Abstract and Applied Analysis

Robust Control, Optimization, and Applications to Markovian Jumping Systems

Guest Editors: Shuping He, Zhengguang Wu, Hao Shen, Yanyan Yin, and Quanxin Zhu

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Editorial

Robust Control, Optimization, and Applications to Markovian Jumping Systems

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Markovian jumping systems have arisen naturally in the mathematical modeling of phenomena spanning disciplines in the social sciences, natural sciences, and engineering. This kind of stochastic dynamical systems can be employed to model the dynamics when parameters are subject to random abrupt changes due to sudden environment changes, subsystem switching, system noises, executor faults, and so forth. Much attention has been given to modeling, optimization, and real applications of such stochastic dynamical systems in the literature in recent years. As the advanced control and optimization will provide a basis for the design and application of such stochastic systems, these advanced techniques would result in substantial and sustainable benefits. The accepted papers in this special issue include stochastic stability, stabilization, stochastic control optimization, system modeling and identification methods, predictive control, signal processing, robust filtering, multiagent systems, networked control systems, time-delayed systems, neural networks, the Takagi-Sugeno fuzzy systems, simulated annealing, and fault detection methods.

We have accepted thirty-six papers in this special issue. In the published papers, eight consider the stability and stabilization problems of stochastic systems. There are fourteen papers which discuss the problems of the controller design and relevant optimization algorithms. Six articles study the system modeling and identification methods. One paper focuses on the fault detection for wireless networked control systems with stochastic uncertainties and multiple time delays, and seven consider the state estimation and filtering problems.

The problems of stochastic stability and stabilization problems of Markovian jumping systems have been extensively studied by many researchers, and many relevant results have been made. The paper entitled "*Sufficient conditions on the exponential stability of neutral stochastic differential equations with time-varying delays*" by Y. Tian and B. Chen considers the exponential stability in almost sure sense of the neutral stochastic differential equations with time-varying delays and the paper entitled "*Delay-dependent robust exponential stability and H_∞ analysis for a class of uncertain Markovian jumping system with multiple delays*" by J. Xia deals with the problem of robust exponential stability and H_∞ performance analysis for a class of uncertain Markovian jumping systems with multiple delays. The paper entitled "*On input-to-state stability of impulsive stochastic systems with time delays*" by F. Yao et al. is concerned with pth moment input-to-state stability and stochastic input-to-state stability of impulsive stochastic systems with time delays. The paper entitled "*Absolute stability of a class of nonlinear singular systems with time delay*" by H.-B. Zeng et al. studies the absolute stability for a class of nonlinear singular systems with time delay. The paper entitled "*Analysis and design of*

networked control systems with random Markovian delays and uncertain transition probabilities" by L. Qiu et al. focuses on the stability issue of discrete-time networked control systems with random Markovian delays and uncertain transition probabilities. For the stochastic stabilization aspects, the paper entitled "*Finite-time boundedness for a class of delayed Markovian jumping neural networks with partly unknown transition probabilities*" by L. Liang is concerned with the problem of finite-time boundedness for a class of delayed Markovian jumping neural networks with partly unknown transition probabilities, and the paper entitled "*Output feedback adaptive stabilization of uncertain nonholonomic systems*" by Y. Wu et al. investigates the problem of output feedback adaptive stabilization control design for a class of nonholonomic chained systems. The paper entitled "*Robust exponential stabilization of stochastic delay interval recurrent neural networks with distributed parameters and Markovian jumping by using periodically intermittent control*" by J. Hu et al. considers a class of stochastic delay recurrent neural networks with distributed parameters and Markovian jumping.

In recent years, the research on control optimization for stochastic dynamic systems has received more and more attention. For the H_∞ control aspects, the paper entitled "*Finite-time H_∞ control for discrete-time Markov jump systems with actuator saturation*" by B. Li and J. Zhao presents the finite-time control problem for discrete-time Markov jump systems subject to saturating actuators, and the paper entitled "*Finite-time H_∞ control for a class of discrete-time Markov jump systems with actuator saturation via dynamic antiwindup design*" by J. Zhao et al. deals with the finite-time control problem for discrete-time Markov jump systems subject to saturating actuators. The paper entitled "*Stochastic finite-time H_∞ performance analysis of continuous-time systems with random abrupt changes*" by B. Wang proposes the problem of H_∞ control performance analysis of continuous-time systems with random abrupt changes. The paper entitled "*Robust finite-time H_∞ control for nonlinear Markovian jump systems with time delay under partially known transition probabilities*" by D. Yang and G. Zong considers the problem of robust finite-time H_∞ control for a class of nonlinear Markovian jump systems with time delay under partially known transition probabilities. The paper entitled "*Nonfragile H_∞ control for stochastic systems with Markovian jumping parameters and random packet losses*" by J. Wang and K. Zhang is concerned with the nonfragile H_∞ control problem for stochastic systems with Markovian jumping parameters and random packet losses. The paper entitled "*Finite-time control for Markovian jump systems with polytopic uncertain transition description and actuator saturation*" by Z. Tang addresses the finite-time L_2-L_∞ control problems for Markovian jump systems with time-varying delays, actuator saturation, and polytopic uncertain transition description; the paper entitled "*Resilient robust finite-time L_2-L_∞ controller design for uncertain neutral system with mixed time-varying delays*" by X. Chen and S. He proposes the delay-dependent resilient robust finite-time L_2-L_∞ control problem of uncertain neutral time-delayed system. For the predictive control aspects, the paper entitled "*A simplified predictive control of constrained Markov jump system with mixed uncertainties*" by Y. Yin et al. designs a simplified model predictive control algorithm for discrete-time Markov jump systems with mixed uncertainties; the paper entitled "*Predictive function optimization control for a class of hydraulic servo vibration systems*" by X. Feng et al. is concerned with the problem of predictive function control for a class of hydraulic vibration servo control systems. The paper entitled "*Global multivariable control of permanent magnet synchronous motor for mechanical elastic energy storage system under multiclass nonharmonic external disturbances*" by Y. Yu et al. proposes a global multivariable control algorithm based on nonlinear internal model principle under multiclass external disturbances; the paper entitled "*Feedforward and feedback control performance assessment for nonlinear systems*" by Z. Wang and J. Chen proposes a performance assessment method for nonlinear feedforward and feedback control systems. For the multiagent systems control aspects, the paper entitled "*Output feedback control for couple-group consensus of multiagent systems*" by H. Zhao et al. deals with the couple-group consensus problem for multiagent systems via output feedback control; the paper entitled "*Asynchronous gossip-based gradient-free method for multiagent optimization*" by D. Yuan considers the constrained multiagent optimization problem; and the paper entitled "*Control of multiagent systems: a stochastic pinning viewpoint*" G. Wang develops a stochastic pinning approach for multiagent systems to guarantee such systems being almost surely stable.

Over the past few decades, the signal processing, estimation, and filtering problems have long been the mainstream of research topics. For the filtering problems, the paper entitled "*Structural stiffness identification based on the extended Kalman filter research*" by F. Wang et al. develops an extended Kalman filter to identify the structural stiffness parameters, the paper entitled "*Improved robust H_∞ filtering approach for nonlinear systems*" by J. Chen and H. Sun presents an improved design approach of robust H_∞ filter for a class of nonlinear systems described by the Takagi-Sugeno fuzzy model, and the paper entitled "*Delay-dependent robust L_2-L_∞ filtering for a class of fuzzy stochastic systems*" by Z. Li and X. Yang is concerned with the L_2-L_∞ filtering problem for a kind of Takagi-Sugeno's fuzzy stochastic system with time-varying delay and parameter uncertainties. For the state estimation problems, the paper entitled "*State estimation for wireless network control system with stochastic uncertainty and time delay based on sliding mode observer*" by P. Guo et al. considers the state estimation problems for a kind of wireless network control system with stochastic uncertainty and time delay, whereas the paper entitled "*State estimation for time-delay systems with Markov jump parameters and missing measurements*" by Y. Tan et al. is concerned with the state estimation problem for a class of time-delay systems with Markovian jump parameters and missing measurements. The paper entitled "*Constants within error estimates for Legendre-Galerkin spectral approximations of control-constrained optimal control problems*" by J. Zhou addresses the explicit formulae of constants within a posteriori error estimate for optimal control problems, and the paper entitled "*Optimal state estimation for discrete-time Markov jump systems with missing observations*" by Q. Sun et al. is concerned with the

optimal linear estimation for a class of direct-time Markov jump systems with missing observations. The paper entitled "*Fault detection for wireless networked control systems with stochastic uncertainties and multiple time delays*" by L. Rong et al. investigates the fault detection problem for a class of wireless networked control systems.

As is well known that the system modeling and identification are very useful in engineering applications, the paper entitled "*Integration by parts and martingale representation for a Markov chain*" by T. K. Siu derives the integration-byparts formulas for functions of fundamental jump processes relating to a continuous-time, finite-state Markov chain using the Bismut measure change approach. The paper entitled "*Generalized mutual synchronization between two controlled interdependent networks*" by Q. Xu et al. focuses on the generalized mutual synchronization between two controlled interdependent networks. The paper entitled "*Cascading dynamics of heterogenous scale-free networks with recovery mechanism*" by S. Li et al. defines five kinds of weighting strategies to assign the external resources for recovering the edges from cascading failures in heterogeneous scale-free networks. The paper entitled "*Portfolio strategy of financial market with regime switching driven by geometric Lévy process*" by L. Zhou and Z. Wang studies the problem of a portfolio strategy for financial market with regime switching driven by the geometric Lévy process. The paper entitled "*Gsanular space reduction to a β multigranulation fuzzy rough set*" by J. Zhou et al. further generalizes β multigranulation rough set to fuzzy environment. The paper entitled "*Boundary recognition by simulating a diffusion process in wireless sensor networks*" by D. Gu et al. proposes a distributed algorithm for boundary recognition in wireless sensor networks.

Although the selected topics and published papers are not a comprehensive representation of stochastic Markovian jumping systems, the authors represent the rich and many-faceted knowledge and we still hope the reader will find our special issue very useful.

Acknowledgments

We first express our great appreciation to all the authors of this special issue for their high quality contributions. All the reviewers' efforts in reviewing the papers are also very greatly acknowledged.

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Research Article

Boundary Recognition by Simulating a Diffusion Process in Wireless Sensor Networks

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Wireless sensor networks (WSN) are becoming increasingly promising in practice. As the predeployment design and optimization are usually unpractical in random deployment scenarios, the global optimum of the WSN's performance is achievable only if the topology dependent self-organizing process acquires the overview of the WSN, in which the boundary is the most important. The idea of this paper comes from the fact that contours only break on the geometrical boundary and the WSN are discrete sampling systems of real environments. By simulating a diffusion process in discrete form, the end point of semi-contours suggests the boundary nodes of a WSN. The simulation cases show the algorithm is well worked in WSN with average degree higher than 10. The boundary recognition could be very valuable for other algorithms dedicated to optimize the overall performance of WSN.

1. Introduction

There are some areas where we have interests in what is happening, but environments are hostile for a man or too costly to sending a man for the duty. Wireless sensor networks (WSN), which are usually at low cost and self-organized, are appropriate for those tasks [1, 2].

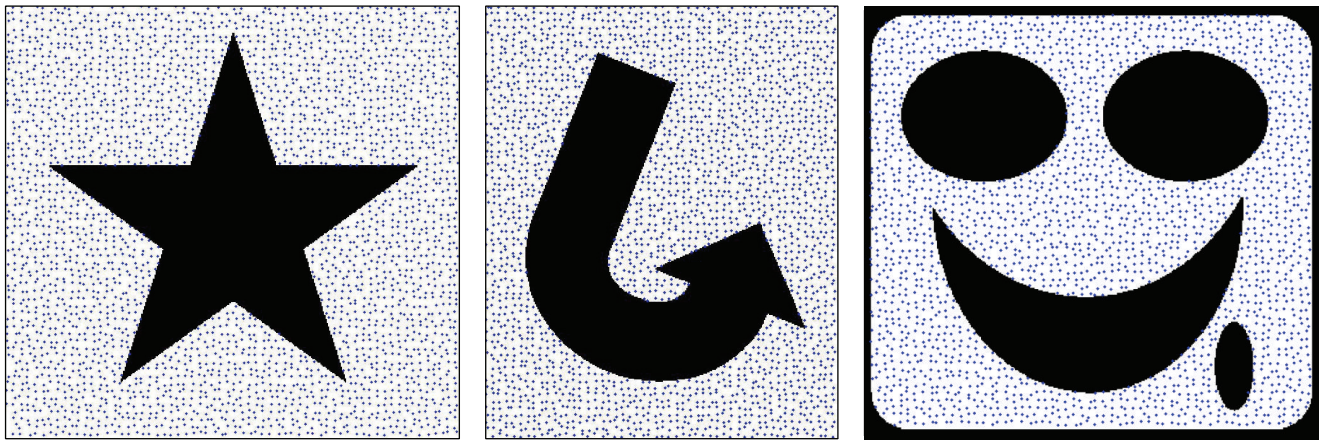
Some existed algorithms assume that the sensor fields are convex in shape [3, 4]. However, such assumption is not always fulfilled. The situation of a particular interested area is often unknown. It is quite possible that the area contains some regions with poor accessibility, such as unforeseen obstacles and/or holes [5]. Thus, applying those protocols may lead to a degraded performance or suffer a failure result. Thus, recognizing the geometry of the field should be the first step of organizing WSN, which is deployed in an unknown field, to try to achieve better performances. Plotting the boundary is the most basic measure to describe a geometric shape and probably the best one. In this paper, we study the problem of revealing the global geometric feature of the sensor field, in particular, recognizing the sensor nodes on the boundary.

Our viewpoint is to regard the WSN as a discrete sampling of the geometric environment. This is inspired by the fact that the WSN are used for providing intense monitoring of the environment. So, the boundaries of the sensor field usually represent the physical boundary of the underlying environments, such as walls of buildings and changes of topography. More importantly, newly appeared boundaries, which means a majority of local sensors are off duty due to destruction or power deficient, could be an indicator of emergency. For example, a wild fire in forest damages all sensors in fire line and also creates new boundary in the sensor field. An inner boundary is also an important indicator of the unhealthiness of the network, such as insufficient connectivity and coverage, revealing the locations where additional sensor nodes are required.

Furthermore, bottleneck recognition [6, 7], which is vital for precise schedule over WSN, requires boundary information. And, in coverage problem, the coverage intensity near the network boundary attracts a lot of research interest [8, 9]. So, boundary recognition provides useful information for other WSN applications.

2. Previous Works and Assumptions

It is always easy to find the boundary when an overview is offered. For example, in Figure 1, the white area represents the



(a) (b) (c)

Figure 1: Sensor nodes deployed in geometric areas.

field and the dots represent the sensor nodes. The task should be easy if we can have a glance at one of these pictures, because the overview is provided for human brain. However, in WSN, such centralized process of acquiring the overview needs lots of communications to collect all connection information in the whole networks. The cost of doing so in WSN is extremely high in both energy and time. So, decentralized algorithm or distributed ones are required.

There are some distributed algorithms trying to recognize the boundary in the literature. They can be classified into three categories by their basic ideas: geometric-based algorithms, statistical algorithms, and topological-based algorithms.

The geometric-based algorithms assume that a node of WSN realizes the exact locations of itself and the nodes in its neighborhood. Fang proposed the algorithm based on the fact that a data packet can only get stuck in a node at boundary in a geographical forwarding [10, 11]. So some boundary nodes are identified. Repeating the process of such geographical forwarding starting from different beacon nodes eventually discovers almost the complete boundary cycles. The idea is nice and clear. However, the information of location depends on locating algorithm or locating device such as GPS system. Locating algorithm certainly consumes some energy and the locating error may lead to boundary error. While the locating device is usually an energy hunger. And more, sweeping over the whole network again and again consumes lots of energy. So, the geometric based algorithm recognizes the boundary at a high cost.

The information of nodes' location definitely benefits the boundary recognition. However, the boundary recognition is also needed in the WSN which do not have the ability of locating. So statistical algorithms and topological based algorithms are developed for such WSN.

Statistical algorithm assumes that the nodes are uniformly randomly deployed in sensor field. Fekete proposed an algorithm with such assumption [12]. The idea is inspired by the law of large number. According to the law, the average of the results obtained from a large number of trials should be close to the expected value and will tend to become closer as more trials are performed. In his algorithm, the deployment of other nodes is regarded as a "trial," and the ratio of neighboring area and the whole sensing area is regarded as "expected value." So, if lots of nodes are deployed, the number of neighbor nodes should be "total trials" × "expected value." Thus, a node should have a number of neighbors that is close to the average degree (the average number of neighboring nodes in the whole network), unless it is on a boundary. This is because the neighboring area of a boundary node is much smaller than an interior node. The algorithm does not require any location information and gets good result in WSN with high average degree. However, the requirement of density is unrealistic: the average degree should be close or over 100 [13]. In practice, the network is often so sparse that the number of "trials" is not big enough to make the results close to "expected value."

Topological based algorithms assume that a node knows only which other nodes are connected directly [14–18]. This assumption is similar to that in this paper; especially we are inspired by Funke's approach [17]. In this method, a group of beacons are randomly selected first. Then, after flooding, all nodes in WSN are given a "distant," which is the hop count to the nearest beacon. In this way, there are many iso-contours of "distant" in the WSN. Finally, the nodes where the iso-contours break are marked as boundary nodes. The simulation of the algorithm shows that some interior nodes are faultily identified. This is because the value of "distant" is measured in integer; randomness of deployment may cause the "distant" of a particular node vary from x to $x + 1$ or $x - 1$. Such phenomenon in WSN makes it possible that interior node is faultily identified, especially in sparse networks. Wang proposed another topological based algorithm [13]. It is reported that the complete sequences of boundary nodes are identified. However, this method can only be used in the scenario that the WSN have topology holes in them. The algorithm does not find any boundary nodes in WSN which is simply connected (without holes). Furthermore, if there are some nodes faultily recognized

already (this is never avoided completely), the final process, which connects distributed boundary nodes to a sequence in order to decrease the missing identification, could be a disaster as faulty identification increases massively. In a recent paper, [18] proposes another topological algorithm, which achieves good hole detection result. But its complexity is higher than the algorithm proposed in this paper.

This paper proposes a distributed algorithm for recognizing the boundary of WSN, using only direct connection information. We do not assume that any location information, distance information, or angular information is collected.

This paper is based on the following assumptions:

(1) the nodes in WSN are provided with limited computation ability, energy, and memory;

(2) the communication range of a node is much greater than sensing range; so, the average degree is reasonable if the sensing field is well covered;

(3) the nodes are uniformly randomly deployed in the sensing field;

(4) the nodes are deployed in a closed area;

(5) the sensing data are not required; that is, the algorithm does not require any positioning information about the nodes.

The basic idea of this paper comes from an intense observation of a gas diffusion process in a closed space. We are motivated by the fact that some features of concentration field suggest the boundary of a closed space and then realize that the boundary of WSN can be recognized by simulating similar process.

3. An Observation of Mass Diffusion

3.1. The Process of Mass Diffusion. Consider the following scenario. Bounded space G is filled with inactive gas. For some reasons, another type inactive gas α is generated at constant rate at $Y_1, Y_2, \ldots, Y_n$, which are inside the space G. As time goes, gas α gradually spreads everywhere in the space. This process is a typical diffusion process. Let us observe that the concentration C of gas α varies from time t and position $P(x, y, z)$ intensely in this process:

$$C = C(P, t). \tag{1}$$

Equation (1) is continuous and two-order differentiable mathematically.

Fick's first law relates the diffusive flux to the concentration under the assumption of steady state. It postulates that the flux goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient (spatial derivative):

$$\vec{j} = -D\nabla C, \tag{2}$$

where $\vec{j}$ is the diffusion flux [mol·m^{-2}·s^{-1}]. $\vec{j}$ measures the amount of substance that flows through a small area during a small time interval. D is the diffusion coefficient or diffusivity in dimensions [m^2·s^{-1}]. ∇C is the concentration gradient [mol·m^{-4}].

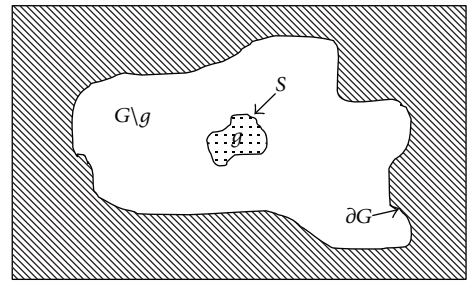


Figure 2: Inside object G, interior region g bounded by surface S.

Equation (2) is the differential form of Fick's first law, which shows how diffusive flux behaves locally. By integrating (2) over an infinitesimal surface S, the integral form of Fick's fist law is derived:

$$\frac{dJ}{dt} = -D \iint_S \nabla C \overrightarrow{dA}, \tag{3}$$

where dJ/dt is the amount of substance transferred per unit time [mol·s^{-1}] and $\overrightarrow{dA}$ is an oriented surface area element [m^2]. The direction is to the outward normal of the element.

Equation (3) describes how substance transfers through a surface.

If the surface S is a non-self-intersecting continuous closed surface as shown in Figure 2, Jordan-Brouwer separation theorem asserts that the surface S divides the object G (a 3-dimensional bounded closed domain) into an "interior" region g, bounded by surface S, and an "exterior" region $G \setminus g$, which consists of all other parts, so that any continuous path ends in different regions intersects S somewhere. Hence, all substance exchange between interior region g and exterior $G \setminus g$ flows through S.

The net amount of substance that flows into G in a small time interval $[t_1, t_2]$ can be derived from (3) by integrating over t:

$$\begin{aligned} J &= \int_{t_1}^{t_2} \oiint_S D\nabla C \, \overrightarrow{dA} \, dt \\ &= \int_{t_1}^{t_2} \iiint_g \left[\frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) \right. \\ &\qquad \left. + \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right)\right] dx\, dy\, dz\, dt. \end{aligned} \tag{4}$$

In the scenario we are observing, there are a group of sources $Y = \{Y_1, Y_2, \ldots, Y_n\}$, which generate gas α at constant rate:

$$F(P, t) = \begin{cases} \text{constant} & P \in Y \\ 0 & P \notin Y. \end{cases} \tag{5}$$

Thus, the net gain of gas α in g in time interval $[t_1, t_2]$ is the sum of the gas that is generated in g and that flows into g:

$$\int_{t_1}^{t_2} \iiint_g \left[\frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right) + F(x,y,z)\right] dx\,dy\,dz\,dt. \tag{6}$$

Meanwhile, the net gain of gas α should also be described by integrating the concentration change over time and space:

$$\int_{t_1}^{t_2} \iiint_g \frac{\partial C}{\partial t} dx\,dy\,dz\,dt. \tag{7}$$

By the law of mass conservation, (6) equals (7) for any space g and time interval $[t_1, t_2]$:

$$\frac{\partial C}{\partial t} = D\Delta C + F, \tag{8}$$

where Δ is the Laplace operator.

3.2. Analysis in Mathematics. Considering the case introduced above, there is not any gas α in G at the very beginning:

$$C = 0 \quad \text{on } G \times \{0\}. \tag{9}$$

And, as the space G is bounded, gas α does not spread outside G. Thus, at the boundary of space ∂G

$$\frac{\partial C(P,t)}{\partial n} = 0 \quad \text{on } \partial G \times (0,t], \tag{10}$$

where n is the outward normal of ∂G.

For the differential equation (8), initial condition and boundary condition are given as (9) and (10). So, there is a unique solution for any time and any spot in G. Therefore, we can simulate the diffusion process discussed above and determine the concentration value everywhere at any time.

In previous works, the boundary of the space G is always known. How about if the boundary exists but we do not know where it is? Shall we find the boundary by observing the concentration field?

Given time t, there is a concentration distribution of α in G. Denote $I(v) = \{P \mid C(P) = v\}$, $I_{\text{less}}(v) = \{P \mid C(P) < v\}$, and $I_{\text{more}}(v) = \{P \mid C(P) > v\}$, where v is nonextreme concentration value. So, $I(v)$, $I_{\text{less}}(v)$, and $I_{\text{more}}(v)$ are all nonempty sets. $I(v)$ is an iso-contour of concentration value v. $I_{\text{less}}(v)$ and $I_{\text{more}}(v)$ are two sets of points where concentration values are less or more than v.

There is at least one point $P_v \in I(v)$ on any path connecting $I_{\text{less}}(v)$ and $I_{\text{more}}(v)$.

Proof. Let $P_l \in I_{\text{less}}(v)$ and $P_m \in I_{\text{more}}(v)$. Let $\varphi : [0,1] \to \mathbb{R}^3$, such that $\varphi(0) = P_l$, $\varphi(1) = P_m$, and the restriction of φ to $[0,1]$ is injective. That is, φ is a non-self-intersecting continuous curved line segment which ends with P_l and P_m. Assume, if possible, $\exists\varphi$, such that $I(v) \cap \varphi = \emptyset$.

If so, for all $x \in [0,1]$, $C(\varphi(x)) \neq v$. As $\varphi \subset G$ and $C(P)$ is a continuous function on G, $C(\varphi(x))$ is a continuous function about x on $[0,1]$. $C(\varphi(0)) = C(P_l) < v$ and $C(\varphi(1)) = C(P_m) > v$; by the intermediate value theorem, $\exists x \in [0,1]$ such that $C(\varphi(x)) = v$, contradicting for all $x \in [0,1]$, $C(\varphi(x)) \neq v$.

Therefore, for all φ, $\exists x \in [0,1]$ such that $\varphi(x) \in I(v)$. □

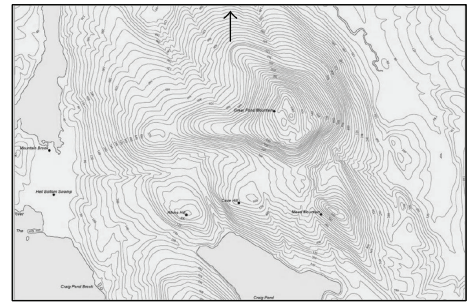


Figure 3: A map with contour [19].

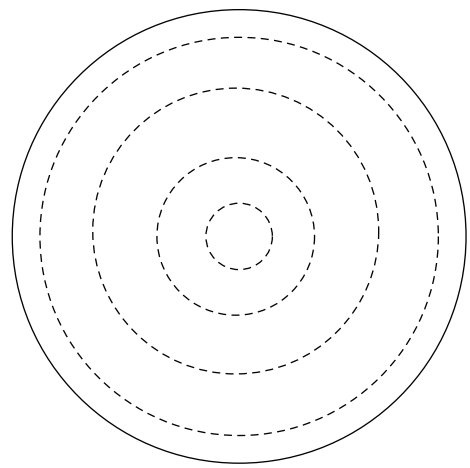


Figure 4: Iso-contours when source is at the center of a circle.

Consequently, iso-contour never breaks in G; otherwise, there should have been paths connecting $I_{\text{less}}(v)$ and $I_{\text{more}}(v)$. Therefore, iso-contour either is closed surface or breaks on the boundary of space G. That is, $\partial I(v) \subset \partial G$. In particular, if G is a 2D space, iso-contour either is closed curve or ends on the boundary of G. Thus, "P is the endpoint of an iso-contour." $\Rightarrow$ "P is on the boundary of G."

The result is encouraging. However, we should notice that "P is on the boundary of G." $\not\Rightarrow$ "P is the endpoint of an iso-contour."

In a map with contour, it is possible that contour is a tangent curve to the boundary of the map at somewhere, as the arrow points to in Figure 3. In this case, the union of end points of all contour is almost equal to the complete boundary of G, missing very few isolated points. But in the worst case, the contour is a tangent curve to the boundary everywhere. Here is an example. The space G is circle or sphere in shape, and the source of gas α is exactly at the center. In this case, the outmost iso-contour meets the boundary exactly as shown in Figure 4.

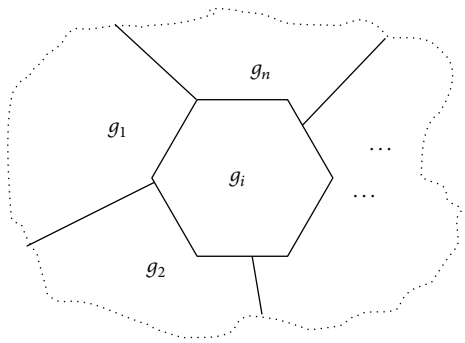


FIGURE 5: Subspace g_i surrounded by $g_1, g_2, \ldots, g_n$.

Although the worst case happens in small probability, we should and can avoid it by generating another type of inactive gas α' at different position and observing the iso-contour $I'(v)$ of α'. This way, the worst case is avoided and the difference between $\bigcup_v(\partial I(v) \cup \partial I'(v))$ and ∂G is even smaller. More generally, if there are n types of inactive gas generating in space G at different positions, the union of end points of all iso-contours and all types is almost equal to the boundary of G:

$$\partial G \approx \bigcup_v \bigcup_n \partial I_n(v). \tag{11}$$

4. Simulation of Diffusion in WSN for Boundary Recognition

4.1. Discrete Form of Diffusion. A space G is uniformly divided into M subspaces. $G = \bigcup_{i=0}^{M} g_i$, $g_i \cap g_j = \emptyset$, $i \neq j$. As G is uniformly divided, denote A as joint area of adjacent subspace, x as distance between them, and V as volume of a subspace. Consider the diffusion process in subspace g_i, which is surrounded by $g_1, g_2, \ldots, g_n$, (see Figure 5).

Similar to (3), the net gain from adjacent subspace by diffusion is

$$\frac{dJ_i}{dt} = \sum_{j=1}^{n} \frac{DA\left(C_j - C_i\right)}{x}. \tag{12}$$

And the change of concentration is due to diffusion in adjacent subspace and the source effect:

$$\frac{dC_i}{dt} = F + \sum_{j=1}^{n} \frac{DA\left(C_j - C_i\right)}{Vx}, \tag{13}$$

where

$$F = \begin{cases} \text{positive constant}, & \text{if there is source of gas in } g_i \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

So,

$$\lim_{\Delta t \to 0} \frac{C_i(t+\Delta t) - C_i(t)}{\Delta t} = F + \frac{DA}{Vx} \sum_{j=1}^{n} \left(C_j(t) - C_i(t)\right)$$

$$\lim_{\Delta t \to 0} C_i(t+\Delta t) = \lim_{\Delta t \to 0} \left[F\Delta t + \frac{DA\Delta t}{Vx} \sum_{j=1}^{n} C_j(t) + \left(1 - \frac{nDA\Delta t}{Vx}\right) C_i(t) \right]. \tag{15}$$

Let $E_i(t) = (1/n)\sum_{j=1}^{n} C_j(t)$ as average concentration of g_i's adjacent subspace and $k = nDA\Delta t/Vx$. If $\Delta t \to 0$, then $k \in (0,1)$ and

$$C_i(t+\Delta t) \approx kE_i(t) + (1-k)C_i(t) + F\Delta t. \tag{16}$$

The concentration of g_i after time interval Δt is a weighted average of current concentration of g_i and its surroundings, plus a positive constant if there is source of gas in it.

4.2. Simulating Diffusion in WSN. In a WSN application, a lot of sensor nodes are deployed in a sensing area. Our viewpoint is to regard the WSN as a discrete sampling of the environment. Every sensor node is a sample of local area. So, we virtually start a simulation of multigas diffusion process.

Assuming that w types of gas are spreading in the area, the local concentration of them at sensor node N_i is $C_i = [C_i^1, C_i^2, \ldots, C_i^w]$. The nodes that can communicate directly with N_i represent the adjacent subspace. Randomly, select w groups of nodes as diffusion source of w types of gas.

At the very beginning, for all i, $C_i(0) = 0$. Then the diffusion process starts.

At time t, sensor node N_i broadcasts its current concentration vector $C_i(t)$. Its 1-hop neighbors will receive this data package. Meanwhile, N_i receives the data packages from its neighbors $C_1(t), C_2(t), \ldots, C_n(t)$. Applying (16) and when $\Delta t = 1$,

$$C_i(t+1) = (1-k)C_i(t) + kE_i(t) + F, \tag{17}$$

where

$$E_i(t) = \frac{1}{n}\sum_{j=1}^{n} C_j(t). \tag{18}$$

$F = [\underbrace{0, \ldots, 0}_{u-1}, c, \underbrace{0, \ldots, 0}_{w-u}]$ if N_i is selected as source of the uth type of gas. $F = 0$ if N_i is not source of any type.

4.3. Semi-Iso-Contours and Their End Points. After repeating several times, the diffusion process spreads virtual gas everywhere in the sensor field. Figure 6(a) shows the concentration distribution of one type of gas in a sensor field. Then, we can draw semi-iso-contours. The reason why we call it semi-iso-contour rather than iso-contour is that there are seldom absolute equalities in such discrete sampling system. Therefore, approximately equality is employed instead. The

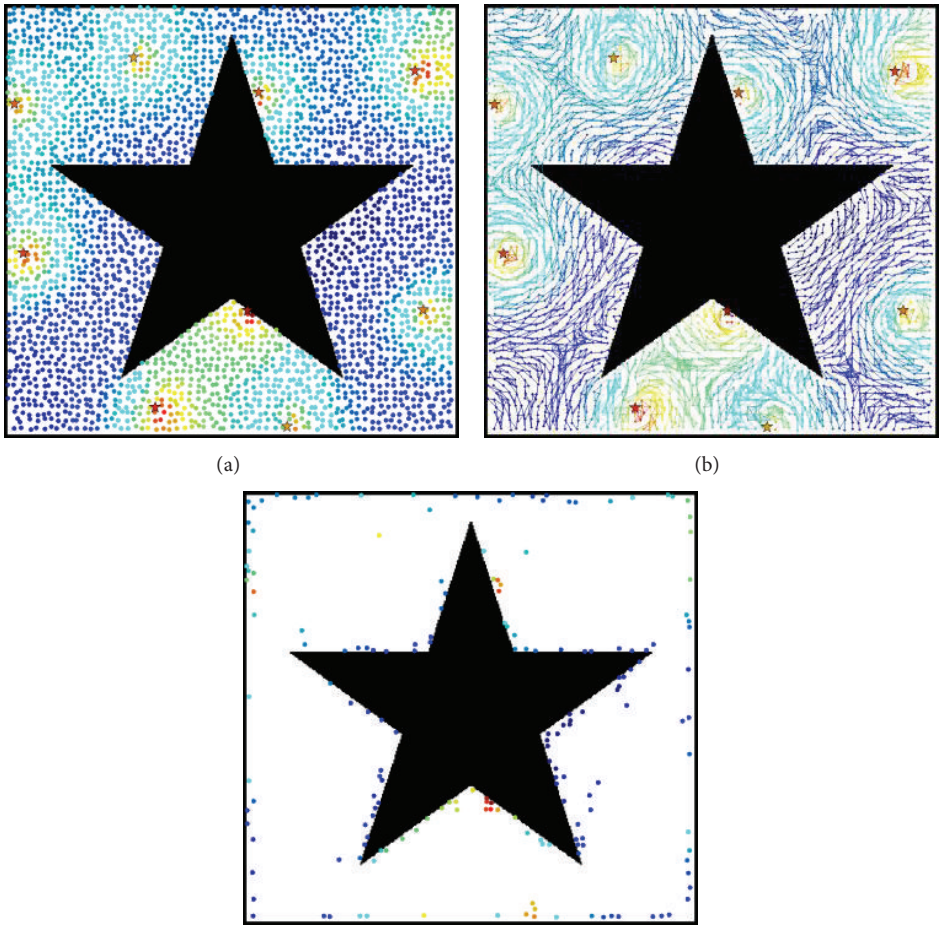


Figure 6: A concentration map and semi-iso-contours. The virtual sources are marked as star. The hotter colour represents higher concentration value, while the cooler colour represents lower concentration value.

criteria we used for approximately equality in 1-hop neighbors are that $C_i^u \approx C_j^u$ if and only if $|C_i^u - C_j^u| < 0.3 \max_{k=0}^{n} |C_i^u - C_k^u|$. Figure 6(b) shows semi-iso-contours in the sensor field. And Figure 6(c) displays all end points of semi-iso-contours.

The end points of semi-iso-contours roughly show the boundary of the sensor field in Figure 6(c). But, there are both some faulty recognitions and miss recognitions. This is because the WSN are a discrete sampling system rather than a continuous physical system. To increase the quality of boundary recognition, we shall use information from other types of gas. In our simulation, $w = 10$; that is, 10 types of virtual gas are spreading simultaneously. Figure 7 show semi-iso-contours and their end points of the other 9 types of virtual gas.

4.4. Final Results of Boundary Recognition. Reading Figure 7, we can conclude that the inner nodes are much less probable to be the end points of semi-iso-contours than the boundary nodes. So, if a node is an end point of semi-iso-contour for multiple times in different types of virtual gas, it is very possible that it is located at the boundary of the WSN. When we pick all nodes that are end points at least 3 times out of 10, the boundary recognition is shown in Figure 8.

4.5. Complexity Analysis. Our approach for boundary recognition consists of 3 steps as follows:

(1) simulating the process of diffusion;

(2) drawing semi-iso-contours;

(3) determining whether to be an end point or not.

The 1st step repeats multiple times of communication in neighborhood and calculation. In each repeat, every node should communicate with all its 1-hop neighbors and update w dimensional vector $C_i(t)$ to $C_i(t+1)$. This is $O(nw)$ in time, where n is the number of 1-hop neighbors. The process should repeat $O(h)$ times in order to guarantee that all nodes are affected by virtual diffusion, where h is the maximum hop counts between 2 nodes in the sensor field. h is decided by the range of the sensor field and the communication range of sensor nodes, which is constant after deployment. So the time complexity for the first step is $O(nw)O(1) = O(nw)$.

The 2nd step requires a comparison in neighborhood. That is $O(nw)$ in time.

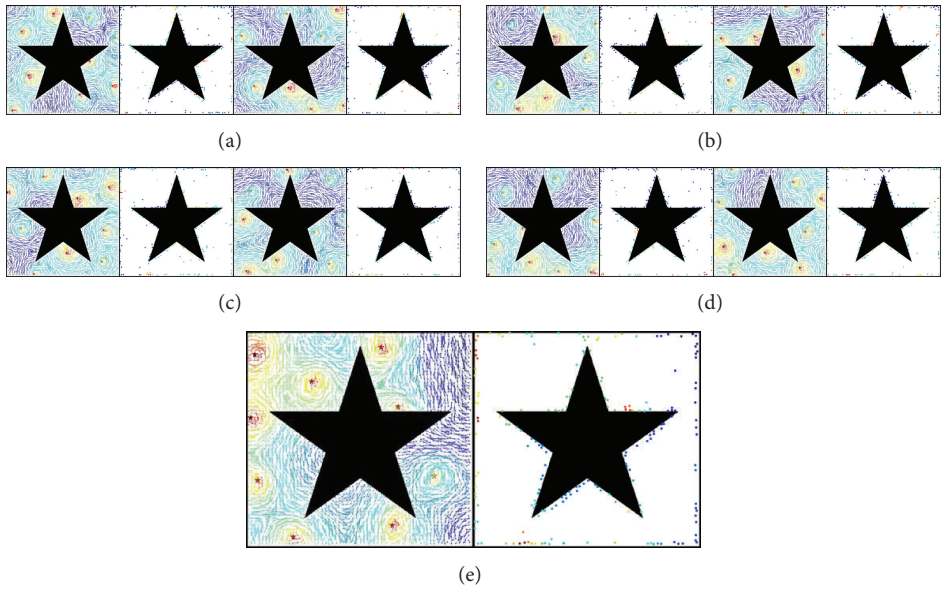


Figure 7: Semi-iso-contours and their end points of the other 9 types of virtual gas.

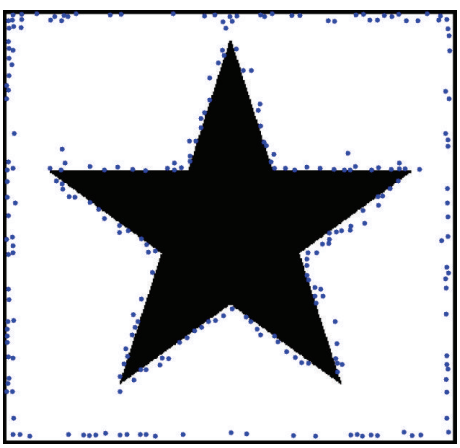


Figure 8: Final result of boundary recognition.

The 3rd step requires a count over w types. That is $O(w)$ in time.

So, the overall complexity in time is $O(nw)$.

In all three steps, the nodes record current concentration value of adjacent nodes and itself. All historical data are discarded. So the overall complexity in memory is $O(nw)$.

A recent paper proposes a topological based algorithm [18], which achieves good hole detection result. In this paper, Dijkstra's shortest path algorithm is used to construct manifold, so the total complexity is at least $O(N^2)$, where N is the number of all sensors in the field. In comparison, the algorithm we proposed is much less complex.

5. Case Study

The algorithm discussed above is applied in different sensor fields. The results are shown in Figure 9.

In all these cases, the nodes that are recognized as boundary nodes generally cover the geometrical boundary of the sensor fields. A few inner nodes, which are at least 1-hop range away from actual geometrical boundary, are faultily identified. Table 1 is a statistic of faulty recognition.

It is predictable that the result of boundary recognition is better if the average degree is higher, because when $n \to \infty$, the discrete sampling system tends to continuous system. In the other hand, the sparseness of WSN challenges the algorithm proposed in this paper. The result in a same area as case (a) in Figure 9 with lower average degree is shown in Figure 10. And the relation between faulty rate and average degree is shown in Figure 11 faulty rate versus average degree.

When faulty rate increases up to 5% or higher (Figure 10(b)), the recognition result is worthless. Thus, the algorithm proposed in this paper should be only applied in the WSN with average degree higher than 10. Funke tests the algorithm in sparse WSN [17]. The testing area is a circle hole in square. The comparison shown in Figure 12 indicates that the result of our algorithm at average degree of 10 is comparable with Funke's result at average of 18 and is much better than Funke's result at average degree of 10.

6. Conclusion

In this paper, a distributed algorithm for boundary recognition in WSN is proposed. The idea comes from the facts that iso-contours only break on the geometrical boundary and the WSN is a discrete sampling system of real environment. Then, we virtually start a diffusion process to create concentration gradient field in WSN, and finally the nodes that are often identified as end points of semi-iso-contours are regarded as boundary nodes. The simulation results show that the algorithm works well for the WSN with average degree over 10. Further, as diffusion in 3D space is well studied,

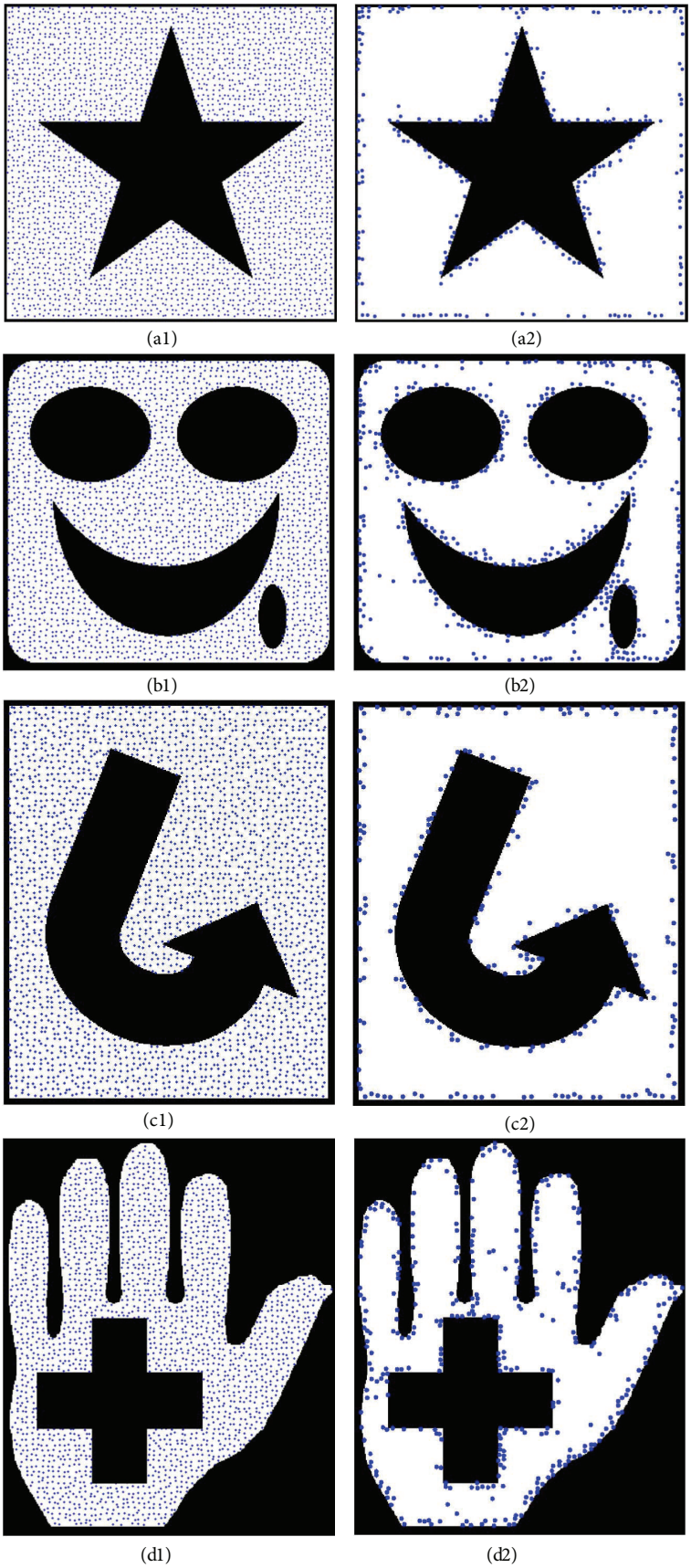


Figure 9: Continued.

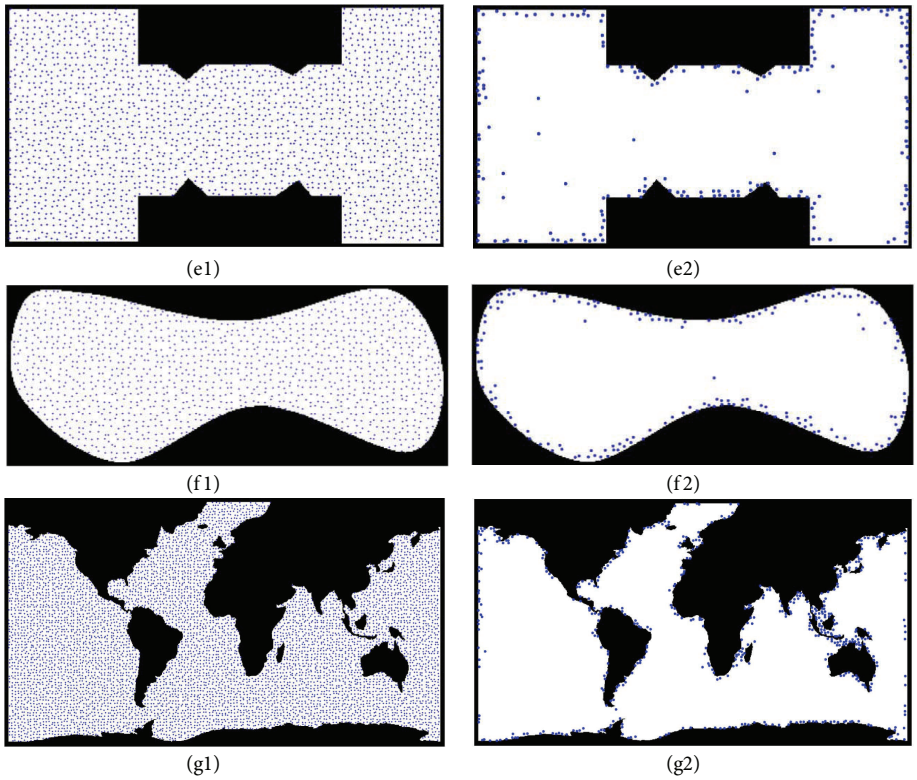


Figure 9: Boundary recognition in multiple cases. (a) 3023 nodes with average degree 13.1; (b) 2094 nodes with average degree 12.6; (c) 2381 nodes with average degree 13.0; (d) 2115 nodes with average degree 12.5; (e) 2024 nodes with average degree 13.4; (f) 1311 nodes with average degree 13.2; (g) 6811 nodes with average degree 13.3.

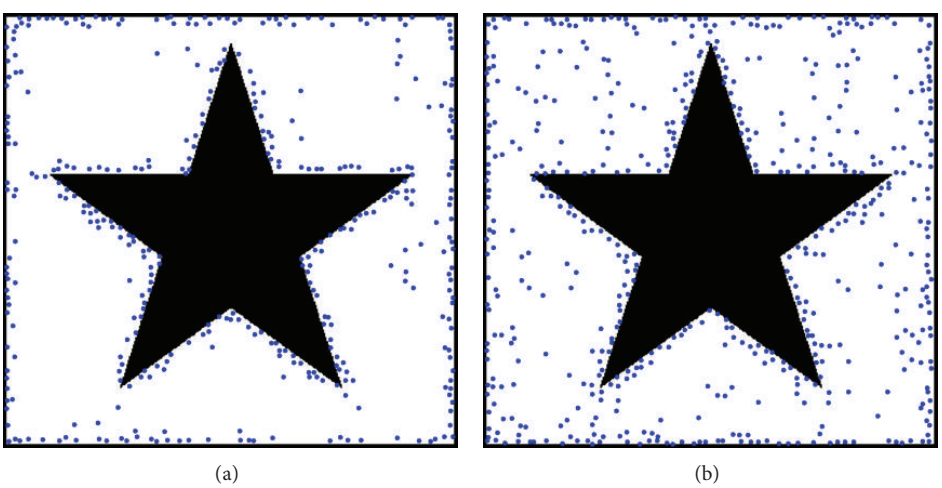


Figure 10: Boundary recognition in sparse WSN. (a) 10.9 in average degree and (b) 8.9 in average degree.

Table 1: Statistics of faulty recognition.

	Figure 9(a)	Figure 9(b)	Figure 9(c)	Figure 9(d)	Figure 9(e)	Figure 9(f)	Figure 9(g)
Number of nodes	3023	2094	2381	2115	2024	1311	6811
Inner nodes	2468	1402	1866	1448	1738	1065	5618
Faulty recognition	9	25	2	18	13	9	2
Faulty rate (%)	0.36	1.78	0.08	1.24	0.7	0.85	0.04

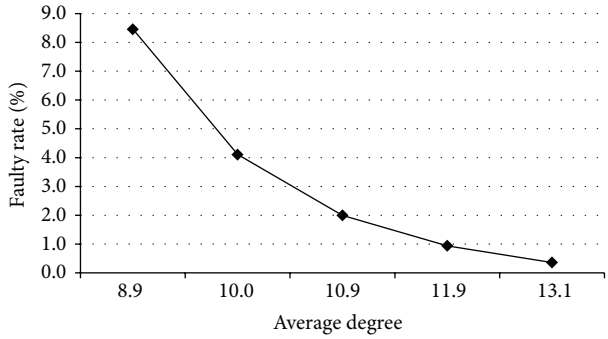


Figure 11: Faulty rate versus average degree.

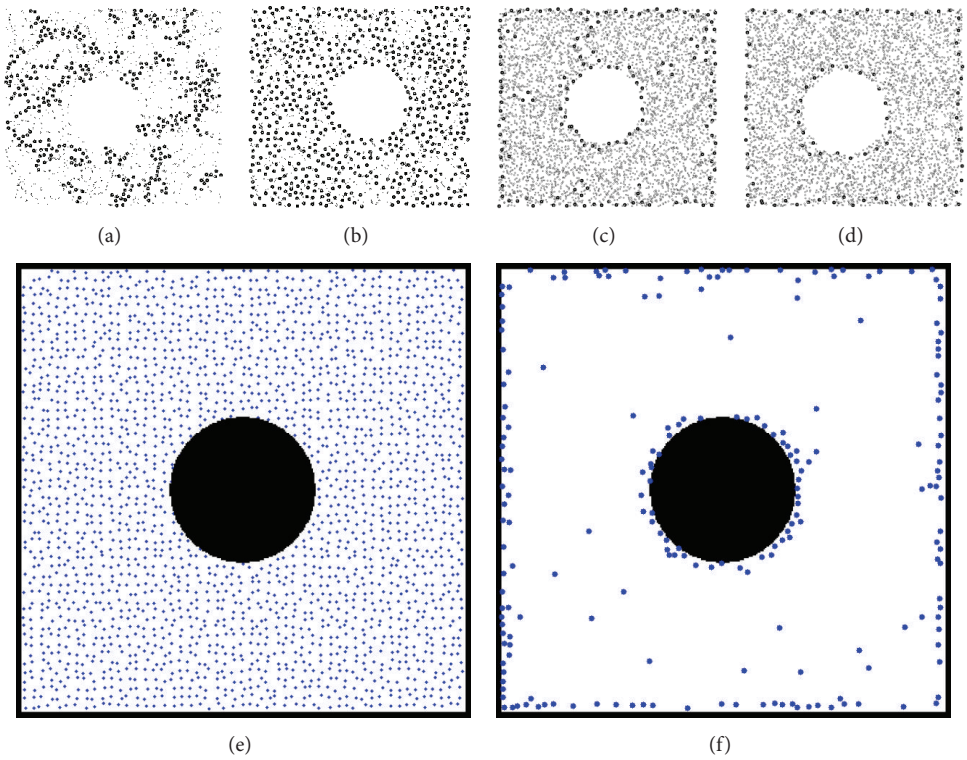


Figure 12: A comparison between algorithms. (a–d) Funke's algorithm with average degree at 5 in (a), 10 in (b), 18 in (c), and 39 in (d). The black dots are identified as boundary nodes, while gray ones are inner nodes. (e)–(f) Our algorithm with average degree at 10.

our algorithm is potentially to be improved to recognize boundary of a 3D WSN, which is also a hot research topic [8, 20].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Integration by Parts and Martingale Representation for a Markov Chain

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Integration-by-parts formulas for functions of fundamental jump processes relating to a continuous-time, finite-state Markov chain are derived using Bismut's change of measures approach to Malliavin calculus. New expressions for the integrands in stochastic integrals corresponding to representations of martingales for the fundamental jump processes are derived using the integration-by-parts formulas. These results are then applied to hedge contingent claims in a Markov chain financial market, which provides a practical motivation for the developments of the integration-by-parts formulas and the martingale representations.

1. Introduction

Integration by parts is at the heart of Malliavin calculus and its applications. It is deemed to be useful in mathematical finance, stochastic filtering and control as well as the theory of partial differential equations. Particularly, in mathematical finance, an integration-by-parts formula is useful in hedging contingent claims, numerical computations of Greeks, and portfolio optimization; see, for example, Benth et al. [1], León et al. [2], Imkeller [3], and Fournié et al. [4, 5], amongst others. Indeed, integration-by-parts formulas are one of the key results in a number of works on Malliavin calculus for stochastic differential equations driven by Wiener processes and jump processes. Some examples are Bismut [6], Bichteler et al. [7], Bass and Cranston [8], Norris [9], and Elliott and Tsoi [10, 11] to name a few. These authors adopted the approach to Malliavin calculus pioneered by Bismut [6], where an integration-by-parts formula was established by first considering a "small" perturbation of the original process and then compensating the effect of the perturbation by Girsanov's change of measure. For an excellent account of Malliavin calculus and its applications, one may refer to, for example, Nualart [12], Privault [13], and di Nunno et al. [14].

Markov chain is an important mathematical tool in probability theory and has vast applications in diverse fields. For example, in finance and actuarial science, there has been an interest in pricing contingent claims under Markov chain markets; see, for example, Norberg [15] and Elliott and Kopp [16] for bond pricing in a Markov chain market, Song et al. [17] for pricing options in a multivariate Markov chain market, Elliott et al. [18] and van der Hoek and Elliott [19, 20] for pricing options in Markov chain markets, and Norberg [21] and Koller [22] for pricing insurance products in Markov chain models. In statistics, particularly in nonlinear time series analysis, Markov chain plays an important role in studying the stochastic stability and ergodicity of stochastic difference equations; see, for example, Tong [23]. Markov chain also plays an important role in stochastic filtering and control. There is a large amount of literature on the use of Markov chain and related stochastic processes in stochastic filtering and control. Some recent literature is Shen et al. [24], He and Liu [25, 26], Zhang et al. [27], He [28], Siu [29], Ellliott and Siu [30], and Wu et al. [31], amongst others. The monograph by Elliott et al. [32] provided discussions on hidden Markov models and their applications in various fields such as signal processing and image processing. The monographs by Yin and Zhang [33, 34] provided discussions on the theories and applications of discrete-time and continuous-time Markov chain, respectively. A recent monograph by Ching et al. [35] presented applications of Markov chain in diverse fields such as manufacturing systems, marketing, and finance.

It appears that in the finance and actuarial science literature much attention has been given to pricing contingent claims in Markov chain markets. It seems that relatively less attention has been paid to hedging contingent claims in Markov chain markets. An integration-by-parts formula is a useful tool for hedging contingent claims. It seems that the literature mainly focuses on developing and applying integration-by-parts formulas in the cases of Wiener processes, Lévy processes, and single jump processes (see, e.g., Elliott and Tsoi [10, 11], Nualart [12], Privault [13], and di Nunno et al. [14]). An integration-by-parts formula in the case of a Markov chain seems lacking. Motivated by the hedging problem in Markov chain markets, it may be of interest to derive an integration-by-parts formula which is useful for hedging contingent claims in Markov chain markets.

In this paper, we derive integration-by-parts formulas for functions of a family of fundamental jump processes relating to a continuous-time, finite-state Markov chain using the Bismut measure change approach. The formulas are derived by considering "small" perturbations to the jump intensity parameters of the fundamental jump processes, which are then compensated by Girsanov's measure change. Using the integration-by-parts formulas, new expressions for the integrands in representations of martingales for the fundamental jump processes are derived. Firstly, we consider a function of the terminal values of the fundamental jump processes. Then, the results are extended to a function of the integrals with respect to the whole paths of the fundamental jump processes. The function of the path integrals may be considered a canonical form of a random variable which is measurable with respect to filtration generated by the whole path of the Markov chain. No infinite-dimensional calculus of variations is involved in the derivations. Indeed, only finite-dimensional calculus is adopted. The martingale representation results derived here may be useful for hedging contingent claims in the Markov chain financial market developed by Norberg [21], where the dynamics of share prices were driven by the basic martingales of the fundamental jump processes relating to a continuous-time, finite-state Markov chain.

The rest of the paper is organized as follows. Section 2 describes the Markov chain, the fundamental jump processes, and the basic martingales relating to the chain. Section 3 derives the integration-by-parts formula for a function of the terminal values of the fundamental jump processes. In Section 4, the expression of the integrand in the martingale representation is obtained. The results are then extended to a function of the integrals of the whole paths of the fundamental jump processes in Section 5. An application of the martingale representation result to hedging contingent claims in the Markov chain financial market of Norberg [21] is given in Section 6. Section 7 summarizes the paper and suggests some potential topics for future research.

2. Markov Chain, Fundamental Jump Processes and Basic Martingales

The aim of this section is to present some known results in Markov chain, its fundamental jump processes and basic martingales which are relevant to the later developments.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite time horizon $\mathcal{T} := [0, T]$, where $T < \infty$. Let $\mathbf{X} := \{\mathbf{X}(t) \mid t \in \mathcal{T}\}$ be a continuous-time, finite-state Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$. As in Elliott et al. [32], we suppose that the state space of the chain $\mathbf{X}$ is a finite set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ in $\mathfrak{R}^N$, where the jth component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. The space $\mathcal{E}$ is called the canonical state space of $\mathbf{X}$.

To specify the probability laws of the chain $\mathbf{X}$, we define a family of rate matrices, or intensity matrices, $\{\mathbf{A}(t) \mid t \in \mathcal{T}\}$ under $\mathbb{P}$, where, for each $t \in \mathcal{T}$, $\mathbf{A}(t) := [a_{ij}(t)]_{i,j=1,2,\ldots,N}$. For each $i, j = 1, 2, \ldots, N$ with $i \neq j$ and each $t \in \mathcal{T}$, $a_{ij}(t)$ is the instantaneous transition intensity of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ at time t. Note that for each $i, j = 1, 2, \ldots, N$ and each $t \in \mathcal{T}$,

(1) $a_{ij}(t) \geq 0$, for $i \neq j$;

(2) $\sum_{j=1}^{N} a_{ij}(t) = 0$, so $a_{ii}(t) \leq 0$.

We suppose here that, for each $i, j = 1, 2, \ldots, N$, $a_{ij}(t)$ is a bounded and deterministic function of time t.

Let $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) \mid t \in \mathcal{T}\}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by the chain $\mathbf{X}$. Note that $\mathbb{F}^{\mathbf{X}}$ is right-continuous. Then with the canonical state space of the chain $\mathbf{X}$, Elliott et al. [32] obtained the following semimartingale dynamics for $\mathbf{X}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u-)\,\mathbf{X}(u)\,du + \mathbf{M}(t), \quad t \in \mathcal{T}. \tag{1}$$

Here $\mathbf{M} := \{\mathbf{M}(t) \mid t \in \mathcal{T}\}$ is an $\mathfrak{R}^N$-valued, square-integrable, $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

For each $i, k = 1, 2, \ldots, N$ with $i \neq k$, let $J_{ik} := \{J_{ik}(t) \mid t \in \mathcal{T}\}$, where $J_{ik}(t)$ counts the number of transitions of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_k$ up to and including time t. That is,

$$J_{ik}(t) := \sum_{0<s\leq t} \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle \mathbf{X}(s), \mathbf{e}_k \rangle. \tag{2}$$

$\{J_{ik} \mid i, k = 1, 2, \ldots, N, i \neq k\}$ is called a family of fundamental jump processes relating to the chain $\mathbf{X}$; $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathfrak{R}^N$.

Define, for each $i, k = 1, 2, \ldots, N$ with $i \neq k$, a process $M_{ik} := \{M_{ik}(t) \mid t \in \mathcal{T}\}$ by putting

$$M_{ik}(t) := \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i \rangle \langle d\mathbf{M}(s), \mathbf{e}_k \rangle. \tag{3}$$

Then it is obvious from the definition that M_{ik}, $i, k = 1, 2, \ldots, N$, are $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingales and $\{M_{ik} \mid i, k = 1, 2, \ldots, N, i \neq k\}$ is called a family of basic martingales. Indeed these martingales are orthogonal, purely discontinuous, and square-integrable. Furthermore, $M_{ik}(0) = 0$.

The following lemma gives the semimartingale decomposition for J_{ik}. This result is standard (see, e.g., Elliott [36] and Elliott et al. [32]).

Lemma 1. *For each $i, k = 1, 2, \ldots, N$ with $i \neq k$ and each $t \in \mathcal{T}$,*

$$J_{ik}(t) = \int_0^t a_{ik}(s) \langle \mathbf{X}(s), \mathbf{e}_i \rangle\, ds + M_{ik}(t). \tag{4}$$

Proof. The proof of this lemma is standard. For the sake of completeness, we present the proof here:

$$\begin{aligned} J_{ik}(t) &:= \sum_{0<s\le t} \langle \mathbf{X}(s-), \mathbf{e}_i\rangle \langle \mathbf{X}(s), \mathbf{e}_k\rangle \\ &= \sum_{0<s\le t} \langle \mathbf{X}(s-), \mathbf{e}_i\rangle \langle \Delta\mathbf{X}(s), \mathbf{e}_k\rangle \\ &= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i\rangle \langle d\mathbf{X}(s), \mathbf{e}_k\rangle \\ &= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i\rangle \langle \mathbf{A}(s)\mathbf{X}(s-), \mathbf{e}_k\rangle ds \\ &\quad + \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i\rangle \langle d\mathbf{M}(s), \mathbf{e}_k\rangle \\ &= \int_0^t a_{ik}(s) \langle \mathbf{X}(s-), \mathbf{e}_i\rangle ds + M_{ik}(t) \\ &= \int_0^t a_{ik}(s) \langle \mathbf{X}(s), \mathbf{e}_i\rangle ds + M_{ik}(t). \end{aligned} \tag{5}$$

The last equality is due to the fact that the set of all jump times of the chain $\mathbf{X}$ has zero "dt"-measure. □

From Lemma 1 and the definition of M_{ik},

$$M_{ik}(t) := J_{ik}(t) - \int_0^t a_{ik}(s) \langle \mathbf{X}(s), \mathbf{e}_i\rangle ds, \quad t \in \mathcal{T}, \tag{6}$$

is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale. Consequently, under $\mathbb{P}$, $\{a_{ik}(t)\langle \mathbf{X}(t), \mathbf{e}_i\rangle \mid t \in \mathcal{T}\}$ is the intensity process of J_{ik}.

3. Integration by Parts for Functions of Fundamental Jump Processes

In this section we first present small perturbations to the jump intensities of the fundamental jump processes and then compensate the perturbations by a Girsanov-type measure change. The integration-by-parts formula for a "suitable" function of the terminal values of the fundamental jump processes is then derived by differentiation. The techniques used to derive the integration-by-parts formula here are adapted to those used in Elliott and Tsoi [10] for deriving an integration-by-parts formula for a single jump process. It seems that the origin of these techniques may be traced back to the work of Bismut [6].

For each $i, k = 1, 2, \ldots, N$ with $i \ne k$, let $\eta_{ik} := \{\eta_{ik}(t) \mid t \in \mathcal{T}\}$ be a nonnegative, $\mathbb{P}$-a.s. bounded, $\mathbb{F}^{\mathbf{X}}$-predictable process. Then for an arbitrarily small $\epsilon > 0$, we define a small "stochastic" perturbation $a^{\epsilon}_{ik}(t)$ to $a_{ik}(t)$ in the direction $\eta_{ik}(t)$ by putting

$$a^{\epsilon}_{ik}(t) := (1 + \epsilon\eta_{ik}(t))\, a_{ik}(t). \tag{7}$$

We then take

$$a^{\epsilon}_{ii}(t) := -\sum_{k=1, i\ne k}^{N} a^{\epsilon}_{ik}(t), \tag{8}$$

so that

$$\sum_{k=1}^{N} a^{\epsilon}_{ik}(t) = 0. \tag{9}$$

Note that, for each $t \in \mathcal{T}$, $\eta_{ik}(t) > 0$ and $\epsilon > 0$, so $a^{\epsilon}_{ik}(t) \ge 0$, $i \ne k$, and $a^{\epsilon}_{ii}(t) \le 0$.

Define, for each $i, k = 1, 2, \ldots, N$ with $i \ne k$, the jump process $J^{\epsilon}_{ik} := \{J^{\epsilon}_{ik}(t) \mid t \in \mathcal{T}\}$ by putting

$$J^{\epsilon}_{ik}(t) := \int_0^t a^{\epsilon}_{ik}(t) \langle \mathbf{X}(u), \mathbf{e}_i\rangle du + M_{ik}(t), \tag{10}$$

where $M_{ik}(t)$ is defined in Section 2 as follows:

$$M_{ik}(t) := \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_i\rangle \langle d\mathbf{M}(s), \mathbf{e}_k\rangle. \tag{11}$$

By definition,

$$M_{ik}(t) = J^{\epsilon}_{ik}(t) - \int_0^t a^{\epsilon}_{ik}(t) \langle \mathbf{X}(u), \mathbf{e}_i\rangle du, \quad t \in \mathcal{T}, \tag{12}$$

is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale. Consequently, J^{ϵ}_{ik} has the intensity process $\{a^{\epsilon}_{ik}(t)\langle \mathbf{X}(t), \mathbf{e}_i\rangle \mid t \in \mathcal{T}\}$ under $\mathbb{P}$ and it is related to J_{ik} as follows:

$$J^{\epsilon}_{ik}(t) := J_{ik}(t) + \epsilon \int_0^t \eta_{ik}(u)\, a_{ik}(u) \langle \mathbf{X}(u), \mathbf{e}_i\rangle du. \tag{13}$$

To simplify the notation, write $\lambda_{ik}(t) := \eta_{ik}(t)a_{ik}(t)\langle \mathbf{X}(t), \mathbf{e}_i\rangle$, for each $i, k = 1, 2, \ldots, N$ with $i \ne k$ and each $t \in \mathcal{T}$. Then

$$J^{\epsilon}_{ik}(t) = J_{ik}(t) + \epsilon \int_0^t \lambda_{ik}(u)\, du. \tag{14}$$

The process J^{ϵ}_{ik} is called a perturbed process of the fundamental jump process J_{ik}, so we have a family of perturbed processes $\{J^{\epsilon}_{ik} \mid i, k = 1, 2, \ldots, N, i \ne k\}$ corresponding to the family of the fundamental jump processes $\{J_{ik} \mid i, k = 1, 2, \ldots, N, i \ne k\}$.

For each $i, k = 1, 2, \ldots, N$ with $i \ne k$ and each $t \in \mathcal{T}$, let

$$\theta^{\epsilon}_{ik}(t) := -\frac{\epsilon\eta_{ik}(t)}{1 + \epsilon\eta_{ik}(t)}. \tag{15}$$

Define, for each $t \in \mathcal{T}$,

$$Z^{\epsilon}(t) := \sum_{i,k=1, i\ne k}^{N} \int_0^t \theta^{\epsilon}_{ik}(u-)\, dM_{ik}(u). \tag{16}$$

Consider an $\mathbb{F}^{\mathbf{X}}$-adapted process $\Lambda^{\epsilon} := \{\Lambda^{\epsilon}(t) \mid t \in \mathcal{T}\}$ defined by setting

$$\Lambda^{\epsilon}(t) = 1 + \int_0^t \Lambda^{\epsilon}(u-)\, dZ^{\epsilon}(u). \tag{17}$$

Then by Elliott [37] (see Theorem 13.5 therein),

$$\Lambda^{\epsilon}(t) = \mathcal{E}(Z^{\epsilon})(t) = \prod_{0<u\le t} (1 + \Delta Z^{\epsilon}(u)), \tag{18}$$

where $\mathscr{E}(Z^\epsilon) := \{\mathscr{E}(Z^\epsilon)(t) \mid t \in \mathscr{T}\}$ is the stochastic exponential of the process Z^ϵ; $\Delta Z^\epsilon(t) := Z^\epsilon(t) - Z^\epsilon(t-)$.

Then, for each $t \in \mathscr{T}$,

$$\Lambda^\epsilon(t) = \exp\left(\sum_{i,k=1,i\neq k}^{N} \int_0^t \left[\ln\left(1+\theta_{ik}^\epsilon(u)\right) - \theta_{ik}^\epsilon(u)\right] \times a_{ik}^\epsilon(u)\left\langle \mathbf{X}(u), \mathbf{e}_i\right\rangle du + \sum_{i,k=1,i\neq k}^{N} \int_0^t \ln\left(1+\theta_{ik}^\epsilon(u)\right) dM_{ik}(u)\right). \tag{19}$$

Note that by definition $\theta_{ik}^\epsilon(t) > -1$, for each $t \in \mathscr{T}$, so the process $\Lambda^\epsilon := \{\Lambda^\epsilon(t) \mid t \in \mathscr{T}\}$ is strictly positive. Furthermore, Λ^ϵ is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

A new probability measure $\mathbb{P}^\epsilon$ equivalent to $\mathbb{P}$ on $\mathscr{F}^{\mathbf{X}}(T)$ is now defined by putting

$$\left.\frac{d\mathbb{P}^\epsilon}{d\mathbb{P}}\right|_{\mathscr{F}^{\mathbf{X}}(T)} := \Lambda^\epsilon(T). \tag{20}$$

The following lemma will be used to derive the integration-by-parts formula.

Lemma 2. *The $\mathbb{P}^\epsilon$-law of J_{ik}^ϵ, $i,k = 1,2,\ldots,N$ with $i \neq k$, is equal to the $\mathbb{P}$-law of J_{ik}, $i,k = 1,2,\ldots,N$ with $i \neq k$.*

Proof. By a version of Girsanov's theorem, the process $M_{ik}^\epsilon := \{M_{ik}^\epsilon(t) \mid t \in \mathscr{T}\}$ defined by

$$M_{ik}^\epsilon(t) := J_{ik}^\epsilon(t) - \int_0^t \left(1+\theta_{ik}^\epsilon(u)\right) a_{ik}^\epsilon(u)\left\langle \mathbf{X}(u), \mathbf{e}_i\right\rangle du, \quad t \in \mathscr{T}, \tag{21}$$

is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P}^\epsilon)$-martingale. Note that

$$\left(1+\theta_{ik}^\epsilon(t)\right) a_{ik}^\epsilon(t) = a_{ik}(t), \tag{22}$$

so J_{ik}^ϵ has the intensity process $\{a_{ik}(t)\langle \mathbf{X}(t), \mathbf{e}_i\rangle \mid t \in \mathscr{T}\}$ under $\mathbb{P}^\epsilon$. This is the same as the intensity process of J_{ik} under $\mathbb{P}$. □

Remark 3. The $(\mathbb{F}^{\mathbf{X}}, \mathbb{P}^\epsilon)$-martingale M_{ik}^ϵ defined in the proof of Lemma 2 is related to the $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale M_{ik} as follows:

$$M_{ik}^\epsilon(t) = M_{ik}(t) - \epsilon \int_0^t \lambda_{ik}(u)\, du, \quad t \in \mathscr{T}. \tag{23}$$

To simplify our notation and illustrate the main idea, we consider the situation where the chain $\mathbf{X}$ has two states. In this case, the family of fundamental jump processes relating to the chain is $\{J_{12}, J_{21}\}$ and its corresponding perturbed processes are $\{J_{12}^\epsilon, J_{21}^\epsilon\}$.

Let $G : \mathfrak{R}^2 \to \mathfrak{R}$ be any measurable, integrable, and differentiable function. Note that from Lemma 2 the $\mathbb{P}^\epsilon$-law of $(J_{12}^\epsilon(T), J_{21}^\epsilon(T))$ is the same as the $\mathbb{P}$-law of $(J_{12}(T), J_{21}(T))$. Consequently,

$$E\left[G\left(J_{12}(T), J_{21}(T)\right)\right] = E^\epsilon\left[G\left(J_{12}^\epsilon(T), J_{21}^\epsilon(T)\right)\right]. \tag{24}$$

Here E and E^ϵ are expectations under $\mathbb{P}$ and $\mathbb{P}^\epsilon$, respectively.

Write

$$\begin{aligned} \mathbf{J}(T) &:= \left(J_{12}(T), J_{21}(T)\right)' \in \mathfrak{R}^2, \\ \mathbf{J}^\epsilon(T) &:= \left(J_{12}^\epsilon(T), J_{21}^\epsilon(T)\right)' \in \mathfrak{R}^2, \end{aligned} \tag{25}$$

where $\mathbf{y}'$ is the transpose of a vector, or a matrix, $\mathbf{y}$.

Define the following gradient of G with respect to $\mathbf{x} := (x_1, x_2)' \in \mathfrak{R}^2$:

$$D_{\mathbf{x}} G(\mathbf{x}) = \left(\frac{\partial}{\partial x_1} G(\mathbf{x}), \frac{\partial}{\partial x_2} G(\mathbf{x})\right)' \in \mathfrak{R}^2. \tag{26}$$

Then the following theorem gives the integration-by-parts formula.

Theorem 4. *For each $t \in \mathscr{T}$, let*

$$\begin{aligned} \overline{\mathbf{M}}(t) &:= \left(M_{12}(t), M_{21}(t)\right)' \in \mathfrak{R}^2, \\ \boldsymbol{\eta}(t) &:= \left(\eta_{12}(t), \eta_{21}(t)\right)' \in \mathfrak{R}^2. \end{aligned} \tag{27}$$

Write, for each $t \in \mathscr{T}$,

$$\varphi(t) := \left(\int_0^t \lambda_{12}(u)\, du, \int_0^t \lambda_{21}(u)\, du\right)' \in \mathfrak{R}^2. \tag{28}$$

Then for any measurable, integrable, and differentiable function $G : \mathfrak{R}^2 \to \mathfrak{R}$,

$$E\left[\left\langle D_{\mathbf{x}} G(\mathbf{J}(T)), \varphi(T)\right\rangle\right] = E\left[G(\mathbf{J}(T)) \int_0^T \boldsymbol{\eta}'(u)\, d\overline{\mathbf{M}}(u)\right]. \tag{29}$$

Proof. By a version of Bayes' rule,

$$E\left[G(\mathbf{J}(T))\right] = E^\epsilon\left[G\left(\mathbf{J}^\epsilon(T)\right)\right] = E\left[\Lambda^\epsilon(T)\, G\left(\mathbf{J}^\epsilon(T)\right)\right]. \tag{30}$$

Differentiating both sides with respect to ϵ and setting $\epsilon = 0$ give

$$E\left[\left.\frac{\partial}{\partial\epsilon}\Lambda^\epsilon(T)\right|_{\epsilon=0} G\left(\mathbf{J}^\epsilon(T)\right)\Big|_{\epsilon=0}\right] + E\left[\Lambda^\epsilon(T)\Big|_{\epsilon=0}\left\langle D_{\mathbf{x}} G\left(\mathbf{J}^\epsilon(T)\right), \frac{\partial}{\partial\epsilon}\mathbf{J}^\epsilon(T)\right\rangle\Big|_{\epsilon=0}\right] = 0. \tag{31}$$

It is obvious that $\Lambda^\epsilon(T)|_{\epsilon=0} = 1$ and that $\mathbf{J}^\epsilon(T)|_{\epsilon=0} = \mathbf{J}(T)$. Consequently,

$$E\left[\left.\frac{\partial}{\partial\epsilon}\Lambda^\epsilon(T)\right|_{\epsilon=0} G(\mathbf{J}(T))\right] + E\left[\left\langle D_{\mathbf{x}} G(\mathbf{J}(T)), \frac{\partial}{\partial\epsilon}\mathbf{J}^\epsilon(T)\right\rangle\Big|_{\epsilon=0}\right] = 0. \tag{32}$$

Now

$$\frac{\partial}{\partial\epsilon}\mathbf{J}^{\epsilon}(T)\Big|_{\epsilon=0} = \left(\int_0^T \lambda_{12}(u)\,du, \int_0^T \lambda_{21}(u)\,du\right)' = \varphi(T),$$

$$\frac{\partial}{\partial\epsilon}\Lambda^{\epsilon}(T) = \Lambda^{\epsilon}(T) \times \left[\sum_{i,k=1,i\neq k}^{2}\int_0^T \ln\left(\frac{1}{1+\epsilon\eta_{ik}(u)}\right)\eta_{ik}(u)\,a_{ik}(u)\,I_{\{\mathbf{X}(u)=\mathbf{e}_i\}}du - \sum_{i,k=1,i\neq k}^{2}\int_0^T \frac{\eta_{ik}(u)}{(1+\epsilon\eta_{ik}(u))^2}dM_{ik}(u)\right]. \tag{33}$$

Then

$$\frac{\partial}{\partial\epsilon}\Lambda^{\epsilon}(T)\Big|_{\epsilon=0} = -\sum_{i,k=1,i\neq k}^{2}\int_0^T \eta_{ik}(u)\,dM_{ik}(u) = -\int_0^T \boldsymbol{\eta}'(u)\,d\overline{\mathbf{M}}(u). \tag{34}$$

Hence the result follows. □

The following two integration-by-parts formulas are immediate consequences of Theorem 4.

Corollary 5. *For any measurable, integrable, and differentiable function* $G : \mathfrak{R}^2 \to \mathfrak{R}$,

$$E\left[\frac{\partial}{\partial x_1}G(\mathbf{J}(T))\int_0^T \lambda_{12}(u)\,du\right] = E\left[G(\mathbf{J}(T))\int_0^T \eta_{12}(u)\,dM_{12}(u)\right]. \tag{35}$$

Proof. The result follows by putting $\eta_{21}(t) = 0$, for all $t \in \mathcal{T}$, in Theorem 4. □

Corollary 6. *For any measurable, integrable, and differentiable function* $G : \mathfrak{R}^2 \to \mathfrak{R}$,

$$E\left[\frac{\partial}{\partial x_2}G(\mathbf{J}(T))\int_0^T \lambda_{21}(u)\,du\right] = E\left[G(\mathbf{J}(T))\int_0^T \eta_{21}(u)\,dM_{21}(u)\right]. \tag{36}$$

Proof. The result follows by putting $\eta_{12}(t) = 0$, for all $t \in \mathcal{T}$, in Theorem 4. □

Remark 7. The integration-by-parts formula in Corollary 5 (Corollary 6) may be interpreted as an integration-by-parts formula obtained by perturbing the intensity $\{a_{12}(t) \mid t \in \mathcal{T}\}$ ($\{a_{21}(t) \mid t \in \mathcal{T}\}$) along the direction η_{12} (η_{21}).

Remark 8. In Elliott and Kohlmann [38], an integration-by-parts formula for functions of jump processes was developed. Using the concept of stochastic flows, the integration-by-parts formula was derived for functions of the terminal values of jump processes. An advantage of the approach by Elliott and Kohlmann [38] is that the integration-by-parts formula was derived without using infinite-dimensional calculus. The integration-by-parts formula for functions of the terminal values of jump processes has an important application. Elliott and Kohlmann [38] demonstrated how this integration-by-parts formula may be applied to establish the existence and smoothness of the density of a jump process. This is a key area of application of Malliavin calculus. Using the method in Elliott and Kohlmann [38], the integration-by-parts formula in Theorem 4 may be used to establish the existence and uniqueness of the densities of some stochastic processes depending on the fundamental jump processes relating to the chain. This may represent an interesting topic for future research.

Remark 9. In the Markov chain financial market of Norberg [21], the dynamics of share prices are described by the fundamental jump processes relating to a continuous-time, finite-state Markov chain. The integration-by-parts formula in Theorem 4 may be used to hedge contingent claims whose payoffs depend on the terminal values of the share prices in the continuous-time Markov chain market of Norberg [21]. We will discuss this in some detail in Section 6.

4. Martingale Representation Using Integration by Parts

Martingale representation is one of the fundamental results in stochastic analysis and calculus. It has many significant applications in diverse fields such as mathematical finance, stochastic filtering, and control. A crucial question in a martingale representation is to determine the integrand in the representation. This question is of primary importance in many applications of martingale representations. The Clark-Haussmann-Ocone-Karatzas formula was developed to address this question in the case of a Wiener space (see Clark [39], Haussmann [40], Ocone [41], Ocone and Karatzas [42], and Karatzas et al. [43]). Elliott and Kohlmann [44] pioneered the use of stochastic flows to identify the integrand in a stochastic integral in a martingale representation under a Markov diffusion setting. Elliott and Kohlmann [38] extended the approach in Elliott and Kohlmann [44] to the case of a Markov jump process. Elliott and Tsoi [10, 11] adopted integration-by-parts formulas to derive integrands in martingale representations in a single jump process and a Poisson process, respectively. Aase et al. [45] adopted a white-noise approach to Malliavin calculus to establish a white-noise generalization of the Clark-Haussmann-Ocone-Karatzas formula in the cases of multidimensional Gaussian white noise, multidimensional Poisson white noise, and their combination. Di Nunno et al. [46] adopted a chaos expansion approach to Malliavin calculus to establish a white-noise generalization of the Clark-Haussmann-Ocone-Karatzas formula for Lévy processes.

In this section, we apply the integration-by-parts formula obtained in the last section to derive the integrand in a martingale representation for a function of the terminal values of the fundamental jump processes. Though the techniques to be used here are similar to those adopted in Elliott and Tsoi

[10, 11], it seems that the formulas of the integrand derived here appear to be new. Again to simplify our notation, we consider here the two-regime Markov chain presented in Section 3.

Note that the filtration $\mathbb{F}^{\mathbf{X}}$ generated by the chain $\mathbf{X}$ is the same as the filtration generated by the family of fundamental jump processes $\{J_{12}, J_{21}\}$. Then we state the following martingale representation result which was due to Brémaud [47].

Theorem 10. *For any real-valued, square-integrable* $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$*-martingale* $L := \{L(t) \mid t \in \mathcal{T}\}$,

$$L(T) := E[L(T)] + \int_0^T \gamma'(u)\, d\overline{\mathbf{M}}(u), \tag{37}$$

for some $\mathfrak{R}^2$*-valued,* $\mathbb{F}^{\mathbf{X}}$*-predictable process* $\{\gamma(t) \mid t \in \mathcal{T}\}$.

Furthermore, we need the following expression for the predictable quadratic variation $\{\langle \mathbf{M}, \mathbf{M}\rangle(t) \mid t \in \mathcal{T}\}$ of $\mathbf{M} := \{\mathbf{M}(t) \mid t \in \mathcal{T}\}$, which was derived in Elliott et al. [32].

Lemma 11. *Let* $\mathbf{diag}[\mathbf{y}]$ *be a diagonal matrix with the diagonal elements being given by the components in a vector* $\mathbf{y}$. *For each* $t \in \mathcal{T}$,

$$\begin{aligned}\langle \mathbf{M}, \mathbf{M}\rangle(t) = \int_0^t \Big(&\mathbf{diag}[\mathbf{A}(u)\mathbf{X}(u)] - \mathbf{diag}[\mathbf{X}(u)]\mathbf{A}'(u) \\ &-\mathbf{A}(u)\,\mathbf{diag}[\mathbf{X}(u)]\Big)\, du.\end{aligned} \tag{38}$$

To simplify our notation, let $\{\mathbf{f}(t) \mid t \in \mathcal{T}\}$ be a matrix-valued process defined as follows:

$$\begin{aligned}\mathbf{f}(t) := {}&\mathbf{diag}[\mathbf{A}(t)\mathbf{X}(t)] - \mathbf{diag}[\mathbf{X}(t)]\mathbf{A}'(t) \\ &- \mathbf{A}(t)\,\mathbf{diag}[\mathbf{X}(t)] \in \mathfrak{R}^2 \otimes \mathfrak{R}^2.\end{aligned} \tag{39}$$

Note that $\{\mathbf{f}(t) \mid t \in \mathcal{T}\}$ is the density process of the measure $d\langle \mathbf{M}, \mathbf{M}\rangle(t)$ with respect to the Lebesgue measure dt on $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ and $d\langle \mathbf{M}, \mathbf{M}\rangle(t)$ is absolutely continuous with respect to dt, where $\mathcal{B}(\mathcal{T})$ is the Borel σ-field generated by open subsets of $\mathcal{T}$.

Then

$$\langle \mathbf{M}, \mathbf{M}\rangle(t) = \int_0^t \mathbf{f}(u)\, du. \tag{40}$$

The following lemma will be used to derive the expressions for the integrand in the martingale representation.

Lemma 12. *For each* $i, k = 1, 2$ *with* $i \neq k$, *the predictable quadratic variation of* M_{ik}, *namely* $\{\langle M_{ik}, M_{ik}\rangle(t) \mid t \in \mathcal{T}\}$, *is given by*

$$\langle M_{ik}, M_{ik}\rangle(t) = \int_0^t \langle \mathbf{X}(u), \mathbf{e}_i\rangle\, \mathbf{e}_k' \mathbf{f}(u)\, \mathbf{e}_k du \in \mathfrak{R}. \tag{41}$$

Proof. Recall that

$$\begin{aligned}M_{ik}(t) &:= \int_0^t \langle \mathbf{X}(u-), \mathbf{e}_i\rangle \langle d\mathbf{M}(u), \mathbf{e}_k\rangle \\ &= \int_0^t \langle \mathbf{X}(u-), \mathbf{e}_i\rangle\, \mathbf{e}_k' d\mathbf{M}(u).\end{aligned} \tag{42}$$

Then

$$\begin{aligned}&\langle M_{ik}, M_{ik}\rangle(t) \\ &\quad= \int_0^t \langle \mathbf{X}(u-), \mathbf{e}_i\rangle\, \mathbf{e}_k' d\langle \mathbf{M}, \mathbf{M}\rangle(u)\, \mathbf{e}_k \langle \mathbf{X}(u-), \mathbf{e}_i\rangle \\ &\quad= \int_0^t \langle \mathbf{X}(u-), \mathbf{e}_i\rangle\, \mathbf{e}_k' \mathbf{f}(u)\, \mathbf{e}_k du \\ &\quad= \int_0^t \langle \mathbf{X}(u), \mathbf{e}_i\rangle\, \mathbf{e}_k' \mathbf{f}(u)\, \mathbf{e}_k du.\end{aligned} \tag{43}$$

The last equality follows from the fact that the set of all jump times of the chain $\mathbf{X}$ has "dt"-measure zero. □

By the martingale representation presented in Theorem 10,

$$G(\mathbf{J}(T)) = E[G(\mathbf{J}(T))] + \int_0^T \gamma'(u)\, d\overline{\mathbf{M}}(u), \tag{44}$$

for some $\mathbb{F}^{\mathbf{X}}$-predictable process $\gamma := \{\gamma(t) \mid t \in \mathcal{T}\}$.

It can be supposed that $E[G(\mathbf{J}(T))] = 0$ by subtraction. Then

$$G(\mathbf{J}(T)) = \int_0^T \gamma'(u)\, d\overline{\mathbf{M}}(u). \tag{45}$$

The integrand γ is then determined in the following theorem. Though the techniques used in the proof of the following theorem are similar to those used in Proposition 3.5 of Elliott and Tsoi [11], the expressions for the integrand presented below appear to be new.

Theorem 13. *Suppose that* $a_{12}(t), a_{21}(t) > 0$ *for each* $t \in \mathcal{T}$. *Then the integrand* $\gamma := \{\gamma(t) \mid t \in \mathcal{T}\}$, *where* $\gamma(t) := (\gamma_1(t), \gamma_2(t))' \in \mathfrak{R}^2$, *is determined by*

$$\begin{aligned}\gamma_1(t) &= E\left[\frac{\partial}{\partial x_1} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(t-)\right] \frac{a_{12}(t)}{a_{21}(t)}, \\ &\qquad \text{on the set } \{\mathbf{X}(t) = \mathbf{e}_1\}, \\ \gamma_2(t) &= E\left[\frac{\partial}{\partial x_2} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(t-)\right] \frac{a_{21}(t)}{a_{12}(t)}, \\ &\qquad \text{on the set } \{\mathbf{X}(t) = \mathbf{e}_2\}.\end{aligned} \tag{46}$$

Proof. We only give the proof for the integrand $\gamma_1(t)$ since the integrand $\gamma_2(t)$ can be derived similarly. Firstly, by the martingale representation for $G(\mathbf{J}(T))$, Lemma 12, and the orthogonality of M_{12} and M_{21},

$$\begin{aligned}&E\left[G(\mathbf{J}(T)) \int_0^T \eta_{12}(u)\, dM_{12}(u)\right] \\ &\quad= E\left[\left(\int_0^T \gamma'(u)\, d\overline{\mathbf{M}}(u)\right)\left(\int_0^T \eta_{12}(u)\, dM_{12}(u)\right)\right] \\ &\quad= E\left[\int_0^T \gamma_1(u)\, \eta_{12}(u)\, d\langle M_{12}, M_{12}\rangle(u)\right] \\ &\quad= E\left[\int_0^T \gamma_1(u)\, \eta_{12}(u)\, \mathbf{e}_2' \mathbf{f}(u)\, \mathbf{e}_2 I_{\{\mathbf{X}(u-) = \mathbf{e}_1\}} du\right].\end{aligned} \tag{47}$$

Then using the integration-by-parts formula in Corollary 5,

$$E\left[\int_0^T \frac{\partial}{\partial x_1} G(\mathbf{J}(T))\,\lambda_{12}(u)\,du\right] = E\left[\int_0^T \gamma_1(u)\,\eta_{12}(u)\,\mathbf{e}_2'\mathbf{f}(u)\,\mathbf{e}_2 I_{\{\mathbf{X}(u-)=\mathbf{e}_1\}}du\right]. \tag{48}$$

For each $u \in \mathcal{T}$, let

$$\psi(u) := \frac{\partial}{\partial x_1} G(\mathbf{J}(T))\,a_{12}(u)\,I_{\{\mathbf{X}(u-)=\mathbf{e}_1\}}. \tag{49}$$

Then there exists an $\mathbb{F}^{\mathbf{X}}$-predictable projection $\{\psi^*(u) \mid u \in \mathcal{T}\}$ of $\{\psi(u) \mid u \in \mathcal{T}\}$ such that, for each $u \in \mathcal{T}$,

$$\psi^*(u) = E\left[\psi(u) \mid \mathcal{F}^{\mathbf{X}}(u-)\right], \quad \mathbb{P}\text{-a.s.}, \tag{50}$$

so that

$$\psi^*(u) = E\left[\frac{\partial}{\partial x_1} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(u-)\right] a_{12}(u)\,I_{\{\mathbf{X}(u-)=\mathbf{e}_1\}}, \quad \mathbb{P}\text{-a.s.} \tag{51}$$

Furthermore, for any $\mathbb{F}^{\mathbf{X}}$-predictable process $\{K(u) \mid u \in \mathcal{T}\}$,

$$E[K(u)\,\psi(u)] = E\left[K(u)\,E\left[\psi(u) \mid \mathcal{F}^{\mathbf{X}}(u-)\right]\right] = E[K(u)\,\psi^*(u)]. \tag{52}$$

Write $\mathcal{H}$ for the family of subsets of $\mathcal{T}\times\Omega$ of the forms $\{0\}\times F_0$ and $(u,t]\times F_u$, where $F_0 \in \mathcal{F}^{\mathbf{X}}(0)$ and $F_u \in \mathcal{F}^{\mathbf{X}}(u)$ for $0 \le u < t \le T$. Note that the predictable σ-field on the product space $\mathcal{T}\times\Omega$ with respect to $\mathbb{F}^{\mathbf{X}}$ is generated by $\mathcal{H}$.

We now take $\eta_{12} = I_{\{0\}\times F_0}$ or $\eta_{12} = I_{(u,t]\times F_u}$, where $I_{\{0\}\times F_0}$ and $I_{(u,t]\times F_u}$ are the indicator functions of the events $\{0\}\times F_0$ and $(u,t]\times F_u$, respectively. Then the integration-by-parts formula in Corollary 5 holds for this η_{12}. Consequently, the following equality holds for all η_{12}'s which are indicators of sets in $\mathcal{H}$:

$$E\left[\int_0^T \eta_{12}(u)\,\psi^*(u)\,du\right] = E\left[\int_0^T \eta_{12}(u)\,\gamma_1(u)\,\mathbf{e}_2'\mathbf{f}(u)\,\mathbf{e}_2 I_{\{\mathbf{X}(u-)=\mathbf{e}_1\}}du\right]. \tag{53}$$

Since the set of all jump times of the chain $\mathbf{X}$ has "dt"-measure zero,

$$E\left[\int_0^T \eta_{12}(u)\,\psi^*(u)\,du\right] = E\left[\int_0^T \eta_{12}(u)\,\gamma_1(u)\,\mathbf{e}_2'\mathbf{f}(u)\,\mathbf{e}_2 I_{\{\mathbf{X}(u)=\mathbf{e}_1\}}du\right]. \tag{54}$$

On the set $\{\mathbf{X}(u) = \mathbf{e}_1\}$,

$$\begin{aligned}\mathbf{e}_2'\mathbf{f}(u)\,\mathbf{e}_2 &= \mathbf{e}_2'\left(\mathbf{diag}[\mathbf{A}(u)\,\mathbf{e}_1] - \mathbf{diag}[\mathbf{e}_1]\,\mathbf{A}'(u) - \mathbf{A}(u)\,\mathbf{diag}[\mathbf{e}_1]\right)\mathbf{e}_2 \\ &= a_{21}(u).\end{aligned} \tag{55}$$

Consequently, for all η_{12}'s which are indicators of sets in $\mathcal{H}$,

$$E\left[\int_0^T \eta_{12}(u)\,\psi^*(u)\,du\right] = E\left[\int_0^T \eta_{12}(u)\,\gamma_1(u)\,a_{21}(u)\,I_{\{\mathbf{X}(u)=\mathbf{e}_1\}}du\right]. \tag{56}$$

Note that

(1) $\mathcal{H}$ generates the $\mathbb{F}^{\mathbf{X}}$-predictable σ-field on the product space $\mathcal{T}\times\Omega$;

(2) the processes $\{\gamma_1(u) \mid u \in \mathcal{T}\}$ and $\{\psi^*(u) \mid u \in \mathcal{T}\}$ are $\mathbb{F}^{\mathbf{X}}$-predictable.

Then

$$\psi^*(u) = \gamma_1(u)\,a_{21}(u)\,I_{\{\mathbf{X}(u)=\mathbf{e}_1\}}, \quad \text{for almost all } (u,\omega) \in \mathcal{T}\times\Omega. \tag{57}$$

Consequently, for almost all $(u,\omega) \in \mathcal{T}\times\Omega$,

$$E\left[\frac{\partial}{\partial x_1} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(u-)\right] a_{12}(u)\,I_{\{\mathbf{X}(u)=\mathbf{e}_1\}} = \gamma_1(u)\,a_{21}(u)\,I_{\{\mathbf{X}(u)=\mathbf{e}_1\}}. \tag{58}$$

Then,

$$\gamma_1(u) = E\left[\frac{\partial}{\partial x_1} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(u-)\right] \frac{a_{12}(u)}{a_{21}(u)}, \tag{59}$$

on the set $\{\mathbf{X}(u) = \mathbf{e}_1\}$. □

5. An Extension to a Function of Path Integrals

The integration-by-parts formulas and the martingale representation developed in the previous sections are now extended to a function of the integrals with respect to the whole paths of the fundamental jump processes relating to the chain $\mathbf{X}$. This function may be considered a canonical form of an $\mathcal{F}^{\mathbf{X}}(T)$-measurable random variable.

Consider an $\mathcal{F}^{\mathbf{X}}(T)$-measurable random variable H which is of the following canonical form:

$$H := h\left(\int_0^T \eta_{12}(t)\,dJ_{12}(t), \int_0^T \eta_{21}(t)\,dJ_{21}(t)\right), \tag{60}$$

where $h : \mathfrak{R}^2 \to \mathfrak{R}$ is any measurable, integrable, and differentiable function. Note that H depends on the whole paths of the fundamental jump processes relating to the chain $\mathbf{X}$; η_{12} and η_{21} are nonnegative, $\mathbb{P}$-a.s. bounded, $\mathbb{F}^{\mathbf{X}}$-predictable processes as defined in Section 3.

We now define some notation. Write

$$I_{12}(T) := \int_0^T \eta_{12}(t)\,dJ_{12}(t), \quad I_{21}(T) := \int_0^T \eta_{21}(t)\,dJ_{21}(t), \quad \mathbf{I}(T) := (I_{12}(T), I_{21}(T))' \in \mathfrak{R}^2. \tag{61}$$

Then

$$H = h(\mathbf{I}(T)). \tag{62}$$

The following theorem gives an extension to the integration-by-parts formula presented in Theorem 4 for the function h.

Theorem 14. *For each $t \in \mathcal{T}$, let*

$$\begin{gathered}\widetilde{\lambda}_{12}(t) := \eta_{12}(t)\lambda_{12}(t), \qquad \widetilde{\lambda}_{21}(t) := \eta_{21}(t)\lambda_{21}(t), \\ \widetilde{\varphi}(t) := \left(\int_0^t \widetilde{\lambda}_{12}(u)\,du, \int_0^t \widetilde{\lambda}_{21}(u)\,du\right) \in \mathfrak{R}^2.\end{gathered} \tag{63}$$

Then

$$E\left[\left\langle D_{\mathbf{x}} h(\mathbf{I}(T)), \widetilde{\varphi}(T)\right\rangle\right] = E\left[h(\mathbf{I}(T))\int_0^T \boldsymbol{\eta}'(u)\,d\overline{\mathbf{M}}(u)\right]. \tag{64}$$

Proof. The proof of this theorem resembles that of Theorem 4. We only give some key steps. For each $\epsilon > 0$, let

$$I_{12}^{\epsilon}(T) := \int_0^T \eta_{12}(t)\,dJ_{12}^{\epsilon}(t), \qquad I_{21}^{\epsilon}(T) := \int_0^T \eta_{21}(t)\,dJ_{21}^{\epsilon}(t). \tag{65}$$

Write

$$\mathbf{I}^{\epsilon}(T) := \left(I_{12}^{\epsilon}(T), I_{21}^{\epsilon}(T)\right)' \in \mathfrak{R}^2. \tag{66}$$

By Lemma 2, the $\mathbb{P}^{\epsilon}$-probability law of $\mathbf{I}^{\epsilon}(T)$ is the same as the $\mathbb{P}$-law of $\mathbf{I}(T)$. Then

$$E[h(\mathbf{I}(T))] = E^{\epsilon}\left[h\left(\mathbf{I}^{\epsilon}(T)\right)\right] = E\left[\Lambda^{\epsilon}(T)\,h\left(\mathbf{I}^{\epsilon}(T)\right)\right]. \tag{67}$$

Differentiating with respect to ϵ and setting $\epsilon = 0$ give

$$\begin{aligned}&E\left[\frac{\partial}{\partial\epsilon}\Lambda^{\epsilon}(T)\Big|_{\epsilon=0} h\left(\mathbf{I}^{\epsilon}(T)\right)\Big|_{\epsilon=0}\right] \\ &\quad + E\left[\Lambda^{\epsilon}(T)\Big|_{\epsilon=0}\left\langle D_{\mathbf{x}} h\left(\mathbf{I}^{\epsilon}(T)\right), \frac{\partial}{\partial\epsilon}\mathbf{I}^{\epsilon}(T)\right\rangle\Big|_{\epsilon=0}\right] = 0.\end{aligned} \tag{68}$$

Then the result follows by noting that

$$\frac{\partial}{\partial\epsilon}\mathbf{I}^{\epsilon}(T)\Big|_{\epsilon=0} = \left(\int_0^T \widetilde{\lambda}_{12}(t)\,dt, \int_0^T \widetilde{\lambda}_{21}(t)\,dt\right)' = \widetilde{\varphi}(T). \tag{69}$$

□

Similarly, the following corollaries are direct consequences of Theorem 14.

Corollary 15. *For any measurable, integrable, and differentiable function $h : \mathfrak{R}^2 \to \mathfrak{R}$,*

$$\begin{aligned}&E\left[\frac{\partial}{\partial x_1} h(\mathbf{I}(T))\int_0^T \widetilde{\lambda}_{12}(u)\,du\right] \\ &\quad = E\left[h(\mathbf{I}(T))\int_0^T \eta_{12}(u)\,dM_{12}(u)\right].\end{aligned} \tag{70}$$

Corollary 16. *For any measurable, integrable, and differentiable function $h : \mathfrak{R}^2 \to \mathfrak{R}$,*

$$\begin{aligned}&E\left[\frac{\partial}{\partial x_2} h(\mathbf{I}(T))\int_0^T \widetilde{\lambda}_{21}(u)\,du\right] \\ &\quad = E\left[h(\mathbf{I}(T))\int_0^T \eta_{21}(u)\,dM_{21}(u)\right].\end{aligned} \tag{71}$$

We now extend the martingale representation in Section 3 to the function $H := h(\mathbf{I}(T))$ of the path integrals. By the martingale representation in Theorem 10,

$$h(\mathbf{I}(T)) = E[h(\mathbf{I}(T))] + \int_0^T \widetilde{\gamma}'(u)\,d\overline{\mathbf{M}}(u), \tag{72}$$

for some $\mathbb{F}^{\mathbf{X}}$-predictable process $\widetilde{\gamma} := \{\widetilde{\gamma}(t) \mid t \in \mathcal{T}\}$.

Again by subtraction we assume that $E[h(\mathbf{I}(T))] = 0$. Then

$$h(\mathbf{I}(T)) = \int_0^T \widetilde{\gamma}'(u)\,d\overline{\mathbf{M}}(u). \tag{73}$$

The following theorem gives an expression for the integrand in the martingale representation for $h(\mathbf{I}(T))$.

Theorem 17. *Suppose that $a_{12}(t), a_{21}(t) > 0$ for each $t \in \mathcal{T}$. Then the integrand $\widetilde{\gamma} := \{\widetilde{\gamma}(t) \mid t \in \mathcal{T}\}$, where $\widetilde{\gamma}(t) := (\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t))' \in \mathfrak{R}^2$, is determined by*

$$\begin{aligned}\widetilde{\gamma}_1(t) &= E\left[\frac{\partial}{\partial x_1} h(\mathbf{I}(T)) \mid \mathcal{F}^{\mathbf{X}}(t-)\right]\frac{a_{12}(t)}{a_{21}(t)}, \\ &\qquad \text{on the set } \{\mathbf{X}(t) = \mathbf{e}_1\}, \\ \widetilde{\gamma}_2(t) &= E\left[\frac{\partial}{\partial x_2} h(\mathbf{I}(T)) \mid \mathcal{F}^{\mathbf{X}}(t-)\right]\frac{a_{21}(t)}{a_{12}(t)}, \\ &\qquad \text{on the set } \{\mathbf{X}(t) = \mathbf{e}_2\}.\end{aligned} \tag{74}$$

Proof. The proof resembles that of Theorem 13. We only need to note the fact that, for all η_{12}'s which are indicators of sets in $\mathcal{H}$, $\eta_{12}^2 = \eta_{12}$. □

6. An Application to Hedging Contingent Claims

In this section we will discuss an application of the martingale representation result derived in Section 4 to hedge contingent claims in the Markov chain financial market of Norberg [21]. Here we consider a simplified version of the Markov chain market of Norberg [21], where there are two risky shares, namely, S_1 and S_2, and the Markov chain has only two states. We also suppose that the market interest rate is zero. In this case, as in Norberg [21], the (discounted) price processes of the two risky shares $\{S_1(t) \mid t \in \mathcal{T}\}$ and $\{S_2(t) \mid t \in \mathcal{T}\}$ under a risk-neutral probability, say $\mathbb{P}$, are governed by

$$\begin{aligned}dS_i(t) &= S_i(t-)\left(\left(\exp\left(\beta_{12}^i\right) - 1\right)dM_{12}(t)\right. \\ &\qquad \left. + \left(\exp\left(\beta_{21}^i\right) - 1\right)dM_{21}(t)\right), \\ S_i(0) &= s_i > 0, \quad i = 1, 2,\end{aligned} \tag{75}$$

where β_{12}^i and β_{21}^i, for $i = 1, 2$, are non-zero constants; $\{M_{12}(t) \mid t \in \mathcal{T}\}$ and $\{M_{21}(t) \mid t \in \mathcal{T}\}$ are $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingales. Note that the two risky shares are correlated since their price dynamics depend on M_{12} and M_{21}.

For each $i = 1, 2$, let $\alpha_{12}^i = \exp(\beta_{12}^i) - 1$ and let $\alpha_{21}^i = \exp(\beta_{21}^i) - 1$. Then, as in Norberg [21], under the risk-neutral measure $\mathbb{P}$, the (discounted) terminal prices $S_1(T)$ and $S_2(T)$ of the shares are given by

$$\begin{aligned} S_i(T) = s_i \exp\Bigg(&-\alpha_{12}^i \int_0^T a_{12}(t) \langle \mathbf{X}(t-), \mathbf{e}_1 \rangle \, dt \\ &- \alpha_{21}^i \int_0^T a_{21}(t) \langle \mathbf{X}(t-), \mathbf{e}_2 \rangle \, dt \\ &+ \beta_{12}^i J_{12}(T) + \beta_{21}^i J_{21}(T) \Bigg), \qquad i = 1, 2. \end{aligned} \tag{76}$$

Consequently, the vector of the (discounted) terminal prices of the shares $\mathbf{S}(T) := (S_1(T), S_2(T))$ is a function of $\mathbf{J}(T) := (J_{12}(T), J_{21}(T))$.

We now consider a contingent claim H written on the two correlated risky shares S_1 and S_2 whose payoff at maturity T is a function of $\mathbf{S}(T)$, say $H(\mathbf{S}(T))$. Two practical examples of contingent claims having payoffs of this form are an exchange option, which is also called a Margrabe option, and a quanto option.

Note that the payoffs of the Margrable option and the quanto option may not be differentiable functions of $\mathbf{S}(T)$. To apply the martingale representation result in Section 4 to derive the hedging quantities for the Margrable option and the quanto option, we need to consider approximations of $H(\mathbf{S}(T))$ by some "smooth" or differentiable payoff functions of $\mathbf{S}(T)$. In the sequel, we suppose that, with a slight abuse of notation, $H(\mathbf{S}(T))$ is such a "smooth" or differentiable payoff function of $\mathbf{S}(T)$.

Then, the payoff $H(\mathbf{S}(T))$ can be written as

$$H(\mathbf{S}(T)) = G(\mathbf{J}(T)), \tag{77}$$

for some "suitable" measurable, differentiable and integrable function $G : \mathfrak{R}^2 \to \mathfrak{R}$.

Define, for each $t \in \mathcal{T}$, a (2×2)-matrix $\Sigma(t)$ by

$$\begin{aligned} \Sigma(t) &:= \begin{pmatrix} S_1(t-)\left(\exp\left(\beta_{12}^1\right) - 1\right) & S_1(t-)\left(\exp\left(\beta_{21}^1\right) - 1\right) \\ S_2(t-)\left(\exp\left(\beta_{12}^2\right) - 1\right) & S_2(t-)\left(\exp\left(\beta_{21}^2\right) - 1\right) \end{pmatrix} \\ &= \begin{pmatrix} S_1(t-)\alpha_{12}^1 & S_1(t-)\alpha_{21}^1 \\ S_2(t-)\alpha_{12}^2 & S_2(t-)\alpha_{21}^2 \end{pmatrix}. \end{aligned} \tag{78}$$

Then the price processes of the two risky shares S_1 and S_2 under the risk-neutral measure $\mathbb{P}$ are governed by the following vector-valued stochastic differential equation:

$$d\mathbf{S}(t) = \Sigma(t) \, d\overline{\mathbf{M}}(t), \tag{79}$$

where $\overline{\mathbf{M}}(t) := (M_{12}(t), M_{21}(t))$ as defined in Theorem 4.

Suppose $\alpha_{12}^1\alpha_{21}^2 \neq \alpha_{21}^1\alpha_{12}^2$. Then, the inverse $\Sigma^{-1}(t)$ of $\Sigma(t)$ exists and is given by

$$\Sigma^{-1}(t) = \frac{1}{\alpha_{12}^1\alpha_{21}^2 - \alpha_{21}^1\alpha_{12}^2} \begin{pmatrix} \dfrac{\alpha_{21}^2}{S_1(t-)} & -\dfrac{\alpha_{21}^1}{S_2(t-)} \\ -\dfrac{\alpha_{12}^2}{S_1(t-)} & \dfrac{\alpha_{12}^1}{S_2(t-)} \end{pmatrix}. \tag{80}$$

Consequently,

$$d\overline{\mathbf{M}}(t) = \Sigma^{-1}(t) \, d\mathbf{S}(t). \tag{81}$$

By the martingale representation in Theorem 10,

$$\begin{aligned} H(\mathbf{S}(T)) &= G(\mathbf{J}(T)) \\ &= E[G(\mathbf{J}(T))] + \int_0^T \gamma'(u) \, d\overline{\mathbf{M}}(u) \\ &= E[H(\mathbf{S}(T))] + \int_0^T \gamma'(u) \Sigma^{-1}(u) \, d\mathbf{S}(u) \\ &= E[H(\mathbf{S}(T))] \\ &\quad + \int_0^T \left(\frac{\gamma_1(u)\alpha_{21}^2 - \gamma_2(u)\alpha_{12}^2}{S_1(u-)} \right) dS_1(u) \\ &\quad + \int_0^T \left(\frac{\gamma_2(u)\alpha_{12}^1 - \gamma_1(u)\alpha_{21}^1}{S_2(u-)} \right) dS_2(u). \end{aligned} \tag{82}$$

Then the claim $H(\mathbf{S}(T))$ can be hedged perfectly by constructing a dynamic portfolio which invests $(\gamma_1(t)\alpha_{21}^2 - \gamma_2(t)\alpha_{12}^2)/S_1(t-)$ units of the risky share S_1 and $(\gamma_2(t)\alpha_{12}^1 - \gamma_1(t)\alpha_{21}^1)/S_2(t-)$ units of the risky share S_2 at time t, for each $t \in \mathcal{T}$. The initial investment of the portfolio is $E[H(\mathbf{S}(T))]$, which is the initial price of the claim $H(\mathbf{S}(T))$. Using Theorem 13, $\gamma_1(t)$ and $\gamma_2(t)$ are determined as

$$\begin{aligned} \gamma_1(t) &= E\left[\frac{\partial}{\partial x_1} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(t-)\right] \frac{a_{12}(t)}{a_{21}(t)}, \\ &\qquad \text{on the set } \{\mathbf{X}(t) = \mathbf{e}_1\}, \\ \gamma_2(t) &= E\left[\frac{\partial}{\partial x_2} G(\mathbf{J}(T)) \mid \mathcal{F}^{\mathbf{X}}(t-)\right] \frac{a_{21}(t)}{a_{12}(t)}, \\ &\qquad \text{on the set } \{\mathbf{X}(t) = \mathbf{e}_2\}. \end{aligned} \tag{83}$$

We only illustrate here the use of the martingale representation result in Section 4 to hedge contingent claims whose payoffs depend only on the terminal prices of the risky shares in the Markov chain market. The martingale representation result in Section 5 may be used to hedge contingent claims with more general payoff structures in the Markov chain market.

7. Conclusion

An integration-by-parts formula for a function of the terminal values of the fundamental jump processes relating to a Markov chain was first established using the Bismut approach

to Malliavin calculus. The formula was then applied to derive a new expression for the integrand in a stochastic integral in a martingale representation. The results were then extended to functions of the integrals with respect to the whole paths of the fundamental jump processes. These functions may be regarded as random variables of canonical forms. Only finite-dimensional calculus was needed in the derivations. Though some complex notations may be involved, the results presented here may be extended to the case of a general N-state Markov chain where a set of fundamental jump processes $\{J_{ik}(t) \mid t \in \mathcal{T}\}$, $i, k = 1, 2, \ldots, N$, $i \neq k$, is used. We applied the martingale representation result derived here to hedge a contingent claim written on two correlated risky shares in the Markov chain financial market of Norberg [21].

There are several future research directions based on the results developed in this paper which may be of theoretical and practical interests. The results may be applied to study the existence and uniqueness of densities of jump processes relating to a Markov chain. It seems that this problem is of fundamental importance in filtering and control theory of hidden Markov chains. Martingale representations play an important role in filtering and control. It may be interesting to explore the applications of the martingale representations developed in this paper in filtering and control for stochastic processes relating to Markov chains. The monograph by Elliott et al. [32] provided some discussions on the filtering and control of hidden Markov chains. The results developed here may be extended to develop Malliavin calculus for stochastic differential equations driven by a continuous-time, finite-state Markov chain and Markov regime-switching stochastic differential equations. It may be of practical interest to further explore the use of the martingale representation results developed here to hedge modern insurance products, such as unit-linked insurance products and longevity bonds in the Markov chain market of Norberg [21]. In Bielecki et al. [48], the valuation of credit derivatives in a Markov chain model was discussed. It may be of practical interest to explore the application of the martingale representation results developed here to hedge credit derivatives in the Markov chain model discussed in Bielecki et al. [48].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Global Multivariable Control of Permanent Magnet Synchronous Motor for Mechanical Elastic Energy Storage System under Multiclass Nonharmonic External Disturbances

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For the technology of mechanical elastic energy storage utilizing spiral torsion springs as the energy storage media presented previously, a global multivariable control algorithm based on nonlinear internal model principle under multiclass external disturbances is proposed. The nonlinear external disturbances with nonharmonic periodic characteristics are generated by multiclass nonlinear external systems. New equations of nonlinear internal model are designed to estimate the multiclass external disturbances. On the basis of constructing the control law of nominal system, a state feedback controller is designed to guarantee the closed-loop system globally uniformly bounded, and a Lyapunov function is constructed to theoretically prove the global uniform boundedness of the multivariable closed-loop system signals. The simulation results verify the correctness and effectiveness of the presented algorithm.

1. Introduction

Energy storage technologies have a great practical significance for the solution of new energy interconnecting to the grid, peak regulation, frequency modulation, and stability control [1, 2]. Based on lucubration of mechanical elastic energy storage (MEES), a new way of MEES method applying spiral torsion springs (STS) as the energy storage material is proposed [3]. Due to the advantages such as large power density, high efficiency, great electromagnetic torque, small volume, and fast response speed, permanent magnet synchronous motor (PMSM) is selected as the actuator for MEES system, just as in many other servo systems [4]. One of the key technologies to be solved for MEES is the nonlinear control owing to its electromechanical coupling properties and nonlinear characteristics of PMSM [5]. When PMSM based MEES system runs in energy storage, the increasing load torque with the tightening of STS is unfavorable for the operation of servo system. In addition, due to the nonlinearities, strong coupling, and time variation of PMSM, especially the existence of the nonlinear external disturbances, the conventional PID controller is difficult to satisfy the requirement of high precision control [6]. Hence, the other control methods should be introduced, just as nonlinear control [7–10], adaptive control [11], state feedback control [12], and so forth.

One of the core questions in control field is to guarantee asymptotical stability of unforced close-loop system, implement the asymptotical tracking of system output for given trajectories, and reject exogenous disturbances [13–16]. The control problems for servo system are also called the output regulation problems; the problems of disturbance rejection under the framework of output regulation have earned extensive attention in recent years [17, 18]. The preexisting papers mostly assumed that the exosystem generating disturbances is linear, neutral, and stable; that is to say, disturbance rejection under sinusoidal perturbation is frequently studied. For instance, literatures [16, 17, 19] deal with the problems

of sinusoidal perturbation rejection in known and unknown frequency, respectively. Nevertheless, the nonharmonic disturbances generated by nonlinear exosystems, which will make practical servo systems, generators, and power flexible mechanisms produce noise and precision reduction [20], are harmful to these practical running systems. However, how to tackle with disturbances generated by nonlinear exosystems is rarely involved [21]. Therefore, rejection of these harmful oscillations is essential to guarantee that the systems operate stably under nonlinear external disturbances.

Another central issue in control field is to extend the established control theory to more complex and generalized circumstances. In [22], disturbance rejection under a class of exosystems for a single-input nonlinear system is studied, in which the dependent external system is a single model, and the researched system is a single-input system. In [23], a global harmonic rejection algorithm for multivariable system is proposed; however, the rejection of the disturbances in the paper aims at the standard sinusoidal components. For the nonlinearity and complexity of PMSM based MEES system, the focus of the paper is to extend the previous works in [22, 23] to a generalized multivariable input field of nonlinear systems under multiclass nonlinear exosystems with the application of the constructed model for PMSM based MEES system in [5] and design a controller to suppress multiclass external nonharmonic disturbances in PSMS based MEES system.

The mainly theoretical contribution of the paper is to propose a global multivariable disturbance control method to reject multiclass nonlinear external nonharmonic disturbances generated by multiclass nonlinear exosystems for general multivariable nonlinear system, and the presented control algorithm is employed to regulate an actual nonlinear system. The validity and effectiveness of the proposed method are testified by the simulation results.

The organization of the paper is as follows. It starts with an introduction of the research status of disturbance rejection and points out the significance of rejection of nonharmonic external disturbances in Section 1. The mathematical model of PMSM based MEES system is constructed and the formulation of control problem in the paper is given in Section 2. In Section 3, the nonlinear internal models simulating the multiclass external nonharmonic disturbances are presented based on internal model principle. Nonlinear multivariable state feedback controller ensuring the closed-loop system globally bounded is demonstrated in Section 4. The verifications of the proposed algorithm by means of numerical simulations are shown in Section 5. Ultimately, Section 6 sums up the conclusions of the research and puts forward the work in the future.

2. Problem Formulation

2.1. Mathematical Model of PMSM Based MEES System. For the convenience of understanding and reutilization, the model of the whole system for PMSM based MEES system proposed in [3, 5] is shown in Figure 1, where gear case is simplified as a model of spring mass damper with multidegree

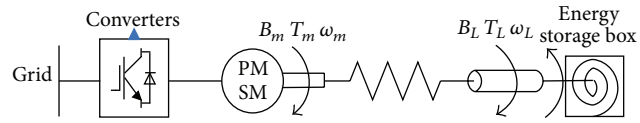

Figure 1: The model of PMSM based MEES system.

of freedom, B_m and B_L denote the damping coefficients of the motor and elastic shaft, respectively, T_m and T_L represent the output torque of the motor and main shaft, respectively, and ω_m and ω_L symbolize the angular velocity of the motor and main shaft of energy storage box, respectively. STS is installed in energy storage box.

In the process of energy storage, PMSM runs in the state of electric motor. For PMSM, assume that the inductance of d-axis L_d is equal to the inductance of q-axis L_q. Accordingly, the mathematical model for PMSM in dq rotating coordinates can be expressed as follows [24]:

$$\begin{aligned} \frac{di_d}{dt} &= -\frac{R_s}{L_d} i_d + p i_q \omega_m + \frac{1}{L_d} u_d, \\ \frac{di_q}{dt} &= -\frac{R_s}{L_q} i_q - p i_d \omega_m - \frac{p\varphi_f}{L_q} \omega_m + \frac{1}{L_q} u_q, \\ \frac{d\omega_m}{dt} &= \frac{p\varphi_f}{J_m} i_q - \frac{B_m}{J_m} \omega_m - \frac{1}{J_m} T_m, \end{aligned} \tag{1}$$

where i_d and i_q denote the current components of stator in dq axis, respectively, u_d and u_q display the voltage components of stator in dq axis, respectively, and R_s, ω_m, φ_f, p, and J_m represent the resistance of stator, angular velocity, flux linkage of rotor, numbers of pole pairs, and moment of inertia of rotor, respectively.

The ratio of gear case is assumed to be r; without regard to the power loss of gear case, the relationship between torque and angular velocity on both sides of gear case can be expressed as

$$T_m = \frac{T_L}{r}, \qquad \omega_m = \omega_L \cdot r. \tag{2}$$

Suppose the outer end of STS to be fixed in spring box as V type, and in terms of the national standard *design and calculation of spiral torsion spring (JB/T7366-1994)* in China, the torque of STS with rectangle cross section can be written as

$$T_L = k \frac{Ebh^3}{6l} n, \tag{3}$$

where n denotes the working turns of spring, E, l, b, and h represent the modulus of elasticity, length, width, and thickness of STS, respectively, and k indicates mass coefficient of spring.

Due to the large mass and high inertia of MEES system, the working rotation velocities for the main shaft of energy storage box and the rotor of PMSM are both assumed to be invariable. Hence, the relationship between the angular

velocity ω_L and the working turns n for the main shaft of energy storage box can be written as follows:

$$n = \frac{\omega_L t}{2\pi}, \tag{4}$$

where t represents the time.

Equation (4) is substituted into (3); the relationship between the torque T_L of STS and angular velocity ω_L of the main shaft can be given by the following formula:

$$T_L = k\frac{Ebh^3\omega_L}{12\pi l}t. \tag{5}$$

For a given STS, (5) shows that the torque T_L of STS is proportional to the time t if angular velocity ω_L is a constant.

The mathematical model for the whole system of PMSM based MEES can be obtained by combining differential equation (1) with algebraic equations (2) and (5).

2.2. Control Problem Formulation. Consider the multivariable nonlinear system with a standard affine form under multiclass disturbances:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\left(\mathbf{u}_i - \mathbf{v}_i(\mathbf{w})\right), \quad 1 \le i \le m, \tag{6}$$

where $\mathbf{x} \in \mathbf{R}^n$ are the state vectors, $\mathbf{u}_i \in \mathbf{R}$ are the control inputs, $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_i(\mathbf{x})$ are the known smooth vector fields, $\mathbf{v}_i(\mathbf{w})$ are the nonlinear disturbance inputs, and $\mathbf{w} \in \mathbf{R}^q$ indicate the external signals generated by the nonlinear exosystem shown as follows:

$$\dot{\mathbf{w}} = \mathbf{s}_i(\mathbf{w}), \quad 1 \le i \le m. \tag{7}$$

If the nonlinear disturbance inputs are ignored, for system (6), its nominal system can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\mathbf{u}_i, \quad 1 \le i \le m. \tag{8}$$

The essence of solving stability problems for a multivariable input system is to convert these problems into the stability problems of multiple single-input systems [25].

Assumption 1. For system (8), there exists a control law of state feedback $\boldsymbol{\alpha}_i(\mathbf{x})$ making the nominal close-loop system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\boldsymbol{\alpha}_i(\mathbf{x})$ asymptotically stabilize at the origin. Therefore, there exists a Lyapunov function $\mathbf{V}(\mathbf{x})$ satisfying

$$\begin{gathered}\underline{d}(\|\mathbf{x}\|) \le \mathbf{V}(\mathbf{x}) \le \overline{d}(\|\mathbf{x}\|), \\ \frac{\partial \mathbf{V}(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\boldsymbol{\alpha}_i(\mathbf{x})\right) \le -d_0(\|\mathbf{x}\|), \\ \left|\frac{\partial \mathbf{V}(\mathbf{x})}{\partial \mathbf{x}}\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\right|^2 \le d_0(\|\mathbf{x}\|),\end{gathered} \tag{9}$$

where $\underline{d}$, $\overline{d}$, and d_0 are all of K_∞ class functions.

Assumption 2. The trajectory of the vector field for the nonlinear exosystem (7) is bounded.

Remark 3. The functions, which meet Assumption 2, include harmonic functions and limit cycles of nonlinear dynamic systems. For instance, the famous Van der Pol circuit can be written as

$$\begin{aligned}\dot{w}_1 &= w_2 - \varsigma\left(\frac{1}{3}w_1^3 - w_1\right), \\ \dot{w}_2 &= -w_1,\end{aligned} \tag{10}$$

where ς denotes a parameter to adjust the period of current or voltage. The eigenvalues of Jacobian matrix for (10) at the origin are $(1/2)(\varsigma \pm \sqrt{\varsigma^2 - 4})$. If $0 < \varsigma \le 2$, the eigenvalues have positive real parts; if $\varsigma > 2$, the eigenvalues are positive real numbers. Consequently, as long as $\varsigma > 0$, the equilibrium points at the origin of (10) are unstable and there exists a bounded limit cycle [20].

Assumption 4. There exists a smooth function $\mathbf{r}_i(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}^q$ making

$$\frac{\partial \mathbf{r}_i(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}_i(\mathbf{x}) = \mathbf{K}_i, \quad 1 \le i \le m, \tag{11}$$

where $K_i \in \mathbf{R}^q$ is a nonzero constant vector.

The problem to be solved in the paper can be described as in the following definition.

Definition 5. For any given compact subset $\mathbf{D}_\mathbf{w} \in \mathbf{R}^q$, a state feedback controller $\mathbf{u}_i$ can be found to make the solution of the close-loop system (6) exist and be bounded, and $\lim_{t\to\infty}\mathbf{x}(t) = 0$ under arbitrary initial conditions for all $\mathbf{w}(\mathbf{0}) \in \mathbf{D}_\mathbf{w}$ and $t \ge 0$.

3. Multiclass Nonlinear Internal Models Design

In the paper, internal model principle (IMC) is utilized to reject the multiclass disturbances. Disturbances rejection by IMC belongs to indirect suppression algorithm. Hence, appropriate internal model equations should be firstly established to estimate the inputting nonlinear disturbances. Because the exosystem discussed in the paper is nonlinear, the internal model equations established should also be nonlinear. Therefore, Assumption 6 is introduced as follows.

Assumption 6. For the nonlinear exosystem (7), when $1 \le i \le m$, there exists an immersion system being depicted as follows:

$$\dot{\boldsymbol{\eta}}_i = \mathbf{F}_i\boldsymbol{\eta}_i + \mathbf{G}_i\gamma_i\left(\mathbf{J}_i\boldsymbol{\eta}_i\right), \qquad \mathbf{v}_i(\mathbf{w}) = \mathbf{H}_i\boldsymbol{\eta}_i, \tag{12}$$

where $\boldsymbol{\eta}_i \in \mathbf{R}^r$, $\mathbf{F}_i, \mathbf{G}_i, \mathbf{H}_i, \mathbf{J}_i$ are matrices with certain dimensions, the matrix pair $(\mathbf{F}_i, \mathbf{H}_i)$ is observable, and there exists a positive definite matrix $\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}$ making the following formula hold:

$$\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i + \left(\mathbf{J}_i\right)^T = 0, \tag{13}$$

and the nonlinear function $\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i)$ can be expressed as

$$\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) = \begin{bmatrix} \gamma_i^1\left(\sum_{j=1}^{r}\mathbf{J}_i^{1j}\boldsymbol{\eta}_i^{1j}\right) \\ \vdots \\ \gamma_i^m\left(\sum_{j=1}^{r}\mathbf{J}_i^{mj}\boldsymbol{\eta}_i^{mj}\right) \end{bmatrix} \tag{14}$$

and satisfies $(s_1 - s_2)^T(\gamma_i(s_1) - \gamma_i(s_2)) \geq 0$.

Consequently, the multiclass nonlinear internal model equations can be designed as follows:

$$\begin{aligned} \dot{\hat{\boldsymbol{\eta}}}_i &= (\mathbf{F}_i - \mathbf{K}_i\mathbf{H}_i)(\hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x})) + \mathbf{G}_i\gamma_i(\mathbf{J}_i(\hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x}))) \\ &\quad + \mathbf{K}_i\mathbf{u}_i + \frac{\partial \mathbf{r}_i(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_i(\mathbf{x}), \end{aligned} \tag{15}$$

where $\mathbf{K}_i \in \mathbf{R}^q$ satisfies Assumption 4 and makes $\mathbf{F}_{i0} = \mathbf{F}_i - \mathbf{K}_i\mathbf{H}_i$ a Hurwitz matrix; hence there exist positive definite matrices $\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}$ and $\mathbf{Q}_{\hat{\boldsymbol{\eta}}_i}$ satisfying the following equation:

$$\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{F}_{i0} + \mathbf{F}_{i0}^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i} = -\mathbf{Q}_{\hat{\boldsymbol{\eta}}_i}. \tag{16}$$

Define an auxiliary error $\mathbf{e}_i$ as follows:

$$\mathbf{e}_i = \boldsymbol{\eta}_i - \hat{\boldsymbol{\eta}}_i + \mathbf{r}_i(\mathbf{x}), \tag{17}$$

and derivative of (17) along with (6), (12), and (15) is given by

$$\begin{aligned} \dot{\mathbf{e}}_i &= \dot{\boldsymbol{\eta}}_i - \dot{\hat{\boldsymbol{\eta}}}_i + \frac{\partial \mathbf{r}_i(\mathbf{x})}{\partial \mathbf{x}}(\mathbf{f}_i(\mathbf{x}) + \mathbf{g}_i(\mathbf{x})(\mathbf{u}_i - \mathbf{v}_i(\mathbf{w}))) \\ &= \mathbf{F}_i\boldsymbol{\eta}_i + \mathbf{G}_i\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - (\mathbf{F}_i - \mathbf{K}_i\mathbf{H}_i)(\hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x})) \\ &\quad - \mathbf{G}_i\gamma_i(\mathbf{J}_i(\hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x}))) - \mathbf{K}_i\mathbf{u}_i - \frac{\partial \mathbf{r}_i(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_i(\mathbf{x}) \\ &= \mathbf{F}_{i0}\mathbf{e}_i + \mathbf{G}_i(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i))). \end{aligned} \tag{18}$$

4. State Feedback Controller Design

In terms of the nonlinear internal models shown in (15) and Assumption 1, the state feedback controller can be designed as follows:

$$\mathbf{u}_i = \boldsymbol{\alpha}_i(\mathbf{x}) + \mathbf{H}_i(\hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x})), \tag{19}$$

where $\boldsymbol{\alpha}_i(\mathbf{x})$ is a controller being able to stabilize the nominal system (8).

Construct a Lyapunov function as follows:

$$W = V(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{e}_i^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{e}_i. \tag{20}$$

Derivative of Lyapunov function W along with system (6) and auxiliary error (18), we can obtain

$$\begin{aligned} \dot{W} &= \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})(\mathbf{u}_i - \mathbf{v}_i(\mathbf{w}))\right) \\ &\quad + \sum_{i=1}^{m}\mathbf{e}_i^T\left(\left(\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{F}_{i0} + \mathbf{F}_{i0}^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\right)\mathbf{e}_i\right. \\ &\qquad \left. + 2\mathbf{e}_i^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i)))\right) \\ &= \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{f}_i(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\boldsymbol{\alpha}_i(\mathbf{x})\right) \\ &\quad + \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\mathbf{H}_i(\hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x})) \\ &\quad - \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\mathbf{H}_i(\mathbf{e}_i + \hat{\boldsymbol{\eta}}_i - \mathbf{r}_i(\mathbf{x})) - \sum_{i=1}^{m}\mathbf{e}_i^T\mathbf{Q}_{\hat{\boldsymbol{\eta}}_i}\mathbf{e}_i \\ &\quad + \sum_{i=1}^{m}2\mathbf{e}_i^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i))) \\ &\leq \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{f}_i(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\boldsymbol{\alpha}_i(\mathbf{x})\right) \\ &\quad - \sum_{i=1}^{m}\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}_i(\mathbf{x})\mathbf{H}_i\mathbf{e}_i - \sum_{i=1}^{m}\lambda_{\min}\left(\mathbf{Q}_{\hat{\boldsymbol{\eta}}_i}\right)\|\mathbf{e}_i\|^2 \\ &\quad + \sum_{i=1}^{m}2\mathbf{e}_i^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i))) \\ &\leq -d_0(\|\mathbf{x}\|) + \sum_{i=1}^{m}\left|\frac{\partial V_i(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}_i(\mathbf{x})\right|\|\mathbf{H}_i\mathbf{e}_i\| \\ &\quad - \sum_{i=1}^{m}\lambda_{\min}\left(\mathbf{Q}_{\hat{\boldsymbol{\eta}}_i}\right)\|\mathbf{e}_i\|^2 \\ &\quad + \sum_{i=1}^{m}2\mathbf{e}_i^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i))), \end{aligned} \tag{21}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a certain matrix.

In terms of Assumption 6, we can obtain $\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i = -(\mathbf{J}_i)^T$; hence

$$\begin{aligned} &\sum_{i=1}^{m}2\mathbf{e}_i^T\mathbf{P}_{\hat{\boldsymbol{\eta}}_i}\mathbf{G}_i(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i))) \\ &= \sum_{i=1}^{m}(-2(\mathbf{J}_i\boldsymbol{\eta}_i - \mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i)))^T(\gamma_i(\mathbf{J}_i\boldsymbol{\eta}_i) - \gamma_i(\mathbf{J}_i(\boldsymbol{\eta}_i - \mathbf{e}_i))) \leq 0. \end{aligned} \tag{22}$$

Applying permanent establishment inequality $2ab \leq ca^2 + c^{-1}b^2$ (choosing $c = 2$) to the second term of (21), we obtain

$$\sum_{i=1}^{m} \left| \frac{\partial V_i(\mathbf{x})}{\partial \mathbf{x}} \mathbf{g}_i(\mathbf{x}) \right| \|\mathbf{H}_i \mathbf{e}_i\| \leq \sum_{i=1}^{m} \left(\left| \frac{\partial V_i(\mathbf{x})}{\partial \mathbf{x}} \mathbf{g}_i(\mathbf{x}) \right|^2 + \frac{1}{4} \|\mathbf{H}_i\|^2 \|\mathbf{e}_i\|^2 \right). \tag{23}$$

Substitute (22) and (23) into (21) and combine (23) with the application of Assumption 1; we obtain

$$\dot{W} \leq -\sum_{i=1}^{m} \left(\lambda_{\min}\left(\mathbf{Q}_{\hat{\eta}_i}\right) - \frac{1}{4}\|\mathbf{H}_i\|^2 \right) \|\mathbf{e}_i\|^2. \tag{24}$$

Choose appropriate $\mathbf{Q}_{\hat{\eta}_i}$ and $\mathbf{H}_i$ to satisfy

$$d_i = \lambda_{\min}\left(\mathbf{Q}_{\hat{\eta}_i}\right) - \frac{1}{4}\|\mathbf{H}_i\|^2 > 0; \tag{25}$$

namely,

$$\dot{W} \leq -\sum_{i=1}^{m} d_i \|\mathbf{e}_i\|^2. \tag{26}$$

By above knowable, all the variables are bounded. Combine with the application of the invariant set theorem, it can be obtained that $\lim_{t\to\infty}\mathbf{x}(t) = 0$ and $\lim_{t\to\infty}\mathbf{e}_i = 0$. Therefore, we give a theorem as follows.

Theorem 7. *There exist positive definite matrices $\mathbf{P}_{\hat{\eta}_i}$ and $\mathbf{Q}_{\hat{\eta}_i}$ satisfying* (13) *and* (16), *nonzero vector $\mathbf{K}_i \in \mathbf{R}^q$ makes $\mathbf{F}_{i0} = \mathbf{F}_i - \mathbf{K}_i\mathbf{H}_i$ be Hurwitz, and* (25) *holds as well. Hence, for the multivariable nonlinear system* (6) *and multiclass exosystem* (7) *satisfying Assumption 1 to Assumption 6, the multiclass nonlinear internal models* (15) *and control inputs* (19) *are able to make the close-loop system globally uniformly hounded, and* $\lim_{t\to\infty}\mathbf{x}(t) = 0$.

Remark 8. Theorem 7 redescribes Definition 5 in essence; furthermore, it provides a feasible way to find a state feedback controller $\mathbf{u}_i$ to stabilize the close-loop system (6) to reference trajectories. In addition, the selection of positive definite matrices $\mathbf{P}_{\hat{\eta}_i}$ and $\mathbf{Q}_{\hat{\eta}_i}$ and nonzero vector $\mathbf{K}_i$ is to design multiclass nonlinear internal models (see (15)) to simulate the external nonharmonic disturbances produced by nonlinear exosystem shown in (7). The control inputs (19) are the state feedback controller described in Definition 5.

5. Numerical Simulation and Analysis

5.1. Description of Simulation Parameters. The verification of the proposed algorithm in the paper is performed by means of numerical simulation in a 0.018 kWh/1.1 kW PMSM based MEES system. The specific parameters of the MEES system are shown as follows: the rating torque of PMSM $T_{\mathrm{m}} = 5.0$ N.m, number of pole-pairs $p = 4$, flux linkage of rotor $\varphi_f = 0.18$ Wb, resistance of stator $R_s = 1.95\,\Omega$, inductances of d-axis and q-axis $L_d = L_q = 0.0115$ H, moment of inertia of rotor $J_m = 0.008$ kg·m^2, ratio of gear case = 40 : 1, the angular velocity of main shaft $\omega_L = 15$ r/min, and damping coefficients of the motor $B_m = 0.01$ N/rad/s.

5.2. Analysis and Discussion of Simulation Results. Considering the multiclass nonlinear disturbances, the mathematical model for the whole system of PMSM based MEES system is converted into the form of (6), and the ultimate result is shown in (27). Equation (27) indicates that the nonlinear model of MEES system is a two-variable input system, which is unable to be dealt with by a single-input algorithm. In addition, (27) includes multiclass nonlinear disturbances $\mathbf{v}_i(\mathbf{w})$, and the rejection algorithm handling a single disturbance cannot address the problem of multiclass disturbances rejection as well:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{2} \mathbf{g}_i(\mathbf{x})\left(\mathbf{u}_i - \mathbf{v}_i(\mathbf{w})\right), \tag{27}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} i_d & \omega_m & i_q \end{bmatrix}^T$,

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -\dfrac{R_s}{L_d}x_1 + px_2x_3 \\ \dfrac{p\varphi_f}{J_m}x_3 - \dfrac{B_m}{J_m}x_2 - \dfrac{1}{J_m}T_m \\ -\dfrac{R_s}{L_q}x_3 - px_2x_1 - \dfrac{p\varphi_f}{L_q}x_2 \end{bmatrix}, \tag{28}$$

$$\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix} \dfrac{1}{L_d} & 0 \end{bmatrix}^T, \qquad \mathbf{g}_2(\mathbf{x}) = \begin{bmatrix} \dfrac{1}{L_q} \end{bmatrix},$$

and the control input $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T = \begin{bmatrix} u_d & u_q \end{bmatrix}^T$.

If the nonlinear disturbances $\mathbf{v}_i(\mathbf{w})$ are ignored in (27), the remaining system in (29) is the nominal system for MEES system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{2} \mathbf{g}_i(\mathbf{x})\,\mathbf{u}_i. \tag{29}$$

For the sake of convenience, the inputs of nonlinear external disturbances v_1 and v_2 are both generated by Van der Pol circuit described in (10) with $\varsigma = 2$, which produces bounded limit cycles. Consequently, Assumption 2 holds.

Suppose that v_1 and v_2 are immersed in the current components of d-axis and q-axis, respectively, and $v_1 = w_1$, $v_2 = w_1 - w_2$. Hence, the difference of the external disturbances represents the fact that the original system isimmersed in multiclass nonlinear disturbance signals.

For v_1, the matrix parameters appearing in (12) are chosen as

$$F_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \qquad J_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix},$$
$$H_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \gamma_1^1(s) = \frac{1}{3}s^3, \qquad \gamma_1^2(s) = 0, \tag{30}$$
$$P_{\hat{\eta}_1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{bmatrix};$$

for $v_2 = w_1 - w_2$, the matrix parameters appearing in (12) are selected as

$$F_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix},$$
$$H_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}, \qquad \gamma_2^1(s) = \frac{1}{3}s^3, \qquad \gamma_2^2(s) = 0, \tag{31}$$
$$P_{\hat{\eta}_2} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{bmatrix},$$

and hence Assumption 6 is satisfied.

Assume that c_1, c_2, and c_3 all are certain positive constants; the control law for the nominal system (29) is designed as

$$\boldsymbol{\alpha}(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\alpha}_1(\mathbf{x}) \\ \boldsymbol{\alpha}_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} -L_d\left(c_1 x_1 + \frac{L_q}{L_d} p x_2 x_3\right) \\ L_q\left(-\frac{p\varphi_f}{J_m} c_2 (x_2 - \omega_{\text{ref}}) + \frac{R_s}{L_q} x_3 + \frac{L_d}{L_q} p x_2 x_1 + \frac{p\varphi_f}{L_q} x_2 \right. \\ \left. -c_3\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right)\right) \end{bmatrix}, \tag{32}$$

where ω_{ref} denotes the angular velocity reference of rotor for PMSM. It can be verified that $\boldsymbol{\alpha}(\mathbf{x})$ can stabilize the nominal system (29) without disturbances, owing to the fact that the stabilization process of (29) is not the emphasis of the paper; hence the detailed deductions are omitted.

Let

$$\mathbf{V}(\mathbf{x}) = \frac{1}{2}x_1^2 + \frac{1}{2}c_2(x_2 - \omega_{\text{ref}})^2 + \frac{1}{2}\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right)^2; \tag{33}$$

after calculations and arrangements, we obtain

$$\begin{aligned} \frac{\partial \mathbf{V}(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^{2}\mathbf{g}_i(\mathbf{x})\boldsymbol{\alpha}_i\right) &= -\left(c_1 + \frac{R_s}{L_d}\right)x_1^2 - \frac{\varphi_f}{J_m}c_2(x_2 - \omega_{\text{ref}})^2 \\ &\quad - c_3\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right)^2 \\ &= -(c_1 + 169.5652)x_1^2 - 225c_2(x_2 - \omega_{\text{ref}})^2 \\ &\quad - c_3\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right)^2, \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{\partial \mathbf{V}(\mathbf{x})}{\partial \mathbf{x}}\sum_{i=1}^{2}\mathbf{g}_i(\mathbf{x}) &= \frac{1}{L_d}x_1 + \frac{1}{L_q}\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right) \\ &= 86.9565x_1 + 86.9565\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right). \end{aligned} \tag{35}$$

Supposing that

$$\mathbf{x}' = \left[x_1(x_2 - \omega_{\text{ref}})\left(x_3 - \frac{J_m}{p\varphi_f}\left(\frac{\varphi_f}{J_m}\omega_{\text{ref}} + \frac{1}{J_m}T_m\right)\right)\right]^T, \tag{36}$$

according to (33), (34), and (35), and choosing $c_1 = 8000$, $c_2 = 40$, and $c_3 = 8000$, we obtain

$$\frac{1}{2}\|\mathbf{x}'\|^2 \le \mathbf{V}(\mathbf{x}) \le 20\|\mathbf{x}'\|^2, \tag{37}$$

$$\frac{\partial \mathbf{V}(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^{2}\mathbf{g}_i(\mathbf{x})\boldsymbol{\alpha}_i\right) \le -7562\|\mathbf{x}'\|^2, \tag{38}$$

$$\left|\frac{\partial \mathbf{V}(\mathbf{x})}{\partial \mathbf{x}}\sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})\right|^2 \le 7562\|\mathbf{x}'\|^2. \tag{39}$$

Hence, Assumption 1 holds.

Choose

$$r_1(\mathbf{x}) = \begin{bmatrix} 9L_d x_1 & 0 \end{bmatrix}^T, \qquad r_2(\mathbf{x}) = \begin{bmatrix} 9L_q x_3 & 0 \end{bmatrix}^T. \tag{40}$$

Consequently,

$$K_1 = \frac{\partial r_1(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix} 9 & 0 \end{bmatrix}^T,$$
$$K_2 = \frac{\partial r_2(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}_2(\mathbf{x}) = \begin{bmatrix} 9 & 0 \end{bmatrix}^T. \tag{41}$$

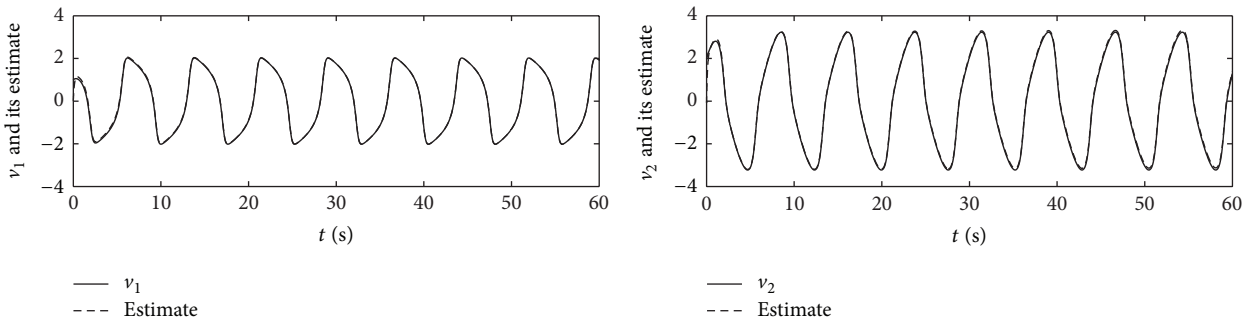

Figure 2: Nonlinear disturbance inputs v and their estimates.

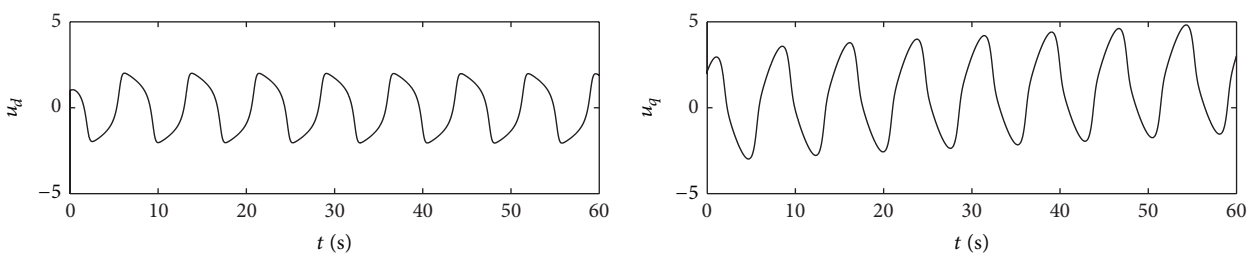

Figure 3: System control inputs u.

Therefore, Assumption 4 is satisfied. In addition, with the application of (41), $\mathbf{F}_{i0}$ and $\mathbf{Q}_{\hat{\eta}_i}$ in (16) can be shown as follows by means of some calculations:

$$F_{10} = \begin{bmatrix} -7 & 1 \\ -1 & 0 \end{bmatrix}, \qquad Q_{\hat{\eta}_1} = \begin{bmatrix} \frac{11}{2} & -10 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix},$$
$$F_{20} = \begin{bmatrix} -7 & 10 \\ -1 & 0 \end{bmatrix}, \qquad Q_{\hat{\eta}_2} = \begin{bmatrix} \frac{11}{2} & -\frac{29}{2} \\ -4 & 15 \end{bmatrix}. \tag{42}$$

The above analysis has verified that MEES system (27) and system (10) of the external disturbances satisfy all the conditions required by Theorem 7. Consequently, based on the multivariable disturbances rejection algorithm proposed in the paper, the multiclass nonlinear internal models and state feedback controller are designed as follows:

$$\begin{aligned}
\dot{\hat{\eta}}_1 &= -7\hat{\eta}_1 + 10\hat{\eta}_2 - 16.8255x_1 + 0.414x_2x_3 \\
&\quad - \frac{2}{3}\left(\hat{\eta}_1 - 0.1035x_1\right)^3 + 9u_1, \\
\dot{\hat{\eta}}_2 &= -\hat{\eta}_1 + 0.1035x_1, \\
\dot{\hat{\eta}}_3 &= -7\hat{\eta}_3 + \hat{\eta}_4 - 16.8255x_3 - 6.48x_2 - 0.414x_2x_1 \\
&\quad - \frac{2}{3}\left(\hat{\eta}_3 - 0.1035x_3\right)^3 + 9u_2, \\
\dot{\hat{\eta}}_4 &= -\hat{\eta}_3 + 0.1035x_3, \\
u_1 &= -0.2185x_1 - 0.046x_2x_3 + \hat{\eta}_1, \\
u_2 &= -90.1535x_3 + 247251.84 + 0.046x_2x_1 \\
&\quad - 413.28x_2 + 127.7788T_m + \hat{\eta}_3 - \hat{\eta}_4.
\end{aligned} \tag{43}$$

The numerical simulations are conducted in Matlab environment. The whole simulation time is set as 60 s with the sampling interval 0.001 s; let the initial condition of the simulation be $x(0) = [0.1 \;\; 0 \;\; 1.0]$, $\hat{\eta}(0) = [0 \;\; 0 \;\; 0 \;\; 0]$, and $w(0) = [1 \;\; -1]$. The reference values of the d-axis current i_d and angular velocity of the motor ω_m are selected as 0 and 600 r/min, respectively. The q-axis current i_q tracks the change of the torque of STS in energy storage. The simulation results are shown in Figures 2, 3, and 4. Figure 2 demonstrates the multiclass nonlinear disturbance inputs and their estimations, from which it can be seen that the multiclass nonlinear disturbances acting on the different state variables in a multivariable input system are successfully estimated relying on the internal models designed. Figure 3 displays the control inputs of the system in dq axis under the existence of multiclass nonlinear external disturbances. Figure 4 describes the system states, which indicates that the system achieves the asymptotical tracking for the reference signals rapidly and the multiclass nonlinear disturbances are completely suppressed. Hence, the multivariable controller designed in the paper has a good control performance.

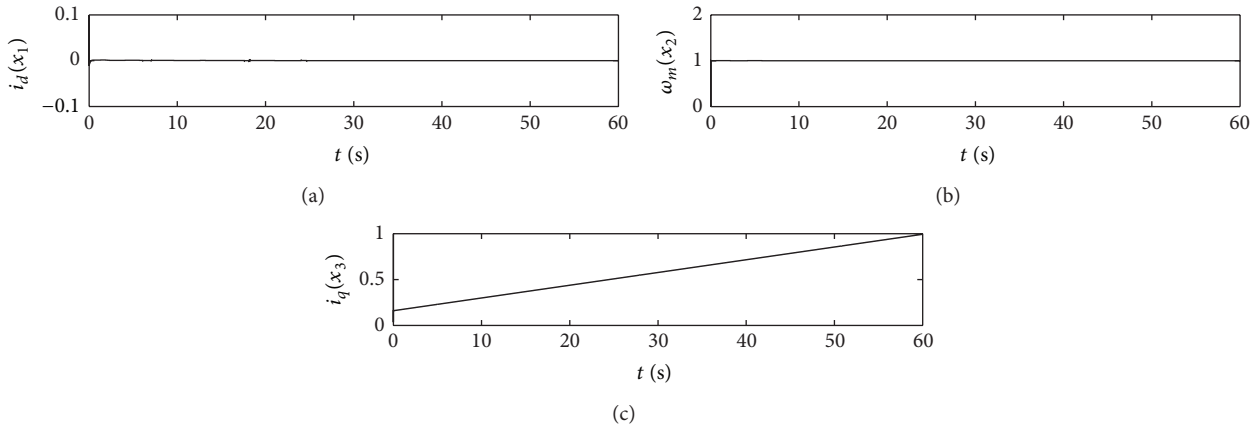

Figure 4: System states x.

6. Conclusion

In light of the strong coupling and nonlinear characteristics of PMSM based MEES system, a global multiclass nonharmonic disturbances rejection method for general multivariable nonlinear system under multiclasses nonlinear exosystems is proposed in the paper. For multiclass nonlinear external disturbances with different periodic bounded nonharmonic characteristics, different nonlinear internal model equations are designed. Based on design of control law for nominal system, a state feedback controller for original system is presented and a Lyapunov function is established to theoretically testify the global boundedness of all signals in multivariable close-loop system. The simulation results show that the multiclass different nonlinear disturbance inputs are all completely rejected and the close-loop system can track the reference signals promptly. Consequently, high accuracy servo control for PMSM based MEES system is realized.

In addition to PMSM based MEES system, many other practical engineering systems, including turbine motor, generator, power flexible manipulator, and communication circuit, are frequently affected by the nonharmonic disturbances generated by the external nonlinear exosystems. As the most famous and typical nonlinear circuit, Van der Pol circuit researched in the paper will excite nonharmonic disturbances and make the system mentioned above produce nonharmonic forced oscillation. The algorithm presented in the paper can eliminate the harmful oscillation and improve the stability for these practical systems.

The critical points of the output regulation problem under nonharmonic disturbances are to model the nonlinear exosystems and propose reasonable algorithm to stabilize the closed-loop system. In the future, the proposed algorithm in the paper can be able to be extended to uncertainly multivariable systems and unknown external signals; correspondingly, the innovative control technologies should be researched to cope with the more complex and generalized circumstances.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Structural Stiffness Identification Based on the Extended Kalman Filter Research

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For the response acquisition of the structure section measuring points, the method of identifying the structural stiffness parameters is developed by using the extended Kalman filter. The state equation of structural system parameter is a nonlinear equation. Dispersing the structural dynamic equation by using Newmark-β method, the state transition matrix of discrete state equation is deduced and the solution of discrete state equation is simplified. The numerical simulation shows that the error of structural recognition doesnot exceed 5% when the noise level is 3%. It meets the requirements of the error limit of the engineering structure, which indicates that the derivation described in this paper has the robustness for the structural stiffness recognition. Shear structure parameter identification examples illustrate its applicability, and the method can also be used to identify physical parameters of large structure.

1. Introduction

The dynamic response of structure which is developed in recent years is used to recognize the structural damage. This method is based on the structural dynamic parameter. The civil engineering structure will accumulate damage because of the collision, environment corrosion, material aging, long-term effects of load, and the fatigue. The local damage will lead to the destruction of the whole structure, which will result in serious engineering accident. The diagnostic techniques of structural damage have been studied since the 1970s in order to guarantee the structural safety and reduce the economic loss [1]. The aim of structural damage recognition is to find the position and the degree of structural damage, which provides foundation for the follow-up assessment of structural safety [2]. Damage identification is based on structural vibration, and the basic principle is structural modal parameter (natural frequency, mode shape, etc.) as the function of the structural physical characteristics (mass, damping, and stiffness), and so the change of physical characteristics will cause the change in system dynamic response [3–5].

Another important property which the ideal damage identification method should have is to be able to distinguish the differences of the two deviations caused by structural modeling error and structural damage. How to explain the structural security status and damage degrees by virtue of the information from measurements is still a scientific theory which is to be improved. The structural damage identification method based on the changes of vibration characteristics has been adopted to research for decades. Because the structural vibration modal parameters (such as frequency, mode shapes, and modal damping) are the functions of structural physical parameters (such as mass, stiffness, and damping), changes in the structural physical parameters will inevitably lead to change in structural vibration modal parameters, which is the basic principle of structural damage identification. Damage identification is usually divided into three levels: to judge the occurrence of damage; to determine the location of damage; to solve the extent of damage [6–9]. The early damage identification method generally determines the occurrence of damage by the changes of the frequency before and after the damage. Later, it has been gradually developed by using various modal testing information (such as displacement mode,

strain mode, and frequency response function) for accurate damage positioning and measurement [10]. The structural damage identification techniques have been combined with modern modal measurement and modern numerical analysis method, and they are playing an important role in the field of civil engineering [11–14].

In terms of algorithms, it usually takes optimization [15, 16] or intelligent algorithm [17–19] and other methods to determine the degree of structural damage. Uncertainty widely exists in practical engineering, and the theory and algorithm of uncertain optimization research is significant for the system. In the optimization method of uncertainty, many studies have been done for the uncertainty analysis and solution strategy [20–23]. Due to the fact that there is a lot of uncertain information during the research process of geotechnical engineering, it is difficult for deterministic models to conclude the complicated mechanical property of geotechnical engineering. Data [24] develops a variety of nondeterministic methods on the basis of deterministic back analysis. As the neural network can reflect any nonlinear systems without knowing the nonlinear physical properties of systems, nonlinear dynamical systems of nonparametric research based on the neural network are increasingly developing [25–28]. Data [29] introduces the principle of SDLV and puts forward the precise SDLV damage localization method based on the success of rod damage identification. As for the unreliable results of structural damage identification caused by large and complex structures and a serious shortage of measurement information, data [30] comes up with damage identification methods of partial main frequency substructure. There are noises in both structural model and measurement response which lead to the numerical instability of structural damage identification. Tikhonov regularization method is a common method to improve ill-conditioned matrix. By introducing a smooth function to Tikhonov penalty function, data [31] improves the impact of noise on structural damage. It is an effective way to study the effects of noise on structural damage identification by means of probability. Data [32] presents the information fusion techniques based on Bayesian theory, which is used to improve the accuracy of structural damage identification results. Structural monitoring can only monitor partial measuring points, while the random damage locating vector method can produce better recognition results for truss bridges and steel frame structures [33]. For the problems of inaccurate damage identification of symmetric structure, data [34] proposes the theory of mobile additional mass to change symmetry of the structure.

Physical parameters identification is one of the main research contents of structural health monitoring. According to the change of physical parameters, especially the stiffness, we are able to identify the structure damage, as well as the damage degree and location. In this paper, the natural excitation technique and the extended Kalman filter algorithm are used in shear structure by adopting time domain identification method, and a new method of physical parameter identification based on environmental excitation is put forward to identify the interlaminar stiffness of the shear structure. Numerical simulation results show that the proposed method can well identify structural parameters. With the increase of noise level, convergence time of identified value to the true value elongates and error increases gradually but within the acceptable scope of the project, which shows algorithm has certain robustness to noise.

2. EKF Principle of Structural Stiffness Identification

The structural equation of motion under seismic excitations can be expressed as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}(\boldsymbol{\theta})\,\mathbf{x} = -\mathbf{M}\ddot{\mathbf{x}}_g, \tag{1}$$

$\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}(\boldsymbol{\theta})$ represent mass matrix of $n \times n$ dimensional, damping matrix, and stiffness matrix. Damping matrix $\mathbf{C}$ uses Rayleigh damping; $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$, and $\mathbf{x}$ are the acceleration of the structure, speed, and displacement response; $\ddot{\mathbf{x}}_g$ is ground motion acceleration; n is the degree of structure freedom; $\boldsymbol{\theta}$ is structural stiffness parameters to be identified, which dimension is m.

At the time of k and $k+1$,

$$\mathbf{M}\ddot{\mathbf{x}}_k + \mathbf{C}\dot{\mathbf{x}}_k + \mathbf{K}(\boldsymbol{\theta})\,\mathbf{x}_k = \mathbf{F}_k, \tag{2}$$

$$\mathbf{M}\ddot{\mathbf{x}}_{k+1} + \mathbf{C}\dot{\mathbf{x}}_{k+1} + \mathbf{K}(\boldsymbol{\theta})\,\mathbf{x}_{k+1} = \mathbf{F}_{k+1}. \tag{3}$$

According to Newmark-β method, at the time of $k+1$, velocity and displacement can be expressed as

$$\begin{aligned} \dot{\mathbf{x}}_{k+1} &= \dot{\mathbf{x}}_k + \frac{\Delta t}{2}\left(\ddot{\mathbf{x}}_k + \ddot{\mathbf{x}}_{k+1}\right), \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \dot{\mathbf{x}}_k \Delta t + \frac{\Delta t^2}{4}\left(\ddot{\mathbf{x}}_k + \ddot{\mathbf{x}}_{k+1}\right), \end{aligned} \tag{4}$$

Δt is discrete time interval.

Substituting (4) into (3) and deducing $\ddot{\mathbf{x}}_{k+1}$, we can deduce the following:

$$\ddot{\mathbf{x}}_{k+1} = \mathbf{A}_1\left[\mathbf{F}_{k+1} - \mathbf{K}(\boldsymbol{\theta})\,\mathbf{x}_k - \mathbf{A}_2\dot{\mathbf{x}}_k - \mathbf{A}_3\ddot{\mathbf{x}}_k\right], \tag{5}$$

$$\begin{aligned} \mathbf{A}_1 &= \left(\frac{\Delta t^2}{4}\mathbf{K}(\boldsymbol{\theta}) + \frac{\Delta t}{2}\mathbf{C} + \mathbf{M}\right)^{-1}, \\ \mathbf{A}_2 &= \left(\Delta t\mathbf{K}(\boldsymbol{\theta}) + \mathbf{C}\right), \\ \mathbf{A}_3 &= \left(\frac{\Delta t^2}{4}\mathbf{K}(\boldsymbol{\theta}) + \frac{\Delta t}{2}\mathbf{C}\right), \end{aligned} \tag{6}$$

$\ddot{\mathbf{x}}_k$ can be obtained through equation (2) as follows:

$$\ddot{\mathbf{x}}_k = \mathbf{M}^{-1}\mathbf{F}_k - \mathbf{M}^{-1}\mathbf{K}(\boldsymbol{\theta})\,\mathbf{x}_k - \mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{x}}_k. \tag{7}$$

Combining (5) and (7), we can deduce the following:

$$\begin{aligned} \left(\ddot{\mathbf{x}}_{k+1} + \ddot{\mathbf{x}}_k\right) &= \mathbf{A}_1\mathbf{F}_{k+1} - \mathbf{A}_1\mathbf{K}(\boldsymbol{\theta})\,\mathbf{x}_k - \mathbf{A}_1\mathbf{A}_2\dot{\mathbf{x}}_k \\ &\quad - \left(\mathbf{A}_1\mathbf{A}_3 - \mathbf{I}\right)\ddot{\mathbf{x}}_k \\ &= \mathbf{a}_k^f + \mathbf{a}_k^d\mathbf{x}_k + \mathbf{a}_k^v\dot{\mathbf{x}}_k, \end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{a}_k^f &= \mathbf{A}_1\mathbf{F}_{k+1} - [\mathbf{A}_1\mathbf{A}_3 - \mathbf{I}]\,\mathbf{M}^{-1}\mathbf{F}_k,\\ \mathbf{a}_k^d &= -\mathbf{A}_1\mathbf{K}(\boldsymbol{\theta}) + [\mathbf{A}_1\mathbf{A}_3 - \mathbf{I}]\,\mathbf{M}^{-1}\mathbf{K}(\boldsymbol{\theta}),\\ \mathbf{a}_k^v &= -\mathbf{A}_1\mathbf{A}_2 + [\mathbf{A}_1\mathbf{A}_3 - \mathbf{I}]\,\mathbf{M}^{-1}\mathbf{C}.\end{aligned} \tag{9}$$

Substituting (8) into (4), we can deduce the following formula:

$$\begin{aligned}\dot{\mathbf{x}}_{k+1} &= \dot{\mathbf{x}}_k + \frac{\Delta t}{2}\left(\mathbf{a}_k^f + \mathbf{a}_k^d\mathbf{x}_k + \mathbf{a}_k^v\dot{\mathbf{x}}_k\right)\\ &= \frac{\Delta t}{2}\mathbf{a}_k^f + \frac{\Delta t}{2}\mathbf{a}_k^d\mathbf{x}_k + \left(\frac{\Delta t}{2}\mathbf{a}_k^v + \mathbf{I}\right)\dot{\mathbf{x}}_k,\\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \dot{\mathbf{x}}_k\Delta t + \frac{\Delta t^2}{4}\left(\mathbf{a}_k^f + \mathbf{a}_k^d\mathbf{x}_k + \mathbf{a}_k^v\dot{\mathbf{x}}_k\right)\\ &= \frac{\Delta t^2}{4}\mathbf{a}_k^f + \left(\frac{\Delta t^2}{4}\mathbf{a}_k^d + \mathbf{I}\right)\mathbf{x}_k + \left(\frac{\Delta t^2}{4}\mathbf{a}_k^v + \Delta t\mathbf{I}\right)\dot{\mathbf{x}}_k.\end{aligned} \tag{10}$$

Write formula (10) in the matrix form

$$\begin{aligned}\begin{Bmatrix}\mathbf{x}_{k+1}\\ \dot{\mathbf{x}}_{k+1}\end{Bmatrix} &= \begin{bmatrix}\left(\frac{\Delta t^2}{4}\mathbf{a}_k^d + \mathbf{I}\right) & \left(\frac{\Delta t^2}{4}\mathbf{a}_k^v + \Delta t\mathbf{I}\right)\\ \frac{\Delta t}{2}\mathbf{a}_k^d & \left(\frac{\Delta t}{2}\mathbf{a}_k^v + \mathbf{I}\right)\end{bmatrix}\begin{Bmatrix}\mathbf{x}_k\\ \dot{\mathbf{x}}_k\end{Bmatrix}\\ &\quad + \begin{Bmatrix}\frac{\Delta t^2}{4}\mathbf{I}\\ \frac{\Delta t}{2}\mathbf{I}\end{Bmatrix}\mathbf{a}_k^f,\end{aligned} \tag{11}$$

$\mathbf{I}$ is the unit matrix of $n \times n$.

Using the method of Newmark-β, we can transform formula (1) into discrete equation (11).

Let

$$\mathbf{y}_k = \{\mathbf{x}_k \;\; \dot{\mathbf{x}}_k \;\; \boldsymbol{\theta}_k\}^T. \tag{12}$$

Then structural equation of state can be rewritten as

$$\begin{aligned}\mathbf{y}_{k+1} &= \begin{bmatrix}\left(\frac{\Delta t^2}{4}\mathbf{a}_k^d(\boldsymbol{\theta}) + \mathbf{I}\right) & \left(\frac{\Delta t^2}{4}\mathbf{a}_k^v(\boldsymbol{\theta}) + \Delta t\mathbf{I}\right) & \mathbf{0}_{n\times m}\\ \frac{\Delta t}{2}\mathbf{a}_k^d(\boldsymbol{\theta}) & \left(\frac{\Delta t}{2}\mathbf{a}_k^v(\boldsymbol{\theta}) + \mathbf{I}\right) & \mathbf{0}_{n\times m}\\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m}\end{bmatrix}\mathbf{y}_k\\ &\quad + \begin{Bmatrix}\frac{\Delta t^2}{4}\mathbf{I}\\ \frac{\Delta t}{2}\mathbf{I}\\ \mathbf{0}_{m\times n}\end{Bmatrix}\mathbf{a}_k^f.\end{aligned} \tag{13}$$

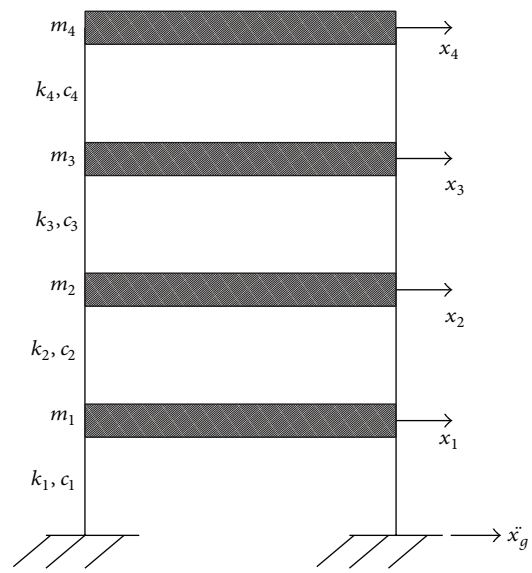


Figure 1: Structure diagram.

Table 1: Structural parameters.

i	1	2	3	4
m_i (kg)	900	675	675	450
k_i (Nm^{-1})	12000	11000	10000	9000

Let

$$\begin{aligned}\boldsymbol{\Phi}_k &= \begin{bmatrix}\left(\frac{\Delta t^2}{4}\mathbf{a}_k^d(\boldsymbol{\theta}) + \mathbf{I}\right) & \left(\frac{\Delta t^2}{4}\mathbf{a}_k^v(\boldsymbol{\theta}) + \Delta t\mathbf{I}\right) & \mathbf{0}_{n\times m}\\ \frac{\Delta t}{2}\mathbf{a}_k^d(\boldsymbol{\theta}) & \left(\frac{\Delta t}{2}\mathbf{a}_k^v(\boldsymbol{\theta}) + \mathbf{I}\right) & \mathbf{0}_{n\times m}\\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m}\end{bmatrix},\\ \boldsymbol{\Gamma}_k &= \left\{\frac{\Delta t^2}{4}\mathbf{I} \;\; \frac{\Delta t}{2}\mathbf{I} \;\; \mathbf{0}_{m\times n}\right\}^T.\end{aligned} \tag{14}$$

If the process noise exists in the system, (13) can be rewritten as

$$\mathbf{y}_{k+1} = \boldsymbol{\Phi}_k\mathbf{y}_k + \boldsymbol{\Gamma}_k\boldsymbol{\alpha}_k^f + \boldsymbol{\omega}_k, \tag{15}$$

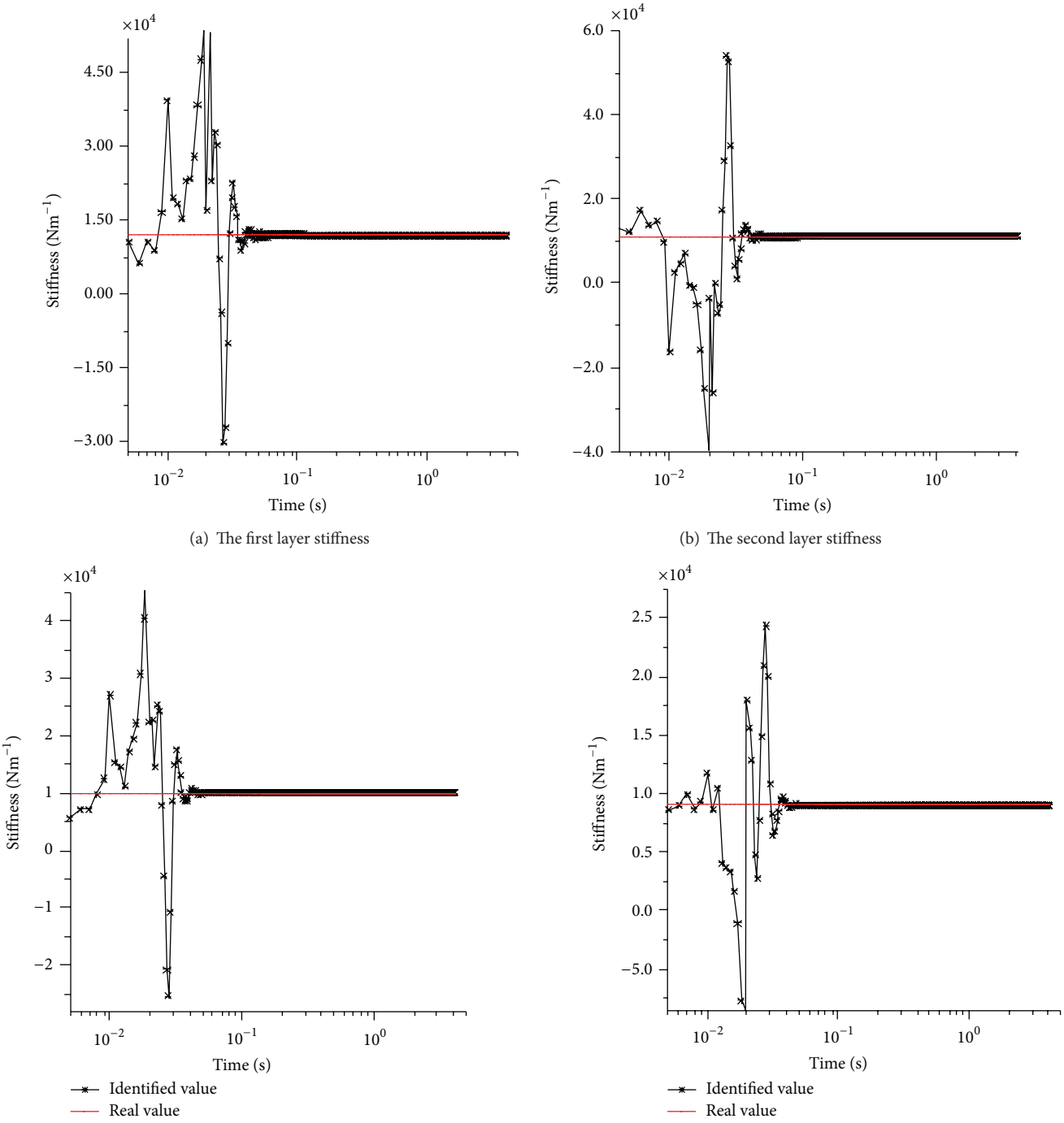
$\boldsymbol{\omega}_k$ is the process noise of system.

The supplementary system observation equation is

$$\mathbf{z}_{k+1} = \mathbf{H}_k\mathbf{y}_k + \boldsymbol{\upsilon}_k, \tag{16}$$

$\mathbf{H}_k$ is the observation matrix of system and $\boldsymbol{\upsilon}_k$ is the observation noise of the system.

Assuming that process noise and observation noise are independent of each other, the covariance matrix of process noise and observation noise is $\mathbf{N} = \mathbf{0}$. Given the system initial value $\mathbf{y}_0$, the initial value of process noise covariance matrix $\mathbf{P}_0$, and observation noise covariance matrix $\mathbf{R}$, discrete augmented state vector $\widehat{\mathbf{y}}_{k+1}$ and covariance matrix $\widehat{\mathbf{P}}_{k+1}$ can be estimated according to the extended Kalman filter method.



(a) The first layer stiffness

(b) The second layer stiffness

(c) The third layer stiffness

(d) The fourth layer stiffness

Figure 2: Structural stiffness identification result without noise.

(1) According to formula (15), calculate and predict the state vector $\widetilde{\mathbf{y}}_{k+1}$.

(2) Calculation of covariance prediction equation is

$$\widetilde{\mathbf{P}}_{k+1} = \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k^T. \tag{17}$$

(3) The gain matrix is

$$\mathbf{K}_{k+1} = \widetilde{\mathbf{P}}_{k+1} \mathbf{H}_{k+1}^T \left(\mathbf{H}_{k+1} \widetilde{\mathbf{P}}_{k+1} \mathbf{H}_{k+1}^T + \mathbf{R}\right)^{-1}. \tag{18}$$

(4) State filtering equation is

$$\widehat{\mathbf{x}}_{k+1} = \widetilde{\mathbf{x}}_{k+1} + \mathbf{K}_{k+1} \left(\mathbf{z}_{k+1} - \mathbf{H}_{k+1} \widetilde{\mathbf{x}}_{k+1}\right). \tag{19}$$

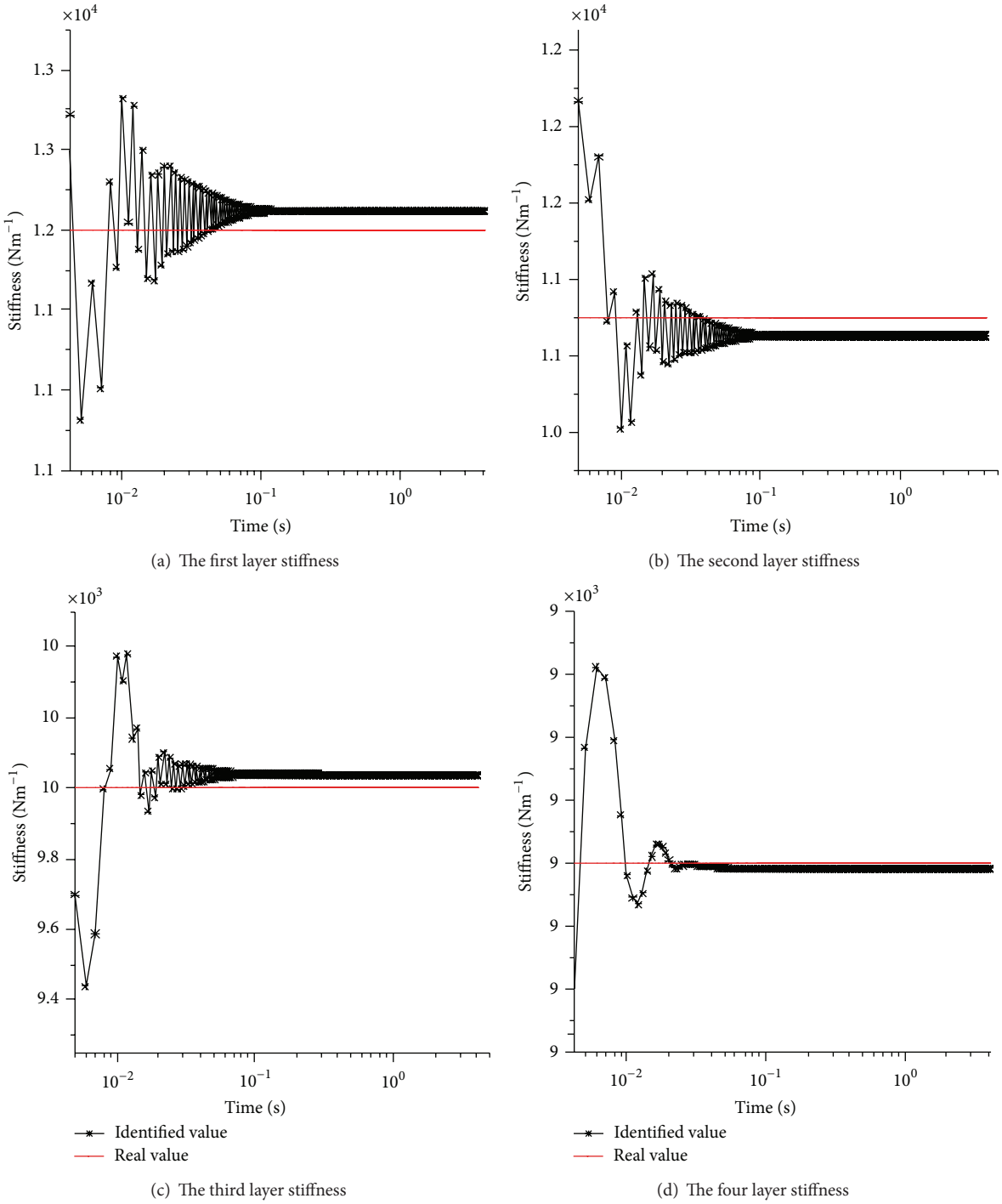


Figure 3: Stiffness identification results when SNR is 40.

