$

$$\mathbf{M}_{22} = \mathbf{M}_{x} \tag{17}$$

$$\mathbf{M}_{33} = \mathbf{M}_{y} \tag{18}$$

$$\mathbf{M}_{21} = \mathbf{M}_{12}{}^{T} = -\mathbf{M}_{xy}\mathbf{B} \tag{19}$$

$$\mathbf{M}_{31} = \mathbf{M}_{13}{}^{T} = \mathbf{S}_{y}{}^{T} + \mathbf{M}_{xy}{}^{T}\mathbf{A} \tag{20}$$

$$\mathbf{G}_{23} = -\mathbf{G}_{32}^{T} = -2\dot{\theta}\mathbf{M}_{xy} \tag{21}$$

$$\mathbf{K}_{22} = \mathbf{K}_{1} - \dot{\theta}^{2}\mathbf{M}_{x} \tag{22}$$

$$\mathbf{K}_{33} = \mathbf{K}_{2} - \dot{\theta}^{2}\left(\mathbf{M}_{y} - \underline{\mathbf{C}}\right) \tag{23}$$

Table 1: The dimensionless natural frequencies of the uniform beam in transversal bending vibration with different γ ($\delta = 0$, $\alpha = \beta = 1$).

γ	First natural frequency		Second natural frequency		Third natural frequency	
	Present	Paper [13]	Present	Paper [13]	Present	Paper [13]
0	3.51601	3.51602	22.0345	22.0345	61.6973	61.6972
1	3.54323	3.54324	22.1585	22.1585	61.8338	61.8337
2	3.62181	3.62182	22.5264	22.5264	62.2413	62.2412
3	3.74361	3.74361	23.1268	23.1268	62.9144	62.9144
4	3.89804	3.89804	23.9423	23.9423	63.8443	63.8443
5	4.07471	4.07471	24.9520	24.9520	65.0196	65.0195
6	4.26498	4.26498	26.1328	26.1328	66.4266	66.4265
7	4.46252	4.46251	27.4623	27.4623	68.0502	68.0501
8	4.66305	4.66305	28.9191	28.9191	69.8748	69.8746
9	4.86398	4.86398	30.4841	30.4841	71.8842	71.8840
10	5.06384	5.06384	32.1408	32.1408	74.0628	74.0626

$$Q_\theta = -2\dot{\theta}\left[\mathbf{S}_x\dot{\mathbf{A}} + \mathbf{A}^T\mathbf{M}_x\dot{\mathbf{A}} + \mathbf{B}^T\mathbf{M}_y\dot{\mathbf{B}} - \underline{\mathbf{B}^T\mathrm{C}\dot{\mathbf{B}}}\right] \tag{24}$$

$$\mathbf{Q}_1 = \dot{\theta}^2\mathbf{S}_x^T. \tag{25}$$

Equation (15) is called the FOAC dynamic equations. The parts with single underline in (16), (23), and (24) are called one-order coupling terms. In the traditional ZOAC model, these parts are always ignored.

3.1. Transversal Bending Vibration Analysis Ignoring Longitudinal Deformation Effect. The transversal bending vibration analysis will be studied in this section. For simplifying analysis, the angular speed of the large overall motion is assumed to be constant as $\ddot{\theta} = 0$. Compared to the transversal deformation, the longitudinal deformation has little influence on the transversal bending vibration. So, the transversal bending vibration equation of the system with constant angular speed can be simplified as

$$\mathbf{M}_{33}\ddot{\mathbf{B}} + \mathbf{K}_{33}\mathbf{B} = \mathbf{0}. \tag{26}$$

For convenience of discussion, dimensionless parameters are used in the present analysis and they are defined as

$$\begin{aligned} &\varsigma = \frac{t}{T}, \qquad \xi = \frac{x}{L}, \qquad \boldsymbol{\kappa}_2 = \frac{\mathbf{B}}{L}, \qquad \delta = \frac{a}{L}, \\ &\gamma = T\dot{\theta}, \qquad \alpha = \frac{b_1}{b_2}, \qquad \beta = \frac{h_1}{h_2}, \end{aligned} \tag{27}$$

where $T = (\rho \mathrm{S}_0 \mathrm{L}^4/EI_0)^{1/2}$ and $S_0 = b_2h_2$, $I_0 = (1/12)b_2{}^3h_2$. And δ, γ, α, and β will denote the hub radius ratio, the angular speed ratio, the width ratio, and the height ratio, respectively.

The dimensionless form of the free vibration equation is finally obtained by using the dimensionless variables and parameters defined in (27)

$$\overline{\mathbf{M}_{33}}\ddot{\boldsymbol{\kappa}}_2 + \overline{\mathbf{K}_{33}}\boldsymbol{\kappa}_2 = 0, \tag{28}$$

where

$$\begin{aligned} \overline{\mathbf{M}_{33}} &= \int_0^1 [(1-\xi)+\xi\alpha][(1-\xi)+\xi\beta]\,\mathbf{\Phi}_y{}^T(\xi)\,\mathbf{\Phi}_y(\xi)\,d\xi \\ \overline{\mathbf{K}_{33}} &= \int_0^1 [(1-\xi)+\xi\alpha]^3[(1-\xi)+\xi\beta]\,\mathbf{\Phi}_y''^T(\xi)\,\mathbf{\Phi}_y''(\xi)\,d\xi \\ &\quad - \gamma^2\int_0^1 [(1-\xi)+\xi\alpha][(1-\xi)+\xi\beta]\,\mathbf{\Phi}_y{}^T(\xi)\,\mathbf{\Phi}_y(\xi)\,d\xi \\ &\quad + \underline{\gamma^2\int_0^1 [(1-\xi)+\xi\alpha][(1-\xi)+\xi\beta](\delta+\xi)\,\overline{\mathbf{H}}(\xi)\,d\xi} \\ \overline{\mathbf{H}}(\xi) &= \int_0^\xi \left(\frac{\partial\mathbf{\Phi}_y(\eta)}{\partial\eta}\right)^T\left(\frac{\partial\mathbf{\Phi}_y(\eta)}{\partial\eta}\right)d\eta. \end{aligned} \tag{29}$$

Let $\boldsymbol{\kappa}_2 = \mathbf{e}^{\lambda\omega\varsigma}\mathbf{\Theta}$; we can get the eigenvalue equation

$$\omega^2\overline{\mathbf{M}_{33}}\mathbf{\Theta} = \overline{\mathbf{K}_{33}}\mathbf{\Theta}, \tag{30}$$

where λ is imaginary, ω is the dimensionless natural frequency, and $\mathbf{\Theta}$ is a constant column matrix.

Table 1 shows the dimensionless natural frequencies of the uniform beam without hub in the transversal bending vibration under different γ, where $\delta = 0$, and $\alpha = \beta = 1$, truncate the first four modes when calculating. From Table 1, we can see that the dimensionless natural frequencies here are the same as the results of paper [13], using another modeling method. That proves the correctness of this model. So we will truncate the first four modes when calculating in the following analysis. Additionally, we can see that the dimensionless natural frequencies increase as the angular speed ratio γ increases.

In Table 2, two models are compared, one is the FOAC model in this paper, and the other is the traditional ZOAC model (without the single underline parts). Also, the beam is considered to be a uniform one for simplifying analysis, where $\delta = \alpha = \beta = 1$. From (28), we can find that the two

Table 2: The dimensionless natural frequencies of the uniform beam in the transversal bending vibration with different models ($\delta = \alpha = \beta = 1$).

γ	First natural frequency		Second natural frequency		Third natural frequency	
	FOAC model	ZOAC model	FOAC model	ZOAC model	FOAC model	ZOAC model
0	3.51601	3.51601	22.0345	22.0345	61.6973	61.6973
2	4.40063	2.89177	23.2807	21.9435	63.0368	61.6649
10	13.2934	9.36149i	43.3038	19.6346	88.9661	60.8815
50	62.8131	49.8762i	183.989	44.8829i	321.035	36.1463

Table 3: The dimensionless natural frequencies of the tapered beam in the transversal bending vibration with different δ ($\alpha = \beta = 0.5$).

δ	First natural frequency		Second natural frequency		Third natural frequency	
	$\gamma = 2$	$\gamma = 10$	$\gamma = 2$	$\gamma = 10$	$\gamma = 2$	$\gamma = 10$
0	4.75279	6.52741	19.9938	28.3386	49.3721	59.7309
0.5	5.10793	11.2473	20.3400	33.8886	49.7335	66.8171
1	5.43934	14.4910	20.6802	38.6215	50.0924	73.2230
5	7.57406	29.5438	23.2193	64.8417	52.8764	111.989
10	9.57836	41.2091	26.0391	87.0114	56.1639	146.690

Table 4: The first dimensionless natural frequencies in the transversal bending vibration with different α ($\delta = \beta = 1$).

α	$\gamma = 2$		$\gamma = 10$	
	First natural frequency	Change (%)	First natural frequency	Change (%)
0	6.20400		15.8566	
0.3	4.99493	19	14.2455	10
0.5	4.72715	5.4	13.8222	3.0
0.7	4.56147	3.5	13.5539	1.9
1	4.40063	3.5	13.2934	1.9

models are same when $\gamma = 0$. So the natural frequencies are equal when $\gamma = 0$. When γ increasing, the natural frequencies of the two models become discriminative. When the angular speed of the large overall motion is low ($\gamma = 2$), the natural frequencies of the two models have a small distinction. But when the large overall motion is at high speed, the natural frequencies of the ZOAC model will be imaginary, which is impractical, while the natural frequencies of the FOAC model are still real. So we can say that the traditional ZOAC model without the coupling deformation is inapplicable when the large overall motion of the beam is at high speed.

Table 3 shows the dimensionless natural frequencies of the tapered beam in the transversal bending vibration under different δ, where $\alpha = \beta = 0.5$. It is obvious that the dimensionless natural frequency is proportional to the hub radius ratio δ, especially when the large overall motion is at high speed.

Table 4 shows the first dimensionless natural frequencies of the tapered beam with different α when $\delta = \beta = 1$. As we can see, the first natural frequency is in inverse proportion to the width ratio α. When α changes from 0 to 0.3, the first natural frequency will have a great change. Compared to the change of the first natural frequency when $\gamma = 2$, the change of the first natural frequency is much smaller when $\gamma = 10$.

Table 5 shows the first dimensionless natural frequencies of the tapered beam with different β when $\delta = \alpha = 1$. Also, the first natural frequency is in inverse proportion to the height ratio β. As we can see, the changes of the first natural frequency under different β have the same variation as the changes of the first natural frequency under different α.

3.2. Transversal Bending Vibration Analysis Including Longitudinal Deformation Effect. The transversal bending vibration analysis including longitudinal deformation effect will be studied in this section. Also, the angular speed of the large overall motion is assumed to be constant as $\ddot{\theta} = 0$, and $\mathbf{Q}_1$ is neglected here for the free vibration analysis. So, the free vibration equation of the system with constant angular speed can be formed as

$$\begin{pmatrix} \mathbf{M}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{33} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{A}} \\ \ddot{\mathbf{B}} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{G}_{23} \\ \mathbf{G}_{32} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{A}} \\ \dot{\mathbf{B}} \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = \mathbf{0}. \tag{31}$$

Other dimensionless parameters are defined as

$$\boldsymbol{\kappa}_1 = \frac{\mathbf{A}}{L}, \qquad \varepsilon = \left(\frac{S_0 L^2}{I_0} \right)^{1/2}, \tag{32}$$

Table 5: The first dimensionless natural frequencies in the transversal bending vibration with different β ($\delta = \alpha = 1$).

β	$\gamma = 2$		$\gamma = 10$	
	First natural frequency	Change (%)	First natural frequency	Change (%)
0	7.74540	27	16.2564	12
0.3	5.65629	9.8	14.3832	3.4
0.5	5.09972	6.9	13.8975	2.2
0.7	4.74692	7.3	13.5916	2.2
1	4.40063		13.2934	

where ε is proportional to the length to thickness ratio of a beam, called the slenderness ratio. Now, the dimensionless form of the free vibration equation can be obtained as

$$\begin{pmatrix} \overline{\mathbf{M}_{22}} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{M}_{33}} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{\kappa}}_1 \\ \ddot{\boldsymbol{\kappa}}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \overline{\mathbf{G}_{23}} \\ \overline{\mathbf{G}_{32}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{\kappa}}_1 \\ \dot{\boldsymbol{\kappa}}_2 \end{pmatrix} + \begin{pmatrix} \overline{\mathbf{K}_{22}} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{K}_{33}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{pmatrix} = \mathbf{0}, \tag{33}$$

where

$$\overline{\mathbf{M}_{22}} = \int_0^1 [(1-\xi)+\xi\alpha]\,[(1-\xi)+\xi\beta]\,\boldsymbol{\Phi}_x{}^T(\xi)\,\boldsymbol{\Phi}_x(\xi)\,d\xi \tag{34}$$

$$\begin{aligned} \overline{\mathbf{G}_{23}} &= -\overline{\mathbf{G}_{32}}^T \\ &= -2\gamma \int_0^1 [(1-\xi)+\xi\alpha]\,[(1-\xi)+\xi\beta]\,\boldsymbol{\Phi}_x{}^T(\xi)\,\boldsymbol{\Phi}_y(\xi)\,d\xi \end{aligned} \tag{35}$$

$$\begin{aligned} \overline{\mathbf{K}_{22}} &= \varepsilon^2 \int_0^1 [(1-\xi)+\xi\alpha]\,[(1-\xi)+\xi\beta]\,\boldsymbol{\Phi}_x'^T(\xi)\,\boldsymbol{\Phi}_x'(\xi)\,d\xi \\ &\quad - \gamma^2 \int_0^1 [(1-\xi)+\xi\alpha]\,[(1-\xi)+\xi\beta] \\ &\qquad \times \boldsymbol{\Phi}_x{}^T(\xi)\,\boldsymbol{\Phi}_x(\xi)\,d\xi. \end{aligned} \tag{36}$$

Equation (31) can also be transformed into the following form:

$$\mathbf{M}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{0}, \tag{37}$$

where

$$\mathbf{M} = \begin{pmatrix} \overline{\mathbf{M}_{22}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{M}_{33}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} \mathbf{0} & \overline{\mathbf{G}_{23}} & \overline{\mathbf{K}_{22}} & \mathbf{0} \\ \overline{\mathbf{G}_{32}} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{K}_{33}} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{38}$$

$$\mathbf{z} = \begin{pmatrix} \dot{\boldsymbol{\kappa}}_1 \\ \dot{\boldsymbol{\kappa}}_2 \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{pmatrix}.$$

Let $\mathbf{z} = \mathbf{Z}e^{\sigma\varsigma}$; we can get

$$(\sigma\,\mathbf{M} + \mathbf{K})\,\mathbf{Z} = \mathbf{0}, \tag{39}$$

where σ is the complex eigenvalue.

Table 6 shows the comparison of the first transversal natural frequencies with and without longitudinal deformation effect when $\alpha = \beta = 0.5$ and $\varepsilon = 70$. The value of $\varepsilon = 70$ guarantees the assumptions of Euler-Bernoulli beam. From (35), we can see that matrix $\overline{\mathbf{G}_{32}}$ is proportional to the angular speed ratio γ. So, the larger γ is, the greater the influence of the longitudinal deformation on the transversal bending vibration is. When the beam is immobile as $\gamma = 0$, (33) can be degenerated to (28). But when the beam has large overall motion, the first natural frequency including longitudinal deformation will be smaller than that ignoring longitudinal deformation. Especially when the large overall motion is at very high speed ($\gamma = 50$), the error will be significant. So, we can get a conclusion that the longitudinal deformation will also have an obvious influence on the transversal bending vibration when the large overall motion is at very high speed.

4. High-Order Approximation Coupling (HOAC) Model

Equation (11) is the rigid-flexible coupling dynamic equation of the system, which is also called the high-order approximation coupling (HOAC) dynamic equation. The parts with double underlines in (13) are called the high-order coupling terms. These terms are always ignored in the FOAC dynamic

Table 6: Comparison of the first transversal natural frequencies with and without longitudinal effect ($\alpha = \beta = 0.5$, $\varepsilon = 70$).

δ	γ	Longitudinal deformation ignored	Longitudinal deformation included	Error (%)
0	0	4.62520	4.62520	0
	2	4.75279	4.75107	0.036
	10	6.52741	6.46607	0.94
	50	14.8390	11.8649	20
1	0	4.62520	4.62520	0
	2	5.43934	5.43737	0.036
	10	14.4910	14.3536	0.95
	50	66.4377	51.7201	22

(a) $\tau_0 = 36\,\mathrm{N{\cdot}m}$ (b) $\tau_0 = 136\,\mathrm{N{\cdot}m}$

Figure 4: The tip deformation of the tapered beam.

Table 7: Data of the hub-tapered beam system.

Property	Symbol	Value
Effective length of the beam	L	8 m
Tip width of the tapered beam	b_1	1.5×10^{-2} m
Bottom width of the tapered beam	b_2	2.5×10^{-2} m
Tip height of the tapered beam	h_1	1.5×10^{-2} m
Bottom height of the tapered beam	h_2	2.5×10^{-2} m
Mass density of the beam	ρ	$2.7667 \times 10^3\,\mathrm{kg{\cdot}m^{-3}}$
Elastic modulus of the beam	E	$6.8952 \times 10^{10}\,\mathrm{N{\cdot}m^{-2}}$
Radius of the hub	a	1 m
Rotary inertia of the hub	J_{oh}	$5\,\mathrm{kg{\cdot}m^2}$

equations for convenience. But they will have a great influence on the dynamics of the system with large deformation. Here, two examples will be given to approve it.

4.1. Beam Undergoing Prescribed Translation and Rotation. A tapered beam made of aluminum with the dimensions and material properties given in Table 7 is used. The boundary conditions of a cantilever beam are used in the simulations. And the torque applied at the hub τ can be expressed as

$$\tau(t) = \begin{cases} \tau_0 \sin\left(\dfrac{2\pi}{T} t\right) & 0 \le t \le T \\ 0 & t > T, \end{cases} \tag{40}$$

where $T = 2s$.

Figure 4(a) gives the tip deformations of the tapered beam with HOAC model and FOAC model when $\tau_0 = 36\,\mathrm{N{\cdot}m}$. When the torque is applied on the hub, the max tip deformation is 0.56 m, which is only a small deformation to the effective length of the beam. In this case, the tip deformations of the HOAC model and the FOAC model are exactly coincidental, while Figure 4(b) gives the tip deformations of the two models when $\tau_0 = 136\,\mathrm{N{\cdot}m}$. When the torque applying on the hub, the max tip deformation of the HOAC model reached 2.32 m, this is obviously a large deformation to the effective length of the beam. So, in this case, the result of the FOAC model becomes divergent as shown in Figure 4(b). On the other hand, the result of the HOAC model is still convergent. Then, we can say that the FOAC model is inapplicable in the situation of large deformation, while the HOAC

Figure 5: The deformation in rotating frame.

Figure 6: The motion in inertial reference frame.

model is applicable. That is because the high-order terms with double underlines respected to $w_c(x,t)$ were ignored in the FOAC modeling.

The deformation of the tapered beam in the rotating frame $o' - \mathbf{i}'\mathbf{j}'$, with HOAC model when $\tau_0 = 136\,\mathrm{N{\cdot}m}$, is shown in Figure 5. And the motion in the inertial reference frame $o - \mathbf{ij}$ is shown in Figure 6.

In order to research the dynamic response of the tapered beam, two ratios are defined here, the width ratio $R_b = b_1/b_2$ and the height ratio $R_h = h_1/h_2$. And two situations are assumed here, on the assumption that the weight of the tapered beam stays invariable.

(1) Suppose the height of the beam keeps uniform, as $h_1 = h_2 = 0.02\,\mathrm{m}$. Then, we will get

$$(1 + R_b)\, b_2 = 0.04. \tag{41}$$

(2) Suppose the width of the beam keeps uniform, as $b_1 = b_2 = 0.02\,\mathrm{m}$. Then, we will get

$$(1 + R_h)\, h_2 = 0.04. \tag{42}$$

Figure 7 gives the max tip deformations and the response frequency of the tapered beam with different R_b and R_h. It is apparent that the max tip deformation increases as R_b and R_h increases, while the response frequency decreases as R_b and R_h increases. Also, we will find that the change of R_b will have a greater influence on the dynamics of the tapered beam on our assumption (the weight of the tapered beam keeps invariable). But in any case, all R_b and R_h will have a great influence on the dynamics of the tapered beam.