(5) Error covariance filtering equation is

$$\mathbf{P}_{k+1} = \widetilde{\mathbf{P}}_{k+1} - \mathbf{K}_{k+1}\mathbf{H}_{k+1}\widetilde{\mathbf{P}}_{k+1}. \tag{20}$$

3. The Numerical Simulation

To validate the effectiveness of the algorithm, a four-story shear structure is considered, as shown in Figure 1. Structure damping is the Rayleigh damping; quality factor is 6.984×10^{-3} and stiffness coefficient is 9.390×10^{-4}; the rest parameters of the structure are shown in Table 1. The input of basement is elecentro wave. Use the method of time-history analysis to calculate structural response and gather each layer displacement response as measurements to identify stiffness between the layers.

Three working conditions in this paper are considered, which are measurements without noise, measurements with 1% of noise, and measurements with 3% of noise. All the

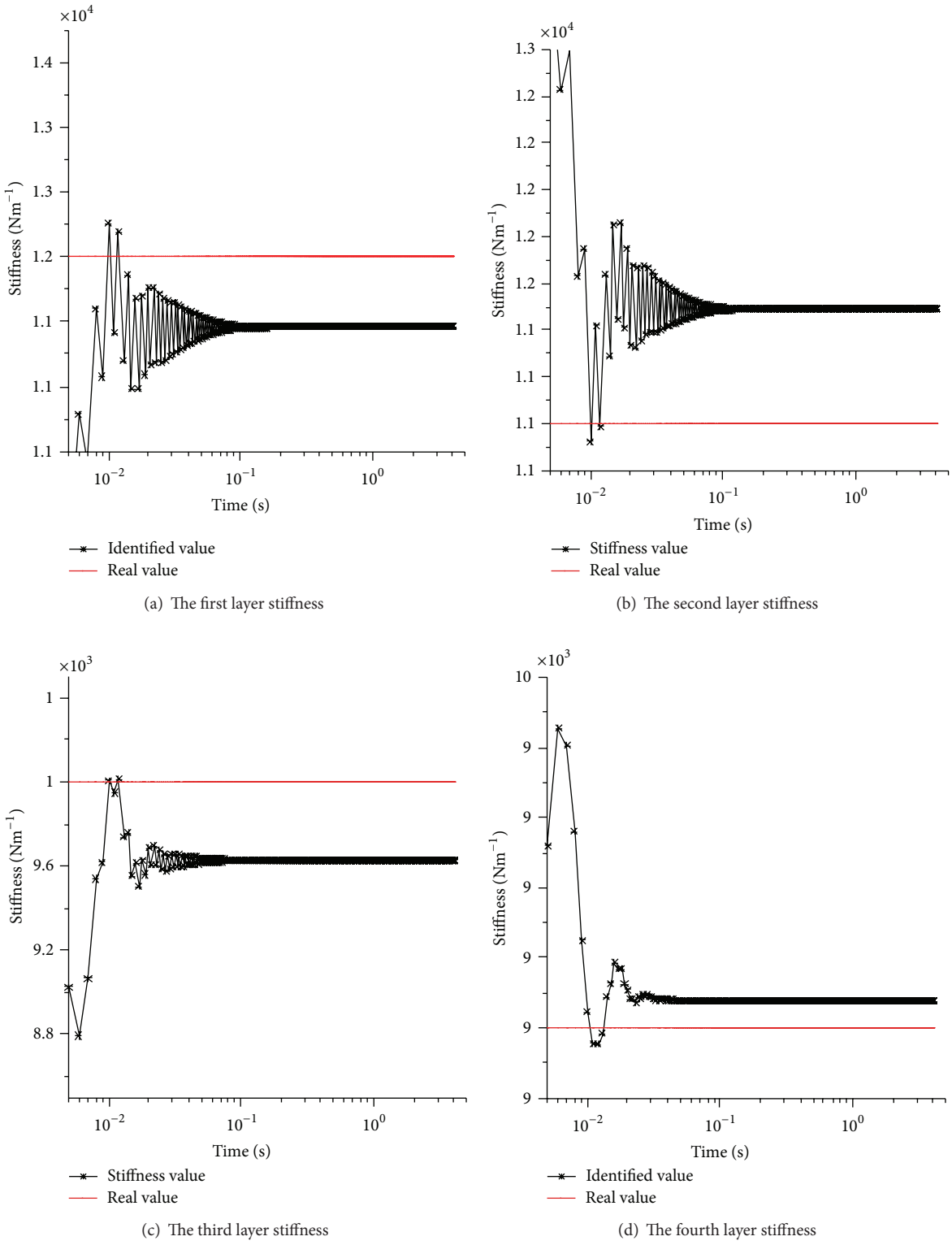


Figure 4: Stiffness identification results when SNR is 30.

noises are zero mean white noise. Figure 2 shows the structure stiffness identification value without noise, from which we can see that identified value of stiffness converges to the true value quickly. Figure 3 shows the identification result of measurements with 1% of noise. Due to the effect of the noise, there is a certain error between identified value and true value. Table 2 shows the error level; the maximum error is within the 1% with the situation of 1% of noise. Figure 4

Table 2: Structural stiffness identification results.

Stiffness	Noise level					
	No noise	Error level %	SNR = 40	Error level %	SNR = 30	Error level %
K_1	12000	0	12118	0.98	11459	−4.51
K_2	11000	0	10904	−0.87	11491	4.46
K_3	10000	0	10039	0.39	9628	−3.72
K_4	9000	0	8991	−0.10	9038	0.42

shows the measurements with 3% of noise. From Table 2, we can see that the maximum error is 4.6%, and it is within the acceptable range of the engineering.

4. Conclusion

The paper using the method of Newmark-β disperses the equations of motion and deduces the state transition equation containing the stiffness parameters to be identified. By the extended Kalman filter algorithm to identify the stiffness parameters, the conclusion can be drown as follows.

(i) Expanding order state equation is nonlinear state equation of state variables, and the state transition matrix is generally through the partial derivative of the nonlinear equation, which solving process is complicated. This paper directly deduces the state transition matrix by Newmark-β method, and the result is concise and intuitive.

(ii) Noise affects the identification precision. When there is no noise in acquisition response, the identification stiffness converges to the true stiffness precisely. With the noise increases, the error of identification precision increases too. When the SNR amounts to forty, the error is within 1%, and, when the SAR is thirty, the error is within 5%. The speed of error increasing surpasses that of noise increasing.

(iii) Noise affects the speed of identification. When there is no noise in the acquisition response, the algorithm convergence accesses the truth value. And, when noise increases, the speed of algorithm convergence becomes slow. In the practical condition that noise has affected acquisition response, the time length of acquisition should be guaranteed so that the algorithm converges to the stable value, which can ensure reliable identification results.

(iv) From the results of numerical simulation, the proposed algorithm has different identification precisions to the various layers of shear structure, which is in the increasing condition from the bottom to the top. The incentive of structure is under the affection of earthquake, and greater response of structural layer can be obtained from the higher ground; the different responses between the layers will affect algorithm identification precision.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Finite-Time Control for Markovian Jump Systems with Polytopic Uncertain Transition Description and Actuator Saturation

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The problem of finite-time L_2-L_∞ control for Markovian jump systems (MJS) is investigated. The systems considered time-varying delays, actuator saturation, and polytopic uncertain transition description. The purpose of this paper is to design a state feedback controller such that the system is finite-time bounded (FTB) and a prescribed L_2-L_∞ disturbance attenuation level during a specified time interval is guaranteed. Based on the Lyapunov method, a linear matrix inequality (LMI) optimization problem is formulated to design the delayed feedback controller which satisfies the given attenuation level. Finally, illustrative examples show that the proposed conditions are effective for the design of robust state feedback controller.

1. Introduction

In the aspect of modeling practical systems with abrupt random changes, such as manufacturing system, telecommunication, and economic systems, MJS have powerful ability. MJS have been extensively studied during the past decades and many systematic results have been obtained [1–3]. The peak-to-peak filtering problem was studied for a class of Markov jump systems with uncertain parameters in [4]. A robust H_2 state feedback controller for continuous-time Markov jump linear systems subject to polytopic-type parameter uncertainty was designed in [5]. In [6], the authors address the stabilization problem for single-input Markov jump linear systems via mode-dependent quantized state feedback for control.

Actuator saturation which can lead to poor performance of the closed-loop system is another active research area. In practical situations, it may be encountered sometimes. How to preserve the closed-loop system performance in the case of actuator saturation would be more meaningful. In [7], the H_∞ control problem for discrete-time singular Markov jump systems with actuator saturation was considered. In [8] the stochastic stabilization problem for a class of Markov jump linear systems subject to actuator saturation was considered.

In some practical applications, the behavior of the system over a finite-time interval is mainly considered. Finite-time stable (FTS) and Lyapunov asymptotic stability are independent concepts. The concept of FTS was first introduced in [9]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. FTS of linear time-varying systems was considered in [10]. Sufficient conditions for the solvability of both the state and the output feedback problems are stated. Amato [11] provided a necessary and sufficient condition for the FTS of linear-varying systems with jumps. Recently, robust finite-time H_∞ control of jump systems was dealt with in [12–14]. In [15], the problems of finite-time stability analysis were investigated for a class of Markovian switching stochastic systems. To the best of authors' knowledge, however, the problem of finite-time L_2-L_∞ performance for discrete-time MJS with imprecise transition probabilities and time-varying delays has not been well addressed, which motivates our work.

This paper deals with this problem. More specifically, the actuator is saturation. By using the Lyapunov-Krasovskii functional, a new sufficient condition for stochastic asymptotic stability with finite-time L_2-L_∞ performance is derived in terms of LMI. Based on this, the existence condition of

the desired performance which guarantees finite-time stability and an L_2-L_∞ performance of the MJS is presented. A numerical example is provided to show the effectiveness of the proposed results.

Throughout the paper, if not explicitly stated, matrices are assumed to have compatible dimensions. The notation $W > (\geq, <, \leq) 0$ is used to denote a symmetric positive definite (positive semidefinite, negative, negative semidefinite) matrix. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ represent the minimum and maximum eigenvalues of the corresponding matrix, respectively. I is the identity matrix with compatible dimensions. $\|\cdot\|$ refers to the Euclidean norm of vectors and $E[\cdot]$ stands for the mathematical expectation. For a symmetric block matrix, "$*$" is used as an ellipsis for the terms that are obtained by symmetry.

2. Problem Statement and Preliminaries

Consider a discrete-time MJS with actuator saturation and delay in the state. Let the system dynamics be described by the following:

$$\begin{aligned} x(k+1) &= A_{\theta 1}(r_k)\, x(k) + A_{\theta 2}(r_k)\, x(k-d) \\ &\quad + B_{\theta 1}(r_k)\, \sigma(u_k) + B_{\theta 2}(r_k)\, w_k, \\ z(k) &= C_{\theta 1}(r_k)\, x(k) + C_{\theta 2}(r_k)\, x(k-d) + D_{\theta 1}(r_k)\, w_k, \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the system state, $z_k \in R^n$ is the system output, $u_k \in R^m$ is the control input, $w_k \in R^q$ is the disturbance input which belongs to $L_2[0,\infty)$ and $\sum_{k=0}^{\infty} w_k^T w_k < \kappa^2$, and κ is a given positive scalar. $A_{\theta 1}(r_k), A_{\theta 2}(r_k), B_{\theta 1}(r_k), B_{\theta 2}(r_k), C_{\theta 1}(r_k), D_{\theta 1}(r_k)$, and $D_{\theta 2}(r_k)$ are appropriately dimensioned real-valued matrices, which belong to the part of convex polyhedron $\Phi(r_k)$:

$$\begin{aligned} &\Phi(r_k) \\ &= \Bigg\{ \sum_{l=1}^{L} \theta_l \left[A_{l1}(r_k), A_{l2}(r_k), B_{l1}(r_k), \right. \\ &\qquad B_{l2}(r_k), C_{l1}(r_k), C_{l2}(r_k), \\ &\qquad \left. D_{l1}(r_k), D_{l2}(r_k) \right], \sum_{l=1}^{L} \theta_l = 1, \theta_l \geq 0 \Bigg\}, \end{aligned} \tag{2}$$

where $A_{l1}(r_k), A_{l2}(r_k), B_{l1}(r_k), B_{l2}(r_k), C_{l1}(r_k), C_{l2}(r_k)$, and $D_{l1}(r_k)$ are matrix functions of the random jumping process $\{r_k\}$ (Figure 1), which is a discrete-time Markov chain taking values in a finite set $\Omega = \{1, 2, \ldots, S\}$ with transition probabilities:

$$P\{r_{k+1} = j \mid r_k = i\} = \pi_{ij}. \tag{3}$$

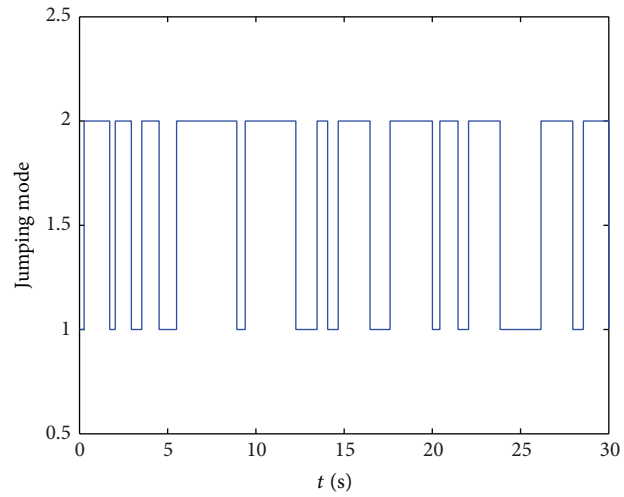

Figure 1: Jumping mode.

Here $\pi_{ij} \geq 0$ and for any $i, j \in \Omega$, $\sum_{j=1}^{s} \pi_{ij} = 1$. Assuming that the transition probability π_{ij} is not exactly known, a certain range can only be given

$$\begin{aligned} &[\pi_v(i,1), \pi_v(i,2), \ldots, \pi_v(i,S)] \\ &\quad = \sum_{m=1}^{M} v_m [\pi_m(i,1), \pi_m(i,2), \ldots, \pi_m(i,S)], \end{aligned} \tag{4}$$

where $v = [v_1 \cdots v_M]^T \in R^M$ and $\sum_{m=1}^{M} v_m = 1$, and the transition probability belongs to the following convex polyhedron:

$$\aleph(r_k = i) = \mathrm{Co}\left\{ \begin{matrix} [\pi_1(i,1), \pi_1(i,2), \ldots, \pi_1(i,N)] \\ [\pi_M(i,1), \pi_M(i,2), \ldots, \pi_M(i,N)] \end{matrix} \right\}. \tag{5}$$

When the system operates in the ith mode ($r_k = i$), for simplicity, the matrices $A_{\theta 1}(r_k), A_{\theta 2}(r_k), B_{\theta 1}(r_k), B_{\theta 2}(r_k), C_{\theta 1}(r_k)$, and $D_{\theta 1}(r_k)$ are denoted as $A_{\theta 1i}, A_{\theta 2i}, B_{\theta 1i}, B_{\theta 2i}, C_{\theta 1i}$, and $D_{\theta 1i}$, respectively. d is a positive integer denoting the constant delay of the system state (Figures 2 and 3).

In system (1), $\sigma(\cdot) : R^m \to R^m$ is the vector-valued standard saturation function defined as follows:

$$\sigma(u) = [\sigma(u_1), \sigma(u_2), \ldots, \sigma(u_m)]^T, \tag{6}$$

where $\sigma(u_\theta) = \mathrm{sign}(u_\theta) \min\{1, |u_\theta|\}$. It is assumed that system (1) is completely controllable. A mode-dependent controller is considered here with the following form:

$$\sigma(u(k)) = \sigma(K_i x(k)), \tag{7}$$

where $K_i \in R^{m \times n}$ ($\forall r_k = i \in \Omega$) is the controller gain to be determined.

Let M be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose each element of M is M_j, $j = 1, \ldots, 2^m$, and denote $M_j^- = I - M_j$. Note that M_j^- is also an element of M if $M_j \in M$. Let h_{ij} be the jth row of the matrix H_i, and define the symmetric polyhedron by $\varphi(H_i) = \{x(t) \in R^n : |f_{ij} x(t)| \leq 1, i = 1, 2, \ldots, m\}$.

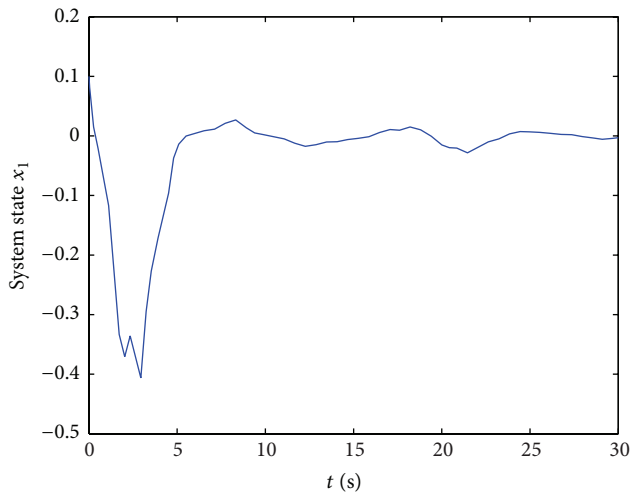

FIGURE 2: Response of the system state x_1.

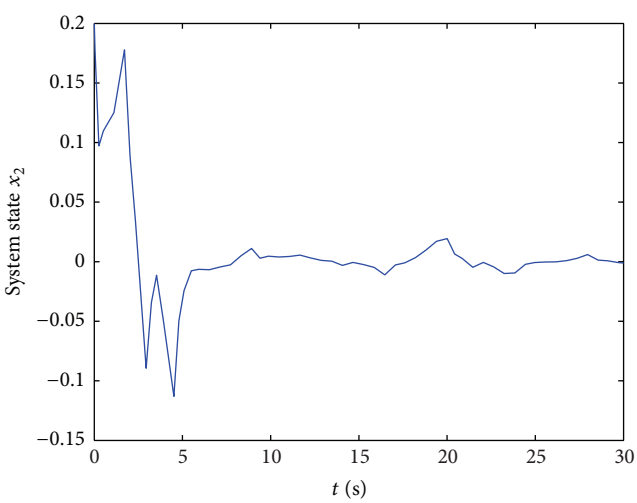

FIGURE 3: Response of the system state x_2.

Lemma 1 (see [8]). *Let $K_i, H_i \in R^{m\times n}$ be given matrix. For $x(t) \in R^n$, if $x(t) \in \varphi(H_i)$, then*

$$\sigma\left(K_i x(t)\right) = \sum_{r=1}^{2^m} \zeta_r \left(M_r K_i + M_r^- H_i\right) x(t), \tag{8}$$

where $0 \le \zeta_r \le 1$, $\sum_{r=1}^{2^m} \zeta_r = 1$.

By the connection of (6), (7) and (8), the following closed-loop MJS are obtained:

$$\begin{aligned} x(k+1) = {} & \Bigg(A_{\theta 1}(r_k) + B_{\theta 1}(r_k) \\ & \times \sum_{r=1}^{2^m} \zeta_r \left(M_r K_i + M_r^- H_i\right) \Bigg) x(k) \\ & + A_{\theta 2}(r_k) x(k-d) + B_{\theta 2}(r_k) w_k. \end{aligned} \tag{9}$$

To describe the main objective of this note more precisely, let us now introduce the following definition for the underlying system.

Definition 2 (see [13]). Given a time constant $T > 0$, the MJS

$$x(k+1) = A_{\theta 1}(r_k) x(k) + A_{\theta 2}(r_k) x(k-d) \tag{10}$$

are said to be FTS with respect to $(\hbar_1 \ \ \hbar_2 \ \ T \ \ R_i)$, if there exist positive matrix $R_i > 0$, scalars $\hbar_1 > 0$ and $\hbar_2 > 0$, and

$$\begin{aligned} & E\left\{x^T(k_1) R_i x(k_1)\right\} \\ & \quad \le \hbar_1 \Longrightarrow E\left\{x^T(k_2) R_i x(k_2)\right\} \\ & \quad \le \hbar_2, \quad k_1 \in \{-h, \ldots, 0\}, \ k_2 \in \{1, 2, \ldots, T\}. \end{aligned} \tag{11}$$

Definition 3 (see [13]). Given a time constant $T > 0$, the MJS

$$x(k+1) = A_{\theta 1}(r_k) x(k) + A_{\theta 2}(r_k) x(k-d) + B_{\theta 2}(r_k) w_k \tag{12}$$

are said to be finite-time bounded (FTB) with respect to $(\hbar_1 \ \ \hbar_2 \ \ T \ \ R_i)$, if there exist positive matrix $R_i > 0$ and scalars $\hbar_1 > 0$ and $\hbar_2 > 0$, and satisfied (11).

In general, FTB and FTS are different. If there is external disturbance in systems, the concept of FTB is used. Conversely, FTS is addressed.

The objective of this paper is to design a delayed feedback controller which satisfies the given attenuation level of system (1). The design procedure is given in the next section.

Definition 4. The time-delay MJS (1) is said to be finite-time L_2-L_∞ control with respect to $(\hbar_1 \ \ \hbar_2 \ \ T \ \ R_i)$ and performance γ, where $R_i > 0$, $\gamma > 0$, $\hbar_1 > 0$, and $\hbar_2 > 0$, if the time-delay MJS (1) is stochastically FTB and under the zero-initial condition the output $z(k)$ satisfies

$$\|z(k)\|_\infty < \gamma \|w(k)\|_2 \tag{13}$$

for all nonzero $w(k) \in L_2[0, \infty)$ subject to the zero-initial condition.

3. Main Results

In this section, firstly stochastic FTB analysis of nominal time-delay MJS (1) is provided. Then, these results will be extended to the MJS (1) with actuator saturation and uncertain transition probability. LMI conditions are established.

Lemma 5. *System (1) with $\sigma(u_k) \equiv 0$ is stochastic FTB with respect to $(\hbar_1 \ \ \hbar_2 \ \ d \ \ R_i \ \ N)$; if for scalars $\varsigma \ge 1$, $\hbar_1 > 0$, and $\hbar_2 > 0$, there exist symmetric matrices $R_i > 0$ $(i \in \Omega)$*

and $Q_i > 0$ $(i \in \Omega)$, such that the following matrix inequalities hold:

$$\Lambda = \begin{bmatrix} A_{l1i}^T \overline{P}_i A_{l1i} - P_i + Q & * & * \\ A_{l2i}^T \overline{P}_i A_{l1i} & -Q + A_{l2i}^T \overline{P}_i A_{l2i} & * \\ B_{l2i}^T \overline{P}_i A_{l1i} & B_{l2i}^T \overline{P}_i A_{l2i} & B_{l2i}^T \overline{P}_i B_{l2i} - I \end{bmatrix} < 0 \tag{14}$$

$$\varsigma^k \left[\lambda_{\max}\left(\widetilde{P}_{r(0)}\right) + \lambda_{\max}(Q) \cdot d\right] c_1 \le c_2 \cdot \lambda_{\min}\left(\widetilde{P}_i\right), \tag{15}$$

where $\overline{P}_i = \sum_{j=1}^{S} \pi_{vij} P_j$.

Proof. Choose the following Lyapunov functional:

$$V(k) = x^T(k) P_i x(k) + \sum_{n=k-d}^{k-1} x^T(n) Q x(n). \tag{16}$$

The proof of Lemma 5 is divided into two parts. In the first part, the following inequality is obtained:

$$E\{V(k)\} < \varsigma^k E\{V(0)\} + \varsigma^k w^T(k) w(k). \tag{17}$$

Then, we compute

$$\begin{aligned} \Delta V(k) &= E\{V(k+1)\} - V(k) \\ &= \sum_{j=1}^{S} \pi_{vij} x^T(k+1) P_j x(k+1) - x^T(k) P_i x(k) + \sum_{j=1}^{S} \pi_{vij} x^T(k) Q x(k) - x^T(k-d) Q x(k-d) \\ &= \vartheta^T(k) \begin{bmatrix} A_{\theta1i}^T \overline{P}_i A_{\theta1i} - P_i + Q & * & * \\ A_{\theta2i}^T \overline{P}_i A_{\theta1i} & -Q + A_{\theta2i}^T \overline{P}_i A_{\theta2i} & * \\ B_{\theta2i}^T \overline{P}_i A_{\theta1i} & B_{\theta2i}^T \overline{P}_i A_{\theta2i} & B_{\theta2i}^T \overline{P}_i B_{\theta2i} \end{bmatrix} \vartheta(k) \\ &= \vartheta^T(k) \left[\left(\sum_{l=1}^{L} \theta_l\right)\left(\sum_{l=1}^{L} \theta_l\right) \begin{bmatrix} A_{l1i}^T \overline{P}_i A_{l1i} - P_i + Q & * & * \\ A_{l2i}^T \overline{P}_i A_{l1i} & -Q + A_{l2i}^T \overline{P}_i A_{l2i} & * \\ B_{l2i}^T \overline{P}_i A_{l1i} & B_{l2i}^T \overline{P}_i A_{l2i} & B_{l2i}^T \overline{P}_i B_{l2i} \end{bmatrix} \right] \vartheta(k), \end{aligned} \tag{18}$$

where $\vartheta(k) = \left[x(k) \quad x(k-d) \quad w(k)\right]$.

Note condition (14); it follows that

$$\begin{aligned} E\{V(k+1)\} - V(k) < {} & (\varsigma - 1) V(k) \\ & + w^T(k) w(k), \quad \varsigma \ge 1. \end{aligned} \tag{19}$$

Therefore, we obtain that

$$E\{V(k+1)\} < \varsigma V(k) + w^T(k) w(k). \tag{20}$$

That is,

$$\begin{aligned} E\{V(x(1), r(1))\} &< \varsigma V(x(0), r(0)) + w^T(k) w(k) \\ &\vdots \\ E\{V(x(k+1), r(k+1))\} &< \varsigma E\{V(x(k), r(k))\} + w^T(k) w(k). \end{aligned} \tag{21}$$

By recursive,

$$E\{V(k)\} < \varsigma^k E\{V(0)\} + E\left\{\sum_{\tau=0}^{k-1} \varsigma^{k-\tau-1} w^T(\tau) w(\tau)\right\}. \tag{22}$$

Then the inequality in (17) is obtained.

In the second part, stochastic FTB is established:

$$\begin{aligned} E\{V(k)\} &= E\left\{x^T(k) P_i x(k) + \sum_{n=k-d}^{k-1} x^T(n) Q x(n)\right\} \\ &\ge \lambda_{\min}\left(\widetilde{P}_i\right) E\left\{x^T(k) R x(k)\right\}. \end{aligned} \tag{23}$$

On the other hand,

$$\begin{aligned} \varsigma^k E\{V(0)\} \le \varsigma^k \Bigg[& \lambda_{\max}\left(\widetilde{P}_{r(0)}\right) E\left\{x^T(0) R x(0)\right\} \\ & + \lambda_{\max}(Q) E\left\{\sum_{n=-d}^{-1} x^T(n) R x(n)\right\} \Bigg]. \end{aligned} \tag{24}$$

From Definition 2, we have

$$\varsigma^k E\{V(0)\} \le \varsigma^k \left\{\left[\lambda_{\max}\left(\widetilde{P}_{r(0)}\right) + \lambda_{\max}(Q) \cdot d\right]\right\} \hbar_1 + \varsigma^k. \tag{25}$$

By (23) and (25), we know

$$\begin{aligned} &E\left\{x^T(k)Rx(k)\right\} \\ &\quad\le \frac{\varsigma^k\left\{\left[\lambda_{\max}\left(\widetilde{P}_{r(0)}\right)+\lambda_{\max}(Q)\cdot d\right]+1\right\}\hbar_1}{\lambda_{\min}\left(\widetilde{P}_i\right)} \\ &\quad\le \hbar_2. \end{aligned} \tag{26}$$

This completes the proof. □

Theorem 6. *System* (1) *is finite-time L_2-L_∞ control and satisfies the given lever γ with respect to $(\hbar_1\ \ \hbar_2\ \ d\ \ R_i\ \ N)$; if for scalars $\varsigma \ge 1$, $\hbar_1 > 0$, and $\hbar_2 > 0$, there exist symmetric matrices $R_i > 0$ $(i \in \Omega)$ and $Q > 0$, such that the following matrix inequalities hold:*

$$\begin{aligned} &\Theta_1 \\ &= \begin{bmatrix} A_{l1i}^T\overline{P}_iA_{l1i}-P_i+Q & * & * \\ A_{l2i}^T\overline{P}_iA_{l1i} & -Q+A_{l2i}^T\overline{P}_iA_{l2i} & * \\ B_{l2i}^T\overline{P}_iA_{l1i} & B_{l2i}^T\overline{P}_iA_{l2i} & B_{l2i}^T\overline{P}_iB_{l2i}-I \end{bmatrix} \\ &< 0, \end{aligned} \tag{27}$$

$$\Theta_2 = \begin{bmatrix} -P_i & * & * & * \\ 0 & -Q & * & * \\ 0 & 0 & -I & * \\ C_{l1k} & C_{l2k} & D_{l2k} & -\gamma^2 I \end{bmatrix} \tag{28}$$

$$\varsigma^k\left[\lambda_{\max}\left(\widetilde{P}_{r(0)}\right)+\lambda_{\max}(Q)\cdot d\right]c_1 \le c_2\cdot\lambda_{\min}\left(\widetilde{P}_i\right).$$

Proof. System (1) with $\sigma(u_k) \equiv 0$ is FTB according to Lemma 5 and inequality (27).

Subsequently, to establish the energy-to-peak performance for the system (1), assume that the initial values for the plant are zeros and consider the following function:

$$\aleph := E\{V(k)\} - \sum_{i=0}^{k-1} w_i^T w_i. \tag{29}$$

For any nonzero $w_k \in l_2[0,\infty)$ and $k > 0$, it follows from (18) that

$$\begin{aligned} \aleph &:= E\left\{\sum_{i=0}^{k-1}\Delta V(i) - \sum_{i=0}^{k-1} w_i^T w_i\right\} \\ &= \vartheta^T(k)\Theta_1\vartheta(k). \end{aligned} \tag{30}$$

It follows from (27) that $E\{V(k)\} < \sum_{i=0}^{k-1} w_i^T w_i$.

For all the time instants $k > 0$, the expectation of the output can be evaluated as

$$\begin{aligned} E\left\{z_k^T z_k\right\} &= E\left\{\vartheta^T(k)\left[C_{\theta1k}\ \ C_{\theta2k}\ \ D_{\theta1k}\right]^T\right. \\ &\qquad \left.\times\left[C_{\theta1k}\ \ C_{\theta2k}\ \ D_{\theta1k}\right]\vartheta(K)\right\} \\ &< E\left\{\gamma^2\vartheta^T(k)\begin{bmatrix} P_i & 0 & 0 \\ * & \overline{Q}_i & 0 \\ * & * & I \end{bmatrix}\vartheta(k)\right\} \\ &< \gamma^2 E\left\{\sum_{i=0}^{k} w_i^T w_i\right\} < \gamma^2 E\left\{\sum_{i=0}^{\infty} w_i^T w_i\right\} \\ &= \gamma^2\|w\|_2^2. \end{aligned} \tag{31}$$

Applying Definition 4, the statement of Theorem 6 is true. □

Theorem 7. *Consider the uncertain time-delay system* (1); *there exists a state feedback controller $\sigma(K_ix(t))$ such that the uncertain time-delay system* (1) *is finite-time L_2-L_∞ control with respect to $(\hbar_1\ \ \hbar_2\ \ d\ \ R_i\ \ N)$, if the following LMIs hold:*

$$\Lambda_1 = \begin{bmatrix} -X_i & 0 & 0 & \varepsilon_{14} & \cdots & \varepsilon_{16} & X_i \\ * & -R & 0 & \varepsilon_{24} & \cdots & \varepsilon_{26} & 0 \\ * & * & -I & \varepsilon_{34} & \cdots & \varepsilon_{36} & 0 \\ * & * & * & \varepsilon_{44} & \cdots & 0 & 0 \\ * & * & * & * & \ddots & 0 & 0 \\ * & * & * & * & * & \varepsilon_{66} & 0 \\ * & * & * & * & * & * & -R \end{bmatrix} < 0, \tag{32}$$

$$\Lambda_2 = \begin{bmatrix} -P_i & * & * & * \\ 0 & -Q & * & * \\ 0 & 0 & -I & * \\ C_{l1k} & C_{l2k} & D_{l2k} & -\gamma^2 I \end{bmatrix} < 0, \tag{33}$$

$$\varsigma^k\left[\lambda_{\max}\left(\widetilde{P}_{r(0)}\right)+\lambda_{\max}(Q)\cdot d\right]c_1 \le c_2\cdot\lambda_{\min}\left(\widetilde{P}_i\right),$$

where $\varepsilon_{14} = \sqrt{\pi_{i1}}(A_{l1i}+B_{l1i}(M_rY_i+M_r^-Z_i)$, $\varepsilon_{24} = \sqrt{\pi_{i1}}RA_{l2i}$, $\varepsilon_{34} = \sqrt{\pi_{i1}}RB_{l2i}$, $\varepsilon_{44} = -X_1$, $\varepsilon_{16} = \sqrt{\pi_{iS}}(A_{l1i}+B_{l1i}(M_rY_i+M_r^-Z_i)_i)$, $\varepsilon_{26} = \sqrt{\pi_{iS}}RA_{l2i}$, $\varepsilon_{36} = \sqrt{\pi_{iS}}RB_{l2i}$, and $\varepsilon_{66} = -X_S$.

The state feedback controller is designed as $\sigma(K_ix(t)) = \sum_{r=1}^{2^m}\zeta_r(M_rK_i+M_r^-H_i)x(t)$.

Proof. Noting condition (27) and $\overline{P}_i = \Gamma_i\kappa\Gamma_i^T$, where $\kappa = \mathrm{diag}\{P_1,\ldots,P_S\}$, $\Gamma_i = [\sqrt{p_{i1}}I,\ldots,\sqrt{p_{iS}}I]$ thus Θ_1 can be rewritten as

$$\begin{aligned} \Theta_1 &= \begin{bmatrix} -P_i+Q & * & * \\ 0 & -Q & * \\ 0 & 0 & -I \end{bmatrix} \\ &\quad + \begin{bmatrix} A_{l1i}^T \\ A_{l2i}^T \\ B_{l2i}^T \end{bmatrix}\kappa\left[A_{l1i}\ \ A_{l2i}\ \ B_{l2i}\right] < 0. \end{aligned} \tag{34}$$

Using Schur complement, it can be obtained

$$\begin{bmatrix} -P+Q & 0 & 0 & A_{l1i}^T\Gamma_i \\ * & -Q & 0 & A_{l2i}^T\Gamma_i \\ * & * & -I & B_{l2i}^T\Gamma_i \\ * & * & * & -\kappa^{-1} \end{bmatrix} < 0. \tag{35}$$

Let $X_i = P_i^{-1}$, $R = Q^{-1}$, $Y_i = K_iX_i$, and $Z_i = H_iX_i$. Pre- and postmultiplying (35) by $\mathrm{diag}\{X_i, R, I, I\}$ and then using Schur complement, then inequality (32) is obtained. Implying Theorem 6, we can conclude that the corresponding closed-loop system is finite-time L_2-L_∞ control. This completes the proof. □

4. Numeral Example

To illustrate the proposed results, a numerical example is considered for finite-time L_2-L_∞ control. The system is described by (1) and assumed to have two modes; $\Omega = \{1, 2\}$. The mode switching is governed by a Markov chain that has the following transition probability matrix:

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}. \tag{36}$$

The system matrices are as follows:

$$\begin{aligned} A_{111} = A_{211} &= \begin{bmatrix} 0.3 & 0.102 \\ -0.663 & 0.3 \end{bmatrix}, \\ A_{112} = A_{212} &= \begin{bmatrix} 0.8 & 0.0539 \\ -0.8655 & 0.8 \end{bmatrix}, \\ A_{121} = A_{221} &= \begin{bmatrix} 0.5 & 0.06 \\ -0.843 & 0.5 \end{bmatrix}, \\ A_{122} = A_{222} &= \begin{bmatrix} 0.9 & 0.0766 \\ -0.7661 & 0.9 \end{bmatrix}, \\ B_{111} = B_{211} &= \begin{bmatrix} 0.0005 \\ 0.0539 \end{bmatrix}, \\ B_{212} = B_{112} &= \begin{bmatrix} 0.005 \\ 0.1078 \end{bmatrix}, \\ B_{121} = B_{221} &= \begin{bmatrix} 0.0045 \\ 0.0539 \end{bmatrix}, \\ B_{122} = B_{222} &= \begin{bmatrix} 0.0045 \\ 0.1078 \end{bmatrix}, \\ C_{111} = C_{211} = C_{112} = C_{212} &= \begin{bmatrix} 0 & 0.2 \end{bmatrix}, \\ C_{121} = C_{221} = C_{122} = C_{222} &= \begin{bmatrix} 0.3 & 0 \end{bmatrix}, \\ D_{111} = D_{211} = D_{112} = D_{212} &= 0.3. \end{aligned} \tag{37}$$

Assume L_2-L_∞ performance of level $\gamma = 0.3$; by applying Theorem 7, we can explicitly compute the optimally achievable closed-loop L_2-L_∞ performance γ from Theorem 7 as $\gamma = 0.2056$. Response of the system state is depicted in Figures 2 and 3.

5. Conclusion

The problem of finite-time L_2-L_∞ control for MJS has been studied. By using the Lyapunov functional approach, a sufficient condition is derived such that the closed-loop MJS are stochastic FTB and satisfy the given level. The controller can be obtained by using the exiting LMI optimization techniques. Finally, numerical and simulation results demonstrate the effectiveness of the results of the paper.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Resilient Robust Finite-Time L_2-L_∞ Controller Design for Uncertain Neutral System with Mixed Time-Varying Delays

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The delay-dependent resilient robust finite-time L_2-L_∞ control problem of uncertain neutral time-delayed system is studied. The disturbance input is assumed to be energy bounded and the time delays are time-varying. Based on the Lyapunov function approach and linear matrix inequalities (LMIs) techniques, a state feedback controller is designed to guarantee that the resulted closed-loop system is finite-time bounded for all uncertainties and to satisfy a given L_2-L_∞ constraint condition. Simulation results illustrate the validity of the proposed approach.

1. Introduction

Dynamical systems with time delays and uncertain parameters have been of considerable interest over the past decades. In fact, time delays are always the important source of system instability and poor performance [1–4]. As a special class of time-delay systems, the neutral type time-delayed system has also received some attention in recent years. This time-delayed system contains time delays both in its state and in the derivative of its states. Moreover, neutral time-delayed systems are frequently encountered in many dynamics, such as automatic control, distributed network system containing lossless transmission line, heat exchangers, and population ecology. Various analysis approaches have been utilized to find stability criteria and control design conditions for asymptotic stability of neutral time delays [5–10].

It is now worth pointing out that the control performances mentioned above concern the desired behavior of control dynamics over an infinite-time interval and it always deals with the asymptotic property of system trajectories. For controlling a dynamical system, it can meet the requirements of asymptotic stability, but it will not reflect the transient characteristics. Asymptotic stability is unable to satisfy the transient requirements of industrial production if there exists large amount of overshoot, oscillation change, and nonlinear disturbance within a finite-time interval. To deal with this transient performance of control dynamics, Dorato gave the concept of finite-time stability [11] (or short-time stability) in the early 1960s. Then, the relevant concepts of finite-time bounded (FTB) [12], finite-time stabilization [13], finite-time H_∞ control [14], and finite-time L_2-L_∞ [15] control have been revisited in form of linear matrix inequalities (LMIs) techniques. And this transient performance is widely applied to time-delay systems, uncertain systems, nonlinear systems, stochastic systems, and so forth. However, to the best of our knowledge, very few results in the literature consider the related control problems of neutral time-varying delays in the finite-time interval.

On the other hand, the L_2-L_∞ performance has attracted considerable attention as an important performance evaluation index when it was first proposed in 1989 [16]. In engineering practice, although the study of the impact of noise and delay on the system performance is important, the extremum problem of the controlled output cannot be ignored, because the controlled output should be controlled within a certain range. In control theory and engineering application, the L_2-L_∞ control has very important significance that lies in its performance index which can control the output value minimization. Unfortunately, up to now, the theme of L_2-L_∞ control design of uncertain neutral systems with time-varying delays has received little attention.

Motivated by the above discussion, this paper focuses on the problem of finite-time L_2-L_∞ controller design for a class

of neutral systems with mixed time-varying delays and uncertainties. By constructing a suitable Lyapunov function, the sufficient conditions are derived that closed-loop controlled system is FTB and satisfies the given finite-time interval induced L_2-L_∞ norm of the operator from the unknown disturbance to the output. We also show that the L_2-L_∞ controller designing problem can be dealt with by solving a set of coupled LMIs. Finally, a numerical example illustrates the effectiveness of the developed techniques.

2. Problem Statement

Consider the following neutral time-delayed system with uncertainties:

$$\Sigma_0 : \begin{cases} \dot{\mathbf{x}}(t) - (\mathbf{C} + \Delta\mathbf{C}(t))\dot{\mathbf{x}}(t-\tau(t)) = (\mathbf{A} + \Delta\mathbf{A}(t))\mathbf{x}(t) \\ \qquad + (\mathbf{A}_d + \Delta\mathbf{A}_d(t))\mathbf{x}(t-h(t)) + \mathbf{B}\mathbf{u}(t) \\ \qquad + (\mathbf{D} + \Delta\mathbf{D}(t))\mathbf{w}(t) \\ \mathbf{y}(t) = (\mathbf{F} + \Delta\mathbf{F}(t))\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) \\ \mathbf{x}(t_0 + \theta) = \boldsymbol{\phi}(\theta), \quad \theta \in [-\max\{h,\tau\}, 0], t_0 = 0, \end{cases} \tag{1}$$

where $\mathbf{x}(t) \in \mathbf{R}^n$ is the state, $\mathbf{u}(t) \in \mathbf{R}^m$ is the controlled input, $\mathbf{y}(t) \in \mathbf{R}^q$ is the controlled output, and $\mathbf{w}(t) \in \mathbf{R}^p$ is the disturbance input that belongs to $L_2[0, +\infty)$ and for a given positive number δ and constant time T, the following form is satisfied:

$$\int_0^T \mathbf{w}^T(t)\mathbf{w}(t)\,dt \le \delta, \quad \delta \ge 0. \tag{2}$$

$h(t)$ and $\tau(t)$ are time-varying delays and satisfy

$$\begin{aligned} 0 \le h(t) \le h, \qquad \dot{h}(t) \le h_d, \\ 0 \le \tau(t) \le \tau, \qquad \dot{\tau}(t) \le \tau_d < 1, \end{aligned} \tag{3}$$

where h, τ, h_d, and τ_d are constant scalars. $\boldsymbol{\phi}(\theta) \in L_2[-\max\{h,\tau\}, 0]$ is the continuous initial function. $\mathbf{A}$, $\mathbf{A}_d$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{F} \in \mathbf{R}^{n\times n}$ are known constant matrices, and $\Delta\mathbf{A}(t)$, $\Delta\mathbf{A}_d(t)$, $\Delta\mathbf{C}(t)$, $\Delta\mathbf{D}(t)$, and $\Delta\mathbf{F}(t)$ are unknown time-variant matrices representing the norm-bounded parameter uncertainties and satisfy the following form:

$$\begin{aligned} &[\Delta\mathbf{A}(t) \;\; \Delta\mathbf{A}_d(t) \;\; \Delta\mathbf{C}(t) \;\; \Delta\mathbf{D}(t)] \\ &\quad = \mathbf{M}_1\boldsymbol{\sigma}(t)[\mathbf{H}_1 \;\; \mathbf{H}_2 \;\; \mathbf{H}_3 \;\; \mathbf{H}_4], \end{aligned} \tag{4}$$

$$\Delta\mathbf{F}(t) = \mathbf{M}_2\boldsymbol{\sigma}(t)\mathbf{H}_1, \tag{5}$$

where $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, and $\mathbf{H}_4$ are known real matrices with suitable dimension and $\boldsymbol{\sigma}(t)$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$\boldsymbol{\sigma}^T(t)\boldsymbol{\sigma}(t) \le \mathbf{I}. \tag{6}$$

In this paper, we consider the state feedback controller as follows:

$$\mathbf{u}(t) = (\mathbf{K} + \Delta\mathbf{K}(t))\mathbf{x}(t), \tag{7}$$

where $\mathbf{K}$ is the unknown controller gain and $\Delta\mathbf{K}(t)$ is the time-varying controller gain which satisfies

$$\Delta\mathbf{K}(t) = \mathbf{N}\boldsymbol{\eta}(t)\mathbf{S}, \qquad \boldsymbol{\eta}^T(t)\boldsymbol{\eta}(t) \le \mathbf{I}. \tag{8}$$

Then, we can get the following closed-loop control system:

$$\Sigma: \begin{cases} \dot{\mathbf{x}}(t) - \overline{\mathbf{C}}\dot{\mathbf{x}}(t-\tau(t)) = \widehat{\mathbf{A}}\mathbf{x}(t) \\ \qquad + \overline{\mathbf{A}}_d\mathbf{x}(t-h(t)) + \overline{\mathbf{D}}\mathbf{w}(t) \\ \mathbf{y}(t) = \widehat{\mathbf{F}}\mathbf{x}(t) \\ \mathbf{x}(t_0 + \theta) = \boldsymbol{\phi}(\theta), \quad \theta \in [-\max\{h,\tau\}, 0], t_0 = 0, \end{cases} \tag{9}$$

where $\widehat{\mathbf{A}} = \overline{\mathbf{A}} + \Delta\overline{\mathbf{A}}(t)$, $\overline{\mathbf{A}} = \mathbf{A} + \mathbf{B}\mathbf{K}$, $\Delta\overline{\mathbf{A}}(t) = \Delta\mathbf{A}(t) + \mathbf{B}\Delta\mathbf{K}(t)$, $\overline{\mathbf{A}}_d = \mathbf{A}_d + \Delta\mathbf{A}_d(t)$, $\overline{\mathbf{C}} = \mathbf{C} + \Delta\mathbf{C}(t)$, $\overline{\mathbf{D}} = \mathbf{D} + \Delta\mathbf{D}(t)$, $\widehat{\mathbf{F}} = \overline{\mathbf{F}} + \Delta\overline{\mathbf{F}}(t)$, $\overline{\mathbf{F}} = \mathbf{F} + \mathbf{G}\mathbf{K}$, and $\Delta\overline{\mathbf{F}}(t) = \Delta\mathbf{F}(t) + \mathbf{G}\Delta\mathbf{K}(t)$.

The main purpose of this paper is to design an appropriate resilient state feedback controller (7), such that the closed-loop control system Σ is finite-time bounded and satisfies the given performance index constraints.

Before proceeding with the study, we give the relevant definitions and lemmas first.

Definition 1. For given positive scalars c_1, δ, and T and a symmetrical positive determined matrix $\mathbf{R}$, the closed-loop system Σ is robust finite-time bounded (FTB) with respect to $(c_1, c_2, \delta, \mathbf{R}, T)$, if there exists a positive constant c_2 with $c_2 > c_1$, such that, for all the external disturbances $\mathbf{w}(t)$ satisfying condition (2), the following formula is satisfied:

$$\boldsymbol{\phi}^T(\theta)\mathbf{R}\boldsymbol{\phi}(\theta) \le c_1 \Longrightarrow \mathbf{x}^T(t)\mathbf{R}\mathbf{x}(t) < c_2, \quad \forall t \in [0, T]. \tag{10}$$

Remark 2. If the disturbance input is not present in the closed-loop system, that is, $\mathbf{w}(t) = 0$, the concept of FTB will reduce into finite-time stability (FTS). It is worth mentioning that Lyapunov stability and finite-time stability are two different concepts. The former is largely known to the control characteristic in infinite-time interval, but the latter concerns the boundedness analysis of the controlled states within a finite-time interval. Obviously, a finite-time stable system may not be Lyapunov stochastically stable and vice versa.

Definition 3. The state feedback controller in the form of (7) is considered as a robust finite-time L_2-L_∞ controller for the closed-loop system Σ, if the system Σ is FTB with respect to $(c_1, c_2, \delta, \mathbf{R}, T)$ and under the zero initial condition, there exist two positive scalars γ and T for all disturbance which satisfy condition (2), such that

$$\|\mathbf{y}(t)\|_\infty^2 \le \gamma^2\|\mathbf{w}(t)\|_2^2, \tag{11}$$

where $\|\mathbf{y}(t)\|_\infty^2 = \sup_{t\in[0,T]}|\mathbf{y}^T(t)\mathbf{y}(t)|$, $\|\mathbf{w}(t)\|_2^2 = \int_0^T \mathbf{w}^T(t)\mathbf{w}(t)dt$.

Lemma 4 (see [17]). *For any real positive scalars* α, β *(where* $\alpha > \beta$*) and a positive definite symmetric matrix* $\mathbf{S}$, *then the following inequality holds for a vector function* $\boldsymbol{\omega} : [\beta, \alpha] \to \mathbf{R}^n$ *which can let the integrals converge:*

$$\begin{aligned}&\left(\int_{\beta}^{\alpha} \boldsymbol{\omega}(\boldsymbol{\sigma})\, d\sigma\right)^T \mathbf{S} \left(\int_{\beta}^{\alpha} \boldsymbol{\omega}(\boldsymbol{\sigma})\, d\sigma\right) \\ &\quad \le (\alpha - \beta) \left(\int_{\beta}^{\alpha} \boldsymbol{\omega}^T(\boldsymbol{\sigma})\, \mathbf{S} \boldsymbol{\omega}(\boldsymbol{\sigma})\, d\sigma\right).\end{aligned} \tag{12}$$

Lemma 5 (see [17]). *For any positive scalar* h *and positive definite symmetric matrix* $\mathbf{S}$, *the following inequality is satisfied:*

$$\begin{aligned}&\frac{2}{h^2}\left(\int_{-h}^{0}\int_{t+\theta}^{t} \boldsymbol{\omega}(\boldsymbol{\sigma})\, d\sigma\, d\theta\right)^T \mathbf{S} \left(\int_{-h}^{0}\int_{t+\theta}^{t} \boldsymbol{\omega}(\boldsymbol{\sigma})\, d\sigma\, d\theta\right) \\ &\quad \le \int_{-h}^{0}\int_{t+\theta}^{t} \boldsymbol{\omega}^T(\boldsymbol{\sigma})\, \mathbf{S}\boldsymbol{\omega}(\sigma)\, d\sigma\, d\theta.\end{aligned} \tag{13}$$

Lemma 6 (see [15]). *For any given appropriate dimension matrix* $\mathbf{H}$ *and* $\mathbf{E}$, *if there exists a matrix* $\mathbf{W}(t)$ *which satisfied* $\mathbf{W}^T(t)\mathbf{W}(t) \le \mathbf{I}$ *and a scalar* $\varepsilon > 0$, *then*

$$\mathbf{H}\mathbf{W}(t)\,\mathbf{E} + \mathbf{E}^T\mathbf{W}^{\mathbf{T}}(t)\,\mathbf{H}^T \le \varepsilon^{-1}\mathbf{H}\mathbf{H}^T + \varepsilon\mathbf{E}^T\mathbf{E}. \tag{14}$$

3. Main Results

In this section, our main purpose is to solve the design problem of a resilient robust finite-time L_2-L_∞ controller for a class of uncertain neutral systems with mixed time-varying delays.

Theorem 7. *Given positive scalars* c_1, δ, T, *and* α, *positive definite symmetric matrix* $\mathbf{R}$, *and time-delay parameters* $h > 0$, $h_d > 0$, $\tau > 0$, *and* $\tau_d > 0$, *the closed-loop system* Σ *is FTB with respect to* $(c_1, c_2, \delta, \mathbf{R}, T)$, *if there exist positive scalars* $\lambda_i, i = 1, 2, \ldots, 6$, c_2, *and symmetric positive definite matrices* $\mathbf{P}_i, i = 1, 2, \ldots, 6$, $\mathbf{Q}_i, i = 1, 2, \ldots, 4$, *and* $\mathbf{W}_i, i = 1, 2, \ldots, 6$, *such that*

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\Pi}_1 & \boldsymbol{\Pi}_2 & \boldsymbol{\Pi}_3 \\ * & \boldsymbol{\Pi}_4 & \boldsymbol{\Pi}_5 \\ * & * & \boldsymbol{\Pi}_6 \end{bmatrix} < 0, \tag{15}$$

$$c_1\left[\lambda_2 + h\lambda_3 + h\lambda_4 + \tau\lambda_5 + \tau\lambda_6\right] + \delta\left(1 - e^{-\alpha T}\right) < \lambda_1 c_2 e^{-\alpha T}, \tag{16}$$

where

$$\boldsymbol{\Pi}_1 = \left[\boldsymbol{\Pi}_{ij}\right]_{7\times 7},$$

$$\begin{aligned}\boldsymbol{\Pi}_{11} = {} & \widehat{\mathbf{A}}^T\mathbf{P}_1 + \mathbf{P}_1\widehat{\mathbf{A}} + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5 \\ & + \mathbf{W}_1 + \mathbf{W}_3 + \mathbf{W}_4 + \mathbf{W}_6 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T \\ & - \alpha\mathbf{P}_1 - \frac{\alpha}{\tau}\mathbf{P}_6 - 2\alpha\mathbf{Q}_1 - 2\alpha\mathbf{Q}_2 - 2\alpha\mathbf{Q}_3 - 2\alpha\mathbf{Q}_4,\end{aligned}$$

$$\boldsymbol{\Pi}_{12} = \mathbf{P}_1\overline{\mathbf{A}}_d - \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T,$$

$$\boldsymbol{\Pi}_{13} = \mathbf{P}_1\overline{\mathbf{C}} + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T,$$

$$\boldsymbol{\Pi}_{14} = -\mathbf{W}_4 + \mathbf{W}_5 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T + \frac{\alpha}{\tau}\mathbf{P}_6,$$

$$\boldsymbol{\Pi}_{15} = -\mathbf{W}_2 - \mathbf{W}_3 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T,$$

$$\boldsymbol{\Pi}_{16} = -\mathbf{W}_5 - \mathbf{W}_6 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T,$$

$$\boldsymbol{\Pi}_{17} = \mathbf{P}_1\overline{\mathbf{D}},$$

$$\boldsymbol{\Pi}_{22} = -\left(1 - h_d\right)\mathbf{P}_2 - \mathbf{W}_1 + \mathbf{W}_2 - \mathbf{W}_1^T + \mathbf{W}_2^T,$$

$$\boldsymbol{\Pi}_{23} = -\mathbf{W}_1^T + \mathbf{W}_2^T,$$

$$\boldsymbol{\Pi}_{24} = -\mathbf{W}_4 + \mathbf{W}_5 - \mathbf{W}_1^T + \mathbf{W}_2^T,$$

$$\boldsymbol{\Pi}_{25} = -\mathbf{W}_2 - \mathbf{W}_3 - \mathbf{W}_1^T + \mathbf{W}_2^T,$$

$$\boldsymbol{\Pi}_{26} = -\mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_1^T + \mathbf{W}_2^T,$$

$$\boldsymbol{\Pi}_{27} = 0,$$

$$\boldsymbol{\Pi}_{33} = -\left(1 - \tau_d\right)\mathbf{P}_6,$$

$$\boldsymbol{\Pi}_{34} = -\mathbf{W}_4 + \mathbf{W}_5,$$

$$\boldsymbol{\Pi}_{35} = -\mathbf{W}_2 - \mathbf{W}_3,$$

$$\boldsymbol{\Pi}_{36} = -\mathbf{W}_5 - \mathbf{W}_6,$$

$$\boldsymbol{\Pi}_{37} = 0,$$

$$\boldsymbol{\Pi}_{44} = -\left(1 - \tau_d\right)\mathbf{P}_4 - \mathbf{W}_4 + \mathbf{W}_5 - \mathbf{W}_4^T + \mathbf{W}_5^T - \frac{\alpha}{\tau}\mathbf{P}_6,$$

$$\boldsymbol{\Pi}_{45} = -\mathbf{W}_2 - \mathbf{W}_3 - \mathbf{W}_4^T + \mathbf{W}_5^T,$$

$$\boldsymbol{\Pi}_{46} = -\mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_4^T + \mathbf{W}_5^T,$$

$$\boldsymbol{\Pi}_{47} = 0,$$

$$\boldsymbol{\Pi}_{55} = -\mathbf{P}_3 - \mathbf{W}_2 - \mathbf{W}_3 - \mathbf{W}_2^T - \mathbf{W}_3^T,$$

$$\boldsymbol{\Pi}_{56} = -\mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_2^T - \mathbf{W}_3^T,$$

$$\boldsymbol{\Pi}_{57} = 0,$$

$$\boldsymbol{\Pi}_{66} = -\mathbf{P}_5 - \mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_5^T - \mathbf{W}_6^T,$$

$$\boldsymbol{\Pi}_{67} = 0,$$

$$\boldsymbol{\Pi}_{77} = -\alpha\mathbf{I},$$

$$\boldsymbol{\Pi}_2 = \begin{bmatrix} h\mathbf{W}_1 & h\mathbf{W}_2 & h\mathbf{W}_3 & \tau\mathbf{W}_4 & \tau\mathbf{W}_5 & \tau\mathbf{W}_6 \\ h\mathbf{W}_1 & h\mathbf{W}_2 & h\mathbf{W}_3 & \tau\mathbf{W}_4 & \tau\mathbf{W}_5 & \tau\mathbf{W}_6 \\ h\mathbf{W}_1 & h\mathbf{W}_2 & h\mathbf{W}_3 & \tau\mathbf{W}_4 & \tau\mathbf{W}_5 & \tau\mathbf{W}_6 \\ h\mathbf{W}_1 & h\mathbf{W}_2 & h\mathbf{W}_3 & \tau\mathbf{W}_4 & \tau\mathbf{W}_5 & \tau\mathbf{W}_6 \\ h\mathbf{W}_1 & h\mathbf{W}_2 & h\mathbf{W}_3 & \tau\mathbf{W}_4 & \tau\mathbf{W}_5 & \tau\mathbf{W}_6 \\ h\mathbf{W}_1 & h\mathbf{W}_2 & h\mathbf{W}_3 & \tau\mathbf{W}_4 & \tau\mathbf{W}_5 & \tau\mathbf{W}_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{\Pi}_3 = \begin{bmatrix} \widehat{\mathbf{A}}^T\mathbf{P}_6 & h\widehat{\mathbf{A}}^T\mathbf{Q}_1 & h\widehat{\mathbf{A}}^T\mathbf{Q}_2 & \tau\widehat{\mathbf{A}}^T\mathbf{Q}_3 & \tau\widehat{\mathbf{A}}^T\mathbf{Q}_4 \\ \overline{\mathbf{A}}_d^T\mathbf{P}_6 & h\overline{\mathbf{A}}_d^T\mathbf{Q}_1 & h\overline{\mathbf{A}}_d^T\mathbf{Q}_2 & \tau\overline{\mathbf{A}}_d^T\mathbf{Q}_3 & \tau\overline{\mathbf{A}}_d^T\mathbf{Q}_4 \\ \overline{\mathbf{C}}^T\mathbf{P}_6 & h\overline{\mathbf{C}}^T\mathbf{Q}_1 & h\overline{\mathbf{C}}^T\mathbf{Q}_2 & \tau\overline{\mathbf{C}}^T\mathbf{Q}_3 & \tau\overline{\mathbf{C}}^T\mathbf{Q}_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \overline{\mathbf{D}}^T\mathbf{P}_6 & h\overline{\mathbf{D}}^T\mathbf{Q}_1 & h\overline{\mathbf{D}}^T\mathbf{Q}_2 & \tau\overline{\mathbf{D}}^T\mathbf{Q}_3 & \tau\overline{\mathbf{D}}^T\mathbf{Q}_4 \end{bmatrix},$$

$$\mathbf{\Pi}_4 = \operatorname{diag}\{-h\mathbf{Q}_1 \quad -h\mathbf{Q}_1 \quad -h\mathbf{Q}_2 \quad -\tau\mathbf{Q}_3 \quad -\tau\mathbf{Q}_3 \quad -\tau\mathbf{Q}_4\},$$

$$\mathbf{\Pi}_5 = [0]_{6\times 5},$$

$$\mathbf{\Pi}_6 = \operatorname{diag}\{-\mathbf{P}_6 \quad -h\mathbf{Q}_1 \quad -h\mathbf{Q}_2 \quad -\tau\mathbf{Q}_3 \quad -\tau\mathbf{Q}_3\}. \tag{17}$$

Proof. Construct a positive definite Lyapunov function as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \tag{18}$$

where

$$\begin{aligned} V_1(t) &= \mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t), \\ V_2(t) &= \int_{t-h(t)}^{t} \mathbf{x}^T(s)\mathbf{P}_2\mathbf{x}(s)\,ds + \int_{t-h}^{t} \mathbf{x}^T(s)\mathbf{P}_3\mathbf{x}(s)\,ds, \\ V_3(t) &= \int_{t-\tau(t)}^{t} \mathbf{x}^T(s)\mathbf{P}_4\mathbf{x}(s)\,ds + \int_{t-\tau}^{t} \mathbf{x}^T(s)\mathbf{P}_5\mathbf{x}(s)\,ds, \\ V_4(t) &= \int_{t-\tau(t)}^{t} \dot{\mathbf{x}}^T(s)\mathbf{P}_6\dot{\mathbf{x}}(s)\,ds, \\ V_5(t) &= \int_{-h}^{0}\int_{t+\theta}^{t} \dot{\mathbf{x}}^T(s)\left(\mathbf{Q}_1+\mathbf{Q}_2\right)\dot{\mathbf{x}}(s)\,ds\,d\theta \\ &\quad + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\mathbf{x}}^T(s)\left(\mathbf{Q}_3+\mathbf{Q}_4\right)\dot{\mathbf{x}}(s)\,ds\,d\theta. \end{aligned} \tag{19}$$

We take the time derivative of $V(t)$ along the trajectory of system Σ and it yields the following:

$$\begin{aligned} \dot{V}_1(t) &= \mathbf{x}^T(t)\left(\mathbf{P}_1\widehat{\mathbf{A}} + \widehat{\mathbf{A}}^T\mathbf{P}_1\right)\mathbf{x}(t) \\ &\quad + \mathbf{x}^T(t)\mathbf{P}_1\overline{\mathbf{A}}_d\mathbf{x}(t-h(t)) \\ &\quad + \mathbf{x}^T(t)\mathbf{P}_1\overline{\mathbf{C}}\dot{\mathbf{x}}(t-\tau(t)) \\ &\quad + \mathbf{x}^T(t)\mathbf{P}_1\overline{\mathbf{D}}\mathbf{w}(t) + \mathbf{x}^T(t-h(t))\overline{\mathbf{A}}_d^T\mathbf{P}_1\mathbf{x}(t) \\ &\quad + \dot{\mathbf{x}}^T(t-\tau(t))\overline{\mathbf{C}}^T\mathbf{P}_1\mathbf{x}(t) + \boldsymbol{\omega}^T(t)\overline{\mathbf{D}}^T\mathbf{P}_1\mathbf{x}(t), \\ \dot{V}_2(t) &\le \mathbf{x}^T(t)\left(\mathbf{P}_2+\mathbf{P}_3\right)\mathbf{x}(t) \\ &\quad - \left(1-h_d\right)\mathbf{x}^T(t-h(t))\mathbf{P}_2\mathbf{x}(t-h(t)) \\ &\quad - \mathbf{x}^T(t-h)\mathbf{P}_3\mathbf{x}(t-h), \\ \dot{V}_3(t) &\le \mathbf{x}^T(t)\left(\mathbf{P}_4+\mathbf{P}_5\right)\mathbf{x}(t) \\ &\quad - \left(1-\tau_d\right)\mathbf{x}^T(t-\tau(t))\mathbf{P}_4\mathbf{x}(t-\tau(t)) \\ &\quad - \mathbf{x}^T(t-\tau)\mathbf{P}_5\mathbf{x}(t-\tau), \\ \dot{V}_4(t) &\le \dot{\mathbf{x}}^T(t)\mathbf{P}_6\dot{\mathbf{x}}(t) \\ &\quad - \left(1-\tau_d\right)\dot{\mathbf{x}}^T(t-\tau(t))\mathbf{P}_6\dot{\mathbf{x}}(t-\tau(t)), \\ \dot{V}_5(t) &= \dot{\mathbf{x}}^T(t)\left(h\left(\mathbf{Q}_1+\mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3+\mathbf{Q}_4\right)\right)\dot{\mathbf{x}}(t) \\ &\quad - \int_{t-h(t)}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_1\dot{\mathbf{x}}(s)\,ds - \int_{t-h}^{t-h(t)} \dot{\mathbf{x}}^T(s)\mathbf{Q}_1\dot{\mathbf{x}}(s)\,ds \\ &\quad - \int_{t-h}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_2\dot{\mathbf{x}}(s)\,ds - \int_{t-\tau(t)}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_3\dot{\mathbf{x}}(s)\,ds \\ &\quad - \int_{t-\tau}^{t-\tau(t)} \dot{\mathbf{x}}^T(s)\mathbf{Q}_3\dot{\mathbf{x}}(s)\,ds - \int_{t-\tau}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)\,ds. \end{aligned} \tag{20}$$

For any symmetric positive definite matrices $\mathbf{W}_i, i = 1, 2, \ldots, 6$, the following equations are satisfied according to Leibniz-Newton lemma:

$$\begin{aligned} 2\boldsymbol{\zeta}^T(t)\mathbf{W}_1\left[\mathbf{x}(t) - \mathbf{x}(t-h(t)) - \int_{t-h(t)}^{t} \dot{\mathbf{x}}(s)\,ds\right] &= 0, \\ 2\boldsymbol{\zeta}^T(t)\mathbf{W}_2\left[\mathbf{x}(t-h(t)) - \mathbf{x}(t-h) - \int_{t-h}^{t-h(t)} \dot{\mathbf{x}}(s)\,ds\right] &= 0, \\ 2\boldsymbol{\zeta}^T(t)\mathbf{W}_3\left[\mathbf{x}(t) - \mathbf{x}(t-h) - \int_{t-h}^{t} \dot{\mathbf{x}}(s)\,ds\right] &= 0, \\ 2\boldsymbol{\zeta}^T(t)\mathbf{W}_4\left[\mathbf{x}(t) - \mathbf{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{\mathbf{x}}(s)\,ds\right] &= 0, \\ 2\boldsymbol{\zeta}^T(t)\mathbf{W}_5\left[\mathbf{x}(t-\tau(t)) - \mathbf{x}(t-\tau) - \int_{t-\tau}^{t-\tau(t)} \dot{\mathbf{x}}(s)\,ds\right] &= 0, \\ 2\boldsymbol{\zeta}^T(t)\mathbf{W}_6\left[\mathbf{x}(t) - \mathbf{x}(t-\tau) - \int_{t-\tau}^{t} \dot{\mathbf{x}}(s)\,ds\right] &= 0, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \boldsymbol{\zeta}(t) &= \left[\mathbf{x}^T(t) \quad \mathbf{x}^T(t-h(t)) \quad \dot{\mathbf{x}}^T(t-\tau(t)) \quad \mathbf{x}^T(t-\tau(t)) \quad \mathbf{x}^T(t-h) \quad \mathbf{x}^T(t-\tau)\right]^T, \\ \boldsymbol{\xi}(t) &= \left[\boldsymbol{\zeta}^T(t) \quad \mathbf{w}^T(t)\right]^T. \end{aligned} \tag{22}$$

According to (20)-(21), we can obtain

$$\begin{aligned}
\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \dot{V}_5(t) \\
&\le \boldsymbol{\xi}^T(t)\boldsymbol{\Omega}_1\boldsymbol{\xi}(t) - \int_{t-h(t)}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_1\dot{\mathbf{x}}(s)\,ds \\
&\quad - \int_{t-h}^{t-h(t)} \dot{\mathbf{x}}^T(s)\mathbf{Q}_1\dot{\mathbf{x}}(s)\,ds - \int_{t-h}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_2\dot{\mathbf{x}}(s)\,ds \\
&\quad - \int_{t-\tau(t)}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_3\dot{\mathbf{x}}(s)\,ds - \int_{t-\tau}^{t-\tau(t)} \dot{\mathbf{x}}^T(s)\mathbf{Q}_3\dot{\mathbf{x}}(s)\,ds \\
&\quad - \int_{t-\tau}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)\,ds \\
&\quad + 2\boldsymbol{\zeta}^T(t)\mathbf{W}_1\left[\mathbf{x}(t) - \mathbf{x}(t-h(t)) - \int_{t-h(t)}^{t}\dot{\mathbf{x}}(s)\,ds\right] \\
&\quad + 2\boldsymbol{\zeta}^T(t)\mathbf{W}_2\left[\mathbf{x}(t-h(t)) - \mathbf{x}(t-h) - \int_{t-h}^{t-h(t)}\dot{\mathbf{x}}(s)\,ds\right] \\
&\quad + 2\boldsymbol{\zeta}^T(t)\mathbf{W}_3\left[\mathbf{x}(t) - \mathbf{x}(t-h) - \int_{t-h}^{t}\dot{\mathbf{x}}(s)\,ds\right] \\
&\quad + 2\boldsymbol{\zeta}^T(t)\mathbf{W}_4\left[\mathbf{x}(t) - \mathbf{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{\mathbf{x}}(s)\,ds\right] \\
&\quad + 2\boldsymbol{\zeta}^T(t)\mathbf{W}_5\left[\mathbf{x}(t-\tau(t)) - \mathbf{x}(t-\tau) - \int_{t-\tau}^{t-\tau(t)}\dot{\mathbf{x}}(s)\,ds\right] \\
&\quad + 2\boldsymbol{\zeta}^T(t)\mathbf{W}_6\left[\mathbf{x}(t) - \mathbf{x}(t-\tau) - \int_{t-\tau}^{t}\dot{\mathbf{x}}(s)\,ds\right] \\
&= \boldsymbol{\xi}^T(t)\boldsymbol{\Omega}_1\boldsymbol{\xi}(t) + \boldsymbol{\zeta}^T(t)\boldsymbol{\Omega}_2\boldsymbol{\zeta}(t) \\
&\quad - \int_{t-h(t)}^{t}\left(\boldsymbol{\zeta}^T(t)\mathbf{W}_1 + \dot{\mathbf{x}}^T(s)\mathbf{Q}_1\right) \times \mathbf{Q}_1^{-1}\left(\mathbf{W}_1^T\boldsymbol{\zeta}(t) + \mathbf{Q}_1\dot{\mathbf{x}}(s)\right)ds \\
&\quad - \int_{t-h}^{t-h(t)}\left(\boldsymbol{\zeta}^T(t)\mathbf{W}_2 + \dot{\mathbf{x}}^T(s)\mathbf{Q}_1\right) \times \mathbf{Q}_1^{-1}\left(\mathbf{W}_2^T\boldsymbol{\zeta}(t) + \mathbf{Q}_1\dot{\mathbf{x}}(s)\right)ds \\
&\quad - \int_{t-h}^{t}\left(\boldsymbol{\zeta}^T(t)\mathbf{W}_3 + \dot{\mathbf{x}}^T(s)\mathbf{Q}_2\right) \times \mathbf{Q}_2^{-1}\left(\mathbf{W}_3^T\boldsymbol{\zeta}(t) + \mathbf{Q}_2\dot{\mathbf{x}}(s)\right)ds \\
&\quad - \int_{t-\tau(t)}^{t}\left(\boldsymbol{\zeta}^T(t)\mathbf{W}_4 + \dot{\mathbf{x}}^T(s)\mathbf{Q}_3\right) \times \mathbf{Q}_3^{-1}\left(\mathbf{W}_4^T\boldsymbol{\zeta}(t) + \mathbf{Q}_3\dot{\mathbf{x}}(s)\right)ds \\
&\quad - \int_{t-\tau}^{t-\tau(t)}\left(\boldsymbol{\zeta}^T(t)\mathbf{W}_5 + \dot{\mathbf{x}}^T(s)\mathbf{Q}_3\right) \times \mathbf{Q}_3^{-1}\left(\mathbf{W}_5^T\boldsymbol{\zeta}(t) + \mathbf{Q}_3\dot{\mathbf{x}}(s)\right)ds \\
&\quad - \int_{t-\tau}^{t}\left(\boldsymbol{\zeta}^T(t)\mathbf{W}_6 + \dot{\mathbf{x}}^T(s)\mathbf{Q}_4\right) \times \mathbf{Q}_4^{-1}\left(\mathbf{W}_6^T\boldsymbol{\zeta}(t) + \mathbf{Q}_4\dot{\mathbf{x}}(s)\right)ds.
\end{aligned} \tag{23}$$

Since $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, and $\mathbf{Q}_4$ are positive definite symmetric matrices, we have

$$\dot{V}(t) \le \boldsymbol{\xi}^T(t)\boldsymbol{\Omega}_1\boldsymbol{\xi}(t) + \boldsymbol{\zeta}^T(t)\boldsymbol{\Omega}_2\boldsymbol{\zeta}(t), \tag{24}$$

where

$$\begin{aligned}
\boldsymbol{\Omega}_1 &= \left[\boldsymbol{\Omega}_{ij}\right]_{7\times 7}, \\
\boldsymbol{\Omega}_{11} &= \widehat{\mathbf{A}}^T\mathbf{P}_1 + \mathbf{P}_1\widehat{\mathbf{A}} + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5 \\
&\quad + \widehat{\mathbf{A}}^T\left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\widehat{\mathbf{A}} \\
&\quad + \mathbf{W}_1 + \mathbf{W}_3 + \mathbf{W}_4 + \mathbf{W}_6 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \\
\boldsymbol{\Omega}_{12} &= \mathbf{P}_1\overline{\mathbf{A}}_d + \widehat{\mathbf{A}}^T\left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{A}}_d \\
&\quad - \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \\
\boldsymbol{\Omega}_{13} &= \mathbf{P}_1\overline{\mathbf{C}} + \widehat{\mathbf{A}}^T\left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{C}} \\
&\quad + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \\
\boldsymbol{\Omega}_{14} &= -\mathbf{W}_4 + \mathbf{W}_5 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \\
\boldsymbol{\Omega}_{15} &= -\mathbf{W}_2 - \mathbf{W}_3 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \\
\boldsymbol{\Omega}_{16} &= -\mathbf{W}_5 - \mathbf{W}_6 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \\
\boldsymbol{\Omega}_{17} &= \mathbf{P}_1\overline{\mathbf{D}} + \widehat{\mathbf{A}}^T\left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{D}}, \\
\boldsymbol{\Omega}_{22} &= -\left(1 - h_d\right)\mathbf{P}_2 \\
&\quad + \overline{\mathbf{A}}_d^T\left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{A}}_d \\
&\quad - \mathbf{W}_1 + \mathbf{W}_2 - \mathbf{W}_1^T + \mathbf{W}_2^T, \\
\boldsymbol{\Omega}_{23} &= \overline{\mathbf{A}}_d^T\left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{C}} \\
&\quad - \mathbf{W}_1^T + \mathbf{W}_2^T, \\
\boldsymbol{\Omega}_{24} &= -\mathbf{W}_4 + \mathbf{W}_5 - \mathbf{W}_1^T + \mathbf{W}_2^T, \\
\boldsymbol{\Omega}_{25} &= -\mathbf{W}_2 - \mathbf{W}_3 - \mathbf{W}_1^T + \mathbf{W}_2^T, \\
\boldsymbol{\Omega}_{26} &= -\mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_1^T + \mathbf{W}_2^T,
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Omega}_{27} &= \overline{\mathbf{A}}_d^T \left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{D}},\\
\mathbf{\Omega}_{33} &= -\left(1-\tau_d\right)\mathbf{P}_6\\
&\quad + \overline{\mathbf{C}}^T \left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{C}},\\
\mathbf{\Omega}_{34} &= -\mathbf{W}_4 + \mathbf{W}_5,\\
\mathbf{\Omega}_{35} &= -\mathbf{W}_2 - \mathbf{W}_3,\\
\mathbf{\Omega}_{36} &= -\mathbf{W}_5 - \mathbf{W}_6,\\
\mathbf{\Omega}_{37} &= \overline{\mathbf{C}}^T \left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{D}},\\
\mathbf{\Omega}_{44} &= -\left(1-\tau_d\right)\mathbf{P}_4 - \mathbf{W}_4 + \mathbf{W}_5 - \mathbf{W}_4^T + \mathbf{W}_5^T,\\
\mathbf{\Omega}_{45} &= -\mathbf{W}_2 - \mathbf{W}_3 - \mathbf{W}_4^T + \mathbf{W}_5^T,\\
\mathbf{\Omega}_{46} &= -\mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_4^T + \mathbf{W}_5^T,\\
\mathbf{\Omega}_{47} &= 0,\\
\mathbf{\Omega}_{55} &= -\mathbf{P}_3 - \mathbf{W}_2 - \mathbf{W}_3 - \mathbf{W}_2^T - \mathbf{W}_3^T,\\
\mathbf{\Omega}_{56} &= -\mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_2^T - \mathbf{W}_3^T,\\
\mathbf{\Omega}_{57} &= 0,\\
\mathbf{\Omega}_{66} &= -\mathbf{P}_5 - \mathbf{W}_5 - \mathbf{W}_6 - \mathbf{W}_5^T - \mathbf{W}_6^T,\\
\mathbf{\Omega}_{67} &= 0,\\
\mathbf{\Omega}_{77} &= \overline{\mathbf{D}}^T \left(\mathbf{P}_6 + h\left(\mathbf{Q}_1 + \mathbf{Q}_2\right) + \tau\left(\mathbf{Q}_3 + \mathbf{Q}_4\right)\right)\overline{\mathbf{D}},\\
\mathbf{\Omega}_2 &= h\mathbf{W}_1\mathbf{Q}_1^{-1}\mathbf{W}_1^T + h\mathbf{W}_2\mathbf{Q}_1^{-1}\mathbf{W}_3^T + h\mathbf{W}_3\mathbf{Q}_2^{-1}\mathbf{W}_3^T\\
&\quad + \tau\mathbf{W}_4\mathbf{Q}_3^{-1}\mathbf{W}_4^T + \tau\mathbf{W}_5\mathbf{Q}_3^{-1}\mathbf{W}_5^T + \tau\mathbf{W}_6\mathbf{Q}_4^{-1}\mathbf{W}_6^T.
\end{aligned} \tag{25}$$

Recalling formula (24) and Lemmas 4 and 5 and using Schur complement, we can get

$$\dot{V}(t) - \alpha V(t) - \alpha\mathbf{w}^T(t)\mathbf{w}(t) \le \boldsymbol{\xi}^T(t)\mathbf{\Pi}\boldsymbol{\xi}(t) < 0; \tag{26}$$

that is,

$$\dot{V}(t) < \alpha V(t) + \alpha\mathbf{w}^T(t)\mathbf{w}(t). \tag{27}$$

Pre- and postmultiplying (27) by $e^{-\alpha t}$, we have

$$\frac{d}{dt}\left(e^{-\alpha t}V(t)\right) < \alpha e^{-\alpha t}\mathbf{w}^T(t)\mathbf{w}(t). \tag{28}$$

Then integrating the aforementioned inequality from 0 to t, where $t \in [0, T]$, it yields

$$e^{-\alpha t}V(t) - V(0) < \alpha\int_0^t e^{-\alpha\tau}\mathbf{w}^T(\tau)\mathbf{w}(\tau)\,d\tau. \tag{29}$$

Considering condition (2), (29) can be simplified as

$$\begin{aligned}
V(t) &< e^{\alpha t}\left[V(0) + \alpha\int_0^t e^{-\alpha\tau}\mathbf{w}^T(\tau)\mathbf{w}(\tau)\,d\tau\right]\\
&< e^{\alpha T}\left[V(0) + \delta\left(1 - e^{-\alpha T}\right)\right].
\end{aligned} \tag{30}$$

On the other hand,

$$\begin{aligned}
V(t) &\ge V_1(t) = \mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) \ge \lambda_{\min}\left(\widetilde{\mathbf{P}}_1\right)\mathbf{x}^T(t)\mathbf{R}\mathbf{x}(t),\\
V(0) &\le \boldsymbol{\phi}^T(\theta)\mathbf{P}_1\boldsymbol{\phi}(\theta) + h\boldsymbol{\phi}^T(\theta)\mathbf{P}_2\boldsymbol{\phi}(\theta)\\
&\quad + h\boldsymbol{\phi}^T(\theta)\mathbf{P}_3\boldsymbol{\phi}(\theta) + \tau\boldsymbol{\phi}^T(\theta)\mathbf{P}_4\boldsymbol{\phi}(\theta)\\
&\quad + \tau\boldsymbol{\phi}^T(\theta)\mathbf{P}_5\boldsymbol{\phi}(\theta)\\
&\le \lambda_{\max}\left(\widetilde{\mathbf{P}}_1\right)\boldsymbol{\phi}^T(\theta)\mathbf{R}\boldsymbol{\phi}(\theta) + h\lambda_{\max}\left(\widetilde{\mathbf{P}}_2\right)\boldsymbol{\phi}^T(\theta)\mathbf{R}\boldsymbol{\phi}(\theta)\\
&\quad + h\lambda_{\max}\left(\widetilde{\mathbf{P}}_3\right)\boldsymbol{\phi}^T(\theta)\mathbf{R}\boldsymbol{\phi}(\theta)\\
&\quad + \tau\lambda_{\max}\left(\widetilde{\mathbf{P}}_4\right)\boldsymbol{\phi}^T(\theta)\mathbf{R}\boldsymbol{\phi}(\theta)\\
&\quad + \tau\lambda_{\max}\left(\widetilde{\mathbf{P}}_5\right)\boldsymbol{\phi}^T(\theta)\mathbf{R}\boldsymbol{\phi}(\theta)\\
&\le \lambda_{\max}\left(\widetilde{\mathbf{P}}_1\right)c_1 + h\lambda_{\max}\left(\widetilde{\mathbf{P}}_2\right)c_1 + h\lambda_{\max}\left(\widetilde{\mathbf{P}}_3\right)c_1\\
&\quad + \tau\lambda_{\max}\left(\widetilde{\mathbf{P}}_4\right)c_1 + \tau\lambda_{\max}\left(\widetilde{\mathbf{P}}_5\right)c_1.
\end{aligned} \tag{31}$$

Then, formula (27) can be written as

$$\begin{aligned}
&\mathbf{x}^T(t)\mathbf{R}\mathbf{x}(t)\\
&\quad \le \frac{c_1\left[\lambda_2 + h\lambda_3 + h\lambda_4 + \tau\lambda_5 + \tau\lambda_6\right] + \delta\left(1 - e^{-\alpha T}\right)}{\lambda_1 e^{-\alpha T}},
\end{aligned} \tag{32}$$

which can be guaranteed by condition (16). This completes the proof. □

According to Theorem 7, we will obtain the resilient robust finite-time L_2-L_∞ controller for a class of uncertain neutral system with mixed time-varying delays.

Theorem 8. *Given positive scalars c_1, T, δ, and α, positive definite symmetric matrix $\mathbf{R}$, and time-delay parameters $h > 0$, $h_d > 0$, $\tau > 0$, and $\tau_d > 0$, the closed-loop neutral system Σ is FTB with respect to $(c_1, c_2, \delta, \mathbf{R}, T)$ and satisfies the cost function* (11) *for all admissible disturbance $\mathbf{w}(t)$, if there exist positive scalars c_2 and β and symmetric positive definite matrices $\mathbf{P}_i, i = 1, 2, \ldots, 6$, $\mathbf{Q}_i, i = 1, 2, \ldots, 4$, $\mathbf{W}_i, i = 1, 2, \ldots, 6$, such that conditions* (15) *and* (16) *and the following LMI hold:*

$$\mathbf{\Psi} = \begin{bmatrix} -\mathbf{P}_1 & \widehat{\mathbf{F}}^T \\ * & -\beta\mathbf{I} \end{bmatrix} < 0. \tag{33}$$

Proof. Similar to the proof of Theorem 7, (29) can be rewritten as

$$e^{-\alpha t}V(t) < \alpha\int_0^t e^{-\alpha\tau}\mathbf{w}^T(\tau)\mathbf{w}(\tau)\,d\tau. \tag{34}$$

Then, we have

$$\mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) \le V(t) < \alpha e^{\alpha T}\int_0^t \mathbf{w}^T(\tau)\mathbf{w}(\tau)\,d\tau. \tag{35}$$

From (33), we can obviously get

$$\widehat{\mathbf{F}}^T\widehat{\mathbf{F}} < \beta\mathbf{P}_1. \tag{36}$$

Considering system Σ, we have

$$\mathbf{y}^T(t)\mathbf{y}(t) = \left[\widehat{\mathbf{F}}\mathbf{x}(t)\right]^T\left[\widehat{\mathbf{F}}\mathbf{x}(t)\right] = \mathbf{x}^T(t)\widehat{\mathbf{F}}^T\widehat{\mathbf{F}}\mathbf{x}(t). \tag{37}$$

Combining (35)–(37), we can obtain

$$\mathbf{x}^T(t)\widehat{\mathbf{F}}^T\widehat{\mathbf{F}}\mathbf{x}(t) \le \beta\mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) \le \beta\alpha e^{\alpha T}\int_0^t \mathbf{w}^T(\tau)\mathbf{w}(\tau)\,d\tau; \tag{38}$$

that is,

$$\mathbf{y}^T(t)\mathbf{y}(t) \le \beta\alpha e^{\alpha T}\int_0^t \mathbf{w}^T(\tau)\mathbf{w}(\tau)\,d\tau. \tag{39}$$

Letting $\gamma^2 = \beta\alpha e^{\alpha T}$, we have $\|\mathbf{y}(t)\|_\infty^2 < \gamma^2\|\mathbf{w}(t)\|_2^2$. This completes the proof. □

Theorem 9. *Given positive scalars c_1, T, δ, and α, positive definite symmetric matrix $\mathbf{R}$, and time-delay parameters $h > 0$, $h_d > 0$, $\tau > 0$, and $\tau_d > 0$, the closed-loop neutral system Σ is FTB with respect to $(c_1, c_2, \delta, \mathbf{R}, T)$, satisfies the cost function (11) for all admissible disturbance $\mathbf{w}(t)$, and exists as a state feedback controller in the form of (7) with $\mathbf{K} = \mathbf{U}\mathbf{P}_1^{-1}$, if there exist positive scalars c_2, β, ε_i, $i = 1, 2, \ldots, 4$, and μ_i, $i = 1, 2, \ldots, 5$, and symmetric positive definite matrices L_i, $i = 1, 2, \ldots, 10$, T_i, $i = 1, 2, \ldots, 6$, $\overline{\mathbf{Q}}_i$, $i = 1, 2, \ldots, 4$, $\mathbf{P}_i$, $i = 2, 3, \ldots, 5$, $\overline{\mathbf{P}}_6$, and $\mathbf{U}$, such that the following LMIs are feasible:*

$$\widetilde{\mathbf{\Pi}} = \begin{bmatrix} \widetilde{\mathbf{\Pi}}_1 & \widetilde{\mathbf{\Pi}}_2 & \widetilde{\mathbf{\Pi}}_3 & \widetilde{\mathbf{\Pi}}_7 \\ * & \widetilde{\mathbf{\Pi}}_4 & \widetilde{\mathbf{\Pi}}_5 & \widetilde{\mathbf{\Pi}}_8 \\ * & * & \widetilde{\mathbf{\Pi}}_6 & \widetilde{\mathbf{\Pi}}_9 \\ * & * & * & \widetilde{\mathbf{\Pi}}_{10} \end{bmatrix} < 0, \tag{40}$$

$$\widetilde{\mathbf{\Psi}} = \begin{bmatrix} -\mathbf{L}_1 & \mathbf{L}_1\mathbf{F}^T + \mathbf{U}^T\mathbf{G}^T & \mathbf{L}_1\mathbf{H}_1^T & \mathbf{L}_1\mathbf{S}^T \\ * & \widetilde{\mathbf{\Psi}}_{22} & 0 & 0 \\ * & * & -\varepsilon_3\mathbf{I} & 0 \\ * & * & * & -\varepsilon_4\mathbf{I} \end{bmatrix} < 0, \tag{41}$$

$$\mu_1\mathbf{R}^{-1} < \mathbf{L}_1 < \mathbf{R}^{-1}, \tag{42}$$

$$0 < \mathbf{P}_2 < \mu_2\mathbf{R}, \tag{43}$$

$$0 < \mathbf{P}_3 < \mu_3\mathbf{R}, \tag{44}$$

$$0 < \mathbf{P}_4 < \mu_4\mathbf{R}, \tag{45}$$

$$0 < \mathbf{P}_5 < \mu_5\mathbf{R}, \tag{46}$$

$$\begin{bmatrix} c_1\left[h(\mu_2+\mu_3)+\tau(\mu_4+\mu_5)\right]+\delta\left(1-e^{-\alpha T}\right)-c_2e^{-\alpha T} & \sqrt{c_1} \\ * & -\mu_1 \end{bmatrix} < 0, \tag{47}$$

where

$$\widetilde{\mathbf{\Pi}}_1 = \left[\widetilde{\mathbf{\Pi}}_{ij}\right]_{7\times 7}, \qquad \widetilde{\mathbf{\Pi}}_{11} = \mathbf{L}_1\mathbf{A}^T + \mathbf{A}\mathbf{L}_1 + \mathbf{U}^T\mathbf{B}^T + \mathbf{B}\mathbf{U} + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4 + \mathbf{L}_5 + \mathbf{T}_1 + \mathbf{T}_3 + \mathbf{T}_4 + \mathbf{T}_6$$
$$+ \mathbf{T}_1^T + \mathbf{T}_3^T + \mathbf{T}_4^T + \mathbf{T}_6^T - \alpha\mathbf{L}_1 - \frac{\alpha}{\tau}\mathbf{L}_6 - 2\alpha\mathbf{L}_7 - 2\alpha\mathbf{L}_8 - 2\alpha\mathbf{L}_9 - 2\alpha\mathbf{L}_{10},$$

$$\widetilde{\mathbf{\Pi}}_{12} = \mathbf{A}_d\mathbf{L}_1 - \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_1^T + \mathbf{T}_3^T + \mathbf{T}_4^T + \mathbf{T}_6^T, \qquad \widetilde{\mathbf{\Pi}}_{13} = \mathbf{C}\mathbf{L}_1 + \mathbf{T}_1^T + \mathbf{T}_3^T + \mathbf{T}_4^T + \mathbf{T}_6^T,$$

$$\widetilde{\mathbf{\Pi}}_{14} = -\mathbf{T}_4 + \mathbf{T}_5 + \mathbf{T}_1^T + \mathbf{T}_3^T + \mathbf{T}_4^T + \mathbf{T}_6^T + \frac{\alpha}{\tau}\mathbf{L}_6, \qquad \widetilde{\mathbf{\Pi}}_{15} = -\mathbf{T}_2 - \mathbf{T}_3 + \mathbf{T}_1^T + \mathbf{T}_3^T + \mathbf{T}_4^T + \mathbf{T}_6^T,$$

$$\widetilde{\mathbf{\Pi}}_{16} = -\mathbf{T}_5 - \mathbf{T}_6 + \mathbf{T}_1^T + \mathbf{T}_3^T + \mathbf{T}_4^T + \mathbf{T}_6^T, \qquad \widetilde{\mathbf{\Pi}}_{17} = \mathbf{D}, \qquad \widetilde{\mathbf{\Pi}}_{22} = -\left(1-h_d\right)\mathbf{L}_2 - \mathbf{T}_1 + \mathbf{T}_2 - \mathbf{T}_1^T + \mathbf{T}_2^T,$$

$$\widetilde{\mathbf{\Pi}}_{23} = -\mathbf{T}_1^T + \mathbf{T}_2^T, \qquad \widetilde{\mathbf{\Pi}}_{24} = -\mathbf{T}_4 + \mathbf{T}_5 - \mathbf{T}_1^T + \mathbf{T}_2^T, \qquad \widetilde{\mathbf{\Pi}}_{25} = -\mathbf{T}_2 - \mathbf{T}_3 - \mathbf{T}_1^T + \mathbf{T}_2^T,$$

$$\widetilde{\mathbf{\Pi}}_{26} = -\mathbf{T}_5 - \mathbf{T}_6 - \mathbf{T}_1^T + \mathbf{T}_2^T, \qquad \widetilde{\mathbf{\Pi}}_{27} = 0, \qquad \widetilde{\mathbf{\Pi}}_{33} = -\left(1-\tau_d\right)\mathbf{L}_6, \qquad \widetilde{\mathbf{\Pi}}_{34} = -\mathbf{T}_4 + \mathbf{T}_5,$$

$$\widetilde{\mathbf{\Pi}}_{35} = -\mathbf{T}_2 - \mathbf{T}_3, \qquad \widetilde{\mathbf{\Pi}}_{36} = -\mathbf{T}_5 - \mathbf{T}_6, \qquad \widetilde{\mathbf{\Pi}}_{37} = 0, \qquad \widetilde{\mathbf{\Pi}}_{44} = -\left(1-\tau_d\right)\mathbf{L}_4 - \mathbf{T}_4 + \mathbf{T}_5 - \mathbf{T}_4^T + \mathbf{T}_5^T - \frac{\alpha}{\tau}\mathbf{L}_6,$$

$$\widetilde{\mathbf{\Pi}}_{45} = -\mathbf{T}_2 - \mathbf{T}_3 - \mathbf{T}_4^T + \mathbf{T}_5^T, \qquad \widetilde{\mathbf{\Pi}}_{46} = -\mathbf{T}_5 - \mathbf{T}_6 - \mathbf{T}_4^T + \mathbf{T}_5^T, \qquad \widetilde{\mathbf{\Pi}}_{47} = 0,$$

$$\widetilde{\mathbf{\Pi}}_{55} = -\mathbf{L}_3 - \mathbf{T}_2 - \mathbf{T}_3 - \mathbf{T}_2^T - \mathbf{T}_3^T, \qquad \widetilde{\mathbf{\Pi}}_{56} = -\mathbf{T}_5 - \mathbf{T}_6 - \mathbf{T}_2^T - \mathbf{T}_3^T, \qquad \widetilde{\mathbf{\Pi}}_{57} = 0,$$

$$\widetilde{\mathbf{\Pi}}_{66} = -\mathbf{L}_5 - \mathbf{T}_5 - \mathbf{T}_6 - \mathbf{T}_5^T - \mathbf{T}_6^T, \qquad \widetilde{\mathbf{\Pi}}_{67} = 0, \qquad \widetilde{\mathbf{\Pi}}_{77} = -\alpha\mathbf{I},$$

$$\widetilde{\mathbf{\Pi}}_2 = \begin{bmatrix} h\mathbf{T}_1 & h\mathbf{T}_2 & h\mathbf{T}_3 & \tau\mathbf{T}_4 & \tau\mathbf{T}_5 & \tau\mathbf{T}_6 \\ h\mathbf{T}_1 & h\mathbf{T}_2 & h\mathbf{T}_3 & \tau\mathbf{T}_4 & \tau\mathbf{T}_5 & \tau\mathbf{T}_6 \\ h\mathbf{T}_1 & h\mathbf{T}_2 & h\mathbf{T}_3 & \tau\mathbf{T}_4 & \tau\mathbf{T}_5 & \tau\mathbf{T}_6 \\ h\mathbf{T}_1 & h\mathbf{T}_2 & h\mathbf{T}_3 & \tau\mathbf{T}_4 & \tau\mathbf{T}_5 & \tau\mathbf{T}_6 \\ h\mathbf{T}_1 & h\mathbf{T}_2 & h\mathbf{T}_3 & \tau\mathbf{T}_4 & \tau\mathbf{T}_5 & \tau\mathbf{T}_6 \\ h\mathbf{T}_1 & h\mathbf{T}_2 & h\mathbf{T}_3 & \tau\mathbf{T}_4 & \tau\mathbf{T}_5 & \tau\mathbf{T}_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{\mathbf{\Pi}}_3 = \begin{bmatrix} \mathbf{L}_1\mathbf{A}^T + \mathbf{U}^T\mathbf{B}^T & h\mathbf{L}_1\mathbf{A}^T + h\mathbf{U}^T\mathbf{B}^T & h\mathbf{L}_1\mathbf{A}^T + h\mathbf{U}^T\mathbf{B}^T & \tau\mathbf{L}_1\mathbf{A}^T + \tau\mathbf{U}^T\mathbf{B}^T & \tau\mathbf{L}_1\mathbf{A}^T + \tau\mathbf{U}^T\mathbf{B}^T \\ \mathbf{L}_1\mathbf{A}_d^T & h\mathbf{L}_1\mathbf{A}_d^T & h\mathbf{L}_1\mathbf{A}_d^T & \tau\mathbf{L}_1\mathbf{A}_d^T & \tau\mathbf{L}_1\mathbf{A}_d^T \\ \mathbf{L}_1\mathbf{C}^T & h\mathbf{C}^T\mathbf{Q}_1 & h\mathbf{C}^T\mathbf{Q}_2 & \tau\mathbf{C}^T\mathbf{Q}_3 & \tau\mathbf{C}^T\mathbf{Q}_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbf{D}^T & h\mathbf{D}^T & h\mathbf{D}^T & \tau\mathbf{D}^T & \tau\mathbf{D}^T \end{bmatrix},$$

$$\widetilde{\mathbf{\Pi}}_4 = \operatorname{diag}\left\{-h\mathbf{L}_7 \;\; -h\mathbf{L}_7 \;\; -h\mathbf{L}_8 \;\; -\tau\mathbf{L}_9 \;\; -\tau\mathbf{L}_9 \;\; -\tau\mathbf{L}_{10}\right\}, \qquad \widetilde{\mathbf{\Pi}}_5 = [0]_{6\times 5},$$

$$\widetilde{\mathbf{\Pi}}_6 = \operatorname{diag}\left\{-\overline{\mathbf{P}}_6 \;\; -h\overline{\mathbf{Q}}_1 \;\; -h\overline{\mathbf{Q}}_2 \;\; -\tau\overline{\mathbf{Q}}_3 \;\; -\tau\overline{\mathbf{Q}}_3\right\}, \qquad \widetilde{\mathbf{\Pi}}_7 = \begin{bmatrix} \mathbf{L}_1\mathbf{H}_1^T & \mathbf{L}_1\mathbf{S}^T & \varepsilon_1\mathbf{M}_1 & \varepsilon_2\mathbf{BN} \\ \mathbf{L}_1\mathbf{H}_2^T & 0 & 0 & 0 \\ \mathbf{L}_1\mathbf{H}_3^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{L}_1\mathbf{H}_4^T & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{\mathbf{\Pi}}_8 = [0]_{6\times 4}, \qquad \widetilde{\mathbf{\Pi}}_{10} = \operatorname{diag}\left\{-\varepsilon_1\mathbf{I} \;\; -\varepsilon_1\mathbf{I} \;\; -\varepsilon_2\mathbf{I} \;\; -\varepsilon_2\mathbf{I}\right\}, \qquad \widetilde{\mathbf{\Psi}}_{22} = -\beta\mathbf{I} + \varepsilon_3\mathbf{M}^T\mathbf{M}_2^T + \varepsilon_4\mathbf{G}\mathbf{N}^T\mathbf{N}\mathbf{G}^T. \tag{48}$$

Proof. Replacing $\widehat{\mathbf{A}}$, $\overline{\mathbf{A}}_d$, $\overline{\mathbf{C}}$, and $\overline{\mathbf{D}}$ in (15) with $\widehat{\mathbf{A}} = \overline{\mathbf{A}} + \Delta\overline{\mathbf{A}}(t)$, $\overline{\mathbf{A}} = \mathbf{A} + \mathbf{BK}$, $\Delta\overline{\mathbf{A}}(t) = \Delta\mathbf{A}(t) + \mathbf{B}\Delta\mathbf{K}(t)$, $\overline{\mathbf{A}}_d = \mathbf{A}_d + \Delta\mathbf{A}_d(t)$, $\overline{\mathbf{C}} = \mathbf{C} + \Delta\mathbf{C}(t)$, and $\overline{\mathbf{D}} = \mathbf{D} + \Delta\mathbf{D}(t)$, respectively, we have

$$\mathbf{\Pi} = \overline{\mathbf{\Pi}} + \Delta\overline{\mathbf{\Pi}} < 0, \tag{49}$$

where

$$\overline{\mathbf{\Pi}} = \begin{bmatrix} \overline{\mathbf{\Pi}}_1 & \mathbf{\Pi}_2 & \overline{\mathbf{\Pi}}_3 \\ * & \mathbf{\Pi}_4 & \mathbf{\Pi}_5 \\ * & * & \mathbf{\Pi}_6 \end{bmatrix} < 0, \qquad \overline{\mathbf{\Pi}}_1 = \left[\overline{\mathbf{\Pi}}_{ij}\right]_{7\times 7}, \qquad \overline{\mathbf{\Pi}}_1 = \begin{bmatrix} \overline{\mathbf{\Pi}}_{11} & \overline{\mathbf{\Pi}}_{12} & \overline{\mathbf{\Pi}}_{13} & \mathbf{\Pi}_{14} & \mathbf{\Pi}_{15} & \mathbf{\Pi}_{16} & \mathbf{P}_1\mathbf{D} \\ * & \mathbf{\Pi}_{22} & \mathbf{\Pi}_{23} & \mathbf{\Pi}_{24} & \mathbf{\Pi}_{25} & \mathbf{\Pi}_{26} & \mathbf{\Pi}_{27} \\ * & * & \mathbf{\Pi}_{33} & \mathbf{\Pi}_{34} & \mathbf{\Pi}_{35} & \mathbf{\Pi}_{36} & \mathbf{\Pi}_{37} \\ * & * & * & \mathbf{\Pi}_{44} & \mathbf{\Pi}_{45} & \mathbf{\Pi}_{46} & \mathbf{\Pi}_{47} \\ * & * & * & * & \mathbf{\Pi}_{55} & \mathbf{\Pi}_{56} & \mathbf{\Pi}_{57} \\ * & * & * & * & * & \mathbf{\Pi}_{66} & \mathbf{\Pi}_{67} \\ * & * & * & * & * & * & \mathbf{\Pi}_{77} \end{bmatrix},$$

$$\begin{aligned} \overline{\mathbf{\Pi}}_{11} = {} & \overline{\mathbf{A}}^T\mathbf{P}_1 + \mathbf{P}_1\overline{\mathbf{A}} + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5 + \mathbf{W}_1 + \mathbf{W}_3 + \mathbf{W}_4 + \mathbf{W}_6 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T \\ & - \alpha\mathbf{P}_1 - \frac{\alpha}{\tau}\mathbf{P}_6 - 2\alpha\mathbf{Q}_1 - 2\alpha\mathbf{Q}_2 - 2\alpha\mathbf{Q}_3 - 2\alpha\mathbf{Q}_4, \\ \overline{\mathbf{\Pi}}_{12} = {} & \mathbf{P}_1\mathbf{A}_d - \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \qquad \overline{\mathbf{\Pi}}_{13} = \mathbf{P}_1\mathbf{C} + \mathbf{W}_1^T + \mathbf{W}_3^T + \mathbf{W}_4^T + \mathbf{W}_6^T, \end{aligned}$$

$$\overline{\mathbf{\Pi}}_3 = \begin{bmatrix} \overline{\mathbf{A}}^T\mathbf{P}_6 & h\overline{\mathbf{A}}^T\mathbf{Q}_1 & h\overline{\mathbf{A}}^T\mathbf{Q}_2 & \tau\overline{\mathbf{A}}^T\mathbf{Q}_3 & \tau\overline{\mathbf{A}}^T\mathbf{Q}_4 \\ \mathbf{A}_d^T\mathbf{P}_6 & h\mathbf{A}_d^T\mathbf{Q}_1 & h\mathbf{A}_d^T\mathbf{Q}_2 & \tau\mathbf{A}_d^T\mathbf{Q}_3 & \tau\mathbf{A}_d^T\mathbf{Q}_4 \\ \mathbf{C}^T\mathbf{P}_6 & h\mathbf{C}^T\mathbf{Q}_1 & h\mathbf{C}^T\mathbf{Q}_2 & \tau\mathbf{C}^T\mathbf{Q}_3 & \tau\mathbf{C}^T\mathbf{Q}_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbf{D}^T\mathbf{P}_6 & h\mathbf{D}^T\mathbf{Q}_1 & h\mathbf{D}^T\mathbf{Q}_2 & \tau\mathbf{D}^T\mathbf{Q}_3 & \tau\mathbf{D}^T\mathbf{Q}_4 \end{bmatrix}, \qquad \Delta\overline{\mathbf{\Pi}} = \begin{bmatrix} \Delta\overline{\mathbf{\Pi}}_1 & 0 & \Delta\overline{\mathbf{\Pi}}_3 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} < 0,$$

$$\Delta\overline{\mathbf{\Pi}}_1 = \begin{bmatrix} \Delta\overline{\mathbf{A}}^T(t)\mathbf{P}_1 + \mathbf{P}_1\Delta\overline{\mathbf{A}}(t) & \mathbf{P}_1\Delta\overline{\mathbf{A}}_d(t) & \mathbf{P}_1\Delta\overline{\mathbf{C}}(t) & 0 & 0 & 0 & \mathbf{P}_1\Delta\overline{\mathbf{D}}(t) \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix},$$

$$\Delta\overline{\mathbf{\Pi}}_3 = \begin{bmatrix} \Delta\overline{\mathbf{A}}^T(t)\mathbf{P}_6 & h\Delta\overline{\mathbf{A}}^T(t)\mathbf{Q}_1 & h\Delta\overline{\mathbf{A}}^T(t)\mathbf{Q}_2 & \tau\Delta\overline{\mathbf{A}}^T(t)\mathbf{Q}_3 & \tau\Delta\overline{\mathbf{A}}^T(t)\mathbf{Q}_4 \\ \Delta\mathbf{A}_d^T(t)\mathbf{P}_6 & h\Delta\mathbf{A}_d^T(t)\mathbf{Q}_1 & h\Delta\mathbf{A}_d^T(t)\mathbf{Q}_2 & \tau\Delta\mathbf{A}_d^T(t)\mathbf{Q}_3 & \tau\Delta\mathbf{A}_d^T(t)\mathbf{Q}_4 \\ \Delta\mathbf{C}^T(t)\mathbf{P}_6 & h\Delta\mathbf{C}^T(t)\mathbf{Q}_1 & h\Delta\mathbf{C}^T(t)\mathbf{Q}_2 & \tau\Delta\mathbf{C}^T(t)\mathbf{Q}_3 & \tau\Delta\mathbf{C}^T(t)\mathbf{Q}_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Delta\mathbf{D}^T(t)\mathbf{P}_6 & h\Delta\mathbf{D}^T(t)\mathbf{Q}_1 & h\Delta\mathbf{D}^T(t)\mathbf{Q}_2 & \tau\Delta\mathbf{D}^T(t)\mathbf{Q}_3 & \tau\Delta\mathbf{D}^T(t)\mathbf{Q}_4 \end{bmatrix}, \tag{50}$$

bring formulas (4) and (8) into $\Delta\overline{\mathbf{\Pi}}$, and according to Lemma 6, we have

$$\mathbf{\Gamma}_{d1}\boldsymbol{\sigma}(t)\mathbf{\Gamma}_{e1} + \left(\mathbf{\Gamma}_{d1}\boldsymbol{\sigma}(t)\mathbf{\Gamma}_{e1}\right)^T \le \varepsilon_1\mathbf{\Gamma}_{d1}\mathbf{\Gamma}_{d1}^T + \varepsilon_1^{-1}\mathbf{\Gamma}_{e1}^T\mathbf{\Gamma}_{e1},$$

$$\mathbf{\Gamma}_{d2}\boldsymbol{\eta}(t)\mathbf{\Gamma}_{e2} + \left(\mathbf{\Gamma}_{d2}\boldsymbol{\eta}(t)\mathbf{\Gamma}_{e2}\right)^T \le \varepsilon_2\mathbf{\Gamma}_{d2}\mathbf{\Gamma}_{d2}^T + \varepsilon_2^{-1}\mathbf{\Gamma}_{e2}^T\mathbf{\Gamma}_{e2}, \tag{51}$$

where

$$\begin{gathered} \mathbf{\Gamma}_{d1} = \left[\mathbf{M}_1^T\mathbf{P}_1 \;\; 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 \;\; \mathbf{M}_1^T\mathbf{P}_6 \;\; h\mathbf{M}_1^T\mathbf{Q}_1 \;\; h\mathbf{M}_1^T\mathbf{Q}_2 \;\; \tau\mathbf{M}_1^T\mathbf{Q}_3 \;\; \tau\mathbf{M}_1^T\mathbf{Q}_4\right]^T, \\ \mathbf{\Gamma}_{e1} = \left[\mathbf{H}_1 \;\; \mathbf{H}_2 \;\; \mathbf{H}_3 \;\; 0\;0\;0 \;\; \mathbf{H}_4 \;\; 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\right], \\ \mathbf{\Gamma}_{d2} = \left[\mathbf{N}^T\mathbf{B}^T\mathbf{P}_1 \;\; 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 \;\; \mathbf{N}^T\mathbf{B}^T\mathbf{P}_6 \;\; h\mathbf{N}^T\mathbf{B}^T\mathbf{Q}_1 \;\; h\mathbf{N}^T\mathbf{B}^T\mathbf{Q}_2 \;\; \tau\mathbf{N}^T\mathbf{B}^T\mathbf{Q}_3 \;\; \tau\mathbf{N}^T\mathbf{B}^T\mathbf{Q}_4\right]^T, \\ \mathbf{\Gamma}_{e2} = \left[\mathbf{S} \;\; 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\right]. \end{gathered} \tag{52}$$

Considering

$$\begin{aligned} \Delta\overline{\mathbf{\Pi}} &= \mathbf{\Gamma}_{d1}\boldsymbol{\sigma}(t)\mathbf{\Gamma}_{e1} + \left(\mathbf{\Gamma}_{d1}\boldsymbol{\sigma}(t)\mathbf{\Gamma}_{e1}\right)^T \\ &\quad + \mathbf{\Gamma}_{d2}\boldsymbol{\eta}(t)\mathbf{\Gamma}_{e2} + \left(\mathbf{\Gamma}_{d2}\boldsymbol{\eta}(t)\mathbf{\Gamma}_{e2}\right)^T \\ &\le \varepsilon_1\mathbf{\Gamma}_{d1}\mathbf{\Gamma}_{d1}^T + \varepsilon_1^{-1}\mathbf{\Gamma}_{e1}^T\mathbf{\Gamma}_{e1} + \varepsilon_2\mathbf{\Gamma}_{d2}\mathbf{\Gamma}_{d2}^T + \varepsilon_2^{-1}\mathbf{\Gamma}_{e2}^T\mathbf{\Gamma}_{e2}, \end{aligned} \tag{53}$$

(49) can be guaranteed by

$$\overline{\mathbf{\Pi}} + \varepsilon_1\mathbf{\Gamma}_{d1}\mathbf{\Gamma}_{d1}^T + \varepsilon_1^{-1}\mathbf{\Gamma}_{e1}^T\mathbf{\Gamma}_{e1} + \varepsilon_2\mathbf{\Gamma}_{d2}\mathbf{\Gamma}_{d2}^T + \varepsilon_2^{-1}\mathbf{\Gamma}_{e2}^T\mathbf{\Gamma}_{e2} < 0. \tag{54}$$

Using Schur complement, equality (54) can be rewritten as

$$\widehat{\mathbf{\Pi}} = \begin{bmatrix} \overline{\mathbf{\Pi}}_1 & \mathbf{\Pi}_2 & \overline{\mathbf{\Pi}}_3 & \widehat{\mathbf{\Pi}}_7 \\ * & \mathbf{\Pi}_4 & \overline{\mathbf{\Pi}}_5 & \widehat{\mathbf{\Pi}}_8 \\ * & * & \overline{\mathbf{\Pi}}_6 & \widehat{\mathbf{\Pi}}_9 \\ * & * & * & \widehat{\mathbf{\Pi}}_{10} \end{bmatrix} < 0, \tag{55}$$

where

$$\widehat{\mathbf{\Pi}}_7 = \begin{bmatrix} \mathbf{S}_1^T & \mathbf{N}^T & \varepsilon_1\mathbf{P}_1\mathbf{M}_1 & \varepsilon_2\mathbf{P}_1\mathbf{BN} \\ \mathbf{S}_2^T & 0 & 0 & 0 \\ \mathbf{S}_3^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{S}_4^T & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\mathbf{\Pi}}_9 = \begin{bmatrix} 0 & 0 & \varepsilon_1\mathbf{P}_6\mathbf{M}_1 & \varepsilon_2\mathbf{P}_6\mathbf{BN} \\ 0 & 0 & h\varepsilon_1\mathbf{Q}_1\mathbf{M}_1 & h\varepsilon_2\mathbf{Q}_1\mathbf{BN} \\ 0 & 0 & h\varepsilon_1\mathbf{Q}_2\mathbf{M}_1 & h\varepsilon_2\mathbf{Q}_2\mathbf{BN} \\ 0 & 0 & \tau\varepsilon_1\mathbf{Q}_3\mathbf{M}_1 & \tau\varepsilon_2\mathbf{Q}_3\mathbf{BN} \\ 0 & 0 & \tau\varepsilon_1\mathbf{Q}_4\mathbf{M}_1 & \tau\varepsilon_2\mathbf{Q}_4\mathbf{BN} \end{bmatrix},$$

$$\widehat{\mathbf{\Pi}}_8 = [0]_{6\times 4},$$
$$\widehat{\mathbf{\Pi}}_{10} = \operatorname{diag}\{-\varepsilon_1\mathbf{I} \quad -\varepsilon_2\mathbf{I} \quad -\varepsilon_1\mathbf{I} \quad -\varepsilon_2\mathbf{I}\}. \tag{56}$$

Letting $\mathbf{L}_1 = \mathbf{P}_1^{-1}$, $\mathbf{L}_2 = \mathbf{L}_1\mathbf{P}_2\mathbf{L}_1$, $\mathbf{L}_3 = \mathbf{L}_1\mathbf{P}_3\mathbf{L}_1$, $\mathbf{L}_4 = \mathbf{L}_1\mathbf{P}_4\mathbf{L}_1$, $\mathbf{L}_5 = \mathbf{L}_1\mathbf{P}_5\mathbf{L}_1$, $\mathbf{L}_6 = \mathbf{L}_1\mathbf{P}_6\mathbf{L}_1$, $\mathbf{L}_7 = \mathbf{L}_1\mathbf{Q}_1\mathbf{L}_1$, $\mathbf{L}_8 = \mathbf{L}_1\mathbf{Q}_2\mathbf{L}_1$, $\mathbf{L}_9 = \mathbf{L}_1\mathbf{Q}_3\mathbf{L}_1$, $\mathbf{L}_{10} = \mathbf{L}_1\mathbf{Q}_4\mathbf{L}_1$, $\mathbf{T}_1 = \mathbf{L}_1\mathbf{W}_1\mathbf{L}_1$, $\mathbf{T}_2 = \mathbf{L}_1\mathbf{W}_2\mathbf{L}_1$, $\mathbf{T}_3 = \mathbf{L}_1\mathbf{W}_3\mathbf{L}_1$, $\mathbf{T}_3 = \mathbf{L}_1\mathbf{W}_3\mathbf{L}_1$, $\mathbf{T}_4 = \mathbf{L}_1\mathbf{W}_4\mathbf{L}_1$, $\mathbf{T}_5 = \mathbf{L}_1\mathbf{W}_5\mathbf{L}_1$, $\mathbf{T}_6 = \mathbf{L}_1\mathbf{W}_6\mathbf{L}_1$, $\overline{\mathbf{P}}_6 = \mathbf{P}_6^{-1}$, $\overline{\mathbf{Q}}_1 = \mathbf{Q}_1^{-1}$, $\overline{\mathbf{Q}}_2 = \mathbf{Q}_2^{-1}$, $\overline{\mathbf{Q}}_3 = \mathbf{Q}_3^{-1}$, $\overline{\mathbf{Q}}_4 = \mathbf{Q}_4^{-1}$, and $\mathbf{U} = \mathbf{KL}_1$, we can obtain condition (40) by pre- and postmultiplying inequality (55) by block-diagonal matrix

$$\operatorname{diag}\{\mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{I} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_1^{-1} \quad \mathbf{P}_6^{-1} \quad \mathbf{Q}_1^{-1} \quad \mathbf{Q}_2^{-1} \quad \mathbf{Q}_3^{-1} \quad \mathbf{Q}_4^{-1} \quad \mathbf{I} \quad \mathbf{I} \quad \mathbf{I} \quad \mathbf{I}\}. \tag{57}$$

Next, we will prove that condition (33) is equivalent to (41).

Considering

$$\mathbf{\Psi} = \overline{\mathbf{\Psi}} + \Delta\overline{\mathbf{\Psi}} < 0, \tag{58}$$

where

$$\overline{\mathbf{\Psi}} = \begin{bmatrix} -\mathbf{P}_1 & \overline{\mathbf{F}}^T \\ * & -\beta\mathbf{I} \end{bmatrix}, \qquad \Delta\overline{\mathbf{\Psi}} = \begin{bmatrix} 0 & \Delta\overline{\mathbf{F}}^T(t) \\ * & 0 \end{bmatrix}, \tag{59}$$

combining with formulas (5) and (8), and using Schur complement, we have

$$\begin{aligned} \Delta\overline{\mathbf{\Psi}} &= \mathbf{\Gamma}_{d3}\boldsymbol{\sigma}(t)\mathbf{\Gamma}_{e3} + \left(\mathbf{\Gamma}_{d3}\boldsymbol{\sigma}(t)\mathbf{\Gamma}_{e3}\right)^T \\ &\quad + \mathbf{\Gamma}_{d4}\boldsymbol{\eta}(t)\mathbf{\Gamma}_{e4} + \left(\mathbf{\Gamma}_{d4}\boldsymbol{\eta}(t)\mathbf{\Gamma}_{e4}\right)^T \\ &\le \varepsilon_3\mathbf{\Gamma}_{d3}\mathbf{\Gamma}_{d3}^T + \varepsilon_3^{-1}\mathbf{\Gamma}_{e3}^T\mathbf{\Gamma}_{e3} + \varepsilon_4\mathbf{\Gamma}_{d4}\mathbf{\Gamma}_{d4}^T + \varepsilon_4^{-1}\mathbf{\Gamma}_{e4}^T\mathbf{\Gamma}_{e4}, \end{aligned} \tag{60}$$

where $\mathbf{\Gamma}_{d3} = \begin{bmatrix} 0 & \mathbf{M}_2^T \end{bmatrix}^T$, $\mathbf{\Gamma}_{e3} = \begin{bmatrix} \mathbf{H}_1 & 0 \end{bmatrix}$, $\mathbf{\Gamma}_{d4} = \begin{bmatrix} 0 & \mathbf{N}^T\mathbf{G}^T \end{bmatrix}^T$, and $\mathbf{\Gamma}_{e4} = \begin{bmatrix} \mathbf{S} & 0 \end{bmatrix}$.

Then, we can get the following inequality which ensures (58):

$$\overline{\mathbf{\Psi}} + \varepsilon_3\mathbf{\Gamma}_{d3}\mathbf{\Gamma}_{d3}^T + \varepsilon_3^{-1}\mathbf{\Gamma}_{e3}^T\mathbf{\Gamma}_{e3} + \varepsilon_4\mathbf{\Gamma}_{d4}\mathbf{\Gamma}_{d4}^T + \varepsilon_4^{-1}\mathbf{\Gamma}_{e4}^T\mathbf{\Gamma}_{e4} < 0, \tag{61}$$

Using the Schur complement, equality (61) can be rewritten as

$$\widehat{\mathbf{\Psi}} = \begin{bmatrix} -\mathbf{P}_1 & \mathbf{F}^T + \mathbf{K}^T\mathbf{G}^T & \mathbf{H}_1^T & \mathbf{S}^T \\ * & \widetilde{\mathbf{\Psi}}_{22} & 0 & 0 \\ * & * & -\varepsilon_3\mathbf{I} & 0 \\ * & * & * & -\varepsilon_4\mathbf{I} \end{bmatrix} < 0. \tag{62}$$

Then, we can obtain condition (40) by pre- and post-multiplying inequality (62) by block-diagonal matrix $\operatorname{diag}\{\mathbf{P}_1^{-1} \quad \mathbf{I} \quad \mathbf{I} \quad \mathbf{I}\}$.

Denoting $\widetilde{\mathbf{L}}_1 = \mathbf{R}^{1/2}\mathbf{L}_1\mathbf{R}^{1/2}$, $\widetilde{\mathbf{P}}_2 = \mathbf{R}^{-1/2}\mathbf{P}_2\mathbf{R}^{-1/2}$, $\widetilde{\mathbf{P}}_3 = \mathbf{R}^{-1/2}\mathbf{P}_3\mathbf{R}^{-1/2}$, $\widetilde{\mathbf{P}}_4 = \mathbf{R}^{-1/2}\mathbf{P}_4\mathbf{R}^{-1/2}$, and $\widetilde{\mathbf{P}}_5 = \mathbf{R}^{-1/2}\mathbf{P}_5\mathbf{R}^{-1/2}$, we know that condition (16) is equivalent to (47) according to conditions (42)–(46). This completes the proof. □

4. Simulation Example

In this part, we consider a class of neutral time-varying delayed systems with parameters described as

$$\begin{gathered} \mathbf{A} = \begin{bmatrix} 1.5 & 0.2 \\ 2.1 & 0.9 \end{bmatrix}, \qquad \mathbf{A}_d = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1.0 \\ 0.8 \end{bmatrix}, \\ \mathbf{C} = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \mathbf{D} = \begin{bmatrix} 0.1 & 0.2 \\ -0.2 & 0.1 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 1.5 & 1.7 \\ 0.2 & 0.9 \end{bmatrix}, \\ \mathbf{G} = \begin{bmatrix} 2 \\ -1.5 \end{bmatrix}, \qquad \mathbf{M}_1 = \begin{bmatrix} 1.1 \\ -0.7 \end{bmatrix}, \qquad \mathbf{M}_2 = \begin{bmatrix} 0.8 \\ -0.4 \end{bmatrix}, \\ \mathbf{H}_1 = \begin{bmatrix} 1.4 & 0.8 \end{bmatrix}, \qquad \mathbf{H}_2 = \begin{bmatrix} 0.4 & 1.1 \end{bmatrix}, \\ \mathbf{H}_3 = \begin{bmatrix} 0.7 & 0.2 \end{bmatrix}, \qquad \mathbf{H}_4 = \begin{bmatrix} 0.5 & 1.3 \end{bmatrix}, \\ \mathbf{N} = [0.2], \qquad \mathbf{S} = \begin{bmatrix} 0.2 & 0.6 \end{bmatrix}. \end{gathered} \tag{63}$$

In this note, we choose the initial values for $c_1 = 1$, $T = 5$, $\alpha = 0.3$, and $\delta = 1.0$ and the upper bounds on the delays are $\tau = 0.8$, $h = 0.5$, $h_d = 0.9$, and $\tau_d = 0.9$. By using the LMI toolbox in MATLAB to solve LMIs (40)–(47), we can get the finite-time L_2-L_∞ controller gain as follows:

$$\begin{gathered} \mathbf{L}_1 = \begin{bmatrix} 0.6515 & -0.1789 \\ -0.1789 & 0.3827 \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} -0.3115 & -0.0343 \end{bmatrix}, \\ \mathbf{K} = \mathbf{UL}_1^{-1} = \begin{bmatrix} -0.5768 & -0.3593 \end{bmatrix}, \end{gathered} \tag{64}$$

with constraint conditions $\beta = 14.7085$, $\gamma = 0.9923$, and $c_2 = 124.6975$.

Selecting $h(t) = 0.9/(1+t^2)$, $\tau(t) = 0.11/(3+t^2)$, $\boldsymbol{\sigma}(t) = (0.9/(1+t^2))\mathbf{I}$, $\boldsymbol{\eta}(t) = (1.5/(1+t^2))\mathbf{I}$, and $h(t) = 0.9/(1+t^2)$,

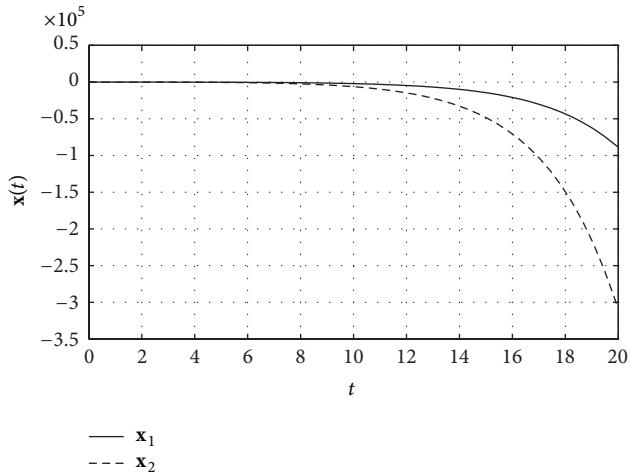

Figure 1: The trajectories of open-loop controlled system state $\mathbf{x}(t)$.

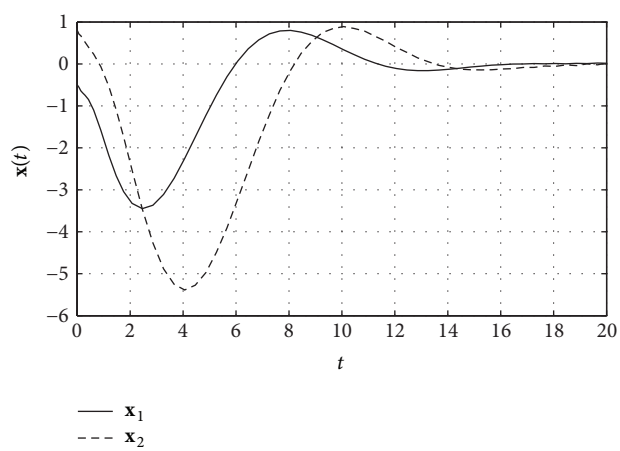

Figure 2: The trajectories of closed-loop controlled system state $\mathbf{x}(t)$.

$t \in [0, 20]$, and setting the initial states $\mathbf{x}_0 = [-0.5 \quad 0.8]^T$ and $\mathbf{w}_0 = [0.04 \quad 0.08]^T$, we have the open-loop controlled system state simulation graph and the trajectories of closed-loop controlled system state and output as shown in Figures 1, 2, and 3, respectively. Figure 4 shows the evolution of function $\mathbf{x}^T(t)\mathbf{R}\mathbf{x}(t)$ $(t \in [0, 20])$ of the uncertain neutral time-delayed system Σ_0. Based on comparison between result in Figure 1 and result in Figure 2, we noted that the design finite-time L_2-L_∞ controller can make the closed-loop controlled system achieve FTB.

5. Conclusions

This paper studied the delay-dependent resilient robust finite-time L_2-L_∞ control problem for a class of uncertain neutral time-delayed system with mixed time-varying delays. A state feedback controller is designed by using LMI technique and free weighting matrices, such that the closed-loop

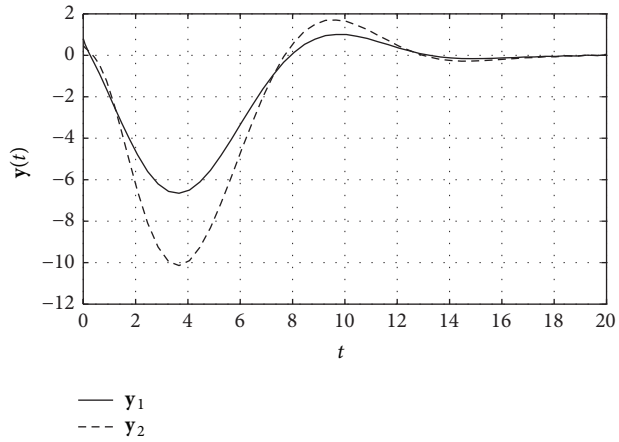

Figure 3: The trajectories of closed-loop controlled system output $\mathbf{y}(t)$.

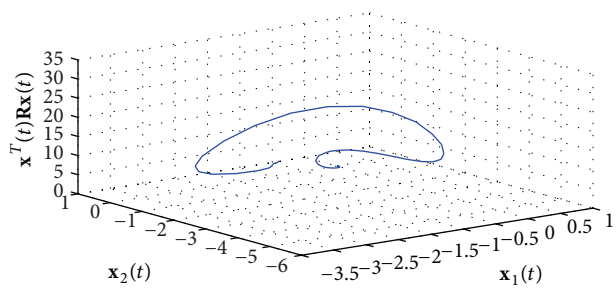

Figure 4: The graph of $\mathbf{x}^T(t)\mathbf{R}\mathbf{x}(t)$ $(t \in [0, T])$ of closed-loop controlled system.

controlled system is FTB and satisfies the input-output L_2-L_∞ performance matrices. The simulation results verify the effectiveness of the design method. We will consider the finite-time observer for neutral time-delayed system in the future.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Output Feedback Adaptive Stabilization of Uncertain Nonholonomic Systems

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This paper investigates the problem of output feedback adaptive stabilization control design for a class of nonholonomic chained systems with uncertainties, involving virtual control coefficients, unknown nonlinear parameters, and unknown time delays. The objective is to design a robust nonlinear output-feedback switching controller, which can guarantee the stabilization of the closed loop systems. An observer and an estimator are employed for states and parameters estimates, respectively. A constructive controller design procedure is proposed by applying input-state scaling transformation, parameter separation technique, and backstepping recursive approach. Simulation results are provided to show the effectiveness of the proposed method.

1. Introduction

The control and feedback stabilization problems of nonholonomic systems have been widely studied by many researchers. It is well known that control of nonholonomic systems is extremely challenging, largely due to the impossibility of asymptotically stabilizing nonholonomic systems via smooth time-invariant state feedback, a well-recognized fact pointed out in [1, 2]. In order to overcome this obstruction, a number of approaches have been proposed for the problem, which mainly include discontinuous feedback, time-varying feedback, and hybrid stabilization. The discontinuous feedback stabilization was first proposed by [3], and then further discussion was made in [4–7]; especially an elegant discontinuous coordinate transformation approach is proposed in [5] for the stabilization problem of nonholonomic systems. Meanwhile, the smooth time-varying feedback control strategies also have drawn much attention [8–11].

As pointed out in [9], many nonlinear mechanical systems with nonholonomic constraints can be transformed, either locally or globally, to the nonholonomic systems in the so-called chained form. So far, there have been a number of controller design approaches [8–25] for such chained nonholonomic systems. Recently, adaptive control strategies have been proposed to stabilize the nonholonomic systems. For instance, the problem of adaptive state-feedback control is studied in [15–19], while output feedback controller design in [20–24]. Considering the actual modeling perspective, time delay should be taken into account. The problem of state feedback stabilization is studied for the delayed nonholonomic systems in [25, 26]. However, the virtual control coefficients and unknown parameter vector are not considered in its system models. Here, an iterative controller design method will be proposed for the output feedback adaptive stabilization of the concerned delayed nonholomic systems.

In this paper, we study a class of chained nonholonomic systems with strong nonlinear drifts, and the problem of adaptive output-feedback stabilization for the concerned nonholonomic systems is investigated. The constructive design method proposed in this note is based on a combined application of the input scaling technique, the backstepping

recursive approach, and the novel Lyapunov-Krasovskii functionals. The switching control strategy for the first subsystem is employed to achieve the asymptotic stabilization.

The rest of this paper is organized as follows. In Section 2, the problem formulation and some preliminary knowledge are given. Section 3 presents the controller design procedure and stability analysis. Section 4 gives the switching control strategy. In Section 5, numerical simulations testify to the effectiveness of the proposed method, and Section 6 summarizes the paper.

2. Problem Formulation and Preliminaries

In this paper, we deal with a class of nonholonomic systems described by

$$\begin{aligned}
\dot{x}_0(t) &= d_0 u_0(t) + \phi_0(t, x_0(t)),\\
\dot{x}_1(t) &= d_1 u_0(t) x_2(t) + \varphi_1(u_0(t), y(t), y(t-\tau_1))\\
&\quad + \phi_1(t, u_0(t), x_0(t), x(t), \theta),\\
&\vdots\\
\dot{x}_{n-1}(t) &= d_{n-1} u_0(t) x_n(t) + \varphi_{n-1}(u_0(t), y(t), y(t-\tau_{n-1}))\\
&\quad + \phi_{n-1}(t, u_0(t), x_0(t), x(t), \theta),\\
\dot{x}_n(t) &= d_n u_1(t) + \varphi_n(u_0(t), y(t), y(t-\tau_n))\\
&\quad + \phi_n(t, u_0(t), x_0(t), x(t), \theta),\\
y(t) &= [x_0(t), x_1(t)]^T,
\end{aligned} \tag{1}$$

where $[x_0(t), x(t)]^T = [x_0(t), x_1(t), \ldots, x_n(t)]^T \in R^{n+1}$, $u(t) = [u_0(t), u_1(t)]^T \in R^2$, and $y(t) \in R^2$ are system states, control input, and measurable output, respectively; $\theta \in R^m$ is an unknown parameter vector; ϕ_0 (known) and ϕ_i $(1 \le i \le n)$ (unknown) denote the possible modeling error and neglected dynamics; φ_i $(1 \le i \le n)$ are known modeled dynamics, which contain output delays; τ_i $(1 \le i \le n)$ are unknown constants, and d_i $(0 \le i \le n)$ referred to the respective virtual control coefficients.

In this paper, we make the following assumptions on the virtual control directions d_i and nonlinear functions φ_i, ϕ_i in system (1).

Assumption 1. d_0 is a known constant and the sign of $\overline{d}_n$ is known, where $\overline{d}_n = d_1 d_2 \cdots d_n$.

Assumption 2. There exist known smooth nonnegative functions $\overline{\phi}_o$ and $\overline{\phi}_i$ $(1 \le i \le n)$ such that

$$\begin{aligned}
&\phi_0(t, x_0(t)) = x_0 \overline{\phi}_0(x_0(t)),\\
&|\phi_i(t, u_0(t), x_0(t), x(t), \theta)|\\
&\quad \le |x_1| \overline{\phi}_i(u_0(t), x_0(t), x_1(t), \theta).
\end{aligned} \tag{2}$$

for all $(t, u_0(t), x_0(t), x(t), \theta) \in R_+ \times R \times R \times R^n \times R^m$.

Assumption 3. For every $1 \le i \le n$, the nonlinear function φ_i satisfies inequality

$$\begin{aligned}
|\varphi_i(u_0(t), y(t), y(t-\tau_i))| &\le |x_1(t)| \psi_i(u_0(t), y(t))\\
&\quad + |x_1(t) x_1(t-\tau_i)|\\
&\quad \times \overline{\varphi}_i(u_0(t), y(t), y(t-\tau_i)),
\end{aligned} \tag{3}$$

in which $\overline{\varphi}_i$ and ψ_i are known smooth nonnegative nonlinear functions.

Remark 4. Compared with some existing literatures in recent years, the structure of our concerned system (1) is more general. For instance, in [15], it is assumed that not only the virtual control directions $d_i = 1$ and the dynamics ϕ_i satisfy $\phi_i = \widetilde{\phi}_i^T \theta$, but also the modeled dynamics φ_i do not exist. In [22], the virtual control coefficients and time delays have not been considered, and the expression $\phi_i = \widetilde{\phi}_i^T \theta$ is also required. While $d_i = 1$ and φ_i and unknown parameters θ are not existent, system (1) degenerates to the one studied in [21]. When $\varphi_i = 0$, together with $\phi_i = \widetilde{\phi}_i^T \theta$, system (1) becomes the considered system in [23].

Remark 5. Note that here we only use the sign of $\overline{d}_n = d_1 d_2 \cdots d_n$ without any knowledge of individual virtual control direction d_i $(1 \le i \le n)$. Moreover, Assumptions 2 and 3 are imposed on the nonlinear functions ϕ_i and φ_i, respectively. In fact, if the modeled dynamics φ_i do not involve time delays, inequality (3) is reduced into

$$|\varphi_i(u_0(t), y(t))| \le |x_1(t)| \psi_i(u_0(t), y(t)). \tag{4}$$

It can be seen that the above inequality condition is used in some existing literatures, such as [20, 21], and so on.

Our object of this paper is to design adaptive output feedback control laws under Assumptions 1–3, such that the system states $(x_0(t), x(t))$ converge to zero, while other signals of the closed-loop system are bounded. The designed control laws can be expressed in the following form:

$$\begin{gathered}
u_0 = \mu_0(y(t)), \qquad u_1 = \mu_1(y(t), \nu(t)),\\
\dot{\nu}(t) = \varrho(y(t), \nu(t)).
\end{gathered} \tag{5}$$

Next, we list some lemmas which will be applied in the coming controller design.

Lemma 6 (see [27]). *For any real-valued continuous function $f(x, y)$, where $x \in R^n, y \in R^m$, there are smooth functions $a(x) \ge 0, b(y) \ge 0, c(x) \ge 1, d(y) \ge 1$ such that*

$$|f(x, y)| \le a(x) + b(y), \quad |f(x, y)| \le c(x) d(y). \tag{6}$$

Lemma 7 (see [19]). *For any continuous function $\mu_0(t)$ there exist two strictly positive real numerates $p_{\min}$ and $p_{\max}$ such that the unique solution $P(t)$ of the following matrix differential equation:*

$$\dot{P} = P(A - \mu_0(t)L)^T + (A - \mu_0(t)L)P - PC^TCP + I, \quad P(0) = P_0 > 0, \tag{7}$$

satisfies $p_{\min} I \le P(t) \le p_{\max} I,\ t \ge 0.$

By Lemma 6 and Assumption 1, we know that there exist smooth functions $\omega_i \ge 1$, and $\zeta_i \ge 1$ such that

$$\begin{aligned} &|\phi_i(t, u_0(t), x_0(t), x(t), \theta)| \\ &\quad \le |x_1| \omega_i(u_0(t), x_0(t), x_1(t)) \zeta_i(\theta). \end{aligned} \tag{8}$$

Furthermore, we denote $\vartheta = \sum_{i=1}^{n} \zeta_i(\theta)$; then it yields

$$\begin{aligned} &|\phi_i(t, u_0(t), x_0(t), x(t), \theta)| \\ &\quad \le |x_1| \omega_i(u_0(t), x_0(t), x_1(t)) \vartheta. \end{aligned} \tag{9}$$

3. Output Feedback Adaptive Stabilization Control Design

In this paper, we design control laws $u_0(t)$ and $u_1(t)$ separately to globally asymptotically stabilize the system (1). According to the structure of system (1), we can see that when $x_0(t)$ converges to zero, $x_i(t)$ $(1 \le i \le n)$ will be uncontrollable. A widely used method to design control law $u_1(t)$ is to introduce a discontinuous input scaling transformation (12). On the other hand, the control directions d_i are unknown; then we should employ another coordinate transformation to overcome the obstacle.

3.1. State Coordinate Transformation. Firstly, we design the coordinate transformation as follows:

$$\overline{x}_i(t) = \overline{d}_{i-1} x_i(t), \quad 1 \le i \le n, \tag{10}$$

where $\overline{d}_0 = 1$ and $\overline{d}_{i-1} = d_1 d_2 \cdots d_{i-1}$ $(1 \le i \le n+1)$. Then, the system (1) can be transformed into

$$\begin{aligned} \dot{x}_0(t) &= d_0 u_0(t) + \phi_0(t, x_0(t)), \\ \dot{\overline{x}}_1(t) &= u_0(t)\overline{x}_2(t) + \varphi_1(u_0(t), y(t), y(t-\tau_1)) \\ &\quad + \phi_1(t, u_0(t), x_0(t), x(t), \theta), \\ &\vdots \\ \dot{\overline{x}}_n(t) &= \overline{d}_n u_1(t) + \overline{d}_{n-1}\varphi_n(u_0(t), y(t), y(t-\tau_n)) \\ &\quad + \overline{d}_{n-1}\phi_n(t, u_0(t), x_0(t), x(t), \theta). \end{aligned} \tag{11}$$

Next, the following input-state scaling discontinuous transformation is introduced:

$$z_i(t) = \frac{\overline{x}_i(t)}{u_0^{n-i}(t)}, \quad 1 \le i \le n. \tag{12}$$

Under the new $z(t)$-coordinates, the $\overline{x}(t)$-subsystem (10) is changed into

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) - (n-1)\frac{\dot{u}_0(t)}{u_0(t)} z_1(t) \\ &\quad + \frac{1}{u_0^{n-1}(t)}(\varphi_1 + \phi_1), \\ \dot{z}_i(t) &= z_{i+1}(t) - (n-i)\frac{\dot{u}_0(t)}{u_0(t)} z_i(t) \\ &\quad + \frac{1}{u_0^{n-i}(t)}(\overline{d}_{i-1}\varphi_i + \overline{d}_{i-1}\phi_i), \\ \dot{z}_n(t) &= \overline{d}_n u_1(t) + \overline{d}_{n-1}\varphi_n + \overline{d}_{n-1}\phi_n. \end{aligned} \tag{13}$$

Next, we can design the control laws $u_0(t)$ and $u_1(t)$ to asymptotically stabilize the states $x_0(t)$ and $z(t)$, respectively. Rewrite system (13) in the compact form

$$\dot{z}(t) = \left(A - L\frac{\dot{u}_0(t)}{u_0(t)}\right) z(t) + Bu_1(t) + \Psi + \Phi, \tag{14}$$

where

$$\begin{gathered} A = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} n-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \\ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \overline{d}_n \end{bmatrix}, \qquad \Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_n \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} \end{gathered} \tag{15}$$

with

$$\begin{aligned} \Psi_i &= \overline{d}_{i-1} \frac{\varphi_i(u_0(t), y(t), y(t-\tau_i))}{u_0^{n-i}(t)}, \\ \Phi_i &= \overline{d}_{i-1} \frac{\phi_i^d(t, u_0(t), x_0(t), x(t), \theta)}{u_0^{n-i}(t)}. \end{aligned} \tag{16}$$

In order to obtain the estimation for the nonlinear functions Ψ_i and Φ_i, the following lemmas are given.

Lemma 8. *For every $1 \le i \le n$, there exists smooth nonnegative function $\widetilde{\omega}_i(u_0(t), x_0(t), z_1(t))$ such that*

$$|\Phi_i| \le |\overline{d}_{i-1}| \cdot |z_1(t)| \widetilde{\omega}_i(u_0(t), x_0(t), z_1(t)) \vartheta. \tag{17}$$

Lemma 9. *For every $1 \le i \le n$, there exist smooth nonnegative functions $\widetilde{\psi}_i, \widetilde{\varphi}_i, f_{i1}, f_{i2}$ such that*

$$\begin{aligned} |\Psi_i| &\le |\overline{d}_{i-1}| \cdot |z_1(t) z_1(t-\tau_i)| \\ &\quad \times \widetilde{\varphi}_i(u_0(t), u_0(t-\tau_i), y(t), y(t-\tau_i)) \\ &\quad + |\overline{d}_{i-1}| \cdot |z_1(t)| \widetilde{\psi}_i(u_0(t), y(t)) \\ &\le |\overline{d}_{i-1}| \cdot |z_1(t) z_1(t-\tau_i)| f_{i1}(u_0(t), y(t)) \end{aligned}$$

$$\times f_{i2}\left(u_0\left(t-\tau_i\right), y\left(t-\tau_i\right)\right) + \left|\overline{d}_{i-1}\right| \cdot \left|z_1(t)\right| \widetilde{\psi}_i\left(u_0(t), y(t)\right). \tag{18}$$

Remark 10. By lemmas and assumptions before, Lemmas 8 and 9 can be derived easily, and then the proof is omitted.

3.2. Observer Design. Define the following filter/estimator:

$$\dot{\xi}_0(t) = \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)\xi_0(t) + PC^T\left(y(t) - C\xi_0(t)\right), \tag{19}$$

$$\dot{\upsilon}(t) = \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)\upsilon(t) + e_n u_1(t), \tag{20}$$

$$\dot{P} = P\left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)^T + \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)P - PC^TCP + I, \tag{21}$$

where $y(t) = z_1(t), e_n = [0,\ldots,1]^T, \xi_0 = [\xi_{01},\ldots,\xi_{0n}]^T, \upsilon = [\upsilon_1,\ldots,\upsilon_n]^T$, $A_0 = A - KC, C = [1,0,\ldots,0]$, $K = [k_1,\ldots,k_n]^T$, and k_i $(1 \le i \le n)$ are design parameters to be determined later. Let $\widehat{z}(t) = \xi_0(t) + \overline{d}_n\upsilon, \sigma(t) = z(t) - \overline{d}_n\upsilon(t)$; then, the estimation error $\varepsilon(t) = z(t) - \widehat{z}(t)$ and the newly defined parameter $\sigma(t)$ satisfy the dynamical equations

$$\begin{aligned}\dot{\varepsilon}(t) &= \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)} - PC^TC\right)\varepsilon(t) \\ &\quad + \left(K - PC^T\right)z_1(t) + PC^TC\sigma(t) + \Psi + \Phi, \\ \dot{\sigma}(t) &= \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)\sigma(t) + Kz_1(t) + \Psi + \Phi.\end{aligned} \tag{22}$$

3.3. Control Design. In this section, the intergrator backstepping approach will be used to design the control laws $u_0(t)$ and $u_1(t)$ subject to $x_0(t_0) \neq 0$. The case that the initial condition $x_0(t_0) = 0$ will be treated in Section 4.

Step 0. At this step, control law $u_0(t)$ will be designed, which is essential to guarantee the effectiveness of the subsequent steps. For the $x_0(t)$-subsystem, choose the control $u_0(t)$ as follows:

$$u_0(t) = -\lambda_0 x_0(t) - \lambda_0 x_0(t)\overline{\phi}_0\left(x_0(t)\right), \tag{23}$$

where λ_0 is a constant satisfying $\lambda_0 d_0 > 1$. Introduce the Lyapunov function candidate $V_0 = (1/2)x_0^2(t)$, and the time derivative of V_0 satisfies

$$\begin{aligned}\dot{V}_0 &= -\lambda_0 d_0 x_0^2(t) - \lambda_0 d_0 x_0^2(t)\overline{\phi}_0\left(x_0(t)\right) \\ &\quad + x_0(t)\phi_0\left(t, x_0(t)\right) \\ &\le -\lambda_0 d_0 x_0^2(t) \triangleq -c_0 x_0^2(t),\end{aligned} \tag{24}$$

where $c_0 = \lambda_0 d_0 > 1$. This indicates that $x_0(t)$ converges to zero exponentially.

Since $\overline{\phi}_0(x_0(t))$ is a smooth function, then there exist a constant $M_0 > 1$, such that $|\overline{\phi}_0(x_0(t))| \le M_0$ for $|x_0(t)| \le 1$. Therefore, the following inequality is true with $|x_0(t)| \le 1$:

$$\dot{V}_0 \ge -\left(\lambda_0 d_0 + \lambda_0 d_0 M_0 + M_0\right)x_0^2(t) \triangleq -\rho x_0^2(t), \tag{25}$$

which implies that when $|x_0(t)| \le 1$, the state $x_0(t)$ converges to zero with a rate less than a certain constant ρ. It is $x_0(t)$ which does not become zero in any time instant. Therefore, the adopted input-state scaling discontinuous transformation in (12) is effective.

According to the design of control law $u_0(t)$ in (23), it can be computed that

$$\begin{aligned}\frac{\dot{u}_0(t)}{u_0(t)} &= -\lambda_0 d_0 - \left(\lambda_0 d_0 - 1\right)\overline{\phi}_0\left(x_0(t)\right) \\ &\quad - \lambda_0 d_0 x_0(t)\frac{\partial\overline{\phi}_0\left(x_0(t)\right)}{\partial x_0(t)} \\ &\quad + \frac{x_0(t)\overline{\phi}_0\left(x_0(t)\right)}{1+\overline{\phi}_0\left(x_0(t)\right)}\frac{\partial\overline{\phi}_0\left(x_0(t)\right)}{\partial x_0(t)} \\ &\triangleq \beta + \widetilde{\phi}_0\left(x_0(t)\right),\end{aligned} \tag{26}$$

where $\beta = -\lambda_0 d_0$ and $\widetilde{\phi}_0 = -(\lambda_0 d_0 - 1)\overline{\phi}_0(x_0(t)) - \lambda_0 d_0 x_0(t)(\partial\overline{\phi}_0(x_0(t))/\partial x_0(t)) + (x_0(t)\overline{\phi}_0(x_0(t))/(1 + \overline{\phi}_0(x_0(t))))(\partial\overline{\phi}_0(x_0(t))/\partial x_0(t))$.

Remark 11. From (26), we know that β is a constant and $\widetilde{\phi}_0(x_0(t))$ is a function with respect to $x_0(t)$. Moreover, we can conclude that $\widetilde{\phi}_0(x_0(t))$ is smooth because $\overline{\phi}_0(x_0(t))$ is a nonnegative smooth function.

Denote $A_1 = A_0 - KC - L\beta$; we can choose appropriate design parameters k_i $(1 \le i \le n)$ such that A_1 is Hurwitz. Then there exists a positive definite matrix Q satisfying $QA_1 + A_1^TQ = -\mu I$, and μ is a positive constant.

Step 1. For $z_1(t)$-subsystem in (13),

$$\begin{aligned}\dot{z}_1(t) &= z_2(t) - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t) \\ &\quad + \frac{1}{u_0^{n-1}(t)}\left(\varphi_1 + \phi_1\right) \\ &= \varepsilon_2(t) + \xi_{02}(t) + \overline{d}_n\upsilon_2(t) \\ &\quad - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t) + \Psi_1 + \Phi_1,\end{aligned} \tag{27}$$

let $\eta_1(t) = z_1(t)$, and $\eta_2(t) = \upsilon_2(t) - \alpha_1$. Introduce the following Lyapunov functional:

$$V_1 = \overline{V}_1 + \widetilde{V}_1, \tag{28}$$

where

$$\begin{aligned}\overline{V}_1 &= \varepsilon^T(t)P^{-1}\varepsilon(t)+\sigma^T(t)Q\sigma(t)\\&\quad+\frac{1}{2}\eta_1^2(t)+\frac{\left|\overline{d}_n\right|}{2}\widetilde{\Theta}_1^T\widetilde{\Theta}_1\\\widetilde{V}_1 &= \left(4\ell_1+\delta_2\|Q\|^2\right)\sum_{j=1}^{n}\int_{t-\tau_j}^{t}\eta_1^4(\sigma)f_{j2}^4\left(u_0(\sigma),\ y(\sigma)\right)d\sigma\\&\quad+\frac{n}{2}\int_{t-\tau_1}^{t}\eta_1^2(\sigma)f_{12}^2\left(u_0(\sigma),y(\sigma)\right)d\sigma,\end{aligned}\tag{29}$$

with ℓ_1,δ_2 being positive constants to be designed; $\widetilde{\Theta}_1=\Theta_1-\widehat{\Theta}_1$, where Θ_1 is an unknown parameter vector to be specified later, and $\widehat{\Theta}_1$ is an estimate of Θ_1.

Associated with (22) and (27), the time derivatives of $\overline{V}_1$ and $\widetilde{V}_1$ can be calculated, respectively, that

$$\begin{aligned}\dot{\overline{V}}_1 &= 2\varepsilon^T(t)P^{-1}\left(A_0-L\frac{\dot{u}_0(t)}{u_0(t)}-PC^TC\right)\varepsilon(t)\\&\quad+2\varepsilon^T(t)P^{-1}\left(K-PC^T\right)z_1(t)\\&\quad+2\varepsilon^T(t)C^TC\sigma(t)+2\varepsilon^T(t)P^{-1}\Psi\\&\quad+2\varepsilon^T(t)P^{-1}\Phi+2\sigma^T(t)Q\left(A_0-L\frac{\dot{u}_0(t)}{u_0(t)}\right)\sigma(t)\\&\quad+2\sigma^T(t)QKz_1(t)+2\sigma^T(t)Q\Psi\\&\quad+2\sigma^T(t)Q\Phi-2\varepsilon^T(t)\left(A_0-L\frac{\dot{u}_0(t)}{u_0(t)}\right)^TP^{-1}\varepsilon(t)\\&\quad+2\varepsilon^T(t)C^TC\varepsilon(t)-\varepsilon^T(t)P^{-2}\varepsilon(t)\\&\quad+\eta_1(t)\left[\varepsilon_2(t)+\xi_{02}(t)+\overline{d}_nv_2(t)-(n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right.\\&\qquad\left.+\Psi_1+\Phi_1\right]-\left|\overline{d}_n\right|\widetilde{\Theta}_1^T\dot{\widehat{\Theta}}_1\\&= -\varepsilon^T(t)P^{-2}\varepsilon(t)-\mu\sigma^T(t)\sigma(t)\\&\quad+2\varepsilon^T(t)P^{-1}\left(K-PC^T\right)z_1(t)+2\varepsilon^T(t)P^{-1}\Psi\\&\quad+2\varepsilon^T(t)P^{-1}\Phi+2\varepsilon^T(t)C^TC\sigma(t)\\&\quad-2\sigma^T(t)QL\widetilde{\phi}\left(x_0(t)\right)\sigma(t)+2\sigma^T(t)QKz_1(t)\\&\quad+2\sigma^T(t)Q\Psi+2\sigma^T(t)Q\Phi+\eta_1(t)\Psi_1+\eta_1(t)\Phi_1\\&\quad-(n-1)\frac{\dot{u}_0(t)}{u_0(t)}\eta_1^2(t)+\eta_1(t)\varepsilon_2(t)-\varepsilon^T(t)C^TC\varepsilon(t)\\&\quad+\eta_1(t)\left[\xi_{02}(t)+\overline{d}_nv_2(t)\right]-\left|\overline{d}_n\right|\widetilde{\Theta}_1^T\dot{\widehat{\Theta}}_1,\end{aligned}\tag{30}$$

$$\begin{aligned}\dot{\widetilde{V}}_1 &= \left(4\ell_1+\delta_2\|Q\|^2\right)\sum_{j=1}^{n}\eta_1^4(t)f_{j2}^4\left(u_0(t),y(t)\right)+\frac{n}{2}\eta_1^2(t)\\&\quad\times f_{12}^2\left(u_0(t),y(t)\right)-\left(4\ell_1+\delta_2\|Q\|^2\right)\sum_{j=1}^{n}\eta_1^4\left(t-\tau_j\right)\\&\quad\times f_{j2}^4\left(u_0\left(t-\tau_j\right),y\left(t-\tau_j\right)\right)-\frac{n}{2}\eta_1^2(t-\tau_1)\\&\quad\times f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right).\end{aligned}\tag{31}$$

For some terms on the right-hand side of (30), the following estimations (32)–(34) should be conducted. Firstly, by Lemma 8 and Young's inequality, we can obtain that there exist positive constants ℓ_1,δ_1 to make the following inequalities hold:

$$\begin{aligned}\eta_1(t)\Phi_1 &\le \eta_1^2(t)+\frac{1}{4}\eta_1^2(t)\widetilde{\omega}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\vartheta^2\\&\le \eta_1^2(t)+\frac{1}{4}\eta_1^2(t)\widetilde{\omega}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\vartheta_1,\\2\varepsilon^T(t)P^{-1}\Phi &\le \frac{1}{4\ell_1}\varepsilon^T(t)P^{-2}\varepsilon(t)\\&\quad+4\ell_1\sum_{j=1}^{n}\eta_1^2(t)\widetilde{\omega}_j^2\left(u_0(t),x_0(t),z_1(t)\right)\overline{d}_{j-1}^2\vartheta^2\\&\le \frac{1}{4\ell_1}\varepsilon^T(t)P^{-2}\varepsilon(t)+4\ell_1\sum_{j=1}^{n}\eta_1^2(t)\\&\quad\times\widetilde{\omega}_j^2\left(u_0(t),x_0(t),z_1(t)\right)\vartheta_1,\\2\sigma^T(t)Q\Phi &\le \frac{1}{\delta_1}\sigma^T(t)\sigma(t)\\&\quad+\delta_1\|Q\|^2\sum_{j=1}^{n}\eta_1^2(t)\widetilde{\omega}_j^2\left(u_0(t),x_0(t),z_1(t)\right)\vartheta_1,\end{aligned}\tag{32}$$

where $\vartheta_1=\vartheta^2+\sum_{j=1}^{n-1}\overline{d}_j^2\vartheta^2$. Next, employ Lemma 9 and Young's inequality, and we have

$$\begin{aligned}&\eta_1(t)\Psi_1\\&\quad\le \eta_1^2(t)\widetilde{\psi}_1\left(u_0(t),y(t)\right)+\frac{1}{2}\eta_1^4(t)f_{11}^2\left(u_0(t),y(t)\right)\\&\qquad+\frac{1}{2}\eta_1^2(t-\tau_1)f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right),\\&2\varepsilon^T(t)P^{-1}\Psi\\&\quad\le \frac{1}{4\ell_1}\varepsilon^T(t)P^{-2}\varepsilon(t)+4\ell_1\sum_{j=1}^{n}\Psi_j^2\\&\quad\le \frac{1}{4\ell_1}\varepsilon^T(t)P^{-2}\varepsilon(t)\end{aligned}$$

$$
\begin{aligned}
&+ 8\ell_1 \sum_{j=1}^{n} \eta_1^2(t)\, \widetilde{\psi}_j^2(u_0(t), y(t))\, \overline{d}_{j-1}^2 \\
&+ 4\ell_1 \sum_{j=1}^{n} \eta_1^4(t-\tau_j)\, f_{j2}^4(u_0(t-\tau_j), y(t-\tau_j)) \\
&+ 4\ell_1 \sum_{j=1}^{n} \eta_1^4(t)\, f_{j1}^4(u_0(t), y(t))\, \overline{d}_{j-1}^4 \\
\le\ & \frac{1}{4\ell_1} \varepsilon^T(t) P^{-2} \varepsilon(t) \\
&+ 8\ell_1 \sum_{j=1}^{n} \eta_1^2(t)\, \widetilde{\psi}_j^2(u_0(t), y(t))\, d \\
&+ 4\ell_1 \sum_{j=1}^{n} \eta_1^4(t-\tau_j)\, f_{j2}^4(u_0(t-\tau_j), y(t-\tau_j)) \\
&+ 4\ell_1 \sum_{j=1}^{n} \eta_1^4(t)\, f_{j1}^4(u_0(t), y(t))\, d, \\
2\sigma^T(t) Q\Psi \\
\le\ & \frac{1}{\delta_2} \sigma^T(t)\sigma(t) + 2\delta_2 \|Q\|^2 \sum_{j=1}^{n} \eta_1^2(t)\, \widetilde{\psi}_j^2(u_0(t), y(t))\, d \\
&+ \delta_2 \|Q\|^2 \sum_{j=1}^{n} \eta_1^4(t-\tau_j)\, f_{j2}^4(u_0(t-\tau_j), y(t-\tau_j)) \\
&+ \delta_2 \|Q\|^2 \sum_{j=1}^{n} \eta_1^4(t)\, f_{j1}^4(u_0(t), y(t))\, d,
\end{aligned}
\tag{33}
$$

where $d = 1 + \sum_{j=1}^{n-1} \overline{d}_j^2 + \sum_{j=1}^{n-1} \overline{d}_j^4$, and δ_2 is a positive constant.

By completing the square, the following estimations are also true:

$$
\begin{aligned}
\eta_1(t)\varepsilon_2(t) &\le \frac{1}{4\ell_1} \varepsilon^T(t) P^{-2} \varepsilon(t) + \ell_1 P_{\max}^2 \eta_1^2(t), \\
2\varepsilon^T(t) P^{-1} K z_1(t) &\le \frac{1}{4\ell_1} \varepsilon^T(t) P^{-2} \varepsilon(t) + 4\ell_1 K^T K \eta_1^2(t), \\
-2\varepsilon^T(t) C^T z_1(t) &\le \frac{1}{2} \varepsilon^T(t) C^T C \varepsilon(t) + 2\eta_1^2(t), \\
2\varepsilon^T(t) C^T C \sigma(t) &\le \frac{1}{2} \varepsilon^T(t) C^T C \varepsilon(t) + 2\sigma^T(t)\sigma(t), \\
2\sigma^T(t) QKz_1(t) &\le \sigma^T(t)\sigma(t) + K^T Q^T Q K \eta_1^2(t).
\end{aligned}
\tag{34}
$$

Substitute (31)–(34) into $\dot{V}_1$, it yields

$$
\begin{aligned}
\dot{V}_1 &= \dot{\overline{V}}_1 + \dot{\widetilde{V}}_1 \\
&\le -\left(1 - \frac{1}{\ell_1}\right) \varepsilon^T(t) P^{-2} \varepsilon(t) \\
&\quad - \overline{c}_1 \eta_1^2(t) - (n-1)\, \widetilde{\phi}(x_0(t))\, \eta_1^2(t) \\
&\quad - \overline{\mu} \sigma^T(t)\sigma(t) - \widetilde{\phi}(x_0(t))\, \sigma^T(t)[QL + LQ]\sigma(t) \\
&\quad - \left|\overline{d}_n\right| \widetilde{\Theta}_1^T \dot{\widehat{\Theta}}_1 \\
&\quad - \frac{n-1}{2} \eta_1^2(t-\tau_1)\, f_{12}^2(u_0(t-\tau_1), y(t-\tau_1)) \\
&\quad + \overline{d}_n \eta_1(t) \left[\Theta_1^T \Upsilon_1 + \upsilon_2(t)\right],
\end{aligned}
\tag{35}
$$

where $\overline{\mu} = \mu - 1/\delta_1 - 1/\delta_2 - 3$, $\overline{c}_1 = c_1 - 3 - K^T Q^T Q K - 4\ell_1 K^T K - \ell_1 P_{\max}^2 + (n-1)\beta$, $\Theta_1^T = (1/\overline{d}_n)[1, d, \vartheta_1]$, and $\Upsilon_1 = [\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}]^T$ with

$$
\begin{aligned}
\Upsilon_{11} &= c_1 \eta_1(t) + \xi_{02}(t) \\
&\quad + \eta_1(t)\, \widetilde{\psi}_1(u_0(t), y(t)) + \frac{1}{2} \eta_1^3(t)\, f_{11}^2(u_0(t), y(t)) \\
&\quad + \left(4\ell_1 + \delta_2 \|Q\|^2\right) \sum_{j=1}^{n} \eta_1^3(t)\, f_{j2}^4(u_0(t), y(t)) \\
&\quad + \frac{n}{2} \eta_1(t)\, f_{12}^2(u_0(t), y(t)), \\
\Upsilon_{12} &= 8\ell_1 \sum_{j=1}^{n} \eta_1(t)\, \widetilde{\psi}_j^2(u_0(t), y(t)) \\
&\quad + \left(4\ell_1 + \delta_2 \|Q\|^2\right) \sum_{j=1}^{n} \eta_1^3(t)\, f_{j1}^4(u_0(t), y(t)), \\
\Upsilon_{13} &= \left(4\ell_1 + \delta_1 \|Q\|^2\right) \sum_{j=1}^{n} \eta_1(t)\, \widetilde{\omega}_j^2(u_0(t), x_0(t), z_1(t)) \\
&\quad + \frac{1}{4} \eta_1(t)\, \widetilde{\omega}_1^2(u_0(t), x_0(t), z_1(t)).
\end{aligned}
\tag{36}
$$

Choose the virtual control function α_1 and the adaptation law of $\widehat{\Theta}_1$ as follows:

$$
\alpha_1 = -\widehat{\Theta}_1^T \Upsilon_1, \tag{37}
$$

$$
\dot{\widehat{\Theta}}_1 = \operatorname{sign}\left(\overline{d}_n\right) \Upsilon_1 \eta_1(t). \tag{38}
$$

Notice that $\overline{d}_n\eta_1(t)\eta_2(t) \le \eta_1^2(t) + (\overline{d}_n^2/4)\eta_2^2(t)$, then it follows from (35)–(38) that

$$\begin{aligned}\dot{V}_1 \le & -\left(1-\frac{1}{\ell_1}\right)\varepsilon^T(t)P^{-2}\varepsilon(t) - \overline{\mu}\sigma^T(t)\sigma(t) \\ & -(\overline{c}_1-1)\eta_1^2(t) - (n-1)\widetilde{\phi}(x_0(t))\eta_1^2(t) \\ & -\widetilde{\phi}(x_0(t))\sigma^T(t)[QL+LQ]\sigma(t) - \frac{n-1}{2} \\ & \times\eta_1^2(t-\tau_1)f_{12}^2(u_0(t-\tau_1), y(t-\tau_1)) + \frac{\overline{d}_n^2}{4}\eta_2^2(t).\end{aligned} \tag{39}$$

Step 2. Introduce the new variable $\eta_3(t) = v_3(t) - \alpha_2$, where α_2 is regarded as the virtual control input, and take the Lyapunov functional as

$$V_2 = V_1 + \frac{1}{2}\eta_2^2(t) + \frac{1}{2}\widetilde{\Theta}_2^T\widetilde{\Theta}_2, \tag{40}$$

where $\widetilde{\Theta}_2 = \Theta_2 - \widehat{\Theta}_2$, Θ_2 is an unknown parameter vector to be defined later, and $\widehat{\Theta}_2$ is an estimate of Θ_2. Then, combined with (20), (37), and (39), we have

$$\begin{aligned}\dot{V}_2 = \dot{V}_1 + \eta_2(t)\Big\{ & -k_2v_1(t) - (n-2)\beta v_2(t) \\ & -(n-2)\widetilde{\phi}_0(x_0(t))v_2(t) + \eta_3(t) + \alpha_2 \\ & -\frac{\partial\alpha_1}{\partial\widehat{\Theta}_1^T}\dot{\widehat{\Theta}}_1 - \frac{\partial\alpha_1}{\partial\xi_{02}}\dot{\xi}_{02} - \frac{\partial\alpha_1}{\partial x_0}\dot{x}_0 - \frac{\partial\alpha_1}{\partial u_0}\dot{u}_0 \\ & -\frac{\partial\alpha_1}{\partial z_1}\left[\xi_{02} - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right] \\ & -\frac{\partial\alpha_1}{\partial z_1}\varepsilon_2(t) - \frac{\partial\alpha_1}{\partial z_1}\Psi_1 - \frac{\partial\alpha_1}{\partial z_1}\Phi_1 \\ & -\frac{\partial\alpha_1}{\partial z_1}\overline{d}_n v_2(t)\Big\} - \widetilde{\Theta}_2^T\dot{\widehat{\Theta}}_2.\end{aligned} \tag{41}$$

Using Lemmas 8 and 9 and Young's inequality, the following inequalities hold:

$$\begin{aligned}& -\frac{\partial\alpha_1}{\partial z_1}\eta_2(t)\Psi_1 \\ & \le \frac{1}{2}\eta_1^2(t) + \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\widetilde{\psi}_1^2(u_0(t), x_0(t), z_1(t))\eta_2^2(t) \\ & \quad + \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\eta_1^2(t)f_{11}^2(u_0(t), y(t))\eta_2^2(t) \\ & \quad + \frac{1}{2}\eta_1^2(t-\tau_1)f_{12}^2(u_0(t-\tau_1), y(t-\tau_1)), \\ & -\frac{\partial\alpha_1}{\partial z_1}\eta_2(t)\Phi_1 \\ & \le \frac{1}{2}\eta_1^2(t) + \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\widetilde{\omega}_1^2(u_0(t), x_0(t), z_1(t))\eta_2^2(t)\vartheta^2, \\ & -\frac{\partial\alpha_1}{\partial z_1}\eta_2(t)\varepsilon_2(t) \\ & \le \frac{1}{\ell_2}\varepsilon^T(t)P^{-2}\varepsilon(t) + \frac{\ell_2}{4}p_{\max}^2\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\eta_2^2(t).\end{aligned} \tag{42}$$

By the above inequalities, we get

$$\begin{aligned}\dot{V}_2 \le & -\left(1-\frac{1}{\ell_1}-\frac{1}{\ell_2}\right)\varepsilon^T(t)P^{-2}\varepsilon(t) \\ & -\overline{\mu}\sigma^T(t)\sigma(t) - (\overline{c}_1-2)\eta_1^2(t) \\ & -(n-1)\widetilde{\phi}(x_0(t))\eta_1^2(t) \\ & -\widetilde{\phi}(x_0(t))\sigma^T(t)[QL+LQ]\sigma(t) \\ & -\frac{n-2}{2}\eta_1^2(t-\tau_1)f_{12}^2(u_0(t-\tau_1), y(t-\tau_1)) + \eta_2(t) \\ & \times\Big\{-k_2v_1(t) - (n-2)\beta v_2(t) - (n-2)\widetilde{\phi}(x_0(t)) \\ & \times v_2(t) - \frac{\partial\alpha_1}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1 - \frac{\partial\alpha_1}{\partial\xi_{02}}\dot{\xi}_{02} - \frac{\partial\alpha_1}{\partial x_0}\dot{x}_0 + \eta_3(t) \\ & -\frac{\partial\alpha_1}{\partial u_0}\dot{u}_0 - \frac{\partial\alpha_1}{\partial z_1}\left[\xi_{02} - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right] \\ & +\frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\eta_1^2(t)f_{11}^2(u_0(t), y(t))\eta_2(t) \\ & +\alpha_2 + \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\widetilde{\psi}_1^2(u_0(t), x_0(t), z_1(t))\eta_2(t) \\ & +\frac{\ell_2}{4}p_{\max}^2\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\eta_2(t) + \Theta_2^T\Upsilon_2\Big\} - \widetilde{\Theta}_2^T\dot{\widehat{\Theta}}_2,\end{aligned} \tag{43}$$

where $\Theta_2^T = [\vartheta^2, \overline{d}_n^2, \overline{d}_n]$ and $\Upsilon_2 = [(1/2)(\partial\alpha_1/\partial z_1)^2\widetilde{\omega}_1^2\eta_2(t),\ \eta_2(t)/4, -(\partial\alpha_1/\partial z_1)v_2(t)]^T$. By taking the adaptation law $\dot{\widehat{\Theta}}_2 = \Upsilon_2\eta_2(t)$ and the virtual control function α_2 as

$$\begin{aligned}\alpha_2 = & -c_2\eta_2(t) + k_2v_1(t) \\ & +(n-2)\beta v_2(t) + (n-2)\widetilde{\phi}(x_0(t))v_2(t) \\ & +\frac{\partial\alpha_1}{\partial z_1}\left[\xi_{02} - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right] + \frac{\partial\alpha_1}{\partial x_0}\dot{x}_0 \\ & +\frac{\partial\alpha_1}{\partial u_0}\dot{u}_0 + \frac{\partial\alpha_1}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1 + \frac{\partial\alpha_1}{\partial\xi_{02}}\dot{\xi}_{02}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\eta_1^2(t)\,f_{11}^2\left(u_0(t),y(t)\right)\eta_2(t)\\
&-\widehat{\Theta}_2^T\Upsilon_2-\frac{\ell_2}{4}p_{\max}^2\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\eta_2(t)\\
&-\frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1}\right)^2\widetilde{\psi}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\eta_2(t),
\end{aligned}
\tag{44}
$$

we can obtain

$$
\begin{aligned}
\dot{V}_2\le&-\left(1-\frac{1}{\ell_1}-\frac{1}{\ell_2}\right)\varepsilon^T(t)P^{-2}\varepsilon(t)\\
&-\overline{\mu}\sigma^T(t)\sigma(t)-\left(\overline{c}_1-2\right)\eta_1^2(t)-c_2\eta_2^2(t)\\
&-(n-1)\widetilde{\phi}\left(x_0(t)\right)\eta_1^2(t)\\
&-\widetilde{\phi}\left(x_0(t)\right)\sigma^T(t)\left[QL+LQ\right]\sigma(t)+\eta_2(t)\eta_3(t)\\
&-\frac{n-2}{2}\eta_1^2(t-\tau_1)f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right).
\end{aligned}
\tag{45}
$$

Step 3. Define that $\eta_4(t)=v_4(t)-\alpha_3$, where α_3 is the virtual control input, and consider the following Lyapunov functional:

$$
V_3=V_2+\frac{1}{2}\eta_3^2(t)+\frac{1}{2}\widetilde{\Theta}_3^T\widetilde{\Theta}_3.
\tag{46}
$$

The time derivative of V_3 along the estimator system (20) satisfies

$$
\begin{aligned}
\dot{V}_3=&\dot{V}_2+\eta_3(t)\\
&\times\Big\{-k_3v_1(t)-(n-3)\beta v_3(t)\\
&-(n-3)\widetilde{\phi}\left(x_0(t)\right)v_3(t)+\eta_4(t)\\
&+\alpha_3-\frac{\partial\alpha_2}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1-\frac{\partial\alpha_2}{\partial\widehat{\Theta}_2}\dot{\widehat{\Theta}}_2-\frac{\partial\alpha_2}{\partial\xi_{02}}\dot{\xi}_{02}-\frac{\partial\alpha_2}{\partial x_0}\dot{x}_0\\
&-\frac{\partial\alpha_2}{\partial u_0}\dot{u}_0-\frac{\partial\alpha_1}{\partial z_1}\left[\xi_{02}-(n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right]\\
&-\frac{\partial\alpha_2}{\partial v_1}\dot{v}_1-\frac{\partial\alpha_2}{\partial v_2}\dot{v}_2-\frac{\partial\alpha_1}{\partial z_1}\varepsilon_2(t)-\frac{\partial\alpha_2}{\partial z_1}\Psi_1-\frac{\partial\alpha_2}{\partial z_1}\Phi_1\\
&-\frac{\partial\alpha_2}{\partial z_1}\overline{d}_nv_2(t)\Big\}-\widetilde{\Theta}_3^T\dot{\widetilde{\Theta}}_3.
\end{aligned}
\tag{47}
$$

By similar conduction method in (42), we have

$$
\begin{aligned}
&-\frac{\partial\alpha_2}{\partial z_1}\eta_3(t)\Psi_1\\
&\quad\le\frac{1}{2}\eta_1^2(t)+\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\widetilde{\psi}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\eta_3^2(t)\\
&\qquad+\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\eta_1^2(t)\,f_{11}^2\left(u_0(t),y(t)\right)\eta_3^2(t)\\
&\qquad+\frac{1}{2}\eta_1^2(t-\tau_1)\,f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right),\\
&-\frac{\partial\alpha_2}{\partial z_1}\eta_3(t)\Phi_1\\
&\quad\le\frac{1}{2}\eta_1^2(t)+\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\widetilde{\omega}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\eta_3^2(t)\,\vartheta^2,\\
&-\frac{\partial\alpha_2}{\partial z_1}\eta_3(t)\varepsilon_2(t)\\
&\quad\le\frac{1}{\ell_3}\varepsilon^T(t)P^{-2}\varepsilon(t)+\frac{\ell_3}{4}p_{\max}^2\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\eta_3^2(t),
\end{aligned}
\tag{48}
$$

where $\ell_3>0$ is a scalar. Based on (48), it yields

$$
\begin{aligned}
\dot{V}_3\le&-\left(1-\frac{1}{\ell_1}-\frac{1}{\ell_2}-\frac{1}{\ell_3}\right)\varepsilon^T(t)P^{-2}\varepsilon(t)-\overline{\mu}\sigma^T(t)\sigma(t)\\
&-\left(\overline{c}_1-3\right)\eta_1^2(t)-(n-1)\widetilde{\phi}\left(x_0(t)\right)\eta_1^2(t)\\
&-\widetilde{\phi}\left(x_0(t)\right)\sigma^T(t)\left[QL+LQ\right]\sigma(t)\\
&-\frac{n-3}{2}\eta_1^2(t-\tau_1)\,f_{12}^2\left(y(t-\tau_1)\right)\\
&+\eta_3(t)\Big\{\eta_2(t)-k_3v_1(t)-(n-3)\beta v_3(t)-(n-3)\\
&\qquad\times\widetilde{\phi}\left(x_0(t)\right)v_3(t)+\eta_4(t)+\alpha_3-\frac{\partial\alpha_2}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1\\
&\qquad-\frac{\partial\alpha_2}{\partial\widehat{\Theta}_2}\dot{\widehat{\Theta}}_2-\frac{\partial\alpha_1}{\partial\xi_{02}}\dot{\xi}_{02}-\frac{\partial\alpha_1}{\partial x_0}\dot{x}_0-\frac{\partial\alpha_1}{\partial u_0}\dot{u}_0\\
&\qquad-\frac{\partial\alpha_1}{\partial z_1}\left[\xi_{02}-(n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right]\\
&\qquad-\frac{\partial\alpha_2}{\partial v_1}\dot{v}_1-\frac{\partial\alpha_2}{\partial v_2}\dot{v}_2+\frac{\ell_3}{4}p_{\max}^2\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\eta_3(t)\\
&\qquad+\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\eta_1^2(t)\,f_{11}^2\left(u_0(t),y(t)\right)\eta_3(t)
\end{aligned}
$$

$$+\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\widetilde{\psi}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\times\eta_3(t)+\Theta_3^T\Upsilon_3\Big\}-\widetilde{\Theta}_3^T\dot{\widehat{\Theta}}_3, \tag{49}$$

where $\Theta_3^T=[\vartheta^2,\overline{d}_n]$ and $\Upsilon_3=[(1/2)(\partial\alpha_2/\partial z_1)^2\widetilde{\omega}_1^2\eta_3(t), -(\partial\alpha_2/\partial z_1)v_2(t)]^T$. Choose the tuning function $\pi_3\Upsilon_3\eta_3(t)$, and the virtual control function α_3 as follows:

$$\begin{aligned}\alpha_3=&-c_3\eta_3(t)-\eta_2(t)+k_3v_1(t)\\&+(n-3)\beta v_3(t)+(n-3)\widetilde{\phi}\left(x_0(t)\right)v_3(t)+\frac{\partial\alpha_2}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1\\&+\frac{\partial\alpha_2}{\partial\widehat{\Theta}_2}\dot{\widehat{\Theta}}_2+\frac{\partial\alpha_2}{\partial\xi_{02}}\dot{\xi}_{02}+\frac{\partial\alpha_2}{\partial x_0}\dot{x}_0+\frac{\partial\alpha_2}{\partial u_0}\dot{u}_0\\&+\frac{\partial\alpha_2}{\partial z_1}\left[\xi_{02}-(n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right]+\frac{\partial\alpha_2}{\partial v_1}\dot{v}_1\\&+\frac{\partial\alpha_2}{\partial v_2}\dot{v}_2-\frac{\ell_3}{4}p_{\max}^2\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\eta_3(t)\\&-\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\eta_1^2(t)f_{11}^2\left(u_0(t),y(t)\right)\eta_3(t)-\widehat{\Theta}_3^T\Upsilon_3\\&-\frac{1}{2}\left(\frac{\partial\alpha_2}{\partial z_1}\right)^2\widetilde{\psi}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\eta_3(t).\end{aligned} \tag{50}$$

Under the virtual control function α_3 and the tuning function π_3 defined above, the derivative of V_3 becomes that

$$\begin{aligned}\dot{V}_3\le&-\left(1-\frac{1}{\ell_1}-\frac{1}{\ell_2}-\frac{1}{\ell_3}\right)\varepsilon^T(t)P^{-2}\varepsilon(t)\\&-\left(\overline{c}_1-3\right)\eta_1^2(t)-c_2\eta_2^2(t)-c_3\eta_3^2(t)\\&-\overline{\mu}\sigma^T(t)\sigma(t)-\widetilde{\phi}\left(x_0(t)\right)\sigma^T(t)[QL+LQ]\sigma(t)\\&-(n-1)\widetilde{\phi}\left(x_0(t)\right)\eta_1^2(t)+\eta_3(t)\eta_4(t)-\widetilde{\Theta}_3^T\left(\dot{\Theta}_3-\pi_3\right)\\&-\frac{n-3}{2}\eta_1^2(t-\tau_1)f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right).\end{aligned} \tag{51}$$

Step i ($4\le i\le n$). Assume that, at Step i–1, a virtual control function α_{i-1}, a tuning function π_{i-1}, and a Lyapunov functional V_{i-1} have been designed in such a way that

$$\begin{aligned}\dot{V}_{i-1}\le&-\left(1-\sum_{j=1}^{i-1}\frac{1}{\ell_j}\right)\varepsilon^T(t)P^{-2}\varepsilon(t)\\&-\left(\overline{c}_1-i+1\right)\eta_1^2(t)-\sum_{j=2}^{i-1}c_j\eta_j^2(t)\\&-\overline{\mu}\sigma^T(t)\sigma(t)+\eta_{i-1}(t)\eta_i(t)-\widetilde{\phi}\left(x_0(t)\right)\sigma^T(t)\\&\times[QL+LQ]\sigma(t)-\frac{n-i+1}{2}\\&\times\eta_1^2(t-\tau_1)f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right)\\&-\widetilde{\Theta}_3^T\left(\dot{\Theta}_3-\pi_{i-1}\right)-(n-1)\widetilde{\phi}\left(x_0(t)\right)\eta_1^2(t)\\&-\sum_{j=3}^{i-2}\frac{\partial\alpha_j}{\partial\widehat{\Theta}_3}\left(\dot{\widehat{\Theta}}_3-\pi_{i-1}\right)\eta_{j+1}(t).\end{aligned} \tag{52}$$

Let $\eta_{i+1}(t)=v_{i+1}(t)-\alpha_i$, where α_i is regarded as the virtual control input, and choose Lyapunov functional as

$$V_i=V_{i-1}+\frac{1}{2}\eta_i^2(t). \tag{53}$$

Based on (52), the time derivative of V_i satisfies

$$\begin{aligned}\dot{V}_i=\dot{V}_{i-1}+\eta_i(t)\Big\{&-k_iv_1(t)-(n-i)\beta v_i(t)\\&-(n-i)\widetilde{\phi}\left(x_0(t)\right)v_i(t)+\eta_{i+1}(t)\\&-\frac{\partial\alpha_{i-1}}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1-\frac{\partial\alpha_{i-1}}{\partial\widehat{\Theta}_2}\dot{\widehat{\Theta}}_2-\frac{\partial\alpha_{i-1}}{\partial\widehat{\Theta}_3}\dot{\widehat{\Theta}}_3\\&-\frac{\partial\alpha_{i-1}}{\partial z_1}\left[\xi_{02}-(n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right]\\&+\alpha_i-\frac{\partial\alpha_{i-1}}{\partial u_0}\dot{u}_0-\frac{\partial\alpha_{i-1}}{\partial\xi_{02}}\dot{\xi}_{02}\\&-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial v_j}\dot{v}_j-\frac{\partial\alpha_{i-1}}{\partial z_1}\varepsilon_2(t)-\frac{\partial\alpha_{i-1}}{\partial z_1}\Psi_1\\&-\frac{\partial\alpha_{i-1}}{\partial x_0}\dot{x}_0-\frac{\partial\alpha_{i-1}}{\partial z_1}\Phi_1-\frac{\partial\alpha_{i-1}}{\partial z_1}\overline{d}_nv_2(t)\Big\}.\end{aligned} \tag{54}$$

Next, we estimate the following terms in the right-hand side of (53) by Lemmas 8 and 9 and Young's inequality as follows:

$$\begin{aligned}&-\frac{\partial\alpha_{i-1}}{\partial z_1}\eta_i(t)\Psi_1\\&\le\frac{1}{2}\eta_1^2(t)+\frac{1}{2}\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2\widetilde{\psi}_1^2\left(u_0(t),x_0(t),z_1(t)\right)\eta_i^2(t)\\&\quad+\frac{1}{2}\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2\eta_1^2(t)f_{11}^2\left(u_0(t),y(t)\right)\eta_i^2(t)\\&\quad+\frac{1}{2}\eta_1^2(t-\tau_1)f_{12}^2\left(u_0(t-\tau_1),y(t-\tau_1)\right),\\&-\frac{\partial\alpha_{i-1}}{\partial z_1}\eta_i(t)\Phi_1\end{aligned}$$

$$\leq \frac{1}{2}\eta_1^2(t) + \frac{1}{2}\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2 \tilde{\omega}_1^2 \eta_i^2(t)\vartheta^2,$$

$$-\frac{\partial\alpha_{i-1}}{\partial z_1}\eta_i(t)\varepsilon_2(t)$$

$$\leq \frac{1}{\ell_i}\varepsilon^T(t)P^{-2}\varepsilon(t) + \frac{\ell_i}{4}p_{\max}^2\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2\eta_i^2(t). \tag{55}$$

Choosing the virtual control function α_i as

$$\begin{aligned}\alpha_i = {} & -c_i\eta_i(t) - \eta_{i-1}(t) + k_i v_1(t) \\ & + (n-i)\beta v_i(t) + (n-i)\tilde{\phi}(x_0(t))v_i(t) \\ & + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial v_j}\dot{v}_j + \frac{\partial\alpha_{i-1}}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1 + \frac{\partial\alpha_{i-1}}{\partial\widehat{\Theta}_2}\dot{\widehat{\Theta}}_2 \\ & + \frac{\partial\alpha_{i-1}}{\partial x_0}\dot{x}_0 + \sum_{j=3}^{i-2}\frac{\partial\alpha_j}{\partial\widehat{\Theta}_3}\Upsilon_i\eta_{j+1}(t) + \frac{\partial\alpha_{i-1}}{\partial u_0}\dot{u}_0 \\ & + \frac{\partial\alpha_{i-1}}{\partial z_1}\left[\xi_{02} - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right] \\ & - \frac{\ell_i}{4}p_{\max}^2\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2\eta_i(t) + \frac{\partial\alpha_{i-1}}{\partial\widehat{\Theta}_3}\pi_i \\ & - \frac{1}{2}\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2\eta_1^2(t)f_{11}^2(y(t))\eta_i(t) + \frac{\partial\alpha_{i-1}}{\partial\xi_{02}}\dot{\xi}_{02} - \widehat{\Theta}_3^T\Upsilon_i,\end{aligned} \tag{56}$$

and the tuning function $\pi_i = \pi_{i-1} + \Upsilon_i\eta_i(t)$ with $\Upsilon_i = [(1/2)(\partial\alpha_{i-1}/\partial z_1)^2\tilde{\omega}_1^2\eta_i(t),\ -(\partial\alpha_{i-1}/\partial z_1)v_2(t)]^T$. Then, we can show that

$$\begin{aligned}\dot{V}_i \leq {} & -\left(1 - \sum_{j=1}^{i}\frac{1}{\ell_j}\right)\varepsilon^T(t)P^{-2}\varepsilon(t) \\ & - (\bar{c}_1 - i)\eta_1^2(t) - \sum_{j=2}^{i}c_j\eta_j^2(t) \\ & - \bar{\mu}\sigma^T(t)\sigma(t) - \tilde{\phi}(x_0(t))\sigma^T(t)[QL + LQ]\sigma(t) \\ & - (n-1)\times\tilde{\phi}(x_0(t))\eta_1^2(t) \\ & - \widetilde{\Theta}_3^T(\dot{\Theta}_3 - \pi_i) - \sum_{j=3}^{i-1}\frac{\partial\alpha_j}{\partial\widehat{\Theta}_3}\eta_{j+1}(t)\left(\dot{\widehat{\Theta}}_3 - \pi_i\right) \\ & - \frac{n-i}{2}\eta_1^2(t-\tau_1)f_{12}^2(u_0(t-\tau_1), y(t-\tau_1)) \\ & + \eta_i(t)\eta_{i+1}(t).\end{aligned} \tag{57}$$

At the last step ($i = n$), the true input $u_1(t)$ will be designed on the basis of the virtual control $\alpha_i's$ and the Lyapunov function V_{n-1} introduced before.