4.2. Single Pendulum with Gravity. To illustrate the effect of large deformation, a flexible tapered pendulum with free falling is investigated as shown in Figure 8. The pendulum is assumed to fall under the effect of gravity.

To a flexible tapered pendulum with free falling, the gravitational potential energy should be included. So, the gravitational potential energy will be added to the system potential energy

$$V_g = -\mathbf{g}^T \mathbf{W} \int_0^L \rho S(x)\, \mathbf{r} dx = -\mathbf{g}^T \mathbf{W} \mathbf{R}, \tag{43}$$

where $\mathbf{g}$ is the gravitational acceleration array and $\mathbf{W}$ is the homogeneous transfer matrix. $\mathbf{g}$, $\mathbf{W}$, and $\mathbf{R}$ can be expressed as

$$\mathbf{g}^T = \left[g_x, g_y\right] \qquad \mathbf{W} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$\mathbf{R} = \int_0^L \rho S \mathbf{r} dx. \tag{44}$$

Since the FOAC model could not be used in the large deformation problems, paper [16] investigated the application of an absolute nodal coordinate formulation in the coupling dynamics of flexible beams with large deformation. The simulation of a flexible pendulum indicated that the absolute nodal coordinate formulation is suited for the beams with large deformation, while the FOAC model cannot be used in that case. Here, we will simulate the example of Figure 4 in paper [16] by using the HOAC model. In reference to paper [16], the flexible pendulum is a uniform one, $\rho = 2.76667 \times 10^3\,\mathrm{kg.m^{-3}}$, $L = 1.8\,\mathrm{m}$, $S(x) = 2.5 \times 10^{-4}\,\mathrm{m^2}$, $I(x) = 1.3 \times 10^{-9}\,\mathrm{m^4}$, and $E = 6.8952 \times 10^9\,\mathrm{N{\cdot}m^{-2}}$.

Figure 9 shows the tip deformation of the flexible pendulum with large deformation. As we can see, the result of the FOAC model is divergent, while the result of the HOAC model is nearly the same as the result of paper [16]. This proves the validity of the HOAC model and the applicability in the large deformation situation. As we know, the absolute nodal coordinate method cannot distinguish the rigid body motion from the elastic deformation of the flexible body. Small deformation problems also should be treated as large deformation problems, while the FOAC model can only deal with the small deformation problems. However, the HOAC model proposed in this paper can make up for the deficiency of those two methods.

To further prove the applicability of the HOAC model in the large deformation situation, a quiet flexible tapered pendulum will be studied here. The dimensions and material properties are still as shown in Table 7, except $E = 6.8952 \times 10^5\,\mathrm{N{\cdot}m^{-2}}$, a = 0 m.

Figure 7: The dynamic impact of R_b, R_h.

Figure 8: The flexible tapered pendulum.

Figure 9: The tip deformation of the uniform pendulum.

Figure 10: The deformed shape at different times when $E = 6.8952 \times 10^5\ \mathrm{N \cdot m^{-2}}$.

Figure 10 shows the deformed shape of the tapered pendulum with HOAC model at different time steps when $E = 6.8952 \times 10^5\ \mathrm{N \cdot m^{-2}}$. As we can see, the deformation is very large at any time step. If we use the FOAC or ZOAC models, the simulation will be divergent.

Figure 11 shows the energy change of the tapered pendulum. Curve 1 is the kinetic energy, curve 2 is the elastic potential energy, curve 3 is the gravitational potential energy, and curve 4 is the total energy. Obviously, as the pendulum falls, the kinetic energy increases at the beginning stage. But when the pendulum passes someplace, the kinetic energy will decrease. Also, the elastic potential energy changes the same law as the kinetic energy, while the gravitational potential energy decreases before the pendulum falls to the lowest point. When the pendulum passes the lowest point, the gravitational potential energy will increase. Anyway, the total energy keeps invariant at any time.

Figure 12 shows the deformed shape of the tapered pendulum with HOAC model at different time steps when $E = 6.8952 \times 10^{10}\ \mathrm{N \cdot m^{-2}}$. As we can see, the deformation is very

Figure 11: The energy change of the tapered pendulum when $E = 6.8952 \times 10^5\ \mathrm{N{\cdot}m^{-2}}$.

Figure 12: The deformed shape at different times when $E = 6.8952 \times 10^{10}\ \mathrm{N{\cdot}m^{-2}}$.

Figure 13: The energy change when $E = 6.8952 \times 10^5\ \mathrm{N{\cdot}m^{-2}}$ and $E = 6.8952 \times 10^{10}\ \mathrm{N{\cdot}m^{-2}}$.

small at any time step. So, the elastic potential energy is almost 0 (Figure 13). However, the elastic potential energy is very significant when $E = 6.8952 \times 10^5\ \mathrm{N{\cdot}m^{-2}}$. Also, there are some differences between the kinetic energies when $E = 6.8952 \times 10^5\ \mathrm{N{\cdot}m^{-2}}$ and $E = 6.8952 \times 10^{10}\ \mathrm{N{\cdot}m^{-2}}$. The kinetic energy change is more complex when large deformation occurs.

5. Conclusion

A new rigid-flexible coupling model of a hub-tapered beam system, which is also called the high-order approximation coupling (HOAC) model, was established in this paper. The HOAC model could not only be used in the case of small deformation, but also in the case of large deformation, while the previous FOAC model could only be used in the case of small deformation. Based on the HOAC model, the motion and the dynamic response of the tapered beam were studied.

Also, the natural frequencies of the FOAC model were studied through the transversal bending vibration analysis. The dimensionless natural frequencies were shown to increase as the angular speed ratio γ and the hub radius ratio δ increased. However, they decreased as the width ratio α and the height ratio β increased. When the large overall motion was at very high speed, the longitudinal deformation would also have a significant influence on the transversal bending vibration.

Acknowledgments

This work was financially supported by the National Natural Science Foundation of China (nos. 11302192, 10772085, 11132007, and 11272155) and the 333 Project of Jiangsu Province (BRA2011172), for which the authors are grateful.

References

[1] T. R. Kane, R. R. Ryan, and A. K. Banerjee, "Dynamics of a cantilever beam attached to a moving base," *Journal of Guidance, Control, and Dynamics*, vol. 10, no. 2, pp. 139–151, 1987.

[2] A. K. Banerjee and J. M. Dickens, "Dynamics of an arbitrary flexible body in large rotation and translation," *Journal of Guidance, Control, and Dynamics*, vol. 13, no. 2, pp. 221–227, 1990.

[3] A. Q. Liu and K. M. Liew, "Non-linear substructure approach for dynamic analysis of rigid-flexible multibody systems," *Computer Methods in Applied Mechanics and Engineering*, vol. 114, no. 3-4, pp. 379–396, 1994.

[4] I. Sharf, "Geometric stiffening in multibody dynamics formulations," *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 4, pp. 882–890, 1995.

[5] D. J. Zhang and R. L. Huston, "On dynamic stiffening of flexible bodies having high angular velocity," *Mechanics of Structures and Machines*, vol. 24, no. 3, pp. 313–329, 1996.

[6] H. H. Yoo and S. H. Shin, "Vibration analysis of rotating cantilever beams," *Journal of Sound and Vibration*, vol. 212, no. 5, pp. 807–808, 1998.

[7] J. Y. Liu and J. Z. Hong, "Dynamic modeling and modal truncation approach for a high-speed rotating elastic beam," *Archive of Applied Mechanics*, vol. 72, no. 8, pp. 554–563, 2002.

[8] H. Yang, J. Hong, and Z. Yu, "Dynamics modelling of a flexible hub-beam system with a tip mass," *Journal of Sound and Vibration*, vol. 266, no. 4, pp. 759–774, 2003.

[9] G.-P. Cai, J.-Z. Hong, and S. X. Yang, "Dynamic analysis of a flexible hub-beam system with tip mass," *Mechanics Research Communications*, vol. 32, no. 2, pp. 173–190, 2005.

[10] G.-P. Cai and C. W. Lim, "Dynamics studies of a flexible hub-beam system with significant damping effect," *Journal of Sound and Vibration*, vol. 318, no. 1-2, pp. 1–17, 2008.

[11] J. Y. Liu and J. Z. Hong, "Geometric stiffening effect on rigid-flexible coupling dynamics of an elastic beam," *Journal of Sound and Vibration*, vol. 278, no. 4-5, pp. 1147–1162, 2004.

[12] D.-G. Zhang and J.-B. Yu, "Dynamical analysis of a flexible cantilever beam with large overall motions," *Journal of Vibration Engineering*, vol. 19, no. 4, pp. 475–480, 2006.

[13] J. Cheng, H. Xu, and A. Yan, "Frequency analysis of a rotating cantilever beam using assumed mode method with coupling effect," *Mechanics Based Design of Structures and Machines*, vol. 34, no. 1, pp. 25–47, 2006.

[14] J. R. Banerjee, H. Su, and D. R. Jackson, "Free vibration of rotating tapered beams using the dynamic stiffness method," *Journal of Sound and Vibration*, vol. 298, no. 4-5, pp. 1034–1054, 2006.

[15] Y.-A. Huang and Z.-C. Deng, "Dynamics analysis of the hub and tapered beam coupled system with tip mass," *Chinese Journal of Computational Mechanics*, vol. 24, no. 1, pp. 14–49, 2007.

[16] B. Li and J.-Y. Liu, "Application of absolute nodal coordination formulation in flexible beams with large deformation," *Journal of Shanghai Jiaotong University*, vol. 39, no. 5, pp. 827–831, 2005.

[17] A. A. Shabana, "Finite element incremental approach and exact rigid body inertia," *Journal of Mechanical Design*, vol. 118, no. 2, pp. 171–178, 1996.

[18] M. A. Omar and A. A. Shabana, "A two-dimensional shear deformable beam for large rotation and deformation problems," *Journal of Sound and Vibration*, vol. 243, no. 3, pp. 565–576, 2001.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 784525, 10 pages
http://dx.doi.org/10.1155/2013/784525

Research Article

Analysis of Elastic-Plastic Responses of a New Single-Story Frame-Bent Structure during an Earthquake Based on the Transfer Matrix Method of Multibody System

Jianguo Ding,[1] Yuwei Dai,[1,2] and Zhi Qiao[1]

[1] *School of Sciences, Nanjing University of Science and Technology, Nanjing 210094, China*
[2] *International Century Architecture Design Co., Ltd., No. 11 Taihu Middle Road, Changzhou 213000, China*

Correspondence should be addressed to Jianguo Ding; nustdjg@163.com

Received 5 June 2013; Accepted 25 August 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Jianguo Ding et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Finite element methods, such as the ANSYS, are widely used as traditional structural dynamic analysis methods. Due to the low calculation efficiency of the ANSYS, there is a hindrance factor in using the finite element method to compute the elastic-plastic responses of special engineering structures. In this paper, we detail a pioneer study in which the transfer matrix method of multibody system (MS-TMM) was introduced to analyze the dynamic responses of civil engineering structures. Although the responses obtained by the MS-TMM are similar to the responses obtained by the ANSYS, the ANSYS requires a greater amount of computation time. The results indicate that the MS-TMM can satisfy the requirements of calculation efficiency and calculation accuracy.

1. Introduction

Negative Gaussian curvature board is the scientific name for a prestressed concrete saddle plate, which is commonly known as a saddle plate. The saddle plate is a prestressed element that is composed of a combination of beam and plate. The saddle plate is widely used in frame-bent industrial structures in southern China [1]. In frame-bent structures, a framework is used in the longitudinal direction, and the saddle plates are used on the roof [2]. Although seismic analyses of single-story industrial buildings with saddle plates are rare, they possess practical significance.

Finite element methods, such as the ANSYS, are widely used as traditional structural dynamic analysis methods. However, the calculation efficiency of the ANSYS is low, which has prevented its application in special engineering fields [3]. With social and technological advancements, the demand for calculation accuracy and calculation efficiency has increased, which remains a contradiction and creates a time-delay problem in active vibration control technology. Therefore, a vibration analysis method that works efficiently and accurately for engineers and researchers is needed.

The transfer matrix method of multibody system (MS-TMM) was developed by Rui et al. in 2008 [4]. The premise of MS-TMM is the decomposition of global dynamics into substructures and joints. The joint is involved in the progress of docking and transfer [5]. Compared with the classical transfer matrix method, this method offers advantages such as high programming, a low order for the system matrix, and high computational efficiency, and global dynamics equations of the system are not required [6]. Compared with the finite element method, the transfer matrix method of multibody system can reduce computing time and improve computational efficiency. As a result, the MS-TMM has been widely used in areas such as aerospace and the manufacture of weapons and vehicles in recent years [7]. However, the application of MS-TMM for solving structural dynamic responses remains in the initial stage.

In this paper, we introduced the MS-TMM for the analysis of the elastic-plastic responses of a new single-story

frame-bent structure during rarely occurring earthquakes. A comparison of the results obtained by the MS-TMM and ANSYS is discussed.

2. Mechanical Model

In the new single-story frame-bent structure, precast prestressed saddle plates are used on the roof, a framework is used in the longitudinal direction, and saddle plates are connected to the longitudinal beams. Because the mass and stiffness distribution of the new single-story frame-bent structure are average in the longitudinal direction, a representative frame-bent model of the structure is selected to study its responses during rarely occurring earthquakes. Based on the saddle plates connected to the longitudinal frame beams in the structure, a model of the links is simplified to a model of the planar hinges because a rotational degree of freedom and a transverse displacement degree of freedom exist between the hinges. The saddle plates are considered to be the model of rigid bodies because minimal transverse deformation occurs under the action of transverse loads. The columns in the new single-story frame-bent structure are simplified to a model of elements, such as particles and massless beams $(1, 2, \ldots, j)$, whose deformations are only considered in the transverse direction. And in structural engineering, the foundation is usually assumed as a rigid body. The mechanical model of a new single-story frame-bent structure during an earthquake can be simplified as shown in Figure 1, including particles $(2, 4, 6, \ldots, j)$ and $(2, 4, 6, \ldots, j')$, massless beams $(1, 3, 5, \ldots, j-1)$, planar hinges $(1')$, and rigid bodies $(2')$. A rigid body contains three degrees of freedom, including translations in the x and y directions, as well as a rotation around the z-axis.

3. The Transfer Matrix of a New Single-Story Frame-Bent Structure Based on the MS-TMM

We define the state vector of the element in plane motion as

$$Z = \left[x, y, \theta_z, m_z, q_x, q_y, 1\right]^T, \tag{1}$$

where x, y, θ_z, m_z, q_x, and q_y denote the displacements in the x and y directions, angular displacement around z, internal moment, internal force in the x direction, and internal force in the y direction, respectively.

The transfer process is derived as follows.

The side column on the left is as

$$\begin{aligned}
\mathbf{Z}_{2,1} &= \mathbf{U}_1\mathbf{Z}_{0,1},\\
\mathbf{Z}_{2,3} &= \mathbf{U}_2\mathbf{Z}_{2,1},\\
&\vdots\\
&\vdots\\
\mathbf{Z}_{j,j-1} &= \mathbf{U}_{j-1}\mathbf{Z}_{j-2,j-1},\\
\mathbf{Z}_{j,1'} &= \mathbf{H}\mathbf{U}_j\mathbf{Z}_{j,j-1},\\
\mathbf{Z}_{2',1'} &= \mathbf{U}_{1'}\mathbf{Z}_{j,1'},\\
\mathbf{Z}_{2',3'} &= \mathbf{U}_{2'}\mathbf{Z}_{2',1'},\\
\mathbf{Z}_{j',3'} &= \mathbf{U}_{3'}\mathbf{Z}_{2',3'}.
\end{aligned} \tag{2}$$

The middle column is as

$$\begin{aligned}
\mathbf{Z}_{2,1} &= \mathbf{U}_1\mathbf{Z}_{0',1},\\
\mathbf{Z}_{2,3} &= \mathbf{U}_2\mathbf{Z}_{2,1},\\
&\vdots\\
&\vdots\\
\mathbf{Z}_{j',j-1} &= \mathbf{U}_{j-1}\mathbf{Z}_{j-2,j-1}.
\end{aligned} \tag{3}$$

The side column on the right is as

$$\begin{aligned}
\mathbf{Z}_{2',1'} &= \mathbf{U}_{1'}\mathbf{Z}_{j',1'},\\
\mathbf{Z}_{2',3'} &= \mathbf{U}_{2'}\mathbf{Z}_{2',1'},\\
\mathbf{Z}_{j,3'} &= \mathbf{U}_{3'}\mathbf{Z}_{2',3'},\\
\mathbf{Z}_{j,j-1} &= \mathbf{H}\mathbf{U}_j\mathbf{Z}_{j,3'},\\
\mathbf{Z}_{j-2,j-1} &= \mathbf{U}_{j-1}\mathbf{Z}_{j,j-1},\\
&\vdots\\
&\vdots\\
\mathbf{Z}_{2,1} &= \mathbf{U}_2\mathbf{Z}_{2,3},\\
\mathbf{Z}_{0'',1} &= \mathbf{U}_1\mathbf{Z}_{2,1}.
\end{aligned} \tag{4}$$

In the same coordinate, we obtain

$$\begin{aligned}
\mathbf{Z}_{j,1'} &= \mathbf{H}\mathbf{U}_j\ \mathbf{Z}_{j,j-1}\ \left(\theta = -90^\circ\right),\\
\mathbf{Z}_{j,j-1} &= \mathbf{H}\mathbf{U}_j\mathbf{Z}_{j,3'}\ \left(\theta = -90^\circ\right),
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathbf{H} &= \begin{bmatrix} \mathbf{H}_{1,1} & \mathbf{O}_{3\times4} \\ \mathbf{O}_{4\times3} & \mathbf{H}_{2,2} \end{bmatrix}, \qquad \mathbf{H}_{1,1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\\
\mathbf{H}_{2,2} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{6}$$

Figure 1: The mechanical model of a single-story frame-bent building structure.