The actual control input $u_1(t)$ can be designed as

$$\begin{aligned}u_1(t) = {} & -c_n\eta_n(t) - \eta_{n-1}(t) + k_n v_1(t) \\ & + \frac{\partial\alpha_{n-1}}{\partial\widehat{\Theta}_1}\dot{\widehat{\Theta}}_1 + \frac{\partial\alpha_{n-1}}{\partial\widehat{\Theta}_2}\dot{\widehat{\Theta}}_2 + \frac{\partial\alpha_{n-1}}{\partial\xi_{02}}\dot{\xi}_{02} \\ & + \frac{\partial\alpha_{n-1}}{\partial x_0}\dot{x}_0 + \frac{\partial\alpha_{n-1}}{\partial u_0}\dot{u}_0 \\ & + \frac{\partial\alpha_{n-1}}{\partial z_1}\left[\xi_{02} - (n-1)\frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right] - \widehat{\Theta}_3^T\Upsilon_n \\ & + \frac{\partial\alpha_{n-1}}{\partial\widehat{\Theta}_3}\pi_n - \frac{1}{2}\left(\frac{\partial\alpha_{n-1}}{\partial z_1}\right)^2\eta_1^2(t)f_{11}^2(y(t))\eta_n(t) \\ & - \frac{\ell_n}{4}p_{\max}^2\left(\frac{\partial\alpha_{n-1}}{\partial z_1}\right)^2\eta_n(t) \\ & + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial v_j}\dot{v}_j + \sum_{j=3}^{n-2}\frac{\partial\alpha_j}{\partial\widehat{\Theta}_3}\Upsilon_n\eta_{j+1}(t),\end{aligned} \tag{58}$$

and the update law $\widehat{\Theta}_3 = \pi_n$ with $\pi_n = \pi_{n-1} + \Upsilon_n\eta_n(t)$ and $\Upsilon_n = [(1/2)(\partial\alpha_{n-1}/\partial z_1)^2\tilde{\omega}_1^2\eta_n(t), -(\partial\alpha_{n-1}/\partial z_1)v_2(t)]^T$. Eventually, it can be achieved that

$$\begin{aligned}\dot{V}_n \leq {} & -\left(1 - \sum_{j=1}^{n}\frac{1}{\ell_j}\right)\varepsilon^T(t)P^{-2}\varepsilon(t) \\ & - \bar{\mu}\sigma^T(t)\sigma(t) - (\bar{c}_1 - n)\eta_1^2(t) - \sum_{j=2}^{n}c_j\eta_j^2(t) - (n-1) \\ & \times\tilde{\phi}(x_0(t))\eta_1^2(t) - \tilde{\phi}(x_0(t))\sigma^T(t)[QL + LQ]\sigma(t).\end{aligned} \tag{59}$$

3.4. Stability Analysis. Notice that $\tilde{\phi}(x_0(t))$ tends to zero as $x_0(t)$ converges to origin, and $\delta_1, \delta_2, \ell_i, c_i$ $(1 \leq i \leq n)$ in (59) are positive design parameters. Therefore, by an appropriate parameter choice, there exist positive constants $\lambda_i > 0$ $(1 \leq i \leq n+2)$ such that

$$\begin{aligned}\dot{V}_n \leq {} & -\sum_{j=1}^{n}\lambda_j\eta_j^2(t) - \lambda_{n+1}\varepsilon^T(t)P^{-2}\varepsilon(t) \\ & - \lambda_{n+2}\sigma^T(t)\sigma(t).\end{aligned} \tag{60}$$

It can be seen that $\eta_i(t), \varepsilon(t), \sigma(t), \widetilde{\Theta}_1, \widetilde{\Theta}_2, \widetilde{\Theta}_3$ are bounded. Since θ and d_i are unknown bounded parameters, $\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_1$ are bounded. According to estimator equations (19)–(21), it can be deduced that the boundedness of $z_1(t) = \eta_1(t)$ guarantees the boundedness of $\xi_0(t)$, and then $v_1(t) = (1/\bar{d}_n)(z_1(t) - \sigma_1(t))$ and α_1 are also bounded. By similar analysis, we can conclude that all signals of the closed loop system are bounded.

By LaSalle invariant Theorem, it further achieves that $\eta_i(t), \varepsilon(t), \sigma(t), \widetilde{\Theta}_1, \widetilde{\Theta}_2, \widetilde{\Theta}_3 \to 0$ as $t \to \infty$. By the controller

design procedure, we get that $\xi_0(t), v(t), \alpha_i, u_1(t)$ asymptotically tend to zero. Then, the definitions $\hat{z}(t) = \xi_0(t) + \bar{d}_n v(t)$ and $z(t) = \varepsilon(t) + \hat{z}(t)$ show the asymptotical convergence of $\hat{z}(t)$ and $z(t)$. Finally, from the transformations (10) and (12), we know $x_i(t) = (1/\bar{d}_n)u_0^{n-i}(t)z_i(t)$, which indicates that the states $x_i(t)$ asymptotically converge to zero with the initial condition $x_0(t_0) \neq 0$.

For purposes of analysis, we can rewrite the system (14) as follows:

$$\dot{z}(t) = \left(A_1 - L\bar{\phi}_0(x_0(t))\right)z(t) + Kz_1(t) + Bu_1(t) + \Psi + \Phi. \tag{61}$$

To solve the above differential equation, we have

$$\begin{aligned} z(t) &= e^{(A_1 - L\bar{\phi}_0(x_0(t)))t} z(t_0) \\ &\quad + \int_{t_0}^{t} e^{(A_1 - L\bar{\phi}_0(x_0(t)))(t-s)} \left[Kz_1(s) + Bu_1(s) + \Psi + \Phi\right] ds. \end{aligned} \tag{62}$$

Notice that $A_1 = A - KC - L\beta$ is *Hurwitz*, and $\bar{\phi}_0(x_0(t))$ tends to zero as $x_0(t) \to 0$, then by Lemmas 8 and 9, there exist constants $\varrho_1 > 0, \varrho_2 > 0$ such that

$$\begin{aligned} |z(t)| &\leq \varrho_1 e^{-\varrho_2 t} |z(t_0)| \\ &\quad + \int_{t_0}^{t} \varrho_1 e^{-\varrho_2(t-s)} \left[\|K\| \cdot |z_1(s)| + \|B\| \cdot |u_1(s)| + \|\Psi\| + \|\Phi\|\right] ds \\ &\leq \varrho_1 e^{-\varrho_2 t} |z(t_0)| \\ &\quad + \varrho_1 e^{-\varrho_2 t} \int_{t_0}^{t} e^{\varrho_2 s} \left[\|K\| \cdot |z_1(s)| + \|B\| \cdot |u_1(s)| + |z_1(s)| \widetilde{G}_1 + |z_1(s)| \widetilde{G}_2\right] ds, \end{aligned} \tag{63}$$

where $\widetilde{G}_1$ is a nonnegative smooth function of $d_i, u_0(s), u_0(s-\tau_i), y(s), y(s-\tau_i)$, and $\widetilde{G}_2$ is a nonnegative smooth function of $d_i, u_0(s), x_0(s), z_1(s), \vartheta$.

Since $x_0(t), x_1(t), u_0(t)$ and the system parameters are all bounded, then $\widetilde{G}_1, \widetilde{G}_2$ in (63) are also bounded. Employing the convergence of $x_0(t), z_1(t), u_1(t)$, we can get that $z(t)$-system is globally asymptotically convergent. From the introduced transformations before, it can be deduced that system (1) is also asymptotically convergent. Now, we can express the following theorem.

Theorem 12. *For system* (1)*, under Assumptions 1–3, if the control strategies* (23) *and* (58) *are applied with an appropriate choice of the design parameters, the global asymptotic stabilization of the closed loop system is achieved for* $x_0(t_0) \neq 0$.

In the next section, we will deal with the stability analysis of the closed loop as long as the initial condition $x_0(t_0)$ is zero.

4. Switching Controller

Several switching controllers have been proposed in some existing literatures. As well known, the choice of a constant feedback for $u_0(t)$ may lead to a finite escape. In this note, the following switching category can be designed for the stabilization of system (1) with the initial sate $x_0(t_0) = 0$. Choosing controller $u_0(t)$ as

$$u_0(t) = \operatorname{sign}(d_0)\, u_0^*, \quad \text{when } |x_0(t)| \leq \varrho_3 < x_0^*, \tag{64}$$

where $u_0^* > 0$ and $\varrho_3 > 0$ are constants.

Since $x_0(t_0) = 0$, then $\dot{x}_0(t_0)$ with $u_0(t)$ can be deduced

$$\dot{x}_0(t_0) = |d_0| u_0^* + \phi(t, x_0(t_0)) = |d_0| u_0^* > 0, \tag{65}$$

then during the initial small time period, $x_0(t)$ is increasing and satisfies $|x_0(t)| + |x_0(t)|\bar{\phi}_0(x_0(t)) < |d_0|u_0^*$.

Choose x_0^* that satisfy

$$|x_0^*| + |x_0^*| \bar{\phi}_0(x_0^*) = |d_0| u_0^*. \tag{66}$$

Obviously, $x_0(t)$ is increasing when $x_0(t) \leq x_0^*$. When $|x_0(t)| \leq \varrho < x_0^*$, choose the controller $u_0(t) = \operatorname{sign}(d_0)u_0^*$, and the controller $u_1(t)$ can be designed according to the simple nonlinear backstepping iterative approach. Since $|x_0(t)| > \varrho_3$, at t_s, we switch the control laws $u_0(t)$ and $u_1(t)$ into (23) and (58), respectively.

Theorem 13. *For system* (1)*, under Assumptions 1–3, if above switching control strategy is applied with an appropriate choice of the design parameters, then the closed-loop system is globally asymptotic regulated at the origin for* $x_0(t_0) = 0$.

5. Simulation Example

In this section, a numerical example will be given to illustrate that the proposed systematic control law design method is effective. Consider the following system:

$$\begin{aligned} \dot{x}_0(t) &= d_0 u_0(t) + x_0(t)^3, \\ \dot{x}_1(t) &= d_1 u_0(t) x_2(t) + \frac{1}{2}\ln\left(1 + x_1^2(t)\right) e^{x_0(t)} \\ &\quad \times x_1^2(t-0.3) + x_1(t)\theta_1^{x_1(t)}, \\ \dot{x}_2(t) &= d_2 u_1(t) + x_1(t) e^{x_0(t-0.2)} \\ &\quad \times x_1^3(t-0.2) + \ln\left(1 + (\theta_2 x_2(t))^2\right), \\ y(t) &= [x_0(t), x_1(t)]^T, \end{aligned} \tag{67}$$

where d_0, d_1, d_2 are virtual control directions with d_1, d_2 unknown and d_0 known, and the sign of $\bar{d}_2 = d_1 d_2$ is also known. θ_1, θ_2 are unknown bounded parameters. Next, we consider to design the controller $u_0(t)$ and $u_1(t)$ to asymptotically stabilize system (67) by the measurable

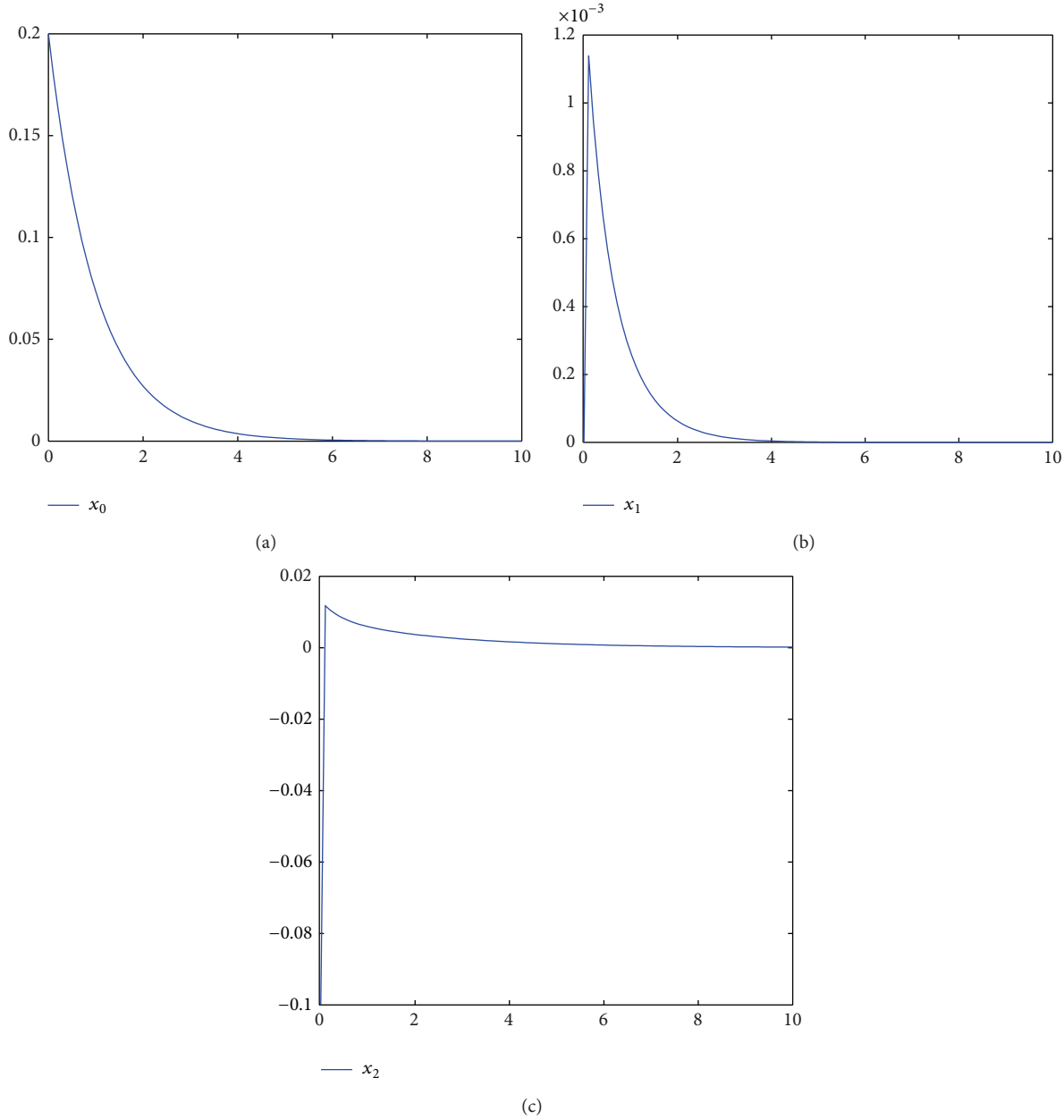

Figure 1: States $x_0(t), x_1(t), x_2(t)$.

output. We assume that $x_0(t_0) \neq 0$ and make the following estimation for some nonlinear terms in system (67):

$$\begin{aligned} x_1(t)\,\theta_1^{x_1(t)} &\leq |x_1(t)|\, e^{(1/2)x_1^2(t)}\vartheta, \\ \ln\left(1 + (\theta_2 x_2(t))^2\right) &\leq |x_1(t)|\,\vartheta, \end{aligned} \tag{68}$$

where $\vartheta = e^{(1/2)\ln^2\theta_1} + |\theta_2|$.

Firstly, we introduce the following transformation:

$$\overline{x}_1(t) = x_1(t), \qquad \overline{x}_2(t) = d_1 x_2(t), \tag{69}$$

and then the system (67) can be rewritten as

$$\begin{aligned} \dot{x}_0(t) &= d_0 u_0(t) + x_0(t)^3, \\ \dot{\overline{x}}_1(t) &= u_0(t)\,\overline{x}_2(t) + \frac{1}{2}\ln\left(1 + x_1^2(t)\right) e^{x_0(t)} \\ &\quad \times x_1^2(t-0.3) + x_1(t)\,\theta_1^{x_1(t)}, \\ \dot{\overline{x}}_2(t) &= \overline{d}_2 u_1(t) + d_1 x_1(t)\, e^{x_0(t-0.2)} \\ &\quad \times x_1^3(t-0.2) + d_1 \ln\left(1 + (\theta_2 x_2(t))^2\right), \end{aligned} \tag{70}$$

where $\overline{d}_2 = d_1 d_2$, and assume that the sign of $\overline{d}_2$ is known.

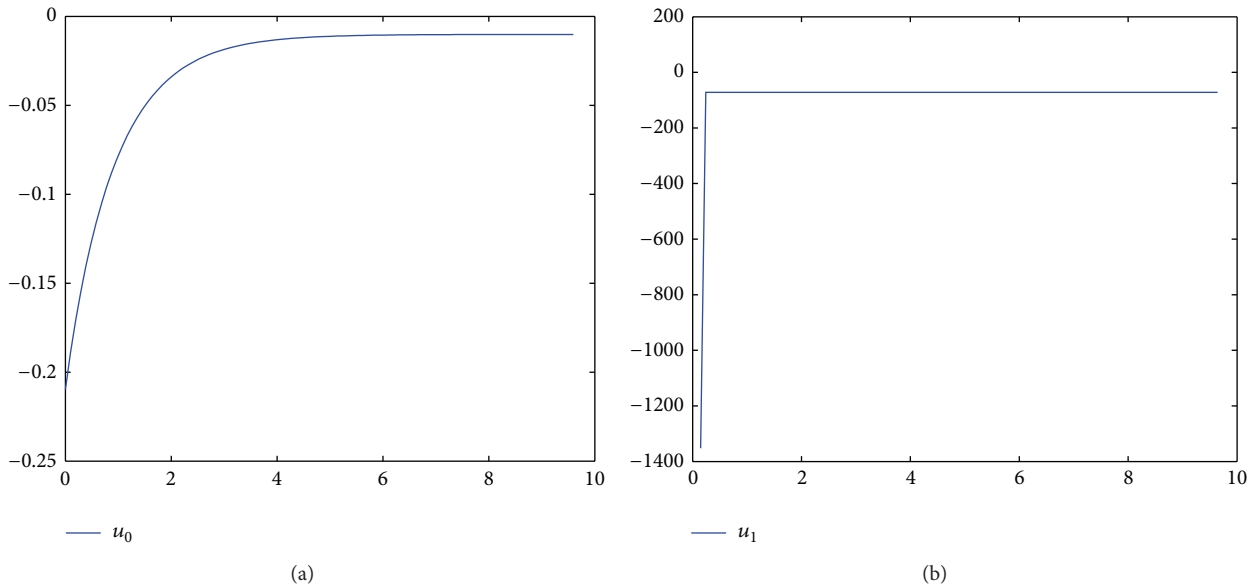

FIGURE 2: Controllers $u_0(t)$ and $u_1(t)$.

Next, make the following input scaling transformation for $\overline{x}(t)$-system:

$$z_1(t) = \frac{\overline{x}_1(t)}{u_0(t)}, \qquad z_2(t) = \overline{x}_2(t), \tag{71}$$

and then the transformed system is

$$\dot{z}(t) = \left(A - L\frac{\dot{u}_0(t)}{u_0(t)}\right) z(t) + Bu_1(t) + \Psi + \Phi, \tag{72}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ \overline{d}_2 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \\ \Psi_1 &= \frac{\ln\left(1 + x_1^2(t)\right) e^{x_0(t)} x_1^2(t-0.3)}{2u_0(t)}, \\ \Psi_i &= d_1 x_1(t) e^{x_0(t-0.2)} x_1^3(t-0.2), \\ \Phi_1 &= \frac{x_1(t)\theta_1^{x_1(t)}}{u_0(t)}, \\ \Phi_2 &= d_1 \ln\left(1 + (\theta_2 x_2(t))^2\right) . \frac{\dot{u}_0(t)}{u_0(t)}. \end{aligned} \tag{73}$$

Design the following controller $u_0(t)$:

$$u_0(t) = -c_0 x_0(t) - c_0 x_0(t)^3, \tag{74}$$

and then $\dot{u}_0(t)/u_0(t)$ can be calculated as follows:

$$\frac{\dot{u}_0(t)}{u_0(t)} = -c_0 d_0 - 3c_0 d_0 x_0(t) + \frac{x_0^2(t) + 3x_0^4(t)}{1 + x_0^2(t)}. \tag{75}$$

For system (72), constructing the following estimator:

$$\begin{aligned} \dot{\xi}_0(t) &= \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)\xi_0(t) + PC^T\left(y(t) - C\xi_0(t)\right), \\ \dot{\upsilon}(t) &= \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)\upsilon(t) + e_n u_1(t), \\ \dot{P} &= P\left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)^T + \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)P - PC^TCP + I, \end{aligned} \tag{76}$$

where $y(t) = z_1(t), e_n = [0,1]^T, \xi_0 = [\xi_{01}, \xi_{02}]^T, \upsilon = [\upsilon_1, \upsilon_2]^T$, $A_0 = A - KC$, $C = [1,0]$, and $K = [k_1, k_2]^T$. The design of k_1, k_2 can guarantee that $A_1 = A_0 - KC - L\beta$ is *Hurwitz*. It is further achieved that there exists plosive definite matrix Q satisfying $QA_1 + A_1^TQ = -\mu I$, in which $\mu > 0$ is a constant. Denote $\widehat{z}(t) = \xi_0(t) + \overline{d}_n\upsilon$, $\sigma(t) = z(t) - \overline{d}_n\upsilon(t)$ and $\varepsilon(t) = z(t) - \widehat{z}(t)$, and then the observation error $\varepsilon(t)$ and parameter invariable $\sigma(t)$ satisfy

$$\begin{aligned} \dot{\varepsilon}(t) &= \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)} - PC^TC\right)\varepsilon(t) \\ &\quad + \left(K - PC^T\right)z_1(t) + PC^TC\sigma(t) + \Psi + \Phi, \\ \dot{\sigma}(t) &= \left(A_0 - L\frac{\dot{u}_0(t)}{u_0(t)}\right)\sigma(t) + Kz_1(t) + \Psi + \Phi. \end{aligned} \tag{77}$$

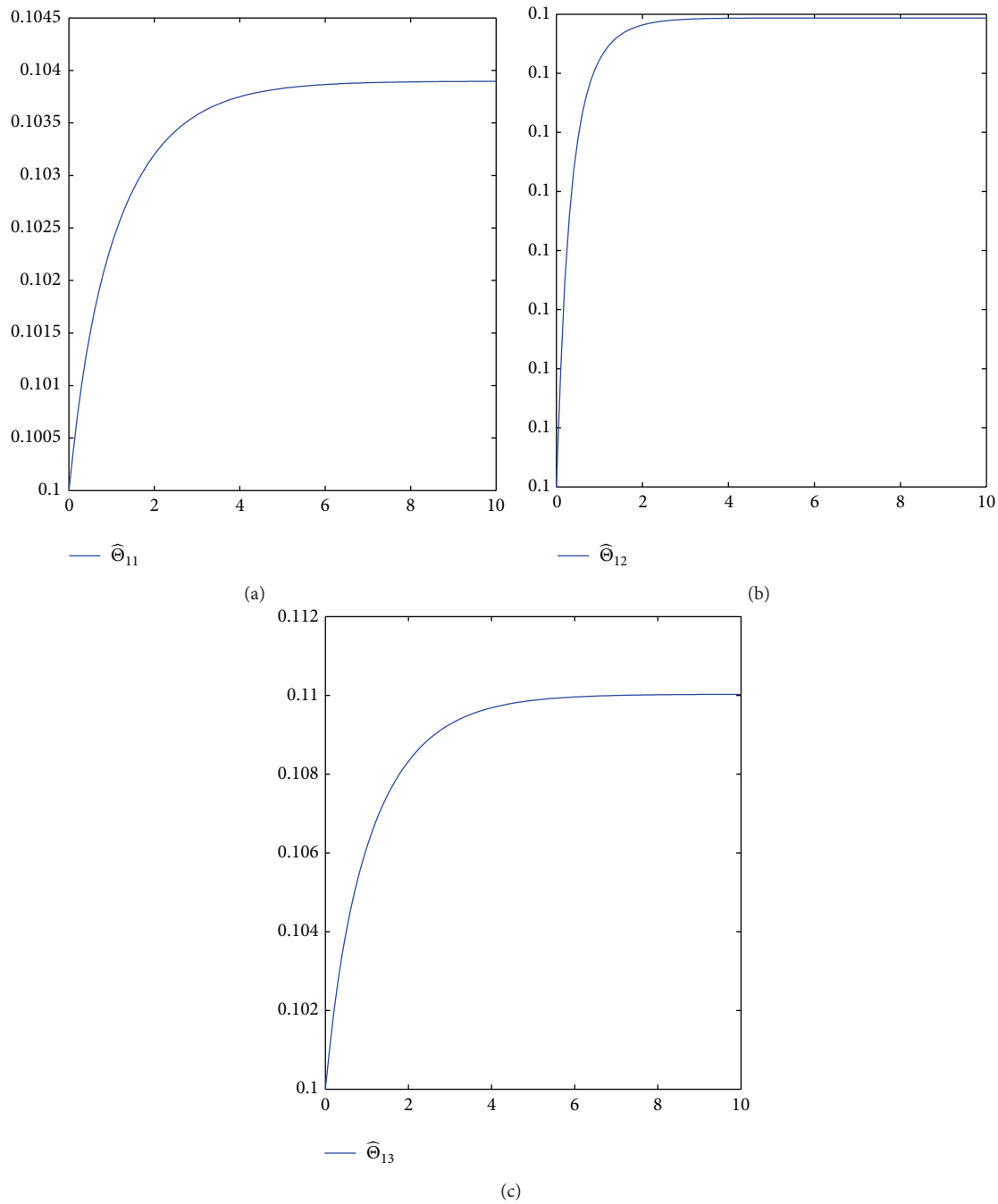

FIGURE 3: Parameters $\widehat{\Theta}_{11}, \widehat{\Theta}_{12}, \widehat{\Theta}_{13}$.

Define the invariable that $\eta_1(t) = z_1(t), \eta_2(t) = \upsilon_2(t) - \alpha_1$. According to the iterative procedure in Section 3, we can design the virtual control function and controller $u_1(t)$ as

$$\begin{aligned}
\alpha_1 &= -\widehat{\Theta}^T \Upsilon_1 = -\left[\widehat{\Theta}_{11}, \widehat{\Theta}_{12}, \widehat{\Theta}_{13}\right]\left[\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}\right]^T, \\
u_1(t) &= -c_2\eta_2(t) + k_2\upsilon_1(t) + \frac{\partial\alpha_1}{\partial\widehat{\Theta}_1^T}\dot{\widehat{\Theta}}_1 + \frac{\partial\alpha_1}{\partial\xi_{02}}\dot{\xi}_{02} \\
&\quad + \frac{\partial\alpha_1}{\partial u_0(t)}\dot{u}_0(t) + \frac{\partial\alpha_1}{\partial z_1(t)}\left[\xi_{02} - \frac{\dot{u}_0(t)}{u_0(t)}z_1(t)\right] \\
&\quad - \frac{\ell_2}{4}\left(P_{12}^2 + P_{22}^2\right)\left(\frac{\partial\alpha_1}{\partial z_1(t)}\right)^2\eta_2(t) \\
&\quad - \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial z_1(t)}\right)^2 e^{2x_0(t)}\eta_1^2(t)\eta_2(t) - \widehat{\Theta}_2^T\Upsilon_2,
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
\Upsilon_{11} &= c_1\eta_1(t) + \xi_{02}(t) + \frac{1}{2}\eta_1^3(t)e^{2x_0(t)} \\
&\quad + \left[\frac{\ell_1}{8} + \frac{\delta_1}{32}\|Q\|^2\right]\eta_1^7(t)u_0^8(t)
\end{aligned}$$

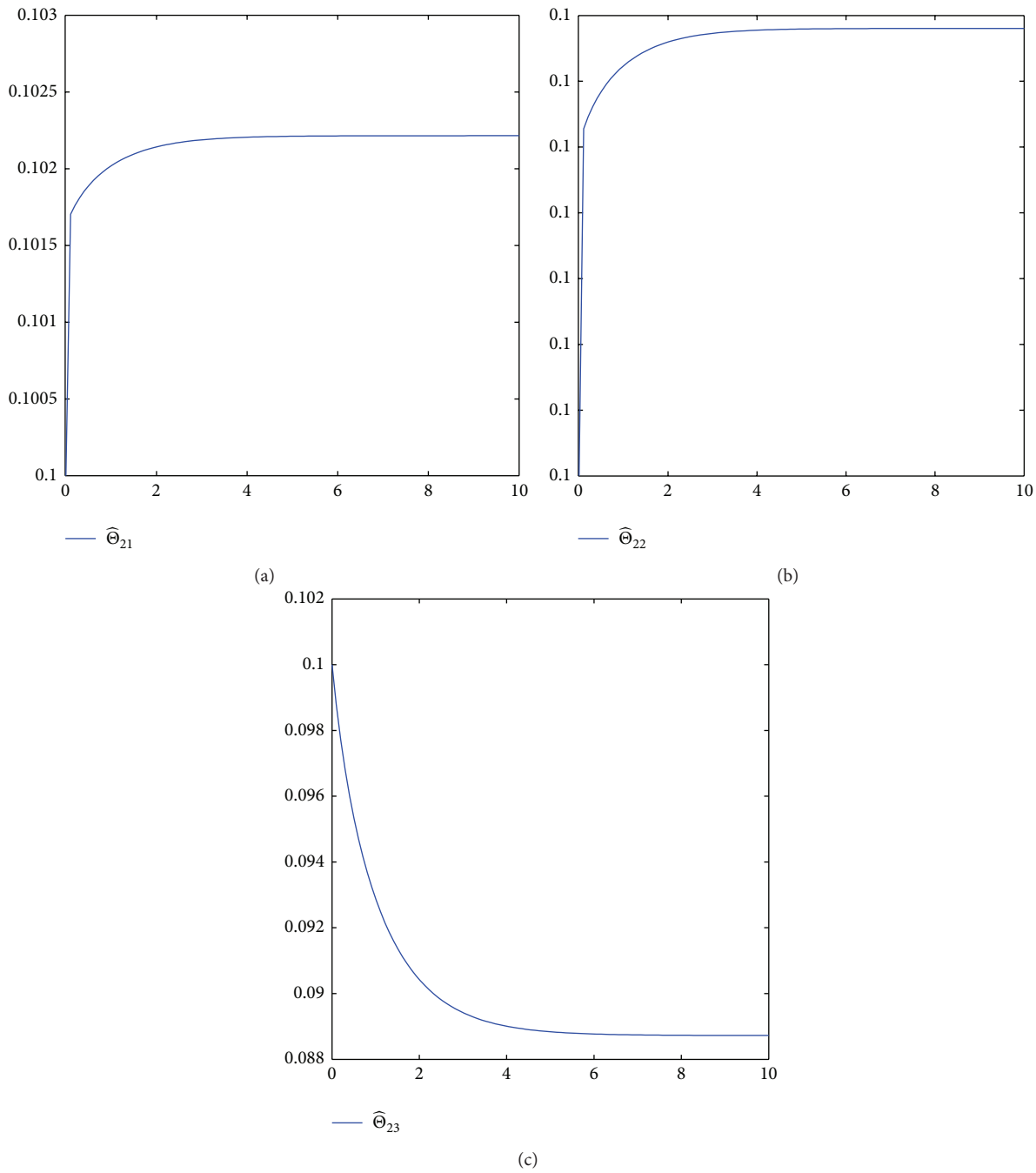

FIGURE 4: Parameters $\widehat{\Theta}_{21}, \widehat{\Theta}_{22}, \widehat{\Theta}_{23}$.

$$+\left[2\ell_1+\frac{\delta_1}{2}\|Q\|^2\right]\eta_1^{11}(t)\,e^{4x_0(t)}u_0^{12}(t)+\frac{1}{4}\eta_1^3(t)\,u_0^4(t),$$

$$\Upsilon_{12}=2\ell_1\eta_1^3(t)\,e^{4x_0(t)}+2\ell_1\eta_1^3(t)\,u_0^4(t)+\frac{\delta_1}{2}\|Q\|^2\eta_1^3(t)\,e^{4x_0(t)}+\frac{\delta_1}{2}\|Q\|^2\eta_1^3(t)\,u_0^4(t),$$

$$\Upsilon_{13}=\left[\frac{1}{4}+4\ell_1+\delta_2\|Q\|^2\right]\eta_1(t)\,e^{z_1^2(t)u_0^2(t)}+\left[4\ell_1+\delta_2\|Q\|^2\right]\eta_1(t)\,u_0^2(t),$$

$$\widehat{\Theta}_2^T=\left[\widehat{\Theta}_{21},\widehat{\Theta}_{22},\widehat{\Theta}_{23}\right],$$

$$\Upsilon_2=\left[\frac{1}{4}\left(\frac{\partial\alpha_1}{\partial z_1(t)}\right)^2 e^{z_1^2(t)u_0^2(t)}\eta_2(t),\frac{1}{4}\eta_2(t),-\frac{\partial\alpha_1}{\partial z_1(t)}\upsilon_2(t)\right]^T. \tag{79}$$

The adaption laws of the parameter invariable in controller $u_1(t)$ are chosen as

$$\dot{\widehat{\Theta}}_1=\operatorname{sign}\left(\overline{d}_2\right)\Upsilon_1\eta_1(t),\qquad \dot{\widehat{\Theta}}_2=\Upsilon_2\eta_2(t). \tag{80}$$

For simulation use, we pick the unknown parameters $d_1 = 1.5, d_2 = 2.5, \theta_1 = \theta_2 = 0.5$. In addition, we take the other controller design parameters as $c_0 = 1$, $c_1 = 130$, $c_2 = 2$, $k_1 = 4$, $k_2 = 1$, $\ell_1 = 2$, $\ell_2 = 3, \delta_1 = \delta_2 = 4$. Moreover, The initial state condition is $[0.2, 0, \ -0.1]^T$. Simulation results are shown in Figures 1, 2, 3, and 4. It is obvious that the states $x_0(t), x_1(t), x_2(t)$ and control input $u_0(t), u_1(t)$ converge to zero, and the parameters estimation invariable tend to constants.

6. Conclusion

The output-feedback adaptive stabilization was investigated for a class of nonholonomic systems with unknown virtual control coefficients, nonlinear uncertainties, and unknown time delays. In order to overcome the difficulties, we introduce suitable transformation and novel Lyapunov-Krasovskii functionals, and then a recursive technique is given to design the adaptive controller. To make the input-state scaling transformation effective, the switching control strategy is employed to achieve the asymptotic stabilization.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publishing of this paper.

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Research Article

Sufficient Conditions on the Exponential Stability of Neutral Stochastic Differential Equations with Time-Varying Delays

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The exponential stability is investigated for neutral stochastic differential equations with time-varying delays. Based on the Lyapunov stability theory and linear matrix inequalities (LMIs) technique, some delay-dependent criteria are established to guarantee the exponential stability in almost sure sense. Finally a numerical example is provided to illustrate the feasibility of the result.

1. Introduction

Neutral differential equations are well-known models from many areas of science and engineering, wherein, quite often the future state of such systems depends not only on the present state but also involves derivatives with delays. Deterministic neutral differential equations were originally introduced by Hale and Meyer [1] and discussed in Hale and Lunel (see [2]) and Kolmanovskii et al. (for details, see also [3, 4]), among others. Motivated by chemical engineering systems as well as theory of aeroelasticity, stochastic neutral delay systems have been intensively studied over recent year [5–9]. Mao initiated the study of exponential stability of neutral stochastic differential delay equations in [5], while [9] incorporated Razumikhinis approach in neutral stochastic functional differential equations to investigate the stability problem. It is pointed out in Section 5 [10] that the conditions imposed in [5, 9] make the theory not applicable to the delay equation.

More recently, Luo et al. [6] proposed new criteria on exponential stability of neutral stochastic delay differential equations. In [11, 12], Milošević investigated the almost sure exponential stability of a class of highly nonlinear neutral stochastic differential equations with time-dependent delay, and some sufficient conditions were given for the considered systems. However, when the exponential stability of the neutral system with time-delay is considered, one always assumes that the derivative of the delay function is less than 1 (e.g., [6]). Meanwhile, the delay-independent conditions in [6, 10] are restricted when the delay is small. On the other hand, some results are proposed on stochastic Markovian jumping systems (e.g., [13–20]) and finite-time problems of stochastic systems (e.g., [18–22]), which can provide some useful methods and techniques for the neutral stochastic systems. This paper aims to develop the exponential stability in almost sure sense of the neutral stochastic differential equations with time-varying delays. Under the weaker assumptions that the derivative of time delay is less than some constant, sufficient conditions for the exponential stability are given in terms of linear matrix inequality (LMI) based on Lyapunov stability theory, which can be checked easily by MATLAB LMI Toolbox.

The paper is organized as follows. In the remainder of this section we recall some preliminaries, mainly from [5]. In Section 3 we state the main results on exponential stability.

Section 4 will provide numerical examples to illustrate the feasibility and effectiveness of the results, and the conclusion will be made in Section 5.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and F_0 containing all P-null sets). Let $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))^n$ be m-dimensional Wiener process defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. A^T stands for the transpose of the vector or matrix A. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{A^T A}$. $a \vee b$ denotes $\max\{a, b\}$. $\lambda_{\max}(\cdot), \lambda_{\min}(\cdot)$ are maximum eigenvalue and minimum eigenvalue, respectively.

Consider the following n-dimensional neutral stochastic differential delay equations with time-varying delays:

$$\begin{aligned} &\mathrm{d}\left[x(t) - G(x(t-\delta(t)))\right] \\ &\quad = f(x(t), x(t-\delta(t)), t)\,\mathrm{d}t \\ &\qquad + g(x(t), x(t-\delta(t)), t)\,\mathrm{d}w(t) \\ &x(t) = \xi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, and $G \in C(\mathbb{R}^n, \mathbb{R}^n)$. The functions $\delta(t) : \mathbb{R}_+ \to [0, \tau]$ are continuously differentiable such that $0 \leq \delta(t) \leq \delta, \dot{\delta}(t) \leq \bar{\delta}$. Let $C([-\tau, 0], \mathbb{R}^n)$ denote the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. Let $C^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable $C([-\tau, 0], \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : \tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^2 < \infty$. To guarantee the existence and uniqueness of the solution, we first list the following hypotheses.

(H_1) Both the functionals f and g satisfy the uniform Lipschitz conditions. That is, there is a diagonal positive matrix $L = \operatorname{diag}\{L_1, L_2, \ldots, L_n\}$ such that

$$\begin{aligned} &\left|f(x, y, t) - f(\bar{x}, \bar{y}, t)\right| \vee \left|g(x, y, t) - g(\bar{x}, \bar{y}, t)\right| \\ &\qquad \leq \left|L(x - \bar{x})\right| + \left|L(y - \bar{y})\right| \end{aligned} \tag{2}$$

for all $t \geq 0$ and those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

(H_2) There is a constant $k \in (0, 1)$ such that for all $\phi_1, \phi_2 \in C^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$

$$\left|G(\phi_1) - G(\phi_2)\right|^2 \leq k \sup_{-\tau\leq\theta\leq 0} \left|\phi_1(\theta) - \phi_2(\theta)\right|^2. \tag{3}$$

It is well known (see, e.g., [3]) that under hypotheses H_1, H_2 (1) has a unique continuous solution on $t \geq -\tau$.

To obtain sufficient conditions on almost sure exponential stability, the following lemmas and definition are given.

Lemma 1 (see [23]). *For any positive definite constant matrix $M \in \mathbb{R}^{n\times n}$, scalar $r > 0$, and vector function $f(\cdot) : [0, r] \to \mathbb{R}^n$ such that the integrations in the following are well defined, then the following inequality holds:*

$$\left(\int_0^r f(s)ds\right)^T M \left(\int_0^r f(s)\,ds\right) \leq r \int_0^r f^T(s) M f(s)\,ds. \tag{4}$$

The following semimartingale convergence theorem will play an important role in the later parts.

Lemma 2 (see [24]). *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processed on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ς be a nonnegative $\mathcal{F}_0$-measurable random variable. Define*

$$X(t) = \varsigma + A(t) - U(t) + M(t) \quad \text{for } t \geq 0. \tag{5}$$

If $X(t)$ is nonnegative, then

$$\begin{aligned} \left\{\lim_{t\to\infty} A(t) < \infty\right\} &\subset \left\{\lim_{t\to\infty} X(t) < \infty\right\} \\ &\quad \cap \left\{\lim_{t\to\infty} U(t) < \infty\right\} \quad \text{a.s.}, \end{aligned} \tag{6}$$

where $B \subset D$ a.s. means $P(B \cap D) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s., then for almost all $w \in \Omega$,

$$\lim_{t\to\infty} X(t) < \infty, \qquad \lim_{t\to\infty} U(t) < \infty; \tag{7}$$

that is, both $X(t)$ and $U(t)$ converge to finite random variables.

Definition 3 (see [25]). The equilibrium of solution $\{x(t), t \geq 0\}$ of (1) is said to be almost sure exponentially stable if there exists a constant $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq -\varepsilon \quad \text{a.s.} \tag{8}$$

for any bounded initial condition ξ.

3. Main Results

Theorem 4. *Let hypotheses H_1, H_2 hold. System (1) is almost sure exponentially stable, if there exists positive definite matrix such that the following LMI holds*

$$\Omega = \begin{pmatrix} \Xi_{11} & N_2^T - N_1^T & N_3^T + U^T & N_4^T & -N_1^T + N_5^T & 0 \\ * & \Xi_{22} & -N_3^T & -N_4^T & -N_2^T - N_5^T & 0 \\ * & * & \Xi_{33} & 0 & -N_3^T & U \\ * & * & * & \Xi_{44} & -N_4^T & 0 \\ * & * & * & * & -\frac{1}{\delta}R - 2N_5^T & 0 \\ * & * & * & * & * & -\varepsilon I \end{pmatrix} < 0, \tag{9}$$

where

$$\begin{aligned} \Xi_{11} &= \beta P + e^{\beta\delta} Q + P + 2N_1^T, \\ \Xi_{22} &= \left(-\left(1-\overline{\delta}\right) e^{\beta\delta} \vee -\left(1-\overline{\delta}\right)\right) Q - 2N_2, \\ \Xi_{33} &= -2U + e^{\beta\delta} S + 2\lambda_{\max}(P) L^T L \\ &\quad + \frac{2}{\beta}\lambda_{\max}(R)\left(e^{\beta\delta} - 1\right) L_1, \\ \Xi_{44} &= -\left(1-\overline{\delta}\right) S + 2\lambda_{\max}(P) L^T L \\ &\quad + \frac{2}{\beta}\lambda_{\max}(R)\left(e^{\beta\delta} - 1\right) L_1 + \varepsilon k I. \end{aligned} \tag{10}$$

Proof. To confirm that the stochastic neutral differential equation (1) is mean-square exponentially stable with decay rate β, we define a Lyapunov-Krasovskii functional $V(x(t), t)$ as follows:

$$\begin{aligned} V(x(t), t) &= e^{\beta t} \rho^T(t) P \rho(t) \\ &\quad + \int_{t-\delta(t)}^{t} e^{\beta(s+\delta)} \rho^T(s) Q \rho(s)\, \mathrm{d}s \\ &\quad + \int_{-\delta}^{0} \int_{t+\theta}^{t} e^{\beta(s-\theta)} f^T(x(s), y(s), s) \\ &\qquad \times R f(x(s), y(s), s)\, \mathrm{d}s\, \mathrm{d}\theta \\ &\quad + \int_{t-\delta(t)}^{t} e^{\beta(s+\delta)} x^T(s) S x(s)\, \mathrm{d}s. \end{aligned} \tag{11}$$

For simplicity, let $y(t) = x(t-\delta(t))$, $\rho(t) = x(t) - G(y(t))$. By generalizing Ito's formula, we have that

$$EV(\rho(t), t) = EV(\rho(0), 0) + \int_0^t \mathscr{L}V(x(s), y(s), s)\, \mathrm{d}s. \tag{12}$$

Then, the derivative of $V(\rho(t), t)$ along the solution of (1) gives

$$\begin{aligned} &LV(x(t), y(t), t) \\ &\quad = \beta e^{\beta t} \rho^T(t) P \rho(t) + 2e^{\beta t} \rho(t)^T P f(x(t), y(t), t) \\ &\qquad + e^{\beta t} g^T(x(t), y(t), t) P g(x(t), y(t), t) \\ &\qquad + e^{\beta(t+\delta)} \rho(t) Q \rho(t) \\ &\qquad - e^{\beta(t-\delta(t))+\delta} \rho^T(t-\delta(t)) Q \rho(t-\delta(t)) \left(1 - \dot{\delta}(t)\right) \\ &\qquad + \frac{1}{\beta} e^{\beta t}\left(e^{\beta\delta} - 1\right) f^T(x(t), y(t), t) R f(x(t), y(t), t) \\ &\qquad - e^{\beta t} \int_{t-\delta}^{t} f^T(x(s), y(s), s) R f(x(s), y(s), s)\, \mathrm{d}s \\ &\qquad + e^{\beta(t+\delta)} x(t) S x(t) - e^{\beta(t-\delta(t))+\delta} \\ &\qquad \times x^T(t-\delta(t)) S x(t-\delta(t)) \left(1 - \dot{\delta}(t)\right). \end{aligned} \tag{13}$$

Note that, from system (1) and Newton-Leibniz formula, we have

$$\begin{aligned} M &= \left(\rho(t) - \rho(t-\delta(t)) - \int_{t-\delta(t)}^{t} f(x(s), y(s), s)\, \mathrm{d}s \right. \\ &\quad \left. - \int_{t-\delta(t)}^{t} g(x(s), y(s), s)\, \mathrm{d}w(s)\right) = 0. \end{aligned} \tag{14}$$

By calculation, it is clear that

$$\begin{aligned} &f^T(x(t), y(t), t) R f(x(t), y(t), t) \\ &\quad \le \lambda_{\max}(R) f^T(x(t), y(t), t) f(x(t), y(t), t) \\ &\quad \le \lambda_{\max}(R) \left|f(x(t), y(t), t)\right|^2 \\ &\quad \le 2\lambda_{\max}(R)\left(x^T(t) L^T L x(t) + x^T(t-\delta(t)) L^T L y(t)\right), \end{aligned} \tag{15}$$

and then by which, we have

$$\begin{aligned}
&2\rho^T(t)Pf(x(t),y(t),t)\\
&\quad\le \rho^T(t)P^T\rho(t)\\
&\qquad+ f^T(x(t),y(t),t)Pf(x(t),y(t),t)\\
&\quad\le \rho^T(t)P^T\rho(t)\\
&\qquad+ 2\lambda_{\max}(P)\left(x^T(t)L^TLx(t)+y^T(t)L^TLy(t)\right),\\
&-e^{\beta(t-\delta(t))+\delta}\rho^T(t)Q\rho(t)\left(1-\dot{\delta}(t)\right)\\
&\qquad\le -\left(1-\overline{\delta}\right)e^{\beta t}\rho^T(t)Q\rho(t).\\
&-e^{\beta(t-\delta(t))+\delta}x^T(t)Sx(t)\left(1-\dot{\delta}(t)\right)\\
&\qquad\le -\left(1-\overline{\delta}\right)e^{\beta t}x^T(t)Sx(t).
\end{aligned}\tag{16}$$

Moreover, by Lemma 2, one can get

$$\begin{aligned}
&-\int_{t-\delta}^{t} f^T(x(s),y(s),s)Rf(x(s),y(s),s)\,\mathrm{d}s\\
&\quad\le -\int_{t-\delta(t)}^{t} f^T(x(s),y(s),s)Rf(x(s),y(s),s)\,\mathrm{d}s\\
&\quad\le -\frac{1}{\delta}\left(\int_{t-\delta(t)}^{t} f(x(s),y(s),s)\mathrm{d}s\right)^T\\
&\qquad\times R\left(\int_{t-\delta(t)}^{t} f(x(s),y(s),s)\,\mathrm{d}s\right).
\end{aligned}\tag{17}$$

Letting $L_1 = L^TL$ and substituting (14)–(17) into (13) yield

$$\begin{aligned}
&LV(x(t),y(t),t)\\
&\quad\le e^{\beta t}\Big\{\beta\rho^T(t)P\rho(t)+\rho^T(t)P^T\rho(t)\\
&\qquad+2\lambda_{\max}(P)x^T(t)L_1x(t)\\
&\qquad+2\lambda_{\max}(P)y^T(t)L_1y(t)\\
&\qquad+\left(-\left(1-\overline{\delta}\right)e^{\beta\delta}\vee-\left(1-\overline{\delta}\right)\right)\rho^T(t)Q\rho(t)\\
&\qquad+e^{\beta\delta}\rho^T(t)Q_1\rho(t)+2\lambda_{\max}\frac{1}{\beta}\left(e^{\beta\delta}-1\right)\\
&\qquad\times\left(x^T(t)L_1x(t)+y^T(t)L_1y(t)\right)\\
&\qquad-\frac{1}{\delta}\left(\int_{t-\delta(t)}^{t} f(x(s),y(s),s)\mathrm{d}s\right)^T\\
&\qquad\times R\left(\int_{t-\delta(t)}^{t} f(x(s),y(s),s)\,\mathrm{d}s\right)\Big\}.
\end{aligned}\tag{18}$$

Furthermore, from (14), it follows that

$$\begin{aligned}
A &= 2\eta^T\left(N_1^T,N_2^T,N_3^T,N_4^T,N_5^T\right)^T M = 0,\\
B &= 2x^T(t)U\left[\rho(t)-x(t)+G\left(y(t)\right)\right] = 0,
\end{aligned}\tag{19}$$

where $\eta = (\rho^T, \rho^T(t-\delta(t)), x^T(t), y^T(t), (\int_{t-\delta(t)}^{t} f(x,y,s)\,\mathrm{d}s)^T)^T$, and N_i $(1 \le i \le 5)$, U are matrices with compatible dimensions.

It can be shown that

$$\begin{aligned}
&\int_0^t \mathscr{L}V(x(s),y(s),s)\,\mathrm{d}s\\
&\qquad+e^{\beta t}(A+B)+e^{\beta t}M^TP_1\left(M(t)+\int_{t-\delta(t)}^{t} g(x,y,s)\,\mathrm{d}s\right)\\
&\quad\le e^{\beta t}\Big\{\beta\rho^T(t)P\rho(t)+\rho^T(t)P^T\rho(t)\\
&\qquad+x^T(t)L_1x(t)+e^{\beta\delta}\rho^T(t)Q\rho(t)\\
&\qquad+y^T(t)L_1y(t)+2\lambda_{\max}(P)x^T(t)L_1x(t)\\
&\qquad+2\lambda_{\max}y^T(t)L_1y(t)\\
&\qquad-\left(1-\overline{\delta}\right)\rho^T(t)Q\rho(t)+e^{\beta\delta}x^T(t)Sx(t)\\
&\qquad-\left(1-\overline{\delta}\right)x^T(t)Sx(t)+2\lambda_{\max}\frac{1}{\beta}\left(e^{\beta\delta}-1\right)\\
&\qquad\times\left(x^T(t)L_1x(t)+y^T(t)L_1y(t)\right)\\
&\qquad-\frac{1}{\delta}\left(\int_{t-\delta(t)}^{t} f(x(s),y(s),s)\mathrm{d}s\right)^T\\
&\qquad\times R\left(\int_{t-\delta(t)}^{t} f(x(s),y(s),s)\,\mathrm{d}s\right)\Big\}\\
&\quad\le e^{\beta t}\left\{\eta^T\widetilde{\Omega}\eta+\varepsilon^{-1}x^T(t)UU^Tx(t)\right\},
\end{aligned}\tag{20}$$

where Ω is defined as

$$\widetilde{\Omega} = \begin{pmatrix}
\Xi_{11} & N_2^T-N_1^T & N_3^T+U^T & N_4^T & -N_1^T+N_5^T\\
* & \Xi_{22} & -N_3^T & -N_4^T & -N_2^T-N_5^T\\
* & * & \Xi_{33} & 0 & -N_3^T\\
* & * & * & \Xi_{44} & -N_4^T\\
* & * & * & * & -\frac{1}{\delta}R-2N_5^T
\end{pmatrix}.\tag{21}$$

By Schur complement, we know that $\eta^T\widetilde{\Omega}\eta + \varepsilon^{-1}x^T(t)UU^Tx(t) < 0$. On the other hand, it follows that

$$V(\rho(t),t) = V(\rho(0),0) + \int_0^t \mathscr{L}V(x(s),y(s),s)\,\mathrm{d}s + \int_0^t 2e^{\beta s}x^s(t)\,g(x(s),y(s),s)\,\mathrm{d}\omega(s). \tag{22}$$

Note that ξ is bounded and V, G are continuous; then $V(\rho(0))$ must be nonnegative bounded. Moreover, $\mathscr{L}V(x,y,t) \le 0$ can be obtained directly:

$$V(\rho(t),t) \le V(\rho(0),0) + \int_0^t 2e^{\beta t}x^T(s)\,g(x(s),y(s),s)\,\mathrm{d}\omega(s). \tag{23}$$

By applying Lemma 2 to (23), one sees that

$$\lim_{t\to\infty}\sup V(\rho(t),t) < \infty; \tag{24}$$

hence there exists a positive random variable ζ satisfying

$$\sup_{0\le t<\infty} e^{\beta t}\left|x(t) - G(y(t))\right|^2 \le \zeta. \tag{25}$$

Since, for any $\varepsilon_3 \in (0,1)$

$$\left|x(t) - G(y(t))\right|^2 \ge \left(1-\varepsilon_3^{-1}\right)|x(t)|^2 - (\varepsilon_3 - 1)\left|G(y(t))\right|^2, \tag{26}$$

we must have

$$\begin{aligned}\sup_{0\le t\le T} e^{\beta t}|x(t)|^2 &\le \zeta + \frac{k^2}{\varepsilon_3}\sup_{0\le t\le T} e^{\beta t}\left|y(t)\right|^2 \\ &\le \zeta + k^2e^{\beta\tau}\|\xi\|^2 + \frac{k^2}{\varepsilon_3}e^{\beta\tau}\sup_{0\le t\le T} e^{\beta t}|x(t)|^2.\end{aligned} \tag{27}$$

From the above inequality (26), it yields the desired result

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\beta}{2}. \tag{28}$$

That completes the proof. □

4. Example

In this section, a numerical example will be given to illustrate that the proposed method is effective.

Example 1. Consider the following system:

$$\begin{aligned} d\left[x_1(t) - 0.1x_2(t-\delta(t))\right] &= -x_1(t)\,x_2(t-\delta(t))\,\mathrm{d}t \\ &\quad + x_1(t)\sin^2(t-\delta(t))\,\mathrm{d}\omega(t), \\ d\left[x_2(t) - 0.1x_1(t-\delta(t))\right] &= -x_2(t)\,x_1(t-\delta(t))\,\mathrm{d}t \\ &\quad + x_2(t)\cos^2(t-\delta(t))\,\mathrm{d}\omega(t), \end{aligned} \tag{29}$$

where the delay function is defined as $\delta(t) = (1/4)\sin(t), t > 0$. It is obvious that (29) satisfies the assumptions H_1 and H_2, and here $L = I$, $k = 0.1$. Moreover, since $\dot{\delta} = (1/4)\cos(t)$, then $\overline{\delta} = 1/4$.

According to Theorem 4 and employing MATLAB LMI Toolbox, it is relatively easy to deduce that the neutral stochastic differential equation (29) is almost sure exponentially stable.

Remark 5. Comparing with some existing sufficient criteria for neural stochastic differential equations (e.g., [6, 11, 12]), the obtained result is given in terms of linear matrix inequality (LMI), which can be easily checked by MATLAB LMI Toolbox.

5. Conclusion

The exponential stability is investigated for a class of neutral stochastic differential equations with time-varying delays. In order to overcome the difficulties, we introduce suitable Lyapunov functionals and employ linear matrix inequalities (LMIs) technique, and then a delay-dependent criteria are given to check the almost sure exponential stability of the concerned equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

State Estimation for Wireless Network Control System with Stochastic Uncertainty and Time Delay Based on Sliding Mode Observer

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State estimation problem is considered for a kind of wireless network control system with stochastic uncertainty and time delay. A sliding mode observer is designed for the system under the situation that no missing measurement occurs and system uncertainty happens in a stochastic way. The observer designed for the system can guarantee the system states will be driven onto the sliding surface under control law, and the sliding motion of system states on sliding surface will be stable. By constructing proper Lyapunov-Krasovskii functional, sufficient conditions are acquired via linear matrix inequality. Finally, simulation result is employed to show the effectiveness of the proposed method.

1. Introduction

During the past decades, state estimation problem is a hot issue in academical field, and researchers made fruitful research on both theoretical research and practical applications [1–4]. At the same time, research and applications in wireless network control system (WiNCS) attracted much interest from scholars [5–9]. Compared with wired network control system, WiNCS is more convenient in integrated wiring and could be arranged in hazardous area people cannot easily reach. Sensor nodes are distributed in advance to collect information and transmit signal back via signal channel through multihop technology, until they reach users' terminal applications. Because of its low cost, mobility character, and convenience for using, WiNCS is widely used in hospital monitoring, military and urban affairs, and some other important cases [10–13]. However, as the whole system is becoming complicated and integrated, people want to keep track of useful measured values, and a rich body of literature has appeared on the topic of state estimation or observer-based design; see [14–16] and the references therein.

As a special class of complex networks, WiNCS has its own characteristics due mainly to large numbers of nodes distributed over the region of interest. In this case, sensor nodes may be in mobile motion, which is attributed to the uncertainty in system state, and much effort has been made on this issue [17–22]. For distributed state estimation problem, each sensor node in WiNCS estimates system state not only from its own measurement but also from its neighboring nodes, so some unpredictable factors which happen in a stochastic way will enhance the complexity in state estimation [23–25]. Reference [26] studied synchronization problem for Markovian jump neural networks with time-varying delay and variable samplings. Stochastic stability of the error system was guaranteed by two criteria, and mode-independent controller was designed based on the maximum sampling interval. Reference [27] constructed proper Lyapunov-Krasovskii functional and studied the adaptive fault estimation problems

for stochastic Markovian jump systems (MJSs), combined with time delays. Adaptive fault detection observer was designed and the sufficient condition was proposed. Reference [28] concentrated on the proportional-integral control problems of stochastic Markovian jump systems (MJSs) with uncertain parameters. It transformed the controller design problem to an output feedback control problem, and a sufficient condition was proposed via LMI.

Nodes receive measurement information from other nodes in WiNCS which is time consuming, so time delay cannot be avoided in the research of WiNCS, and many scholars devoted to this problem [29–33]. Reference [34] considered the problem that both had discrete and distributed delays, and the delay dependent passivity condition was acquired. Reference [35] investigated the problem of sampled-data extended dissipative control for uncertain Markov jump systems. By proposing a new integral inequality, a novel exponential stability criterion and an extended dissipativity condition were established. In addition, a sufficient condition for desired mode-independent sampled-data controller was also obtained. Reference [36] dealt with the average consensus problem in directed networks of agents with switching topology and time delay. It proved that all the agents could reach average consensus under a proper time delay if the topology of agents was weakly connected and balanced.

Motivated by previous research stated above, our target is to deal with state estimation problem in wireless network control system, which contains stochastic uncertainty and time delay. A sliding mode observer is designed in two steps; by the use of Lyapunov stability theory, sufficient conditions are proposed to make sure that system states can be driven onto the sliding surface within finite time and make stable sliding motion on sliding surface.

The rest of the paper is organized as follows. In Section 2, the state estimation problem of WiNCS is formulated and some useful lemmas are introduced. In Section 3, we make our designing process in two steps. Besides, the gain of observer is acquired by LMI. An illustrated example is given in Section 4 to demonstrate the effectiveness of the proposed method. Finally, we give our conclusions in Section 5.

2. Problem Formulation

In this paper, we consider the following discrete stochastic and time delay model:

$$\begin{aligned} x(k+1) &= \left(A+\alpha_k \Delta A\right) x(k) + A_d \sum_{i=1}^{N} x(k-i) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state vector, $y(k) \in \mathbb{R}^n$ is the measured output, ΔA is internal perturbation arising from uncertain factors, $u(k)$ is system input, A, A_d, B, and C are constant matrices with appropriate dimensions, and α_k is Bernoulli distributed white sequences governed by

$$\begin{aligned} \operatorname{Prob}\{\alpha_k = 1\} &= \mathbb{E}\{\alpha_k\} = \alpha, \\ \operatorname{Prob}\{\alpha_k = 0\} &= 1 - \mathbb{E}\{\alpha_k\} = 1 - \alpha, \end{aligned} \tag{2}$$

where $\alpha \in [0,1]$ is a known constant.

Remark 1. Most nodes in WiNCS may be in dynamic motion; they collect information from areas of interest. However, they may be affected by external environment, such as temperature, humidity, and topography, or linkage status inside the system, so system uncertainty may happen in a stochastic way, which increases the complexity of the system.

We make the following assumptions for system model (1).

Assumption 2. The perturbation parameter of the system satisfies

$$\Delta A = GD(k)H. \tag{3}$$

Respectively, G and H are known constant matrix, $D(k)$ is time-delay uncertain matrix, yet Lebesgue-measurable, and $D^k(t)D(k) \le I$.

Assumption 3. $C^T C$ is full rank.

Assumption 4. All the states of the system can be measured and no missing measurement occurs.

We construct the following sliding mode observer:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + A_d \sum_{i=1}^{N} \hat{x}(k-i) \\ &\quad + Bu(k) + L\left[y(k) - \hat{y}(k)\right] + w(k), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{4}$$

where L is the gain of observer to be designed and $w(k)$ is nonlinear item in observer.

So the state error $e_x(k)$ and output error $e_y(k)$ of the system are as follows:

$$\begin{aligned} e_x(k) &= x(k) - \hat{x}(k), \\ e_y(k) &= y(k) - \hat{y}(k). \end{aligned} \tag{5}$$

In this case, the system error model is

$$\begin{aligned} e_x(k+1) &= (A - KC)e_x(k) + (\alpha_k - \alpha)\Delta A x(k) \\ &\quad + \alpha \Delta A x(k) + A_d \sum_{i=1}^{N} e_x(k-i) - w(k), \\ e_y(k+1) &= Ce_x(k). \end{aligned} \tag{6}$$

We define the sliding surface $s(k)$ as

$$s(k) = e_y(k). \tag{7}$$

Our target in this paper is to find the gain of observer L and the nonlinear input $w(k)$ that can drive system state onto the sliding surface within finite time and, in addition, make sure that sliding motion of system states on sliding surface is stable. Two conditions should be satisfied when designing sliding mode observer [37]:

(1) system error model is asymptotically stable when $s(k+1) = s(k) = 0$;

(2) sliding mode manifold satisfies $\|s(k+1)\| < \|s(k)\|$.

Remark 5. We set system output error as sliding surface; the benefit is, condition (1) guarantees that sliding motion on sliding surface is stable, and condition (2) ensures that system state can be driven onto sliding surface within finite time. In this case, we convert state estimation problem into sliding mode observer design problem.

Besides, some useful and important lemmas that will be used in deriving out results are introduced below.

Lemma 6. *Let $Y = Y^T$, D, E, and $F(t)$ be real matrix of proper dimensions, and $F^T(t)F(t) \le I$; then inequality $Y + DFE + (DFE)^T < 0$ holds if there exists a constant ε, which makes the following equation hold:*

$$Y + \varepsilon DD^T + \varepsilon^{-1}E^TE < 0. \tag{8}$$

Lemma 7 (Schur complement). *Given a symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, where S_{11} is $r \times r$ dimensional, the following three conditions are equivalent:*

(1) $S < 0$;

(2) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;

(3) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Lemma 8. *For any $x, y \in R^n$, $\mu > 0$, the following equation holds:*

$$2x^Ty \le \mu x^Tx + \frac{1}{\mu}y^Ty. \tag{9}$$

3. Main Results

In this section, two theorems will be given. The first one ensures that system states can reach the sliding surface within finite time.

Theorem 9. *For system error model (6) which meets Assumptions 2, 3, and 4, and $w(k) = w_1(k) + w_2(k)$, where $w_1(k) = A_d \sum_{i=1}^N e_x(k-i)$, $w_2(k) = GHx(k)$, system states will be driven onto the sliding surface within finite time if there exists a general matrix $L \in R^{n_x \times n_y}$ making (10) hold:*

$$\begin{bmatrix} -C^TC & \sqrt{3+\alpha}(A-LC)^TC^T \\ * & -I \end{bmatrix} < 0. \tag{10}$$

Proof. Considering $s(k+1) = e_y(k+1) = Ce_x(k+1)$ and $\|s(k+1)\| < \|s(k)\|$, we have

$$\begin{aligned}
&\mathbb{E}\left\{s^T(k+1)s(k+1) - s^T(k)s(k)\right\} \\
&\quad = \mathbb{E}\left\{e_y^T(k+1)e_y(k+1) - e_y^T(k)e_y(k)\right\} \\
&\quad = e_x^T(k)\left[(A-LC)^TC^TC(A-LC) - C^TC\right]e_x(k) \\
&\qquad + 2\alpha e_x^T(k)(A-LC)^TC^TC\Delta Ax(k) \\
&\qquad + 2e_x^T(A-LC)^TC^TCA_d\sum_{i=1}^N e_x(k-i) \\
&\qquad + 2\alpha x^T(k)\Delta A^TC^TCA_d\sum_{i=1}^N e_x(k-i) \\
&\qquad + 2e_x^T(k)(A-LC)^TC^TCw(k) \\
&\qquad + 2\sum_{i=1}^N e_x^T(k-i)A_d^TC^TCw(k) \\
&\qquad + 2\alpha x^T(k)\Delta A^TC^TCw(k) + \alpha x^T(k)\Delta A^TC^TC\Delta Ax(k) \\
&\qquad + \sum_{i=1}^N e_x^T(k-i)A_d^TC^TC\sum_{i=1}^N e_x(k-i) + w^T(k)C^TCw(k).
\end{aligned} \tag{11}$$

According to Lemma 8, we have

$$\begin{aligned}
&2\alpha e_x^T(k)(A-LC)^TC^TC\Delta Ax(k) \\
&\quad \le \alpha e_x^T(k)(A-LC)^TC^TC(A-LC)e_x(k) \\
&\qquad + \alpha x^T(k)\Delta A^TC^TC\Delta Ax(k),
\end{aligned} \tag{12}$$

$$\begin{aligned}
&2e_x(k)^T(A-LC)^TC^TCA_d\sum_{i=1}^N e_x(k)(k-i) \\
&\quad \le e_x(k)^T(A-LC)^TC^TC(A-LC)e_x(k) \\
&\qquad + \sum_{i=1}^N e_x(k)^T(k-i)A_d^TC^TCA_d\sum_{i=1}^N e_x(k)(k-i),
\end{aligned} \tag{13}$$

$$\begin{aligned}
&2\alpha x^T(k)\Delta A^TC^TCA_d\sum_{i=1}^N e_x(k-i) \\
&\quad \le \alpha x^T(k)\Delta A^TC^TC\Delta Ax(k) \\
&\qquad + \alpha\sum_{i=1}^N e_x^T(k-i)A_d^TC^TCA_d\sum_{i=1}^N e_x(k-i),
\end{aligned} \tag{14}$$

$$\begin{aligned}
&2e_x^T(k)(A-LC)^TC^TCw(k)\\
&\quad\le e_x^T(k)(A-LC)^TC^TC(A-LC)e_x(k)\\
&\qquad+w^T(k)C^TCw(k),
\end{aligned}\tag{15}$$

$$\begin{aligned}
&2\sum_{i=1}^{N}e_x^T(k-i)A_d^TC^TCw(k)\\
&\quad\le\sum_{i=1}^{N}e_x^T(k-i)A_d^TC^TCA_d\sum_{i=1}^{N}e_x(k-i)+w^T(k)C^TCw(k),
\end{aligned}\tag{16}$$

$$\begin{aligned}
&2\alpha x^T(k)\Delta A^TC^TCw(k)\\
&\quad\le\alpha x^T(k)\Delta A^TC^TC\Delta Ax(k)+\alpha w^T(k)C^TCw(k).
\end{aligned}\tag{17}$$

Substituting (12) to (17) into (11), we have

$$\begin{aligned}
&\mathbb{E}\left\{s^T(k+1)s(k+1)-s^T(k)s(k)\right\}\\
&\quad\le e_x^T(k)\left[(3+\alpha)(A-LC)^TC^TC(A-LC)-C^TC\right]e_x(k)\\
&\qquad+x^T(k)\left[(3+\alpha)\Delta A^TC^TC\Delta A\right]x(k)\\
&\sum_{i=1}^{N}e_x^T(k-i)\left[(3+\alpha)A_d^TC^TCA_d^T\right]\\
&\qquad\times\sum_{i=1}^{N}e_x(k-i)+(3+\alpha)w^T(k)C^TCw(k)\\
&\quad=e_x^T(k)\left[(3+\alpha)(A-LC)^T\ C^TC(A-LC)-C^TC\right]e_x(k)\\
&\qquad+\left[\Delta Ax(k)-w_2(k)\right]^T(3+\alpha)C^TC\left[\Delta Ax(k)-w_2(k)\right]\\
&\qquad+\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right]^T\\
&\qquad\times(3+\alpha)C^TC\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right]\\
&\quad\le e_x^T(k)\left[(3+\alpha)(A-LC)^T\right.\\
&\qquad\qquad\left.\times C^TC(A-LC)-C^TC\right]e_x(k)\\
&\qquad+\left[GHx(k)-w_2(k)\right]^T(3+\alpha)C^TC\left[GHx(k)-w_2(k)\right]\\
&\qquad+\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right]^T\\
&\qquad\times(3+\alpha)C^TC\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right].
\end{aligned}\tag{18}$$

We make $w_1(k)=A_d\sum_{i=1}^{N}e_x(k-i)$, $w_2(k)=GHx(k)$, so we have

$$\begin{aligned}
&\mathbb{E}\left\{s^T(k+1)s(k+1)-s^T(k)s(k)\right\}\\
&\quad\le e_x^T(k)\left[(3+\alpha)(A-LC)^TC^TC(A-LC)-C^TC\right]e_x(k)\\
&\quad\le 0.
\end{aligned}\tag{19}$$

By Schur complement, inequality (19) is equivalent to

$$\begin{bmatrix}-C^TC & \sqrt{3+\alpha}(A-LC)^TC^T\\ * & -I\end{bmatrix}<0.\tag{20}$$

The proof of Theorem 9 is complete. □

Next, we will prove that sliding motion on sliding surface is stable.

Theorem 10. *For system error model* (6) *which meets Assumptions 2, 3, and 4, and* $w(k)=w_1(k)+w_2(k)$, *where* $w_1(k)=A_d\sum_{i=1}^{N}e_x(k-i)$, $w_2(k)=GHx(k)$, *sliding motion of system states on sliding surface will be stable if there exists a general matrix* $L\in R^{n_x\times n_y}$ *and* $P\in R^{n_x\times n_x}$ *making* (21) *hold:*

$$\begin{bmatrix}-P & \sqrt{3+\alpha}(A-LC)^TP\\ * & -P\end{bmatrix}<0.\tag{21}$$

Proof. We construct the following Lyapunov-Krasovskii functional:

$$V(k)=e_x^T(k)Pe_x(k).\tag{22}$$

Making difference of $V(k)$, we have

$$\begin{aligned}
&\mathbb{E}\left\{e_x^T(k+1)Pe_x(k+1)-e_x^T(k)Pe_x(k)\right\}\\
&\quad=e_x^T(k)\left[(A-LC)^TP(A-LC)-P\right]e_x(k)\\
&\qquad+2\alpha e_x^T(k)(A-LC)^TP\Delta Ax(k)\\
&\qquad+2e_x^T(A-LC)^TPA_d\sum_{i=1}^{N}e_x(k-i)\\
&\qquad+2\alpha x^T(k)\Delta A^TPA_d\sum_{i=1}^{N}e_x(k-i)\\
&\qquad+2e_x^T(k)(A-LC)^TPw(k)\\
&\qquad+2\sum_{i=1}^{N}e_x^T(k-i)A_d^TPw(k)+2\alpha x^T(k)\Delta A^TPw(k)\\
&\qquad+\alpha x^T(k)\Delta A^TP\Delta Ax(k)\\
&\qquad+\sum_{i=1}^{N}e_x^T(k-i)A_d^TP\sum_{i=1}^{N}e_x(k-i)+w^T(k)Pw(k).
\end{aligned}\tag{23}$$

According to Lemma 8, we have

$$\begin{aligned}
&2\alpha e_x^T(k)(A-LC)^T P\Delta A x(k)\\
&\quad\le \alpha e_x^T(k)(A-LC)^T P(A-LC)e_x(k)\\
&\qquad+\alpha x^T(k)\Delta A^T P\Delta A x(k),
\end{aligned}\tag{24}$$

$$\begin{aligned}
&2e_x(k)^T(A-LC)^T PA_d\sum_{i=1}^{N}e_x(k)(k-i)\\
&\quad\le e_x(k)^T(A-LC)^T P(A-LC)e_x(k)\\
&\qquad+\sum_{i=1}^{N}e_x^T(k-i)A_d^T PA_d\sum_{i=1}^{N}e_x(k-i),
\end{aligned}\tag{25}$$

$$\begin{aligned}
&2\alpha x^T(k)\Delta A^T PA_d\sum_{i=1}^{N}e_x(k-i)\\
&\quad\le \alpha x^T(k)\Delta A^T P\Delta A x(k)\\
&\qquad+\alpha\sum_{i=1}^{N}e_x^T(k-i)A_d^T PA_d\sum_{i=1}^{N}e_x(k-i),
\end{aligned}\tag{26}$$

$$\begin{aligned}
&2e_x^T(k)(A-LC)^T Pw(k)\\
&\quad\le e_x^T(k)(A-LC)^T P(A-LC)e_x(k)+w^T(k)Pw(k),
\end{aligned}\tag{27}$$

$$\begin{aligned}
&2\sum_{i=1}^{N}e_x^T(k-i)A_d^T Pw(k)\\
&\quad\le \sum_{i=1}^{N}e_x^T(k-i)A_d^T PA_d\\
&\qquad\times\sum_{i=1}^{N}e_x(k-i)+w^T(k)Pw(k),
\end{aligned}\tag{28}$$

$$\begin{aligned}
&2\alpha x^T(k)\Delta A^T Pw(k)\\
&\quad\le \alpha x^T(k)\Delta A^T P\Delta A x(k)+\alpha w^T(k)Pw(k).
\end{aligned}\tag{29}$$

Substituting (24) to (29) into (23), we have

$$\begin{aligned}
&\mathbb{E}\left\{e_x^T(k+1)Pe_x(k+1)-e_x^T(k)Pe_x(k)\right\}\\
&\quad\le e_x^T(k)\left[(3+\alpha)(A-LC)^T P(A-LC)-P\right]e_x(k)\\
&\qquad+x^T(k)\left[(3+\alpha)\Delta A^T P\Delta A\right]x(k)\\
&\qquad\sum_{i=1}^{N}e_x^T(k-i)\left[(3+\alpha)A_d^T PA_d^T\right]\\
&\qquad\times\sum_{i=1}^{N}e_x(k-i)+(3+\alpha)w^T(k)Pw(k)\\
&\quad= e_x^T(k)\left[(3+\alpha)(A-LC)^T P(A-LC)-P\right]e_x(k)\\
&\qquad+\left[\Delta Ax(k)-w_2(k)\right]^T(3+\alpha)P\left[\Delta Ax(k)-w_2(k)\right]\\
&\qquad+\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right]^T\\
&\qquad\times(3+\alpha)P\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right]\\
&\quad\le e_x^T(k)\left[(3+\alpha)(A-LC)^T P(A-LC)-P\right]e_x(k)\\
&\qquad+\left[GHx(k)-w_2(k)\right]^T(3+\alpha)P\left[GHx(k)-w_2(k)\right]\\
&\qquad+\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right]^T\\
&\qquad\times(3+\alpha)P\left[A_d\sum_{i=1}^{N}e_x(k-i)-w_1(k)\right].
\end{aligned}\tag{30}$$

We make $w_1(k)=A_d\sum_{i=1}^{N}e_x(k-i)$, $w_2(k)=GHx(k)$, so we have

$$\begin{aligned}
&\mathbb{E}\left\{e_x^T(k+1)Pe_x(k+1)-e_x^T(k)Pe_x(k)\right\}\\
&\quad\le e_x^T(k)\left[(3+\alpha)(A-LC)^T P(A-LC)-P\right]e_x(k)\\
&\quad\le 0.
\end{aligned}\tag{31}$$

By Schur complement again, inequality (31) is equivalent to

$$\begin{bmatrix}-P & \sqrt{3+\alpha}(A-LC)^T P\\ * & -P\end{bmatrix}<0.\tag{32}$$

The proof of Theorem 10 is complete. □

4. Numerical Simulations

In this section, a numerical simulation is given for testing the theorems developed in this paper. Consider the system model (1), where

$$A=\begin{bmatrix}6.18 & 0 & -3\\ 5.64 & 1.005 & 3.15\\ 2.97 & 0 & 3.3\end{bmatrix},$$

$$A_d=\begin{bmatrix}-3.38 & 0.15 & 0\\ -0.285 & 0.93 & 0.48\\ -0.555 & 1.17 & 2.235\end{bmatrix},$$

$$C=\begin{bmatrix}2.8800 & 1.1048 & 0.0150\\ 1.3095 & 0.7680 & 0.0195\\ 0.0930 & 0.0315 & 1.0800\end{bmatrix},\qquad B=\begin{bmatrix}1.2\\ 0.7\\ 1\end{bmatrix},$$

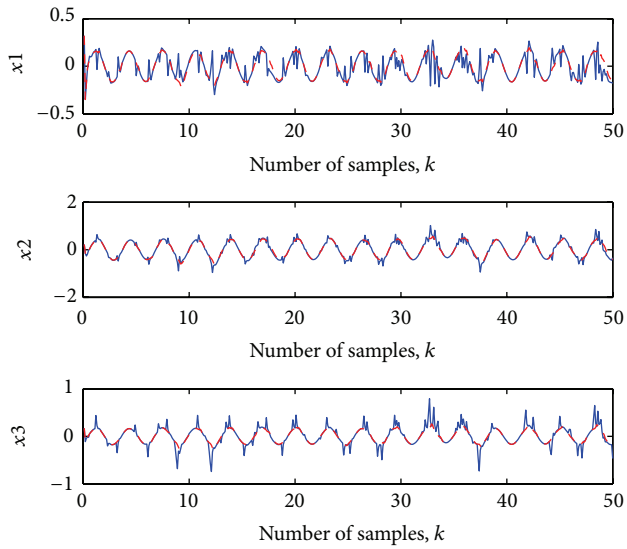

Figure 1: Trajectories of system states and observed value.

$$G = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0 & 0.23 & -0.76 \\ 0.87 & 0.5 & -0.32 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.184 & 0.112 & 0.23 \\ 0.097 & -0.16 & -0.156 \\ -0.277 & -0.069 & -0.152 \end{bmatrix},$$

$$D(k) = \begin{bmatrix} 0.8\sin(0.7k) & 0 & 0 \\ 0 & 0.8\sin(0.7k) & 0 \\ 0 & 0 & 0.8\sin(0.7k) \end{bmatrix},$$

$$u(k) = \sin(0.3k), \qquad \alpha = 0.15. \tag{33}$$

The initial states are $x(k) = [1.3 \quad 0.427 \; -0.92]^T$; according to Theorems 9 and 10, we have

$$L = \begin{bmatrix} 6.3100 & -8.9663 & -2.7035 \\ 3.8252 & -4.3148 & 2.9414 \\ 2.8593 & -4.2400 & 3.0924 \end{bmatrix},$$
$$P = 10^5 \times \begin{bmatrix} 1.3987 & 0 & 0 \\ 0 & 1.3987 & 0 \\ 0 & 0 & 1.3987 \end{bmatrix}. \tag{34}$$

Simulation results are shown in Figure 1; red line denotes real states value and blue line denotes observed value. From the figure, we can see that the estimated value tracks the real value very well, which can fully demonstrate the effectiveness of the proposed method. States error is shown in Figure 2, which is much smaller than observer design method proposed in [38], shown in Figure 3; this is because slide mode control is insensitive to uncertainty and external disturbance. However, we can see from Figure 2 that when system states are at turning point, states error becomes larger; this is because chattering is inevitable in slide mode control, which is our future work.

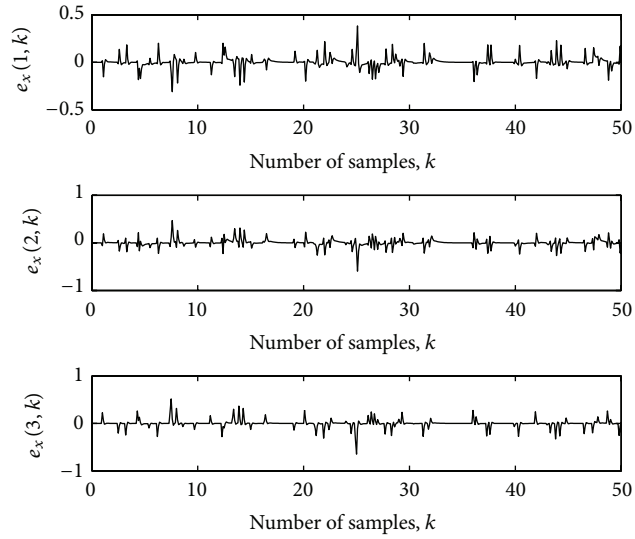

Figure 2: Trajectories of system states error.

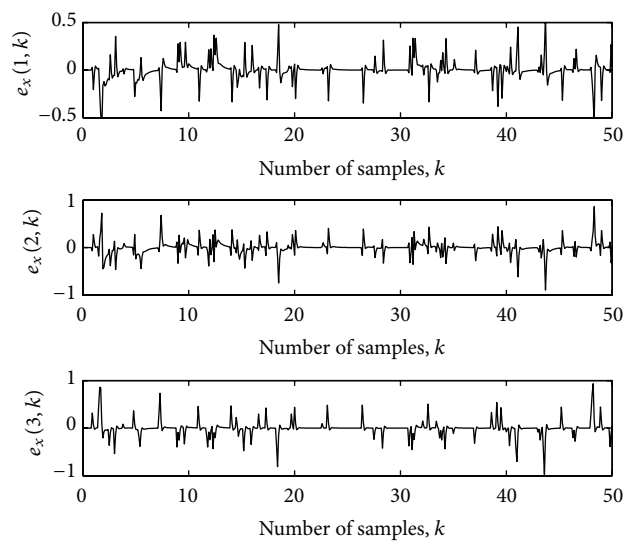

Figure 3: Trajectories of system states error by applying method proposed in [38].

5. Conclusion

This paper considers state estimation problem in wireless network control system with stochastic uncertainty and time delay. System uncertainty is assumed to occur in a stochastic way, which increases system complexity, and time delay is also considered. A sliding mode observer is designed in two steps; by constructing proper Lyapunov-Krasovskii functional, sufficient condition is acquired via linear matrix inequality. Finally, simulation result is employed to show the effectiveness of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Fault Detection for Wireless Networked Control Systems with Stochastic Uncertainties and Multiple Time Delays

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The fault detection problem for a class of wireless networked control systems is investigated. A Bernoulli distributed parameter is introduced in modeling the system dynamics; moreover, multiple time delays arising in the communication are taken into account. The detection observer for tracking the system states is designed, which generates both the state errors and the output errors. By adopting the linear matrix inequality method, a sufficient condition for the stability of wireless networked control systems with stochastic uncertainties and multiple time delays is proposed, and the gain of the fault detection observer is obtained. Finally, an illustrated example is provided to show that the observer designed in this paper tracks the system states well when there is no fault in the systems; however, when fault happens, the observer residual signal rises rapidly and the fault can be quickly detected, which demonstrate the effectiveness of the theoretical results.

1. Introduction

The network technology has received compelling attention during the past decades [1]. The networked control system, which plays an important role in modern industry such as the car industry and the health care, can usually be classified into the wired networked control system (WNCS) and the wireless networked control system (WiNCS) [2, 3]. Compared with WNCS, WiNCS is a comparatively new technology, which is widely used in military, monitory, and other complex situations. In WiNCS, large numbers of sensor nodes are arranged in the region of interest; due to the characteristics of the wireless communication, information flows among nodes are dynamic. As the structure of WiNCS becomes increasingly modular, system faults may result in fatal damage to the whole system. As a result, the fault detection problem for WiNCS deserves to be investigated.

The fault detection problem for WiNCS has been studied extensively in recent years [4–10]. Reference [7] considers the fault detection problem for WiNCS with access constraints and random packet dropouts. The residual generation is carried out based on a deterministic formulation and a residual evaluation is proposed by considering the random packet dropouts. In addition, with the help of Chebyshev's inequality, the fault detection threshold value is obtained. Reference [8] investigates the fault detection problem for a class of linear time invariant systems with limited network quality of services (QoS). The probabilistic switching between different situations is required to obey a homogeneous Markovian chain. In [9], the adaptive observer-based fault estimation problem is studied; by exploring the augmented matrix, error dynamic systems are transferred to Markov jumping systems. In [10], the T-S fuzzy model is adopted to describe the system model, with the benefit that the exact value of network-induced delay is not necessarily known; a fuzzy observer-based approach for the fault detection is developed.

In WiNCS, the sensor nodes gather information from the plant and pass the information to the designed controller via the bus structure. However, as the sensor nodes may be influenced by several unexpected factors such as temperature and moisture, not all the sensor nodes are in the working status. When some nodes are not working, the information flow may be transmitted from other signal channels, which arouses in the uncertainties of WiNCS. On the other hand,

as the information transmission is time consuming, time delays arise naturally in WiNCS. The fault detection problem for wireless networked control systems with both stochastic uncertainties and multiple time delays has not been considered in the literature, which motivates the work in this paper. In this paper, we investigate the fault detection for a class of wireless networked control systems. A Bernoulli distributed parameter is introduced in modeling the system dynamics and the detection observer for tracking the system states is designed. By adopting the linear matrix inequality method, a sufficient condition for the stability of wireless networked control systems with stochastic uncertainties and multiple time delays is proposed, and the gain of the fault detection observer is obtained. Finally, an example is given to show the effectiveness of proposed method.

The rest of this paper is organized as follows. In Section 2, we provide the problem formulation. In Section 3, the sufficient condition for the stability of WiNCS is obtained by the linear matrix inequality method. An illustrative example is given in Section 4 and Section 5 is a brief conclusion of this paper.

2. Problem Formulation

Consider a class of WiNCS. The system model is given as follows:

$$\begin{aligned} x(k+1) &= \left(A+\alpha_k\widetilde{A}\right)x(k)+A_d\sum_{i=1}^{N}x(k-i)+Bu(k)+E_f f(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbf{R}^{n_x}$ denotes the state without delays, $x(k-i) \in \mathbf{R}^{n_x}$ denotes the delayed state of the system, $u(k) \in \mathbf{R}^{n_u}$ denotes the system input, $f(k) \in \mathbf{R}^{n_f}$ denotes the fault of the system, $y(k) \in \mathbf{R}^{n_y}$ denotes the system output, α_k is the stochastic variable, and A, $\widetilde{A}$, A_d, B, E_f, and C are constant matrices with appropriate dimensions. The control law $u(k)$ has the following form:

$$u(k) = Kx(k), \tag{2}$$

where K is the parameter matrix.

In this paper, the stochastic variable α_k is assumed to be a Bernoulli distributed sequence, which represents whether the communication environment changes or not at each nonnegative integer time k. We assume that $\alpha_k = 0$ when there is no change in communication environment, and $\alpha_k = 1$ when the network environment changes. Moreover, the following equations hold:

$$\begin{aligned} P\{\alpha_k = 1\} &= \alpha, \\ P\{\alpha_k = 0\} &= 1-\alpha, \end{aligned} \tag{3}$$

where $\alpha \in [0,1]$ is a given constant.

In order to generate a residual signal, we design the fault detection observer for model (1) as follows:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + A_d\sum_{i=1}^{N}\hat{x}(k-i) \\ &\quad + BK\hat{x}(k) + L\left[y(k)-\hat{y}(k)\right], \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{4}$$

where $\hat{x}(k)$ and $\hat{y}(k)$ are the state and the output of the observer and L is the parameter of the observer to be designed.

Let

$$\begin{aligned} e_x(k) &= x(k)-\hat{x}(k), \\ e_y(k) &= y(k)-\hat{y}(k). \end{aligned} \tag{5}$$

The error model of the system is given as

$$\begin{aligned} e_x(k+1) &= (A+BK-LC)e_x(k) + (\alpha_k-\alpha)\widetilde{A}x(k) \\ &\quad + \alpha\widetilde{A}x(k) + A_d\sum_{i=1}^{N}e_x(k-i), \\ e_y(k) &= Ce_x(k). \end{aligned} \tag{6}$$

Next, we define the system residual as $r(k) = e_y(k)$. Note that if there is no fault in the system, the residual is close to zero. We set up the residual evaluation function J and the fault threshold J_{th} as

$$\begin{aligned} J(k) &= \sqrt{\sum_{k=1}^{N} r^T(k)\,r(k)}, \\ J_{\text{th}} &= \sup J(k). \end{aligned} \tag{7}$$

By comparing J and J_{th}, it can be decided whether the fault happens or not:

$$\begin{aligned} J \le J_{\text{th}} &\quad \text{No fault happens}, \\ J > J_{\text{th}} &\quad \text{Fault happens}. \end{aligned} \tag{8}$$

Lemma 1 (Schur complement). *For matrices A, B, and C, $A + B^TCB < 0$ equals*

$$\begin{bmatrix} A & B^T \\ B & -C^{-1} \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -C^{-1} & B \\ B^T & A \end{bmatrix} < 0. \tag{9}$$

3. Main Results

In this section, a fault detection observer will be designed based on the Lyapunov stability theory. A sufficient condition for the stability of the system is obtained by the linear matrix inequalities.

Theorem 2. *The error system* (6) *is quadratically stable if there exist matrices $P > 0$, $Q > 0$, and $R > 0$ with appropriate dimensions satisfying the following inequality:*

$$W' = \begin{pmatrix} W'_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ * & W'_{22} & 0 & W_{24} & W_{25} \\ * & * & W'_{33} & 0 & W_{35} \\ * & * & * & W'_{44} & W_{45} \\ * & * & * & * & W_{55} \end{pmatrix} < 0, \tag{10}$$

where

$$\begin{aligned}
W'_{11} &= \alpha(A+BK)^T P\widetilde{A} + \alpha\widetilde{A}^T P(A+BK) - P + NQ, \\
W_{12} &= \alpha\widetilde{A}^T P(A+BK) - \alpha\widetilde{A}^T RC, \\
W_{13} &= (A+BK)^T PA_d + \alpha\widetilde{A}^T PA_d, \\
W_{14} &= \alpha\widetilde{A}^T PA_d, \\
W_{15} &= \begin{bmatrix} (A+BK)^T P & \sqrt{2}\overline{\alpha}\widetilde{A}^T P & \sqrt{2}\overline{\alpha}A^T P & 0 & 0 & 0 \end{bmatrix}, \\
W'_{22} &= NQ - P, \\
W_{24} &= (A+BK)^T PA_d - (RC)^T A_d, \\
W_{25} &= \begin{bmatrix} 0 & 0 & 0 & (A+BK)^T P - (RC)^T & 0 & 0 \end{bmatrix}, \\
W'_{33} &= -\frac{2}{(1+N)N}Q, \\
W_{35} &= \begin{bmatrix} 0 & 0 & 0 & 0 & A_d^T P & 0 \end{bmatrix}, \\
W'_{44} &= -\frac{2}{(1+N)N}Q, \\
W_{45} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_d^T P \end{bmatrix}, \\
W_{55} &= \operatorname{diag}\{-P, -P, -P, -P, -P, -P\}, \\
\overline{\alpha} &= \sqrt{\alpha(1-\alpha)}, \\
R &= L^T P.
\end{aligned} \tag{11}$$

Proof. We choose the following Lyapunov function:

$$V(k) = V_1(k) + V_2(k). \tag{12}$$

where

$$\begin{aligned}
V_1(k) &= e_x^T(k) Pe_x(k) + x^T(k) Px(k), \\
V_2(k) &= \sum_{i=1}^{N}\sum_{l=k-i}^{k-1}\left\{e_x^T(k) Qe_x(k) + x^T(k) Qx(k)\right\}.
\end{aligned} \tag{13}$$

Calculate the difference of (12) along system (6); we have

$$\begin{aligned}
E\Delta V_1(k) &= e_x^T(k+1) Pe_x(k+1) + x^T(k+1) Px(x+1) \\
&\quad - e_x^T(k) Pe_x(k) - x^T(k) Px(k) \\
&= e_x^T(k)(A-LC+BK)^T P(A-LC+BK) e_x(k) \\
&\quad + \alpha e_x^T(k)(A-LC+BK)^T P\widetilde{A}x(k) \\
&\quad + e_x^T(k)(A-LC+BK)^T PA_d\sum_{i=1}^{N} e_x(k-i) \\
&\quad + \alpha(1-\alpha)x^T(k)\widetilde{A}^T P\widetilde{A}x(k) \\
&\quad + \alpha x^T(k)\widetilde{A}^T P(A-LC+BK) e_x(k) \\
&\quad + \alpha^2 x^T(k)\widetilde{A}^T P\widetilde{A}x(k) \\
&\quad + \alpha x^T(k)\widetilde{A}^T PA_d\sum_{i=1}^{N} e_x(k-i) \\
&\quad + \sum_{i=1}^{N} e_x^T(k-i) A_d^T P(A-LC+BK) e_x(k) \\
&\quad + \alpha\sum_{i=1}^{N} e_x^T(k-i) A_d^T P\widetilde{A}x(k) \\
&\quad + \sum_{i=1}^{N} e_x^T(k-i) A_d^T PA_d\sum_{i=1}^{N} e_x(k-i) \\
&\quad - e_x^T(k) Pe_x(k) \\
&\quad + x^T(k)(A+BK)^T P(A+BK) x(k) \\
&\quad + \alpha x^T(k)(A+BK)^T P\widetilde{A}x(k) \\
&\quad + x^T(k)(A+BK)^T PA_d\sum_{i=1}^{N} x(k-i) \\
&\quad + \alpha(1-\alpha)x^T(k)\widetilde{A}^T P\widetilde{A}x(k) \\
&\quad + \alpha x^T(k)\widetilde{A}^T P(A+BK) x(k) \\
&\quad + \alpha^2 x^T(k)\widetilde{A}^T P\widetilde{A}x(k) \\
&\quad + \alpha x^T(k)\widetilde{A}^T PA_d\sum_{i=1}^{N} x(k-i) \\
&\quad + \sum_{i=1}^{N} x^T(k-i) A_d^T P(A+BK) x(k) \\
&\quad + \alpha\sum_{i=1}^{N} x^T(k-i) A_d^T P\widetilde{A}x(k)
\end{aligned}$$

$$+\sum_{i=1}^{N} x^T(k-i) A_d^T P A_d \sum_{i=1}^{N} x(k-i) - x^T(k) P x(k). \tag{14}$$

According to Jensen's inequality, we have

$$\begin{aligned} E\Delta V_2(k) &= \sum_{i=1}^{N}\left\{\sum_{l=k+1-i}^{k} e_x^T(l) Q e_x(l) - \sum_{l=k-i}^{k-1} e_x^T(l) Q e_x(l) + \sum_{l=k+1-i}^{k} x^T(l) Q x(l) - \sum_{l=k-i}^{k-1} x^T(l) Q x(l)\right\} \\ &= \sum_{i=1}^{N}\left\{e_x^T(k) Q e_x(k) - e_x^T(k-i) Q e_x(k-i) + x^T(k) Q x(k) - x^T(k-i) Q x(k-i)\right\} \\ &\le N e_x^T(k) Q e_x(k) + N x^T(k) Q x(k) - \frac{2}{(1+N)N}\sum_{i=1}^{N} e_x^T(k-i) Q \sum_{i=1}^{N} e_x(k-i) \\ &\quad - \frac{2}{(1+N)N}\sum_{i=1}^{N} x^T(k-i) Q \sum_{i=1}^{N} x(k-i). \end{aligned} \tag{15}$$

Substituting $E\Delta V_1(k)$ and $E\Delta V_2(k)$ into (12), we have

$$\begin{aligned} E\Delta V(k) &= E\Delta V_1(k) + E\Delta_2 V(k) \\ &= Z^T(k) W Z(k), \end{aligned} \tag{16}$$

where

$$Z(k) = \left[x^T(k) \quad e_x^T(k) \quad \sum_{i=1}^{N} x^T(k-i) \quad \sum_{i=1}^{N} e_x^T(k-i)\right]^T,$$

$$W = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ * & W_{22} & 0 & W_{24} \\ * & * & W_{33} & 0 \\ * & * & * & W_{44} \end{pmatrix},$$

$$\begin{aligned} W_{11} &= (A+BK)^T P (A+BK) + \alpha(A+BK)^T P \widetilde{A} + 2\alpha(1-\alpha)\widetilde{A}^T P \widetilde{A}, \\ &\quad + \alpha \widetilde{A}^T P (A+BK) + 2\alpha^2 \widetilde{A}^T P \widetilde{A} - P + NQ, \\ W_{22} &= (A-LC+BK)^T P (A-LC+BK) - P + NQ, \\ W_{33} &= A_d^T P A_d - \frac{2}{(1+N)N} Q, \\ W_{44} &= A_d^T P A_d - \frac{2}{(1+N)N} Q, \end{aligned} \tag{17}$$

and W_{12}, W_{13}, W_{14}, and W_{24} are the same as in (10).

According to the Lyapunov stability theory, the error system is stable if

$$W < 0. \tag{18}$$

According to Lemma 1, (18) is equivalent to

$$W' = \begin{pmatrix} W'_{11} & W_{12} & W_{13} & W_{14} & W'_{15} \\ * & W'_{22} & 0 & W_{24} & W'_{25} \\ * & * & W'_{33} & 0 & W'_{35} \\ * & * & * & W'_{44} & W'_{45} \\ * & * & * & * & W'_{55} \end{pmatrix} < 0, \tag{19}$$

where

$$\begin{aligned} W'_{15} &= \left[(A+BK)^T \quad \sqrt{2}\overline{\alpha}\widetilde{A}^T \quad \sqrt{2}\overline{\alpha}A^T \quad 0 \quad 0 \quad 0\right], \\ W'_{25} &= \left[0 \quad 0 \quad 0 \quad (A-LC+BK)^T \quad 0 \quad 0\right], \\ W'_{35} &= \left[0 \quad 0 \quad 0 \quad 0 \quad A_d^T \quad 0\right], \\ W'_{45} &= \left[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad A_d^T\right], \\ W'_{55} &= \operatorname{diag}\left\{-P^{-1}, -P^{-1}, -P^{-1}, -P^{-1}, -P^{-1}, -P^{-1}\right\}. \end{aligned} \tag{20}$$

We set $R = L^T P$ and multiply diag$\{I, I, I, I, P, P, P, P, P\}$ on both sides of (19). Then we can get inequality (10). The proof is completed. □

Remark 3. As is well known, the randomly occurring phenomena have been extensively investigated in recent years. In this paper, we consider the case where the communication environment is affected by some factors in a probabilistic way described by Bernoulli random variable α_k.

Remark 4. In practice, many systems have stochastic Markovian jumping dynamics [11–15]. Future research efforts will be devoted to the fault detection for wireless sensor networks with stochastic Markovian jumping dynamics.

4. An Illustrative Example

In this section, we will provide a numerical example to illustrate the effectiveness of the theoreticalresults.

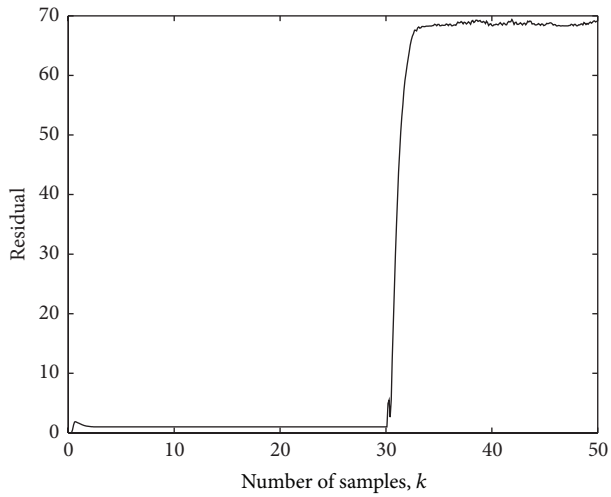

Figure 1: System residual signal with fault.

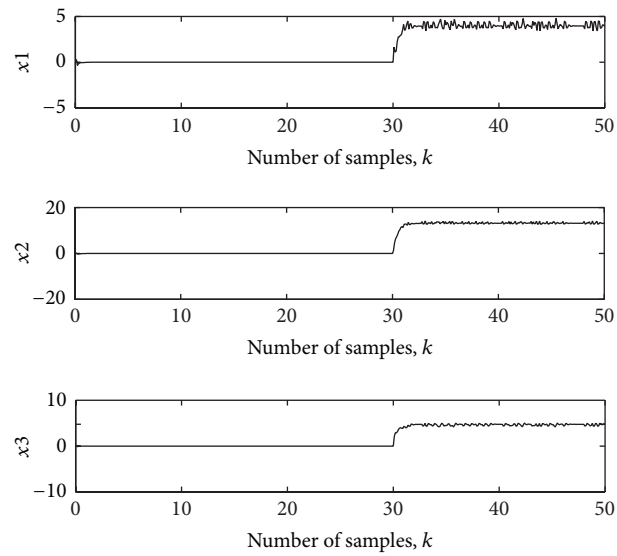

Figure 2: System fault in different states.

Example 1. Consider system (1), where

$$A = \begin{pmatrix} 0.4985 & 0.4849 & -0.7260 \\ 0.4550 & 0.4441 & 1.1011 \\ 0.2396 & 0 & 0.2662 \end{pmatrix},$$

$$A_d = \begin{pmatrix} -0.2807 & 0.0121 & 0 \\ -0.0230 & 0.0750 & 0.0387 \\ -0.0448 & 0.0944 & 0.1803 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.0883 \\ 0.0544 \\ 0.0835 \end{pmatrix}, \tag{21}$$

$$C = \begin{pmatrix} 2.3232 & 0.8912 & 0.0121 \\ 1.0563 & 0.6195 & 0.0157 \\ 0.0750 & 0.0254 & 0.8712 \end{pmatrix},$$

$$\widetilde{A} = \begin{pmatrix} -0.0102 & 0.0239 & 0.0101 \\ 0 & -0.1018 & 0.1201 \\ 0.2120 & -0.0349 & 0.0002 \end{pmatrix},$$

$$K = \begin{pmatrix} 275.7123 & 245.9043 & -350.6332 \end{pmatrix}.$$

We set $\alpha = 0.2$, and P, Q, and R can be obtained as

$$P = \begin{pmatrix} 131.2345 & 28.2728 & 7.8883 \\ 28.2728 & 110.5035 & -5.9995 \\ 7.8883 & -5.9995 & 196.6286 \end{pmatrix},$$

$$Q = \begin{pmatrix} 15.7873 & -0.0697 & -0.0521 \\ -0.0697 & 15.2671 & 0.1465 \\ -0.0521 & 0.1465 & 15.3377 \end{pmatrix}, \tag{22}$$

$$R = \begin{pmatrix} -46.0191 & -51.8770 & 71.2995 \\ 170.4173 & 158.7889 & -128.9558 \\ -89.4728 & 102.9384 & 27.9875 \end{pmatrix}.$$

Moreover, we have

$$L = \begin{pmatrix} -0.2917 & 1.0973 & -0.9504 \\ -0.3751 & 1.1201 & 1.1865 \\ 0.3629 & -0.6657 & 0.2167 \end{pmatrix}. \tag{23}$$

When fault happens at sampling time 30, we can see that residual signal rises quickly, which indicates that fault occurs, as shown in Figure 1. In addition, we can clearly see different fault happens in every state from Figure 2.

5. Conclusion

In this paper, we discussed the fault detection problem for WiNCS with both stochastic uncertainties and multiple time delays. By adopting the Lyapunov method, a sufficient condition for the stability of the system is provided, and the gain of observer is also acquired. Finally, simulation results show the effectiveness of theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Constants within Error Estimates for Legendre-Galerkin Spectral Approximations of Control-Constrained Optimal Control Problems

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Explicit formulae of constants within the a *posteriori* error estimate for optimal control problems are investigated with Legendre-Galerkin spectral methods. The constrained set is put on the control variable. For simpleness, one-dimensional bounded domain is taken. Meanwhile, the corresponding a *posteriori* error indicator is established with explicit constants.

1. Introduction

Recently, spectral method has been extended to approximate the discretization of partial differential equations for design optimization, engineering design, and other engineering computations. It provides higher accurate approximations with a relatively small number of unknowns if the solution is smooth; see [1]. There have been extensive researches on finite element methods for optimal control problems, which focus on control-constrained problems; see [2–8]. The authors [9] studied state-constrained optimal control problems with finite element methods. However, there are few works on optimal control problems with spectral methods.

In order to get a numerical solution with acceptable accuracy, spectral methods only increase the degree of basis where the error indicator is larger than the *a posteriori* error indicator, while the finite element methods refine meshes (see [10]). There have been lots of papers concerning on *a posteriori* error estimates for h-version finite element methods, but not for spectral methods. Guo [11] got a reliable and efficient error indicator for p-version finite element method in one dimension with a certain weight. Zhou and Yang [12] deduced a simple error indicator for spectral Galerkin methods. In [13], the authors investigated Legendre-Galerkin spectral method for optimal control problems with integral constraint for state in one-dimensional bounded domain. It is difficult to obtain optimal *a posteriori* error estimates. Thus, if one gets the constants within upper bound *a posteriori* error estimates, it is easy to ensure the degree of polynomials to get an acceptable accuracy.

In this paper, the control-constrained optimal control problems are solved with Legendre-Galerkin spectral methods, and constants within upper bound of the *a posteriori* error indicator, which can be used to decide the least unknowns for acceptable accuracy, are proposed. By introducing auxiliary systems, explicit formulae of the constants within the *a posteriori* error estimates are obtained.

The outline of this paper is as follows. In Section 2, the model problem and its Legendre-Galerkin spectral approximations are listed. In Section 3, the constants within the *a posteriori* error estimates are investigated in details, and the explicit formulae are obtained. The conclusions are given in Section 4.

2. A Model Problem and Its Legendre-Galerkin Spectral Approximations

Throughout this paper, we focus on $I = (-1, 1)$ and adopt the standard notations $W^{m,p}$ for Sobolev spaces with the norm $\|\cdot\|_{W^{m,p}}$ and the seminorm $|\cdot|_{W^{m,p}}$; see [14]. Specially, we set $W_0^{m,p} = \{w \in W^{m,p} : w|_{\partial I} = 0\}$. If $p = 2$, we denote $W^{m,2}$ and $W_0^{m,2}$ by H^m and H_0^1, respectively.

The problem in which we are interest is the following distributed convex optimal control problem with integral constraint on the control variable:

$$\min_{u\in K} \quad J(u,y)=\frac{1}{2}\int_I (y-y_d)^2+\frac{\alpha}{2}\int_I u^2, \tag{1}$$

$$\text{subject to} \quad -y''=f+u \quad \text{in } I, \qquad y|_{\partial I}=0, \tag{2}$$

where $K=\{w\in L^2(I):\int_I w\geq 0\}$, and the control variable $u\in U=L^2(I)$, the state variable $y\in V=H_0^1(I)$, and $y_d\in L^2(I)$ is the observation.

In order to assure existence and regularity of the solution, we assume that f and y_d are infinitely smooth functions; α is a given positive constant, for simplicity, we set $\alpha=1$. It is well-known that (1) has a unique solution (see [5, 15]).

Now, we introduce the weak formula of (1). We give some basic notations which will be used in the sequel. Let

$$(v,w)=\int_I vw, \quad \forall v,\ w\in L^2(I), \qquad a(v,w)=\int_I v'w', \quad \forall v,\ w\in H_0^1(I). \tag{3}$$

Hence, the state equation (2) reduces to

$$a(y,w)=(f+u,w), \quad \forall w\in H_0^1(I). \tag{4}$$

Then, (1) can be rewritten as follows: find (u,y) such that

$$(\mathscr{P})\begin{cases}\min\limits_{u\in K} & J(u,y)=\frac{1}{2}\int_I (y-y_d)^2+\frac{1}{2}\int_I u^2,\\ \text{s.t.} & a(y(u),w)=(f+u,w), \quad \forall w\in V.\end{cases} \tag{5}$$

We recall following optimality conditions of the optimal control problem (for the details, please refer to [8, 15]): (1) has a unique solution (y,u). Meanwhile, (y,u) is the solution of (1) if and only if there is a costate $p\in V$ such that the triplet (y,p,u) satisfies the following optimal conditions:

$$a(y,w)=(f+u,w), \quad \forall w\in V, \qquad a(q,p)=(y-y_d,q), \quad \forall q\in V, \qquad (u+p,v-u)\geq 0, \quad \forall v\in K\subset U. \tag{6}$$

Let $\mathscr{P}_N(I)=\{\text{polynomials of degree}\leqslant N \text{ on } I\}$ and let $V_N=\mathscr{P}_N\cap H_0^1(I)$. One may expand the discrete polynomial spaces as

$$V_N=\operatorname{span}\{\phi_1(x),\phi_2(x),\ldots,\phi_N(x)\}\subset V, \qquad U_N=\mathscr{P}_N(I)\cap U, \qquad K_N=\mathscr{P}_N(I)\cap K. \tag{7}$$

One prefers to choose appropriate bases of V_N such that the resulting linear system is as simple as possible. Following [16], we choose the basis functions as

$$\phi_i(x)=c_i\left(L_{i-1}(x)-L_{i+1}(x)\right), \quad c_i=\frac{1}{\sqrt{4i+2}}, \quad i=1,2,\ldots,N, \tag{8}$$

where $L_r(x)$ denotes the r-th degree Legendre polynomial. Then, Galerkin spectral approximations of (5) read as follows: find (u_N,y_N) such that

$$(\mathscr{P}^N)\begin{cases}\min\limits_{u_N\in K\subset U_N} & J(u_N,y_N)=\frac{1}{2}\int_I (y_N-y_d)^2+\frac{1}{2}\int_I u_N^2,\\ \text{s.t.} & a(y_N,w_N)=(f+u_N,w_N), \quad \forall w_N\in V_N.\end{cases} \tag{9}$$

It is obvious that (9) has a solution(y_N,u_N) and (y_N,u_N) is the solution if and only if there is a costate $p_N\in V_N$ satisfies the triplet (y_N,p_N,u_N) such that

$$\begin{aligned} a(y_N,w_N)&=(f+u_N,w_N), \quad \forall w_N\in V_N,\\ a(q_N,p_N)&=(y_N-y_d,q_N), \quad \forall q_N\in V_N,\\ (u_N+p_N,v_N-u_N)&\geq 0, \quad \forall v_N\in K_N.\end{aligned} \tag{10}$$

Now, we are at the point to analyse the relationship between the optimal control and costate, which reads as follows:

$$u=\max\{0,\overline{p}\}-p, \tag{11}$$

where $\overline{p}$ denotes the integral average on I of the costate p (see [2]). Thus, for Galerkin spectral approximations, it follows that there holds

$$u_N=\max\{0,\overline{p}_N\}-p_N. \tag{12}$$

Let

$$\begin{aligned} J(u)&=\frac{1}{2}\int_I (y-y_d)^2+\frac{1}{2}\int_I u^2,\\ J_N(u_N)&=\frac{1}{2}\int_I (y_N-y_d)^2+\frac{1}{2}\int_I u_N^2.\end{aligned} \tag{13}$$

It is clear that $J(\cdot)$ is uniformly convex. Then, there exits a $c_0>0$ independent of N, such that

$$\left(J'(u)-J'(u_N),u-u_N\right)\geq c_0\|u-u_N\|_{0,I}^2. \tag{14}$$

3. Constants within the *a Posteriori* Error Estimates

In this section, we calculate all constants within the *a posteriori* error estimates. Firstly, we analyze the constant in Poincaré inequality.

For $I=(-1,1)$, we recall the Poincaré inequality with L^2-norm as (see [17])

$$\|v\|_{0,I}\leq\frac{|I|}{2}\|v'\|_{0,I}. \tag{15}$$

Now, we are at the point to investigate all of constants in details. We introduce an auxiliary state $y(u_N)\in H_0^1(I)$, which satisfies

$$a(y(u_N),w)=(f+u_N,w), \quad \forall w\in H_0^1(I). \tag{16}$$

Subtracting (16) from (5), we get

$$a(y-y(u_N),w)=(u-u_N,w), \quad \forall w\in H_0^1(I). \tag{17}$$

Let $w = y(u_N) - y \in H_0^1(\Omega)$. It is clear that

$$a\left(y\left(u_N\right)-y, y\left(u_N\right)-y\right)=\left(u_N-u, y\left(u_N\right)-y\right), \quad (18)$$

and then there hold

$$\begin{aligned}\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I}^2 &\leq \left\|u_N-u\right\|_{0,I}\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I} \\ &\leq \frac{|I|}{2}\left\|u_N-u\right\|_{0,I}\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I},\end{aligned} \quad (19)$$

which means that

$$\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I} \leq \frac{|I|}{2}\left\|u_N-u\right\|_{0,I}. \quad (20)$$

Hence,

$$\begin{aligned}&\left\|y\left(u_N\right)-y\right\|_{1,I} \\ &\leq\left(\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I}^2+\left(\frac{|I|}{2}\right)^2\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I}^2\right)^{1/2} \\ &=\left(1+\left(\frac{|I|}{2}\right)^2\right)^{1/2}\left\|\left(y\left(u_N\right)-y\right)'\right\|_{0,I}.\end{aligned} \quad (21)$$

So, we can easily obtain that

$$\left\|y\left(u_N\right)-y\right\|_{1,I} \leq\left(1+\left(\frac{|I|}{2}\right)^2\right)^{1/2} \frac{|I|}{2}\left\|u_N-u\right\|_{0,I}. \quad (22)$$

We denote by c_1 the constant in (22), and then

$$c_1=\left(1+\left(\frac{|I|}{2}\right)^2\right)^{1/2} \frac{|I|}{2}. \quad (23)$$

Here, we recall the following orthogonal projection operator: for any $v \in L^2(I)$, $\mathbb{P}_N : L^2(I) \mapsto V_N$ satisfies:

$$\left(\mathbb{P}_N v-v, w_N\right)=0 \quad \forall w_N \in V_N. \quad (24)$$

Lemma 1. *For all $v \in H^\sigma(I)$ $(\sigma \geq 0)$, one has*

$$\left\|\mathbb{P}_N v-v\right\|_{0,I} \leq c_2 N^{-\sigma}\|v\|_{\sigma,I}, \quad (25)$$

where $c_2 = 2\sqrt{2}$.

We denote by $y(u_N)$ and $p(u_N)$ two intermediate variables, and there hold

$$\begin{aligned}\left(J'(u), v\right)&=(u+p, v), \\ \left(J_N'\left(u_N\right), v\right)&=\left(u_N+p_N, v\right), \\ \left(J'\left(u_N\right), v\right)&=\left(u_N+p\left(u_N\right), v\right).\end{aligned} \quad (26)$$

Using (6), (10) and (14), for $\forall v_N = \mathbb{P}_N v$, we have

$$\begin{aligned}&c_0\left\|u-u_N\right\|_{0,I} \\ &\quad\leq\left(J'(u)-J'\left(u_N\right), u-u_N\right) \\ &\quad\leq-\left(J'\left(u_N\right), u-u_N\right) \\ &\quad=\left(J_N'\left(u_N\right), u_N-u\right)+\left(J_N'\left(u_N\right)-J'\left(u_N\right), u-u_N\right) \\ &\quad\leq\left(J_N'\left(u_N\right), v_N-u\right)+\left(J_N'\left(u_N\right)-J'\left(u_N\right), u-u_N\right) \\ &\quad=\left(J_N'\left(u_N\right)-J'\left(u_N\right), u-u_N\right)=\left(p_N-p\left(u_N\right), u-u_N\right) \\ &\quad\leq\left\|p_N-p(u_N)\right\|_{0,I}\left\|u-u_N\right\|_{0,I},\end{aligned} \quad (27)$$

which means that

$$\left\|u-u_N\right\|_{0,I} \leq \frac{1}{c_0}\left\|p_N-p\left(u_N\right)\right\|_{0,I}. \quad (28)$$

Now, we are at the point to derive the constant for $\|y_N - y(u_N)\|_{1,I}$. Let $E^y = y_N - y(u_N)$ and $E_I^y = \mathbb{P}_N E^y \in V_N$. Then

$$\begin{aligned}&\left\|y_N-y\left(u_N\right)\right\|_{1,I}^2 \\ &=\left\|E^y\right\|_{1,I}^2 \leq\left(1+\left(\frac{|I|}{2}\right)^2\right) a\left(E^y, E^y\right) \\ &=\left(1+\left(\frac{|I|}{2}\right)^2\right) a\left(E^y-E_I^y, E^y\right) \\ &=\left(1+\left(\frac{|I|}{2}\right)^2\right)\left(f+u_N+y_N'', E^y-E_I^y\right) \\ &\leq\left(1+\left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\left\|f+u_N+y_N''\right\|_{0,I} \cdot\left\|E^y\right\|_{1,I},\end{aligned} \quad (29)$$

which is equivalent to

$$\left\|y_N-y\left(u_N\right)\right\|_{1,I} \leq\left(1+\left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\left\|f+u_N+y_N''\right\|_{0,I}. \quad (30)$$

Hence,

$$\left\|y_N-y\left(u_N\right)\right\|_{1,I} \leq c_3 N^{-1}\left\|f+u_N+y_N''\right\|_{0,I}, \quad (31)$$

where

$$c_3=\left(1+\left(\frac{|I|}{2}\right)^2\right) c_2. \quad (32)$$

Likewise, we derive the constant for $\|p_N - p(u_N)\|_{1,I}$. Similarly, let $E^p = p_N - p(u_N)$ and $E_I^p = \mathbb{P}_N E^p \in V_N$. Then

$$\begin{aligned}
\|p_N - p(u_N)\|_{1,I}^2 = \|E^p\|_{1,I}^2 &\le \left(1 + \left(\frac{|I|}{2}\right)^2\right) a(E^p, E^p) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(a\left(E^p, E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(a\left(p(u_N) - p_N, E^p - E_I^p\right)\right. \\
&\qquad \left. + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(\left(-p''(u_N), E^p - E_I^p\right)\right. \\
&\qquad \left. + \left(p_N'', E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(\left(y_N - y_d + p_N'', E^p - E_I^p\right)\right. \\
&\qquad \left. + \left(y(u_N) - y_N, E^p\right)\right) \\
&\le \left(1 + \left(\frac{|I|}{2}\right)^2\right)\|E^p\|_{1,I}\left\{c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I}\right. \\
&\qquad \left. + \|y_N - y(u_N)\|_{0,I}\right\}.
\end{aligned} \tag{33}$$

We deduce that

$$\begin{aligned}
&\|p_N - p(u_N)\|_{1,I} \\
&\quad \le \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left\{c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I}\right. \\
&\qquad \left. + \|y_N - y(u_N)\|_{0,I}\right\}.
\end{aligned} \tag{34}$$

Combining all of the above analyses, we derive that

$$\begin{aligned}
&\|u - u_N\|_{0,I} + \|y - y_N\|_{1,I} + \|p - p_N\|_{1,I} \\
&\quad \le \|u - u_N\|_{0,I} + \|y - y(u_N)\|_{1,I} + \|y_N - y(u_N)\|_{1,I} \\
&\qquad + \|p - p(u_N)\|_{1,I} + \|p_N - p(u_N)\|_{1,I} \\
&\quad = \|u - u_N\|_{0,I} + \|y_N - y(u_N)\|_{1,I} + \|p_N - p(u_N)\|_{1,I} \\
&\qquad + \|y - y(u_N)\|_{1,I} + \|p - p(u_N)\|_{1,I} \\
&\quad \le \|u - u_N\|_{0,I} + \|y_N - y(u_N)\|_{1,I} + \|p_N - p(u_N)\|_{1,I} \\
&\qquad + \|y - y(u_N)\|_{1,I} + c_1\|y - y(u_N)\|_{0,I} \\
&\quad \le \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I} \\
&\qquad + \left(1 + \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right)\right) c_3 N^{-1} \\
&\qquad \times \|f + u_N + y_N''\|_{0,I},
\end{aligned} \tag{35}$$

which means that

$$\begin{aligned}
&\|u - u_N\|_{0,I} + \|p - p_N\|_{1,I} + \|y - y_N\|_{1,I} \\
&\quad \le \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I} \\
&\qquad + \left(1 + \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right)\right) c_3 N^{-1} \\
&\qquad \times \|f + u_N + y_N''\|_{0,I}.
\end{aligned} \tag{36}$$

For $|I| = 2$, there holds

$$\|u - u_N\|_{0,I} + \|p - p_N\|_{1,I} + \|y - y_N\|_{1,I} \le \eta, \tag{37}$$

where the *a posteriori* error indicator η is defined as

$$\begin{aligned}
\eta &= 4\sqrt{2}\left(1 + \frac{3 + \sqrt{2}}{c_0}\right) N^{-1}\|y_N - y_d + p_N''\|_{0,I} \\
&\quad + 4\sqrt{2}\left(3 + \frac{6 + 2\sqrt{2}}{c_0}\right) N^{-1}\|f + u_N + y_N''\|_{0,I}.
\end{aligned} \tag{38}$$

4. Conclusion

This paper discussed the explicit formulae of constants in the upper bound of the *a posteriori* error estimate for optimal control problems with Legendre-Galerkin spectral methods in one-dimensional bounded domain. Thus, with those formulae, it is easy to choose a suitable degree of polynomials to obtain acceptable accuracy. In the future, we are going to discuss the corresponding constants in the lower bound of the *a posteriori* error indicator.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Robust Exponential Stabilization of Stochastic Delay Interval Recurrent Neural Networks with Distributed Parameters and Markovian Jumping by Using Periodically Intermittent Control

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We consider a class of stochastic delay recurrent neural networks with distributed parameters and Markovian jumping. It is assumed that the coefficients in these neural networks belong to the interval matrices. Several sufficient conditions ensuring robust exponential stabilization are derived by using periodically intermittent control and Lyapunov functional. The obtained results are very easy to verify and implement, and improve the existing results. Finally, an example with numerical simulations is given to illustrate the presented criteria.

1. Introduction

In recent decades, neural network dynamics has been widely studied by many authors due to the fact that neural network dynamics can be applied to associate memory, signal processing, pattern classification, and quadratic optimization. Liao and Mao [1, 2] investigated the stability of stochastic neural network for the first time in 1996. By Razumikhin-type theorems, the stability of stochastic neural networks with variable delays was considered [3]. Considering electrons moving in the asymptotic electromagnetic field, the diffusion phenomena could not be ignored. Luo et al. [4] gave several algebra criteria for stochastic Hopfield neural networks with distributed parameters by using average Lyapunov function. The asymptotic stability of stochastic reaction- diffusion systems was also established in [5]. The asymptotic behavior of several classes of neural networks with reaction-diffusion terms has been reported in [6–9]. Hu et al. [10] discussed the exponential stability and synchronization of delay neural networks with reaction-diffusion terms by impulsive control.

However, the parameters in neural networks are always some uncertainty and error. Taking these uncertainty and error into account, Xu et al. [11] investigated stochastic exponential robust stability of interval neural networks with reaction-diffusion terms and mixed delays by applying the vector Lyapunov function method and M-matrix theory. Wang and Gao [12] studied global exponential robust stability of reaction-diffusion interval neural networks with time-varying delays by means of the topological degree theory and Lyapunov functional method. And, a sufficient condition was presented for robust global exponential stability of interval reaction-diffusion Hopfield neural networks with distributed delays by constructing Lyapunov functional and utilizing some inequality techniques [13].

The neural networks driven by continuous-time Markov Chains have been also used to model many practical neural networks because they may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The exponential stability and stabilization of recurrent neural networks with Markovian jumping were discussed in [14–20]. Robust stability of stochastic delayed additive neural networks with Markov jumping was investigated in [21]. Mao

[22] studied the stability of stochastic delay interval system with Markovian jumping by linear matrix inequality.

Many control approaches have been developed to stable and synchronized system such as impulsive control [23] and intermittent control [24–29]. Gan [24–26] revealed exponential synchronization of three classes of stochastic delay neural networks via periodically intermittent control. Hu et al. [27, 28] investigated exponential stabilization and synchronization of delay neural networks. Huang et al. [29] studied stabilization of delayed chaotic neural networks by periodically intermittent control.

In this paper, we will consider a class of stochastic delay interval recurrent neural networks with distributed parameters and Markovian switching whose active functions are more general than the Lipschitz continuous active function [24–26] and the monotone active function [27–29]. By the average Lyapunov functional and periodically intermittent control, several sufficient conditions ensuring robust exponential stabilization are given. Therefore, the organization of this paper is as follows. Some preliminaries and introduction are given in Section 2. In Section 3, robust exponential stabilization of these stochastic neural networks is proved. An example with numerical simulation is given to illustrate the effectiveness of the obtained results in Section 4.

2. Preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right-continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $\mathbb{R}^m$ be the m-dimensional Euclidean space and let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^m$, $\mathbb{R}_+ = [0, +\infty)$ and $\tau > 0$. Assuming that $\Omega_0 \subset \mathbb{R}^m$ is a bounded compact set with smooth boundary $\partial\Omega_0$ and mes $\Omega_0 > 0$ in space $\mathbb{R}^m$. Let $C([-\tau, 0] \times \Omega_0; \mathbb{R}^n)$ denote the family of continuous function $\phi(t, x)$ from $[-\tau, 0] \times \Omega_0$ to $\mathbb{R}^n$ with $\|\phi\| = \sup_{-\tau\leq t\leq 0, x\in\Omega_0} |\phi(t, x)|$. Denote by $C^b_{\mathscr{F}_0}([-\tau, 0] \times \Omega_0, \mathbb{R}^n)$ the family of all bounded, $\mathscr{F}_0$-measurable, $C([-\tau, 0]\times\Omega_0; \mathbb{R}^n)$-valued random variables. Let $W(t), t \geq 0$ be n-dimension Brownian motion defined on the probability space. Let $r(t), t \geq 0$ be right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (r_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases} \tag{1}$$

where $\Delta > 0$. Here, $r_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}. \tag{2}$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is well known that almost every sample path of $r(t)$ is right-continuous step function with a finite number of simple jumps in any finite subinterval $\mathbb{R}_+$.

In this paper, we consider a class of stochastic delay interval recurrent neural networks with distributed parameters and Markovian jumping:

$$\begin{aligned} du_i(t,x) = \Bigg\{ & \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}(r(t)) \frac{\partial u_i(t,x)}{\partial x_k}\right) \\ & - a_i(r(t))\, u_i(t,x) \\ & + \sum_{j=1}^{n} b_{ij}(r(t))\, f_j\left(u_j(t,x)\right) \\ & + \sum_{j=1}^{n} c_{ij}(r(t))\, g_j\left(u_j\left(t-\tau_{ij}(t)\right), x\right) \Bigg\} dt \\ & + \sum_{j=1}^{n} h_{ij}\left(u_j(t,x), u_j\left(t-\tau_{ij}(t), x\right)\right) dW_j(t), \end{aligned} \tag{3}$$

for $t \geq 0$, $i = 1, 2, \ldots, n$, where $n \geq 2$ denotes the number of neurons in neural networks. $x = (x_1, x_2, \ldots, x_m)^T \in \Omega_0 \subset \mathbb{R}^m$ is the space variable, $\Omega_0 = \{x = (x_1, x_2, \ldots, x_m)^T \| x_k| < \theta_k, k = 1, 2, \ldots, m\}$ is a bounded compact set with smooth boundary $\partial\Omega_0$, and mes $\Omega_0 > 0$ in space $\mathbb{R}^m$. $u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_n(t, x))^T$ corresponds to the state variable of the ith neural in space x and at time t. $D_{ik}(r(t)) \geq 0$ denotes the transmission diffusion operator along the ith neuron. $a_i(r(t)) > 0$ denotes the changing time constant or passive decay rate of the ith neuron. $b_{ij}(r(t))$ and $c_{ij}(r(t))$ denote the connection weight and the delayed connection weight of the jth neuron on the ith neuron, respectively. $\tau_{ij}(t)$ corresponds to the transmission delay and satisfies $0 \leq \tau_{ij}(t) \leq \tau$, $\dot{\tau}_{ij}(t) \leq \tau_0 < 1$ for all $t \geq 0$ (τ, τ_0 is a constant, $i, j = 1, 2, \ldots, n$). $h_{ij}(\cdot, \cdot)$ denotes stochastic perturbation function to the neuron.

The boundary condition of system (3),

$$u(t,x)|_{\partial\Omega_0} = 0, \quad (t,x) \in [-\tau, +\infty) \times \partial\Omega_0, \ i = 1, 2, \ldots, n. \tag{4}$$

The initial value of system (3),

$$u(t,x) = \phi_i(t,x), \quad (t,x) \in [-\tau, 0) \times \Omega_0, \ i = 1, 2, \ldots, n. \tag{5}$$

Moreover, $A(r(t)) = \operatorname{diag}(a_1(r(t)), \ldots, a_n(r(t)))$, $B(r(t)) = (b_{ij}(r(t)))_{n\times n}$, and $C(r(t)) = (c_{ij}(r(t)))_{n\times n}$ are the interval connection weight matrix for each value of $r(t)$ in $\mathbb{S}$ with the initial value $r(0) = r_0$; $D(r(t)) = (b_{ij}(r(t)))_{n\times m}$ is interval transmission diffusion operator matrix for each value of $r(t)$ in $\mathbb{S}$ with the initial value $r(0) = r_0$.

For convenience, we give the following notions that for $r(t) = l$ in $\mathbb{S}$:

$$\begin{aligned}
A^* &= \{A(l) = \operatorname{diag}(a_1(l), \ldots, a_n(l)) : \underline{A}(l) \le A(l) \le \overline{A}(l), \\
&\quad \text{i.e., } \underline{a}_i(l) \le a_i(l) \le \overline{a}_i(l), i = 1, 2, \ldots, n\}; \\
B^* &= \{B(l) = (b_{ij}(l))_{n\times n} : \underline{B}(l) \le B(l) \le \overline{B}(l), \\
&\quad \text{i.e., } \underline{b}_{ij}(l) \le b_{ij}(l) \le \overline{b}_{ij}(l), \ i, j = 1, 2, \ldots, n\}; \\
C^* &= \{C(l) = (c_{ij}(l))_{n\times n} : \underline{C}(l) \le C(l) \le \overline{C}(l), \\
&\quad \text{i.e., } \underline{c}_{ij}(l) \le c_{ij}(l) \le \overline{c}_{ij}(l), \ i, j = 1, 2, \ldots, n\}; \\
D^* &= \{D(l) = (d_{ij}(l))_{n\times n} : \underline{D}(l) \le D(l) \le \overline{D}(l), \\
&\quad \text{i.e., } \underline{d}_{ij}(l) \le d_{ij}(l) \le \overline{d}_{ij}(l), \\
&\quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m\}.
\end{aligned} \tag{6}$$

Definition 1. The stochastic vector $u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))^T$ is called the solution of system (3)–(5), if it satisfies the following conditions:

(i) $u(t,x)$ is adapted to $\{\mathcal{F}_t\}_{t\ge 0}$;

(ii) for every $T_0 \in \mathbb{R}_+$, $u(t,x) \in C^b_{\mathcal{F}_0}([0,T_0]\times\Omega_0;\mathbb{R}^n)$ and

$$\mathbb{E}\left(\max_{x\in\Omega_0}\int_0^{T_0}\left[|u(t,x)|^2 + |\nabla u(t,x)|^2\right]dt\right) < +\infty; \tag{7}$$

(iii) for every $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\int_{\Omega_0} u_i(t,x)\,dx \\
&= \int_{\Omega_0}\phi_i(0,x)\,dx + \int_{\Omega_0}\int_0^t \sum_{k=1}^m \frac{\partial}{\partial x_k} \\
&\quad \times \left(D_{ik}(r(s))\frac{\partial u_i(s,x)}{\partial x_k}\right)dsdx \\
&\quad + \int_{\Omega_0}\int_0^t \Bigg[-a_i(r(s))u_i(s,x) + \sum_{j=1}^n b_{ij}(r(s))f_j(u_j(s,x)) \\
&\qquad + \sum_{j=1}^n c_{ij}(r(s))g_j(u_j(s-\tau_{ij}(s),x))\Bigg]dsdx \\
&\quad + \int_{\Omega_0}\int_0^t \sum_{j=1}^n h_{ij}(u_j(s,x), u_j(s-\tau_{ij}(s),x))\,dW_j(s)\,dx, \\
&\qquad (t,x) \in [0,T_0]\times\Omega_0,
\end{aligned} \tag{8}$$

so it holds as $\mathbb{P}$-a.s., $i = 1, 2, \ldots, n$.

Definition 2. System (3)–(5) is called robust exponential stable in pth moment for any $A(l) \in A^*, B(l) \in B^*, C(l) \in C^*, D(l) \in D^*, l \in \mathbb{S}$ if the solution $u(t,x)$ of system (3)–(5) satisfies

$$\limsup_{t\to+\infty}\frac{1}{t}\log\left(\mathbb{E}\|u(t,x)\|^p\right) < 0, \tag{9}$$

where $\|u(t,x)\| = (\int_{\Omega_0}|u(t,x)|^p dx)^{1/p}$, $(t,x) \in \mathbb{R}_+ \times \Omega_0$.

To assure the existence and uniqueness of the solution to system (3)–(5) (see, [30, 31]), we give the following assumptions:

(H1) for $i = 1, 2, \ldots n$, $\forall s_1, s_2 \in \mathbb{R}$, the neuron activation functions f_i, g_i are bounded, $f_i(0) = g_i(0) = 0$, and satisfy

$$\begin{aligned}
L_i^- &\le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le L_i^+, \\
N_i^- &\le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le N_i^+,
\end{aligned} \tag{10}$$

where $s_1 \ne s_2$, and $L_i^-, L_i^+, N_i^-, N_i^+$ are constants.

(H2) For $i, j = 1, 2, \ldots, n$, $\forall s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in \mathbb{R}$, there exists positive constant σ_{ij}, such that

$$\left|h_{ij}(s_1, s_2) - h_{ij}(\tilde{s}_1, \tilde{s}_2)\right|^2 \le \sigma_{ij}\left(|s_1 - \tilde{s}_1|^2 + |s_2 - \tilde{s}_2|^2\right) \tag{11}$$

and $h_{ij}(0,0) = 0$.

(H3) Time-varying delay function $\tau_{ij}(\cdot) : [0,+\infty) \to [0,+\infty)$ $(i, j = 1, 2, \ldots, n)$ satisfies

$$0 \le \tau_{ij}(t) \le \tau, \qquad \dot{\tau}_{ij}(t) \le \tau_0 \le 1, \tag{12}$$

for $t \ge 0$, where τ and τ_0 are constants.

It is well known, if the parameters or time-varying delay in neural networks is appropriately chosen, neural networks may lead to some phenomena such as instability, divergence, oscillation, chaos [32, 33].

In order to stabilize the origin of system (3)–(5), we introduce the following periodically intermittent controller:

$$v_i(t,x) = \begin{cases} \sum_{j=1}^n k_{ij}u_j(t,x), & MT \le t < MT + \delta, \\ 0, & MT + \delta \le t < (M+1)T, \end{cases} \tag{13}$$

where $M = 0, 1, 2, \ldots$ and k_{ij} is the control gains for $i, j = 1, 2, \ldots, n$, T denotes the control period, and $0 < \delta < T$ is called the control width.

Then, system (3) under the periodically intermittent controller (13) is described by the following equations:

$$\begin{aligned} du_i(t,x) = \Bigg\{ & \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}(l)\frac{\partial u_i(t,x)}{\partial x_k}\right) \\ & - a_i(r(t))u_i(t,x) + \sum_{j=1}^{n} b_{ij}(r(t))f_j\left(u_j(t,x)\right) \\ & + \sum_{j=1}^{n} c_{ij}(r(t))g_j\left(u_j\left(t-\tau_{ij}(t)\right),x\right) \\ & + \sum_{j=1}^{n} k_{ij}u_j(t,x)\Bigg\}dt \\ & + \sum_{j=1}^{n} h_{ij}\left(u_j(t,x),u_j\left(t-\tau_{ij}(t),x\right)\right)dW_j(t) \\ & MT \le t < MT+\delta, \\ du_i(t,x) = \Bigg\{ & \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}(l)\frac{\partial u_i(t,x)}{\partial x_k}\right) \\ & - a_i(r(t))u_i(t,x) + \sum_{j=1}^{n} b_{ij}(r(t))f_j\left(u_j(t,x)\right) \\ & + \sum_{j=1}^{n} c_{ij}(r(t))g_j\left(u_j\left(t-\tau_{ij}(t)\right),x\right)\Bigg\}dt \\ & + \sum_{j=1}^{n} h_{ij}\left(u_j(t,x),u_j\left(t-\tau_{ij}(t),x\right)\right)dW_j(t) \\ & MT+\delta \le t < (M+1)T. \end{aligned} \tag{14}$$

Lemma 3 (see [10]). *Let $p \ge 2$ be a positive integer, let θ_k be a positive constant, let Ω_0 be a cube $|x_k| \le \theta_k$ for $k = 1, 2, \ldots, m$, and let $u(x)$ be a real-valued function belonging to $C^1(\Omega_0)$ which vanish on the boundary $\partial\Omega_0$; that is, $u(x)|_{\partial\Omega_0} = 0$; then*

$$\int_{\Omega_0} |u(x)|^p dx \le \frac{p^2\theta_k^2}{4}\int_{\Omega_0} |u(x)|^{p-2}\left|\frac{\partial u}{\partial x_k}\right|^2 dx. \tag{15}$$

3. Robust Exponential Stabilization

In this section, we design suitable T, δ, and k_{ij} such that system (3)–(5) under the external controller (13) can realize robust exponential stability in pth moment. For convenience, we give some denotations as follows:

$$\begin{aligned} \lambda_i = \min_{l\in\mathbb{S}}\mu_l \Bigg\{ & \sum_{k=1}^{m}\frac{4(p-1)\underline{D}_{ik}}{p\theta_k^2} + pa_i^* \\ & - \sum_{j=1}^{n}\sum_{\ell=1}^{p-1}\left(\tilde{b}_{ij}^{p\alpha_{\ell ij}}\tilde{L}_j^{p\beta_{\ell ij}} + \tilde{c}_{ij}^{p\xi_{\ell ij}}\tilde{N}_j^{p\zeta_{\ell ij}}\right) \\ & - \frac{p-1}{2}\sum_{j=1}^{n}\sum_{\ell=1}^{p-2}\left(\sigma_{ij}^{p\epsilon_{\ell ij}} + \sigma_{ij}^{p\omega_{\ell ij}}\right) \\ & - \sum_{j=1}^{n}\Bigg(\tilde{b}_{ji}^{p\alpha_{pji}}\tilde{L}_i^{p\beta_{pji}} \\ & + \tilde{c}_{ji}^{p\xi_{pji}}\tilde{N}_i^{p\zeta_{pji}} \\ & + \frac{p-1}{2}\left(\sigma_{ji}^{p\epsilon_{(p-1)ji}} + \sigma_{ji}^{p\epsilon_{pji}}\right)\Bigg)\Bigg\}, \end{aligned} \tag{16}$$

$$\begin{aligned} \kappa_i = \min_{l\in\mathbb{S}}\mu_l \Bigg\{ & \sum_{k=1}^{m}\frac{4(p-1)\underline{D}_{ik}}{p\theta_k^2} + pa_i^* \\ & - \sum_{j=1}^{n}\sum_{\ell=1}^{p-1}\left(\tilde{b}_{ij}^{p\bar{\alpha}_{\ell ij}}\tilde{L}_j^{p\bar{\beta}_{\ell ij}} + \tilde{c}_{ij}^{p\bar{\xi}_{\ell ij}}\tilde{N}_j^{p\bar{\zeta}_{\ell ij}}\right) \\ & - \frac{p-1}{2}\sum_{j=1}^{n}\sum_{\ell=1}^{p-2}\left(\sigma_{ij}^{p\bar{\epsilon}_{\ell ij}} + \sigma_{ij}^{p\bar{\omega}_{\ell ij}}\right) \\ & - \sum_{j=1}^{n}\Bigg(\tilde{b}_{ji}^{p\bar{\alpha}_{pji}}\tilde{L}_i^{p\bar{\beta}_{pji}} + \tilde{c}_{ji}^{p\bar{\xi}_{pji}}\tilde{N}_i^{p\bar{\zeta}_{pji}} \\ & + \frac{p-1}{2}\left(\bar{\sigma}_{ji}^{p\bar{\epsilon}_{(p-1)ji}} + \sigma_{ji}^{p\bar{\epsilon}_{pji}}\right)\Bigg)\Bigg\}, \end{aligned} \tag{17}$$

$$\begin{aligned} \nu_i = \max_{l\in\mathbb{S}}\mu_l \Bigg[& pk_{ii} + \sum_{\substack{j=1\\ j\ne i}}^{n}\sum_{\ell=1}^{p-1}\left|k_{ij}\right|^{p\eta_{\ell ij}^*} \\ & + \sum_{\substack{j=1\\ j\ne i}}^{n}\left|k_{ji}\right|^{p\eta_{pji}^*}\Bigg], \end{aligned} \tag{18}$$

$$\begin{aligned} \eta_i = \max_{l\in\mathbb{S}}\mu_l \sum_{j=1}^{n}\Bigg[& \frac{p-1}{2}\left(\sigma_{ji}^{\omega_{(p-1)ji}} + \sigma_{ji}^{\omega_{pji}}\right) \\ & + \tilde{c}_{ji}^{p\xi_{pji}}\tilde{N}_i^{p\zeta_{pji}}\Bigg], \end{aligned} \tag{19}$$

where $a_i^* = \min_{l\in\mathbb{S}} \underline{a}_i(l)$, $\widetilde{b}_{ij} = \max_{l\in\mathbb{S}} \check{b}_{ij}(l)$, $\check{b}_{ij}(l) = \max\{|\underline{b}_{ij}(l)|, |\overline{b}_{ij}(l)|\}$, $\widetilde{c}_{ij} = \max_{l\in\mathbb{S}} \check{c}_{ij}(l)$,

$$\begin{aligned} &\check{c}_{ij}(l) = \max\left\{\left|\underline{c}_{ij}(l)\right|, \left|\overline{c}_{ij}(l)\right|\right\}, \\ &\widetilde{L}_j = \max\left\{\left|L_j^-\right|, \left|L_j^+\right|\right\}, \qquad \widetilde{N}_j = \max\left\{\left|N_j^-\right|, \left|N_j^+\right|\right\}, \end{aligned} \tag{20}$$

$\mu_l > 0$, and $\alpha_{\ell ij}$, $\beta_{\ell ij}$, $\xi_{\ell ij}$, $\zeta_{\ell ij}$, $\epsilon_{\ell ij}$, $\omega_{\ell ij}$, $\eta_{\ell ij}^*$, $\overline{\alpha}_{\ell ij}$, $\overline{\beta}_{\ell ij}$, $\overline{\xi}_{\ell ij}$, $\overline{\zeta}_{\ell ij}$, $\overline{\epsilon}_{\ell ij}$, and $\overline{\omega}_{\ell ij}$ are nonnegative constants, satisfying

$$\begin{aligned} \sum_{\ell=1}^{p}\alpha_{\ell ij} &= \sum_{\ell=1}^{p}\beta_{\ell ij} = \sum_{\ell=1}^{p}\xi_{\ell ij} = \sum_{\ell=1}^{p}\zeta_{\ell ij} \\ &= \sum_{\ell=1}^{p}\epsilon_{\ell ij} = \sum_{\ell=1}^{p}\omega_{\ell ij} = \sum_{\ell=1}^{p}\overline{\alpha}_{\ell ij} = \sum_{\ell=1}^{p}\overline{\beta}_{\ell ij} \\ &= \sum_{\ell=1}^{p}\overline{\xi}_{\ell ij} = \sum_{\ell=1}^{p}\overline{\zeta}_{\ell ij} = \sum_{\ell=1}^{p}\overline{\epsilon}_{\ell ij} = \sum_{\ell=1}^{p}\overline{\omega}_{\ell ij} \\ &= \sum_{\ell=1}^{p}\eta_{\ell ij}^* = 1, \qquad \underline{D}_{ik} = \min_{l\in\mathbb{S}} \underline{D}_{ik}(l). \end{aligned} \tag{21}$$

In the following, we give an assumption:

(H4) $\lambda_i - \nu_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \eta_i/(1-\tau_0) > 0$ and there exists $\rho_i > 0$ such that

$$\kappa_i + \rho_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \frac{\eta_i}{1-\tau_0} > 0, \quad i = 1, 2, \ldots, n. \tag{22}$$

We consider the function

$$H_i(\check{\varepsilon}_i) = \lambda_i - \nu_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \check{\varepsilon}_i \max_{l\in\mathbb{S}} \mu_l - \frac{\eta_i e^{\check{\varepsilon}_i \tau}}{1-\tau_0}, \quad i = 1, 2, \ldots, n. \tag{23}$$

It is easy to see that

$$\begin{aligned} H_i'(\check{\varepsilon}_i) &= -\max_{l\in\mathbb{S}} \mu_l - \frac{\tau\eta_i e^{\check{\varepsilon}_i \tau}}{1-\tau_0} < 0, \\ H_i(0) &= \lambda_i - \nu_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \frac{\eta_i}{1-\tau_0} > 0. \end{aligned} \tag{24}$$

On the other hand, $H_i(\check{\varepsilon}_i)$ is continuous on $[0, +\infty)$, and $F_i(\check{\varepsilon}_i) \to -\infty$ as $\check{\varepsilon}_i \to +\infty$. Then there exists a positive constant $\check{\varepsilon}_i^*$ such that $H_i(\check{\varepsilon}_i^*) \geq 0$ and $H_i(\check{\varepsilon}_i) > 0$, for $\check{\varepsilon}_i \in (0, \check{\varepsilon}_i^*)$.

Let $\check{\varepsilon} = \min_{1\leq i\leq n}\{\check{\varepsilon}_i^*\}$; then we have

$$H_i(\check{\varepsilon}) = \lambda_i - \nu_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \check{\varepsilon} \max_{l\in\mathbb{S}} \mu_l - \frac{\eta_i}{1-\tau_0} e^{\check{\varepsilon}\tau} \geq 0. \tag{25}$$

In similar, there exists a positive constant $\widehat{\varepsilon} > 0$, such that

$$F_i(\widehat{\varepsilon}) = k_i + \rho_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \widehat{\varepsilon} \max_{l\in\mathbb{S}} \mu_l - \frac{\eta_i}{1-\tau_0} e^{\widehat{\varepsilon}\tau} \geq 0. \tag{26}$$

Let $\varepsilon = \min\{\check{\varepsilon}, \widehat{\varepsilon}\}$; we have

$$H_i(\varepsilon) > 0, \quad F_i(\varepsilon) > 0, \quad i = 1, 2, \ldots, n. \tag{27}$$

We give another assumption:

(H5) $\varepsilon - \rho(T-\delta)/\overline{\mu}T > 0$, where $\rho = \max_{1\leq i\leq n} \rho_i$, $\overline{\mu} = \min_{l\in\mathbb{S}}\{\mu_l\}$.

Theorem 4. *Under assumptions (H1)–(H5), the origin of system* (3)–(5) *under periodically intermittent controller* (13) *is robust exponentially stable in pth moment.*

Proof. Let us define the average Lyapunov-Krasovskii functional (see [4]) $V_1 : C([0, +\infty) \times \Omega_0, \mathbb{R}^n) \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$V_1(u(t,x), r(t), t) = \int_{\Omega_0} V(u(t,x), r(t), t)\, dx \tag{28}$$

with

$$\begin{aligned} &V(u(t,x), r(t), t) \\ &= \mu_{r(t)} \sum_{i=1}^{n} e^{\varepsilon t} |u_i(t,x)|^p + \frac{e^{\varepsilon\tau}}{1-\tau_0} \sum_{i=1}^{n} \eta_i \int_{t-\tau_{ij}(t)}^{t} e^{\varepsilon s} |u_i(s,x)|^p ds, \end{aligned} \tag{29}$$

where $\mu_{r(t)} > 0$.

By the generalized Itô formula (see [31]), we have

$$\begin{aligned} &\mathbb{E}V_1(u(t,x), r(t), t) \\ &= \mathbb{E}V_1(\phi, r(0), 0) + \mathbb{E}\int_0^t \int_{\Omega_0} \mathscr{L}V(u(s,x), r(s), s)\, dx ds. \end{aligned} \tag{30}$$

By Lemma 3.1 in [22] and $r(t) = l$, we get that for $(t,x) \in [MT, MT+\delta) \times \Omega_0$

$$\begin{aligned} &\mathscr{L}V(u(t,x), l, t) \\ &= \varepsilon\mu_l \sum_{i=1}^{n} e^{\varepsilon t} |u_i(t,x)|^p + p\mu_l e^{\varepsilon t} \sum_{i=1}^{n} |u_i(t,x)|^{p-1} \\ &\quad \times \left\{ \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}(l) \frac{\partial u_i(t,x)}{\partial x_k}\right) - a_i(l) u_i(t,x) \right. \\ &\qquad + \sum_{j=1}^{n} b_{ij}(l) f_j(u_j(t,x)) \\ &\qquad + \sum_{j=1}^{n} c_{ij}(l) g_j(u_j(t-\tau_{ij}(t), x)) \\ &\qquad \left. + \sum_{j=1}^{n} k_{ij} u_j(t,x) \right\} \end{aligned}$$

$$
\begin{aligned}
&+\mu_l e^{\varepsilon t}\frac{p(p-1)}{2}\sum_{i=1}^{n}\left|u_i(t,x)\right|^{p-2}\\
&\times\sum_{j=1}^{n}h_{ij}^{2}\left(u_j(t,x),u_j\left(t-\tau_{ij}(t),x\right)\right)\\
&+\sum_{q=1}^{N}\gamma_{lq}\mu_q e^{\varepsilon t}\sum_{i=1}^{n}\left|u_i(t,x)\right|^{p}\\
&+\frac{e^{\varepsilon\tau}}{1-\tau_0}\sum_{i=1}^{n}\eta_i e^{\varepsilon t}\left|u_i(t,x)\right|^{p}\\
&-\frac{e^{\varepsilon\tau}}{1-\tau_0}\sum_{i=1}^{n}\eta_i e^{\varepsilon(t-\tau_{ij}(t))}\\
&\times\left|u_i\left(t-\tau_{ij}(x),x\right)\right|^{p}\left(1-\dot{\tau}_{ij}(t)\right)\\
&+\sum_{q=1}^{N}\gamma_{lq}\frac{e^{\varepsilon\tau}}{1-\tau_0}\sum_{i=1}^{n}\eta_i\int_{t-\tau_{ij}(t)}^{t}e^{\varepsilon s}\left|u_i(s,x)\right|^{p}ds.
\end{aligned}
\tag{31}
$$

By the fundamental inequality $|a+b|\le|a|+|b|$, we have

$$
\begin{aligned}
&\mathscr{L}V(u(t,x),l,t)\\
&\le\sum_{i=1}^{n}e^{\varepsilon t}\left\{\varepsilon\mu_l\left|u_i(t,x)\right|^{p}+p\mu_l\left|u_i(t,x)\right|^{p-1}\right.\\
&\qquad\times\left(\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{ik}(l)\right)\frac{\partial u_i(t,x)}{\partial x_k}\right)\\
&\qquad+\left(k_{ii}-a_i(l)\right)p\mu_l\left|u_i(t,x)\right|^{p}\\
&\qquad+p\mu_l\left|u_i(t,x)\right|^{p-1}\left(\sum_{j=1}^{n}\left|b_{ij}(l)\right|\left|f_j\left(u_j(t,x)\right)\right|\right)\\
&\qquad+p\mu_l\left|u_i(t,x)\right|^{p-1}\\
&\qquad\times\left(\sum_{j=1}^{n}\left|c_{ij}(l)\right|\left|g_j\left(\mu_j\left(t-\tau_{ij}(t)\right),x\right)\right|\right)\\
&\qquad+\sum_{\substack{j=1\\ j\ne i}}^{n}p\mu_l\left|u_i(t,x)\right|^{p-1}\left|k_{ij}\right|\left|\mu_j(t,x)\right|\\
&\qquad+\mu_l\frac{p(p-1)}{2}\left|u_i(t,x)\right|^{p-2}\\
&\qquad\times\sum_{j=1}^{n}\sigma_{ij}\left(\left|u_j(t,x)\right|^{2}+\left|u_j\left(t-\tau_{ij}(t),x\right)\right|^{2}\right)\\
&\qquad+\sum_{q=1}^{N}\gamma_{lq}\mu_q\left|u_i(t,x)\right|^{p}\\
&\qquad+\frac{e^{\varepsilon\tau}}{1-\tau_0}\eta_i\left|u_i(t,x)\right|^{p}\\
&\qquad\left.-\eta_i\left|u_i\left(t-\tau_{ij}(t),x\right)\right|^{p}\right\},
\end{aligned}
\tag{32}
$$

where we use $\sum_{q=1}^{N}\gamma_{lq}=0$.

By using the fundamental inequality $a_1^p+a_2^p+\cdots+a_p^p\ge pa_1a_2,\ldots,a_p$ $(a_i\ge0, i=1,2,\ldots,p)$, we have

$$
\begin{aligned}
&p\mu_l\left|u_i(t,x)\right|^{p-1}\left(\sum_{j=1}^{n}\left|b_{ij}(l)\right|\left|f_j\left(u_j(t,x)\right)\right|\right)\\
&\le\mu_l\sum_{j=1}^{n}p\left|u_i(t,x)\right|^{p-1}\widetilde{b}_{ij}\widetilde{L}_j\left|u_j(t,x)\right|\\
&=\mu_l\sum_{j=1}^{n}p\left[\prod_{\ell=1}^{p-1}\widetilde{b}_{ij}^{\alpha_{\ell ij}}\widetilde{L}_j^{\beta_{\ell ij}}\left|u_i(t,x)\right|\right]\\
&\times\left(\widetilde{b}_{ij}^{\alpha_{pij}}\widetilde{L}_j^{\beta_{pij}}\left|u_j(t,x)\right|\right)\\
&\le\mu_l\sum_{j=1}^{n}\sum_{\ell=1}^{p-1}\widetilde{b}_{ij}^{p\alpha_{\ell ij}}\widetilde{L}_j^{p\beta_{\ell ij}}\left|u_i(t,x)\right|^{p}\\
&+\mu_l\sum_{j=1}^{n}\widetilde{b}_{ij}^{p\alpha_{pij}}\widetilde{L}_j^{p\beta_{pij}}\left|u_j(t,x)\right|^{p},\\
&p\mu_l\left|u_i(t,x)\right|^{p-1}\\
&\quad\times\left(\sum_{j=1}^{n}\left|c_{ij}(l)\right|\left|g_j\left(u_j\left(t-\tau_{ij}(t),x\right)\right)\right|\right)\\
&\le p\mu_l\sum_{j=1}^{n}\left|u_i(t,x)\right|^{p-1}\widetilde{c}_{ij}\widetilde{N}_j\left|u_j\left(t-\tau_{ij}(t),x\right)\right|\\
&=\mu_l\sum_{j=1}^{n}p\left[\prod_{\ell=1}^{p-1}\widetilde{c}_{ij}^{\xi_{\ell ij}}\widetilde{N}_j^{\zeta_{\ell ij}}\left|u_j(t,x)\right|\right]\\
&\quad\times\left(\widetilde{c}_{ij}^{\xi_{pij}}\widetilde{N}_j^{\zeta_{pij}}\left|u_j\left(t-\tau_{ij}(t),x\right)\right|\right)\\
&\le\mu_l\sum_{j=1}^{n}\sum_{\ell=1}^{p-1}\widetilde{c}_{ij}^{p\xi_{\ell ij}}\widetilde{N}_j^{p\zeta_{\ell ij}}\left|u_j(t,x)\right|^{p}\\
&\quad+\mu_l\sum_{j=1}^{n}\widetilde{c}_{ij}^{p\xi_{pij}}\widetilde{N}_j^{p\zeta_{pij}}\left|u_j\left(t-\tau_{ij}(t),x\right)\right|^{p}.
\end{aligned}
\tag{33}
$$

Similarly, we have

$$\begin{aligned}
&\mu_l \sum_{\substack{j=1\\ j\neq i}}^{n} p\left|u_i(t,x)\right|^{p-1}\left|k_{ij}\right|\left|u_j(t,x)\right| \\
&= \mu_l \sum_{\substack{j=1\\ j\neq i}}^{n} p\left[\prod_{\ell=1}^{p-1}\left|k_{ij}\right|^{\eta^*_{\ell ij}}\left|u_i(t,x)\right|\right] \\
&\quad \times\left(\left|k_{ij}\right|^{\eta^*_{pij}}\left|u_j(t,x)\right|\right) \\
&\le \mu_l \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\ell=1}^{p-1}\left|k_{ij}\right|^{p\eta^*_{\ell ij}}\left|u_i(t,x)\right|^{p} \\
&\quad + \mu_l \sum_{\substack{j=1\\ j\neq i}}^{n}\left|k_{ij}\right|^{p\eta^*_{pij}}\left|u_j(t,x)\right|^{p},
\end{aligned}$$

$$\begin{aligned}
&\mu_l \frac{p(p-1)}{2}\left|u_i(t,x)\right|^{p-2} \\
&\quad \times \sum_{j=1}^{n}\sigma_{ij}\left|u_j(t,x)\right|^{2} \\
&= \mu_l \frac{p-1}{2}\sum_{j=1}^{n} p\left[\prod_{\ell=1}^{p-2}\left|\sigma_{ij}\right|^{\epsilon_{\ell ij}}\left|u_i(t,x)\right|\right] \\
&\quad \times \left(\left|\sigma_{ij}\right|^{\epsilon_{(p-1)ij}}\left|u_j(t,x)\right|\right)\left(\left|\sigma_{ij}\right|^{\epsilon_{pij}}\left|u_j(t,x)\right|\right) \\
&\le \mu_l \frac{p-1}{2}\sum_{j=1}^{n}\sum_{\ell=1}^{p-2}\left|\sigma_{ij}\right|^{p\epsilon_{\ell ij}}\left|u_i(t,x)\right|^{p} \\
&\quad + \mu_l \frac{p-1}{2}\sum_{j=1}^{n}\left(\left|\sigma_{ij}\right|^{p\epsilon_{(p-1)ij}} + \left|\sigma_{ij}\right|^{p\epsilon_{pij}}\right) \\
&\quad \times \left|u_j(t,x)\right|^{p}.
\end{aligned} \tag{34}$$

Further, we also have

$$\begin{aligned}
&\mu_l \frac{p(p-1)}{2}\left|u_i(t,x)\right|^{p-2} \\
&\quad \times \sum_{j=1}^{n}\sigma_{ij}\left|u_j\left(t-\tau_{ij}(t),x\right)\right|^{2} \\
&= \mu_l \frac{p-1}{2}\sum_{j=1}^{n} p\left[\prod_{\ell=1}^{p-2}\sigma_{ij}^{\omega_{\ell ij}}\left|u_i(t,x)\right|\right] \\
&\quad \times \left(\sigma_{ij}^{\omega_{(p-1)ij}}\left|u_j\left(t-\tau_{ij}(t),x\right)\right|\right) \\
&\quad \times \left(\sigma_{ij}^{\omega_{pij}}\left|u_j\left(t-\tau_{ij}(t),x\right)\right|\right) \\
&\le \mu_l \frac{p-1}{2}\sum_{j=1}^{n}\sum_{\ell=1}^{p-2}\left|\sigma_{ij}\right|^{p\omega_{\ell ij}}\left|u_i(t,x)\right|^{p} \\
&\quad + \mu_l \frac{p-1}{2}\sum_{j=1}^{n}\left(\sigma_{ij}^{\omega_{(p-1)ij}} + \sigma_{ij}^{\omega_{pij}}\right) \\
&\quad \times \left|u_j\left(t-\tau_{ij}(t),x\right)\right|^{p}.
\end{aligned} \tag{35}$$

Substituting (33)–(35) into (32), we obtain

$$\begin{aligned}
&\mathscr{L}V(u(t,x),l,t) \\
&\le \sum_{i=1}^{n} e^{\varepsilon t}\left\{\left[\mu_l + p\mu_l k_{ii} - p\mu_l a_i^* \right.\right. \\
&\qquad + \sum_{q=1}^{N}\gamma_{lq}\mu_q + \frac{e^{\varepsilon\tau}\eta_i}{1-\tau_0} \\
&\qquad + \mu_l \sum_{j=1}^{n}\sum_{\ell=1}^{p-1}\left(\tilde{b}_{ij}^{p\alpha_{\ell ij}}\tilde{L}_j^{p\beta_{\ell ij}} + \tilde{c}_{ij}^{p\xi_{\ell ij}}\tilde{N}_j^{p\zeta_{\ell ij}}\right) \\
&\qquad + \mu_l \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\ell=1}^{p-1}\left|k_{ij}\right|^{p\eta^*_{\ell ij}} \\
&\qquad \left. + \mu_l \frac{p-1}{2}\sum_{j=1}^{n}\sum_{\ell=1}^{p-2}\left(\sigma_{ij}^{p\epsilon_{\ell ij}} + \sigma_{ij}^{p\omega_{\ell ij}}\right)\right] \\
&\qquad \times \left|u_i(t,x)\right|^{p} \\
&\qquad + \mu_l \sum_{j=1}^{n}\left[\tilde{b}_{ij}^{p\alpha_{pij}}\tilde{L}_j^{p\beta_{pij}} + \tilde{c}_{ij}^{p\xi_{pij}}\tilde{N}_j^{p\zeta_{pij}}\right. \\
&\qquad\qquad \left. + \frac{p-1}{2}\left(\sigma_{ij}^{p\epsilon_{(p-1)ij}} + \sigma_{ij}^{p\epsilon_{pij}}\right)\right] \\
&\qquad \times \left|u_i(t,x)\right|^{p} + \mu_l \sum_{\substack{j=1\\ j\neq i}}^{n}\left|k_{ij}\right|^{p\eta^*_{pij}}\left|u_j(t,x)\right|^{p} \\
&\qquad + \mu_l \frac{p-1}{2}\sum_{j=1}^{n}\left(\sigma_{ij}^{\omega_{(p-1)ij}} + \sigma_{ij}^{\omega_{pij}}\right) \\
&\qquad \times \left|u_j\left(t-\tau_{ij}(t),x\right)\right|^{p} \\
&\qquad + \mu_l \sum_{j=1}^{n}\tilde{c}_{ij}^{p\xi_{pij}}\tilde{N}_j^{p\zeta_{pij}}\left|u_j\left(t-\tau_{ij}(t),x\right)\right|^{p} \\
&\qquad \left. - \eta_i\left|u_i\left(t-\tau_{ij}(t),x\right)\right|^{p}\right\} \\
&\quad + \sum_{i=1}^{n} e^{\varepsilon t} p\mu_l\left|u_i(t,x)\right|^{p-1}
\end{aligned}$$

$$\times \left(\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}(l)\frac{\partial u_i(t,x)}{\partial x_k}\right)\right)$$
$$\leq \sum_{i=1}^{n} e^{\varepsilon t}\left\{\varepsilon \max_{l\in\mathbb{S}}\mu_l + \nu_i - \overline{\lambda}_i + \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q + \frac{e^{\varepsilon\tau}\eta_i}{1-\tau_0}\right\}$$
$$\times \left|u_i(t,x)\right|^p + \sum_{i=1}^{n} e^{\varepsilon t} p\mu_l \left|u_i(t,x)\right|^{p-1}$$
$$\times \left(\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}(l)\frac{\partial u_i(t,x)}{\partial x_k}\right)\right), \tag{36}$$

where

$$\begin{aligned}\overline{\lambda}_i = \min_{l\in\mathbb{S}}\mu_l \Bigg\{ & pa_i^* - \sum_{j=1}^{n}\sum_{\ell=1}^{p-1}\left(\widetilde{b}_{ij}^{p\alpha_{\ell ij}}\widetilde{L}_j^{p\beta_{\ell ij}} + \widetilde{c}_{ij}^{p\xi_{\ell ij}}\widetilde{N}_j^{p\zeta_{\ell ij}}\right) \\ & - \frac{p-1}{2}\sum_{j=1}^{n}\sum_{\ell=1}^{p-2}\left(\sigma_{ij}^{p\epsilon_{\ell ij}} + \sigma_{ij}^{p\omega_{\ell ij}}\right) \\ & - \sum_{j=1}^{n}\Bigg(\widetilde{b}_{ji}^{p\alpha_{pji}}\widetilde{L}_i^{p\beta_{pji}} + \widetilde{c}_{ji}^{p\xi_{pji}}\widetilde{N}_i^{p\zeta_{pji}} \\ & + \frac{p-1}{2}\left(\sigma_{ji}^{p\epsilon_{(p-1)ji}} + \sigma_{ji}^{p\epsilon_{pji}}\right)\Bigg)\Bigg\}.\end{aligned} \tag{37}$$

Substituting (36) into (30), we obtain

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t) \\ &\leq \mathbb{E}V_1(\phi,r(0),0) \\ &\quad - \mathbb{E}\int_0^t\int_{\Omega_0}\sum_{i=1}^{n} e^{\varepsilon s}\Bigg[\overline{\lambda}_i - \varepsilon\max_{l\in\mathbb{S}}\mu_l - \nu_i \\ &\qquad - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - \frac{e^{\varepsilon\tau}\eta_i}{1-\tau_0}\Bigg] \\ &\qquad \times \left|u_i(s,x)\right|^p dxds \\ &\quad + \mathbb{E}\int_0^t\sum_{i=1}^{n} e^{\varepsilon s}\int_{\Omega_0} p\mu_{r(s)}\left|u_i(s,x)\right|^{p-1} \\ &\quad \times\left(\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{ik}(r(s))\frac{\partial u_i(s,x)}{\partial x_k}\right)\right)dxds.\end{aligned} \tag{38}$$

By Lemma 3 and the boundary condition (4), we have

$$\begin{aligned}&\int_{\Omega_0} p\left|u_i(s,x)\right|^{p-1} \\ &\quad \times\left(\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{ik}(r(s))\frac{\partial u_i(s,x)}{\partial x_k}\right)\right)dx \\ &\leq -\sum_{k=1}^{m}\frac{4(p-1)D_{ik}(r(s))}{p\theta_k^2}\int_{\Omega_0}\left|u_i(s,x)\right|^p dx \\ &\leq -\sum_{k=1}^{m}\frac{4(p-1)\underline{D}_{ik}}{p\theta_k^2}\int_{\Omega_0}\left|u_i(s,x)\right|^p dx.\end{aligned} \tag{39}$$

Substituting these into (38), we get

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t) \\ &\leq \mathbb{E}V_1(\phi,r(0),0) \\ &\quad - \mathbb{E}\int_0^t\int_{\Omega_0}\sum_{i=1}^{n} e^{\varepsilon s}\Bigg[\overline{\lambda}_i + \min_{l\in\mathbb{S}}\mu_l\sum_{k=1}^{m}\frac{4(p-1)\underline{D}_{ik}}{p\theta_k^2} \\ &\qquad - \varepsilon\max_{l\in\mathbb{S}}\mu_l - \nu_i - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q \\ &\qquad - \frac{e^{\varepsilon\tau}\eta_i}{1-\tau_0}\Bigg]\left|u_i(s,x)\right|^p dxds \\ &\leq \mathbb{E}V_1(\phi,r(0),0) \\ &\quad - \mathbb{E}\int_0^t\int_{\Omega_0}\sum_{i=1}^{n} e^{\varepsilon s}\Bigg[\lambda_i - \varepsilon\max_{l\in\mathbb{S}}\mu_l - \nu_i \\ &\qquad - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - \frac{e^{\varepsilon\tau}\eta_i}{1-\tau_0}\Bigg] \\ &\quad \times\left|u_i(s,x)\right|^p dxds \\ &\leq \mathbb{E}V_1(\phi,r(0),0)\ (t,x)\in[MT,MT+\delta)\times\Omega_0.\end{aligned} \tag{40}$$

Similarly, for $(t,x)\in[MT+\delta,(M+1)T)\times\Omega_0$, we can obtain

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t) \\ &\leq \mathbb{E}V_1(u(MT+\delta,x),r(MT+\delta),MT+\delta) \\ &\quad - \mathbb{E}\int_0^t\int_{\Omega_0}\sum_{i=1}^{n} e^{\varepsilon s}\Bigg[\kappa_i + \rho_i - \varepsilon\max_{l\in\mathbb{S}}\mu_l \\ &\qquad - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - \frac{e^{\varepsilon\tau}\eta_i}{1-\tau_0}\Bigg] \\ &\quad \times\left|u_i(s,x)\right|^p dxds \\ &\quad + \mathbb{E}\int_0^t\int_{\Omega_0}\sum_{i=1}^{n} e^{\varepsilon s}\rho_i\left|u_i(s,x)\right|^p dxds \\ &\leq \mathbb{E}V_1(u(MT+\delta,x),r(MT+\delta),MT+\delta) \\ &\quad + \frac{\rho}{\overline{\mu}}\mathbb{E}\int_0^t V_1(u(s,x),r(s),s)\,ds,\end{aligned} \tag{41}$$

where $\rho = \max_{1\leq i\leq n}\rho_i$, $\overline{\mu} = \min_{l\in\mathbb{S}}\mu_l$.