The frame-bent structure is decomposed into numerous chain systems; thus, the transfer equation is expressed as

$$\begin{aligned} \mathbf{Z}_{j',3'} &= \mathbf{U}_{\mathrm{I}}\mathbf{Z}_{0,1}, \\ \mathbf{Z}_{j',j-1} &= \mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1}, \\ \mathbf{Z}_{0'',1} &= \mathbf{U}_{\mathrm{III}}\mathbf{Z}_{j',1'}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathbf{U}_{\mathrm{I}} &= \mathbf{U}_{3'}\mathbf{U}_{2'}\mathbf{U}_{1'}\mathbf{H}\mathbf{U}_{j}\mathbf{U}_{j-1}\cdots\mathbf{U}_{2}\mathbf{U}_{1}, \\ \mathbf{U}_{\mathrm{II}} &= \mathbf{U}_{j-1}\mathbf{U}_{j-2}\cdots\mathbf{U}_{2}\mathbf{U}_{1}, \\ \mathbf{U}_{\mathrm{III}} &= \mathbf{U}_{1}\mathbf{U}_{2}\mathbf{U}_{j-1}\mathbf{H}\mathbf{U}_{j}\mathbf{U}_{3'}\mathbf{U}_{2'}\mathbf{U}_{1'}. \end{aligned} \tag{8}$$

The element numbered as j' is a particle with two inputs and one output. The state vectors of these two inputs consist of $\mathbf{Z}_{j',3'}$ and $\mathbf{Z}_{j',j-1}$. Referring to (1), we define the state vector of input point as

$$\begin{aligned} \mathbf{Z}_{I,j'} = \big[&\mathbf{X}_{j',3'}, \mathbf{Y}_{j',3'}, \mathbf{\Theta}_{zj',3'}, \mathbf{M}_{zj',3'}, \mathbf{Q}_{xj',3'}, \\ &\mathbf{Q}_{yj',3'}, \mathbf{M}_{zj',j-1}, \mathbf{Q}_{xj',j-1}, \mathbf{Q}_{yj',j-1}, 1\big]^{T}, \end{aligned} \tag{9}$$

$$\mathbf{Z}_{I,j'} = \mathbf{E}_1\mathbf{Z}_{j',3'} + \mathbf{E}_2\mathbf{Z}_{j',j-1}, \tag{10}$$

where

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_6 & \mathbf{O}_{6\times1} \\ \mathbf{O}_{4\times6} & \mathbf{O}_{4\times1} \end{bmatrix}, \qquad \mathbf{E}_2 = \begin{bmatrix} \mathbf{O}_{6\times3} & \mathbf{O}_{6\times4} \\ \mathbf{O}_{4\times3} & \mathbf{I}_4 \end{bmatrix}. \tag{11}$$

Therefore, the transfer equation of the element j' is expressed as follows:

$$\mathbf{Z}_{j',1'} = \mathbf{U}_{j'}\mathbf{Z}_{I,j''}. \tag{12}$$

$$\begin{aligned} \mathbf{Z}_{j',1'} &= \mathbf{U}_{j'}\mathbf{E}_1\mathbf{Z}_{j',3'} + \mathbf{U}_{j'}\mathbf{E}_2\mathbf{Z}_{j',j-1}. \\ &= \mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}}\mathbf{Z}_{0,1} + \mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1}. \end{aligned} \tag{13}$$

$$\begin{aligned} \mathbf{Z}_{0'',1} = \mathbf{U}_{\mathrm{III}}\mathbf{Z}_{j',1'} &= \mathbf{U}_{\mathrm{III}}\big(\mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}}\mathbf{Z}_{0,1} \\ &\quad + \mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1}\big) \\ &= \mathbf{U}_{\mathrm{III}}\mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}}\mathbf{Z}_{0,1} + \mathbf{U}_{\mathrm{III}}\mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1}. \end{aligned} \tag{14}$$

$$\mathbf{U}_{\mathrm{III}}\mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}}\mathbf{Z}_{0,1} + \mathbf{U}_{\mathrm{III}}\mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1} - \mathbf{Z}_{0'',1} = 0. \tag{15}$$

For j', all displacements are equivalent; we obtain

$$\mathbf{E}_3\mathbf{Z}_{j',3'} = \mathbf{E}_3\mathbf{Z}_{j',j-1} = \mathbf{E}_3\mathbf{Z}_{j',1'}, \tag{16}$$

where $\mathbf{E}_3 = [\mathbf{I}_3 \;\; \mathbf{O}_{3\times4}]$.

By combining (7), (10), (12), and (16), we obtain the following equations:

$$\begin{aligned} &\mathbf{E}_3\mathbf{U}_{\mathrm{I}}\mathbf{Z}_{0,1} - \mathbf{E}_3\mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1} = 0, \\ &\big(\mathbf{E}_3\mathbf{U}_{\mathrm{I}} - \mathbf{E}_3\mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}}\big)\mathbf{Z}_{0,1} - \mathbf{E}_3\mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}}\mathbf{Z}_{0',1} = 0. \end{aligned} \tag{17}$$

By combining (15), and (17), we obtain

$$\mathbf{Z}_{\mathrm{all}}\mathbf{U}_{\mathrm{all}} = 0, \tag{18}$$

where

$$\begin{aligned} \mathbf{U}_{\mathrm{all}} &= \begin{bmatrix} \mathbf{U}_{\mathrm{III}}\mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}} & \mathbf{U}_{\mathrm{III}}\mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}} & -\mathbf{I}_7 \\ \mathbf{E}_3\mathbf{U}_{\mathrm{I}} & -\mathbf{E}_3\mathbf{U}_{\mathrm{II}} & \mathbf{O}_{3\times7} \\ \mathbf{E}_3\mathbf{U}_{\mathrm{I}} - \mathbf{E}_3\mathbf{U}_{j'}\mathbf{E}_1\mathbf{U}_{\mathrm{I}} & -\mathbf{E}_3\mathbf{U}_{j'}\mathbf{E}_2\mathbf{U}_{\mathrm{II}} & \mathbf{O}_{3\times7} \end{bmatrix}, \\ \mathbf{Z}_{\mathrm{all}} &= \begin{bmatrix} \mathbf{Z}_{0,1} \\ \mathbf{Z}_{0',1} \\ \mathbf{Z}_{0'',1} \end{bmatrix}. \end{aligned} \tag{19}$$

The boundary conditions are listed as follows:

$$\begin{aligned} \mathbf{Z}_{0,1} &= \big[0, 0, 0, M_{z0}, Q_{x0}, Q_{y0}, 1\big]^{T}, \\ \mathbf{Z}_{0',1} &= \big[0, 0, 0, M_{z0'}, Q_{x0'}, Q_{y0'}, 1\big]^{T}, \\ \mathbf{Z}_{0'',1} &= \big[0, 0, 0, M_{z0''}, Q_{x0''}, Q_{y0''}, 1\big]^{T}. \end{aligned} \tag{20}$$

The detailed steps for solving the elastic-plastic responses of a new single-story frame-bent structure during rarely occurring earthquakes by the MS-TMM are shown in Figure 2.

All transfer matrixes of the elements are illustrated in (21) through (26). The transfer matrix of a rigid body in plane motion is expressed as

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & -y_{IO}(t_{i-1}) & 0 & 0 & 0 & b_1G_1 - b_2G_2 \\ 0 & 1 & x_{IO}(t_{i-1}) & 0 & 0 & 0 & b_1G_2 + b_2G_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_{41} & u_{42} & u_{43} & 1 & u_{45} & u_{46} & u_{47} \\ -(m_rA + c_\zeta C + k) & 0 & (m_rA + c_\zeta C + k)\, y_{IC}(t_{i-1}) & 0 & 1 & 0 & u_{57} \\ 0 & -(m_rA + c_\zeta C + k) & -(m_rA + c_\zeta C + k)\, x_{IC}(t_{i-1}) & 0 & 0 & 1 & u_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

where m_r is the mass of the rigid body and c_ζ is the damping of the structure

$$\begin{aligned}
u_{41} &= (m_rA + c_\zeta C + k)\, y_{IO} - m_rAy_{IC}, \\
u_{42} &= -(m_rA + c_\zeta C + k)\, x_{IO} + m_rAx_{IC}, \\
u_{43} &= (m_rA + c_\zeta C + k)\, x_{IC}(t_{i-1})\, x_{IO} \\
&\quad - (m_rA + c_\zeta C + k)\, y_{IC}(t_{i-1})\, y_{IO} + JA, \\
u_{45} &= -y_{IO}, \qquad u_{46} = x_{IO}, \\
u_{47} &= -m_c - u_{57}y_{IO} + u_{67}x_{IO} + JB_\theta \\
&\quad + (m_rB_{yI} - f_{y,c})\, x_{IC} + (f_{x,c} - m_rB_{xI})\, y_{IC}, \\
u_{57} &= -(m_rA + c_\zeta C + k)(c_{c1}G_1 - c_{c2}G_2) \\
&\quad - m_rB_{xc} - c_\zeta D_{xc} + f_{x,c}, \\
u_{67} &= -(m_rA + c_\zeta C + k)(c_{c1}G_1 - c_{c2}G_2) \\
&\quad - m_rB_{yc} - c_\zeta D_{yc} + f_{y,c}, \\
x_{IC}(t_{i-1}) &= \frac{(c_{c1}C_I - c_{c2}S_I)}{t_{i-1},}, \\
y_{IC}(t_{i-1}) &= \frac{(c_{c1}S_I + c_{c2}C_I)}{t_{i-1}}, \\
x_{IO}(t_{i-1}) &= \frac{(b_1C_I - b_2S_I)}{t_{i-1}}, \\
y_{IO}(t_{i-1}) &= \frac{(b_1S_I + b_2C_I)}{t_{i-1}}, \\
G_1 &= \cos\theta(t_{i-1}) \left\{1 - \frac{1}{2}\left[\dot\theta(t_{i-1})\,\Delta t\right]^2\right\} \\
&\quad + \theta(t_{i-1}) \sin\theta(t_{i-1}), \\
G_2 &= \sin\theta(t_{i-1}) \left\{1 - \frac{1}{2}\left[\dot\theta(t_{i-1})\,\Delta t\right]^2\right\} \\
&\quad + \theta(t_{i-1}) \cos\theta(t_{i-1}),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&A = \frac{1}{\alpha\Delta t^2}, \qquad B = A\left[-x_{t_{i-1}} - \dot x_{t_{i-1}}\Delta t - \left(\frac{1}{2} - \alpha\right)\ddot x_{t_{i-1}}\Delta t^2\right], \\
&C = \beta\Delta tA, \qquad D = \dot x_{t_{i-1}} + (1-\beta)\,\Delta t\ddot x_{t_{i-1}} + \beta\Delta tB \\
&\qquad\qquad \left(\alpha = \frac{1}{2},\ \beta = \frac{1}{4}\right).
\end{aligned} \tag{23}$$

When the earthquake acceleration is $\ddot x_g(t_{i-1})$ at $t = t_{i-1}$, $f_{x,c} = -m_r\ddot x_g(t_{i-1})$. The transfer matrix of a massless elastic beam in lateral vibration is as follows:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & l & \frac{l^2}{2EI} & 0 & \frac{l^3}{6EI} & \frac{f_il^3}{24EI} \\ 0 & 0 & 1 & \frac{l}{EI} & 0 & \frac{l^2}{2EI} & \frac{f_il^2}{6EI} \\ 0 & 0 & 0 & 1 & 0 & l & \frac{f_il}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & f_i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

The transfer matrix of the particle is expressed as

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -(mA + c_\zeta C + k) & 0 & 0 & 0 & 1 & -mB_{yI} + c_\zeta D_{yI} + f_{y,c} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

where m is the mass of the particle and c_ζ is the damping of the structure.

Figure 2: The process of solving the elastic-plastic responses of structures by the MS-TMM.

When the earthquake acceleration is $\ddot{x}_g(t_{i-1})$ at $t = t_{i-1}$, $f_{y,c} = -m\ddot{x}_g\,(t_{i-1})$. The transfer matrix of the planar hinge is expressed as

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{k_x} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{k_y} & 0 \\ 0 & 0 & 1 & \frac{1}{k'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

where k_x, and k_y are the spring stiffness in the x, y directions, respectively, and k' is the stiffness of the rotational spring around z.

The process of solving the elastic-plastic responses of a new single-story frame-bent structure during rarely occurring earthquake is as follows (Figure 2).

4. The Elastic-Plastic Restoring Force Model of the Structure

Because the structure exhibits elastic plasticity during rarely occurring earthquakes, the stiffness varies with time; thus, it is important to establish relations between the restoring force and the displacement. Currently, the commonly used models are the double restoring force model and the three linear restoring force models. The double model is the simpler model. To reflect the properties of the concrete structure in the elastic-plastic region, the model of the trilinear restoring force is selected for this paper, as shown in Figure 3 [8].

In Figure 3, point 1 is the cracking point, F_c and x_c are the corresponding force and displacement, stage 01 is elastic, and k_1 is the elastic stiffness; point 2 is the yielding point, F_y and x_y are the corresponding force and displacement, stage 12 represents cracking to yielding, and k_2 is the cracking stiffness; after yielding, k_3 is the yielding stiffness; the secant stiffness k_4 is the unloading stiffness; in stage 12, when unloading to zero, the straight line extends to the cracking point for the first reverse loading; in stage 23, when unloading to zero, the straight line extends to the yielding point for the first reverse loading; for subsequent reverse loadings, the

Figure 3: The model of the trilinear restoring force.

Figure 4: Processing of the three categories of inflection points.

straight line extends to the point of maximum deformation prior to the first cycle.

In the model of the stiffness degradation for the three linear restoring forces, the variation in stiffness is represented by the intersecting point of the two straight lines; this type of intersecting point is named as the inflection point, for example, points 1, 2, 3, and 4 in Figure 3. Some errors exist in the inflection point; if we do not continue the processing to the inflection point, additional errors will cause distortion of the computational results due to the distortion of the restoring force model. Therefore, we employ a shortened integral step $p\Delta t(0 < p < 1)$ at the inflection point to ensure that the restoring force curve is located precisely in the allowable error range at the end of the shortened integral step and that it continues to the integral in the new interval $(t + p\Delta t, t+\Delta t)$. Generally, we employ the method of approximate interpolation to determine p. A different inflection point p corresponds to a different corresponding method. Inflection points are generally divided into three categories. The first category comprises the transition points, in which steep lines transition to slow lines, such as points 1, 2, and 5 in Figure 3. The second category comprises transition points, in which slow lines transition to steep lines, such as points 3, 6, 8, and 9 in Figure 3. The third category comprises points for which the restoring forces change sign, such as points 4, 7, and 10 in Figure 3.

(1) Determine the value of the first category p.

For the two points shown in Figure 4, the Taylor series is used to expand at time t. Equation (27) can be obtained as follows:

$$x_2 = x\left(t + p\Delta t\right) = x\left(t\right) + \dot{x}\left(t\right) p\Delta t. \tag{27}$$

According to formula (27), p can be determined as follows:

$$p = \frac{x\left(t + p\Delta t\right) - x\left(t\right)}{\dot{x}\left(t\right)\Delta t} = \frac{x_2 - x_1}{\dot{x}_1 \Delta t} = \frac{x_2 - x_1}{x_3 - x_1}. \tag{28}$$

After p has been determined, we can calculate the speed and acceleration of point 2 as

$$\begin{aligned} \dot{x}_2 &= \dot{x}_1 + p\left(\dot{x}_3 - \dot{x}_1\right), \\ \ddot{x}_2 &= \ddot{x}_1 + p\left(\ddot{x}_3 - \ddot{x}_1\right). \end{aligned} \tag{29}$$

(2) Determine the value of the second category p.

Researchers have concluded that the second category p, such as point 5, does not require processing. Because the skeleton curve is relatively rough, the pursuit of this subtle precision is insignificant.

(3) Determine the value of the third category p.

For point 5 in Figure 4 as follows:

$$p = \frac{F_9}{F_9 - F_7}. \tag{30}$$

5. An Engineering Example

A new single-story bent-frame structure with two spans and equal heights is selected as an engineering example. Because the lateral stiffness and mass of each bent-frame structure in the longitudinal direction are nearly equivalent, the transverse seismic action is approximately equal. Thus, the original spatial structure can be simplified to a model of planar structure under the lateral action of an earthquake. Consider a typical bent-frame structure as an example; the calculating diagram is shown in Figure 1 ($j = 6$). The type of saddle plate is YMB − 15 − 3.0; the transversal span is 15 m; the grade of loads is I; the snow load is 0.35 KN/m^2; the strength grade of concrete in the columns is C30; the longitudinal span is 9 m; the section size of the RC columns is 600 mm × 800 mm; the thickness of the wall is 240 mm; the transversal spring stiffness k_x and the stiffness of the

rotational spring k' between a saddle plate and a longitudinal beam are 1.68×10^8 N/m, 1.59×10^6 N · m, respectively; the elastic modulus E is 3×10^{10} N/m^2; the elastic stiffness of a column in longitudinal frame k_1 is 7.68×10^8 N/m; the cracking stiffness k_2, the yielding stiffness k_3, and the secant stiffness k_4 are 3.16×10^8 N/m, 2.3×10^8 N/m, and 3.46×10^8 N/m, respectively; the cracking displacement x_c is 6.3 mm; the yielding displacement x_y is 21.8 mm; the mass of saddle plate $m_{2'}$ is 25.14×10^3 kg; the masses of particles are $m_2 = m_4 = m_6 = 5.93 \times 10^3$ kg. The seismic fortification intensity is 7, the site type is II, and the classification of design earthquake is the 1st classification.

To study the responses of a new single-story bent-frame structure during rarely occurring earthquakes, two natural earthquake waves (the El Centro earthquake wave and the Taft earthquake wave) and one artificial earthquake wave (the Nanjing earthquake wave) are chosen. The duration time is 10 s. According to the seismic codes of the PROC, we adjust the peak value of the earthquake waves to 220 cm/s^2 for the rarely occurring earthquake.

5.1. Analysis of the Elastic-Plastic Displacement of a New Single-Story Bent-Frame Structure during Rarely Occurring Earthquakes Is Based on the MS-TMM. The responses of a new single-story bent-frame structure during rarely occurring earthquakes are computed using the MS-TMM (as shown in Figure 5) and the finite element software ANSYS (as shown in Figure 6). When the responses of a new single-story bent-frame structure during rarely occurring earthquakes are computed by using the finite element software ANSYS, the BEAM198 and SHELL63 are chosen as the models of a beam or column and a saddle plate, respectively. Because the foundation is assumed to be a rigid body, the link model at the bottom of a column corresponds to the fixed constraint.

As illustrated in Figures 5 and 6, based on the MS-TMM, the maximum elastic-plastic displacement on the top of the new single-story bent-frame structure under the activity of El Centro Wave, Taft Wave, and Nanjing Wave is 47.93 mm at $t = 2.78$ s, 44.41 mm at $t = 4.48$ s, and 46.65 mm at $t = 8.00$ s, respectively. And based on ANSYS, those are 43.13 mm at $t = 2.76$ s, 46.12 mm at $t = 4.46$ s, and 42.3 mm at $t = 7.96$ s, respectively. Based on the MS-TMM, the maximum elastic-plastic story drift angles under the activity of three earthquake waves are 1/188, 1/203, and 1/193, respectively. And based on ANSYS, those are 1/195, 1/213, and 1/205, respectively. The average maximum elastic-plastic displacement on the top of the structure got by both methods are 46.33 mm and 43.58 mm under the activity of three earthquake waves, respectively; the average maximum elastic-plastic story drift angles got by both methods are 1/194 and 1/205 under the activity of three earthquake waves, respectively. Because determinate regulations regarding elastic-plastic story drift displacement angle of the frame-bent structure are nonexistent, we refer to the concrete frame structure, in which the limit of an elastic-plastic story drift angle is 1/50. Thus, the frame-bent structure satisfies the requirement of the seismic codes of the PROC.

Figure 5: The displacement time history curves for the top of the structure during rarely occurring earthquakes (based on the MS-TMM).

Figure 6: The displacement time history curves for the top of the structure during rarely occurring earthquakes (based on the ANSYS).

Evidently, the elastic-plastic displacement trajectories on the top of a new single-story bent-frame structure obtained by the MS-TMM and the ANSYS under the actions of the three earthquake waves are very close, as shown in Figures 5 and 6.

The maximum displacement on the top of a new single-story bent-frame structure obtained by MS-TMM and ANSYS under the actions of the three earthquake waves is shown in Table 1; their errors are also listed.

As listed in Table 1, the error for the average maximum elastic-plastic displacement on the top of a new single-story bent-frame structure under the actions of three earthquake waves using the MS-TMM and ANSYS is 5.66%, which is similar to the results of engineering computations. The MS-TMM required only 100 s of post-processing time, but the ANSYS required more than 1800 s of postprocessing time to solve the maximum elastic-plastic displacement on the top of the structure. Because the transition matrixes and the transmission equations do not require a second derivation, they are selected directly from the library of transition matrixes; the transfer matrix method of multibody system is convenient. Computations of the elastic-plastic displacement during rarely occurring earthquakes with the

Table 1: The maximum elastic-plastic displacement for the top of a new single-story bent-frame structure during rarely occurring earthquakes.

	El Centro Wave	Taft Wave	Nanjing Wave	Average value
MS-TMM (mm)	47.93	44.41	46.65	46.33
ANSYS (mm)	43.13	46.12	42.30	43.85
Error	11.13%	3.71%	10.28%	5.66%

Figure 7: The shear force time history curves for the bottom of the side column during rarely occurring earthquakes (based on the MS-TMM).

Figure 8: The shear force time history curves for the bottom of the side column during rarely occurring earthquakes (based on ANSYS).

Figure 9: The shear force time history curves for the bottom of the middle column during rarely occurring earthquakes (based on the MS-TMM).

Figure 10: The shear force time history curves for the bottom of the middle column during rarely occurring earthquakes (Based on ANSYS).

MS-TMM exhibit more advantages than the computations by the ANSYS.

5.2. Analysis of the Base Shear Force of a New Single-Story Bent-Frame Structure during Rarely Occurring Earthquakes Based on the MS-TMM. In the design of seismic structures, note that the beams are damaged prior to the columns, which is named the concept of strong columns and weak beams, because the destruction of beams belongs to broken members and the destruction of columns endangers the entire structure. Therefore, we analyze the maximum shear force at the bottom of the column as a control target.

As illustrated in Figures 7 and 8, based on the MS-TMM, the maximum shear forces at the bottom of the side column under the activity of El Centro Wave, Taft Wave, and Nanjing Wave are 119.82 kN at $t = 4.44$ s, 97.78 kN at $t = 4.48$ s, and 111.14 kN at $t = 7.98$ s, respectively. And based on ANSYS, those are 104.42 kN at $t = 4.46$ s, 93.83 kN at $t = 4.40$ s, and 101.85 kN at $t = 7.92$ s, respectively. The average maximum shear forces at the bottom of the side column under the actions of the three earthquake waves got by the MS-TMM and ANSYS are 109.58 kN and 100.1 KN, respectively.

Evidently, shear force trajectories at the bottom of the side column of a new single-story bent-frame structure obtained by the MS-TMM and the ANSYS under the actions of the three earthquake waves are very close, as shown in Figures 7 and 8.

Table 2: The maximum shear forces at the bottom of the side column in a new single-story bent-frame structure during rarely occurring earthquakes.

	El Centro wave	Taft Wave	Nanjing Wave	Average value
MS-TMM (kN)	119.82	97.78	111.14	109.58
ANSYS (kN)	104.42	93.83	101.85	100.10
Error	14.75%	4.21%	9.12%	9.50%

Table 3: The maximum shear forces at the bottom of the middle column in a new single-story bent-frame structure during rarely occurring earthquakes.

	El Centro Wave	Taft Wave	Nanjing Wave	Average value
MS-TMM (kN)	125.81	104.67	116.70	115.73
ANSYS (kN)	112.80	101.30	110.30	108.10
Error	6.22%	3.33%	5.80%	7.06%

The maximum shear forces at the bottom of the side column obtained by the MS-TMM and ANSYS under the actions of the three earthquake waves are shown in Table 2; their errors are also listed.

As shown in Table 2, the error between the average maximum shear force at the bottom of the side column obtained by MS-TMM and the average maximum shear force at the bottom of the side column obtained by ANSYS is 9.5%, which is similar to the results of engineering computations.

As illustrated in Figures 9 and 10, based on the MS-TMM, the maximum shear forces at the bottom of the middle column under the activity of El Centro Wave, Taft Wave, and Nanjing Wave are 125.81 kN at $t = 4.46$ s, 104.67 kN at $t = 4.50$ s, and 116.70 kN at $t = 7.94$ s, respectively. And based on ANSYS, those are 112.8 kN at $t = 4.42$ s, 101.3 kN at $t = 4.48$ s, and 110.3 kN at $t = 7.94$ s, respectively. The average maximum shear forces at the bottom of the side column under the actions of the three earthquake waves got by the MS-TMM and ANSYS are 115.73 kN and 108.1 KN, respectively.

Evidently, shear force trajectories at the bottom of the middle column of a new single-story bent-frame structure obtained by the MS-TMM and the ANSYS under the actions of the three earthquake waves are very close, as shown in Figures 9 and 10.

The maximum shear forces at the bottom of the middle column obtained by MS-TMM and ANSYS under the actions of the three earthquake waves are shown in Table 3; their errors are also listed.

As shown in Table 3, the error between the average maximum shear force at the bottom of the middle column obtained by the MS-TMM and the average maximum shear force at the bottom of the middle column obtained by ANSYS is 7.06%. The results are nearly equivalent.

Because the speed of calculations by the MS-TMM for computing the base shear forces of a new single-story bent-frame structure during rarely occurring earthquakes is 10 times faster than the speed of calculations by the ANSYS; the MS-TMM has been proved to be extremely efficient in analyzing the seismic responses of structures, especially for active vibration controls of structures. The MS-TMM provides a feasible direction in the solution of the time lag.

6. Conclusions

In this paper, we conducted a pioneer study that included the analysis of the elastic-plastic responses of structures using the transfer matrix method of multibody systems (MS-TMM). We studied the seismic performance of a single-story frame-bent building structure during rarely occurring earthquakes and compared the calculation speed based on the MS-TMM with the calculation speed based on the finite element method ANSYS. We obtained the following conclusions.

(1) The transfer matrix method of multibody system (MS-TMM) can be introduced to the dynamic analysis of civil engineering structures. The method has been proved to be substantially more efficient than other methods, such as the finite element method ANSYS.

(2) The error between the average maximum elastic-plastic displacement on the top of a new single-story bent-frame structure under the actions of three earthquake waves obtained by MS-TMM and ANSYS is 5.66%; this value is similar to the results of engineering computations. The MS-TMM required only 100 s to solve the maximum elastic-plastic displacement on the top of the structure, whereas ANSYS required more than 1800 s.

(3) The error between the average maximum shear force at the bottom of the middle column obtained by the MS-TMM and the average maximum shear force at the bottom of the middle column obtained by the ANSYS is 7.06%; the variation in error for the average maximum shear force on the side column is 9.5%. The results are nearly equivalent. Because the calculation speed of the MS-TMM is 10 times faster than the calculation speed of the ANSYS; the MS-TMM has been proved to be highly efficient for analyzing the dynamic responses of structures.

References

[1] Y. W. Dai, J. G. Ding, C. X. Wang, G. P. Zhang, and L. Chen, "Seismic analysis on single-storey industrial buildings with

roof of saddle plate in different span," *Applied Mechanics and Materials*, vol. 193, pp. 1270–1277, 2012.

[2] J. Ding and Z. Qiao, "Study on seismic performance of a new multi-layer frame-bent industrial building by pushover analysis," *Applied Mechanics and Materials*, vol. 193, pp. 1038–1047, 2012.

[3] B. Rong, X. Rui, G. Wang, and F. Yang, "New Efficient method for dynamic modeling and simulation of flexible multibody systems moving in plane," *Multibody System Dynamics*, vol. 24, no. 2, pp. 181–200, 2010.

[4] X. Rui, L. Yun, Y. Lu, B. He, and G. Wang, *Transfer Matrix Method of Multibody System and Its Application*, Science Press, Beijing, China, 2008.

[5] B. He, X. Rui, and H. Zhang, "Transfer matrix method for natural vibration analysis of tree system," *Mathematical Problems in Engineering*, vol. 2012, Article ID 393204, 19 pages, 2012.

[6] L. K. Abbas, X. Rui, and P. Marzocca, "Panel flutter analysis of plate element based on the absolute nodal coordinate formulation," *Multibody System Dynamics*, vol. 27, no. 2, pp. 135–152, 2012.

[7] Z.-H. Zhan, X.-T. Rui, B. Rong, F.-F. Yang, and G.-P. Wang, "Design of active vibration control for launcher of multiple launch rocket system," *Journal of Multi-Body Dynamics*, vol. 225, no. 3, pp. 280–293, 2011.

[8] X. Lu, G. Jin, and X. Wu, *The Theory and Application of Finite Element Method for Reinforced Concrete Structure-Nonlinear*, Tongji University Press, Shanghai, China, 2002.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 284782, 11 pages
http://dx.doi.org/10.1155/2013/284782

Research Article

Cooperative Search by Combining Simulated and Real Robots in a Swarm under the View of Multibody System Dynamics

Qirong Tang and Peter Eberhard

Institute of Engineering and Computational Mechanics, University of Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany

Correspondence should be addressed to Peter Eberhard; peter.eberhard@itm.uni-stuttgart.de

Received 26 June 2013; Accepted 13 August 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Q. Tang and P. Eberhard. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents a new approach for cooperative search of a robot swarm. After modeling the robot, the mechanical Particle Swarm Optimization method is conducted based on physical robot properties. Benefiting from the effective localization and navigation by sensor data fusion, a mixed robot swarm which contains both simulated and real robots is then successfully used for searching a target cooperatively. With the promising results from experiments based on different scenarios, the feasibility, the interaction of real and simulated robots, the fault tolerance, and also the scalability of the proposed method are investigated.

1. Introduction

Nowadays robots are used more and more since they can do various tasks instead of humans. Among the variety of existing robots, swarms of mobile robots receive growing attention by researchers. A number of large international projects have been conducted, for example, SWARM-Bots, SWARMMANOID, and SYMBARION [1]. Swarms of mobile robots have many applications, for example, for efficient collection of oil spill [2, 3], for light or odorant detection [4–6], for searching survivors after an earthquake [7], and for assisting humans to handle nuclear leakage [8]. Readers may notice from these applications that cooperative search is quite representative for the swarm robotics research since it covers most of the technical points. Many researches focus on cooperative search; however, this is still at an early stage in spite of its fast development. Many critical issues have not been solved yet, including, for example, the challenge in controlling and guiding robots systematically and the transfer to workable strategies for real robots. From a technical perspective, for cooperative search of robot swarms, key points are performing collaboration, showing fault tolerance, sharing information, distributing the work, and realizing scalability. These are also the advantages of a mobile robot swarm in contrast to a single mobile robot. Neither the traditional methods such as artificial potential fields, A^* or D^*, nor new methods like genetic algorithms and ant colony algorithms can produce satisfactory results. In recent years, the comparatively new stochastic Particle Swarm Optimization (PSO) algorithms have attracted researchers' attention. Many research groups use PSO and its variants for swarm mobile robots search; see, for example, [9–11]. Unfortunately, these publications have shown shortcomings for cooperative search. For example, [9] used PSO for multirobot search; however, its focus is on optimizing PSO parameters and it does not consider the scalability for a large number of robots. The work in [10] gives an ad hoc application using a PSO based method for group robots search. Its solutions are valid for a specific robot which has only limited motion ability. At the moment similar researches mostly stop at a theoretical stage and are verified only by simulation. Some researchers have used PSO on real robots reluctantly, but they have limitations such as poor scalability and inadequate search ability. Thus, this study develops a systematic cooperative search method which contains sufficient search ability, scalability, and fault tolerance, considers robot physical properties, and qualifies for real-simulated robots interaction.

Section 2 builds kinematic and dynamic models of the used robot. Section 3 designs the cooperative search algorithm. Experiments performed on a mixed robot swarm for different cases are shown in Section 4, while conclusions are given in Section 5.

Figure 1: Festo Robotino and its main components (photo from Robotino manual [12]).

Figure 2: Robotino controller head and its components.

2. Kinematic and Dynamic Models

2.1. The Investigated Omnidirectional Mobile Robot. Our research object is the Festo Robotino which is an omnidirectional mobile robot. Its appearance and main components are shown in Figure 1, while Figure 2 shows the robot control unit used for realizing control and software running.

2.2. Kinematic Model. The kinematic model of a Robotino is based on the robot's structure and kinematic behavior. Since it is omnidirectional, it has the ability to move in any direction no matter what the current orientation is. This mainly results from its three special wheels; see Figure 3. The three wheel units has an angle of 120° between each other; see Figure 4. The origin of the local robot frame $[x_l \quad y_l]$; see Figure 4, coincides with the center of the robot base and is set to coincide with the origin of the global frame $[x_g \quad y_g]$ at the beginning but rotates the robot by angle ϕ_g. The three lateral velocities are marked as $v_{l,i}, i = 1, 2, 3$ which also can be considered as free or passive velocities. The drive forces are defined by f_1, f_2, and f_3. To describe the location of the three wheels relative to the local frame, we introduce the angles α_i where $\alpha_1 = 0^\circ$, $\alpha_2 = 120^\circ$, and $\alpha_3 = 240^\circ$. Thus, the relationship between the global velocity $[\dot{x}_g \quad \dot{y}_g \quad \dot{\phi}_g]$ of the robot and the driven translational velocities v_i of three wheels is governed by

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} -\sin(\phi_g+\alpha_1) & \cos(\phi_g+\alpha_1) & R \\ -\sin(\phi_g+\alpha_2) & \cos(\phi_g+\alpha_2) & R \\ -\sin(\phi_g+\alpha_3) & \cos(\phi_g+\alpha_3) & R \end{bmatrix}}_{\mathbf{S}_g} \cdot \underbrace{\begin{bmatrix} \dot{x}_g \\ \dot{y}_g \\ \dot{\phi}_g \end{bmatrix}}_{\dot{\mathbf{u}}_g}. \tag{1}$$

Here R is the distance from the robot center to the center of wheel; the matrix $\mathbf{S}_g$ is the geometrical relationship between the global velocity $\dot{\mathbf{u}}_g$ and the translational velocity $\mathbf{v}$ which can also be expressed by the wheel's angular velocities ω_i multiplied by the wheel radius r; that is,

$$v_i = r\omega_i. \tag{2}$$

Inserting (2) into (1) and reformulation yields

$$\boldsymbol{\omega} = [\omega_1 \quad \omega_2 \quad \omega_3]^T = \frac{1}{r}\mathbf{S}_g \cdot \dot{\mathbf{u}}_g. \tag{3}$$

If $\mathbf{S}_g$ is split into a constant part $\mathbf{S}_l$ with the known parameters α_1, α_2, and α_3 and a variable part $\mathbf{T}_{\text{rot}}^{-1}(\phi_g)$, the kinematic model is

$$\underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{\boldsymbol{\omega}} = \frac{1}{r}\underbrace{\begin{bmatrix} 0 & 1 & R \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & R \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & R \end{bmatrix}}_{\mathbf{S}_l} \cdot \underbrace{\begin{bmatrix} \cos(\phi_g) & \sin(\phi_g) & 0 \\ -\sin(\phi_g) & \cos(\phi_g) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}_{\text{rot}}^{-1}} \cdot \underbrace{\begin{bmatrix} \dot{x}_g \\ \dot{y}_g \\ \dot{\phi}_g \end{bmatrix}}_{\dot{\mathbf{u}}_g}. \tag{4}$$

Here $\mathbf{T}_{\text{rot}}$ is the orthogonal rotation matrix between the global and the local frames where $\mathbf{T}_{\text{rot}}^{-1} = \mathbf{T}_{\text{rot}}^T$. In addition, here $\det(\mathbf{T}_{\text{rot}}) = 1$.

2.3. Dynamic Model. Each of Robotino's three wheels is driven by its own DC motor. Wheel and motor shafts are linked by a gearbox and a gear belt with a total of 16 : 1 reduction ratio to enhance torque and reduce rotational velocity, see the drivetrain in Figure 5. Due to Newton's second law of motion, the planar mobile robot motion can be described by

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}}_{\mathbf{M}} \cdot \underbrace{\begin{bmatrix} \ddot{x}_g \\ \ddot{y}_g \\ \ddot{\phi}_g \end{bmatrix}}_{\ddot{\mathbf{u}}_g} = \underbrace{\begin{bmatrix} f_{g,x} \\ f_{g,y} \\ l_{g,\phi} \end{bmatrix}}_{\mathbf{f}_g}. \tag{5}$$

Here $\mathbf{f}_g$ is the generalized global force acting on the robot, $\mathbf{M}$ is a mass matrix where m is the mass and J is the rotational

(a) Whole wheel unit (b) One of the two slides

Figure 3: Illustration of the special wheel and its motion mechanism.

Figure 4: Geometry and kinematic relations of Robotino base.

Figure 5: Robotino drivetrain.

moment of inertia of the robot, and $\ddot{\mathbf{u}}_g$ is the generalized acceleration. For $\mathbf{f}_g$ and the robot local force $\mathbf{f}_l$ one has

$$\underbrace{\begin{bmatrix} f_{g,x} \\ f_{g,y} \\ l_{g,\phi} \end{bmatrix}}_{\mathbf{f}_g} = \underbrace{\begin{bmatrix} \cos(\phi_g) & -\sin(\phi_g) & 0 \\ \sin(\phi_g) & \cos(\phi_g) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}_{\text{rot}}} \cdot \underbrace{\begin{bmatrix} f_{l,x} \\ f_{l,y} \\ l_{l,\phi} \end{bmatrix}}_{\mathbf{f}_l}, \tag{6}$$

and the connection of $\mathbf{f}_l$ to the wheel drag force $\mathbf{f}_{\text{drag}}$ is

$$\underbrace{\begin{bmatrix} f_{l,x} \\ f_{l,y} \\ l_{l,\phi} \end{bmatrix}}_{\mathbf{f}_l} = \underbrace{\begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ R & R & R \end{bmatrix}}_{\mathbf{S}_l^T} \cdot \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{\mathbf{f}_{\text{drag}}}. \tag{7}$$

Now, substituting (6) and (7) into (5) yields

$$\mathbf{f}_{\text{drag}} = \left(\mathbf{S}_l^T\right)^{-1} \cdot \mathbf{T}_{\text{rot}}^{-1} \cdot \mathbf{M} \cdot \ddot{\mathbf{u}}_g. \tag{8}$$

In the following, the Robotino's motor dynamic is investigated as well as its relation to (8). Figure 6 shows the simplified circuit diagram of the used DC motor. Based on this and Kirchhoff's voltage law, we get for the current transient changes $\dot{c}_i = dc_i/dt$ $(i = 1, 2, 3)$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}}_{\mathbf{U}} = R_\Omega \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\mathbf{c}} + L \underbrace{\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \end{bmatrix}}_{\dot{\mathbf{c}}} + \underbrace{\begin{bmatrix} U_{m,1} \\ U_{m,2} \\ U_{m,3} \end{bmatrix}}_{\mathbf{U}_m}, \tag{9}$$

where $\mathbf{U}$ and $\mathbf{U}_m$ are the input voltage acting on the motors and the induced voltage from the motor winding,

respectively. The constants R_Ω and L represent resistance and inductance of the motor, respectively. According to Faraday's law of induction we have

$$\mathbf{U}_m = k_m \boldsymbol{\omega}_m, \tag{10}$$

where k_m is the back electromotive force constant which depends on the build-up of the motor, and $\boldsymbol{\omega}_m$ are the motors' angular velocities. The DC motor's torque relation is

$$\mathbf{l}_{\text{load}} = k_t \mathbf{c} - J_m \dot{\boldsymbol{\omega}}_m - k_l \boldsymbol{\omega}_m, \tag{11}$$

where k_t and k_l are the torque constant and the viscous friction coefficient, respectively. The loading torque vector $\mathbf{l}_{\text{load}}$ is responsible for dragging the wheels, and J_m describes the motors rotational moment of inertia. Furthermore, one has

$$n\mathbf{l}_{\text{load}} = r\mathbf{f}_{\text{drag}}, \qquad \boldsymbol{\omega}_m = n\boldsymbol{\omega}, \tag{12}$$

where $n = 16$ is the gear ratio. When placing (12) into (11) one obtains

$$\mathbf{f}_{\text{drag}} = \frac{k_t n}{r}\mathbf{c} - \frac{J_m n^2}{r}\dot{\boldsymbol{\omega}} - \frac{k_l n^2}{r}\boldsymbol{\omega}. \tag{13}$$

Substituting the kinematic model (4) and its time derivative

$$\dot{\boldsymbol{\omega}} = \frac{1}{r}\mathbf{S}_l \cdot \left(\mathbf{T}_{\text{rot}}^T\right)' \cdot \dot{\mathbf{u}}_g + \frac{1}{r}\mathbf{S}_l \cdot \mathbf{T}_{\text{rot}}^T \cdot \ddot{\mathbf{u}}_g, \tag{14}$$

into (13) and comparing with (8) yield the current equation

$$\begin{aligned}\frac{k_t n}{r}\mathbf{c} = {} & \left(\left(\mathbf{S}_l^T\right)^{-1} \cdot \mathbf{T}_{\text{rot}}^T \cdot \mathbf{M} + \frac{J_m n^2}{r^2}\mathbf{S}_l \cdot \mathbf{T}_{\text{rot}}^T\right) \cdot \ddot{\mathbf{u}}_g \\ & + \left(\frac{J_m n^2}{r^2}\mathbf{S}_l \cdot \left(\mathbf{T}_{\text{rot}}^T\right)' + \frac{k_l n^2}{r^2}\mathbf{S}_l \cdot \mathbf{T}_{\text{rot}}^T\right) \cdot \dot{\mathbf{u}}_g.\end{aligned} \tag{15}$$

Undetermined parameters in (15) can be read or calculated from the motors characteristics in Figure 7. Thus, with (15) the required current and further the actual applied voltage are calculated for resulting robot motions.