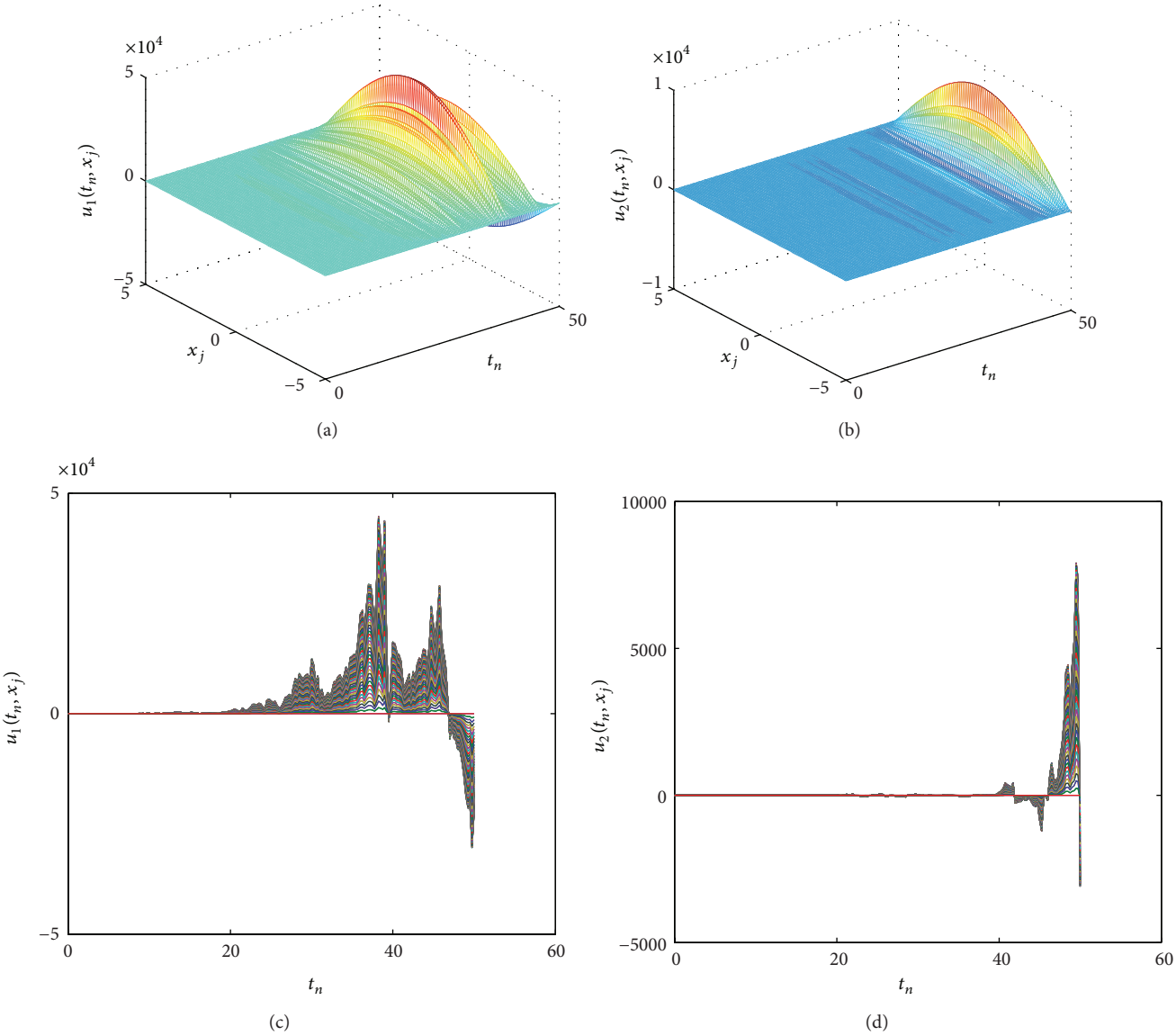

Figure 1: Surface curves and state trajectories for system (54) in model 1.

By the Gronwall inequality, we have

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t)\\&\quad\le \mathbb{E}V_1(u(MT+\delta,x),\\&\qquad r(MT+\delta),MT+\delta)\,e^{(\rho/\overline{\mu})(t-MT-\delta)}.\end{aligned}\tag{42}$$

Combining (40) and (42), we summarize that,

(I) for $(t,x)\in[0,\delta)\times\Omega_0$, from (40), we have

$$\mathbb{E}V_1(u(t,x),r(t),t)\le \mathbb{E}V_1(u(0,x),r(0),0).\tag{43}$$

(II) For $(t,x)\in[\delta,T)\times\Omega_0$, from (42), we get

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t)\\&\quad\le \mathbb{E}V_1(u(0,x),r(0),0)\,e^{(\rho/\overline{\mu})(t-\delta)}.\end{aligned}\tag{44}$$

(III) For $(t,x)\in[T,T+\delta)\times\Omega_0$, from (40), we have

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t)\\&\quad\le \mathbb{E}V_1(u(T,x),r(T),T)\\&\quad\le \mathbb{E}V_1(u(0,x),r(0),0)\,e^{(\rho/\overline{\mu})(T-\delta)}.\end{aligned}\tag{45}$$

(IV) For $(t,x)\in[T+\delta,2T)\times\Omega_0$, from (42), we have

$$\begin{aligned}&\mathbb{E}V_1(u(t,x),r(t),t)\\&\quad\le \mathbb{E}V_1(u(T+\delta,x),r(T+\delta),T+\delta)\,e^{(\rho/\overline{\mu})(t-T-\delta)}\\&\quad\le \mathbb{E}V_1(u(0,x),r(0),0)\,e^{(\rho/\overline{\mu})(t-2\delta)}.\end{aligned}\tag{46}$$

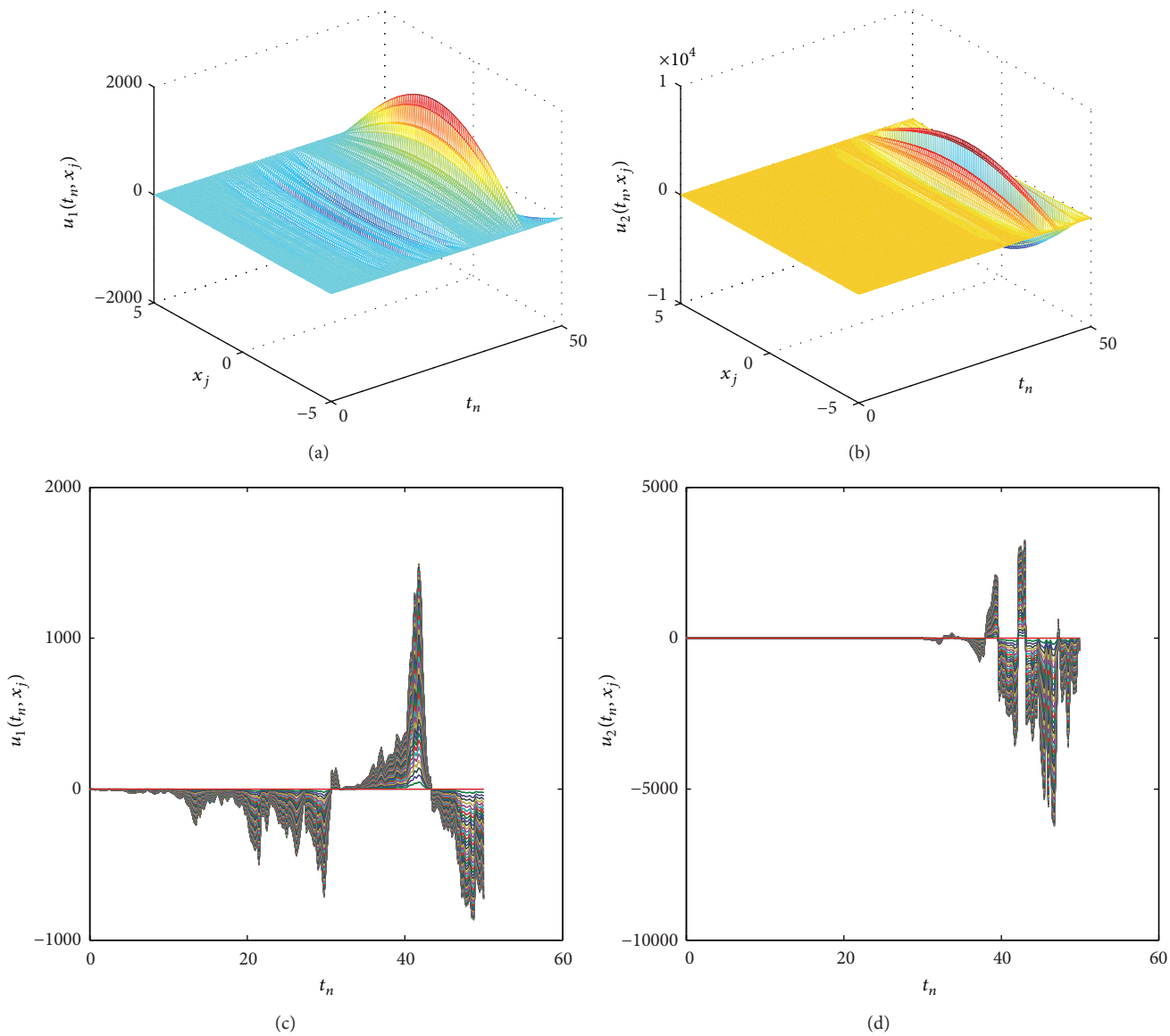

Figure 2: Surface curves and state trajectories for system (54) in model 2.

Repeating the above procedure, we obtain that, for $(t, x) \in [MT, MT + \delta)$, $M \leq (t/T)$,

$$\begin{aligned} &\mathbb{E}V_1(u(t,x), r(t), t) \\ &\quad \leq \mathbb{E}V_1(u(MT,x), r(MT), MT) \\ &\quad \leq \mathbb{E}V_1(u(0,x), r(0), 0)\, e^{(\rho(T-\delta)/\overline{\mu}T)t}. \end{aligned} \tag{47}$$

Moreover, for $(t, x) \in [MT + \delta, (M+1)T) \times \Omega_0$, $(t/T) < M + 1$,

$$\begin{aligned} &\mathbb{E}V_1(u(t,x), r(t), t) \\ &\leq \mathbb{E}V_1\left(\mu(MT+\delta, x), r(MT+\delta), MT+\delta\right) e^{(\rho/\overline{\mu})(t-MT-\delta)} \\ &\leq \mathbb{E}V_1(u(0,x), r(0), 0)\, e^{(\rho/\overline{\mu})((T-\delta)/T)t}. \end{aligned} \tag{48}$$

Hence, for any $(t, x) \in [0, +\infty) \times \Omega_0$, we always have

$$\mathbb{E}V_1(u(t,x), r(t), t) \leq \mathbb{E}V_1(u(0,x), r(0), 0)\, e^{(\rho/\overline{\mu})((T-\delta)/T)t}. \tag{49}$$

By (28) and (49), we have

$$\begin{aligned} &e^{\varepsilon t}\overline{\mu}\mathbb{E}\int_{\Omega_0}\sum_{i=1}^{n}|u_i(t,x)|^p dx \\ &\leq \mathbb{E}V_1(u(t,x), r(t), t) \\ &\leq \mathbb{E}V_1(u(0,x), r(0), 0)\, e^{(\rho/\overline{\mu})((T-\delta)/T)t}. \end{aligned} \tag{50}$$

Note that

$$\begin{aligned}
&\mathbb{E}V_1(u(0,x), r(0), 0)\\
&= \mathbb{E}\int_{\Omega_0}\mu_l\sum_{i=1}^{n}|u_i(0,x)|^p dx\\
&\quad + \frac{e^{\varepsilon\tau}}{1-\tau_0}\mathbb{E}\int_{\Omega_0}\sum_{i=1}^{n}\eta_i\int_{-\tau_{ij}(0)}^{0}e^{\varepsilon s}|u_i(s,x)|^p ds dx\\
&\le \max_{l\in\mathbb{S}}\mu_l\mathbb{E}\int_{\Omega_0}\sum_{i=1}^{n}|\phi_i(0,x)|^p dx\\
&\quad + \sup_{-\tau\le s\le 0}\left[\left(\max_{1\le i\le n}\eta_i\right)\frac{\tau e^{\varepsilon\tau}}{1-\tau_0}\mathbb{E}\int_{\Omega_0}\sum_{i=1}^{n}|\phi_i(s,x)|^p dx\right]\\
&= M_0.
\end{aligned} \tag{51}$$

Under assumption (H5), the assertion of Theorem 4 follows from (50) and (51). □

Corollary 5. *Under assumptions (H1)–(H3), the origin of system* (3)–(5) *under periodically intermittent control* (13) *is robust exponentially stable in pth moment if the following conditions hold:*

(I) $\nu_i < 0, \lambda_i - \nu_i - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - (\eta_i/(1-\tau_0)) > 0, i = 1,2,\ldots,n,$

(II) $\varepsilon - (\overline{\nu}(T-\delta)/(\overline{\mu}T)) > 0$, *where* $\overline{\nu} = \max_{1\le i\le n}\{|\nu_i|\}$, $\overline{\mu} = \min_{l\in\mathbb{S}}\mu_l$.

Proof. In Theorem 4, let $\alpha_{\ell ij} = \overline{\alpha}_{\ell ij}$, $\beta_{\ell ij} = \overline{\beta}_{\ell ij}$, $\xi_{\ell ij} = \overline{\xi}_{\ell ij}$, $\zeta_{\ell ij} = \overline{\zeta}_{\ell ij}$, $\epsilon_{\ell ij} = \overline{\epsilon}_{\ell ij}$, $\omega_{\ell ij} = \overline{\omega}_{\ell ij}$ for all $\ell = 1,2,\ldots,p, i,j = 1,2,\ldots,n$; then $\lambda_i = \kappa_i$. Under condition (i), select $\rho_i = -\nu_i$, and Corollary 5 holds immediately from Theorem 4.

In Theorem 4, we choose $\alpha_{\ell ij} = \overline{\alpha}_{\ell ij} = \beta_{\ell ij} = \overline{\beta}_{\ell ij} = \xi_{\ell ij} = \overline{\xi}_{\ell ij} = \zeta_{\ell ij} = \overline{\zeta}_{\ell ij} = \eta^* = \epsilon_{\ell ij} = \overline{\epsilon}_{\ell ij} = \omega_{\ell ij} = \overline{\omega}_{\ell ij} = 1/p$ for $\ell = 1,2,\ldots,p$, and $i,j = 1,2,\ldots,n$; then

$$\begin{aligned}
\widetilde{\lambda}_i &= \widetilde{\kappa}_i\\
&= \min_{l\in\mathbb{S}}\mu_l\left\{\sum_{k=1}^{m}\frac{4(p-1)\underline{D}_{ik}}{p\theta_k^2} + pa_i^*\right.\\
&\qquad - (p-1)\sum_{j=1}^{n}\left(\widetilde{b}_{ij}\widetilde{L}_j + \widetilde{c}_{ij}\widetilde{N}_j\right)\\
&\qquad - (p-1)(p-2)\sum_{j=1}^{n}\sigma_{ij}\\
&\qquad \left. - \sum_{j=1}^{n}\left(\widetilde{b}_{ji}\widetilde{L}_j + \widetilde{c}_{ji}\widetilde{N}_j + (p-1)\sigma_{ji}\right)\right\},\\
\widetilde{\nu}_i &= \max_{l\in\mathbb{S}}\mu_l\left\{pk_{ii} + (p-1)\right.\\
&\qquad \left.\times\sum_{\substack{j=1\\ j\ne i}}^{n}|k_{ij}| + \sum_{\substack{j=1\\ j\ne i}}^{n}|k_{ji}|\right\},\\
\widetilde{\eta}_i &= \max_{l\in\mathbb{S}}\mu_l\sum_{j=1}^{n}\left[(p-1)\sigma_{ji} + \widetilde{c}_{ji}\widetilde{N}_j\right].
\end{aligned} \tag{52}$$

Then, as the proof of Theorem 4, we have the following. □

Corollary 6. *Under assumptions (H1)–(H3), the origin of system* (3)–(5) *under periodically intermittent control* (13) *is robust exponentially stable in pth moment if the following conditions hold:*

(I) $\widetilde{\lambda}_i - \widetilde{\nu}_i - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - (\widetilde{\eta}_i/(1-\tau_0)) > 0, i = 1,2,\ldots,n,$

(II) *there exists* $\widetilde{\rho}_i > 0$, *such that* $\widetilde{\lambda}_i + \widetilde{\rho}_i - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - \widetilde{\eta}_i/(1-\tau_0) > 0$, $i = 1,2,\ldots,n,$

(III) $\widetilde{\varepsilon} - (\widetilde{\rho}(T-\delta)/\overline{\mu}T) > 0$, *where* $\widetilde{\rho} = \max_{1\le i\le n}\widetilde{\rho}_i$, $\overline{\mu} = \min_{l\in\mathbb{S}}\{\mu_l\}$.

Combining Corollary 5 and Corollary 6, we have the following.

Corollary 7. *Under assumptions (H1)–(H3), the origin of system*(3)–(5) *under periodically intermittent control* (13) *is robust exponentially stable in pth moment if the following conditions hold:*

(I) $\widetilde{\nu}_i < 0, \widetilde{\lambda}_i - \widetilde{\nu}_i - \max_{l\in\mathbb{S}}\sum_{q=1}^{N}\gamma_{lq}\mu_q - (\widetilde{\eta}_i/(1-\tau_0)) > 0,$ $i = 1,2,\ldots,n,$

(II) $\widetilde{\varepsilon} - (\widehat{\nu}(T-\delta)/\overline{\mu}T) > 0$, *where* $\widehat{\nu} = \max_{0\le i\le n}\{|\widetilde{\nu}_i|\}, \overline{\mu} = \min_{l\in\mathbb{S}}\{\mu_l\}$.

Remark 8. By constructing an average Lyapunov function, the stabilization of stochastic Hopfield neural networks with distributed parameters was studied in [4]. The feedback controller in [4] was designed as the compound function of the state and activation function. Therefore, the feedback controller may be the nonlinear functions. However, we see in this paper that the control width is greater than the time delay and the periodically intermittent controller is linear and practical.

Remark 9. In [11–13, 24–26], robust exponential stability and exponential synchronization of some classes of neural networks with reaction-diffusion terms were discussed. The activation function satisfies Lipschitz condition. In fact, the activation function may be not monotone. But, from assumption (H1) in this paper, the activation functions include the monotone functions. So the results of this paper are less conservational and more general.

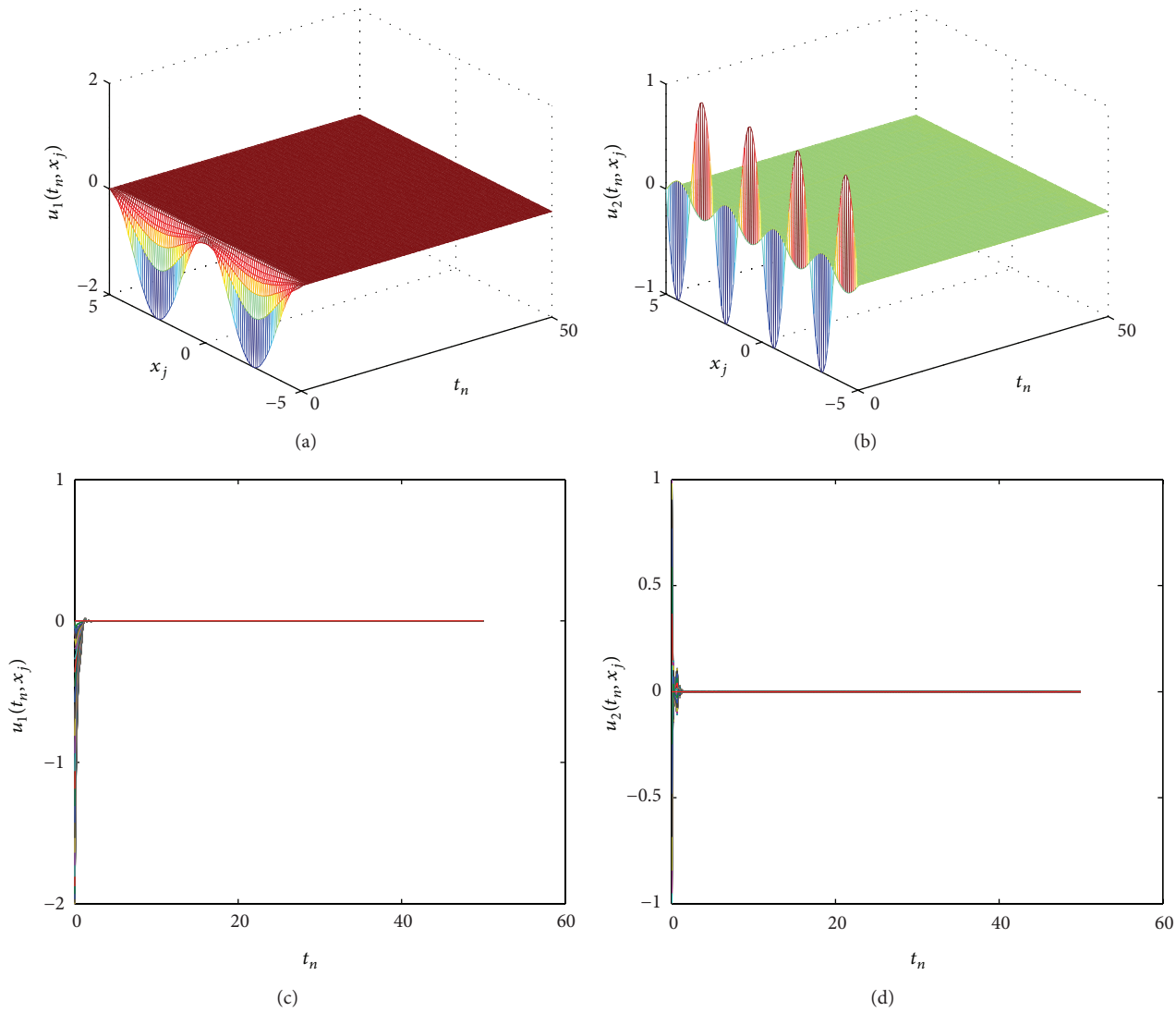

FIGURE 3: Surface curves and state trajectories for system (54) in model 1 under periodically intermittent control (13), $T = 5$ and $\delta = 4.7$.

Remark 10. In [27, 28], the periodically intermittent controller was designed to stabilization and synchronization of two classes of neural networks, where the activation functions satisfy

$$0 < \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le L_i^+, \qquad 0 < \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le N_i^+. \tag{53}$$

In fact, they need $L_i^+ \ge L_i^- > 0$, $N_i^+ \ge N_i^- > 0$ in assumption (H1) of this paper. Obviously, the assumption (H1) of this paper is weaker than those of papers [27, 28].

Remark 11. In this paper, if the transmission delay $\tau_{ij}(t)$ is not continuous and differential, we can give new sufficient conditions ensuring robust exponential stabilization and antisynchronization for system (3)–(5) by applying linear matrix inequality (LMI) technique and periodically intermittent control. We will give the topics in future research.

4. Numerical Example

In this section, we give an example with numerical simulations to illustrate our result in the preceding section.

Example 1. Consider the 2-dimensional stochastic interval recurrent neural networks with two models as follows:

$$\begin{aligned} du_1(t,x) = \Bigg\{ & D_1(r(t)) \frac{\partial^2 u_1(t,x)}{\partial x^2} - a_1(r(t)) u_1(t,x) \\ & + \sum_{j=1}^{2} b_{1j}(r(t)) f_j(u_j(t,x)) \\ & + \sum_{j=1}^{2} c_{1j}(r(t)) g_j(u_j(t - \tau_{1j}(t), x)) \Bigg\} dt \end{aligned}$$

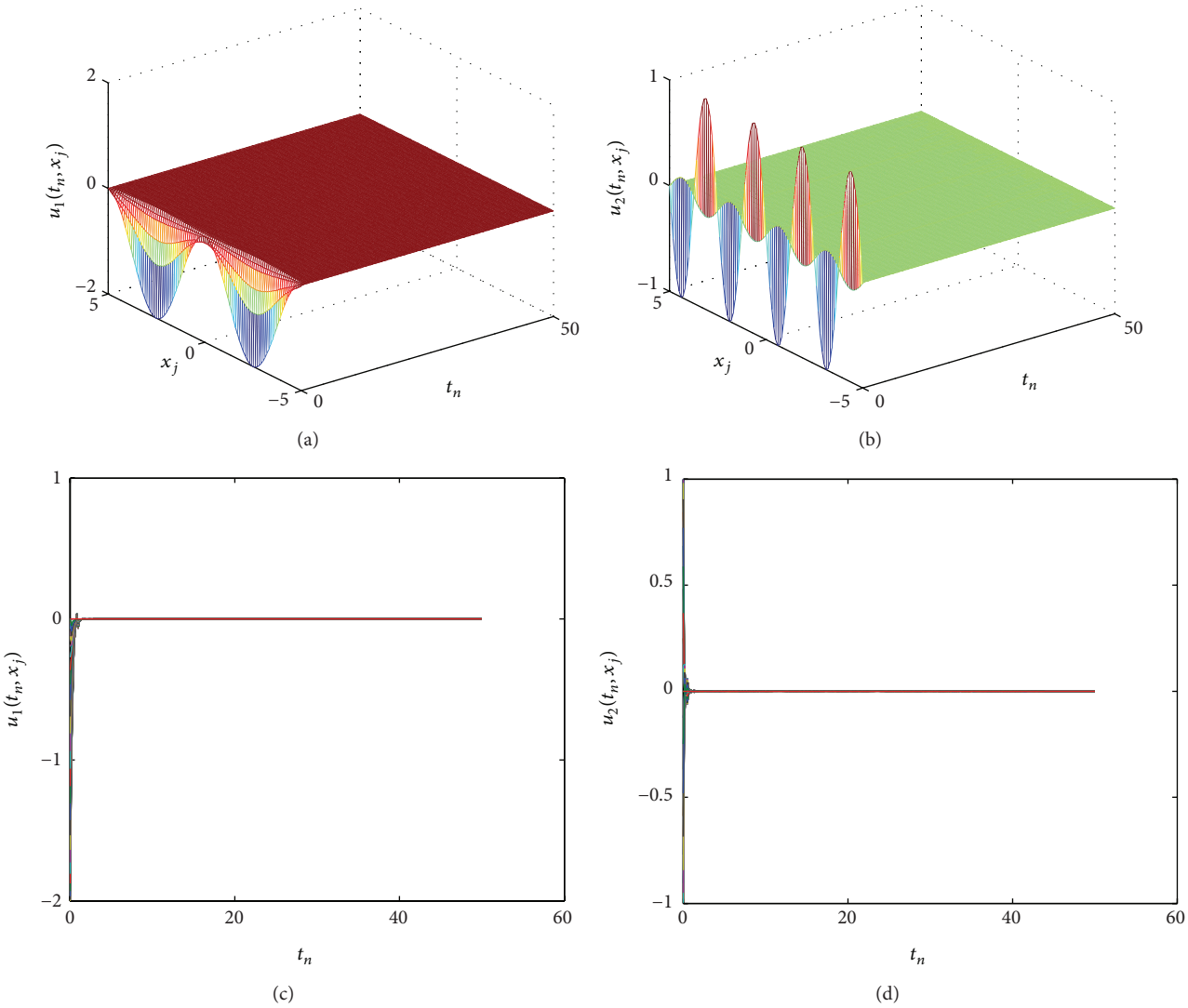

FIGURE 4: Surface curves and state trajectories for system (54) in model 2 under periodically intermittent control (13), $T = 5$ and $\delta = 4.7$.

$$+\sum_{j=1}^{2} h_{1j}\left(u_j(t,x), u_j\left(t-\tau_{1j}(t), x\right)\right) dW_j(t),$$

$$du_2(t,x) = \left\{D_2(r(t))\frac{\partial^2 u_2(t,x)}{\partial x^2} - a_2(r(t))u_2(t,x)\right.$$
$$+\sum_{j=1}^{2} b_{2j}(r(t)) f_j\left(u_j(t,x)\right) + \sum_{j=1}^{2} c_{2j}(r(t))$$
$$\left.\times g_j\left(u_j\left(t-\tau_{2j}(t), x\right)\right)\right\} dt$$
$$+\sum_{j=1}^{2} h_{2j}\left(u_j(t,x), u_j\left(t-\tau_{2j}(t), x\right)\right) dW_j(t), \tag{54}$$

with the boundary conditions $u_1(t,0) = u_2(t,0) = u_1(t,2) = u_2(t,2) = 0$, $t \geq -1$ and the initial value $u_1(t,x) = e^t(\cos(2\pi x) - 1)$, $u_2(t,x) = e^{2t}\sin(4\pi x)$, $t \in [-1,0] \times \Omega_0$, where $\tau_{ij}(t) = (e^t/(1+e^t))$, $\Omega_0 = [-5,5] \in \mathbb{R}$, and the generator of the Markov chain

$$\Gamma = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$f_i(u_i) = \frac{3}{4}\sin(u_i) + \frac{1}{4}u_i,$$

$$g_i(u_i) = \frac{1}{2}\left(|u_i+1| - |u_i-1|\right),$$

$$h_{11}\left(u_1(t,x), u_1\left(t-\tau_{11}(t), x\right)\right)$$
$$= 0.1u_1(t,x) + 0.2u_1\left(t-\tau_{11}(t), x\right),$$

$$h_{12}\left(u_2(t,x), u_2\left(t-\tau_{12}(t), x\right)\right)$$
$$= h_{21}\left(u_1(t,x), u_1\left(t-\tau_{21}(t)\right), x\right) = 0,$$

$$h_{22}\left(u_2(t,x), u_2\left(t-\tau_{22}(t), x\right)\right) = 0.3u_2(t,x) + 0.4u_2\left(t-\tau_{22}(t), x\right). \tag{55}$$

We assume that the interval matrices are the same in every model; let

$$\begin{aligned} D(r(t)) &= \begin{pmatrix} D_1(r(t)) & \\ & D_2(r(t)) \end{pmatrix} = \begin{pmatrix} [1,2] & \\ & [2,3] \end{pmatrix}, \\ A(r(t)) &= \begin{pmatrix} a_1(r(t)) & \\ & a_2(r(t)) \end{pmatrix} \\ &= \begin{pmatrix} [0.28, 0.42] & \\ & [0.18, 0.35] \end{pmatrix}, \\ B(r(t)) &= \begin{pmatrix} B_{11}(r(t)) & B_{12}(r(t)) \\ B_{21}(r(t)) & B_{22}(r(t)) \end{pmatrix} = \begin{pmatrix} [4,5] & \\ & [3,4] \end{pmatrix}, \\ C(r(t)) &= \begin{pmatrix} C_{11}(r(t)) & C_{12}(r(t)) \\ C_{21}(r(t)) & C_{22}(r(t)) \end{pmatrix} \\ &= \begin{pmatrix} [0.2, 0.3] & \\ & [0.3, 0.4] \end{pmatrix}. \end{aligned} \tag{56}$$

The surface curves and state trajectories of system (54) in model (1) and model (2) are given, respectively, as shown in Figures 1 and 2. They exhibit instability behavior.

Let $p = 2, \mu_1 = 2, \mu_2 = 4$. By simple calculation, we obtain

$$\begin{aligned} &L_i^- = -\frac{1}{2}, \qquad L_i^+ = 1, \qquad N_i^- = 0, \\ &N_i^+ = 1, \qquad \tau = 1, \qquad \tau_0 = \frac{1}{4} \\ &\widetilde{\lambda}_1 = \widetilde{\kappa}_1 = -40.16, \qquad \widetilde{\eta}_1 = 1.52 \\ &\widetilde{\lambda}_2 = \widetilde{\kappa}_2 = -34.4, \qquad \widetilde{\eta}_2 = 2.88. \end{aligned} \tag{57}$$

Now, we consider the periodically intermittent control (13), where the parameters are given as follows:

$$k_{11} = -10, \qquad k_{22} = -10, \qquad k_{12} = 0, \qquad k_{21} = 0. \tag{58}$$

Then $\widetilde{\nu}_1 = -80$ and $\widetilde{\nu}_2 = -80$,

$$\begin{aligned} &\widetilde{\lambda}_1 - \widetilde{\nu}_1 - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \frac{\widetilde{\eta}_1}{1-\tau_0} > 0, \\ &\widetilde{\lambda}_2 - \widetilde{\nu}_2 - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \frac{\widetilde{\eta}_2}{1-\tau_0} > 0. \end{aligned} \tag{59}$$

From $H_i(\widetilde{\varepsilon}_i) = \widetilde{\lambda}_i - \widetilde{\nu}_i - \max_{l\in\mathbb{S}} \sum_{q=1}^{N} \gamma_{lq}\mu_q - \widetilde{\varepsilon}_i \max_{l\in\mathbb{S}} \mu_l - (\widetilde{\eta}_i/(1-\tau_0))e^{\widetilde{\varepsilon}_i\tau} = 0, i = 1, 2$, we have

$$\widetilde{\varepsilon} = \min\{\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2\} = \min\{2.605, 2.413\} = 2.413. \tag{60}$$

Choose $T = 5$, from Corollary 7, $\delta = 4.7$. Then the origin of system (54) under the periodically intermittent controller (13) is robust exponentially stable in mean square. The surface curves and state trajectories in model (1) and model (2) are given, respectively, as shown in Figures 3 and 4.

Conflict of Interests

The authors declare no competing financial interests.

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Research Article

Improved Robust H_∞ Filtering Approach for Nonlinear Systems

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An improved design approach of robust H_∞ filter for a class of nonlinear systems described by the Takagi-Sugeno (T-S) fuzzy model is considered. By introducing a free matrix variable, a new sufficient condition for the existence of robust H_∞ filter is derived. This condition guarantees that the filtering error system is robustly asymptotically stable and a prescribed H_∞ performance is satisfied for all admissible uncertainties. Particularly, the solution of filter parameters which are independent of the Lyapunov matrix can be transformed into a feasibility problem in terms of linear matrix inequalities (LMIs). Finally, a numerical example illustrates that the proposed filter design procedure is effective.

1. Introductions

In recent years, when the external disturbance and the statistical properties of the measurement noise are unknown, using H_∞ filtering approach to estimate the states of a linear system becomes one of the focuses on the estimated theoretical research, and some useful research results [1–4] are obtained. However, how to design an effective filter for nonlinear systems is still a very difficult problem. Over the past two decades, there has been a rapidly growing interest in fuzzy control of nonlinear systems. In particular, the fuzzy model proposed by Takagi and Sugeno [5] receives a great deal of attention. And it indicates that this type of fuzzy model has a good approximation performance for the complex nonlinear systems, so some scholars attempt to apply this fuzzy model to design H_∞ filter for nonlinear systems. Feng et al. [6] were prior scholars to study the filter for nonlinear systems by using T-S fuzzy model and linear matrix inequality (LMI) techniques. For a class of discrete nonlinear dynamic systems, Tseng and Chen [7] and Pan et al. [8] studied a fuzzy H_∞ filtering problem. After that, Tseng [9, 10] and Tian et al. [11] discussed the design problem of robust H_∞ fuzzy filter for a class of continuous nonlinear systems. Moreover, the above-obtained results were extended to the fuzzy H_∞ filter or robust H_∞ filter design for nonlinear systems with time delay [12–15]. In addition, H_∞ filtering approach is also applied to Markovian jump systems [16], nonlinear interconnected systems [17], chaotic systems [18], and networked nonlinear systems [19] for the discrete-time case and stochastic systems [20] and singular systems [21] for the continuous-time case. Nevertheless, in the above-mentioned results, the solving process of filter parameters is related to the Lyapunov matrix, which will more or less bring some conservative to the results. The reason is that most of the existence conditions of filter are sufficient conditions; if the Lyapunov matrix cannot be found, then the filter parameters which maybe exist cannot be constructed. For this reason, de Oliveira et al. [22] proposed a novel filter design method by introducing free matrices to the framework of the quadratic Lyapunov function. By means of decoupling the relations between the Lyapunov matrix and the system matrix, the conservative of the results will be reduced. But due to the restrictions of LMI characteristics, this method can only be applied to the discrete systems [17, 23, 24]. Lately, Apkarian et al. [25] extended this idea to the linear continuous systems with the aid of Projection Theorem. And this idea has been used in other fields [26–28]. Unfortunately, to the best of our knowledge, this idea has not yet been introduced to the design of robust H_∞ filter for the uncertain continuous nonlinear systems.

Taking into account the above-mentioned results, this paper will discuss a new design method of robust H_∞ filter for a class of uncertain nonlinear systems. Firstly, the

T-S fuzzy model is employed to represent the nonlinear systems. Then, on the basis of the bounded real lemma of continuous systems, a new criterion for the existence of the improved robust H_∞ filter is obtained via introducing a free matrix variable. Based on this criterion, the solution of the filter parameters independent of the Lyapunov matrix can be obtained. Combined with the linear matrix inequality techniques, the filter design problem can be transformed into a feasibility problem of a set of linear matrix inequalities. Finally, a simulation example will be given to verify the validity of the proposed method.

2. Problem Formulation

Consider a class of uncertain nonlinear systems described by the following T-S fuzzy models.

Plant Rule i:

$$\begin{aligned}
&\text{IF } v_1(t) \text{ is } M_{i1} \text{ and} \cdots \text{and } v_p(t) \text{ is } M_{ip} \\
&\text{THEN } \dot{x}(t) = \left(A_i + \Delta A_i(t)\right) x(t) + B_i w(t), \\
&\qquad y(t) = \left(C_i + \Delta C_i(t)\right) x(t) + D_i w(t), \\
&\qquad z(t) = L_i x(t), \\
&\qquad x_0 = x(0), \quad i = 1, 2, \ldots, r,
\end{aligned} \tag{1}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector, $y(t) \in \mathfrak{R}^m$ is the measured output, $z(t) \in \mathfrak{R}^l$ is the signal to be estimated, and $w(t) \in L_2^q[0, \infty)$ is the noise signal vector (including process and measurement noises). x_0 is the initial state condition of the system, which is considered to be known and, without loss of generality, assumed to be zero. $v_1(t), \ldots, v_p(t)$ are the premise variables, M_{ij} $(j = 1, 2, \ldots, p)$ is the fuzzy set, and r is the number of IF-THEN rules. A_i, B_i, C_i, D_i, and L_i are known real constant matrices with appropriate dimensions of the ith subsystem, respectively. The uncertain time-varying matrices $\Delta A_i(t)$ and $\Delta C_i(t)$ represent the parameter uncertainties in the system model and are assumed to be norm-bounded of the following forms:

$$\begin{aligned}
\Delta A_i(t) &= E_{ai} F(t) H_{ai}, \\
\Delta C_i(t) &= E_{ci} F(t) H_{ci}, \\
i &= 1, 2, \ldots, r,
\end{aligned} \tag{2}$$

where E_{ai}, E_{ci}, H_{ai}, and H_{ci} are known constant matrices of appropriate dimensions, which reflect the structural information of uncertainty, and $F(t)$ is an uncertainty matrix function with Lebesgue measurable elements and satisfies

$$F^{\mathrm{T}}(t) F(t) \le I. \tag{3}$$

By using the weighted average method for defuzzification, the uncertain fuzzy dynamic model for the system (1) can be inferred as follows:

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(v(t)) \left[\left(A_i + \Delta A_i(t)\right) x(t) + B_i w(t)\right], \\
y(t) &= \sum_{i=1}^{r} h_i(v(t)) \left[\left(C_i + \Delta C_i(t)\right) x(t) + D_i w(t)\right], \\
z(t) &= \sum_{i=1}^{r} h_i(v(t)) L_i x(t),
\end{aligned} \tag{4}$$

where $v(t) = [v_1(t), v_2(t), \ldots, v_p(t)]^{\mathrm{T}}$, and

$$\begin{aligned}
h_i(v(t)) &= \frac{\mu_i(v(t))}{\sum_{i=1}^{r} \mu_i(v(t))}, \\
\mu_i(v(t)) &= \prod_{j=1}^{p} M_{ij}\left(v_j(t)\right), \\
i &= 1, 2, \ldots, r,
\end{aligned} \tag{5}$$

in which $M_{ij}(v_j(t))$ is the grade of membership of $v_j(t)$ in the fuzzy set M_{ij}, while $\mu_i(v(t))$ is the grade of membership of the ith rule.

In general, it is assumed that $\mu_i(v(t)) \ge 0$, $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} \mu_i(v(t)) > 0$. Therefore, it is easy to obtain that $h_i(v(t)) \ge 0$, $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} h_i(v(t)) = 1$.

Based on the T-S fuzzy models (1), the full-order filter is constructed as follows.

Filter Rule i:

$$\begin{aligned}
&\text{IF } v_1(t) \text{ is } M_{i1} \text{ and} \cdots \text{and } v_p(t) \text{ is } M_{ip} \\
&\text{THEN } \dot{\hat{x}}(t) = A_{fi} \hat{x}(t) + B_{fi} y(t), \\
&\qquad \hat{z}(t) = C_{fi} \hat{x}(t) + D_{fi} y(t), \quad i = 1, 2, \ldots, r,
\end{aligned} \tag{6}$$

where $\hat{x}(t) \in \mathfrak{R}^n$ is the state vector of filter and $\hat{z}(t) \in \mathfrak{R}^l$ is an estimate value of the filter output. The matrices A_{fi}, B_{fi}, C_{fi}, and D_{fi} are filter parameters to be determined. Here, it is assumed that the initial condition of filter is $\hat{x}_0 = 0$. Then the whole fuzzy filter can be expressed as

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{r} h_i(v(t)) \left[A_{fi} \hat{x}(t) + B_{fi} y(t)\right], \\
\hat{z}(t) &= \sum_{i=1}^{r} h_i(v(t)) \left[C_{fi} \hat{x}(t) + D_{fi} y(t)\right].
\end{aligned} \tag{7}$$

Set the state variable as $\tilde{x}(t) = [x^{\mathrm{T}}(t) \hat{x}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and estimated error as $\tilde{z}(t) = z(t) - \hat{z}(t)$. Then the filtering error

dynamic equation inferred from formulas (4) and (7) can be described as follows:

$$\begin{aligned}\dot{\tilde{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(v(t))\, h_j(v(t)) \\ &\quad \times \left[\left(\widetilde{A}_{ij} + \Delta\widetilde{A}_{ij}(t)\right)\tilde{x}(t) + \widetilde{B}_{ij} w(t)\right], \\ \tilde{z}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(v(t))\, h_j(v(t)) \\ &\quad \times \left[\left(\widetilde{C}_{ij} + \Delta\widetilde{C}_{ij}(t)\right)\tilde{x}(t) + \widetilde{D}_{ij} w(t)\right],\end{aligned} \tag{8}$$

where

$$\begin{aligned}\widetilde{A}_{ij} &= \begin{bmatrix} A_j & 0 \\ B_{fi}C_j & A_{fi}\end{bmatrix}, \qquad \widetilde{B}_{ij} = \begin{bmatrix} B_j \\ B_{fi} \quad D_j\end{bmatrix}, \\ \widetilde{C}_{ij} &= \left[L_j - D_{fi}C_j \quad -C_{fi}\right], \qquad \widetilde{D}_{ij} = -D_{fi}D_j, \\ \Delta\widetilde{A}_{ij}(t) &= \begin{bmatrix} \Delta A_j(t) & 0 \\ B_{fi}\Delta C_j(t) & 0\end{bmatrix}, \\ \Delta\widetilde{C}_{ij}(t) &= \left[-D_{fi}\Delta C_j(t) \quad 0\right].\end{aligned} \tag{9}$$

Let $H_{\tilde{z}w}(s)$ be the transfer function from the disturbance input $w(t)$ to the estimation error $\tilde{z}(t)$. Then the robust H_∞ filter design problem considered in this paper can be described as follows: for a given constant $\gamma > 0$, find a H_∞ full-order filter in the form of (7) so that the filtering error dynamic system (8) is robustly asymptotically stable and the H_∞ norm of the transfer function $H_{\tilde{z}w}(s)$ is less than the given constant γ; that is, $\|H_{\tilde{z}w}(s)\|_\infty < \gamma$ is satisfied. Here, constant γ is called a prescribed H_∞ performance level.

For brevity, the functions $h_i(v(t))$ will be replaced by h_i in the subsequence, and $\Delta A_i(t)$, $\Delta C_i(t)$, $\Delta\widetilde{A}_{ij}(t)$, $\Delta\widetilde{C}_{ij}(t)$ will be replaced by ΔA_i, ΔC_i, $\Delta\widetilde{A}_{ij}$, $\Delta\widetilde{C}_{ij}$.

3. Robust H_∞ Filter Design

According to the bounded real lemma of continuous-time systems, this section firstly gives a sufficient condition for the existence of robust H_∞ filter for the uncertain fuzzy system (4). That is, for a given constant $\gamma > 0$, the filtering error system (8) is robustly asymptotically stable and satisfies $\|H_{\tilde{z}w}(s)\|_\infty < \gamma$, if there exists a symmetric positive definite matrix $P \in \mathfrak{R}^{2n\times 2n}$, such that the following matrix inequality holds:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \begin{bmatrix} \left(\widetilde{A}_{ij} + \Delta\widetilde{A}_{ij}\right)^{\mathrm{T}} P + P\left(\widetilde{A}_{ij} + \Delta\widetilde{A}_{ij}\right) & P\widetilde{B}_{ij} & \left(\widetilde{C}_{ij} + \Delta\widetilde{C}_{ij}\right)^{\mathrm{T}} \\ \widetilde{B}_{ij}^{\mathrm{T}} P & -\gamma^2 I & \widetilde{D}_{ij}^{\mathrm{T}} \\ \widetilde{C}_{ij} + \Delta\widetilde{C}_{ij} & \widetilde{D}_{ij} & -I \end{bmatrix} < 0. \tag{10}$$

With the aid of the basic idea of [25], an improved robust H_∞ filter design method is obtained by introducing a free matrix variable, in which the Lyapunov function matrix and the filtering error system matrix are separated. Then the filter parameters to be determined can be solved independently of the Lyapunov function matrix. This kind of processing method can reduce the conservatism of the results.

Theorem 1. *For a given constant $\gamma > 0$, the filtering error system (8) is robustly asymptotically stable and satisfies $\|H_{\tilde{z}w}(s)\|_\infty < \gamma$, if there exist a symmetric positive definite matrix $P \in \mathfrak{R}^{2n\times 2n}$ and matrix $V \in \mathfrak{R}^{2n\times 2n}$, such that the following matrix inequality holds:*

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \begin{bmatrix} -\left(V + V^{\mathrm{T}}\right) & V^{\mathrm{T}}\left(\widetilde{A}_{ij} + \Delta\widetilde{A}_{ij}\right) + P & V^{\mathrm{T}}\widetilde{B}_{ij} & 0 & V^{\mathrm{T}} \\ * & -P & 0 & \left(\widetilde{C}_{ij} + \Delta\widetilde{C}_{ij}\right)^{\mathrm{T}} & 0 \\ * & * & -\gamma^2 I & \widetilde{D}_{ij}^{\mathrm{T}} & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -P \end{bmatrix} < 0. \tag{11}$$

Proof. Rewrite the matrix inequality (11) in the following form:,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \left[\Phi + \widetilde{M}^{\mathrm{T}} V^{\mathrm{T}} \widetilde{N} + \widetilde{N}^{\mathrm{T}} V \widetilde{M}\right] < 0, \tag{12}$$

where

$$\Phi = \begin{bmatrix} 0 & P & 0 & 0 & 0 \\ * & -P & 0 & \left(\widetilde{C}_{ij} + \Delta\widetilde{C}_{ij}\right)^{\mathrm{T}} & 0 \\ * & * & -\gamma^2 I & \widetilde{D}_{ij}^{\mathrm{T}} & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -P \end{bmatrix},$$

$$\widetilde{M} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\widetilde{N} = \begin{bmatrix} -I & \widetilde{A}_{ij} + \Delta\widetilde{A}_{ij} & \widetilde{B}_{ij} & 0 & I \end{bmatrix}. \tag{13}$$

□

According to the Projection Theorem [25], inequality (12) is equivalent to the following inequality; that is,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \begin{bmatrix} P\left(\widetilde{A}_{ij} + \Delta\widetilde{A}_{ij}\right) + \left(\widetilde{A}_{ij} + \Delta\widetilde{A}_{ij}\right)^{\mathrm{T}} P - P & P\widetilde{B}_{ij} & \left(\widetilde{C}_{ij} + \Delta\widetilde{C}_{ij}\right)^{\mathrm{T}} & P \\ \widetilde{B}_{ij}^{\mathrm{T}} P & -\gamma^2 I & \widetilde{D}_{ij}^{\mathrm{T}} & 0 \\ \widetilde{C}_{ij} + \Delta\widetilde{C}_{ij} & \widetilde{D}_{ij} & -I & 0 \\ P & 0 & 0 & -P \end{bmatrix} < 0. \tag{14}$$

Applying the Schur complement, it is easy to know that inequality (10) can be deduced from the above inequality. That is to say, inequality (11) is a sufficient condition of the establishment of inequality (10), which can guarantee the filtering error system (8) is robustly asymptotically stable and satisfies the prescribed H_∞ performance level.

Take into account that inequality (11) is a nonlinear matrix inequality on the matrix variables (P, A_{fi}, B_{fi}, C_{fi}, D_{fi}, $i = 1, 2, \ldots, r$), so it is very difficult to solve these variables directly. In this end, the variable substitution method will be utilized in the following derivation to transform inequality (11) into the form of linear matrix inequalities. Then the parameters of robust H_∞ filter can be easily achieved by applying the MATLAB LMI toolbox.

Lemma 2 (see [29]). *Given matrices Y, H, and E of appropriate dimensions, where Y is symmetric, then the inequality $Y + HFE + E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}} < 0$ holds for all F satisfying $F^{\mathrm{T}}F \leq I$, if and only if there exists a constant $\varepsilon > 0$ such that the equality $Y + \varepsilon HH^{\mathrm{T}} + \varepsilon^{-1}E^{\mathrm{T}}E < 0$ holds.*

Let matrices V, V^{-1}, and P be partitioned as follows:

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \qquad V^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$
$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}, \tag{15}$$

where $V_{11}, W_{11}, P_{11} \in \mathfrak{R}^{n\times n}$.

Then introduce the following nonsingular matrices:

$$\Pi_1 = \begin{bmatrix} V_{11} & I \\ V_{21} & 0 \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} I & W_{11} \\ 0 & W_{21} \end{bmatrix}. \tag{16}$$

Obviously, the equation $V\Pi_2 = \Pi_1$ holds.

Denote $T_1 = \mathrm{diag}\{\Pi_2, \Pi_2, I, I, \Pi_2\}$, $\overline{P} = \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{12}^{\mathrm{T}} & \overline{P}_{22} \end{bmatrix} = \Pi_2^{\mathrm{T}} P \Pi_2$. Let inequality (11) be pre- and postmultiplied by T_1^{T} and T_1, respectively, and substitute the expression of the matrix variables $\widetilde{A}_{ij}$, $\widetilde{B}_{ij}$, $\widetilde{C}_{ij}$, and $\widetilde{D}_{ij}$. The following matrix inequality can be obtained:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & 0 & V_{11}^{\mathrm{T}} & \Xi_{18} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & B_j & 0 & I & W_{11} \\ * & * & -\overline{P}_{11} & -\overline{P}_{12} & 0 & \Xi_{36} & 0 & 0 \\ * & * & * & -\overline{P}_{22} & 0 & \Xi_{46} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & -D_j^{\mathrm{T}} D_{fi}^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & -\overline{P}_{11} & -\overline{P}_{12} \\ * & * & * & * & * & * & * & -\overline{P}_{22} \end{bmatrix} < 0, \tag{17}$$

where

$$\Xi_{11} = -V_{11} - V_{11}^{\mathrm{T}}, \quad \Xi_{12} = -I - V_{11}^{\mathrm{T}} W_{11} - V_{21}^{\mathrm{T}} W_{21},$$
$$\Xi_{13} = \overline{P}_{11} + V_{11}^{\mathrm{T}}\left(A_j + \Delta A_j\right) + V_{21}^{\mathrm{T}} B_{fi}\left(C_j + \Delta C_j\right),$$
$$\Xi_{14} = \overline{P}_{12} + V_{11}^{\mathrm{T}}\left(A_j + \Delta A_j\right) W_{11} + V_{21}^{\mathrm{T}} B_{fi}\left(C_j + \Delta C_j\right) W_{11} + V_{21}^{\mathrm{T}} A_{fi} W_{21},$$
$$\Xi_{15} = V_{11}^{\mathrm{T}} B_j + V_{21}^{\mathrm{T}} B_{fi} D_j,$$
$$\Xi_{18} = V_{11}^{\mathrm{T}} W_{11} + V_{21}^{\mathrm{T}} W_{21}, \quad \Xi_{22} = -W_{11} - W_{11}^{\mathrm{T}},$$
$$\Xi_{23} = \overline{P}_{12}^{\mathrm{T}} + A_j + \Delta A_j, \quad \Xi_{24} = \overline{P}_{22} + \left(A_j + \Delta A_j\right) W_{11},$$
$$\Xi_{36} = L_j^{\mathrm{T}} - \left(C_j + \Delta C_j\right)^{\mathrm{T}} D_{fi}^{\mathrm{T}},$$
$$\Xi_{46} = W_{11}^{\mathrm{T}} L_j^{\mathrm{T}} - W_{11}^{\mathrm{T}}\left(C_j + \Delta C_j\right)^{\mathrm{T}} D_{fi}^{\mathrm{T}} - W_{21}^{\mathrm{T}} C_{fi}^{\mathrm{T}}. \tag{18}$$

Moreover, denote $T_2 = \mathrm{diag}\{I, W_{11}^{-1}, I, W_{11}^{-1}, I, I, I, W_{11}^{-1}\}$. Similarly, multiply inequality (17) by T_2^{T} on the left and by T_2 on the right. At the same time, let

$$\widetilde{P} = \begin{bmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{12}^{\mathrm{T}} & \widetilde{P}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & W_{11}^{-1} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{12}^{\mathrm{T}} & \overline{P}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_{11}^{-1} \end{bmatrix},$$
$$Q = W_{11}^{-1}, \qquad R = V_{21}^{\mathrm{T}} W_{21} Q,$$
$$X_i = V_{21}^{\mathrm{T}} A_{fi} W_{21} Q,$$
$$Y_i = V_{21}^{\mathrm{T}} B_{fi}, \qquad Z_i = C_{fi} W_{21} Q. \tag{19}$$

Then inequality (17) can be equivalent to the following form:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \begin{bmatrix} \Xi_{11} & \widetilde{\Xi}_{12} & \widetilde{\Xi}_{13} & \widetilde{\Xi}_{14} & \widetilde{\Xi}_{15} & 0 & V_{11}^{\mathrm{T}} & V_{11}^{\mathrm{T}} + R \\ * & \widetilde{\Xi}_{22} & \widetilde{\Xi}_{23} & \widetilde{\Xi}_{24} & Q^{\mathrm{T}} B_j & 0 & Q^{\mathrm{T}} & Q^{\mathrm{T}} \\ * & * & -\widetilde{P}_{11} & -\widetilde{P}_{12} & 0 & \Xi_{36} & 0 & 0 \\ * & * & * & -\widetilde{P}_{22} & 0 & \widetilde{\Xi}_{46} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & -D_j^{\mathrm{T}} D_{fi}^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & -\widetilde{P}_{11} & -\widetilde{P}_{12} \\ * & * & * & * & * & * & * & -\widetilde{P}_{22} \end{bmatrix} < 0, \tag{20}$$

where

$$\begin{aligned}
\widetilde{\Xi}_{12} &= -Q - V_{11}^{\mathrm{T}} - R, \\
\widetilde{\Xi}_{13} &= \widetilde{P}_{11} + V_{11}^{\mathrm{T}} \left(A_j + \Delta A_j\right) + Y_i \left(C_j + \Delta C_j\right), \\
\widetilde{\Xi}_{14} &= \widetilde{P}_{12} + V_{11}^{\mathrm{T}} \left(A_j + \Delta A_j\right) + Y_i \left(C_j + \Delta C_j\right) + X_i, \\
\widetilde{\Xi}_{15} &= V_{11}^{\mathrm{T}} B_j + Y_i D_j, \qquad \widetilde{\Xi}_{22} = -Q - Q^{\mathrm{T}} \\
\widetilde{\Xi}_{23} &= \widetilde{P}_{12}^{\mathrm{T}} + Q^{\mathrm{T}} \left(A_j + \Delta A_j\right), \\
\widetilde{\Xi}_{24} &= \widetilde{P}_{22} + Q^{\mathrm{T}} \left(A_j + \Delta A_j\right), \\
\widetilde{\Xi}_{46} &= L_j^{\mathrm{T}} - \left(C_j + \Delta C_j\right)^{\mathrm{T}} D_{fi}^{\mathrm{T}} - Z_i^{\mathrm{T}}.
\end{aligned} \tag{21}$$

In the following, by substituting expression (2) of the uncertain matrices $\Delta A_i(t)$ and $\Delta C_i(t)$ into the matrix inequality (20), it can be obtained that

$$\begin{aligned}
\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \big[& \widehat{\Xi}_{ij} + E_{1j} F(t) H_{1j} + H_{1j}^{\mathrm{T}} F^{\mathrm{T}}(t) E_{1j}^{\mathrm{T}} \\
& + E_{2ij} F(t) H_{2j} + H_{2j}^{\mathrm{T}} F^{\mathrm{T}}(t) E_{2ij}^{\mathrm{T}} \big] < 0,
\end{aligned} \tag{22}$$

where

$$\widehat{\Xi}_{ij} = \begin{bmatrix} \Xi_{11} & \widetilde{\Xi}_{12} & \widehat{\Xi}_{13} & \widehat{\Xi}_{14} & \widetilde{\Xi}_{15} & 0 & V_{11}^{\mathrm{T}} & V_{11}^{\mathrm{T}} + R \\ * & \widetilde{\Xi}_{22} & \widehat{\Xi}_{23} & \widehat{\Xi}_{24} & Q^{\mathrm{T}} B_j & 0 & Q^{\mathrm{T}} & Q^{\mathrm{T}} \\ * & * & -\widetilde{P}_{11} & -\widetilde{P}_{12} & 0 & \Xi_{36} & 0 & 0 \\ * & * & * & -\widetilde{P}_{22} & 0 & \widehat{\Xi}_{46} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & -D_j^{\mathrm{T}} D_{fi}^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & -\widetilde{P}_{11} & -\widetilde{P}_{12} \\ * & * & * & * & * & * & * & -\widetilde{P}_{22} \end{bmatrix},$$

$$\begin{aligned}
\widehat{\Xi}_{13} &= \widetilde{P}_{11} + V_{11}^{\mathrm{T}} A_j + Y_i C_j, \\
\widehat{\Xi}_{23} &= \widetilde{P}_{12}^{\mathrm{T}} + Q^{\mathrm{T}} A_j, \qquad \widehat{\Xi}_{14} = \widetilde{P}_{12} + V_{11}^{\mathrm{T}} A_j + Y_i C_j + X_i, \\
\widehat{\Xi}_{24} &= \widetilde{P}_{22} + Q^{\mathrm{T}} A_j, \qquad \widehat{\Xi}_{46} = L_j^{\mathrm{T}} - C_j^{\mathrm{T}} D_{fi}^{\mathrm{T}} - Z_i^{\mathrm{T}}, \\
E_{1j}^{\mathrm{T}} &= \left[E_{aj}^{\mathrm{T}} V_{11} \;\; E_{aj}^{\mathrm{T}} Q \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right], \\
H_{1j} &= \left[0 \;\; 0 \;\; H_{aj} \;\; H_{aj} \;\; 0 \;\; 0 \;\; 0 \;\; 0\right], \\
E_{2ij}^{\mathrm{T}} &= \left[E_{cj}^{\mathrm{T}} Y_i^{\mathrm{T}} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -E_{cj}^{\mathrm{T}} D_{fi}^{\mathrm{T}} \;\; 0 \;\; 0\right], \\
H_{2j} &= \left[0 \;\; 0 \;\; H_{cj} \;\; H_{cj} \;\; 0 \;\; 0 \;\; 0 \;\; 0\right].
\end{aligned} \tag{23}$$

According to Lemma 2, the matrix inequality (22) holds for all admissible uncertainty matrices $F(t)$ satisfying condition (3), if and only if there exist constants $\varepsilon_{1ij} > 0$ and

$\varepsilon_{2ij} > 0$, $i, j = 1, 2, \ldots, r$, such that the following matrix inequality holds:

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \left[\widehat{\Xi}_{ij} + \varepsilon_{1ij} H_{1j}^{\mathrm{T}} H_{1j} + \varepsilon_{1ij}^{-1} E_{1j} E_{1j}^{\mathrm{T}} \right. \\
&\qquad \left. + \varepsilon_{2ij} H_{2j}^{\mathrm{T}} H_{2j} + \varepsilon_{2ij}^{-1} E_{2ij} E_{2ij}^{\mathrm{T}}\right] \\
&= \sum_{i=1}^{r} h_i^2 \left[\widehat{\Xi}_{ii} + \varepsilon_{1ii} H_{1i}^{\mathrm{T}} H_{1i} + \varepsilon_{1ii}^{-1} E_{1i} E_{1i}^{\mathrm{T}} \right. \\
&\qquad \left. + \varepsilon_{2ii} H_{2i}^{\mathrm{T}} H_{2i} + \varepsilon_{2ii}^{-1} E_{2ii} E_{2ii}^{\mathrm{T}}\right] \\
&\quad + \sum_{i=1}^{r}\sum_{i<j}^{r} h_i h_j \left[\widehat{\Xi}_{ij} + \varepsilon_{1ij} H_{1j}^{\mathrm{T}} H_{1j} + \varepsilon_{1ij}^{-1} E_{1j} E_{1j}^{\mathrm{T}} \right. \\
&\qquad \left. + \varepsilon_{2ij} H_{2j}^{\mathrm{T}} H_{2j} + \varepsilon_{2ij}^{-1} E_{2ij} E_{2ij}^{\mathrm{T}} + \widehat{\Xi}_{ji} \right. \\
&\qquad + \varepsilon_{1ji} H_{1i}^{\mathrm{T}} H_{1i} + \varepsilon_{1ji}^{-1} E_{1i} E_{1i}^{\mathrm{T}} \\
&\qquad \left. + \varepsilon_{2ji} H_{2i}^{\mathrm{T}} H_{2i} + \varepsilon_{2ji}^{-1} E_{2ji} E_{2ji}^{\mathrm{T}}\right] < 0.
\end{aligned} \tag{24}$$

Applying the Schur complement lemma to the above matrix inequality, the following conclusion can be reached from the above deduction.

Theorem 3. *For a given constant $\gamma > 0$, the filtering error system* (8) *is robustly asymptotically stable and satisfies $\|H_{\tilde{z}w}(s)\|_\infty < \gamma$, if there exist constant $\varepsilon_{1ij} > 0$, $\varepsilon_{2ij} > 0$, symmetric positive definite matrix $\widetilde{P}_{11}$, $\widetilde{P}_{22}$, and matrices $\widetilde{P}_{12}$, V_{11}, Q, R, X_i, Y_i, Z_i, D_{fi}, $i, j = 1, 2, \ldots, r$, such that for all admissible uncertainties* (3) *the following linear matrix inequalities hold:*

$$\begin{gathered}
\begin{bmatrix} \widehat{\Xi}_{ii} + \varepsilon_{1ii} H_{1i}^{\mathrm{T}} H_{1i} + \varepsilon_{2ii} H_{2i}^{\mathrm{T}} H_{2i} & E_{1i} & E_{2ii} \\ E_{1i}^{\mathrm{T}} & -\varepsilon_{1ii} I & 0 \\ E_{2ii}^{\mathrm{T}} & 0 & -\varepsilon_{2ii} I \end{bmatrix} < 0, \quad i = 1, 2, \ldots, r, \\
\begin{bmatrix} \begin{pmatrix} \widehat{\Xi}_{ij} + \widehat{\Xi}_{ji} + \varepsilon_{1ij} H_{1j}^{\mathrm{T}} H_{1j} + \varepsilon_{1ji} H_{1i}^{\mathrm{T}} H_{1i} \\ + \varepsilon_{2ij} H_{2j}^{\mathrm{T}} H_{2j} + \varepsilon_{2ji} H_{2i}^{\mathrm{T}} H_{2i} \end{pmatrix} & E_{1j} & E_{2ij} & E_{1i} & E_{2ji} \\ E_{1j}^{\mathrm{T}} & -\varepsilon_{1ij} I & 0 & 0 & 0 \\ E_{2ij}^{\mathrm{T}} & 0 & -\varepsilon_{2ij} I & 0 & 0 \\ E_{1i}^{\mathrm{T}} & 0 & 0 & -\varepsilon_{1ji} I & 0 \\ E_{2ji}^{\mathrm{T}} & 0 & 0 & 0 & -\varepsilon_{2ji} I \end{bmatrix} < 0, \quad i < j.
\end{gathered} \tag{25}$$

Using the matrix relations of formula (19) and the equivalence of the transfer function, filter parameter matrices are given as follows:

$$\begin{aligned}
&A_{fi} = R^{-1} X_i, \qquad B_{fi} = R^{-1} Y_i, \\
&C_{fi} = Z_i, \qquad D_{fi} = D_{fi}, \\
&\qquad\qquad i = 1, 2, \ldots, r.
\end{aligned} \tag{26}$$

Set $\rho = \gamma^2$, and an optimization problem about robust H_∞ filter can be described in the following:

$$\begin{aligned}
&\min \quad \rho \\
&\text{s.t.} \quad (25).
\end{aligned} \tag{27}$$

Thus, the obtained filter can be called an optimal robust H_∞ filter of the uncertain fuzzy system (4), and the corresponding optimal disturbance attenuation level is $\gamma^* = \sqrt{\rho}$.

4. Numerical Example

In this section, a numerical example will be given to illustrate the effectiveness of robust H_∞ filtering approach developed in the previous section (see Figure 1) [30].

According to the literature [30], Figure 1 can be described by the following state equations:

$$\begin{aligned}
\dot{x}_1(t) &= -0.1 x_1(t) - 0.5 x_1^3(t) + 50 x_2(t), \\
\dot{x}_2(t) &= -x_1(t) - 10 x_2(t) + w(t), \\
y(t) &= x_1(t) + w(t), \\
z(t) &= x_1(t),
\end{aligned} \tag{28}$$

where $x_1(t) = v_C(t)$ is capacitor voltage and $x_2(t) = i_L(t)$ is inductor current.

Assume that the state variable $x_1(t)$ satisfies $|x_1(t)| \le 3$. In order to simplify the calculation, two fuzzy rules will be used to approximate the nonlinear system (28).

Plant Rule 1:

$$\begin{aligned}
&\text{IF } x_1(t) \text{ is } M_1(x_1(t)), \\
&\text{THEN } \dot{x}(t) = (A_1 + \Delta A_1(t)) x(t) + B_1 w(t). \\
&\qquad y(t) = (C_1 + \Delta C_1(t)) x(t) + D_1 w(t), \\
&\qquad z(t) = L_1 x(t).
\end{aligned} \tag{29}$$

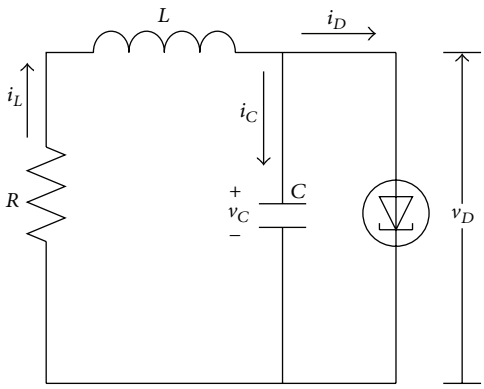

Figure 1: Tunnel diode circuit system.

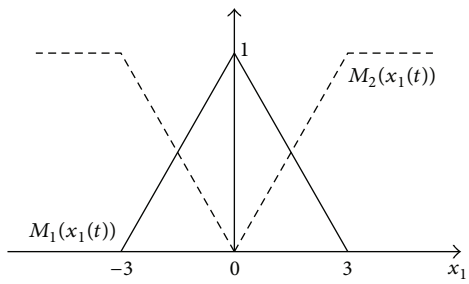

Figure 2: Fuzzy membership functions.

Plant Rule 2:

$$\begin{aligned} &\text{IF } x_1(t) \text{ is } M_2(x_1(t)), \\ &\text{THEN } \dot{x}(t) = (A_2 + \Delta A_2(t))\, x(t) + B_2 w(t) \\ &\qquad y(t) = (C_2 + \Delta C_2(t))\, x(t) + D_2 w(t), \\ &\qquad z(t) = L_2 x(t), \end{aligned} \tag{30}$$

where model parameters are given below:

$$\begin{aligned} &A_1 = \begin{bmatrix} -0.1 & 50 \\ -1 & -10 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -4.6 & 50 \\ -1 & -10 \end{bmatrix} \\ &B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ &D_1 = D_2 = 1, \qquad L_1 = L_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ &E_{a1} = E_{a2} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \qquad H_{a1} = H_{a2} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \\ &E_{c1} = E_{c2} = 1, \qquad H_{c1} = H_{c2} = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}. \end{aligned} \tag{31}$$

And the fuzzy membership functions corresponding to the above two rules are given in Figure 2.

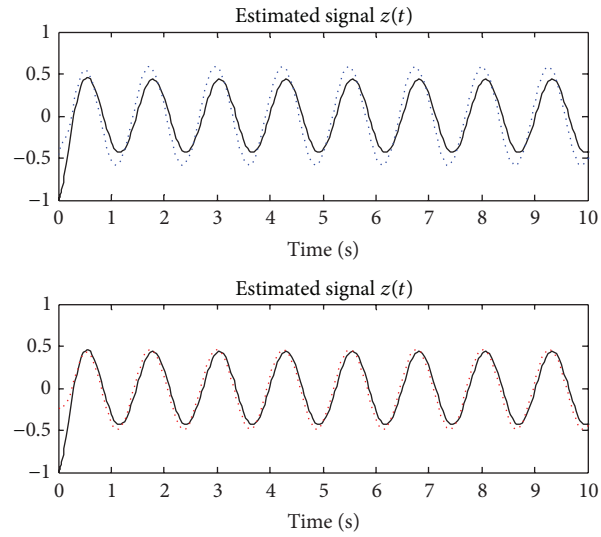

Figure 3: Filtering results for $w(t) = 0.5\sin(5t)$.

By giving the H_∞ performance level $\gamma = 1$ and constructing the fuzzy filter (7), by solving linear matrix inequalities (25), the modified filter parameters can be obtained as follows:

$$\begin{aligned} &A_{f1} = \begin{bmatrix} -1.6780 & 41.9421 \\ -1.8564 & -9.1733 \end{bmatrix}, \\ &A_{f2} = \begin{bmatrix} -5.0817 & 43.5742 \\ -1.8467 & -9.1707 \end{bmatrix}, \\ &B_{f1} = \begin{bmatrix} 1.0785 \\ 0.9305 \end{bmatrix}, \qquad B_{f2} = \begin{bmatrix} -0.1095 \\ 0.9196 \end{bmatrix}, \\ &C_{f1} = \begin{bmatrix} 0.6950 & -0.3013 \end{bmatrix}, \\ &C_{f2} = \begin{bmatrix} 0.7422 & -0.3745 \end{bmatrix}, \\ &D_{f1} = 0.2516, \qquad D_{f2} = 0.2566. \end{aligned} \tag{32}$$

Assume that the initial state of system is $x_0 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$, the initial state of filter is $\hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$, and the uncertain matrix is selected as $F(t) = \sin(t)$. Apply the above-obtained filter to the system (28) for filtering simulation. When the exogenous interference noise is set as $w(t) = 0.5\sin(5t)$, the simulation results are shown in Figure 3, in which the blue dotted line indicates the case without introducing a free matrix variable, while the red dotted line represents the case with introducing a free matrix variable. Similarly, Figure 4 shows the filtering results when the noise is a random noise with zero mean and variance of 0.01. Obviously, from the simulation results, it can be seen that the filtering results with introducing a free matrix variable are better than those of not introducing, and the former makes the system have a higher error estimation accuracy.

Moreover, by solving the optimization problem (27), the minimum disturbance attenuation level is obtained as $\gamma^* = 5.1 \times 10^{-7}$. By comparison, the result without introducing

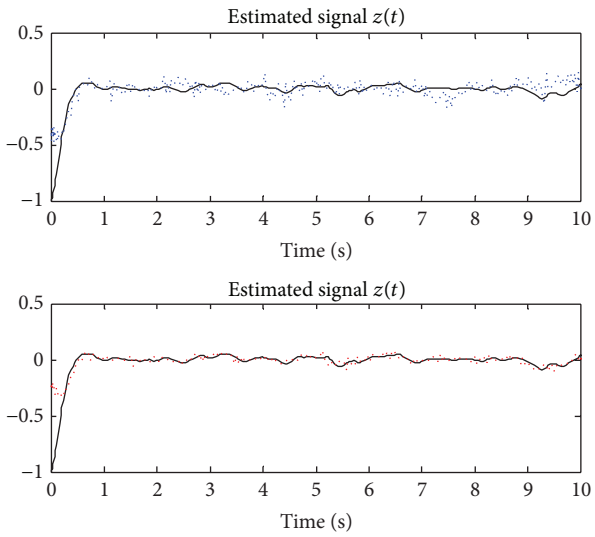

Figure 4: Filtering results for random noise interference.

a free matrix variable is also given as $\gamma^* = 3.3 \times 10^{-6}$. Thus it can be seen that the system can obtain lower disturbance attenuation level by introducing a free matrix variable.

5. Conclusions

This paper successfully extends the ideology of literature [25] to robust H_∞ filter design for a class of uncertain nonlinear systems. By introducing a free matrix variable, this paper gives a new systematic design methodology of robust H_∞ filter. In particular, the filter parameters can be designed independent of the Lyapunov matrix. This method can decouple between the Lyapunov matrix and the system matrix, so it can reduce the conservatism of the system to a certain extent. The solution of filter can be converted into a standard LMI problem. From the simulation results, it can be seen that the improved filter has the lower conservatism and the higher estimation accuracy, which is useful for engineering application.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publishing of this paper.

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Research Article

Output Feedback Control for Couple-Group Consensus of Multiagent Systems

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This paper deals with the couple-group consensus problem for multiagent systems via output feedback control. Both continuous- and discrete-time cases are considered. The consensus problems are converted into the stability problem of the error systems by the system transformation. We obtain two necessary and sufficient conditions of couple-group consensus in different forms for each case. Two different algorithms are used to design the control gains for continuous- and discrete-time case, respectively. Finally, simulation examples are given to show the effectiveness of the proposed results.

1. Introduction

During the past decade, consensus problem of multiagent system has attracted a lot of attentions in control area [1–11]. It is mainly due to its wide applications in practice, such as sensor networks, unmanned aerial vehicles, and robotics. In [2], the authors studied the consensus seeking problem of multiagent systems with dynamically changing interaction topologies, where both discrete and continuous consensus algorithms were considered. In [9], the authors studied the containment problem of linear multiagent systems, where a pinning control strategy was designed for a part of agents such that all the agents can achieve a consensus with the leader asymptotically. A second-order consensus problem for multiagent systems with nonlinear dynamics and directed topologies was studied in [10]. More works about consensus problem were surveyed in [11].

Sometimes the interaction topology does not have a spanning tree, while it contains two or more subgraphs which include a spanning tree, respectively. In this case, some researchers studied the group consensus problem [12–15]. In [12], the authors studied the group consensus problem of multiagent systems with switching topologies. The group consensus was proved to be equivalent to the asymptotical stability of a class of switched linear systems by a double-tree-form transformation. In [13], two different kinds of consensus protocols were given to deal with the group consensus problem for double-integrator dynamic multiagent systems. In [15], the sampled-data control method was employed to deal with the group consensus problem for multiagent systems, where the interaction topology is undirected.

Sometimes the system states are not known completely, while the output of the systems is measurable. The output will be used to design the controller for this case, that is, output feedback controller. Recently, the output feedback control problems have been reported in a lot of literature [16–19]. In [16], the output feedback robust stabilization problem for a class of jump linear system was studied. In [17], the authors studied the finite-time stabilization of continuous-time linear systems via dynamic output feedback. In [18], the Lyapunov-Metzler inequalities were used to study the dynamic output feedback control problem of switched linear systems. Very recently, the method based on output feedback control has been used to analyze the networked systems [20–23]. In [20], the consensusability of a class of linear multiagent systems was studied, where the agent updates its information by using the neighbor's output. In [21], the output regulation theory was used to study the output consensus problems for heterogeneous uncertain linear multiagent systems. In [22], by using appropriate coordinate transformation, a new consensus algorithm via dynamic output feedback control for multiagent systems was studied. While in [23], the joint

effects of agent dynamic and network topology on the consensusability of linear discrete-time multiagent systems via relative output feedback were studied.

Motivated by the aforementioned works, we will investigate the couple-group consensus problems for multiagent systems via output feedback control. The systems considered include both continuous-time case and discrete-time case. We convert the couple-group consensus problems of multiagent systems into the stability problems of the error systems by a system transformation. Based on linear system theory, some necessary and sufficient conditions for couple-group consensus are obtained. For continuous-time case, the algorithm based on homotopy method is given to compute the allowable control gain. For discrete-time case, the algorithm based on cone complementary linearization method is given to compute the allowable control gain.

Notation. Let $\mathbb{R}$ and $\mathbb{N}$ represent, respectively, the real number set and the nonnegative integer set. Denote the spectral radius of the matrix M by $\rho(M)$. Suppose that $A, B \in \mathbb{R}^{p\times p}$. Let $A \succeq B$ (resp., $A \succ B$) denote that $A - B$ is symmetric positive semidefinite (resp., symmetric positive definite). I_n denotes the $n \times n$ identity matrix. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent, respectively, the real part and imaginary part of a number. Let $\mathbf{0}$ denote zero matrix with appropriate dimensions.

2. Preliminaries and Problem Formulation

Graph Theory. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph of order n, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ and $\mathcal{E}$ represent the node set and the edge set, respectively. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix associated with $\mathcal{G}$, where $a_{ij} > 0$ if $(v_i, v_j) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. An edge $(v_i, v_j) \in \mathcal{E}$ if agent j can obtain the information from agent i. We say agent i is a neighbor of agent j. Let $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$ denote the neighbor set of agent i. The (nonsymmetrical) Laplacian matrix $\mathcal{L}$ associated with $\mathcal{A}$ and hence $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$, where $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, for all $i \neq j$. A directed path is a sequence of edges in a directed graph in the form of $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots$, where $v_{i_k} \in \mathcal{V}$. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A directed spanning tree of $\mathcal{G}$ is a directed tree that contains all nodes of $\mathcal{G}$. A directed graph has or contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph; that is, there exists at least one node having a directed path to all of the other nodes.

Suppose that the multiagent systems considered consist of $n+m$ agents. In this paper, we will consider both continuous-time case and discrete-time case. We assume that the first n agents achieve a consistent state while the last m agents achieve another consistent state. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote the topology of multiagent system considered. Denote $\mathcal{I}_1 = \{1, 2, \ldots, n\}$, $\mathcal{I}_2 = \{n+1, n+2, \ldots, n+m\}$. Let $\mathcal{V}_1 = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{V}_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\}$ represent the first n agents and the last m agents, respectively. Then, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \Phi$. In addition, let $N_{1i} = \{v_j \in \mathcal{V}_1 : (v_i, v_j) \in \mathcal{E}\}$ and $N_{2i} = \{v_j \in \mathcal{V}_2 : (v_i, v_j) \in \mathcal{E}\}$.

For continuous-time case, the ith agent's dynamics are as follows:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \\ i &= 1, \ldots, n+m, \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state, $u_i(t) \in \mathbb{R}$ is the control input, and $y_i(t) \in \mathbb{R}$ is the output. $A, B, C \in \mathbb{R}$ are the system coefficients.

For discrete-time case, the ith agent's dynamics are as follows:

$$\begin{aligned} x_i[k+1] &= Ax_i[k] + Bu_i[k], \\ y_i[k] &= Cx_i[k], \\ i &= 1, \ldots, n+m, \end{aligned} \tag{2}$$

where $x_i[k] \in \mathbb{R}$ is the state, $u_i[k] \in \mathbb{R}$ is the control input, and $y_i[k] \in \mathbb{R}$ is the output. $A, B, C \in \mathbb{R}$ are the system coefficients.

Sometimes the agent's state is difficult to obtain, while the output is measurable. Our main purpose in this paper is to design consensus algorithm based on the output such that the multiagent systems can achieve couple-group consensus. We consider the following consensus algorithms for continuous-time case and discrete-time case, respectively.

Continuous-Time Case. Consider

$$u_i(t) = \begin{cases} \alpha\left[\sum_{j\in N_{1i}} a_{ij}\left(y_j(t) - y_i(t)\right) + \sum_{j\in N_{2i}} a_{ij} y_j(t)\right] & \forall i \in \mathcal{I}_1, \\ \alpha\left[\sum_{j\in N_{1i}} a_{ij} y_j(t) + \sum_{j\in N_{2i}} a_{ij}\left(y_j(t) - y_i(t)\right)\right] & \forall i \in \mathcal{I}_2, \end{cases} \tag{3}$$

where $a_{ij} \geq 0$ for all $i, j \in \mathcal{I}_1$, $a_{ij} \geq 0$ for all $i, j \in \mathcal{I}_2$, and $a_{ij} \in \mathbb{R}$ for all $(v_i, v_j) \in \mathcal{E}_o = \{(i, j) : i \in \mathcal{I}_1, j \in \mathcal{I}_2\} \cup \{(i, j) : j \in \mathcal{I}_1, i \in \mathcal{I}_2\}$. α is the control gain to be designed.

Discrete-Time Case. Consider

$$u_i[k] = \begin{cases} \gamma\left[\sum_{j\in N_{1i}} a_{ij}\left(y_j[k] - y_i[k]\right) + \sum_{j\in N_{2i}} a_{ij} y_j[k]\right] & \forall i \in \mathcal{I}_1, \\ \gamma\left[\sum_{j\in N_{1i}} a_{ij} y_j[k] + \sum_{j\in N_{2i}} a_{ij}\left(y_j[k] - y_i[k]\right)\right] & \forall i \in \mathcal{I}_2, \end{cases} \tag{4}$$

where $a_{ij} \geq 0$ for all $i, j \in \mathcal{I}_1$, $a_{ij} \geq 0$ for all $i, j \in \mathcal{I}_2$, and $a_{ij} \in \mathbb{R}$ for all $(v_i, v_j) \in \mathcal{E}_o = \{(i, j) : i \in \mathcal{I}_1, j \in \mathcal{I}_2\} \cup \{(i, j) : j \in \mathcal{I}_1, i \in \mathcal{I}_2\}$. γ is the control gain to be designed. In addition, we suppose the algorithms in (3) and (4) satisfy similar assumption to that of [24].

Assumption 1. (1): $\sum_{j=n+1}^{n+m} a_{ij} = 0$ for all $i \in \mathcal{I}_1$; (2): $\sum_{j=1}^{n} a_{ij} = 0$ for all $i \in \mathcal{I}_2$.

Assumption 2. The subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have a directed spanning tree, respectively.

Denote $X(t) \triangleq [x_1(t), \ldots, x_{n+m}(t)]^T$. Using (3) in (1) yields

$$\dot{X}(t) = \begin{bmatrix} AI_n - \alpha BC\mathcal{L}_1 & \alpha BC\Omega_1 \\ \alpha BC\Omega_2 & AI_m - \alpha BC\mathcal{L}_2 \end{bmatrix} X(t), \quad (5)$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Laplacian matrices corresponding to subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, and

$$\Omega_1 = \begin{bmatrix} a_{1(n+1)} & a_{1(n+2)} & \cdots & a_{1(n+m)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n(n+1)} & a_{n(n+2)} & \cdots & a_{n(n+m)} \end{bmatrix},$$
$$\Omega_2 = \begin{bmatrix} a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{(n+m)1} & a_{(n+m)2} & \cdots & a_{(n+m)n} \end{bmatrix}. \quad (6)$$

Denote $X[k] \triangleq [x_1[k], \ldots, x_{n+m}[k]]^T$. Using (4) in (2) yields

$$X[k+1] = \begin{bmatrix} AI_n - \alpha BC\mathcal{L}_1 & \alpha BC\Omega_1 \\ \alpha BC\Omega_2 & AI_m - \alpha BC\mathcal{L}_2 \end{bmatrix} X[k], \quad (7)$$

where $\mathcal{L}_1, \mathcal{L}_2, \Omega_1$, and Ω_2 are the same as that of continuous-time case.

Remark 3. The group consensus problem of continuous time multiagent systems was studied in [12, 13, 15]. In [14], the authors studied the group consensus problem for discrete-time multiagent systems. However, the couple-group consensus problem for the multiagent systems with stochastic switching topologies has not been researched. In addition, our method in this paper is based on the output feedback control, which is different from the existing results.

Our main purpose is to give the conditions for couple-group consensus. We next convert the consensus problem of multiagent system into the stability problem of the error systems. Before giving the main results, the following definitions and lemma are needed.

Definition 4 (see [24]). The multiagent system in (5) is said to achieve couple-group consensus if the states of agents satisfy (i) $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, for all $i, j \in \mathcal{I}_1$ and (ii) $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, for all $i, j \in \mathcal{I}_2$.

Definition 5 (see [24]). The multiagent system in (7) is said to achieve couple-group consensus if the states of agents satisfy (i) $\lim_{k\to\infty}\|x_i[k] - x_j[k]\| = 0$, for all $i, j \in \mathcal{I}_1$ and (ii) $\lim_{k\to\infty}\|x_i[k] - x_j[k]\| = 0$, for all $i, j \in \mathcal{I}_2$.

Lemma 6 (see [25] Schur complements). *Consider a hermitian matrix Q such that $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$. Then, $Q \succ 0$ if and only if*

$$Q_{22} \succ 0,$$
$$Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T \succ 0 \quad (8)$$

or

$$Q_{11} \succ 0,$$
$$Q_{22} - Q_{12}^T Q_{11}^{-1} Q_{12} \succ 0. \quad (9)$$

3. Main Results

In this section, we will give the main results of this paper.

3.1. Continuous-Time Case. Let

$$z_i(t) \triangleq x_i(t) - x_n(t), \quad i = 1, \ldots, n-1,$$
$$z_j(t) \triangleq x_j(t) - x_{n+m}(t), \quad j = n+1, \ldots, n+m-1, \quad (10)$$
$$Z(t) \triangleq [z_1(t), \ldots, z_{n-1}(t), z_{n+1}(t), \ldots, z_{n+m-1}(t)]^T.$$

Then by some computations, we obtain the error systems as follows:

$$\begin{aligned} \dot{Z}(t) &= \begin{bmatrix} AI_{n-1} - \alpha BC\widetilde{\mathcal{L}}_1 & \alpha BC\widetilde{\Omega}_1 \\ \alpha BC\widetilde{\Omega}_2 & AI_{m-1} - \alpha BC\widetilde{\mathcal{L}}_2 \end{bmatrix} Z(t) \\ &\triangleq F_c Z(t) \\ &= \left(AI_{n+m-2} + \alpha BC\widetilde{F}_c\right) Z(t), \end{aligned} \quad (11)$$

where

$$\widetilde{F}_c = \begin{bmatrix} -\widetilde{\mathcal{L}}_1 & \widetilde{\Omega}_1 \\ \widetilde{\Omega}_2 & -\widetilde{\mathcal{L}}_2 \end{bmatrix},$$
$$\widetilde{\mathcal{L}}_1 = \begin{bmatrix} l_{11} - l_{n1} & \cdots & l_{1(n-1)} - l_{n(n-1)} \\ \vdots & \cdots & \vdots \\ l_{(n-1)1} - l_{n1} & \cdots & l_{(n-1)(n-1)} - l_{n(n-1)} \end{bmatrix},$$
$$\widetilde{\mathcal{L}}_2 = \begin{bmatrix} l_{(n+1)1} - l_{(n+m)1} & \cdots & l_{(n+1)(n+m-1)} - l_{(n+m)(n+m-1)} \\ \vdots & \cdots & \vdots \\ l_{(n+m-1)1} - l_{(n+m)1} & \cdots & l_{(n+m-1)(n+m-1)} - l_{(n+m)(n+m-1)} \end{bmatrix},$$
$$\widetilde{\Omega}_1 = \begin{bmatrix} a_{1(n+1)} - a_{n(n+m)} & \cdots & a_{1(n+m-1)} - a_{n(n+m-1)} \\ \vdots & \cdots & \vdots \\ a_{(n-1)(n+1)} - a_{n(n+m)} & \cdots & a_{(n-1)(n+m-1)} - a_{n(n+m-1)} \end{bmatrix},$$
$$\widetilde{\Omega}_2 = \begin{bmatrix} a_{(n+1)1} - a_{(n+m)1} & \cdots & a_{(n+1)(n-1)} - a_{(n+m)(n-1)} \\ \vdots & \cdots & \vdots \\ a_{(n+m-1)1} - a_{(n+m)1} & \cdots & a_{(n+m-1)(n-1)} - a_{(n+m)(n-1)} \end{bmatrix}. \quad (12)$$

Here we have used Assumption 1 and the property of Laplacian matrix.

Now the couple-group consensus problem of (5) has been converted into the stability problem of error system (11). We next give our main results.

Theorem 7. *The multiagent system* (5) *can achieve couple-group consensus asymptotically if and only if α satisfies $A + \alpha BC\,\mathrm{Re}(\mu_i) < 0$, where μ_i $(i = 1, \ldots, n+m-2)$ is the ith eigenvalue of $\widetilde{F}_c$.*

Proof. According to the aforementioned discussion, we know that the multiagent systems (5) can achieve couple-group consensus asymptotically if and only if the error system (11) is asymptotically stable. It follows from linear system theory [26] that system (11) which is asymptotically stable is equivalent to all eigenvalues of F_c having negative real parts. Denote the ith eigenvalues of F_c and $\widetilde{F}_c$, respectively, by η_i and μ_i $(i = 1, \ldots, n+m-2)$. Then, $\eta_i = A + \alpha BC\mu_i$. $\mathrm{Re}(\eta_i) < 0$ is equivalent to $A + \alpha BC\,\mathrm{Re}(\mu_i) < 0$. This completes the proof. □

Remark 8. Theorem 7 provides a necessary and sufficient condition of couple-group consensus for multiagent system (5). According to linear system theory, we know that system (11) which is asymptotically stable is equivalent to that in which there exists a positive matrix P such that $PF_c + F_c^T P \prec 0$. Hence, we can get another condition of couple-group consensus for multiagent systems (5).

Theorem 9. *The multiagent system* (5) *can achieve couple-group consensus asymptotically if and only if there exists a positive definite matrix P such that*

$$2AP + \alpha BC\left(P\widetilde{F}_c + \widetilde{F}_c^T P\right) \prec 0 \tag{13}$$

holds.

Proof. The proof is straightforward; here is omitted. □

Remark 10. Theorem 9 gives a necessary and sufficient condition for couple-group consensus in forms of matrix inequality. However, the matrix inequality in (13) is nonlinear with regard to variables α and P. Here we provide a numerical algorithm based on homotopy method to solve this problem. The similar method can be found in [22, 27, 28].

Algorithm 11. Consider the following

Step 1. Introduce a real number λ varying from 0 to 1, and construct a matrix function

$$H(\alpha, P, \lambda) = (1-\lambda) F_1(P) + \lambda F_2(\alpha, P) \tag{14}$$

with $F_1(P) = 2AP$, $F_2(P) = \alpha BC(P\widetilde{F}_c + \widetilde{F}_c^T P)$.

Step 2 (Set $\lambda = 0$). Compute the initial value of P_0 by solving the LMI $H(\alpha, P, 0) \prec \mathbf{0}$.

Step 3. Increase λ by some homotopy path, such as $\lambda = (k/M)$ $(k = 1, 2, \ldots, M)$. M is a large positive integer, for example, $M = 1000$. Compute α^0 by solving LMI $H(\alpha, P_0, 1/M) \prec \mathbf{0}$.

Step 4. Increase λ by the same homotopy path as Step 3. Compute P_1 by solving LMI $H(\alpha^0, P, 2/M) \prec \mathbf{0}$.

Step 5. Repeat Steps 3 and 4 until λ reaches Step 1.

3.2. Discrete-Time Case. Similar to continuous-time case, we can get the similar results for discrete-time case.

Let

$$\begin{aligned} z_i[k] &\triangleq x_i[k] - x_n[k], \quad i = 1, \ldots, n-1, \\ z_j[k] &\triangleq x_j[k] - x_{n+m}[k], \quad j = n+1, \ldots, n+m-1, \\ Z[k] &\triangleq \left[z_1[k], \ldots, z_{n-1}[k], z_{n+1}[k], \ldots, z_{n+m-1}[k]\right]^T. \end{aligned} \tag{15}$$

Then by some computations, we obtain the error systems as follows:

$$\begin{aligned} &Z[k+1] \\ &\quad = \begin{bmatrix} AI_{n-1} - \alpha BC\widetilde{\mathscr{L}}_1 & \alpha BC\widetilde{\Omega}_1 \\ \alpha BC\widetilde{\Omega}_2 & AI_{m-1} - \alpha BC\widetilde{\mathscr{L}}_2 \end{bmatrix} Z[k] \\ &\quad \triangleq F_d Z[k] \\ &\quad = \left(AI_{n+m-2} + \alpha BC\widetilde{F}_d\right) Z[k], \end{aligned} \tag{16}$$

where

$$\begin{aligned} \widetilde{F}_d &= \begin{bmatrix} -\widetilde{\mathscr{L}}_1 & \widetilde{\Omega}_1 \\ \widetilde{\Omega}_2 & -\widetilde{\mathscr{L}}_2 \end{bmatrix}, \\ \widetilde{\mathscr{L}}_1 &= \begin{bmatrix} l_{11} - l_{n1} & \cdots & l_{1(n-1)} - l_{n(n-1)} \\ \vdots & \cdots & \vdots \\ l_{(n-1)1} - l_{n1} & \cdots & l_{(n-1)(n-1)} - l_{n(n-1)} \end{bmatrix}, \\ \widetilde{\mathscr{L}}_2 &= \begin{bmatrix} l_{(n+1)1} - l_{(n+m)1} & \cdots & l_{(n+1)(n+m-1)} - l_{(n+m)(n+m-1)} \\ \vdots & \cdots & \vdots \\ l_{(n+m-1)1} - l_{(n+m)1} & \cdots & l_{(n+m-1)(n+m-1)} - l_{(n+m)(n+m-1)} \end{bmatrix}, \\ \widetilde{\Omega}_1 &= \begin{bmatrix} a_{1(n+1)} - a_{n(n+m)} & \cdots & a_{1(n+m-1)} - a_{n(n+m-1)} \\ \vdots & \cdots & \vdots \\ a_{(n-1)(n+1)} - a_{n(n+m)} & \cdots & a_{(n-1)(n+m-1)} - a_{n(n+m-1)} \end{bmatrix}, \\ \widetilde{\Omega}_2 &= \begin{bmatrix} a_{(n+1)1} - a_{(n+m)1} & \cdots & a_{(n+1)(n-1)} - a_{(n+m)(n-1)} \\ \vdots & \cdots & \vdots \\ a_{(n+m-1)1} - a_{(n+m)1} & \cdots & a_{(n+m-1)(n-1)} - a_{(n+m)(n-1)} \end{bmatrix}. \end{aligned} \tag{17}$$

Here we have used Assumption 1 and the property of Laplacian matrix.

Now the couple-group consensus problem of (7) has been converted into the stability problem of error system (16). We next give our main results.

Theorem 12. *The multiagent system* (7) *can achieve couple-group consensus asymptotically if and only if γ and (A, B, C) satisfy*

$$\begin{aligned} &-\frac{A}{BC}\cos(\mu_i) - \frac{1}{B^2C^2|\mu_i|^2} \\ &\quad \times \sqrt{B^2C^2\left[A^2\,\mathrm{Re}^2(\mu_i) + (1-A^2)|\mu_i|^2\right]} \\ &\quad < \alpha \end{aligned}$$

$$< -\frac{A}{BC}\cos(\mu_i) + \frac{1}{B^2C^2|\mu_i|^2} \times \sqrt{B^2C^2\left[A^2\operatorname{Re}^2(\mu_i) + (1-A^2)|\mu_i|^2\right]},$$
$$A^2\left(\operatorname{Re}^2(\mu_i) - |\mu_i|^2\right) + |\mu_i|^2 > 0, \tag{18}$$

where μ_i $(i = 1, \ldots, n+m-2)$ is the ith eigenvalue of $\widetilde{F}_c$.

Proof. According to the aforementioned discussion, we know that the multiagent systems (7) can achieve couple-group consensus asymptotically if and only if the error system (16) is asymptotically stable. It follows from linear system theory [26] that system (16) which is asymptotically stable is equivalent to all eigenvalues of F_c being within the unit circle. Denote the ith eigenvalues of $\widetilde{F}_d$ by μ_i $(i = 1, \ldots, n+m-2)$. Then $\rho(F_d) < 1$ is equivalent to $(A + \alpha BC\operatorname{Re}(\mu_i))^2 + (\alpha BC\operatorname{Im}(\mu_i))^2 < 1$. That is,

$$B^2C^2|\mu_i|^2\alpha^2 + 2ABC\operatorname{Re}(\mu_i)\alpha + A^2 - 1 < 0. \tag{19}$$

By some computations, we know that if the conditions in (18) hold, then the inequality (19) is solvable. This completes the proof. □

Remark 13. Theorem 12 provides a necessary and sufficient condition of couple-group consensus for multiagent system (7). According to linear system theory, we know that system (16) which is asymptotically stable is equivalent to that in which there exists a positive matrix P such that $P - F_d^T P F_d \succ 0$. Hence, we can get another condition of couple-group consensus for multiagent systems (7).

Theorem 14. *The multiagent system (7) can achieve couple-group consensus asymptotically if and only if there exist positive definite matrices P, Q and scalar γ such that the following LMI*

$$\begin{bmatrix} P & F_d^T \\ F_d & Q \end{bmatrix} \succ \mathbf{0} \tag{20}$$

holds with the constraint $P^{-1} = Q$. Here F_d is defined in (16).

Proof. According to the discussion in Remark 13, and by using Schur complement lemma (Lemma 6) and letting $Q \triangleq P^{-1}$, the proof can be obtained. This completes the proof. □

Remark 15. Theorem 14 provides a necessary and sufficient condition of couple-group consensus for multiagent systems (7). We can get γ by solving LMI in (20) with constrain $P^{-1} = Q$. The cone complementarity linearization (CCL) method can be used to solve this problem [19, 29]. We next summarize the algorithm as follows.

Algorithm 16. Consider the following

Step 1. Find a feasible point of LMI (20) γ^0, P^0, Q^0, set $k = 0$. If there are none, exit.

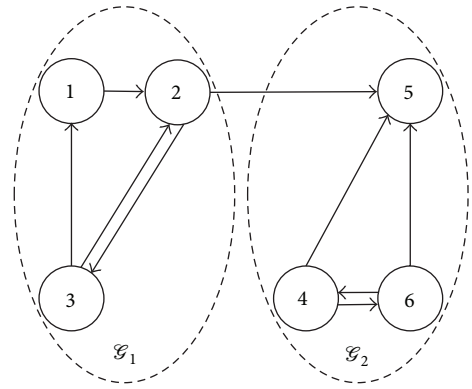

Figure 1: Topology $\mathcal{G}$.

Step 2. Find γ^{k+1}, P^{k+1}, Q^{k+1} by solving the convex minimization problem

$$t_k = \min\left\{\operatorname{tr}\left(PQ^k + QP^k\right)\right\} \tag{21}$$

s.t.

$$\begin{bmatrix} P & F_d^T \\ F_d & Q \end{bmatrix} \succ \mathbf{0}, \qquad \begin{bmatrix} P & I_{n+m-2} \\ I_{n+m-2} & Q \end{bmatrix} \succeq \mathbf{0}. \tag{22}$$

Step 3. If $t_k = 2(n+m-2)$, end this algorithm, and the feasible γ is given by $\gamma = \gamma^{k+1}$. Otherwise, set $k = k+1$ and go to Step 2.

4. Simulation Examples

In this section, two examples will be given to show the usefulness of the theoretical results. For simplicity, we let $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$. On the other hand, we suppose that a_{ij} takes values in a set $\{-1, 0, 1\}$ for v_i, v_j belonging to different node sets, respectively.

Example 1. This example is for continuous-time multiagent systems. The interaction topology is as shown in Figure 1, which includes six nodes. It can be seen that the graph contains two subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$. $(v_1, v_2, v_3) \in \mathcal{G}_1$, $(v_4, v_5, v_6) \in \mathcal{G}_2$. Each of them has a directed spanning tree. Let $A = 0.3$, $B = 0.8$, and $C = -0.9$. By solving the optimization problem in Algorithm 11, we obtain $\alpha = 0.6304$ and

$$Q = \begin{bmatrix} 32.3767 & -2.6303 & 0 & 0 \\ -2.6303 & 16.5170 & 0 & 0 \\ 0 & 0 & 21.9891 & 0 \\ 0 & 0 & 0 & 21.9891 \end{bmatrix}. \tag{23}$$

The state trajectories of the agents are as shown in Figure 2. It can be seen that the agents belonging to $\mathcal{G}_1$ and $\mathcal{G}_2$ achieve two different consistent states, respectively.

Example 2. This example is for discrete-time multiagent systems. For simplicity, we suppose that the interaction topology is the same as that of continuous-time case, that is, $\mathcal{G}$. Let

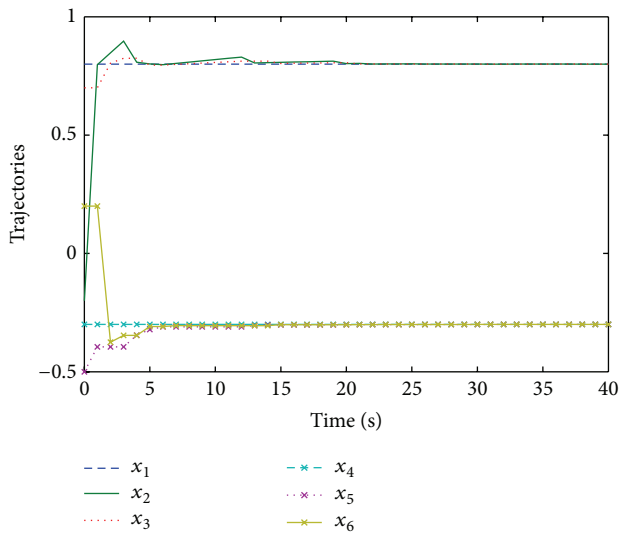

Figure 2: State trajectories for continuous-time case.

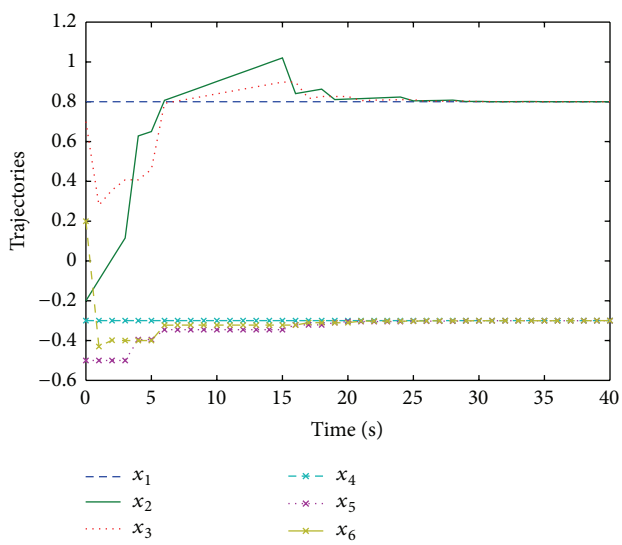

Figure 3: State trajectories for discrete-time case.

$A = 0.4$, $B = 0.7$, and $C = -0.8$. By solving the optimization problem in Algorithm 16, we obtain $\gamma = -0.2531$ and

$$P = \begin{bmatrix} 1.7451 & -0.8085 & -1.2822 & 0.5011 \\ -0.8085 & 4.2494 & 6.6476 & -1.5457 \\ -1.2822 & 6.6476 & 13.4860 & -3.2780 \\ 0.5011 & -1.5457 & -3.2780 & 1.9709 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0.6426 & 0.1246 & -0.0273 & -0.1111 \\ 0.1246 & 1.0569 & -0.5293 & -0.0832 \\ -0.0273 & -0.5293 & 0.3915 & 0.2430 \\ -0.1111 & -0.0832 & 0.2430 & 0.8746 \end{bmatrix}. \quad (24)$$

Figure 3 shows the consensus results.

5. Conclusion

In this paper, we have studied the couple-group consensus problems for both continuous-time and discrete-time multiagent systems via output feedback control. By a system transformation, the consensus problems of multiagent systems have been converted into the stability problems of the error systems. Some necessary and sufficient conditions of couple-group consensus for multiagent systems have been obtained. Two algorithms have been given to compute the allowable control gains. The effectiveness of the proposed results has been shown by the simulation examples.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Feedforward and Feedback Control Performance Assessment for Nonlinear Systems

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A performance assessment method for nonlinear feedforward and feedback control systems is proposed in this paper. First, the existence of minimum variance performance bound for two nonlinear systems with different structures is analyzed, and the closed-loop model of nonlinear system is obtained with the help of iterative orthogonal least squares identification method. Then, the technology of variance analysis is introduced to establish the variance contributions due to both disturbances and controller. A nonlinear performance index for the feedforward and feedback control systems is estimated using an ANOVA-like variance decomposition method. Finally, a meaningful example is simulated to show the effectiveness of our method.

1. Introduction

The technology of control performance assessment (CPA) has attracted much attention in recent years, due to the extensive application of automatic control systems in industrial area. CPA is a management tool to maintain efficient operation performance of automation systems. The main aim is to evaluate the performance of control loops in control systems, diagnose the reason of poor performance, and present effective proposals for improvement once the control performance of a running controller cannot meet the desired requirements.

The study of CPA began to blossom some 20 years ago with the pioneering work by Harris [1]; he proposed a linear performance index based on minimum variance benchmark. Desborough and Harris [2] proposed a normalized performance index for assessment of linear SISO controller performance, which can be estimated by linear regression methods. Stanfelj et al. [3] presented a method that utilized autocorrelation and cross correlation functions for monitoring and diagnosing the cause of poor performance of feedforward and feedback control systems. Desborough and Harris [4] developed a performance assessment algorithm based on variance table to investigate the variance contributions due to disturbances and controllers for a linear feedforward and feedback system. Harris et al. [5] developed a method for assessing the performance of linear MIMO control systems, and this method requires an estimate of the process interactor matrix that characterizes the dead-time structure. Almost at the same time, Huang et al. [6] developed a new approach based on filtering and correlation (FCOR) analysis of the process output and filtered data, which can be used to estimate the controller performance of a general class of linear MIMO processes. Subsequently, Huang et al. [7] developed a method for the performance assessment of linear multivariate feedback plus feedforward control systems using minimum variance control as the benchmark. CPA theoretical issues have been reported by several literatures, such as the references published by Qin [8], Huang and Shah [9], and Jelali [10].

Although the field of CPA has received much attention in theory and engineering in recent years [11–14], the most previous studies are focused on linear systems. In real applications, the industrial processes are naturally nonlinear systems. The estimation of the minimum variance performance lower bound (MVPLB) and the performance index using the linear control performance assessment techniques may be distorted by these nonlinearities. Due to the internal complexity and lack of effective mathematical tools, far less

has been written on the CPA methods for nonlinear systems. For a special class of nonlinear SISO processes that can be described by the superposition of a nonlinear dynamic model and additive linear disturbance, Harris and Yu [15] presented a method to estimate the MVPLB using closed-loop data. Continuing this idea, estimates of the MVPLB for the moderate valve stiction cases are proposed by Yu et al. [16]. Yu et al. [17] proposed a new CPA performance index for general nonlinear SISO models based on an ANOVA-like variance decomposition method. This new performance index is not based on the MVPLB, but it can be used to estimate the MVPLB for some nonlinear systems detailed are discussed in [15]. Considering the process nonlinearity and valve stiction nonlinearity in control system, Zhang [18] proposed some CPA methods for nonlinear systems based on minimum variance benchmark. Yu et al. [19] extended CPA to nonlinear MIMO systems. However, in order to make the problem tractable, they restrict the system structure to be a model with additive linear disturbances and where the nonlinearity is in the form of valve stiction.

In spite of the fact that multivariate control schemes are justified from an economic and quality improvement standpoint, the univariate controllers are the mostly used controllers in practical applications. The performance of these SISO control schemes can be enhanced by including feedforward elements. In this paper, we study the performance assessment for nonlinear feedforward and feedback control systems. The objective of our work is to estimate the MVPLB for this nonlinear system and analyze the contribution of each controller for the overall performance bound. This study has an important guiding significance for the adjustment and design of the actual control system. Two common situations are often encountered in pragmatic feedforward and feedback control systems. The first case is, although a feedforward variable can be measured, it is not used in the control systems; in such situation, the result of CPA for nonlinear feedforward and feedback control systems can provide an estimation of the variance reductions if feedforward controller is considered. In the other case, a feedforward variable is both measured and used in a feedforward and feedback control scheme, and then, the performance of the individual controllers can be assessed by the result of this paper, such that we can determine which controller should be principally adjusted to improve the performance of feedforward and feedback control systems.

Based on some methods for the performance assessment of linear feedforward and feedback control systems, this paper is an extension to nonlinear systems. The outline of this paper is organized as follows. As a prerequisite, the performance assessment of linear feedforward and feedback systems is discussed in Section 2. In Section 3, the existence of MVPLB for nonlinear feedforward and feedback systems is analyzed. In Section 4, a description of the ANOVA-like variance decomposition method is given and a new performance index of nonlinear control systems is proposed. Finally, a simulation is made to illustrate the proposed methodology in Section 5, and this is followed by a conclusion in Section 6.

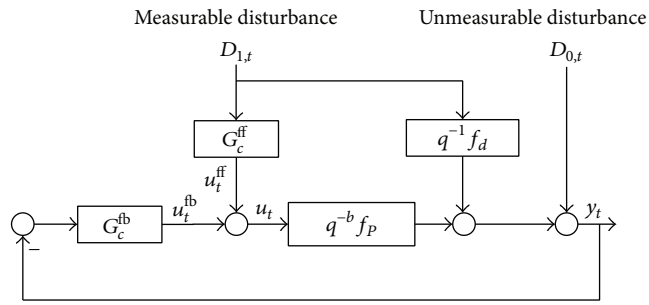

Figure 1: Schematic of feedforward and feedback control system.

2. Analysis of Variance in Linear Feedforward and Feedback Control System

A structural schematic of general feedforward and feedback control system is given in Figure 1, where y_t is output variable of the process, u_t is manipulated variable which is adjusted by summing the outputs from the feedback controller u_t^{fb} and feedforward controller u_f^{ff}. G_c^{fb} is feedback controller transfer function, and G_c^{ff} is feedforward controller transfer function. $q^{-b}f_P$ represents the process model that may be linear or nonlinear. b is the number of whole periods of process delay. $q^{-l}f_dD_{1,t}$ represents the effect that the measured disturbance $D_{1,t}$ has on the process output, and l is the number of periods of delay it takes for a change in $D_{1,t}$ to begin to affect the output. In linear systems, $q^{-l}f_dD_{1,t}$ is often expressed by transfer function as $q^{-l}N_dD_{1,t}$. $D_{0,t}$ and $D_{1,t}$ represent the unmeasured and measured disturbances, respectively. In this paper, work is based on the assumption that there is no cross correlation among the unmeasured and measured disturbances, and this is reasonable for many industrial processes.

In linear systems, the delay-free process model f_P can be represented by the following equation:

$$f_P = \frac{\omega\left(q^{-1}\right)}{\delta\left(q^{-1}\right)}, \tag{1}$$

where $\omega(q^{-1})$ and $\delta(q^{-1})$ are stable polynomials in the backshift operator q^{-1}. Disturbances $D_{0,t}$ and $D_{1,t}$ are represented by autoregressive integrated moving average (ARIMA) time series models:

$$D_{i,t} = \frac{\theta_i\left(q^{-1}\right)}{\varphi_i\left(q^{-1}\right)\nabla^{d_i}}\alpha_{i,t}, \quad i = 0, 1. \tag{2}$$

$\{\alpha_{i,t}\}$ is a sequence of independently and identically distributed random variables with mean zero and constant variance σ_i^2. $\theta_i(q^{-1})$ and $\varphi_i(q^{-1})$ are monic and stable polynomials. The difference operator is defined as $\nabla \stackrel{\mathrm{def}}{=} (1 - q^{-1})$, and d_i is the degree of differencing. The linear feedforward

and feedback control system can be modeled as the sum of two disturbances and a linear transfer function:

$$y_t = q^{-b}\frac{\omega\left(q^{-1}\right)}{\delta\left(q^{-1}\right)}u_t + D_{0,t} + q^{-l}N_d D_{1,t}. \tag{3}$$

Substituting the feedforward and feedback controller representation into above equation and multiplying both sides by q^d and collecting terms

$$y_{t+b} = \frac{\omega\left(q^{-1}\right)}{\delta\left(q^{-1}\right)}u_t^{\text{fb}} + D_{0,t+b} + \frac{\omega\left(q^{-1}\right)}{\delta\left(q^{-1}\right)}u_t^{\text{ff}} + q^{-l}N_d D_{1,t+b}. \tag{4}$$

In an analogous manner to the minimum variance feedback controller, the design of minimum variance feedforward and feedback controller can be derived. The research result of Desborough and Harris [4] reported that the linear closed-loop system can be described in terms of the unmeasured disturbance driving force and the measured feedforward variable. We do the similar work, which yields

$$\begin{aligned} y_{t+b} &= \frac{\omega\left(q^{-1}\right)}{\delta\left(q^{-1}\right)}\left(-G_c^{\text{fb}}y_t\right) + \frac{\theta_0\left(q^{-1}\right)}{\varphi_0\left(q^{-1}\right)\nabla^{d_0}}\alpha_{0,t+b} \\ &\quad + \frac{\omega\left(q^{-1}\right)}{\delta\left(q^{-1}\right)}G_c^{\text{ff}}D_{1,t} + q^{-l}N_d D_{1,t+b} \\ &= \frac{\theta_0\left(q^{-1}\right)/\varphi_0\left(q^{-1}\right)\nabla^{d_0}}{1+q^{-b}\left[\omega\left(q^{-1}\right)/\delta\left(q^{-1}\right)\right]G_c^{\text{fb}}}\alpha_{0,t+b} \\ &\quad + \frac{q^{-b}G_P\left(q^{-1}\right)G_c^{\text{ff}} + q^{-l}N_d}{1+q^{-b}\left[\omega\left(q^{-1}\right)/\delta\left(q^{-1}\right)\right]G_c^{\text{fb}}}D_{1,t+b} \\ &= \psi_0\left(q^{-1}\right)\alpha_{0,t+b} + \psi_1\left(q^{-1}\right)D_{1,t+b}, \end{aligned} \tag{5}$$

where $\psi_0(q^{-1})$ is the closed-loop transfer function between y_t and the driving force for the unmeasured disturbance. $\psi_1(q^{-1})$ is the closed-loop transfer function between y_t and measured feedforward variable $D_{1,t}$. Alternatively, the process can be described in terms of the driving forces alone:

$$y_t = \psi_0\left(q^{-1}\right)\alpha_{0,t+b} + \psi_1\left(q^{-1}\right)\alpha_{1,t+b}. \tag{6}$$

Each of the closed-loop transfer functions in (6) can be expanded in a convergent power series in q^{-1}:

$$\psi_i\left(q^{-1}\right) = \sum_{h=0}^{\infty}\psi_{i,h}q^{-h}. \tag{7}$$

This expansion is obtained by writing each transfer function as a ratio of polynomials q^{-1} and then dividing the numerator into the denominator using polynomial long division. Then the process output can be extended as

$$y_{t+b} = y_{0,t+b} + y_{1,t+b} = \sum_{h=0}^{\infty}\psi_{0,h}q^{-h}\alpha_{0,t+b} + \sum_{h=0}^{\infty}\psi_{1,h}q^{-h}\alpha_{1,t+b}. \tag{8}$$

The term $y_{0,t+b}$ is the contribution of unmeasured disturbance $D_{0,t}$ to the process output; it can be written as

$$\begin{aligned} y_{0,t+b} &= \left(1+\psi_{0,1}q^{-1}+\cdots+\psi_{0,b-1}q^{-(b-1)}\right)\alpha_{0,t+b} \\ &\quad + \left(\psi_{0,b}q^{-b}+\psi_{0,b+1}q^{-(b+1)}+\cdots\right)\alpha_{1,t+b} \\ &= e_{0,t+b/t} + \sum_{h=b}^{\infty}\psi_{0,h}q^{-h}\alpha_{0,t+b} = e_{0,t+b/t} + y_{0,t+b}^{\text{fb}}. \end{aligned} \tag{9}$$

The first term $e_{0,t+b/t}$ in above function is recognized as the prediction error, which is independent of the second term. The second term is the contribution to the process output $y_{0,t+b}$ which arises from the nonoptimality of the control associated with the unmeasured disturbance, and it is also a function of the process dynamics, the unmeasured disturbance, and the feedback controller only.

In a similar manner, the contribution of the measured disturbance $D_{1,t}$ to the process output can be written as

$$\begin{aligned} y_{1,t+b} &= e_{1,t+b/t} + \left(\psi_{1,b}q^{-b}+\cdots+\psi_{1,b+l-1}q^{-(b+l-1)}\right)\alpha_{1,t+b} \\ &\quad + \left(\psi_{1,b+l}q^{-(b+l)}+\psi_{1,b+l+1}q^{-(b+l+1)}+\cdots\right)\alpha_{1,t+b} \\ &= e_{1,t+b/t} + \sum_{h=b}^{b+l-1}\psi_{1,h}q^{-h}\alpha_{1,t+b} \\ &\quad + \sum_{h=b+l}^{\infty}\psi_{1,h}q^{-h}\alpha_{1,t+b} = e_{1,t+b/t} + y_{1,t+b}^{\text{ff}} + y_{1,t+b}^{\text{ff\&fb}}, \end{aligned} \tag{10}$$

where

$$e_{1,t+b/t} = \begin{cases} 0, & l \geq b \\ \left(\underbrace{\psi_{1,0}q^{0}+\cdots+\psi_{1,l-1}q^{-(l-1)}}_{0} + \psi_{1,l}q^{-l} \right. \\ \left. \quad + \psi_{1,l+1}q^{-(l+1)}+\cdots+\psi_{1,b-1}q^{-(b-1)}\right)\alpha_{1,t+b} & l < b. \end{cases} \tag{11}$$

In (10), the first term $e_{1,t+b/t}$ is the prediction error for the measured disturbance, and it is independent of the second and third terms. The second term is the contribution to the output $y_{1,t+b}$ which arises from the nonoptimality of the feedforward controller only, and the third term is the contribution which arises from the combined effect of the nonoptimality of the feedforward controller and the feedback controller.

Since it has been assumed that the measured and unmeasured disturbances are not cross correlated, the prediction error $e_{0,t+b/t}$ and $e_{1,t+b-l/t}$ are independent of all the controllers. Then, the process output under minimum variance control is given by the sum of the individual error in forecasting the effect of the disturbances:

$$y_{t+b}^{\text{mv}} = e_{0,t+b/t} + e_{1,t+b-l/t}, \tag{12}$$

and the MVPLB can be written as

$$\sigma_{\text{mv}}^2 = \text{var}\left(e_{0,t+b/t}\right) + \text{var}\left(e_{1,t+b/t}\right). \tag{13}$$

3. MVPLB of Nonlinear Feedforward and Feedback Control System

Due to the effect of various factors such as complexity of nonlinear behavior and challenges in model determination and parameter estimation, far less has been written to extend the methods for performance assessment to nonlinear systems. In order to simplify the analysis and without loss of generality, the problem of estimation for minimum variance performance bound for nonlinear feedforward and feedback systems is given in two aspects.

First, we only assume that the process model has a nonlinear representation in the structural schematic Figure 1, and this is not very restrictive in many applications. Then, the closed output b-steps into the future of the nonlinear system can be expressed as

$$y_{t+b} = f_P\left(u_t^*\right) + D_{0,t+b} + N_d D_{1,t+b-l}, \tag{14}$$

where the notation $f_P(\cdot)$ denotes a nonlinear function of process model, and the superscript $*$ is used to represent the vector collecting the immediate historical values; that is, $u_t^* \overset{\text{def}}{=} (u_{t-1}, \ldots, u_{t-n_u})$. Decomposing the unmeasured disturbance $D_{0,t+b}$ into a prediction error and a prediction

$$D_{0,t+b} = e_{0,t+b/t} + \widehat{D}_{0,t+b/t}, \tag{15}$$

the prediction $\widehat{D}_{0,t+b/t}$ is the b-step ahead minimum mean square error prediction for the value of the unmeasured disturbance b steps into the future. The effects of the measured feedforward variables are also decomposed into a prediction error and a prediction

$$N_d D_{1,t+b-l} = e_{1,t+b-l/t} + \widehat{D}_{1,t+b-l/t}. \tag{16}$$

The prediction $\widehat{D}_{1,t+b-l/t}$ is the $b-l$ step ahead minimum mean square error prediction for the value of the measured disturbance $b-l$ steps into the future. Note that if l is greater than or equal to b, then $e_{1,t+b-l/t} = 0$. This implies that there is no prediction error since we exactly know the future value of the effect on the process of the measured disturbance.

The minimum variance control law is found by minimizing the mean square error of the output:

$$\begin{aligned} y_{t+b} &= f_P\left(u_t^*\right) + D_{0,t+b} + N_d D_{1,t+b-l} \\ &= f_P\left(u_t^*\right) + e_{0,t+b/t} + \widehat{D}_{0,t+b/t} + e_{1,t+b-l/t} + \widehat{D}_{1,t+b-l/t} \\ &= \underbrace{e_{0,t+b/t} + e_{1,t+b-l/t}}_{\text{term 1}} + \underbrace{f_P\left(u_t^*\right) + \widehat{D}_{0,t+b/t} + \widehat{D}_{1,t+b-l/t}}_{\text{term 2}}. \end{aligned} \tag{17}$$

It follows from this formula that the minimum variance controller (MVC) set the manipulated variables to exactly cancel the predictions; that is,

$$f_P\left(u_t^{\text{fb}}, u_t^{\text{ff}}\right) + \widehat{D}_{0,t+b/t} + \widehat{D}_{1,t+b-l/t} = 0. \tag{18}$$

Then the process output under this control scheme can be denoted by

$$y_{t+b}^{\text{mv}} = e_{0,t+b/t} + \begin{cases} 0, & l \geq b \\ e_{1,t+b-l/t}, & l < b. \end{cases} \tag{19}$$

As we have assumed that there is no cross correlation among the unmeasured and measured disturbances, the prediction errors $e_{0,t+b/t}$ and $e_{1,t+b-l/t}$ are independent and unrelated with controller parameters. Then, the MVPLB of closed-loop output is

$$\begin{aligned} \sigma_{\text{mv}}^2 &= \operatorname{var}\left(e_{0,t+b/t}\right) + \operatorname{var}\left(e_{1,t+b-l/t}\right) \\ &= \left(1 + \psi_{0,1}'^2 + \cdots + \psi_{0,b-1}'^2\right)\sigma_0^2 \\ &\quad + \begin{cases} 0 & l \geq b \\ \left(\psi_{1,0}'^2 + \psi_{1,1}'^2 + \cdots + \psi_{1,b-l-1}'^2\right)\sigma_1^2 & l < b. \end{cases} \end{aligned} \tag{20}$$

From above derivation, we can conclude that the MVPLB of nonlinear feedforward and feedback system is identical to that of linear system. The difference is that it is possible to adopt different controllers for obtaining same minimum variance.

Second, a more general form of nonlinear feedforward and feedback control systems is considered:

$$y_t = \underbrace{q^{-b} f_P(u_t^*)}_{\text{nonlinear}} + \underbrace{\widetilde{D}_{0,t}}_{\text{nonlinear}} + \underbrace{\widetilde{D}_{1,t-l}}_{\text{nonlinear}}, \tag{21}$$

where the terms $\widetilde{D}_{0,t}$ and $\widetilde{D}_{1,t-l}$ are called output disturbances which represent the effect that the unmeasured and measured disturbances have on the process output, respectively. They are also nonlinear and can be represented by nonlinear ARMA model as

$$\begin{aligned} \widetilde{D}_{0,t} &= f_{0,D}\left(\widetilde{D}_{0,t-1}^*, \alpha_{0,t-1}^*\right) + \alpha_{0,t}, \\ \widetilde{D}_{1,t} &= f_d D_{1,t} = f_{1,D}\left(\widetilde{D}_{1,t-1}^*, \alpha_{1,t-1}^*\right) + \alpha_{1,t}. \end{aligned} \tag{22}$$

Further, we assume that the output disturbance admits a representation of the form

$$\gamma_i\left(q^{-1}\right)\nabla^{d_i}\widetilde{D}_{i,t} = \underbrace{\sum_{k=1}^{m}\theta_{i,k}\alpha_{i,t-k} + \sum_{k_1=1}^{m}\sum_{k_2=k_1}^{m}\theta_{i,k_1k_2}\alpha_{i,t-k_1}\alpha_{t-k_2} + \cdots + \sum_{k_1=1}^{m}\cdots\sum_{k_k=k_{k-1}}^{m}\theta_{i,k_1\cdots k_k}\alpha_{i,t-k_1}\cdots\alpha_{i,t-k_k}}_{f_{i,D}(\alpha_{i,t-1}^*)} + \alpha_{i,t}, \tag{23}$$

where $\{\alpha_{i,t}\}$ is a white noise sequence with mean $\mu_{i,\alpha}$ and variance $\sigma^2_{i,\alpha}$, and $\gamma_i(q^{-1})$ is monic and stable polynomial, and we also assume that the disturbance model is invertible. Multiply both sides by q^b and substitute for all values of y_{t+b-i}, $i = 1, \ldots, b-1$, in (21):

$$\begin{aligned} y_{t+b} = {} & f_P(u_t^*) + \sum_j^{b-1} \tau_{0,j}\left(f_{0,D}\left(\alpha^*_{0,t+b-1-j}\right) + \alpha_{0,t+b-j}\right) \\ & + K_{0,b}\left(\widetilde{D}_{0,t}, \alpha^*_{0,t}\right) \\ & + \sum_j^{b-l-1} \tau_{1,j}\left(f_{1,D}\left(\alpha^*_{1,t+b-l-1-j}\right) + \alpha_{1,t+b-l-j}\right) \\ & + K_{1,b}\left(\widetilde{D}_{1,t}, \alpha^*_{1,t}\right), \end{aligned} \tag{24}$$

where $\tau_{i,j}$ is the jth impulse coefficient of $[\gamma_i(q^{-1})\nabla^{d_i}]^{-1}$, $i = 0$ or 1. $K_{i,b}(\widetilde{D}_{i,t}, \alpha^*_{i,t})$ is a remainder term that is obtained by successive substitutions. The unmeasured output disturbance is represented as

$$\begin{aligned} \widetilde{D}_{0,t+b} = {} & \sum_j^{b-1} \tau_{0,j}\left(f_{0,D}\left(\alpha^*_{0,t+b-1-j}\right) + \alpha_{0,t+b-j}\right) \\ & + K_{0,b}\left(\widetilde{D}_{0,t}, \alpha^*_{0,t}\right). \end{aligned} \tag{25}$$

According to the definition of conditional expectation, the b-step ahead prediction is

$$\begin{aligned} \widehat{\widetilde{D}}_{0,t+b/t} = {} & E\left\{\sum_j^{b-1} \tau_{0,j}\left(f_{0,D}\left(\alpha^*_{0,t+b-1-j}\right) + \alpha_{0,t+b-j}\right) \mid I_t\right\} \\ & + E\left\{K_{0,b}\left(\widetilde{D}_{0,t}, \alpha^*_{0,t}\right)\right\} \\ = {} & E\left\{\sum_j^{b-1} \tau_{0,j}\left(f_{0,D}\left(\alpha^*_{0,t+b-1-j}\right) + \alpha_{0,t+b-j}\right) \mid I_t\right\} \\ & + K_{0,b}\left(\widetilde{D}_{0,t}, \alpha^*_{0,t}\right). \end{aligned} \tag{26}$$

Now in the aforementioned equation, we know

$$\begin{aligned} & E\{\alpha_{0,t+k} \mid I_t\} = \mu_{0,\alpha}, \quad k = 1, \ldots .b, \\ & E\{\alpha_{0,t-k} \mid I_t\} = \alpha_{0,t-k} = \widetilde{D}_{0,t-k} - \widehat{\widetilde{D}}_{0,t-k/t-k-1}, \quad k \geq 0, \\ & E\left\{f_{0,D}\left(\alpha^*_{0,t+k}\right) \mid I_t\right\} \\ & = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{0,D}\left(\alpha^*_{0,t+k}\right) \\ & \qquad \times p_0\left(\alpha_{0,t+k}, \ldots, \alpha_{0,t+1}\right) d\alpha_{0,t+k} \cdots d\alpha_{0,t+1}, \end{aligned} \tag{27}$$

where $p_0(\alpha_{0,t+k}, \ldots, \alpha_{0,t+1})$ is the joint distribution of $\alpha_{0,t+k} \cdots \alpha_{0,t+1}$. Then the prediction error for the unmeasured output disturbance is

$$\begin{aligned} \widetilde{e}_{0,t+b/t} = {} & \widetilde{D}_{0,t+b} - \widehat{\widetilde{D}}_{0,t+b/t} \\ = {} & \sum_{j=0}^{b-1} \tau_{0,j}\left(f_{0,D}\left(\alpha^*_{0,t+b-1-j}\right) - E\left\{f_{0,D}\left(\alpha^*_{0,t+b-1-j}\right) \mid I_t\right\}\right. \\ & \left. + \alpha_{0,t+b-j} - \mu_{0,\alpha}\right). \end{aligned} \tag{28}$$

In a same manner, the prediction error for the measured output disturbance is

$$\begin{aligned} \widetilde{e}_{1,t+b-l/t} = {} & \widetilde{D}_{1,t+b-l} - \widehat{\widetilde{D}}_{1,t+b-l/t} \\ = {} & \begin{cases} 0, & l \geq b \\ \displaystyle\sum_{j=0}^{b-l-1} \tau_{1,j}\left(f_{1,D}\left(\alpha^*_{1,t+b-l-1-j}\right)\right. & \\ \qquad \left. - E\left\{f_{1,D}\left(\alpha^*_{1,t+b-l-1-j}\right) \mid I_t\right\}\right. & \\ \qquad \left. + \alpha_{1,t+b-l-j} - \mu_{1,\alpha}\right), & l < b. \end{cases} \end{aligned} \tag{29}$$

The process output can be written as

$$y_{t+b} = f_P(u_t^*) + \widehat{\widetilde{D}}_{0,t+b/t} + \widehat{\widetilde{D}}_{1,t+b-l/t} + \widetilde{e}_{0,t+b/t} + \widetilde{e}_{1,t+b-l/t}. \tag{30}$$

If it is possible to find the control action at time t such that

$$f_P(u_t^*) + \widehat{\widetilde{D}}_{0,t+b/t} + \widehat{\widetilde{D}}_{1,t+b-l/t} = 0, \tag{31}$$

then the resulting controller is the minimum variance controller. It may not be possible to implement a minimum variance controller due to the various reasons. For instance, it may lead to excessive manipulated variable action and may not be robust to modeling errors. However, the output variance set by minimum variance provides a theoretical lower bound on the system output and can be used as a useful guide for controller assessment.

The process output under minimum variance control is given by the sum of the individual error in predicting the effect of the disturbances:

$$y^{\mathrm{mv}}_{t+b} = \widetilde{e}_{0,t+b/t} + \begin{cases} 0, & l \geq b \\ \widetilde{e}_{1,t+b-l/t}, & l < b. \end{cases} \tag{32}$$

It should be pointed out that the terms $\widetilde{e}_{0,t+b/t}$ and $\widetilde{e}_{1,t+b-l/t}$ are very complicated functions, and they may not be expanded in convergent time series as that in linear systems. Therefore, it is difficult to estimate the MVPLB from the closed-loop operation data of feedforward and feedback control system by using traditional linear regression method. But we can get a conclusion that the MVPLB does not depend on the manipulated variable and only related with the most recent b past unmeasured disturbance driving force and $b - l$ past measured disturbance driving force.

4. ANOVA-Based Performance Assessment of Nonlinear Feedforward and Feedback Control System

Analysis of variance (ANOVA) methods are a class of statistical methods that are useful in process systems engineering. Its primary task is to decompose the variance of a response variable into contributions arising from the inputs and assess the magnitude and significance of each of their contributions. Historically, the ANOVA variance decomposition techniques were used to provide variance analysis for nonlinear systems with the multidisturbance sources [20].

For the output of a static system such as $Y = f(X_1, X_2, \ldots, X_P)$, the relative importance of the independent inputs can be quantified by the fractional variance, and this can be calculated using an ANOVA-like decomposition formula [21]:

$$\mathrm{Var}\,[Y] = \sum_i V_i + \sum_i \sum_{j>i} V_{ij} + \cdots + V_{12\cdots p}, \tag{33}$$

where $V_i = \mathrm{Var}[E[Y \mid X_i = x_i]]$, $V_i = \mathrm{Var}[E[Y \mid X_i = x_i]]$, $V_{ij} = \mathrm{Var}[E[Y \mid X_i = x_i, X_j = x_j]] - \mathrm{Var}[E[Y \mid X_i = x_i]] - \mathrm{Var}[E[Y \mid X_j = x_j]]$ and so on. $E[Y \mid X_i = x_i]$ denotes the expectation of Y conditional on X_i when fixing the value x_i, and V stands for variance over all the possible values of x_i. In the same way, if we partition the variable set $(X_1, X_2, \ldots, X_P)$ into two groups: $U_1 = (X_1, \ldots, X_k)$ and $U_2 = (X_{P-k+1}, \ldots, X_P)$, then the variance of $Y = f(U_1, U_2)$ can be decomposed into $V[Y] = V_{U_1} + V_{U_2} + V_{U_1U_2}$.

For the nonlinear feedforward and feedback control systems described by Figure 1, we separate the disturbance entering the system after time 0, say $[\alpha_{0,t+b}, \alpha_{0,t+b-1}, \ldots, \alpha_{0,1}, \alpha_{1,t+b-l}, \alpha_{1,t+b-l-1}, \ldots, \alpha_{1,1}]$, into two groups: $x_1 = [\alpha_{0,t+b}, \ldots, \alpha_{0,t+1}, \alpha_{1,t+b-l}, \ldots, \alpha_{1,t+1}]$ and $x_2 = [\alpha_{0,t}, \ldots, \alpha_{0,1}, \alpha_{1,t}, \ldots, \alpha_{1,1}]$. The first group includes all the disturbances entering the system after time t and the second group includes all the disturbances entering the system up to and including time t and including time t starting from the initial time $t = 0$. Now, we are interested in determining the sensitivity of output y_{t+b} variations of two vector series x_1 and x_2. Since the future behavior of y_{t+b} is possibly dependent on initial conditions due to the nonlinearity, the initial condition must be considered before using the ANOVA-like decomposition equation. Using the well-known variance decomposition theorem, the variance of y_{t+b} can be decomposed into two terms:

$$V\left[y_{t+b}\right] = E_{I_0}\left[V_x\left[y_{t+b} \mid I_0\right]\right] + V_{I_0}\left[E_x\left[y_{t+b} \mid I_0\right]\right], \tag{34}$$

where $x = [x_1, x_2]$ denotes all of disturbances entering the system from time 1 to time $t + b$ and I_0 denotes initial conditions. The first term in above equation is the fractional contribution to the variance of y_{t+b} from the disturbance signal and the interaction between disturbance and the initial condition. The second term is the fractional contribution to the output solely due to the uncertainties in the initial condition. Given the initial condition I_0, conditional variance $V_x[y_{t+b} \mid I_0]$ can be decomposed as

$$V_x \mid I_0 = V_x\left[y_{t+b} \mid I_0\right] = V_1 \mid I_0 + V_2 \mid I_0 + V_{12} \mid I_0, \tag{35}$$

where $V_1 \mid I_0 = V_{x1}[E_{x2}[y_{t+b} \mid (x_1, I_0)]]$, $V_2 \mid I_0 = V_{x2}[E_{x1}[y_{t+b} \mid (x_2, I_0)]]$, and $V_{12} \mid I_0 = V_x[E_x[y_{t+b} \mid (x, I_0)]] - V_1 \mid I_0 - V_2 \mid I_0$. $E_{I_0}[V_1 \mid I_0]$ denotes the main effect of x_1 on the $V[y_{t+b}]$. $E_{I_0}[V_2 \mid I_0]$ denotes the interaction contributing to the $V[y_{t+b}]$ that is not accounted for the main effects of x_1 and x_2. Consequently, a suitable performance index can be constructed by referring to Harris index:

$$\eta_t = \frac{E_{I_0}\left[V_1 \mid I_0\right]}{\mathrm{Var}\left[y_{t+b}\right]}. \tag{36}$$

If the nonlinear model is stationary, then the distribution of $\lim_{t\to\infty} y_{t+b}$ can reach an equilibrium. For linear time series, this limiting distribution is independent of initial condition. But for a stationary nonlinear model, the limiting distribution may depend on the initial condition. Therefore, the performance index η_t will depend on the initial condition. If the distribution of $\lim_{t\to\infty} y_{t+b}$ does not depend on the initial conditions, the process is termed ergodic. In actual industry, the cases that processes are strongly nonergodic are more pathological than common cases. For an ergodic nonlinear system, $V_{I_0}[E_x[Y_{t+b}|I_0]]$ in (34) will be zero for $t \to \infty$, and the variance decomposition can be expressed when $t \to \infty$ as

$$\mathrm{Var}\left[y_{t+b}\right] = E_{I_0}\left[V_1 \mid I_0 + V_2 \mid I_0 + V_{12} \mid I_0\right] = V_1 + V_2 + V_{12}, \tag{37}$$

where $V_1 = V_{x1}[E_{x2}[y_{t+b} \mid x_1]]$, $V_2 = V_{x2}[E_{x1}[y_{t+b} \mid x_2]]$, and $V_{12} = V[y_{t+b}] - V_1 - V_2$. The performance index will turn into

$$\lim_{t\to\infty} \eta_t = \lim_{t\to\infty} \frac{V_1}{\mathrm{Var}\left[y_{t+b}\right]}. \tag{38}$$

Generally, we will approximate the infinite limit in above equation by some suitably large value η_M.

In Section 3, we conclude that the MVPLB of nonlinear feedforward and feedback control systems is existent and only related with the most recent b past unmeasured disturbance driving force and $b - l$ past measured disturbance driving force. Moreover, we have $x_1 = [\alpha_{0,t+b}, \ldots, \alpha_{0,t+1}, \alpha_{1,t+b-l}, \ldots, \alpha_{1,t+1}]$, so η_t just is the minimum variance performance index of the nonlinear feedforward and feedback control systems.

For the computation of the performance index, the principal task is to estimate the closed-loop model of nonlinear feedforward and feedback control system. Firstly, the measured feedforward variable transfer function, given in (2), must be estimated. Using the linear regression techniques and past values of $D_{1,t}$. The model of measured disturbance can be estimated by

$$D_{1,t} = \sum_{i=1}^{J_D} \lambda_i D_{1,t-i} + \widehat{\alpha}_{1,t}. \tag{39}$$

$\widehat{\alpha}_{1,t}$ is an estimate of the independent driving force for measured disturbance. If the process is controlled by a linear or nonlinear feedforward and feedback controller such as

$u_t = g(y_t, \ldots, y_{t-n_y})$, then the output of closed-loop system can be written as

$$\begin{aligned} y_{t+b} &= f_1\left(y_t, \ldots, y_{t-n_y}, \alpha_{0,t+b}, \ldots, \alpha_{0,t-n_0}, D_{1,t+b-l}, \ldots, D_{1,t-n_D}\right) \\ &= f_2\left(y_t, \ldots, y_{t-n_y}, \alpha_{0,t+b}, \ldots, \alpha_{0,t-n_0}, \widehat{\alpha}_{1,t+b-l}, \ldots, \widehat{\alpha}_{1,t-n_1}\right). \end{aligned} \tag{40}$$

According to the existing knowledge, any continuous $f(\cdot)$ can be arbitrarily well approximated by polynomial models. Therefore, expanding $f_2(\cdot)$ in above equation as a polynomial of degree l gives the representation

$$\begin{aligned} y_{t+b} = \varepsilon_0 &+ \sum_{i_1=1}^{n} \varepsilon_{i_1} x_{i_1,t} + \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \varepsilon_{i_1 i_2} x_{i_1,t} x_{i_2,t} + \cdots \\ &+ \sum_{i_1=1}^{n} \cdots \sum_{i_l=i_{l-1}}^{n} \varepsilon_{i_1 \cdots i_l} x_{i_1,t} \cdots x_{i_l,t} + \xi_t, \end{aligned} \tag{41}$$

where

$$n = n_y + n_0 + n_1, \tag{42}$$

and $x_{1,t} = y_t$, $x_{2,t} = y_{t-1}, \ldots, x_{n_y,t} = y_{t-n_y}, x_{n_y+1,t} = \alpha_{0,t+b}, \ldots, x_{n_y+n_0,t} = \alpha_{0,t-n_0}$, and $x_{n_y+n_0+1,t} = \alpha_{1,t+b-l}, \ldots, x_{n,t} = \alpha_{1,t-n_1}$. Moreover, the output of closed-loop system can be written as a linear regression model:

$$y_{t+b} = \sum_{i=1}^{M} p_{i,t} \varepsilon_i + \xi_t, \quad t = 1, \ldots, N, \tag{43}$$

where N is the data length, the $p_{i,t}$ are monomials of $x_{1,t}$ to $x_{n,t}$ up to degree l, $p_{1,t} = 1$ corresponding to a constant term, ξ_t is some modeling error, and the ε_i, $i = 1, \ldots, M$, are unknown parameters to be estimated. Then above equation can be written in the matrix form

$$\mathbf{Y} = \mathbf{P}\Theta + \mathbf{E}, \tag{44}$$

where

$$\begin{aligned} \mathbf{Y} &= \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}^T = \begin{bmatrix} p_{1,1} & \cdots & p_{M,1} \\ \vdots & \ddots & \vdots \\ p_{1,N} & \cdots & p_{M,N} \end{bmatrix}, \\ \mathbf{\Theta} &= \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_M \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix}. \end{aligned} \tag{45}$$

In reality, as parameters n_y, n_0, and n_1 are unknown, we must consider the combined problem of structure selection and parameter estimation. To avoid losing significant terms which must be included in the final model, we are forced to consider the full model set at the beginning of the identification and then to select a subset from full model set and find the corresponding parameter. The orthogonal least squares (OLS) method [22] can be used to determine the order and estimate the parameters of the model. Denote

$$\widetilde{P}^{(0)} = \left[P \ : \ Y\right]. \tag{46}$$

After a series of Householder transformations $H^{(i)}$, $i = 1, \ldots, k-1$ have been successively applied to $\widetilde{P}^{(0)}$; it is transformed to

$$\widetilde{P}^{(k-1)} = \left[\widetilde{R}_{k-1} \ \ \widetilde{p}_k^{(k-1)} \ \cdots \ \widetilde{p}_M^{(k-1)} \ : \ Y^{(K-1)}\right], \tag{47}$$

where $\widetilde{R}_{k-1} = (R_{k-1} \quad 0)^T$, $\widetilde{p}_k^{(k-1)} = (\widetilde{p}_{1,k}^{(k-1)}, \ldots, \widetilde{p}_{N,k}^{(k-1)})^T$, and $Y^{(k-1)} = (y_1^{(k-1)}, \ldots, y_N^{(k-1)})^T$, and R_{k-1} is the $(k-1) \times (k-1)$ upper triangular matrix. Further denote

$$a_j^{(k)} = \left(\sum_{i=k}^{N} \left(\widetilde{p}_{i,j}^{(k-1)}\right)^2\right)^{1/2}; \qquad b_j^{(k)} = \sum_{i=k}^{N} \widetilde{p}_{i,j}^{(k-1)} y_i^{(k-1)}, \qquad j = k, \ldots, M. \tag{48}$$

Assume that the maximum of $(b_j^{(k)}/a_j^{(k)})^2$, $j = k, \ldots, M$, is achieved at $j = j_m$. Then interchange the j_mth column of $\widetilde{p}_k^{(k-1)}$ with the kth column. The procedure is terminated at M_sth stage when

$$1 - \sum_{i=1}^{M_s} \frac{\left(y_j^{(M_s)}\right)^2}{\langle Y, Y \rangle} \le \rho, \quad \text{or} \quad M_s = M, \tag{49}$$

where $\rho (0 < \rho \le 1)$ is a desired tolerance. Using backward substitution, the subset model parameter estimate Θ_s is computed from

$$R_{M_s} \Theta_s = \left[y_1^{(M_s)} \cdots y_{M_s}^{(M_s)}\right]^T. \tag{50}$$

In addition, since the terms of unmeasured disturbance driving force are generally unmeasured, the identification will require an iterative approach. The identification procedures can be clarified as follows.

Step 1. Set the initial sequence $\alpha_{0,t}$ by fitting a linear model or setting the $\alpha_{0,t}$ to zero, and set iteration number $i = 1$.

Step 2. Identify the nonlinear model and get the prediction errors or residuals $\xi_k^{[i]}$, $k = 1, \ldots N$.

Step 3. If certain identification criteria are achieved, then the program jumps to Step 6. Otherwise, Step 4 is run.

Step 4. Replace the initial sequence by the prediction errors or residuals.

Step 5. Set iteration number $i = i + 1$ and return to Step 2.

Step 6. End of program.

Once the parameters of the closed-loop model are estimated, Monte Carlo (MC) method may be used to compute

Table 1: The obtained model coefficients and minimum variance by linear estimation method.

Terms	$b = 3, l = 5$			$b = 5, l = 3$		
	Real values	Estimated values	Variances	Real values	Estimated values	Variances
1	1	1	0	1	1	0
$\psi_{0,1}$	1.60	1.4977	0.0118	1.60	1.3811	0.0172
$\psi_{0,2}$	1.76	1.4321	0.0396	1.76	1.2671	0.0466
$\psi_{0,3}$	×	×	×	1.54	1.0338	0.0801
$\psi_{0,4}$	×	×	×	1.05	0.6427	0.0825
σ_0^2	0.05	0.1208	0.0040	0.05	0.1712	0.0230
$\psi_{1,0}$	×	×	×	×	×	×
$\psi_{1,1}$	×	×	×	×	×	×
$\psi_{1,2}$	×	×	×	×	×	×
$\psi_{1,3}$	×	×	×	1	1.0005	0.0234
$\psi_{1,4}$	×	×	×	0.3	0.2849	0.0588
σ_1^2	0.1	0.0997	$2.1861e-004$	0.1	0.0989	$2.0217e-004$
σ_{mv}^2	**0.3329**	**0.7038**	**0.0919**	**0.6156**	**1.1688**	**0.2958**

the variance decomposition. Firstly, two random vectors, $\dot{x}^{(k)} = [\dot{x}_1^{(k)}, \dot{x}_2^{(k)}]_{N_t\times 1}$ and $\ddot{x}^{(k)} = [\ddot{x}_1^{(k)}, \ddot{x}_2^{(k)}]_{N_t\times 1}$, are generated, which are two sets of N_{mc} simulation of multi-dimensional inputs that have the requisite distribution. N_t denotes memory length of the model. Then, the mean and variance of y_{t+b} given the initial condition I_0 can be calculated by

$$\begin{aligned} \widehat{\bar{y}}_{t+b} \mid I_0 &\cong \frac{1}{N}\sum_{k=1}^{N_{\mathrm{mc}}} f_2\left(\dot{x}^{(k)}\right) \mid I_0; \\ \widehat{V}_x \mid I_0 &\cong \frac{1}{N}\sum_{k=1}^{N_{\mathrm{mc}}}\left(f_2\left(\dot{x}^{(k)}\right) \mid I_0\right)^2 - \left(\widehat{\bar{y}}_{t+b} \mid I_0\right)^2. \end{aligned} \tag{51}$$

The partial variances can be estimated as

$$\begin{aligned} \widehat{V}_1 \mid I_0 &\cong \frac{1}{N}\sum_{k=1}^{N_{\mathrm{mc}}} f_2\left(\dot{x}_1^{(k)}, \dot{x}_2^{(k)}\right) f_2\left(\dot{x}_1^{(k)}, \ddot{x}_2^{(k)}\right) - \left(\widehat{\bar{y}}_{t+b} \mid I_0\right)^2, \\ \widehat{V}_2 \mid I_0 &\cong \frac{1}{N}\sum_{k=1}^{N_{\mathrm{mc}}} f_2\left(\dot{x}_1^{(k)}, \dot{x}_2^{(k)}\right) f_2\left(\ddot{x}_1^{(k)}, \dot{x}_2^{(k)}\right) - \left(\widehat{\bar{y}}_{t+b} \mid I_0\right)^2, \\ \widehat{V}_{12} \mid I_0 &\cong \widehat{V}_x \mid I_0 - \widehat{V}_1 \mid I_0 - \widehat{V}_2 \mid I_0. \end{aligned} \tag{52}$$

To calculate the $\widehat{V}_1 \mid I_0$ with the different initial conditions, the average of these values can be used as the estimates of $E_{I_0}[V_1 \mid I_0]$, and the performance index of nonlinear feedforward and feedback control system can be obtained.

5. Simulation Study

This section presents a simulation experiment to show the effectiveness of the proposed strategy. The model of nonlinear feedforward and feedback control system that we have chosen is expressed as

$$y_t = f\left(u_{t-b}^*\right) + D_{0,t} + q^{-3}\left(1 - 0.6q^{-1}\right) D_{1,t}, \tag{53}$$

where $f(u_{t-b}^*)$ is process model represented by a nonlinear polynomial:

$$\begin{aligned} f\left(u_{t-b}^*\right) = {} & 0.2u_{t-3} + 0.3u_{t-4} + u_{t-5} + 0.8u_{t-3}^2 \\ & + 0.8u_{t-3}u_{t-4} - 0.7u_{t-4}^2 - 0.5u_{t-5}^2 - 0.5u_{t-3}u_{t-5}. \end{aligned} \tag{54}$$

The measured and unmeasured disturbances are, respectively, given by

$$D_{0,t} = \frac{1}{1 - 1.6q^{-1} + 0.8q^{-2}}\alpha_{0.t}, \qquad D_{1,t} = \frac{1}{1 - 0.9q^{-1}}\alpha_{1,t}, \tag{55}$$

where $\{\alpha_{0,t}\}$ and $\{\alpha_{1,t}\}$ are sequences of independent and identically distributed normal variables with mean zero, and the variances are, respectively, 0.05 and 0.1.

Assume that the process is presently being controlled about a fixed set point by a simple proportional feedforward controller in addition to an integral feedback controller. The manipulated variable is given by

$$u_t = -0.1D_{1,t} - \frac{0.3 - 0.2q^{-1}}{1 - q^{-1}} y_t. \tag{56}$$

Two closed-loop signal curves of different time-delay conditions $b = 3,\ l = 5$ and $b = 5,\ l = 3$ are shown in Figure 2. Then, the traditional linear regression method is applied to estimate the MVPLB for nonlinear forward feedback control system. The estimated values of model parameters and MVPLB are shown in Table 1, where the model orders are $J_0 = 7$, $J_1 = 7$, and $J_D = 1$ by applying AIC criterion and the values are calculated by 100 times' statistics. It can be seen that the estimated value of model parameters and MVPLB by traditional linear regression method has larger deviation, which is always larger than the real value. This implies the excessive estimation.

It is necessary to identify the model of closed-loop system to estimate the minimum variance performance index of the

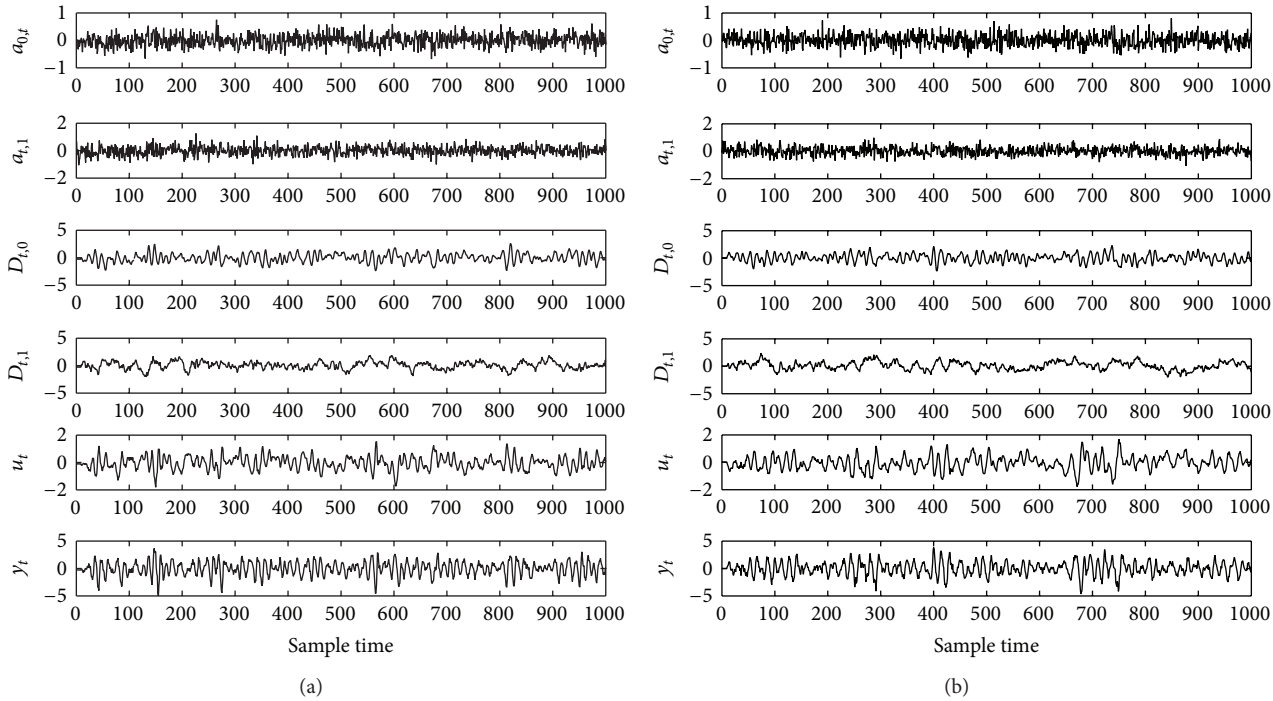

Figure 2: 1000 samples for the closed-loop nonlinear feedforward and feedback system subjected to measured and unmeasured disturbances. (a) $b = 3, l = 5$; (b) $b = 5, l = 3$.

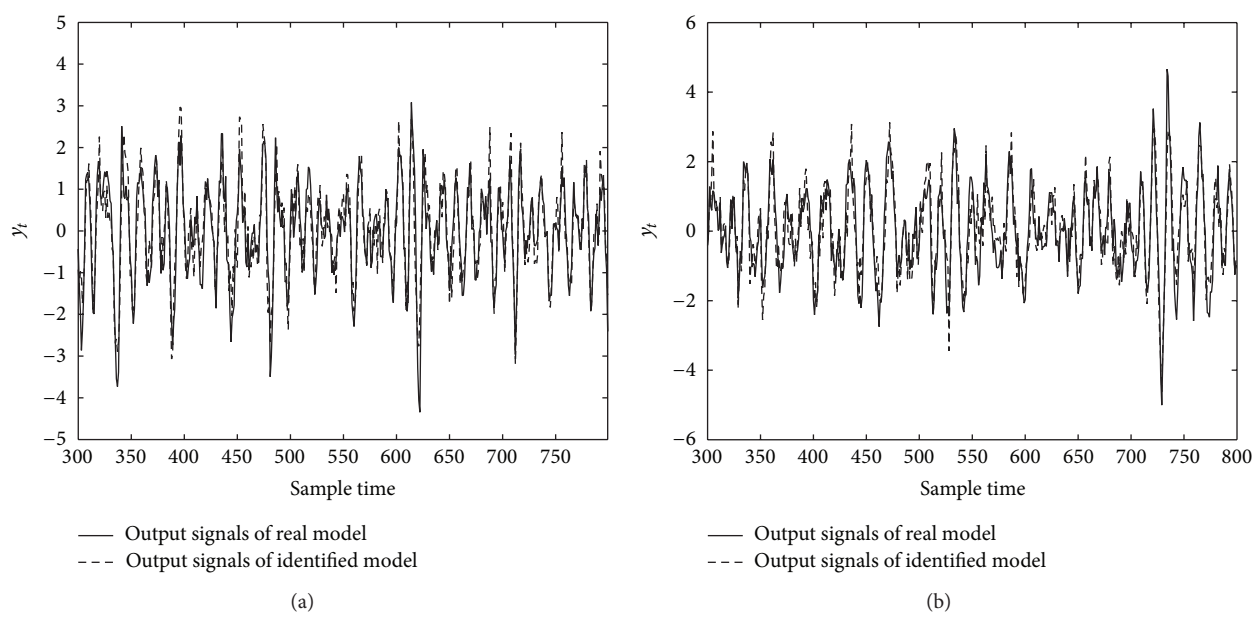

Figure 3: Output signals of identified model comparing with actual model. (a) $b = 3, l = 5$; (b) $b = 5, l = 3$.

nonlinear system. First, we collect the disturbance signals which can be measured and then apply the linear regression method to fit the curve to obtain the parameter of the white noise. Furthermore, we use iterative orthogonal least square method to identify the closed-loop model. The comparison for the output signal of identified model and actual model under two different time delays is shown in Figure 3. We can see the identified model can well approximate to the real nonlinear model.

It is noted that the output variance of nonlinear system is also related to the initial value. Thus, to see whether the resulting controller performance based on variance decomposition method includes the influence of the initial value or not, the output variation of closed-loop system during the period $t = 1, 2, \ldots, 40$ is shown in Figure 4. It can be seen that when $t > 20$, the distribution of the system output tends to be stable; thus we get the conclusion that the output has nothing to do with the initial value.

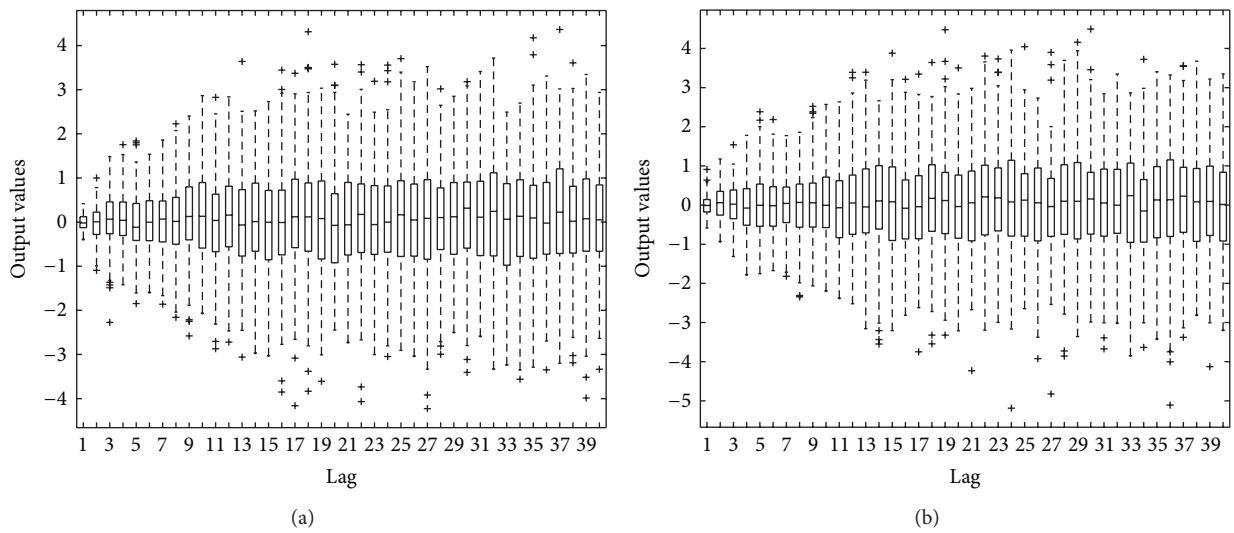

Figure 4: Box plots for closed-loop system output on memory length $t = 1, \ldots, 40$. (a) $b = 3, l = 5$; (b) $b = 5, l = 3$.

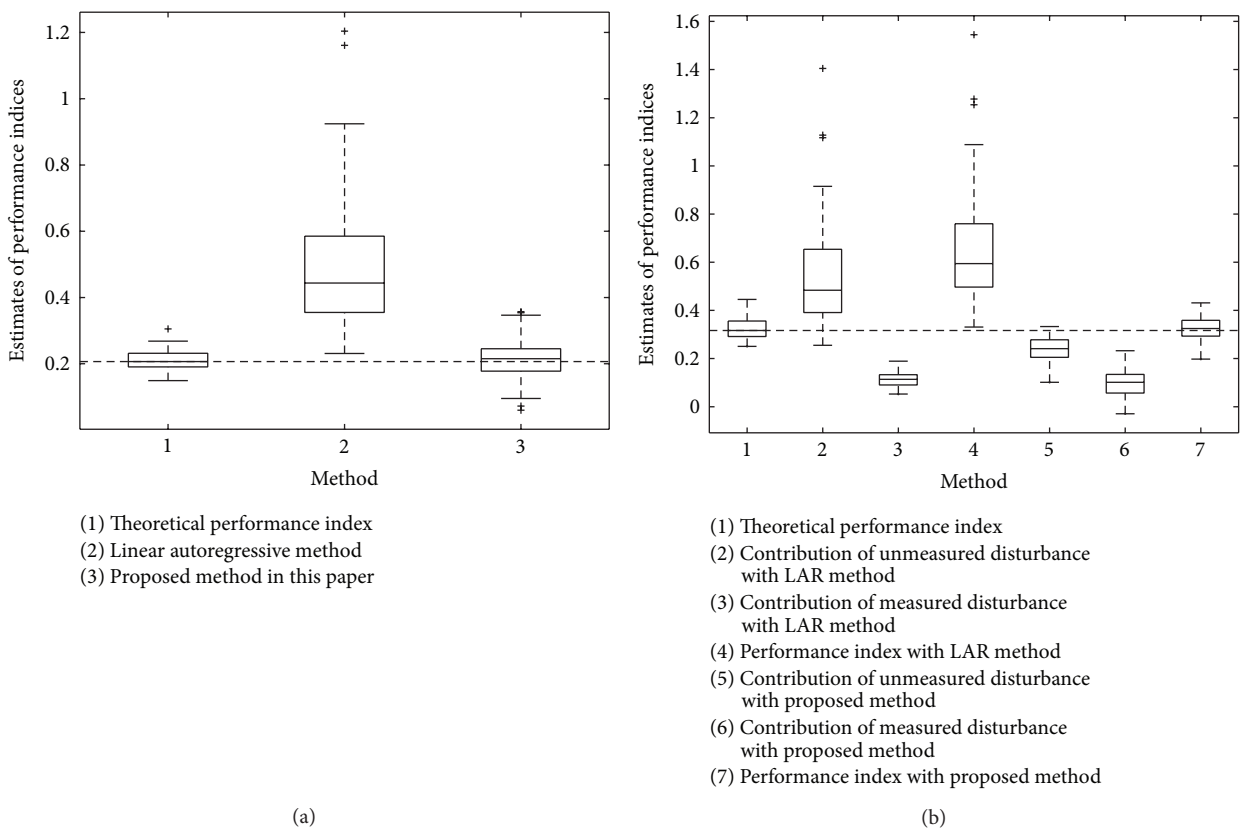

Figure 5: Box plots of the estimates of the minimum variance lower bound for the nonlinear feedforward and feedback control system. (a) $b = 3, l = 5$; (b) $b = 5, l = 3$.

Selecting an appropriate memory length of 40 and applying 100 times' Monte Carlo experiments, the box plots of MVPLB estimates with time delay $b = 3$, $l = 5$, and $b = 5$, $l = 3$ by applying variance decomposition method proposed by this paper and traditional linear estimation method can be seen in Figure 5. In Figure 5(a), the first column gives theoretical performance index for nonlinear system with time delay $b = 3$, $l = 5$, and the second column and third one, respectively, show the estimates of performance index by traditional linear method and that by the method in this paper. In Figure 5(b), the first column gives theoretical performance index for nonlinear system with time delay $b = 5$, $l = 3$, and the fourth column and seventh column, respectively, show the estimates of performance index by traditional linear

method and that by the method in this paper. The second and third column, respectively, show the contributions of performance index of the unmeasured and measured disturbance signal applying traditional linear method. The fifth and sixth column show the contributions of performance index of the unmeasured and measured disturbance signal applying the method proposed by this paper, respectively. From the plot, we can see that the estimates of performance index using our new method are more close to the theoretical value than that using traditional linear method and get the conclusion that it is effective to estimate the MVPLB of nonlinear forward and feedback system by applying the CPA method based on variance decomposition method.

Remarks. (i) The MVPLB of this nonlinear feedback and forward control system can be decomposed into the best possible bounds for each of the controllers. According to the variance contributions of the unmeasured and measured disturbance, we can confirm the degree of controller performance by the feedback controller and the feedforward controller.

(ii) When the feedforward delay exceeds the feedback delay, there is no error in predicting of the future disturbance by using given information at current time. In such case, the overall MVPLB is only the contribution of unmeasured disturbance. This is the reason why only three columns are included in Figure 5.

(iii) This new nonlinear CPA method requires only observable signals and crude estimates of the process delay and another delay that it takes for a change in measured feedforward variable to begin to affect the output.

(iv) The proposed method needs to estimate the closed-loop nonlinear model, and the identification of the closed-loop model will directly affect the estimates of the MVPLB.

6. Conclusions

The problem of control performance assessment for nonlinear feedforward and feedback system is investigated in this paper. We provide a method based on the variance decomposition to estimate the MVPLB for two classes of nonlinear feedforward and feedback control system. When the time delay of the process and measured disturbance are known, the performance index based on minimum variance benchmark can be estimated by the data from the closed-loop system; the simulation shows the effectiveness of the proposed approach. More specifically, the assumption of one measured disturbance is also suitable for the multimeasured disturbance cases; thus the method in this paper can be extended from SISO to MISO.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Predictive Function Optimization Control for a Class of Hydraulic Servo Vibration Systems

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This paper is concerned with the problem of predictive function control (PFC) for a class of hydraulic vibration servo control systems. Our aim is to design a new advanced control strategy such that the control system can track trajectory in a fast and accurate way. For this end, the mathematical model of the hydraulic vibration servo control system is firstly studied. By analyzing the nonlinear, time-varying, and model structure uncertainty features of the objects, the desired control strategy is presented based on PFC. Finally, the simulation results show that our proposed method is effective and can be used to improve the tracking speed, accuracy, and robustness.

1. Introduction

Servo system is a kind of vibration which can make the actuators change rule and the action of vibration control system according to the input signal. Hydraulic control has many advantages, such as quick response speed, high precision of vibration, and online-adjustable vibration parameters. It is not surprising that hydraulic control has been widely used in many areas, for example, the engineering construction, mechanical processing, agricultural machinery, and other industrial and agricultural production processes [1, 2]. By using crushing, piling, drilling, the work of hydraulic engineering machinery and screening, grinding, polishing, dusting, casting production technology of hydraulic vibration equipment, and so forth. Due to the complexity of working environment, hydraulic servo vibration systems are a kind of typical unknown uncertainty systems, where exist the large internal parameter changes and external load disturbance. A foundational question is, how to establish accurate mathematical model? Such a question has increased the difficulty of the control system design. In this case, the conventional linear time-invariant combination of PID control scheme is not suitable owing to the lack of the ability of quick disturbance rejection and difficulty of coordinating the contradiction between speediness and stability and achieving the robustness of the system [3, 4].

It is well known that predictive control is suitable for the case, when the controlled object is not easy to build accurate mathematical model and also has complex industrial production process, such as petroleum, chemical industry, metallurgy, and other areas of the process industry [5]. With the development of the theory and application aspects, predictive control technology has made great development [6–9]. Predictive functional control (PFC) method [10–12] is developed on the basis of the principle of predictive control of a novel predictive control algorithm. In view of the advantages of model predictive control, it enhances the regularity of input control by introducing basis functions to improve the quickness and accuracy. The adaptive predictive functional controller has been designed based on the stability of the Laguerre model to solve the induction motor efficiency. By solving optimization problem of the maximum torque current ratio control [10] and employing feedforward compensation decoupling design idea, the considered system is decomposed into two with measurable disturbances of single into a single subsystem. The simulation experimental results

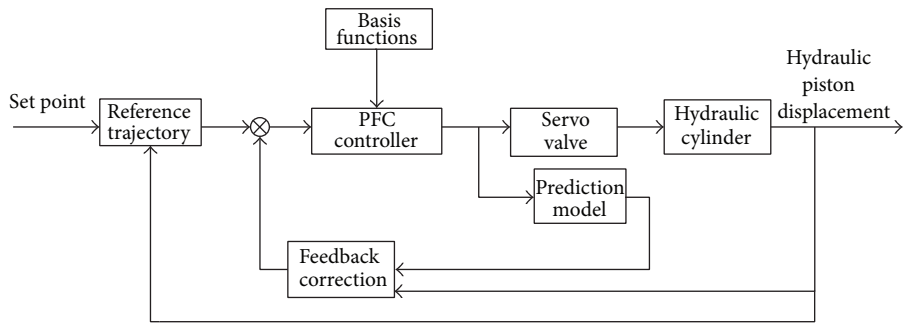

Figure 1: Block scheme of predictive function control (PFC).

show that compared with traditional PI current controller, the controller has high tracking precision, fast response speed, strong anti-interference ability, and the good control effect.

Time delays are frequently encountered in a variety of dynamic systems [13–16]. For the large pure time-delay system, there was a kind of inner loop PI control and outer loop using a scale factor self-tuning fuzzy incremental predictive functional control strategy, which was applied to the circuit and the main form of the object in general controlled object in [11]. A large number of simulation experiments indicated that this method was better than the other methods; even in the severe model mismatch case, it still has very strong robustness and anti-interference ability. In [12], Kautz function approximation was used to get the characterization process state space equation of object; an adaptive predictive functional controller based on Kautz was designed. In [17], the design method of the typical predictive functional controller of servo control system was given by employing the basic principle and characteristics of predictive functional control. However, to the best of our knowledge, by using the PFC method, there are few attempts that have been made to cope with the rapid servo system. Such a question has not been fully studied.

In view of the above analysis, this paper deals with the problem of hydraulic servo vibration control in engineering practice. Based on the analysis of the mathematical model of hydraulic servo vibration control system, the controlled object of nonlinear, time-varying, and uncertainty model structure, and characteristics of control system for quick tracking research, an optimal control scheme is presented in light of the predictive functional control algorithm. The effectiveness of the given PFC method is verified by MATLAB simulation which is applied to the effectiveness of the hydraulic servo system of vibration control.

2. Predictive Functional Control Algorithm Design

Predictive function control belongs to model predictive control. Compared to the traditional model predictive control, it not only has three basic characteristics (forecasting model, rolling optimization, and feedback correction) of generally predictive control but also has its own characteristics. In order to emphasize the control input, each moment added control input is structured and seen as a linear combination of a number of preselected basis functions. By using the known processes of these basis functions, the weights of the objective function can be obtained from the optimization calculation of basis functions to get the corresponding control amount. In the hydraulic vibration servo system, operating quantity is the hydraulic servo valve opening and the output is the displacement of hydraulic piston rod. By changing the opening of the hydraulic servo valve to regulate the amount of oil hydraulic cylinder, the output displacement hydraulic piston rod is controlled. In this paper, the design of PFC control project such as Figure 1, show that PFC major compositions include: choice of basis functions, the reference trajectory prediction model, error compensation, and rolling optimization of several parts.

2.1. The Choice of Basis Functions. In the hydraulic servo predictive functional control algorithm, the control input mechanism is an important factor affecting system performance [18, 19], and consequently the role of the newly added control can be described as a linear combination of a number of basis functions:

$$u(k+i) = \sum_{n=1}^{N} \mu_n f_n(i), \tag{1}$$

where $i = 0, 1, \ldots, P-1$ and f_n $(n = 1, \ldots, N)$ are basis functions, μ_n are the linear combination coefficients, $f_n(i)$ mean the value of the base functions when $t = iT$, and P represents the length of time domain prediction optimization. The selection of basis function is based on the nature of the controlled object and the requirements of desired trajectory. For example, the base functions often take a step, ramp, or exponential function. Depending on the selected basis function, the output response of the object is calculated by means of off-line.

2.2. Model Predictions. The output displacement of hydraulic servo predictive model predictive function control $y_m(k)$ is composed of model free output displacement $y_1(k)$ and model output displacement function $y_f(k)$. Model free output represents the output of the model, which is determined by the control measured amount in the past instead of

the current time and the future. The expression is given as follows:

$$y_1(k) = F(x(k)). \tag{2}$$

In this expression, F means the mathematical expression of object prediction model and $x(k)$ means the information which is known at time k. Model function output $y_f(k)$ stands for the new model response after adding controls at the present time, which is another part of the output. In the hydraulic servo predictive functional control, controlling the structure of the input is both the key to ensure the control performance and the difference between the model predictive control and other control methods. The displacement output of future model function can be expressed as

$$y_f(k+i) = \sum_{n=1}^{N} \mu_n g_n(i), \quad i = 1, \ldots, P, \tag{3}$$

where $g_n(i)$ is the model output by $f_n(i)$; displacement model output of hydraulic servo can be calculated by the following formula:

$$y_m(k) = y_1(k) + y_f(k). \tag{4}$$

2.3. Reference Trajectories. Predictive functional control is the same as MAC, in the control process making the process output tracking reference trajectories gradually prevent dramatic changes of controlled quantity and overshoot phenomenon. For the hydraulic servo control system, reference trajectories can be as a first-order exponential form like

$$y_r(k+i) = c(k+i) - \lambda^i \left(c(k) - y_p(k)\right). \tag{5}$$

In $\lambda = e^{(-T_s/T_r)}$, T_s represents the sampling period, T_r means the reference trajectories time constant, and $c(k)$ is the value set.