3. Design of the Cooperative Search Algorithm Based on Particle Swarm Optimization

3.1. The Basic Particle Swarm Optimization Model. Particle Swarm Optimization (PSO) is usually considered to be inspired from swarms of birds and schools of fish. For example, in Figure 8 a swarm of birds is shown in which each bird is taken as an adaptive agent and can get information from the environment and other agents.

The computational algorithm of PSO was first introduced by Kennedy and Eberhart in [14]. In PSO, a swarm consists of N_p members called particles, and each possesses information about position and velocity in the work space. The particles are generated with an initial random position and velocity within a defined range. The recursions of basic PSO use the vectors $\Delta\mathbf{x}$ and $\mathbf{x}$ to denote the particles "velocity" and

Figure 6: Circuit diagram of the DC motor used in the Robotino.

actual position, respectively. The so-called "velocity" of the ith particle at the $(s+1)$th iteration can be updated by

$$\Delta\mathbf{x}_i^{s+1} = \omega\Delta\mathbf{x}_i^s + c_1 r_{i,1}^s\left(\mathbf{x}_{i,\text{self}}^{\text{best},s} - \mathbf{x}_i^s\right) + c_2 r_{i,2}^s\left(\mathbf{x}_{\text{swarm}}^{\text{best},s} - \mathbf{x}_i^s\right), \tag{16}$$

and the position update is done by

$$\mathbf{x}_i^{s+1} = \mathbf{x}_i^s + \Delta\mathbf{x}_i^{s+1}. \tag{17}$$

The term $\omega\Delta\mathbf{x}_i^s$ in (16) is the inertia part which contains the coefficient ω. The factors c_1 as well as c_2 in (16) are used for cognitive and social scaling which are generated from the attractions to the previous personal best position $\mathbf{x}_{i,\text{self}}^{\text{best},s}$ and the entire swarm best position $\mathbf{x}_{\text{swarm}}^{\text{best},s}$, respectively. The factors $r_{i,1}^s$ and $r_{i,2}^s$ are uniformly distributed random variables both range in $[0, 1]$. In (17), the "velocity" actually corresponds to a displacement $\Delta\mathbf{x}_i^{s+1} = h\dot{\mathbf{x}}_i^s$ where h is the time step and $\dot{\mathbf{x}}_i^s$ is the mechanical velocity. However, in PSO literature h is usually set to be one second. Thus, for all N_p particles, the basic PSO algorithm may be summarized as

$$\begin{aligned}\begin{bmatrix}\mathbf{x}^{s+1}\\ \dot{\mathbf{x}}^{s+1}\end{bmatrix} = {} & \begin{bmatrix}\mathbf{x}^s\\ \omega\dot{\mathbf{x}}^s\end{bmatrix} \\ & + \begin{bmatrix}\dot{\mathbf{x}}^s \\ c_1\mathbf{r}_1^s \cdot \left(\mathbf{x}_{\text{self}}^{\text{best},s} - \mathbf{x}^s\right) + c_2\mathbf{r}_2^s \cdot \left(\hat{\mathbf{x}}_{\text{swarm}}^{\text{best},s} - \mathbf{x}^s\right)\end{bmatrix},\end{aligned} \tag{18}$$

where

$$\mathbf{r}_i^s = \operatorname{diag}\left(r_{1,i}^s\mathbf{I}_q, r_{2,i}^s\mathbf{I}_q, \ldots, r_{N_p,i}^s\mathbf{I}_q\right), \quad i \in \{1, 2\} \tag{19}$$

are generated independently for every iteration. Here, q is the generalized dimension. Considering the case with 3 generalized dimensions, $\mathbf{I}_q = \mathbf{I}_3$ is a 3×3 unit matrix. Similarly the matrices

$$\mathbf{x}_i^s = \begin{bmatrix}x_{1,i}^s\\ x_{2,i}^s\\ x_{3,i}^s\end{bmatrix}, \qquad \mathbf{x}_{i,\text{self}}^{\text{best},s} = \begin{bmatrix}x_{1,i,\text{self}}^{\text{best},s}\\ x_{2,i,\text{self}}^{\text{best},s}\\ x_{3,i,\text{self}}^{\text{best},s}\end{bmatrix},$$

Figure 7: Characteristic diagram of the DC motor-GR 42 × 25 DC motor used on Robotino (data from information of Dunkermotoren [13]).

Figure 8: Swarm of birds (photo taken at Seine riverside in Paris, France, 2011).

$$\mathbf{x}_{\text{swarm}}^{\text{best},s} = \begin{bmatrix} x_{1,\text{swarm}}^{\text{best},s} \\ x_{2,\text{swarm}}^{\text{best},s} \\ x_{3,\text{swarm}}^{\text{best},s} \end{bmatrix}, \quad i = 1, 2, \ldots, N_p,$$

$$\mathbf{x}^s = \begin{bmatrix} \mathbf{x}_1^s \\ \mathbf{x}_2^s \\ \vdots \\ \mathbf{x}_i^s \\ \vdots \\ \mathbf{x}_{N_p}^s \end{bmatrix}, \quad \mathbf{x}_{\text{self}}^{\text{best},s} = \begin{bmatrix} \mathbf{x}_{1,\text{self}}^{\text{best},s} \\ \mathbf{x}_{2,\text{self}}^{\text{best},s} \\ \vdots \\ \mathbf{x}_{i,\text{self}}^{\text{best},s} \\ \vdots \\ \mathbf{x}_{N_p,\text{self}}^{\text{best},s} \end{bmatrix},$$

$$\widehat{\mathbf{x}}_{\text{swarm}}^{\text{best},s} = \begin{bmatrix} \mathbf{x}_{\text{swarm}}^{\text{best},s} \\ \mathbf{x}_{\text{swarm}}^{\text{best},s} \\ \vdots \\ \mathbf{x}_{\text{swarm}}^{\text{best},s} \end{bmatrix}, \quad \mathbf{x}^s, \mathbf{x}_{\text{self}}^{\text{best},s}, \widehat{\mathbf{x}}_{\text{swarm}}^{\text{best},s} \in \mathbb{R}^{3N_p \times 1} \tag{20}$$

are defined. For more details one can refer to [15].

3.2. Extension to Mechanical Particle Swarm Optimization. In recent years, the PSO has shown its distinctive advantages for guiding robot swarm motions, for example, in cooperative search. The particles in PSO share some information during the search according to their update formulae and try to find the minimum of an objective function. This is quite similar to a robot swarm searching for a target in an environment according to some search algorithms. If one considers that each particle represents one robot, then the particle motions are determined not only by the PSO update but also by robots mechanical properties and their drives. Unfortunately, the PSO itself is only an optimization tool. Many researches like [4, 9, 11, 16] work without much attention to the robot properties, not to mention including the properties into the guiding algorithms. Most of them are still using PSO as an optimization tool during the search processes. It can be predicted that if they perform motions with real robots, extensive control is required to realize the swarm motions and to solve the difficulties caused by the ill consideration.

Actually, the physical properties of each member in the robotic swarm can seriously affect the performance of the whole swarm system since each individual is a mechatronical system. Thus, the physical background and physical laws must be considered to generate reasonable search trajectories. In this work, we trace back to the PSO algorithm and consider the mechanical properties of robots such as mass, inertia, and forces. This is a big difference between our research and others', which results in our algorithm named mechanical Particle Swarm Optimization. In addition, we assume that the required forces on a robot in mechanical PSO are caused by other robots; that is, the whole robot swarm is a virtually linked multibody system. Physically, the forces are created by corresponding robot actuators.

Considering a real mobile robot i, it moves in a time step Δt by

$$\begin{bmatrix} \mathbf{x}_i^{s+1} \\ \dot{\mathbf{x}}_i^{s+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_i^s \\ \dot{\mathbf{x}}_i^s \end{bmatrix} + \Delta t \begin{bmatrix} \dot{\mathbf{x}}_i^s \\ \ddot{\mathbf{x}}_i^s \end{bmatrix}, \tag{21}$$

where the robot position $\mathbf{x}_i^s$ is given in absolute coordinates, and the simple Euler forward integration method is used.

Here $\ddot{\mathbf{x}}_i^s$ is the generalized acceleration of robot i at time step s, and $\dot{\mathbf{x}}_i^s$ is its velocity.

In multibody dynamics, one knows that all the external influences acting on a rigid body i can be synthesized by resultant forces and torques when our robots work on a plane ground. Due to the Newton-Euler equations, the motion of robot i can be described by

$$\begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & J_i \end{bmatrix} \cdot \begin{bmatrix} a_{i,x} \\ a_{i,y} \\ \alpha_i \end{bmatrix} = \begin{bmatrix} f_{i,x} \\ f_{i,y} \\ l_i \end{bmatrix}, \tag{22}$$

where J_i is the rotational moment of inertia, $a_{i,x}$ and $a_{i,y}$ are the resultant accelerations in x and y directions, respectively, while $f_{i,x}$ and $f_{i,y}$ are the corresponding forces. The α_i represents the planar rotational acceleration, and l_i is the corresponding torque. Thus, the generalized planar acceleration for robot i can be written as

$$\ddot{\mathbf{x}}_i = \left[a_{i,x} \;\; a_{i,y} \;\; \alpha_i\right]^T. \tag{23}$$

If one defines the generalized acceleration

$$\ddot{\mathbf{x}} = \left[\ddot{\mathbf{x}}_1 \;\; \ddot{\mathbf{x}}_2 \;\; \cdots \;\; \ddot{\mathbf{x}}_i \;\; \cdots \;\; \ddot{\mathbf{x}}_{N_p}\right]^T \in \mathbb{R}^{3N_p \times 1} \tag{24}$$

for all N_p robots and $\mathbf{k}$ as the term resulting from Euler equations, as well as a definition that $\mathbf{q}$ contains the information of forces and moments, then the motion of the entire swarm is described by

$$\mathbf{M} \cdot \ddot{\mathbf{x}} + \mathbf{k} = \mathbf{q} \quad \text{or} \quad \ddot{\mathbf{x}} = \mathbf{M}^{-1} \cdot (\mathbf{q} - \mathbf{k}) = \mathbf{M}^{-1} \cdot \mathbf{F}, \tag{25}$$

where $\mathbf{F}$ contains the generalized forces. Here, the multibody system is a free system since no links or joints are involved. Thus,

$$\mathbf{M} = \operatorname{diag}\left(m_1, m_1, J_1, m_2, m_2, J_2, \ldots, m_{N_p}, m_{N_p}, J_{N_p}\right) \tag{26}$$

possesses both the information of masses and rotational inertias of all the robots in the swarm. With (21) and (25), the motion of all N_p robots over time can be computed by

$$\begin{bmatrix} \mathbf{x}^{s+1} \\ \dot{\mathbf{x}}^{s+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{s} \\ \dot{\mathbf{x}}^{s} \end{bmatrix} + \Delta t \begin{bmatrix} \dot{\mathbf{x}}^{s} \\ \mathbf{M}^{-1} \cdot \mathbf{F}^{s} \end{bmatrix}. \tag{27}$$

The used simple Euler forward algorithm is neither very stable nor very efficient. However, these aspects are not critical here, because for signals provided from sensors of real robots, the introduced inaccuracies are even worse.

Next, the essential connection between the mechanical motion of robots and the mechanical PSO will be made. At the moment, the robots are only affected by forces such that no torques appear from other robots; that is, $l_i = 0$. Of course the robot itself can provide its own torques to perform rotations. In other words, the trajectories generated from mechanical PSO only include the two translational motions. There are mainly two reasons to design it like this. One important reason is that we want to benefit from the omnidirectional feature of the robot so as to reduce the consumption of energy. In Figure 9 one can see four basic motion modes for the robot which starts from its initial pose to the desired final pose not strictly along a preferred trajectory.

Figure 9: Robot omnidirectional feature.

The motions in Figures 9(a) and 9(b) are preferred since our researched robot is omnidirectional, and these two modes can save energy. Nonetheless, it is still difficult to run mode (b) directly due to the challenges met in robot position control since the nonlinearity of this robot motion becomes significant when the robot orientation ϕ_g is changing simultaneously. The reason behind this is that the orientation ϕ_g is contained in many entries of the system matrix when describing this robot model in the state space for designing proper controllers. The changing orientation will play an important role in contributing the nonlinearity. Thus, we choose mode (a) as a trade-off using the omnidirectional feature of the robot. The whole trajectory is formed by many small segments. Although the robot does not have the torque affection from other robots, some additional rotation adjustments from the robot itself still can be performed if necessary. In analogy to (18), the force $\mathbf{F}^s$ is assumed to consist of three parts,

$$\begin{gathered} \mathbf{f}_1^s = -\mathbf{h}_{f_1}^s \cdot \left(\mathbf{x}^s - \mathbf{x}_{\text{self}}^{\text{best},s}\right), \\ \mathbf{f}_2^s = -\mathbf{h}_{f_2}^s \cdot \left(\mathbf{x}^s - \hat{\mathbf{x}}_{\text{swarm}}^{\text{best},s}\right), \qquad \mathbf{f}_3^s = -\mathbf{h}_{f_3}^s \cdot \dot{\mathbf{x}}^s, \end{gathered} \tag{28}$$

where $\mathbf{h}_{f_1}^s, \mathbf{h}_{f_2}^s$, and $\mathbf{h}_{f_3}^s$ are force factor matrices. Here, the swarm mode $\hat{\mathbf{x}}_{\text{swarm}}^{\text{best},s}$ can be adjusted to a neighborhood mode $\mathbf{x}_{\text{nhood}}^{\text{best},s}$ when the swarm size is big which brings difficulties for the robots communication. Force $\mathbf{f}_1^s$ has the physical meaning of an attraction force by the last self best position, $\mathbf{f}_2^s$ represents the attraction force by neighborhood or swarm best position, and $\mathbf{f}_3^s$ is a kind of inertia force which is proportional to its last velocity and counteracts a change in

direction. Thus, the motion (27) of the entire swarm robotic system is governed by

$$\begin{bmatrix} \mathbf{x}^{s+1} \\ \dot{\mathbf{x}}^{s+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{s} \\ \left(\mathbf{I}_{3N_p} - \Delta t \mathbf{M}^{-1} \cdot \mathbf{h}^{s}_{f_3}\right) \cdot \dot{\mathbf{x}}^{s} \end{bmatrix} + \Delta t \begin{bmatrix} \dot{\mathbf{x}}^{s} \\ \mathbf{M}^{-1} \cdot \mathbf{h}^{s}_{f_1} \cdot \left(\mathbf{x}^{\text{best},s}_{\text{self}} - \mathbf{x}^{s}\right) \\ +\mathbf{M}^{-1} \cdot \mathbf{h}^{s}_{f_2} \cdot \left(\hat{\mathbf{x}}^{\text{best},s}_{\text{swarm}} - \mathbf{x}^{s}\right) \end{bmatrix}. \tag{29}$$

Comparing the developed mechanical PSO model (29) for generating trajectories for swarm mobile robots cooperative search to the basic PSO model (18), one can see corresponding relationships as follows:

$$\Delta t \mathbf{M}^{-1} \cdot \mathbf{h}^{s}_{f_1} \longleftrightarrow c_1 \mathbf{r}^{s}_1, \qquad \Delta t \mathbf{M}^{-1} \cdot \mathbf{h}^{s}_{f_2} \longleftrightarrow c_2 \mathbf{r}^{s}_2, \qquad \mathbf{I}_{3N_p} - \Delta t \mathbf{M}^{-1} \cdot \mathbf{h}^{s}_{f_3} \longleftrightarrow \omega. \tag{30}$$

The mass matrix $\mathbf{M}$ can be different and can even be changed during motion for each robot which enhances the flexibility. Random effects are included in $\mathbf{h}^{s}_{f_1}$, $\mathbf{h}^{s}_{f_2}$, and $\mathbf{h}^{s}_{f_3}$, and all forces must be created equivalently by local actuators in the robots.

4. Experiments with a Mixed Robot Swarm

4.1. General Experiment Specifications. The experimental environment is a 3 m × 3 m square, and the origin of the coordinate system is set at the center. Thus, both for x- and y-directions the experimental range is [−1.5 m 1.5 m]; see Figure 10. Obstacles and robots in the real environment have a one-to-one counterpart in the simulation. We use a red circle for representing a simulated robot and a gray circle for the one which uses position and velocity information from a real physical robot. The searched target is set at (−0.5 m, 0.45 m) and represented by a red pentagon. However, the robots do not know it. Five real obstacles are also shown in Figure 10. The searching robots are guided by mechanical PSO step by step; that is, the robots are tracking the mechanical PSO generated trajectories online based on the derived robot kinematic and dynamic models under the help of robot position control [17]. An obstacle avoidance module which is able to avoid both robot-obstacle and robot-robot collisions is activated if there is the danger of collisions. The obstacle avoidance module has a higher priority than mechanical PSO; that is, the latter should be collision free. Worth to be mentioned is that the simulation and real experiment are synchronized which means we can realize simulated robots and real robots in a mixed search simultaneously.

Since the robots are searching for a target cooperatively in the environment and the target is described in a general way, the performance of the fitness function in mechanical PSO is represented by the distances between current robots and target. Nevertheless, the target is unknown to the robots. The only change required for instantiation is to change the performance type in the fitness function. For example, in the case of light source search, the distances should be replaced by the strength of the detected light from each robot.

(a) Simulation environment

(b) Real experimental environment

Figure 10: Environment for robot swarm search.

Initial robot poses can be changed and can be arbitrary. However, the pose is required by mechanical PSO for its update. Therefore, the evaluation center should be informed about the initial pose although the robots themselves do not need to know. If the robot has the ability to self-localize its initial position with sufficient accuracy, the user can randomly place the robot to the searched environment without the necessity of manually measuring its pose. In this investigation we directly measure the global initial pose of the "randomly placed robot" since the used robot lacks a sufficiently accurate self-localization of its initial pose. However, our mechanical PSO is designed for robots with arbitrary initial poses.

4.2. Verify Real-Simulated Interactions and Fault Tolerance. This experiment focuses on the real-simulated robots interaction and the fault tolerance ability of the developed mechanical PSO for robots cooperative search. Thus, during the search process, one simulated robot is artificially set to lose

(a) Phase 1: initial status

(b) Phase 2: meet failed robot 4 and interaction

(c) Phase 3: interaction and avoid failed robot 4

(d) Phase 4: final status

Figure 11: Four critical phases of an interaction and fault tolerance experiment.

Figure 12: Robot trajectories of an interaction and fault tolerance experiment.

its wireless signal and stop there. Totally four robots with two real robots 1 and 3 and two simulated ones 2 and 4 are used in this experiment.

Four representative states, that is, the initial status, the status when robot 1 meets and interacts with the simulated failed robot 4, the status when robot 1 avoids the failed robot and passes the challenging narrow way, and the final status of one of the experiments are demonstrated in Figure 11. The robot group uses 189 s for the pure search process, that is, without counting sensor execution and delay time from real robots. From Figure 11 one can see that robot 4 fails after phase 2 during the mission. It stops at one of the obstacle's corners. This leads to the interactions between the real robot 1 and the simulated robot 4. Robot 1 needs to avoid robot 4 while keeping collision free to the obstacles at its left and right sides. This is a big challenge for robot 1 since it needs to pass a narrow channel due to the blocking from the failed robot. In this case the mechanical PSO should have an affordable search ability, and the real robot should have sufficient localization accuracy.

Robot 1 and robot 3 have real-real interactions when they are close to the target. We ran this kind of experiment many times, and positive results were obtained. Some of the trajectories are shown in Figure 12 with the computed and real trajectories being compared. From the convincing trajectory figure one can see that our developed mechanical PSO is valid for such a challenging case. Robot 1 successfully passes the narrow way and approaches the target while it does not collide to the failed robot and the obstacles nearby.

The robots interaction, fault tolerance, and mutual avoidance are also shown easily by the velocity curves in Figures 13 and 14 for robot 1 and robot 4, respectively. Both figures have a good agreement to the trajectories in Figure 12. From Figure 13 one can see that robot 1 has several critical time domains (r1–r4 marked with green circles) where this real robot interacts with the failed robot and avoids obstacles. In the area of r1, robot 1 and robot 4 have the first interaction since the velocities in x- and y-directions of robot 1 go through the zero point. In the r2 area, robot 1 has two very short time interactions with robot 4 since before that moment robot 1 moves at positive x-direction along one obstacle while robot 4 meets the same obstacle (left box) and tries to move at positive y-direction with almost zero velocity in x-direction after "negotiating" with the obstacle. The two robots then meet at the obstacle corner and the interaction happens. At the area of r3, robot 4 has already stopped for a while, so robot 1 has to run a negative y velocity since it was following robot 4

FIGURE 13: Robot 1 global velocity change.

FIGURE 14: Robot 4 global velocity change.

along the vertical boundary of the nearby obstacle (left box). Thus, the time of r3 is also a little bit delayed compared to the time of s3. In the area of r4, robot 1 is trying to pass the challenging narrow channel; thus it has several fluctuations. Accordingly, robot 4 has the s1–s3 feature points. Robot 4 is artificially set to fail after about 100 s running. From Figures 13 and 14 we also can see that the velocity of the simulated robot is much smoother than that of the real robot. This is mainly due to the slippage of the real robot.

(a) Phase 1: initial status

(b) Phase 2: handling obstacles

(c) Phase 3: interaction at left up corner

(d) Phase 4: all robots avoided obstacles

(e) Phase 5: swarm cooperation

(f) Phase 6: final status

FIGURE 15: Critical phases of one of the runs with mixed simulated and real robots in a swarm.

Figure 16: Trajectories of one of the runs with mixed simulated and real robots.

Another very interesting feature worth to be emphasized is the powerful search ability of mechanical PSO. After the failure of robot 4, only three robots remain as "particle swarm" and still possess the swarm advantage. This confirms that our method for the robots cooperative search is running in a systematic way and is not completely dependent on one or several of the robots.

4.3. Cooperative Search by a Robot Swarm Mixed with Simulated and Real Robots. A very natural further step is to increase the number of robots for cooperative search, that is, to set up a bigger robot swarm which is also mixed of simulated and real robots. This is a comprehensive verification; besides the feasibility, it also aims to verify the scalability and flexibility of the investigated methods. In this experiment, four simulated robots 5–8 and four real robots 1–4 are used which are marked with numbers in Figure 15. The number of used robots is quite flexible; however, due to spatial restrictions in this laboratory scenario, only eight robots are included here.

Some important phases of one of the runs of this mixed swarm cooperative search experiment are presented in Figure 15. The initial phase of this run is shown in Figure 15(a) where eight robots are distributed in the search environment. In the second phase, most of the robots meet or yet get closer to obstacles, and these robots are trying to deal with nearby obstacles; see Figure 15(b).

The first interaction between simulated and real robots happens at the left-upper corner in Figure 15(c). In phase 4, Figure 15(d), one can see that all the robots successfully avoid obstacles and approach the target. Swarm cooperation is basically demonstrated in Figure 15(e) where the robots are performing coordination for finding a reasonable position. Eventually, after consuming 165 s for the search algorithm, its final status is recorded in phase 6; see Figure 15(f).

For this robot swarm cooperative search experiment, trajectories of robots are also recorded in Figure 16 which further confirms the applicability of the developed methods.

Comparing the two experiments, the second one increases the number of robots for both simulated and real robots. However, the pure search time is reduced in the second experiment since more robots have a better search ability. But due to the real robot execution and sensor delay time, the total time used for the second experiment is slightly increased. Nevertheless, as long as the robot number increases further, this trend will disappear since our solution is partially distributed.

By performing search experiments with a mixed set of simulated and real robots, the purpose of this investigation is verified. It is a good choice to enhance the search ability of the robot swarm by this method. Furthermore, it provides a nice way for the investigation of simulation and practical experiments interactively; that is, it bridges tightly the simulation side and the experiment side.

5. Conclusions

This work investigates cooperative search of a robot swarm. Kinematic and dynamic modeling of the used robot is performed to provide the basis for the control plant. Then the mechanical PSO algorithm which includes robot physical properties is described for generating search trajectories of the robots. This gives a powerful search ability. Those considered forces between the robot members are used in a virtually linked multibody system. With convincing results from experiments, the goal of hardware swarm mobile robots searching for a target collaboratively is achieved. Performing cooperative search by mixed simulated and real robots in a swarm is another very challenging investigation of this work and is one of its novelties. It shows the scalability and flexibility of a robot swarm. The whole scheme was instantiated in a planar case; however, our method can be extended in the future to a 3D case and to include torques as well. In addition, the guiding mechanical PSO method itself does not rely on specific models of the used robots. Thus, our guiding scheme is qualified also for heterogeneous robot models. According to the obtained encouraging results from experiments, one can claim that the investigated method shows its validity, feasibility, reliability (fault tolerance), qualification for real-simulated robots interaction, and scalability. Based on these, many further experiments are currently done.

Acknowledgments

Part of this work is supported by the Cluster of Excellence Simulation Technology (SimTech) in Stuttgart, Germany, which is greatly acknowledged. The publication is supported

by the German Research Foundation (DFG) within the funding programme Open Access Publishing. All these supports are highly appreciated.

References

[1] R. Bogue, "Swarm intelligence and robotics," *Industrial Robot*, vol. 35, no. 6, pp. 488–495, 2008.

[2] N. M. P. Kakalis and Y. Ventikos, "Robotic swarm concept for efficient oil spill confrontation," *Journal of Hazardous Materials*, vol. 154, no. 1–3, pp. 880–887, 2008.

[3] D. Fritsch, *Steuerung selbstorganisierender Multi-Roboter-Systeme für dynamische Sammelaufgaben am Beispiel der Bekämpfung maritimer Ölverschmutzungen [Ph.D. thesis]*, University of Stuttgart, Stuttgart, Germany, 2009.

[4] J. M. Hereford, M. Siebold, and S. Nichols, "Using the particle swarm optimization algorithm for robotic search applications," in *Proceedings of the IEEE Swarm Intelligence Symposium (SIS '07)*, pp. 53–59, Honolulu, Hawaii, USA, April 2007.

[5] A. T. Hayes, A. Martinoli, and R. M. Goodman, "Swarm robotic odor localization: off-line optimization and validation with real robots," *Robotica*, vol. 21, no. 4, pp. 427–441, 2003.

[6] W. Jatmiko, K. Sekiyama, and T. Fukuda, "A PSO-based mobile robot for odor source localization in dynamic advection-diffusion with obstacles environment: theory, simulation and measurement," *IEEE Computational Intelligence Magazine*, vol. 2, no. 2, pp. 37–51, 2007.

[7] E. Rohmer, T. Yoshida, K. Ohno, K. Nagatani, S. Tadokoro, and E. Konayagi, "Quince: a collaborative mobile robotic platform for rescue robots research and development," in *Proceedings of the 5th International Conference on Advanced Mechatronics*, pp. 225–230, Osaka, Japan, October 2010.

[8] R. Quinn, "US Sending Robots to Fukushima," 2011, http://www.newser.com.

[9] S. Doctor, G. K. Venayagamoorthy, and V. G. Gudise, "Optimal PSO for collective robotic search applications," in *Proceedings of the Congress on Evolutionary Computation (CEC '04)*, pp. 1390–1395, June 2004.

[10] J. M. Hereford and M. Siebold, "Multi-robot search using a physicallyembedded Particle Swarm Optimization," *International Journal of Computational Intelligence Research*, vol. 4, no. 2, pp. 197–209, 2008.

[11] J. Pugh, L. Segapelli, and A. Martinoli, "Applying aspects of multirobot search to Particle Swarm Optimization," in *Proceedings of the 5th International Workshop on Ant Colony Optimization and Swarm Intelligence*, pp. 506–507, Brussels, Belgium, September 2006.

[12] M. Bliesener, C. Weber, K. Kling, U. Karras, and D. Zitzmann, *Festo Robotino Manual*, Festo Didactic GmbH & Co. KG, Denkendorf, Germany, 2007.

[13] Dunkermotoren, *Permanent Magnet DC-Motors*, 2009.

[14] J. Kennedy and R. Eberhart, "Particle swarm optimization," in *Proceedings of the IEEE International Conference on Neural Networks*, pp. 1942–1948, December 1995.

[15] Q. Tang and P. Eberhard, "A PSO-based algorithm designed for a swarm of mobile robots," *Structural and Multidisciplinary Optimization*, vol. 44, no. 4, pp. 483–498, 2011.

[16] S. Xue and J. Zeng, "Controlling swarm robots for target search in parallel and asynchronously," *International Journal of Modelling, Identification and Control*, vol. 8, no. 4, pp. 353–360, 2009.

[17] P. Eberhard and Q. Tang, "Sensor data fusion for the localization and position control of one kind of omnidirectional mobile robots," in *Multibody System Dynamics, Robotics and Control*, J. Gerstmayr and H. Gattringer, Eds., pp. 45–74, Springer, Wien, Austria, 2013.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 362317, 8 pages
http://dx.doi.org/10.1155/2013/362317

Research Article

Global Impact Dynamic Modeling and Verification of a Flexible Beam with Large Overall Motion

Yuechen Duan,[1] Dingguo Zhang,[2] and Jiazhen Hong[3]

[1] *School of Mechanical Engineering, Zhengzhou University, Zhengzhou, Henan 450001, China*
[2] *School of Sciences, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China*
[3] *Department of Engineering Mechanics, Shanghai Jiao Tong University, Shanghai 200240, China*

Correspondence should be addressed to Dingguo Zhang; zhangdg419@mail.njust.edu.cn

Received 20 June 2013; Accepted 14 August 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 Yuechen Duan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The theoretical modeling, numerical simulation, and experimental verification for the impact dynamics of a flexible beam with large overall motion are researched. Based on the rigid-flexible coupling dynamic theory of flexible multibody system, the rigid-flexible coupling dynamic equations of the beam are established. On the use of the continuous contact force method (CCFM) and the contact constraint method (CCM), the respective global impact dynamic equations of the system are derived to achieve dynamic transformation and solution in different stages including impact and unimpact status. The experimental study on the impact dynamics of the flexible beam is investigated, and the dynamic response in the impact process is obtained. The experimental results are compared with the results of the impact dynamic simulation and the finite element method (FEM) simulation. The system's dynamic behaviors in the impact process are analyzed, and the accuracy of the two impact dynamic theories is verified.

1. Introduction

In engineering fields, there are a lot of noncontinuous dynamic problems between complex multibody systems represented by contact and impact, such as landing impact of an aeroplane and wheel/rail contact problem [1, 2]. The impact process will make a great effect on the system's dynamic behavior due to its short duration of time and strong intensity of force. Impact has become an important factor that cannot be ignored in system analysis and control, and it brings great difficulties for the dynamic modeling and numerical simulation of multibody systems [3, 4].

A reasonable dynamic modeling approach is the research foundation for the impact dynamics of flexible multibody systems. According to different assumptions on the impact process, the impact dynamic modeling methods for flexible multibody systems can be summarized as the following three types as discussed by [5, 6]: the impulse-momentum method (IMM), the continuous contact force method (CCFM), and the contact constraint method (CCM).

The impulse-momentum method is an approximate approach which is based on the classic impact theory for rigid bodies, and the impact is considered to finish instantaneously. The state of motion of the system after impact is acquired by calculating the generalized impulse-momentum equations and the coefficient of restitution equations, however, the dynamic responses in the impact process cannot be obtained [7, 8]. The continuous contact force method assumes that the impact force is caused by the local contact deformation, and it is an approximate method that replaces the complex deformation of the contact region by the contact force element. Researchers have made a series of studies on the impact dynamics of multibody systems by using different continuous contact force models [9–11]. The contact constraint method considers the contact effect as the contact constraints which are based on the kinematic conditions of the impact process. The kinematic variables of the system and the contact force are obtained by calculating the dynamic equations within the constraint equations. And the contact constraint method has also been widely used as discussed by [12–14].

Figure 1: Flexible beam with large overall motion.

Only limited experimental researches on the impact dynamics of flexible multibody systems can be found, because the impact process is a transient dynamic behavior with very short duration of time and it raises greater requirements for the response speed of the experimental instruments and the design of the experimental scheme. Seifried et al. [15] investigated the radial impact of a steel sphere impacting with a half-circular aluminum plate both numerically and experimentally, and Louge and Adams [16] observed oblique impacts of a hard aluminum oxide sphere on a thick elastoplastic polycarbonate plate by recording stroboscopic photographs of the sphere trajectory and spin.

In order to describe the global rigid-flexible coupling impact dynamic behaviors of a flexible beam with large overall motion accurately, the dynamics of the flexible beam impacting the fixed-rigid surface will be studied. The unimpact dynamic equations of the beam will be established by using the rigid-flexible coupling theory. The impact dynamic equations will be derived on the use of the continuous contact force model and the contact constraint model, respectively, and the characteristics of the two models will be analyzed. The experimental research of the impact dynamics of the flexible beam will be investigated. And the simulation results using the two impact models will be compared with the experimental results and the FEM results to verify the accuracy of the impact modeling methods.

2. Equations of Motion

The impact dynamics of a flexible cantilever beam rotating about joint O are studied, as shown in Figure 1. A point P on the beam impacts with a point Q on the rigid O-XZ plane. The beam is with the length L, Young's modulus E, the cross-sectional moment of inertia I, the cross-sectional area S, and the volume density ρ. The driving torque at the rotating joint is τ. An inertial coordinate system OXY is established over the shaft O of the flexible beam, with the base vectors $\mathbf{i}$ and $\mathbf{j}$. And a floating coordinate system Oxy is established over O simultaneously, with the base vectors $\mathbf{i}'$ and $\mathbf{j}'$, where the x-axis is along with the neutral axis of the beam when undeformed. The rotational angle θ from the inertial coordinate system to the floating coordinate system is used to describe the large overall motion of the beam, and the deformation motion is described in the floating coordinate system.

The flexible beam is assumed to be a slender Euler-Bernoulli beam. The description of the beam's deformation is shown in Figure 1. In the floating coordinate system, a point $N(x, 0)$ on the beam when undeformed moves to $N'(x', y')$, then, the vector of the relative deformation of N in the floating coordinate system is

$$\mathbf{u} = u_x(x,t)\,\mathbf{i}' + u_y(x,t)\,\mathbf{j}'. \tag{1}$$

The transverse bending deformation and the longitudinal shortening caused by transverse deformation which is also called the coupling deformation item are considered, while the axial tension and compression deformation are ignored. In (1), $u_y(x,t)$ is the transverse deformation and $u_x(x,t)$ is the coupling deformation of point N. The assumed mode method is used to describe the beam's deformation, and the deformation can be expressed as

$$u_y(x,t) = \mathbf{\Phi}_y(x)\,\mathbf{B}(t), \tag{2}$$

$$u_x(x,t) = -\frac{1}{2}\int_0^x \left(\frac{\partial u_y}{\partial \xi}\right)^2 \mathrm{d}\xi = -\frac{1}{2}\mathbf{B}^T(t)\,\mathbf{H}(x)\,\mathbf{B}(t). \tag{3}$$

In (2) and (3), $\mathbf{\Phi}_y(x)$ is the row vector of the transverse modal functions, $\mathbf{B}(t)$ is the column vector of the transverse modal coordinates, and $\mathbf{H}(x)$ is the coupling shape function matrix.

u_x in (3) is the nonlinear second-order coupling item of the deformation [17], which is ignored in the traditional zeroth-order dynamic model. When the large overall motion is at high speed, the coupling deformation will have a significant effect on the system's dynamic behavior. By considering the coupling deformation item, the rigid-flexible coupling dynamic model mentioned here is a more accurate model.

The position vector of point N in the inertial coordinate system after deformation is

$$\mathbf{r} = \mathbf{W}(\mathbf{r}_0 + \mathbf{u}), \tag{4}$$

where $\mathbf{r}_0$ is the position vector of point N in the floating coordinate system before deformation and $\mathbf{W}$ is the direction cosine matrix from the inertial coordinate system to the floating coordinate system.

And the velocity vector of point N in the floating coordinate system is as follows:

$$\dot{\mathbf{r}} = \dot{\mathbf{W}}(\mathbf{r}_0 + \mathbf{u}) + \mathbf{W}\dot{\mathbf{u}}. \tag{5}$$

Then, the energy status and the state of motion of the entire beam can be calculated. The kinetic energy T of the system is

$$T = \frac{1}{2}\int_V \rho\dot{\mathbf{r}}^T\dot{\mathbf{r}}\,\mathrm{d}V = \frac{1}{2}\int_0^L \rho S\left(\dot{\mathbf{r}}^T\dot{\mathbf{r}}\right)\mathrm{d}x. \tag{6}$$

And the potential energy V of the system can be expressed as

$$V = V_E + V_G, \tag{7}$$

where V_E is the elastic potential energy and V_G is the gravitational potential energy, considered as the following:

$$V_E = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 u_y}{\partial x^2}\right)^2 \mathrm{d}x = \frac{1}{2}\int_0^L EI\left(\mathbf{B}^T\mathbf{\Phi}_y''^T\mathbf{\Phi}_y''\mathbf{B}\right)\mathrm{d}x, \tag{8a}$$

$$V_G = \int_0^L \mathrm{d}V_G = \int_0^L -\left(\rho S\mathbf{g}^T\mathbf{W}\left(\mathbf{r}_0+\mathbf{u}\right)\right)\mathrm{d}x = -\mathbf{g}^T\mathbf{W}\mathbf{R}. \tag{8b}$$

In which $\mathbf{g}$ is the column vector of the gravity acceleration.

The system is holonomic and $\mathbf{q} = [\theta, \mathbf{B}^T]^T$ are the generalized coordinates of the system. Let $\mathbf{F}_q$ be the generalized force corresponding with the nonconservative forces, which is only related to the driving torque τ at the joint. Substitute the system's kinetic energy and potential energy into the Lagrange equations as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}} + \mathbf{F}_\mathbf{q}. \tag{9}$$

After complex derivation, as discussed by [18], the unimpact rigid-flexible coupling dynamic equations of the system can be obtained as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}^{\mathrm{I}}, \tag{10}$$

where $\mathbf{M}$ is the generalized mass matrix, and $\mathbf{Q}^{\mathrm{I}}$ is the unimpact generalized force array of the system.

3. Impact Dynamic Modeling

For the impact dynamics of a flexible beam with large overall motion, the large overall motion, the small deformation motion, the impact process, and the coupling effect of them need to be fully considered. The continuous contact force method and the contact constraint method are used, respectively, in the impact dynamic modeling of the system. For both methods, the corresponding impact solving part can be added to the unimpact dynamic equations of the system programmatically and conveniently. The following assumptions for the impact models are introduced: the impact is point-to-point impact, and the size and inertia of the local impacting area do not change during the impact process.

3.1. Continuous Contact Force Method (CCFM). For CCFM, it is assumed that the small deformation of flexible bodies is within the range of linear elasticity, and the impact force is caused by the local contact deformation. The penetrating displacement and velocity between the impacting bodies are chosen as the calculating parameters for the impact force. It is an approximation method that replaces the complex deformation of the contact area with elastic or elastic-plastic force elements. The advantage of the continuous contact force method is that the impact force in the entire impact process can be calculated, and the conversion from unimpact process to impact process is continuous and smooth. Thus, the method is widely used in engineering fields, especially in elastic impact at low speed. However, the contact parameters of CCFM do not have a unified form and are difficult to obtain. The method is mostly based on the quasistatic contact theories, and the influence of the large overall motion and the stress wave is not considered. Meanwhile, the assumption that the impacting bodies penetrate into each other does not meet the physical reality.

Figure 2: Nonlinear spring damper model.

In this paper, the nonlinear spring damper model is selected as the CCFM model. The elastic contact force is used to reflect the restoring force, and the nonlinear damping force is used to reflect the energy loss in the impact process, as shown in Figure 2. The elastic contact force is along with the normal direction of impact and is always a compression force. The direction of the nonlinear damping force is opposite to the direction of the relative velocity of the impacting bodies.