2.4. Rolling Optimization. Optimization objective of hydraulic servo predictive model predictive function control can be expressed as

$$J = \min \left\{ \sum_{i=P_1}^{P_2} \left[y_r(k+h_i) - y_p(k+h_i) \right]^2 \right\}, \tag{6}$$
$$y_p(k+i) = y_m(k+i) + e(k+i),$$

where P_1 and P_2 are, respectively, the minimum and maximum of the optimizing time domain, $y_p(k+i)$ represents the forecast process output displacement of hydraulic servo, $e(k+i)$ is the future error of the process displacement, and $y_m(k+i)$ is the displacement output of the model at time $k+i$.

2.5. Error Prediction and Compensation. Because of the actual control process model mismatch affected by nonlinear characteristics as well as other uncertainties, the displacement of the predicted value will deviate from the actual value. In the control system, the displacement error between hydraulic servo object and model input is sent to the predictor. Therefore it will be found as the feedforward, which is input to reference trajectory for compensation. And the future forecast displacement error is

$$e(k+i) = y(k) - y_m(k), \tag{7}$$

where $y_m(k)$ represents the model output displacement at time k.

For the next $n+i$ times prediction of displacement error in PFC algorithm, in order to improve accuracy, the polynomial fitting error is employed which is estimated based on a known time value:

$$\begin{aligned} e(n+i) &= e(n) + \sum_{l=1}^{l_2} e_l(n) i^l \\ &= y_p(n) - y_m(n) + \sum_{l=1}^{l_2} \beta_l(n) i^l, \quad (i = 1, 2, \ldots, L). \end{aligned} \tag{8}$$

Among them, $e(n+i)$ is the displacement prediction error between hydraulic servo system and model at time $n+i$; it is composed of a corrected error and error at time n. This process ($L \geq l_2 \geq 1$, $L \geq l_2 \geq 1$, $L \geq l_2$) is called self-compensation.

In PFC control algorithm, prediction horizon length P, basis function $f_n(i)$, and time coefficient of reference trajectories λ are the important parameters of controller designed. The choice of basis functions can broadly determine control accuracy, stability and robustness of the control mainly determined by the range of the prediction horizon, and the reference trajectories major impact on the dynamic response of the control system. For the impact of the system, a different design has different emphases. Therefore, it can quickly adjust the parameters according to the specific performance requirements to shorten the setting time which is a major advantage of the PFC control.

3. The Mathematical Modeling of Hydraulic Servo System

Hydraulic servo vibration system is the use of the variations in the oil pressure flow to deliver hydraulic energy and directly produce piston reciprocating cycle. Variation of pressure in the oil flow is dependent on the hydraulic vibration equipment in the process of vibration motion parameters (such as velocity, acceleration, and amplitude) or liquid parameters (such as pressure, flow, etc.) change as a feedback signal to control. Due to the advantages of high control accuracy, stiffness big, fast response speed, high speed startup, inverse kinematics, and so forth, the hydraulic control system can form of light weight, small volume, accelerating ability, quick action, and high control precision of control system, to drive the high power load. Therefore, the hydraulic servo system has been more and more widely used in the agricultural engineering machinery and equipment and production. The hydraulic servo system of the vibration control block scheme is shown in Figure 2.

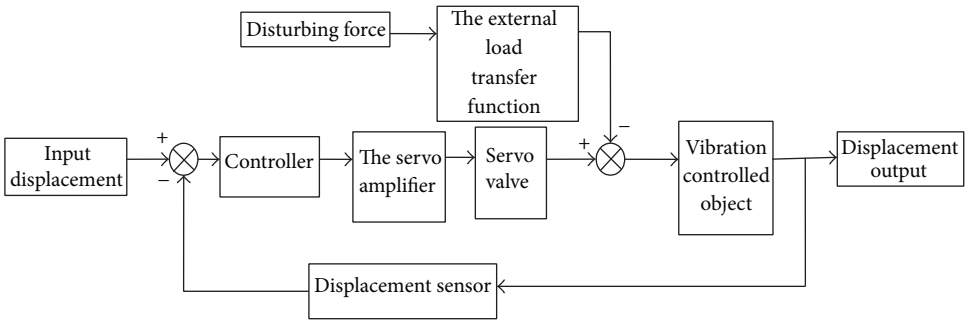

Figure 2: Block scheme of the hydraulic servo control system.

3.1. Basic Equation of Hydraulic Servo Valve. The hydraulic servo valve is an extremely complex closed-loop control system, which usually is as the input signal. According to physical characteristics to establish the output flow of linearized equation for [17]

$$Q_L = Q_{\text{svo}} - K_c P_L, \tag{9}$$

where $Q_{\text{svo}} = K_{\text{sv}} I_c$ and I_c stands for the input current signal, Q_{svo} is the servo valve light flow, K_{sv} is the servo valve static flow of amplification coefficient, K_c stands for the pressure of the servo valve flow amplification coefficient, and P_L is for load pressure.

In view of the characteristic analysis of the hydraulic system, the servo valve has often the very high response characteristics; the dynamic can be ignored when compared with the hydraulic power components, so we can approximately regard it as a proportion of link [14]. Consider

$$\frac{Q_{\text{svo}}}{I_c} = \frac{K_{\text{sv}}}{1 + (s/w_{\text{sv}})}. \tag{10}$$

3.2. Basic Equation of Servo Amplifier and Displacement Sensor. We can approximate the servo amplifier and displacement sensor link as proportion link; then

$$\begin{aligned} I_c &= K_p U, \\ y &= K_s x_p. \end{aligned} \tag{11}$$

By Laplace transformation,

$$Y = K_s X_p, \tag{12}$$

where y is the actual measured output by displacement sensor, K_s stands for displacement sensor amplifier gain, K_p stands for power amplifier amplification gain, and U is the output of the controller instructions.

3.3. Determining the Transfer Function. The flow gain of servo valve is as follows:

$$K_{\text{sv}} = \frac{q_n}{I_n}. \tag{13}$$

The transfer function of the servo valve is as follows:

$$G_{\text{sv}}(s) = \frac{Q_{\text{sv}}}{I_c} = \frac{K_{\text{sv}}}{(s^2/w_{\text{sv}}^2) + (2\xi_{\text{sv}}/w_{\text{sv}})\, s + 1}. \tag{14}$$

For transfer function of cylinder piston displacement output X_p for

$$G_V(s) = \frac{Q_L}{i} = \frac{1/A_p}{s\left((s^2/w_h^2) + (2\xi_h s/w_h) + 1\right)}. \tag{15}$$

For transfer function of external disturbance load FL input to the transfer function of cylinder piston displacement output X_p for

$$G_L(s) = \frac{X_p(s)}{F_L(s)} = \frac{-\left(K_{ce}/A_p^2\right)\left(1 + (V_l/4\beta_e K_{ce})\, s\right)}{s\left((s^2/w_h^2) + (2\xi_h s/w_h) + 1\right)}. \tag{16}$$

Thus, determining the system block diagram is shown in Figure 3.

Note that $K_f = A_p s$. So, the system open loop transfer function is

$$\begin{aligned} G_k(s) &= K_p G_{\text{sv}}(s)\, G_v(s) \\ &= \frac{1/A_p}{s\left((s^2/w_h^2) + (2\xi_h/w_h)\, s + 1\right)} \times K_p K_{\text{sv}} K_s. \end{aligned} \tag{17}$$

4. Simulation Result and Analysis

The hydraulic vibration, which makes use of liquid pressure, realizes vibration object sinusoidal movement up and down in the power system and servo valve for pressure control. Because of the larger vibration and impact, hydraulic vibration which is suitable for high temperature and high pressure, such as underwater environment, not only can be used for drilling, crushing, piling, and drilling, such as hydraulic pressure road engineering machinery homework tasks, but also can be used in farm, furrowing delisting, crop cultivation and harvesting, and water conservancy irrigation and agriculture engineering field [20, 21]. However, such hydraulic vibration mechanical equipment generally requires the hydraulic control system to drive the controlled according

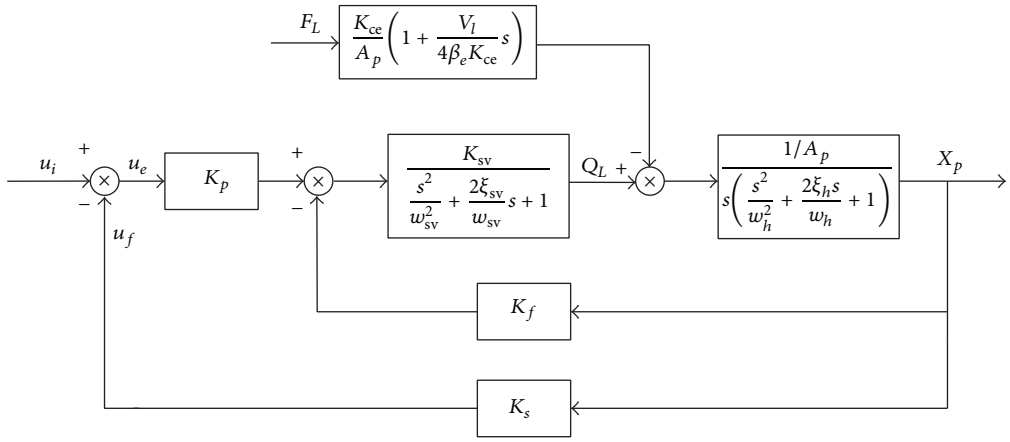

Figure 3: Block scheme of the hydraulic servo system.

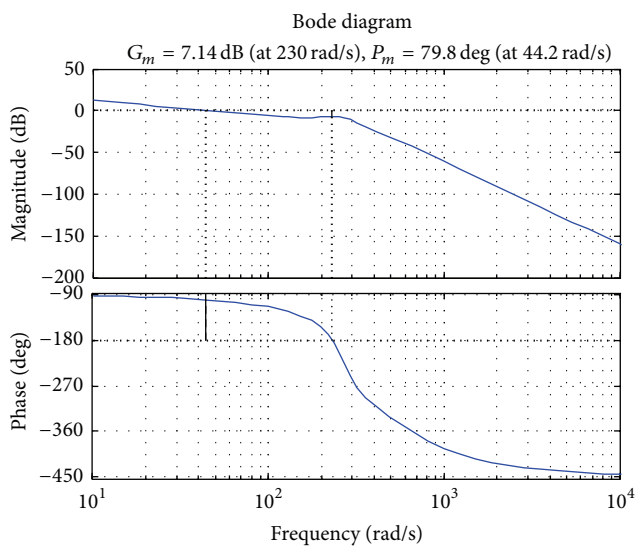

Figure 4: The amplitude and phase frequency characteristics of the servo system.

to the given amplitude, frequency, and the sine of the up and down reciprocating movement. This paper selects the continuous casting mould hydraulic vibration technology as an example for discussion [22]; calculation results of related parameters can be obtained as follows: $A_p = 7.91 \times 10^{-3}\ \mathrm{m}^3$, $K_p = 0.00\,1\ \mathrm{A/V}$, $K_{sv} = 2.5 \times 10^{-2}\ \mathrm{m}^3/(\mathrm{s{\cdot}A})$, $w_h = 38\ \mathrm{rad/s}$, $\zeta_h = 0.25$, and $K_s = 200\ \mathrm{v/m}$. Put them into formula (17), using model conversion function tf2ss(num,den) of MATLAB, we can get the state space model for

$$\dot{x} = \begin{bmatrix} 2.8349 & -2.735 & 0.9002 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \qquad (18)$$
$$y = \begin{bmatrix} 0.0005 & 0.0019 & 0.0005 \end{bmatrix} x.$$

Through the study of the amplitude-phase frequency stability analysis of system, we obtain the system phase frequency bode diagram as shown in Figure 4.

As is shown in the figure, we can draw that the crossover frequency $w_c = 44.2\ \mathrm{rad/s}$, phase margin $\gamma = 79.8^\circ$; when phase frequency, through -180°, $w_g = 230\ \mathrm{rad/s}$, amplitude margin Kg (dB) $= 7.14\ \mathrm{dB}$. We can also conclude that the system dynamic performance is poorer and the precision of tracking curve is not high when crossing frequency w_c is small.

In order to study the tracking performance of hydraulic servo vibration system, this paper selects the sine signal $R(t) = 7 * \sin(5\pi t)$ as a set point trajectory; the validity of the method of PFC can be verified via the MATLAB simulation to compare the control effect of PID and PFC. As shown in Figure 6. When using traditional PID controller to control, taking Z-N method [23], the PID gain parameters $k_p = 9.7$, $k_i = 2.5\ \mathrm{s}$, and $k_d = 0.4\ \mathrm{s}$. In view of the servo vibration system state space model [23], using MATLAB simulation tools, writing M file simulation program, PFC simulation prediction optimization time domain is 20, the reference trajectory of time constant Tr $= 2\ \mathrm{ms}$, and sampling time is

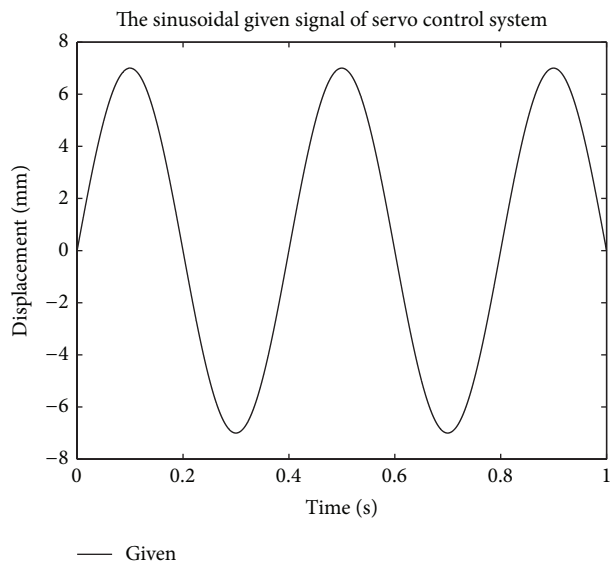

Figure 5: Nonsinusoidal reference signal of the servo control system.

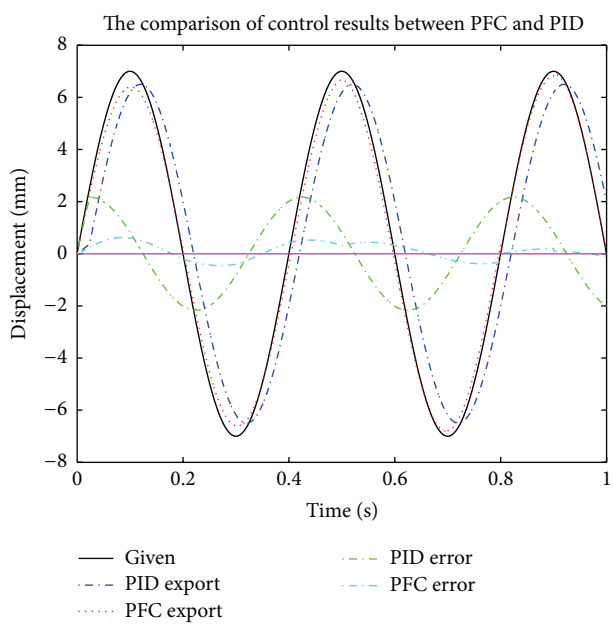

Figure 6: Comparison of PFC and PID control simulation results.

0.6 ms. Selecting index function $f_n(i) = i^{k-1}$ as basis function, linear combination number is 2. After the simulation under the control of predictive function, we get the results in Figures 5 and 6.

In Figure 6 there are the nonsinusoidal velocity response curves of the hydraulic servo system and error curve obtained by PFC and PID control. From Figure 6, as to traditional PID controllers, the output of the system can track the change of the input signal, but there is a certain phase lag and short of waveform completely tracking precision of the system requirements. But the displacement output settings obtained by using predictive functional control are better able to track the trajectory and its control effect is better than conventional PID control methods.

5. Conclusion

In this paper, hydraulic servo vibration system is the specific research object, aiming at the existence and the uncertainty of its larger internal external load disturbance and parameter, based on further study of the characteristics of the hydraulic servo control system. And the predictive functional control method is applied to hydraulic servo vibration system.

The simulation result indicates that the hydraulic servo system, based on the good control quality of predictive functional control such as fast dynamic response, small overshoot, and strong stability, can effectively achieve rapid location tracking. Its control system dynamic and static quality and robustness in the case of time-varying parameters and antijamming capability are superior to the conventional PID control method. The parameter uncertainty has good robustness, with high value of engineering application.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publishing of this paper.

Acknowledgments

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Research Article

Delay-Dependent Robust $L_2 - L_\infty$ Filtering for a Class of Fuzzy Stochastic Systems

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This paper is concerned with the $L_2 - L_\infty$ filtering problem for a kind of Takagi-Sugeno (T-S) fuzzy stochastic system with time-varying delay and parameter uncertainties. Parameter uncertainties in the system are assumed to satisfy global Lipschitz conditions. And the attention of this paper is focused on the stochastically mean-square stability of the filtering error system, and the $L_2 - L_\infty$ performance level of the output error with the disturbance input. The method designed for the delay-dependent filter is developed based on linear matrix inequalities. Finally, the effectiveness of the proposed method is substantiated with an illustrative example.

1. Introduction

It is well known that many phenomena in engineering have unavoidable uncertain factors that are modeled by the stochastic differential equation. And in recent years, the stochastic system has been widely studied. A great number of investigations on stochastic systems have been reported in the literature. For example, the adaptive back stepping controller has been addressed in [1, 2] for stochastic nonlinear systems in a strict-feedback form. When the time delay appears, [3, 4] have investigated the stability of the time-delay stochastic neutral networks; controllers under different performance levels have been designed for the stochastic system in [5–7] for the delay-dependent controller, H_∞ output feedback controller, and $L_2 - L_\infty$ controller, respectively. And [8–14] have studied the controlling and filtering problem for stochastic jumping systems. However, the results mentioned above are only suitable for the nonlinear systems which have exact known nonlinear dynamics models. As an efficient technique to linearize the nonlinear differential equations, T-S fuzzy model [15] can offer a good way to represent the nonlinear dynamics models.

By using T-S fuzzy model, nonlinear systems turn into linear input-output relations which could be handled easily by appropriate fuzzy sets. This method can be seen in the stirred tank reactor system in [16] and the truck trailer system in [17]. Nowadays, the researches of T-S fuzzy system have grown into a great number. A lot of results have been reported in the literature. For example, the stability and control problem of T-S fuzzy systems have been investigated in [18–22] and the references therein.

On the other hand, state estimation has been found in many practical applications and it has been extensively studied over decades. It aims at estimating the unavailable state variables or their combination for the given system [23, 24]. As a branch of state estimation theory, the filtering problem has become an important research field. The H_∞ filtering problem for the T-S fuzzy system has been addressed in [25–30]; [31–33] have considered the $L_2 - L_\infty$ filtering problem for delayed T-S fuzzy systems with different method. Moreover, robust filters are investigated in [34–36] for stochastic nonlinear systems.

Following above discussion, T-S fuzzy model could be used to divide the nonlinear stochastic systems into several subsystems. And during the past decade, many problems have been tackled. Reference [37] deals with the robust fault detection problem for T-S fuzzy stochastic systems. And [38, 39] consider the stabilization for the fuzzy stochastic systems with delays. References [40–43] have studied the control problem for fuzzy stochastic systems. An adaptive fuzzy controller has been designed for stochastic nonlinear systems in [44]. Reference [45] addresses the passivity of the stochastic

T-S fuzzy system. Solutions to fuzzy stochastic differential equations with local martingales have been addressed in [46]. Then recognizing the value of state estimating when state variables are unavailable, it is important to research the filtering problem for T-S fuzzy stochastic systems. However, there are few results available to the best of the authors knowledge, especially the results on L_2-L_∞ filtering problem for the fuzzy stochastic systems.

As a consequence, this paper will focus on the robust fuzzy delay-dependent L_2-L_∞ filter design for a T-S fuzzy stochastic system with time-varying delay and norm-bounded parameter uncertainties by using the Lyapunov-Krasovskii functional technique and some useful free-weighting matrices. The obtained sufficient conditions are expressed in terms of linear matrix inequality (LMI) approach. The remainder of this paper is organized as follows. The filter design problem is formulated in Section 2. And Section 3 gives our main results. In Section 4, a numerical example is shown to illustrate the effectiveness of the proposed methods. Finally, we conclude the paper in Section 5.

Notation. The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition. Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp., positive definite). $\mathbf{R}^n$ denotes the n-dimensional Euclidean space and $\mathbf{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. I stands for an identity matrix of appropriate dimension, while $I_n \in \mathbf{R}^n$ denotes a vector of ones. The notation $*$ is used as an ellipsis for terms that are induced by symmetry. diag(...) stands for a block-diagonal matrix. $|\cdot|$ denotes the Euclidean norm for vectors and $\|\cdot\|$ denotes the spectral norm for matrices. $\mathbf{L}_2[0,\infty)$ represents the space of square-integrable vector functions over $[0,\infty)$. $\mathsf{E}(\cdot)$ stands for the mathematical expectation operator. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation and Preliminaries

Consider the time-delay T-S fuzzy stochastic system with time-varying parameter uncertainties as the following form:

$$\begin{aligned}
(\Sigma) : dx(t) &= \sum_{i=1}^{r}\rho_i(s(t))\{[(A_i+\Delta A_i(t))x(t) \\
&\quad + (A_{di}+\Delta A_{di}(t))x(t-\tau(t))+B_i v(t)]\,dt \\
&\quad + [(H_i+\Delta H_i(t))x(t) \\
&\quad + (H_{di}+\Delta H_{di}(t))x(t-\tau(t))]\,d\omega(t)\}, \\
dy(t) &= \sum_{i=1}^{r}\rho_i(s(t))[C_i x(t)+C_{di}x(t-\tau(t))+D_i v(t)]\,dt, \\
z(t) &= \sum_{i=1}^{r}\rho_i(s(t))[L_i x(t)], \\
x(t) &= \varphi(t), \quad t\in[-h_2,0],
\end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^m$ is the system state; $\varphi(t)$ is a given differential initial function on $[-h_2,0]$; $\omega(t)$ is a scalar zero mean Gaussian white noise process with unit covariance; $y(t) \in \mathbf{R}^n$ is the measured output; $z(t) \in \mathbf{R}^l$ is a signal to be estimated; $v(t) \in \mathbf{R}^s$ is the noise signal which belongs to $\mathscr{L}_2[0,\infty)$; $\tau(t)$ is a continuous differentiable function representing the time-varying delay in $x(t)$, which is assumed to satisfy for all $t \geq 0$,

$$0 \leq h_1 \leq \tau(t) < h_2. \tag{2}$$

In the considered fuzzy stochastic system, A_i, A_{di}, B_i, H_i, H_{di}, C_i, C_{di}, D_i, and L_i are known constant matrices with appropriate dimensions. $\Delta A_i(t)$, $\Delta A_{di}(t)$, $\Delta H_i(t)$, and $\Delta H_{di}(t)$ represent the unknown time-varying parameter uncertainties and are assumed to satisfy

$$\begin{bmatrix}\Delta A_i(t) & \Delta A_{di}(t)\\ \Delta H_i(t) & \Delta H_{di}(t)\end{bmatrix} = \begin{bmatrix}M_{1i}\\ M_{2i}\end{bmatrix} F_i(t)\begin{bmatrix}N_{1i} & N_{2i}\end{bmatrix}, \tag{3}$$

where M_{1i}, M_{2i}, N_{1i}, and N_{2i} are known real constant matrices and the unknown time-varying matrix function satisfying

$$F_i(t)^T F_i(t) \leq I \quad \forall t. \tag{4}$$

And using the fuzzy theory, there always have for all t,

$$\rho_i(s(t)) \geq 0, \quad i=1,2,\ldots,r, \quad \sum_{i=1}^{r}\rho_i(s(t)) = 1. \tag{5}$$

The fuzzy filters we considered are as follows:

$$\begin{aligned}
d\hat{x}(t) &= \sum_{i=1}^{r}\rho_i(s(t))\left[A_{fi}\hat{x}(t)\,dt + B_{fi}dy(t)\right], \\
\hat{z}(t) &= \sum_{i=1}^{r}\rho_i(s(t))\left[L_{fi}\hat{x}(t)\right],
\end{aligned} \tag{6}$$

in which the fuzzy rules have the same representations as in (1). $\hat{x}(t) \in \mathbf{R}^n$ and $\hat{z}(t) \in \mathbf{R}^l$. A_{fi}, B_{fi}, and L_{fi} are the filters needed to be determined.

Remark 1. It is worth to mention that there are two approaches for the filter design in fuzzy systems. The implementation of the filter could be chosen to depend on or not depend on the fuzzy rules when the fuzzy model is available or not. And it is obvious to see that the former filter related to the fuzzy rules is less conserve and more complex. So we assume that the fuzzy is known here, which means the fuzzy-rule-dependent filter is investigated in this paper as in (6).

Let $\xi(t) = [x(t)^T \quad \hat{x}(t)^T]^T$ and $e(t) = z(t) - \hat{z}(t)$. And the filtering error dynamic system can be written as

$$\begin{aligned}
(\tilde{\Sigma}) : d\xi(t) &= \left[(\tilde{A}+\Delta\tilde{A}(t))\xi(t) + (\tilde{A}_d+\Delta\tilde{A}_d(t))K\xi(t-\tau(t)) \right. \\
&\quad \left. + \tilde{B}v(t)\right]dt \\
&\quad + \left[(\tilde{H}+\Delta\tilde{H}(t))\xi(t) + (\tilde{H}_d+\Delta\tilde{H}_d(t)) \right. \\
&\quad \left. \times K\xi(t-\tau(t))\right]d\omega(t), \\
e(t) &= \tilde{L}\xi(t),
\end{aligned} \tag{7}$$

where

$$\widetilde{A} = \begin{bmatrix} \overline{A} & 0 \\ \overline{B}_f\overline{C} & \overline{A}_f \end{bmatrix}, \qquad \widetilde{A}_d = \begin{bmatrix} \overline{A}_d \\ \overline{B}_f\overline{C}_{d1} \end{bmatrix},$$

$$\widetilde{H} = \begin{bmatrix} \overline{H} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta\widetilde{A}(t) = \begin{bmatrix} \Delta\overline{A}(t) & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Delta\widetilde{A}_d(t) = \begin{bmatrix} \Delta\overline{A}_d(t) \\ 0 \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} \overline{B} \\ \overline{B}_f\overline{D} \end{bmatrix},$$

$$\Delta\widetilde{H}(t) = \begin{bmatrix} \Delta\overline{H}(t) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta\widetilde{H}_d(t) = \begin{bmatrix} \Delta\overline{H}_d(t) \\ 0 \end{bmatrix},$$

$$\widetilde{H}_d = \begin{bmatrix} \overline{H}_d \\ 0 \end{bmatrix}, \qquad \overline{A} = \sum_{i=1}^{r}\rho_i(s(t))A_i,$$

$$\overline{A}_d = \sum_{i=1}^{r}\rho_i(s(t))A_{di}, \qquad \overline{C} = \sum_{i=1}^{r}\rho_i(s(t))C_i,$$

$$\overline{H} = \sum_{i=1}^{r}\rho_i(s(t))H_i, \qquad \overline{H}_d = \sum_{i=1}^{r}\rho_i(s(t))H_{di},$$

$$\overline{C}_d = \sum_{i=1}^{r}\rho_i(s(t))C_{di}, \qquad \overline{B} = \sum_{i=1}^{r}\rho_i(s(t))B_i,$$

$$\overline{D} = \sum_{i=1}^{r}\rho_i(s(t))D_i, \qquad \widetilde{L} = \begin{bmatrix} \overline{L} & -\overline{L}_f \end{bmatrix},$$

$$\overline{A}_f = \sum_{i=1}^{r}\rho_i(s(t))A_{fi}, \qquad \overline{B}_f = \sum_{i=1}^{r}\rho_i(s(t))B_{fi},$$

$$\overline{L}_f = \sum_{i=1}^{r}\rho_i(s(t))L_{fi}, \qquad \overline{L} = \sum_{i=1}^{r}\rho_i(s(t))L_i,$$

$$\overline{L}_d = \sum_{i=1}^{r}\rho_i(s(t))L_{di}, \qquad K = \begin{bmatrix} I & 0 \end{bmatrix},$$

$$\Delta\overline{A}(t) = \sum_{i=1}^{r}\rho_i(s(t))\Delta A_i(t),$$

$$\Delta\overline{A}_d(t) = \sum_{i=1}^{r}\rho_i(s(t))\Delta A_{di}(t),$$

$$\Delta\overline{H}(t) = \sum_{i=1}^{r}\rho_i(s(t))\Delta H_i(t),$$

$$\Delta\overline{H}_d(t) = \sum_{i=1}^{r}\rho_i(s(t))\Delta H_{di}(t). \tag{8}$$

We intend to design sets of fuzzy filters in the form of (6) in this paper, such that for any scalar $0 \le h_1 < h_2$ and a prescribed level of noise attenuation $\gamma > 0$, the filtering error system $(\widetilde{\Sigma})$ could be mean square stable. Moreover, the error system $(\widetilde{\Sigma})$ satisfies $L_2 - L_\infty$ performance.

Throughout the paper, we adopt the following definitions and lemmas, which help to complete the proof of the main results.

Definition 2. The system (Σ) is said to be robust stochastic mean-square stable if there exists $\delta(\varepsilon) > 0$ for any $\varepsilon > 0$ such that

$$\mathsf{E}\left(\|x(t)\|^2\right) < \varepsilon, \quad t > 0, \tag{9}$$

when $\sup_{-h\le s\le 0}\mathsf{E}(\|\varphi(s)\|^2) < \delta(\varepsilon)$, for any uncertain variables. And in addition,

$$\lim_{t\to\infty}\mathsf{E}\left(\|x(t)\|^2\right) = 0, \tag{10}$$

for any initial conditions.

Definition 3. The robust stochastic mean-square stable system $(\widetilde{\Sigma})$ is said to satisfy the $L_2 - L_\infty$ performance, for the given scalar $\gamma > 0$ and any nonzero $v(t) \in L_2[0,\infty)$, and the system $(\widetilde{\Sigma})$ satisfies

$$\|e(t)\|_\infty < \gamma\|v(t)\|_2, \tag{11}$$

and for any uncertain variables, where

$$\|e(t)\|_\infty^2 := \sup_t e(t)^T e(t). \tag{12}$$

Lemma 4. *For the given matrices M, N, F with $F^TF \le I$ and positive scalar $\varepsilon > 0$, the following inequality holds:*

$$MFN + (MFN)^T \le \varepsilon MM^T + \varepsilon^{-1}N^TN. \tag{13}$$

3. Robust Stochastic Stabile

First, we define the following variables for convenience:

$$\begin{aligned}\Phi(t) &= \left(\widetilde{A} + \Delta\widetilde{A}(t)\right)\xi(t) + \left(\widetilde{A}_d + \Delta\widetilde{A}_d(t)\right)K\xi(t-\tau(t)) \\ &\quad + \widetilde{B}v(t), \\ g(t) &= \left(\widetilde{H} + \Delta\widetilde{H}(t)\right)\xi(t) + \left(\widetilde{H}_d + \Delta\widetilde{H}_d(t)\right)K\xi(t-\tau(t)).\end{aligned} \tag{14}$$

Theorem 5. *The filtering error system $(\widetilde{\Sigma})$ is robust stochastic mean square stable and (11) is satisfied for any time-varying delay $0 \le h_1 \le \tau(t) < h_2$, if there exist matrices $P = P^T > 0$, $R = R^T > 0$, $Q_i = Q_i^T > 0$, $Z_i = Z_i^T > 0$, T_{1i}, T_{2i}, $i = 1, 2$, such that the following matrix inequalities hold:*

$$\begin{bmatrix} P & \widetilde{L}^T \\ \widetilde{L} & \gamma^2 I \end{bmatrix} > 0, \qquad \Psi = \begin{bmatrix} \Omega & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0, \tag{15}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & \Omega_{14} & 0 & P\widetilde{B} \\ * & \Omega_{22} & 0 & \Omega_{24} & 0 & 0 \\ * & * & \Omega_{33} & \Omega_{34} & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 \\ * & * & * & * & * & -I \end{bmatrix},$$

$$\begin{aligned}
\Psi_{12} &= \left[\widetilde{T}_1 \;\; \widetilde{T}_2 \;\; h_{21}\widetilde{T}_1 \;\; h_{21}\widetilde{T}_2 \;\; h_{21}\breve{A}^T K^T Z_1 \;\; h_{21}\breve{H}^T K^T Z_2 \;\; \breve{H}P\right],\\
\Psi_{22} &= \operatorname{diag}\{-Z_2, -Z_2, -h_{21}Z_1, -h_{21}Z_1, -h_{21}Z_1,\\
&\qquad -h_{21}Z_2, -P\},\\
\Omega_{11} &= P\left(\widetilde{A} + \Delta\widetilde{A}(t)\right) + \left(\widetilde{A} + \Delta\widetilde{A}(t)\right)^T P\\
&\quad + K^T\left(Q_1 + Q_2 + (h_2 - h_1)R\right)K,\\
\Omega_{14} &= P\left(\widetilde{A}_d + \Delta\widetilde{A}_d(t)\right),\\
\Omega_{22} &= -Q_1 + T_1 + T_1^T, \qquad \Omega_{24} = -T_1 + T_1,\\
\Omega_{33} &= -Q_2 - T_2 - T_2^T, \qquad \Omega_{34} = T_2 - T_2^T,\\
\Omega_{44} &= -T_1 - T_1^T + T_2 + T_2^T,\\
\Omega_{55} &= \frac{-R}{(h_2 - h_1)},\\
\widetilde{T}_1 &= \left[0 \;\; T_1^T \;\; 0 \;\; T_1^T \;\; 0 \;\; 0\right]^T,\\
\widetilde{T}_2 &= \left[0 \;\; 0 \;\; T_2^T \;\; T_2^T \;\; 0 \;\; 0\right]^T,\\
\breve{A} &= \left[\widetilde{A}^T + \Delta\widetilde{A}^T(t) \;\; 0 \;\; 0 \;\; \widetilde{A}_d^T + \Delta\widetilde{A}_d^T(t) \;\; 0 \;\; \widetilde{B}^T\right]^T,\\
\breve{H} &= \left[\widetilde{H}^T + \Delta\widetilde{H}^T(t) \;\; 0 \;\; 0 \;\; \widetilde{H}_d^T + \Delta\widetilde{H}_d^T(t) \;\; 0 \;\; 0\right]^T,\\
h_{21} &= h_2 - h_1.
\end{aligned} \tag{16}$$

Proof. Define the following Lyapunov-Krasovskii candidate for system $(\widetilde{\Sigma})$:

$$\begin{aligned}
V(\xi(t), t) &= \xi^T(t)P\xi(t) + \int_{t-h_1}^{t} \xi^T(s)K^T Q_1 K\xi(s)\,ds\\
&\quad + \int_{t-h_2}^{t} \xi^T(s)K^T Q_2 K\xi(s)\,ds\\
&\quad + \int_{-h_2}^{-h_1}\int_{t+\beta}^{t} \Phi^T(s)K^T Z_1 K\Phi(s)\,ds\,d\beta\\
&\quad + \int_{-h_2}^{-h_1}\int_{t+\beta}^{t} g^T(s)K^T Z_2 K g(s)\\
&\quad + \int_{-h_2}^{-h_1}\int_{t+\beta}^{t} \xi^T(s)K^T RK\xi(s)\,ds\,d\beta.
\end{aligned} \tag{17}$$

When $v(t) = 0$,

$$dV(\xi(t), t) = \mathbf{L}V(\xi(t), t) + 2\xi^T(t)Pg(t)\,d\omega(t). \tag{18}$$

By using the Newton-Leibnitz formula, the following equations can be got for any matrices T_1, T_2 with appropriate dimensions:

$$\begin{aligned}
&2\eta^T(t)\overline{T}_1 K\left[\xi(t-h_1) - \xi(t-\tau(t)) - \int_{t-\tau(t)}^{t-h_1}\Phi(s)\,ds\right.\\
&\qquad \left. - \int_{t-\tau(t)}^{t-h_1} g(s)\,d\omega(s)\right] = 0,\\
&2\eta^T(t)\overline{T}_2 K\left[\xi(t-\tau(t)) - \xi(t-h_2) - \int_{t-h_2}^{t-\tau(t)}\Phi(s)\,ds\right.\\
&\qquad \left. - \int_{t-h_2}^{t-\tau(t)} g(s)\,d\omega(s)\right] = 0,\\
&(\tau(t) - h_1)\eta^T(t)\overline{T}_1 Z_1^{-1}\overline{T}_1^T\eta(t)\\
&\qquad - \int_{t-\tau(t)}^{t-h_1}\eta^T(t)\overline{T}_1 Z_1^{-1}\overline{T}_1^T\eta(t)\,ds = 0,\\
&(h_2 - \tau(t))\eta^T(t)\overline{T}_2 Z_1^{-1}\overline{T}_2^T\eta(t)\\
&\qquad - \int_{t-h_2}^{t-\tau(t)}\eta^T(t)\overline{T}_2 Z_1^{-1}\overline{T}_2^T\eta(t)\,ds = 0,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\overline{T}_1 &= \left[0 \;\; T_1^T \;\; 0 \;\; T_1^T \;\; 0\right]^T,\\
\overline{T}_2 &= \left[0 \;\; 0 \;\; T_2^T \;\; T_2^T \;\; 0\right]^T.
\end{aligned} \tag{20}$$

And $\eta(t)$ is a new vector defined as follows:

$$\eta^T(t) = \left[\xi^T(t) \;\; \xi^T(t-h_1)K^T \;\; \xi^T(t-h_2)K^T \;\; \xi^T(t-\tau(t))K^T \;\; \left(\int_{t-h_2}^{t-h_1}\xi(s)^T ds\right)K^T\right]. \tag{21}$$

By the above formulas (19) and Lemma 4, we can deduce that

$$\begin{aligned}
&\mathbf{L}V(\xi(t), t)\\
&\quad = 2\xi^T(t)P\Phi(t) + g^T(t)Pg(t) + \xi^T(t)K^T Q_1 K\xi(t)\\
&\qquad + \xi^T(t)K^T Q_2 K\xi(t) - \xi^T(t-h_1)K^T Q_1 K\xi(t-h_1)\\
&\qquad - \xi^T(t-h_2)K^T Q_2 K\xi(t-h_2) + h_{21}\Phi^T(t)K^T Z_1 K\Phi(t)\\
&\qquad + h_{21}g(t)^T K^T Z_2 K g(t) - \int_{t-h_2}^{t-h_1}\xi^T(s)K^T RK\xi(s)\,ds\\
&\qquad - \int_{t-h_2}^{t-h_1}\Phi^T(s)K^T Z_1 K\Phi(s)\,ds\\
&\qquad - \int_{t-h_2}^{t-h_1} g^T(s)K^T Z_2 K g(s)\,ds + h_{21}\Phi^T(t)K^T RK\Phi(t)\\
&\quad \le \eta^T(t)\left[\overline{\Omega} + h_{21}\overline{T}_1 Z_1^{-1}\overline{T}_1^T + \widehat{H}\left(K^T(h_2 - h_1)Z_2 K + P\right)\widehat{H}\right.\\
&\qquad + \widehat{A}K^T h_{21}Z_1 K\widehat{A}^T + h_{21}\overline{T}_2 Z_1^{-1}\overline{T}_2^T\\
&\qquad \left. + \overline{T}_1 Z_2^{-1}\overline{T}_1^T + \overline{T}_2 Z_2^{-1}\overline{T}_2^T\right]\eta(t)
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t-\tau(t)}^{t-h_1}\left[\eta^T(t)\overline{T}_1+\Phi^T(s)K^TZ_1\right]Z_1^{-1}\\
&\qquad\times\left[Z_1K\Phi(s)+\overline{T}_1^T\eta(t)\right]ds\\
&-\int_{t-h_2}^{t-\tau(t)}\left[\eta^T(t)\overline{T}_2+\Phi^T(s)K^TZ_1\right]Z_1^{-1}\\
&\qquad\times\left[Z_1K\Phi(s)+\overline{T}_2^T\eta(t)\right]ds\\
&+\left(\int_{t-\tau(t)}^{t-h_1}g(s)d\omega(s)\right)^TK^TZ_2K\left(\int_{t-\tau(t)}^{t-h_1}g(s)\,d\omega(s)\right)\\
&+\left(\int_{t-h_2}^{t-\tau(t)}g(s)d\omega(s)\right)^TK^TZ_2K\left(\int_{t-h_2}^{t-\tau(t)}g(s)\,d\omega(s)\right)\\
&-\int_{t-\tau(t)}^{t-h_1}g^T(s)K^TZ_2Kg(s)\,ds\\
&-\int_{t-h_2}^{t-\tau(t)}g^T(s)K^TZ_2Kg(s)\,ds,
\end{aligned}
\tag{22}
$$

where

$$
\overline{\Omega}=\begin{bmatrix}\Omega_{11}&0&0&\Omega_{14}&0\\ *&\Omega_{22}&0&\Omega_{24}&0\\ *&*&\Omega_{33}&\Omega_{34}&0\\ *&*&*&\Omega_{44}&0\\ *&*&*&*&\Omega_{55}\end{bmatrix}
\tag{23}
$$

$$
\begin{aligned}
\widehat{A}&=\left[\left(\widetilde{A}+\Delta\widetilde{A}(t)\right)^T\quad 0\quad 0\quad\left(\widetilde{A}_d+\Delta\widetilde{A}_d(t)\right)^T\quad 0\right]^T,\\
\widehat{H}&=\left[\left(\widetilde{H}+\Delta\widetilde{H}(t)\right)^T\quad 0\quad 0\quad\left(\widetilde{H}_d+\Delta\widetilde{H}_d(t)\right)^T\quad 0\right]^T.
\end{aligned}
$$

During the analysis, it can be seen that

$$
\begin{aligned}
&\left(\int_{t-\tau(t)}^{t-h_1}g(s)d\omega(s)\right)^TK^TZ_2K\left(\int_{t-\tau(t)}^{t-h_1}g(s)\,d\omega(s)\right)\\
&+\left(\int_{t-h_2}^{t-\tau(t)}g(s)\,d\omega(s)\right)^TK^TZ_2K\left(\int_{t-h_2}^{t-\tau(t)}g(s)\,d\omega(s)\right)\\
&-\int_{t-\tau(t)}^{t-h_1}g^T(s)K^TZ_2Kg(s)\,ds\\
&-\int_{t-h_2}^{t-\tau(t)}g^T(s)K^TZ_2Kg(s)\,ds=0,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&-\int_{t-\tau(t)}^{t-h_1}\left[\eta^T(t)\overline{T}_1+\Phi^T(s)K^TZ_1\right]Z_1^{-1}\\
&\qquad\times\left[Z_1K\Phi(s)+\overline{T}_1^T\eta(t)\right]ds\\
&-\int_{t-h_2}^{t-\tau(t)}\left[\eta^T(t)\overline{T}_2+\Phi^T(s)K^TZ_1\right]Z_1^{-1}\\
&\qquad\times\left[Z_1K\Phi(s)+\overline{T}_2^T\eta(t)\right]ds<0.
\end{aligned}
\tag{25}
$$

And applying the Schur complement to (15), we can derive the following inequality with $v(t)=0$:

$$
\begin{aligned}
&\overline{\Omega}+h_{21}\overline{T}_1Z_1^{-1}\overline{T}_1^T+\widehat{H}\left(K^Th_{21}Z_2K+P\right)\widehat{H}+\overline{T}_1Z_2^{-1}\overline{T}_1^T\\
&\quad+\widehat{A}K^Th_{21}Z_1K\widehat{A}^T+h_{21}\overline{T}_2Z_1^{-1}\overline{T}_2^T+\overline{T}_2Z_2^{-1}\overline{T}_2^T<0.
\end{aligned}
\tag{26}
$$

From (22)–(26), we can get that

$$
\mathbf{L}V(\xi(t),t)<0,
\tag{27}
$$

which ensures that system $(\widetilde{\Sigma})$ with $v(t)=0$ is robustly stochastically stable according to Definition 2 and [47]. By Itô's formula, it is easy to derive

$$
\mathsf{E}(V(\xi(t),t))=\mathsf{E}\left(\int_0^t\mathsf{L}V(\xi(s),s)\,ds\right).
\tag{28}
$$

Now we establish the L_2-L_∞ performance of the filtering error system $(\widetilde{\Sigma})$. It is easy to obtain

$$
\begin{aligned}
&\mathbf{L}V(\xi(t),t)-\omega(t)^T\omega(t)\\
&\quad\le\overline{\eta}^T(t)\left[\Omega+h_{21}\widetilde{T}_1Z_1^{-1}\widetilde{T}_1^T+\check{H}\left(K^Th_{21}Z_2K+P\right)\check{H}\right.\\
&\qquad+\check{A}K^Th_{21}Z_1K\check{A}^T+h_{21}\widetilde{T}_2Z_1^{-1}\widetilde{T}_2^T\\
&\qquad\left.+\widetilde{T}_1Z_2^{-1}\widetilde{T}_1^T+\widetilde{T}_2Z_2^{-1}\widetilde{T}_2^T\right]\overline{\eta}(t).
\end{aligned}
\tag{29}
$$

Then applying the Schur complement formula to (15), we can get

$$
\begin{aligned}
&\overline{\eta}^T(t)\left[\Omega+(h_2-h_1)\widetilde{T}_1Z_1^{-1}\widetilde{T}_1^T+\check{H}\left(K^Th_{21}Z_2K+P\right)\check{H}\right.\\
&\qquad+\widetilde{T}_2Z_2^{-1}\widetilde{T}_2^T+\check{A}K^Th_{21}Z_1K\check{A}^T+h_{21}\widetilde{T}_2Z_1^{-1}\widetilde{T}_2^T\\
&\qquad\left.+\widetilde{T}_1Z_2^{-1}\widetilde{T}_1^T\right]\overline{\eta}(t)<0,
\end{aligned}
\tag{30}
$$

for all $t>0$, where

$$
\overline{\eta}^T(t)=\left[\xi^T(t)\quad\xi^T(t-h_1)K^T\quad\xi^T(t-h_2)K^T\quad\xi^T(t-\tau(t))K^T\quad\left(\int_{t-h_2}^{t-h_1}\xi(s)^Tds\right)K^T\quad v(t)\right].
\tag{31}
$$

Therefore, for all $\overline{\eta}(t) \neq 0$, $\mathbf{L}V(\xi(t),t) - \omega(t)^T\omega(t) < 0$, which means

$$\xi^T(t)P\xi(t) \leq V(\xi(t),t) < \int_0^t \omega(s)^T\omega(s)\,ds. \tag{32}$$

Then using the Schur complement to the first formula in (15), we have $\widetilde{L}^T\widetilde{L} < \gamma^2 P$, which guarantees

$$\begin{aligned} &e(t)^T e(t) - \xi^T(t)\widetilde{L}^T\widetilde{L}\xi(t) \\ &\quad < \gamma^2\xi^T(t)P\xi(t) < \gamma^2\int_0^t \omega(s)^T\omega(s)\,ds \\ &\quad \leq \gamma^2\int_0^\infty \omega(s)^T\omega(s)\,ds. \end{aligned} \tag{33}$$

Therefore, $\|e\|_\infty < \gamma\|\omega\|_2$ for any zero mean Gaussian white noise process $\omega(t)$ with unit covariance. □

Remark 6. The system we studied is a time-varying delay system containing the information of both the lower bound and the upper bound of time delay. By such a consideration, delay-dependent result is more reliable and approaches to reality that not all the delays begin with 0 moment.

Remark 7. It is worth mentioning that Theorem 5 can be easily extended to investigate the robust H_∞ filtering design problem for the systems $(\widetilde{\Sigma})$ with parameter uncertainties.

Now we are in a position to present a sufficient condition for the solvability of robust $L_2 - L_\infty$ filtering problem.

Theorem 8. *Consider the uncertain T-S fuzzy stochastic time-varying delay system (Σ) and a constant scalar $\gamma > 0$. The robust $L_2 - L_\infty$ filtering problem is solvable if there exist scalars $\varepsilon_i > 0$ and matrices $W > 0$, $X > 0$, $R > 0$, $Q_i > 0$, $Z_i > 0$, T_{1i}, T_{2i}, $i = 1,2$; Φ_{1i}, Φ_{2i}, Φ_{3i}, Φ_{4i}, $1 \leq i \leq r$, $\{\Upsilon_i = \Upsilon_i^T, 1 \leq i \leq r\}$, $\{\Delta_{ij}, 1 \leq i < j \leq r\}$, and such that the following LMIs hold:*

$$X - W > 0, \tag{34}$$

$$\begin{bmatrix} \Upsilon_1 & \Delta_{12} & \cdots & \Delta_{1r} \\ * & \Upsilon_2 & \cdots & \Delta_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Upsilon_r \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} \Gamma_{ii} - \Upsilon_i + \varepsilon_i\Xi_i\Xi_i^T & \Theta_i \\ * & -\varepsilon_i \end{bmatrix} < 0, \quad (1 \leq i \leq r), \tag{36}$$

$$\begin{bmatrix} \Gamma_{ij} + \Gamma_{ji} - \Delta_{ij} - \Delta_{ij}^T + \varepsilon_i\Xi_i\Xi_i^T + \varepsilon_j\Xi_j\Xi_j^T & \Theta_i & \Theta_j \\ * & -\varepsilon_i & 0 \\ * & * & -\varepsilon_i \end{bmatrix} < 0, \quad (1 \leq i < j \leq r), \tag{37}$$

$$\begin{bmatrix} W & W & L_j^T - \Phi_{3i}^T \\ * & X & L_j^T \\ * & * & \gamma^2 I \end{bmatrix} > 0, \quad (1 \leq i \leq r,\ 1 \leq j \leq r), \tag{38}$$

where

$$\begin{gathered} \Theta_i^T = \begin{bmatrix} M_{1i}^T W & M_{1i}^T X & \mathbf{0}_{1*9} & h_{21}M_{1i}^T Z_1 & h_{21}M_{2i}^T Z_2 & M_{2i}^T X & M_{2i}^T \end{bmatrix}, \\ \Xi_i^T = \begin{bmatrix} N_{1i}^T & N_{1i}^T & 0 & 0 & N_{2i}^T & \mathbf{0}_{10*1} \end{bmatrix}, \\ \Gamma_{ij} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ * & \Gamma_{22} & \mathbf{0}_{4*4} \\ * & * & \Gamma_{33} \end{bmatrix}, \\ \Gamma_{11} = \begin{bmatrix} G_{11} & G_{12} & 0 & 0 & G_{15} & 0 & G_{17} \\ * & G_{22} & 0 & 0 & G_{25} & 0 & G_{27} \\ * & * & G_{33} & 0 & G_{35} & 0 & 0 \\ * & * & * & G_{44} & G_{45} & 0 & 0 \\ * & * & * & * & G_{55} & 0 & 0 \\ * & * & * & * & * & -\dfrac{R}{h_{21}} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix}, \\ G_{11} = WA_i + A_i^T W + Q_1 + Q_2 + h_{21}R, \\ G_{12} = WA_i + A_i^T X + C_j^T\Phi_{1i}^T + \Phi_{2i}^T + Q_1 + Q_2 + h_{21}R, \\ G_{15} = WA_{di}, \qquad G_{17} = WB_i, \\ G_{22} = X^T A_i + \Phi_{1i}C_j + C_j^T\Phi_{1i}^T + A_i^T X + Q_1 + Q_2 + h_{21}R, \\ G_{25} = X^T A_{di} + \Phi_{1i}C_{dj}, \\ G_{27} = X^T B_i + \Phi_{1i}D_j, \\ G_{33} = -Q_1 + T_1 + T_1^T, \qquad G_{35} = -T_1 + T_1^T, \\ G_{44} = -Q_2 - T_2 - T_2^T, \qquad G_{45} = T_2 - T_2^T, \\ G_{55} = -T_1 - T_1^T + T_2 + T_2^T, \\ \Gamma_{12} = \begin{bmatrix} \widetilde{T}_1 & \widetilde{T}_2 & h_{21}\widetilde{T}_1 & h_{21}\widetilde{T}_2 \end{bmatrix}, \\ \Gamma_{22} = \operatorname{diag}\{-Z_2, -Z_2, -h_{21}Z_1, -h_{21}Z_1\}, \\ \Gamma_{13} = \begin{bmatrix} h_{21}A_i^T Z_1 & h_{21}H_i^T Z_2 & H_i^T X & H_i^T \\ h_{21}A_i^T Z_1 & h_{21}H_i^T Z_2 & H_i^T X & H_i^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{21}A_{di}^T Z_1 & h_{21}H_{di}^T Z_2 & H_{di}^T X & H_{di}^T \\ 0 & 0 & 0 & 0 \\ h_{21}B_i^T Z_1 & 0 & 0 & 0 \end{bmatrix}, \\ \Gamma_{33} = \operatorname{diag}\left\{-h_{21}Z_1, -h_{21}Z_2, \begin{bmatrix} -X & -I \\ * & -\Phi_{4i} \end{bmatrix}\right\}. \end{gathered} \tag{39}$$

When the LMIs (34)–(38) are feasible, the time-dependent filter we desired here can be chosen as

$$\begin{aligned} &A_{fi} = \sigma^{-1}\Phi_{2i}W^{-1}\beta^{-T}, \qquad B_{fi} = \sigma^{-1}\Phi_{1i}, \\ &L_{fi} = \Phi_{3i}W^{-1}\beta^{-T}, \quad i = 1,\ldots,r, \end{aligned} \tag{40}$$

where σ and β are nonsingular matrices satisfying $\sigma\beta^T = I - XW^{-1}$.

Proof. Similar to [33], we know that $I - XW^{-1}$ is nonsingular. Therefore, there always exist nonsingular matrices σ and β

such that $\sigma\beta^T = I - XW^{-1}$ holds. Then we define the nonsingular matrices Λ_1 and Λ_2 as follows:

$$\Lambda_1 = \begin{bmatrix} W^{-1} & I \\ \beta^T & o \end{bmatrix}; \qquad \Lambda_2 = \begin{bmatrix} I & X \\ 0 & \sigma^T \end{bmatrix}. \tag{41}$$

Define $U = \Lambda_2\Lambda_1^{-1}$. Then there is

$$U = \begin{bmatrix} X & \sigma \\ \sigma^T & \beta^{-1}W^{-1}(X_W)W^{-1}\beta^{-T} \end{bmatrix} > 0. \tag{42}$$

Now using Lemma 4 and recalling (36), we can deduce that

$$\begin{aligned}
\Lambda &= \sum_{i=1}^{r}\rho_i^2(s(t))\left[\Gamma_{ii} + \Theta_i F_i(t)\Xi_i^T + \Xi_i F_i(t)\Theta_i^T\right] \\
&\quad + \sum_{i=1}^{r}\sum_{j>i}^{r}\rho_i(s(t))\rho_j(s(t)) \\
&\qquad \times \left[\Gamma_{ij} + \Theta_i F_i(t)\Xi_i^T + \Xi_i F_i(t)\Theta_i^T + \Gamma_{ji}\right. \\
&\qquad\quad \left. + \Theta_i F_j(t)\Xi_j^T + \Xi_j F_j(t)\Theta_j^T\right] \\
&< \sum_{i=1}^{r}\rho_i^2(s(t))\left[\Gamma_{ii} + \varepsilon_i^{-1}\Theta_i\Theta_i^T + \varepsilon_i\Xi_i\Xi_i^T\right] \\
&\quad + \sum_{i=1}^{r}\sum_{j>i}^{r}\rho_i(s(t))\rho_j(s(t)) \\
&\qquad \times \left[\Gamma ij + \varepsilon_i^{-1}\Theta\Theta_i^T + \varepsilon_i\Xi_i\Xi_i^T + \Gamma_{ji} + \varepsilon_j^{-1}\Theta\Theta_j^T\right. \\
&\qquad\quad \left. + \varepsilon_j\Xi_j\Xi_j^T\right] \\
&< \sum_{i=1}^{r}\rho_i^2(s(t))\Upsilon_i + \sum_{i=1}^{r}\sum_{j>i}^{r}\rho_i(s(t))\rho_j(s(t))\left[\Delta_{ij} + \Delta_{ji}^T\right] \\
&= \begin{bmatrix} \rho_1(s(t))I \\ \rho_2(s(t))I \\ \vdots \\ \rho_r(s(t))I \end{bmatrix}^T \begin{bmatrix} \Upsilon_1 & \Delta_{12} & \cdots & \Delta_{1r} \\ * & \Delta_2 & \cdots & \Delta_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Upsilon_r \end{bmatrix} \begin{bmatrix} \rho_1(s(t))I \\ \rho_2(s(t))I \\ \vdots \\ \rho_r(s(t))I \end{bmatrix} < 0.
\end{aligned} \tag{43}$$

We can deduce that

$$\begin{aligned}
&\left\{\operatorname{diag}\left(\Lambda_2^{-T}\begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}, I, \ldots, \begin{bmatrix} \sigma^{-T} & 0 \\ 0 & I \end{bmatrix}\right)\right\}\Lambda \\
&\left\{\operatorname{diag}\left(\Lambda_2^{-T}\begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}, I, \ldots, \begin{bmatrix} \sigma^{-1} & 0 \\ 0 & I \end{bmatrix}\right)\right\} \\
&\quad = \begin{bmatrix} \Omega & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0,
\end{aligned} \tag{44}$$

which is equivalent to (15). Therefore, it is easy to see that the condition in Theorem 5 and the LMIs in (34)–(37) are equivalent. Finally, it can be concluded that the filtering error system $(\widetilde{\Sigma})$ is stochastically stable with $L_2 - L_\infty$ performance level γ. □

Remark 9. The desired $L_2 - L_\infty$ filters can be constructed by solving the LMIs in (34)–(38), which can be implemented by using standard numerical algorithms, and no tuning of parameters will be involved.

Remark 10. In the proof of above Theorem, we adopt (25), (26), and Newton-Leibnitz formula to reduce the conservatism. Moreover, the results obtained in Theorems 5 and 8 can be further extended based on fuzzy or piecewise Lyapunov-Krasovskii function.

4. Numerical Example

In this section, a numerical example is provided to show the effectiveness of the results obtained in the previous section.

Example 1. Consider the T-S fuzzy stochastic system $(\widetilde{\Sigma})$ with model parameters given as follows:

$$\begin{aligned}
A_1 &= \begin{bmatrix} -2.3 & 0 \\ 0.2 & -1.1 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} -0.2 & 0.2 \\ -0.16 & -0.18 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -2.1 & 0.1 \\ 0.1 & -1.4 \end{bmatrix}, \\
H_1 &= \begin{bmatrix} -0.4 & 0.1 \\ 0.3 & -0.5 \end{bmatrix}, \qquad H_{d1} = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.05 \end{bmatrix}, \\
H_2 &= \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.5 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1 & -0.4 \end{bmatrix}, \qquad C_{d1} = \begin{bmatrix} -0.4 & -0.1 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} -0.2 & 0.4 \end{bmatrix}, \qquad C_{d2} = \begin{bmatrix} -0.4 & 0.5 \end{bmatrix}, \\
L_1 &= \begin{bmatrix} 1.5 & -0.6 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} -0.3 & 0.2 \end{bmatrix}, \\
D_1 &= 0.2, \qquad D_2 = -0.2, \\
B_1 &= \begin{bmatrix} 0.9 \\ -0.2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix}, \\
A_{d2} &= \begin{bmatrix} -0.18 & 0 \\ -0.22 & -0.24 \end{bmatrix}, \qquad H_{d2} = \begin{bmatrix} -0.05 & 0.01 \\ 0.03 & -0.04 \end{bmatrix}.
\end{aligned} \tag{45}$$

And the parameter uncertainties are shown as:

$$\begin{aligned}
M_{11} &= \begin{bmatrix} 0.1 & 0.2 \\ -0.5 & 0.1 \end{bmatrix}, \qquad M_{12} = \begin{bmatrix} -0.2 & 0.1 \\ 0.3 & -0.1 \end{bmatrix}, \\
M_{21} &= \begin{bmatrix} 0.8 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}, \\
N_{11} &= \begin{bmatrix} 0 & -0.3 \\ 0.1 & -0.2 \end{bmatrix}, \qquad N_{21} = \begin{bmatrix} -0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \\
M_{22} &= \begin{bmatrix} -0.1 & 0.2 \\ 0.4 & -0.2 \end{bmatrix}, \\
N_{12} &= \begin{bmatrix} -0.5 & 0 \\ 0.2 & -0.3 \end{bmatrix}, \qquad N_{22} = \begin{bmatrix} 0 & -0.2 \\ 0 & 0.1 \end{bmatrix}.
\end{aligned} \tag{46}$$

The membership functions are

$$\begin{aligned}
h_1(x_1(t)) &= \left(1 - \frac{3x_1}{1 + \exp(6x_1(t) + 2)}\right), \\
h_2(x_1(t)) &= 1 - h_1(x_1(t)).
\end{aligned} \tag{47}$$

By using the Matlab LMI Control Toolbox, we have the robust $L_2 - L_\infty$ filtering problem which is solvable to Theorem 8. It can be calculated that for any $0 < h_1(t) \leq 3$,

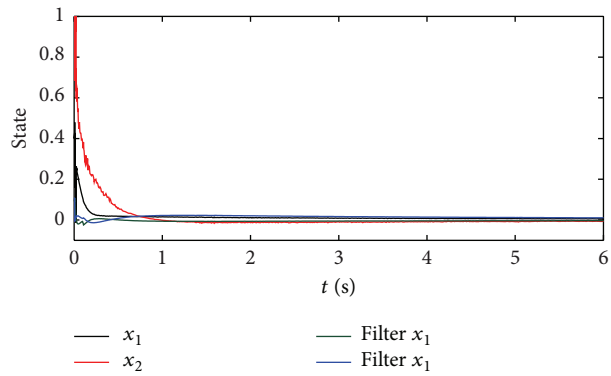

FIGURE 1: State responses of $x(t)$ and $\hat{x}(t)$.

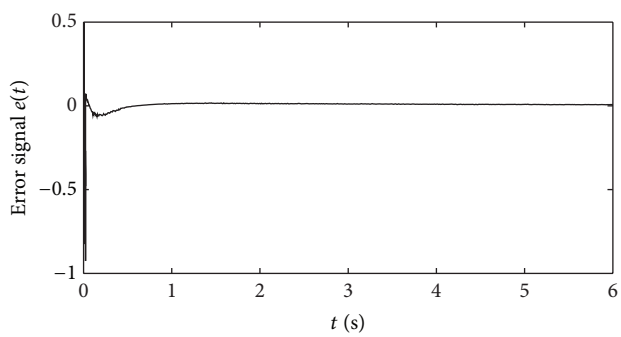

FIGURE 2: Responses of the error signal $e(t)$.

$0 < h_2(t) \leq 8$, $\gamma = 0.42$ the robust $L_2 - L_\infty$ filtering problem can be solved. A desired fuzzy filter can be constructed as in the form of (6) with

$$\begin{aligned} A_{f1} &= \begin{bmatrix} -5.4320 & 0.4511 \\ 1.8159 & -1.5495 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} -8.1142 & 3.4902 \\ 2.9687 & -5.9058 \end{bmatrix}, \\ B_{f1} &= \begin{bmatrix} -1.0301 \\ 0.1040 \end{bmatrix}, \qquad B_{f2} = \begin{bmatrix} -1.0171 \\ 0.0415 \end{bmatrix}, \\ L_{f1} &= \begin{bmatrix} -0.3063 & -0.0422 \end{bmatrix}, \\ L_{f2} &= \begin{bmatrix} -0.2667 & -0.0422 \end{bmatrix}. \end{aligned} \tag{48}$$

The simulation results of the state response of the plant and the filter are given in Figure 1, where the initial condition is $x_0(t) = [0.4 \quad 2.5]^T$, $\hat{x}_0(t) = [0.1 \quad 0.1]^T$. Figure 2 shows the simulation results of the signal $e(t)$, and the exogenous disturbance input $v(t)$ is given by $v(t) = 12/(5 + 2t)$, $t \geq 0$, which belongs to $\mathscr{L}_2[0, \infty)$.

5. Conclusion

This paper considers the robust $L_2 - L_\infty$ filter design problem for the uncertain T-S fuzzy stochastic system with time-varying delay. An LMI approach has been developed to design the fuzzy filter ensuring not only the robust stochastic mean-square stability but also a prescribed $L_2 - L_\infty$ performance level of the filtering error system for all admissible uncertainties. A numerical example has been provided to show the effectiveness of the proposed filter design methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Analysis and Design of Networked Control Systems with Random Markovian Delays and Uncertain Transition Probabilities

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This paper focuses on the stability issue of discrete-time networked control systems with random Markovian delays and uncertain transition probabilities, wherein the random time delays exist in the sensor-to-controller and controller-to-actuator. The resulting closed-loop system is modeled as a discrete-time Markovian delays system governed by two Markov chains. Using Lyapunov stability theory, a result is established on the Markovian structure and ensured that the closed-loop system is stochastically stable. A simulation example illustrates the validity and feasibility of the results.

1. Introduction

Networked control systems (NCS) find many successful applications in power grids, manufacturing plants, vehicles, aircrafts, spacecrafts, remote surgery, and so on [1]. Compared with the traditional control systems, the use of the communication networks brings many advantages such as low cost, reduced weight, and simple installation and maintenance, as well as high efficiency, flexibility, and reliability. However, inserting communication networks into feedback control loops has also resulted in several interesting and challenging issues, such as packet dropouts [2], time delays [3–10], quantization [11], time-varying transmission intervals [12], distributed synchronization [13], or some of the constraints considered simultaneously [14–17], which make the analysis and design of NCS complex. These imperfections block the way of harvesting reliable NCS by implementing existing control techniques [18]. To overcome these drawbacks, significant attention has been paid to the NCS research ranging from system identification and stability analysis to controller and filter designs. See the survey papers [1, 19, 20] and the references therein.

The network-induced time delays are known to be the major challenges in NCS, which may be potential causes for the deteriorating performance or instability of NCS. Consequently, numerous works have been conducted on the time-delay issue in the past years. For example, in [21], the mixed H_2/H_∞ control issue of NCS with random time delays has been investigated based on Markovian jump linear systems method. In [9], the stability problem of NCS with uncertain time-varying delays has been investigated. The stability and stabilization of NCS with random time delay usually use Markovian jump linear systems (MJLS) approach, and, recently, many significant achievements have been obtained for MJLS in [22–26]. However, most of the approaches for NCS based on Markovian jump systems framework assumed that the Markovian transition probabilities are known a priori, which severely limit the utility of the Markov model. Furthermore, such assumption may not hold true especially in the case where networked control is

applied to the remote plants. Recently, the H_∞ filter problem for a class of uncertain Markovian jumping systems with bounded transition probabilities has been investigated in [27], but the well-established results cannot be directly used to NCS. To the best of the authors' knowledge, up to now, very limited efforts have been devoted to studying the system with uncertain transition probability matrices for NCS, which motivates our investigation.

In this paper, we address the analysis and design of NCS with random time delays modeled by Markov chains in forward sensor-to-controller (S-C) and feedback controller-to-actuator (C-A) communication links and with the uncertain transition probability matrices. The main contributions of this paper are highlighted as follows. (i) A model is proposed for NCS with random Markovian delays and uncertain transition probability matrices. (ii) The system modeled will be more generalized and avoid the ideal assumption that the transition probabilities are known a priori. (iii) New criteria for stability are obtained based on a Lyapunov approach. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed control scheme for NCS with random time delays and uncertain transition probability matrices.

The remainder of this paper is organized as follows. A model with Markovian delays and uncertain transition probabilities is obtained in Section 2. The main results are obtained based on a Lyapunov approach and the linear matrix inequalities technique in Section 3. Section 4 presents the simulation results. Finally, the conclusions are provided in Section 5.

Notations. Matrices are assumed to have appropriate dimensions. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The notations $A > 0$ ($A < 0$) indicate that A is a real symmetric positive (negative) definite matrix. I and 0 denote the identity matrix and the zero matrix with appropriate dimensions, respectively. Superscripts "T" and "-1" stand for the matrix transposition and the matrix inverse, respectively. $\mathbb{E}[\cdot]$ stands for the mathematical expectation and $\mathrm{diag}\{A, B\}$ stands for a block-diagonal matrix of A and B. I and 0 denote the identity matrix and zero matrix with appropriate dimensions, respectively. $\mathrm{sym}\{A\}$ denotes the expression $A + A^T$, and $*$ means symmetric terms in symmetric entries.

2. NCS Model

The framework of networked control systems is depicted in Figure 1. The plant, sensor, controller, and actuator are spatially distributed and closed through a network. Random time delays exist in both of S-C and C-A.

The plant is described by the following discrete-time linear time-invariant plant model:

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state vector and $u(k) \in \mathbb{R}^m$ is the control input. A and B are known real constant matrices with appropriate dimensions.

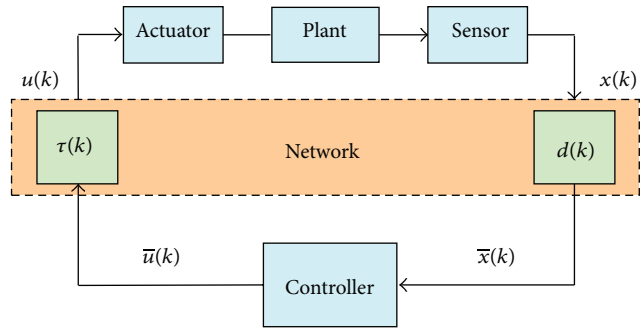

Figure 1: Diagram of a NCS with time delays.

For this system, we will consider a state feedback controller as follows:

$$\overline{u}(k) = K\overline{x}(k), \tag{2}$$

where K is the state feedback controller gain.

Random S-C and C-A time delays are $d(k)$ and $\tau(k)$, respectively. $d(k)$ and $\tau(k)$ are assumed be bounded; that is, $0 \le \underline{d} \le d(k) \le \overline{d}$, $0 \le \underline{\tau} \le \tau(k) \le \overline{\tau}$, where $\underline{d} = \min\{d(k)\}$, $\overline{d} = \max\{d(k)\}$, $\underline{\tau} = \min\{\tau(k)\}$, and $\overline{\tau} = \max\{\tau(k)\}$. One way to model delays $d(k)$ and $\tau(k)$ is by using the finite-state Markov chains presented in [21]. The main advantages of the Markov model considering the dependence between delays are that the current time delays in real networks delays are frequently related to the previous delays. In this paper $d(k)$ and $\tau(k)$ are modeled as two homogeneous Markov chains.

By substituting controller (2) to plant (1), we obtain a closed-loop system as follows:

$$x(k+1) = Ax(k) + BKx(k-\eta_k), \tag{3}$$

where $\eta_k = \tau(k) + d(k-\tau(k))$.

In system (3), $\{d(k), k \in \mathbb{Z}\}$ and $\{\tau(k), k \in \mathbb{Z}\}$ are two finite state discrete-time homogeneous Markov chains with values in the finite sets $S_1 = \{0, \ldots, s_1\}$ and $S_2 = \{0, \ldots, s_2\}$ with the uncertain transition probability matrices $\widehat{\pi}$ and $\widehat{\lambda}$. $\widehat{\pi} = \{\widehat{\pi}_{ij}\}$ and $\widehat{\lambda} = \{\widehat{\lambda}_{nm}\}$ denote the uncertain transition probability matrices of Markov chain $d(k)$ and $\tau(k)$, respectively, with probabilities $\widehat{\pi}_{ij}$ and $\widehat{\lambda}_{nm}$, which are defined by

$$\begin{aligned} \Pr\{d(k+1) = j \mid d(k) = i\} &= \widehat{\pi}_{ij}, \\ \Pr\{\tau(k+1) = n \mid \tau(k) = m\} &= \widehat{\lambda}_{mn}, \end{aligned} \tag{4}$$

where $\Pr\{\widehat{d}_0 = i\} = \widehat{\pi}_i \ge 0$, $\Pr\{\widehat{\tau}_0 = m\} = \widehat{\lambda}_m \ge 0$ and $\sum_{j=0, j\ne i}^{s_1} \widehat{\pi}_{ij} = 1 - \widehat{\pi}_{ii}$, $\sum_{n=0, n\ne m}^{s_2} \widehat{\lambda}_{mn} = 1 - \widehat{\lambda}_{mm}$ for all $\{i, j\} \in S_1$ and $\{m, n\} \in S_2$. The transition probability matrices $\widehat{\pi} \triangleq [\widehat{\pi}_{ij}]$ and $\widehat{\pi} \triangleq [\widehat{\lambda}_{mn}]$ are unknown a priori but belong to the following bounded compact set:

$$\widehat{\pi} = \pi + \Delta\pi, \qquad \widehat{\lambda} = \lambda + \Delta\lambda, \tag{5}$$

where $\pi \triangleq [\pi_{ij}]$ $(i, j \in S_1)$ and $\lambda \triangleq [\lambda_{mn}]$ $(m, n \in S_2)$ are known constant matrices. $\Delta\pi \triangleq [\Delta\pi_{ij}]$ $(i, j \in S_1)$ and $\Delta\lambda \triangleq$

$[\Delta\lambda_{ij}]$ $(i, j \in S_2)$ denote the uncertainty in the transition probability matrices, where $\Delta\pi$ and $\Delta\lambda$ satisfy

$$\sum_{j=0, j\neq i}^{s_1} \Delta\pi_{ij} = -\Delta\pi_{ij} \quad (i, j \in S_1),$$
$$\sum_{n=0, n\neq m}^{s_2} \Delta\lambda_{mn} = -\Delta\lambda_{mn} \quad (m, n \in S_2), \tag{6}$$

where $0 \leq |\Delta\pi_{ij}| \leq \varepsilon_{ij}$, $0 \leq |\Delta\lambda_{mn}| \leq \varepsilon_{mn}$, and ε_{ij} and ε_{mn} are the known small scalar for all $(i, j \in S_1)$ and $(m, n \in S_2)$, respectively.

Remark 1. Closed-loop system (3) is a linear system with the Markovian delays $d(k)$ and $\tau(k)$, which describe the behavior of the S-C and C-A random time delays, and with the uncertain transition probabilities.

Remark 2. The uncertain transition probabilities $\widehat{\pi}$ and $\widehat{\lambda}$ contain the certain terms π and λ, and the uncertain terms $\Delta\pi$ and $\Delta\lambda$, respectively. The uncertain terms $\Delta\pi$ and $\Delta\lambda$ are bounded, and the sums of the elements in each row are zeros.

3. Stability Analysis and Controller Design

By applying a Lyapunov approach and a linear matrix inequality technique, this section provides sufficient conditions for the stochastic stability and the synthesis of state feedback controller design of the system (3).

Definition 3 (see [21]). The closed-loop system (3) is said to be stochastically stable if, for every finite $x_0 = x(0)$, initial mode $d_0 = d(0) \in S_1$ and $\tau_0 = \tau(0) \in S_2$, there exists a finite $\mathscr{W} > 0$ such that

$$\mathbb{E}\left\{\sum_{k=0}^{\infty} \|x(k)\|^2 \mid x_0, d_0, \tau_0\right\} < x_0^T \mathscr{W} x_0. \tag{7}$$

Theorem 4. *For the system* (3)*, random but bounded scalars* $d(k) \in [\underline{d}\ \ \overline{d}]$ *and* $\tau(k) \in [\underline{\tau}\ \ \overline{\tau}]$*. If, for each mode* $\{i, j\} \in S_1$ *and* $\{m, n\} \in S_2$ *and matrices* $P_{i,m} > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$*, and* $R_2 > 0$, $\mathscr{M}_s = [\mathscr{M}_{1s}\ \ \mathscr{M}_{2s}\ \ \mathscr{M}_{3s}]$ *and* K *exist that satisfy the following matrix inequalities:*

$$\Gamma_{i,m} = \begin{bmatrix} -R_1^{-1} & 0 & \Xi_1 \\ * & -R_2^{-1} & \Xi_2 \\ * & * & \Xi_3 \end{bmatrix} < 0, \tag{8}$$

where

$$\Xi_1 = [\overline{t}(A - I)\ \ 0\ \ \overline{t}BK\ \ 0\ \ 0],$$
$$\Xi_2 = [(\overline{t} - \underline{t})(A - I)\ \ 0\ \ (\overline{t} - \underline{t})BK\ \ 0\ \ 0],$$
$$\Xi_3 = \Upsilon + \mathrm{sym}\overline{\mathscr{M}}^T \Omega,$$
$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \overline{P}_{i,m} & R_1 & 0 & 0 \\ * & \Upsilon_{22} & 0 & 0 & 0 \\ * & * & \Upsilon_{33} & R_1 + R_2 & R_2 \\ * & * & * & \Upsilon_{44} & 0 \\ * & * & * & * & \Upsilon_{55} \end{bmatrix},$$
$$\Upsilon_{11} = \overline{P}_{i,m} - P_{i,m} - R_1 + Q_1 + Q_2 + (\overline{t} - \underline{t} + 1)Q_3,$$
$$\Upsilon_{22} = \overline{P}_{i,m}, \qquad \Upsilon_{33} = -Q_3 - 2R_1 - 2R_2,$$
$$\Upsilon_{44} = -Q_1 - R_1 - R_2, \qquad \Upsilon_{55} = -Q_2 - R_2,$$
$$\overline{\mathscr{M}} = [\mathscr{M}_{1s}\ \ \mathscr{M}_{2s}\ \ \mathscr{M}_{3s}\ \ 0\ \ 0],$$
$$\Omega = [A - I\ \ -I\ \ BK\ \ 0\ \ 0],$$
$$\overline{P}_{i,m} = \sum_{j=0}^{s_1}\sum_{n=0}^{s_2} \widehat{\pi}_{ij}\widehat{\lambda}_{mn}P_{j,n},$$
$$\overline{t} = \overline{d} + \overline{\tau}, \quad \underline{t} = \underline{d} + \underline{\tau}, \tag{9}$$

and $\widehat{\pi}_{ij}$ *and* $\widehat{\lambda}_{mn}$ *are defined in* (4) *and* (5).

Then the closed-loop system (3) is stochastically stable.

Proof. For the closed-loop system (3), the stochastic Lyapunov functional candidate is constructed as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{10}$$

with

$$V_1(k) = x(k)^T P(d(k), \tau(k))\, x(k),$$
$$V_2(k) = \sum_{l=k-\overline{t}}^{k-1} x(l)^T Q_1 x(l) + \sum_{l=k-\underline{t}}^{k-1} x(l)^T Q_2 x(l),$$
$$V_3(k) = \sum_{\theta=-\overline{t}+2}^{-\underline{t}+1}\sum_{l=k+\theta-1}^{k-1} x(l)^T Q_3 x(l) + \sum_{l=k-\eta_k}^{k-1} x(l)^T Q_3 x(l),$$
$$V_4(k) = \sum_{\theta=-\overline{t}+1}^{0}\sum_{l=k+\theta-1}^{k-1} \overline{t}\delta(l)^T R_1 \delta(l) + \sum_{\theta=-\overline{t}+1}^{-\underline{t}}\sum_{l=k+\theta-1}^{k-1} (\overline{t} - \underline{t})\delta(l)^T R_2 \delta(l), \tag{11}$$

where $P(d(k), \tau(k)) > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$, and $R_2 > 0$.

Let $\delta(l) = x(l+1) - x(l)$, noting that $x(k+1) = Ax(k) + BKx(k - \eta_k)$. Then $0 = (A - I)x(k) - \delta(k) + BKx(k - \eta_k)$. For simplicity, we will use the following notations:

$\zeta(k) = \left[x(k)^T \quad \delta(k)^T \quad x(k-\eta_k)^T\right]^T$. Then, for any weighting matrices $\mathscr{M}_s$ with compatible dimensions (and let $\mathscr{M}_s = [\mathscr{M}_{1s} \quad \mathscr{M}_{2s} \quad \mathscr{M}_{3s}]$), we have $2\zeta(k)^T\mathscr{M}_s^T((A-I)x(k) - \delta(k) + BKx(k-\eta_k)) = 0$. Along the trajectory of the solution of the closed-loop system (3), we obtain

$$\begin{aligned}\mathbb{E}\left[\Delta V_1(k)\right] = {} & \mathbb{E}\left\{[x(k)+\delta(k)]^T\overline{P}_{i,m}[x(k)+\delta(k)]\right\} \\ & - x(k)^T P_{i,m} x(k) + 2\zeta(k)^T\mathscr{M}_s^T \\ & \times\left((A-I)x(k) + BK(k-\eta_k) - \delta(k)\right),\end{aligned} \tag{12}$$

$$\begin{aligned}\mathbb{E}\left[\Delta V_2(k)\right] = {} & x(k)^T(Q_1+Q_2)x(k) \\ & - x(k-\bar{t})^T Q_1 x(k-\bar{t}) \\ & - x(k-\underline{t})^T Q_2 x(k-\underline{t}),\end{aligned} \tag{13}$$

$$\begin{aligned}\mathbb{E}\left[\Delta V_3(k)\right] \le {} & (\bar{t}-\underline{t}+1)x(k)^T Q_3 x(k) \\ & - x(k-\eta_k)^T Q_3 x(k-\eta_k),\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbb{E}\left[\Delta V_4(k)\right] = {} & \bar{t}^2\delta(k)^T R_1\delta(k) + (\bar{t}-\underline{t})^2\delta(k)^T R_2\delta(k) \\ & - \sum_{l=k-\bar{t}}^{k-1}\bar{t}\delta(l)^T R_1\delta(l) \\ & - \sum_{l=k-\bar{t}}^{k-\underline{t}-1}(\bar{t}-\underline{t})\delta(l)^T R_2\delta(l).\end{aligned} \tag{15}$$

By Jensen's inequality, we can obtain

$$\begin{aligned}& - \sum_{l=k-\bar{t}}^{k-1}\bar{t}\delta(l)^T R_1\delta(l) \\ & = -\left(\sum_{l=k-\bar{t}}^{k-\eta_k-1} + \sum_{l=k-\eta_k}^{k-1}\right)(\bar{t}-\eta_k+\eta_k)\delta(l)^T R_1\delta(l) \\ & \le -\left((\bar{t}-\eta_k)\sum_{l=k-\bar{t}}^{k-\eta_k-1}\delta^T(l)R_1\delta(l) + \eta_k\sum_{l=k-\eta_k}^{k-1}\delta^T(l)R_1\delta(l)\right) \\ & \le -\left(\left(\sum_{l=k-\bar{t}}^{k-\eta_k-1}\delta(l)\right)^T R_1\left(\sum_{l=k-\bar{t}}^{k-\eta_k-1}\delta(l)\right)\right. \\ & \qquad \left. + \left(\sum_{l=k-\eta_k}^{k-1}\delta(l)\right)^T R_1\left(\sum_{l=k-\eta_k}^{k-1}\delta(l)\right)\right) \\ & = -(x(k-\eta_k) - x(k-\bar{t}))^T R_1(x(k-\eta_k) - x(k-\bar{t})) \\ & \quad - (x(k) - x(k-\eta_k))^T R_1(x(k) - x(k-\eta_k)).\end{aligned} \tag{16}$$

Similarly, we have

$$\begin{aligned}& - \sum_{l=k-\bar{t}}^{k-\underline{t}-1}(\bar{t}-\underline{t})\delta(l)^T R_2\delta(l) \\ & \le -\left(\sum_{l=k-\bar{t}}^{k-\eta_k-1}\delta(l)\right)^T R_2\left(\sum_{l=k-\bar{t}}^{k-\eta_k-1}\delta(l)\right) \\ & \quad + \left(\sum_{l=k-\eta_k}^{k-\underline{t}-1}\delta(l)\right)^T R_2\left(\sum_{l=k-\eta_k}^{k-\underline{t}-1}\delta(l)\right) \\ & = -(x(k-\eta_k) - x(k-\bar{t}))^T R_2(x(k-\eta_k) - x(k-\bar{t})) \\ & \quad - (x(k-\underline{t}) - x(k-\eta_k))^T R_2(x(k-\underline{t}) - x(k-\eta_k)).\end{aligned} \tag{17}$$

By substituting (16) and (17) to (15) and then combining (12), (13), and (14), we have

$$\begin{aligned}& \mathbb{E}[\Delta V] \\ & \le \xi(k)^T\left\{\Xi_3 + \left[\bar{t}(A-I) \quad 0 \quad \bar{t}BK \quad 0 \quad 0\right]^T\right. \\ & \qquad \times R_1\left[\bar{t}(A-I) \quad 0 \quad \bar{t}BK \quad 0 \quad 0\right] \\ & \qquad + \left[(\bar{t}-\underline{t})(A-I) \quad 0 \quad (\bar{t}-\underline{t})BK \quad 0 \quad 0\right]^T \\ & \qquad \left.\times R_2\left[(\bar{t}-\underline{t})(A-I) \quad 0 \quad (\bar{t}-\underline{t})BK \quad 0 \quad 0\right]\right\}\xi(k) \\ & = \xi(k)^T\Gamma_{i,m}\xi(k),\end{aligned} \tag{18}$$

where $\xi(k) = \left[\zeta(k)^T \quad x(k-\bar{t})^T \quad x(k-\underline{t})^T\right]^T$. By using the Schur complement, (8) guarantees that $\Gamma_{i,m} < 0$. Therefore,

$$\mathbb{E}[\Delta V] \le -\boldsymbol{\lambda}_{\min}(-\Gamma_{i,m})\xi(k)^T\xi(k) \le -\boldsymbol{\eta}x(k)^T x(k), \tag{19}$$

where $\boldsymbol{\lambda}_{\min}(-\Gamma_{i,m})$ denotes the minimal eigenvalue of $-\Gamma_{i,m}$ and $\boldsymbol{\eta} = \inf\{\lambda_{\min}(-\Gamma_{i,m})\}$. From (19), it follows that, for any $t > 0$,

$$\mathbb{E}[V(k+1)] - \mathbb{E}[V(0)] \le -\boldsymbol{\eta}\sum_{k=0}^{t}\mathbb{E}\left[x(k)^T x(k)\right]. \tag{20}$$

Furthermore

$$\sum_{k=0}^{t}\mathbb{E}\left[x(k)^T x(k)\right] \le \frac{1}{\boldsymbol{\eta}}\mathbb{E}[V(0)]. \tag{21}$$

By taking $t \to \infty$ as the limit, we obtain

$$\sum_{k=0}^{\infty}\mathbb{E}\left[x(k)^T x(k)\right] \le \frac{1}{\boldsymbol{\eta}}\mathbb{E}[V(0)] = \frac{1}{\boldsymbol{\eta}}x_0^T P(d_0,\tau_0)x_0 < \infty. \tag{22}$$

According to Definition 3, the closed-loop system (3) exhibits stochastic stability for all uncertain transition probability matrices. □

Theorem 4 gives a sufficient condition for the stochastic stability of the system (3). However, it should be noted that the controller gain K cannot be obtained according to the condition in (8) because of the nonlinear terms R_1^{-1} and R_2^{-1} and the uncertain terms $\Delta\pi$ and $\Delta\lambda$. To handle this problem, the equivalent LMI conditions are given as follows.

Before proceeding further, we provide the following lemma that will play a significant role in processing the uncertainty terms $\Delta\pi$ and $\Delta\lambda$ of uncertain transition probability matrices $\hat{\pi}$ and $\hat{\lambda}$.

Lemma 5 (see [28]). *For any vectors of $a, b \in \mathscr{R}^n$ and positive matrix $Z \in \mathscr{R}^{n_Z, n_Z}$, the following holds:*

$$2a^T b \le a^T Z a + b^T Z^{-1} b. \tag{23}$$

Theorem 6. *For the system* (3)*, the random but bounded scalars $d(k) \in \left[\underline{d}\ \ \overline{d}\right]$ and $\tau(k) \in \left[\underline{\tau}\ \ \overline{\tau}\right]$. If, for each mode $\{i, j\} \in S_1$ and $\{m, n\} \in S_2$, the tuning parameters $\varphi_1 > 0$ and $\varphi_2 > 0$, the scalars $\varepsilon_{ij} > 0$ and $\varepsilon_{nm} > 0$, and matrices $\widehat{P}_{i,m} > 0$, $X > 0$, $\widehat{Q}_1 > 0$, $\widehat{Q}_2 > 0$, $\widehat{Q}_3 > 0$, $\widehat{R}_1 > 0$, $\widehat{R}_2 > 0$, $Z_i > 0$, $Z_m > 0$, and $Z_{j,n} > 0$, and Y exist such that*

$$\begin{bmatrix} \Theta_{11} & 0 & 0 & \Theta_{14} \\ * & \Theta_{22} & 0 & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{bmatrix} < 0, \tag{24}$$

where

$$\Theta_{11} = -\operatorname{sym}\{X\} + \widehat{R}_1, \qquad \Theta_{22} = -\operatorname{sym}\{X\} + \widehat{R}_2,$$

$$\Theta_{33} = \operatorname{diag}\left\{-\widehat{Z}_m, -\widehat{Z}_i, -\widehat{Z}_{j,n}\right\},$$

$$\Theta_{14} = \left[\bar{t}(AX - X)\ \ 0\ \ \bar{t}BY\ \ 0\ \ 0\right],$$

$$\Theta_{24} = \left[(\bar{t} - \underline{t})(AX - X)\ \ 0\ \ (\bar{t} - \underline{t})BY\ \ 0\ \ 0\right],$$

$$\Theta_{34} = \begin{bmatrix} \Psi_1 & \Psi_1 & 0 & 0 & 0 \\ \Psi_2 & \Psi_2 & 0 & 0 & 0 \\ \Psi_3 & \Psi_3 & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{Z}_m = \operatorname{diag}\underbrace{\{Z_m, \ldots, Z_m\}}_{s_1(s_2-1)}, \qquad \widehat{Z}_i = \operatorname{diag}\underbrace{\{Z_i, \ldots, Z_i\}}_{(s_1-1)s_2},$$

$$\widehat{Z}_{j,n} = \operatorname{diag}\underbrace{\{Z_{j,n}, \ldots, Z_{j,n}\}}_{s_1 s_2},$$

$$\Psi_1 = \begin{bmatrix} \sqrt{\pi_{i1}}\Delta\widehat{P}_{1,m} \\ \cdots \\ \sqrt{\pi_{is_1}}\Delta\widehat{P}_{s_1,m} \end{bmatrix}, \qquad \Psi_2 = \begin{bmatrix} \sqrt{\lambda_{m1}}\Delta\widehat{P}_{i,1} \\ \cdots \\ \sqrt{\lambda_{ms_2}}\Delta\widehat{P}_{i,s_2} \end{bmatrix},$$

$$\Psi_3 = \begin{bmatrix} \widehat{P}_{1,1} \\ \cdots \\ \widehat{P}_{s_1,s_2} \end{bmatrix},$$

$$\Delta\widehat{P}_{j,m} = \begin{bmatrix} \widehat{P}_{j,1} - \widehat{P}_{j,m} \\ \cdots \\ \widehat{P}_{j,m-1} - \widehat{P}_{j,m} \\ \widehat{P}_{j,m+1} - \widehat{P}_{j,m} \\ \cdots \\ \widehat{P}_{j,s_2} - \widehat{P}_{j,m} \end{bmatrix}, \qquad \Delta\widehat{P}_{i,n} = \begin{bmatrix} \widehat{P}_{1,n} - \widehat{P}_{j,n} \\ \cdots \\ \widehat{P}_{i-1,n} - \widehat{P}_{j,n} \\ \widehat{P}_{i+1,n} - \widehat{P}_{j,n} \\ \cdots \\ \widehat{P}_{s_1,n} - \widehat{P}_{j,n} \end{bmatrix},$$

$$\Theta_{44} = \operatorname{sym}\left\{\Pi^T\widehat{\Omega}\right\} + \begin{bmatrix} \widehat{\Phi}_{11} & \vartheta_1\circledast & \vartheta_2\widehat{R}_1 & 0 & 0 \\ * & \vartheta_1^2\circledast & 0 & 0 & 0 \\ * & * & \vartheta_2^2\widehat{\Phi}_{33} & \vartheta_2\left(\widehat{R}_1 + \widehat{R}_2\right) & \vartheta_2\widehat{R}_2 \\ * & * & * & \widehat{\Phi}_{44} & 0 \\ * & * & * & * & \widehat{\Phi}_{55} \end{bmatrix},$$

$$\widehat{\Phi}_{11} = \circledast - \widehat{P}_{i,m} - \widehat{R}_1 + \widehat{Q}_1 + \widehat{Q}_2 + (\bar{t} - \underline{t} + 1)\widehat{Q}_3,$$

$$\widehat{\Phi}_{33} = -\widehat{Q}_3 - 2\widehat{R}_1 - 2\widehat{R}_2, \qquad \widehat{\Phi}_{44} = -\widehat{Q}_1 - \widehat{R}_1 - \widehat{R}_2,$$

$$\widehat{\Phi}_{55} = -\widehat{Q}_2 - \widehat{R}_2,$$

$$\circledast = \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\left(\pi_{ij}\lambda_{mn}\widehat{P}_{j,n} + \frac{\left(\varepsilon_{ij}\varepsilon_{mn}\right)^2}{4}Z_{j,n}\right) + \sum_{j=0}^{s_1}\sum_{n=0,n\neq m}^{s_2}\frac{\pi_{ij}\varepsilon_{mn}^2}{4}Z_m + \sum_{j=0,j\neq i}^{s_1}\sum_{n=0}^{s_2}\frac{\lambda_{mn}\varepsilon_{ij}^2}{4}Z_i,$$

$$\Pi = \left[I\ \ I\ \ I\ \ 0\ \ 0\right],$$

$$\widehat{\Omega} = \left[AX - X\ \ -\vartheta_1 X\ \ \vartheta_2 BY\ \ 0\ \ 0\right]. \tag{25}$$

Then the closed-loop system (3) *is stochastically stable and the controller $\overline{u}(k) = K\overline{x}(k) = YX^{-1}\overline{x}(k)$ is a state feedback controller of the system* (3).

Proof. Let $\overline{\Delta}_i = \operatorname{diag}\{I, I, X_{i,m}, \vartheta_1 X_{i,m}, \vartheta_2 X_{i,m}, X_{i,m}, X_{i,m}\}$, $\mathscr{M}_{1s}^{-1} = X_{i,m}$, $\mathscr{M}_{2s}^{-1} = \vartheta_1 X_{i,m}$, and $\mathscr{M}_{3s}^{-1} = \vartheta_2 X_{i,m}$, where $\vartheta_1 > 0$ and $\vartheta_2 > 0$ are known tuning parameters. We restrict $X_{i,m}$ to be the same for all $\{i, m\}$ (namely, $X_{i,m} = X$) and give the notations as

$$\widehat{P}_{i,m} = X^T P_{i,m} X, \qquad \widehat{\overline{P}}_{i,m} = X^T \overline{P}_{i,m} X, \qquad \widehat{R}_1 = X^T R_1 X,$$

$$\widehat{R}_2 = X^T R_2 X,$$

$$\widehat{Q}_1 = X^T Q_2 X, \qquad \widehat{Q}_2 = X^T Q_2 X, \qquad \widehat{Q}_3 = X^T Q_3 X. \tag{26}$$

Pre- and postmultiplying $\overline{\Delta}_i^T$ and $\overline{\Delta}_i$ to (8), respectively, we have

$$\begin{bmatrix} -X\widehat{R}_1^{-1}X^T & 0 & \widehat{\Xi}_1 \\ * & -X\widehat{R}_2^{-1}X^T & \widehat{\Xi}_2 \\ * & * & \widehat{\Xi}_3 \end{bmatrix} < 0, \tag{27}$$

where

$$\widehat{\Xi}_1 = \left[\bar{t}\,(AX - X) \;\; 0 \;\; \bar{t}BY \;\; 0 \;\; 0\right],$$

$$\widehat{\Xi}_2 = \left[(\bar{t} - \underline{t})\,(AX - X) \;\; 0 \;\; (\bar{t} - \underline{t})\,BY \;\; 0 \;\; 0\right],$$

$$\widehat{\Xi}_3 = \widetilde{\Phi} + \mathrm{sym}\Pi^T\widehat{\Omega},$$

$$\widetilde{\Phi} = \begin{bmatrix} \widetilde{\Phi}_{11} & \vartheta_1\widehat{\overline{P}}_{i,m} & \vartheta_2\widehat{R}_1 & 0 & 0 \\ * & \vartheta_1^2\widehat{\overline{P}}_{i,m} & 0 & 0 & 0 \\ * & * & \vartheta_2^2\widehat{\Phi}_{33} & \vartheta_2\left(\widehat{R}_1 + \widehat{R}_2\right) & \vartheta_2\widehat{R}_2 \\ * & * & * & \widehat{\Phi}_{44} & 0 \\ * & * & * & * & \widehat{\Phi}_{55} \end{bmatrix},$$

$$\widetilde{\Phi}_{11} = \widehat{\overline{P}}_{i,m} - \widehat{P}_{i,m} - \widehat{R}_1 + \widehat{Q}_1 + \widehat{Q}_2 + (\bar{t} - \underline{t} + 1)\,\widehat{Q}_3,$$

$$\overline{P}_{i,m} = \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\widehat{\pi}_{ij}\widehat{\lambda}_{mn}\widehat{P}_{j,n}, \tag{28}$$

where $\bar{t}$ and $\underline{t}$ are defined in Theorem 4 and $\widehat{\Phi}_{33}$, $\widehat{\Phi}_{44}$, and $\widehat{\Phi}_{55}$ are defined in Theorem 6.

According to the assumption on uncertain transition probabilities $\widehat{\pi}$ and $\widehat{\lambda}$ and the fact that $\sum_{j=0,j\neq i}^{s_1}\Delta\pi_{ij} = -\Delta\pi_{ii}$ and $\sum_{n=0,n\neq m}^{s_2}\Delta\lambda_{mn} = -\Delta\lambda_{mm}$, one has

$$\begin{aligned} \widehat{\overline{P}}_{i,m} &= \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\widehat{\pi}_{ij}\widehat{\lambda}_{mn}\widehat{P}_{j,n} \\ &= \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\left(\pi_{ij} + \Delta\pi_{ij}\right)\left(\lambda_{mn} + \Delta\lambda_{mn}\right)\widehat{P}_{j,n} \\ &= \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\pi_{ij}\lambda_{mn}\widehat{P}_{j,n} + \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\pi_{ij}\Delta\lambda_{mn}\widehat{P}_{j,n} \\ &\quad + \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\Delta\pi_{ij}\lambda_{mn}\widehat{P}_{j,n} + \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\Delta\pi_{ij}\Delta\lambda_{mn}\widehat{P}_{j,n}. \end{aligned} \tag{29}$$

Note that

$$\begin{aligned} &\sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\pi_{ij}\Delta\lambda_{mn}\widehat{P}_{j,n} \\ &\quad = \sum_{j=0}^{s_1}\pi_{ij}\left(\sum_{n=0,n\neq m}^{s_2}\Delta\lambda_{mn}\widehat{P}_{j,n} + \Delta\lambda_{mm}\widehat{P}_{j,m}\right) \\ &\quad = \sum_{j=0}^{s_1}\pi_{ij}\sum_{n=0,n\neq m}^{s_2}\Delta\lambda_{mn}\left(\widehat{P}_{j,n} - \widehat{P}_{j,m}\right). \end{aligned} \tag{30}$$

By Lemma 5 and the fact that $|\Delta\lambda_{mn}| \le \varepsilon_{mn}$, we have

$$\begin{aligned} &\sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\pi_{ij}\Delta\lambda_{mn}\widehat{P}_{j,n} \\ &\quad \le \sum_{j=0}^{s_1}\pi_{ij}\sum_{n=0,n\neq m}^{s_2}\left(\frac{1}{4}\varepsilon_{mn}^2 Z_m + \left(\widehat{P}_{j,n} - \widehat{P}_{j,m}\right)^T \right. \\ &\qquad \left. \times Z_m^{-1}\left(\widehat{P}_{j,n} - \widehat{P}_{j,m}\right)\right). \end{aligned} \tag{31}$$

Similarly, we have

$$\begin{aligned} &\sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\Delta\pi_{ij}\lambda_{mn}\widehat{P}_{j,n} \\ &\quad \le \sum_{j=0,j\neq i}^{s_1}\sum_{n=0}^{s_2}\lambda_{mn} \\ &\qquad \times\left(\frac{1}{4}\varepsilon_{ij}^2 Z_i + \left(\widehat{P}_{j,n} - \widehat{P}_{i,n}\right)^T Z_i^{-1}\left(\widehat{P}_{j,n} - \widehat{P}_{i,n}\right)\right). \\ &\sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\Delta\pi_{ij}\Delta\lambda_{mn}\widehat{P}_{j,n} \\ &\quad \le \sum_{j=0}^{s_1}\sum_{n=0}^{s_2}\left(\frac{1}{4}\varepsilon_{ij}^2\varepsilon_{mn}^2 Z_{j,n} + \widehat{P}_{j,n}^T Z_{j,n}^{-1}\widehat{P}_{j,n}\right). \end{aligned} \tag{32}$$

Note that, for any matrix X, we have $XW^{-1}X^T \ge \mathrm{sym}\{X\} - W$ for $W = \widehat{R}_1$ and $W = \widehat{R}_2$. Combining (29), (31), and (32) and by the Schur complement, (24) can be yielded easily from (27); this completes the proof of Theorem 6. □

4. Numerical Example

In this section, we illustrate our results through an example. We apply the results in Theorem 6 to a simple inverted pendulum system [5] shown in Figure 2, which is a two-order unstable system. The state variables are $\left[\varphi \;\; \dot{\varphi}\right]^T$, where φ is the angular position of the pendulum. The parameters used are $m = 0.1$ kg and $L = 1$ m, without friction surfaces. The sampling time is $T_s = 0.05$ s. The plant matrices are given by

$$A = \begin{bmatrix} 1.0123 & 0.0502 \\ 0.4920 & 1.0123 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.0125 \\ 0.5020 \end{bmatrix}. \tag{33}$$

We assume that the stochastic Markovian jumping S-C delay $d(k) \in \{0, 1\}$ and C-A delay $\tau(k) \in \{0, 1, 2\}$ and their uncertain transition probability matrices are given as follows:

$$\pi = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}, \qquad \lambda = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.2 & 0.4 \end{bmatrix},$$

$$\Delta\pi = \begin{bmatrix} 0.02 & -0.02 \\ -0.01 & 0.01 \end{bmatrix}, \qquad \Delta\lambda = \begin{bmatrix} 0.03 & -0.03 & 0 \\ -0.02 & 0.01 & 0.01 \\ -0.03 & 0.02 & 0.01 \end{bmatrix}. \tag{34}$$

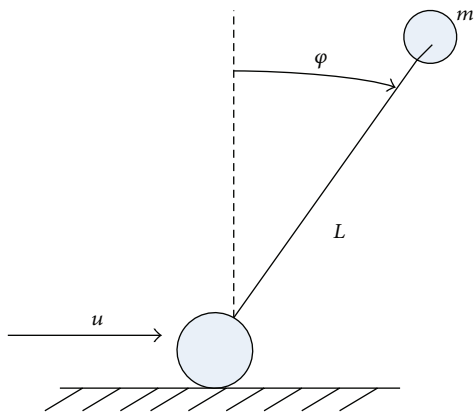

FIGURE 2: A simple inverted pendulum.

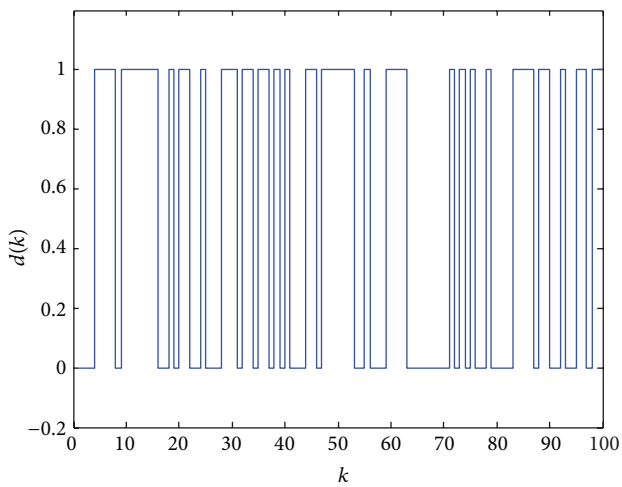

FIGURE 3: Values of the S-C delay $d(k)$.

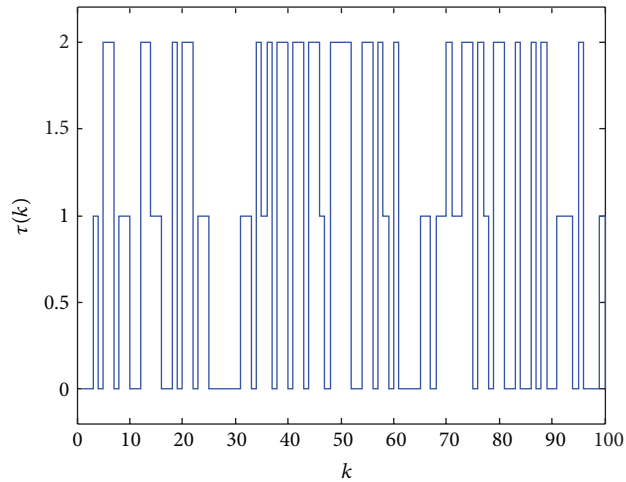

FIGURE 4: Values of the C-A delay $\tau(k)$.

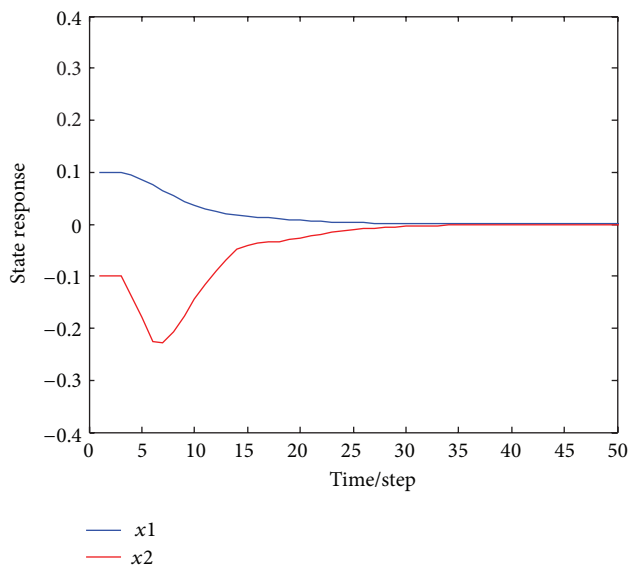

FIGURE 5: State trajectories under K.

The eigenvalues of A are 1.1695 and 0.8551. Therefore, the discrete-time model is unstable.

Figures 3 and 4 show part of the simulation of the Markov chains mode. The initial conditions are as follows: $d(0) = 0$, $\tau(0) = 0$, and $x(0) = [0.1 \quad -0.1]^T$. By Theorem 6, when $\varepsilon_{ij} = 0.02$, $\varepsilon_{mn} = 0.03$, $\vartheta_1 = 0.09$, and $\vartheta_2 = 12$, we can obtain the gain matrix K of state feedback controller (2) which is constructed as

$$K = YX^{-1} = \begin{bmatrix} -0.1046 & -0.1177 \end{bmatrix} \begin{bmatrix} 0.1636 & -0.3923 \\ 0.3923 & 1.4211 \end{bmatrix}^{-1} = \begin{bmatrix} -2.4757 & -0.7662 \end{bmatrix}. \tag{35}$$

The state trajectories of the system (3) are shown in Figure 5, where two curves represent state trajectories under the controller gains K. Figure 5 also indicates that the system (3) is stochastically stable.

Remark 7. In this example, the uncertain transition probabilities are given as a discrete probability distribution function. When the uncertain transition probability is given as a continuous probability distribution function, we can use the H_2 norm of the continuous probability distribution function as the upper bound to simulation.

5. Conclusions

The state feedback stabilization problem for a class of NCS with the S-C and C-A random time delays is investigated in this paper. The resulting closed-loop NCS is modeled as a linear system with uncertain Markovian transition probabilities. New sufficient conditions on stochastic stability and stabilization are obtained by Lyapunov stability theory and linear matrix inequalities method. An example is presented to illustrate the effectiveness of the approach. Although only the time-delay issue for NCS is addressed in this paper, the method can be extended to the NCS with the random packet dropouts, time delays, and packet dropouts and to the MJLS with the uncertain Markovian transition probabilities.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Finite-Time H_∞ Control for Discrete-Time Markov Jump Systems with Actuator Saturation

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This paper investigates the finite-time control problem for discrete-time Markov jump systems subject to saturating actuators. A finite-state Markovian process is given to govern the transition of the jumping parameters. The finite-time H_∞ controller via state feedback is designed to guarantee that the resulting system is mean-square locally asymptotically finite-time stabilizable. Based on stochastic finite-time stability analysis, sufficient conditions that ensure stochastic control performance of discrete-time Markov jump systems are derived in the form of linear matrix inequalities. Finally, a numerical example is provided to illustrate the effectiveness of the proposed approach.

1. Introduction

During the past several decades, the issue of finite-time control has drawn increasing attention of academic researchers in the area of control field, and various results have been reported. To this end, a considerable amount of research has been carried out; see Hong et al. [1]; He and Liu [2, 3]; Li et al. [4]; Song et al. [5]; Lan et al. [6]. Among the proposed solutions, state feedback control is an important approach to improve finite-time control performance. For instance, by using both state feedback and dynamic output feedback control, finite-time control of the robot system is studied in Hong et al. [7]. Furthermore, based on dynamic observer-based state feedback and Lyapunov-Krasovskii functional approach, the finite-time H_∞ control problem for time-delay nonlinear jump systems was addressed in the work of He and Liu [2, 3, 8].

On the other hand, more and more attention has been paid to the study of actuator saturation due to its practical and theoretical importance. Therefore, various approaches were investigated to handle systems with actuator saturation, such as in the work of Cao and Lin [9]; the stability of discrete-time systems with actuator saturation was analyzed by a saturation-dependent Lyapunov function. By introducing a time-varying sliding surface, the robust stabilization problem of linear unstable plants with saturating actuators was studied in Corradini and Orlando [10]. Furthermore, the controller design method of Markov jumping systems subject to actuator saturation was presented in Liu et al. [11]. Via dynamic anti-windup fuzzy design, the robust stabilization problem of state delayed T-S fuzzy systems with input saturation was proposed in Song et al. [12]. Other results can refer to [13–17] and references therein.

It is well known that the control problem of Markov jump systems has also been extensively studied and a large variety of control problems have been widely investigated, for instance, the stabilization of Markov jump systems with time delays [18–24], robust control [25], control of singular Markov systems [26], control of discrete-time stochastic Markov jump systems [27, 28], and fuzzy dissipative control for nonlinear Markovian jump systems [29]. Furthermore, robust stability for uncertain delayed neural networks with Markov jumping parameters was analyzed in Li et al. [30]. Robust H_∞ filter was designed for uncertain discrete Markov jump singular systems with mode-dependent time delay in Ma and Boukas [31]. Delay-dependent robust stabilization problem for uncertain stochastic switching systems with distributed delays was studied in Shen et al. [32]. Via retarded output feedback, passivity-based control problem for Markov jump systems was addressed in Shen et al. [33]. Observer based finite-time H_∞ control problem of discrete-time Markov jump systems was studied in Zhang and

Liu [34]. However, to the best of our knowledge, the problem of finite-time stabilization of discrete-time stochastic systems has not been fully investigated and it is the main purpose of our study.

In this paper, the attention is focused on the finite-time H_∞ control problem of discrete-time Markov jump systems with actuator saturation. A state feedback controller is designed to ensure the stochastic finite-time boundedness and stochastic finite-time stabilization of the resulting closed-loop system for all admissible disturbances. The desired controller can be designed via solving a convex optimization problem. Finally, a numerical example is employed to show the effectiveness of the proposed method.

Notation. Throughout the paper, for symmetric matrices X and Y, the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp., positive definite). I is the identity matrix with appropriate dimension. The notation N^T represents the transpose of the matrix N; $\lambda_{\max}(M)$ (resp., $\lambda_{\min}(M)$) means the largest (resp., smallest) eigenvalue of the matrix M; $(\Omega, \mathscr{F}, \mathscr{P})$ is a probability space; Ω is the sample space, $\mathscr{F}$ is the σ-algebra of subsets of the sample space, and $\mathscr{P}$ is the probability measure on $\mathscr{F}$; $\mathscr{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathscr{P}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The symbol $*$ is used to denote a matrix which can be inferred by symmetry, $\mathrm{He}\{A\} = A^T + A$.

2. Preliminaries and Problem Description

2.1. Preliminaries. Throughout this paper, we will use the following definitions and lemmas.

Lemma 1 (see [12]). *For the matrix K_i and the system Σ, the appropriate matrix $L_i \in \mathbb{R}^{m\times n}$ is given if $x(k)$ is in the set $D(u_o)$, where $D(u_o)$ is defined as follows:*

$$D(u_o) = \{x(k) \in \mathbb{R}^n;\ -u_{0(k)} \leq (K_{i(k)} - L_{i(k)})\, x(t) \leq u_{0(k)},\ u_{0(k)} > 0,\ k = 1, \ldots, m\}; \tag{1}$$

then for any diagonal positive matrix $T \in \mathbb{R}^{m\times m}$, we derive

$$\psi(u(k))^T T\left(\psi(u(k)) - L_i x(k)\right) \leq 0. \tag{2}$$

Lemma 2 (see [32]). *For the given symmetric matrix $S \in \mathbb{R}^{(n+m)\times(n+m)}$:*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, \tag{3}$$

where $S_{11} \in \mathbb{R}^{n\times n}$, $S_{12} \in \mathbb{R}^{n\times m}$, and $S_{22} \in \mathbb{R}^{m\times m}$; the following conditions are equivalent:

(1) $S < 0$,

(2) $S_{11} < 0,\ S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

(3) $S_{22} < 0,\ S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Definition 3 (see [34]). The resulting closed-loop system (12) is stochastic finite-time stable (SFTB) with respect to $(\delta_x, \epsilon, R_i, N, d)$ with $0 < \delta_x < \epsilon$, $R_i > 0$, and $N \in Z_{k\geq 0}$ if there exists state feedback controller such that

$$E\left\{x^T(0) R_i x(0)\right\} \leq \delta_x^2 \Longrightarrow E\left\{x^T(k) R_i x(k)\right\} < \epsilon^2, \quad \forall k \in \{1, 2, \ldots, N\}. \tag{4}$$

Definition 4 (see [34]). The resulting closed-loop system (12) is said to be stochastic H_∞ finite-time stable via state feedback with respect to $(\delta_x, \epsilon, R_i, N, \gamma, d)$ with $0 < \delta_x < \epsilon$, $R_i > 0$, $\gamma > 0$, and $N \in Z_{k\geq 0}$ if the system (11)-(12) is SFTB with respect to $(\delta_x, \epsilon, R_i, N, \gamma, d)$ and under the zero-initial condition the output $z(k)$ satisfies

$$E\left\{\Sigma_{j=0}^N z^T(j) z(j)\right\} \leq \gamma^2 E\left\{\Sigma_{j=0}^N w^T(j) w(j)\right\}, \tag{5}$$

for any nonzero $w(k)$ which satisfies (10), where γ is a prescribed positive scalar. Moreover, the state feedback controller (11) is called H_∞ controller of MJS (12).

2.2. Problem Description. Consider the following discrete-time Markov jump system (Σ) in the probability space $(\Omega, \mathscr{F}, \mathscr{P})$:

$$\begin{aligned} x(k+1) &= A(r(k))\, x(k) + B(r(k))\, \mathrm{sat}(u(k)) + G(r(k))\, w(k), \\ z(k) &= C(r(k))\, x(k) + D_1(r(k))\, \mathrm{sat}(u(k)) + D_2(r(k))\, w(k), \end{aligned} \tag{6}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $z(k) \in \mathbb{R}^l$ is the controlled output, and $\mathrm{sat}(u(k)) \in \mathbb{R}^m$ is the saturated control input. $w(k) \in L_2^p[0 + \infty)$ is the external disturbances. $\{r(k)\}$ is a discrete-time Markov process and takes values from a finite set $S = \{1, 2, \ldots, \mathscr{N}\}$ with transition probabilities given by

$$\Pr(r_{k+1} = j \mid r_k = i) = \pi_{ij}, \tag{7}$$

where $\pi_{ij} \geq 0$, for $\forall j, i \in S$, and $\Sigma_{j\in S}\pi_{ij} = 1$. Moreover, the transition rates matrix of the system (Σ) is defined by

$$\begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1\mathscr{N}} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2\mathscr{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{\mathscr{N}1} & \pi_{\mathscr{N}2} & \cdots & \pi_{\mathscr{N}\mathscr{N}} \end{bmatrix}. \tag{8}$$

The inputs of the plant are supposed to be bounded as follows:

$$-u_{0(k)} \leq u_{(k)} \leq u_{0(k)}, \quad u_{0(k)} > 0, \quad k = 1, \ldots, m. \tag{9}$$

For the system (Σ), to simplify the notation, we denote $A_i = A(r(k))$ for each $r(k) = i \in S$, and the other symbols are similarly denoted. A_i, B_i, G_i, C_i, D_{1i}, and D_{2i} are known mode-dependent constant matrices with appropriate dimensions.

Assumption 5 (see [34]). The external disturbance $w(k)$ is varying and satisfies the following constraint condition:

$$\Sigma_{k=0}^{T} w(k)^T w(k) \le d, \quad d \ge 0. \tag{10}$$

For the system (Σ), we construct the following state feedback controller:

$$u(k) = K(r(k))\, x(k). \tag{11}$$

Then, the resulting closed-loop discrete-time Markov jump system (MJS) is as follows:

$$\begin{aligned} x(k+1) &= (A_i + B_i K_i)\, x(k) + B_i \psi(u(k)) + G_i w(k), \\ z(k) &= (C_i + D_{1i} K_i)\, x(k) + D_{1i} \psi(u(k)) + D_{2i} w(k), \end{aligned} \tag{12}$$

where $\psi(u(k)) = \mathrm{sat}(u(k)) - u(k)$.

3. Main Results

In this section, we investigate the design of a state feedback controller which guarantees the locally finite-time stabilizable of the resulting closed-loop system. Some sufficient conditions and the method of designing state feedback controller are given.

Theorem 6. *For each $r(k) = i \in S$, there exists a feedback controller $u(k) = K_i x(k)$, $K_i = Y_i X_i^{-1}$, such that the resulting closed-loop system* (12) *is SFTB with respect to $(\delta_x, \epsilon, R_i, N, d)$ with $0 < \delta_x < \epsilon$, if there exist scalars $\mu \ge 0$, $\sigma_1 \ge 0$, and $\sigma_2 \ge 0$, three sets of mode-dependent symmetric matrices $X_i > 0$, $J_i > 0$, and $Q_i > 0$, and two sets of mode-dependent matrices Y_i and $\bar{L}_i = L_i X_i$, such that the following conditions hold:*

$$\begin{bmatrix} -\mu X_i & 0 & \bar{L}_i^T & \bar{L}_{1i}^T \\ * & -Q_i & 0 & L_{2i}^T \\ * & * & -2J_i & L_{3i}^T \\ * & * & * & -W \end{bmatrix} < 0, \tag{13}$$

$$\begin{bmatrix} \sigma_2 d^2 - \mu^{-N}\epsilon^2 & * \\ \delta_x & -\sigma_1 \end{bmatrix} < 0, \tag{14}$$

$$\begin{bmatrix} X_i & * \\ Y_i + \bar{L}_i & u_{0(k)}^2 \end{bmatrix} > 0, \quad k = 1, \ldots, m, \tag{15}$$

$$\sigma_1 R_i^{-1} < X_i < R_i^{-1}, \tag{16}$$

$$0 < Q_i < \sigma_2 I, \tag{17}$$

where

$$\begin{aligned} W &= \mathrm{diag}\{X_1, X_2, \ldots, X_n\}, \\ \bar{L}_{1i}^T &= \left[\sqrt{\pi_{i1}}\left(X_i A_i^T + Y_i^T B^T\right) \quad \sqrt{\pi_{i2}}\left(X_i A_i^T + Y_i^T B^T\right) \quad \cdots \quad \sqrt{\pi_{in}}\left(X_i A_i^T + Y_i^T B^T\right)\right], \\ L_{2i}^T &= \left[\sqrt{\pi_{i1}} G_i^T \quad \sqrt{\pi_{i2}} G_i^T \quad \cdots \quad \sqrt{\pi_{in}} G_i^T\right], \\ L_{3i}^T &= \left[\sqrt{\pi_{i1}} J_i B_i^T \quad \sqrt{\pi_{i2}} J_i B_i^T \quad \cdots \quad \sqrt{\pi_{in}} J_i B_i^T\right]. \end{aligned} \tag{18}$$

Proof. Define the following Lyapunov function for each $\delta(t) = i \in S$:

$$V(k) = x(k)^T P_i x(k). \tag{19}$$

It is readily obtained that

$$\begin{aligned} E\{V(k+1)\} &= E\left\{\Sigma_{j=1}^{n} \pi_{ij} x(k+1)^T P_j x(k+1)\right\} \\ &= \xi(k)^T \left[L_{1i} \;\; L_{2i} \;\; L_{3i}\right]^T \bar{W} \left[L_{1i} \;\; L_{2i} \;\; L_{3i}\right] \xi(k), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \xi(k) &= \left[x(k)^T \quad w(k)^T \quad \psi(k)^T\right], \\ \bar{W} &= \mathrm{diag}\{P_1, P_2, \ldots, P_h\}, \\ L_{1i}^T &= \left[\sqrt{\pi_{i1}}(A_i + B_i K_i)^T \quad \sqrt{\pi_{i2}}(A_i + B_i K_i)^T \quad \cdots \quad \sqrt{\pi_{in}}(A_i + B_i K_i)^T\right], \\ L_{2i}^T &= \left[\sqrt{\pi_{i1}} G_i^T \quad \sqrt{\pi_{i2}} G_i^T \quad \cdots \quad \sqrt{\pi_{in}} G_i^T\right], \\ L_{3i}^T &= \left[\sqrt{\pi_{i1}} B_i^T \quad \sqrt{\pi_{i2}} B_i^T \quad \cdots \quad \sqrt{\pi_{in}} B_i^T\right]. \end{aligned} \tag{21}$$

Then by pre- and postmultiplying (13) by $\mathrm{diag}\{P_i, I, T_i, I\}$ with $P_i = X_i^{-1}$, $T_i = J_i^{-1}$, we have

$$\begin{bmatrix} -\mu P_i & 0 & L_i^T T_i & L_{1i}^T \\ * & -Q_i & 0 & L_{2i}^T \\ * & * & -2T_i & L_{3i}^T \\ * & * & * & -W \end{bmatrix} < 0. \tag{22}$$

By using Schur complement lemma, we derive

$$\begin{aligned} &\xi(k)^T \begin{bmatrix} -\mu P_i & 0 & L_i^T T_i \\ * & -Q_i & 0 \\ * & * & -2T_i \end{bmatrix} \\ &\quad \times \xi(k) + \xi(k)^T \left[L_{1i} \;\; L_{2i} \;\; L_{3i}\right]^T \bar{W} \left[L_{1i} \;\; L_{2i} \;\; L_{3i}\right] \xi(k) < 0. \end{aligned} \tag{23}$$

It follows that

$$\begin{aligned} E\{V(k+1)\} &< \mu x(k)^T P_i x(k) + w(k)^T Q_i w(k) \\ &\quad + 2\psi(k)^T T_i \psi(k) - 2\psi(k)^T T_i L_i x(k). \end{aligned} \tag{24}$$

Since $2\psi(k)^T T_i \psi(k) - 2\psi(k)^T T_i L_i x(k) \le 0$, we get

$$E\{V(k+1)\} < \mu x(k)^T P_i x(k) + w(k)^T Q_i w(k). \quad (25)$$

It is shown that

$$E\{V(k+1)\} < \mu V(k) + \sup_{(i\in S)} \{\lambda_{\max}(Q_i)\} w(k)^T w(k). \quad (26)$$

Then we have

$$\begin{aligned} E\{V(k+1)\} &< \mu E\{V(k)\} \\ &\quad + \sup_{(i\in S)} \{\lambda_{\max}(Q_i)\} E\left\{w(k)^T w(k)\right\}. \end{aligned} \quad (27)$$

Since $\mu \ge 1$, it is easily found that

$$\begin{aligned} E\{V(k+1)\} &< \mu E\{V(0)\} \\ &\quad + \sup_{(i\in S)} \{\lambda_{\max}(Q_i)\} E \\ &\quad \times \left\{\sum_{j=0}^{k-1} \mu^{k-j-1} w(j)^T w(j)\right\} \\ &\le \mu^k E\{V(0)\} + \sup_{(i\in S)} \{\lambda_{\max}(Q_i)\} \mu^k d^2. \end{aligned} \quad (28)$$

Let

$$\bar{P}_i = R_i^{-1/2} P_i R_i^{-1/2}, \quad (29)$$

and noting that

$$E\left\{x^T(0) R_i x(0)\right\} \le \delta_x^2, \quad (30)$$

it can be verified that

$$\begin{aligned} E\{V(0)\} &= E\left\{x^T(0) P_i x(0)\right\} = E\left\{x^T(0) R_i^{1/2} \bar{P}_i R_{i^{1/2}} x(0)\right\} \\ &\le \sup_{i\in S} \left\{\lambda_{\max}\left(\bar{P}_i\right)\right\} E\left\{x^T(0) R_i x(0)\right\} \\ &\le \sup_{i\in S} \left\{\lambda_{\max}\left(\bar{P}_i\right)\right\} \delta_x^2. \end{aligned} \quad (31)$$

Similarly, for all $i \in S$, we can obtain

$$\begin{aligned} E\{V(k)\} &= E\left\{x^T(k) P_i x(k)\right\} \\ &= E\left\{x^T(k) R_i^{1/2} \bar{P}_i R_{i^{1/2}} x(k)\right\} \\ &\ge \inf_{i\in S} \{\lambda_{\min}\}\left(\bar{P}_i\right) E\left\{x^T(k) R_i x(k)\right\}. \end{aligned} \quad (32)$$

Then it is not difficult to find that

$$\begin{aligned} &E\left\{x^T(k) R_i x(k)\right\} \\ &\quad < \frac{\sup_{i\in S}\left\{\lambda_{\max}\left(\bar{P}_i\right)\right\} \mu^k \delta_x^2 + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} \mu^k d^2}{\inf_{i\in S}\{\lambda_{\min}\}\left(\bar{P}_i\right)}, \end{aligned} \quad (33)$$

which implies

$$\frac{\sup_{i\in S}\left\{\lambda_{\max}\left(\bar{P}_i\right)\right\} \mu^k \delta_x^2 + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} \mu^k d^2}{\inf_{i\in S}\{\lambda_{\min}\}\left(\bar{P}_i\right)} < \epsilon^2. \quad (34)$$

Then, one can obtain that

$$\begin{aligned} &\sup_{i\in S}\left\{\lambda_{\max}\left(\bar{P}_i\right)\right\} \delta_x^2 \\ &\quad + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} d^2 < \inf_{i\in S}\{\lambda_{\min}\}\left(\bar{P}_i\right) \mu^{-N} \epsilon^2. \end{aligned} \quad (35)$$

Set

$$\begin{aligned} X_i &= P_i^{-1}, \\ \sigma_1 R_i^{-1} &< X_i < R_i^{-1}, \\ 0 &< Q_i < \sigma_2 I; \end{aligned} \quad (36)$$

it is easy to see that

$$\sigma_1^{-1} \delta_x^2 + \sigma_2 d^2 < \mu^{-N} \epsilon^2. \quad (37)$$

It is obvious that (37) is equivalent to (14). Based on Lemma 1, it is easy to obtain condition (15). This completes the proof. □

Theorem 7. *For each $r(k) = i \in S$, there exists a feedback controller $u(k) = K_i x(k)$, $K_i = Y_i X_i^{-1}$, such that the resulting closed-loop system (12) is said to be stochastic H_∞ finite-time stable via state feedback with respect to $(\delta_x, \epsilon, R_i, N, \gamma, d)$, if there exist three scalars $\mu \ge 0$, $\sigma_1 \ge 0$, and $\gamma \ge 0$, two sets of mode-dependent symmetric matrices $X_i > 0$ and $J_i > 0$, and two sets of mode-dependent matrices Y_i and $\bar{L}_i = L_i X_i$, such that the following conditions hold:*

$$\begin{bmatrix} -\mu X_i & 0 & \bar{L}_i & \bar{L}_{1i} & Y_i^T D_i^T + X_i C_i^T \\ * & -\gamma^2 \mu^{-N} I & 0 & L_{2i}^T & D_{2i}^T \\ * & * & -2J_i & L_{3i}^T & D_{1i}^T \\ * & * & * & -W & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \quad (38)$$

$$\begin{bmatrix} \mu^{-N}\left(d^2\gamma^2 - \epsilon^2\right) & * \\ \delta_x & -\sigma_1 \end{bmatrix} < 0, \quad (39)$$

$$\begin{bmatrix} X_i & * \\ Y_i + \bar{L}_i & u_{0(k)}^2 \end{bmatrix} > 0, \qquad k = 1, \ldots, m, \quad (40)$$

$$\sigma_1 R_i^{-1} < X_i < R_i^{-1}. \quad (41)$$

Proof. Choose the similar Lyapunov function as Theorem 6 and denote

$$\begin{aligned} &\Pi(x(k), w(k), r(k) = i) \\ &\quad = E\{V(k+1)\} - \mu V(k) + z(k)^T z(k) \\ &\quad - \gamma^2 \mu^{-N} w(k)^T w(k). \end{aligned} \quad (42)$$

Thus, in the light of Theorem 6, we have

$$\begin{aligned}
&\Pi\left(x(k), w(k), r_k = i\right) \\
&\quad \le \xi(k)^T \left[L_{1i} \;\; L_{2i} \;\; L_{3i}\right]^T \bar{W} \left[L_{1i} \;\; L_{2i} \;\; L_{3i}\right] \xi(k) \\
&\qquad + \xi(k)^T \left[C_i + D_{1i}K_i \;\; D_{2i} \;\; D_{3i}\right]^T \\
&\qquad \times \left[C_i + D_{1i}K_i \;\; D_{2i} \;\; D_{3i}\right] \xi(k) \\
&\qquad + \xi^T(k) \begin{bmatrix} -\mu X_i & 0 & L_i^T T_i \\ * & -\gamma^2 \mu^{-N} I & 0 \\ * & * & -2T_i \end{bmatrix} \xi(k).
\end{aligned} \tag{43}$$

Then by pre- and postmultiplying (38) by diag$\{P_i, I, T_i, I\}$ and considering Schur complement Lemma and (43), we derive

$$\Pi\left(x(k), w(k), r(k) = i\right) < 0, \tag{44}$$

holds for all $r_k = i \in S$. According to (44), one can obtain that

$$\begin{aligned}
E\{V(k+1)\} &< \mu E\{V(k)\} \\
&\quad - E\left\{z(k)^T z(k)\right\} + \gamma^2 \mu^{-N} E\left\{w(k)^T w(k)\right\}.
\end{aligned} \tag{45}$$

Then, we have

$$\begin{aligned}
E\{V(k)\} &< \mu^k E\{V(0)\} \\
&\quad - \Sigma_{j=0}^{k-1} \mu^{k-j-1} E\left\{z(j)^T z(j)\right\} \\
&\quad + \gamma^2 \mu^{-N} E\left\{\Sigma_{j=0}^{k-1} \mu^{k-j-1} w(j)^T w(j)\right\}.
\end{aligned} \tag{46}$$

Under the zero-value initial condition and noting that $V(k) \ge 0$, for all $K \in Z_{k\ge 0}$, it is shown that

$$\begin{aligned}
&\Sigma_{j=0}^{k-1} \mu^{k-j-1} E\left\{z(j)^T z(j)\right\} \\
&\quad < \gamma^2 \mu^{-N} E\left\{\Sigma_{j=0}^{k-1} \mu^{k-j-1} w(j)^T w(j)\right\}.
\end{aligned} \tag{47}$$

Since $\mu \ge 1$ and from (47), we have

$$\begin{aligned}
E\left\{\Sigma_{j=0}^{N} z(j)^T z(j)\right\} &= \Sigma_{j=0}^{N} E\left\{z(j)^T z(j)\right\} \\
&\le \Sigma_{j=0}^{N} E\left\{\mu^{N-j} z(j)^T z(j)\right\} \\
&\le \gamma^2 \mu^{-N} E\left\{\Sigma_{j=0}^{N} \mu^{N-j} w(j)^T w(j)\right\} \\
&\le \gamma^2 E\left\{\Sigma_{j=0}^{N} w(j)^T w(j)\right\}.
\end{aligned} \tag{48}$$

The following proof is similar to the process of Zhang and Liu [34].

Since $\epsilon(P_i, 1) \subset D(u_0)$, it follows that

$$\begin{bmatrix} P_i & * \\ K_i + L_i & u_{0(k)}^2 \end{bmatrix} > 0, \quad k = 1, \ldots, m, \tag{49}$$

and then by pre- and postmultiplying (49) by diag$(X_i, I,)$ and its transpose, respectively, we derive condition (40). This completes the proof. □

Remark 8. For the given scalars $(\delta_x, \epsilon, R_i, N, \gamma, d)$, we can take γ^2 as the optimized variable to obtain an optimized finite-time stabilized controller. The attenuation lever γ^2 can be reduced to the minimum possible value such that LMIs (38)–(41) hold. The optimization problem can be described as follows:

$$\begin{aligned}
&\min \quad \rho \\
&\qquad \left(X_i \;\; Y_i \;\; K_i \;\; \delta_x\right) \\
&\text{s.t.} \quad \text{LMIs (38)–(41) with } \rho = \gamma^2.
\end{aligned} \tag{50}$$

4. Illustrative Examples

In this section, a numerical example is provided to demonstrate the effectiveness of the proposed method. Consider the following systems with four operation modes.

Mode 1. Consider

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 1.8 & 0.6 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{51}$$

Mode 2. Consider

$$A_2 = \begin{bmatrix} 1.2 & 1 \\ 0.8 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{52}$$

Mode 3. Consider

$$A_3 = \begin{bmatrix} 0.76 & -2.28 \\ 0.80 & -0.96 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad G_3 = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}. \tag{53}$$

Mode 4. Consider

$$A_4 = \begin{bmatrix} 1.28 & -0.38 \\ 0.80 & -0.88 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix}, \qquad G_4 = \begin{bmatrix} 0.8 \\ 0 \end{bmatrix}. \tag{54}$$

The transition rate matrix is given as follows:

$$\begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.3 & 0.4 \end{bmatrix}. \tag{55}$$

In this case, we choose the initial values for $R_i = I_2$, $i = 1, 2, 3, 4$, $\delta_x = 1$, $N = 5$, $\alpha = 10^{-10}$, $\mu = 2.5$, and $d = 1$; Theorem 6 yields to $\epsilon = 36.2671$, $\sigma_1 = 0.4906$, $\sigma_2 = 13.7421$, and the bounds of the input saturation $u_0 = 0.05$.

Based on Theorem 7, we derive

$$\begin{aligned}
K_1 &= \left[-0.7723 \;\; 0.5862\right], \qquad K_2 = \left[-0.1021 \;\; -0.0506\right], \\
K_3 &= \left[-0.8706 \;\; 0.5591\right], \qquad K_4 = \left[-0.3019 \;\; -0.3635\right].
\end{aligned} \tag{56}$$

Remark 9. The figures are given on the last page. In this part Figure 1 is r_k of the jump rates; Figure 2 shows the states of the open-loop Markovian jump system; Figure 3 shows the states

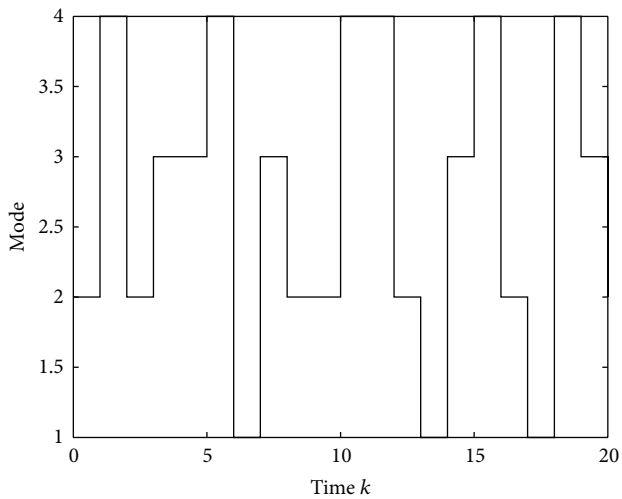

FIGURE 1: r_k of jump rates.

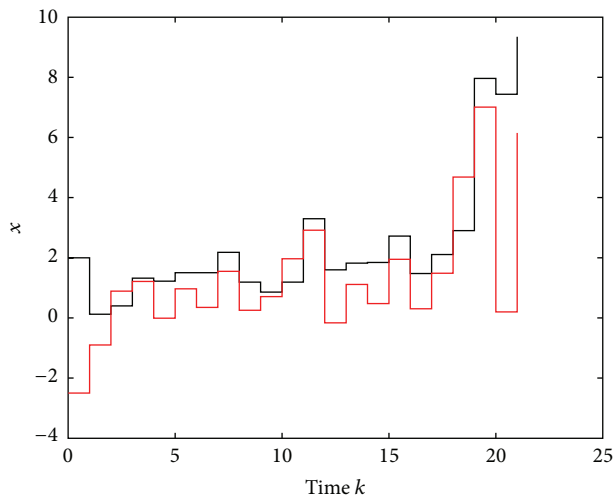

FIGURE 2: $x(k)$ of open-loop system.

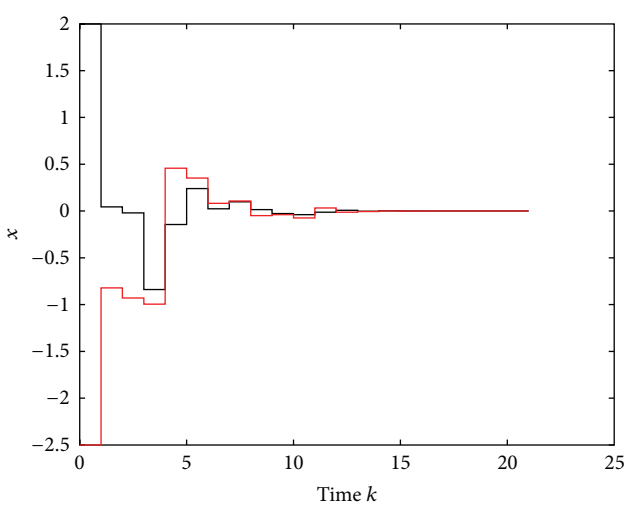

FIGURE 3: $x(k)$ of closed-loop system.

of the closed-loop Markovian jump system. By applying the controller studied in this paper to the closed-loop plant, it is obviously noticed that x_1 and x_2 converge to zero quickly. Based on the figures provided, the controller we designed guarantees that the resulting closed-loop systems are mean-square locally asymptotically finite-time stabilizable.

5. Conclusions

This paper considers the finite-time H_∞ stabilization problem for a class of discrete-time Markov jump systems with input saturation. The finite-time H_∞ controller via state feedback is designed to guarantee the stochastic finite-time boundedness and stochastic finite-time stabilization of the considered closed-loop system for all admissible disturbances. Based on stochastic finite-time stability analysis, sufficient conditions are derived in the form of linear matrix inequalities. Finally, simulation results are given to illustrate the effectiveness of the proposed approach. In the future, we will study the finite-time stabilization problem for a class of Markov jump systems with constrained input and time delay.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Control of Multiagent Systems: A Stochastic Pinning Viewpoint

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A stochastic pinning approach for multiagent systems is developed, which guarantees such systems being almost surely stable. It is seen that the pinning is closely related to being a Bernoulli variable. It has been proved for the first time that a series of systems can be stabilized by a Brownian noise perturbation in terms of a pinning scheme. A new terminology named "stochastic pinning control" is introduced to describe the given pinning algorithm. Additionally, two general cases that the expectation of the Bernoulli variable with bounded uncertainty or being unknown are studied. Finally, two simulation examples are provided to demonstrate the effectiveness of the proposed methods.

1. Introduction

Due to the broad applications in cooperative control of unmanned air vehicles, formation control of mobile robots, sensor networks, and cooperative surveillance, multiagent problems have drawn a lot of attention. In particular, multiagent coordination with multiple leaders becomes more and more important, which forces a group of agents into a specific target region. Because of the spatial distribution of actuators and sensors, it is of high cost or even impossible in practice to implement a centralized controller. Instead, distributed control emerged to be a promising tool for coordination of multiagent systems, which is usually to design a controller to every subsystem. During the past years, many important results have been reported in [1–6]. Many natural and man-made systems, such as ecosystems, internet, Word Wide Web, social networks, and power grids, are described by it. In recent years, the analysis and control of complex behaviors in complex networks have become a hot topic across many fields such as in [7–11]. Especially, synchronization related to being the most important collective behavior of complex networks, such as ER random, small-world, and scale-free complex networks [12–14], has been extensively studied. Via introducing a Bernoulli stochastic variable describing the random switching of controllers, the distributed synchronization of complex networks was studied in [15, 16].

Due to the complexity of complex networks, it is usually not easy to control a complex network by adding controllers to all nodes. Instead, pinning control only uses a small number of controllers. In this sense, it is said that pinning control is a promising method, which can efficiently reduce the number of controllers. The pinning control strategy for linear coupled networks was investigated in [17, 18], in which two different pinning strategies, random pinning and special pinning methods, were theoretically and numerically compared. During the past decades, a lot of results on synchronization of various complex networks by pinning control have emerged, for example, [19–25]. By searching such references on pinning control, it is concluded that all the pinning methods are realized by a kind of regular controllers in terms of in the drift section of a system. However, it is possible to design a controller to stabilize a stochastic system almost surely which is unable to be stabilized in mean-square sense. Based on these facts, it is asked that can we design a pinning controller referred to be a Brownian noise perturbation to stabilize multiagent systems? To the best of authors' knowledge, the control problem of multiagent systems via a pinning controller only in the diffusion part has not yet been investigated, which motivates the current research.

In this paper, the control problem of multiagent systems is firstly considered by a stochastic pinning viewpoint. In

contrast to the existing results of pinning control methods, the main contributions of this paper are as follows. (1) The control of multiagent systems is firstly realized by a pinning control method in terms of a Brownian noise perturbation. (2) In order to achieve this goal, new pinning control in terms of stochastic pinning control (SPC) is developed, in which the Bernoulli variable plays an important role in SPC. (3) More general cases such as the expectation of Bernoulli variable with uncertainty and being unknown are considered respectively. (4) The relationship among the expectation, the pinning fraction, and the pinning control gain for both random and special pinning control is demonstrated in detail.

Notation. $\mathbb{R}^n$ denotes the n dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. $|\cdot|$ denotes the Euclidean norm. $\mathscr{E}\{\cdot\}$ is the expectation operator with respect to some probability measure. $\otimes$ is the Kronecker product. In symmetric block matrices, we use "$*$" as an ellipsis for the terms induced by symmetry, diag$\{\cdots\}$ for a block-diagonal matrix. $\mathbb{S} \triangleq \{1, 2, \ldots, N\} = \mathbb{S}_l \bigcup \overline{\mathbb{S}}_l$, where $\mathbb{S}_l = \{1, 2, \ldots, l\}$ and $\overline{\mathbb{S}}_l = \mathbb{S} - \mathbb{S}_l$.

2. Problem Formulation

Consider a multiagent system consisting of N agents, such as in [6], the model of agent, $i \in \mathbb{S}$, is described as

$$\dot{x}_i = f_i(x_i, t), \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector, $f_i(x_i, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the inherent nonlinear dynamic. However, it can be also be dealt with by the fuzzy method [26, 27]. In this paper, the nonlinear term $f_i(x_i, t)$ is treated directly, and an assumption is needed here.

Assumption 1. Nonlinear function $f_i(x_i, t)$ is assumed to satisfy the following condition:

$$x_i^T f_i(x_i, t) \le \theta_i x_i^T x_i, \quad \forall x_i \in \mathbb{R}^n, \ \forall t \ge 0, \tag{2}$$

where $\theta_i \ge 0$ is a given constant.

In this paper, a new pinning control method for system (1) in terms of stochastic pinning control is proposed as

$$u_i = c\sum_{j=1}^{N} g_{ij}x_j - ck_i x_i, \quad i \in \mathbb{S}_l, \tag{3}$$

$$u_i = c\sum_{j=1}^{N} g_{ij}x_j - c\alpha(t) k_i x_i, \quad i \in \overline{\mathbb{S}}_l, \tag{4}$$

respectively, where c is the coupling strength, $G = (g_{ij}) \in \mathbb{R}^{N\times N}$ is the coupling matrix with $g_{ij} \ge 0$, $i \ne j$, $g_{ii} = -\sum_{j=1, j\ne i}^{N} g_{ij}$, which is an irreducible matrix, and $k_i \in \mathbb{R}^{n\times n}$ is the control gain. Bernoulli variable $\alpha(t)$ in (4) is described as

$$\alpha(t) = 1 \text{ or } 0 \tag{5}$$

whose expectation is $\mathscr{E}\{\alpha(t)\} = \alpha$.

In this paper, such controllers will be used as a pinning controller in terms of Brownian noise perturbation. As a result, we have the closed-loop system as

$$dx_i = f_i(x_i, t)\,dt + c\left(\sum_{j=1}^{N} g_{ij}x_j - k_i x_i\right) d\omega(t), \quad i \in \mathbb{S}_l, \tag{6}$$

$$dx_i = f_i(x_i, t)\,dt + c\left(\sum_{j=1}^{N} g_{ij}x_j - \alpha(t) k_i x_i\right) d\omega(t), \quad i \in \overline{\mathbb{S}}_l, \tag{7}$$

where $\omega(t) \in \mathbb{R}^n$ is a n-dimensional Brownian motion or Wiener process. Let $x = \left[x_1^T \ \cdots \ x_N^T\right]^T$, $f = \left[f_1^T \ \cdots \ f_N^T\right]^T$, and $K_{\alpha(t)} = \text{diag}\{k_1, \ldots, k_l, \alpha(t)k_{l+1}, \ldots, \alpha(t)k_N\}$; one has $G_1 = G - K_{\alpha(t)=1}$ and $G_2 = G - K_{\alpha(t)=0}$, respectively. Without of loss generality, in this paper it is assumed that $k_i = k_j = k > 0$, for all $i, j \in \mathbb{S}$. Based on these notations, we have

$$dx = f(x, t)\,dt + c\widehat{G}(\eta(t))\,x d\omega(t), \tag{8}$$

where $\widehat{G}(\eta(t)) = G(\eta(t)) \otimes I_n$. Operation mode $\{\eta(t), t \ge 0\}$ takes values in set $\mathbb{B} = \{1, 2\}$ and is described as

$$\eta(t) = \begin{cases} 1, & \text{if } \alpha(t) = 1 \\ 2, & \text{if } \alpha(t) = 0, \end{cases} \tag{9}$$

whose probabilities are $\Pr\{\eta(t) = 1\} = \alpha$ and $\Pr\{\eta(t) = 2\} = 1 - \alpha$, respectively.

Remark 2. It is worth pointing out that the proposed pinning control method is different from the existing pinning methods. Firstly, the pinning problem of this paper is realized by a Brownian noise perturbation, which cannot be solved by the usual analysis methods. Secondly, the introduced Bernoulli variable $\alpha(t)$ plays an important role in achieving the pinning control in terms of Brownian noise control. Such differences embody the property of stochastic pinning control. That is, when $\alpha(t) = 0$, only controller (6) works and is a pinning controller. On the contrary, if $\alpha(t) = 1$, the desired controller becomes a distributed controller, which is not pinning control in fact. In this sense, it is said that the developed pinning control is a stochastic control method.

Remark 3. It should be remarked that this framework is necessary to achieve pinning control through Brownian noise perturbations. If there is no $\alpha(t)$ in (7), the underlying systems become (6) with $i \in \mathbb{S}_l$ and (1) with $i \in \overline{\mathbb{S}}_l$, which can be obtained by applying the usual pinning methods directly. Unfortunately, it is concluded that this pinning framework is very hard to realize the desired object.

The reason will be explained later. On the other hand, when $\alpha(t) \equiv 1$, we have (6) only, which is seen as a distributed controller instead of a pinning controller. Thus, it is claimed that the given pinning control algorithm bridges the traditional pinning control and distributed control. In addition, $K_{\alpha(t)}$ can also choose other forms, such as $K_{\alpha(t),\beta(t)} = \operatorname{diag}\{\alpha(t)k_1, \ldots, \alpha(t)k_l, \beta(t)k_{l+1}, \ldots, \beta(t)k_N\}$, where both $\alpha(t)$ and $\beta(t)$ are dependent or independent Bernoulli variables.

Definition 4. The equilibrium of system (8) is said to be almost surely exponentially stable if for any $x_0 \in \mathbb{R}^n$

$$\lim_{t\to\infty} \sup \frac{1}{t} \log\left(\left|x\left(t, x_0\right)\right|\right) < 0 \quad \text{a.s.} \tag{10}$$

3. Main Results

Theorem 5. *For given scalars θ and α, the equilibrium of system* (8) *is almost surely exponentially stable, if there exists $k > 0$ such that*

$$2\theta + \alpha c^2 \Omega_1 + (1-\alpha) c^2 \Omega_2 < 0, \tag{11}$$

where $\Omega_i = \lambda_{\max}(G_i^T G_i) - 2\lambda_{\max}^2(G_i)$, $i \in \mathbb{B}$.

Proof. For any given initial condition $x_0 \neq 0$, it is known that $x(t) \triangleq x(t; x_0)$ will never reach zero with probability one, and by Itô formula, it is obtained that

$$\begin{aligned}
&d\left[\log\left(|x|^2\right)\right] \\
&= \frac{\left(2x^T f(x,t) + c^2 x^T \widehat{G}_i^T \widehat{G}_i x\right) dt + 2cx^T \widehat{G}_i x d\omega(t)}{|x|^2} \\
&\quad - \frac{2c^2 \left|x^T \widehat{G}_i x\right|^2}{|x|^4} dt \\
&= \frac{\left(2x^T f(x,t) + c^2 x^T \left(G_i^T G_i\right) \otimes I_n x\right) dt + 2cx^T \widehat{G}_i x d\omega(t)}{|x|^2} \\
&\quad - \frac{2c^2 \left|x^T \widehat{G}_i x\right|^2}{|x|^4} dt \\
&\le \left(2\theta + c^2 \Omega_i\right) dt + \frac{2cx^T \widehat{G}_i x}{|x|^2} d\omega(t).
\end{aligned} \tag{12}$$

Then, it is obtained that

$$\begin{aligned}
\log\left(|x(t)|^2\right) &\le \log\left(|x_0|^2\right) \\
&\quad + \int_0^t \left(2\theta + c^2 \Omega\left(\eta(s)\right)\right) ds + M(t),
\end{aligned} \tag{13}$$

where $M(t) = \int_0^t ((2cx^T(s)\widehat{G}(\eta(s))x(s))/|x(s)|^2) d\omega(s)$ is a continuous martingale vanishing at $t = 0$. Taking into account (9), it is seen that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \int_0^t \left(2\theta + c^2 \Omega\left(\eta(s)\right)\right) ds \\
&\quad = 2\theta + \alpha c^2 \Omega_1 + (1-\alpha) c^2 \Omega_2 \quad \text{a.s.}
\end{aligned} \tag{14}$$

On the other hand, it is concluded that for any finite k given in G_i, there exists a positive scalar ρ_i that the quadratic variation of $M(t)$ is

$$\begin{aligned}
\langle M(t), M(t)\rangle &= \int_0^t \frac{4c^2 \left|x^T(s) \widehat{G}\left(\eta(s)\right) x(s)\right|^2}{|x(s)|^4} ds \\
&\le 4t \max_{i\in\mathbb{B}} \rho_i.
\end{aligned} \tag{15}$$

Applying the strong law of the large numbers to $M(t)$, one has

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.} \tag{16}$$

Based on (14) and (16), we conclude that

$$\begin{aligned}
&\lim_{t\to\infty} \sup \frac{1}{t} \log\left(\left|x\left(t, x_0\right)\right|\right) \\
&\quad \le 2\theta + \alpha c^2 \Omega_1 + (1-\alpha) c^2 \Omega_2 < 0 \quad \text{a.s.}
\end{aligned} \tag{17}$$

which is guaranteed by (11). This completes the proof. □

Remark 6. By Theorem 5, it is known that if condition (11) holds, one can stabilize system (1) by a pinning control tactic in terms of Brownian noise perturbation. However, condition (11) is impossible or hard to be satisfied if controller (7) is with $\alpha(t) \equiv 0$. If $\alpha(t) \equiv 0$, condition (11) becomes $2\theta + c^2\Omega_2 < 0$, where pinning controller is same as the traditional pinning controller. Unfortunately, it is said that $2\theta + c^2\Omega_2 < 0$ with $\theta > 0$ is impossible or hard to be satisfied because of $\Omega_2 > 0$. That is, due to the property of G in (3) or (4), we have $\lambda_{\max}(G_2) < 0$, and $\lambda_{\max}^2(G_2) = (\lambda_{\max}(G_2))^2$. It usually results in $\lambda_{\max}(G_2^T G_2) \gg 2\lambda_{\max}^2(G_2)$. In this sense, when $\alpha(t) \equiv 0$, it is difficult to realize the pinning control goal of this paper.

From Theorem 5, it is seen that the expectation α plays an important role in SPC which should be given exactly. In some applications, it is very hard or of high cost to obtain α exactly. Instead, only its estimation $\tilde{\alpha}$ is available. Then, it is natural and important to study how to realize SPC when α is uncertain. If there exists an uncertainty in α, we will use its estimation $\tilde{\alpha}$. It is described as

$$\Delta\alpha = \alpha - \tilde{\alpha}, \quad \tilde{\alpha} \in [0, 1], \tag{18}$$

where admissible uncertainty $\Delta\alpha \in [-\epsilon, \epsilon]$ with $\epsilon \in [0, 1]$. Then, we have the following theorem.

Theorem 7. *For given scalars θ and $\tilde{\alpha}$, the equilibrium of system* (8) *is robust almost surely exponentially stable for any admissible uncertainty* (18)*, if there exist $k > 0$ and $\mu > 0$ such that*

$$2\theta + \tilde{\alpha}c^2\Omega_1 + (1-\tilde{\alpha})c^2\Omega_2 + 0.25\epsilon^2c^4\mu + (\Omega_1-\Omega_2)^2\mu^{-1} < 0. \tag{19}$$

Proof. Based on the proof of Theorem 5, it is obtained that the change of α only takes place in (14); that is

$$\alpha c^2\Omega_1 + (1-\alpha)c^2\Omega_2 = \tilde{\alpha}c^2\Omega_1 + (1-\tilde{\alpha})c^2\Omega_2 + \Delta\alpha c^2(\Omega_1-\Omega_2) < 0. \tag{20}$$

For $\Delta\alpha c^2(\Omega_1-\Omega_2)$ with any $\mu > 0$, it is seen that

$$\Delta\alpha c^2(\Omega_1-\Omega_2) \le 0.25(\Delta\alpha)^2c^4\mu + (\Omega_1-\Omega_2)^2\mu^{-1}. \tag{21}$$

Taking into account (19) and (21), one has (11). That completes the proof. □

It is seen that the conditions of Theorems 5 and 7 are not LMIs, which are not solved directly. In the following, another condition in terms of LMIs with equation constraints is proposed, which could be solved easily.

Theorem 8. *For given scalars θ and α, the equilibrium of system* (8) *is almost surely exponentially stable, if there exist $k > 0$, $\delta_i > 0$, $\gamma_i > 0$, $\overline{\gamma}_i > 0$, $\sigma_i > 0$, and $\overline{\sigma}_i > 0$, such that the following LMIs hold for all $i \in \mathbb{B}$:*

$$2\theta + \alpha c^2\overline{\Omega}_1 + (1-\alpha)c^2\overline{\Omega}_2 < 0 \tag{22}$$

$$\begin{bmatrix} -\delta_i I & G_i^T \\ G_i & -I \end{bmatrix} \le 0 \tag{23}$$

$$\begin{bmatrix} -\overline{\sigma}_i & \overline{\gamma}_i \\ \overline{\gamma}_i & -1 \end{bmatrix} \le 0 \tag{24}$$

$$\gamma_i\overline{\gamma}_i = 1, \qquad \sigma_i\overline{\sigma}_i = 1 \tag{25}$$

either

$$G_i + G_i^T \ge 2\gamma_i I \tag{26}$$

or

$$G_i + G_i^T \le -2\gamma_i I, \tag{27}$$

where $\overline{\Omega}_i = \delta_i - 2\sigma_i$.

Proof. Based on (11), it is known that if there are $\delta_i > 0$ and $\gamma_i > 0$ such that

$$\lambda_{\max}\left(G_i^TG_i\right) \le \delta_i, \qquad \lambda_{\max}^2(G_i) \ge \gamma_i^2, \tag{28}$$

hold, which are guaranteed by

$$G_i^TG_i \le \delta_i I \qquad \lambda_{\max}(G_i) \ge \gamma_i I \tag{29}$$

or

$$\lambda_{\max}(G_i) \le -\gamma_i I, \tag{30}$$

respectively. Based on these, it is obvious that (28) can be obtained by (23), (26), or (27). Then, we have (11) which is insured by

$$2\theta + \alpha c^2\left(\delta_1 - 2\gamma_1^2\right) + (1-\alpha)c^2\left(\delta_2 - 2\gamma_2^2\right) < 0, \tag{31}$$

where δ_i and γ_i should be determined. Because of nonlinear term γ_i^2 in (31), it cannot be solved by LMI tool box directly. By introducing a variable σ_i satisfying $\gamma_i^2 \ge \sigma_i$, it is concluded that it is equivalent to $\sigma_i^{-1} \ge \gamma_i^{-2}$. By Schur complement and condition (25), one has (22)–(25) implying (11). This completes the proof. □

If the expectation α is unknown, how to achieve a useful SPC is another general case. For this case, we have the following theorem.

Theorem 9. *For a given scalar θ, the equilibrium of system* (8) *is almost surely exponentially stable, if there exists $k > 0$ such that*

$$\Omega_1 + \Omega_2 < 0. \tag{32}$$

In this case, the expectation α can be unknown, but it should be satisfied

$$\alpha > \frac{\Omega_2}{\Omega_2 - \Omega_1}. \tag{33}$$

Proof. Based on the proof of Theorem 5, one can easily have the equilibrium of system (8) almost surely exponentially stable, if there exists $k > 0$ such that

$$2\theta + c^2\alpha\Omega_1 + c^2(1-\alpha)\Omega_2 < 0, \tag{34}$$

which is equivalent to

$$2\theta + c^2\left[\alpha(\Omega_1-\Omega_2) + \Omega_2\right] < 0. \tag{35}$$

If α is unknown but satisfies (33), one could always choose a sufficiently large scalar c such that (35) holds. This completes the proof. □

Corollary 10. *For a given scalar θ, the equilibrium of system* (8) *is almost surely exponentially stable, if there exists $k > 0$ such that $\Omega_1 < 0$ and* (32) *hold. In this case, the expectation α can be unknown, but it should be satisfied $\alpha \in (0.5, 1]$.*

Proof. Similar to the proof of Theorem 9, (34) can be rewritten to be

$$2\theta + (2\alpha - 1) c^2 \Omega_1 + (1 - \alpha) c^2 (\Omega_1 + \Omega_2) < 0, \tag{36}$$

which could be guaranteed by

$$2\theta + (2\alpha - 1) c^2 \Omega_1 < 0, \tag{37}$$

$$(1 - \alpha) c^2 (\Omega_1 + \Omega_2) \leq 0. \tag{38}$$

Since $\Omega_1 < 0$ and $\alpha \in (0.5, 1]$, it is obtained that there is always a sufficiently large constant c such that (37) holds. On the other hand, under the conditions of α and $c > 0$, we have (33) implying (38) directly. This completes the proof. □

It is claimed that the key idea of SPC described by (3) and (4) can be used to construct a pinning controller in the drift section. That is

$$\begin{aligned} \dot{x}_i &= f_i(x_i, t) + c\left(\sum_{j=1}^{N} g_{ij} x_j - \alpha(t) k_i x_i\right), \quad i \in \mathbb{S}_l, \\ \dot{x}_i &= f_i(x_i, t) + c\sum_{j=1}^{N} g_{ij} x_j, \quad i \in \overline{\mathbb{S}}_l. \end{aligned} \tag{39}$$

Let $x = \left[x_1^T \cdots x_N^T\right]^T$, $f = \left[f_1^T \cdots f_N^T\right]^T$, and $K = \operatorname{diag}\{k_1, \ldots, k_l, 0, \ldots, 0\}$; one has

$$\dot{x} = f(x, t) + c(G - \alpha(t) K) \otimes I_n x. \tag{40}$$

It is rewritten to be

$$\dot{x} = f(x, t) + c\widetilde{G} \otimes I_n x + c(\alpha(t) - \alpha) K \otimes I_n x, \tag{41}$$

where $\widetilde{G} = G + \alpha K$. Without of loss generality, in matrix K, it is also assumed that $k_i = k_j = k > 0$, for all $i, j \in \mathbb{S}_l$. Since $\alpha(t)$ is a Bernoulli variable, it is known that $\mathscr{E}\{\alpha(t) - \alpha\} = 0$.

Theorem 11. *For given scalars θ and α, the equilibrium of system* (40) *is exponentially mean-square stable, if there exists $k > 0$ such that*

$$2\theta + c\lambda_{\max}(\widetilde{G}) < 0. \tag{42}$$

Proof. Choose the following Lyapunov function:

$$V(x(t), t) = x^T(t) x(t). \tag{43}$$

Based on (42), we have

$$\begin{aligned} \mathscr{L}V(x(t), t) &\leq x^T(t)\left(2\theta I_{Nn} + c\widetilde{G} \otimes I_n\right) x(t) \\ &\leq \left(2\theta + c\lambda_{\max}\widetilde{G}\right) x^T(t) x(t) < 0. \end{aligned} \tag{44}$$

Then, there is always a sufficient small scalar $\mu > 0$ such that

$$\mathscr{L}V(x(t), t) \leq -\mu x^T(t) x(t) < 0. \tag{45}$$

By Dynkin's formula, it is obtained that for $T > 0$

$$\begin{aligned} &\mathscr{E}\left(x^T(T) x(T)\right) - \mathscr{E}\left(x^T(0) x(0)\right) \\ &\quad \leq -\mu \int_0^T x^T(s) x(s)\, ds < 0. \end{aligned} \tag{46}$$

Applying the Gronwall-Bellman lemma to (46), one gets

$$\mathscr{E}\left(x^T(t) x(t)\right) \leq |x(0)|^2 \exp(-\mu t). \tag{47}$$

This completes the proof. □

Remark 12. It should be pointed out that the pinning controller in the drift section is also different from the existing methods such as [17, 18, 20, 21, 28]. It is said that the pinning method of this paper is a stochastic algorithm, where the expectation plays an important role. Compared with the traditional pinning methods, the desired pinning controller is not necessary implemented online, which is added to some nodes in terms of probability α. The correlation among the expectation α, the pinning fraction δ, and the control gain k is firstly illustrated in Theorem 11, which is also shown by numerical examples.

Similarly, when the expectation α is uncertain and satisfies (18), we have the following result.

Theorem 13. *For given scalars θ and $\tilde{\alpha}$, the equilibrium of system* (40) *is robust exponentially mean-square stable for any admissible uncertainty* (18)*, if there exist $k > 0$ and $\mu > 0$ such that*

$$2\theta + c\lambda_{\max}(\widehat{G}) + c\epsilon k < 0, \tag{48}$$

where $\widehat{G} = G + \tilde{\alpha} K$.

Proof. Based on the proof of Theorem 11 and taking in (48), it is obtained that (44) is rewritten as

$$\begin{aligned} &\mathscr{L}V(x(t), t) \\ &\quad \leq x^T(t)\left(2\theta I_{Nn} + c\widetilde{G} \otimes I_n\right) x(t) \\ &\quad = x^T(t)\left(2\theta I_{Nn} + c\widehat{G} \otimes I_n + c\Delta\alpha K \otimes I_n\right) x(t) \\ &\quad \leq x^T(t)\left(2\theta I_{Nn} + c\widehat{G} \otimes I_n + c\epsilon k I_{Nn}\right) x(t) < 0, \end{aligned} \tag{49}$$

where K is defined in (41). Obviously, it is known that (48) implies (49). This completes the proof. □

When α is unknown, we have the following theorem.

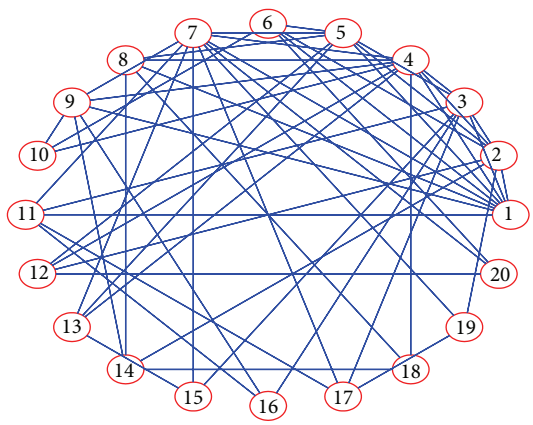

FIGURE 1: Connection of the closed-loop system.

Theorem 14. *For a given scalar θ, the equilibrium of system* (40) *is exponentially mean-square stable, if there exists $k > 0$ such that*

$$\lambda_{\max}(G + K) < 0. \tag{50}$$

In this case, there is an SPC such that (40) *is exponentially mean-square stable with unknown α.*

Proof. Based on Theorem 11, the equilibrium of system (40) is exponentially mean-square stable, if there exists $k > 0$ such that

$$2\theta I_{Nn} + c(G + \alpha K) \otimes I_n < 0, \tag{51}$$

which is equivalent to

$$2\theta I_{Nn} + c\alpha(G + K) \otimes I_n + c(1 - \alpha) G \otimes I_n < 0. \tag{52}$$

By the definition of G, it is known that $\lambda_i(G) \leq 0, i = 1, \ldots, N$. Because of $c > 0$ and $\alpha \in [0, 1]$, it is concluded that (52) is guaranteed by

$$2\theta + c\alpha\lambda_{\max}(G + K) < 0. \tag{53}$$

If (50) satisfies, one could always choose a sufficiently large scalar c such that (53) holds. This completes the proof. □

4. Numerical Examples

In this section, two numerical examples are used to demonstrate the utility of the proposed method.

Example 15. Without loss of generality, consider a multiagent system with 20 agents, whose agent is described as

$$\begin{aligned} \dot{x}_1 &= 0.2x_1 + x_2, \\ \dot{x}_2 &= -x_1 - 0.5x_2. \end{aligned} \tag{54}$$

In this example, such multiagent system will be stabilized by a stochastic pinning controller whose connection of agents in terms of scale-free network is simulated in Figure 1. Then, the coupling matrix can be obtained from Figure 1 directly.

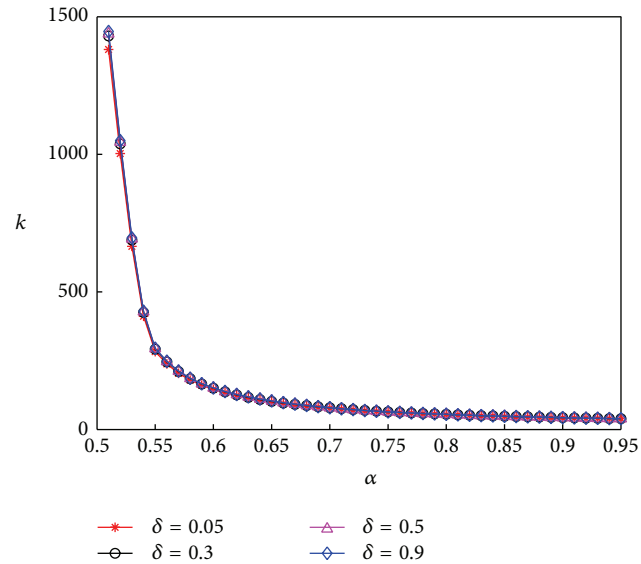

FIGURE 2: Correlation between k and α with given different δ.

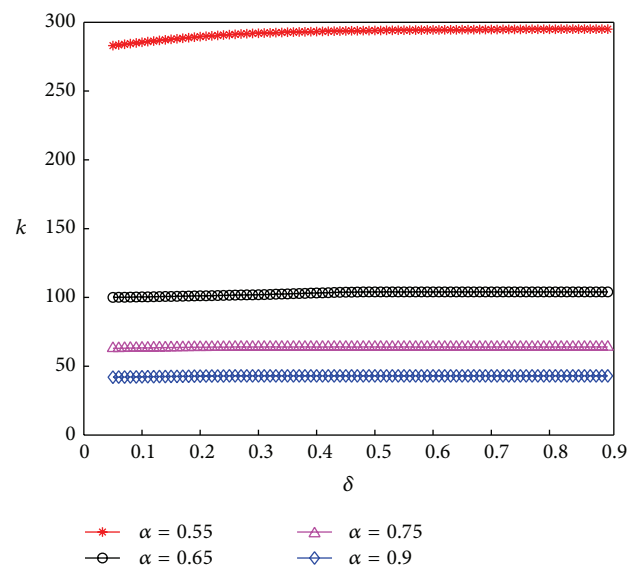

FIGURE 3: Correlation between k and δ with given different α.

From system (54), it is obtained that θ_i can be chosen as $\theta = 1.37$. The correlations among the expectation α, the pinning fraction δ, and the pinning control gain k are given in Theorem 5, which are demonstrated in Figures 2 and 3, respectively. In this paper, the special pinning control means (3) takes place in the nodes with more degrees. From Figure 2, it is seen that for given δ, larger expectation α results in smaller control gain k. When α is small, such as $\alpha \leq 0.7$, the curve of k along with α changes sharply, while the other section is gentle. On the other hand, the change of k along with δ under given α is shown in Figure 3. By this simulation, one knows that larger δ results in larger k. Different from Figure 2, it is seen that the whole curve is gentle. If $\delta = 0.1$, $\alpha = 0.6$, by Theorem 5, we have $k_{\min} = 150$ with $k > 0$.

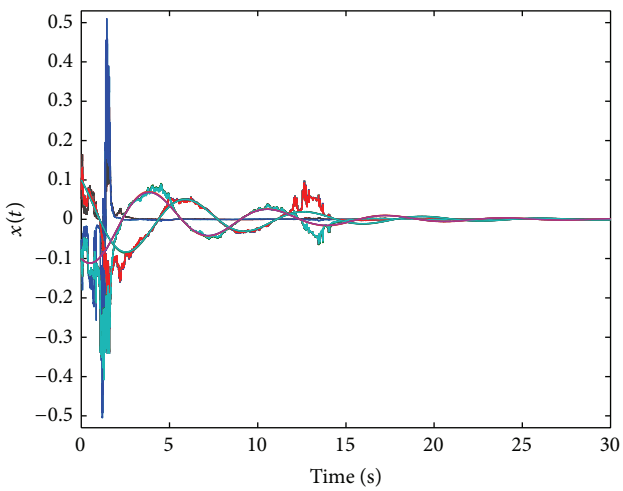

FIGURE 4: Simulation of the closed-loop system by SPC.

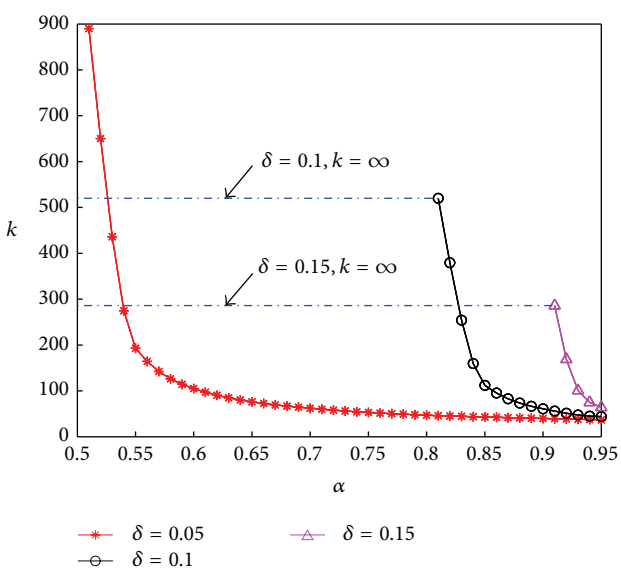

FIGURE 5: Correlation between k and α with given different δ.

Let initial condition of system (54) be $x_0 = [0.1 \quad -0.1]^T$; the state response of the closed-loop system is given in Figure 4, which is stable and demonstrates that the desired SPC is effective.

When the stochastic pinning controller is realized by random pinning control in terms of (3) taking place in any nodes, one has the following results, which are given in Figures 5 and 6, respectively. Considering Figures 2 and 5, it is concluded that control gain k in both of them becomes larger when α takes larger values. Especially, from Figure 5, it is further obtained that in some cases with larger pinning fraction δ, smaller α results in no solution to k. On the other hand, in Figure 6, one has that for some given values of α such as $\alpha = 0.91$, $\alpha = 0.93$ and $\alpha = 0.95$, there is no solution to control gain k in terms of random pinning control when δ satisfies $\delta > 0.15$ and $N * \delta$ should be an integer number. On the contrary, even if $\delta = 0.9$, we also have the

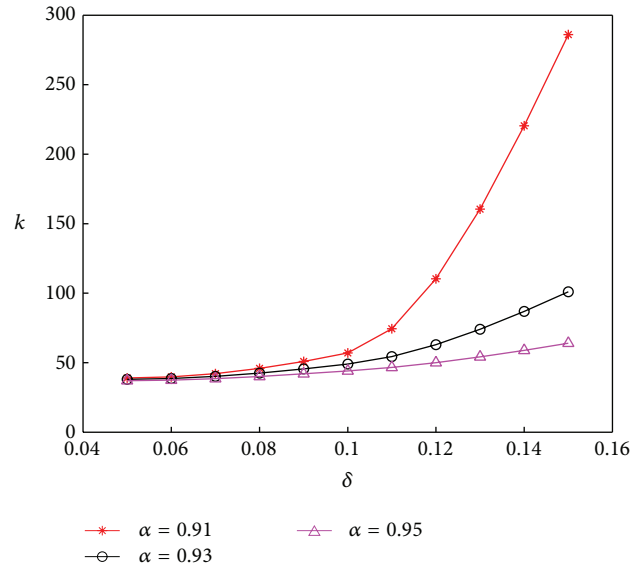

FIGURE 6: Correlation between k and δ with given different α.

control gain of special pinning controller. In this sense, it is said that special pinning control is better. For a given α, when pinning fraction δ becomes larger, larger control gain k is needed no matter which pinning control algorithm is selected. That means if one wants to pin a multiagent system by exploiting SPC described by (3) and (4) in terms of more agents controlled directly, he should provide a larger control gain k. Moreover, there is an interesting phenomenon in Figure 2 with $\delta = 0.05$ and Figure 5 with $\delta = 0.05$. That is, for the same α, the gain of random pinning controller is smaller than one of special pinning controller. This phenomenon can be explained if the two pinning methods are effective, because of special pinning control pinning more nodes due to the controlled nodes more "important", it needs its control gain k larger.

Example 16. Consider a dynamical node of complex network is a Chua's chaotic circuit described by

$$\begin{aligned} \dot{x}_1(t) &= \beta(-x_1 + x_2 - \varsigma(x_1)) \\ \dot{x}_2(t) &= x_1 - x_2 + x_3 \\ \dot{x}_3(t) &= -\gamma x_2, \end{aligned} \tag{55}$$

where $\varsigma(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$. When the parameters are $\beta = 10$, $\gamma = 18$, $a = -4/3$, and $b = -3/4$, Chua's system has a chaotic attractor shown in Figure 7. By computation, one has $\theta = 5.1623$ in view of Assumption 1. Suppose an undirected network consisting of $N = 20$ nodes in terms of small word network, where the connection is given in Figure 8. Similarly, its coupling matrix is easily obtained from Figure 8.

By Theorem 11 with coupling strength $c = 85$, one has the relationship among the expectation α, the pinning fraction δ, and the pinning control gain k in terms of special pinning control, which are given Figures 9 and 10, respectively. From

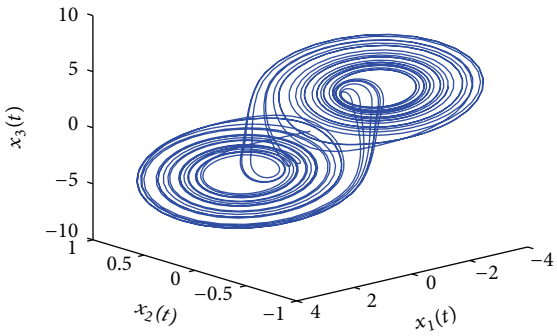

FIGURE 7: Chaotic orbits of Chua's circuit.