The nonlinear spring damper model describes the contact-impact process based on the Hertz contact theory, and the nonlinear damping force is added artificially. The expression of the total normal impact force is

$$F = \left(F_k + F_d\right)\cdot L\left(\delta\right), \qquad L\left(\delta\right) = \begin{cases} 0, & \delta \geq 0, \\ 1, & \delta < 0, \end{cases} \tag{11}$$

where δ is the penetrating capacity between the impacting bodies which is negative when the impacting bodies penetrate into each other, $L(\delta)$ is a logic function judging contact due to δ, F_k is the normal elastic contact force, and F_d is the nonlinear damping force. As discussed by [10], the concrete expressions are selected as follows:

$$F_k = K|\delta|^{3/2}, \qquad F_d = -CK|\delta|^{3/2}\dot{\delta} = -CF_k\dot{\delta}, \qquad C = \frac{3\left(1-e^2\right)}{4v_0}. \tag{12}$$

In (12), $\dot{\delta}$ is the penetrating velocity between the impacting bodies, K is the contact stiffness coefficient depending on the material and geometry characteristics of the contact surface, C is the damping coefficient, e is the coefficient of

restitution, and v_0 is the initial relative velocity between the impacting points.

The concept of the impact force potential energy is introduced [19], and it is used to calculate the generalized force corresponding with the impact force. As a result, the influence of the impact force on the system's dynamics exists only on the generalized force item. Thus, the impact dynamic equations of the system are established by adding the generalized impact force item to the unimpact dynamic equations (10), while the original generalized mass matrix $\mathbf{M}$ and the unimpact generalized force array $\mathbf{Q}^{\mathrm{I}}$ have no changes. Similar to the form of the gravitational potential energy, the impact force potential energy V_F is

$$V_F = -\mathbf{n}_F^T F \mathbf{W} \mathbf{h}_F. \tag{13}$$

In which, $\mathbf{n}_F$ is the vector of the impacting direction in the inertial coordinate system, $\mathbf{h}_F$ is the position vector of the impacting point in the floating coordinate system, $\mathbf{W}$ is the direction cosine matrix from the inertial coordinate system to the floating coordinate system, and F is the magnitude of the impact force. The impact force potential energy makes an effect only in the impact process and will be 0 in the remaining time because the impact force is 0.

The impact dynamic equations of the system can be established as follows:

$$\begin{gathered} \mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}, \\ \mathbf{Q} = \mathbf{Q}^{\mathrm{I}} + \mathbf{Q}^{\mathrm{II}}. \end{gathered} \tag{14}$$

In (14), the expressions of the generalized coordinate array $\mathbf{q}$, the generalized mass matrix $\mathbf{M}$, and the unimpact generalized force array $\mathbf{Q}^{\mathrm{I}}$ are the same as before, and $\mathbf{Q}^{\mathrm{II}}$ is the generalized impact force array, considered as the following:

$$\mathbf{Q}^{\mathrm{II}} = -\frac{\partial V_F}{\partial \mathbf{q}} = \mathbf{D}F, \tag{15}$$

where $\mathbf{D}$ is an array.

3.2. Contact Constraint Method (CCM). For CCM, contact-impact related constraint equations are added to the system's dynamic equations in the impact process to solve the impact dynamic problem, which means that the two impacting points P and Q on the two impacting bodies keep the same position in the impact process, as shown in Figure 3. Then the impact dynamics are solved by adding the unilateral contact constraint equations to the original unimpact dynamic equations, and the impact dynamic equations of the system turn from ordinary differential equations (ODEs) into differential algebraic equations (DAEs). The assumptions that the impact finishes instantaneously and the impacting bodies penetrate into each other are not needed for CCM. The essence of the method is that the contact constraints are imposed on the system according to the kinematic conditions of the impact process, and the impact force and the system's dynamic response are obtained by calculating the global dynamic equations rather than given artificially before impact. With

Figure 3: Contact constraint model.

respect to IMM which cannot calculate the impact force as well as CCFM which has difficulties in obtaining the contact parameters, the CCM has the advantages in that the entire impact process can be calculated and the contact-related parameters by adding artificially are not needed. However, the state of motion of the system mutates at the initial impact moment due to the fact that the system turns from unconstraint status into constraint status instantaneously. And it brings difficulties to the numerical calculation as a result of the jump and discontinuity phenomenon.

The system's degree of freedom is n and the number of contact constraint equations in the impact process is s. Then, the contact constraint equations in the form of position, velocity, and acceleration can be written as

$$\mathbf{C}(\mathbf{q}, t) = \mathbf{r}_P - \mathbf{r}_Q = \mathbf{0}, \tag{16}$$

$$\dot{\mathbf{C}} = \mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}} + \mathbf{C}_{\mathbf{t}} = \mathbf{0}, \tag{17}$$

$$\ddot{\mathbf{C}} = \mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}} + \left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}} + \mathbf{C}_{tt} = \mathbf{0}. \tag{18}$$

In (16) to (18), the position constraint equation $\mathbf{C}$, and the velocity constraint equation $\dot{\mathbf{C}}$, the acceleration constraint equation $\ddot{\mathbf{C}}$ are s-order arrays; $\mathbf{q}$ is an n-order generalized coordinate array; and $\mathbf{r}_P$ and $\mathbf{r}_Q$ are the positions of the impacting points. For concise presentation, define $\mathbf{C}_{\mathbf{q}} = \partial\mathbf{C}(\mathbf{q}, t)/\partial\mathbf{q}$ and $\mathbf{C}_t = \partial\mathbf{C}(\mathbf{q}, t)/\partial t$, same as below.

The contact constraint equation in the acceleration form (18) can also be written as

$$\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}} = -\left(\mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{C}_{tt} \triangleq \boldsymbol{\gamma}. \tag{19}$$

Then, the impact dynamic equations of the system are

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{q}}^T \\ \mathbf{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \boldsymbol{\gamma} \end{bmatrix}, \tag{20}$$

where $\mathbf{C}_{\mathbf{q}} \in R^{s\times n}$ is the Jacobian matrix of contact constraints and $\boldsymbol{\lambda}$ is the Lagrange multiplier array which represents the contact constraint forces.

The numerical violation and calculation difficulty may occur at the initial impact moment due to the uncoordination of motion by adding contact constraints directly. Therefore, a method is needed to achieve kinematic conversion from unimpact status to impact status at the initial impact moment, to realize motion coordination and to get the initial impact conditions by adding contact constraints [14]. It is difficult to obtain the initial impact conditions, and scholars do not have a unified opinion on them by far. Some methods for obtaining the initial impact conditions have been reported, such as

getting the mutated value of velocity according to the first-order discontinuity in an elastic continuum in the stress wave propagation of flexible bodies, and getting the coordination conditions of velocity using the local assumption in the substructure method.

The IMM is chosen to obtain the initial impact conditions in this paper. The IMM is used once on the system at the initial impact moment, and the coefficient of restitution e is set to be 0. Then, the initial impact conditions are obtained, and the kinematic conversion from unimpact status to impact status is achieved. After IMM is used, the impacting point P at the beam bonds to the impact surface naturally after the normal velocity mutates, and the conditions for adding contact constraints are met.

The impact dynamic equations of IMM are shown in [8] as follows:

$$\begin{aligned} \mathbf{M}\Delta\dot{\mathbf{q}} &= \mathbf{D}I, \\ \mathbf{D}^T\Delta\dot{\mathbf{q}} &= -(1+e)\,v_0, \end{aligned} \tag{21}$$

where $\Delta\dot{\mathbf{q}}$ is the increment array of the generalized velocities, e is the coefficient of restitution, v_0 is the initial relative velocity between the impacting points, and $\mathbf{D}$ is the array shown in (15).

At the initial impact moment, (21) is used once for calculation and e is set to be 0. Then, the kinematic conversion from unimpact status to impact status and the motion coordination are achieved without other assumptions. After the calculation, the generalized coordinates of the system do not change, and the generalized velocities of the system mutate at the moment. And the initial conditions for using CCM can be established as

$$\begin{aligned} \dot{\mathbf{q}}|_{t^+} &= \dot{\mathbf{q}}|_{t^-} + \Delta\dot{\mathbf{q}}, \\ \mathbf{q}|_{t^+} &= \mathbf{q}|_{t^-}. \end{aligned} \tag{22}$$

4. Experimental Verification

The experiment that a flexible beam with large rotary motion impacts with a rigid hemisphere is carried out to verify the accuracy of the above two impact dynamic modeling theories.

The impact of flexible bodies is completed in a very short time; therefore, the response measurement on the impact process raises high requirements for the performance of the experimental platform and the accuracy of the measuring instruments. The experimental testing system of the impact of flexible multibody system consists of the test bench, laser vibrometer, quartz piezoelectric force sensor, conditioning amplifiers, calibrator of vibration sensor, electric power, Polytec Scanning Vibrometer (PSV) software system, and LABIVIEW processing software system. As shown in Figure 4, the laser vibrometer is used to acquire the velocity signal and the piezoelectric force sensor is used to acquire the force signal during impact. The beginning and ending of the impact is determined by the sign of the impact force. The experimental system does not need additional equipment, which reduces the interference in the experimental system.

Figure 4: Experimental schematic diagram.

Figure 5: Test bench.

As shown in Figure 5, the test bench consists of the aluminum frame, flexible beam, cast iron base, sensors fixed standoffs, clamping devices, position-limit beam, auxiliary wires, and fixing equipment. The initial angle and the impact velocity of the beam are controlled by the position of the position-limit beam.

The experimental subject is a flexible beam with large rotary motion, and the free-fall flexible beam impacts with the aluminum hemisphere of the force sensor. The impact force response at the impacting position and the normal velocity response of three points on the beam are measured. The aluminum beam with cylinder cross-section has the length $L = 0.7\,\mathrm{m}$, cross-sectional radius $R = 0.0145\,\mathrm{m}$, area moment of inertia $I = 3.47e-8\,\mathrm{m}^4$, mass density $\rho = 2767\,\mathrm{kg/m}^3$, Young's modulus $E = 68.95\,\mathrm{GPa}$, and Poisson's

ratio $\mu = 0.3$. The impacting position $L_p = 0.585$ m and the normal velocity of the impacting point at the initial impact moment $v_0 = -0.4$ m/s.

5. Simulation and Results Comparison

The impact dynamic simulation of a flexible beam corresponding to the experiment is carried out by using CCFM and CCM, respectively. The simulation results are compared with the experimental results and the FEM results by LS-DYNA which is widely used on calculating transient dynamic problems such as impact, to verify the accuracy and credibility of the impact dynamic solving methods.

When the simulation time is set to be 0.1 s and the time step is set to be $1.0e-6$ s, the CPU solving time is 19.9 s by using CCFM and 22.4 s by using CCM. It is the larger number of dynamic equations of CCM and the longer calculating time for DAEs which lead to the longer solving time for CCM.

The impact force and the normal velocity of the impacting point which is the measuring point 1 in Figure 4, obtained by CCFM, experiment, and LS-DYNA are compared, as shown in Figures 6 and 7. As seen from Figure 6, the impact force calculated by CCFM is closer to the experimental result, and the changing trends of the impact force are basically identical. However, the impact force calculated by FEM has a larger difference with the experimental result. The impact force increases from 0 at the start of impact and reduces to 0 at the end of impact, and the entire impact process lasts for 0.6 ms approximately. The impact force has two peaks: the maximum peak is 689 N (CCFM) and 584 N (experiment), and the second peak is 524 N (CCFM) and 297 N (experiment). The error of the maximum peak of the impact force is about 17.98% for CCFM and experiment.

In Figure 7, the normal velocity response of the impacting point calculated by CCFM after impact has some difference with the experimental result, but the overall changing trend is basically identical. The normal velocity of the impacting point changes immediately at the start of impact, and it reduces to 0 then increases reversely. The flexible beam is at the maximum compression state when the impact force reaches its maximum and the normal velocity of the impacting point becomes 0, which agrees with the actual situation of CCFM.

The impact force and the normal velocity of the impacting point obtained by CCM, experiment, and LS-DYNA are compared, as shown in Figures 8 and 9. As seen from Figure 8, the impact force calculated by CCM is closer to the experimental result, and the changing trends of the impact force are basically identical. The maximum peak of the impact force is 625 N (CCM) and 584 N (experiment), and the error of the maximum peak is about 7.02% for CCM and experiment, which is smaller than the CCFM. However, the number of the peak of the impact force obtained by CCM is more than that of the experimental result. It means that the impact force changes more dramatically, and the impact duration of time is a little shorter for CCM.

In Figure 9, the normal velocity response of the impacting point calculated by CCM after impact has some difference with the experimental result, but the overall changing trend is basically the same. The normal velocity of the impacting point mutates to 0 at the initial impact moment when establishing the initial impact conditions by using IMM, which agrees with the actual situation of CCM.

Figure 6: Comparison of impact force (CCFM).

Figure 7: Comparison of normal velocity (CCFM).

From Figures 6 to 9, it is known that CCFM and CCM are credible for solving the impact dynamics of flexible multibody system. The reasons which lead to the differences between numerical simulation and experimental results may include but not limited to that the force sensor is not an ideal rigid body and the damping influence is not considered. In the commercial FEM software, it is that the rigid-flexible coupling effect is ignored which makes an important role in the impact dynamics and there are some accuracy problems

Figure 8: Comparison of impact force (CCM).

Figure 9: Comparison of normal velocity (CCM).

in the penalty function contact algorithm which leads to the huge difference of FEM.

6. Conclusion

In this paper, the theoretical modeling, numerical simulation, and experimental verification for the impact dynamics of a flexible beam with large overall motion are studied. Firstly, the effect of the coupling deformation on the dynamics of the system is considered, and the unimpact rigid-flexible coupling dynamic equations of the system are established. The CCFM based on the nonlinear spring damper model, and the CCM based on the impact kinematic constraints are selected to be the impact dynamic solving methods. The impact dynamic equations of the system are derived, respectively, and advantages and weaknesses of the two types of impact dynamic solving methods are analyzed. Based on the theoretical modeling, the impact experimental verification of the flexible beam is carried out and the simulation results are compared with the experimental results and the FEM simulation results. The variation of the dynamic behavior of the system is revealed during the impact process, and the accuracy and credibility of two impact dynamic solving methods are verified.

Acknowledgments

The paper is supported by the National Science Foundations of China (nos. 11132007, 11272155) and the 333 Project of Jiangsu Province (no. BRA2011172).

References

[1] F. Pfeiffer and C. Glocker, "Contacts in multibody systems," *Journal of Applied Mathematics and Mechanics*, vol. 64, no. 5, pp. 773–782, 2000.

[2] A. A. Shabana, K. E. Zaazaa, J. L. Escalona, and J. R. Sany, "Development of elastic force model for wheel/rail contact problems," *Journal of Sound and Vibration*, vol. 269, no. 1-2, pp. 295–325, 2004.

[3] W. Schiehlen, "Research trends in multibody system dynamics," *Multibody System Dynamics*, vol. 18, no. 1, pp. 3–13, 2007.

[4] A. A. Shabana, "Flexible multibody dynamics: review of past and recent developments," *Multibody System Dynamics*, vol. 1, no. 2, pp. 189–222, 1997.

[5] W. J. Stronge, *Impact Mechanics*, Cambridge University Press, Cambridge, UK, 2000.

[6] Y. A. Khulief, "Modeling of impact in multibody systems: an overview," *Journal of Computational and Nonlinear Dynamics*, vol. 8, pp. 1–15, 2013.

[7] A. S. Yigit, A. G. Ulsoy, and R. A. Scott, "Dynamics of a radially rotating beam with impact, part 1 and 2," *Journal of Vibration, Acoustics, Stress, and Reliability in Design*, vol. 112, no. 1, pp. 65–77, 1990.

[8] D. G. Zhang and J. Angeles, "Impact dynamics of flexible-joint robots," *Computers and Structures*, vol. 83, no. 1, pp. 25–33, 2005.

[9] Y. A. Khulief and A. A. Shabana, "A continuous force model for the impact analysis of flexible multibody systems," *Mechanism and Machine Theory*, vol. 22, no. 3, pp. 213–224, 1987.

[10] H. M. Lankarani and P. E. Nikravesh, "Contact force model with hysteresis damping for impact analysis of multibody systems," *Journal of Mechanisms, Transmissions, and Automation in Design*, vol. 112, no. 3, pp. 369–376, 1990.

[11] R. W. G. Anderson, A. D. Long, and T. Serre, "Phenomenological continuous contact-impact modelling for multibody simulations of pedestrian-vehicle contact interactions based on experimental data," *Nonlinear Dynamics*, vol. 58, no. 1-2, pp. 199–208, 2009.

[12] S.-C. Wu and E. J. Haug, "Substructure technique for dynamics of flexible mechanical systems with contact-impact," *Journal of Mechanical Design*, vol. 112, no. 3, pp. 390–398, 1990.

[13] S. S. Sandhu and J. McPhee, "Partitioned dynamic simulation of multibody systems," *Journal of Computational and Nonlinear Dynamics*, vol. 5, no. 3, pp. 1–7, 2010.

[14] F. X. Dong, J. Z. Hong, K. Zhu, and Z. Y. Yu, "Initial conditions of impact dynamics," *Journal of Shanghai Jiaotong University*, vol. 15, no. 3, pp. 368–371, 2010.

[15] R. Seifried, B. Hu, and P. Eberhard, "Numerical and experimental investigation of radial impacts on a half-circular plate," *Multibody System Dynamics*, vol. 9, no. 3, pp. 265–281, 2003.

[16] M. Y. Louge and M. E. Adams, "Anomalous behavior of normal kinematic restitution in the oblique impacts of a hard sphere on an elastoplastic plate," *Physical Review E*, vol. 65, no. 2, pp. 6–11, 2002.

[17] P. Shi, J. McPhee, and G. R. Heppler, "A deformation field for Euler-Bernoulli beams with applications to flexible multibody dynamics," *Multibody System Dynamics*, vol. 5, no. 1, pp. 79–104, 2001.

[18] D. G. Zhang, "Recursive Lagrangian dynamic modeling and simulation of multi-link spatial flexible manipulator arms," *Applied Mathematics and Mechanics*, vol. 30, no. 10, pp. 1283–1294, 2009.

[19] Y. C. Duan and D. G. Zhang, "Rigid-flexible coupling dynamics of a flexible robot with impact," *Advanced Materials Research*, vol. 199-200, pp. 243–250, 2011.

Hindawi Publishing Corporation
Advances in Mechanical Engineering
Volume 2013, Article ID 327379, 7 pages
http://dx.doi.org/10.1155/2013/327379

Research Article

Evaluating Energy Flux in Vibrofluidized Granular Bed

N. A. Sheikh,[1] S. Manzoor,[2] and U. Ghani[3]

[1] *High Performance Computing Lab., Department of Mechanical Engineering, Mohammad Ali Jinnah University, Islamabad, Pakistan*
[2] *Department of Mechanical Engineering, University of Engineering & Technology, Taxila, Pakistan*
[3] *Wolfson School of Mechanical & Manufacturing, Loughborough University, Loughborough, LE11 3TU, UK*

Correspondence should be addressed to N. A. Sheikh; ndahmed@gmail.com

Received 7 May 2013; Accepted 19 July 2013

Academic Editor: Xiaoting Rui

Copyright © 2013 N. A. Sheikh et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Granular flows require sustained input of energy for fluidization. A level of fluidization depends on the amount of heat flux provided to the flow. In general, the dissipation of the grains upon interaction balances the heat inputs and the resultant flow patterns can be described using hydrodynamic models. However, with the increase in packing fraction, the heat fluxes prediction of the cell increases. Here, a comparison is made for the proposed theoretical models against the MD simulations data. It is observed that the variation of packing fraction in the granular cell influences the heat flux at the base. For the elastic grain-base interaction, the predictions vary appreciably compared to MD simulations, suggesting the need to accurately model the velocity distribution of grains for averaging.

1. Introduction

Granular flows are of particular interest due to their vast and diverse applications as well as their intricate nature and dynamics. One of the major aspects of these flows is the dissipative nature of granular material. A granular material requires continual input of energy for a sustained flow. The amount of heat flux injected at the boundaries strongly influences the flow physics of granular materials especially in the case of vibrated beds. While simulating these flows, the behaviour of granular material at or due to the solid surface is an integral part of the solution for the entire flow field. Considerable efforts have been made to understand the influence of solid surface on granular flow physics, both experimentally and analytically. The experimental work of Savage and Sayed [1] shows the effect of rough walls. Profound effects of boundaries on the granular flow properties have been reported [2–5], which indicate that the drive surfaces are equally important as the granular material itself in determining the results of the test.