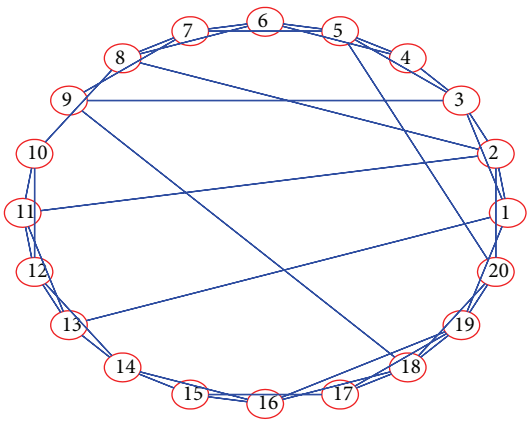

FIGURE 8: Connection of the complex network system.

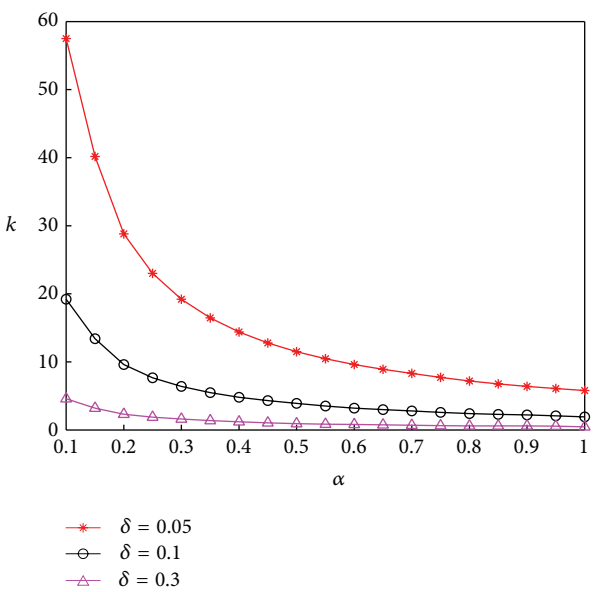

FIGURE 9: Correlation between k and α with given different δ.

such simulations, it is seen that larger expectation α results in smaller control gain k with given pinning fraction δ, while larger pinning fraction δ also results in smaller k with given expectation δ. This property is same as that in Example 15. Let initial condition of system (55) be $x_0 = [0.1 \quad -0.1 \quad 0.2]^T$

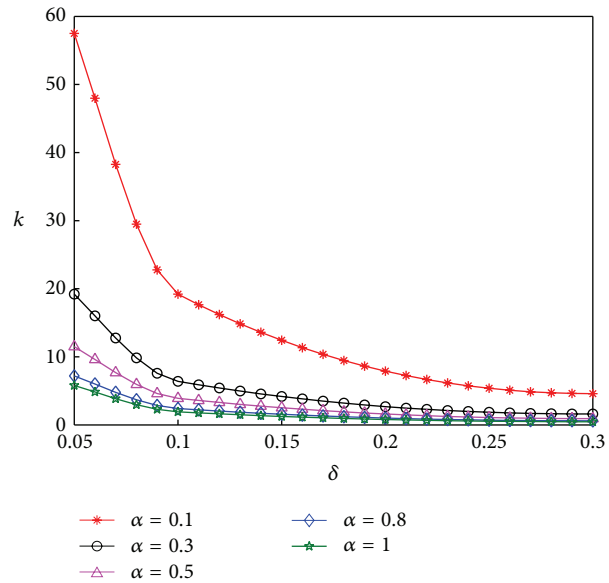

FIGURE 10: Correlation between k and δ with given different α.

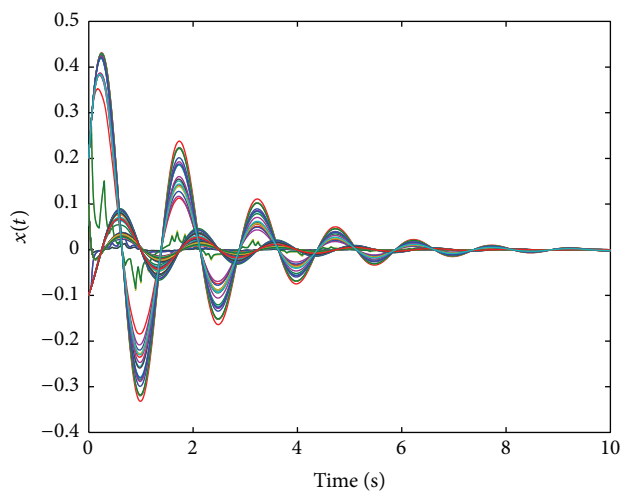

FIGURE 11: Simulation of the closed-loop system.

and $\delta = 0.1$, $\alpha = 0.7$, one has $k_{\min} = 2.8$ with $k > 0$ by Theorem 11. The state curve of the closed-loop system is given in Figure 11. From Figure 11, it is said that the desired pinning controller is useful. If the desired pinning controller is realized by random pinning control, we also have the corresponding simulations of correlation among α, δ, and k. Such relationships are demonstrated in Figures 12 and 13, respectively, which are quite different to the above cases. That is, the array of curves $\delta = 0.05$, $\delta = 0.1$, and $\delta = 0.3$ in Figure 12 is different from those in Figures 9, 2, and 5, though there is also a consistency that larger α leads to smaller k. Accordingly, a phenomenon different from Figures 3, 6, and 10 is shown in Figure 13. For a given α, it is seen that the value change of k is not in accordance with δ. Such differences come from the properties of complex network and random pinning control.

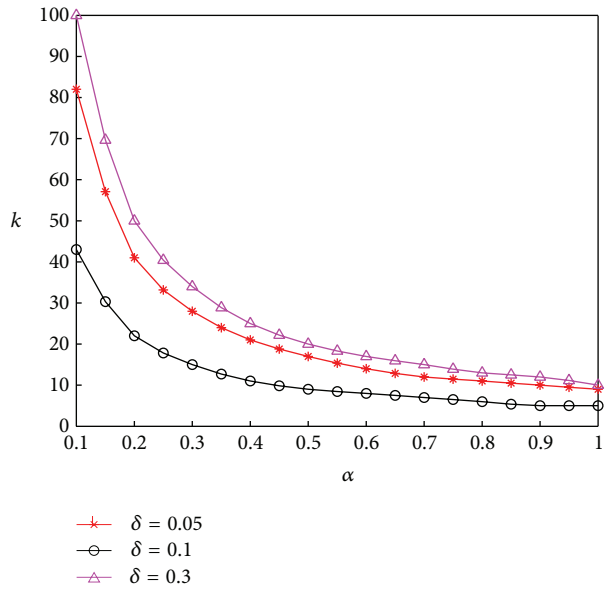

Figure 12: Correlation between k and α with given different δ.

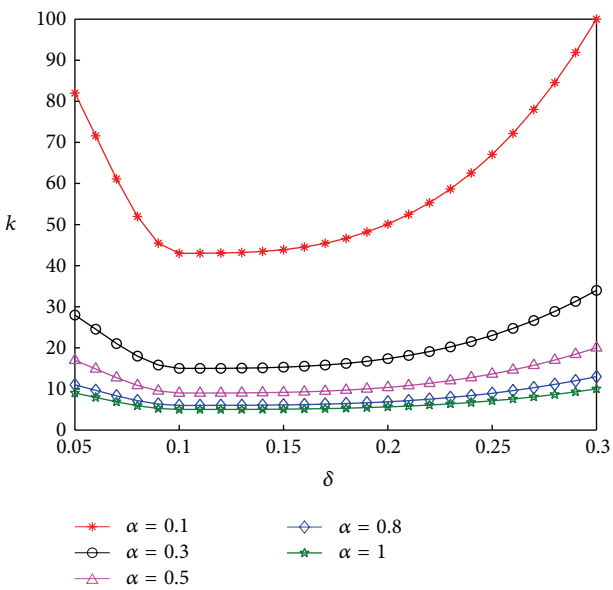

Figure 13: Correlation between k and δ with given different α.

5. Conclusions

In this paper, a new pinning method with a stochastic pinning viewpoint is proposed to investigate the control problem of multiagent systems. It has been shown that a fraction of controllers added to nodes in terms of Brownian noise perturbations can stabilize the underlying systems, whose control method is defined as "stochastic pinning control." It is also seen that the Bernoulli variable plays an essential role in realizing SPC. Based on the given method, new sufficient conditions of the expectation with uncertainty and being unknown are also established. Finally, the utility of the developed theory is illustrated by numerical examples. In this paper, there is no delay in the underlying system. When there is time delay in the controller such as [29], one may design a similar stochastic pinning controller with time delay, which will be our further topics.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Optimal State Estimation for Discrete-Time Markov Jump Systems with Missing Observations

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This paper is concerned with the optimal linear estimation for a class of direct-time Markov jump systems with missing observations. An observer-based approach of fault detection and isolation (FDI) is investigated as a detection mechanic of fault case. For systems with known information, a conditional prediction of observations is applied and fault observations are replaced and isolated; then, an FDI linear minimum mean square error estimation (LMMSE) can be developed by comprehensive utilizing of the correct information offered by systems. A recursive equation of filtering based on the geometric arguments can be obtained. Meanwhile, a stability of the state estimator will be guaranteed under appropriate assumption.

1. Introduction

Discrete-time Markov jump linear systems (MJLSs) are basically linear discrete-time systems with discretional parameters evolving with a finite-state Markov chain. It can be used in modeling systems with abrupt structures, for example, those which may be found in signal processing, fault detection [1, 2], and subsystem switching. One classical application is maneuvering target tracking, in which signals of interest are modeled by using MJLSs [3]. In these fields, the problems of state estimation for MJLSs play an essential role in recovering some desired variables from given noisy observations for output variables. However, many approaches of achieving the state estimation of MJLSs include the generalized pseudo-Bayesian (GPB) algorithm [4, 5], the interacting multiple model (IMM) filtering [6], stochastic sampling based methods [7, 8], and LMMSE filter. Those methods are different from each other in their estimation criteria and means [2, 9–12]. Among them, LMMSE filter has been well studied for MJLSs in many of literary works [9].

On the other hand, since applications of sensors networks are becoming ubiquitous in practical systems, wireless or wireline communication channels are essential for data communication. Examples are offered ranging from advanced aircraft, spacecraft to manufacturing process. As communication channels are time varying and unreliable, the phenomena of random time delays and random packet dropout usually occur in these networked systems. Hence, more and more attention has been paid to systems with observer-based fault during the past years. For example, studies on optimal recursive filter for systems with intermittent observations can be traced back to Nahi [13], whose work assumed that uncertainty of observations is independent and identically distributed. Afterwards, by using linear matrix inequalities (LMIs) techniques, the L_2-L_∞ performance, H_∞ performance, finite-time L_2-L_∞ performance, and finite-time H_∞ performance have been well studied for solving filtering and control problems occurring in stochastic systems with uncertain elements [14–21]. In [22–25], the stability analysis of random Riccati equation arising from Kalman filtering with intermittent observations was investigated elaborately. H_∞ filtering algorithm [26–28] has been developed for discrete systems with random packet losses in [29, 30]. In [31], a robust filtering algorithm was developed for state estimation of MJLSs with random missing observation by applying basic IMM approach and H_∞ technique. Reference [32] dealt with the fault detection filtering (FDF) design within stochastic H_∞ filtering frame for a class of discrete-time nonlinear Markov jump systems with lost measurements.

Although the aforementioned references give efficient and practical tools to deal with the filtering problems

for systems with package dropout, the results given by the methods constructed based on LMIs techniques are sometimes too conservative. What is more, IMM approach mentioned a priori requires online calculations. Inspired by the effectiveness of LMMSE mechanic used in solving state estimation problem of MJLSs with random time delays in [33], the problem of state estimation of MJLSs with random missing observations is formulated into LMMSE filtering frame. This frame can lead to a time-varying linear filter easy to implement. At the same time, most calculations can be performed off-line.

Aiming at solving the issue of uncertain observations in MJLSs, this paper provides a heuristic method for detecting the fault in process of transmitting observation. An approach of fault detection and isolation (FDI) [32, 34] for a class of MJLSs with missing observations will be investigated. The key point of FDI is to construct the residual generator and determine the residual evaluation function and the threshold. Then, by comparing the value of the evaluation function with the prescribed threshold, we will make judgment whether an alarm of fault is generated. The situation of uncertainties of observation can be naturally and conveniently reflected. With knowing the information of the faulty case, a conditional prediction of observation will be obtained, which can be used as replacement of the faulty one. At this time, we can utilize the optimal state estimator of pervious instant and parameters for constructing observer of system to estimate the observation at current time. By this way, we can skip and avoid the fault observation.

Accordingly, by applying the basic FDI approach and basic LMMSE algorithm, an FDI-LMMSE filtering algorithm is developed for state estimation of MJLSs with random missing observation. In order to solve the optimal estimation problem, the measurements' loss process is modeled as a Bernoulli distributed white sequence taking values from 0 to 1 randomly. The estimation problem is then reformulated as an optimal linear filtering of a class of MJLSs, which have random missing observation and necessary model compensation, via state augmentation [35–38]. A recursive filtering is formulated in terms of Riccati difference equations. At the same time, we will show that estimator is stable under necessary assumptions in this paper.

This paper is organized as follows. Section 2 gives the problem formulation. A recursive optimal solution is given in Section 3. Its stability is discussed in Section 4. In Section 5, a numerical example is shown to explain the effectiveness of approach proposed in our paper. At last, the conclusions are drawn in Section 6.

2. Problem Formulation

On the stochastic basis $(\Omega, \mathscr{F}_k, \{\mathscr{F}_k\}, P)$, considering the following jump Markov linear system model:

$$\begin{aligned} x_{k+1} &= A(r_k)x_k + B(r_k)(a(r_k) + w_k) \\ y_k &= \gamma_k C(r_k)x_k + D(r_k)v_k, \end{aligned} \tag{1}$$

where $\{x_k \in R^n\}$ is continuous-valued based-state sequence with known initial distribution $x_0 = \mathscr{N}(x_0; \bar{x}_0, \Sigma_0)$. $a(r_k) \in R^n$ is assumed to be known time-varying constant to each value of r_k. $\{y_k \in R^s\}$ is the noisy observation sequence. $\{w_k \in R^n\}$ is the noisy observation sequence with distribution $w_k \sim N(w_k;\ 0, Q)$. $\{v_k \in R^s\}$ is a white measurement noise sequence independent of the process noise with distribution $v_k \sim N(v_k;\ 0, R)$.

Remark 1. $a(r_k)$ is a compensation between practical systems and models applied in this paper.

$\{r_k\}$ is the unknown discrete-valued Markov chain with a finite-state space $N = \{1, 2, \ldots, N\}$. The transition probability matrix is $\Pi = [\pi_{ij}]_{N\times N}$, where $i, j \in N$. We set $\mu_i(k) := P(r_k = i)$. The basic variables w_k, v_k, x_0 and the modal-state sequence r_k are assumed to be mutually independent for all k. $A(r_k)$, $B(r_k)$, $C(r_k)$, $D(r_k)$ are assumed to be known time-varying system matrices to each value of r_k. For notational simplicity, the following notations and definitions hold in the rest of the paper:

$$\begin{gathered} r_k^j = \{r_k = j\}, \quad A_j = A\left(r_k^j\right), \quad B_j = B\left(r_k^j\right), \\ C_j = C\left(r_k^j\right), \quad D_j = D\left(r_k^j\right). \end{gathered} \tag{2}$$

In this paper, consider that the observations are sent to the estimator via a Gilbert-Elliot channel, where the packet arrival is modeled using a binary random variable $\{\gamma_k\}$, with probability $P(\gamma_k = 1) = \eta$, and with γ_k independent of γ_s if $k \neq s$. Let γ_k be independent of w_k, v_k, x_k; that is, according to this model, the measurement equation consists of noise alone or noise plus signal, depending on whether γ_k is 0 or 1.

Notation 1. Some notations which we will use throughout the paper should be presented first. We will denote by $R^{m\times n}$ the space of $m \times n$ real matrices and by R^m the space of m-dimensional real vectors. The superscript T indicates transpose of a matrix. For a collection of N matrices $D_1, \ldots, D_N$, with $D_j \in R^{m\times n}$, $\mathrm{diag}\{D_j\} \in R^{Nm\times Nn}$ represents the diagonal matrix formed by D_j in the diagonal.

Notation 2. Define $H^n = \{X = (X_1 \cdots X_N); X_i \in R^{n\times n}, i \in N\}$ and $H^{+n} = \{X = (X_1 \cdots X_N);\ X_i \geq 0, i \in N\}$. For $X = (X_1 \cdots X_N) \in H^{+n}$, $V = (V_1 \cdots V_N) \in H^{+n}$, if $X \geq V$ for each $i \in N$, we have $X_i \geq V_i$.

3. Recursive Optimal Solution

In this section, a solution to the optimal estimator will be presented via the projection theory and the state augmentation in the Hilbert space.

3.1. Preliminaries. First, we denote by $\mathscr{L}(y^k)$ the linear space spanned by the observation $y^k = \{y_k^T, \ldots, y_0^T\}$. If $\theta = \sum_{i=1}^k \xi(i)^T y_i$ for some $\xi(i) \in R^m$, $i = 1, \ldots, k$, the random variable $\theta \in \mathscr{L}(y^k)$.

Let $1_{\{r_k=j\}}$ represent an indicator of Markov process, which is defined as follows:

$$z_j(k) \triangleq x_k 1_{\{r_k=j\}} \in R^n \tag{3}$$

$$z(k) \triangleq \begin{pmatrix} z_1(k) \\ \vdots \\ z_N(k) \end{pmatrix} \in R^{Nn}. \tag{4}$$

And call $\hat{z}(k) = E(z(k))$. Define also $\hat{z}(k \mid k-1)$ as the projection of $z(k)$ onto the linear space $\mathscr{L}(y^k)$ and

$$\tilde{z}(k \mid k-1) \triangleq z(k) - \hat{z}(k \mid k-1). \tag{5}$$

Then, we first define the following second-moment matrices associated with the aforementioned variables. They play key roles in deriving the covariance matrices of the estimator errors and optimal estimator:

$$\begin{aligned} Z_i(k) &\triangleq E\left\{z_i(k)(z_i(k))^T\right\} \in \mathbf{B}(R^n), \\ Z(k) &\triangleq E\left\{z(k)(z(k))^T\right\} \in \mathbf{B}\left(R^{Nn}\right), \\ \widehat{Z}(k \mid l) &\triangleq E\left\{\hat{z}(k \mid l)(\hat{z}(k \mid l))^T\right\} \in \mathbf{B}\left(R^{Nn}\right), \\ \widetilde{Z}(k \mid l) &\triangleq E\left\{\tilde{z}(k \mid l)(\tilde{z}(k \mid l))^T\right\} \in \mathbf{B}\left(R^{Nn}\right). \end{aligned} \tag{6}$$

Considering the following augment matrices:

$$\begin{aligned} A(k) &\triangleq \begin{bmatrix} \pi_{11}(k)A_1(k) & \cdots & \pi_{1N}(k)A_N(k) \\ \vdots & \ddots & \vdots \\ \pi_{N1}(k)A_1(k) & \cdots & \pi_{NN}(k)A_N(k) \end{bmatrix} \\ &\in \mathbf{B}\left(R^{Nn}\right) \\ D(k) &\triangleq \left[D_1(k)\mu_1(k)^{1/2} \cdots D_N(k)\mu_N(k)^{1/2}\right] \\ &\in \mathbf{B}\left(R^{Ns}, R^s\right), \\ C(k) &\triangleq [C_1 \ \cdots \ C_N] \in \mathbf{B}\left(R^{Nn}, R^s\right), \\ \bar{a} &\triangleq [a_1, \ldots, a_N], \\ B(k) &\triangleq \operatorname{diag}\left[\left[\left(\pi_k^{1j}\mu_k^1\right)^{1/2} B_1 \cdots \left(\pi_k^{Nj}\mu_k^N\right)^{1/2} B_N\right]\right] \\ &\in \mathbf{B}\left(R^{N^2 n}, R^{Nn}\right) \end{aligned} \tag{7}$$

then system can be described as follows:

$$\begin{aligned} z(k+1) &= A(k)z(k) + B(k)(\bar{a} + w(k)) \\ y(k) &= \gamma_k C(k)z(k) + D(k)v(k). \end{aligned} \tag{8}$$

Note that $y(k) = y_k$.

Assumption 2. For all k, $B(k)\bar{a}\bar{a}^T B(k)^T \gg P$, where P convergency value of $\widetilde{Z}(k \mid k-1)$, which will be given in Section 3.

3.2. Optimal Estimator. From geometric arguments in [39], the LMMSE filter for MJLSs with uncertain observations can be derived in this section. The following lemmas present necessary and sufficient conditions on derivation of FDI-LMMSE filtering.

Lemma 3. *For any given time instant k, one has*

$$\begin{aligned} Z_j(k+1) &= \sum_{i=1}^{m} \pi_{ij} A_i(k) Z_i(k) A_i^T(k) \\ &\quad + \sum_{i=1}^{N} \pi_{ij}\mu_{k-1}^i B_i\left(a_i a_i^T + Q\right) B_i^T \\ Z(k) &= \operatorname{diag}\left[Z_j(k)\right], \end{aligned} \tag{9}$$

where $Z_i(0) = \mu_0^i X_0$.

Proof. For any given instant k, we have from (8) that

$$\begin{aligned} Z_j(k+1) &= E\left[z_j(k+1)z_j^T(k+1)\right] \\ &= \sum_{i=1}^{m} \pi_{ij} A_i(k) Z_i(k) A_i^T(k) \\ &\quad + \sum_{i=1}^{N} \pi_{ij}\mu_{k-1}^i B_i\left(a_i a_i^T + Q\right) B_i^T. \end{aligned} \tag{10}$$

Recalling that $X_0 = E[\bar{x}_0 \bar{x}_0^T]$, initial covariance matrix $Z_i(0) = \mu_0^i X_0$.

To derive the optimal filter, we first define the innovation sequence as

$$\tilde{y}(k) = y(k) - \hat{y}(k \mid k-1), \tag{11}$$

where conditional prediction $\hat{y}(k \mid k-1)$ is the projection of $y(k)$ onto the linear space of $\mathscr{L}(y^{k-1})$. Consider

$$\hat{y}(k \mid k-1) = C(k)\hat{z}(k \mid k-1). \tag{12}$$

Then, according to (4) and (8), the generated residual will be obtained as

$$\tilde{y}(k \mid k-1) = \begin{cases} C(k)\tilde{z}(k \mid k-1) + D(k)v(k), & \gamma_k = 1, \\ D(k)v(k) - C(k)\hat{z}(k \mid k-1), & \gamma_k = 0. \end{cases} \tag{13}$$

In the following, an FDI scheme will be constructed, which can detect whether observation at instant k is lost. In this paper, we choose the following mean square of the residual as the residual evaluation function to measure the energy of the residual:

$$S_k = E\left(\tilde{y}(k \mid k-1)\tilde{y}(k \mid k-1)^T\right). \tag{14}$$

From (12)-(13) we get that

$$\begin{aligned} &E\left(\tilde{y}(k \mid k-1)\tilde{y}(k \mid k-1)^T\right) \\ &\quad = \begin{cases} C(k)\widetilde{Z}(k \mid k-1)C(k)^T + D(k)RD(k)^T, & \gamma_k = 1, \\ C(k)\widehat{Z}(k \mid k-1)C(k)^T + D(k)RD(k)^T, & \gamma_k = 0. \end{cases} \end{aligned} \tag{15}$$

Suppose that $\widetilde{Z}(k \mid k-1)$ is convergent to P at the instant k, from Assumption 2, we have that

$$B(k-1)\,\overline{a}\,\overline{a}^T B(k-1)^T \gg \widetilde{Z}(k \mid k-1). \tag{16}$$

If $\gamma_k = 1$, we have that

$$\begin{aligned} S_k^{\gamma_k=1} &= C(k)\,\widetilde{Z}(k \mid k-1)\,C(k)^T + D(k)\,RD(k)^T \\ &\ll C(k)\,B(k-1)\,\overline{a}\,\overline{a}^T B(k-1)^T C(k)^T + D(k)\,RD(k)^T. \end{aligned} \tag{17}$$

If $\gamma_k = 0$,

$$S_k^{\gamma_k=0} = C(k)\,\widehat{Z}(k \mid k-1)\,C(k)^T + D(k)\,RD(k)^T. \tag{18}$$

From (8),

$$\begin{aligned} \widehat{Z}(k \mid k-1) &= A(k-1)\,\widehat{Z}(k-1 \mid k-1)\,A(k-1)^T \\ &\quad + B(k-1)\,\overline{a}\,\overline{a}^T B(k-1)^T \\ &\geq B(k-1)\,\overline{a}\,\overline{a}^T B(k-1)^T; \end{aligned} \tag{19}$$

we have that

$$S_k^{\gamma_k=0} \geq C(k)\,B(k-1)\,\overline{a}\,\overline{a}^T B(k-1)^T C(k)^T + D(k)\,RD(k)^T. \tag{20}$$

The FDI scheme in the following lemma will play a key role in deriving the main results of this paper. □

Lemma 4. *With above derivation, we can decide whether the observations of system were lost and detect the lost information at instant k according to the following rule:*

$$\begin{aligned} S_k > S_{th} &\Longrightarrow \gamma_k = 0 \\ S_k \leq S_{th} &\Longrightarrow \gamma_k = 1, \end{aligned} \tag{21}$$

where

$$S_{th} = C(k)\,B(k-1)\,\overline{a}\,\overline{a}^T B(k-1)^T C(k)^T + D(k)\,RD(k)^T. \tag{22}$$

With the fault being detected, the missing information γ_k can be taken into consideration when designing the FDI-LMMSE filter. The fault observation can be replaced and isolated by $\widehat{y}(k \mid k-1)$. By the above approach, we can skip the error information at the instant k and use the correct information of pervious instant $k-1$ to estimate the value of $\widehat{x}(k \mid k)$ state at instant k directly.

Theorem 5. *Consider the system represented by* (8). *Then the LMMSE $\widehat{x}_{k|k}$ is given by*

$$\widehat{x}_{k|k} = \sum_{i=1}^{N} \widehat{z}_i(k \mid k), \tag{23}$$

where $\widehat{z}^c(k \mid k)$ satisfies the recursive equation

$$\begin{aligned} \widehat{z}_{\gamma_k}(k \mid k) &= \widehat{z}(k \mid k-1) + \gamma_k \widetilde{Z}(k \mid k-1)\,C(k)^T \\ &\quad \times \left[C(k)\widetilde{Z}(k \mid k-1)C(k)^T + D(k)RD(k)^T\right]^{-1} \\ &\quad \times \left(C(k)\,\widetilde{z}(k \mid k-1) + D(r_k)\,v(k)\right) \end{aligned} \tag{24}$$

$$\widehat{z}(k+1 \mid k) = A(k)\,\widehat{z}(k \mid k) + B(k)\,\overline{a}, \tag{25}$$

where $\widehat{z}(0 \mid -1) = [\mu_0^1 x_0, \ldots, \mu_0^N x_0]^T$.

Proof. Recall that observation estimator is given by (12).

Now, $\widetilde{y}(k \mid k-1)$ can be rewritten as the following equation:

$$\widetilde{y}(k \mid k-1) = \gamma_k\left[C(k)\,\widetilde{z}(k \mid k-1) + D(k)\,v(k)\right]. \tag{26}$$

Considering the geometric argument as in [39], the estimator $\widehat{z}(k \mid k-1)$ satisfies the following equations:

$$\widehat{z}(k \mid k-1) = E\left(z(k)\left(y^{k-1}\right)^T\right)\operatorname{cov}\left(\left(y^{k-1}\right)^{-1} y^{k-1}\right) \tag{27}$$

$$\begin{aligned} \widehat{z}(k \mid k) &= \widehat{z}(k \mid k-1) + E\left(\widetilde{z}(k)\,\widetilde{y}(k \mid k-1)^T\right) \\ &\quad \times E\left(\widetilde{y}(k \mid k-1)\,\widetilde{y}(k \mid k-1)^T\right)^{-1} \\ &\quad \times \left(y(k) - \widehat{y}(k \mid k-1)\right). \end{aligned} \tag{28}$$

From (26), we get that

$$E\left(\widetilde{z}(k)\,\widetilde{y}(k \mid k-1)^T\right) = \gamma_k\left[\widetilde{Z}(k \mid k-1)\,C(k)^T\right]. \tag{29}$$

Because v_k is independent of $\{r_k, y(k-1)\}$, we have that

$$\begin{aligned} &\left\langle \alpha^T D(r_k)\,v_k; \beta^T y(k-1)\right\rangle \\ &\quad = E\left(\alpha^T D(r_k)\,v_k \beta^T y(k-1)\right) \\ &\quad = E\left(v_k^T\right) E\left[\alpha^T D(r_k)\,\beta^T y(k-1)\right] \\ &\quad = 0, \end{aligned} \tag{30}$$

showing that $D(k)v_k$ is orthogonal to $\mathscr{L}(y^{k-1})$. Similar reasoning shows the orthogonality between $D(k)v_k$ and $\widetilde{z}(k \mid k-1)$. Recalling that $\widehat{z}(k \mid k-1) \in \mathscr{L}(y^{k-1})$ and $\widetilde{z}(k \mid k-1)$ are orthogonal to $\mathscr{L}(y^{k-1})$, we can obtain that $\widetilde{z}(k \mid k-1)$ is orthogonal to $\widehat{z}(k \mid k-1)$. Then, from (27), the result can be obtained as follows:

$$\begin{aligned} \widehat{z}_j(k \mid k-1) &= E\left(z_j(k)\left(y^{k-1}\right)^T\right)\operatorname{cov}\left(\left(y^{k-1}\right)^{-1} y^{k-1}\right) \\ &= \sum_{i=1}^{N} \pi_{ij} A_i \widehat{z}_i(k-1 \mid k-1) + \sum_{i=1}^{N} \pi_{ij} \mu_{k-1}^i B_i a_i. \end{aligned} \tag{31}$$

From (11), (28) and (26), (29), we get that

$$\begin{aligned}\hat{z}(k \mid k) &= \hat{z}(k \mid k-1) + \gamma_k \widetilde{Z}(k \mid k-1) C(k)^T \\ &\quad \times \left[C(k)\widetilde{Z}(k \mid k-1)C(k)^T + D(k)RD(k)^T\right]^{-1} \\ &\quad \times \left(C(k)\tilde{z}(k \mid k-1) + D(r_k)v(k)\right).\end{aligned} \tag{32}$$

The positive-semidefinite matrices $\widetilde{Z}(k \mid k-1)$ are obtained from

$$\widetilde{Z}(k \mid k-1) = Z(k) - \widehat{Z}(k \mid k-1). \tag{33}$$

And the recursive equation about $\widehat{Z}(k \mid k-1)$ is given as follows:

$$\begin{aligned}\widehat{Z}_{\gamma_k}(k \mid k) &= \widehat{Z}_{\gamma_k}(k \mid k-1) \\ &\quad + \gamma_k^2 \widehat{Z}_{\gamma_k}(k \mid k-1) C(k)^T \\ &\quad \times \left(C(k)\widetilde{Z}_{\gamma_k}(k \mid k-1)C(k)^T + D(k)RD(k)^T\right)^{-1} \\ &\quad \times C(k)\widehat{Z}_{\gamma_k}(k \mid k-1)\end{aligned} \tag{34}$$

$$\begin{aligned}\widehat{Z}_{\gamma_k}(k \mid k-1) &= A(k-1)\widehat{Z}_{\gamma_{k-1}}(k-1 \mid k-1)A(k-1)^T \\ &\quad + B(k-1)\bar{a}\bar{a}^T B(k-1)^T,\end{aligned}$$

where $\widehat{Z}(0 \mid -1) = z(0 \mid -1)z(0 \mid -1)^T$.

$\widetilde{Z}(k+1 \mid k)$ can be derived directly as a recursive Riccati equation in the following derivation. In the following, we denote the linear operator

$$\Psi(\cdot, k) : H^n \longrightarrow B\left(R^{Nn}\right) \tag{35}$$

by $\Gamma(k)$, in which $\Psi(\cdot, k)$ is

$$\begin{aligned}\Psi(\Gamma(k)) &= \operatorname{diag}\left[\sum_{i=1}^{N} \pi_{ij} A_i Z_i(k) A_i^T\right] \\ &\quad - A(k)\left(\operatorname{diag}\left[Z_i(k)\right]\right)A(k)^T \geq 0.\end{aligned} \tag{36}$$

□

Theorem 6. *$\widetilde{Z}(k+1 \mid k)$ satisfies the following recursive Riccati equation:*

$$\begin{aligned}\widetilde{Z}(k+1 \mid k) &= A(k)\widetilde{Z}(k \mid k-1)A(k)^T \\ &\quad + \Psi(\Gamma(k), k) + B(k)QB(k)^T \\ &\quad - \gamma_k^2 A(k)\widetilde{Z}(k \mid k-1)C(k)^T \\ &\quad \times \left[C(k)\widetilde{Z}(k \mid k-1)C(k)^T + D(k)RD(k)^T\right]^{-1} \\ &\quad \times C(k)\widetilde{Z}(k \mid k-1)A(k)^T,\end{aligned} \tag{37}$$

where $\Gamma(k) = (Z_1(k), Z_2(k), \ldots, Z_N(k))$ is given by the recursive equation (9) from Lemma 3.

Unlike the classical case, the sequence $\{\widetilde{Z}(k+1 \mid k)\}_{k \in Z_+}$ is now random, which result from its dependence on the random sequence $\{\gamma_k\}_{k \in Z_+}$.

Proof. Rewrite state equation in (8) as follows:

$$z(k+1) = A(k)z(k) + M(k+1)z(k) + B(k)\bar{a} + \vartheta(k), \tag{38}$$

where

$$\begin{aligned}M(k+1, j) &= \left[m_1(k+1, j) \cdots m_N(k+1, j)\right], \\ m_i(k+1, j) &= \left(1_{\{r_{k+1}=j\}} - \pi_{ij}\right) A_i 1_{\{r_k=i\}}, \\ M(k+1) &= \begin{bmatrix} M(k+1, 1) \\ \vdots \\ M(k+1, N) \end{bmatrix}, \\ \vartheta(k) &= \begin{bmatrix} 1_{\{r_{k+1}=1\}} B_1 w(k) \\ \vdots \\ 1_{\{r_{k+1}=N\}} B_N w(k) \end{bmatrix}.\end{aligned} \tag{39}$$

From (32), we define

$$\begin{aligned}T(k) &= -A(k)\widetilde{Z}(k \mid k-1)C(k)^T \\ &\quad \times \left[C(k)\widetilde{Z}(k \mid k-1)C(k)^T + D(k)RD(k)^T\right]^{-1}.\end{aligned} \tag{40}$$

From (25) and (32), we have that

$$\begin{aligned}\hat{z}_{\gamma_k}(k+1 \mid k) &= A(k)\hat{z}(k \mid k-1) \\ &\quad + \gamma_k T(k) C(k) \tilde{z}(k \mid k-1) \\ &\quad + B(k)\bar{a} + \gamma_k T(k) D(k) w_k.\end{aligned} \tag{41}$$

Then from (41) and (38), we get that

$$\begin{aligned}\tilde{z}_{\gamma_k}(k+1 \mid k) &= A(k)\tilde{z}(k \mid k-1) \\ &\quad + \gamma_k T(k) C(k) \tilde{z}(k \mid k-1) \\ &\quad + M(k)z(k) + \vartheta(k) + \gamma_k T(k) D(k) w_k.\end{aligned} \tag{42}$$

Therefore, at this point, we obtain the recursive equation for $\widetilde{Z}(k \mid k-1)$ as follows:

$$\begin{aligned}\widetilde{Z}_{\gamma_k}(k+1 \mid k) &= \left(A(k) + \gamma_k T(k) C(k)\right) \\ &\quad \times \widetilde{Z}(k \mid k-1)\left(A(k) + \gamma_k T(k)C(k)\right)^T \\ &\quad + E\left(M(k+1)z(k)z(k)^T M(k+1)^T\right) \\ &\quad + E\left(\vartheta(k)\vartheta(k)^T\right) \\ &\quad + \gamma_k^2 T(k) D(k) RD(k)^T T(k)^T.\end{aligned} \tag{43}$$

By a series of algebraic manipulations, we have

$$E\left(M(k+1)z(k)z(k)^T M(k+1)^T\right) = \Psi(\Gamma(k),k)$$
$$E\left(\vartheta(k)\vartheta(k)^T\right) = B(k)QB(k)^T. \tag{44}$$

Substituting (44) into (43) yields the recursive equation for $\widetilde{Z}(k \mid k-1)$ as

$$\begin{aligned}\widetilde{Z}_{\gamma_k}(k+1 \mid k) = {} & \left(A(k)+\gamma_k T(k)C(k)\right)\widetilde{Z}(k \mid k-1) \\ & \times \left(A(k)+\gamma_k T(k)C(k)\right)^T \\ & + \Psi(\Gamma(k),k) + B(k)QB(k)^T \\ & + \gamma_k^2 T(k)D(k)RD(k)^T T(k)^T.\end{aligned} \tag{45}$$

□

4. Stability of the State Estimator

As we all see, the intermittent observations are the source of potential instability. From Theorem 6, however, the error covariance matrix obtained from the LMMSE can be rewritten in terms of a recursive Riccati equation of γ_k. In this section, based on that following assumptions hold, we show that the proposed estimator is stable as provided in our paper.

Assumption 7. $\{r_k, k = 0, 1\cdots\}$ is assumed to be ergodic Markov chain.

Assumption 8. System (1) is mean square stable (MSS) according to the definition in [35].

First, (37) describes a recursive Riccati equation for $\widetilde{Z}(k+1 \mid k)$. We should establish now its convergence when $k \to \infty$. It follows from Assumption 2 that $\lim_{k\to\infty} P(r_k = i)$ exists and it is independent of r_0. We define

$$\mu_i = \lim_{k\to\infty} P(r_k = i) = \lim_{k\to\infty} \mu_i(k). \tag{46}$$

We redefine the matrix as follows:

$$\begin{gathered} A \triangleq \begin{pmatrix} \pi_{11}A_1 & \cdots & \pi_{1N}A_N \\ \vdots & \ddots & \vdots \\ \pi_{N1}A_1 & \cdots & \pi_{NN}A_N \end{pmatrix}, \\ B \triangleq \operatorname{diag}\left[\left[\left(\pi_{1j}\mu_1\right)^{1/2}B_1 \cdots \left(\pi_{Nj}\mu_N^N\right)^{1/2}B_N\right]\right], \\ C \triangleq \left[C_1, C_2, \ldots, C_N\right], \qquad D \triangleq \left[D_1\mu_1^{1/2}\cdots D_N\mu_N^{1/2}\right]. \end{gathered} \tag{47}$$

Then, we give the following facts and lemmas for system, which will be used in the proof of stability of the covariance matrix of estimation error.

With regard to Assumptions 2 and 7 and Proposition 3.36 in [35], $\Gamma(k) \to \Gamma$ as $k \to \infty$, where $\Gamma = \{Z_1, Z_2, \ldots, Z_N\}$ is the unique solution that satisfies

$$Z_j = \sum_{i=1}^{N} \pi_{ij}\left(A_i Z_i A_i^T + \mu_i B_i\left(a_i a_i^T + Q\right)B_i^T\right). \tag{48}$$

Then we have $\inf_{l\geq k}\mu_i(l) > 0$ holding for all $i \in \mathbf{N}$ (since $\exists l$, we have $\mu_i(l) \to \mu_i > 0$ as $k \to \infty$). Defining $\alpha_i(k) = \inf_{l\geq k}\mu_i(k+l)$, then we get

$$\mu_i(k+\kappa) \geq \alpha_i(k) \geq \alpha_i(k-1), \quad k = 1, 2, \ldots;\ i \in \mathbf{N}. \tag{49}$$

At the same time, $\alpha_i(k) \to \mu_i$ $(k \to \infty)$ exponentially fast.

From (37), as $k \to \infty$, we obtain the mean state covariance as follows:

$$\begin{aligned}\overline{\widetilde{Z}}(k+1 \mid k) &= \lim_{k\to\infty} \widetilde{Z}(k+1 \mid k) \\ &= \lim_{k\to\infty} E\left[\widetilde{Z}(k+1 \mid k)\right] \\ &= A\overline{\widetilde{Z}}(k \mid k-1)A^T + \Psi(\Gamma(k)) \\ &\quad + B(k)QB(k)^T - \eta\overline{\widetilde{Z}}(k \mid k-1)C^T \\ &\quad \times \left[C\overline{\widetilde{Z}}(k \mid k-1)C^T + D(k)RD(k)^T\right]^{-1} \\ &\quad \times C\overline{\widetilde{Z}}(k \mid k-1)A^T.\end{aligned} \tag{50}$$

An operator is introduced for any positive-semidefinite matrix X as follows:

$$\mathcal{T} = -AXC^T\left(CXC^T + DRD^T\right)^{-1}. \tag{51}$$

Then

$$\mathcal{T}(k) = -A\overline{\widetilde{Z}}(k \mid k-1)C^T\left(C\overline{\widetilde{Z}}(k \mid k-1)C^T + DRD^T\right)^{-1}. \tag{52}$$

Define now $\bar{\Gamma}(k) = (\bar{Z}_1(k), \ldots, \bar{Z}_N(k)) \in H^{n+}$ with $\bar{Z}_i(0) = 0$, $j \in \mathbf{N}^+$, and

$$\bar{Z}_j(k+1) = \sum_{i=1}^{N} \pi_{ij}\left(A_i\bar{Z}_i(k)A^T + \alpha_i(k)B_iQB_i^T\right). \tag{53}$$

Lemma 9 (see [35]). $\bar{\Gamma}(k) \xrightarrow{k\to\infty} \Gamma$ *and for each* $k = 0, 1, 2, \ldots$, *one can get that*

$$\Gamma(k+\kappa) \geq \bar{\Gamma}(k) \geq \bar{\Gamma}(k-1). \tag{54}$$

Now, one defines

$$\begin{aligned}\Upsilon(k+1) = {} & A\Upsilon(k)A^T + \operatorname{diag}\left[\sum_{i=1}^{N}\alpha_i(k)\pi_{ij}B_iQB_i^T\right] + \Psi\left(\bar{\Gamma}\right) \\ & - \eta A\Upsilon(k)C^T\left(C\Upsilon(k)C^T + \bar{D}(k)R\bar{D}(k)^T\right)^{-1} \\ & \times CP_1(k)A^T,\end{aligned} \tag{55}$$

where $\Upsilon(0) = 0$, $\bar{D}(k) = [D_1\alpha_1(k)^{1/2}\cdots D_N\alpha_N(k)^{1/2}]$.

From the definition of κ *and condition of* $DRD^T > 0$, *one notices that the inverse of* $C\Upsilon(k)C^T + DRD^T$ *exists.*

Lemma 10. *For each $k = 0, 1 \cdots$, one gets that*

$$\Upsilon(k) \le \Upsilon(k+1) \le \overline{\widetilde{Z}}(k+1+\kappa \mid k+\kappa). \tag{56}$$

Proof. In order to deduce (56), we define

$$\mathscr{M}(k) = -A\Upsilon(k)C^T\left(C\Upsilon(k)C^T + \bar{D}(k)R\bar{D}(k)^T\right)^{-1}. \tag{57}$$

Then, if $\Upsilon(k) \le \overline{\widetilde{Z}}(k+\kappa \mid k+\kappa-1)$,

$$\begin{aligned}
&\Upsilon(k+1)\\
&\quad= \left(A + \sqrt{\eta}\mathscr{T}(k+\kappa)C\right)\Upsilon(k)\left(A + \sqrt{\eta}\mathscr{T}(k+\kappa)C\right)^T\\
&\qquad+ \Psi\left(\bar{\Gamma}(k)\right) + \eta\mathscr{T}(k+\kappa)\bar{D}(k)R\bar{D}(k)\mathscr{T}(k+\kappa)^T\\
&\qquad- \eta\left(\mathscr{T}(k+\kappa) - \mathscr{M}(k)\right)\left(C\Upsilon(k)C^T + \bar{D}(k)R\bar{D}(k)\right)\\
&\qquad\times\left(\mathscr{T}(k+\kappa) - \mathscr{M}(k)\right)^T + \operatorname{diag}\left[\sum_{i=1}^{N}\alpha_i(k)\pi_{ij}B_iQB_i^T\right]\\
&\quad\le \left(A + \sqrt{\eta}\mathscr{T}(k+\kappa)C\right)\overline{\widetilde{Z}}(k+\kappa \mid k+\kappa-1)\\
&\qquad\times\left(A + \sqrt{\eta}\mathscr{T}(k+\kappa)C\right)^T\\
&\qquad+ \operatorname{diag}\left[\sum_{i=1}^{N}\mu_i(k+\kappa)\pi_{ij}B_iQB_i^T\right] + \Psi\left(\bar{\Gamma}(k+\kappa)\right)\\
&\qquad+ \eta\mathscr{T}(k+\kappa)D(k+\kappa)RD(k+\kappa)\mathscr{T}(k+\kappa)^T\\
&\quad= \overline{\widetilde{Z}}(k+1+\kappa \mid k+\kappa).
\end{aligned} \tag{58}$$

Obviously, when $\Upsilon(0) = 0 \le \overline{\widetilde{Z}}(\kappa \mid \kappa-1)$, it yields $\Upsilon(k) \le \overline{\widetilde{Z}}(k+\kappa \mid k+\kappa-1)$, $k = 0, 1, 2 \cdots$. Similarly if $\Upsilon(k-1) \le \Upsilon(k)$, based on (49) and (54), we have

$$\begin{aligned}
&\Upsilon(k)\\
&\quad= \left(A + \sqrt{\eta}\mathscr{M}(k)C\right)\Upsilon(k-1)\left(A + \sqrt{\eta}\mathscr{M}(k)C\right)^T\\
&\qquad+ \Psi\left(\bar{\Gamma}(k)\right) + \eta\mathscr{M}(k)\bar{D}(k)R\bar{D}(k)\mathscr{M}(k)^T\\
&\qquad+ \eta\left(\mathscr{M}(k) - \mathscr{M}(k-1)\right)\\
&\qquad\times\left(C\Upsilon(k)C^T + \bar{D}(k)R\bar{D}(k)\right)\left(\mathscr{M}(k) - \mathscr{M}(k-1)\right)^T\\
&\qquad+ \operatorname{diag}\left[\sum_{i=1}^{N}\alpha_i(k-1)\pi_{ij}B_iQB_i^T\right]\\
&\quad\le \left(A + \sqrt{\eta}\mathscr{M}(k)C\right)\Upsilon(k)\left(A + \sqrt{\eta}\mathscr{M}(k)C\right)^T + \Psi\left(\bar{\Gamma}(k)\right)\\
&\qquad+ \operatorname{diag}\left[\sum_{i=1}^{N}\alpha_i(k)\pi_{ij}B_iQB_i^T\right]\\
&\qquad+ \eta\mathscr{M}(k)\bar{D}(k)R\bar{D}(k)\mathscr{M}(k)^T\\
&\quad= \Upsilon(k+1).
\end{aligned} \tag{59}$$

Since $\Upsilon(0) = 0 \le \Upsilon(1)$, the induction argument is completed for $\Upsilon(k) \le \Upsilon(k+1)$. □

Theorem 11. *Suppose that Assumptions 7 and 8 hold. Consider that the algebraic Riccati equation*

$$P = APA^T + \Psi(\Gamma) + BQB^T - \eta APC\left[CPC^T + DRD\right]^{-1}CPA^T \tag{60}$$

satisfies (48), *where* $\Gamma = \{Z_1, Z_2, \ldots, Z_N\}$. *Then, there exists a unique nonnegative definite solution P to* (60). $r_\sigma(A) \le 1$, *and for any* $\Gamma(0) = \{Z_1(0), \ldots, Z_N(0)\}$, $Z_i(0) \ge 0$, $i = 1 \cdots N$, *and* $\overline{\widetilde{Z}}(0 \mid -1) = \widetilde{Z}(0 \mid -1) \ge 0$, *one has* $\overline{\widetilde{Z}}(k+1 \mid k)$ *given by* (50) *satisfying* $\overline{\widetilde{Z}}(k+1 \mid k) \to P$, $k \to \infty$.

Proof. Due to MSS of 5.38 [35], we have from Proposition 3.6 in chapter 3 [35] that $r_\sigma(A) < 1$. According to the standard results for algebraic Riccati equation there is a unique positive-semidefinite solution $P \in B(R^{Nn})$ to (60). And moreover $r_\sigma(A + \sqrt{\eta}\mathscr{T}(P)C) < 1$.

From Theorem 11, we get that P satisfied

$$\begin{aligned}
P &= \left(A + \sqrt{\eta}\mathscr{T}(P)C\right)P\left(A + \sqrt{\eta}\mathscr{T}(P)C\right)^T\\
&\quad+ \Psi(\Gamma) + BQB^T + \eta\mathscr{T}(P)DRD^T\mathscr{T}(P)^T.
\end{aligned} \tag{61}$$

Define $P(0) = \overline{\widetilde{Z}}(0 \mid -1) = \widetilde{Z}(0 \mid -1)$ and

$$\begin{aligned}
P(k+1) &= \left(A + \sqrt{\eta}\mathscr{T}(P)C\right)P(k)\left(A + \sqrt{\eta}\mathscr{T}(P)C\right)^T\\
&\quad+ \Psi(\Gamma(k)) + B(k)QB(k)^T\\
&\quad+ \eta\mathscr{T}(P)D(k)RD(k)^T\mathscr{T}(P)^T.
\end{aligned} \tag{62}$$

Then (50) can be rewritten as

$$\begin{aligned}
&\overline{\widetilde{Z}}(k+1 \mid k)\\
&\quad= \left(A + \sqrt{\eta}\mathscr{T}(P)C\right)\overline{\widetilde{Z}}(k \mid k-1)\left(A + \sqrt{\eta}\mathscr{T}(P)C\right)^T\\
&\qquad+ \Psi(\Gamma(k)) + B(k)QB(k)^T\\
&\qquad+ \eta\mathscr{T}(P)D(k)RD(k)\mathscr{T}(P)^T - \eta\left(\mathscr{T}(k) - \mathscr{T}(P)\right)\\
&\qquad\times\left[C\overline{\widetilde{Z}}(k \mid k-1)C^T + D(k)RD(k)^T\right]\\
&\qquad\times\left(\mathscr{T}(k) - \mathscr{T}(P)\right)^T.
\end{aligned} \tag{63}$$

Suppose that $P(k) \geq \overline{\overline{Z}}(k \mid k-1)$, we have that

$$\begin{aligned} P(k+1) - \overline{\overline{Z}}(k+1 \mid k) &= \left(A + \sqrt{\eta}\mathcal{T}(P)C\right) \\ &\quad \times \left(P(k) - \overline{\overline{Z}}(k \mid k-1)\right)\left(A + \sqrt{\eta}\mathcal{T}(P)C\right)^T \\ &\quad + \eta\left(\mathcal{T}(k) - \mathcal{T}(P)\right) \\ &\quad \times \left(C\overline{\overline{Z}}(k \mid k-1)C^T + D(k)RD(k)^T\right) \\ &\quad \times \left(\mathcal{T}(k) - \mathcal{T}(P)\right)^T. \end{aligned} \tag{64}$$

By definition, $P(0) = \widetilde{Z}(0 \mid 1)$. Suppose that $P(k) \geq \overline{\overline{Z}}(k \mid k-1)$. From (64), we have that $P(k+1) \geq \overline{\overline{Z}}(k+1 \mid k)$. Therefore we have shown by induction that $P(k) \geq \overline{\overline{Z}}(k \mid k-1)$ for all $k = 0, 1, 2 \cdots$. From MSS and ergodicity of the Markov chain we have that $\Gamma(k) \xrightarrow{k\to\infty} \Gamma$, $D(k) \xrightarrow{k\to\infty} D$, and $B(k) \xrightarrow{k\to\infty} B$ exponentially fast. From $r_\sigma(A+\sqrt{\eta}\mathcal{T}(P)C) < 1$ and same reasoning as in the proof of proposition 3.36 in [35] we have that $P(k) \to P$ as $k \to \infty$, where P satisfies

$$\begin{aligned} \bar{P} = &\left(A + \sqrt{\eta}\mathcal{T}(P)C\right)\bar{P}\left(A + \sqrt{\eta}\mathcal{T}(P)C\right)^T \\ &+ \Psi(\Gamma) + BQB^T + \eta\mathcal{T}(P)DRD^T\mathcal{T}(P)^T. \end{aligned} \tag{65}$$

And P is the unique solution to (65). Recalling that P satisfies (62), we get that P is also a solution to (65) and from uniqueness, $\bar{P} = P$. Then, we obtain that

$$\overline{\overline{Z}}(k \mid k-1) \leq P. \tag{66}$$

And $P(k) \to P$. From (66) and (56) in Lemma 10 it follows that $0 \leq \Upsilon(k) \leq \Upsilon(k+1) \leq P(k+1+\kappa)$. And thus we can conclude that $\Upsilon(k) \to \Upsilon$ whenever $k \to \infty$ for some $\Upsilon \geq 0$. Moreover, from the fact that $\alpha_i(k) \xrightarrow{k\to\infty} \mu_i$ and $\bar{\Gamma}(k) \xrightarrow{k\to\infty} \Gamma$, we have that Υ satisfies (60).

From uniqueness of the positive-semidefinite solution to (60), we can conclude that $\Upsilon = P$. From (66) and (56), $\Upsilon(k) \leq \overline{\overline{Z}}(k+\kappa \mid k+\kappa-1) \leq P(k+\kappa)$ and since $\Upsilon(k) \to P$ and $P(k) \to P$ as $k \to \infty$, we get that $\overline{\overline{Z}}(k \mid k-1) \xrightarrow{k\to\infty} P$. □

The upper bound P for the error covariance matrix to a stationary value for linear minimum mean square error (LMMSE) estimation can be easily obtained. It is described that if the system is MSS and the missing information is detected, then the error covariance matrix will converge to the unique nonnegative definite solution of an algebraic Riccati equation associated with the problem.

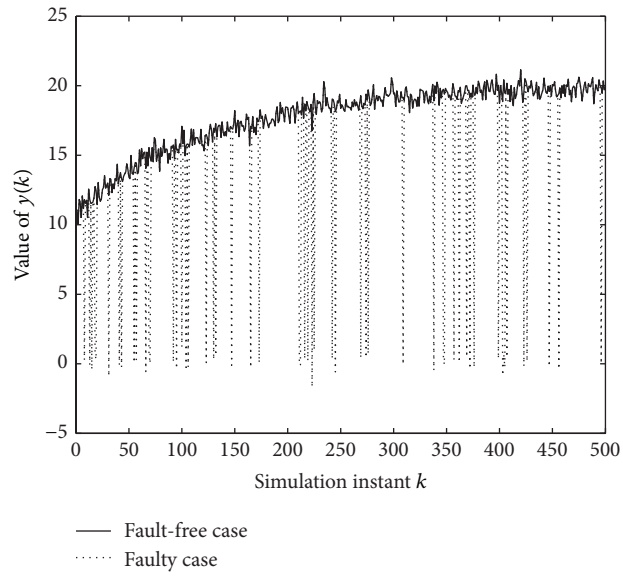

Figure 1: Value of observations.

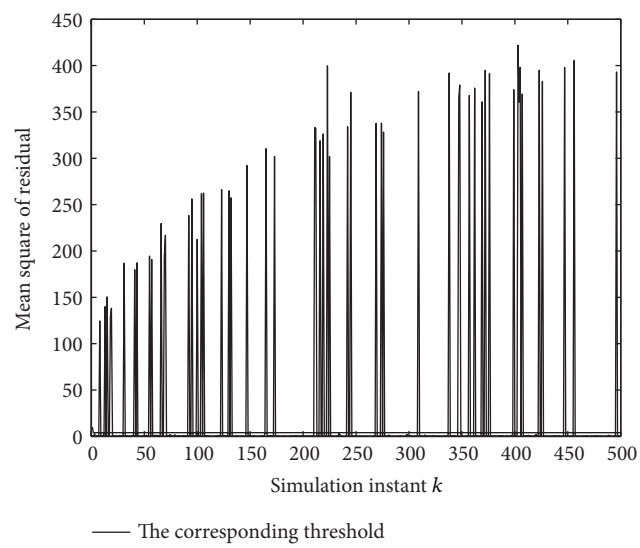

Figure 2: Mean square of residual.

5. Numerical Example

In order to evaluate the performance of our method, in this section, we are going to use a scalar MJLS described by the following equations:

$$x_{k+1} = A_{r_k}x_k + B_{r_k}\left(a_{r_k} + w_k\right)$$

$$y_k = \gamma_k C_{r_k}x_k + D_{r_k}v_k$$

$$A_1 = \begin{pmatrix} 1 & 0.995 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0.99 \\ 0 & 1 \end{pmatrix},$$

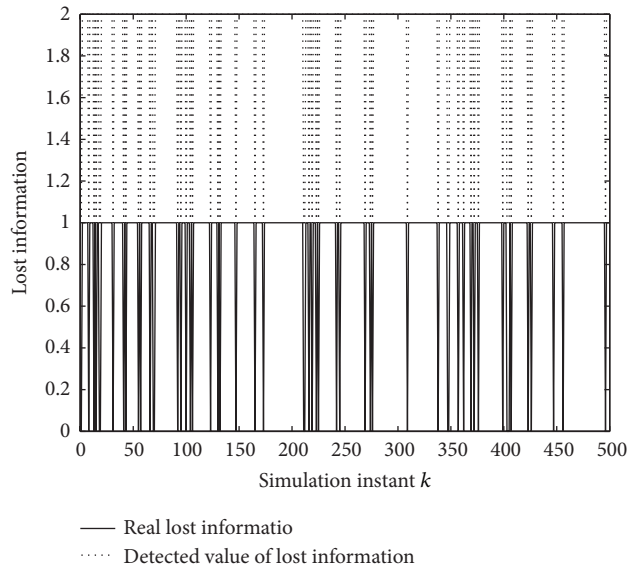

Figure 3: Detection of fault.

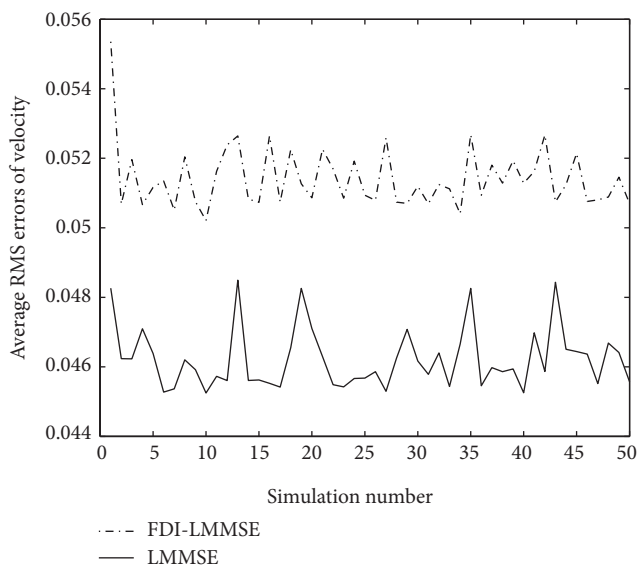

Figure 4: Average RMS target position error.

$$B_1 = B_2 = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \qquad C_1 = C_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$D_1 = D_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \qquad a_1 = 1, \qquad a_2 = 2, \tag{67}$$

where $x_k(1,1)$, $x_k(2,1)$, and a_k denote the target position, velocity, and acceleration, respectively. The initial state x_0 is normally distributed with mean 10 and variance 1. $r_k \in \{1,2\}$, and w_k, v_k are independent white noise sequences with

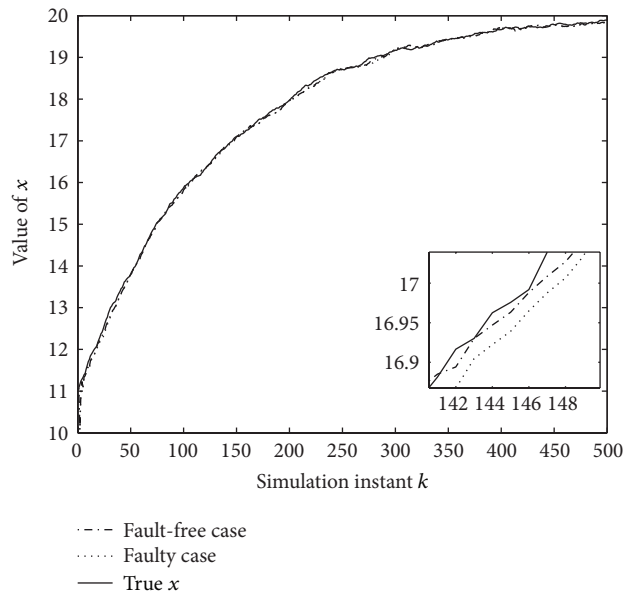

Figure 5: Comparison between state estimators of $x(1)$ under no faulty case and faulty case, respectively, and real state value.

covariance of 0.1^2, and $\mu_0^1 = \mu_0^2 = 0.5$. The transition probability matrix for the finite-state Markov chain is

$$\Pi = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix},$$
$$P(\gamma_k = 1) = 0.9, \qquad P(\gamma_k = 0) = 0.1. \tag{68}$$

To assess the performance of algorithms, the average root mean square (RMS) error based on H times Monte-Carlo simulation is defined as

$$\mathrm{RMS} = \frac{1}{H}\frac{1}{T}\sum_{i=1}^{H}\sum_{k=1}^{T}\left[\left(x_k^i - \hat{x}_k^i\right)^2\right]^{1/2}, \tag{69}$$

where the time step T is chosen as 500, $H = 50$.

The simulation results are obtained as follows. Figure 5 presents the real states and their estimators subject to fault-free case and faulty case, respectively, based on the given path. Figure 1 shows the observations with lost data from unreliable channel and observations from reliable channel. As the proposed algorithm can be thought of as a generalization of the well-known LMMSE filtering, we denote it by FDI-LMMSE filtering in the simulation. The RMS in the position of FDI-LMMSE filtering in the faulty case is compared with that of LMMSE filtering in the fault-free case in the Figure 4. It can be shown in Figures 2 and 3 that the residual can deliver fault alarms soon after the fault occurs. From the simulation results, we can see that the obtained linear estimator for systems with random missing data are tracking well to the real state value, which is the estimation scheme proposed in this paper produces good performance.

6. Conclusions

This paper has addressed the estimation problem for MJLSs with random missing data. Random missing data introduced by the network is modeled as Bernoulli distribution variable. By usage of an observer-based FDI as a residual generator, the design of FDI-LMMSE filter has been formulated in the framework of LMMSE filtering. Complete analytical solution has been obtained by solving the recursive Riccati equations. It has been proved from theorem derivation and a numerical example simulation that the proposed state estimator is effective.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

State Estimation for Time-Delay Systems with Markov Jump Parameters and Missing Measurements

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This paper is concerned with the state estimation problem for a class of time-delay systems with Markovian jump parameters and missing measurements, considering the fact that data missing may occur in the process of transmission and its failure rates are governed by random variables satisfying certain probabilistic distribution. By employing a new Lyapunov function and using the convexity property of the matrix inequality, a sufficient condition for the existence of the desired state estimator for Markovian jump systems with missing measurements can be achieved by solving some linear matrix inequalities, which can be easily facilitated by using the standard numerical software. Furthermore, the gain of state estimator can also be derived based on the known conditions. Finally, a numerical example is exploited to demonstrate the effectiveness of the proposed method.

1. Introduction

As a class of multimodal systems, Markovian jump systems (MJSs) have received considerable attention in the past two decades [1–6]. The system parameters usually jump in a finite mode set, in which the transitions among different modes are governed by a Markov chain. Due to the fact that many dynamical systems subject to random abrupt variations can be modeled by MJSs, many applications of MJSs can be showed, such as power systems, failure prone manufacturing systems, communication systems, biochemical systems with diverse changes of environmental conditions, and economy system. Quite a number of useful results have been extensively studied, such as stability and stabilization, robust control, optimal control, H_∞ control, synchronization, H_∞ filtering, and sliding mode control [7–19]. For example, the author in [7] studied the problem of unbiased estimation of Markov jump systems with distributed delays, and sufficient conditions are obtained for the unbiased H_∞ filtering scheme to MJSs by stochastic Lyapunov-Krasovskii functional framework. The author in [8] considered robust H_∞ control problems for stochastic fuzzy neutral MJSs with parameters uncertainties and multiple time delays, and a sufficient condition and H_∞ control criteria are formulated in the form of linear matrix inequalities by selecting appropriate Lyapunov functions. In term of the peak-to-peak filtering problem for a class of MJSs with uncertain parameters, the author in [9] investigated it. Sufficient conditions that the solution of the peak-to-peak filter existed are given by using the constructed Lyapunov functional and linear matrix inequalities. More details on this topic can be found in [20] and the references therein.

In recent years, due to the fact that, for many practical state estimation applications, the problem of state estimation with linear or nonlinear time-delay systems has received much attention, it is of great significance to estimate systems states and then utilize the estimated systems states to achieve certain design objectives. At the same time, in the procedure of state estimator design, time delays cannot be neglected and their existence often results in a poor performance. Some nice results on state estimation for time-delay systems have been showed in the literature [21–23]. Meanwhile, some state estimation problem for JMSs has been hot topics so that many important results have been reported in the literature

[16, 24, 25]. The author in [16] studied the state estimation and sliding-mode control problems for continuous-time Markovian jump singular systems with unmeasured states. The author in [24] concerned the problem of H_∞ estimation for a class of Markov jump linear systems (MJLSs) with time-varying transition probabilities in discrete-time domain. In [25] efficient simulation-based algorithms called particle filters were used to solve the optimal state estimates for a class of jump Markov linear systems. The author in [26] considered state estimation for Markovian jumping delayed continuous-time recurrent neural networks, where only matrix parameters were mode-dependent. Different from the studies [26, 27] studied state estimation problem for a class of discrete-time neural networks with Markovian jumping parameters and mode-dependent mixed time delay, where the discrete and distributed delays were mode-dependent.

Recently, Liu et al. [28] studied the H_∞ filter design for Markovian jump systems with time-varying delays. However, these papers do not consider the data missing of sensor in the process of transmission. Motivated by the idea of above papers, we will investigate the problem of state estimation for Markovian jump systems with both time delays and missing measurements. This work is not a simple extension of [28] to MJSs. Our main difficulties come from the state estimator design and missing measurements analysis for the MJSs. Thus, how to design an appropriate state estimator and how to establish a sufficient condition for the existence of the desired state estimator derived are the key problems to be solved. Based on the above analysis, in this paper, we studied state estimator design for MJSs with both missing measurements and time delays via employing a new Lyapunov function and using the convexity property of the matrix inequality. With the proposed method, we established a sufficient condition for the existence of the desired state estimator. Furthermore, the problem of state estimator design is studied; that is, an observer is designed for the MJSs with missing measurements to estimate the states.

In this paper, the problem of state estimator design for MJSs with interval time-varying delay is narrated. A new Lyapunov function is established to obtain less conservative results, in which the lower and upper delay bound of interval time-varying delay is included. Based on above analysis, the item $\int_{t-\tau_{(t)}}^{t} e^T(s)Q_2(\theta_t)e(s)ds$ can depart into two parts to deal with, respectively, and the convexity of the matrix functions is used to avoid the conservative caused by enlarging $\tau(t)$ to τ_M in the deriving results.

The rest of this paper is organized as follows. Section 2 presents the problem statement and preliminaries. An LMI-based sufficient condition for the existence of the desired state estimator derived is proposed in Section 3. A numerical example is provided in Section 4 and we conclude this paper in Section 5.

$\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of $n \times m$ real matrices; the superscript "T" represents matrix transposition; $\|\cdot\|$ represents the Euclidean vector norm or the induced matrix 2-norm as appropriate; I is the identity matrix of appropriate dimension. $\mathbb{E}\{x\}$ represents the expectation of x when x is a stochastic variable. $\left[\begin{smallmatrix} A & * \\ B & C \end{smallmatrix}\right]$ denote a symmetric matrix, where $*$ denotes the entries implied by symmetry, for a matrix B and two symmetric matrices A and C. The notation $X > 0$ (resp., $X \geq 0$), for $X \in \mathbb{R}^{n\times n}$, means that the matrix X is real symmetric positive definite (resp., positive semidefinite).

2. Problem Statement and Preliminaries

Fix a probability space (Ω, $\mathscr{F}$, and $\mathscr{P}$) and consider the following class of uncertain linear stochastic systems with Markovian jump parameters and time-varying delays:

$$\begin{aligned}
\dot{x}(t) &= A(\theta_t)x(t) + A_d(\theta_t)x(t-\tau(t)) + A_\omega(\theta_t)\omega(t),\\
y(t) &= C(\theta_t)x(t) + C_d(\theta_t)x(t-\tau(t)) + C_\omega(\theta_t)\omega(t),\\
z(t) &= L(\theta_t)x(t) + L_d(\theta_t)x(t-\tau(t)) + L_\omega(\theta_t)\omega(t),\\
x(t) &= \phi(t), \quad \forall t \in [-\tau_M, -\tau_m].
\end{aligned} \tag{1}$$

$x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^r$ is the measurement vector, $z(t) \in \mathbb{R}^p$ is the signal to be estimated, $\omega(t) \in L_2[0,\infty)$ is the exogenous disturbance signal, and $\{\theta_t\}$ is a continuous-time Markovian process which has right continuous trajectories and takes values in a finite set $\mathscr{S} = \{1, 2, \ldots, \mathscr{N}\}$ with stationary transition probabilities:

$$\operatorname{Prob}\{\theta_{t+h} = j \mid \theta_t = i\} = \begin{cases} \pi_{ij}h + o(h), & i \neq j,\\ 1 + \pi_{ii}h + o(h), & i = j, \end{cases} \tag{2}$$

where $h > 0$, $\lim_{h\to 0}(o(h)/h) = 0$, and $\pi_{ij} \geq 0$, for $j \neq i$ is the transition rate from mode i at time t to the mode j at time $t + h$ and

$$\pi_{ii} = -\sum_{j=1,\, j\neq i}^{N} \pi_{ij}. \tag{3}$$

In the system (1), the time delay $\tau(t)$ is a time-varying continuous function satisfying the following assumption:

$$0 \leq \tau_m \leq \tau(t) \leq \tau_M < \infty, \quad \dot{\tau}(t) \leq \mu, \quad \forall t > 0, \tag{4}$$

where τ_M is the upper bound and τ_m is the lower bound of the communication delay, and μ is the upper bound of change rate of communication delay.

When considering the data missing in the sensor channel, the actual output of sensor measurements in system (1) can be described as

$$\begin{aligned}
\tilde{y}(t) &= \Xi y(t)\\
&= \Xi\left[C(\theta_t)x(t) + C_d(\theta_t)x(t-\tau(t)) + C_\omega(\theta_t)\omega(t)\right],
\end{aligned} \tag{5}$$

where $\Xi = \operatorname{diag}\{\xi_1, \xi_2, \ldots, \xi_m\} = \sum_{i=1}^{m} \xi_i K_i$, $K_i = \operatorname{diag}\{\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{m-i}\}$, and ξ_i ($i = 1, 2, \ldots, m$) are unrelated stochastic variables taking values in $[0, 1]$. The mathematical expectation and variance of ξ_i are $\bar{\xi}_i$ and σ_i^2, respectively.

Remark 1. It can be seen from (5) that stochastic Ξ is introduced to reflect the unreliable sensors, which describes the status of the whole sensor and has been extensively studied in the literature such as [29–33]. Generally speaking, different sensor has different failure rate. So it is reasonable to assume that the failure rate for each individual sensor satisfies individual probabilistic distribution, and the elements ξ_i ($i = 1, 2, \ldots, m$) of the random matrix Ξ correspond to the status of the ith sensor. At one moment, if $\xi_i = 1$, it indicates that the ith sensor is well working; if $\xi_i = 0$, it indicates that ith sensor fails completely or data missing in the sensor channel; if $\xi_i \in (0, 1)$, it means that the ith sensor fails partly. Therefore, while $\Xi = \operatorname{diag}\{1, 1, \ldots, 1\}$, it means the status of the whole sensor is in good working condition. Thus the model which we will establish in this paper is more general.

In this paper, considering the data missing of sensor in the process of information communication and based on the measurement $\tilde{y}(t)$, we consider the following state estimator for system (1):

$$\begin{aligned}
\dot{\hat{x}}(t) &= A(\theta_t)\hat{x}(t) + A_d(\theta_t)\hat{x}(t-\tau(t)) \\
&\quad + G(\theta_t)(y_1(t) - \tilde{y}(t)), \\
\hat{y}(t) &= C(\theta_t)\hat{x}(t) + C_d(\theta_t)\hat{x}(t-\tau(t)), \\
\hat{z}(t) &= L(\theta_t)\hat{x}(t) + L_d(\theta_t)\hat{x}(t-\tau(t)),
\end{aligned} \tag{6}$$

where $y_1(t) = \Xi\hat{y}(t) = \Xi[C(\theta_t)\hat{x}(t) + C_d(\theta_t)\hat{x}(t-\tau(t))]$.

Remark 2. Similar to (5), we also consider the data missing of sensor in the process of information communication for the system (6) of state estimation.

The set $\mathcal{S}$ contains the various operation modes of system (1) and, for each possible value of $\theta_t = i$, $i \in \mathcal{S}$, the matrices connected with "ith mode" will be denoted by

$$\begin{aligned}
&A_i := A(\theta_t = i), \qquad A_{di} := A_d(\theta_t = i), \\
&A_{\omega i} := A_\omega(\theta_t = i), \\
&C_i := C(\theta_t = i), \qquad C_{di} := C_d(\theta_t = i), \\
&C_{\omega i} := C_\omega(\theta_t = i), \\
&L_i := L(\theta_t = i), \qquad L_{di} := L_d(\theta_t = i), \\
&L_{\omega i} := L_\omega(\theta_t = i),
\end{aligned} \tag{7}$$

where $A_i, A_{di}, A_{\omega i}, C_i, C_{di}, C_{\omega i}, L_i, L_{di}$, and $L_{\omega i}$ are constant matrices for any $i \in \mathcal{S}$. In this paper we assume that the jumping process $\{\theta_t\}$ is accessible; that is, the operation mode of system (1) is known for every $t \geq 0$.

Set the estimation error $e(t) = \hat{x}(t) - x(t)$ and $\tilde{z}(t) = \hat{z}(t) - z(t)$. Then the following error dynamics of the state estimation system will be showed as follows:

$$\begin{aligned}
\dot{e}(t) &= \bar{A}_i e(t) + \bar{A}_{di} e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t) \\
&\quad + G_i(\Xi - \bar{\Xi}) C_i e(t) + G_i(\Xi - \bar{\Xi}) C_{di} e(t-\tau(t)) \\
&\quad - G_i(\Xi - \bar{\Xi}) C_{\omega i}\omega(t), \\
\tilde{z}(t) &= L_i e(t) + L_{di} e(t-\tau(t)) - L_{\omega i}\omega(t),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&\bar{A}_i = A_i + G_i\bar{\Xi}C_i, \qquad \bar{A}_{di} = A_{di} + G_i\bar{\Xi}C_{di}, \\
&\bar{A}_{\omega i} = -A_{\omega i} - G_i\bar{\Xi}C_{\omega i}.
\end{aligned} \tag{9}$$

The state estimation problem which is addressed in this paper is to design a state estimator in form of (8) such that

(i) the estimation error system (8) with $\omega(t) = 0$ is exponentially stable;

(ii) the H_∞ performance $\|\tilde{z}(t)\|_2 < \gamma\|\omega\|_2$ is sure for all nonzero $\omega(t) \in L_2[0, \infty)$ and a prescribed $\gamma > 0$ under the condition $e(t) = 0$, for all $t \in [-\tau_M, -\tau_m]$.

Before giving the main results, the following lemmas and definitions are needed in the proof of our main results.

Lemma 3 (see [34]). *For any constant matrix $R \in \mathbb{R}$, $R = R^T > 0$, vector function $x : [-\tau_M, 0] \to \mathbb{R}^n$, and constant $\tau_M > 0$ such that the following integration is well defined; then the following inequality holds:*

$$\begin{aligned}
&-\tau_M \int_{t-\tau_M}^{t} \dot{x}^T(s) R \dot{x}(s)\, ds \\
&\quad \leq \begin{bmatrix} x(t) \\ x(t-\tau_M) \end{bmatrix}^T \begin{bmatrix} -R & R \\ R & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_M) \end{bmatrix}.
\end{aligned} \tag{10}$$

Lemma 4 (see [35]). *Suppose Ξ_1, Ξ_2, and Ω are constant matrices of appropriate dimensions, $0 \leq \tau_m \leq \tau(t) \leq \tau_M$; then*

$$(\tau(t) - \tau_m)\Xi_1 + (\tau_M - \tau(t))\Xi_2 + \Omega < 0 \tag{11}$$

if and only if the following two inequalities hold:

$$\begin{aligned}
&(\tau_M - \tau_m)\Xi_1 + \Omega < 0, \\
&(\tau_M - \tau_m)\Xi_2 + \Omega < 0.
\end{aligned} \tag{12}$$

Definition 5. The system (8) is considered to be exponentially stable in the mean-square sense (EMSS), if there exist constants $\lambda > 0$, $\alpha > 0$, such that $t > 0$:

$$E\left\{\|x(t)\|^2\right\} \leq \alpha e^{-\lambda t} \sup_{-\tau_M < s < 0} \left\{\|\phi(s)\|^2\right\}. \tag{13}$$

Definition 6. For a given function $V : C^b_{F_0}([-\tau_M, 0], R^n) \times S \to R$, its infinitesimal operator $\mathscr{L}$ [36] is defined as

$$\mathscr{L}V(x_t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left[E\left(V\left(x_{t+\Delta} \mid x_t \right) - V\left(x_t \right) \right) \right]. \tag{14}$$

3. Main Results

Theorem 7. *For some given constants $0 \le \tau_m \le \tau_M$ and γ, the system (8) is exponentially mean-square stable (EMSS) with a prescribed H_∞ performance γ, if there exist $P_i > 0$, $Q_0 > 0$, $Q_1 > 0$, $Q_{2i} > 0$, $R_0 > 0$, $R_1 > 0$, $Z_1 > 0$, $Z_2 > 0$, $M_{ik} > 0$, and $N_{ik} > 0$ $(i \in \mathcal{S}, k = 1, 2, \ldots, 5)$ with appropriate dimensions, so that the following matrix inequalities hold:*

$$\Psi = \begin{bmatrix} \Psi_{11} & * & * & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * & * & * \\ \Psi_{41}(s) & \Psi_{42}(s) & 0 & -R_1 & * & * \\ \Psi_{51} & \Psi_{52} & 0 & 0 & \Psi_{55} & * \\ \Psi_{61} & \Psi_{62} & 0 & 0 & 0 & \Psi_{66} \end{bmatrix} < 0, \quad s = 1, 2, \tag{15}$$

$$\sum_{j=1}^{N} \pi_{ij} Q_{2j} \le Z_k, \quad k = 1, 2, \tag{16}$$

where

$$\begin{aligned}
\Psi_{11} &= P_i \bar{A}_i + \bar{A}_i^T P_i + Q_0 + Q_1 + Q_{2i} - R_0 + \tau_m Z_1 + \delta Z_2 + \sum_{j=1}^{N} \pi_{ij} P_j, \\
\Psi_{21} &= \left[P_i^T \bar{A}_{di} - M_{i1} + N_{i1}, R_0^T + M_{i1}, -N_{i1}, P_i^T \bar{A}_{\omega i} \right]^T, \\
\Psi_{22} &= \begin{bmatrix} -(1-\mu) Q_{2i} - M_{i2} - M_{i2}^T + N_{i2} + N_{i2}^T & * & * & * \\ -M_{i3} + M_{i2}^T + N_{i3} & -Q_0 - R_0 + M_{i3} + M_{i3}^T & * & * \\ -M_{i4} + N_{i4} - N_{i2}^T & M_{i4} - N_{i3}^T & -Q_1 - N_{i4} - N_{i4}^T & * \\ -M_{i5} + N_{i5} & M_{i5} & -N_{i5} & -\gamma^2 I \end{bmatrix}, \\
\Psi_{31} &= \begin{bmatrix} \tau_m R_0 \bar{A}_i \\ \sqrt{\delta} R_1 \bar{A}_i \\ L_i \end{bmatrix}, \qquad \Psi_{32} = \begin{bmatrix} \tau_m R_0 \bar{A}_{di} & 0 & 0 & \tau_m R_0 \bar{A}_{\omega i} \\ \sqrt{\delta} R_1 \bar{A}_{di} & 0 & 0 & \sqrt{\delta} R_1 \bar{A}_{\omega i} \\ L_{di} & 0 & 0 & -L_{\omega i} \end{bmatrix}, \qquad \Psi_{33} = \operatorname{diag}\{-R_0, -R_1, -I\}, \\
\Psi_{41}(1) &= \sqrt{\delta} M_{i1}^T, \qquad \Psi_{41}(2) = \sqrt{\delta} N_{i1}^T, \qquad \delta = \tau_M - \tau_m, \\
\Psi_{42}(1) &= \left[\sqrt{\delta} M_{i2}^T \ \ \sqrt{\delta} M_{i3}^T \ \ \sqrt{\delta} M_{i4}^T \ \ \sqrt{\delta} M_{i5}^T \right], \qquad \Psi_{42}(2) = \left[\sqrt{\delta} N_{i2}^T \ \ \sqrt{\delta} N_{i3}^T \ \ \sqrt{\delta} N_{i4}^T \ \ \sqrt{\delta} N_{i5}^T \right], \\
\Psi_{51} &= \left[\tau_m \sigma_1 C_i^T K_1^T G_i^T R_0^T, \tau_m \sigma_2 C_i^T K_2^T G_i^T R_0^T, \ldots, \tau_m \sigma_m C_i^T K_m^T G_i^T R_0^T, 0, \ldots, 0 \right]^T, \\
\Psi_{52} &= \left[\Theta_{5d}, 0, 0, \Theta_{5\omega} \right], \qquad \Psi_{55} = \operatorname{diag}\{-R_0, -R_0, \ldots, -R_0\}, \\
\Psi_{61} &= \left[\sqrt{\delta} \sigma_1 C_i^T K_1^T G_i^T R_1^T, \sqrt{\delta} \sigma_2 C_i^T K_2^T G_i^T R_1^T, \ldots, \sqrt{\delta} \sigma_m C_i^T K_m^T G_i^T R_1^T, 0, \ldots, 0 \right]^T, \\
\Psi_{62} &= \left[\Theta_{6d}, 0, 0, \Theta_{6\omega} \right], \qquad \Psi_{66} = \operatorname{diag}\{-R_1, -R_1, \ldots, -R_1\}, \\
\Theta_{5d} &= \left[0, \tau_m \sigma_1 C_{di}^T K_1^T G_i^T R_0^T, \tau_m \sigma_2 C_{di}^T K_2^T G_i^T R_0^T, \ldots, \tau_m \sigma_m C_{di}^T K_m^T G_i^T R_0^T, 0, \ldots, 0 \right]^T, \\
\Theta_{5\omega} &= \left[0, \ldots, 0, -\tau_m \sigma_1 C_{\omega i}^T K_1^T G_i^T R_0^T, -\tau_m \sigma_2 C_{\omega i}^T K_2^T G_i^T R_0^T, \ldots, -\tau_m \sigma_m C_{\omega i}^T K_m^T G_i^T R_0^T \right]^T, \\
\Theta_{6d} &= \left[\sqrt{\delta} \sigma_1 C_{di}^T K_1^T G_i^T R_1^T, \sqrt{\delta} \sigma_2 C_{di}^T K_2^T G_i^T R_1^T, \ldots, \sqrt{\delta} \sigma_m C_{di}^T K_m^T G_i^T R_1^T, 0, \ldots, 0 \right]^T, \\
\Theta_{6\omega} &= \left[0, \ldots, 0, -\sqrt{\delta} \sigma_1 C_{\omega i}^T K_1^T G_i^T R_1^T, -\sqrt{\delta} \sigma_2 C_{\omega i}^T K_2^T G_i^T R_1^T, \ldots, -\sqrt{\delta} \sigma_m C_{\omega i}^T K_m^T G_i^T R_1^T \right]^T,
\end{aligned} \tag{17}$$

Proof. Introduce a new vector

$$\zeta^T(t) = \left[e^T(t) \ \ e^T(t - \tau(t)) \ \ e^T(t - \tau_m) \ \ e^T(t - \tau_M) \ \ \omega^T(t) \right] \tag{18}$$

and two matrices

$$\begin{aligned}
\Gamma_1 &= \left[\bar{A}_i \ \ \bar{A}_{di} \ \ 0 \ \ 0 \ \ \bar{A}_{\omega i} \right], \\
\Gamma_2 &= \left[L_i \ \ L_{di} \ \ 0 \ \ 0 \ \ -L_{\omega i} \right].
\end{aligned} \tag{19}$$

The system (8) can be rewritten as

$$\begin{aligned}\dot{e}(k) &= \Gamma_1\zeta(t) + G_i\left(\Xi - \bar{\Xi}\right)C_i e(t) \\ &\quad + G_i\left(\Xi - \bar{\Xi}\right)C_{di}e(t-\tau(t)) - G_i\left(\Xi - \bar{\Xi}\right)C_{\omega i}\omega(t), \\ \tilde{z}(t) &= \Gamma_2\zeta(t).\end{aligned} \tag{20}$$

Let $x_t(s) = x(t+s), (-\tau(t) \le s \le 0)$. Then, the same as [37], $\{(x_t, \theta_t), t \ge 0\}$ is a Markov process. Choose the following Lyapunov functional candidate:

$$V(x_t, \theta_t) = \sum_{i=1}^{4} V_i(x_t, \theta_t), \tag{21}$$

where

$$\begin{aligned}V_1(x_t, \theta_t) &= e^T(t)P(\theta_t)e(t), \\ V_2(x_t, \theta_t) &= \int_{t-\tau_m}^{t} e^T(s)Q_0e(s)\,ds + \int_{t-\tau_M}^{t} e^T(s)Q_1e(s)\,ds \\ &\quad + \int_{t-\tau_{(t)}}^{t} e^T(s)Q_2(\theta_t)e(s)\,ds, \\ V_3(x_t, \theta_t) &= \tau_m\int_{t-\tau_m}^{t}\int_{s}^{t}\dot{e}^T(v)R_0\dot{e}(v)\,dv\,ds \\ &\quad + \int_{t-\tau_M}^{t-\tau_m}\int_{s}^{t}\dot{e}^T(v)R_1\dot{e}(v)\,dv\,ds, \\ V_4(x_t, \theta_t) &= \int_{t-\tau_m}^{t}\int_{s}^{t} e^T(v)Z_1e(v)\,dv\,ds \\ &\quad + \int_{t-\tau_M}^{t-\tau_m}\int_{s}^{t} e^T(v)Z_2e(v)\,dv\,ds.\end{aligned} \tag{22}$$

Let $\mathscr{L}$ be the weak infinite generator of the random process $\{x_t, \theta_t\}$. Then, for each $\theta_t = i,\ i \in \mathcal{S}$, taking expectation on it, we obtain

$$\begin{aligned}&\mathbb{E}\{\mathscr{L}V(x_t, \theta_t)\} \\ &\quad \le e^T(t)\Bigg(2P_i\bar{A}_i \\ &\qquad + \sum_{j=1}^{N}\pi_{ij}P_j + Q_0 + Q_1 + Q_{2i} + \tau_m Z_1 + \delta Z_2\Bigg)e(t) \\ &\qquad + 2e^T(t)P_i\bar{A}_{di}e(t-\tau(t)) + 2e^T(t)P_i\bar{A}_{\omega i}\omega(t) \\ &\qquad - e^T(t-\tau_m)Q_0e(t-\tau_m) - \int_{t-\tau_m}^{t} e^T(s)Z_1e(s)\,ds \\ &\qquad + \mathbb{E}\left\{\delta\dot{e}^T(t)R_1\dot{e}(t)\right\} - \tau_m\int_{t-\tau_m}^{t}\dot{e}^T(s)R_0\dot{e}(s)\,ds \\ &\qquad - (1-\mu)e^T(t-\tau(t))Q_{2i}e(t-\tau(t)) \\ &\qquad + \int_{t-\tau_{(t)}}^{t} e^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)e(s)\,ds \\ &\qquad + \mathbb{E}\left\{\tau_m^2\dot{e}^T(t)R_0\dot{e}(t)\right\} - \int_{t-\tau_M}^{t-\tau_m}\dot{e}^T(s)R_1\dot{e}(s)\,ds \\ &\qquad - e^T(t-\tau_M)Q_1e(t-\tau_M) - \int_{t-\tau_M}^{t-\tau_m} e^T(s)Z_2e(s)\,ds.\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}&\int_{t-\tau_{(t)}}^{t} e^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)e(s)\,ds \\ &\quad = \int_{t-\tau_{(t)}}^{t-\tau_m} e^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)e(s)\,ds \\ &\qquad + \int_{t-\tau_m}^{t} e^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)e(s)\,ds.\end{aligned} \tag{24}$$

From (16) and (24), we can derive that

$$\begin{aligned}&\int_{t-\tau_{(t)}}^{t} e^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)e(s)\,ds - \int_{t-\tau_m}^{t} e^T(s)Z_1e(s)\,ds \\ &\quad - \int_{t-\tau_M}^{t-\tau_m} e^T(s)Z_2e(s)\,ds \\ &\quad = \int_{t-\tau_m}^{t} e^T(s)\left[\sum_{j=1}^{N}\pi_{ij}Q_{2j} - Z_1\right]e(s)\,ds \\ &\qquad + \int_{t-\tau_{(t)}}^{t-\tau_m} e^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)e(s)\,ds \\ &\qquad - \int_{t-\tau_M}^{t-\tau_m} e^T(s)Z_2e(s)\,ds \\ &\quad \le \int_{t-\tau_m}^{t} e^T(s)\left[\sum_{j=1}^{N}\pi_{ij}Q_{2j} - Z_1\right]e(s)\,ds \\ &\qquad + \int_{t-\tau_{(t)}}^{t-\tau_m} e^T(s)\left[\sum_{j=1}^{N}\pi_{ij}Q_{2j} - Z_2\right]e(s)\,ds < 0.\end{aligned} \tag{25}$$

It follows from Lemma 3 that

$$-\tau_m \int_{t-\tau_m}^{t} \dot{e}^T(s) R_0 \dot{e}(s)\, ds \le \begin{bmatrix} e(t) \\ e(t-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R_0 & R_0 \\ R_0 & -R_0 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau_m) \end{bmatrix}. \tag{26}$$

Combining ((23), (25), and (26)) and introducing slack matrices M_i, N_i, $i = 1, 2 \ldots 5$, we obtain

$$\begin{aligned}
&\mathbb{E}\{\mathscr{L}V(x_t,\theta_t)\} - \gamma^2\omega^T(t)\omega(t) + \tilde{z}^T(t)\tilde{z}(t) \\
&\le e^T(t)\Bigg(2P_i\bar{A}_i + \sum_{j=1}^{N}\pi_{ij}P_j + Q_0 + Q_1 + Q_{2i} \\
&\qquad + \tau_m Z_1 + \delta Z_2\Bigg)e(t) \\
&\quad + 2e^T(t)P_i\bar{A}_{di}e(t-\tau(t)) + 2e^T(t)P_i\bar{A}_{\omega i}\omega(t) \\
&\quad + \begin{bmatrix} e(t) \\ e(t-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R_0 & R_0 \\ R_0 & -R_0 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau_m) \end{bmatrix} \\
&\quad - e^T(t-\tau_M)Q_1 e(t-\tau_M) \\
&\quad - (1-\mu)e^T(t-\tau(t))Q_{2i}e(t-\tau(t)) \\
&\quad + \mathbb{E}\left\{\tau_m^2\dot{e}^T(t)R_0\dot{e}(t)\right\} - \int_{t-\tau_M}^{t-\tau_m}\dot{e}^T(s)R_1\dot{e}(s)\,ds \\
&\quad + \mathbb{E}\left\{\delta\dot{e}^T(t)R_1\dot{e}(t)\right\} - \gamma^2\omega^T(t)\omega(t) + \zeta(t)^T\Gamma_2^T\Gamma_2\zeta(t) \\
&\quad + 2\zeta^T(t)M_i\left[e(t-\tau_m) - e(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_m}\dot{e}(s)\,ds\right] \\
&\quad - e^T(t-\tau_m)Q_0 e(t-\tau_m) \\
&\quad + 2\zeta^T(t)N_i\Bigg[e(t-\tau(t)) - e(t-\tau_M) \\
&\qquad - \int_{t-\tau_M}^{t-\tau(t)}\dot{e}(s)\,ds\Bigg],
\end{aligned} \tag{27}$$

where $M_i^T = \begin{bmatrix} M_{i1}^T & M_{i2}^T & M_{i3}^T & M_{i4}^T & M_{i5}^T \end{bmatrix}$, $N_i^T = \begin{bmatrix} N_{i1}^T & N_{i2}^T & N_{i3}^T & N_{i4}^T & N_{i5}^T \end{bmatrix}$.

Note that

$$\begin{aligned}
&-2\zeta^T(t)M_i\int_{t-\tau(t)}^{t-\tau_m}\dot{e}(s)\,ds \\
&\quad \le \int_{t-\tau(t)}^{t-\tau_m}\dot{e}^T(s)R_1\dot{e}(s)\,ds \\
&\qquad + (\tau(t)-\tau_m)\zeta^T(t)M_iR_1^{-1}M_i^T\zeta(t), \\
&-2\zeta^T(t)N_i\int_{t-\tau_M}^{t-\tau(t)}\dot{e}(s)\,ds \\
&\quad \le \int_{t-\tau_M}^{t-\tau(t)}\dot{e}^T(s)R_1\dot{e}(s)\,ds \\
&\qquad + (\tau_M-\tau(t))\zeta^T(t)N_iR_1^{-1}N_i^T\zeta(t), \\
&\mathbb{E}\left\{\delta\dot{e}^T(t)R_1\dot{e}(t)\right\} \\
&\quad = \delta\left[\bar{A}_ie(t) + \bar{A}_{di}e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right]^T \\
&\qquad \times R_1\left[\bar{A}_ie(t) + \bar{A}_{di}e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right] \\
&\qquad + \delta e^T(t)C_i^T\sum_{i=1}^{m}\sigma_i^2K_i^TG_i^TR_1G_iK_iC_ie(t) \\
&\qquad + \delta e^T(t-\tau(t))C_{di}^T\sum_{i=1}^{m}\sigma_i^2K_i^TG_i^TR_1G_iK_iC_{di}e(t-\tau(t)) \\
&\qquad - \delta\omega^T(t)C_{\omega i}^T\sum_{i=1}^{m}\sigma_i^2K_i^TG_i^TR_1G_iK_iC_{\omega i}\omega(t),
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\mathbb{E}\left\{\tau_m^2\dot{e}^T(t)R_0\dot{e}(t)\right\} \\
&\quad = \tau_m^2\left[\bar{A}_ie(t) + \bar{A}_{di}e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right]^T \\
&\qquad \times R_0\left[\bar{A}_ie(t) + \bar{A}_{di}e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right] \\
&\qquad + \tau_m^2e^T(t)C_i^T\sum_{i=1}^{m}\sigma_i^2K_i^TG_i^TR_0G_iK_iC_ie(t) \\
&\qquad + \tau_m^2e^T(t-\tau(t))C_{di}^T\sum_{i=1}^{m}\sigma_i^2K_i^TG_i^TR_0G_iK_iC_{di}e(t-\tau(t)) \\
&\qquad - \tau_m^2\omega^T(t)C_{\omega i}^T\sum_{i=1}^{m}\sigma_i^2K_i^TG_i^TR_0G_iK_iC_{\omega i}\omega(t).
\end{aligned} \tag{29}$$

Combining (27)–(29), we can obtain

$$\begin{aligned}
&\mathbb{E}\{\mathscr{L}V(x_t,\theta_t)\} - \gamma^2\omega^T(t)\omega(t) + \tilde{z}^T(t)\tilde{z}(t) \\
&\le e^T(t)\Bigg(2P_i\bar{A}_i + \sum_{j=1}^{N}\pi_{ij}P_j + Q_0 + Q_1 + Q_{2i} + \tau_m Z_1
\end{aligned}$$

$$
\begin{aligned}
&+\delta Z_2 \Big) e(t) + 2e^T(t) P_i \bar{A}_{di} e(t-\tau(t)) \\
&+ 2e^T(t) P_i \bar{A}_{\omega i} \omega(t) \\
&+ \delta e^T(t-\tau(t)) C_{di}^T \sum_{i=1}^{m} \sigma_i^2 K_i^T G_i^T R_1 G_i K_i C_{di} e(t-\tau(t)) \\
&- e^T(t-\tau_m) Q_0 e(t-\tau_m) - e^T(t-\tau_M) Q_1 e(t-\tau_M) \\
&- (1-\mu) e^T(t-\tau(t)) Q_{2i} e(t-\tau(t)) \\
&+ \delta e^T(t) C_i^T \sum_{i=1}^{m} \sigma_i^2 K_i^T G_i^T R_1 G_i K_i C_i e(t) \\
&+ \delta \left[\bar{A}_i e(t) + \bar{A}_{di} e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right]^T \\
&\times R_1 \left[\bar{A}_i e(t) + \bar{A}_{di} e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right] \\
&- \delta \omega^T(t) C_{\omega i}^T \sum_{i=1}^{m} \sigma_i^2 K_i^T G_i^T R_1 G_i K_i C_{\omega i} \omega(t) \\
&+ \tau_m^2 e^T(t) C_i^T \sum_{i=1}^{m} \sigma_i^2 K_i^T G_i^T R_0 G_i K_i C_i e(t) \\
&+ \tau_m^2 e^T(t-\tau(t)) C_{di}^T \sum_{i=1}^{m} \sigma_i^2 K_i^T G_i^T R_0 G_i K_i C_{di} e(t-\tau(t)) \\
&- \tau_m^2 \omega^T(t) C_{\omega i}^T \sum_{i=1}^{m} \sigma_i^2 K_i^T G_i^T R_0 G_i K_i C_{\omega i} \omega(t) \\
&+ \tau_m^2 \left[\bar{A}_i e(t) + \bar{A}_{di} e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right]^T \\
&\times R_0 \left[\bar{A}_i e(t) + \bar{A}_{di} e(t-\tau(t)) + \bar{A}_{\omega i}\omega(t)\right] \\
&+ \begin{bmatrix} e(t) \\ e(t-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R_0 & R_0 \\ R_0 & -R_0 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau_m) \end{bmatrix} \\
&- \gamma^2 \omega^T(t)\omega(t) + \zeta(t)^T \Gamma_2^T \Gamma_2 \zeta(t) \\
&+ 2\zeta^T(t) M_i \left[x(t-\tau_m) - x(t-\tau(t))\right] \\
&+ 2\zeta^T(t) N_i \left[x(t-\tau(t)) - x(t-\tau_M)\right] \\
&+ (\tau(t)-\tau_m) \zeta^T(t) M_i R_1^{-1} M_i^T \zeta(t) \\
&+ (\tau_M - \tau(t)) \zeta^T(t) N_i R_1^{-1} N_i^T \zeta(t).
\end{aligned} \tag{30}
$$

By using Lemma 4 and Schur complement, it is easy to see that (15) and $s = 1, 2$ are sufficient conditions to guarantee

$$
\mathbb{E}\{\mathscr{L}V(x_t,\theta_t)\} - \gamma^2 \omega^T(t)\omega(t) + \tilde{z}^T(t)\tilde{z}(t) < 0. \tag{31}
$$

Then, the following inequality can be concluded:

$$
\mathbb{E}\{\mathscr{L}V(x_t,i,t)\} < -\lambda_{\min}(\Psi)\, \mathbb{E}\left\{\zeta^T(t)\zeta(t)\right\}. \tag{32}
$$

Define a new function as

$$
W(x_t,i,t) = e^{\epsilon t} V(x_t,i,t). \tag{33}
$$

Its infinitesimal operator $\mathscr{L}$ is given by

$$
\mathscr{W}(x_t,i,t) = \epsilon e^{\epsilon t} V(x_t,i,t) + e^{\epsilon t} \mathscr{L}V(x_t,i,t). \tag{34}
$$

By the generalized Itô formula [36], we can obtain from (34) that

$$
\begin{aligned}
&\mathbb{E}\{W(x_t,i,t)\} - \mathbb{E}\{W(x_0,i)\} \\
&\quad = \int_0^t \epsilon e^{\epsilon s} \mathbb{E}\{V(x_s,i)\}\, ds + \int_0^t e^{\epsilon s} \mathbb{E}\{\mathscr{L}V(x_s,i)\}\, ds.
\end{aligned} \tag{35}
$$

Then, similar to the method of [1], we can see that there exists a positive number α such that for $t > 0$

$$
\mathbb{E}\{V(x_t,i,t)\} \le \alpha \sup_{-\tau_M \le s \le 0} \left\{\|\phi(s)\|^2\right\} e^{-\epsilon t}. \tag{36}
$$

Since $V(x_t,i,t) \ge \{\lambda_{\min}(P_i)\} x^T(t) x(t)$, it can be shown from (36) that for $t \ge 0$

$$
\mathbb{E}\left\{x^T(t)x(t)\right\} \le \bar{\alpha}^{-\epsilon t} \sup_{-\tau_M \le s \le 0} \left\{\|\phi(s)\|^2\right\}, \tag{37}
$$

where $\bar{\alpha} = \alpha/(\lambda_{\min} P_i)$. Recalling Definition 5, the proof can be completed. □

Remark 8. In the above proof, a new Lyapunov function is constructed, and the term $\int_{t-\tau_{(t)}}^{t} x^T(s)(\sum_{j=1}^{N} \pi_{ij} Q_{2j}) x(s) ds$ in (25) is separated into two parts. It is easy to see that this method is less conservative than the ones in the literature [5, 38].

Remark 9. A delay-dependent stochastic stability condition for MJSs with interval time-varying delays is provided in Theorem 7. In the proof of Theorem 7, the convexity property of the matrix inequality is treated in terms of Lemma 4, which need not enlarge $\tau(t)$ to τ_M, so the common existed conservatism caused by this kind of enlargement in [39–42] can be avoided, and thus the conservative result will be decreased.

Theorem 7 established some analysis results. In the following, the problem of state estimator design is to be considered and the following results can be readily obtained from Theorem 7.

Theorem 10. *For some given constants γ and $0 \le \tau_m \le \tau_M$, the augmented system* (8) *is stochastically stable with a prescribed H_∞ performance γ if there exist $P_i > 0$, $Q_0 > 0$, $Q_1 > 0$, $Q_{2i} > 0$, $R_0 > 0$, $R_1 > 0$, $Z_1 > 0$, $Z_2 > 0$, $\bar{G}_i$, M_{ik}, and*

N_{ik} $(i \in \mathcal{S}, k = 1, 2 \ldots, 5)$ with appropriate dimensions so that the following LMIs hold for a given $\varepsilon > 0$:

$$\widehat{\Psi} = \begin{bmatrix} \widehat{\Psi}_{11} & * & * & * & * & * \\ \widehat{\Psi}_{21} & \Psi_{22} & * & * & * & * \\ \widehat{\Psi}_{31} & \widehat{\Psi}_{32} & \widehat{\Psi}_{33} & * & * & * \\ \Psi_{41}(s) & \Psi_{42}(s) & 0 & -R_1 & * & * \\ \widehat{\Psi}_{51} & \widehat{\Psi}_{52} & 0 & 0 & \widehat{\Psi}_{55} & * \\ \widehat{\Psi}_{61} & \widehat{\Psi}_{62} & 0 & 0 & 0 & \widehat{\Psi}_{66} \end{bmatrix} < 0, \quad s = 1, 2, \tag{38}$$

$$\sum_{j=1}^{N} \pi_{ij} Q_{2j} \le Z_k, \quad k = 1, 2, \tag{39}$$

where

$$\begin{aligned}
\widehat{\Psi}_{11} &= P_i A_i + A_i^T P_i + \bar{G}_i C_i + C_i^T \bar{G}_i^T + Q_0 + Q_1 \\
&\quad + Q_{2i} - R_0 + \tau_m Z_1 + \delta Z_2 + \sum_{j=1}^{N} \pi_{ij} P_j, \\
\widehat{\Psi}_{21} &= \left[P_i^T A_{di} + \bar{G}_i C_{di} - M_{i1} + N_{i1}, R_0^T + M_{i1}, \right. \\
&\quad \left. -N_{i1}, -P_i^T A_{\omega i} - \bar{G}_i C_{\omega i} \right]^T, \\
\widehat{\Psi}_{31} &= \begin{bmatrix} \tau_m P_i A_i + \tau_m \bar{G}_i C_{\omega i} \\ \sqrt{\delta} P_i A_i + \sqrt{\delta} \bar{G}_i C_{\omega i} \\ L_i \end{bmatrix}, \\
\widehat{\Psi}_{32} &= \begin{bmatrix} \tau_m P_i A_{di} + \tau_m \bar{G}_i C_{di} & 0 & 0 & -\tau_m P_i A_{\omega i} - \tau_m \bar{G}_i C_{\omega i} \\ \sqrt{\delta} P_i A_{di} + \sqrt{\delta} \bar{G}_i C_{di} & 0 & 0 & -\sqrt{\delta} P_i A_{\omega i} - \sqrt{\delta} \bar{G}_i C_{\omega i} \\ L_{di} & 0 & 0 & -L_{\omega i} \end{bmatrix}, \\
\widehat{\Psi}_{33} &= \operatorname{diag}\left\{ -2\varepsilon P_i + \varepsilon^2 R_0, -2\varepsilon P_i + \varepsilon^2 R_1, -I \right\}, \\
\widehat{\Psi}_{51} &= \left[\tau_m \sigma_1 C_i^T K_1^T \bar{G}_i^T, \tau_m \sigma_2 C_i^T K_2^T \bar{G}_i^T, \ldots, \right. \\
&\quad \left. \tau_m \sigma_m C_i^T K_m^T \bar{G}_i^T, 0, \ldots, 0 \right]^T, \\
\widehat{\Psi}_{52} &= \left[\widehat{\Theta}_{5d}, 0, 0, \widehat{\Theta}_{5\omega} \right], \\
\widehat{\Psi}_{55} &= \operatorname{diag}\left\{ -2\varepsilon P_i + \varepsilon^2 R_0, -2\varepsilon P_i + \varepsilon^2 R_0, \ldots, \right. \\
&\quad \left. -2\varepsilon P_i + \varepsilon^2 R_0 \right\}, \\
\widehat{\Psi}_{61} &= \left[\sqrt{\delta} \sigma_1 C_i^T K_1^T \bar{G}_i^T, \sqrt{\delta} \sigma_2 C_i^T K_2^T \bar{G}_i^T, \ldots, \right. \\
&\quad \left. \sqrt{\delta} \sigma_m C_i^T K_m^T \bar{G}_i^T, 0, \ldots, 0 \right]^T, \\
\widehat{\Psi}_{62} &= \left[\widehat{\Theta}_{6d}, 0, 0, \widehat{\Theta}_{6\omega} \right], \\
\widehat{\Psi}_{66} &= \operatorname{diag}\left\{ -2\varepsilon P_i + \varepsilon^2 R_1, -2\varepsilon P_i + \varepsilon^2 R_1, \ldots, -2\varepsilon P_i + \varepsilon^2 R_1 \right\}, \\
\widehat{\Theta}_{5d} &= \left[0, \tau_m \sigma_1 C_{di}^T K_1^T \bar{G}_i^T, \tau_m \sigma_2 C_{di}^T K_2^T \bar{G}_i^T, \ldots, \right. \\
&\quad \left. \tau_m \sigma_m C_{di}^T K_m^T \bar{G}_i^T, 0, \ldots, 0 \right]^T, \\
\widehat{\Theta}_{5\omega} &= \left[0, \ldots, 0, -\tau_m \sigma_1 C_{\omega i}^T K_1^T \bar{G}_i^T, -\tau_m \sigma_2 C_{\omega i}^T K_2^T \bar{G}_i^T, \ldots, \right. \\
&\quad \left. -\tau_m \sigma_m C_{\omega i}^T K_m^T \bar{G}_i^T \right]^T, \\
\widehat{\Theta}_{6d} &= \left[0, \sqrt{\delta} \sigma_1 C_{di}^T K_1^T \bar{G}_i^T, \sqrt{\delta} \sigma_2 C_{di}^T K_2^T \bar{G}_i^T, \ldots, \right. \\
&\quad \left. \sqrt{\delta} \sigma_m C_{di}^T K_m^T \bar{G}_i^T, 0, \ldots, 0 \right]^T, \\
\widehat{\Theta}_{6\omega} &= \left[0, \ldots, 0, -\sqrt{\delta} \sigma_1 C_{\omega i}^T K_1^T \bar{G}_i^T, -\sqrt{\delta} \sigma_2 C_{\omega i}^T K_2^T \bar{G}_i^T, \ldots, \right. \\
&\quad \left. -\sqrt{\delta} \sigma_m C_{\omega i}^T K_m^T \bar{G}_i^T \right]^T,
\end{aligned} \tag{40}$$

and Ψ_{22}, $\Psi_{41}(s)$, $\Psi_{42}(s)$, and δ are as defined in Theorem 7.

Moreover, the state estimator gain in the form of (6) *is as follows:*

$$G_i = P_i^{-1} \bar{G}_i. \tag{41}$$

Proof. Defining $\bar{G}_i = P_i G_i$, from (15) and using Schur complement, the matrix inequality (15) holds if and only if

$$\breve{\Psi} = \begin{bmatrix} \widehat{\Psi}_{11} & * & * & * & * & * \\ \widehat{\Psi}_{21} & \Psi_{22} & * & * & * & * \\ \widehat{\Psi}_{31} & \widehat{\Psi}_{32} & \breve{\Psi}_{33} & * & * & * \\ \Psi_{41}(s) & \Psi_{42}(s) & 0 & -R_1 & * & * \\ \widehat{\Psi}_{51} & \widehat{\Psi}_{52} & 0 & 0 & \widehat{\Psi}_{55} & * \\ \widehat{\Psi}_{61} & \widehat{\Psi}_{62} & 0 & 0 & 0 & \widehat{\Psi}_{66} \end{bmatrix} < 0, \quad s = 1, 2, \tag{42}$$

where

$$\breve{\Psi}_{33} = \operatorname{diag}\left\{ -P_i R_0^{-1} P_i, -P_i R_1^{-1} P_i, -I \right\}. \tag{43}$$

Due to

$$\left(R_k - \varepsilon^{-1} P_i \right) R_k^{-1} \left(R_k - \varepsilon^{-1} P_i \right) \ge 0, \quad i \in S, \ k = 0, 1, \tag{44}$$

we can have

$$-P_i R_k^{-1} P_i \le -2\varepsilon P_i + \varepsilon^2 R_k, \quad i \in S, \ k = 0, 1. \tag{45}$$

Substituting $-P_i R_k^{-1} P_i$ with $-2\varepsilon P_i + \varepsilon^2 R_k$ $(k = 0, 1)$ in (42), we obtain (38), so if (38) holds, we have (15) holds, and from above proof, we know that the desired state estimator gain matrix is $G_i = P_i^{-1}\bar{G}_i$. This completes the proof. □

Remark 11. Inequality (45) is used to bound the term $-P_i R_k^{-1} P_i$. This step can be improved by adopting the cone complementary algorithm [43], which is popular in recent control designs. The scaling parameter $\varepsilon > 0$ here can be used to improve conservatism in Theorem 10. In addition, Theorem 10 shows that for given ε we can obtain the state estimator gain by solving a set of LMIs in (38) and (39).

4. Numerical Example

In this section, well-studied example is considered to illustrate the effectiveness of above approaches proposed and also to explain the proposed method on state estimator design.

Consider linear Markovian jump systems in the form of (1) with two modes. For modes 1 and 2, the dynamics of system with following parameters [28] are described as

$$A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -2.5 & 1 \\ -0.1 & 0.3 & -3.8 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} -0.2 & 0.1 & 0.6 \\ 0.5 & -1 & -0.8 \\ 0 & 1 & -2.5 \end{bmatrix},$$

$$A_{\omega 1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.8 & 0.3 & 0 \end{bmatrix}, \qquad C_{d1} = \begin{bmatrix} 0.2 & -0.3 & -0.6 \end{bmatrix},$$

$$C_{\omega 1} = 0.2,$$

$$L_1 = \begin{bmatrix} 0.5 & -0.1 & 1 \end{bmatrix}, \qquad L_{d1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \qquad L_{\omega 1} = 0,$$

$$A_2 = \begin{bmatrix} -2.5 & 0.5 & -0.1 \\ 0.1 & -3.5 & 0.3 \\ -0.1 & 1 & -2 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0 & -0.3 & 0.6 \\ 0.1 & 0.5 & 0 \\ -0.6 & 1 & -0.8 \end{bmatrix},$$

$$A_{\omega 2} = \begin{bmatrix} -0.6 \\ 0.5 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \end{bmatrix}, \qquad C_{d2} = \begin{bmatrix} 0 & -0.6 & 0.2 \end{bmatrix},$$

$$C_{\omega 2} = 0.5,$$

$$L_2 = \begin{bmatrix} 0 & 1 & 0.6 \end{bmatrix}, \qquad L_{d2} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

$$L_{\omega 2} = 0, \tag{46}$$

Suppose the initial conditions are given by $x(0) = [0.8 \;\; 0.2 \;\; -0.9]^T$, $\hat{x}(0) = [0 \;\; 0.2 \;\; 0]^T$, and the transition probability matrix

$$\pi = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}. \tag{47}$$

By Theorem 10, we get the maximum time delay $\tau_M = 5.9250$ for $\tau_m = 1$, $\mu = 0.5$, $\varepsilon = 10$, and $\gamma = 1.2$. Meanwhile, we can get the fact that the maximum time delay will become larger with decreasing rates of $\tau(t)$ when other variables are fixed. For example, the maximum time delay is $\tau_M = 6.4072$ for $\mu = 0.1$ if other parameters did not change.

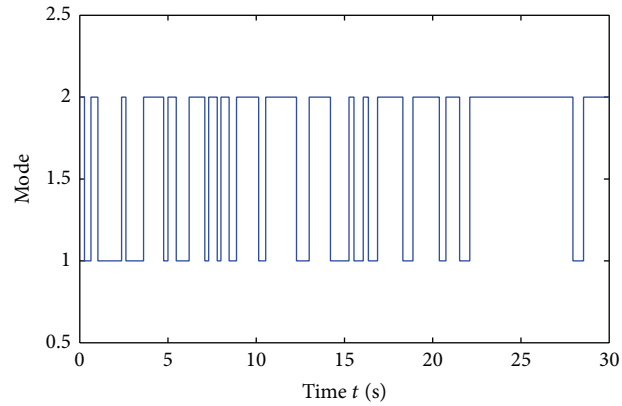

Figure 1: Operation modes.

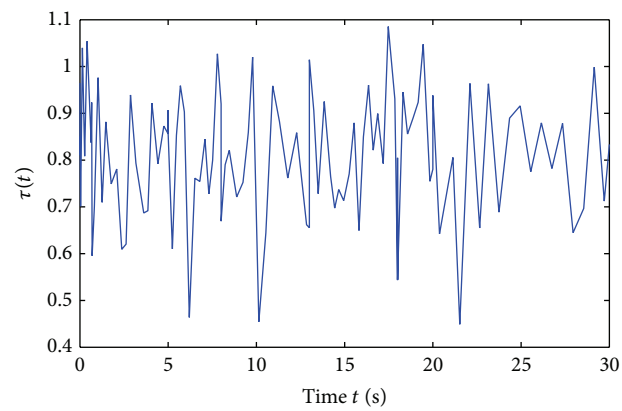

Figure 2: Interval time-varying delay.

The corresponding state estimator gain matrices for $\mu = 0.5$ are given by

$$G_1 = \begin{bmatrix} 0.7370 \\ -1.3432 \\ -3.4025 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.7847 \\ 0.2051 \\ -1.1228 \end{bmatrix}. \tag{48}$$

To illustrate the performance of the designed state estimator, choose the disturbance function as follows:

$$\omega(t) = \begin{cases} -0.425, & 5 < t < 8 \\ 0.375, & 13 < t < 18 \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

With this state estimator, the simulation results are shown in Figures 1, 2, and 3 which show the operation modes of the MJSs, interval time-varying delay, and estimated signal error $\eta(t) = z(t) - \bar{z}(t)$, respectively. From Figures 1, 2, and 3, it can be showed that the designed state estimator performs well.

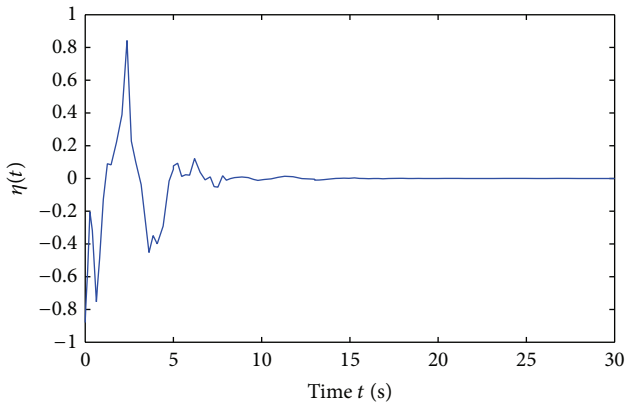

FIGURE 3: Estimated signals error $\eta(t) = z(t) - \tilde{z}(t)$.

5. Conclusions

In this paper, we established the design method of state estimation problem for a class of time-delay systems with Markov jump parameters and missing measurements. By employing a new Lyapunov function method and using the convexity property of the matrix inequality, an LMI-based sufficient condition for the existence of the desired state estimator derived is proposed, which can lead to much less conservative analysis results. Finally, a numerical example has been carried out to show the effectiveness of our obtained results of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Input-to-State Stability of Impulsive Stochastic Systems with Time Delays

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This paper is concerned with pth moment input-to-state stability (p-ISS) and stochastic input-to-state stability (SISS) of impulsive stochastic systems with time delays. Razumikhin-type theorems ensuring p-ISS/SISS are established for the mentioned systems with external input affecting both the continuous and the discrete dynamics. It is shown that when the impulse-free delayed stochastic dynamics are p-ISS/SISS but the impulses are destabilizing, the p-ISS/SISS property of the impulsive stochastic systems can be preserved if the length of the impulsive interval is large enough. In particular, if the impulse-free delayed stochastic dynamics are p-ISS/SISS and the discrete dynamics are marginally stable for the zero input, the impulsive stochastic system is p-ISS/SISS regardless of how often or how seldom the impulses occur. To overcome the difficulties caused by the coexistence of time delays, impulses, and stochastic effects, Razumikhin techniques and piecewise continuous Lyapunov functions as well as stochastic analysis techniques are involved together. An example is provided to illustrate the effectiveness and advantages of our results.

1. Introduction

In practice, the performance of a real control system is affected more or less by uncertainties such as unmodeled dynamics, parameter perturbations, exogenous disturbances, and measurement errors [1]. To describe how solutions behave robustly to external inputs or disturbances, the concept of input-to-state stability (ISS) has been proven useful and effective in this regard. ISS was originally proposed by Sontag [2] for continuous-time systems. In view of its importance in the analysis and synthesis of nonlinear control systems [3–5], ISS and its variants such as local ISS, integral ISS, and exponential-weighted ISS have been investigated quite intensively and extended to different types of dynamical systems, for instance, discrete-time systems [6, 7], switched systems [1, 8–11], network control systems [12], neural networks [13–15], and so forth.

As it is well known, impulsive effect is likely to exist in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time in the fields such as medicine and biology, economics, electronics, and telecommunications [16]. Recently, Hespanha initiated the study of ISS for impulsive systems [17]. It was proved therein that impulsive systems possessing an exponential ISS-Lyapunov function are uniformly ISS over a certain class of impulse time sequences. Since time delay phenomena are often encountered in real world systems and the existence of time delay is a significant cause of instability and deteriorative performance, [18] investigated the ISS property for nonlinear impulsive systems with time delays by using Razumikhin techniques. And [19] was also concerned with ISS of impulsive systems with time delays, where ISS theorems different from those in [18] were established by adopting both Razumikhin techniques and Lyapunov-Krosovskii functional method.

In addition to the time delays and impulse effects, stochastic perturbations are always unavoidable in real systems (see [20–23] and references therein). Impulsive stochastic delayed systems incorporate impulses effects, stochastic perturbations, and time delays in one system simultaneously. During the last decade, there has been extensive interest in the study of force-free delayed impulsive stochastic systems;

we refer to [24–28] and references therein. However, the corresponding theory for impulsive stochastic systems with external inputs has been relatively less developed.

The present paper aims to generalize the ISS results for deterministic delayed impulsive systems to stochastic settings. The pth moment input-to-state stability (p-ISS) and stochastic input-to-state stability (SISS) properties for impulsive stochastic delayed systems with external input affecting both the continuous dynamics and the impulses are investigated and Razumikhin-type theorems guaranteeing the p-ISS/SISS are established. The results indicate that when the delayed continuous stochastic dynamics are p-ISS/SISS and the discrete dynamics are destabilizing, the p-ISS/SISS properties of the original impulsive stochastic systems can be maintained if the length of impulsive interval is large enough. In particular, if the impulse-free delayed stochastic dynamics are p-ISS/SISS and the discrete dynamics are marginally stable for the zero input, the impulsive stochastic system is p-ISS/SISS regardless of how often or how seldom the impulses occur. As a byproduct, the criteria on pth moment global asymptotic stability (p-GAS) and global asymptotical stability in probability (GASiP) are also derived. The initial idea of this paper came from the works for deterministic impulsive delayed systems [18] and impulse-free stochastic systems [1, 29], but its extension to impulsive stochastic delayed systems will be much more challenging due to the simultaneous existence of time delays, impulses, and stochastic effects.

The rest of this paper is organized as follows. In Section 2, some basic notations and definitions used throughout the paper are introduced. In Section 3, criteria ensuring uniform p-ISS/SISS/p-GAS/GASiP are established and applied to linear impulsive stochastic delayed systems. Section 4 provides a numerical example to illustrate the effectiveness and advantages of our results. Finally, Section 5 includes a summary and a discussion of some extensions of the paper.

2. Preliminaries

Throughout this paper, unless otherwise specified, we will employ the following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and let $\mathbb{E}[\cdot]$ be the expectation operator with respect to the given probability measure $\mathbb{P}$. Let $w(t) = (w_1(t), \ldots, w_d(t))^{\mathrm{T}}$ be a d-dimensional Brownian motion defined on the probability space. $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{R}^n$ denotes the n-dimensional real space equipped with the Euclidean norm $|\cdot|$, and $\mathbb{R}^{n\times m}$ denotes the $n \times m$-dimensional real matrix space.

Let $\tau \geqslant 0$ and $PC([-\tau, 0]; \mathbb{R}^n) = \{\varphi : [-\tau, 0] \to \mathbb{R}^n \mid \varphi(t)$ is continuous for all but at most a finite number of points $\bar{t}$, at which $\varphi(\bar{t}^+)$, $\varphi(\bar{t}^-)$ exist and $\varphi(\bar{t}^+) = \varphi(\bar{t})\}$, where $\varphi(\bar{t}^+)$ and $\varphi(\bar{t}^-)$ denote the right-hand and left-hand limits of $\varphi(t)$ at $\bar{t}$, respectively. We equip the linear space $PC([-\tau, 0]; \mathbb{R}^n)$ with the norm $\|\varphi\|$ defined by $\|\varphi\| = \sup\{|\varphi(\theta)| : -\tau \leqslant \theta \leqslant 0\}$. Let $PC^b_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_t$-measurable and bounded $PC([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leqslant \theta \leqslant 0\}$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous and strictly increasing and satisfies $\alpha(0) = 0$; it is of class $\mathcal{K}_\infty$ if in addition $\alpha(s) \to \infty$ as $s \to \infty$. Note that if α is of class $\mathcal{K}_\infty$, then the inverse function α^{-1} is well defined and is also of class $\mathcal{K}_\infty$. $v\mathcal{K}_\infty$ and $c\mathcal{K}_\infty$ are the subsets of $\mathcal{K}_\infty$ functions that are convex and concave, respectively. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \geqslant 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geqslant 0$. The composition of two functions $\phi : A \to B$ and $\psi : B \to C$ is denoted by $\psi \circ \phi : A \to C$.

If A is a vector or a matrix, its transpose is denoted by A^{T}. If P is a square matrix, $P > 0$ ($P \leqslant 0$) means that P is a symmetric positive definite (negative semidefinite) matrix. $\underline{\lambda}(\cdot)$ and $\overline{\lambda}(\cdot)$ represent the minimum and maximum eigenvalues of the corresponding matrix, respectively, and I stands for the identity matrix. The symbol $*$ is used in symmetric matrices to denote the entries which can be inferred by symmetry. Unless explicitly stated, all matrices are assumed to have real entries and compatible dimensions.

We consider the following impulsive stochastic nonlinear system with external inputs:

$$\begin{aligned} \mathrm{d}x &= f(t, x_t, u_c(t))\,\mathrm{d}t + g(t, x_t, u_c(t))\,\mathrm{d}w(t), \\ &\qquad t \neq t_k, \quad t \geqslant t_0, \\ x(t_k) &= I_k(t_k, x(t_k^-), u_d(t_k^-)), \quad k \in \mathbb{N}, \end{aligned} \tag{1}$$

with initial data $x_{t_0} = \{x(t_0 + \theta) : -\tau \leqslant \theta \leqslant 0\} = \xi \in \mathbb{P}C^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$, where $x \in \mathbb{R}^n$ and $x_t = \{x(t + \theta) : -\tau \leqslant \theta \leqslant 0\}$ is regarded as a $PC([-\tau, 0]; \mathbb{R}^n)$-valued random variable; $u_c \in \mathcal{L}^{m_1}_\infty$ is locally essentially bounded external input and $u_d \in \mathcal{L}^{m_2}_\infty$ is the impulsive disturbance input; $\mathcal{L}^m_\infty$ denotes the set of all locally essentially bounded function $u : \mathbb{R}_+ \to \mathbb{R}^m$ with norm $\|u\|_\infty = \operatorname{ess\,sup}_{t\geqslant t_0} |u(t)|$; $\|u\|_{[a,b]} = \operatorname{ess\,sup}_{t\in[a,b]} |u(t)|$; both $f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{m_1} \to \mathbb{R}^n$ and $g : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{m_1} \to \mathbb{R}^{n\times d}$ are uniformly locally Lipschitz with respect to x and u; $I_k : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{m_2} \to \mathbb{R}^n$ represents the impulsive perturbation of x at t_k and satisfies $|I_k(t_k, x, u)| < \infty$; $\{t_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of impulse times. We use $\mathcal{S}_{\min}(\beta)$ and $\mathcal{S}_{\mathrm{all}}$ to denote the class of impulsive time sequences that satisfy $\inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \geqslant \beta$ and the set containing all impulse time sequences, respectively.

Moreover, we assume that $f(t, 0, 0) = g(t, 0, 0) = I_k(t, 0, 0) \equiv 0$ for all $t \geqslant t_0$, $k \in \mathbb{N}$; then system (1) admits a trivial solution $x(t) \equiv 0$. The input pair (u_c, u_d) is said to be admissible, denoted by $(u_c, u_d) \in \mathcal{U}$, if $u_c \in \mathcal{L}^{m_1}_\infty$, $u_d \in \mathcal{L}^{m_2}_\infty$ guarantee the the existence of a unique solution to system (1).

On the foundation of the ISS concepts for impulse-free stochastic systems [1, 29, 30] and those for deterministic impulsive systems [18], we proposed the following definitions for impulsive stochastic delayed systems (1).

Definition 1. For a prescribed sequence $\{t_k\}_{k\in\mathbb{N}}$, system (1) is said to be pth ($p > 0$) moment input-to-state stable (ISS) if there exist functions $\beta \in \mathcal{KL}, \alpha, \gamma_c, \gamma_d \in \mathcal{K}_\infty$ such that,

for every initial condition $\xi \in PC^b_{\mathcal{F}_{t_0}}$ and every input pair $(u_c, u_d) \in \mathcal{U}$,

$$\alpha\left(\mathbb{E}|x(t)|^p\right) \leqslant \beta\left(\mathbb{E}\|\xi\|^p, t-t_0\right) + \gamma_c\left(\|u_c\|_{[t_0,t]}\right) + \gamma_d\left(\max_{t_k\in(t_0,t]}\{|u_d(t_k^-)|\}\right), \quad t \geqslant t_0. \tag{2}$$

Definition 2. For a given sequence $\{t_k\}_{k\in\mathbb{N}}$, system (1) is said to be stochastic input-to-state stable (SISS), if, for any $\varepsilon > 0$, there exist functions $\beta \in \mathcal{KL}$ and $\alpha, \gamma_c, \gamma_d \in \mathcal{K}_\infty$, such that, for every initial condition $\xi \in PC^b_{\mathcal{F}_{t_0}}$ and every input pair $(u_c, u_d) \in \mathcal{U}$,

$$P\left\{\alpha(|x(t)|) < \beta\left(\|\xi\|, t-t_0\right) + \gamma_c\left(\|u_c\|_{[t_0,t]}\right) + \gamma_d\left(\max_{t_k\in(t_0,t]}\{|u_d(t_k^-)|\}\right)\right\} > 1-\varepsilon, \quad t \geqslant t_0. \tag{3}$$

Remark 3. Redefining β and γ_c, γ_d, one can assume that α in (2) or (3) is the identity: if $\alpha(r) \leqslant \beta(s,t) + \gamma_c(v) + \gamma_d(\nu)$ holds, then also $r \leqslant \alpha^{-1}(\beta(s,t) + \gamma_c(v) + \gamma_d(\nu)) \leqslant \alpha^{-1}(3\beta(s,t)) + \alpha^{-1}(3\gamma_c(v)) + \alpha^{-1}(3\gamma_d(\nu))$. We know by Lemma 4.2 in [31] that $\alpha^{-1}(3\beta(\cdot,\cdot)) \in \mathcal{KL}$ and $\alpha^{-1}(3\gamma_c(\cdot))$, $\alpha^{-1}(3\gamma_d(\cdot)) \in \mathcal{K}_\infty$. So estimates of the same type as (2) and (3) but with no "α" are obtained.

In the following, we will define p-GAS and GASiP in the form of $\mathcal{KL}$ function, which present very close analogy to p-ISS and SISS, respectively.

Definition 4. For a prescribed sequence $\{t_k\}_{k\in\mathbb{N}}$, system (1) with input $u_c \equiv 0$, $u_d \equiv 0$ is said to be pth ($p > 0$) moment globally asymptotically stable (GAS) if there exists a function $\beta \in \mathcal{KL}$ such that, for every initial condition $\xi \in PC^b_{\mathcal{F}_{t_0}}$,

$$\mathbb{E}|x(t)|^p \leqslant \beta\left(\mathbb{E}\|\xi\|^p, t-t_0\right), \quad t \geqslant t_0. \tag{4}$$

Definition 5. For a given sequence $\{t_k\}_{k\in\mathbb{N}}$, system (1) with input $u_c \equiv 0$, $u_d \equiv 0$ is said to be globally asymptotically stable in probability (GASiP), if, for any $\varepsilon > 0$, there exists a function $\beta \in \mathcal{KL}$, such that, for every initial condition $\xi \in PC^b_{\mathcal{F}_{t_0}}$,

$$P\left\{|x(t)| < \beta\left(\|\xi\|, t-t_0\right)\right\} > 1-\varepsilon, \quad t \geqslant t_0. \tag{5}$$

Remark 6. By the vanishing of $\gamma_c(s)$ and $\gamma_d(s)$ at $s = 0$, (2) and (3) will degenerate to (4) and (5), respectively, when $u \equiv 0$, which means that p-ISS/SISS of system (1) implies p-GAS/GASiP of the corresponding unforced system.

System (1) is said to be uniformly p-ISS or uniformly SISS over a given class of admissible impulsive time sequences $\mathcal{S}$ if (2) or (3) holds for every sequence in $\mathcal{S}$ with functions α, β, γ_c, and γ_d independent of the choice of the sequence. Uniform p-GAS and uniform GASiP can be defined similarly.

Definition 7 (see [24]). A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be of class v_0 if the following hold true.

(i) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and for each $x, y \in \mathbb{R}^n$, $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, and $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-, x)$ exists.

(ii) $V(t,x)$ is once continuously differentiable in t and twice in x in each of the sets $(t_{k-1}, t_k) \times \mathbb{R}^n$, $k \in \mathbb{N}$.

If $V \in v_0$, define an operator $\mathcal{L}V$ [24] with respect to system (1) by

$$\mathcal{L}V(t,\varphi,u) = V_t(t,\varphi(0)) + V_x(t,x) f(t,\varphi,u) + \frac{1}{2}\operatorname{trace}\left[g^{\mathrm{T}}(t,\varphi,u) V_{xx}(t,x) g(t,\varphi,u)\right], \tag{6}$$

where

$$V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \quad V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right), \quad V_{xx}(t,x) = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}. \tag{7}$$

3. Main Results

In this section, we will develop Lyapunov-Razumikhin methods and establish some criteria which provide sufficient conditions for the p-ISS and SISS properties of impulsive stochastic delayed systems (1).

Theorem 8. *Assume that there exist functions $V \in v_0$, $\chi_1, \chi_2 \in \mathcal{K}_\infty$, $\alpha_1 \in c\mathcal{K}_\infty$, $\alpha_2 \in v\mathcal{K}_\infty$ and scalars $q > 1$, $c > 0$, $\mu \in [1, q)$ such that*

(i) $\alpha_1(|x|^p) \leqslant V(t,x) \leqslant \alpha_2(|x|^p)$;

(ii) $\mathbb{E}\mathcal{L}V(t,\varphi) \leqslant -c\mathbb{E}V(t,\varphi(0)) + \chi_1(|u_c(t)|)$, *for all* $t \geqslant t_0$, $t \neq t_k$ *and* $\varphi \in PC_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ *whenever* $\mathbb{E}V(t+\theta,\varphi(\theta)) \leqslant q\mathbb{E}V(t,\varphi(0))$;

(iii) $\mathbb{E}V(t_k, I_k(t_k^-, x, u_d)) \leqslant \mu\mathbb{E}V(t_k, x) + \chi_2(|u_d|)$.

Then for any given $\rho > 0$ satisfying $\mu e^{-c\rho} < 1$, system (1) *is uniformly p-ISS over $\mathcal{S}_{\min}(\rho)$. In particular, when $\mu = 1$, system* (1) *is uniformly p-ISS over $\mathcal{S}_{all}$.*

Proof. Since $\mu e^{-c\rho} < 1$, then $0 < 1 + e^{-c\rho} - 1/\mu < 1$ and there exists $c' > 0$ such that $c\max\{\mu e^{-c\rho}, 1 + e^{-c\rho} - 1/\mu\} < c' < c$. We choose $\lambda \in (0, c')$ such that $(q/\mu)e^{-\lambda\tau} > 1$, $\mu e^{-(c-\lambda)\rho} \leqslant 1$, $\lambda \leqslant c - \mu(c - c')$. Let $\{t_k\}_{k\in\mathbb{N}}$ be any impulsive time sequence belonging to $\mathcal{S}_{\min}(\rho)$. For simplicity, we write $V(t,x) = V(t)$. Define

$$J(t) = e^{\lambda(t-t_0)}\left[\mathbb{E}V(t) - J_0(t)\right], \quad t \geqslant t_0 - \tau, \tag{8}$$

where $J_0(t) = (1/(c - c'))\chi_1(\|u_c\|_{[t_0,t]}) + \sum_{t_k\in(t_0,t]} e^{-\lambda(t-t_k)}\chi_2(|u_d(t_k^-)|)$ for $t \geqslant t_0$ and $J_0(t) = 0$ for $t \in [t_0 - \tau, t_0]$. We claim that

$$J(t) \leqslant \alpha_2\left(\mathbb{E}\|\xi\|^p\right), \quad t \geqslant t_0. \tag{9}$$

We first prove that (9) holds for $t \in [t_0, t_1)$. By condition (i) and Jensen's inequality, it is easy to see that

$$\begin{aligned} J(t) &= \mathbb{E}V(t)\, e^{\lambda(t-t_0)} \\ &\leqslant \alpha_2\left(\mathbb{E}|x(t)|^p\right) \\ &\leqslant \alpha_2\left(\mathbb{E}\|\xi\|^p\right), \quad t \in [t_0 - \tau, t_0]. \end{aligned} \tag{10}$$

If (9) is not true for $t \in [t_0, t_1)$, there must exist some $t \in [t_0, t_1)$ such that $J(t) > \alpha_2(\mathbb{E}\|\xi\|^p)$. Let $t^* = \inf\{t \in [t_0, t_1) : J(t) > \alpha_2(\mathbb{E}\|\xi\|^p)\}$. Then by the right continuity of $J(t)$ in $t \in [t_0, t_1)$ and noticing (10), we have $t^* \geqslant t_0$ and

$$\begin{aligned} J(t^*) = \alpha_2\left(\mathbb{E}\|\xi\|^p\right), \qquad J(t) \leqslant \alpha_2\left(\mathbb{E}\|\xi\|^p\right), \\ t \in [t_0 - \tau, t^*), \quad D^+J(t^*) > 0. \end{aligned} \tag{11}$$

Because $J(t^*) \geqslant J(t^* + s)$, $s \in [-\tau, 0]$ implies

$$\begin{aligned} \mathbb{E}V(t^*) &\geqslant e^{\lambda s}\mathbb{E}V(t^* + s) - e^{\lambda s}J_0(t^* + s) + J_0(t^*) \\ &\geqslant e^{-\lambda\tau}\mathbb{E}V(t^* + s) > \frac{\mu}{q}\mathbb{E}V(t^* + s) \\ &\geqslant \frac{1}{q}\mathbb{E}V(t^* + s), \quad s \in [-\tau, 0], \end{aligned} \tag{12}$$

it follows from condition (ii) that

$$\mathbb{E}\mathscr{L}V(t^*) \leqslant -c\mathbb{E}V(t^*) + \chi_1\left(|u_c(t^*)|\right). \tag{13}$$

For $\rho > 0$ sufficiently small satisfying $t^* + \rho < t_1$, by the Itô formula, we have

$$\mathbb{E}V(t^* + \rho) - \mathbb{E}V(t^*) = \int_{t^*}^{t^*+\rho} \mathbb{E}\mathscr{L}V(s, x_s)\, ds \tag{14}$$

which yields

$$\limsup_{\rho\to 0^+} \frac{\mathbb{E}V(t^* + \rho) - \mathbb{E}V(t^*)}{\rho} = \limsup_{\rho\to 0^+} \frac{1}{\rho}\int_{t^*}^{t^*+\rho} \mathbb{E}\mathscr{L}V(s)\, ds; \tag{15}$$

that is,

$$D^+\mathbb{E}V(t^*) = \mathbb{E}\mathscr{L}V(t^*) \leqslant -c\mathbb{E}V(t^*) + \chi_1\left(|u_c(t^*)|\right), \tag{16}$$

where $D^+\mathbb{E}V(t) \triangleq \limsup_{\rho\to 0^+}[\mathbb{E}V(t+\rho) - \mathbb{E}V(t)]/\rho$. On the other hand, $J(t^*) = \alpha_2(\mathbb{E}\|\xi\|^p) \geqslant 0$ implies

$$\mathbb{E}V(t^*) \geqslant J_0(t^*). \tag{17}$$

Therefore, from (16) and (17), and noticing $J_0(t) = (1/(c - c'))\chi_1(\|u_c\|_{[t_0,t]})$ and $D^+J_0(t) \geqslant 0$ for $t \in [t_0, t_1)$, we have

$$\begin{aligned} D^+J(t^*) &= e^{\lambda(t^*-t_0)}\left[D^+\mathbb{E}V(t^*) + \lambda\mathbb{E}V(t^*)\right. \\ &\qquad \left. - \lambda J_0(t^*) - D^+J_0(t^*)\right] \\ &\leqslant e^{\lambda(t^*-t_0)}\left[-(c-\lambda)\mathbb{E}V(t^*) + \chi_1\left(|u_c(t^*)|\right)\right. \\ &\qquad \left. -\lambda J_0(t^*)\right] \\ &\leqslant e^{\lambda(t^*-t_0)}\left[-cJ_0(t^*) + \chi_1\left(|u_c(t^*)|\right)\right] \\ &\leqslant \frac{-c'}{c-c'}e^{\lambda(t^*-t_0)}\chi_1\left(\|u_c\|_{[t_0,t^*]}\right) < 0, \end{aligned} \tag{18}$$

which contradicts $D^+J(t^*) > 0$. Therefore, (9) holds for $t \in [t_0 - \tau, t_1)$.

Suppose that (9) holds for $t \in [t_0 - \tau, t_m)$, where $m \geqslant 1$, $m \in \mathbb{N}$. We will prove that (9) holds for $t \in [t_m, t_{m+1})$. To this end, we claim that

$$J_1(t_m^-) \leqslant \alpha_2\left(\mathbb{E}\|\xi\|^p\right), \tag{19}$$

where $J_1(t) = e^{\lambda(t-t_0)}[\mu\mathbb{E}V(t) - J_0(t)]$. If not, then $J_1(t_m^-) > \alpha_2(\mathbb{E}\|\xi\|^p)$. We consider the following two cases.

Case 1. $J_1(t) > \alpha_2(\mathbb{E}\|\xi\|^p)$ for all $t \in [t_{m-1}, t_m)$. It is easy to see that $J_1(t) > \alpha_2(\mathbb{E}\|\xi\|^p) \geqslant J(t + \theta)$ for $t \in [t_{m-1}, t_m)$ and $\theta \in [-\tau, 0]$. It follows that

$$\begin{aligned} \mathbb{E}V(t + \theta) &< e^{-\lambda\theta}\left[\mu\mathbb{E}V(t) - J_0(t)\right] + J_0(t + \theta) \\ &\leqslant e^{\lambda\tau}\mu\mathbb{E}V(t) - e^{-\lambda\theta}J_0(t) + J_0(t + \theta) \\ &< q\mathbb{E}V(t), \quad t \in [t_{m-1}, t_m), \ \theta \in [-\tau, 0]. \end{aligned} \tag{20}$$

The last inequality comes from the fact that $(q/\mu)e^{-\lambda\tau} > 1$, and

$$\begin{aligned} J_0(t + \theta) &= \frac{1}{c - c'}\chi_1\left(\|u_c\|_{[t_0,t+\theta]}\right) \\ &\qquad + \sum_{t_k\in(t_0,t+\theta]} e^{-\lambda(t+\theta-t_k)}\chi_2\left(|u_d(t_k^-)|\right) \\ &\leqslant \frac{e^{-\lambda\theta}}{c - c'}\chi_1\left(\|u_c\|_{[t_0,t]}\right) \\ &\qquad + \sum_{t_k\in(t_0,t]} e^{-\lambda(t+\theta-t_k)}\chi_2\left(|u_d(t_k^-)|\right) \\ &= e^{-\lambda\theta}J_0(t). \end{aligned} \tag{21}$$

By condition (ii), (20) indicates that

$$\mathbb{E}\mathscr{L}V(t) \leqslant -c\mathbb{E}V(t) + \chi_1\left(|u_c(t)|\right), \quad t \in [t_{m-1}, t_m). \tag{22}$$

By Itô's formula, we have

$$\begin{aligned} e^{ct_m}\mathbb{E}V(t_m^-) &= e^{ct_{m-1}}\mathbb{E}V(t_{m-1}) \\ &\quad + \int_{t_{m-1}}^{t_m} e^{cs}\left[c\mathbb{E}V(s) + \mathbb{E}\mathscr{L}V(s)\right] ds \\ &\leqslant e^{ct_{m-1}}\mathbb{E}V(t_{m-1}) \\ &\quad + \int_{t_{m-1}}^{t_m} e^{cs}\chi_1(|u_c(s)|)\, ds \\ &\leqslant e^{ct_{m-1}}\mathbb{E}V(t_{m-1}) \\ &\quad + \frac{1}{c}e^{ct_m}\chi_1\left(\|u_c\|_{[t_0,t_m)}\right); \end{aligned} \tag{23}$$

thus,

$$\mathbb{E}V(t_m^-) \leqslant e^{-c(t_m-t_{m-1})}\mathbb{E}V(t_{m-1}) + \frac{1}{c}\chi_1\left(\|u_c\|_{[t_0,t_m)}\right). \tag{24}$$

On the other hand, $J(t_{m-1}) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)$ implies

$$\begin{aligned} \mathbb{E}V(t_{m-1}) &\leqslant e^{-\lambda(t_{m-1}-t_0)}\alpha_2\left(\mathbb{E}\|\xi\|^p\right) + J_0(t_{m-1}) \\ &\leqslant e^{-\lambda(t_{m-1}-t_0)}\alpha_2\left(\mathbb{E}\|\xi\|^p\right) \\ &\quad + \frac{1}{c-c'}\chi_1\left(\|u_c\|_{[t_0,t_{m-1}]}\right) \\ &\quad + \sum_{k=1}^{m-1} e^{-\lambda(t_{m-1}-t_k)}\chi_2\left(|u_d(t_k^-)|\right). \end{aligned} \tag{25}$$

Substituting (25) into (24) and noticing the fact that $\inf\{t_k - t_{k-1}\} \geqslant \rho$, we have

$$\begin{aligned} \mathbb{E}V(t_m^-) &\leqslant e^{-c(t_m-t_{m-1})-\lambda(t_{m-1}-t_0)}\alpha_2\left(\|\xi\|^p\right) \\ &\quad + \left(\frac{1}{c} + \frac{e^{-c\rho}}{c-c'}\right)\chi_1\left(\|u_c\|_{[t_0,t_m)}\right) \\ &\quad + e^{-c(t_m-t_{m-1})}\sum_{k=1}^{m-1} e^{-\lambda(t_{m-1}-t_k)}\chi_2\left(|u_d(t_k^-)|\right). \end{aligned} \tag{26}$$

Substituting (26) into $J_1(t_m^-)$ yields

$$\begin{aligned} J_1(t_m^-) &= e^{\lambda(t_m-t_0)}\left[\mu\mathbb{E}V(t_m^-) - J_0(t_m^-)\right] \\ &\leqslant \mu e^{\lambda(t_m-t_0)}\Bigg[e^{-c(t_m-t_{m-1})-\lambda(t_{m-1}-t_0)}\alpha_2\left(\mathbb{E}\|\xi\|^p\right) \\ &\quad + \left(\frac{e^{-c\rho}}{c-c'} + \frac{1}{c}\right)\chi_1\left(\|u_c\|_{[t_0,t_m)}\right) \\ &\quad + e^{-c(t_m-t_{m-1})}\sum_{k=1}^{m-1} e^{-\lambda(t_{m-1}-t_k)} \\ &\quad \times\chi_2\left(|u_d(t_k^-)|\right)\Bigg] \\ &\quad - e^{\lambda(t_m-t_0)}\Bigg[\frac{1}{c-c'}\chi_1\left(\|u_c\|_{[t_0,t_m)}\right) \\ &\quad + \sum_{k=1}^{m-1} e^{-\lambda(t_m-t_k)}\chi_2\left(|u_d(t_k^-)|\right)\Bigg] \\ &\leqslant \mu e^{-(c-\lambda)\rho}\alpha_2\left(\mathbb{E}\|\xi\|^p\right) \\ &\quad + e^{\lambda(t_m-t_0)}\left[\mu\left(\frac{e^{-c\rho}}{c-c'} + \frac{1}{c}\right) - \frac{1}{c-c'}\right]\chi_1\left(\|u_c\|_{[t_0,t_m)}\right) \\ &\quad + \left(\mu e^{-(c-\lambda)\rho} - 1\right)\sum_{k=1}^{m-1} e^{\lambda(t_k-t_0)}\chi_2\left(|u_d(t_k^-)|\right) \\ &\leqslant \alpha_2\left(\mathbb{E}\|\xi\|^p\right). \end{aligned} \tag{27}$$

The last inequality holds because $\mu e^{-(c-\lambda)\rho} \leqslant 1$ and $\mu(((e^{-c\rho})/(c-c')) + (1/c)) - (1/(c-c')) \leqslant 0$. This is a contradiction.

Case 2. There exists some $t \in [t_{m-1}, t_m)$ such that $J_1(t) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)$. Set $t' = \sup\{t \in [t_{m-1}, t_m) : J_1(t) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)\}$. Then $J_1(t') = \alpha_2(\mathbb{E}\|\xi\|^p)$ and $J_1(t) > \alpha_2(\mathbb{E}\|\xi\|^p)$ for $t \in (t', t_m)$. Thus, for $t \in [t', t_m)$, $J_1(t) \geqslant \alpha_2(\mathbb{E}\|\xi\|^p) \geqslant J(t+\theta)$ for $\theta \in [-\tau, 0]$. This implies that (20) holds for $\theta \in [-\tau, 0]$, $t \in [t', t_m)$. Thus, by condition (ii),

$$D^+\mathbb{E}V(t) = \mathbb{E}\mathscr{L}V(t) \leqslant -c\mathbb{E}V(t) + \chi_1\left(|u_c(t)|\right), \quad t \in [t', t_m). \tag{28}$$

Hence, noticing the fact that $J_0(t) \geqslant 0$, $D^+J_0(t) \geqslant 0$ for $t \in [t', t_m)$, we have

$$\begin{aligned} D^+J_1(t) &= e^{\lambda(t-t_0)}\left[\lambda\mu\mathbb{E}V(t) - \lambda J_0(t) + \mu D^+\mathbb{E}V(t) - D^+J_0(t)\right] \\ &\leqslant e^{\lambda(t-t_0)}\left[-\mu(c-\lambda)\mathbb{E}V(t) - \lambda J_0(t) + \mu\chi_1\left(|u_c(t)|\right) - D^+J_0(t)\right] \\ &\leqslant \mu e^{\lambda(t-t_0)}\left[-(c-\lambda)\mathbb{E}V(t) + \chi_1\left(|u_c(t)|\right)\right], \quad t \in [t', t_m). \end{aligned} \tag{29}$$

Because $J_1(t) \geqslant \alpha_2(\mathbb{E}\|\xi\|^p) > 0$ for $t \in [t', t_m)$, there holds $\mathbb{E}V(t) > (1/\mu)J_0(t) > (1/\mu(c-c'))\chi_1(\|u_c\|_{[t_0,t]})$ for $t \in [t', t_m)$. Substituting this inequality with (29), and recalling the choice of λ, it follows that

$$D^+J_1(t) \leqslant -e^{\lambda(t-t_0)}\left(\frac{c-\lambda}{c-c'} - \mu\right)\chi_1\left(\|u_c\|_\infty\right) \leqslant 0, \tag{30}$$

which yields the following contradiction: $\alpha_2(\mathbb{E}\|\xi\|^p) < J_1(t_m^-) \leqslant J_1(t') = \alpha_2(\mathbb{E}\|\xi\|^p)$.

Therefore, we have $J_1(t_m^-) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)$. Using condition (iii), we obtain that $J(t_m) \leqslant J_1(t_m^-) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)$. Repeating

the argument used in the proof of $J(t) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)$ for $t \in [t_0, t_1)$, we can get $J(t) \leqslant \alpha_2(\mathbb{E}\|\xi\|^p)$ for $t \in [t_m, t_{m+1})$. By the mathematical induction, we know that (9) holds for all $t \geqslant t_0$. For any given $t \in [t_m, t_{m+1})$, one can get

$$\sum_{t_k \in (t_0,t]} e^{-\lambda(t-t_k)} \chi_2 \left(\left|u_d\left(t_k^-\right)\right|\right) \leqslant \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right). \tag{31}$$

It follows from (9), (31), and the definition of $J(t)$ that

$$\mathbb{E}V(t) \leqslant e^{-\lambda(t-t_0)} \alpha_2\left(\mathbb{E}\|\xi\|^p\right) + \frac{1}{c-c'} \chi_1\left(\|u_c\|_\infty\right) + \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right). \tag{32}$$

By casualty,

$$\mathbb{E}V(t) \leqslant e^{-\lambda(t-t_0)} \alpha_2\left(\mathbb{E}\|\xi\|^p\right) + \frac{1}{c-c'} \chi_1\left(\|u_c\|_{[t_0,t]}\right) + \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right). \tag{33}$$

Then by condition (i) and Jensen's inequality, the required assertion (2) holds with $\beta(r,s) = e^{-\lambda s}\alpha_2(r)$, $\gamma_c(r) = (1/(c-c'))\chi_1(r)$ and $\gamma_d(r) = (1/(1-e^{-\lambda\rho}))\chi_2(r)$. By Lemma 4.2 in [31], it is easy to see that $\beta \in \mathcal{KL}$, $\gamma_c, \gamma_d \in \mathcal{K}_\infty$. As β, γ_c, γ_d are independent of the particular choice of the impulse time sequence, system (1) is uniformly p-ISS over $\mathcal{S}_{\min}(\rho)$.

For the special case $\mu = 1$, $\mu e^{-c\rho} < 1$ holds for any $\rho > 0$, so system (1) is uniformly p-ISS over $\mathcal{S}_{\min}(\rho)$ for any $\rho > 0$. In other words, system (1) is uniformly p-ISS over $\mathcal{S}_{\text{all}}$. The proof is complete. □

Remark 9. When $\mu > 1$, condition (iii) implies that the impulses may be destabilizing. So, in order to maintain the p-ISS property of system (1), the impulse interval is required to be large enough to reduce the effect of the impulses. When $\mu = 1$, the discrete dynamics are marginally stable for the zero input. In this case, the p-ISS of system (1) is not affected by the impulses.

With minor modification to the conditions of Theorem 8, a criterion on SISS can be obtained as follows.

Theorem 10. *Assume that conditions (ii) and (iii) of Theorem 8 hold, while condition (i) is replaced by*

(i)* $\alpha_1(|x|) \leqslant V(t,x) \leqslant \alpha_2(|x|)$,

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. *Then, for any given* $\rho > 0$ *satisfying* $\mu e^{-c\rho} < 1$, *system* (1) *is uniformly SISS over* $\mathcal{S}_{\min}(\rho)$. *In particular, when* $\mu = 1$, *system* (1) *is uniformly SISS over* $\mathcal{S}_{all}$.

Proof. By condition (i*), (10) can be replaced by

$$J(t) = \mathbb{E}V(t)\, e^{\lambda(t-t_0)} \leqslant \mathbb{E}\alpha_2(|x(t)|) \leqslant \mathbb{E}\alpha_2\left(\|\xi\|\right), \quad t \in [t_0 - \tau, t_0]. \tag{34}$$

Then, following the same lines of the proof of Theorem 8, it is easy to see that

$$\mathbb{E}V(t) \leqslant e^{-\lambda(t-t_0)} \mathbb{E}\alpha_2\left(\|\xi\|\right) + \frac{1}{c-c'} \chi_1\left(\|u_c\|_{[t_0,t]}\right) + \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right) \tag{35}$$

holds for all $t \geqslant t_0$. Consequently, by Chebyshev's inequality, it follows that

$$\begin{aligned} P\Bigg\{ & V(t) - e^{-\lambda(t-t_0)} \alpha_2\left(\|\xi\|\right) \\ & \geqslant \delta\Bigg(\frac{1}{c-c'} \chi_1\left(\|u_c\|_{[t_0,t]}\right) + \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right)\Bigg)\Bigg\} \\ & \leqslant \mathbb{E}V(t) - \mathbb{E}\alpha_2\left(\|\xi\|\right) e^{-\lambda(t-t_0)} \\ & \quad \times \Bigg(\delta\Bigg(\frac{1}{c-c'} \chi_1\left(\|u_c\|_{[t_0,t]}\right) + \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right)\Bigg)\Bigg)^{-1} \\ & \leqslant \varepsilon, \end{aligned} \tag{36}$$

where ε can be made arbitrarily small by an appropriate choice of $\delta \in \mathcal{K}_\infty$. That is,

$$\begin{aligned} P\Bigg\{ & V(t) < e^{-\lambda(t-t_0)} \alpha_2\left(\|\xi\|\right) \\ & + \delta\Bigg(\frac{1}{c-c'} \chi_1\left(\|u_c\|_{[t_0,t]}\right) + \frac{1}{1-e^{-\lambda\rho}} \chi_2 \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right)\Bigg)\Bigg\} \\ & > 1 - \varepsilon, \end{aligned} \tag{37}$$

which yields

$$P\Bigg\{ V(t) < \beta\left(\|\xi\|, t-t_0\right) + \gamma_c\left(\|u_c\|_{[t_0,t]}\right) + \gamma_d \left(\max_{t_k \in (t_0,t]} \left\{\left|u_d\left(t_k^-\right)\right|\right\}\right)\Bigg\} > 1 - \varepsilon, \tag{38}$$

where $\beta(r,s) = e^{-\lambda s}\alpha_2(r)$, $\gamma_c(r) = \delta((2/(c-c'))\chi_1(r))$, $\gamma_d(r) = \delta((2/(1-e^{-\lambda\rho}))\chi_2(r))$. By condition (i*), we know that (3) holds. Therefore, system (1) is uniformly SISS over $\mathcal{S}_{\min}(\rho)$ and the proof is complete. □

In view of Definitions 1–5, it is easy to obtain the following criteria on p-GAS and GASiP according to Theorems 8 and 10.

Corollary 11. *Assume that there exist functions $V \in v_0$, $\alpha_1 \in c\mathcal{K}_\infty$, $\alpha_2 \in v\mathcal{K}_\infty$ and scalars $q > 1$, $c > 0$, $\mu \in [1, q)$ such that*

(i) $\alpha_1(|x|^p) \leqslant V(t,x) \leqslant \alpha_2(|x|^p)$;

(ii) $\mathbb{E}\mathscr{L}V(t,\varphi) \leqslant -c\mathbb{E}V(t,\varphi(0))$, *for all* $t \geqslant t_0$, $t \neq t_k$ *and* $\varphi \in PC_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ *whenever* $\mathbb{E}V(t+\theta,\varphi(\theta)) \leqslant q\mathbb{E}V(t,\varphi(0))$;

(iii) $\mathbb{E}V(t_k, I_k(t_k^-, x, u_d)) \leqslant \mu\mathbb{E}V(t_k, x)$.

Then, for any given $\rho > 0$ satisfying $\mu e^{-c\rho}$, system (1) *is uniformly p-GAS over $\mathcal{S}_{\min}(\rho)$. In particular, when $\mu = 1$, system* (1) *is uniformly p-GAS over $\mathcal{S}_{all}$.*

Corollary 12. *Assume that conditions (ii) and (iii) of Corollary 11 hold, while condition (i) is replaced by*

(i)* $\alpha_1(|x|) \leqslant V(t,x) \leqslant \alpha_2(|x|)$,

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Then, for any given $\rho > 0$ satisfying $\mu e^{-c\rho} < 1$, system (1) *is uniformly GASiP over $\mathcal{S}_{\min}(\rho)$. In particular, when $\mu = 1$, system* (1) *is uniformly GASiP over $\mathcal{S}_{all}$.*

Now let us apply the obtained results to the linear impulsive stochastic delayed system with the following form:

$$\begin{aligned} dx &= \left(Ax(t) + A_1 x_\tau + Bu_c(t)\right)dt \\ &\quad + \left(Cx(t) + C_1 x_\tau + Du_c(t)\right)dw, \quad t \neq t_k, \\ x(t_k) &= Ex(t_k^-) + Fu_d(t_k^-), \quad k \in \mathbb{N}, \end{aligned} \tag{39}$$

on $t \geqslant t_0$ with initial data $x_{t_0} = \xi$, where $x \in \mathbb{R}^n$ and $u_c \in \mathscr{L}_\infty^{m_1}$, $u_d \in \mathscr{L}_\infty^{m_2}$ are system state and inputs, respectively; x_τ is short for $x(t-\tau)$; A, A_1, B, C, C_1, D, E, F are constant matrices with appropriate dimensions.

Corollary 13. *Assume that there exist a matrix $P > 0$ and constants $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 1$, $\lambda_5 > 0$ satisfying $\lambda_4 e^{(\lambda_1+\lambda_2 p)\rho} < 1$ such that the following matrix inequalities hold:*

$$\begin{bmatrix} A^TP + PA + C^TPC - \lambda_1 P & PA_1 + C^TPC_1 & PB + C^TPD \\ * & C_1^TPC_1 - \lambda_2 P & C_1^TPD \\ * & * & D^TPD - \lambda_3 I \end{bmatrix} \leqslant 0,$$

$$\begin{bmatrix} E^TPE - \lambda_4 P & E^TPF \\ * & F^TPF - \lambda_5 I \end{bmatrix} \leqslant 0. \tag{40}$$

Then system (39) *is uniformly ISS in mean square and uniformly SISS over $\mathcal{S}_{\min}(\rho)$.*

Proof. We choose the candidate ISS-Lyapunov function $V(t,x) = x^TPx$. By using (40), and in view of the fact that $\lambda_{\min}(P)|x|^2 \leqslant x^TPx \leqslant \lambda_{\max}(P)|x|^2$, we can obtain by simple calculation that

$$\begin{aligned} &\mathscr{L}V(t,x,u) \\ &= \begin{pmatrix} x \\ x_\tau \\ u_c \end{pmatrix}^T \\ &\quad \times \begin{pmatrix} A^TP + PA + C^TPC & PA_1 + C^TPC_1 & PB + C^TPD \\ * & C_1^TPC_1 & C_1^TPD \\ * & * & D^TPD \end{pmatrix} \\ &\quad \times \begin{pmatrix} x \\ x_\tau \\ u_c \end{pmatrix} \\ &\leqslant \lambda_1 x^TPx + \lambda_2 x_\tau^TPx_\tau + \lambda_3 u_c^Tu_c. \end{aligned} \tag{41}$$

So, whenever $\mathbb{E}V(t+\theta) \leqslant q\mathbb{E}V(t)$, we have

$$\mathbb{E}\mathscr{L}V(t,x,u) \leqslant (\lambda_1 + \lambda_2 q)\mathbb{E}V(t,x) + \lambda_3|u_c|^2. \tag{42}$$

On the other hand,

$$\begin{aligned} V(t_k, x(t_k)) &= \begin{pmatrix} x \\ u_d \end{pmatrix}^T \begin{pmatrix} E^TPE & E^TPF \\ * & F^TPF \end{pmatrix} \begin{pmatrix} x \\ u_d \end{pmatrix} \\ &\leqslant \lambda_4 x^TPx + \lambda_5 u_d^Tu_d. \end{aligned} \tag{43}$$

So,

$$\mathbb{E}V(t_k, x(t_k)) \leqslant \lambda_4\mathbb{E}V(t_k^-) + \lambda_5|u_d|^2. \tag{44}$$

It is obvious that all conditions of Theorem 8 are satisfied, with $c = -(\lambda_1 + \lambda_2 p)$ and $\mu = \lambda_4$. Therefore, we conclude by Theorems 8 and 10 that system (39) is uniformly p-ISS and uniformly SISS over $\mathcal{S}_{\min}(\rho)$. □

Remark 14. It is noted that (40) are not linear with the combined variables $(P, \lambda_1, \lambda_2, \lambda_4)$, and, therefore, they are not linear matrix inequalities (LMIs). This makes the computation difficult but also flexible. We can first assign λ_1, λ_2, and λ_4 and then solve LMIs (40) by using the Matlab LMI Toolbox.

4. Illustrative Example

In this section, to illustrate the validity of our results, we give the following linear numerical example. We point out that, due to the effect of the input u, the state $x(t)$ will not converge to zero but will remain bounded (in the sense of mean square or in probability), which is consistent with the definition of p-ISS/SISS.

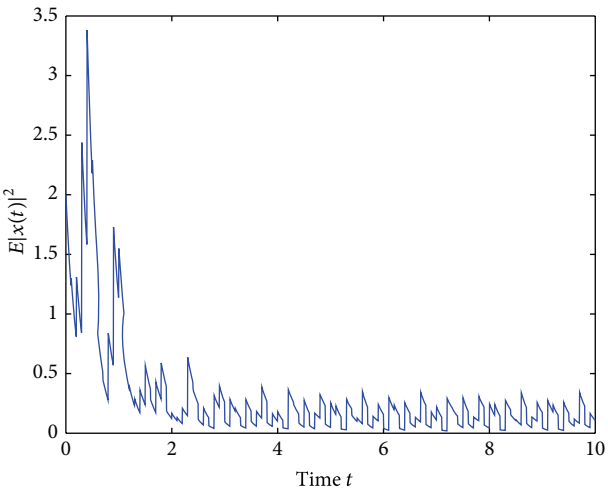

FIGURE 1: The mean square of the solution with external input (2000 samples).

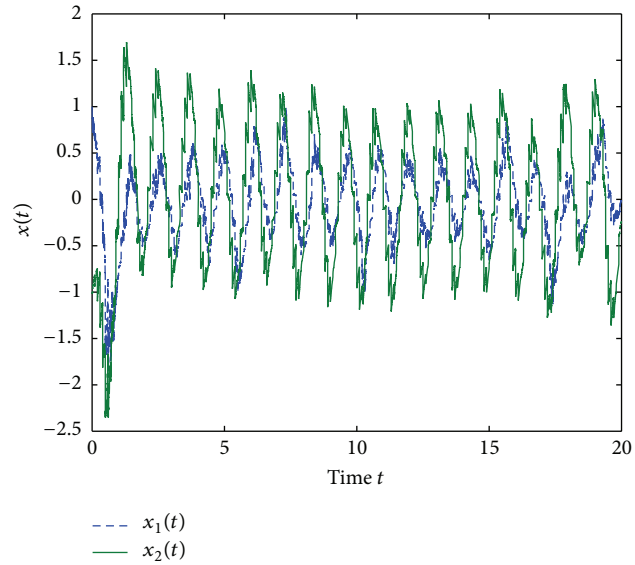

FIGURE 2: The state trajectory of the system with external input (single sample).

Example 1. Consider system (39) with the following parameters:

$$A = \operatorname{diag}(-3, -2.5), \qquad A_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$E = \begin{bmatrix} 1.1 & 0.2 \\ -0.3 & 1.2 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}. \tag{45}$$

Setting $\lambda_1 = -3.8$, $\lambda_2 = 0.2$, $\lambda_4 = 1.4$, and solving LMIs (40) by using the Matlab LMI Toolbox, then

$$P = \begin{bmatrix} 395.5597 & -72.7854 \\ 395.5597 & 269.6470 \end{bmatrix}, \qquad \lambda_3 = 0.9546, \qquad \lambda_5 = 3.5515 \tag{46}$$

is a group of feasible solutions. Choosing $p = 1.41 > \mu = \lambda_4 = 1.4$, $\rho = 0.1$, it is easy to check that all the conditions of Corollary 13 are satisfied, which means that the system is uniform ISS in mean square and uniform SISS for arbitrary sequence of impulse times satisfying $\inf\{t_k - t_{k-1}\} \geqslant 0.1$. The sample path and the mean square of the solution are displayed in Figures 1 and 2, respectively, where $\tau = 0.5$, initial data $\xi(\theta) = [1 \ \ -1]^{\mathrm{T}}$ for $\theta \in [-0.5, 0]$, and impulse interval $t_k - t_{k-1} = 0.1$ and external inputs $u_c(t) = u_d(t) = \sin t$.

As p-ISS/SISS implies p-GAS/GASiP of the corresponding unforced system, we conclude that the system with $u_c = u_d \equiv 0$ is uniform GAS in mean square and GASiP for arbitrary sequence of impulse times satisfying $\inf\{t_k - t_{k-1}\} \geqslant 0.1$. The simulations of the unforced system are shown in Figures 3 and 4.

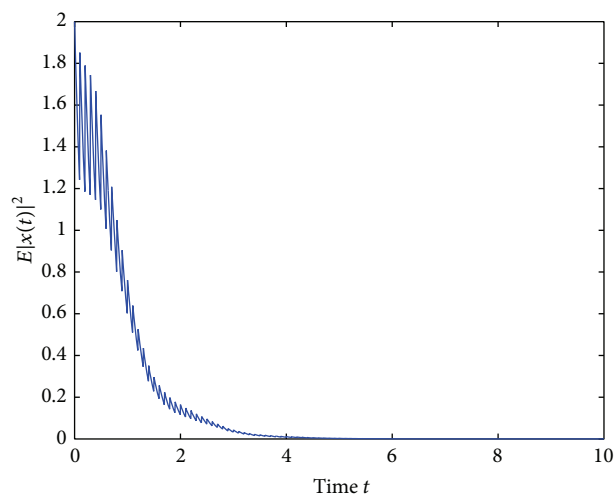

FIGURE 3: The mean square of the solution without external input (2000 samples).

5. Conclusions

This paper has investigated the p-ISS/SISS of impulsive stochastic systems with external inputs. By combining stochastic analysis techniques, piecewise continuous Lyapunov functions, and Razumikhin techniques, sufficient conditions for uniform p-ISS/SISS over a given class of impulse times sequences have been established. As a byproduct, the criteria on p-GAS/GASiP are also derived. For future research, interesting topics may include establishing p-ISS/SISS theorems with stabilizing impulses, as well as p-ISS/SISS analysis by exploring new techniques such as Lyapunov-Krosovskii functional method.

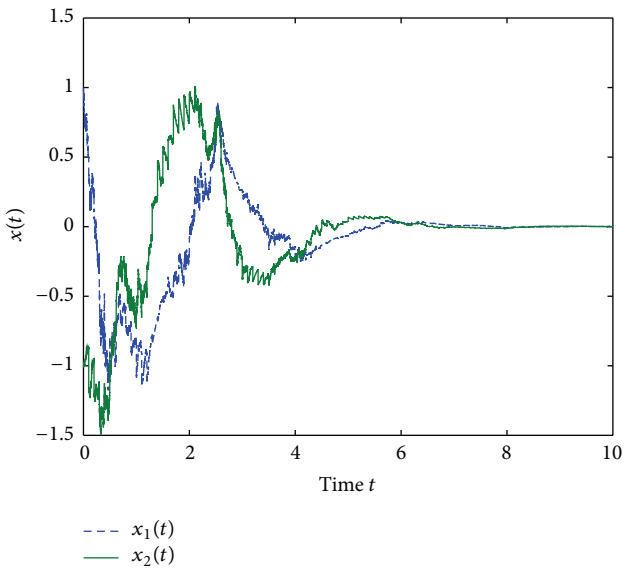

Figure 4: The state trajectory of the system without external input (single sample).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Absolute Stability of a Class of Nonlinear Singular Systems with Time Delay

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This paper deals with the absolute stability for a class of nonlinear singular systems with time delay. By employing a new Lyapunov-Krasovskii functional with the idea of partitioning delay length, improved delay-dependent stability criteria are established. The resulting condition is formulated in terms of linear matrix inequalities (LMIs), which is easy to be verified by exiting LMI optimization algorithms. A numerical example is given to show the effectiveness of the proposed technique and its improvements over the existing results.

1. Introduction

Since the concept of absolute stability and the Lur'e problem were introduced, the absolute stability of Lur'e control systems has received considerable attention and many rich results have been proposed during the last decades [1]. Time delays widely exist in practical systems, which is a source of instability and deteriorated performance [2–4]. Therefore, great efforts have been made to investigate the absolute stability of Lur'e systems with time delay and many results have been achieved [4–9].

Recently, an integral inequality approach was proposed to investigate the Lur'e system with time delay and new absolute stability criteria were obtained [7]. In addition, as it is impossible to reduce the conservatism of the derived conditions by employing simple Lyapunov-Krasovskii functional, some other efforts are made to improve the delay-dependent conditions via introducing new Lyapunov-Krasovskii functionals. For example, improved results for time delay systems were obtained by introducing the augmented Lyapunov-Krasovskii functional [10] and the delay-partitioning Lyapunov-Krasovskii functional [5]. By employing a discretized Lyapunov-Krasovskii functional, new absolute stability condition for a class of nonlinear neutral systems is derived in [11]. Although [11] can achieve less conservative results, the condition was much more complicated than those based on simple Lyapunov-Krasovskii functionals.

On the other hand, singular systems have been extensively studied in the past few years due to the fact that singular systems describe physical systems better than state-space ones [12–15]. Depending on the area of application, these models are also called descriptor systems, semistate systems, differential-algebraic systems, or generalized state-space systems. Therefore, the study of the absolute stability problem for the Lur'e singular system with time delay is of theoretical and practical importance [16].

In this paper, by employing the delay-partitioning approach proposed in [17], we construct a new Lyapunov-Krasovskii functional to investigate the absolute stability of Lur'e singular systems with time delay. Improved delay-dependent absolute stability criteria are presented. The criteria are easy to follow, and those criteria obtained in [16] by using simple Lyapunov-Krasovskii functional are involved in our results. Numerical example is given to demonstrate the advantage of the proposed method.

Notation. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; for a real matrix P, $P > 0$ (resp., $P < 0$) means that P is real symmetric and positive definite (resp., negative definite); I is an identity matrix of appropriate dimensions,

and the symmetric terms in a symmetric matrix are denoted by "$*$."

2. Problem Statement and Preliminaries

Consider the following system with time delay and sector-bounded nonlinearity:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bx(t-h) + Dw(t), \\ z(t) &= Mx(t) + Nx(t-h), \\ w(t) &= -\varphi(t, z(t)), \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system; $w(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^m$ are input vector and output vector, respectively; E, A, B, D, M, N are constant matrices, where E may be singular and it is assumed that rank $E = r \le n$ and that the scalar $h > 0$ is the delay of the system; the initial condition, $\phi(t)$, is a continuous vector-valued function of $t \in [-h, 0]$. $\varphi(t, z(t)) \in \mathbb{R}^m$ is a nonlinear function, which is piecewise continuous in t, globally Lipschitz in $z(t)$, $\varphi(t, 0) = 0$, and satisfies the following sector condition:

$$[\varphi(t, z(t)) - K_1 z(t)]^T [\varphi(t, z(t)) - K_2 z(t)] \le 0, \tag{2}$$

where K_1 and K_2 are constant real matrices and $K = K_2 - K_1$ is a symmetric positive definite matrix. It is customary that such a nonlinear function $\varphi(t, z(t))$ is said to belong to a sector $[K_1, K_2]$.

In this paper, we also investigate the robust absolute stability of the following uncertain system:

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A(t))x(t) + (B + \Delta B(t)) \\ &\quad \times x(t-h) + Dw(t), \\ z(t) &= Mx(t) + Nx(t-h), \\ w(t) &= -\varphi(t, z(t)), \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{3}$$

where the uncertainties are of the form

$$[\Delta A(t) \quad \Delta B(t)] = LF(t)[E_a \quad E_b], \tag{4}$$

where L, E_a, and E_b are constant matrices, and $F(t)$ is a time-varying matrix satisfying

$$F^T(t)F(t) \le I, \quad \forall t. \tag{5}$$

Next, the following definitions and lemmas are introduced, which will be used in the proof of the main results.

Definition 1 (see [12]). (i) The pair (E, A) is said to be regular if $\det(sE - A)$ is not identically zero. (ii) The pair (E, A) is said to be impulse-free if $\deg(\det(sE - A)) = \operatorname{rank} E$.

Definition 2 (see [12]). (i) The nonlinear singular system (1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free. (ii) The nonlinear singular system (1) is said to be globally uniformly asymptotically stable for any nonlinear function $\varphi(t, z(t))$ satisfying (2) if, for any $\epsilon > 0$, there exists a scalar $\delta(\epsilon)$ such that, for any compatible initial conditions $\phi(t)$ satisfying $\sup_{-h \le t \le 0} \|\phi(t)\| \le \delta(\epsilon)$, the solution $x(t)$ of the system (1) satisfies $\|x(t)\| \le \epsilon$ for $t \ge 0$. Furthermore, $\lim_{t \to \infty} x(t) = 0$.

Lemma 3 (see [18]). *Consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$; if φ is uniformly continuous and $\int_0^\infty \varphi(s)ds < \infty$, $\lim_{t \to \infty} \varphi(t) = 0$.*

Lemma 4 (see [19]). *For any symmetric positive-definite matrix $M \in \mathbb{R}^{n \times n}$ and a scalar $\gamma > 0$, if there exists a vector function $\omega(\alpha) : [-\gamma, 0] \to \mathbb{R}^n$ such that the following integrals are well defined, then*

$$\begin{aligned} &-\gamma \int_{-\gamma}^{0} \dot{\omega}(t+\alpha)^T E^T M E \dot{\omega}(t+\alpha)\, d\alpha \\ &\quad \le \begin{bmatrix} \omega(t) \\ \omega(t-\gamma) \end{bmatrix}^T \begin{bmatrix} -E^T M E & E^T M E \\ * & -E^T M E \end{bmatrix} \begin{bmatrix} \omega(t) \\ \omega(t-\gamma) \end{bmatrix}. \end{aligned} \tag{6}$$

Lemma 5 (see [20]). *Let H, E, and $F(t)$ be real matrices of appropriate dimensions with $F(t)$ satisfying $F^T(t)F(t) \le I$. Then, for any scalar $\varepsilon > 0$,*

$$HF(t)E + (HF(t)E)^T \le \varepsilon^{-1} HH^T + \varepsilon E^T E. \tag{7}$$

3. Main Results

Firstly, by means of the loop transformation suggested in [21], it can be concluded that the absolute stability of system (1) in the sector $[K_1, K_2]$ is equivalent to that of the following system in the sector $[0, K_2 - K_1]$:

$$\begin{aligned} E\dot{x}(t) &= \bar{A}x(t) + \bar{B}x(t-h) + Dw(t), \\ z(t) &= Mx(t) + Nx(t-h), \\ w(t) &= -\varphi(t, z(t)), \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{8}$$

where $\bar{A} = A - DK_1M$, $\bar{B} = B - DK_1N$.

Thus, for the absolute stability of system (1), we have the following result.

Theorem 6. *Given integer k and scalar $\tau = h/k > 0$, the system (1) with nonlinear connection function satisfying (2) is absolutely stable in the sector $[K_1, K_2]$ if there exist a scalar $\varepsilon > 0$, matrices*

$$\begin{gathered} P = P^T > 0, \\ Q_a = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1k} \\ * & Q_{22} & \cdots & Q_{2k} \\ * & * & \ddots & \vdots \\ * & * & * & Q_{kk} \end{bmatrix} \ge 0, \\ Z_i = Z_i^T > 0, \quad (i = 1, 2 \ldots, k), \end{gathered} \tag{9}$$

and a matrix S with appropriate dimensions, such that the following LMI holds:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ * & \Phi_{22} & 0 \\ * & * & \Phi_{33} \end{bmatrix} < 0, \tag{10}$$

where

$$\Phi_{11} = \begin{bmatrix} \Pi_1 & \widetilde{P}\bar{B} + E^T Z_1 E & \widetilde{P}D - \varepsilon M^T (K_2 - K_1)^T \\ * & -Q_{kk} - E^T Z_k E & -\varepsilon N^T (K_2 - K_1)^T \\ * & * & -2\varepsilon I \end{bmatrix},$$

$$\Phi_{12} = \begin{bmatrix} E^T Z_1 E + Q_{12} & Q_{13} & \cdots & Q_{1k} \\ -Q_{1k}^T & \cdots & -Q_{(k-2)k}^T & E^T Z_k E - Q_{(k-1)k}^T \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\Pi_1 = \widetilde{P}\bar{A} + \bar{A}^T \widetilde{P} + Q_{11} - E^T Z_1 E, \quad \widetilde{P} = E^T P + SR^T,$$

$$\Phi_{22} = \begin{bmatrix} \Lambda_1 & \bar{\Lambda}_1 & \bar{Q}_{13} & \cdots & \bar{Q}_{1(k-1)} \\ * & \Lambda_2 & \bar{\Lambda}_2 & \cdots & \vdots \\ * & * & \ddots & \ddots & \bar{Q}_{(k-3)(k-1)} \\ * & * & * & \Lambda_{k-2} & \bar{\Lambda}_{k-2} \\ * & * & * & * & \Lambda_{k-1} \end{bmatrix},$$

$$\bar{Q}_{ij} = Q_{(i+1)(j+1)} - Q_{ij}, \quad \Lambda_i = -E^T Z_i E - E^T Z_{i+1} E + \bar{Q}_{ii}, \quad (i, j = 1, 2, \ldots, k-1),$$

$$\bar{\Lambda}_i = E^T Z_{i+1} E + \bar{Q}_{i(i+1)}, \quad (i = 1, 2, \ldots, k-2),$$

$$\Phi_{13} = \tau \Gamma^T \sum_{i=1}^{k} Z_i, \quad \Phi_{33} = -\sum_{i=1}^{k} Z_i, \quad \Gamma = \begin{bmatrix} \bar{A} & \bar{B} & D \end{bmatrix} \tag{11}$$

and $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfying $R^T E = 0$.

Proof. Firstly, dividing the delay h into k equal segments, the length of each segment is denoted as τ; that is, $\tau = h/k$. Choosing a Lyapunov-Krasovskii functional is as follows:

$$\begin{aligned} V(t, x_t) &= x^T(t) E^T P E x(t) + \int_{t-\tau}^{t} \zeta_1^T(s) Q_a \zeta_1(s)\, ds \\ &\quad + \sum_{i=1}^{k} \int_{-i\tau}^{-(i-1)\tau} \int_{t+\theta}^{t} \tau \dot{x}^T(s) E^T Z_i E \dot{x}(s)\, ds\, d\theta, \end{aligned} \tag{12}$$

where

$$P > 0,$$

$$Q_a = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1k} \\ * & Q_{22} & \cdots & Q_{2k} \\ * & * & \ddots & \vdots \\ * & * & * & Q_{kk} \end{bmatrix} \geq 0, \tag{13}$$

$$Z_i > 0, \quad (i = 1, 2, \ldots, k)$$

are to be determined and $\zeta_1(t) = [x^T(t)\ x^T(t-\tau)\ \cdots\ x^T(t-(k-1)\tau)]^T$.

Calculating the derivative of each $V(t, x_t)$ along the solutions of system (8) yields

$$\begin{aligned} \dot{V}(t, x_t) &= x^T(t)\left(E^T P \bar{A} + \bar{A}^T P E\right) x(t) \\ &\quad + 2x^T(t) E^T P \bar{B} x(t-h) + 2x^T(t) E^T P D w(t) \\ &\quad + \zeta_1^T(t) Q_a \zeta_1(t) - \zeta_1^T(t-\tau) Q_a \zeta_1(t-\tau) \\ &\quad + \tau^2 \dot{x}^T(t) \sum_{i=1}^{k} E^T Z_i E \dot{x}(t) \\ &\quad - \sum_{i=1}^{k} \int_{t-i\tau}^{t-(i-1)\tau} \tau \dot{x}^T(s) E^T Z_i E \dot{x}(s)\, ds. \end{aligned} \tag{14}$$

Let $\theta = -\int_{t-i\tau}^{t-(i-1)\tau} \tau \dot{x}^T(s) E^T Z_i E \dot{x}(s) ds$; using Lemma 4, we have

$$\begin{aligned} \theta &\leq \begin{bmatrix} x(t-(i-1)\tau) \\ x(t-i\tau) \end{bmatrix}^T \begin{bmatrix} -E^T Z_i E & E^T Z_i E \\ * & -E^T Z_i E \end{bmatrix} \\ &\quad \times \begin{bmatrix} x(t-(i-1))\tau \\ x(t-i\tau) \end{bmatrix}. \end{aligned} \tag{15}$$

From (1) and (2), for $\varphi(t, z(t)) \in [0, K_2 - K_1]$ and a scalar $\varepsilon > 0$, it can be deduced that

$$\begin{aligned} 0 &\leq -2\varepsilon w^T(t) w(t) - 2\varepsilon w^T(t)(K_2 - K_1) \\ &\quad \times [Mx(t) + Nx(t-h)]. \end{aligned} \tag{16}$$

Noting that $R^T E = 0$, we can deduce

$$0 = R^T \bar{A} x(t) + R^T \bar{B} x(t-h) + R^T D w(t). \tag{17}$$

From (14)–(17), we get

$$\dot{V}(t, x_t) \leq \zeta^T(t) \left[\Psi + \tau^2 \bar{\Gamma}^T \sum_{i=1}^{k} Z_i \bar{\Gamma}\right] \zeta(t), \tag{18}$$

where

$$\Psi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix}, \qquad \bar{\Gamma} = \begin{bmatrix} \bar{A} & \bar{B} & D & 0 \end{bmatrix}$$

$$\zeta(t) = \begin{bmatrix} x^T(t) & x^T(t-h) & w^T(t) & \zeta_2^T(t) \end{bmatrix}^T,$$

$$\zeta_2^T(t) = \begin{bmatrix} x^T(t-\tau) & x^T(t-2\tau) & \cdots & x^T(t-(k-1)\tau) \end{bmatrix}^T. \tag{19}$$

If $\Psi + \tau^2 \bar{\Gamma}^T \sum_{i=1}^{k} Z_i \bar{\Gamma} < 0$, which is equivalent to (10) by Schur complements [22], then $\dot{V}(t, x_t) < 0$ holds.

In what follows, we show that the nonlinear singular system (1) is regular and impulse-free. Since rank $E = r \leq n$, there exist two invertible matrices G and $H \in \mathbb{R}^{n\times n}$ such that

$$\bar{E} = GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{20}$$

Then, R can be parameterized as

$$R = G^T \begin{bmatrix} 0 \\ \bar{\Phi} \end{bmatrix}, \tag{21}$$

where $\bar{\Phi} \in \mathbb{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix.

Like in (20), we define

$$\begin{aligned} \bar{A} &= GAH = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \\ \bar{P} &= G^{-T}PG^{-1} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}, \\ \bar{Z}_i &= G^{-T}Z_iG^{-1} = \begin{bmatrix} \bar{Z}_{i11} & \bar{Z}_{i12} \\ \bar{Z}_{i21} & \bar{Z}_{i22} \end{bmatrix}, \quad (i = 1, 2, \ldots, k), \\ \bar{S} &= H^TS = \begin{bmatrix} \bar{S}_{11} \\ \bar{S}_{21} \end{bmatrix}, \qquad \bar{R} = G^{-T}R = \begin{bmatrix} 0 \\ \bar{\Phi} \end{bmatrix}. \end{aligned} \tag{22}$$

Since $A^T(PE + RS^T) + (E^TP + SR^T)A + Q_{11} - E^TZ_1E < 0$ and $Q_{11} \geq 0$, we can formulate the following inequality easily:

$$\psi = A^T\left(PE + RS^T\right) + \left(E^TP + SR^T\right)A - E^TZ_1E < 0. \tag{23}$$

Pre- and postmultiplying $\psi < 0$ by H^T and H, respectively, yield

$$\begin{aligned} \bar{\psi} &= H^T\psi H = \bar{A}^T\bar{P}\,\bar{E} + \bar{A}^T\bar{R}\,\bar{S}^T + \bar{E}^T\bar{P}\,\bar{A} + \bar{S}\,\bar{R}^T\bar{A} - \bar{E}^T\bar{Z}_1\bar{E} \\ &= \begin{bmatrix} \bar{\psi}_{11} & \bar{\psi}_{12} \\ * & \bar{A}_{22}^T\bar{\Phi}\,\bar{S}_{21}^T + \bar{S}_{21}\bar{\Phi}^T\bar{A}_{22} \end{bmatrix} < 0. \end{aligned} \tag{24}$$

As $\bar{\psi}_{11}$ and $\bar{\psi}_{12}$ are irrelevant to the results of the following discussion, the expressions about these two variables are omitted here. It is easy to deduce from (24) that

$$\bar{A}_{22}^T\bar{\Phi}\,\bar{S}_{21}^T + \bar{S}_{21}\bar{\Phi}^T\bar{A}_{22} < 0 \tag{25}$$

and thus $\bar{A}_{22}$ is nonsingular. Otherwise, supposing A_{22} is singular, there must exist a nonzero vector $\varsigma \in \mathbb{R}^{n-r}$ which ensures that $\bar{A}_{22}\varsigma = 0$. And then it can be concluded that $\varsigma^T(\bar{A}_{22}^T\bar{\Phi}\,\bar{S}_{21}^T + \bar{S}_{21}\bar{\Phi}^T\bar{A}_{22})\varsigma = 0$, and this contradicts (25). So $\bar{A}_{22}$ is nonsingular. Then, it can be shown that

$$\begin{aligned} \det(sE - A) &= \det\left(G^{-1}\right)\det\left(s\bar{E} - \bar{A}\right)\det\left(H^{-1}\right) \\ &= \det\left(G^{-1}\right)\det\left(-\bar{A}_{22}\right) \\ &\quad \times \det\left(sI_r - \left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)\right)\det\left(H^{-1}\right) \end{aligned} \tag{26}$$

which implies that $\det(sE - A)$ is not identically zero and $\deg(\det(sE - A)) = r = \operatorname{rank} E$. Then, the pair of (E, A) is regular and impulse-free, which implies that system (1) is regular and impulse-free.

Defining $\xi(t) = \left[\begin{smallmatrix} \xi_1(t) \\ \xi_2(t) \end{smallmatrix}\right] = H^{-1}x(t)$, then we have

$$\begin{aligned} \bar{\lambda}_1\left\|\xi_1(t)\right\|^2 - V(x(0)) &\leq \xi^T(t)\,\bar{E}^T\bar{P}\,\bar{E}\xi(t) - V(x(0)) \\ &= x^T(t)\,E^TPEx(t) - V(x(0)) \\ &\leq V(x(t)) - V(x(0)) \\ &= \int_0^t \dot{V}(x(s))\,ds \\ &\leq -\bar{\lambda}_2\int_0^t \|x(s)\|^2 ds \\ &\leq -\bar{\lambda}_2\|H\|^2\int_0^t \left\|\xi(s)\right\|^2 ds, \end{aligned} \tag{27}$$

where $\bar{\lambda}_1 = \lambda_{\min}(\bar{P}_{11})$, $\bar{\lambda}_2 = -\lambda_{\max}(\Phi)$.

Taking into account (27), we can deduce that

$$\bar{\lambda}_1\left\|\xi_1(t)\right\|^2 + \bar{\lambda}_2\|H\|^2\int_0^t \left\|\xi(s)\right\|^2 ds \leq V(x(0)). \tag{28}$$

Noting that $\|x(t)\|$ and $\int_0^t \|x(s)\|^2 ds$ are bounded, it follows that $\|\xi(t)\|$ and $\int_0^t \|\xi(s)\|^2 ds$ are bounded; from Lemma 3, one can conclude that $\lim_{t\to\infty}\xi(t) = 0$; thus $\lim_{t\to\infty}x(t) = 0$. According to Definition 2, the singular system (8) is globally uniformly asymptotically stable for $\varphi(t, z(t)) \in [0, K_2 - K_1]$. Thus the singular system (8) is absolutely stable in the sector $[0, K_2 - K_1]$, which is equivalent to the absolute stability of system (1) in the sector $[K_1, K_2]$. This completes the proof. □

For uncertain system (3), substituting $A + LF(t)E_a$ and $B + LF(t)E_b$ for A and B in (10) and utilizing Lemma 5 and Schur complements [22], we have the following result.

Theorem 7. *Given integer k and scalar $\tau = h/k > 0$, the system* (3) *with nonlinear connection function satisfying* (2) *and time-varying structured uncertainties satisfying* (4) *is robustly absolutely stable in the sector $[K_1, K_2]$ if there exist scalars $\varepsilon > 0, \lambda > 0$, matrices*

$$\begin{gathered} P = P^T > 0, \\ Q_a = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1k} \\ * & Q_{22} & \cdots & Q_{2k} \\ * & * & \ddots & \vdots \\ * & * & * & Q_{kk} \end{bmatrix} \geq 0, \\ Z_i = Z_i^T > 0, \quad (i = 1, 2 \ldots, k), \end{gathered} \tag{29}$$

and a matrix S with appropriate dimensions, such that the following LMI holds:

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \widehat{P}L & \lambda\widehat{E} \\ * & \Phi_{22} & 0 & 0 & 0 \\ * & * & \Phi_{33} & \tau\sum_{i=1}^{k} Z_iL & 0 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0, \tag{30}$$

Table 1: Maximum upper bounds of h.

	α					
	0.15	0.5	1	1.5	2.5	3.5
[16, Theorem 3]	2.6556	2.3358	1.9489	1.6352	1.1780	0.8749
Theorem 7, $k = 2$	3.7209	3.2715	2.7269	2.2848	1.6396	1.2103
Theorem 7, $k = 3$	3.9351	3.4592	2.8824	2.4141	1.7303	1.2748
Theorem 7, $k = 4$	4.0114	3.5262	2.9379	2.4600	1.7625	1.2976

where

$$\widehat{P} = \begin{bmatrix} \left(E^T P + SR^T\right) \\ 0 \\ 0 \end{bmatrix}, \qquad \widehat{E} = \begin{bmatrix} E_a^T \\ E_b^T \\ 0 \end{bmatrix} \tag{31}$$

and $R \in \mathbb{R}^{n\times(n-r)}$ *is any matrix with full column rank and satisfying* $R^T E = 0$; $\Phi_{11}, \Phi_{12}, \Phi_{13}, \Phi_{22}, \Phi_{33}$ *are defined in Theorem 6.*

Remark 8. It is worth mentioning that the conservatism is reduced with the increase of k. At the same time, more matrix variables are involved in the corresponding LMI, which will increase the computing complexity.

Remark 9. In [5], some absolute stability conditions have been obtained for Lur'e system with time delay based on a delay-partitioning approach. However, the results proposed in this paper achieve some improvement and are more general than [5]. Let $E = I$, $S = 0$, and $Q_{ij} = 0$ $(i \neq j)$ in (30); Theorem 7 reduces to Theorem 3 in [5].

4. Numerical Example

In this section, we provide a numerical example to demonstrate the effectiveness of the proposed method.

Example 10. Consider uncertain system (3) with the following parameters:

$$\begin{gathered} E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}, \\ B = \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \\ M = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad N = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \\ K_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \\ L = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad \alpha \geq 0, \qquad E_a = E_b = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \end{gathered} \tag{32}$$

In this example, we choose $R = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. For various α, the maximum upper bounds of time delay obtained by Theorem 7 are listed in Table 1 in comparison with those obtained by [16]. It is clear that our approach provides larger stability region than [16]. Furthermore, it is concluded from the table that larger upper bounds of h can be obtained as k increases.

5. Conclusions

The absolute stability problem has been investigated for time delay singular systems with sector-bounded nonlinearity. Some improved conditions have been derived based on the delay-partitioning approach. A numerical example has been given to verify the effectiveness of the proposed methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Asynchronous Gossip-Based Gradient-Free Method for Multiagent Optimization

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This paper considers the constrained multiagent optimization problem. The objective function of the problem is a sum of convex functions, each of which is known by a specific agent only. For solving this problem, we propose an asynchronous distributed method that is based on gradient-free oracles and gossip algorithm. In contrast to the existing work, we do not require that agents be capable of computing the subgradients of their objective functions and coordinating their step size values as well. We prove that with probability 1 the iterates of all agents converge to the same optimal point of the problem, for a diminishing step size.

1. Introduction

In recent years, the problem of solving convex optimization problems over a network has attracted a lot of research attention; see [1–18]. The objective function of the problem is a sum of convex functions, each of which is known by a specific agent only. Such problems arise in many real applications including distributed finite-time optimal rendezvous [2] and distributed regression over sensor networks [5]. The methods that are designed for solving these optimization problems need to be fully distributed; that is, there does not exist a central coordinator.

In this paper, we propose an asynchronous gossip-based gradient-free method for solving the convex optimization problem over a multiagent network. The method is based on the gossip algorithm [19] and the gradient-free oracles [20]. The method is asynchronous in the sense that only one agent communicates at a given time, in contrast to the synchronous methods where all agents communicate simultaneously. Moreover, the method does not rely on the assumption that the information of the subgradients of the objective function is available. As is well known that for a variety of reasons there have been many instances where derivatives of the objective functions are unavailable or computationally expensive to calculate [20, 21].

Literature Review. In [3], the authors study the problem of minimizing a sum of multiple convex functions, each of which is known to one specific agent only. The authors use the average consensus algorithm in the literature on multiagent systems (see, e.g., [19, 22–26]) as a mechanism to develop a distributed subgradient method for solving the optimization problem; the convergence of the method is also given for a constant step size. The authors in [7] further take the global equality and inequality constraints into consideration. The work in [2] proposes a variant of the distributed subgradient method in [3], in which at each iteration several consensus steps are executed, which simplifies the proof of the convergence of the method. Inspired by the work in [2], the authors in [6] further incorporate the global inequality constraints. The aforementioned methods are synchronous because they require that all agents in the network update at the same time. To overcome this limitation, the work in [14] develops an asynchronous distributed algorithm, based on the gossip algorithm. The algorithm is asynchronous in the sense that only one agent communicates at a given time. Moreover, all agents use different step size values and they do not require

any coordination of the agents. In [5], the author further removes the need for bidirectional communications of the asynchronous algorithm in [14]; the convergence of the algorithm is also established. The aforementioned methods or algorithms, however, rely on the assumption that the subgradients of the objective functions are available to each agent, respectively.

By comparison to previous work, the main contributions of this paper are twofold: (i) different from the methods or algorithms considered in existing papers, which rely on computing the subgradients of each agent's objective function, we propose the derivative-free method which is based on utilizing the random gradient-free oracles; (ii) the proposed method is asynchronous, in the sense that all agents use different step size values that do not require any coordination of the agents. We prove that with probability 1 the iterates of all agents converge to the same optimal point of the problem, for a diminishing step size.

Notation and Terminology. Let $\mathbb{R}^d$ be the d-dimensional vector space. We denote the standard inner product on $\mathbb{R}^d$ by $\langle a, b\rangle = \sum_{i=1}^{d} a_i b_i$, for $a, b \in \mathbb{R}^d$. We write $\|x\|$ to denote the Euclidean norm of a vector x and $\Pi_{\mathcal{X}}[x]$ to denote the Euclidean projection of a vector x on $\mathcal{X}$. We use x^{T} to denote the transpose of x. For a matrix P, $[\mathsf{P}]_{ij}$ represents the element in the ith row and jth column of P, and P^{T} represents its transpose. We use $\mathbb{E}[x]$ to denote the expected value of a random variable x. For a function f, its gradient at a point x is represented by $\nabla f(x)$.

2. Problem Formulation

In this section, we start by describing the constrained multiagent optimization problem. Then, we provide some preliminary results on the gossip algorithm that we use in developing the method.

2.1. Constrained Multiagent Optimization. We consider the following constrained multiagent optimization problem:

$$\min_{x\in\mathcal{X}} f(x) \triangleq \sum_{i=1}^{N} f^i(x), \tag{1}$$

where $x \in \mathbb{R}^d$ is a decision vector; $f^i\colon \mathbb{R}^d \to \mathbb{R}$ is the convex objective function of agent i known only by agent i, and we assume that f^i is Lipschitz continuous over $\mathcal{X}$ with Lipschitz constant $L(f^i)$; $\mathcal{X} \subseteq \mathbb{R}^d$ is a nonempty closed convex set. We denote the optimal set of problem (1) by $\mathcal{X}^*$, and we assume that it is nonempty. Note that in problem (1), each function f^i need not be differentiable.

2.2. Gossip Algorithm. The underlying network topology of problem (1) is denoted by $G = (V, E)$, where $V = \{1, \ldots, N\}$ is the node set and E is the set of links $\{i, j\}$ with $i \neq j$ and $\{i, j\} \in E$ only if there is a link between agents i and j. We assume that the network G is fixed, undirected, and connected.

In the paper, we utilize gossip algorithm as a mechanism to design the method. To be specific, at each time instant, agent i is chosen with probability $1/N$, and then with some positive probability, agent i communicates with one of its neighbors agent j. The iterations evolve as follows: for $k \geq 0$,

$$x_{k+1}^i = x_{k+1}^j = \frac{1}{2}x_k^i + \frac{1}{2}x_k^j \tag{2}$$

and for agents s that do not belong to $\{i, j\}$, update

$$x_{k+1}^s = x_k^s. \tag{3}$$

3. Gossip-Based Gradient-Free Method

In this section, motivated by the random gradient-free method in [20] and the gossip algorithm in [19], we present an asynchronous gossip-based gradient-free method for solving problem (1). We use I_{k+1} to denote the index of the agent that is chosen to update at time $k+1$ and J_{k+1} the index of the agent communicating with agent I_{k+1}. The method is given as follows.

Gossip-Based Gradient-Free Method with a Diminishing Step Size

Initialize: choose random $x_0^i \in \mathcal{X}$, $\forall i \in V$.
Iteration ($k \geq 0$):

(i) for $i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}$:

(1) compute $\varphi_{k+1}^i = (1/2)x_k^{\mathsf{I}_{k+1}} + (1/2)x_k^{\mathsf{J}_{k+1}}$;

(2) compute $x_{k+1}^i = \Pi_{\mathcal{X}}[\varphi_{k+1}^i - \sigma_k^i \mathsf{G}_{\mu_k^i}(x_k^i)]$, where $\sigma_k^i = (\Sigma_k^i)^{-1}$, and Σ_k^i denotes the number of updates that agent i has performed until time k, inclusively, and $\mathsf{G}_{\mu^i}(x_k^i)$ is the random gradient-free oracle, given by

$$\mathsf{G}_{\mu_k^i}\left(x_k^i\right) = \frac{f^i\left(x_k^i + \mu_k^i \nu_k^i\right) - f^i\left(x_k^i\right)}{\mu_k^i}\nu_k^i, \tag{4}$$

where $\mu_k^i = \mu\sigma_k^i$, and μ is a positive constant; ν_k^i is a random variable generated locally according to the Gaussian distribution.

(ii) For $i \notin \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}$: $x_{k+1}^i = x_k^i$.

We use $\mathcal{F}_k$ to denote the σ-field generated by the entire history of the random variables to iteration k; that is,

$$\mathcal{F}_k = \left\{x_0^i, i \in V\right\} \cup \left\{\mathsf{I}_{s+1}, \mathsf{J}_{s+1}, \nu_s^{\mathsf{I}_{s+1}}, \nu_s^{\mathsf{J}_{s+1}}; 0 \leq s \leq k-1\right\}, \tag{5}$$

where $\mathcal{F}_0 = \{x_0^i, i \in V\}$.

The method can be presented in a more compact form, by defining the following weight matrix:

$$\mathsf{W}_{k+1} = I - \frac{1}{2}\left(e_{\mathsf{I}_{k+1}} - e_{\mathsf{J}_{k+1}}\right)\left(e_{\mathsf{I}_{k+1}} - e_{\mathsf{J}_{k+1}}\right)^{\mathsf{T}}, \quad k \geq 0, \tag{6}$$

where I is the identity matrix and $e_i \in \mathbb{R}^N$ denotes the ith standard basis vector. It is easy to see that $\mathsf{W}_{k+1} \in \mathbb{R}^{N\times N}$ is doubly stochastic.

Now we can write the method as follows: for all $k \geq 0$ and any $i \in V$,

$$\begin{aligned} \varphi_{k+1}^i &= \sum_{j=1}^N [\mathsf{W}_{k+1}]_{ij} x_k^j, \\ x_{k+1}^i &= \varphi_{k+1}^i + \left[\Pi_{\mathcal{X}}\left[\varphi_{k+1}^i - \sigma_k^i \mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right] - \varphi_{k+1}^i\right] \\ &\quad \times \mathbf{1}_{\{i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}\}}, \end{aligned} \tag{7}$$

where $\mathbf{1}_{\{i\in\{\mathsf{I}_{k+1},\mathsf{J}_{k+1}\}\}}$ is the indicator function of the event $\{i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}\}$. For the gradient-free oracle $\mathsf{G}_{\mu_k^i}(x_k^i)$, we have the following lemma, which is adopted from [20].

Lemma 1. *For each $i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}$ and all $k \geq 0$, one has the following:*

(a) $\mathbb{E}[\mathsf{G}_{\mu_k^i}(x_k^i) \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}] = \nabla f_{\mu_k^i}^i(x_k^i)$, *where* $f_{\mu_k^i}^i(x) = (1/\kappa)\int_{\mathbb{R}^d} f^i(x + \mu_k^i \xi) e^{-(1/2)\|\xi\|^2} d\xi$ *with* $\kappa = \int_{\mathbb{R}^d} e^{-(1/2)\|\xi\|^2} d\xi = (2\pi)^{d/2}$, *and it satisfies:*

$$f^i(x) \leq f_{\mu_k^i}^i(x) \leq f^i(x) + \mu_k^i \sqrt{d} L\left(f^i\right). \tag{8}$$

(b) $\mathbb{E}[\|\mathsf{G}_{\mu_k^i}(x_k^i)\|^2 \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}] \leq (d+4)^2 L^2(f^i)$.

Remark 2. Note that method (7) is asynchronous, in the sense that to implement the method, each agent need not coordinate its step size with the step sizes of its neighbors; the time-varying parameters μ_k^i $(k \geq 0, i \in V)$ share the same feature. In addition, to implement the method (7), the information of subgradients of the objective functions is not needed; however, each agent only needs to make two function evaluations per iteration to get the gradient-free oracle.

Let $\mathcal{E}_k^i = \{i \in \{\mathsf{I}_k, \mathsf{J}_k\}\}$ be the event that agent i updates at time k and π^i the probability of event $\mathcal{E}_k^i$. It is easy to see that

$$\pi^i = \frac{1}{N} + \frac{1}{N}\sum_{j \in N^i} \pi_{ji}, \tag{9}$$

where N^i denotes the set that contains all agents that are neighboring to agent i and $\pi_{ji} > 0$ denotes the probability that agent i is chosen by its neighbor j to communicate. In the paper, we denote $\check{\pi} = \min_{i\in V}\pi^i$ and $\hat{\pi} = \max_{i\in V}\pi^i$, respectively. There is an interesting link between the step size $\sigma_k^i = (\Sigma_k^i)^{-1}$ and the probability π^i that agent i updates.

Lemma 3 (see [17]). *Let $\pi_{\min} = \min_{\{i,j\}\in E}\pi_{ij}$. Let $\sigma_k^i = (\Sigma_k^i)^{-1}$ for all $k \geq 1$ and $i \in V$, and also let e be a scalar such that $0 < e < 1/2$. Then, there exists a large enough $\tilde{k} = \tilde{k}(e, N)$ such that with probability 1 for all $k \geq \tilde{k}$ and $i \in V$,*

(a) $\sigma_k^i \leq 2/k\pi^i$;

(b) $(\sigma_k^i)^2 \leq 4N^2/k^2(1+\pi_{\min})^2$;

(c) $|\sigma_k^i - (1/k\pi^i)| \leq 2/k^{3/2-e}(1+\pi_{\min})^2$.

To establish the convergence of method (7), we also make use of the following lemma.

Lemma 4 (see [5]). *Let $\{u_k\}$, $\{v_k\}$, $\{a_k\}$, and $\{w_k\}$ be nonnegative random sequences such that for all $k \geq 1$, $\mathbb{E}[u_{k+1} \mid F_k] \leq (1+a_k)u_k - v_k + w_k$ with probability 1, where $F_k = \{\{u_s, v_s, a_s, w_s\}; 1 \leq s \leq k\}$. If $\sum_{k=1}^\infty a_k < \infty$ and $\sum_{k=1}^\infty w_k < \infty$ with probability 1, then, with probability 1, the sequence $\{u_k\}$ converges to some random variable and $\sum_{k=1}^\infty v_k < \infty$.*

We now present the main result of the paper, which is given in the following theorem.

Theorem 5. *Let $\{x_k^i\}$, $i \in V$, be the sequences generated by method (7) with $\sigma_k^i = (\Sigma_k^i)^{-1}$ and $\mu_k^i = \mu\sigma_k^i$, where μ is some positive constant. Assume that problem (1) has a nonempty optimal set $\mathcal{X}^*$. Also, assume that the sequence $\{\nu_k^i; i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}\}$ is independent and identically distributed. Then the sequences $\{x_k^i\}$, $i \in V$, converge to the same random point in $\mathcal{X}^*$ with probability 1.*

Proof. For $k \geq 0$ and $i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}$, we have for any $x \in \mathcal{X}$,

$$\begin{aligned} \left\|x_{k+1}^i - x\right\|^2 &= \left\|\Pi_{\mathcal{X}}\left[\varphi_{k+1}^i - \sigma_k^i \mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right] - x\right\|^2 \\ &\leq \left\|\varphi_{k+1}^i - \sigma_k^i \mathsf{G}_{\mu_k^i}\left(x_k^i\right) - x\right\|^2 \\ &\leq \left\|\varphi_{k+1}^i - x\right\|^2 + \left(\sigma_k^i\right)^2 \left\|\mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right\|^2 \\ &\quad - 2\sigma_k^i \left\langle \mathsf{G}_{\mu_k^i}\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle \\ &= \left\|\varphi_{k+1}^i - x\right\|^2 + \left(\sigma_k^i\right)^2 \left\|\mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right\|^2 \\ &\quad - \frac{2}{k\pi^i}\left\langle \mathsf{G}_{\mu_k^i}\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle \\ &\quad - 2\left(\sigma_k^i - \frac{1}{k\pi^i}\right)\left\langle \mathsf{G}_{\mu_k^i}\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle, \end{aligned} \tag{10}$$

where the first inequality follows from the nonexpansive property of the projection operation. For $k \geq \tilde{k}$, by recalling Lemma 3(c), with probability 1 the last term on the right-hand side of (10) can be bounded as follows:

$$\begin{aligned} &-2\left(\sigma_k^i - \frac{1}{k\pi^i}\right)\left\langle \mathsf{G}_{\mu_k^i}\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle \\ &\quad \leq \frac{2}{k^{3/2-e}\left(1+\pi_{\min}\right)^2}\left(\left\|\mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right\|^2 + \left\|\varphi_{k+1}^i - x\right\|^2\right). \end{aligned} \tag{11}$$

Substituting the preceding inequality into (10) gives

$$\begin{aligned} \left\|x_{k+1}^i - x\right\|^2 &\leq \left(1 + \frac{2}{k^{3/2-e}\left(1+\pi_{\min}\right)^2}\right)\left\|\varphi_{k+1}^i - x\right\|^2 \\ &\quad + \left(\left(\sigma_k^i\right)^2 + \frac{2}{k^{3/2-e}\left(1+\pi_{\min}\right)^2}\right)\left\|\mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right\|^2 \\ &\quad - \frac{2}{k\pi^i}\left\langle \mathsf{G}_{\mu_k^i}\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle. \end{aligned} \tag{12}$$

To simplify the notation, we denote $A_k = 2/k^{3/2-e}(1+\pi_{\min})^2$ and $B_k = 4N^2/k^2(1+\pi_{\min})^2 + A_k$; then from Lemma 3(b) and (12) it follows that with probability 1 for all $k \geq \tilde{k}$ and $i \in \{\mathsf{I}_{k+1}, \mathsf{J}_{k+1}\}$,

$$\begin{aligned}\left\|x_{k+1}^i - x\right\|^2 \leq (1+A_k)\left\|\varphi_{k+1}^i - x\right\|^2 + B_k\left\|\mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right\|^2 \\ - \frac{2}{k\pi^i}\left\langle \mathsf{G}_{\mu_k^i}\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle.\end{aligned} \tag{13}$$

Taking the conditional expectation on $\mathcal{F}_k$, I_{k+1} and J_{k+1} jointly yields

$$\begin{aligned}&\mathbb{E}\left[\left\|x_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}\right] \\ &\quad\leq (1+A_k)\left\|\varphi_{k+1}^i - x\right\|^2 \\ &\qquad + B_k\mathbb{E}\left[\left\|\mathsf{G}_{\mu_k^i}\left(x_k^i\right)\right\|^2 \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}\right] \\ &\qquad - \frac{2}{k\pi^i}\left\langle \mathbb{E}\left[\mathsf{G}_{\mu_k^i}\left(x_k^i\right) \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}\right], \varphi_{k+1}^i - x\right\rangle \\ &\quad\leq (1+A_k)\left\|\varphi_{k+1}^i - x\right\|^2 + B_k(d+4)^2L^2\left(f^i\right) \\ &\qquad - \frac{2}{k\pi^i}\left\langle \nabla f_{\mu_k^i}^i\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle,\end{aligned} \tag{14}$$

where the last inequality follows from using Lemma 1. For the last term on the right-hand side of the preceding inequality, we can derive

$$\begin{aligned}&-\frac{2}{k\pi^i}\left\langle \nabla f_{\mu_k^i}^i\left(x_k^i\right), \varphi_{k+1}^i - x\right\rangle \\ &\quad= -\frac{2}{k\pi^i}\left\langle \nabla f_{\mu_k^i}^i\left(x_k^i\right), x_k^i - x\right\rangle \\ &\qquad - \frac{2}{k\pi^i}\left\langle \nabla f_{\mu_k^i}^i\left(x_k^i\right), \varphi_{k+1}^i - x_k^i\right\rangle \\ &\quad\leq -\frac{2}{k\pi^i}\left[f_{\mu_k^i}^i\left(x_k^i\right) - f_{\mu_k^i}^i(x)\right] \\ &\qquad + \frac{2}{k\pi^i}\left\|\nabla f_{\mu_k^i}^i\left(x_k^i\right)\right\|\left\|\varphi_{k+1}^i - x_k^i\right\| \\ &\quad\leq -\frac{2}{k\pi^i}\left[f_{\mu_k^i}^i\left(x_k^i\right) - f_{\mu_k^i}^i(x)\right] \\ &\qquad + \frac{2}{k\pi^i}(d+4)L\left(f^i\right)\left\|\varphi_{k+1}^i - x_k^i\right\|,\end{aligned} \tag{15}$$

where in the last inequality we have use the bound $\|\nabla f_{\mu_k^i}^i(x_k^i)\| \leq (d+4)L(f^i)$, according to Lemma 1. Hence, substituting (15) into (14) yields

$$\begin{aligned}&\mathbb{E}\left[\left\|x_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}\right] \\ &\quad\leq (1+A_k)\left\|\varphi_{k+1}^i - x\right\|^2 + B_k(d+4)^2L^2\left(f^i\right) \\ &\qquad - \frac{2}{k\pi^i}\left[f^i\left(x_k^i\right) - f^i(x)\right] + \frac{2}{k\pi^i}\mu_k^i\sqrt{d}L\left(f^i\right) \\ &\qquad + \frac{2}{k\pi^i}(d+4)L\left(f^i\right)\left\|\varphi_{k+1}^i - x_k^i\right\|,\end{aligned} \tag{16}$$

where we have used the inequalities $f_{\mu_k^i}^i(x_k^i) \geq f^i(x_k^i)$ and $f_{\mu_k^i}^i(x) \leq f^i(x) + \mu_k^i\sqrt{d}L(f^i)$, based on Lemma 1(a). Using the fact that $\mu_k^i = \mu\sigma_k^i$ and Lemma 3(a), we obtain

$$\frac{2}{k\pi^i}\mu_k^i\sqrt{d}L\left(f^i\right) \leq \frac{4\mu}{k^2\left(\pi^i\right)^2}\sqrt{d}L\left(f^i\right) \tag{17}$$

which implies

$$\begin{aligned}&\mathbb{E}\left[\left\|x_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k, \mathsf{I}_{k+1}, \mathsf{J}_{k+1}\right] \\ &\quad\leq (1+A_k)\left\|\varphi_{k+1}^i - x\right\|^2 + B_k(d+4)^2L^2\left(f^i\right) \\ &\qquad + \frac{4\mu}{k^2\left(\pi^i\right)^2}\sqrt{d}L\left(f^i\right) - \frac{2}{k\pi^i}\left[f^i\left(x_k^i\right) - f^i(x)\right] \\ &\qquad + \frac{2}{k\pi^i}(d+4)L\left(f^i\right)\left\|\varphi_{k+1}^i - x_k^i\right\|.\end{aligned} \tag{18}$$

Taking the expectation with respect to $\mathcal{F}_k$ and using the fact the preceding inequality holds with probability π^i, and $x_{k+1}^i = \varphi_{k+1}^i$ with probability $1-\pi^i$, we obtain with probability 1 for all $k \geq \tilde{k}$ and $i \in V$,

$$\begin{aligned}&\mathbb{E}\left[\left\|x_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k\right] \\ &\quad\leq \left(1+\pi^iA_k\right)\mathbb{E}\left[\left\|\varphi_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k\right] \\ &\qquad + \pi^iB_k(d+4)^2L^2\left(f^i\right) \\ &\qquad + \frac{4\mu}{k^2\pi^i}\sqrt{d}L\left(f^i\right) - \frac{2}{k}\left[f^i\left(x_k^i\right) - f^i(x)\right] \\ &\qquad + \frac{2}{k}(d+4)L\left(f^i\right)\mathbb{E}\left[\left\|\varphi_{k+1}^i - x_k^i\right\| \mid \mathcal{F}_k\right].\end{aligned} \tag{19}$$

Summing the above inequality for $i = 1, \ldots, N$, and noting that $\check{\pi} = \min_{i\in V}\pi^i$, $\hat{\pi} = \max_{i\in V}\pi^i$ and denoting $\hat{L}(f) = \max_{i\in V}L(f^i)$, we obtain with probability 1 for all $k \geq \tilde{k}$ and $i \in V$,

$$\begin{aligned}&\sum_{i=1}^N\mathbb{E}\left[\left\|x_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k\right] \\ &\quad\leq (1+\hat{\pi}A_k)\sum_{i=1}^N\mathbb{E}\left[\left\|\varphi_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k\right] \\ &\qquad + N\hat{\pi}B_k(d+4)^2\hat{L}^2(f) + \frac{4N\mu}{k^2\check{\pi}}\sqrt{d}\hat{L}(f) \\ &\qquad - \frac{2}{k}\left[f\left(\overline{x}_k\right) - f(x)\right] + \frac{2}{k}\hat{L}(f)\sum_{i=1}^N\left\|x_k^i - \overline{x}_k\right\| \\ &\qquad + \frac{2}{k}(d+4)\hat{L}(f)\sum_{i=1}^N\mathbb{E}\left[\left\|\varphi_{k+1}^i - x_k^i\right\| \mid \mathcal{F}_k\right],\end{aligned} \tag{20}$$

where $\overline{x}_k = (1/N)\sum_{i=1}^{N} x_k^i$ and we have used the following inequality:

$$\sum_{i=1}^{N}\left[f^i\left(x_k^i\right) - f^i\left(\overline{x}_k\right)\right] \geq -\sum_{i=1}^{N} L\left(f^i\right)\left\|x_k^i - \overline{x}_k\right\| \geq -\widehat{L}\left(f\right)\sum_{i=1}^{N}\left\|x_k^i - \overline{x}_k\right\|. \tag{21}$$

Now by using the definition of the weight matrix W_{k+1} and the convexity of the squared norm it follows that

$$\sum_{i=1}^{N}\mathbb{E}\left[\left\|\varphi_{k+1}^i - x\right\|^2 \mid \mathcal{F}_k\right] \leq \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\mathsf{W}_{k+1}\right]_{ij}\left\|x_k^j - x\right\|^2 = \sum_{j=1}^{N}\left\|x_k^j - x\right\|^2 \tag{22}$$

which yields the final bound for all $k \geq \widetilde{k}$ and $i \in V$ with probability 1:

$$\begin{aligned}\sum_{i=1}^{N}&\mathbb{E}\left[\left\|x_{k+1}^i - x^*\right\|^2 \mid \mathcal{F}_k\right] \\ &\leq \left(1 + \widehat{\pi}A_k\right)\sum_{i=1}^{N}\left\|x_k^i - x^*\right\|^2 + N\widehat{\pi}B_k(d+4)^2\widehat{L}^2\left(f\right) \\ &\quad + \frac{4N\mu}{k^2\widetilde{\pi}}\sqrt{d}\widehat{L}\left(f\right) - \frac{2}{k}\left[f\left(\overline{x}_k\right) - f\left(x^*\right)\right] \\ &\quad + \frac{2}{k}\left(2d+9\right)\widehat{L}\left(f\right)N\max_{i\in V}\left\|x_k^i - \overline{x}_k\right\|,\end{aligned} \tag{23}$$

where $x^* \in \mathcal{X}^*$ and we have used the following inequality:

$$\begin{aligned}\sum_{i=1}^{N}&\mathbb{E}\left[\left\|\varphi_{k+1}^i - x_k^i\right\| \mid \mathcal{F}_k\right] \\ &\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\mathsf{W}_{k+1}\right]_{ij}\left\|x_k^j - x_k^i\right\| \leq 2N\max_{i\in V}\left\|x_k^i - \overline{x}_k\right\|.\end{aligned} \tag{24}$$

Now we are ready to establish the convergence of the method. First, note that

$$\begin{gathered}\sum_{k=1}^{\infty}\widehat{\pi}A_k < \infty, \\ \sum_{k=1}^{\infty}N\widehat{\pi}B_k(d+4)^2\widehat{L}^2\left(f\right) + \sum_{k=1}^{\infty}\frac{4N\mu}{k^2\widetilde{\pi}}\sqrt{d}\widehat{L}\left(f\right) < \infty\end{gathered} \tag{25}$$

which can be easily seen from the explicit expressions for A_k and B_k. For the term $\max_{i\in V}\|x_k^i - \overline{x}_k\|$, we can follow an argument similar to the proof of Lemma 4 in [5] and derive that for each $i \in V$, $\sum_{k=1}^{\infty}(1/k)\|x_k^i - \overline{x}_k\| < \infty$ and $\lim_{k\to\infty}\|x_k^i - \overline{x}_k\| = 0$, which gives

$$\sum_{k=1}^{\infty}\frac{2}{k}\left(2d+9\right)\widehat{L}\left(f\right)N\max_{i\in V}\left\|x_k^i - \overline{x}_k\right\| < \infty. \tag{26}$$

Hence, combining the preceding fact with Lemma 4, which we can obtain with probability 1, the sequence $\{\mathbb{E}[\|x_k^i - x^*\|^2]\}$ converges for any $x^* \in \mathcal{X}^*$, and $\sum_{k=1}^{\infty}(1/k)[f(\overline{x}_k) - f(x^*)] < \infty$ (note that $\overline{x}_k \in \mathcal{X}$, and hence $f(\overline{x}_k) - f(x^*) \geq 0$), which implies

$$\liminf_{k\to\infty} f\left(\overline{x}_k\right) = f\left(x^*\right). \tag{27}$$

This, along with the fact that the sequence $\{\mathbb{E}[\|x_k^i - x^*\|^2]\}$ converges for any $x^* \in \mathcal{X}^*$ and $\lim_{k\to\infty}\|x_k^i - \overline{x}_k\| = 0$, gives our final statement, that is, $\lim_{k\to\infty}x_k^i = x^*$ for all $i \in V$ with probability 1. □

Remark 6. Note that other choices of the parameters μ_k^i ($k \geq 0, i \in V$) are possible. For example, we can set $\mu_k^i = \mu\sqrt{\sigma_k^i}$, for all $k \geq 0$ and any $i \in V$, under which case the convergence of the method (7) can also be established.

Remark 7. In contrast to the subgradient-based methods in [1–3], the implementation of the proposed method does not need the information of subgradients but only the function values. This makes our method suitable for the cases where explicit gradient calculations are computationally infeasible or expensive. In contrast to the gradient-free method in [13], the proposed method is asynchronous and the step sizes do not require any coordination of the agents.

4. Conclusion

In this paper, we have considered the constrained multiagent optimization problem. We present an asynchronous method that is based on the gossip algorithm and the gradient-free oracles for solving the problem. The proposed method removes the need for synchronous communications and the information of the subgradients as well. Finally, we prove that with probability 1 the iterates of all agents converge to the same optimal point of the problem, for a diminishing step size. There are several interesting questions that remain to be explored. For instance, it would be interesting to study the case of constant step size; it would be also interesting to study the effects of message quantization on the proposed method.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Granular Space Reduction to a β Multigranulation Fuzzy Rough Set

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Multigranulation rough set is an extension of classical rough set, and optimistic multigranulation and pessimistic multigranulation are two special cases of it. β multigranulation rough set is a more generalized multigranulation rough set. In this paper, we first introduce fuzzy rough theory into β multigranulation rough set to construct a β multigranulation fuzzy rough set, which can be used to deal with continuous data; then some properties are discussed. Reduction is an important issue of multigranulation rough set, and an algorithm of granular space reduction to β multigranulation fuzzy rough set for preserving positive region is proposed. To test the algorithm, experiments are taken on five UCI data sets with different values of β. The results show the effectiveness of the proposed algorithm.

1. Introduction

Qian et al. [1–3] proposed a multigranulation rough set, which is constructed on a family of granular structures and is different from Pawlak's rough set [4–7]. Qian's multigranulation rough set can be used to approximate an unknown concept through a family of binary relations; each binary relation can generate a granulation space, which may be partition [1], covering [8, 9] or even neighborhood system [10–12] on the universe of discourse.

Qian's rough set includes two basic models, one is optimistic multigranulation rough set and the other is pessimistic multigranulation rough set. The word "optimistic" means that at least one of the granulation spaces can be used for approximating while the word "pessimistic" means that all of the granulation spaces should be used for approximating. In these two models, all of the binary relations, or granulation spaces, are presented simultaneously; therefore, optimistic and pessimistic are two special cases of multigranulation rough set. To get a more suitable model for practical application, Xu et al. [13] proposed a more generalized multigranulation rough set, called β multigranulation rough set that designed by a threshold β for controlling the number of the equivalence classes, which are contained in the target.

In recent years, the multigranulation approach has attracted many researchers' attention [14–18]. Xu et al. generalized multigranulation fuzzy rough sets to tolerance approximation space to construct optimistic and pessimistic multigranulation fuzzy rough sets models [14]. Qian et al. further generalized their optimistic multigranulation rough set into incomplete information system [15]. In [16] Yang et al. introduced fuzzy theory into multigranulation rough set, which employed the T-similarity relations (reflexive, symmetric, and T-transitive) to construct the multigranulation fuzzy rough sets. Therefore, how to generalize multigranulation rough set is an important research field, as we all know how to introduce the fuzzy case to rough set model plays an important role in the development of rough set theory; fuzzy rough set [19] has attracted increasing attention from the domains of machine learning and intelligence data analysis. So, it is not difficult to introduce fuzzy rough set theory into β multigranulation rough set.

Granular space reduction is an important issue of multigranulation rough set and it is recently researched by many scholars [20–24]. In this paper, we focus on the problem to deal with β multigranulation rough set. Hu et al. in [25] proposed a fuzzy-rough attribute reduction. Motivated by this idea, we will introduce fuzzy rough set into β multigranulation rough set to construct β multigranulation fuzzy rough set model and design an algorithm of granular space reduction to β multigranulation fuzzy rough set. The algorithm can be used to granular space reduction in multigranular structures for preserving positive region of β multigranulation fuzzy rough set and will be very useful in big continuous data.

The purpose of this paper is to further generalize β multigranulation rough set to fuzzy environment. To facilitate our discussion, we first present some basic knowledge of rough set in Section 2. In Section 3, β multigranulation fuzzy rough set will be constructed and the properties will be discussed. In Section 4, an algorithm of granular space reduction to β multigranulation fuzzy rough set will be proposed and experiments are taken on five UCI data sets. In Section 5, conclusion is made.

2. Preliminaries

2.1. Rough Sets. Formally, a decision system is an information system $I = \langle U, AT \cup D\rangle$, in which U is a nonempty finite set of objects called the universe of discourse and AT is a nonempty finite set of the condition attributes; D is the set of the decision attributes and $AT \cap D = \varnothing$.

For all $x \in U$, let us denote by $a(x)$ the value that x holds on a ($a \in AT$). For an information system I, one then can describe the relationship between objects through their attributes' values. With respect to a subset of attributes such that $A \subseteq AT$, an indiscernibility relation $\mathrm{IND}(AT)$ may be defined as

$$\mathrm{IND}(AT) = \left\{(x, y) \in U^2 : a(x) = a(y), \forall a \in AT\right\}. \quad (1)$$

The relation $\mathrm{IND}(AT)$ is reflexive, symmetric, and transitive; then $\mathrm{IND}(AT)$ is an equivalence relation.

Definition 1. Let $I = \langle U, AT \cup D\rangle$ be a knowledge base in which $A \subseteq AT$, for all $X \subseteq U$, the lower and upper approximations of X in terms of the equivalence relation $\mathrm{IND}(AT)$ are denoted by $\underline{AT}(X)$ and $\overline{AT}(X)$, respectively:

$$\begin{aligned} \underline{AT}(X) &= \{x \in U : [x]_{AT} \subseteq X\}, \\ \overline{AT}(X) &= \{x \in U : [x]_{AT} \cap X \neq \varnothing\}, \end{aligned} \quad (2)$$

where $[x]_{AT}$ is the equivalence class based on indiscernibility relation $\mathrm{IND}(AT)$ and is denoted as $[x]_{AT} = \{y \in U : (x, y) \in \mathrm{IND}(AT)\}$.

$(\underline{A}(X), \overline{A}(X))$ is referred to as Pawlak's rough set.

2.2. Multigranulation Rough Set. Multigranulation rough set is different from Pawlak's rough set. The former is constructed on a family of the equivalence relations, and the latter is constructed on an equivalence relation. In Qian et al.'s multigranulation rough set theory, two basic models were defined. The first one is the optimistic multigranulation rough set, and the second one is the pessimistic multigranulation rough set.

Definition 2. Let S be an information system in which $A_1, A_2, \ldots, A_m \subseteq AT$; for all $X \subseteq U$, the optimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^m A_i}^O(X)$ and $\overline{\sum_{i=1}^m A_i}^O(X)$, respectively:

$$\begin{aligned} \underline{\sum_{i=1}^m A_i}^O(X) &= \left\{x \in U : [x]_{A_1} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\right\} \\ \overline{\sum_{i=1}^m A_i}^O(X) &= \sim \underline{\sum_{i=1}^m A_i}^O(\sim X), \end{aligned} \quad (3)$$

where $\sim X$ is the complementary set of X.

Definition 3. Let S be an information system in which $A_1, A_2, \ldots, A_m \subseteq AT$; for all $X \subseteq U$, the pessimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^m A_i}^P(X)$ and $\overline{\sum_{i=1}^m A_i}^P(X)$, respectively:

$$\begin{aligned} \underline{\sum_{i=1}^m A_i}^P(X) &= \left\{x \in U : [x]_{A_1} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\right\} \\ \overline{\sum_{i=1}^m A_i}^P(X) &= \sim \underline{\sum_{i=1}^m A_i}^P(\sim X), \end{aligned} \quad (4)$$

where $\sim X$ is the complementary set of X.

2.3. β Multigranulation Rough Set. Optimistic and pessimistic are two special cases of multigranulation rough set. Optimistic case is loose since if only one equivalence class of an object is contained in the target, then such object is included into lower approximation; pessimistic is strict since if all the equivalence classes of an object are contained in the target, then such object is included into lower approximation. To solve this problem, Xu et al. [13] proposed a more generalized multigranulation rough set, called β multigranulation rough set that are designed by a threshold β for controlling the number of the equivalence classes, which are contained in the target.

Definition 4. Let S be a multigranulation decision system; for all $x \in U$ and $X \subseteq U$, the characteristic function is defined as

$$C_X^i(x) = \begin{cases} 1: & [x]_{A_i} \subseteq X \\ 0: & \text{otherwise}, \end{cases} \quad (5)$$

where $A_i \in AT$.

Definition 5. Let S be a multigranulation decision system; for all $X \subseteq U$, the β multigranulation lower and upper approximations of X are denoted by

$$\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) = \left\{x \in U : \frac{\sum_{i=1}^{m} C_X^i(x)}{m} \geq \beta\right\};$$
$$\overline{\sum_{i=1}^{m} A_i}^{\beta}(X) = \left\{x \in U : \frac{\sum_{i=1}^{m}\left(1 - C_{\sim X}^i(x)\right)}{m} \succ 1-\beta\right\}, \tag{6}$$

where $\beta \in (0,1]$. $\sim X$ is the complementary set of X.

$(\underline{\sum_{i=1}^{m} A_i}^{\beta}(X), \overline{\sum_{i=1}^{m} A_i}^{\beta}(X))$ is referred to as β multigranulation rough set of X.

2.4. Fuzzy Rough Set. Fuzzy rough set is a generalization of rough set. It can be used for decision information system to deal with continuous types of conditional attributes. Usually a fuzzy similarity relation is computed by conditional attributes and is employed to measure similarity between two objects, which then develop upper and lower approximations of fuzzy sets. Fuzzy rough set generalize the objects discussed in rough set to fuzzy set and turn the equivalence relation to fuzzy equivalence relation.

Definition 6. Let $U \neq \varnothing$ be a universe of discourse and $\mathscr{R}_A$ a fuzzy similarity relation of U; for all $F \in \mathscr{F}(U)$, the fuzzy lower and upper approximations of F are denoted by

$$\underline{\mathscr{R}_A}(F)(x) = \bigwedge_{y\in U} S\left(1 - \mathscr{R}_A(x,y), F(y)\right),$$
$$\overline{\mathscr{R}_A}(F)(x) = \bigvee_{y\in U} T\left(\mathscr{R}_A(x,y), F(y)\right). \tag{7}$$

In fuzzy rough set, measurement should be introduced to construct fuzzy similarity relation, such as the max-min method. Then fuzzy similarity matrix can be constructed by fuzzy similarity relation; after that, fuzzy equivalent matrix can be constructed in terms of fuzzy similarity matrix by transitive closure method:

$$M(R) = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{2n} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}, \tag{8}$$

where $r_{ij} \in [0,1]$ is the relation value of x_i and x_j.

R is a fuzzy equivalence relation if R satisfies reflectivity, symmetry, and transitivity.

2.5. Multigranulation Fuzzy Rough Set. In [15], Qian et al. introduced the theory of fuzzy set into multigranulation rough set to construct the optimistic and pessimistic multigranulation fuzzy rough sets.

Definition 7. Let S be a fuzzy decision information system, $\widetilde{A_1}, \widetilde{A_2}, \ldots \widetilde{A_m} \subseteq \widetilde{AT}$ are m fuzzy subsets, and D decision attribute; for all $X \subseteq U$, the optimistic multi-granulation fuzzy lower and upper approximations of X are denoted by

$$\underline{\sum_{i=1}^{m} \widetilde{A_i}}^{O}(X) = \left\{x \in U : [x]_{\widetilde{A_1}} \subseteq X \vee [x]_{\widetilde{A_2}} \subseteq X \vee \cdots \vee [x]_{\widetilde{A_m}} \subseteq X\right\}; \tag{9}$$

$$\overline{\sum_{i=1}^{m} \widetilde{A_i}}^{O}(X) = \sim\left(\underline{\sum_{i=1}^{m} \widetilde{A_i}}^{O}(\sim X)\right), \tag{10}$$

where $[x]_{\widetilde{A_i}} = \{y \in U : (x,y) \in \mathrm{IND}(\widetilde{AT})\}$ is the fuzzy equivalent class of x. $\sim X$ is the complementary set of X.

$(\underline{\sum_{i=1}^{m} \widetilde{A_i}}^{O}(X), \overline{\sum_{i=1}^{m} \widetilde{A_i}}^{O}(X))$ is optimistic multigranulation fuzzy rough set.

Definition 8. Let S be a fuzzy decision information system, $\widetilde{A_1}, \widetilde{A_2}, \ldots \widetilde{A_m} \subseteq \widetilde{AT}m$ fuzzy subsets, and D decision attribute; for all $X \subseteq U$, the pessimistic multigranulation fuzzy lower and upper approximations of X are denoted by

$$\underline{\sum_{i=1}^{m} \widetilde{A_i}}^{P}(X) = \left\{x \in U : [x]_{\widetilde{A_1}} \subseteq X \wedge [x]_{\widetilde{A_2}} \subseteq X \wedge \cdots \wedge [x]_{\widetilde{A_m}} \subseteq X\right\};$$
$$\overline{\sum_{i=1}^{m} \widetilde{A_i}}^{P}(X) = \sim\left(\underline{\sum_{i=1}^{m} \widetilde{A_i}}^{P}(\sim X)\right), \tag{11}$$

where $(\underline{\sum_{i=1}^{m} \widetilde{A_i}}^{P}(X), \overline{\sum_{i=1}^{m} \widetilde{A_i}}^{P}(X))$ is pessimistic multigranulation fuzzy rough set. $\sim X$ is the complementary set of X.

3. β Multigranulation Fuzzy Rough Sets

In β multigranulation rough sets, by setting different values of β, we can get different reductions from which the most suitable reduction can be used in next research. The fuzzy rough set is very suitable for big continuous data set. So it is natural to introduce the theory of fuzzy set into β multigranulation rough sets to construct β multigranulation fuzzy rough sets model.

In this section, we will give some definitions of β multigranulation fuzzy rough sets model and discuss some properties of it.

Definition 9. Let S be a fuzzy decision information system, $\widetilde{A_1}, \widetilde{A_2}, \ldots \widetilde{A_m} \subseteq \widetilde{AT}m$ fuzzy subsets, and D is decision attribute; for all $x \in U$ and $X \subseteq U$, the characteristic function is defined as

$$\widetilde{C_X^i}(x) = \begin{cases} 1: & [x]_{\widetilde{A_i}} \subseteq X \\ 0: & \text{otherwise}, \end{cases} \tag{12}$$

where $[x]_{\widetilde{A}_i} = \{y \in U : (x, y) \in \mathrm{IND}(\widetilde{AT})\}$ is the fuzzy equivalent class of x. Then β multigranulation fuzzy rough set is defined as follows.

Definition 10. Let S be a fuzzy decision information system, $\widetilde{A}_1, \widetilde{A}_2, \ldots \widetilde{A}_m \subseteq \widetilde{AT}$ are m fuzzy subsets, and D decision attribute; for all $X \subseteq U$, the β multigranulation fuzzy lower and upper approximations of X are denoted by

$$\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) = \left\{x \in U : \frac{\sum_{i=1}^{m} \widetilde{C_X^i}(x)}{m} \geq \beta\right\}; \tag{13}$$

$$\overline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) = \left\{x \in U : \frac{\sum_{i=1}^{m} \left(1 - \widetilde{C_{\sim X}^i}(x)\right)}{m} \succ 1 - \beta\right\}; \tag{14}$$

where $(\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X), \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X))$ is β multigranulation fuzzy rough set. $\sim X$ is the complementary set of X.

Following Definition 10, we will employ the following denotations:

positive region of X: $\mathrm{POS}_{\widetilde{AT}}^{\beta}(X) = \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X)$;

negative region of X: $\mathrm{NEG}_{\widetilde{AT}}^{\beta}(X) = U - \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X)$;

boundary region of X: $\mathrm{BND}_{\widetilde{AT}}^{\beta}(X) = \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) - \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X)$.

Theorem 11. *Let S be a fuzzy decision information system, $\widetilde{A}_1, \widetilde{A}_2, \ldots \widetilde{A}_m \subseteq \widetilde{AT}$ are m fuzzy subsets, and D decision attribute; for all $X \subseteq U$,*

$$\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{1/m}(X) = \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X), \tag{15}$$

$$\overline{\sum_{i=1}^{m} \widetilde{A}_i}^{1/m}(X) = \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X), \tag{16}$$

$$\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{1}(X) = \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X), \tag{17}$$

$$\overline{\sum_{i=1}^{m} \widetilde{A}_i}^{1}(X) = \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X). \tag{18}$$

Proof. We only prove (15); others can be proven analogously.

For all $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{1/m}(X)$, by (13), there exist $\sum_{i=1}^{m} \widetilde{C_X^i}(x)/m \geq 1/m$, $\sum_{i=1}^{m} \widetilde{C_X^i}(x) \geq 1$; there must be $\widetilde{A}_i \in \widetilde{AT}$ such that $\widetilde{C_X^i}(x) = 1$, from which we can conclude that $[x]_{\widetilde{A}_i} \subseteq X$, $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X)$.

For all $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X)$, by (9), there exists $\widetilde{A}_i \in \widetilde{AT}$ such that $[x]_{\widetilde{A}_i} \subseteq X$. Therefore, by $\widetilde{C_X^i}(x) = 1$ and $\sum_{i=1}^{m} \widetilde{C_X^i}(x) \geq 1$, we can get $\sum_{i=1}^{m} \widetilde{C_X^i}(x)/m \geq 1/m$ such that $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{1/m}(X)$.

Theorem 11 shows that if $\beta = m^{-1}$, β multigranulation fuzzy rough set turns to optimistic multigranulation fuzzy rough set. If $\beta = 1$, β multigranulation fuzzy rough set turns to pessimistic multigranulation fuzzy rough set. Obviously, β multigranulation fuzzy rough set is an extension of optimistic multigranulation fuzzy rough set and pessimistic multigranulation fuzzy rough set. □

Theorem 12. *Let S be a fuzzy decision information system, $\widetilde{A}_1, \widetilde{A}_2, \ldots \widetilde{A}_m \subseteq \widetilde{AT}$ are m fuzzy subsets, and D decision attribute; for all $X \subseteq U$, $\beta \in (0, 1]$, we can get*

$$\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X) \subseteq \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X), \tag{19}$$

$$\overline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X) \subseteq \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) \subseteq \overline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X). \tag{20}$$

Proof. For all $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X)$, by (17), there exists $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{1}(X)$; by (13), there must be $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X)$, such that $\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X) \subseteq \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X)$.

For all $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X)$, by (13), there exists $\sum_{i=1}^{m} \widetilde{C_X^i}(x)/m \geq \beta$; there must be $[x]_{\widetilde{A}_i} \subseteq X$ such that $\widetilde{C_X^i}(x) = 1$; then $\sum_{i=1}^{m} \widetilde{C_X^i}(x)/m \geq 1/m$; by (15), there exists $x \in \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X)$ such that $\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X)$.

So $\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{P}(X) \subseteq \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} \widetilde{A}_i}^{O}(X)$.

Formula (20) can be proven analogously. □

4. Reduction of β Multigranulation Fuzzy Rough Sets

In single granular fuzzy rough set, reduction is a minimal subset of the attributes, which is independent and has the same discernibility power as all of the attributes. The method of preserving the positive region is usually used for attribute reduction. In this paper, we consider each attribute as a granular space. It is natural to introduce this method into β multigranulation fuzzy rough set for granular space reduction.

Definition 13. Let S be a fuzzy decision information system, $\widetilde{A}_1, \widetilde{A}_2, \ldots \widetilde{A}_m \subseteq \widetilde{AT}$ are m fuzzy subsets, D is decision attribute, $U/\mathrm{IND}(D) = \{X_1, X_2, \ldots, X_n\}$ is the partition induced by a set of decision attributes D, and approximation qualities of $U/\mathrm{IND}(D)$ in terms of β multigranulation fuzzy rough set are defined as

$$\gamma\left(\widetilde{AT}, \beta, D\right) = \frac{\left|\bigcup\left\{\underline{\sum_{i=1}^{m} \widetilde{A}_i}^{\beta}\left(X_j\right) : 1 \leq j \leq n\right\}\right|}{|U|}, \tag{21}$$

where, $|X|$ is the cardinal number of set X.

4.1. Significance of Granulation

Definition 14. Let S be a fuzzy decision information system, $\widetilde{A_1}, \widetilde{A_2}, \ldots \widetilde{A_m} \subseteq \widetilde{AT}$ are m fuzzy subsets, D is decision attribute, and $\widetilde{B}$ is a reduction if and only if

(1) $\gamma(\widetilde{B}, \beta, D) = \gamma(\widetilde{AT}, \beta, D)$;

(2) for all $B' \subseteq \widetilde{B}, \gamma(B', \beta, D) \neq \gamma(\widetilde{AT}, \beta, D)$.

By Definition 10, we can get a reduction of S when preserving the approximation quality.

Definition 15. Let S be a fuzzy decision information system, $\widetilde{A_1}, \widetilde{A_2}, \ldots \widetilde{A_m} \subseteq \widetilde{AT}$ are m fuzzy subsets, D is decision attribute, and $\widetilde{B} \subseteq \widetilde{AT}$; for all $\widetilde{A_i} \in \widetilde{B}$, the significance of granulation of $\widetilde{A_i}$ in terms of D is defined as

$$\mathrm{sig}_{\mathrm{in}}\left(\widetilde{A_i}, \widetilde{B}, D\right) = \gamma\left(\widetilde{B}, \beta, D\right) - \gamma\left(\widetilde{B} - \widetilde{A_i}, \beta, D\right), \quad (22)$$

where $\mathrm{sig}_{\mathrm{in}}(\widetilde{A_i}, \widetilde{B}, D)$ represents the changes of the approximation quality if a set of attributes $\widetilde{A_i}$ is eliminated from $\widetilde{AT}$. Also, we can define

$$\mathrm{sig}_{\mathrm{out}}\left(\widetilde{A_i}, \widetilde{B}, D\right) = \gamma\left(\widetilde{B} \cup \widetilde{A_i}, \beta, D\right) - \gamma\left(\widetilde{B}, \beta, D\right) \quad (23)$$

for all $\widetilde{A_i} \in \widetilde{AT} - \widetilde{B}$, $\mathrm{sig}_{\mathrm{out}}(\widetilde{A_i}, \widetilde{B}, D)$ represents the changes of the approximation quality if a set of attributes $\widetilde{A_i}$ is put in $\widetilde{AT}$. These two significances can be used to forward granular structure selection algorithm, and $\mathrm{sig}_{\mathrm{in}}(\widetilde{A_i}, \widetilde{B}, D)$ can determine the significance of every granulation in terms of the approximation quality.

4.2. Granular Space Reduction Algorithm. See Algorithm 1.

4.3. Experiment. To demonstrate the above approach, we use 5 data sets gotten from UCI Repository of Machine Learning databases; the description of the selected data sets is listed in Table 1.

In this experiment, each feature is used to construct a granular structure. All features then correspond to multiple granular structures. For each data set, 5 different β are used; then the different results of granular selection of Table 1 under different values of β are listed in Table 2. In Table 2, "u" means the number of features, "0.001" is the smallest value of β, "1" is the biggest value of β, and "$1/u$" is bigger than 0.01 and smaller than 1.

Through Table 2, we get an interesting outcome that the result of granular selection of each data set is changed with different values of β. The granular selection results are increased with the increased value of β. Take for instance that when $\beta = 0.001$, we get the least number of features; when $\beta = 1$, we get the most number of features. Obviously, when $\beta = 1/u$, it actually represents the optimistic multigranulation fuzzy rough set; when $\beta = 1$, it actually represents pessimistic multigranulation fuzzy rough set, which is too strict for only when all of the granulation spaces satisfy the inclusion condition between the equivalence classes and the target that the object belongs to the lower approximation. in this experiment, there are no reductions can be gotten, which shows that it is difficult to get a satisfied reduction result under pessimistic condition.

Table 1: Data description.

ID	Data set	Samples	Feature
1	biodeg	1055	41
2	Ionosphere	351	34
3	Parkinsons	196	22
4	sonar	208	60
5	wdbc	569	30

Table 2: Granular space selection.

ID	Data set	Granular selection with different β				
		$1/u$	1	0.001	0.005	0.01
1	biodeg	39	41	24	27	31
2	Ionosphere	28	34	26	26	28
3	Parkinsons	15	22	11	11	14
4	sonar	59	60	47	50	55
5	wdbc	25	30	14	21	24

Table 3: Accuracy with neural net.

ID	Data set	Accuracy with different β				
		$1/u$	1	0.001	0.005	0.01
1	biodeg	88.152	87.204	86.161	85.782	86.161
2	Ionosphere	83.191	88.604	85.755	85.755	83.191
3	Parkinsons	75.385	78.462	76.923	76.923	75.385
4	sonar	72.596	73.077	77.885	70.673	75.481
5	wdbc	89.279	89.279	62.742	62.742	74.165

Table 4: Accuracy with decision tree.

ID	Data set	Accuracy with different β				
		$1/u$	1	0.001	0.005	0.01
1	biodeg	94.692	94.123	94.028	94.787	95.64
2	Ionosphere	96.581	99.43	96.866	96.866	95.581
3	Parkinsons	92.308	96.41	95.897	95.897	92.308
4	sonar	98.558	98.558	96.635	96.635	98.558
5	wdbc	97.891	98.77	98.418	98.594	99.297

In order to test the performance of the proposed algorithm and to get a proper β, we employ neural network and decision tree as the validation function. The results are listed in Tables 3 and 4.

We can find in Tables 3 and 4 that when $\beta = 1$, the selected features are just the same as the original data sets, but the accuracy is not always the biggest, such as the result of "biodeg" in Table 4; when $\beta = 0.005$, the accuracy is bigger than $\beta = 1$, which shows that when β is set a suitable value it cannot only reduce redundant granular space but also retain the most useful granular space so as to get better accuracy. The selection of granular space is crucial to the performance

Input: a fuzzy decision information system
$S = \langle U, \widetilde{AT} \cup D \rangle$
Output: a granular space reduction *RED*
Step 1. $\forall \widetilde{A_i} \in \widetilde{AT}$, compute fuzzy similarity matrix
Step 2. $\forall \widetilde{A_i} \in \widetilde{AT}$, compute fuzzy equivalence matrix in terms of the result of *Step 1*, then get fuzzy equivalence class
Step 3. $RED \leftarrow \varnothing$
Step 4. $\forall \widetilde{A_i} \in \widetilde{AT}$, compute $\mathrm{sig}_{\mathrm{in}}(\widetilde{A_i}, \widetilde{AT}, D)$
Step 5. $RED \leftarrow \widetilde{A_j}$, $\mathrm{sig}_{\mathrm{in}}(\widetilde{A_j}, \widetilde{AT}, D) = \max\left\{\mathrm{sig}_{\mathrm{in}}\left(\widetilde{A_i}, \widetilde{AT}, D\right) : \widetilde{A_i} \in \widetilde{AT}\right\}$
Step 6. $\forall \widetilde{A_i} \in \widetilde{AT} - RED$, compute $\mathrm{sig}_{\mathrm{out}}(\widetilde{A_i}, RED, D)$
If
$\mathrm{sig}_{\mathrm{out}}\left(\widetilde{A_k}, RED, D\right) = \max\left\{\mathrm{sig}_{\mathrm{out}}\left(\widetilde{A_i}, RED, D\right) : \widetilde{A_i} \in \widetilde{AT} - RED\right\}$, $RED = RED \cup \{\widetilde{A_k}\}$
Until $\gamma(RED, \beta, D) = \gamma(\widetilde{AT}, \beta, D)$
End
Step 7. return *RED*

Algorithm 1: Find granular space reduction.

of the sequent learning, so the selection should reflex the structure of the data and patterns. Comparing the results of Table 3 with Table 4, we can see that the accuracy of Table 4 is bigger than Table 3; it shows that the accuracy is also related to the performance of classifier.

This experiment shows that the proposed algorithm is more flexible for selecting granular space than optimistic and pessimistic multigranulation fuzzy rough sets, which can use fewer features to get higher accuracy.

5. Conclusions

In this paper, a β multigranulation fuzzy rough set model is proposed, and a corresponding algorithm is proposed; different from other methods, the proposed algorithm is constructed on multigranular spaces, our experiment shows that the granular selection results are increased with the increased value of β, and the algorithm cannot only deal with continuous data but also when β is set properly, the reduction will be suitable to be classified to get a good result. The experiment result shows the effectiveness of our algorithm.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Simplified Predictive Control of Constrained Markov Jump System with Mixed Uncertainties

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A simplified model predictive control algorithm is designed for discrete-time Markov jump systems with mixed uncertainties. The mixed uncertainties include model polytope uncertainty and partly unknown transition probability. The simplified algorithm involves finite steps. Firstly, in the previous steps, a simplified mode-dependent predictive controller is presented to drive the state to the neighbor area around the origin. Then the trajectory of states is driven as expected to the origin by the final-step mode-independent predictive controller. The computational burden is dramatically cut down and thus it costs less time but has the acceptable dynamic performance. Furthermore, the polyhedron invariant set is utilized to enlarge the initial feasible area. The numerical example is provided to illustrate the efficiency of the developed results.

1. Introduction

Hybrid systems are a class of dynamical systems denoted by an interaction between the continuous and discrete dynamics. In control community, the researchers tend to view hybrid systems as continuous state and discrete switching which focuses on the continuous state of dynamic system. Switched systems are a natural result from this point of view. Since switching systems can be applied to model the systems involving abrupt sudden changes which are widely found in the systems of economics and communications as well as manufacturing, more attention has been paid to them (see robust stabilization [1], finite-time analysis [2] and asynchronous switching [3]). When the system model is linear and the switching is driven by Markov process, it leads to Markov jump linear system (MJS). Specifically, MJS presents a stochastic Markov chain to describe the random changes of system parameters or structures, where the dynamic of MJS is switching among the models governed by a finite Markov chain. Due to this superiority, MJS has been widely investigated during the last twenty years. Attractive pioneer works have been obtained (see controller design [4], 2D MJS control [5], peak-to-peak filtering [6], and finite-time control [7, 8]). However, the cases of completely known transition probability (TP) considered in [4–8] are not always achievable since the TP is not easy to be fully accessible (see the delay or packet loss in networked control systems [9]). Thus it is necessary to investigate the partly unknown case [10–12].

On the other hand, the systems in practice are usually subject to input/output constraints. Thus, model predictive control (MPC) is then introduced to solve the problem of MJS with constraints since MPC can explicitly solve the constraints in control action. Successful MPC application in discrete-time MJS can be obtained in [13, 14]. Normally, MPC is reformulated as online quadratic program and results have been reported (see stability [15, 16] and enlarged terminal sets [17]). It should be noted that the online computation in the literature [15–17] leads to heavy computational burden. Thus, the researchers attempted to try a new alternative method to solve the problem. For this reason, explicit MPC [18] is presented. However, when the size of system increases, the time of searching explicit MPC law will also increase sharply.

Based on the above analysis, a simplified MPC design framework is introduced to reduce the burden of online

computation for the constrained MJS with mixed uncertainties. The basic idea is that $(N-1)$ steps of mode-dependent MPC are designed to steer the state to a final neighbour area which includes the origin. Then the final step of robust mode-independent MPC is devised to force the state towards the origin regardless of model uncertainty and transition probability uncertainty. This simplified MPC dramatically reduces the burden of computation with minor performance loss, which implies good balance between the calculation time and dynamical performance. Furthermore, the polyhedron invariant set is applied to further enlarge the initial feasible area.

The construction of the paper is as follows. Section 2 gives the basic dynamical of the system. Section 3 gives the finite-step simplified MPC algorithm and it is formulated as LMIs. Section 4 presents a numerical example to show the efficiency of the results. Section 5 concludes the paper.

Notations. The notations are as follows: R^n denotes a n-dimensional Euclidean space, A^T stands for the transpose of a matrix, $E\{\cdot\}$ denotes the expectation of the stochastic process or vector, a positive-definite matrix is described as $P > 0$, I means the unit matrix with appropriate dimension, and $*$ means the symmetric term in a symmetric matrix.

2. Problem Statement and Preliminaries

The constrained discrete-time MJSs with mixed uncertainties are considered in this paper:

$$\begin{aligned} x_{k+1} &= A(r_k)x_k + B(r_k)u_k, \\ y_k &= C(r_k)x_k, \end{aligned} \tag{1}$$

where $x_k \in R^{n_x}$, $u_k \in R^{n_u}$, $y_k \in R^{n_y}$, respectively, denote the state vector, the input vector, and the controlled output vector. The discrete-time Markov stochastic process $\{r_k, k \geq 0\}$ takes values in a finite set Γ, where Γ contains σ modes of system (1), $\Gamma = \{1, 2, 3, \ldots, \sigma\}$, and r_0 represents the initial mode. The uncertain system model $A(r_k)$ and $B(r_k)$ belong to the model sets

$$\begin{aligned} \Omega(r_k) = \Bigg\{ &[A(r_k), B(r_k)], A(r_k) = \sum_{\iota=1}^{L} \alpha_\iota A_\iota(r_k), \\ &B(r_k) = \sum_{\iota=1}^{L} \alpha_\iota B_\iota(r_k), \sum_{\iota=1}^{L} \alpha_\iota = 1 \Bigg\}. \end{aligned} \tag{2}$$

Inputs and outputs constraints are subject to

$$-u_{\lim} \leq u_k \leq u_{\lim}, \tag{3}$$

$$-y_{\lim} \leq y_k \leq y_{\lim}. \tag{4}$$

The transition probability (TP) matrix is denoted by $\Pi(k) = \{\pi_{ij}(k)\}$, $i, j \in \Gamma$, where $\pi_{ij}(k) = P(r_{k+1} = j \mid r_k = i)$ is the transition probability from mode i at time k to mode j at time $k+1$. The elements in TP matrix satisfy $\pi_{ij}(k) \geq 0$ and $\sum_{j=1}^{\sigma} \pi_{ij}(k) = 1$:

$$\pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1\sigma} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2\sigma} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{\sigma 1} & \pi_{\sigma 2} & \cdots & \pi_{\sigma\sigma} \end{bmatrix}. \tag{5}$$

The uncertain transition probability (TP) implies that some elements in π are unknown; a four-mode transition probability (TP) matrix π may be

$$\pi = \begin{bmatrix} ? & \pi_{12} & ? & ? \\ \pi_{21} & \pi_{22} & ? & ? \\ \pi_{31} & ? & ? & ? \\ ? & ? & \pi_{43} & ? \end{bmatrix}, \tag{6}$$

where "?" represents the inaccessible element in TP matrix. For convenience, we denote $\pi = \pi_{r_k}^k + \pi_{r_k}^{uk}$, for all mode $r_k \in \Gamma$ at sampling time k, if $\pi_{r_k}^k \neq 0$, and redescribe it as $\pi_{r_k}^k = (\kappa_{r_k}^1, \ldots, \kappa_{r_k}^\tau)$, for all $1 \leq l \leq \tau$, where $\kappa_{r_k}^l$ represents the lth exact element in the ith row of π, $\Pi_{r_k}^k = \sum_{j \in \pi_{r_k}^k} \pi_{r_k r_{k+1}}$.

Some preliminaries are introduced before proceeding.

Definition 1 (see [6]). For any initial mode r_0 and state x_0, discrete-time MJS (1) is said to be stochastically stable if

$$\lim_{k \to \infty} E\left\{x_k^T x_k \mid x_0, r_0\right\} \longrightarrow 0. \tag{7}$$

Definition 2. For MJS (1), an ellipsoid set $\Theta = \{x \in R^{n_x} \mid x_k^T P_k(r_k) x_k \leq \gamma_k\}$ associated with the state is said to be asymptotically mode-dependent stable, if the following holds, whenever $x_{k_0} \in \Theta$, then $x_k \in \Theta$ for $k \geq k_0$ and $x_k \to 0$ when $k \to \infty$.

Next, we first derive the online optimal MPC algorithm for system (1). The aim is to minimize the function cost related to worst-case performance and then in Section 4 the corresponding simplified MPC algorithm will be derived. Finally the polyhedron invariant set is applied to further improve the initial feasible district.

3. Simplified MPC Design

3.1. Online Optimal MPC

Theorem 3. *Consider MJS* (1) *with model uncertainties* (2) *and partly unknown TP matrix* (6), *at sampling time k, if there exist a set of matrices $F_k(r_k)$, such that the following holds:*

$$\min_{F_k(r_k)} \max_{A_\iota(r_k), B_\iota(r_k), \pi_{r_k r_{k+1}}, r_k, r_{k+1} \in \Gamma} J_\infty(k) \tag{8}$$

s.t.

$$-u_{\lim} \le u_k \le u_{\lim}\,, \tag{9}$$

$$-y_{\lim} \le y_k \le y_{\lim}\,, \tag{10}$$

$$\begin{aligned} &E\left\{V\left(x_{k+1}, r_{k+1} \mid x_0, r_0\right)\right\} - E\left\{V\left(x_k, r_k \mid x_0, r_0\right)\right\} \\ &\quad \le -E\left\{x_k^T Q\left(r_k\right) x_k + u_k^T R\left(r_k\right) u_k \mid x_0, r_0\right\}. \end{aligned} \tag{11}$$

Then, it decides an upper bound on $J_\infty(k)$, *where* $u_k = F_k(r_k)x_k$, $J_\infty(k) = E\{\sum_{k=0}^{\infty}(x_k^T Q(r_k)x_k + u_k^T R(r_k)u_k) \mid x_0, r_0\}$, $Q(r_k)$, $R(r_k)$ *are positive definite weighting matrices.*

Proof. It is assumed that at the sampling time k, a state-feedback law $u(k+i \mid k) = F_k(r_k)x(k+i \mid k)$, is applied to minimize the worst cost function of J_k; it is easy to show that $V(x_k, r_k \mid x_0, r_0)$ is an upper bound on $J_\infty(k)$. Let $V(x_k) = x_k^T P_k(r_k)x_k$, $P_k(r_k) > 0$, be a quadratic Lyapunov function. For any $[A_i(r_k), B_i(r_k) \in \Omega(r_k)]$, the following constraint holds

$$\begin{aligned} &E\left\{V\left(x_{k+1}, r_{k+1} \mid x_0, r_0\right)\right\} - E\left\{V\left(x_k, r_k \mid x_0, r_0\right)\right\} \\ &\quad \le -E\left\{x_k^T Q\left(r_k\right) x_k + u_k^T R\left(r_k\right) u_k \mid x_0, r_0\right\}. \end{aligned} \tag{12}$$

Summing (12) from $i = 0$ to ∞ on both sides and using the fact $x_{k\to\infty} = 0$ or $V(x_{k\to\infty}) = 0$, we obtain

$$J_\infty(k) \le V\left(x_k, r_k \mid x_0, r_0\right) = x_k^T P_k\left(r_k\right) x_k \tag{13}$$

which implies that $V(x_k, r_k \mid x_0, r_0)$ is an upper bound on $J_\infty(k)$. □

Theorem 4. *Consider MJS* (1) *with polytope model uncertainties* (2) *and partly unknown TP matrix* (4), *if there exist a set of positive definite matrices* $X_k(r_k)$, $Y_k(r_k)$, *such that the following optimization problem* (12) *has an optimal solution:*

$$\min_{F_k(r_k)} \max_{A_i(r_k), B_i(r_k), \pi_{r_k r_{k+1}}, r_k, r_{k+1} \in \Gamma} \gamma_k \tag{14}$$

s.t.

$$\begin{bmatrix} 1 & * \\ x_k & X_k\left(r_k\right) \end{bmatrix} \ge 0, \quad \forall r_k \in \Gamma, r_{k+1} \in \pi_{r_{k+1}}^{uk}, \tag{15}$$

$$\begin{bmatrix} Z & Y_k\left(r_k\right) \\ * & X_k\left(r_k\right) \end{bmatrix} \ge 0, \qquad Z_{tt} \le \left(u_{\lim}^t\right)^2, \tag{16}$$

$$\begin{bmatrix} X_k\left(r_k\right) & * \\ C\left(r_k\right)\theta_l\left(r_k\right) & M \end{bmatrix} \ge 0, \qquad M_{hh} \le \left(y_{\lim}^h\right)^2, \tag{17}$$

$$\begin{bmatrix} X_k\left(r_k\right) & * \\ \theta_l\left(r_k\right) & X_k\left(r_{k+1}\right) \end{bmatrix} \ge 0, \qquad \forall r_{k+1} \in \pi_{r_k}^{uk}, \tag{18}$$

$$\begin{bmatrix} \Pi_{r_k}^k X_k\left(r_k\right) & U^T\left(r_k\right) & X_k\left(r_k\right) Q^{1/2}\left(r_k\right) & Y_k^T\left(r_k\right) Q^{1/2}\left(r_k\right) \\ * & W\left(r_{k+1}\right) & 0 & 0 \\ * & * & \gamma_k I & 0 \\ * & * & * & \gamma_k I \end{bmatrix} \ge 0, \quad \forall r_{k+1} \in \pi_{r_k}^k \tag{19}$$

then, the mode-dependent state-feedback which minimizes the upper bound γ_k *on* $J_\infty(k)$ *and simultaneously stabilizes the closed-loop system within an ellipsoid* $\varepsilon = \{x_k^T X_k^{-1}(r_k)x_k \le 1\}$ *is calculated by* $u(k+i \mid k) = F_k(r_{k+i})x_{k+i|k}$, $F_k(r_{k+i}) = Y_k(r_{k+i})X_k^{-1}(r_{k+i})$, *where* $X_k(r_k) = \gamma_k P_k^{-1}(r_k)$, $\theta_l(r_k) = A_l(r_k)X_k(r_k) + B_l(r_k)Y_k(r_k)$, $U^T(r_k) = [\sqrt{\kappa_{r_k}^1}\theta_l^T(r_k), \ldots, \sqrt{\kappa_{r_k}^\tau}\theta_l^T(r_k)]$, $W(r_{k+1}) = \mathrm{diag}\{X_k(\kappa_{r_k}^1), X_k(\kappa_{r_k}^2), \ldots, X_k(\kappa_{r_k}^\tau)\}$, Z_{tt}, M_{hh}, *respectively, describe the* t*th,* h*th diagonal element of* Z, M, $u_{\lim}^t$ *and* $y_{\lim}^h$, *respectively, describe the* t*th and* h*th element of input and output constraints,* $t = 1, 2, \ldots, n_u$, $h = 1, 2, \ldots, n_y$.

Proof. Let $X_k(r_k) = \gamma_k P_k^{-1}(r_k)$; $J_\infty(k) \le \gamma_k$ in (13) can be solved by the following LMIs:

$$\begin{bmatrix} 1 & * \\ x_k & X_k\left(r_k\right) \end{bmatrix} \ge 0, \qquad \forall r_k \in \Gamma, \quad r_{k+1} \in \pi_{r_{k+1}}^{uk}. \tag{20}$$

The input/output constraints are guaranteed by (16) and (17); the proof is similar to [19]; here we omit the proof. Equation (11) is equivalent to

$$\begin{aligned} \Xi\left(r_k\right) &= P_k\left(r_k\right) - \theta_l^T\left(r_k\right)\left(\sum_{r_{k+1}\in\pi} \pi_{r_k r_{k+1}} P_k\left(r_{k+1}\right)\right) \\ &\quad \times \theta_l\left(r_k\right) - Q\left(r_k\right) - F_k^T\left(r_k\right) R\left(r_k\right) F_k\left(r_k\right) \ge 0. \end{aligned} \tag{21}$$

Since $\sum_{r_{k+1}\in\pi} \pi_{r_k r_{k+1}} = 1$, $\pi_{r_k r_{k+1}} \ge 0$, $\Pi_k^{r_k} = \sum_{r_{k+1}\in\pi_{r_k}^k} \pi_{r_k r_{k+1}}$, it leads to

$$\begin{aligned} \Xi\left(r_k\right) &= \left(\sum_{r_{k+1}\in\pi} \pi_{r_k r_{k+1}}\right) P_k\left(r_k\right) - \theta_l^T\left(r_k\right) \\ &\quad \times \left(\sum_{r_{k+1}\in\pi} \pi_{r_k r_{k+1}} P_k\left(r_{k+1}\right)\right)\theta_l\left(r_k\right) - Q\left(r_k\right) \\ &\quad - F_k^T\left(r_k\right) R\left(r_k\right) F_k\left(r_k\right) \\ &= \Pi_{r_k}^k P\left(r_k\right) - \theta_l^T\left(r_k\right) \\ &\quad \times \left(\sum_{r_{k+1}\in\pi_{r_k}^k} \pi_{r_k r_{k+1}} P_k\left(r_{k+1}\right)\right)\theta_l\left(r_k\right) \\ &\quad + \left(\sum_{r_{k+1}\in\pi_{r_k}^{uk}} \pi_{r_k r_{k+1}}\right) \\ &\quad \times \left(P_k\left(r_k\right) - \theta_l^T\left(r_k\right) P_k\left(r_{k+1}\right)\theta_l\left(r_k\right)\right) \\ &\quad - Q\left(r_k\right) - F_k^T\left(r_k\right) R\left(r_k\right) F_k\left(r_k\right) \ge 0. \end{aligned} \tag{22}$$

One sufficient condition to ensure (22) is

$$\begin{aligned}&\Pi_{r_k}^k P_k(r_k) - \theta_l^T(r_k)\left(\sum_{r_{k+1}\in\pi_{r_k}^k}\pi_{r_k r_{k+1}} P_k(r_{k+1})\right)\\&\quad\times\theta_l(r_k) - Q(r_k) - F_k^T(r_k)R(r_k)F_k(r_k)\geq 0,\\&P_k(r_k) - \theta_l^T(r_k)P_k(r_{k+1})\theta_l(r_k)\geq 0.\end{aligned}\tag{23}$$

Considering the Schur theory complement lemma, (16) and (17) can be derived.

Actually the feedback controller can make the closed-loop system stable in the ellipsoid $\varepsilon = \{x_k^T X_k^{-1}(r_k)x_k \leq 1\}$. Assume that the optimal $P_k^*(r_k)$, $F_k^*(r_k)$ at the moment k are

$$\begin{aligned}P_k^*(r_k) &= \gamma_k^*(X_k^*(r_k))^{-1},\\F_k^*(r_k) &= Y_k^*(X_k^*(r_k))^{-1},\\\theta_{lk}^*(r_k) &= A_l(r_k) + B_l(r_k)F_k^*(r_k),\\\vartheta_{lk}^*(r_k) &= A_l(r_k)X_k^*(r_k) + B_l(r_k)Y_k^*(r_k).\end{aligned}\tag{24}$$

Equations (18) and (19) lead to

$$\begin{aligned}x_k^T P_k^*(r_k)x_k &\geq x_k^T(\theta_{lk}^*(r_k))^T\\&\quad\times\sum_{r_{k+1}\in\pi}\pi_{r_k r_{k+1}}P(r_{k+1})\theta_{lk}^*(r_k)x_k\\&\quad+x_k^T Q(r_k)x_k + x_k^T F_k^T(r_k)R(r_k)F_k(r_k)x_k,\\E\left\{x_k^T P_k^*(r_k)x_k\right\}&\\&\geq E\left\{x_{k+1}^T P_k^*(r_{k+1})x_{k+1}\right\}\\&\quad+x_k^T Q(r_k)x_k + x_k^T F_k^T(r_k)R(r_k)F_k(r_k)x_k.\end{aligned}\tag{25}$$

$P_{k+1}^*(r_{k+1})$ is the optimal value at moment $k+1$; $P_k^*(r_{k+1})$ is a feasible one at moment $k+1$. By the optimum definition,

$$x_{k+1}^T P_k^*(r_{k+1})x_{k+1} \geq x_{k+1}^T P_{k+1}^*(r_{k+1})x_{k+1};\tag{26}$$

then,

$$\begin{aligned}&E\left\{x_k^T P_k^*(r_k)x_k\right\}\\&\quad\geq E\left\{x_{k+1}^T P_{k+1}^*(r_{k+1})x_{k+1}\right\}\\&\quad+x_k^T Q(r_k)x_k + x_k^T F_k^T(r_k)R(r_k)F_k(r_k)x_k.\end{aligned}\tag{27}$$

It is shown that $E\{x_k^T P_k^*(r_k)x_k\}$ decrease strictly as $E\{x_k^T \Phi_k^*(r_k)x_k\} \to 0, k\to\infty$

From Definition 1, the system is stochastically stable. From (27), then

$$E\left\{x_k^T P_k^*(r_k)x_k\right\} \geq E\left\{x_{k+1}^T P_{k+1}^*(r_{k+1})x_{k+1}\right\}.\tag{28}$$

This implies that the ellipsoid is an asymptotically stable invariant one, which completes the proof. □

Corollary 5. *Consider MJS* (1) *with model uncertainties* (2) *and TP matrix* (4) *at current moment k; supposing that there exists a set of positive definite matrices X, Y, such that the following optimization problem has an optimal solution:*

$$\min_{\gamma_k, X, Y}\quad \gamma_k\tag{29}$$

s.t.

$$\begin{aligned}&\begin{bmatrix}1 & *\\x_k & X\end{bmatrix}\geq 0,\\&\forall r_k\in\Gamma,\quad r_{k+1}\in\pi_{r_{k+1}}^{uk},\\&\begin{bmatrix}Z & Y\\* & X\end{bmatrix}\geq 0,\qquad Z_{tt}\leq\left(u_{\lim}^t\right)^2,\\&\begin{bmatrix}X & *\\C(r_k)\theta_l(r_k) & M\end{bmatrix}\geq 0,\qquad M_{hh}\leq\left(y_{\lim}^h\right)^2,\\&\begin{bmatrix}X & *\\\theta_l(r_k) & X\end{bmatrix}\geq 0,\quad\forall r_{k+1}\in\pi_{r_k}^{uk},\\&\begin{bmatrix}\Pi_{r_k}^k X & U^T(r_k) & XQ^{1/2}(r_k) & Y^T Q^{1/2}(r_k)\\* & W(r_{k+1}) & 0 & 0\\* & * & \gamma_k I & 0\\* & * & * & \gamma_k I\end{bmatrix}\geq 0,\\&\qquad\qquad\forall r_{k+1}\in\pi_{r_k}^k\end{aligned}\tag{30}$$

then the mode-independent state-feedback law can minimize the upper bound γ_k on the objective function $J_\infty(k)$ and stabilize the closed-loop system in the ellipsoid $\varepsilon = \{x_k^T X^{-1}x_k \leq 1\}$ and it is obtained by $u(k+i\mid k) = Fx_{k+i|k}$, $F = YX^{-1}$, where $X = \gamma_k P^{-1}$, $\theta_l(r_k) = A_l(r_k)X + B_l(r_k)Y$, $U^T(r_k) = [\sqrt{\kappa_{r_k}^1}\theta_l^T(r_k),\ldots,\sqrt{\kappa_{r_k}^\tau}\theta_l^T(r_k)]$, $W(r_{k+1}) = \operatorname{diag}\{X, X,\ldots, X\}$, Z_{tt} and M_{hh}, respectively, describe the tth and hth diagonal element of Z, M, and $u_{\lim}^t$ and $y_{\lim}^h$, respectively, describe the tth and hth element of input and output constraints, $t = 1, 2,\ldots, n_u$, $h = 1, 2,\ldots, n_y$.

3.2. Simplified MPC Design. In this section, a simplified MPC for uncertain MJS (1) is developed based on the online algorithm in Theorem 4; Figure 1 shows the simplified MPC schematic diagram. Then the simplified mode-independent feedback controller is designed regardless of model uncertainty and TP uncertainty since much more constraints will be nonactive in the neighboring region of origin and this freedom of feasibility is applied to improve the procedure of controller design.

Theorem 6. *Consider uncertain MJS* (1) *associated with an initial state x_0 satisfying $x_0^T Q_0^{-1}(r_0)x_0 \leq 1$; the simplified MPC Algorithm 7 robustly stabilizes the closed-loop system.*

Proof. For the N-step implementation at x_j, $j = 1,\ldots, N$, the selection for x_j in Algorithm 7 implies $Q_{j-1}^{-1}(r_k) < Q_j^{-1}(r_k)$, which means the constructed ellipsoid $\xi_j = \{x \mid$

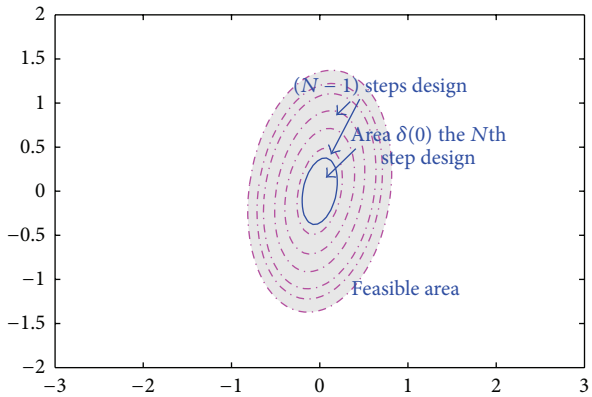

Figure 1: Simplified MPC schematic diagram.

$x^T Q_j^{-1}(r_k)x \le 1\}$ is embedded in ξ_{j-1}, that is, $\xi_j \subset \xi_{j-1}$. For a settled x, $\xi_j = \{x \mid x^T Q_j^{-1}(r_k)x \le 1\}$ is decreasing monotonically associated with j, which guarantees the unique search in the search table for the largest j for $\xi_j = \{x \mid x^T Q_j^{-1}(r_k)x \le 1\}$. If x_k belongs to $\xi_j = \{x \mid x^T Q_j^{-1}(r_k)x \le 1\}$ and $\xi_{j+1} = \{x \mid x^T Q_{j+1}^{-1}(r_k)x > 1\}$, $j = 1, \ldots, N-1$, by applying Theorem 3, the control law $u_k = F_j(r_k)x_k$ will steer the state in ξ_{j-1} to ξ_j. Finally, the controller $u_k = F_N x_k$ (applying Corollary 5) make the state to be in ξ_N and converge to the origin. Furthermore, the LP programming algorithm is utilized to remove redundant constraints [20] and construct a sequence of polyhedral invariant set for MJS and thus enlarge the feasible domain. □

Algorithm 7 (simplified MPC applying polyhedral invariant set). Simplified MPC design is as follows.

(1) Select x_j, $j = 1, \ldots, N$, which satisfy $\varepsilon_{j+1} \subset \varepsilon_j$, $\varepsilon_N = \delta(0)$.

(2) For step 1 to $N-1$, calculate the corresponding mode-dependent gains $\gamma_j(r_k)$, $Q_j(r_k)$, $X_j(r_k)$, $Y_j(r_k)$, $F_j(r_k)$ by applying Theorem 4 and store them in a search table.

(3) For each $F_j(r_k)$, construct the corresponding polyhedral invariant set by the following algorithm: let $S_j(r_k) = [C^T(r_k), -C^T(r_k), F_j^T(r_k), -F_j^T(r_k)]^T$, $d_j(r_k) = [y_{\max}^T(r_k), y_{\min}^T(r_k), u_{\max}^T(r_k), u_{\min}^T(r_k)]^T$. Select row m from $(S_j(r_k), d_j(r_k))$ and then check $\forall j$ if $S_{j,m}(r_k(A_j(r_k) + B_j F_j)(r_k) \le d_{j,m}(r_k))$ is redundant through solving the Linear programming:

$$\begin{aligned} &\max\ \rho_{j,m} \\ &\text{s.t.}\quad \rho_{j,m} = S_{j,m}(r_k)\left(A_j(r_k) + B_j F_j(r_k)\right)x - d_{j,m}(r_k) \\ &S_j(r_k)\,x \le d_j(r_k). \end{aligned} \tag{31}$$

If $\rho_{j,m} > 0$, it implies that the constraint $S_{j,m}(r_k(A_j(r_k) + B_j F_j)(r_k) \le d_{j,m}(r_k))$ is nonredundant; then renew the nonredundant constraints as $S_j(r_k) = [S_j^T(r_k), (S_{j,m}(r_k)(A_j(r_k) + B_j(r_k)F_j(r_k)))^T]^T$, $d_j(r_k) = [d_j(r_k)^T, d_{j,m}(r_k)^T]^T$.

(4) Online implementation: search the state in the search table to fix the needed index $j (j < N)$, decide the smallest polyhedral invariant set $\chi_j(r_k) = \{x \mid S_j(r_k)x \le d_j(r_k)\}$, and finally implement $u_k = F_j(r_k)x_k$.

(5) Online implementation: continue to check if $x \in \chi_N(r_k) = \{x \mid S_N(r_k)x \le d_N(r_k)\}$ is satisfied; if it is true, then apply $u_k = F_N x_k$.

Remark 8. It should be noted that the more approximation of optimality can be obtained as N increases; here N can be chosen according to different prior requirements. Thus, we can adjust the numbers of design step in terms of different requirements.

4. Illustrative Example

Consider the discrete-time MJS with four modes ($\sigma = 4$):

$$\begin{aligned}
A_{11} &= \begin{bmatrix} 1 & 0.1 \\ 0.01 & 0.99 \end{bmatrix}, & B_{11} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{12} &= \begin{bmatrix} 1 & 0.1 \\ 0 & 0.05 \end{bmatrix}, & B_{12} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{21} &= \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.99 \end{bmatrix}, & B_{21} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{22} &= \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.05 \end{bmatrix}, & B_{22} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{31} &= \begin{bmatrix} 1 & 0.1 \\ 0.2 & 0.99 \end{bmatrix}, & B_{31} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{32} &= \begin{bmatrix} 1 & 0.1 \\ 0.15 & 0.1 \end{bmatrix}, & B_{32} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{41} &= \begin{bmatrix} 1 & 0.1 \\ 0.05 & 0.5 \end{bmatrix}, & B_{41} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}, \\
A_{42} &= \begin{bmatrix} 1 & 0.1 \\ 0.05 & 0.1 \end{bmatrix}, & B_{42} &= \begin{bmatrix} 0.1 \\ 0.187 \end{bmatrix}.
\end{aligned} \tag{32}$$

The detailed constraints are $u_{\max} = 2$ and $y_{\max} = 1.5$, initial state x_0 is $[-0.65\ \ 1]^T$, and $C(r_k) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The positive definite weighting matrices are $Q(r_k) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R(r_k) = 0.00002$. The partly unknown TP matrix is randomly generated in Table 1.

Table 1: The partly unknown TP matrix.

1	2	3	4
0.361	?	0.092	?
?	0.090	?	0.248
0.162	0.489	?	?
?	?	0.251	?

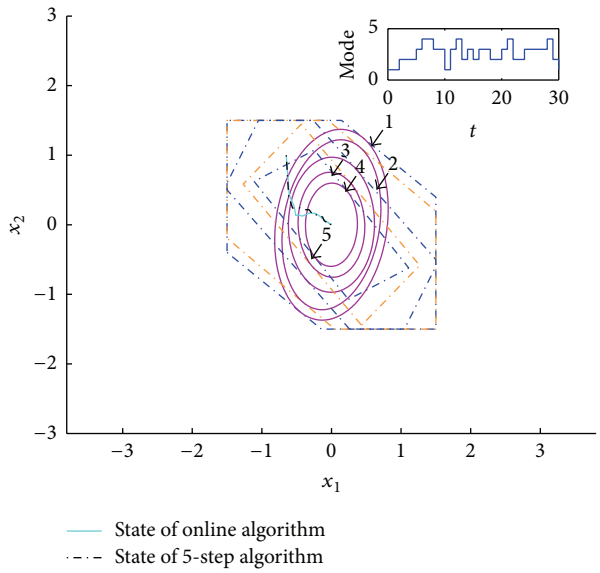

Figure 2: Trajectory of system states.

Table 2: 20 times' average of 30 iterations of system state.

Algorithm	20 times' average	Variance
Online	11.6779 s	0.0069
5-step	0.0055 s	$1.3173e-006$

Here we will show the 5-step example of the proposed Algorithm 7. Firstly, a state set $\{x_j \mid x_j = (0.5, -0.9), (0.4, -0.8), (0.3, -0.7), (0.2, -0.6), (0.1, -0.5)\}$ is designed to compute the corresponding feedback gains $F_j(r_k)$. It is noted that the sequence of states x_j guarantees that the constructed polyhedral invariant sets are embedded, that is, $S_j \subset S_{j-1}$. In this example, the first four mode-dependent feedback laws $F_j(r_k)$, $j = 1, \ldots, 4$ are obtained. When the state goes into the smallest polyhedral invariant set, the final-step (the fifth-step) gain F_5 is designed to steer the state to the origin regardless of model uncertainty and TP uncertainty.

For each chosen x_j in Figure 2, the 5-step ellipsoid invariant sets (purple solid lines) and 5-step polyhedral invariant sets (blue and orange alternant dot dash lines) are illustrated using the numbers 1 to 5. The stabilizable region of polyhedral invariant set constructed by Algorithm 7 is dramatically larger than that of ellipsoid invariant set while the dynamic response of simplified algorithm is comparable with online algorithm.

The results are computed at the same platform (AMD 2.1 GHz, memory 3.0 GB and MATLAB R2010a); the average time and variances of 30 times' running of the system are shown in Table 2. From the table, the burden of computation is significantly reduced by simplified algorithm.

5. Conclusions

The problem of simplified predictive controller design for MJS with mixed uncertainties is investigated. The simplified algorithm drastically reduces the online computational burden with only a little loss of performance. A numerical example is provided to illustrate the validity of the results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Portfolio Strategy of Financial Market with Regime Switching Driven by Geometric Lévy Process

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The problem of a portfolio strategy for financial market with regime switching driven by geometric Lévy process is investigated in this paper. The considered financial market includes one bond and multiple stocks which has few researches up to now. A new and general Black-Scholes (B-S) model is set up, in which the interest rate of the bond, the rate of return, and the volatility of the stocks vary as the market states switching and the stock prices are driven by geometric Lévy process. For the general B-S model of the financial market, a portfolio strategy which is determined by a partial differential equation (PDE) of parabolic type is given by using Itô formula. The PDE is an extension of existing result. The solvability of the PDE is researched by making use of variables transformation. An application of the solvability of the PDE on the European options with the final data is given finally.

1. Introduction

To make a portfolio strategy is to search for a best allocation of wealth among different assets in markets. Taking the European options, for instance, how to distribute the appropriate proportions of each option to maximize total returns at expire time is the core of portfolio strategy problem. There are two points mentioned among the relevant literatures for portfolio selection problems: setting up a market model that approximates to the real financial market and the way of solving it.

Portfolio strategy researches are based on portfolio selection analysis given by Markowitz [1]. Extension of Markowitz's work to the multiperiod model has given by Li and Ng [2] which derived the analytical optimal portfolio policy. These previous researches were assuming that the underlying market has only one state or mode. But the real market might have more than one state and could switch among them. Then, portfolio policies under regime switching have been widely discussed. In a financial market model, the key process S that models the evolution of stock price should be a Brownian motion. Indeed, this can be intuitively justified on the basis of the central limit theorem if one perceives the movement of stocks. The analysis of Øksendal [3] was mainly based on the generalized Black-Scholes model which has two assets $B(t)$ and $S(t)$ as $dB(t) = \rho(t)B(t)dt$ and $dS(t) = \alpha(t)S(t)dt + \beta(t)S(t)dW(t)$, where $W(t)$ is a Brownian motion. In that case, Øksendal formulated optimal selling decision making as an optimal stopping problem and derived a closed-form solution. The underlying problem may be treated as a free boundary value problem, which was extended to incorporate possible regime switching by Guo and Zhang [4] and Pemy et al. [5] with the switching represented by a two-state Markov chain. The rate of return α(t) in the above Black-Scholes models in [4, 5] is a Markov chain which is different from the general one. As an application, Wu and Li [6, 7] have given the strategy of multiperiod mean-variance portfolio selection with regime switching and a stochastic cash flow which depends on the states of a stochastic market following a discrete-time Markov chain. Being put in the Markov jump, Black-Scholes model with regime switching is much closer to the real market.

In recent years, Lévy process as a more general process than Brownian motion has been applied in financial portfolio

optimization. Kallsen [8] gave an optimal portfolio strategy of securities market under exponential Lévy process. More specific than exponential Lévy process, a financial market model with stock price following the geometric Lévy process was discussed by Applebaum [9] in which a Lévy process $X(t)$ and geometric Lévy motion $S(t) = e^{X(t)}$ were introduced. Taking X to be a Lévy process could force our stock prices clearly not moving continuously, and a more realistic approach is that the stock price is allowed to have small jumps in small time intervals. Some applications of financial market driven by Lévy process are taken on life insurance. Vandaele and Vanmaele [10] show the real risk-minimizing hedging strategy for unit-linked life insurance in financial market driven by a Lévy process while Weng [11] has analyzed the constant proportion portfolio insurance by assuming that the risky asset price follows a regime switching exponential Lévy process and obtained the analytical forms of the shortfall probability, expected shortfall and expected gain. Optimizing proportional reinsurance and investment policies in a multidimensional Lévy-driven insurance model is discussed by Bäuerle and Blatter [12]. Moreover, under a generally method, Yuen and Yin [13] have considered the optimal dividend problem for the insurance risk process in a general Lévy process which shows that if the Lévy density is a completely monotone function, then the optimal dividend strategy is a barrier strategy.

Among all the above literatures, those portfolios are always based on one risk-free asset and only one risky asset which may limit the chosen stocks. However, in a real financial market, there always exists more than one risky asset in a portfolio. That is why we are going to extend the single-stock financial market model to a multistock financial market model driven by geometric Lévy process which is more closer to the real market than proposed portfolios cited above. In this paper, we set up a general Black-Scholes model with geometric Lévy process. For the general Black-Scholes model of the financial market, a portfolio strategy which is determined by a partial differential equation (PDE) of parabolic type is given by using Itô formula. The solvability of the PDE is researched by making use of variables transformation. An application of the solvability of the PDE on the European options with the final data is given finally. The contributions of this paper are as follows. (i) The B-S market model is extended into general form in which the interest rate of the bond, the rate of return, and the volatility of the stock vary as the market states switching and the stock prices are driven by geometric Lévy process. (ii) The PDE determining the portfolio strategy and its solvability are extensions of the existing results.

2. Problem Formulation

Assume that $(\Omega, \mathscr{F}, P)$ is a complete probability space and $\{\mathscr{F}_t : t \geq 0\}$ is a nondecreasing family of σ-algebra subfields of $\mathscr{F}$. $\{\alpha(t) : t \geq 0\}$ denotes a Markov chain in $(\Omega, \mathscr{F}, P)$ as the regime of financial market, for example, the bull market or bear market of a stock market. Let $M = \{1, 2, \ldots, m\}$ be the regime space of this Markov chain, and let $\Gamma = (\gamma_{ij})_{m\times m}$ be the transition rate matrix which is satisfying

$$P\{\alpha(t+\Delta) = j \mid \alpha(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & i = j, \end{cases} \tag{1}$$

where $\Delta > 0$ is the increment of time, $\gamma_{ij} \geq 0$ $(i \neq j)$, $\gamma_{ii} = -\sum_{j\neq i, j=1}^{m} \gamma_{ij}$.

In this paper, we consider a financial market model driven by geometric Lévy process. The market consists of one risk-free asset denoted by B and n risky assets denoted by $S_1, S_2, \ldots, S_n$. The price process of these assets obeys the following dynamic equations in which the price process of the risky assets follows the geometric Lévy process; that is,

$$\begin{aligned} dB(t) &= B(t)\, r(t, \alpha(t))\, dt, \qquad B(0) = B_0, \\ dS_k(t) &= S_k(t) \Big[\mu_k(t, \alpha(t))\, dt + \sigma_k(t, \alpha(t))\, dW_k(t) \\ &\qquad + \int_{R-\{0\}} z \widetilde{N}_k(dt, dz) \Big], \\ S_k(0) &= S_k^0 > 0, \end{aligned} \tag{2}$$

where $B(t)$ is the price of B with the interest rate $r(t, \alpha(t))$ and $S_k(t)$ is the price of S_k with the expect rate of return $\mu_k(t, \alpha(t))$ and the volatility $\sigma_k(t, \alpha(t))$, which follow the regime switching of financial market. $S_1(t), S_2(t), \ldots, S_n(t)$ are independent from each other. $W_k(t)$ is the Brownian motion which is independent from $\{\alpha(t) : t \geq 0\}$. $\widetilde{N}_k(\cdot, \cdot)$ is defined as below

$$\widetilde{N}_k(dt, dz) = N_k(dt, dz) - \eta_k(dz)\, dt, \tag{3}$$

where $N_k(dt, dz)$ and $\eta_k(dz)dt$ indicate the number of jumps and average number of jumps within time dt and jump range dz of price process $S_k(t)$, respectively. That is

$$\eta_k(dz)\, dt = \mathbb{E}\left[N_k(dt, dz)\right], \tag{4}$$

where $\mathbb{E}$ is the expectation operator. Moreover, we assume that $N_k(dt, dz)$, $\alpha(t)$, and $W_k(t)$ $(k = 1, 2, \ldots, n)$ are independent of each other.

Remark 1. The finance market model (2) is an extension of the B-S market model in which the interest rate of the bond, the rate of return, and the volatility of the stock vary as the market states switching and the stock prices are driven by geometric Lévy process.

For finance market model (2), we introduce the concept of self-financing portfolio as follows.

Definition 2. A self-financing portfolio $(\varphi, \psi) = (\varphi, \psi_1, \psi_2, \ldots, \psi_n)$ for the financial market model (2) is a series of predictable processes

$$\{\varphi(t)\}_{t\geq 0}, \qquad \{\psi_k(t)\}_{t\geq 0} \quad (k = 1, 2, \ldots, n), \tag{5}$$

that is, for each $T > 0$,

$$\int_0^T |\varphi(s)|^2 ds + \sum_{k=1}^n \int_0^T |\psi_k(s)|^2 ds < \infty, \tag{6}$$

and the corresponding wealth process $\{V(t)\}_{t\geq 0}$, defined by

$$V(t) := \varphi(t) B(t) + \sum_{k=1}^n \psi_k(t) S_k(t), \quad t \geq 0, \tag{7}$$

is an Itô process satisfying

$$dV(t) = \varphi(t) dB(t) + \sum_{k=1}^n \psi_k(t) dS_k(t), \quad t \geq 0. \tag{8}$$

Problem Formulation. In this note, we will propose a portfolio strategy for the financial market model (2) which is determined by a partial differential equation (PDE) of parabolic type by using Itô formula. The solvability of the PDE is researched by making use of variables transformation. Furthermore, the relationship between the solution of the PDE and the wealth process will be discussed.

3. Main Results and Proofs

In this section, we will give the following fundamental results. For the sake of simplification, we write $r(t, \alpha(t))$ as r, $f(t, S(t))$ as f, and so forth.

To obtain the main result, we give the solution of (2) and the characteristic of the derivation (8) of the wealth process.

The exact solutions of $B(t)$ in (2) can be found as follows:

$$B(t) = B(0) \exp\left(\int_0^t r(s, \alpha(s)) ds\right). \tag{9}$$

To solve the second equation in (2) for $S_k(t)$, it follows from the Itô formula that

$$\begin{aligned}
d \ln S_k(t) = {} & \frac{1}{S_k(t)} \left[S_k(t) \mu_k(t, \alpha(t)) dt \right. \\
& \left. + S_k(t) \sigma_k(t, \alpha(t)) dW_k(t)\right] \\
& - \frac{1}{2} \frac{1}{S_k^2(t)} S_k^2(t) \sigma_k^2(t, \alpha(t)) dt \\
& + \int_{R-\{0\}} \left[\ln\left(S_k(t) + zS_k(t)\right) \right. \\
& \left. - \ln\left(S_k(t)\right)\right] \widetilde{N}_k(dt, dz) \\
& + \int_{R-\{0\}} \left[\ln\left(S_k(t) + zS_k(t)\right) - \ln\left(S_k(t)\right) \right. \\
& \left. - zS_k(t) \frac{1}{S_k(t)}\right] \eta_k(dz) dt \\
& + \sum_{j=1}^s \gamma_{ij} \ln\left(S_k(t)\right) \\
= {} & \left[\mu_k(t, \alpha(t)) - \frac{1}{2}\sigma_k^2(t, \alpha(t))\right] dt \\
& + \sigma_k(t, \alpha(t)) dW_k(t) \\
& + \int_{R-\{0\}} \ln(1+z) \widetilde{N}_k(dt, dz) \\
& + \int_{R-\{0\}} [\ln(1+z) - z] \eta_k(dz) dt.
\end{aligned} \tag{10}$$

Integrating both sides of the above equation from 0 to t, we have

$$\begin{aligned}
S_k(t) = S_k^0 \exp\Bigg\{ & \int_0^t \left(\mu_k(s, \alpha(s)) - \frac{1}{2}\sigma_k^2(s, \alpha(s))\right) ds \\
& + \int_0^t \sigma_k(s, \alpha(s)) dW_k(s) \\
& + \int_0^t \int_{R-\{0\}} \ln(1+z) \widetilde{N}_k(ds, dz) \\
& + \int_0^t \int_{R-\{0\}} [\ln(1+z) - z] \eta_k(dz) ds \Bigg\}.
\end{aligned} \tag{11}$$

Proposition 3. *Consider the price model* (2) *of a financial market. If a portfolio* (φ, ψ) *is a self-financing strategy, then the wealth process* $\{V(t)\}_{t\geq 0}$ *defined by* (7) *satisfies*

$$\begin{aligned}
dV(t) = {} & \Bigg\{ r(t, \alpha(t)) V(t) \\
& + \sum_{k=1}^n \psi_k(t) S_k(t) \left[\mu_k(t, \alpha(t)) - r(t, \alpha(t)) \right. \\
& \left. - \int_{R-\{0\}} z\eta_k(dz)\right]\Bigg\} dt \\
& + \sum_{k=1}^n \psi_k(t) S_k(t) \sigma_k(t, \alpha(t)) dW_k(t) \\
& + \sum_{k=1}^n \psi_k(t) S_k(t) \int_{R-\{0\}} zN_k(dt, dz).
\end{aligned} \tag{12}$$

Conversely, consider the model (2) *of a financial market. If a pair* (φ, ψ) *of predictable processes following the wealth process* $\{V(t)\}_{t\geq 0}$ *defined by formula* (7) *satisfies* (12), *then* (φ, ψ) *is a self-financing strategy.*

Proof. Substituting (2) into (8), we have

$$\begin{aligned}
dV(t) &= \varphi(t) dB(t) + \sum_{k=1}^n \psi_k(t) dS_k(t) \\
&= \varphi(t) B(t) r(t, \alpha(t)) dt + \sum_{k=1}^n \psi_k(t) S_k(t)
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\mu_k(t,\alpha(t))\,dt + \sigma_k(t,\alpha(t))\,dW_k(t) \right.\\
&\qquad \left. + \int_{R-\{0\}} z\widetilde{N}_k(dt,dz)\right]\\
&= \left\{\left[V(t) - \sum_{k=1}^{n}\psi_k(t)S_k(t)\right] r(t,\alpha(t)) \right.\\
&\qquad \left. + \sum_{k=1}^{n}\psi_k(t)S_k(t)\mu_k(t,\alpha(t))\right\} dt\\
&\qquad + \sum_{k=1}^{n}\psi_k(t)S_k(t)\sigma_k(t,\alpha(t))\,dW_k(t)\\
&\quad + \sum_{k=1}^{n}\psi_k(t)S_k(t)\int_{R-\{0\}} z\widetilde{N}_k(dt,dz)\\
&= \left\{r(t,\alpha(t))V(t) + \sum_{k=1}^{n}\psi_k(t)S_k(t)\right.\\
&\qquad \times \left[\mu_k(t,\alpha(t)) - r(t,\alpha(t))\right.\\
&\qquad \left.\left. - \int_{R-\{0\}} z\eta_k(dz)\right]\right\} dt\\
&\quad + \sum_{k=1}^{n}\psi_k(t)S_k(t)\sigma_k(t,\alpha(t))\,dW_k(t)\\
&\quad + \sum_{k=1}^{n}\psi_k(t)S_k(t)\int_{R-\{0\}} zN_k(dt,dz),
\end{aligned}
\tag{13}
$$

which is (12).

Conversely, from (2) and (12), we can obtain (8).

This completes the proof of the above proposition. □

Now we give the following fundamental results.

Theorem 4. *Consider the model* (2) *of a financial market. Assume that the portfolio* $(\varphi,\psi_1,\psi_2,\ldots,\psi_n)$ *is a self-financing strategy and* $\{V(t)\}_{t\geq 0}$ *is the wealth process defined by* (7) *and* $\sum_{k=1}^{n}\psi_k S_k\int_{R-\{0\}} z\eta_k(dz) = \sum_{k=1}^{n}\int_{R-\{0\}} z\psi_k S_k\eta_k(dz)$. *If there exists a function* $f(t,S)$ *of* $C^{1,2}$ *class (the set of functions which are once differentiable in t and continuously twice differentiable in S) such that*

$$
\begin{aligned}
V(t) &= f(t,S(t)), \quad t\in[0,T],\\
S(t) &= \left(S_1(t),S_2(t),\ldots,S_n(t)\right),
\end{aligned}
\tag{14}
$$

which holds true, then the portfolio $(\varphi,\psi_1,\psi_2,\ldots,\psi_n)$ *satisfies*

$$
\varphi(t) = \frac{f - (\partial f/\partial S)S^T}{B(t)}, \quad t\geq 0 \tag{15}
$$

$$
\psi(t) = \left(\frac{\partial f}{\partial S_1},\frac{\partial f}{\partial S_2},\ldots,\frac{\partial f}{\partial S_n}\right) = \frac{\partial f}{\partial S}, \quad t\geq 0 \tag{16}
$$

and the function $f(t,S)$ *solves the following backward PDE of parabolic type:*

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + r\sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j = rf,\\
&\qquad t<T,\ S>0.
\end{aligned}
\tag{17}
$$

Moreover, if $V(T) = g(S(T))$, *then the function* $f(t,S)$ *satisfies the following equation:*

$$
f(T,S) = g(S), \quad S>0. \tag{18}
$$

For the converse part, we assume that $T>0$. *If there exists a function* $f(t,S)$ *of* $C^{1,2}$ *class such that* (17) *and* (18) *are satisfied, then the process* (φ,ψ) *defined by* (16) *and* (15) *is a self-financing strategy. The wealth process* $V = \{V(t)\}_{t\in[0,T]}$ *corresponding to* (φ,ψ) *satisfies* (14).

Proof. We proof the direct part of Theorem 4 firstly.

For

$$
V(t) = f(t,S(t)), \tag{19}
$$

by applying the Itô formula, we can infer that

$$
\begin{aligned}
dV(t) &= \frac{\partial f}{\partial t}(t,S(t))\,dt\\
&\quad + \sum_{k=1}^{n}\frac{\partial f}{\partial S_k}(t,S(t))\left(S_k\mu_k dt + S_k\sigma_k dW_k\right)\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}(t,S(t))S_i\sigma_i S_j\sigma_j dt\\
&\quad + \sum_{k=1}^{n}\int_{R-\{0\}}\left(f(t,S+zS) - f(t,S)\right)\widetilde{N}_k(dt,dz)\\
&\quad + \sum_{k=1}^{n}\int_{R-\{0\}}\left[f(t,S+zS) - f(t,S)\right.\\
&\qquad \left. - z\frac{\partial f}{\partial S_k}(t,S)S_k\right]\eta_k(dz)\,dt\\
&\quad + \sum_{j=1}^{m}\gamma_{ij}f(t,S(t))\\
&= \left[\frac{\partial f}{\partial t} + \sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k\mu_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j\right.\\
&\qquad \left. - \sum_{k=1}^{n}\int_{R-\{0\}} z\frac{\partial f}{\partial S_k}S_k\eta_k(dz)\right]dt\\
&\quad + \sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k\sigma_k dW_k\\
&\quad + \sum_{k=1}^{n}\int_{R-\{0\}}\left[f(t,S+zS) - f(t,S)\right]N(dt,dz).
\end{aligned}
\tag{20}
$$

On the other hand, since our strategy is self-financing, the formula (12) is satisfied.

Thus, the rate of return and the volatility in (20) and (12) should be coincided, and hence

$$\sum_{k=1}^{n}\psi_k(t)S_k(t)\sigma_k=\sum_{k=1}^{n}\frac{\partial f}{\partial S_k}(t,S)S_k\sigma_k,$$

$$r(t,\alpha(t))f(t,S)+\sum_{k=1}^{n}\psi_k S_k(\mu_k-r) \tag{21}$$

$$=\frac{\partial f}{\partial t}+\sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k\mu_k+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j.$$

We can easily get $S_k\geq 0$ from (11), which together with the first equation of (21) and the independence of S_k $(k=1,2,\ldots,n)$ yields (16).

From the first equation of (21), (7), and (14), we have

$$r\varphi B=f-\sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k. \tag{22}$$

So that

$$\varphi=\frac{f-\sum_{k=1}^{n}(\partial f/\partial S_k)S_k}{B}=\frac{f-f_S S^T}{B}. \tag{23}$$

Substituting (16) into the second equation of (21), we have

$$rf-\sum_{k=1}^{n}\psi_k S_k r=\frac{\partial f}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j, \tag{24}$$

which is (17).

Conversely, assume that $f=f(t,S)$ is a $C^{1,2}$-class function which is a solution of the PDE (17), and that (φ,ψ) is a process defined by (16) and (15).

Firstly, we will show that a process $V=V(t)$, $t\in[0,T]$, defined by (7) satisfies the equation:

$$V(t)=f(t,S(t)),\quad t\in[0,T]. \tag{25}$$

In fact, substituting formulas (16) and (15) into the right hand side of (7), we have

$$\begin{aligned}V(t)&=\varphi B+\sum_{k=1}^{n}\psi_k S_k=\frac{f-\sum_{k=1}^{n}(\partial f/\partial S_k)S_k}{B}B\\&\quad+\sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k=f,\quad t\geq 0.\end{aligned} \tag{26}$$

This proves (25).

Next, we will show that (φ,ψ) is a self-financing strategy; that is, (12) holds.

By applying the Itô formula to the process V and function f, we have that (20) is satisfied.

Furthermore, by (17),

$$\begin{aligned}&\frac{\partial f}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j=rf-r\sum_{k=1}^{n}S_k\frac{\partial f}{\partial S_k},\\&\frac{\partial f}{\partial t}+\sum_{k=1}^{n}S_k\mu_k\frac{\partial f}{\partial S_k}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j\\&\quad=rf+\sum_{k=1}^{n}(\mu_k-r)S_k\frac{\partial f}{\partial S_k}.\end{aligned} \tag{27}$$

Then, by (25) and (16), we have

$$\begin{aligned}rV+\sum_{k=1}^{n}\psi_k S_k(\mu_k-r)&=\frac{\partial f}{\partial t}+\sum_{k=1}^{n}S_k\mu_k\frac{\partial f}{\partial S_k}\\&\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial S_i\partial S_j}S_i\sigma_i S_j\sigma_j,\end{aligned} \tag{28}$$

$$\sum_{k=1}^{n}\psi_k S_k\sigma_k=\sum_{k=1}^{n}\frac{\partial f}{\partial S_k}S_k\sigma_k. \tag{29}$$

Those together with (16) yield that (20) implies (12). The proof of Theorem 4 is completed. □

Remark 5. In order to determine the portfolio strategy (ϕ,ψ) and obtain the final value $V(t)$, from Theorem 4, we should find the solution of the PDF (17) with the final data (18). This is the key problem in the rest of this section. We have the following result in terms of method of variables transformation.

Theorem 6. *Let $r(t,\alpha(t))$ in (2) be a constant r. The function $f(t,S)$, $t\leq T$, $S>0$ given by the following formula:*

$$\begin{aligned}f(t,S)&=\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\\&\quad\times\sum_{i=1}^{n}\int_{-\infty}^{\infty}e^{-x_i^2/2}g\Big(0,\ldots,0,S_i e^{\sigma_i\sqrt{T-t}x_i-(r-\sigma_i^2/2)(t-T)},\\&\qquad 0,\ldots,0\Big)dx_i,\end{aligned} \tag{30}$$

is a solution of the general Black-Scholes equation (17) with the final data (18).

Proof. We are going to do some equivalent transformations of general B-S equation (17), in order to get an appropriate equivalent equation with analytic solutions. The procedure will be divided into four steps.

Step I. Let

$$\begin{aligned}f(t,S_1,\ldots,S_n)&=e^{r(t-T)}q\Big(t,\ln S_1-\Big(r-\frac{1}{2}\sigma_1^2\Big)(t-T),\ldots,\\&\qquad\ln S_n-\Big(r-\frac{1}{2}\sigma_n^2\Big)(t-T)\Big),\end{aligned} \tag{31}$$

and denote $y_i = \ln S_i - (r - (1/2)\sigma_i^2)(t - T)$ $(i = 1, 2, \ldots, n)$, and then

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{d\left(e^{r(t-T)}\right)}{dt} q + e^{r(t-T)} q_t \\
&= e^{r(t-T)} q \left(\frac{dr}{dt}(t-T) + r\right) \\
&\quad + e^{r(t-T)} \left[q_t - \sum_{i=1}^{n} \frac{\partial q}{\partial y_i}\left(r - \frac{1}{2}\sigma_i^2\right)\right] \\
&= r e^{r(t-T)} q + e^{r(t-T)} q \frac{dr}{dt}(t-T) \\
&\quad + e^{r(t-T)} \left[\frac{\partial q}{\partial t} - \sum_{i=1}^{n} \frac{\partial q}{\partial y_i}\left(r - \frac{1}{2}\sigma_i^2\right)\right], \qquad (32) \\
\frac{\partial f}{\partial S_i} &= e^{r(t-T)} \frac{\partial q}{\partial y_i} \frac{1}{S_i}, \\
\frac{\partial^2 f}{\partial S_i \partial S_j} &= \frac{\partial\left(e^{r(t-T)}\left(\partial q/\partial y_i\right)(1/S_i)\right)}{\partial S_j} \\
&= \begin{cases} e^{r(t-T)} \dfrac{\partial^2 q}{\partial y_i \partial y_j} \dfrac{1}{S_i} \dfrac{1}{S_j}, & i \neq j, \\ e^{r(t-T)} \left(\dfrac{\partial^2 q}{\partial y_i \partial y_i} \dfrac{1}{S_i} \dfrac{1}{S_i} - \dfrac{\partial q}{\partial y_i} \dfrac{1}{S_i^2}\right), & i = j. \end{cases}
\end{aligned}$$

Inserting the above formulas into (17), we get

$$\begin{aligned}
&rf + \frac{dr}{dt}(t-T) q e^{r(t-T)} + e^{r(t-T)} \left[\frac{\partial q}{\partial t} - \sum_{i=1}^{n} \frac{\partial q}{\partial y_i}\left(r - \frac{1}{2}\sigma_i^2\right)\right] \\
&\quad + r \sum_{i=1}^{n} e^{r(t-T)} \frac{\partial q}{\partial y_i} \frac{1}{S_i} S_i + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} e^{r(t-T)} \frac{\partial^2 q}{\partial y_i \partial y_j} \frac{1}{S_i} \frac{1}{S_j} S_i \sigma_i S_j \sigma_j \\
&\quad - \frac{1}{2} \sum_{i=1}^{n} e^{r(t-T)} \frac{\partial q}{\partial y_i} S_i^2 \frac{1}{S_i^2} S_i^2 = rf,
\end{aligned} \qquad (33)$$

which can be simplified as

$$\frac{dr}{dt}(t-T) q + \frac{\partial q}{\partial t} + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 q}{\partial y_i \partial y_j} \sigma_i \sigma_j = 0. \qquad (34)$$

The final data $f(T, S) = g(S)$ can be rewritten as

$$q(T, S) = g\left(e^{S_1}, e^{S_2}, \ldots, e^{S_n}\right). \qquad (35)$$

Step II. We introduce another variable and a new function as follows:

$$\begin{aligned}
&\tau = T - t > 0, \quad t = T - \tau, \ \tau \geq 0, \ t \leq T, \\
&q(t, y) = u(T - t, y) \quad \text{or} \quad u(\tau, y) = q(T - \tau, y).
\end{aligned} \qquad (36)$$

It can be computed that

$$\begin{aligned}
q_t(t, y) &= -u_\tau(T - t, y), \\
\frac{\partial q}{\partial y_i} &= \frac{\partial u}{\partial y_i}(T - t, y), \\
\frac{\partial^2 q}{\partial y_i \partial y_j} &= \frac{\partial^2 u}{\partial y_i \partial y_j}(T - t, y).
\end{aligned} \qquad (37)$$

Substituting the above formulas into (34), we get

$$\begin{aligned}
&\frac{dV}{dt}(t-T) u(T - t, y) - u_\tau(T - t, y) + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial y_i \partial y_j} \sigma_i \sigma_j \\
&\quad = 0, \\
&u(0, y) = g(e^y).
\end{aligned} \qquad (38)$$

Since $r(t, \alpha(t))$ is assumed as a constant r, (38) can be changed into

$$\begin{aligned}
&u_\tau(\tau, y) - \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial y_i \partial y_j} \sigma_i \sigma_j = 0, \\
&u(0, y) = g(e^y).
\end{aligned} \qquad (39)$$

Step III. We claim that the unique solution of (39) is

$$\begin{aligned}
&u(t, y_1, y_2, \ldots, y_n) \\
&\quad = \frac{1}{\sqrt{2\pi\tau}} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \frac{e^{-(y_i - x_i)^2/2\sigma_i^2\tau}}{\sigma_i} g\left(0, \ldots, 0, e^{x_i}, 0, \ldots, 0\right) dx_i.
\end{aligned} \qquad (40)$$

In fact,

$$\begin{aligned}
u_\tau(\tau, y) &= -\frac{1}{2\sqrt{2\pi\tau}\tau} \\
&\quad \times \sum_{i=1}^{n} \int_{-\infty}^{\infty} \frac{e^{-(y_i - x_i)^2/2\sigma_i^2\tau}}{\sigma_i} \\
&\qquad \times g\left(0, \ldots, 0, e^{x_i}, 0, \ldots, 0\right) dx_i \\
&\quad + \frac{1}{\sqrt{2\pi\tau}} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \frac{e^{-(y_i - x_i)^2/2\sigma_i^2\tau}}{\sigma_i} \\
&\qquad \times g\left(0, \ldots, 0, e^{x_i}, 0, \ldots, 0\right) \\
&\qquad \times \frac{(y_i - x_i)^2}{2\sigma_i^2\tau^2} dx_i,
\end{aligned}$$

$$\frac{\partial u}{\partial y_i} = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \frac{e^{-(y_i-x_i)^2/2\sigma_i^2\tau}}{\sigma_i} g\left(0,\ldots,0,e^{x_i},0,\ldots,0\right) \times \left(-\frac{y_i-x_i}{\sigma_i^2\tau}\right) dx_i,$$

$$\frac{\partial^2 u}{\partial y_i \partial y_j} = \begin{cases} 0, & i \neq j, \\ \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \frac{g\left(e^{x_1},\ldots,e^{x_n}\right)}{\sigma_i^2} e^{-(y_i-x_i)^2/2\sigma_i^2\tau} \times \left(\frac{(y_i-x_i)^2}{\sigma_i^4\tau^2} - \frac{1}{\sigma_i^2\tau}\right) \sigma_i^2 dx_i, & i = j. \end{cases} \tag{41}$$

So

$$\begin{aligned} u_\tau(\tau,g) &- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial y_i \partial y_j}\sigma_i\sigma_j \\ &= -\frac{1}{2\sqrt{2\pi\tau}\tau}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \frac{e^{-(y_i-x_i)^2/2\sigma_i^2\tau}}{\sigma_i} \times g\left(0,\ldots,0,e^{x_i},0,\ldots,0\right) dx_i \\ &\quad + \frac{1}{\sqrt{2\pi\tau}}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \frac{e^{-(y_i-x_i)^2/2\sigma_i^2\tau}}{\sigma_i} \times g\left(0,\ldots,0,e^{x_i},0,\ldots,0\right) \frac{(y_i-x_i)^2}{2\sigma_i^2\tau^2} dx_i \\ &\quad - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} \frac{g\left(e^{x_1},\ldots,e^{x_n}\right)}{\sigma_i^2} \times e^{-(y_i-x_i)^2/2\sigma_i^2\tau}\left(\frac{(y_i-x_i)^2}{\sigma_i^4\tau^2} - \frac{1}{\sigma_i^2\tau}\right)\sigma_i^2 dx_i = 0. \end{aligned} \tag{42}$$

Step IV. By introducing a change of variables $z_i = x_i - y_i$, we have $x_i = z_i + y_i$ and $dx_i = dz_i$, where $z_i \in (-\infty,\infty)$. It follows that

$$u(\tau,y) = \frac{1}{\sqrt{2\pi\tau}}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \frac{e^{-(y_i-x_i)^2/2\sigma_i^2\tau}}{\sigma_i} \times g\left(0,\ldots,0,e^{z_i+y_i},0,\ldots,0\right) dz_i. \tag{43}$$

In order to get rid of the denominator $\sigma_i^2\tau$ in the exponent in the above formula, we make another change of variables as

$$z_i = \sigma_i\sqrt{\tau}x_i. \tag{44}$$

So $dz_i = \sigma_i\sqrt{\tau}dx_i$.

Recalling the relationship between q and u described in (36), we therefore have

$$q(t,y) = \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_{-\infty}^{\infty} e^{-x_i^2/2} \times g\left(0,\ldots,0,e^{\sigma_i\sqrt{T-t}x_i+y_i},0,\ldots,0\right) dx_i. \tag{45}$$

Hence, by formula (31), we have

$$f(t,S) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_{-\infty}^{\infty} e^{-x_i^2/2} \times g\left(0,\ldots,0,e^{\sigma_i\sqrt{T-t}x_i+\ln S_i-(r-\sigma_i^2/2)(t-T)}, 0,\ldots,0\right) dx_i. \tag{46}$$

Since $e^{\ln S} = S$, then

$$f(t,S) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_{-\infty}^{\infty} e^{-x_i^2/2} \times g\left(0,\ldots,0,S_i e^{\sigma_i\sqrt{T-t}x_i-(r-\sigma_i^2/2)(t-T)}, 0,\ldots,0\right) dx_i. \tag{47}$$

In this way we proved Theorem 6. □

4. A Financial Example

As an application, we consider the European call option. In Theorem 6, we have given the solution of the general B-S equation (17) which depends on the final data (18); that is, $f(T,s) = g(s)$. More specific, we take the final data $g(s)$ for the European call option as

$$g(S) = g\left(S_1^{-k_1}, S_2^{-k_2},\ldots,S_n^{-k_n}\right) = \sum_{i=1}^{n}(S_i - K_i)^+, \tag{48}$$

where $S_i > 0$ and $K_i > 0$ are the strike price of S_i. Then we have the following corollary from Theorem 6.

Corollary 7. *For the European call option, the solution to the general Black-Scholes value problem* (17) *with the final data* (48) *is given by*

$$f(t,S) = \sum_{i=1}^{n} S_i\Phi\left(-A_i + \sigma_i\sqrt{T-t}\right) - e^{-r(T-t)}\sum_{i=1}^{n} K_i\Phi\left(-A_i\right), \tag{49}$$

where

$$-A_i = \frac{\left(r-\sigma_i^2/2\right)(T-t)+\ln\left(S_i/K_i\right)}{\sigma_i\sqrt{T-t}} =: d_2,$$
$$-A_i+\sigma_i\sqrt{T-t} = \frac{\left(r+\sigma_i^2/2\right)(T-t)+\ln\left(S_i/K_i\right)}{\sigma_i\sqrt{T-t}} =: d_1, \tag{50}$$

that is,

$$f(t,S) = \sum_{i=1}^{n} S_i\Phi\left(d_1\right) - e^{-r(T-t)}\sum_{i=1}^{n} K_i\Phi\left(d_2\right); \tag{51}$$

In particular,

$$f(0,S) = \sum_{i=1}^{n} S_i\Phi\left(d_1\right) - e^{-rT}\sum_{i=1}^{n} K_i\Phi\left(d_2\right). \tag{52}$$

Proof. For a European call option, we infer that

$$S_i e^{\sigma_i\sqrt{T-t}x_i-(r-\sigma_i^2/2)(t-T)} > K_i. \tag{53}$$

Dividing (53) by S_i and taking the ln, we get

$$\sigma_i\sqrt{T-t}x_i - \left(r-\frac{\sigma_i^2}{2}\right)(t-T) > \ln\frac{K_i}{S_i}; \tag{54}$$

that is,

$$x_i > \frac{\ln\left(K_i/S_i\right)-\left(r-\sigma_i^2/2\right)(T-t)}{\sigma_i\sqrt{T-t}} =: A_i. \tag{55}$$

Hence, from (30) and (48), it follows that

$$\begin{aligned}
f(t,S) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \\
&\quad\times\sum_{i=1}^{n}\int_{A_i}^{\infty} e^{-x_i^2/2} S_i e^{\sigma_i\sqrt{T-t}x_i-(r-\sigma_i^2/2)(t-T)}dx_i \\
&\quad - \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n} K_i\int_{A_i}^{\infty} e^{-x_i^2/2}dx_i \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_{A_i}^{\infty} S_i e^{(r-\sigma_i^2/2)(T-t)} e^{-x_i^2/2+\sigma_i\sqrt{T-t}x_i}dx_i \\
&\quad - \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n} K_i\int_{A_i}^{\infty} e^{-x_i^2/2}dx_i \\
&= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_{A_i}^{\infty} S_i e^{-(\sigma_i^2/2)(T-t)} \\
&\quad\times e^{-(1/2)\left(x_i-\sigma_i\sqrt{T-t}\right)^2+(1/2)\sigma_i^2(T-t)}dx_i \\
&\quad - \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n} K_i\int_{A_i}^{\infty} e^{-x_i^2/2}dx_i \\
&= \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{n}\int_{A_i-\sigma_i\sqrt{T-t}}^{\infty} S_i e^{-z^2/2}dz \\
&\quad - \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\sum_{i=1}^{n} K_i\int_{A_i}^{\infty} e^{-x_i^2/2}dx_i \\
&= \sum_{i=1}^{n} S_i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-A_i+\sigma_i\sqrt{T-t}} e^{-x_i^2/2}dx_i \\
&\quad - e^{-r(T-t)}\sum_{i=1}^{n} K_i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-A_i} e^{-x_i^2/2}dx_i \\
&= \sum_{i=1}^{n} S_i\Phi\left(-A_i+\sigma_i\sqrt{T-t}\right) \\
&\quad - e^{-r(T-t)}\sum_{i=1}^{n} K_i\Phi\left(-A_i\right),
\end{aligned} \tag{56}$$

where $\Phi(t)$ is the probability distribution function of a standard Gaussian random variable $N(0,1)$; that is,

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}dx, \quad t\in R. \tag{57}$$

In this way, we have proved Corollary 7. □

Remark 8. The above result is about the European call option. A similar representation to those from the above corollary in the European put option case can be obtained by taking $g(S) = \sum_{i=1}^{n}(K_i - S_i)^+$, $S_i > 0$ for some fixed $K_i > 0$.

5. Conclusion

In this paper, we have considered a financial market model with regime switching driven by geometric Lévy process. This financial market model is based on the multiple risky assets $S_1, S_2, \ldots, S_n$ driven by Lévy process. Itô formula and equivalent transformation methods have been used to solve this complicated financial market model. An example of the portfolio strategy and the final value problem to applying our method to the European call option has been given in the end of this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Finite-Time H_∞ Control for a Class of Discrete-Time Markov Jump Systems with Actuator Saturation via Dynamic Antiwindup Design

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We deal with the finite-time control problem for discrete-time Markov jump systems subject to saturating actuators. A finite-state Markovian process is given to govern the transition of the jumping parameters. A controller designed for unconstrained systems combined with a dynamic antiwindup compensator is given to guarantee that the resulting system is mean-square locally asymptotically finite-time stabilizable. The proposed conditions allow us to find dynamic anti-windup compensator which stabilize the closed-loop systems in the finite-time sense. All these conditions can be expressed in the form of linear matrix inequalities and therefore are numerically tractable, as shown in the example included in the paper.

1. Introduction

It is well known that more and more attention has been paid to the study of actuator saturation due to its practical and theoretical importance. Therefore, various approaches were investigated to handle systems with actuator saturation and dynamic antiwindup approach which is one of the most effective ways to deal with it. To this end, a great number of results have been reported in the literature; see, for example, [1, 2]. Furthermore, the stabilization problem of singular Markovian jump systems with discontinuities and saturation inputs was presented in [3]. Via dynamic antiwindup fuzzy design, the robust stabilization problem of state delayed T-S fuzzy systems with input saturation was proposed in [4].

On the other hand, Markov jump is frequently encountered in many practical systems. Therefore, the study of Markov jump systems has been a hot research topic due to its importance, and many results have been proposed based on various control techniques, such as robust control [5–9], H_∞ control [10, 11], Passivity-based control [12–14], fuzzy dissipative control [15], and neural networks control [14, 16]. Furthermore, observer based finite-time H_∞ control problem of discrete-time Markov jump systems was studied [17].

As it is well known, when dealing with the stability of s system, a distinction should have been made between classical Lyapunov stability and finite-time stability (FTS). Conversely, a system is said to be finite-time stable if, once we fix a time-interval, its state does not exceed some bounds during this time-interval. Some results on FTS have been carried out; see, [18, 19]. Furthermore, finite-time H_∞ filtering problem of time-delay stochastic jump systems with unbiased estimation was proposed in [20]. By applying dynamic observer-based state feedback and the Lyapunov-Krasovskii functional approach, the finite-time H_∞ control problem for time-delay nonlinear jump systems was addressed in the work of He and Liu [21]. However, to the best of our knowledge, the problem of finite-time stabilization of discrete-time stochastic systems subject to actuator saturation has not been fully investigated and it is the main purpose of our study.

In this paper, the attention is focused on the finite-time H_∞ control problem of discrete-time Markov jump systems with actuator saturation based on dynamic anti-windup approach. A controller designed for unconstrained systems combined with a dynamic antiwindup compensator

is given to ensure the stochastic finite-time boundedness and stochastic finite-time stabilization of the resulting closed-loop system for all admissible disturbances. The desired compensator can be designed via solving a convex optimization problem. Finally, a numerical example is employed to show the effectiveness of the proposed method.

Notation 1. Throughout the paper, for symmetric matrices X and Y, the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp., positive definite). I is the identity matrix with appropriate dimension. The notation N^T represents the transpose of the matrix N; $\lambda_{\max}(M)$ (resp., $\lambda_{\min}(M)$) means the largest (resp., smallest) eigenvalue of the matrix M; $(\Omega, \mathscr{F}, \mathscr{P})$ is a probability space; Ω is the sample space; $\mathscr{F}$ is the σ-algebra of subsets of the sample space and $\mathscr{P}$ is the probability measure on $\mathscr{F}$;$\mathscr{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathscr{P}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The symbol $*$ is used to denote a matrix which can be inferred by symmetry. $He\{A\} = A^T + A$.

2. Preliminaries and Problem Description

Consider the following discrete-time Markov jump system (Σ) in the probability space $(\Omega, \mathscr{F}, \mathscr{P})$:

$$\begin{aligned} x_p(k+1) &= A_p(r(k))\, x_p(k) + B_{p,u}(r(k))\, \mathrm{sat}(u(k)) \\ &\quad + B_{p,w}(r(k))\, w(k), \\ y(k) &= C_{p,y}(r(k))\, x_p(k) + D_{p,yw}(r(k))\, w(k), \\ z(k) &= C_{p,z}(r(k))\, x_p(k) + D_{p,zu}(r(k))\, \mathrm{sat}(u(k)) \\ &\quad + D_{p,zw}(r(k))\, w(k), \end{aligned} \tag{1}$$

where $x_p(k) \in \mathbb{R}^{n_p}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, and $\mathrm{sat}(u(k)) \in \mathbb{R}^{n_u}$ is the saturated control input. $w(k) \in L_2^p[0 \quad +\infty)$ is the external disturbances, $y(k) \in \mathbb{R}^{n_y}$ is the measurement output, and $z(k) \in \mathbb{R}^{n_z}$ is the performance output. $\{r(k)\}$ is a discrete-time Markov process and takes values from a finite set $S = \{1, 2, \ldots, \mathcal{N}\}$ with transition probabilities given by

$$\Pr\left(r_{k+1} = j \mid r_k = i\right) = \pi_{ij}, \tag{2}$$

where $\pi_{ij} \geq 0$, for $\forall j, i \in S$, and $\sum_{j \in S} \pi_{ij} = 1$. Moreover, the transition rates matrix of the system (Σ) is defined by

$$\begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1\mathcal{N}} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{\mathcal{N}1} & \pi_{\mathcal{N}2} & \cdots & \pi_{\mathcal{N}\mathcal{N}} \end{bmatrix}. \tag{3}$$

The plant inputs are supposed to be bounded as follows:

$$-u_{0(k)} \leq u_{(k)} \leq u_{0(k)}, \quad u_{0(k)} > 0,\ k = 1, \ldots, m. \tag{4}$$

For the system (Σ), to simplify the notation, we denote $A_{pi} = A_p(r(k))$ for each $r(k) = i \in S$, and the other symbols are similarly denoted. Assume that a linear controller is designed for any $r(k) = i \in S$; then,

$$\begin{aligned} x_c(k+1) &= A_{ci} x_c(k) + B_{cy,i}\, y(k) + B_{cw,i}\, w(k) + v_1, \\ y_c(k) &= C_{ci} x_c(k) + D_{cy,i}\, y(k) + D_{cw,i}\, w(k) + v_2, \end{aligned} \tag{5}$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state and $y_c(k) \in \mathbb{R}^{n_u}$ is the controller output; v_1 and v_2 will be used for antiwindup augmentation. In absence of actuator saturation, the unconstrained closed-loop is formed by setting the following:

$$u = y_c, \qquad v_1 = 0, \qquad v_2 = 0. \tag{6}$$

Assumption 1. The unconstrained closed-loop system (1)–(5) is well posed and internally stable.

In the presence of actuator saturation, the relation between u and y_c is that $u = \mathrm{sat}(y_c)$. To minimize performance degradation caused by saturation, the following antiwindup compensator is designed for the closed-loop systems:

$$\begin{aligned} x_{\mathrm{aw}}(k+1) &= A_{\mathrm{aw},i} x_{\mathrm{aw}}(k) + B_{\mathrm{aw},i}\, \psi\left(y_c(k)\right), \\ v(k) &= C_{\mathrm{aw},i} x_{\mathrm{aw}}(k) + D_{\mathrm{aw},i}\, \psi\left(y_c(k)\right), \end{aligned} \tag{7}$$

where $\psi(y_c(k)) = \mathrm{sat}(y_c(k)) - y_c(k)$. The resulting nonlinear closed-loop system (1), (5), (7) is depicted in Figure 1 and can be represented in the following compact form:

$$\begin{aligned} x(k+1) &= A_i x(k) + B_{qi}\, \psi\left(y_c(k)\right) + B_{wi}\, w(k), \\ y_c(k) &= K_i x(k) + K_{\phi,i}\, \psi\left(y_c(k)\right) + K_{wi}\, w(k), \\ z(k) &= C_{zi} x(k) + D_{zq,i}\, \psi\left(y_c(k)\right) + D_{zw,i}\, w(k), \end{aligned} \tag{8}$$

where

$$\begin{gathered} A_i = \begin{bmatrix} A_{pi} + B_{pu,i} D_{cy,i} C_{py,i} & B_{pu,i} C_{ci} & B_{pu,i} I_2 C_{aw,i} \\ B_{cy,i} C_{py,i} & A_{ci} & I_1 C_{aw,i} \\ 0 & 0 & A_{aw,i} \end{bmatrix}, \\ B_{qi} = \begin{bmatrix} B_{pu,i} I_2 D_{aw,i} + B_{pu,i} \\ I_1 D_{aw,i} \\ B_{aw,i} \end{bmatrix}, \\ B_{wi} = \begin{bmatrix} B_{pw,i} + B_{pu,i} D_{cy,i} D_{p,yw,i} + B_{pu,i} D_{cw,i} \\ B_{cy,i} D_{p,yw,i} + B_{cw,i} \\ 0 \end{bmatrix}, \\ C_{zi} = \left[C_{pz,i} + D_{p,zu,i} D_{ci} C_{py,i} \quad D_{p,zu,i} \quad I_2 C_{aw,i}\right], \\ D_{zq,i} = I_2 D_{aw,i} + D_{p,zu,i}, \\ D_{zw,i} = D_{p,zw,i} + D_{p,zu,i} D_{cw,i} + D_{p,zu,i} D_{ci} D_{p,yw,i}, \\ K_i = \left[D_{cy,i} C_{py,i} \quad C_{ci} \quad I_2 C_{aw,i}\right], \\ K_{\phi,i} = I_2 D_{aw,i}, \qquad K_{w,i} = D_{cw,i} + D_{p,yw,i}, \\ I_1 = \left[I \quad 0\right], \qquad I_2 = \left[0 \quad I\right]. \end{gathered} \tag{9}$$

For this system, we introduce the following definitions and assumption.

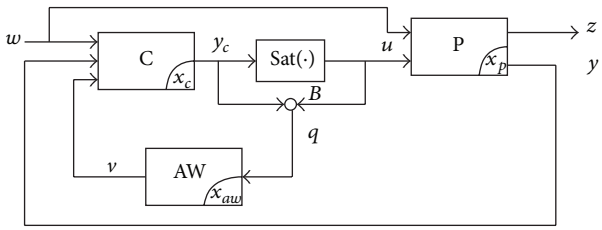

Figure 1: The closed-loop systems with input saturation.

Assumption 2 (see [17]). The external disturbance $w(k)$ is varying and satisfies the following constraint condition:

$$\sum_{k=0}^{T} w(k)^T w(k) \le d, \quad d \ge 0. \tag{10}$$

Definition 3 (see [17]). The resulting closed-loop system (8) is stochastic finite-time stable (SFTB) with respect to $(\delta_x, \epsilon, R_i, N, d)$ with $0 < \delta_x < \epsilon, R_i > 0$, and $N \in Z_{k\ge 0}$, if

$$E\left\{x^T(0) R_i x(0)\right\} \le \delta_x^2 \Longrightarrow E\left\{x^T(k) R_i x(k)\right\} < \epsilon^2, \quad \forall k \in \{1, 2, \ldots, N\}. \tag{11}$$

Definition 4 (see [17]). The resulting closed-loop system (8) is said to be stochastic H_∞ finite-time stable with respect to $(\delta_x, \epsilon, R_i, N, \gamma, d)$ with $0 < \delta_x < \epsilon$, $R_i > 0$, $\gamma > 0$, and $N \in Z_{k\ge 0}$, if the system (8) is SFTB with respect to $(\delta_x, \epsilon, R_i, N, \gamma, d)$, and under the zero-initial condition, the output $z(k)$ satisfies

$$E\left\{\sum_{j=0}^{N} z^T(j) z(j)\right\} \le \gamma^2 E\left\{\sum_{j=0}^{N} w^T(j) w(j)\right\}, \tag{12}$$

for any nonzero $w(k)$ which satisfies (10), where γ is a prescribed positive scalar.

3. Main Results

In this section, we investigate the stabilization analysis of the unconstrained systems and the antiwindup controller design of the resulting closed-loop system. Some sufficient conditions in terms of LMI are given. Before presenting the main results, we give some lemmas as follows.

Lemma 5 (see [4]). *For the closed-loop systems* (8) *with the matrix K_i, the appropriate matrix $L_i \in \mathbb{R}^{m\times n}$ is given, if $x(k)$ is in the set $D(u_o)$, where $D(u_o)$ is defined as follows:*

$$D(u_o) = \{x(k) \in \mathbb{R}^n;\ -u_{0(k)} \le (K_{i(k)} + L_{i(k)}) x(t) \le u_{0(k)}, \quad u_{0(k)} > 0,\ k = 1, \ldots, m\}, \tag{13}$$

then for any diagonal positive matrix $T \in \mathbb{R}^{m\times m}$, one has the following:

$$\psi(u(k))^T T\left(\psi(u(k)) - L_i x(k) + K_{\phi,i}\psi(y_c(k)) + K_{wi} w(k)\right) \le 0. \tag{14}$$

Lemma 6 (see [12]). *For the given symmetric matrix $S \in \mathbb{R}^{(n+m)\times(n+m)}$,*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, \tag{15}$$

where $S_{11} \in \mathbb{R}^{n\times n}$, $S_{12} \in \mathbb{R}^{n\times m}$, and $S_{22} \in \mathbb{R}^{m\times m}$, the following conditions are equivalent:

(1) $S < 0$;

(2) $S_{11} < 0,\ S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;

(3) $S_{22} < 0,\ S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

3.1. Design of Controller. In this section, we design the controller for the unconstrained systems with $v_1 = 0$ and $v_2 = 0$. Combining system (1) with controller (5), we have

$$\begin{aligned} x(k+1) &= A_i x(k) + B_{wi} w(k), \\ z(k) &= C_{zi} x(k) + D_{zw,i} w(k), \end{aligned} \tag{16}$$

where

$$\begin{aligned} A_i &= \begin{bmatrix} A_{pi} + B_{pu,i} D_{cy,i} C_{py,i} & B_{pu,i} C_{ci} \\ B_{cy,i} C_{py,i} & A_{ci} \end{bmatrix}, \\ B_{wi} &= \begin{bmatrix} B_{pw,i} + B_{pu,i} D_{cy,i} D_{p,yw,i} + B_{pu,i} D_{cw,i} \\ B_{cy,i} D_{p,yw,i} + B_{cw,i} \end{bmatrix}. \end{aligned} \tag{17}$$

Theorem 7. *For each $r(k) = i \in S$, the unconstrained system* (16) *is SFTB with respect to $(\delta_x, \epsilon, R_i, N, d)$ with $0 < \delta_x < \epsilon$, if there exist scalars $\mu \ge 0$, $\sigma_1 \ge 0$, $\sigma_2 \ge 0$, and the given $\lambda > 0$, two sets of mode-dependent symmetric positive-defined matrices $\{X_i, i \in S\}$ and $\{Q_i, i \in S\}$, such that the following conditions hold:*

$$\begin{bmatrix} -\mu\lambda I & 0 & L_{1i}^T \\ * & -Q_i & L_{2i}^T \\ * & * & -W \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} \sigma_2 d^2 - \mu^{-N}\epsilon^2 & * \\ \delta_x & -\sigma_1 \end{bmatrix} < 0, \tag{19}$$

$$\lambda X_i < I, \tag{20}$$

$$\sigma_1 R_i^{-1} < X_i < R_i^{-1}, \tag{21}$$

$$0 < Q_i < \sigma_2 I, \tag{22}$$

where

$$W = \operatorname{diag}\{X_1,\ X_2,\ \ldots,\ X_n\},$$
$$\overline{L}_{1i}^T = \left[\sqrt{\pi_{i1}}A_i^T \quad \sqrt{\pi_{i2}}A_i^T \quad \cdots \quad \sqrt{\pi_{in}}A_i^T\right], \tag{23}$$
$$L_{2i}^T = \left[\sqrt{\pi_{i1}}B_{wi}^T \quad \sqrt{\pi_{i2}}B_{wi}^T \quad \cdots \quad \sqrt{\pi_{in}}B_{wi}^T\right].$$

Proof. Define the following Lyapunov function for each $\delta(t) = i \in S$:

$$V(k) = x(k)^T P_i x(k). \tag{24}$$

It is readily obtained that

$$\begin{aligned} E\{V(k+1)\} &= E\left\{\sum_{j=1}^{n}\pi_{ij}x(k+1)^T P_j x(k+1)\right\} \\ &= \xi(k)^T\left[L_{1i} \quad L_{2i}\right]^T \overline{W}\left[L_{1i} \quad L_{2i}\right]\xi(k), \end{aligned} \tag{25}$$

where

$$\begin{aligned} \xi(k) &= \left[x(k)^T \quad w(k)^T\right], \\ \overline{W} &= \operatorname{diag}\{P_1,\ P_2,\ \ldots,\ P_h\}. \end{aligned} \tag{26}$$

By using of Schur complement lemma to (18), and note that $P_i^{-1} = X_i$ and $\lambda X_i < I$, we derive $\lambda I < P_i$; then, we have

$$\begin{aligned} &\xi(k)^T\begin{bmatrix} -\mu\lambda I & 0 \\ * & -Q_i \end{bmatrix}\xi(k) + \xi(k)^T\left[L_{1i} \quad L_{2i}\right]^T\overline{W}\left[L_{1i} \quad L_{2i}\right]\xi(k) \\ &\quad < 0. \end{aligned} \tag{27}$$

It follows that

$$E\{V(k+1)\} < \mu x(k)^T P_i x(k) + w(k)^T Q_i w(k). \tag{28}$$

It is shown that

$$E\{V(k+1)\} < \mu V(k) + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\, w(k)^T w(k). \tag{29}$$

Then we have

$$\begin{aligned} &E\{V(k+1)\} \\ &\quad < \mu E\{V(k)\} + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\, E\left\{w(k)^T w(k)\right\}. \end{aligned} \tag{30}$$

Since $\mu \geq 1$, it is easily found that

$$\begin{aligned} E\{V(k+1)\} &< \mu E\{V(0)\} + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} \\ &\quad \times E\left\{\sum_{j=0}^{k-1}\mu^{k-j-1}w(j)^T w(j)\right\} \\ &\leq \mu^k E\{V(0)\} + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\,\mu^k d^2. \end{aligned} \tag{31}$$

Letting

$$\overline{P}_i = R_i^{-1/2}P_i R_i^{-1/2}, \tag{32}$$

and noting that

$$E\left\{x^T(0)R_i x(0)\right\} \leq \delta_x^2, \tag{33}$$

it can be verified that

$$\begin{aligned} E\{V(0)\} &= E\left\{x^T(0)P_i x(0)\right\} \\ &= E\left\{x^T(0)R_i^{1/2}\overline{P}_i R_{i^{1/2}}x(0)\right\} \\ &\leq \sup_{i\in S}\left\{\lambda_{\max}\left(\overline{P}_i\right)\right\} E\left\{x^T(0)R_i x(0)\right\} \\ &\leq \sup_{i\in S}\left\{\lambda_{\max}\left(\overline{P}_i\right)\right\}\delta_x^2. \end{aligned} \tag{34}$$

Similarly, for all $i \in S$, we can obtain

$$\begin{aligned} E\{V(k)\} &= E\left\{x^T(k)P_i x(k)\right\} \\ &= E\left\{x^T(k)R_i^{1/2}\overline{P}_i R_{i^{1/2}}x(k)\right\} \\ &\geq \inf_{i\in S}\{\lambda_{\min}\}\left(\overline{P}_i\right) E\left\{x^T(k)R_i x(k)\right\}. \end{aligned} \tag{35}$$

Then, it is not difficult to find that

$$\begin{aligned} &E\left\{x^T(k)R_i x(k)\right\} \\ &\quad < \frac{\sup_{i\in S}\left\{\lambda_{\max}\left(\overline{P}_i\right)\right\}\mu^k\delta_x^2 + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\,\mu^k d^2}{\inf_{i\in S}\{\lambda_{\min}\}\left(\overline{P}_i\right)}, \end{aligned} \tag{36}$$

which implies that

$$\frac{\sup_{i\in S}\left\{\lambda_{\max}\left(\overline{P}_i\right)\right\}\mu^k\delta_x^2 + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\,\mu^k d^2}{\inf_{i\in S}\{\lambda_{\min}\}\left(\overline{P}_i\right)} < \epsilon^2. \tag{37}$$

Then, one can obtain that

$$\begin{aligned} &\sup_{i\in S}\left\{\lambda_{\max}\left(\overline{P}_i\right)\right\}\delta_x^2 + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\, d^2 \\ &\quad < \inf_{i\in S}\{\lambda_{\min}\}\left(\overline{P}_i\right)\mu^{-N}\epsilon^2. \end{aligned} \tag{38}$$

Setting

$$\begin{aligned} X_i &= P_i^{-1}, \\ \sigma_1 R_i^{-1} &< X_i < R_i^{-1}, \\ 0 &< Q_i < \sigma_2 I, \end{aligned} \tag{39}$$

it is easy to see that

$$\sigma_1^{-1}\delta_x^2 + \sigma_2 d^2 < \mu^{-N}\epsilon^2. \tag{40}$$

It is obvious that (40) is equivalent to (19).
This completes the proof. □

3.2. Design of Dynamic Antiwindup Compensator

Theorem 8. *For each $r(k) = i \in S$, with antiwindup compensator* (7)*, such that the resulting closed-loop system* (10) *is SFTB with respect to $(\delta_x, \epsilon, R_i, N, d)$ with $0 < \delta_x < \epsilon$, if there exist scalars $\mu \geq 0$, $\sigma_1 \geq 0$, and $\sigma_2 \geq 0$, three sets of mode-dependent symmetric positive-defined matrices $\{X_i, i \in S\}$, $\{Q_i, i \in S\}$ and* diag *positive-defined matrices $\{S_i, i \in S\}$, and two sets of mode-dependent matrices $\{Y_i, i \in S\}$ and $\{\overline{L}_i = L_i X_i, \overline{A}_{aw,i} = A_{aw,i} X_i, \overline{C}_{aw,i} = C_{aw,i} X_i, \overline{B}_{aw,i} = B_{aw,i} S_i, \overline{D}_{aw,i} = D_{aw,i} S_i \ i \in S\}$, such that the following conditions hold:*

$$\begin{bmatrix} -\mu X_i & 0 & \overline{L}_i^T & \overline{L}_{1i}^T \\ * & -Q_i & K_{wi}^T & L_{2i}^T \\ * & * & -2S_i - He\left(\overline{K}_{\varphi,i}\right) & \overline{L}_{3i}^T \\ * & * & * & -W \end{bmatrix} < 0, \tag{41}$$

$$\begin{bmatrix} \sigma_2 d^2 - \mu^{-N}\epsilon^2 & * \\ \delta_x & -\sigma_1 \end{bmatrix} < 0, \tag{42}$$

$$\begin{bmatrix} X_i & * \\ \overline{K}_i + \overline{L}_i & u_{0(k)}^2 \end{bmatrix} > 0, \quad k = 1, \ldots, m, \tag{43}$$

$$\sigma_1 R_i^{-1} < X_i < R_i^{-1}, \tag{44}$$

$$0 < Q_i < \sigma_2 I, \tag{45}$$

where

$$\begin{aligned} W &= \operatorname{diag}\{X_1, X_2, \ldots, X_n\}, \\ \overline{L}_{1i}^T &= \left[\sqrt{\pi_{i1}}\overline{A}_i^T \ \ \sqrt{\pi_{i2}}\overline{A}_i^T \ \cdots \ \sqrt{\pi_{in}}\overline{A}_i^T\right], \\ L_{2i}^T &= \left[\sqrt{\pi_{i1}}B_{wi}^T \ \ \sqrt{\pi_{i2}}B_{wi}^T \ \cdots \ \sqrt{\pi_{in}}B_{wi}^T\right], \\ L_{3i}^T &= \left[\sqrt{\pi_{i1}}\overline{B}_{qi}^T \ \ \sqrt{\pi_{i2}}\overline{B}_{qi}^T \ \cdots \ \sqrt{\pi_{in}}\overline{B}_{qi}^T\right], \end{aligned} \tag{46}$$

with

$$\overline{A}_i = \begin{bmatrix} A_{pi}X_i + B_{pu,i}D_{cy,i}C_{py,i}X_i & B_{pu,i}C_{ci}X_i & B_{pu,i}I_2\overline{C}_{aw,i} \\ B_{cy,i}C_{py,i}X_i & A_{ci}X_i & I_1\overline{C}_{aw,i} \\ 0 & 0 & \overline{A}_{aw,i} \end{bmatrix},$$

$$\overline{B}_{qi} = \begin{bmatrix} B_{pu,i}I_2\overline{D}_{aw,i} + B_{pu,i}X_i \\ I_1\overline{D}_{aw,i} \\ \overline{B}_{aw,i} \end{bmatrix},$$

$$\overline{K}_i = \left[D_{cy,i}C_{py,i}X_i \ \ C_{ci}X_i \ \ I_2\overline{C}_{aw,i}\right], \qquad \overline{K}_{\phi,i} = I_2\overline{D}_{aw,i}. \tag{47}$$

Proof. Define the following Lyapunov function for each $\delta(t) = i \in S$:

$$V(k) = x(k)^T P_i x(k). \tag{48}$$

It is readily obtained that

$$\begin{aligned} E\{V(k+1)\} &= E\left\{\sum_{j=1}^{n}\pi_{ij}x(k+1)^T P_j x(k+1)\right\} \\ &= \xi(k)^T\left[L_{1i} \ \ L_{2i} \ \ L_{3i}\right]^T \overline{W}\left[L_{1i} \ \ L_{2i} \ \ L_{3i}\right]\xi(k), \end{aligned} \tag{49}$$

where

$$\begin{aligned} \xi(k) &= \left[x(k)^T \ \ w(k)^T \ \ \psi(k)^T\right], \\ \overline{W} &= \operatorname{diag}\{P_1, P_2, \ldots, P_h\}, \\ L_{1i}^T &= \left[\sqrt{\pi_{i1}}A_i^T \ \ \sqrt{\pi_{i2}}A_i^T \ \cdots \ \sqrt{\pi_{in}}A_i^T\right], \\ L_{2i}^T &= \left[\sqrt{\pi_{i1}}B_{wi}^T \ \ \sqrt{\pi_{i2}}B_{wi}^T \ \cdots \ \sqrt{\pi_{in}}B_{wi}^T\right], \\ L_{3i}^T &= \left[\sqrt{\pi_{i1}}B_{qi}^T \ \ \sqrt{\pi_{i2}}B_{qi}^T \ \cdots \ \sqrt{\pi_{in}}B_{qi}^T\right]. \end{aligned} \tag{50}$$

Then, by pre- and postmultiplying (41) by $\operatorname{diag}\{P_i, I, T_i, I\}$ with $P_i = X_i^{-1}$, $T_i = S_i^{-1}$, we have

$$\begin{bmatrix} -\mu P_i & 0 & L_i^T T_i & L_{1i}^T \\ * & -Q_i & K_{wi}^T & L_{2i}^T \\ * & * & -2T_i - He\left(T_i K_{\phi,i}\right) & L_{3i}^T \\ * & * & * & -W \end{bmatrix} < 0. \tag{51}$$

By using of Schur complement lemma, we derive

$$\begin{aligned} &\xi(k)^T \begin{bmatrix} -\mu P_i & 0 & L_i^T T_i \\ * & -Q_i & K_{wi}^T \\ * & * & -2T_i - He\left(T_i K_{\phi,i}\right) \end{bmatrix}\xi(k) \\ &\quad + \xi(k)^T\left[L_{1i} \ \ L_{2i} \ \ L_{3i}\right]^T \overline{W}\left[L_{1i} \ \ L_{2i} \ \ L_{3i}\right]\xi(k) < 0. \end{aligned} \tag{52}$$

It follows that

$$\begin{aligned} E\{V(k+1)\} &< \mu x(k)^T P_i x(k) + w(k)^T Q_i w(k) \\ &\quad + \psi(k)^T\left(2T_i + He\left\{T_i K_{\phi,i}\right\}\right)\psi(k) \\ &\quad - 2\psi(k)^T T_i L_i x(k) + 2\psi(k)^T T_i K_{wi} w(k). \end{aligned} \tag{53}$$

Since $\psi(k)^T(2T_i + He\{T_i K_{\phi,i}\})\psi(k) - 2\psi(k)^T T_i L_i x(k) + 2\psi(k)^T T_i K_{wi} w(k) \leq 0$, we get

$$E\{V(k+1)\} < \mu x(k)^T P_i x(k) + w(k)^T Q_i w(k). \tag{54}$$

It is shown that

$$E\{V(k+1)\} < \mu V(k) + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} w(k)^T w(k). \quad (55)$$

Then, we have

$$\begin{aligned} E\{V(k+1)\} &< \mu E\{V(k)\} \\ &\quad + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} E\left\{w(k)^T w(k)\right\}. \end{aligned} \quad (56)$$

Since $\mu \geq 1$, it is easily found that

$$\begin{aligned} E\{V(k+1)\} &< \mu E\{V(0)\} + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\} \\ &\quad \times E\left\{\sum_{j=0}^{k-1}\mu^{k-j-1} w(j)^T w(j)\right\} \\ &\leq \mu^k E\{V(0)\} + \sup_{(i\in S)}\{\lambda_{\max}(Q_i)\}\mu^k d^2. \end{aligned} \quad (57)$$

The following proof is similar to the process of Theorem 7. Based on Lemma 5, it is easy to obtain that

$$\begin{bmatrix} P_i & * \\ K_i + L_i & u_{0(k)}^2 \end{bmatrix} > 0, \quad k = 1, \ldots, m, \quad (58)$$

then pre- and post-multiply (58) by diag$\{X_i, I\}$ which implies (43). This completes the proof. □

Theorem 9. *For each $r(k) = i \in S$, with antiwindup compensator* (7)*, such that the resulting closed-loop system* (10) *is said to be Stochastic H_∞ finite-time stable via state feedback with respect to $(\delta_x, \epsilon, R_i, N, \gamma, d)$, if there exist three scalars $\mu \geq 0$, $\sigma_1 \geq 0$, and $\gamma \geq 0$, two sets of mode-dependent symmetric positive-defined matrices $\{X_i, i \in S\}$ and* diag *matrices $\{S_i, i \in S\}$, and two sets of mode-dependent matrices $\{Y_i, i \in S\}$ and $\{\overline{L}_i = L_i X_i, i \in S\}$, such that the following conditions hold:*

$$\begin{bmatrix} -\mu X_i & 0 & \overline{L}_i^T & \overline{L}_{1i}^T & \overline{C}_{zi}^T \\ * & -\gamma^2\mu^{-N} I & K_{wi}^T & L_{2i}^T & D_{zw,i}^T \\ * & * & -2S_i - He\left(\overline{K}_{\phi,i}\right) & \overline{L}_{3i}^T & \overline{D}_{zq,i}^T \\ * & * & * & -W & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \quad (59)$$

$$\begin{bmatrix} \mu^{-N}\left(d^2\gamma^2 - \epsilon^2\right) & * \\ \delta_x & -\sigma_1 \end{bmatrix} < 0, \quad (60)$$

$$\begin{bmatrix} X_i & * \\ \overline{K}_i + \overline{L}_i & u_{0(k)}^2 \end{bmatrix} > 0, \quad k = 1, \ldots, m, \quad (61)$$

$$\sigma_1 R_i^{-1} < X_i < R_i^{-1}, \quad (62)$$

with

$$\begin{aligned} \overline{C}_{zi} &= \left[\left(C_{pz,i} + D_{p,zu,i} D_{ci} C_{py,i}\right) X_i \;\; D_{p,zu,i} X_i \;\; I_2 \overline{C}_{aw,i}\right], \\ \overline{D}_{zq,i}^T &= I_2 \overline{D}_{aw,i} + D_{p,zu,i}. \end{aligned} \quad (63)$$

Proof. Choose the similar Lyapunov function as Theorem 7 and denote

$$\begin{aligned} &\Pi(x(k), w(k), r(k) = i) \\ &\quad = E\{V(k+1)\} - \mu V(k) + z(k)^T z(k) \\ &\quad - \gamma^2\mu^{-N} w(k)^T w(k). \end{aligned} \quad (64)$$

Thus, in light of Lemma 5, we have

$$\begin{aligned} &\Pi(x(k), w(k), r_k = i) \\ &\quad \leq \xi(k)^T [L_{1i} \;\; L_{2i} \;\; L_{3i}]^T \overline{W} [L_{1i} \;\; L_{2i} \;\; L_{3i}] \xi(k) \\ &\quad + \xi(k)^T [C_{zi} \;\; D_{zw,i} \;\; D_{zq,i}]^T [C_{zi} \;\; D_{zw,i} \;\; D_{zq,i}] \xi(k) \\ &\quad + \xi^T(k) \begin{bmatrix} -\mu X_i & 0 & L_i^T T_i \\ * & -\gamma^2\mu^{-N} I & K_{wi}^T \\ * & * & -2T_i - He\left\{T_i K_{\phi,i}\right\} \end{bmatrix} \xi(k). \end{aligned} \quad (65)$$

Then pre- and postmultiply (59) by diag$\{P_i, I, T_i, I\}$, and considering Schur complement lemma and (65), we derive that

$$\Pi(x(k), w(k), r(k) = i) < 0 \quad (66)$$

holds for all $r_k = i \in S$. According to (66), one can obtain that

$$\begin{aligned} E\{V(k+1)\} &< \mu E\{V(k)\} - E\left\{z(k)^T z(k)\right\} \\ &\quad + \gamma^2\mu^{-N} E\left\{w(k)^T w(k)\right\}. \end{aligned} \quad (67)$$

Then, we have

$$\begin{aligned} E\{V(k)\} &< \mu^k E\{V(0)\} - \sum_{j=0}^{k-1}\mu^{k-j-1} E\left\{z(j)^T z(j)\right\} \\ &\quad + \gamma^2\mu^{-N} E\left\{\sum_{j=0}^{k-1}\mu^{k-j-1} w(j)^T w(j)\right\}. \end{aligned} \quad (68)$$

Under the zero-value initial condition and noting that $V(k) \geq 0$, for all $K \in Z_{k\geq 0}$, it is shown that

$$\sum_{j=0}^{k-1}\mu^{k-j-1} E\left\{z(j)^T z(j)\right\} < \gamma^2\mu^{-N} E\left\{\sum_{j=0}^{k-1}\mu^{k-j-1} w(j)^T w(j)\right\}. \quad (69)$$

Since $\mu \geq 1$ and from (69), we have

$$\begin{aligned} E\left\{\sum_{j=0}^{N} z(j)^T z(j)\right\} &= \sum_{j=0}^{N} E\left\{z(j)^T z(j)\right\} \\ &\leq \sum_{j=0}^{N} E\left\{\mu^{N-j} z(j)^T z(j)\right\} \\ &\leq \gamma^2 \mu^{-N} E\left\{\sum_{j=0}^{N} \mu^{N-j} w(j)^T w(j)\right\} \\ &\leq \gamma^2 E\left\{\sum_{j=0}^{N} w(j)^T w(j)\right\}. \end{aligned} \tag{70}$$

The following proof is similar to the process of Zhang and Liu [17].