Boundary conditions are usually modelled from the first principles of momentum and energy balances at the surface [7]. Thus, by satisfying the expressions for the balance of momentum and energy, the granular boundary effect can be studied. Considerable work can be found on modelling the wall as rough particles which can be treated as granular materials principally, and at high speeds, the effects of shear can be neglected due to slip even leading to instantaneous collision [3–5]. In general, the energy supplied through base vibration E_b is dissipated through particle-particle D_{pp} and particle-side wall collisions D_{pw}. Mathematically,

$$E_b = D_{\mathrm{pp}} + D_{\mathrm{pw}}. \tag{1}$$

The input of energy E_b of the granular cell is through the change in momentum ΔP a particle experiences in encounters with the base. The moving base interacts with the incoming/falling particle causing a change in particle energy ΔE as

$$\Delta E = V\left(t_\delta\right)\Delta P, \tag{2}$$

where $V(t_\delta)$ is the base velocity at time t_δ. Through various methods of averaging, conditions have been obtained that apply to two-dimensional systems of identical disks or three-dimensional systems of identical spheres that interact with boundaries [5]. By assuming that the interaction is completely elastic and instantaneous, averaging over the time, total energy input for sinusoidal wave form can be calculated using [7, 8]:

$$E_b = NmgVf\left(\frac{U}{V}\right), \tag{3}$$

where U is the velocity of the incoming particle, V is the RMS base velocity, N is the number of particles, g is acceleration due to gravity, and m is the mass of single particle. Depending on the unknown function $f(U/V)$, the form of the energy transfer can be modelled for a vibrating surface. Reference [3] showed that the total energy input through vibrating surface can be estimated by considering that each particle possesses mean velocity. In terms of granular temperature T, the proposed heat flux at the vibrating base is given by

$$E_b = \frac{1}{2} NmgV^2 \left(\frac{m}{T}\right)^{1/2}. \tag{4}$$

While for Maxwellian particle velocity distribution, [6] a slightly different relationship for symmetric base profile such as sinusoidal is proposed:

$$E_b = 2NmgV^2 \left(\frac{m}{\pi T}\right)^{1/2}. \tag{5}$$

Both (4) and (5) coincide with (3) with different scale factors depending on the choice of $f(U/V)$. All the formulations rely on the continuum level description of granular temperature and base densities. However, in some scenarios, the predictions based on continuum models are not necessarily correct. At times, the lack of separation of length and/or time scales can lead to break down of continuum order description. Specifically for granular flow in the Knudsen regime or rarefied flow, such description of continuum level state variables is not necessary accurate [8, 9]. For instance, in a vibrofluidized bed, a continual input of energy is required to sustain granular flow usually through the vibrating surface. If the order of the vibration time period matches the mean free time of the particles between collisions, the separation of time scales at the surface can become indistinct. In such a case, assumption of instantaneous collisions may also become susceptible.

It can be concluded that the continuum level description of dissipative granular flows has led to the formation of granular flow models. However, with the increase of number density and/or dissipation, the models tend to fail. Thus, it is necessary to evaluate continuum order description for various flow regimes in order to establish their validity.

2. Vibrating Base-Heat Flux Boundary Condition

The analytical foundation of the heat flux boundary conditions proposed and used in modelling granular beds has largely been developed on the argument of instantaneous binary collisions. Conditions for a steady state of heat flux and momentum balance at a vibrating boundary have successfully been developed [3–6] and implemented in hydrodynamic models [10, 11]. One particularly effective model used in a number of simulations was developed by Richman [4] for bumpy walls and simplifies rather neatly the limiting case of a flat smooth vibrating wall. The results from the Richman method should be directly comparable with Warr's findings [3] in the limits of high frequency excitation. In this latter work, Warr and Huntley showed detailed calculations of the energy transfer from a vibrating boundary to a granular gas. From the mean change in velocity squared of the particles following a collision, they numerically evaluated the energy transfer integral, I_e for a given base velocity magnitude. Keeping in view (3)–(5) and by incorporating momentum balance in the heat balance, I_e can be related to the heat flux through [11] in nondimensional form:

$$J_b^* = \frac{\eta_o G}{(2\pi)^{3/2}} \left(\frac{1}{T^*}\right)^{1/2} I_e^*, \tag{6}$$

where J_b^* is the nondimensional heat flux at the base, η_o is the packing fraction, and T^* is the non-dimensional granular temperature at the base (see Table 1 for details). G is the pair distribution function at the contact encountering the extended volume effect.

Calculations shown by Richman [4] quantified the heat and momentum flux for the rapid granular flows of identical smooth spheres that interact with bumpy boundaries through inelastic collisions. The boundaries were considered to be nonstationary as well and could translate randomly with given specific mean velocities along with deviations about the mean with specified fluctuation velocities. Richman produced expressions for mean energy transfer in all the three orthogonal directions moving with a mean square velocity of $3V^2$ (V^2 for each of the direction), with V being the mean velocity of the base. The relationship includes addition and removal terms scaling with the square of base velocity and granular temperature. The bases of these calculations were upon the Maxwellian velocity distribution functions that relate and describe the velocities of both the flow particles and the boundaries. The rate of exchange was also calculated for the case of linear momentum and kinetic energy between the granular particle and wall particle for a nearly flat surface.

Later, Warr and Huntley showed detailed calculation for a triangular waveform inputting energy [3]. A detailed analysis considering all possible combinations of impacts with possible speeds and positions of a particle upon a vibrating base is provided by Warr and Huntley. The energy was calculated using the mean change in velocity squared of the particles due to collision, and it was evaluated numerically through an energy integral I_e for a given base velocity magnitude. For these calculations, the requirement is on the vibration period which should be much shorter than the mean time between successive collisions made by a given particle and the base. This analysis showed how the energy integral varies with the speed of the incident particles and the velocity of the base while taking into account the coefficient of restitution between the particle and the base. Remarkably, it is shown that high incoming speeds of particles do not necessarily guarantee high energy inputs for small base velocities. In fact, it is also possible for incident particles to lose energy on the average during an impact. This allowed direct comparison to the work of Richman for the mean square velocity of base ($3V^2$) and allowed a calculation of heat flux in a one-dimensional system with flat boundaries.

Kumaran also proposed energy input expressions for vibrated beds [6]. It gives an energy input proportional to the mean square speed of the bed. Although it is easy to

Table 1: Relationship between the dimensional and dimensionless variables.

Dimensionless variable	Dimensional relationship
T^*	T/mgd
z^*	z/d
V^*	$V/\sqrt{(gd)}$
I_e^*	$I_e/(gd)^2$

implement, it does not allow for the possibility of energy extraction from the bed. The nondimensional form of I_e^* for the Kumaran boundary condition can be calculated using [6]:

$$I_e^* = 4T^* \left[(1+e)^2 V^{*2} + 2\left(e^2 - 1\right) T^*\right]. \quad (7)$$

While for Richman, boundary condition I_e^* is given by [3]

$$I_e^* = 8\left(T^* + V^{*2}\right)^{1/2} \left(2V^{*2} - (1-e)\left(T^* + V^{*2}\right)^{1/2} T^{*1/2}\right). \quad (8)$$

In general, the average energy input is proportional to square of nondimensional RMS base velocity V^{*2}. As the base velocity increases, the predictions of the three heat flux models (Richman, Kumaran, and Warr) start to grow. Figure 1 shows the calculation of the energy integral for each of the formulations with $e = 0.9$. We see that at high temperatures (high base velocity), Warr's model is comparable with Richman's calculations, but that at low base temperatures (low base velocity), there are substantial discrepancies. One may notice that predictions using Kumaran are lower than Richman for the given granular temperature range. On the other hand, Warr's model approaches the predictions of Richman's model with the increase of base temperature. At low temperatures, the discrepancy between Warr's model and Richman's model is significant. The discrepancy between Kumaran's and Richman's approach can be related to the additional energy extraction/dissipation term at the base.

In vibrated beds, low granular temperature is often seen along with high packing fractions. The higher densities of granular material especially at higher levels of dissipations may influence the total heat flux at the vibrating boundary. The effect of dissipation during grain-base interaction is phenomenal on the amount of heat flux transfer. However, keeping the effect of grain-base dissipation apart, the effect of high densities is related to the pair distribution function, often modelled using [10, 11]:

$$G = \frac{2-\eta}{2(1-\eta)^3}. \quad (9)$$

The shielding effect due to higher packing fractions results in the likelihood of each grain to interact with base without the influence of its neighbouring. At a given granular temperature, one may observe the heat flux variation with increasing packing fraction in Figure 2 for elastic grain-base interaction. It can be observed that at lower packing fractions and lower temperatures (base velocities), the predictions of Kumaran's model match well with Richman's model. However, as the driving RMS base velocity is increased, the discrepancy between the two models starts to appear especially at higher packing fractions. At higher base velocities (higher temperatures), Kumaran's model predicts fairly high heat flux compared to the model of Richman. The observed discrepancy between the models may lead to significance differences in the flow physics predictions.

Figure 1: Nondimensional heat flux energy integral (log scale) against granular temperature (T^*) at nondimensional base velocity ($V_b^* = 1.74$) for Richman [4], Warr [3], and Kumaran [6].

Figure 2: Nondimensional heat flux against peak packing fraction (η) for models proposed by Richman [4] (*solid line*) and Kumaran [6] (*empty circles*).

3. Steady State Hydrodynamic Model

For a vibrated granular cell, one-dimensional steady state variation in packing fraction or density and granular temperature can be described using

$$\frac{dP}{dz} + \rho g = 0,$$
$$\frac{d}{dz}\left(K\frac{dT}{dz}\right) - D_{\text{pp}} = 0, \tag{10}$$

where ρ is the density, g is the gravity, P is the pressure, K is the thermal conductivity, and D_{pp} is the total dissipation of the granular media. For dilute limits K and D_{pp}, reduce to $K_o = 75/(64\pi d^2)$ and $D_o = 2\pi(6/\pi d)^2$, respectively [10, 11].

The equation of state used for the case is given as

$$p = \rho T\left[1 + 2\left(1 + e\right)G\right]. \tag{11}$$

The equation set (10) requires three conditions to generate the solution. The first one is that the number of particles in the bed is fixed $\int_0^\infty \eta\, dz = N(\pi/6)$, where N is the number of grains per unit area. Secondly that the heat flux condition at the vibrating base is specified, and finally, there is zero heat flux at $z = \infty$. The third boundary condition is the amount of energy imparted at the base. The result shown in Figure 3 uses heat flux estimation of Richman at the vibrating base.

4. Simulation Geometry

The geometry used for the simulation is a vertically vibrated bed with elastic side walls and base. The vibrated bed is filled with smooth hard spheres under vertical vibration in the presence of gravity. The top surface of the cell is kept open. In absence of side wall dissipation, variations in hydrodynamic parameters of the granular flow occur in the vertical direction mainly. The coefficient of restitution between the grains is 0.9. The cell contains 100 grains of uniform diameter $d = 2$ mm under the influence of gravity g. The results for the variation of packing fraction and granulation temperature are compared with event-driven MD simulations.

4.1. MD Simulations. Hard sphere MD simulations are performed to validate the findings of hydrodynamic simulations. These simulations used event-driven scheme with hard sphere model. The ED scheme considers all the collisions as binary and instantaneous only. The base-particle coefficient of restitution was kept at unity to minimize any effects due to the inelasticity of the base. The initial configuration of the cell uses a random distribution of grains in the simulation cell, which is then run for a total of 109 number of collisions to avoid the possibility of biasness with initial placement in the cell. Evaluation of granular temperature and packing fraction and other related parameters is carried out through ensemble averaging over 109 collisions. The hydrodynamic scale trends are generated by mapping the location of individual grains on one-dimensional grid and storing their associated velocities.

For high base velocity, low density, and nearly elastic granular material, the density can be approximated as

$$\rho = \frac{Ng}{T_o}\exp\left(-\frac{gz}{T_o}\right). \tag{12}$$

By equating the energy flux at the bottom plate with the rate of dissipation of energy per unit area of the base, one may find that the base temperature is given by

$$T_o \cong \frac{6}{Nd^2}U^2. \tag{13}$$

It can be noticed that the base temperature is directly dependent on base velocity (temperature) and the packing fraction. With the increase of packing fraction (number of grains in cell), the heat flux variation in the granular cell also varies. Along with heat flux increase at the base, the dissipation also scales ($D_{\text{pp}} \propto T^{3/2}$) [10, 11].

For dilute packing fraction of the cell, the model described using (10) with equation of state can be solved for the granular bed. Figure 3 shows the comparison of packing fraction and nondimensional granular temperature (T^*). The comparison is carried out against the predictions of MD simulations. It can be observed from the variation of packing fraction (Figure 3) that peak packing fraction lies away from $z^* = 0$ (location of base). Naturally, the highest temperature is observed at the base followed by a decreasing profile owing to dissipation. Compared to MD simulations, the comparison is reasonable with prediction of slightly higher peak packing fraction compared to MD simulations. Interestingly, the peak value of packing fraction lies near the lowest value of granular temperature ($\sim z^* = 10$). The region $0 < z^* > 10$ has increasing packing fraction profile and decreasing granular temperature while moving away from the base. This pattern can be visualized as a block of granular material lying in the region of $\sim z^* = 10$, where dissipation rate is at its peak. Below the block of granular material, the grains have rapidly decreasing granular temperature as they move upward.

From the aspect of energy flux imparted through the base, such variation in the packing fraction profile can possibly lead to various types of grain-base collisions. Although the volumetric influence is catered using (9), still the chance for a successive double collision with base (sudden recollision) of a single particle may increase. In such a case, the grains located near the vibrating base experience push back from top material block, immediately after getting kicked from the vibrating surface. This could lead to an extended collision of the grain with the base. An extended collision is the one when the falling grain meets with the base also moving downward. In such a scenario, the resultant direction of motion of the grain will not change. In fact, the grain experiences a secondary collision when the base follows the upward part of the next cycle.

Figure 4 depicts the possible scenarios of grain-base interactions. Figure 4(a) represents the case of extended rapid recollision. The likelihood of this type of collision increases with higher packing fraction compared to case (b) and case (a) in Figure 4. At further higher packing fraction, the peak of the "maximum density" region increases with

Figure 3: Comparison of packing fraction (η) and granular temperature (T^*) variation in 1-dimensional cell for $N = 100$ with MD simulations.

Figure 4: The possible collision scenarios for grain-base interaction.

Figure 5: Moving packing fraction "hump" with the increase of granular cell packing fraction.

a shift in its location (see hydrodynamic predictions in Figure 5), only to increase the chances of sudden extended recollision. At the same time, one may notice that the packing fraction near the base also increases slightly.

Interestingly, the two regions on either sides of the peak packing fraction are fairly similar, that is, the initial steep increase and the following decrease of packing fraction (see Figures 3 and 5). However, the difference in corresponding variation of granular temperature in both regions is phenomenal. The top layer lying above the "block" region of grains has nearly constant granular temperature, though MD simulations suggest slight increase in temperature. The increase of granular temperature away from the vibrating base has

Figure 6: Variation of heat flux in the cell with the increase of granular cell packing fraction.

Figure 7: Variation and comparison of heat flux at the base with the increase of granular cell peak packing fraction at fixed base velocity of $V_b^* = 1.74$.

been previously observed [10]. While the bottom layer has a sharply decreasing granular temperature as the grains move away from the base, the nearly constant granular temperature indicates that the grains in top layer hardly change their energies; that is, there are less collisions between grains at that height, while the rapidly decreasing temperature suggests higher rates of dissipation in the region.

The variation of heat flux throughout the granular bed is also shown in Figure 6. As the packing fraction is increased, the heat flux at the base increases. At the same time, the increased dissipation reduces the fluidization height, resulting in shifting the "maximum density point" towards the base. In such a scenario, one may notice that the calculation of averaged heat flux in vibrated cell relies on the correct description of velocity distribution in the cell. In Figure 7, a comparison of heat flux predicted using MD simulations is drawn against the Kumaran and Richman models for elastic grain-base collision. One may notice that the predictions of the Kumaran as well as the Richman models match reasonably well with the results of MD simulations for low peak packing fractions. However, as the peak packing fraction increases, the deviation between the MD simulations and the Kumaran model becomes apparent. On the other hand, the predictions of heat flux using the Richman model are slightly lower even at low densities. With the increase in the packing fraction, results from MD simulation and the Richman model match somewhat better than Kumaran. However, the discrepancy is still present between the predictions of MD simulations and theoretical models. The results suggest that the theoretical models need improved predictions of velocity distributions of the grains near the base. That may provide better estimation of heat flux at different packing fraction conditions.

5. Conclusion

Granular flow in one-dimensional vibrated cell is modelled using steady state hydrodynamic model. The variation of packing fraction and granular temperature in the cell has dependence on the heat flux imparted to the dissipative media. In the limit of low inelasticity and packing fraction, estimation of heat flux at the vibrating base is predictable. However, with the increase in the granular cell packing fraction, a region of high density granular material is observed away from the base. The region acts like a top shield for the grains moving upward after receiving momentum from the base. With increase in dissipation in the region, the likelihood of grains escaping the block is very low. In such a scenario, the models for the velocity distribution at the base may no longer remain valid and shall be accommodated in the theoretical models. A comparison with MD simulations also shows appreciable deviation from the theoretical predictions. Experimental validation of the presented results can help in revealing the complex flow physics.

Nomenclature

E_b: Energy supplied at base
D_{pp}: Dissipation (particle-particle)
D_{pw}: Dissipation particle-side wall
$V(t_\delta)$: Base velocity at time t_δ
ΔP: Particle change in momentum
U: Velocity of particle
m: Mass of particle
g: Acceleration due to gravity
V: RMS base velocity
N: Number of particles

T: Granular temperature
J_b: Heat flux at base
ρ: Density
η: Packing fraction
η_o: Packing fraction at base
G: Pair function
I_e: Energy transfer integral
e: Coefficient of restitution
P: Pressure
K: Thermal conductivity
D_{pp}: Dissipation of the grains
d: Particle diameter
T_o: Granular temperature at the base
z: Vertical height.

References

[1] S. B. Savage and M. Sayed, "Stresses developed by dry cohesionless granular materials sheared in an annular shear cell," *Journal of Fluid Mechanics*, vol. 142, pp. 391–430, 1984.

[2] M. W. Richman and R. E. Martin, "Confined granular flows induced by identical, parallel, vibrating boundaries," in *Proceedings of the ASME Winter Annual Meeting*, pp. 49–55, December 1993.

[3] S. Warr and J. M. Huntley, "Energy input and scaling laws for a single particle vibrating in one dimension," *Physical Review E*, vol. 52, no. 5, pp. 5596–5601, 1995.

[4] M. W. Richman, "Boundary conditions for granular flows at randomly fluctuating bumpy boundaries," *Mechanics of Materials*, vol. 16, no. 1-2, pp. 211–218, 1993.

[5] J. T. Jenkins and M. Y. Louge, "On the flux of fluctuation energy in a collisional grain flow at a flat, frictional wall," *Physics of Fluids*, vol. 9, no. 10, pp. 2835–2840, 1997.

[6] V. Kumaran, "Temperature of a granular material "fluidized" by external vibrations," *Physical Review E*, vol. 57, no. 5, pp. 5660–5664, 1998.

[7] J. T. Jenkins and M. W. Richman, "Boundary conditions for plane flows of smooth, nearly elastic, circular disks," *Journal of Fluid Mechanics*, vol. 171, pp. 53–69, 1986.

[8] S. McNamara and S. Luding, "Energy flows in vibrated granular media," *Physical Review E*, vol. 58, no. 1, pp. 813–822, 1998.

[9] J. E. Galvin, C. M. Hrenya, and R. D. Wildman, "On the role of the Knudsen layer in rapid granular flows," *Journal of Fluid Mechanics*, vol. 585, pp. 73–92, 2007.

[10] H. Viswanathan, R. D. Wildman, J. M. Huntley, and T. W. Martin, "Comparison of kinetic theory predictions with experimental results for a vibrated three-dimensional granular bed," *Physics of Fluids*, vol. 18, no. 11, Article ID 113302, 2006.

[11] T. W. Martin, J. M. Huntley, and R. D. Wildman, "Hydrodynamic model for a vibrofluidized granular bed," *Journal of Fluid Mechanics*, vol. 535, pp. 325–345, 2005.