Since $\varepsilon(P_i,\ 1) \subset D(u_0)$, it follows that

$$\begin{bmatrix} P_i & * \\ K_i + L_i & u_{0(k)}^2 \end{bmatrix} > 0, \qquad k = 1, \ldots, m, \tag{71}$$

and then pre- and post-multiply (71) by $\mathrm{diag}(X_i,\ I)$ and its transpose, respectively; we derive condition (61). This completes the proof. □

4. Illustrative Examples

In this section, a numerical example is provided to demonstrate the effectiveness of the proposed method. Consider the following systems with four operation modes.

Mode 1 is

$$\begin{gathered} A_{p1} = \begin{bmatrix} 0.75 & -0.75 \\ 1.5 & -1.5 \end{bmatrix}, \qquad B_{pu,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_{pw,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_{py,1} = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}, \qquad C_{pz,1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ D_{pyw,1} = 1, \qquad D_{pzu,1} = 1, \qquad D_{pzw,1} = 0.8. \end{gathered} \tag{72}$$

Mode 2 is

$$\begin{gathered} A_{p2} = \begin{bmatrix} 0.15 & 4.5 \\ 2.10 & -0.4 \end{bmatrix}, \qquad B_{pu,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_{pw,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_{py,2} = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix}, \qquad C_{pz,2} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ D_{pyw,2} = 1, \qquad D_{pzu,2} = 0.9, \qquad D_{pzw,2} = 0.8. \end{gathered} \tag{73}$$

Mode 3 is

$$\begin{gathered} A_{p3} = \begin{bmatrix} 0.24 & 2.50 \\ 1.2 & -2.1 \end{bmatrix}, \qquad B_{pu,3} = \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}, \qquad B_{pw,3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_{py,3} = \begin{bmatrix} -0.1 & 0 \end{bmatrix}, \qquad C_{pz,3} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ D_{pyw,3} = 0.8, \qquad D_{pzu,3} = 1, \qquad D_{pzw,3} = 1.2. \end{gathered} \tag{74}$$

Mode 4 is

$$\begin{gathered} A_{p4} = \begin{bmatrix} 1 & -0.25 \\ 1.5 & -1.5 \end{bmatrix}, \qquad B_{pu,4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_{pw,4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_{py,4} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_{pz,4} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ D_{pyw,4} = 1, \qquad D_{pzu,4} = 0.5, \qquad D_{pzw,4} = 1. \end{gathered} \tag{75}$$

With the given designed controllers,

$$\begin{gathered} A_{c1} = -5.5, \qquad B_{cy,1} = -1, \qquad B_{cw,1} = 1, \\ C_{c1} = -1, \qquad D_{cy,1} = -0.1, \qquad D_{cw,1} = 0.5, \\ A_{c2} = -5, \qquad B_{cy,2} = -0.9, \qquad B_{cw,2} = 1, \\ C_{c2} = -1, \qquad D_{cy,2} = 5.9, \qquad D_{cw,2} = 1, \\ A_{c3} = -4.5, \qquad B_{cy,3} = -1, \qquad B_{cw,3} = 1, \\ C_{c3} = -1.5, \qquad D_{cy,3} = 5.1, \qquad D_{cw,3} = 1, \\ A_{c4} = -7, \qquad B_{cy,4} = -1, \qquad B_{cw,4} = 1, \\ C_{c4} = -1.5, \qquad D_{cy,4} = -2, \qquad D_{cw,4} = 1. \end{gathered} \tag{76}$$

The transition rate matrix is given by the following:

$$\begin{bmatrix} 0.3 & 0.3 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.1 & 0.4 \end{bmatrix}. \tag{77}$$

In this case, we choose the initial values for $R_i = I_2$, $i = 1, 2, 3, 4$, $\delta_x = 1$, $N = 5$, $\alpha = 10^{-10}$, $\mu = 2.5$, $d = 1$, and $w(k) = 0.5(1 + \cos x(k))$; Theorem 7 yields to $\epsilon = 36.2671$, $\sigma_1 = 0.4906$, $\sigma_2 = 13.7421$, and the bounds of the input saturation $u_0 = 0.08$.

Based on Theorem 9, we derive

$$\begin{gathered} A_{aw,1} = -2.67, \qquad A_{aw,2} = -1.86, \\ A_{aw,3} = -1.88, \qquad A_{aw,4} = -2.59, \\ B_{aw,1} = -0.02, \qquad B_{aw,2} = -0.01, \\ B_{aw,3} = -0.01, \qquad B_{aw,4} = -0.02, \\ C_{aw,1} = \begin{bmatrix} 17.27 \\ 0.68 \end{bmatrix}, \qquad C_{aw,2} = \begin{bmatrix} 68.44 \\ -48.31 \end{bmatrix}, \\ C_{aw,3} = \begin{bmatrix} 68.74 \\ -49.29 \end{bmatrix}, \qquad C_{aw,4} = \begin{bmatrix} 17.27 \\ 0.66 \end{bmatrix}, \end{gathered}$$

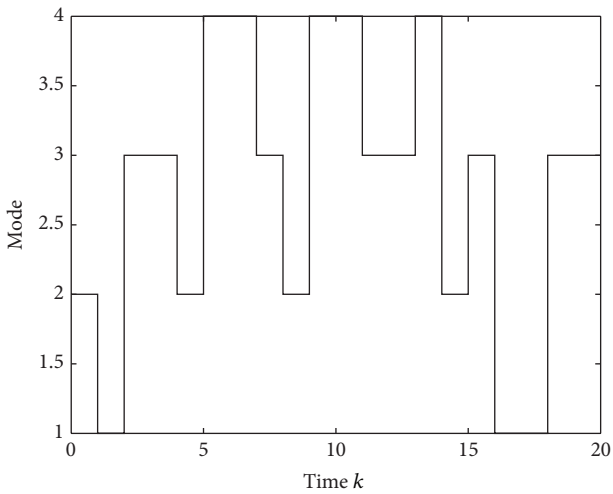

FIGURE 2: r_k of jump rates.

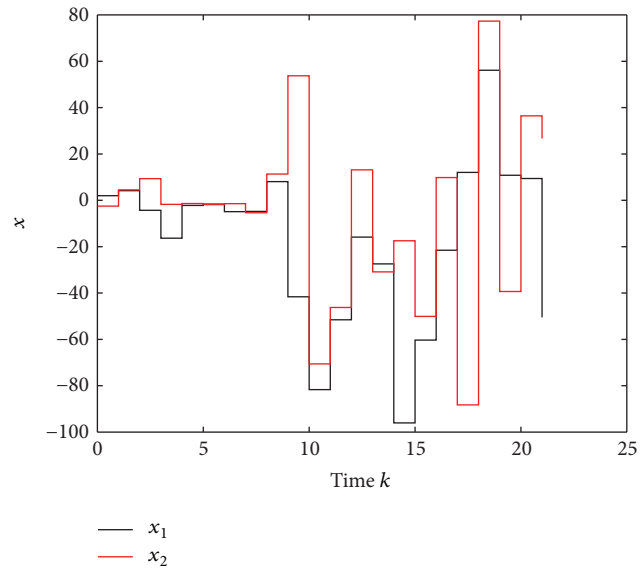

FIGURE 3: $x(k)$ of the system (1)–(5).

$$D_{aw,1} = \begin{bmatrix} 0.21 \\ 0.1 \end{bmatrix}, \qquad D_{aw,2} = \begin{bmatrix} 0.18 \\ -0.1 \end{bmatrix},$$
$$D_{aw,3} = \begin{bmatrix} 0.19 \\ 0 \end{bmatrix}, \qquad D_{aw,4} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}. \tag{78}$$

Remark 10. Figures 2, 3, and 4 are given on the last page. Figure 1 is r_k of the jump rates, Figure 2 and Figure 3 are state response of open and closed-loop system. Based on the figures provided, the controller and the compensator we designed guarantee that the resulting closed-loop systems are mean-square locally asymptotically finite-time stabilizable.

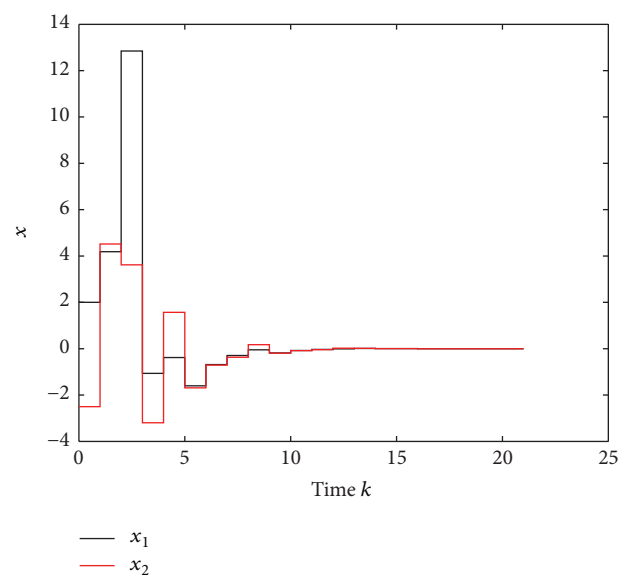

FIGURE 4: $x(k)$ of the closed-loop system (1)–(7).

5. Conclusions and Future Work

In this paper, the finite-time H_∞ stabilization problem for a class of discrete-time Markov jump systems with input saturation has been investigated. Based on stochastic finite-time stability analysis, a controller designed for the unconstrained system with a dynamic antiwindup compensator subject to actuator saturation is given to guarantee the stochastic finite-time boundedness and stochastic finite-time stabilization of the considered closed-loop system for all admissible disturbances. Finally, the effectiveness of the proposed approach has been illustrated by simulation results. The finite-time stabilization problem of Markov jump systems with constrained input and time-delay will be considered in the future work.

Conflict of Interests

The authors declare no conflict of interests.

Acknowledgments

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Research Article

Delay-Dependent Robust Exponential Stability and H_∞ Analysis for a Class of Uncertain Markovian Jumping System with Multiple Delays

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This paper deals with the problem of robust exponential stability and H_∞ performance analysis for a class of uncertain Markovian jumping system with multiple delays. Based on the reciprocally convex approach, some novel delay-dependent stability criteria for the addressed system are derived. At last, numerical examples is given presented to show the effectiveness of the proposed results.

1. Introduction

It is well known that time delay is usually the main reason for instability and poor performance of many practical control systems [1–5]. The stability results for delayed systems can be generally classified into two categories: delay-independent stability criteria and delay-dependent criteria. And the delay-dependent results are often less conservative than the delay-independent ones, especially when the time delays are small. Therefore, much more attention has been focused on study of the delay-dependent stability conditions in recent years. For example, the system transformation method in [6], the descriptor system method in [7], parameter-dependent Lyapunov-Krasovskii functional method in [8], Jensen inequality method in [9], Free-weighting matrix method in [10, 11], integral inequality method in [12], augmented Lyapunov functional method in [13], convex domain method in [14], interval partition method in [15, 16], reciprocally convex method in [17], and so forth. And those approaches have been widely used in the stability analysis for lots of delayed systems in recent years [18–20].

On the other hand, since Markovian jumping systems can model many types of dynamic systems subject to abrupt changes in their structures, such as failure prone manufacturing systems, power systems, and economics systems [21–27], a great deal of results related to stability analysis and synthesis for this class of systems with time delays has been reported in recent years. For example, for the delay-independent results, sufficient conditions for mean squares to stochastic stability were obtained in [28], while exponential stability conditions were proposed in [29]. The robust H_∞ filtering problem was dealt with in [30]. For the delay-dependent ones, the stability and H_∞ control results were presented by resorting to some bounding techniques for some cross terms and using model transformation to the original delay system in [31]. The H_∞ control and Filtering problem were taken into account in [32] using the Free-weighting matrix method. The stability and H_∞ analysis was proposed in [33] with the idea of delay partition. Filtering problem with a new index was considered in [34] using the reciprocally convex method. It is worth mentioned that inspite of the deep study for the delayed stochastic in recent years as mentioned above, there are few papers that consider the problem of stability analysis for uncertain stochastic systems with multiple delays, which motivates our study.

In this paper, the robust exponential stability and H_∞ performance analysis for a class of uncertain Markovian system with multiple time-varying delays is investigated. Some new delay-dependent stability conditions are derived.

And numerical simulation is given to demonstrate the effectiveness of the result.

Notation. Throughout this paper, for symmetric matrices X and Y, the notation $X \geq Y$ (resp., $X > Y$) means that the matrix $X - Y$ is positive semidefinite (resp., positive definite); I is the identity matrix with appropriate dimension; M^T represents the transpose of the matrix M; $\mathscr{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathscr{P}$; $L_2[0,\infty]$ is the space of square-integrable vector functions over $[0,\infty]$; $|\cdot|$ refers to the Euclidean vector norm; $\|\cdot\|_2$ stands for the usual $L_2[0,\infty]$ norm; $(\Omega, \mathscr{F}, \mathscr{P})$ is a probability space with Ω the sample space and $\mathscr{F}$ is the σ-algebra of subsets of the sample space. Matrices, if not explicitly mentioned, are assumed to have compatible dimensions.

2. System Description and Preliminaries

Consider the following uncertain Markovian jumping system with multiple time-varying delays:

$$\begin{aligned}\dot{x}(t) = {} & A_0(r(t),t)\,x(t) \\ & + \sum_{i=1}^{m} A_i(r(t),t)\,x(t-h_i(t)) \\ & + D_1(r(t),t)\,w(t),\end{aligned} \tag{1}$$

$$z(t) = \sum_{i=0}^{m} C_i(r(t))\,x(t-h_i(t)) + D_2(r(t))\,w(t), \tag{2}$$

$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{3}$$

where $x(t) \in R^n$ is the state; $w(t) \in R^p$ is the noise disturbance which is assumed to be an arbitrary signal in $L^2([0,\infty])$; $z(t) \in R^q$ is the signal to be estimated; $r(t)$ is a homogenous stationary Markov chain defined on a complete probability space $\{\Omega, F, P\}$ and taking values in a finite set $S = \{1, 2, \ldots, N\}$ with generator $\Pi = (\lambda_{m,n})$ $(m, n \in S)$ given by

$$\begin{aligned}& P\{r(t+\Delta) = j \mid r(t) = k\} \\ & \quad = \begin{cases} \lambda_{k,j}\Delta + o(\Delta) & \text{if } k \neq j, \\ 1 + \lambda_{k,k}\Delta + o(\Delta) & \text{if } k = j, \end{cases}\end{aligned} \tag{4}$$

where $\Delta > 0$ and $\lim_{\Delta \to 0} o(\Delta)/\Delta = 0$, $\lambda_{k,j} \geq 0$ is the transition rate from k to j if $k \neq j$ and $\lambda_{k,k} = -\sum_{k \neq j} \lambda_{k,j}$. The scalar $h_i(t)$ is the time-varying delay with $0 \leq h_{1i} \leq h_i(t) \leq h_{2i}$, $\dot{h}_i(t) \leq \mu$, $i = 1, 2, \ldots, m$, for any $t > 0$, where h_{1i}, h_{2i}, and μ are positive scalar constants; $\phi(t)$ is the initial function defined in $t \in [-h, 0]$ with $h = \max\{h_{21}, h_{22}, \ldots, h_{2m}\}$; $A_i(r(t))$, $i = 0, 1, \ldots, m$, and $D_1(r(t))$ are matrix functions with time-varying uncertainties described as $A_i(r(t),t) = A_i(r(t)) + \Delta A_i(r(t),t)$, $D_1(r(t),t) = D_1(r(t)) + \Delta D_1(r(t),t)$, where $A_i(r(t))$, $D_1(r(t))$ are known constant matrices, while uncertainties $\Delta A_i(r(t),t)$, $\Delta D_1(r(t),t)$ are assumed to be norm bounded as

$$\begin{aligned}& [\Delta A_i(r(t),t) \;\; \Delta D_1(r(t),t)] \\ & \quad = E(r(t))\,F(r(t),t)\,[H_i(r(t)) \;\; H_d(r(t))], \\ & \qquad i = 0, 1, \ldots, m,\end{aligned} \tag{5}$$

where $E(r(t))$, $H_i(r(t))$, $H_d(r(t))$, and $C_i(r(t))$, $D_2(r(t))$ in (2) are known constant matrices with appropriate dimensions. The unknown matrix functions $F(r(t))$ are having Lebesgue-measurable elements and satisfying

$$F(r(t)) \leq I, \quad \forall t > 0. \tag{6}$$

Remark 1. When $m = 1$, the system with multiple time-varying delays (1)–(3) is actually deduced to the uncertain Markovian jumping system with interval delay, which have been deeply studied in recent years. That is, the obtained results of multiple delayed systems can be directly deduced to the interval delayed systems.

Throughout this paper, we will use the following Definitions and Lemmas.

Definition 2. The uncertain Markovian jumping system with multiple time-varying delays (1)–(3) is said to be robustly exponentially stable in mean square for all admissible uncertainties, if there exist scalars $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for all $t \geq 0$,

$$\|x(t, x_0, t_0)\|^2 \leq \alpha_1 e^{-\alpha_2 t} \sup_{-2h \leq s \leq 0} \left\{\|\phi(s)\|^2\right\}, \tag{7}$$

where $x(t, x_0, t_0)$ is the trivial solution of systems (1)–(3) with $w(t) = 0$.

Definition 3. Given a scalar $\gamma > 0$, uncertain Markovian jumping system with multiple time-varying delays (1)–(3) is said to be robustly exponentially stable with a prescribed H_∞ performance level γ if it is robustly exponentially stable, and under the zero initial condition, satisfies

$$\|z\|_{E_2} < \gamma \|w\|_2, \tag{8}$$

for all admissible uncertainties and nonzero $w(t) \in L_2[0,\infty)$, where

$$\|z\|_{E_2} = \left(\left\{\int_0^\infty \left|z(t)^2\right| dt\right\}\right)^{1/2}. \tag{9}$$

Lemma 4 (see [9]). *For any constant matrix $M \in R^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \to R^m$ such that the integrations in the following are well defined; then*

$$\gamma \int_0^\gamma \omega(\beta)^T M \omega(\beta)\, d\beta \geq \left(\int_0^\gamma \omega(\beta)\, d\beta\right)^T M \int_0^\gamma \omega(\beta)\, d\beta. \tag{10}$$

Lemma 5 (see [35]). *Let A, D, E be real constant matrices with appropriate dimensions; matrix $F(t)$ satisfies $F^T(t)F(t) \leq I$. For any $\varepsilon > 0$, such that $P^{-1} - \varepsilon DD^T > 0$,*

$$DF(t)E + E^T F^T(t) D^T \leq \varepsilon^{-1} DD^T + \varepsilon E^T E. \tag{11}$$

Lemma 6 (see [36]). *Consider system* (1) *with* $0 \le h_{1i} \le h_i(t) \le h_{2i}$, $i = 1, 2, \ldots, m$, *for any matrices* $Z_i \in R^{n\times n}$ *and* $U_i \in R^{n\times n}$ *satisfying* $\begin{bmatrix} Z_i & U_i \\ * & Z_i \end{bmatrix} \ge 0$; *the following inequality holds*

$$-\widehat{d}_i \int_{t-h_{2i}}^{t-h_{1i}} \dot{x}(s)^T Z_i \dot{x}(s)\, ds \le \xi_i(t)^T \Omega_i \xi_i(t), \tag{12}$$

where $\widehat{d}_i = h_{2i} - h_{1i}$, *and*

$$\begin{aligned} \xi_i(t) &= \left[x(t-h_{1i})^T \quad x(t-h_i(t))^T \quad x(t-h_{2i})^T\right]^T, \\ \Omega_i &= \begin{bmatrix} -Z_i & Z_i - U_i & U_i \\ * & -2Z_i + U_i + U_i^T & Z_i - U_i \\ * & * & -Z_i \end{bmatrix}. \end{aligned} \tag{13}$$

3. Main Results

For simplicity, we define

$$\begin{aligned}
\chi(t) &= \left[x(t)^T \quad x(t-h_{11})^T \quad x\left(t-h_1(t)^T\right) \quad x(t-h_{21})^T \quad \cdots \quad x(t-h_{1m})^T \quad x\left(t-h_m(t)^T\right) \quad x(t-h_{2m})^T \quad \dot{x}(t)^T\right]^T, \\
e_1 &= \left[1 \quad 0 \quad \cdots \quad 0\right]^T_{1\times 3m+2}, \\
e_2 &= \left[0 \quad 1 \quad \cdots \quad 0\right]^T_{1\times 3m+2}, \\
&\vdots \\
e_{3m+2} &= \left[0 \quad 0 \quad \cdots \quad 1\right]^T_{1\times 3m+2}, \\
\beta_1 &= \left[1 \quad 0 \quad \cdots \quad 0\right]^T_{1\times 3m+3}, \\
\beta_2 &= \left[0 \quad 1 \quad \cdots \quad 0\right]^T_{1\times 3m+3}, \\
&\vdots \\
\beta_{3m+3} &= \left[0 \quad 0 \quad \cdots \quad 1\right]^T_{1\times 3m+3}, \\
\chi_1(t) &= \left[\chi(t)^T \quad w(t)^T\right]^T.
\end{aligned} \tag{14}$$

3.1. Robust Exponential Stability Analysis. The criteria of the robust exponential stability for the systems (1)–(3) are proposed in the following Theorem.

Theorem 7. *Systems* (1)–(3) *with* $w(t) = 0$ *is robustly exponentially stable if there exist positive matrices* $P_k = P_k^T > 0$, $Q_i = Q_i^T > 0$, $R_{1i} = R_{1i}^T > 0$, $R_{2i} = R_{2i}^T > 0$, $S_i = S_i^T > 0$, $Z_i = Z_i^T > 0$, *any matrices* U_i M_j *with appropriate dimensions satisfying* $\begin{bmatrix} Z_i & U_i \\ * & Z_i \end{bmatrix} \ge 0$, $i = 1, 2, \ldots, m$; $j = 1, 2$, $k = 1, 2, \ldots, N$, *and positive scalars* $\varepsilon > 0$, *such that the following LMI holds*

$$\begin{bmatrix} \Theta_{11k} & \Theta_{12k} \\ * & -\varepsilon I \end{bmatrix} < 0, \quad k = 1, 2, \ldots, N, \tag{15}$$

where

$$\begin{aligned}
\Theta_{11k} &= e_1 P_k e_{3m+2}^T + e_{3m+2} P_k e_1^T \\
&\quad + \left[\sum_{j=1}^{N} e_1 \lambda_{k,j} P_j e_1^T\right] + \left[e_1 \sum_{i=1}^{m} Q_i e_1^T\right] \\
&\quad - \left[\sum_{i=1}^{m} (1-\mu) e_{3i} Q_i e_{3i}^T\right] \\
&\quad + \left[e_1 \sum_{i=1}^{m} (R_{1i} + R_{2i}) e_1^T\right] - \left[\sum_{i=1}^{m} e_{3i-1} R_{1i} e_{3i-1}^T\right] \\
&\quad - \left[\sum_{i=1}^{m} e_{3i+1} R_{2i} e_{3i+1}^T\right] \\
&\quad + \left[\sum_{i=1}^{m} h_{1i}^2 e_{3m+2} S_i e_{3m+2}^T\right] + \left[\sum_{i=1}^{m} \widehat{d}_i^2 e_{3m+2} Z_i e_{3m+2}^T\right] \\
&\quad - \left[\sum_{i=1}^{m} (e_1 - e_{3i-1}) S_i (e_1 - e_{3i-1})^T\right] \\
&\quad + \left(\sum_{i=1}^{m} \operatorname{diag}(0, \Omega_i, 0)\right) \\
&\quad + \left[e_1 M_1 + e_{3m+2} M_2\right]
\end{aligned}$$

$$\begin{aligned}
&\cdot\left[A_0(k)e_1^T + A_1(k)e_3^T + \cdots + A_m(k)e_{3m}^T - e_{3m+2}^T\right] \\
&+\left[e_1A_0^T(k) + e_3A_1^T(k) + \cdots + e_{3m}A_m^T(k) - e_{3m+2}\right] \\
&\times\left[M_1^Te_1^T + M_2^Te_{3m+2}^T\right] \\
&+\varepsilon\left[H_0(k)e_1^T + H_1(k)e_3^T + \cdots + H_m(k)e_{3m}^T\right]^T \\
&\times\left[H_0(k)e_1^T + H_1(k)e_3^T + \cdots + H_m(k)e_{3m}^T\right], \\
\Theta_{12k} &= \left[e_1M_1 + e_{3m+2}M_2\right]E(k).
\end{aligned} \tag{16}$$

And Ω_i $\widehat{d}_i$ *are defined in* (3).

Proof. On one hand, using Lemma 5 and Schur complement lemma to (15), we have

$$\begin{aligned}
\Pi &= e_1P_ke_{3m+2}^T + e_{3m+2}P_ke_1^T \\
&+\left[\sum_{j=1}^{N}e_1\lambda_{k,j}P_je_1^T\right] + \left[e_1\sum_{i=1}^{m}Q_ie_1^T\right] \\
&-\left[\sum_{i=1}^{m}(1-\mu)e_{3i}Q_ie_{3i}^T\right] \\
&+\left[e_1\sum_{i=1}^{m}(R_{1i}+R_{2i})e_1^T\right] - \left[\sum_{i=1}^{m}e_{3i-1}R_{1i}e_{3i-1}^T\right] \\
&-\left[\sum_{i=1}^{m}e_{3i+1}R_{2i}e_{3i+1}^T\right] \\
&+\left[\sum_{i=1}^{m}h_{1i}^2e_{3m+2}S_ie_{3m+2}^T\right] + \left[\sum_{i=1}^{m}\widehat{d}_i^2e_{3m+2}Z_ie_{3m+2}^T\right] \\
&-\left[\sum_{i=1}^{m}(e_1-e_{3i-1})S_i(e_1-e_{3i-1})^T\right] \\
&+\left(\sum_{i=1}^{m}\operatorname{diag}(0,\Omega_i,0)\right) + \left[e_1M_1 + e_{3m+2}M_2\right] \\
&\times\left[A_0(k,t)e_1^T + A_1(k,t)e_3^T + \cdots \right. \\
&\quad\left. +A_m(k,t)e_{3m}^T - e_{3m+2}^T\right] \\
&+\left[A_0(k)e_1^T + A_1(k)e_3^T + \cdots + A_m(k)e_{3m}^T - e_{3m+2}^T\right]^T \\
&\times\left[e_1M_1 + e_{3m+2}M_2\right]^T \\
&\le e_1Pe_{3m+2}^T + e_{3m+2}Pe_1^T \\
&+\left[\sum_{j=1}^{N}e_1P_ke_1^T\right] + \left[e_1\sum_{i=1}^{m}Q_ie_1^T\right] \\
&-\left[\sum_{i=1}^{m}(1-\mu)e_{3i}Q_ie_{3i}^T\right] \\
&+\left[e_1\sum_{i=1}^{m}(R_{1i}+R_{2i})e_1^T\right] - \left[\sum_{i=1}^{m}e_{3i-1}R_{1i}e_{3i-1}^T\right] \\
&-\left[\sum_{i=1}^{m}e_{3i+1}R_{2i}e_{3i+1}^T\right] \\
&+\left[\sum_{i=1}^{m}h_{1i}^2e_{3m+2}S_ie_{3m+2}^T\right] + \left[\sum_{i=1}^{m}\widehat{d}_i^2e_{3m+2}Z_ie_{3m+2}^T\right] \\
&-\left[\sum_{i=1}^{m}(e_1-e_{3i-1})S_i(e_1-e_{3i-1})^T\right] \\
&+\left(\sum_{i=1}^{m}\operatorname{diag}(0,\Omega_i,0)\right) + \left[e_1M_1 + e_{3m+2}M_2\right] \\
&\cdot\left[A_0(k)e_1^T + A_1(k)e_3^T + \cdots + A_m(k)e_{3m}^T - e_{3m+2}^T\right] \\
&+\left[e_1A_0^T(k) + e_3A_1^T(k) + \cdots + e_{3m}A_m^T(k) - e_{3m+2}\right] \\
&\times\left[M_1^Te_1^T + M_2^Te_{3m+2}^T\right] \\
&+\varepsilon\left[H_0(k)e_1^T + H_1(k)e_3^T + \cdots + H_m(k)e_{3m}^T\right]^T \\
&\times\left[H_0(k)e_1^T + H_1(k)e_3^T + \cdots + H_m(k)e_{3m}^T\right] \\
&+\varepsilon^{-1}\left[e_1M_1 + e_{3m+2}M_2\right]E(k)E(k)^T \\
&\times\left[e_1M_1 + e_{3m+2}M_2\right]^T \\
&= \Theta_{11} + \varepsilon^{-1}\Theta_{12}^T\Theta_{12} < 0.
\end{aligned} \tag{17}$$

On the other hand, define a new process $x_t(s) = x(t+s)$, $s \in [-2h, 0]$. Choose a Lyapunov-Krasovskii functional

$$V(x_t,t,r(t)) = \sum_{i=1}^{5}V_i(x_t,t,r(t)), \tag{18}$$

where

$$\begin{aligned}
V_1(x_t,t,r(t)) &= x(t)^TP_{r(t)}x(t), \\
V_2(x_t,t,r(t)) &= \sum_{i=1}^{m}\int_{t-h_i(t)}^{t}x(s)^TQ_ix(s)\,ds, \\
V_3(x_t,t,r(t)) &= \sum_{i=1}^{m}\int_{t-h_{1i}}^{t}x(s)^TR_{1i}x(s)\,ds \\
&\quad+\sum_{i=1}^{m}\int_{t-h_{2i}}^{t}x(s)^TR_{2i}x(s)\,ds, \\
V_4(x_t,t,r(t)) &= \sum_{i=1}^{m}\int_{-h_{1i}}^{0}\int_{t+s}^{t}h_{1i}\dot{x}(\alpha)^TS_i\dot{x}(\alpha)\,d\alpha\,ds, \\
V_5(x_t,t,r(t)) &= \sum_{i=1}^{m}\int_{-h_{2i}}^{-h_{1i}}\int_{t+s}^{t}\widehat{d}_i\dot{x}(\alpha)^TZ_i\dot{x}(\alpha)\,d\alpha\,ds.
\end{aligned} \tag{19}$$

Let $\mathscr{L}$ be the weak infinitesimal generator of the random process $\{x_t, t \geq 0\}$. Then for each $r(t) = k$, $k \in S$, we have

$$\begin{aligned}
&\mathscr{L}V_1(x_t, t, k) \\
&\quad = 2x(t) P_k \dot{x}(t) + \sum_{j=1}^{N} x(t)^T \lambda_{k,j} P_j x(t) \\
&\quad = 2\chi(t)^T \left[e_1 P_k e_{3m+2}^T \right] \chi(t) \\
&\qquad + \chi(t)^T \left[\sum_{j=1}^{N} e_1 \lambda_{k,j} P_j e_1^T \right] \chi(t), \\
&\mathscr{L}V_2(x_t, t, k) \\
&\quad \leq x(t)^T \sum_{i=1}^{m} Q_i x(t) \\
&\qquad - (1-\mu) \sum_{i=1}^{m} x(t - h_i(t))^T Q_i x(t - h_i(t)) \\
&\quad = \chi(t)^T \left[e_1 \sum_{i=1}^{m} Q_i e_1^T \right] \chi(t) \\
&\qquad - \chi(t)^T \left[\sum_{i=1}^{m} (1-\mu) e_{3i} Q_i e_{3i}^T \right] \chi(t), \\
&\mathscr{L}V_3(x_t, t, k) \\
&\quad \leq x(t)^T \left[\sum_{i=1}^{m} (R_{1i} + R_{2i}) \right] x(t) \\
&\qquad - \sum_{i=1}^{m} \left[x(t - h_{1i})^T R_{1i} x(t - h_{1i}) \right. \\
&\qquad\qquad \left. - x(t - h_{2i})^T R_{2i} x(t - h_{2i}) \right] \\
&\quad = \chi(t)^T \left[e_1 \sum_{i=1}^{m} (R_{1i} + R_{2i}) e_1^T \right] \chi(t) \\
&\qquad - \chi(t)^T \left[\sum_{i=1}^{m} e_{3i-1} R_{1i} e_{3i-1}^T \right] \chi(t) \\
&\qquad - \chi(t)^T \left[\sum_{i=1}^{m} e_{3i+1} R_{2i} e_{3i+1}^T \right] \chi(t), \\
&\mathscr{L}V_4(x_t, t, k) \\
&\quad = \sum_{i=1}^{m} h_{1i}^2 \dot{x}(t)^T S_i \dot{x}(t) \\
&\qquad - \sum_{i=1}^{m} \int_{t-h_{1i}}^{t} h_{1i} \dot{x}(s)^T S_i \dot{x}(s)\, ds \\
&\quad = \chi(t)^T \left[\sum_{i=1}^{m} h_{1i}^2 e_{3m+2} S_i e_{3m+2}^T \right] \chi(t) \\
&\qquad - \sum_{i=1}^{m} \int_{t-h_{1i}}^{t} h_{1i} \dot{x}(s)^T S_i \dot{x}(s)\, ds, \\
&\mathscr{L}V_5(x_t, t, k) \\
&\quad = \sum_{i=1}^{m} \hat{d}_i^2 \dot{x}(t)^T Z_i \dot{x}(t) \\
&\qquad - \sum_{i=1}^{m} \int_{t-h_{2i}}^{t-h_{1i}} \hat{d}_i \dot{x}(s)^T Z_i \dot{x}(s)\, ds \\
&\quad = \chi(t)^T \left[\sum_{i=1}^{m} \hat{d}_i^2 e_{3m+2} Z_i e_{3m+2}^T \right] \chi(t) \\
&\qquad - \sum_{i=1}^{m} \int_{t-h_{2i}}^{t-h_{1i}} \hat{d}_i \dot{x}(s)^T Z_i \dot{x}(s)\, ds.
\end{aligned} \tag{20}$$

Applying Lemma 4 to $\mathscr{L}V_4(x_t)$ results in

$$\begin{aligned}
&- \sum_{i=1}^{m} \int_{t-h_{1i}}^{t} h_{1i} \dot{x}(s)^T S_i \dot{x}(s)\, ds \\
&\quad \leq - \sum_{i=1}^{m} \left(\int_{t-h_{1i}}^{t} \dot{x}(s)\, ds \right)^T S_i \left(\int_{t-h_{1i}}^{t} \dot{x}(s)\, ds \right) \\
&\quad = - \sum_{i=1}^{m} \left[x(t) - x(t - h_{1i}) \right]^T S_i \left[x(t) - x(t - h_{1i}) \right] \\
&\quad \leq -\chi(t)^T \sum_{i=1}^{m} \left[(e_1 - e_{3i-1}) S_i (e_1 - e_{3i-1})^T \right] \chi(t),
\end{aligned} \tag{21}$$

and applying Lemma 6 to $\mathscr{L}V_5(x_t, t, k)$, we have that there exists U_i with $\left[\begin{smallmatrix} Z_i & U_i \\ * & Z_i \end{smallmatrix}\right] \geq 0$, $i = 1, 2, \ldots, m$, such that

$$\begin{aligned}
&- \sum_{i=1}^{m} \int_{t-h_{2i}}^{t-h_{1i}} \hat{d}_i f(s)^T Z_i f(s)\, ds \\
&\quad \leq \sum_{i=1}^{m} \xi_i(t)^T \Omega_i \xi_i(t) \\
&\quad = \chi(t)^T \left(\sum_{i=1}^{m} \operatorname{diag}(0, \Omega_i, 0) \right) \chi(t),
\end{aligned} \tag{22}$$

where $\xi_i(t)$ and Ω_i are defined in (13). Meanwhile, we note that

$$\begin{aligned}
&2 \left[x(t)^T M_1 + \dot{x}(t)^T M_2 \right] \\
&\quad \times \left[A_0(k, t) x(t) + \sum_{i=1}^{m} A_i(k, t) x(t - h_i(t)) - \dot{x}(t) \right] = 0,
\end{aligned} \tag{23}$$

that is,

$$\begin{aligned}&2\chi(t)^T\left[e_1M_1+e_{3m+2}M_2\right]\\&\quad\times\left[A_0(k,t)e_1^T+A_1(k,t)e_3^T+\cdots+A_m(k,t)e_{3m}^T-e_{3m+2}^T\right]\\&\quad\times\chi(t)=0.\end{aligned}\tag{24}$$

Then, we can deduce from (19)–(24) that

$$\mathscr{L}V(x_t,t,k)\le\chi(t)^T\Pi\chi(t)<0,\tag{25}$$

where Π is defined in (17). Therefore, by Definition 2 and the results in [37], we have that the system (1) is robustly stable . Now, we will prove the robust stochastic exponential stability in mean square for system (1). Setting $\lambda_0=\lambda_{\min}\{-\Pi\}>0$, we have

$$\mathscr{L}V(x_t,t,k)\le\chi(t)^T\Pi\chi(t)\le-\lambda_0\|x(t)\|^2.\tag{26}$$

Choose $\overline{V}(x_t,t,k)=e^{2\alpha t}V(x_t,t,k)$, where $\alpha>0$; then

$$\begin{aligned}\mathscr{L}\overline{V}(x_t,t,k)&=2ke^{2\alpha t}V(x_t,t,k)+e^{2\alpha t}\mathscr{L}(x_t,t,k)\\&\le 2ke^{2\alpha t}V(x_t,t,k)-\lambda_0e^{2\alpha t}\|x(t)\|^2.\end{aligned}\tag{27}$$

Integrating the above inequality (27), we get

$$\begin{aligned}&\overline{V}(x_t,t,k)\\&\quad\le V(x_0,0,k)\\&\qquad+\int_0^t\left\{2ke^{2\alpha s}V(x_s,s,k)-\lambda_0e^{2\alpha s}\|x(s)\|^2\right\}ds.\end{aligned}\tag{28}$$

From (19), it can be inferred that

$$\begin{aligned}&V(x_s,s,k)\\&\quad\le\lambda_{\max}(P_k)\|x(s)\|^2\\&\qquad+\left[\sum_{i=1}^m\left(\lambda_{\max}(Q_i)+\lambda_{\max}(R_{1i})+\lambda_{\max}(R_{2i})\right)\right]\\&\qquad\times\int_{s-h}^s\|x(v)\|^2dv\\&\qquad+\left[\sum_{i=1}^m h_{1i}\lambda_{\max}(S_i)+d_i\lambda_{\max}(Z_i)\right]\int_{s-h}^s\dot{x}(v)^T\dot{x}(v)\,dv.\end{aligned}\tag{29}$$

Note that

$$\begin{aligned}&\dot{x}(v)^T\dot{x}(v)\\&\quad\le m\left[\lambda_{\max}\left(A_0^T(k,v)A_0(k,v)\right)\|x(v)\|^2\right.\\&\qquad+\lambda_{\max}\left(A_1^T(k,v)A_1(v)\right)\|x(v-h_1(v))\|^2\\&\qquad\left.+\lambda_{\max}\left(A_m^T(k,v)A_m(v)\right)\|x(v-h_m(v))\|^2\right].\end{aligned}\tag{30}$$

We denote $\varrho=m[\lambda_{\max}(A_0^T(k,v)A_0(k,v))+\lambda_{\max}(A_1^T(k,v)A_1(k,v))+\lambda_{\max}(A_m^T(k,v)A_m(k,v))]$; then

$$\int_{s-h}^s\dot{x}(v)^T\dot{x}(v)\,dv\le\varrho\int_{s-2h}^s\|x(v)\|^2dv.\tag{31}$$

From (29) to (31), we obtain

$$V(x_s,s,k)\le\Xi_0\|x(s)\|^2+\Xi_1\int_{s-2h}^s\|x(v)\|^2dv,\tag{32}$$

where

$$\begin{aligned}\Xi_0&=\lambda_{\max}(P_k),\\\Xi_1&=\left[\sum_{i=1}^m\left(\lambda_{\max}(Q_i)+\lambda_{\max}(R_{1i})+\lambda_{\max}(R_{2i})\right)\right]\\&\quad+\varrho\left[\sum_{i=1}^m h_{1i}\lambda_{\max}(S_i)+d_i\lambda_{\max}(Z_i)\right].\end{aligned}\tag{33}$$

By the similar method, we have

$$V(x_0,0,k)\le\theta\sup_{-2h\le s\le0}\left\{\|\phi(s)\|^2\right\},\tag{34}$$

where $\theta=2h\Xi_1$. Therefore, by (28)–(34), we get

$$\begin{aligned}&\overline{V}(x_t,t,k)\\&\quad\le\theta\sup_{-2h\le s\le0}\left\{\|\phi(s)\|^2\right\}\\&\qquad+\int_0^t\left\{2\alpha e^{2\alpha s}\left[\Xi_0\|x(s)\|^2+\Xi_1\int_{s-2h}^s\|x(v)\|^2dv\right]\right.\\&\qquad\left.-\lambda_0e^{2\alpha s}\|x(s)\|^2\right\}ds\\&\quad\le\theta\sup_{-2h\le s\le0}\left\{\|\phi(s)\|^2\right\}\\&\qquad+(2\alpha\Xi_0-\lambda_0)\int_0^te^{2\alpha s}\|x(s)\|^2ds\\&\qquad+e^{2\alpha s}\Xi_1\int_{-2h}^t\|x(v)\|^2dv\\&\quad\le\left(\theta+2he^{2\alpha t}\Xi_1\right)\sup_{-2h\le s\le0}\left\{\|\phi(s)\|^2\right\}\\&\qquad+\left(e^{2\alpha t}\Xi_1+2\alpha\Xi_0-\lambda_0\right)\int_0^t\|x(v)\|^2dv.\end{aligned}\tag{35}$$

Choose $\alpha_0>0$ such that

$$e^{2\alpha_0t}\Xi_1+2\alpha_0\Xi_0-\lambda_0\le0;\tag{36}$$

then

$$\overline{V}(x_t,t,k)\le\left(\theta+2he^{2\alpha_0t}\Xi_1\right)\sup_{-2h\le s\le0}\left\{\|\phi(s)\|^2\right\}.\tag{37}$$

Since $\overline{V}(x_t,t,k) \geq e^{2\alpha_0 t}\lambda_{\min}(P_k)\|x(t)\|^2$, it can be shown from (37) that

$$\|x(t)\|^2 \leq \delta e^{-2\alpha_0 t} \sup_{-2h\leq s\leq 0} \left\{\|\phi(s)\|^2\right\}, \tag{38}$$

where

$$\delta = \frac{\theta + 2he^{2\alpha_0 t}\Xi_1}{\lambda_{\min}(P_k)}, \tag{39}$$

which implies that system (10) is robustly exponentially stable by Definition 2. This completes the proof. □

3.2. Robust H_∞ Exponential Stability Analysis. The criteria of the robust exponential stability with H_∞ performance for the systems (1)–(3) are proposed in the following Theorem.

Theorem 8. *Given a scalar $\gamma > 0$, the systems (1)–(3) are robustly exponentially stable with a prescribed H_∞ performance level γ if there exist matrices $P_k = P_k^T > 0$, $Q_i = Q_i^T > 0$, $R_{1i} = R_{1i}^T > 0$, $R_{2i} = R_{2i}^T > 0$, $S_i = S_i^T > 0$, $Z_i = Z_i^T > 0$, any matrices U_i M_j with appropriate dimensions satisfying $\begin{bmatrix} Z_i & U_i \\ * & Z_i \end{bmatrix} \geq 0$, $i = 1,2,\ldots,m$; $j = 1,2$, $k = 1,2,\ldots,N$, and positive scalars $\varepsilon > 0$, such that the following LMI holds*

$$\begin{bmatrix} \Omega_{11k} & \Omega_{12k} & \Omega_{13k} \\ * & -\varepsilon I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{40}$$

where

$$\begin{aligned}
\Omega_{11k} &= \beta_1 P_k \beta_{3m+2}^T + \beta_{3m+2} P_k \beta_1^T \\
&\quad + \left[\sum_{j=1}^{N} \beta_1 \lambda_{k,j} P_j \beta_1^T\right] + \left[\beta_1 \sum_{i=1}^{m} Q_i \beta_1^T\right] \\
&\quad - \left[\sum_{i=1}^{m} (1-\mu)\beta_{3i} Q_i \beta_{3i}^T\right] \\
&\quad + \left[\beta_1 \sum_{i=1}^{m} (R_{1i} + R_{2i})\beta_1^T\right] - \left[\sum_{i=1}^{m} \beta_{3i-1} R_{1i} \beta_{3i-1}^T\right] \\
&\quad - \left[\sum_{i=1}^{m} \beta_{3i+1} R_{2i} \beta_{3i+1}^T\right] \\
&\quad + \left[\sum_{i=1}^{m} h_{1i}^2 \beta_{3m+2} S_i \beta_{3m+2}^T\right] + \left[\sum_{i=1}^{m} \widehat{d}_i^2 \beta_{3m+2} Z_i \beta_{3m+2}^T\right] \\
&\quad - \left[\sum_{i=1}^{m} (\beta_1 - \beta_{3i-1}) S_i (\beta_1 - \beta_{3i-1})^T\right] \\
&\quad + \left(\sum_{i=1}^{m} \operatorname{diag}(0, \Omega_i, 0)\right) + [\beta_1 M_1 + \beta_{3m+2} M_2] \\
&\quad \cdot \left[A_0(k)\beta_1^T + A_1(k)\beta_3^T + \cdots + A_m(k)\beta_{3m}^T \right. \\
&\qquad \left. - \beta_{3m+2}^T + D_1(k)\beta_{3m+3}^T\right] \\
&\quad + \left[\beta_1 A_0^T(k) + \beta_3 A_1^T(k) + \cdots + \beta_{3m} A_m^T(k) \right. \\
&\qquad \left. - \beta_{3m+2} + \beta_{3m+3} D_1(k)^T\right] \\
&\quad \cdot \left[M_1^T \beta_1^T + M_2^T \beta_{3m+2}^T\right] \\
&\quad + \varepsilon\left[H_0(k)\beta_1^T + H_1(k)\beta_3^T + \cdots + H_m(k)\beta_{3m}^T\right]^T \\
&\quad \times \left[H_0(k)\beta_1^T + H_1(k)\beta_3^T + \cdots + H_m(k)\beta_{3m}^T\right] \\
&\quad - \gamma^2 \beta_{3m+3}\beta_{3m+3}^T. \\
\Omega_{12k} &= [\beta_1 M_1 + \beta_{3m+2} M_2]\, E(k), \\
\Omega_{13k} &= \left[C_0(k)\beta_1^T + C_1(k)\beta_3^T + \cdots + C_m(k)\beta_{3m}^T + D_2(k)\beta_{3m+3}^T\right]^T.
\end{aligned} \tag{41}$$

Proof. Implying Lemma 5 and Schur complement lemma to (40), we obtain

$$\begin{aligned}
\Pi' &= \beta_1 P_k \beta_{3m+2}^T + \beta_{3m+2} P_k \beta_1^T \\
&\quad + \left[\sum_{j=1}^{N} \beta_1 \lambda_{k,j} P_j \beta_1^T\right] + \left[\beta_1 \sum_{i=1}^{m} Q_i \beta_1^T\right] \\
&\quad - \left[\sum_{i=1}^{m} (1-\mu)\beta_{3i} Q_i \beta_{3i}^T\right] \\
&\quad + \left[\beta_1 \sum_{i=1}^{m} (R_{1i} + R_{2i})\beta_1^T\right] - \left[\sum_{i=1}^{m} \beta_{3i-1} R_{1i} \beta_{3i-1}^T\right] \\
&\quad - \left[\sum_{i=1}^{m} \beta_{3i+1} R_{2i} \beta_{3i+1}^T\right] + \left[\sum_{i=1}^{m} h_{1i}^2 \beta_{3m+2} S_i \beta_{3m+2}^T\right] \\
&\quad + \left[\sum_{i=1}^{m} \widehat{d}_i^2 \beta_{3m+2} Z_i \beta_{3m+2}^T\right] \\
&\quad - \left[\sum_{i=1}^{m} (\beta_1 - \beta_{3i-1}) S_i (\beta_1 - \beta_{3i-1})^T\right] \\
&\quad + \left(\sum_{i=1}^{m} \operatorname{diag}(0, \Omega_i, 0)\right) \\
&\quad + [\beta_1 M_1 + \beta_{3m+2} M_2] \\
&\quad \times \left[A_0(k,t)\beta_1^T + A_1(k,t)\beta_3^T + \cdots + A_m(k,t)\beta_{3m}^T \right. \\
&\qquad \left. - \beta_{3m+2}^T + D_1(k,t)\beta_{3m+3}^T\right]
\end{aligned}$$

$$+\left[A_0(k,t)\beta_1^T + A_1(k,t)\beta_3^T + \cdots + A_m(k,t)\beta_{3m}^T - \beta_{3m+2}^T + D_1(k,t)\beta_{3m+3}^T\right]^T \times \left[\beta_1 M_1 + \beta_{3m+2}M_2\right]^T - \gamma^2\beta_{3m+3}\beta_{3m+3}^T. \tag{42}$$

Set

$$J(t) = \left\{\int_0^t \left[z(s)^T z(s) - \gamma^2 w(s)^T w(s)\right]\right\} ds. \tag{43}$$

Then, it is easy to have

$$\begin{aligned} J(t) &= \left\{\int_0^t \left[z(s)^T z(s) - \gamma^2 w(s)^T w(s)\right] + \ell V(x(s), s, k)\right\} ds \\ &\quad - \{V(x(t), t)\} \\ &\le \left\{\int_0^t \left[z(s)^T z(s) - \gamma^2 w(s)^T w(s)\right] + \ell V(x(s), s, k)\right\} ds, \end{aligned} \tag{44}$$

where $V(x(t), t, k)$ is defined in (18). Similar to the proof of Theorem 7, we can obtain

$$z(t)^T z(t) - \gamma^2 w(t)^T w(t) + \ell V(x(t), t) \le \chi_1(t)^T \Pi' \chi_1(t), \tag{45}$$

where Π' is given in (42) and $\chi_1(t)$ is defined in (14). Then, it follows from (40) and (45) that

$$J(t) < 0. \tag{46}$$

This implies that for any nonzero $v(t) \in L_2[0, \infty]$,

$$\|z\|_{E_2} < \gamma \|w\|_2. \tag{47}$$

Therefore, by Definition 3, the system is robustly exponentially stable with a prescribed H_∞ performance level γ. This completes the proof. □

4. Numerical Example

In this section, we provide an example to demonstrate the effectiveness of the proposed method.

Let $m = 2$ and $N = 2$; consider the systems (1)–(3) with parameters as follows.

Mode 1

$$\begin{aligned} &A_0(1) = \begin{bmatrix} -5 & 0 \\ 0.5 & -6 \end{bmatrix}, \quad A_1(1) = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \\ &A_2(1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.5 \end{bmatrix}, \quad D_1(1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \\ &C_0(1) = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad C_1(1) = \begin{bmatrix} -0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \\ &C_2(1) = \begin{bmatrix} -0.1 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \quad D_2(1) = \begin{bmatrix} -0.1 & -0.3 \\ 0.1 & -0.2 \end{bmatrix}, \\ &E(1) = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \quad H_0(1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \\ &H_1(1) = \begin{bmatrix} -0.3 & 0.4 \\ 0.5 & -0.1 \end{bmatrix}, \quad H_2(1) = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}, \\ &H_d(1) = \begin{bmatrix} -0.1 & 0.4 \\ 0.3 & -0.1 \end{bmatrix}, \quad \lambda_{11} = -0.5, \quad \lambda_{12} = 0.5. \end{aligned} \tag{48}$$

Mode 2

$$\begin{aligned} &A_0(2) = \begin{bmatrix} -2 & 0 \\ 1 & -4 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} -1 & 1 \\ 1 & -4 \end{bmatrix}, \\ &A_2(2) = \begin{bmatrix} 0.2 & -0.2 \\ 0 & -0.3 \end{bmatrix}, \quad D_1(2) = \begin{bmatrix} -0.1 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}, \\ &C_0(2) = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & -0.1 \end{bmatrix}, \quad C_1(2) = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.1 \end{bmatrix}, \\ &C_2(2) = \begin{bmatrix} 0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix}, \quad D_2(2) = \begin{bmatrix} 0.1 & 0.2 \\ -0.1 & -0.3 \end{bmatrix}, \\ &E(2) = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -0.3 \end{bmatrix}, \quad H_0(2) = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \\ &H_1(2) = \begin{bmatrix} -0.1 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}, \quad H_2(2) = \begin{bmatrix} 0.1 & 0.1 \\ 0.6 & 0.2 \end{bmatrix}, \\ &H_d(2) = \begin{bmatrix} 0.2 & -0.3 \\ 0.2 & -0.1 \end{bmatrix}, \quad \lambda_{21} = 0.3, \quad \lambda_{22} = -0.3. \end{aligned} \tag{49}$$

And $\gamma = 2$, $\mu = 0.5$, $h_{11} = 0.1$, $h_{21} = 0.4$, $h_{12} = 0.4$, $h_{22} = 0.5$. Then, by solving the LMI (15) with the constraints in Theorem 7, we obtain

$$P_1 = \begin{bmatrix} 59.1376 & 8.2356 \\ 8.2356 & 26.8113 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 229.5038 & 50.5016 \\ 50.5016 & 124.2305 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 37.1817 & -0.5668 \\ -0.5668 & 41.5837 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 28.1504 & 6.9399 \\ 6.9399 & 7.1618 \end{bmatrix},$$

$$R_{11} = \begin{bmatrix} 18.1195 & 0.9123 \\ 0.9123 & 14.7239 \end{bmatrix},$$

$$R_{21} = \begin{bmatrix} 31.6879 & -9.1485 \\ -9.1485 & 54.6448 \end{bmatrix},$$

$$R_{12} = \begin{bmatrix} 31.9976 & 6.7097 \\ 6.7097 & 10.5708 \end{bmatrix},$$

$$R_{22} = \begin{bmatrix} 32.5781 & 6.8947 \\ 6.8947 & 10.6649 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 35.1285 & -0.1159 \\ -0.1159 & 25.3955 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.9787 & 0.2268 \\ 0.2268 & 0.0975 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} 52.7205 & -8.8649 \\ -8.8649 & 70.5952 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 1.7431 & 0.4057 \\ 0.4057 & 0.1638 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 1.7807 & 18.4064 \\ 18.8018 & -53.4269 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} -10.2878 & -1.9164 \\ -1.8946 & -5.3788 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 42.4713 & 5.0586 \\ 8.4912 & 25.0911 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 11.7147 & 1.1250 \\ 1.9353 & 3.9491 \end{bmatrix},$$

$$\varepsilon = 12.8944. \tag{50}$$

If we fix the lower bound of $h_1(t)$ and $h_2(t)$, that is, $h_{12} = 0.3$ and $h_{22} = 0.5$, for the different h_{11}, we can get the upper bounds of h_{21} as in Table 1.

If we fix the lower bound of $h_1(t)$ and $h_2(t)$, that is, $h_{11} = 0.2$ and $h_{21} = 0.6$, for the different h_{12}, we can get the upper bounds of h_{22} as in Table 2.

5. Conclusion

The robust exponential stability and H_∞ performance analysis for uncertain Markovian jumping system with multiple time-varying delays has been investigated based on the reciprocally convex approach. Some new delay-dependent stability conditions are obtained in term of LMIs. Numerical example has been proposed to illustrate the effectiveness of result.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Table 1: The upper bound of h_{21} for different h_{11}.

h_{11}	0.2	0.5	0.7	0.9	1	1.2
The upper bound of h_{21}	0.505	0.617	0.797	0.993	1.093	1.292

Table 2: The upper bound of h_{22} for different h_{12}.

h_{12}	0.3	0.5	0.7	0.9	1	1.2
The upper bound of h_{22}	0.397	0.597	0.797	0.997	1.097	1.297

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Research Article

Generalized Mutual Synchronization between Two Controlled Interdependent Networks

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This paper mainly focuses on the generalized mutual synchronization between two controlled interdependent networks. First, we propose the general model of controlled interdependent networks A and B with time-varying internetwork delays coupling. Then, by constructing Lyapunov functions and utilizing adaptive control technique, some sufficient conditions are established to ensure that the mutual synchronization errors between the state variables of networks A and B can asymptotically converge to zero. Finally, two numerical examples are given to illustrate the effectiveness of the theoretical results and to explore potential application in future smart grid. The simulation results also show how interdependent topologies and internetwork coupling delays influence the mutual synchronizability, which help to design interdependent networks with optimal mutual synchronizability.

1. Introduction

In recent years, extensive efforts have been devoted to understanding the properties of complex networks [1–5]. Particularly, as one of the most interesting and significant collective behaviors in real world, synchronization in complex dynamical networks has received increasing interest owing to its many potential applications in nature, socioeconomic systems, or engineering [6]. In the existing literature, it has been recognized that the network topology plays a significant role in synchronizability of diffusively coupled complex networks [7, 8]. Also, by using some effective control schemes, a variety of synchronization phenomena have been discovered in various complex networks (see [9–18] and relevant references therein). However, the studies mentioned above focused almost exclusively on the inner synchronization inside a single, noninteracting network.

Li et al. [19] studied the outer synchronization (in this paper, we call it mutual synchronization to be defined in Section 2) referring to the synchronization between two or more networks. However, to the best of our knowledge, it can be realized mainly by the open-plus-closed-loop method [19, 20] or based on the drive-response concept [21–27] considering only the intranetwork coupling of network itself. Zheng et al. [28] and Wu et al. [29] further studied the outer synchronization between two complex networks considering two kinds of internetwork coupling, but nevertheless, they both still derived the synchronization criteria based on drive-response concept and did not place the outer synchronization in the context of interdependent networks.

It is well known that many real-world network systems do interact with and depend on each other; for instance, various infrastructures such as transportation, water supply, fuel, and power stations are coupled together; realistic neuronal networks have a clustered structure and they can be viewed as interdependent networks; the epidemic can spread between the coupled networks of the infection layer and the prevention layer; dealing with secure information and cryptography, one can couple two systems to achieve the mutual synchronization, and so forth. Recently, Buldyrev et al. [30] studied the interdependent networks by presenting future smart grid as a real-life example, where the electrical power grid depends on the information network for control and the information network depends on the electrical power grid for their electricity supply. Then, Mei et al. [31] emphasized that it was urgent to research interdependent networks

theory for smart grid. Also, Brummitt et al. [32] demonstrated how interdependence affected cascades of load using a multiple branching process approximation. In a word, efforts have been directed to the cascading failures and robustness of interdependent networks [33–37]. In general, it has been recognized that interdependent topologies, especially interlinking strategy and internetwork coupling strength, play a vital role in cascading behaviors and robustness of interdependent networks. Analogously, this motivates us to attempt to explore the effects of interdependent topologies on the mutual synchronization between two interdependent networks.

Quite recently, Um et al. [38] placed synchronization behavior in the context of interdependent networks, where the one-dimensional regular network is mutually coupled to the WS small-world network. Based on the mean-field analytic approach, it has been revealed that the internetwork coupling and the intranetwork coupling play different roles in the synchronizability of the WS network. However, it is still limited to inner synchronization in one of the two interdependent networks and hence it is necessary and significant to study the mutual synchronization between two controlled interdependent networks.

The major contributions of our work are as follows. First, we propose the general model of two controlled interdependent networks A and B, which take into account not only the intranetwork coupling, but also the time-varying internetwork delays coupling. Second, we place the synchronization in the context of two controlled interdependent networks and study the generalized mutual synchronization of the proposed model. Third, in the numerical examples, to explore the potential application in smart grid, we couple the NW small-world network described by chaotic power system nodes and the scale-free network described by Lorenz chaotic systems following two interdependent interlinking strategies, respectively. Finally, we verify the influences of intranetwork and internetwork coupling and internetwork delays on the controlled mutual synchronizability, which can help to design the optimal interdependent networks.

The remaining part of this paper is organized as follows. Section 2 introduces some useful mathematical preliminaries and proposes the general model of two controlled interdependent networks. The generalized mutual synchronization is investigated and the main theoretical results of this paper are given in Section 3. In Section 4, two numerical examples are provided to explore the potential application in smart grid and to illustrate the correctness and effectiveness of the theoretical results. Finally, some conclusions and further work are given in Section 5.

2. Preliminaries and Model Presentation

2.1. Notations. The standard mathematical notations will be utilized throughout this paper. Let $\mathbb{R} \in (-\infty, +\infty)$, $\mathbb{R}^m$ be the m-dimensional Euclidean space and let $\mathbb{R}^{m\times n}$ be the space of $m \times n$ real matrices; $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ denotes the n-dimensional identity matrix; we use $\mathbf{A}^T$ or $\mathbf{x}^T$ to denote the transpose of the matrix $\mathbf{A}$ or the vector $\mathbf{x}$, respectively; $\lambda_{\max}$ is the maximum eigenvalue of corresponding real symmetric matrix; $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ stand for the 2-norm of the vector $\mathbf{x}$; $\bigotimes$ presents the Kronecker product of two matrices.

2.2. Model of Two Controlled Interdependent Networks. For simplicity and without loss generality, we consider the following model of two controlled interdependent networks (1) and (2) (we call networks A and B, respectively, in this paper) consisting of N identical nodes with time-varying internetwork delays coupling. The dynamical equations for the model of controlled interdependent networks A and B can be given by

$$\dot{\mathbf{x}}_i(t) = f(\mathbf{x}_i(t)) + a^i \sum_{j=1}^{N} a_{ij}\mathbf{\Gamma}_1\mathbf{x}_j(t) + c^i \sum_{j}^{N} c_{ij}\mathbf{\Gamma}_3\mathbf{y}_j(t-\tau_1(t)) \tag{1}$$

$$\begin{aligned}\dot{\mathbf{y}}_i(t) = {} & g(\mathbf{y}_i(t)) + b^i \sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}_2\mathbf{y}_j(t) \\ & + d^i \sum_{j}^{N} d_{ij}\mathbf{\Gamma}_4\mathbf{x}_j(t-\tau_2(t)) + \mathbf{u}_i(t), \quad i = 1, 2, \ldots N,\end{aligned} \tag{2}$$

where $\mathbf{x}_i(t) = (x_{i1}(t), x_{i2}(t), \ldots x_{im}(t))^T \in \mathbb{R}^m$ ($\mathbf{y}_i(t) = (y_{i1}(t), y_{i2}(t), \ldots y_{in}(t))^T \in \mathbb{R}^n$) is the state variable of the ith node in network $A(B)$ at time t; $f : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ ($g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$) is a smooth vector function; $\mathbf{A} = (a_{ij})_{N\times N}$($\mathbf{B} = (b_{ij})_{N\times N}$) stands for the intranetwork coupling matrix describing the topological structure of the network $A(B)$; namely, if there is a connection from node i to node j in network $A(B)$, then $a_{ij}(b_{ij}) = 1$; otherwise, $a_{ij}(b_{ij}) = 0$; however, $\mathbf{C} = (c_{ij})_{N\times N}$ (or $\mathbf{D} = (d_{ij})_{N\times N}$) is the internetwork coupling matrix representing the direct interaction from i in network A to j in network B (or from i in network B to j in network A); that is, if there exists a connection from i in network A to j in network B (or from i in network B to j in network A), then $c_{ij}(d_{ij}) = 1$; otherwise, $c_{ij}(d_{ij}) = 0$; $a^i(b^i)$ and $c^i(d^i)$ are the intranetwork and internetwork coupling strength for node i, respectively; $\mathbf{\Gamma}_1 \in R^{m\times m}$ ($\mathbf{\Gamma}_2 \in \mathbb{R}^{n\times n}, \mathbf{\Gamma}_3 \in \mathbb{R}^{m\times n}, \mathbf{\Gamma}_4 \in \mathbb{R}^{n\times m}$) is an inner coupling matrix describing the interactions between the coupled variables; $\tau_1(t)$, $\tau_2(t)$ are the time-varying internetwork coupling delays between networks A and B, respectively; $\mathbf{u}_i(t) \in R^n$ are the nonlinear controllers to be designed later for the mutual synchronization.

2.3. Mathematical Preliminaries. In order to obtain our theoretical results in Section 3, we introduce some necessary definitions, assumptions, and lemmas.

Definition 1. Let $\varphi_i(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$ ($i = 1, 2, \ldots N$) be the smooth vector functions. We define the generalized mutual synchronization errors as

$$\mathbf{e}_i(t) = \mathbf{y}_i(t) - \varphi_i(\mathbf{x}_i(t)), \quad i = 1, 2, \ldots N. \tag{3}$$

Thus, network A is said to achieve generalized mutual synchronization with network B successfully if

$$\lim_{t\to\infty} \|\mathbf{e}_i(t)\| = 0, \quad i = 1, 2, \ldots N. \tag{4}$$

Assumption 2. Suppose that the vector function $g(\cdot)$ is Lipschitz continuous, namely, for any $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$ and a constant $\mu > 0$, the following inequality holds:

$$\|g(\mathbf{y}) - g(\mathbf{x})\| \leq \mu \|\mathbf{y} - \mathbf{x}\|. \tag{5}$$

Assumption 3. Suppose that the time-varying delays $\tau_1(t)$, $\tau_2(t)$ are continuous differentiable functions with $0 \leq \tau_1(t), \tau_2(t) \leq h < \infty$ and $0 \leq \dot{\tau}_1(t) \leq \varepsilon_1 < 1$. Clearly, this assumption holds for constant $\tau_1(t)$, $\tau_2(t)$.

Remark 4. Assumptions 2 and 3 are both general assumptions, which hold for a broad class of real-world chaotic systems, such as Lorenz system, Chua's oscillator, Chen system, and Lü system [28]. Hence, in the following sections, we always assume that both assumptions hold.

Lemma 5 (see [26]). *If there are any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then the following inequality is true:*

$$\mathbf{x}^T\mathbf{y} \leq \frac{1}{2}\mathbf{x}^T\mathbf{x} + \frac{1}{2}\mathbf{y}^T\mathbf{y}. \tag{6}$$

3. Generalized Mutual Synchronization Criteria

In this section, by designing appropriate adaptive controllers, we can establish some sufficient conditions to insure the generalized mutual synchronization of the proposed general model in Section 2. Obviously, we can deduce some similar criteria for any simple or typical examples from this general model.

Combining (1) and (2) and (3), we can express error system of controlled interdependent networks A and B in terms of

$$\begin{aligned}
\dot{\mathbf{e}}_i(t) &= \dot{\mathbf{y}}_i(t) - \mathbf{J}\dot{\mathbf{x}}_i(t) \\
&= g(\mathbf{y}_i(t)) - \mathbf{J}f(\mathbf{x}_i(t)) + b^i\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}_2\mathbf{e}_j(t) \\
&\quad - c^i\mathbf{J}\sum_{j=1}^{N} c_{ij}\mathbf{\Gamma}_3\mathbf{e}_j(t-\tau_1(t)) \\
&\quad + b^i\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}_2\varphi_j(\mathbf{x}_j(t)) - a^i\mathbf{J}\sum_{j=1}^{N} a_{ij}\mathbf{\Gamma}_1 x_j(t) \\
&\quad + d^i\sum_{j=1}^{N} d_{ij}\mathbf{\Gamma}_4\mathbf{x}_j(t-\tau_2(t)) \\
&\quad - c^i\mathbf{J}\sum_{j=1}^{N} c_{ij}\mathbf{\Gamma}_3\varphi_j(\mathbf{x}_j(t-\tau_1(t))) + \mathbf{u}_i(t), \\
&\qquad i = 1, 2, \ldots N,
\end{aligned} \tag{7}$$

where $\mathbf{J} = D\varphi_i(\mathbf{x}_i)$ is the Jacobian matrix of the function $\varphi_i(\mathbf{x}_i)$.

Remark 6. From (7), one can find that adding appropriate controller to nodes is an alternative method to obtain mutual synchronization between two networks. In this paper, we thus mainly focus on the controlled mutual synchronization between two networks in the general context of two interdependent networks. Therefore, the intranetwork coupling matrices $\mathbf{A}$ and $\mathbf{B}$ and the internetwork coupling matrices $\mathbf{C}$ and $\mathbf{D}$ can be chosen arbitrarily, meaning that it is not necessary for assuming diffusivity, symmetry, or irreducibility of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. In addition, the topology structure, node dynamics, and dimension of state vector of one network can be different from the other.

Remark 7. It is well known that the time delays commonly exist in node dynamics, intranetwork coupling, and internetwork coupling. However, we just consider the time-varying internetwork coupling delays regardless of the others to explore the effects of internetwork coupling behavior on the mutual synchronization. It is noted that many networks of interest, like the Kuramoto model, have nonlinear coupling functions. Similarly, for simplicity, we just consider the linear intranetwork and internetwork coupling.

Theorem 8. *Suppose that Assumptions 2 and 3 hold and that the adaptive controllers* (8) *and the corresponding update laws* (9) *are added to the error system* (7). *Thus, generalized mutual synchronization between controlled interdependent networks A and B with time-varying internetwork delays coupling can be asymptotically realized. Consider*

$$\begin{aligned}
\mathbf{u}_i(t) &= \mathbf{J}f(\mathbf{x}_i(t)) - g(\varphi_i(\mathbf{x}_i(t))) + a^i\mathbf{J}\sum_{j=1}^{N} a_{ij}\mathbf{\Gamma}_1\mathbf{x}_j(t) \\
&\quad - b^i\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}_2\varphi_j(\mathbf{x}_j(t)) \\
&\quad + c^i\mathbf{J}\sum_{j=1}^{N} c_{ij}\mathbf{\Gamma}_3\varphi_j(\mathbf{x}_j(t-\tau_1(t))) \\
&\quad - d^i\sum_{j=1}^{N} d_{ij}\mathbf{\Gamma}_4\mathbf{x}_j(t-\tau_2(t)) - K_i\mathbf{e}_i(t), \\
&\qquad i = 1, 2, \ldots N,
\end{aligned} \tag{8}$$

$$\dot{K}_i = l_i\|\mathbf{e}_i(t)\|^2, \quad i = 1, 2, \ldots N, \tag{9}$$

where K_i are the time-varying feedback gain and l_i are arbitrary positive constants.

Proof. Plugging (8) and (9) into (7), the error dynamical system can be rewritten as

$$\begin{aligned}
\dot{\mathbf{e}}_i(t) &= g(\mathbf{y}_i(t)) - g(\varphi_i(\mathbf{x}_i(t))) + b^i\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}_2\mathbf{e}_j(t) \\
&\quad - c^i\mathbf{J}\sum_{j=1}^{N} c_{ij}\mathbf{\Gamma}_3\mathbf{e}_j(t-\tau_1(t)) - K_i\mathbf{e}_i(t), \quad i = 1, 2, \ldots N.
\end{aligned} \tag{10}$$

Let $\mathbf{e}(t) = (\mathbf{e}_1^T(t), \mathbf{e}_2^T(t), \ldots, \mathbf{e}_N^T(t))^T \in R^{nN}$, and construct a Lyapunov function as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\mathbf{e}_i^T(t)\,\mathbf{e}_i(t) + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{l_i}\left(K_i - \overline{K}\right)^2 + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{1-\varepsilon_1}\int_{t-\tau_1(t)}^{t}\mathbf{e}_i^T(\theta)\,\mathbf{e}_i(\theta)\,d\theta, \tag{11}$$

where $\overline{K}$ is a positive constant large enough to be selected later. Obviously, $V(t) > 0$ for all $\mathbf{e}(t) \neq 0$, meaning that $V(t)$ is positive definite. Calculating the derivative of (11) with respect to time along the solution of the error system (10), together with the updated laws (9), thus, we have

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N}\mathbf{e}_i^T(t)\,\dot{\mathbf{e}}_i(t) + \sum_{i=1}^{N}\frac{1}{l_i}\left(K_i - \overline{K}\right)\dot{K}_i \\
&\quad + \frac{1}{2(1-\varepsilon_1)}\sum_{i=1}^{N}\mathbf{e}_i^T(t)\,\mathbf{e}_i(t) \\
&\quad - \frac{1-\dot{\tau}_1(t)}{2(1-\varepsilon_1)}\sum_{i=1}^{N}\mathbf{e}_i^T(t-\tau_1(t))\,\mathbf{e}_i(t-\tau_1(t)) \\
&= \sum_{i=1}^{N}\mathbf{e}_i^T(t)\left[g(\mathbf{y}_i(t)) - g(\varphi_i(\mathbf{x}_i(t))) - K_i\mathbf{e}_i(t)\right] \\
&\quad + \sum_{i=1}^{N}\left(\frac{1}{2(1-\varepsilon_1)} + K_i - \overline{K}\right)\mathbf{e}_i^T(t)\,\mathbf{e}_i(t) \\
&\quad + \sum_{i=1}^{N}\mathbf{e}_i^T(t)\left[\sum_{j=1}^{N}b^i b_{ij}\mathbf{\Gamma}_2\mathbf{e}_j(t) - \sum_{j=1}^{N}c^i c_{ij}\mathbf{J}\mathbf{\Gamma}_3\mathbf{e}_j(t-\tau_1(t))\right] \\
&\quad - \frac{1-\dot{\tau}_1(t)}{2(1-\varepsilon_1)}\sum_{i=1}^{N}\mathbf{e}_i^T(t-\tau_1(t))\,\mathbf{e}_i(t-\tau_1(t)) \\
&\le \sum_{i=1}^{N}\mathbf{e}_i^T(t)\left(\mu + \frac{1}{2(1-\varepsilon_1)} - \overline{K}\right)\mathbf{e}_i(t) \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{e}_i^T(t)\,b^i b_{ij}\mathbf{\Gamma}_2\mathbf{e}_j(t) \\
&\quad - \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{e}_i^T(t)\,c^i c_{ij}\mathbf{J}\mathbf{\Gamma}_3\mathbf{e}_j(t-\tau_1(t)) \\
&\quad - \frac{1-\dot{\tau}_1(t)}{2(1-\varepsilon_1)}\sum_{i=1}^{N}\mathbf{e}_i^T(t-\tau_1(t))\,\mathbf{e}_i(t-\tau_1(t)).
\end{aligned} \tag{12}$$

Let $\mathbf{E} = \mathbf{G}\otimes\mathbf{\Gamma}_2$; $\mathbf{G} = (b^i b_{ij})_{N\times N}$; $\mathbf{F} = \mathbf{H}\otimes(\mathbf{J}\mathbf{\Gamma}_3)$; $\mathbf{H} = (-c^i c_{ij})_{N\times N}$, thus, we can get

$$\begin{aligned}
\dot{V}(t) &\le \left(\mu + \frac{1}{2(1-\varepsilon_1)} - \overline{K}\right)\mathbf{e}^T(t)\,\mathbf{e}(t) \\
&\quad + \mathbf{e}^T(t)\frac{\mathbf{E}+\mathbf{E}^T}{2}\mathbf{e}(t) + \mathbf{e}^T(t)\,\mathbf{F}\mathbf{e}(t-\tau_1(t)) \\
&\quad - \frac{1-\dot{\tau}_1(t)}{2(1-\varepsilon_1)}\sum_{i=1}^{N}\mathbf{e}_i^T(t-\tau_1(t))\,\mathbf{e}_i(t-\tau_1(t)).
\end{aligned} \tag{13}$$

From Assumption 3, we have

$$\frac{1}{2} - \frac{1-\dot{\tau}_1(t)}{2(1-\varepsilon_1)} \le 0. \tag{14}$$

From Lemma 5, we get

$$\mathbf{e}^T(t)\,\mathbf{F}\mathbf{e}(t-\tau_1(t)) \le \frac{1}{2}\mathbf{e}^T(t)\,\mathbf{F}\mathbf{F}^T\mathbf{e}(t) + \frac{1}{2}\mathbf{e}^T(t-\tau_1(t))\,\mathbf{e}(t-\tau_1(t)). \tag{15}$$

Combining (14) and (15) and (13), we can further get

$$\begin{aligned}
\dot{V}(t) &\le \left(\mu + \frac{1}{2(1-\varepsilon_1)} - \overline{K}\right)\mathbf{e}^T(t)\,\mathbf{e}(t) \\
&\quad + \mathbf{e}^T(t)\frac{\mathbf{E}+\mathbf{E}^T}{2}\mathbf{e}(t) + \frac{1}{2}\mathbf{e}^T(t)\,\mathbf{F}\mathbf{F}^T\mathbf{e}(t) \\
&\quad + \frac{1}{2}\mathbf{e}^T(t-\tau_1(t))\,\mathbf{e}(t-\tau_1(t)) \\
&\quad - \frac{1-\dot{\tau}_1(t)}{2(1-\varepsilon_1)}\frac{1}{2}\mathbf{e}^T(t-\tau_1(t))\,\mathbf{e}(t-\tau_1(t)) \\
&\le \left(\mu + \frac{1}{2(1-\varepsilon_1)} - \overline{K}\right)\mathbf{e}^T(t)\,\mathbf{e}(t) \\
&\quad + \mathbf{e}^T(t)\frac{\mathbf{E}+\mathbf{E}^T}{2}\mathbf{e}(t) + \frac{1}{2}\mathbf{e}^T(t)\,\mathbf{F}\mathbf{F}^T\mathbf{e}(t) \\
&\le \left(\mu + \frac{1}{2(1-\varepsilon_1)} - \overline{K} + \lambda_{\max}\left(\frac{\mathbf{E}+\mathbf{E}^T}{2}\right) + \lambda_{\max}\left(\mathbf{F}\mathbf{F}^T\right)\right)\mathbf{e}^T(t)\,\mathbf{e}(t).
\end{aligned} \tag{16}$$

If we take $\overline{K}$ as

$$\overline{K} \ge \mu + \frac{1}{2(1-\varepsilon_1)} + \lambda_{\max}\left(\frac{\mathbf{E}+\mathbf{E}^T}{2}\right) + \lambda_{\max}\left(\mathbf{F}\mathbf{F}^T\right) + 1, \tag{17}$$

then

$$\dot{V}(t) \le -\mathbf{e}^T(t)\,\mathbf{e}(t) = -\|\mathbf{e}(t)\|^2 \le 0. \tag{18}$$

Clearly, $V(t)$ is nonincreasing and every term of $V(t)$ is bounded. Thus, $\lim_{t\to\infty}V(t)$ tends to a nonnegative value. Since K_i is bounded and increasing (see (8) and (9)), it must also asymptotically converge to a limit. By integrating (18) over 0 to t, we can get $\int_0^t \|\mathbf{e}(\theta)\|^2 d\theta \le -\int_0^t \dot{V}(\theta)d\theta$. Thus, $\lim_{t\to\infty}\int_0^t \|\mathbf{e}(\theta)\|^2 d\theta$ exists and is a nonnegative value. According to Cauchy Criterion, we can obtain $\lim_{t\to+\infty}\int_{t-\tau_1(t)}^t \mathbf{e}_i^T(\theta)\mathbf{e}_i(\theta)d\theta = 0$. Therefore, from the definition of $V(t)$, we can conclude that $\lim_{t\to\infty}\|\mathbf{e}(t)\|^2$ converges to a limited nonnegative constant. Next, we would prove that $\lim_{t\to\infty}\|\mathbf{e}(t)\|^2 = 0$. If this is not true, then $\lim_{t\to+\infty}\|\mathbf{e}(t)\|^2 = \epsilon$ $(\epsilon > 0)$ holds. Obviously, $\|\mathbf{e}(t)\|^2 > \epsilon/2$ holds true for $t \ge \delta > 0$. From (18), we have

$$\dot{V}(t) \le -\|\mathbf{e}(t)\|^2 < -\frac{\epsilon}{2}. \tag{19}$$

Thus, by integrating (19) from H to ∞, we can get

$$\begin{aligned} -V(\delta) \le V(+\infty) - V(\delta) &= \int_\delta^{+\infty} \dot{V}(t)\,dt \\ &< -\int_\delta^{+\infty} \frac{\epsilon}{2}dt = -\infty. \end{aligned} \tag{20}$$

This is a contradiction, and hence $\lim_{t\to+\infty}\|\mathbf{e}(t)\|^2 = 0$; namely, $\lim_{t\to\infty}\|\mathbf{e}_i(t)\| = 0$, $i = 1, 2, \ldots N$. Consequently, the generalized mutual synchronization between controlled interdependent networks A and B is asymptotically obtained by using the proposed adaptive controllers (8) and (9). This completes the proof of the Theorem 8. □

Remark 9. From the proof of the Theorem 8, we know that $V(t)$ is positive definite, $\dot{V}(t)$ is negative definite, and $\lim_{t\to\infty}\mathbf{e}_i(t) = 0$. According to Lyapunov stability theory, we can also get that the synchronization state $\mathbf{e}_i(t) = 0$ is asymptotically stable.

Remark 10. It is noted that (17) is just a sufficient condition, but not the necessary one for the mutual synchronization between controlled interdependent networks A and B.

Based on Theorem 8, we can further obtain some similar synchronization criteria in the following two corollaries.

Corollary 11. *Suppose that Assumptions 2 and 3 hold. If $m = n$ and $\varphi_i(\mathbf{x}_i(t)) = \lambda\mathbf{x}_i(t)$, $\lambda \ne 0$, then projective mutual synchronization between controlled interdependent networks A and B with time-varying internetwork delays coupling can be asymptotically achieved under the following adaptive control schemes:*

$$\begin{aligned} \mathbf{u}_i(t) = {} & \lambda f(\mathbf{x}_i(t)) - g(\lambda x_i(t)) \\ & + \sum_{j=1}^N \lambda\left(a^i a_{ij}\mathbf{\Gamma}_1 - b^i b_{ij}\mathbf{\Gamma}_2\right)\mathbf{x}_j(t) \\ & + \sum_{j=1}^N \lambda^2 c^i c_{ij}\mathbf{\Gamma}_3\mathbf{x}_j(t-\tau_1(t)) \\ & - \sum_{j=1}^N d^i d_{ij}\mathbf{\Gamma}_4\mathbf{x}_j(t-\tau_2(t)) - K_i\mathbf{e}_i(t), \\ & \qquad i = 1, 2, \ldots N, \\ \dot{K}_i = {} & l_i\|\mathbf{e}_i(t)\|^2, \end{aligned} \tag{21}$$

where K_i, l_i have the same implications as those of Theorem 8, respectively.

Corollary 12. *Particularly, in Corollary 11, if $\lambda = \pm 1$, complete mutual synchronization (mutual antisynchronization) between controlled interdependent networks A and B with time-varying internetwork delays coupling can be asymptotically obtained by the adaptive controllers as follows:*

$$\begin{aligned} \mathbf{u}_i(t) = {} & f(\mathbf{x}_i(t)) - g(\pm\mathbf{x}_i(t)) \\ & \pm \sum_{j=1}^N \left(a^i a_{ij}\mathbf{\Gamma}_1 - b^i b_{ij}\mathbf{\Gamma}_2\right)\mathbf{x}_j(t) \\ & + \sum_{j=1}^N c^i c_{ij}\mathbf{\Gamma}_3\mathbf{x}_j(t-\tau_1(t)) \\ & - \sum_{j=1}^N d^i d_{ij}\mathbf{\Gamma}_4\mathbf{x}_j(t-\tau_2(t)) - K_i\mathbf{e}_i(t), \\ & \qquad i = 1, 2, \ldots N, \\ \dot{K}_i = {} & l_i\|\mathbf{e}_i(t)\|^2, \end{aligned} \tag{22}$$

where K_i, l_i have the same implications as those of Theorem 8, respectively.

Remark 13. It is quite natural that Theorem 8 and Corollaries 11 and 12 still hold for some simple cases, such as $\tau_1(t) = 0$, $\tau_2(t) = 0$, $\mathbf{A} = \mathbf{B}$, $\mathbf{C} = \mathbf{D}$, $\mathbf{\Gamma}_1 = \mathbf{\Gamma}_2$, and $\mathbf{\Gamma}_3 = \mathbf{\Gamma}_4$; hence, our model and synchronization methods are applicable to some of the two controlled interdependent networks similar to our model.

Remark 14. Plugging (3) into (10) and (1), we obtain (23) and (24), respectively:

$$\begin{aligned} \dot{\mathbf{e}}_i(t) = {} & g(\mathbf{e}_i(t) + \varphi_i(\mathbf{x}_i(t))) - g(\varphi_i(\mathbf{x}_i(t))) \\ & + \sum_{j=1}^N b^i b_{ij}\mathbf{\Gamma}_2\mathbf{e}_j(t) \end{aligned}$$

$$-\mathbf{J}\sum_{j=1}^{N} c^i c_{ij}\mathbf{\Gamma}_3\mathbf{e}_j\left(t-\tau_1(t)\right)-K_i\mathbf{e}_i(t),$$
$$i=1,2,\ldots N, \tag{23}$$

$$\dot{\mathbf{x}}_i(t)=f\left(\mathbf{x}_i(t)\right)+\sum_{j=1}^{N} a^i a_{ij}\mathbf{\Gamma}_1\mathbf{x}_j(t)$$
$$+\sum_{j}^{N} c^i c_{ij}\mathbf{\Gamma}_3\left(\mathbf{e}_i(t)+\varphi_i\left(\mathbf{x}_i(t)\right)\right), \tag{24}$$
$$i=1,2,\ldots N.$$

Combining (23) and (24), we find that the values of $e_i(t)$ are irrelevant to $\tau_2(t)$, d_{ij}, and d^i under the action of the proposed adaptive controllers (8) and (9). Thus, in the following sections, it is reasonable not to consider the effects of $\tau_2(t)$, d_{ij}, and d^i on the mutual synchronization between controlled interdependent networks A and B.

4. Numerical Simulations and Results

In this section, two numerical examples and their simulations are given to illustrate the correctness and effectiveness of the theoretical results obtained in the previous sections and to identify the factors that influence the mutual synchronizability.

To measure the speed and performance of mutual synchronization process, we define

$$\|\mathbf{e}(t)\|=\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{n} e_{ij}(t)^2}. \tag{25}$$

Actually, $\|\mathbf{e}(t)\|$ is the 2-norm of the synchronization error $\mathbf{e}(t)$, $0<t<+\infty$. Thus, the values of $\|\mathbf{e}(t)\|$ in the initial stage and at the end of simulations imply the mutual synchronization speed and performance, respectively. It should be particularly noted that, in all of the following simulations, the main figures and insets describe the values of $\|\mathbf{e}(t)\|$ during $0\le t<5$ and at the end of simulations ($t=5$), respectively.

Next, to explore the potential application of mutual synchronization in smart grid, we construct network A as NWsmall-world network ($N=50$, $k=3$, $P=0.3$) consisting of identical chaotic power system nodes and network B as scale-free network ($N=50$, $m=m_0=3$) described by Lorenz chaotic systems. The nonlinear function $f(\mathbf{x}_i(t))$ corresponding to chaotic power system nodes [39] is described by

$$f\left(\mathbf{x}_i(t)\right)=\begin{pmatrix} x_{i2}(t) \\ -a_1\sin\left(x_{i1}(t)\right)-b_1x_{i2}(t)+c_1+F\cos\left(d_1t\right)\end{pmatrix}. \tag{26}$$

When taking $a_1=1$, $b_1=0.02$, $c_1=0.2$, $d_1=1$, and $F=0.296$, the above power system nodes are hyperchaotic. The nonlinear function $g(y_i(t))$ involving the Lorenz systems [26] is represented by

$$g\left(\mathbf{y}_i(t)\right)=\begin{pmatrix} a_2\left(y_{i2}(t)-y_{i1}(t)\right) \\ b_2y_{i1}(t)-y_{i2}(t)-y_{i1}(t)\,y_{i3}(t) \\ -c_2y_{i1}(t)+y_{i1}(t)\,y_{i2}(t)\end{pmatrix}. \tag{27}$$

When taking $a_2=10$, $b_2=28$, and $c_2=8/3$, the Lorenz systems are chaotic. As is known to all, the chaotic systems are bounded, thus, $g(\mathbf{y}_i(t))$ satisfies Assumption 2. In the both examples, we arbitrarily select the generalized mapping functions as $\varphi_i(\mathbf{x}_i(t))=(x_{i1}(t),x_{i2}(t),x_{i1}(t)+x_{i2}(t))^T$, meaning that $J=\left(\begin{smallmatrix}1&0\\0&1\\1&1\end{smallmatrix}\right)$; then, we set $l_i=1$, $\mathbf{\Gamma}_1=\mathbf{I}_2$, $\mathbf{\Gamma}_2=\mathbf{I}_3$, $\mathbf{\Gamma}_3=\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)$, $\mathbf{\Gamma}_4=\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)$; next, the initial values $\mathbf{x}_i(0)$ and $\mathbf{y}_i(0)$ can be chosen randomly in $(-1,1)$ and $K_i(0)$ in $(0,1)$; in addition, the internetwork delays will be selected according to the Assumption 3.

For simplicity and for comparing, we further assume that the internetwork coupling links are bidirectional and the coupling strength of each node is equal; that is, $a^i=a$, $b^i=b$, $c^i=c$, and $d^i=d$. From Remark 14, we know that the time evolutions of $\mathbf{e}_i(t)$ are not relevant to $\tau_2(t)$, d_{ij}, and d^i; thus, it is also reasonable to assume $c=d=s$, $\tau_1(t)=\tau_2(t)=\tau(t)$ to simulate the influences of internetwork coupling strength and delays on the mutual synchronizability. Here, we employ the following two interlinking strategies to produce the interdependency matrices $\mathbf{C}$ and $\mathbf{D}$ in the two examples respectively.

(i) One-to-one support dependence interlinking strategy [30] (strategy I for short): node A_i in network A only depends on node B_i in network B and vice versa.

(ii) Multiple support dependence interlinking strategy [37] (strategy II for short): node in network A may randomly depend on more than one node in network B and vice versa.

Example 15. In this example, we generate the interdependency matrices $\mathbf{C}$ and $\mathbf{D}$ following the strategy I and design the adaptive controllers according to Theorem 8. When $a=b=s=1$, $\tau(t)=0.5$, the mutual synchronization errors $\mathbf{e}_i(t)$ are depicted in Figure 1, which shows that controlled interdependent networks A and B can easily achieve the generalized mutual synchronization using the designed controllers. Next, we further simulate the influences of internetwork delays and intranetwork and internetwork coupling strength on the mutual synchronizability between the networks A and B. We fix $a=b=s=1$ and only change the internetwork coupling delays $\tau(t)$; thus, the values of $\|e(t)\|$ for the networks A and B with different $\tau(t)$ are plotted in Figure 2. Similarly, Figures 3, 4, and 5 show the curves of $\|\mathbf{e}(t)\|$ for the networks A and B with fixed parameters $b=s=1$, $\tau(t)=0.5$ and different intranetwork coupling strength a, with $a=s=1$, $\tau(t)=0.5$ and different intranetwork coupling strength b, with $a=b=1$, $\tau(t)=0.5$ and different internetwork coupling strength s, respectively.

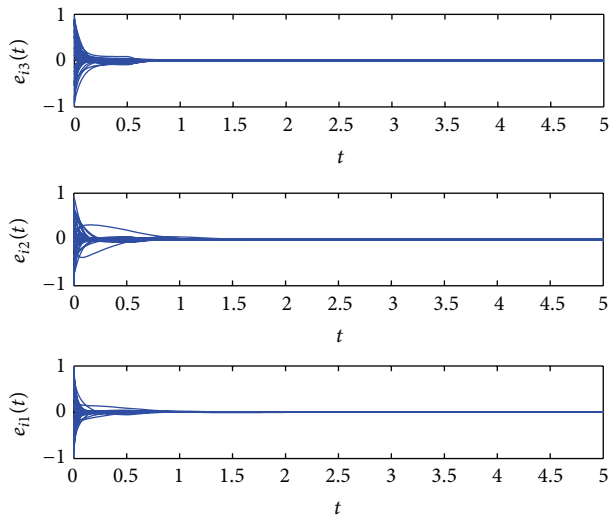

FIGURE 1: The mutual synchronization errors $\mathbf{e}_i(t)$ between the networks A and B interlinked following strategy I with $a = b = s = 1$, $\tau(t) = 0.5$.

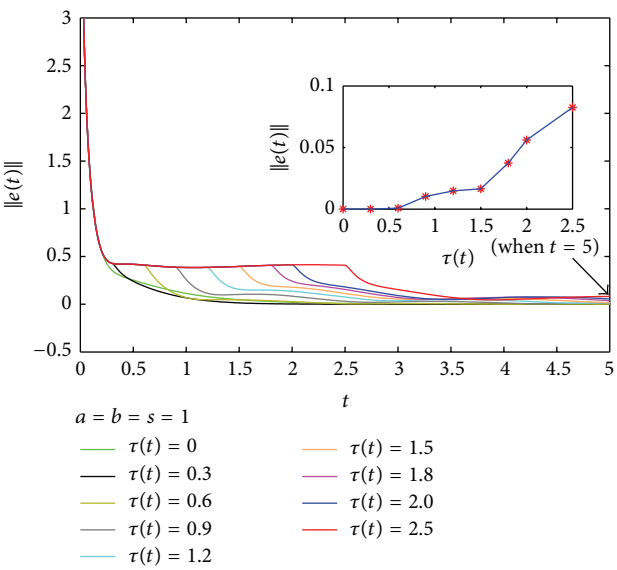

FIGURE 2: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy I with $a = b = s = 1$ and different internetwork delays $\tau(t)$.

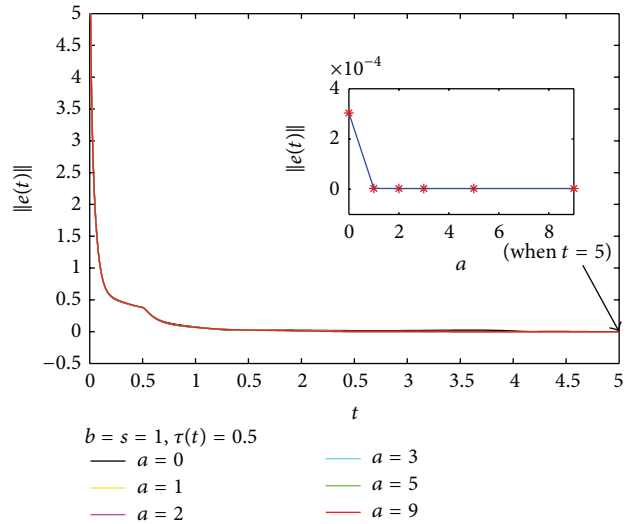

FIGURE 3: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy I with $b = s = 1$, $\tau(t) = 0.5$ and different intranetwork strength a.

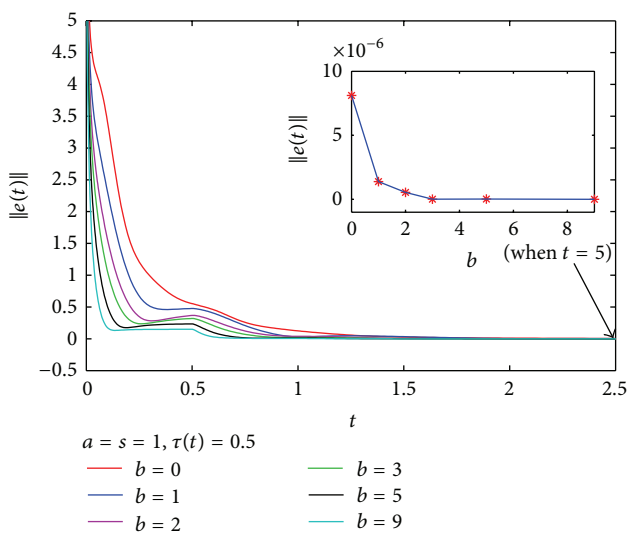

FIGURE 4: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy I with $a = s = 1$, $\tau(t) = 0.5$ and different intranetwork strength b.

Example 16. In this example, we produce the interdependency matrices $\mathbf{C}$ and $\mathbf{D}$ following the strategy II. To measure the effect of the number of interlinking edges on the mutual synchronizability, we define $\langle k \rangle$ as the average number of interlinking edges for each node in network A and the same to network B. We conduct similar simulations as those in Example 15. First, we set $a = b = s = 1$, $\langle k \rangle = 3$, $\tau(t) = e^t/(1 + e^t)$; thus, the time evolutions of the synchronization errors $\mathbf{e}_i(t)$ are depicted in Figure 6, which shows that interdependent networks A and B can achieve the generalized mutual synchronization successfully. Then, Figures 7, 8, 9, and 10, respectively, display the curves of $\|\mathbf{e}(t)\|$ for the networks A and B with fixed parameters $a = b = s = 1$, $\langle k \rangle = 3$ and different internetwork delays $\tau(t)$, with $b = s = 1$, $\langle k \rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$ and different intranetwork coupling strength a, with $a = s = 1$, $\langle k \rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$ and different intranetwork coupling strength b, with $a = b = 1$, $\langle k \rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$ and different internetwork coupling strength s. Finally, we fix the parameters $a = b = s = 1$, $\tau(t) = e^t/(1 + e^t)$; thus, the curves of $\|\mathbf{e}(t)\|$ for the networks A and B with different $\langle k \rangle$ are shown in Figure 11.

From the numerical results, both examples yield coincident tendency as follows, which further affirms our theoretical results. It is observed that the intranetwork coupling a has

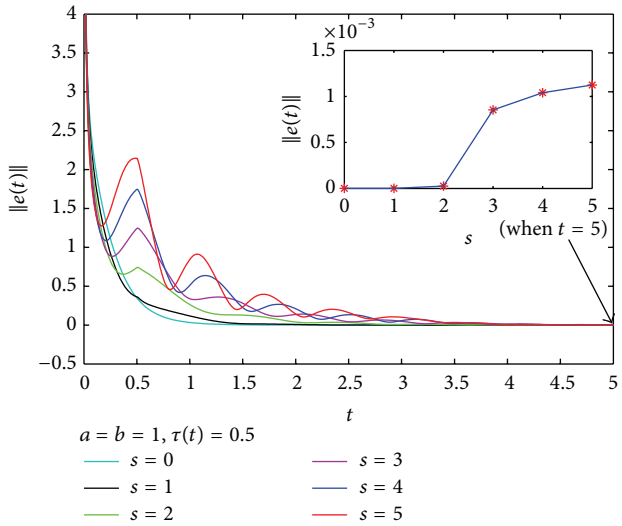

FIGURE 5: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy I with $a = b = 1$, $\tau(t) = 0.5$ and different internetwork strength s.

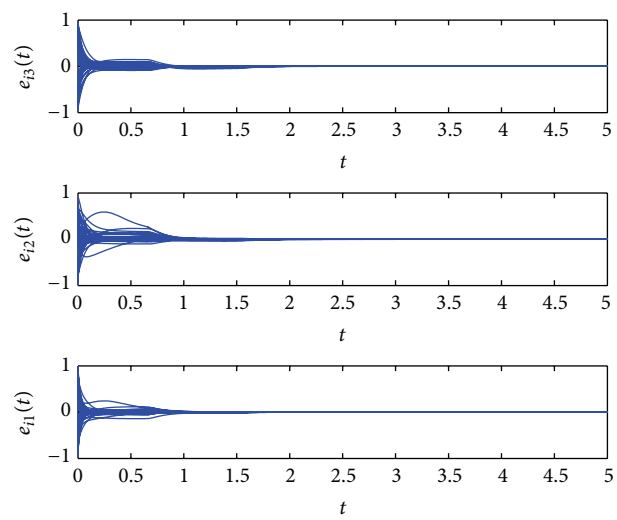

FIGURE 6: The mutual synchronization errors $\mathbf{e}_i(t)$ between interdependent networks A and B interlinked following strategy II with $a = b = s = 1$, $\langle k \rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$.

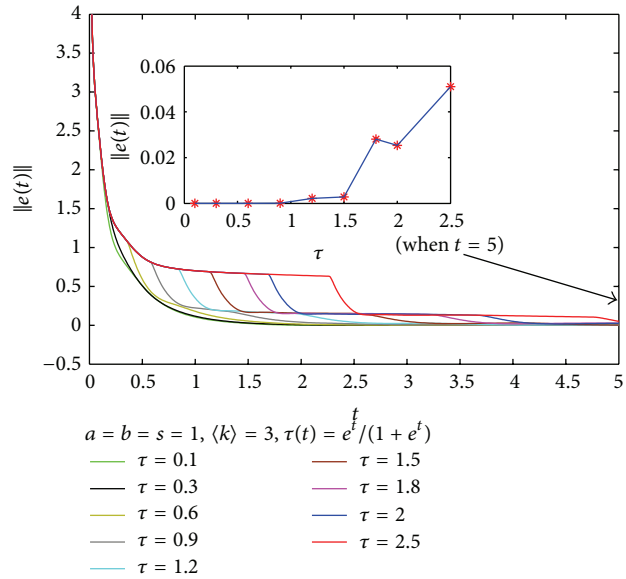

FIGURE 7: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy II with $a = b = s = 1$, $\langle k \rangle = 3$ and different internetwork delays $\tau(t)$.

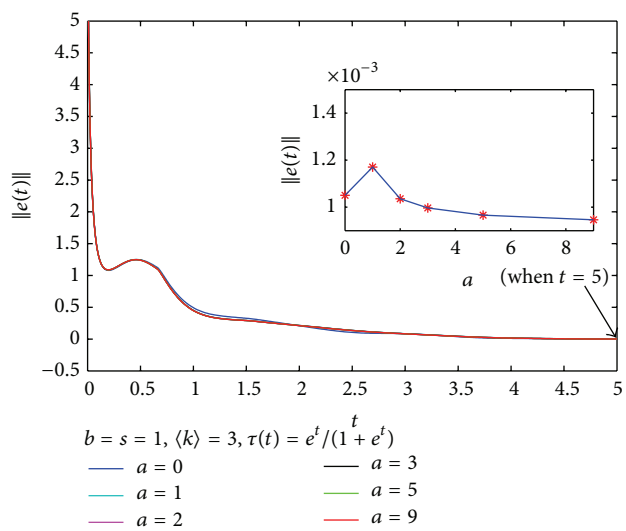

FIGURE 8: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy II with $b = s = 1$, $\langle k \rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$ and different intranetwork strength a.

little influence on mutual synchronization process (shown in Figures 3 and 8 and their insets), and the stronger intranetwork coupling b enhances the mutual synchronizability (shown in Figures 4 and 9 and their insets), while the stronger internetwork coupling worsen the mutual synchronizability (shown in Figures 5 and 10 and their insets). It is also found that the values of $\|\mathbf{e}(t)\|$ both in initial stage and at the end of simulations are increased as the internetwork coupling delay $\tau(t)$ is increased (shown in Figures 2 and 7 and their insets, resp.). In addition, Figure 11 implies that, to some extent, increase of $\langle k \rangle$ is equivalent to the increase of internetwork coupling strength s.

5. Conclusions and Future Work

In this paper, we extend previous research on the outer synchronization between two complex networks to our work on generalized mutual synchronization between two controlled interdependent networks by considering the time-varying internetwork delays coupling. Our model and relevant results are general and can be easily extended to other interdependent networks because there are not any constraints imposed on the intranetwork and internetwork coupling configuration matrices. Based on Lyapunov theory and corresponding mathematical techniques, some sufficient criteria have been

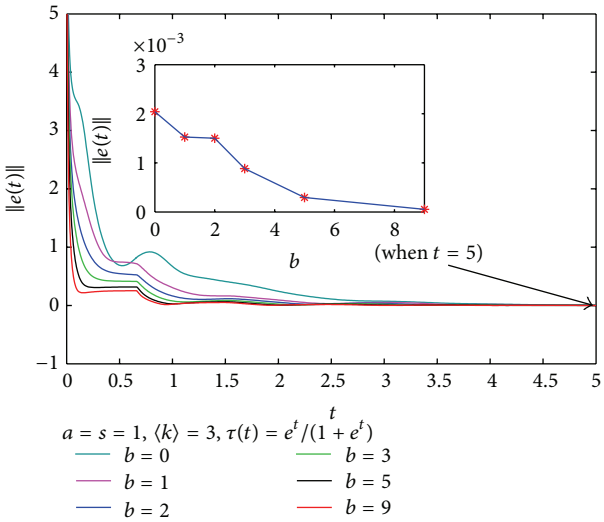

FIGURE 9: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy II with $a = s = 1$, $\langle k\rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$ and different intranetwork strength b.

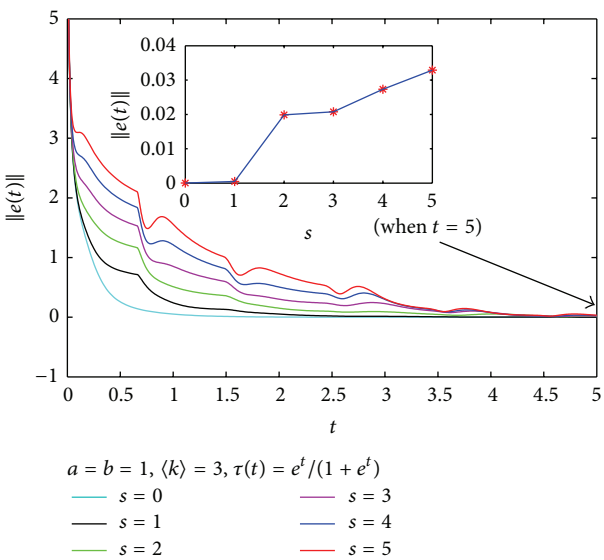

FIGURE 10: The curves of $\|\mathbf{e}(t)\|$ for the networks A and B interlinked following strategy II with $a = b = 1$, $\langle k\rangle = 3$, and $\tau(t) = e^t/(1 + e^t)$ and different internetwork strength s.

derived to guarantee that the proposed interdependent networks model is asymmetrically synchronized. Two numerical examples have been provided to illustrate the feasibility and effectiveness of the theoretical results and to further simulate the effects of internetwork delays, intranetwork and internetwork coupling strength on the mutual controlled synchronizability. In comparison, we find that, under the proposed adaptive controllers, the intranetwork coupling strength enhances the mutual synchronization, while the internetwork coupling delays and coupling strength suppress it. This indicates that the synchronization phenomenon in interdependent networks is different from that in a single

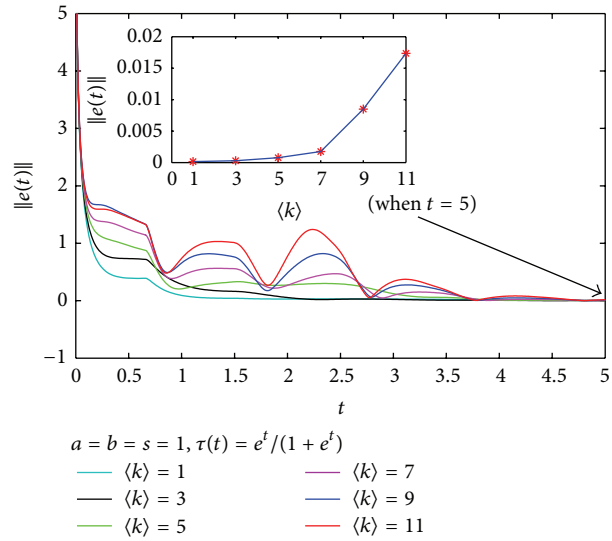

FIGURE 11: The values of $\|e(t)\|$ for the networks A and B interlinked by strategy II with $a = b = s = 1$, $\tau(t) = e^t/(1 + e^t)$ and different $\langle k\rangle$.

network, which highlights the necessity and significance of considering the mutual synchronization in the context of interdependent networks. Thus, with the help of our findings, one can further understand the mutual synchronization phenomenon in two interdependent networks and design interdependent networks with optimal mutual synchronizability for many potential practical applications.

However, the mutual synchronization between two interdependent networks is extremely complex, and we cannot consider all the factors that influence the synchronizability altogether. Also, our theoretical and numerical results are still conservative and the proposed control schemes are still a bit complicated because of the generality of the model. Therefore, how to simplify the control laws and reduce the number of controlled nodes is another important topic and remains to be researched in future. Thus, utilizing the designed controller, one can derive the synchronization conditions based on Lyapunov function approach, which is widely used in dynamic system analysis and design by some recent articles [40–44].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stochastic Finite-Time H_∞ Performance Analysis of Continuous-Time Systems with Random Abrupt Changes

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The problem of H_∞ control performance analysis of continuous-time systems with random abrupt changes is concerned in this paper. By employing an augmented multiple mode-dependent Lyapunov-Krasovskii functional and using some integral inequalities, new sufficient conditions are obtained relating to finite-time bounded and an H_∞ performance index. The finite-time H_∞ control performance problem is solved and desired controller is given to ensure the system trajectory stays within a prescribed bound during a given time interval. At last, two numerical examples are provided to show that our results are less conservative than the existing ones.

1. Introduction

It is well known that Markovian jump systems were introduced when the physical models are always subject to random changes, which can be also regarded as a special class of hybrid systems because of the structures are subject to random abrupt changes [1]. In the recent years, there are a lot of people towards to Markovian jump systems for its widely applications, for example, target tracking, robotics, manufacturing systems, aircraft control, and power systems [2–4]. Markovian jump systems are regarded as a special class of stochastic systems which switches from one to another at different time in the finite operation modes. Many important topics have been studied for Markovian jumping systems such as stability, control synthesis, stabilization, and filter design [5–7].

On the other hand, time delay is very common in practical dynamical systems, for example, networked control systems, chemical processes, communication systems, and so on [8–20]. Therefore, during the past two decades, various research topics have been considered for Markovian jump systems with time-varying delays [8–14]. It worth pointing out that when time delay is small enough in linear Markovian jump systems, the delay-dependent criteria are always less conservative than delay-independent ones. Over the past few years, for Markovian jump systems, many important topics related to delay-dependent have been extensively studied [14, 15].

Generally speaking, finite-time stability is investigated to address these transient performances of control systems in finite-time interval. Up to now, the concept of finite-time stability has been revisited with different systems, and many important results are obtained for finite-time stability and finite-time boundedness [21–26]. However, to the best of authors' knowledge, the stochastic finite-time H_∞ control for Markovian jump systems has not been fully studied. There is some room for next investigation due to the fact that analysis methods in existing references seem still conservative.

The major contribution of this paper is that we introduce a newly Lyapunov-Krasovskii functional for Markovian jump system. Some sufficient conditions are obtained to ensure the finite-time stability and bounded of the closed-loop Markovian jump systems. Compared with traditional methods of MJSs, it is shown the less conservative results can be obtained and the desired H_∞ control performance is obtained by employing mode-dependent Lyapunov functional instead of mode-independent Lyapunov functional. The finite-time bounded criterion can be dealt with in the terms of LMIs. Finally, the effectiveness of the developed techniques is also illustrated by two numerical examples.

2. Preliminaries

Given the probability space (Ω, F, P), where Ω, F, and P represent the sample space, the algebra of events, and the probability measure defined on F, respectively, the following Markovian jump systems over the probability space (Ω, F, P) are considered:

$$\begin{aligned} \dot{x}(t) &= A_{r_t} x(t) + A_{\tau r_t} x\left(t - \tau_{r_t}(t)\right) + B_{r_t} u(t) + D_{r_t} \omega(t), \\ z(t) &= C_{r_t} x(t) + C_{\tau r_t} x\left(t - \tau_{r_t}(t)\right) + F_{r_t} \omega(t), \\ x(t) &= \varphi(t), \quad t = [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathscr{R}^n$ represents the state vector of Markovian jump system, $u(t) \in \mathscr{R}^m$ is the control input, $z(t) \in \mathscr{R}^q$ denotes the controlled output and $\varphi(t), t = [-h, 0]$, where $r_0 \in \mathscr{N}$ is initial condition. $\omega(k) \in \mathscr{R}^q$ denotes the disturbance input which satisfies

$$\int_0^T \omega^{\mathsf{T}}(t)\,\omega(t)\,dt \le d. \tag{2}$$

Firstly, taking value on the finite set $\mathscr{N} = \{1, 2, \ldots, N\}$, let the random form process $\{r_t, t \ge 0\}$ be the stochastic process with transition rate matrix $\Omega = \{\pi_{ij}\}$, $i, j \in \mathscr{N}$ and let the transition probabilities also be denoted as follows:

$$\Pr\left(r_{t+\Delta} = j \mid r_t = i\right) = \rho_{ij} + \pi_{ij}\Delta + o(\Delta), \tag{3}$$

where

$$\varrho_{ij} = \begin{cases} 0, & \text{if } i \ne j \\ 1 & \text{if } i = j, \end{cases} \tag{4}$$

and $\Delta > 0$, $\pi_{ij} \ge 0$, for $i \ne j$, denotes the mode i in time t to time $t + \Delta$ with mode j,

$$-\pi_{ii} = \sum_{j=1, j\ne i}^{N} \pi_{ij} \tag{5}$$

for each mode $i \in \mathscr{N}$, $\lim_{\Delta \to 0_+}(o(\Delta)/\Delta) = 0$. $\tau_i(t)$ denotes the time-varying delay, which satisfies

$$\begin{aligned} 0 < \tau_i(t) &\le h_i < \infty, \\ \dot{\tau}_i(t) &\le \mu_i, \end{aligned} \tag{6}$$

where $h = \max\{h_i, i \in \mathscr{N}\}$ is the given upper bound of time-varying delays $\tau_i(t)$ and $\mu = \max\{\mu_i, i \in \mathscr{N}\}$ is the given upper bound of $\dot{\tau}_i(t)$. All the matrices are known matrices with the appropriate dimension.

In this paper, the objective is to design a state feedback controller as follows:

$$u(t) = K_i x(t), \tag{7}$$

where K_i is the controller gains to be designed.

Definition 1. System (1) is said to be finite-time bounded with respect to (c_1, c_2, T, R, d), if condition (2) and the following inequality hold:

$$\begin{aligned} &\sup_{-h \le \upsilon \le 0} \mathbb{E}\left\{x^{\mathsf{T}}(\upsilon) R x(\upsilon), \dot{x}^{\mathsf{T}}(\upsilon) R \dot{x}(\upsilon)\right\} \le c_1 \\ &\qquad \Longrightarrow \mathbb{E}\left\{x^{\mathsf{T}}(t) R x(t)\right\} < c_2, \quad \forall t \in [0, T], \end{aligned} \tag{8}$$

where $c_2 > c_1 \ge 0$ and $R > 0$.

Definition 2 (see [8]). Considering system (1) with the stochastic Lyapunov function $V(x_t, r_t)$, we get the weak infinitesimal operator as follows:

$$\begin{aligned} \pounds V(x_t, r_t, t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathbb{E}\left\{V\left(x_{t+\Delta t}, r_{t+\Delta t}, t + \Delta t\right)\right\}\right. \\ &\qquad \left. - V(x_t, i, t)\right] \\ &= \frac{\partial}{\partial t} V(x_t, i, t) + \frac{\partial}{\partial x} V(x_t, i, t)\,\dot{x}(t, i) \\ &\quad + \sum_{j=1}^{N} \pi_{ij} V(x_t, j, t). \end{aligned} \tag{9}$$

Definition 3. Given a constant scalar $T > 0$ and for all admissible $\omega(t)$ given in condition (2), if the Markovian jump system (1) is finite-time stochastic bounded and controller outputs satisfy condition (7) with attenuation $\gamma > 0$,

$$\mathbb{E}\left\{\int_0^T z^{\mathsf{T}}(t)\,z(t)\,dt\right\} \le \gamma^2 e^{\eta T} \mathbb{E}\left\{\int_0^T \omega^{\mathsf{T}}(t)\,\omega(t)\,dt\right\}. \tag{10}$$

The Markovian jump system (1) is called the finite-time stochastic bounded with a disturbance attenuation γ.

Lemma 4 (see [27]). *Let f_i: $\mathscr{R}^m \to \mathscr{R}(i = 1, 2, \ldots, N)$ have positive values in an open subset $\mathscr{D}$ of $\mathscr{R}^m$. Then, the reciprocally convex combination of f_i over $\mathscr{D}$ satisfies*

$$\begin{aligned} &\min_{\{\beta_i \mid \beta_i > 0, \sum_i \beta_i = 1\}} \sum_i \frac{1}{\beta_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i \ne j} g_{i,j}(t) \\ &\text{subject to } \left\{g_{i,j} : \mathscr{R}^m \longrightarrow \mathscr{R}, g_{j,i}(t) = g_{i,j}(t),\right. \\ &\qquad \left.\begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \ge 0\right\}. \end{aligned} \tag{11}$$

Lemma 5. *For any constant matrix $M \in \mathcal{R}^{m\times m}$ with $M > 0$, scalars $a < b \le 0$, vector function $x : [a,b] \to \mathcal{R}^m$, such that the integrals in the following are well-defined; then,*

$$-\frac{a^2-b^2}{2}\int_a^b\int_{t+s}^t x^{\mathsf{T}}(s)Mx(s)\,ds\,d\theta \le -\left[\int_a^b\int_{t+s}^t x(s)\,ds\,d\theta\right]^{\mathsf{T}} M\left[\int_a^b\int_{t+s}^t x(s)\,ds\,d\theta\right]. \tag{12}$$

3. Finite-Time H_∞ Performance Analysis

The issue of stability analysis of Markovian jump system (1) subject to $u(t) = 0$ is given firstly. Therefore, the finite-time stability is obtained in this section.

Theorem 6. *System* (1) *is called the finite-time bounded with respect to (c_1, c_2, d, R, T), if there exist matrices*

$$\begin{gathered} P_i > 0, \quad Q_{li} > 0 \quad (l = 1,2), \quad Q > 0, \\ \mathcal{X}_i = \begin{bmatrix} X_{1i} & X_{2i} \\ X_{3i} & X_{4i} \end{bmatrix} > 0, \quad \mathcal{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} > 0, \\ Y_s > 0 \quad (s = 1,2), \quad Z_i > 0, \\ Z > 0, \quad H > 0, \end{gathered} \tag{13}$$

scalars $c_1 < c_2$, $T > 0$, $\lambda_s > 0$, $(s = 1,2,\ldots,12)$, $\eta > 0$, and $\Lambda > 0$, such that $\forall i, j \in \mathcal{N}$ and the inequalities hold as follows:

$$e^{\delta h}\sum_{j=1}^N \pi_{ij}Q_{1j} + e^{\delta h}\sum_{j=1, j\ne i}^N \pi_{ij}Q_{2j} < Q, \tag{14}$$

$$\sum_{j=1}^N \pi_{ij}\mathcal{X}_j - \mathcal{X} < 0, \tag{15}$$

$$\sum_{j=1}^N \pi_{ij}Z_j - Z < 0, \tag{16}$$

$$\begin{bmatrix} \dfrac{\mathcal{X}_i}{h} & \mathcal{S}_i \\ * & \dfrac{\mathcal{X}_i}{h} \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} Y_1 & W_1 \\ * & Y_2 \end{bmatrix} > 0, \tag{18}$$

$$\begin{bmatrix} Y_1 & W_2 \\ * & Y_2 \end{bmatrix} > 0, \tag{19}$$

$$\Xi_i = \begin{bmatrix} \Xi_{11i} & \Xi_{12i} & S_{4i} & \Xi_{14i} & \Xi_{15i} & -S_{3i} + \dfrac{Z_i + Z_i^{\mathsf{T}}}{h} & P_iD_i \\ * & \Xi_{22i} & -S_{4i} + \dfrac{X_{4i}}{h} & 0 & \dfrac{X_{3i}}{2h} + \dfrac{X_{2i}^{\mathsf{T}}}{2h} & S_{3i} - S_{2i}^{\mathsf{T}} - \dfrac{X_{3i}}{2h} - \dfrac{X_{2i}^{\mathsf{T}}}{2h} & 0 \\ * & * & -Q_{1i} - \dfrac{X_{4i}}{h} - W_2 & 0 & 0 & S_{2i}^{\mathsf{T}} + \dfrac{X_{3i}}{2h} + \dfrac{X_{2i}^{\mathsf{T}}}{2h} & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55i} & -S_{1i} - \dfrac{Z_i}{h^2} - \dfrac{Z_i^{\mathsf{T}}}{h^2} & 0 \\ * & * & * & * & * & -\dfrac{X_{1i}}{h} - \dfrac{Z_i}{h^2} - \dfrac{Z_i^{\mathsf{T}}}{h^2} & 0 \\ * & * & * & * & * & * & -\delta H \end{bmatrix} < 0, \tag{20}$$

$$c_1\Lambda + d\delta\lambda_{12}\frac{1}{\eta}\left(1 - e^{-\eta T}\right) < \lambda_1 e^{-\eta T}c_2, \tag{21}$$

where

$$\begin{aligned} \Xi_{11i} = {} & \sum_{j=1}^N \pi_{ij}P_j + \delta P_i + P_iA_i + A_i^{\mathsf{T}}P_i + e^{\delta h}\left(Q_{1i} + Q_{2i}\right) \\ & + hQ + \frac{e^{\delta h - 1}}{\delta}X_{1i} \\ & + \frac{e^{\delta h} - \delta h e^{\delta h} - 1}{\delta^2}X_1 - \frac{X_{4i}}{h} + hY_1 + W_1 - Z_i - Z_i^{\mathsf{T}}, \end{aligned}$$

$$\Xi_{12i} = P_iA_{\tau i} + \frac{X_{4i}}{h} - S_{4i},$$

$$\Xi_{14i} = \frac{e^{\delta h} - 1}{2\delta}\left(X_{2i} + X_{3i}^{\mathsf{T}}\right) + \frac{e^{\delta h} - \delta h e^{\delta h} - 1}{2\delta^2}\left(X_2 + X_3^{\mathsf{T}}\right),$$

$$\Xi_{15i} = -\frac{X_{3i}}{2h} - \frac{X_{2i}^{\mathsf{T}}}{2h} + \frac{Z_i}{h^2} + \frac{Z_i^{\mathsf{T}}}{h^2},$$

$$\Xi_{22i} = -\left(1 - \mu_i\right)Q_{2i} - \frac{2X_{4i}}{h} + S_{4i} + S_{4i}^{\mathsf{T}} - W_1 + W_2,$$

$$\begin{aligned}
\Xi_{44i} &= \frac{e^{\delta h}-1}{\delta}X_{4i} + \frac{e^{\delta h}-\delta h e^{\delta h}-1}{\delta^2}X_{4i} + hY_2 \\
&\quad + \frac{e^{\delta h}-\delta h-1}{\delta^2}Z_i + \frac{h\delta^2 e^{\delta h}+e^{\delta h}+\delta h+1}{\delta^3}Z, \\
\Xi_{55i} &= -\frac{X_{4i}}{h} - \frac{Z_i}{h^2} - \frac{Z_i^{\mathsf{T}}}{h^2}, \\
\Lambda &= \lambda_2 + he^{\delta h}(\lambda_3+\lambda_4) + h^2e^{\delta h}\lambda_5 + h^2e^{\delta h}\lambda_6 \\
&\quad + \frac{1}{2}h^3e^{\delta h}\lambda_7 + h^2e^{\delta h}(\lambda_8+\lambda_9) \\
&\quad + \frac{1}{2}h^3e^{\delta h}\lambda_{10} + \frac{1}{6}h^4e^{\delta h}\lambda_{11}, \\
\lambda_1 &= \max_{i\in\mathcal{N}}\lambda_{\max}(P_i), \qquad \lambda_2 = \max_{i\in\mathcal{N}}\lambda_{\max}(\widetilde{P}_i), \\
\lambda_3 &= \max_{i\in\mathcal{N}}\lambda_{\max}(\widetilde{Q}_{1i}), \qquad \lambda_4 = \max_{i\in\mathcal{N}}\lambda_{\max}(\widetilde{Q}_{2i}), \\
\lambda_5 &= \lambda_{\max}(\widetilde{Q}), \qquad \lambda_6 = \max_{i\in\mathcal{N}}\lambda_{\max}(\widetilde{\mathcal{X}}_i), \\
\lambda_7 &= \max_{i\in\mathcal{N}}\lambda_{\max}(\widetilde{\mathcal{X}}), \qquad \lambda_8 = \lambda_{\max}(\widetilde{Y}_1), \\
\lambda_9 &= \lambda_{\max}(\widetilde{Y}_2), \qquad \lambda_{10} = \max_{i\in\mathcal{N}}\lambda_{\max}(\widetilde{Z}_i), \\
\lambda_{11} &= \lambda_{\max}(\widetilde{Z}), \qquad \lambda_{12} = \lambda_{\max}(H), \\
\widetilde{P}_i &= R^{-(1/2)}P_iR^{-(1/2)}, \\
\widetilde{Q}_{li} &= R^{-(1/2)}Q_{li}R^{-(1/2)} \quad (l=1,2), \\
\widetilde{Q} &= R^{-(1/2)}QR^{-(1/2)}, \\
\widetilde{\mathcal{X}}_i &= R^{-(1/2)}\mathcal{X}_iR^{-(1/2)} \quad (l=1,2), \\
\widetilde{X} &= R^{-(1/2)}XR^{-(1/2)}, \\
\widetilde{Y}_s &= R^{-(1/2)}Y_sR^{-(1/2)} \quad (s=1,2), \\
\widetilde{Z}_i &= R^{-(1/2)}Z_iR^{-(1/2)}, \\
\widetilde{Z} &= R^{-(1/2)}ZR^{-(1/2)}.
\end{aligned} \tag{22}$$

Proof. Firstly, a novel process is defined in this paper as follows:

$$x_t(s) = x(t+s), \quad s\in[-h,0]. \tag{23}$$

Then, the following Lyapunov-Krasovskii functional is considered:

$$V(x_t,r_t,t) = \sum_{l=1}^{5}V_l(x_t,r_t,t), \tag{24}$$

where

$$\begin{aligned}
V_1(x_t,r_t,t) &= x(t)^{\mathsf{T}}e^{\delta t}P_{r_t}x(t), \\
V_2(x_t,r_t,t) &= \int_{t-h}^{t}e^{\delta(s+h)}x^{\mathsf{T}}(s)Q_{1r_t}x(s)\,ds \\
&\quad + \int_{t-\tau_{r_t}(t)}^{t}e^{\delta(s+h)}x^{\mathsf{T}}(s)Q_{2r_t}x(s)\,ds \\
&\quad + \int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s+h)}x^{\mathsf{T}}(s)Qx(s)\,ds\,d\theta, \\
V_3(x_t,r_t,t) &= \int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\eta^{\mathsf{T}}(s)\mathcal{X}_{r_t}\eta(s)\,ds\,d\theta \\
&\quad + \int_{-h}^{0}\int_{\theta}^{0}\int_{t+\upsilon}^{t}e^{\delta(s-\theta)}\eta^{\mathsf{T}}(s)\mathcal{X}\eta(s)\,ds\,d\upsilon\,d\theta, \\
V_4(x_t,r_t,t) &= \int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}x^{\mathsf{T}}(s)Y_1x(s)\,ds\,d\theta \\
&\quad + \int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\dot{x}^{\mathsf{T}}(s)Y_2\dot{x}(s)\,ds\,d\theta, \\
V_5(x_t,r_t,t) &= \int_{-h}^{0}\int_{\theta}^{0}\int_{t+\upsilon}^{t}e^{\delta(s-\theta)}\dot{x}^{\mathsf{T}}(s)Z_{r_t}\dot{x}(s)\,ds\,d\upsilon\,d\theta \\
&\quad + \int_{-h}^{0}\int_{\varsigma}^{0}\int_{\theta}^{0}\int_{t+\upsilon}^{t}e^{\delta(s-\theta)}\dot{x}^{\mathsf{T}}(s) \\
&\qquad \times Z\dot{x}(s)\,ds\,d\upsilon\,d\theta\,d\varsigma,
\end{aligned} \tag{25}$$

where $\eta(t) = [x^{\mathsf{T}}(t), \dot{x}^{\mathsf{T}}(t)]^{\mathsf{T}}$.

Letting i represent the time t, that is, $r_t = i \in \mathcal{N}$, one has

$$\begin{aligned}
\pounds V_1(x_t,i,t) &= \delta e^{\delta t}x^{\mathsf{T}}(t)P_ix(t) + 2e^{\delta t}x^{\mathsf{T}}(t)P_i\dot{x}(t) \\
&\quad + e^{\delta t}x^{\mathsf{T}}(t)\left(\sum_{j=1}^{N}\pi_{ij}P_j\right)x(t) \\
&= e^{\delta t}x^{\mathsf{T}}(t)\left(\sum_{j=1}^{N}\pi_{ij}P_j + \delta P_i\right)x(t) + 2e^{\delta t}x^{\mathsf{T}}(t)P_i \\
&\quad \times (A_ix(t) + A_{\tau i}x(t-\tau_i(t)) + D_i\omega(t)).
\end{aligned} \tag{26}$$

Noting $\pi_{ij} \geq 0$ for $j \neq i$ and $\pi_{ii} \leq 0$, one has

$$\begin{aligned}
\pounds V_2(x_t,i,t) &= e^{\delta t}x^{\mathsf{T}}(t)\left(e^{\delta h}Q_{1i} + e^{\delta h}Q_{2i} + hQ\right)x(t) \\
&\quad - e^{\delta t}x^{\mathsf{T}}(t-h)Q_{1i}x(t-h) \\
&\quad - (1-\dot{\tau}_i(t))e^{\delta(t+h-\tau_i(t))}x^{\mathsf{T}} \\
&\quad \times (t-\tau_i(t))Q_{2i}x(t-\tau_i(t)) \\
&\quad + \int_{t-h}^{t}e^{\delta s}x^{\mathsf{T}}(s)\left(e^{\delta h}\sum_{j=1}^{N}\pi_{ij}Q_{1j} - Q\right)x(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{N}\pi_{ij}\int_{t-\tau_j(t)}^{t} e^{\delta(s+h)}x^{\mathsf{T}}(s)\,Q_{2j}x(s)\,ds\\
&\le e^{\delta t}x^{\mathsf{T}}(t)\left(e^{\delta h}Q_{1i}+e^{\delta h}Q_{2i}+hQ\right)x(t)\\
&\quad-e^{\delta t}x^{\mathsf{T}}(t-h)\,Q_{1i}x(t-h)\\
&\quad-\left(1-\dot{\tau}_i(t)\right)e^{\delta t}x^{\mathsf{T}}\left(t-\tau_i(t)\right)Q_{2i}x\left(t-\tau_i(t)\right)\\
&\quad+\int_{t-h}^{t} e^{\delta s}x^{\mathsf{T}}(s)\left(e^{\delta h}\sum_{j=1}^{N}\pi_{ij}Q_{1j}+e^{\delta h}\right.\\
&\qquad\left.\times\sum_{j=1,j\neq i}^{N}\pi_{ij}Q_{2j}-Q\right)\\
&\qquad\times x(s)\,ds.
\end{aligned}
\tag{27}
$$

It follows from (15) and (28) that

$$
\begin{aligned}
\pounds V_2\left(x_t,i,t\right)&=e^{\delta t}x^{\mathsf{T}}(t)\left(e^{\delta h}Q_{1i}+e^{\delta h}Q_{2i}+hQ\right)x(t)\\
&\quad-e^{\delta t}x^{\mathsf{T}}(t-h)\,Q_{1i}x(t-h)\\
&\quad-\left(1-\dot{\tau}_i(t)\right)e^{\delta t}x^{\mathsf{T}}\left(t-\tau_i(t)\right)Q_{2i}x\left(t-\tau_i(t)\right),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\pounds V_3\left(x_t,i,t\right)&=\int_{-h}^{0}\int_{t+\theta}^{t} e^{\delta(s-\theta)}\eta^{\mathsf{T}}(s)\left(\sum_{j=1}^{N}\pi_{ij}\mathcal{X}_{ij}-\mathcal{X}\right)\\
&\qquad\times\eta(s)\,ds\,d\theta\\
&\quad+e^{\delta t}\eta^{\mathsf{T}}\mathcal{X}_i\eta(t)\int_{-h}^{0}e^{-\delta v}dv\\
&\quad-e^{\delta t}\int_{t-h}^{t}\eta^{\mathsf{T}}(s)\,\mathcal{X}_i\eta(s)\,ds\\
&\quad+e^{\delta t}\eta^{\mathsf{T}}(t)\,\mathcal{X}\eta(t)\int_{-h}^{0}\int_{v}^{0}e^{-\delta v}d\theta\,dv.
\end{aligned}
\tag{29}
$$

By employing Lemma 4, we can obtain that

$$
\begin{aligned}
&-\int_{t-h}^{t}\eta^{\mathsf{T}}(s)\,\mathcal{X}_i\eta(s)\,ds\\
&=-\int_{t-\tau_i(t)}^{t}\eta^{\mathsf{T}}(s)\,\mathcal{X}_i\eta(s)\,ds-\int_{t-h}^{t-\tau_i(t)}\eta^{\mathsf{T}}(s)\,\mathcal{X}_i\eta(s)\,ds\\
&\le-\frac{h}{\tau_i(t)}\left[\int_{t-\tau_i(t)}^{t}\eta(s)\,ds\right]^{\mathsf{T}}\frac{\mathcal{X}_i}{h}\left[\int_{t-\tau_i(t)}^{t}\eta(s)\,ds\right]\\
&\quad-\frac{h}{h-\tau_i(t)}\left[\int_{t-h}^{t-\tau_i(t)}\eta(s)\,ds\right]^{\mathsf{T}}\frac{\mathcal{X}_i}{h}\left[\int_{t-h}^{t-\tau_i(t)}\eta(s)\,ds\right]\\
&\le-\begin{bmatrix}\displaystyle\int_{t-\tau_i(t)}^{t}\eta(s)\,ds\\ \displaystyle\int_{t-h}^{t-\tau_i(t)}\eta(s)\,ds\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\dfrac{\mathcal{X}_i}{h} & \mathcal{S}_i\\ * & \dfrac{\mathcal{X}_i}{h}\end{bmatrix}\begin{bmatrix}\displaystyle\int_{t-\tau_i(t)}^{t}\eta(s)\,ds\\ \displaystyle\int_{t-h}^{t-\tau_i(t)}\eta(s)\,ds\end{bmatrix}.
\end{aligned}
\tag{30}
$$

It follows from (30) and (31) that

$$
\begin{aligned}
V_3\left(x_t,i,t\right)&\le e^{\delta t}\eta^{\mathsf{T}}(t)\\
&\quad\times\left(\frac{e^{\delta h}-1}{\delta}\mathcal{X}_i+\frac{e^{\delta h}-\delta he^{\delta h}-1}{\delta^2}\mathcal{X}\right)\eta(t)\\
&\quad-e^{\delta t}\begin{bmatrix}\displaystyle\int_{t-\tau_i(t)}^{t}\eta(s)\,ds\\ \displaystyle\int_{t-h}^{t-\tau_i(t)}\eta(s)\,ds\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\dfrac{\mathcal{X}_i}{h} & \mathcal{S}_i\\ * & \dfrac{\mathcal{X}_i}{h}\end{bmatrix}\\
&\quad\times\begin{bmatrix}\displaystyle\int_{t-\tau_i(t)}^{t}\eta(s)\,ds\\ \displaystyle\int_{t-h}^{t-\tau_i(t)}\eta(s)\,ds\end{bmatrix}.
\end{aligned}
\tag{31}
$$

Consider

$$
\begin{aligned}
\pounds V_4\left(x_t,i,t\right)&=hx^{\mathsf{T}}(t)\,Y_1x(t)\\
&\quad-\int_{t-h}^{t}x^{\mathsf{T}}(s)\,Y_1x(s)\,ds+h\dot{x}^{\mathsf{T}}(t)\,Y_2\dot{x}(t)\\
&\quad-\int_{t-h}^{t}\dot{x}^{\mathsf{T}}(s)\,Y_2\dot{x}(s)\,ds.
\end{aligned}
\tag{32}
$$

Moreover, the following two zero equalities with any symmetric matrices W_1 and W_2 are considered:

$$
\begin{aligned}
0&=x^{\mathsf{T}}(t)\,W_1x(t)-x^{\mathsf{T}}\left(t-\tau_i(t)\right)W_1x\left(t-\tau_i(t)\right)\\
&\quad-2\int_{t-\tau_i(t)}^{t}x^{\mathsf{T}}(s)\,W_1\dot{x}(s)\,ds,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
0&=x^{\mathsf{T}}\left(t-\tau_i(t)\right)W_2x\left(t-\tau_i(t)\right)\\
&\quad-x^{\mathsf{T}}(t-h)\,W_2x(t-h)\\
&\quad-2\int_{t-h}^{t-\tau_i(t)}x^{\mathsf{T}}(s)\,W_2\dot{x}(s)\,ds.
\end{aligned}
\tag{34}
$$

With the above two zero equalities (34) and (35), an upper bound of $\pounds V_4(x_t,i,t)$ is

$$\begin{aligned}\pounds_4(x_t,i,t) &= x^{\mathsf T}(t)(hY_1+W_1)x(t)+h\dot x^{\mathsf T}(t)Y_2\dot x(t)\\ &\quad+x^{\mathsf T}(t-\tau_i(t))(W_2-W_1)x(t-\tau_i(t))\\ &\quad-x^{\mathsf T}(t-h)W_2x(t-h)\\ &\quad-\int_{t-\tau_i(t)}^{t}\begin{bmatrix}x(s)\\ \dot x(s)\end{bmatrix}^{\mathsf T}\begin{bmatrix}Y_1 & W_1\\ * & Y_2\end{bmatrix}\begin{bmatrix}x(s)\\ \dot x(s)\end{bmatrix}\\ &\quad-\int_{t-h}^{t-\tau_i(t)}\begin{bmatrix}x(s)\\ \dot x(s)\end{bmatrix}^{\mathsf T}\begin{bmatrix}Y_1 & W_1\\ * & Y_2\end{bmatrix}\begin{bmatrix}x(s)\\ \dot x(s)\end{bmatrix}.\end{aligned}\tag{35}$$

From (19), (20), and (36), one can obtain

$$\begin{aligned}\pounds_4(x_t,i,t) &= x^{\mathsf T}(t)(hY_1+W_1)x(t)+h\dot x^{\mathsf T}(t)Y_2\dot x(t)\\ &\quad+x^{\mathsf T}(t-\tau_i(t))(W_2-W_1)x(t-\tau_i(t))\\ &\quad-x^{\mathsf T}(t-h)W_2x(t-h).\end{aligned}\tag{36}$$

Now, $\pounds V_5(x_t,i,t)$ is obtained as follows:

$$\begin{aligned}&\pounds V_5(x_t,i,t)\\ &\quad=\int_{-h}^{0}\int_{\theta}^{0}\int_{t+\upsilon}^{t}e^{\delta(s-\theta)}\dot x^{\mathsf T}(s)\left(\sum_{j=1}^{N}\pi_{ij}Z_j-Z\right)\\ &\qquad\times\dot x(s)\,ds\,d\upsilon\,d\theta\\ &\quad+e^{\delta t}\dot x^{\mathsf T}(t)Z_i\dot x(t)\int_{-h}^{0}\int_{\theta}^{0}e^{-\delta\upsilon}d\upsilon\,d\theta\\ &\quad-e^{\delta t}\int_{-h}^{0}\int_{t+\theta}^{t}\dot x^{\mathsf T}(s)Z_i\dot x(s)\,ds\,d\theta\\ &\quad+e^{\delta t}\dot x^{\mathsf T}(t)Z\dot x(t)\int_{-h}^{0}\int_{\varsigma}^{0}\int_{\theta}^{0}e^{-\delta\theta}d\upsilon\,d\theta\,d\varsigma.\end{aligned}\tag{37}$$

By using Lemma 5, one has

$$\begin{aligned}&-\int_{-h}^{0}\int_{t+\theta}^{t}\dot x^{\mathsf T}(s)Z_i\dot x(s)\,ds\,d\theta\\ &\quad\le-\frac{2}{h^2}\int_{-h}^{0}\int_{t+\theta}^{t}\dot x^{\mathsf T}(s)\,ds\,d\theta Z_i\int_{-h}^{0}\int_{t+\theta}^{t}\dot x(s)\,ds\,d\theta\\ &\quad=-2\left[x(t)-\frac{1}{h}\int_{t-\tau_i(t)}^{t}x(s)\,ds-\frac{1}{h}\int_{t-h}^{t-\tau_i(t)}x(s)\,ds\right]^{\mathsf T}Z_i\\ &\qquad\times\left[x(t)-\frac{1}{h}\int_{t-\tau_i(t)}^{t}x(s)\,ds-\frac{1}{h}\int_{t-h}^{t-\tau_i(t)}x(s)\,ds\right].\end{aligned}\tag{38}$$

Together with (38) and (39), it implies that

$$\begin{aligned}\pounds V_5(x_t,i,t) &\le e^{\delta t}\dot x^{\mathsf T}(t)\frac{e^{\delta h}-\delta h-1}{\delta^2}Z_i\dot x(t)\\ &\quad+e^{\delta t}\dot x^{\mathsf T}(t)\frac{h\delta^2e^{\delta h}+e^{\delta h}+\delta h+1}{\delta^3}Z\dot x(t)\\ &\quad-2e^{\delta t}\left[x(t)-\frac{1}{h}\int_{t-\tau_i(t)}^{t}x(s)\,ds-\frac{1}{h}\int_{t-h}^{t-\tau_i(t)}x(s)\,ds\right]^{\mathsf T}Z_i\\ &\quad\times\left[x(t)-\frac{1}{h}\int_{t-\tau_i(t)}^{t}x(s)\,ds-\frac{1}{h}\int_{t-h}^{t-\tau_i(t)}x(s)\,ds\right].\end{aligned}\tag{39}$$

From (26)–(40), we can eventually obtain

$$\pounds V(x_t,r_t,t)-\delta\omega^{\mathsf T}(t)H\omega(t)\le e^{\delta t}\xi^{\mathsf T}(t)\Xi_i\xi(t),\tag{40}$$

where

$$\begin{aligned}\xi^{\mathsf T}(t)=\Bigg[&x^{\mathsf T}(t),x^{\mathsf T}(t-\tau_i(t)),x^{\mathsf T}(t-h),\int_{t-\tau_i(t)}^{t}x^{\mathsf T}(s)\,ds,\\ &\int_{t-h}^{t-\tau_i(t)}x^{\mathsf T}(s)\,ds,\omega^{\mathsf T}(t)\Bigg].\end{aligned}\tag{41}$$

It follows from (45) that

$$\mathbb{E}\{\pounds V(x_t,r_t,t)\}\le\mathbb{E}[\eta V(x_t,r_t,t)]+\delta\omega^{\mathsf T}(t)H\omega(t).\tag{42}$$

Multiplying the above inequality by $e^{-\eta t}$ yields that

$$\mathbb{E}\left\{\pounds\left[e^{-\eta t}V(x_t,r_t,t)\right]\right\}\le e^{-\eta t}\delta\omega^{\mathsf T}(t)H\omega(t).\tag{43}$$

Integrating the inequality from 0 to t, we have

$$\begin{aligned}&e^{-\eta t}\mathbb{E}[V(x_t,r_t,t)]-\mathbb{E}[V(x_0,r_0,0)]\\ &\quad\le\delta\int_0^t e^{-\eta s}\omega^{\mathsf T}(s)H\omega(s)\,ds.\end{aligned}\tag{44}$$

Denoting $\widetilde P_i=R^{-(1/2)}P_iR^{-(1/2)}$, $\widetilde Q_i=R^{-(1/2)}Q_iR^{-(1/2)}$, $\widetilde Q=R^{-(1/2)}QR^{-(1/2)}$, $\widetilde{\mathscr{X}}_i=R^{-(1/2)}\mathscr{X}_iR^{-(1/2)}$, $\widetilde{\mathscr{X}}=R^{-(1/2)}\mathscr{X}R^{-(1/2)}$, $\widetilde Y_i=R^{-(1/2)}Y_iR^{-(1/2)}$, $\widetilde Y=R^{-(1/2)}YR^{-(1/2)}$, $\widetilde Z_i=R^{-(1/2)}Z_iR^{-(1/2)}$, and $\widetilde Z=R^{-(1/2)}ZR^{-(1/2)}$ yields that

$$\begin{aligned}&\mathbb{E}[V(x_0,r_0,0)]\\ &\quad\le\max_{i\in\mathscr{N}}\lambda_{\max}(\widetilde P_i)x^{\mathsf T}(0)Rx(0)\\ &\qquad+\left(\max_{i\in\mathscr{N}}\lambda_{\max}(Q_{1i})+\max_{i\in\mathscr{N}}\lambda_{\max}(Q_{2i})\right)e^{\delta h}\end{aligned}$$

$$
\begin{aligned}
&\times \int_{-h}^{0} e^{\delta s} x^{\mathsf{T}}(s) R x(s)\, ds \\
&+ e^{\delta h} \lambda_{\max}(Q) \int_{-h}^{0} \int_{\theta}^{0} e^{\delta s} x^{\mathsf{T}}(s) R x(s)\, ds \\
&+ e^{\delta h} \max_{i\in\mathcal{N}} \lambda_{\max}(\mathcal{X}_i) \int_{-h}^{0} \int_{\theta}^{0} e^{-\delta\theta} \eta^{\mathsf{T}}(s) R \eta(s)\, ds\, d\theta \\
&+ e^{\delta h} \lambda_{\max}(\mathcal{X}) \int_{-h}^{0} \int_{\theta}^{0} \int_{v}^{0} e^{-\delta\theta} \eta^{\mathsf{T}}(s) R \eta(s)\, ds\, d\theta\, dv \\
&+ \left(\lambda_{\max}(Y_1) + \lambda_{\max}(Y_2)\right) e^{\delta h} \\
&\times \int_{-h}^{0} \int_{\theta}^{0} e^{-\delta\theta} x^{\mathsf{T}}(s) R x(s)\, ds\, d\theta \\
&+ e^{\delta h} \max_{i\in\mathcal{N}} \lambda_{\max}(Z_i) \int_{-h}^{0} \int_{\theta}^{0} \int_{v}^{0} e^{-\delta\theta} x^{\mathsf{T}}(s) R x(s)\, ds\, d\theta \\
&+ e^{\delta h} \lambda_{\max}(Z) \int_{-h}^{0} \int_{\varsigma}^{0} \int_{\theta}^{0} \int_{v}^{0} e^{-\delta\theta} x^{\mathsf{T}}(s) R x(s)\, ds\, d\theta\, d\varsigma \\
\le\ & \Bigg\{ \max_{i\in\mathcal{N}} \lambda_{\max}\left(\widetilde{P}_i\right) + h e^{\delta h} \\
&\times \Bigg(\max_{i\in\mathcal{N}} \lambda_{\max}\left(Q_{1i} + \max_{i\in\mathcal{N}} \lambda_{\max}(Q_{2i})\right) \\
&+ h^2 e^{\delta h} \lambda_{\max}(Z) + h^2 e^{\delta h} \max_{i\in\mathcal{N}} \lambda_{\max}\left(\widetilde{\mathcal{X}}_i\right) \\
&+ \frac{1}{2} h^3 e^{\delta h} \lambda_{\max}\left(\widetilde{\mathcal{X}}\right) + h^2 e^{\delta h} \left(\lambda_{\max}(Y_1) + \lambda_{\max}(Y_2)\right) \\
&+ \frac{1}{2} h^3 e^{\delta h} \max_{i\in\mathcal{N}} \lambda_{\max}(Z_i) + \frac{1}{6} h^4 e^{\delta h} \lambda_{\max}(Z) \Bigg\} \\
&\times \sup_{-h\le s\le 0} \left\{ x^{\mathsf{T}}(s) R x(s), \dot{x}^{\mathsf{T}}(s) R \dot{x}(s) \right\} = c_1 \Lambda.
\end{aligned}
\tag{45}
$$

For scalars $\eta > 0$ and $T \ge t \ge 0$, (46) turns out to be

$$
\begin{aligned}
\mathbb{E}\left[V(x_t, r_t, t)\right] &\le \mathbb{E}\left[e^{\eta t} V(x_0, r_0, 0)\right] \\
&\quad + e^{\eta t} \delta \int_0^t e^{-\eta s} \omega^{\mathsf{T}}(s) H \omega(s)\, ds \\
&\le e^{\eta T} c_1 \Lambda + d\delta e^{\eta T} \lambda_{\max}(H) \int_0^T e^{-\eta s} ds \\
&\le e^{\eta T} \left\{ c_1 \Lambda + d\delta \lambda_{12} \frac{1}{\eta} \left(1 - e^{-\eta T}\right) \right\}.
\end{aligned}
\tag{46}
$$

To illustrate the bounded, (26) takes the following form:

$$
\begin{aligned}
&\mathbb{E}\left[V(x_t, r_t, t)\right] \ge \mathbb{E}\left[x^{\mathsf{T}}(t) e^{\lambda t} P_i x(t)\right] \\
&\ge \max_{i\in\mathcal{N}} \lambda_{\min}(P_i)\, \mathbb{E}\left[x^{\mathsf{T}}(t) R x(t)\right] = \lambda_1 \mathbb{E}\left[x^{\mathsf{T}}(t) R x(t)\right].
\end{aligned}
\tag{47}
$$

From inequalities (46)–(48), one has

$$
\mathbb{E}\left[x^{\mathsf{T}}(t) R x(t)\right] \le \frac{e^{\eta T}}{\lambda_1} \left\{ c_1 \Lambda + d\delta \lambda_{12} \frac{1}{\eta} \left(1 - e^{-\eta T}\right) \right\}. \tag{48}
$$

Finally, inequalities (24) and (49) guarantee that

$$
\mathbb{E}\left[x^{\mathsf{T}}(t) R x(t)\right] < c_2. \tag{49}
$$

Therefore, the Markovian jump system (1) is finite-time stochastic bounded with respect to (c_1, c_2, d, R, T). □

Remark 7. It should be noted that $\tau_i(t)$ and $\dot{\tau}_i(t)$ may, respectively, get the different upper bound due to the fact that condition (6) holds. However, $\tau_i(t)$ and $\dot{\tau}_i(t)$ always lead to conservativeness for $\tau_i(t) \le h = \max\{h_i, i \in \mathcal{N}\}$ and $\dot{\tau}_i(t) \le \mu = \max\{\mu_i, i \in \mathcal{N}\}$ in [14–18], and this case can be improved with employing the different Lyapunov-Krasovskii functional (26).

Remark 8. It should be pointed out that, in Theorem 6, the novelty of the Lyapunov functional (26) lies in the following: (i) triple-integral terms $V_3(x_t, r_t, t)$ and $V_5(x_t, r_t, t)$ and four-integral term $V_5(x_t, r_t, t)$ are introduced and (ii) the distinct Lyapunov matrices $(P_i, Q_{1i}, Q_{2i}, \mathcal{X}_i, Z_i)$ are chosen for different system modes i $(i = 1, 2, \ldots, N)$.

For the condition $r_t = i$, the Markovian jump system given in this paper is followed by

$$
\begin{aligned}
\dot{x}(t) &= \overline{A}_i x(t) + A_{\tau i} x\left(t - \tau_{r_t}(t)\right) + D_i \omega(t), \\
z(t) &= C_i x(t) + C_{\tau i} x\left(t - \tau_i(t)\right) + F_i \omega(t),
\end{aligned}
\tag{50}
$$

where

$$
\overline{A}_i = A_i + B_i K_i. \tag{51}
$$

Theorem 9. *System* (53) *is finite-time stochastic bounded with respect to* (c_1, c_2, d, R, T) *with a disturbance attenuation, if there exist matrices*

$$
\begin{gathered}
P_i > 0, \quad Q_{li} > 0 \quad (l = 1, 2), \quad Q > 0, \\
\mathcal{X}_i = \begin{bmatrix} X_{1i} & X_{2i} \\ X_{3i} & X_{4i} \end{bmatrix} > 0, \quad \mathcal{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} > 0, \\
Y_s > 0 \quad (s = 1, 2), \quad Z_i > 0, \quad Z > 0,
\end{gathered}
\tag{52}
$$

scalars $c_1 < c_2$, $T > 0$, $\lambda_s > 0$, $(s = 1, 2, \ldots, 12)$, $\eta > 0$ *and* $\Lambda > 0$, *such that for all* $i, j \in \mathcal{N}$, (15)–(20) *and the following inequalities hold:*

$$\Sigma_i = \begin{bmatrix} \Sigma_{11i} & \Xi_{12i} & S_{4i} & \Xi_{14i} & \Xi_{15i} & -S_{3i} + \dfrac{Z_i + Z_i^{\mathsf{T}}}{h} & P_i D_i & C_i^{\mathsf{T}} \\ * & \Xi_{22i} & -S_{4i} + \dfrac{X_{4i}}{h} & 0 & \dfrac{X_{3i}}{2h} + \dfrac{X_{2i}^{\mathsf{T}}}{2h} & S_{3i} - S_{2i}^{\mathsf{T}} - \dfrac{X_{3i}}{2h} - \dfrac{X_{2i}^{\mathsf{T}}}{2h} & 0 & C_{\tau i}^{\mathsf{T}} \\ * & * & -Q_{1i} - \dfrac{X_{4i}}{h} - W_2 & 0 & 0 & S_{2i}^{\mathsf{T}} + \dfrac{X_{3i}}{2h} + \dfrac{X_{2i}^{\mathsf{T}}}{2h} & 0 & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55i} & -S_{1i} - \dfrac{Z_i}{h^2} - \dfrac{Z_i^{\mathsf{T}}}{h^2} & 0 & 0 \\ * & * & * & * & * & -\dfrac{X_{1i}}{h} - \dfrac{Z_i}{h^2} - \dfrac{Z_i^{\mathsf{T}}}{h^2} & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & F_i^{\mathsf{T}} \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{53}$$

$$c_1 \Lambda + d\gamma^2 \frac{1}{\eta}\left(1 - e^{-\eta T}\right) < \lambda_1 e^{-\eta T} c_2, \tag{54}$$

where

$$\begin{aligned} \Sigma_{11i} = {} & \sum_{j=1}^{N} \pi_{ij} P_j + \delta P_i + P_i \overline{A}_i + \overline{A}_i^{\mathsf{T}} P_i \\ & + e^{\delta h}\left(Q_{1i} + Q_{2i}\right) + hQ + \frac{e^{\delta h - 1}}{\delta} X_{1i} \\ & + \frac{e^{\delta h} - \delta h e^{\delta h} - 1}{\delta^2} X_1 - \frac{X_{4i}}{h} + hY_1 + W_1 - Z_i - Z_i^{\mathsf{T}}. \end{aligned} \tag{55}$$

Proof. Considering the Lyapunov-Krasovskii functional in Theorem 6 and from Schur's Lemma, it turns out to be

$$\begin{aligned} & \pounds V\left(x_t, i, t\right) + z^{\mathsf{T}}(t) z(t) - \gamma^2 \omega^{\mathsf{T}}(t) \omega(t) \\ & \quad \le \xi^{\mathsf{T}}(t) \Theta_i\left(\mu_{pi}, h_{qi}\right) \xi(t). \end{aligned} \tag{56}$$

Thanks to (54), we have

$$\begin{aligned} \mathbb{E}\left\{\pounds V\left(x_t, i, t\right)\right\} \le {} & \mathbb{E}\left[\eta V\left(x_t, i, t\right)\right] \\ & + \gamma^2 \omega^{\mathsf{T}}(t) \omega(t) - \mathbb{E}\left[z^{\mathsf{T}}(t) z(t)\right]. \end{aligned} \tag{57}$$

Multiplying the (58) by $e^{-\eta t}$, (58) can be written as

$$\begin{aligned} & \mathbb{E}\left\{\pounds\left[e^{-\eta t} V\left(x_t, i, t\right)\right]\right\} \\ & \quad \le e^{-\eta t}\left[\gamma^2 \omega^{\mathsf{T}}(t) \omega(t) - z^{\mathsf{T}}(t) z(t)\right]. \end{aligned} \tag{58}$$

Under the condition of zero initial and $\mathbb{E}[V(x_t, i, t)] > 0$, one has

$$\begin{aligned} & \int_0^T e^{-\eta t}\left[\gamma^2 \omega^{\mathsf{T}}(t) \omega(t) - z^{\mathsf{T}}(t) z(t)\right] dt \\ & \quad \le \mathbb{E}\left\{\int_0^T \pounds\left[e^{-\eta t} V\left(x_t, i, t\right)\right] dt\right\} \le V\left(x_0, r_0, 0\right) = 0. \end{aligned} \tag{59}$$

Using the Dynkin formula, it results that

$$\mathbb{E}\left[\int_0^T e^{-\eta \upsilon} z^{\mathsf{T}}(\upsilon) z(\upsilon) d\upsilon\right] \le \gamma^2 \mathbb{E}\left[\int_0^T e^{-\eta \upsilon} \omega^{\mathsf{T}}(\upsilon) \omega(\upsilon) d\upsilon\right]. \tag{60}$$

Finally, it is easy to obtains that

$$\mathbb{E}\left[\int_0^T z^{\mathsf{T}}(\upsilon) z(\upsilon) d\upsilon\right] \le \gamma^2 e^{\eta T} \mathbb{E}\left[\int_0^T \omega^{\mathsf{T}}(\upsilon) \omega(\upsilon) d\upsilon\right]. \tag{61}$$

Therefore, the Markovian jump system (53) is finite-time stochastic bounded with an performance γ. □

4. Finite-Time H_∞ Control

Theorem 10. *System* (53) *is finite-time stochastic bounded with respects to* (c_1, c_2, d, R, T) *with an disturbance attenuation, if there exists matrices*

$$\begin{gathered} P_i > 0, \quad \overline{P}_i, Q_{li} > 0 \quad (l = 1, 2), \quad \overline{K}_i, Q > 0, \\ \mathcal{X}_i = \begin{bmatrix} X_{1i} & X_{2i} \\ X_{3i} & X_{4i} \end{bmatrix} > 0, \quad \mathcal{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} > 0, \\ Y_s > 0 \quad (s = 1, 2), \quad Z_i > 0, \quad Z > 0, \end{gathered} \tag{62}$$

scalars $c_1 < c_2$, $T > 0$, $\lambda_s > 0$, $(s = 1, 2, \ldots, 12)$, $\eta > 0$ *and* $\Lambda > 0$, *such that for all* $i, j \in \mathcal{N}$, (15)–(20) *and the following inequalities hold:*

$$\begin{bmatrix} \Theta_{11i} & \Xi_{12i} & S_{4i} & \Xi_{14i} & \Xi_{15i} & -S_{3i}+\dfrac{Z_i+Z_i^{\mathsf{T}}}{h} & P_iD_i & C_i^{\mathsf{T}} \\ * & \Xi_{22i} & -S_{4i}+\dfrac{X_{4i}}{h} & 0 & \dfrac{X_{3i}}{2h}+\dfrac{X_{2i}^{\mathsf{T}}}{2h} & S_{3i}-S_{2i}^{\mathsf{T}}-\dfrac{X_{3i}}{2h}-\dfrac{X_{2i}^{\mathsf{T}}}{2h} & 0 & C_{\tau i}^{\mathsf{T}} \\ * & * & -Q_{1i}-\dfrac{X_{4i}}{h}-W_2 & 0 & 0 & S_{2i}^{\mathsf{T}}+\dfrac{X_{3i}}{2h}+\dfrac{X_{2i}^{\mathsf{T}}}{2h} & 0 & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55i} & -S_{1i}-\dfrac{Z_i}{h^2}-\dfrac{Z_i^{\mathsf{T}}}{h^2} & 0 & 0 \\ * & * & * & * & * & -\dfrac{X_{1i}}{h}-\dfrac{Z_i}{h^2}-\dfrac{Z_i^{\mathsf{T}}}{h^2} & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{63}$$

$$P_iB_i = B_i\overline{P}_i \tag{64}$$

$$c_1\Lambda + d\gamma^2\frac{1}{\eta}\left(1-e^{-\eta T}\right) < \lambda_1 e^{-\eta T}c_2, \tag{65}$$

where

$$\begin{aligned}\Theta_{11i} = {} & \sum_{j=1}^{N}\pi_{ij}P_j + \delta P_i + P_iA_i + B_i\overline{K}_i \\ & + A_i^{\mathsf{T}}P_i + \overline{K}_i^{\mathsf{T}}B_i^{\mathsf{T}} + e^{\delta h}\left(Q_{1i}+Q_{2i}\right) \\ & + hQ + \frac{e^{\delta h-1}}{\delta}X_{1i} + \frac{e^{\delta h}-\delta h e^{\delta h}-1}{\delta^2}X_1 \\ & - \frac{X_{4i}}{h} + hY_1 + W_1 - Z_i - Z_i^{\mathsf{T}},\end{aligned} \tag{66}$$

and the state feedback gain matrices considered in this paper could be designed as follows:

$$K_i = \overline{P}_i^{-1}\overline{K}_i, \quad \forall i = 1,2,\ldots,N. \tag{67}$$

Proof. This proof can be completed in view of Theorem 9 with $P_iB_i = B_i\overline{P}_i$ and $\overline{P}_iK_i = \overline{K}_i$. □

5. Illustrative Example

Example 1. Considering the following example with parameters

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.9 & 0.5 \\ -0.32 & -0.8 \end{bmatrix}, \qquad A_{\tau 1} = \begin{bmatrix} -0.5 & -0.3 \\ 0.3 & -0.2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -1.05 & 0.8 \\ -0.15 & -1.3 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0.6 & -0.4 \\ 0.35 & -0.41 \end{bmatrix}, \end{aligned} \tag{68}$$

the transition probabilities matrix is given as follows:

$$\Omega = \begin{bmatrix} -0.2 & 0.2 \\ 0.8 & -0.8 \end{bmatrix}. \tag{69}$$

Given the different upper bounds of h and δ, the results of the maximum upper bound of decay rates δ and maximum values of h for different time delays are obtained in Tables 1 and 2, respectively. This example indicates fully that the method proposed in the paper plays an important role in reducing conservatism. It can be also seen that our results in this paper show significant improvement over the results obtained in [11, 12]. This clearly shows that our results have less conservatism in the above two cases.

Example 2. Consider the Markovian jump system (1) where

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.8 & 1.5 \\ 2 & 3 \end{bmatrix}, \qquad A_{\tau 1} = \begin{bmatrix} -0.45 & 1 \\ -0.5 & 2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -1 & 0.2 \\ 0.5 & -0.1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad C_{\tau 1} = \begin{bmatrix} 0.03 & 0 \\ 0.01 & 0.02 \end{bmatrix}, \\ E_1 &= \begin{bmatrix} 0.02 & 0 \\ 0.01 & 0.01 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0.01 \\ 0.001 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -2 & 1.2 \\ 1 & 4 \end{bmatrix}, \qquad A_{\tau 2} = \begin{bmatrix} -1 & 1.2 \\ 0 & -0.5 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} -1 & 1 \\ 0.5 & -2 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.1 & 0.02 \\ 0 & 0.1 \end{bmatrix}, \qquad C_{\tau 2} = \begin{bmatrix} 0.02 & 0 \\ 0.1 & 0.02 \end{bmatrix}, \\ E_2 &= \begin{bmatrix} 0.04 & 0 \\ 0.1 & 0.01 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.04 \\ 0.01 \end{bmatrix}, \end{aligned} \tag{70}$$

and corresponding transition rate matrix is

$$\Omega = \begin{bmatrix} -1.2 & 1.2 \\ 1 & -1 \end{bmatrix}. \tag{71}$$

Table 1: Comparison of the upper bounds of the decay rate for different delays.

	$h = 0.2$	$h = 0.5$	$h = 0.8$	$h = 1$	$h = 1.2$
[12]	1.2683	1.0064	0.7962	0.6838	0.5900
[11]	1.3618	1.1769	0.9420	0.7694	0.6261
Theorem 6	1.3622	1.1771	0.9426	0.7696	0.6263

Table 2: Comparison of the allowable values of time delay h for different decay rates.

	$\delta = 0.6$	$\delta = 0.8$	$\delta = 1$	$\delta = 1.2$	$\delta = 1.4$
[12]	1.1768	0.7938	0.5081	0.2731	0.0657
[11]	1.2435	0.9626	0.7368	0.4651	0.1302
Theorem 6	1.2441	0.9630	0.7372	0.4655	0.1304

Assuming that $R = I, T = 2, c_1 = 1$, and $d = 0.01$, by suing LMI toolbox, Theorem 10 provides the following controller gains:

$$K_1 = \begin{bmatrix} -11.5351 & -8.1210 \\ 13.2230 & 10.5612 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -21.2123 & 15.5613 \\ -23.2318 & 16.4518 \end{bmatrix}. \tag{72}$$

6. Conclusions

We have presented the problems of finite-time stochastic H_∞ performance analysis of continuous-time systems with random abrupt changes in this paper. By using a different Lyapunov-Krasovskii functional, several sufficient conditions are provided to ensure the Markovian jump system is finite-time stochastic bounded. The controller gains can be dealt with by LMIs toolbox and optimization techniques. At last, two numerical examples are proposed to illustrate the effective and advantage of the developed theories.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Robust Finite-Time H_∞ Control for Nonlinear Markovian Jump Systems with Time Delay under Partially Known Transition Probabilities

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This paper is concerned with the problem of robust finite-time H_∞ control for a class of nonlinear Markovian jump systems with time delay under partially known transition probabilities. Firstly, for the nominal nonlinear Markovian jump systems, sufficient conditions are proposed to ensure finite-time boundedness, H_∞ finite-time boundedness, and finite-time H_∞ state feedback stabilization, respectively. Then, a robust finite-time H_∞ state feedback controller is designed, which, for all admissible uncertainties, guarantees the H_∞ finite-time boundedness of the corresponding closed-loop system. All the conditions are presented in terms of strict linear matrix inequalities. Finally a numerical example is provided to demonstrate the effectiveness of all the results.

1. Introduction

Markovian jump systems, a class of hybrid dynamical systems, which consists of an indexed family of continuous or discrete-time subsystems and a set of Markovian chain that orchestrates the switching between them at stochastic time instants, have received extensive attention over the past few decades [1, 2]. Many real world processes, such as economic systems [3], manufacturing systems [4], electric power systems [5], and communication systems [6], may be modeled as Markovian jump systems when any malfunction of sensors or actuators cause a jump behavior in process performance. Recently, nonlinear Markovian jump systems have been extensively applied and developed in various disciplines of science and engineering, and a great number of excellent works have been developed [7–9].

Generally speaking, the behavior of nonlinear Markovian jump systems is determined by the transition probabilities in the jumping process. Usually, it is assumed that the information on transition probabilities was completely known. However, transition probabilities may be partially known for some real systems. For example, the networked control systems can be modeled by nonlinear Markovian jump systems with partially known transition probabilities when the packet dropouts or channel delays occur [10]. In addition, there are few results about the known bounds of transition probability rates or the fixed connection weighting matrices [11, 12]. Therefore, it is reasonable to study Markovian jump systems with partially known transition probabilities, especially, when it is difficult to measure the bounds of transition probability rates. It stimulates the research interests of the author.

Uncertainties and time delay frequently occur in various engineering systems, which usually is a source of instability and often causes undesirable performance and even makes the system out of control [14, 15]. Therefore, time delay systems with robustness have received an increasing attention among the control community [16–18]. On the other hand, one may be interested in not only system stability but also a bound of system trajectories over a fixed short time [19]. For instance, for the problem of robot arm control [7], when the robot works under different environmental conditions with changing payloads, it requests that the angle position of the arm should not exceed some threshold in a prescribed time interval. Meanwhile, the scholars attach more importance to the H_∞ control problem, which is to find a stable controller such that the disturbance attenuation level γ is below a prescribed level. There are a great number of useful and interesting results about H_∞ control problem for linear and nonlinear Markovian jump systems in the literature [20–25]. To the best of our knowledge, the synthesis issue of

robust finite-time H_∞ control for nonlinear Markovian jump systems with time delay under partially known transition probabilities has not been fully investigated until now, which motivates us to carry out the present study.

In this paper, we investigate the problem of robust finite-time H_∞ control for nonlinear Markovian jump systems with time delay under partially known transition probabilities. The main contributions lie in the fact that some tractable sufficient conditions are provided to ensure H_∞ finite-time boundedness or finite-time H_∞ state feedback stabilization. A robust finite-time H_∞ state feedback controller is designed, which guarantees the H_∞ finite-time boundedness of the closed-loop system. Seeking computational convenience, all the conditions are cast in the format of linear matrix inequalities. Finally, a numerical example is provided to demonstrate the effectiveness of the main results.

Notations. Throughout this paper, the notations used are fairly standard. For real symmetric matrices A and B, the notation $A \geq B$ (resp., $A > B$) means that the matrix A-B is positive semi-definite (resp., positive definite). A^T represents the transpose matrix of A, and A^{-1} represents the inverse matrix of A. $\lambda_{\max}(B)$ ($\lambda_{\min}(B)$) is the maximum (resp., minimum) eigenvalue of a matrix B. $\operatorname{diag}\{A \quad B\}$ represents the block diagonal matrix of A and B. I is the unit matrix with appropriate dimensions, and the term of symmetry is stated by the asterisk $*$ in a matrix. $\mathbb{R}^n$ stands for the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and $\mathscr{M} = \{1, 2, \ldots, N\}$ means a set of positive numbers. $\| * \|$ denotes the Euclidean norm of vectors. $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation of the stochastic process or vector. $L_2^n[0, +\infty)$ is the space of n-dimensional square integrable function vector over $[0, +\infty)$.

2. Problem Formulation and Preliminaries

Give a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where Ω is the sample space, $\mathscr{F}$ is the algebra of events, and $\mathscr{P}$ is the probability measure defined on $\mathscr{F}$. The random process $\{r_t, t \geq 0\}$ is a Markovian stochastic process taking values in a finite set $\mathscr{M} = \{1, 2, \ldots, N\}$ with the transition probability rate matrix $\Pi = \{\pi_{ij}\}$, $i, j \in \mathscr{M}$, and the transition probability from mode i at time t to mode j at time $t + \Delta t$ is expressed as

$$P\{r_{t+\Delta t} = j \mid r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & i = j, \end{cases} \tag{1}$$

with the transition probability rates $\pi_{ij} \geq 0$, for $i, j \in \mathscr{M}$, $i \neq j$, and $\sum_{j=1, i\neq j}^{N} \pi_{ij} = -\pi_{ii}$, where $\Delta t > 0$, and $\lim_{\Delta t \to 0}(o(\Delta t)/\Delta t) = 0$.

Consider the following nonlinear Markovian jump system with time delay in the probability space $(\Omega, \mathscr{F}, \mathscr{P})$:

$$\begin{aligned} \dot{x}(t) = {} & (A(r_t) + \Delta A(r_t))\, x(t) \\ & + (A_d(r_t) + \Delta A_d(r_t))\, x(t-\tau) \\ & + (B(r_t) + \Delta B(r_t))\, u(t) + G(r_t)\, w(t) \\ & + f(r_t, x(t), x(t-\tau)), \\ z(t) = {} & C(r_t)\, x(t) + C_d(r_t)\, x(t-\tau) + D(r_t)\, u(t) \\ & + E(r_t)\, w(t), \\ x(t) = {} & \varphi(t), \quad t \in [-\tau \quad 0], \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in L_2^n[0, +\infty)$ is an arbitrary external disturbance, $z(t) \in \mathbb{R}^l$ is the control output, $\varphi(t)$ represents a vector-valued initial function, and $\tau \in \mathbb{R}^+$ is the constant delay. $f(\cdot,\cdot,\cdot)$: $\mathscr{M} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is an unknown nonlinear function. $A(r_t)$, $A_d(r_t)$, $B(r_t)$, $G(r_t)$, $C(r_t)$, $C_d(r_t)$, $D(r_t)$, and $E(r_t)$ are known mode-dependent constant matrices with appropriate dimensions. $\Delta A(r_t)$, $\Delta A_d(r_t)$, and $\Delta B(r_t)$ are unknown matrices, denoting the uncertainties in the system, and the uncertainties are time-varying but norm bounded uncertainties satisfying

$$\begin{aligned} \Delta A(r_t) &= M_1(r_t)\, F(t, r_t)\, N_1(r_t), \\ \Delta B(r_t) &= M_2(r_t)\, F(t, r_t)\, N_2(r_t), \\ \Delta A_d(r_t) &= M_3(r_t)\, F(t, r_t)\, N_3(r_t), \end{aligned} \tag{3}$$

where $M_1(r_t)$, $N_1(r_t)$, $M_2(r_t)$, $N_2(r_t)$, $M_3(r_t)$, and $N_3(r_t)$ are known mode-dependent matrices with appropriate dimensions and $F(t, r_t)$ is the time-varying unknown matrix function with Lebesgue norm measurable elements satisfying

$$F(t, r_t)^T F(t, r_t) \leq I. \tag{4}$$

Consider the following state feedback controller:

$$u(t) = K(r_t)\, x(t) + K_d(r_t)\, x(t-\tau), \tag{5}$$

where $K(r_t)$ and $K_d(r_t)$ are the state feedback gains to be designed. Then the closed-loop system is as follows:

$$\begin{aligned} \dot{x}(t) = {} & (A(r_t) + \Delta A(r_t) + B(r_t)\, K(r_t) \\ & + \Delta B(r_t)\, K(r_t))\, x(t) \\ & + (A_d(r_t) + \Delta A_d(r_t) + B(r_t)\, K_d(r_t) \\ & + \Delta B(r_t)\, K_d(r_t))\, x(t-\tau) \\ & + G(r_t)\, w(t) + f(r_t, x(t), x(t-\tau)), \\ z(t) = {} & (C(r_t) + D(r_t)\, K(r_t))\, x(t) \\ & + (C_d(r_t) + D(r_t)\, K_d(r_t))\, x(t-\tau) \\ & + E(r_t)\, w(t), \\ x(t) = {} & \varphi(t), \quad t \in [-\tau \quad 0]. \end{aligned} \tag{6}$$

For notational simplicity, when $r(t) = i, i \in \mathscr{M}$, $A(r_t)$, $A_d(r_t)$, $B(r_t)$, $G(r_t)$, $K(r_t)$, $K_d(r_t)$, $C(r_t)$, $C_d(r_t)$, $D(r_t)$, $E(r_t)$, $\Delta A(r_t)$, $\Delta B(r_t)$, $M_1(r_t)$, $N_1(r_t)$, $M_2(r_t)$, $N_2(r_t)$, $M_3(r_t)$, $N_3(r_t)$, and $f(r_t, x(t), x(t-\tau))$ are, respectively, denoted as A_i, A_{di}, B_i, G_i, K_i, K_{di}, C_i, C_{di}, D_i, E_i, ΔA_i, ΔB_i, M_{1i}, N_{1i}, M_{2i}, N_{2i}, M_{3i}, N_{3i}, and $f_i(x(t), x(t-\tau))$.

In addition, the transition probability rates are considered to be partially known; that is, some elements in matrix $\Pi = \{\pi_{ij}\}$ are unknown. For instance, for system (2) with four subsystems, the transition probability rate matrix Π may be as

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & ? & ? \\ ? & ? & \pi_{23} & \pi_{24} \\ \pi_{31} & ? & \pi_{33} & ? \\ \pi_{41} & ? & ? & ? \end{bmatrix}, \tag{7}$$

where "?" represents the unknown transition probability rate. $\forall i \in \mathscr{M}$, we denote $\mathscr{M} = L_k^i \cup L_{uk}^i$, and

$$\begin{aligned} L_k^i &\triangleq \left\{ j : \pi_{ij} \text{ is known, for } j \in \mathscr{M} \right\}, \\ L_{uk}^i &\triangleq \left\{ j : \pi_{ij} \text{ is unknown, for } j \in \mathscr{M} \right\}. \end{aligned} \tag{8}$$

Moreover, if $L_k^i \neq \mathbf{0}$, it is further described as

$$L_k^i = \left\{ k_1^i, k_2^i, \ldots k_m^i \right\}, \quad 1 \le m \le \mathscr{M}, \tag{9}$$

where $k_m^i \in \mathscr{M}$ represents the mth known transition probability rate of the set L_k^i in the ith row of the transition probability rate matrix Π.

Remark 1. When $L_{uk}^i = \mathbf{0}$, $L_k^i = \mathscr{M}$, it is reduced to the case where the transition probability rates of the Markovian jump process $\{r_t, t \ge 0\}$ are completely known. When $L_k^i = \mathbf{0}$, $L_{uk}^i = \mathscr{M}$, it means that the transition probability rates of the Markovian jump process $\{r_t, t \ge 0\}$ are completely unknown. Mixing the above two aspects, here, a general form is considered.

In this paper, the following assumptions, definitions, and lemmas play an important role in our later development.

Assumption 2. The external disturbance $w(t)$ is varying and satisfies the constraint condition:

$$\int_{t_0}^{T} w^T(s)\, w(s)\, ds \le d, \quad d \ge 0. \tag{10}$$

Assumption 3. $\forall i \in \mathscr{M}$, $f_i(0,0) = 0$, and $f_i(x(t), x(t-\tau))$ satisfies the following inequality

$$\begin{aligned} &\left\| f_i(x(t), x(t-\tau)) \right\|^2 \\ &\quad \le \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} F_{11}^i & F_{12}^i \\ * & F_{22}^i \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \end{aligned} \tag{11}$$

where

$$F_i := \begin{bmatrix} F_{11}^i & F_{12}^i \\ * & F_{22}^i \end{bmatrix} \ge 0. \tag{12}$$

Definition 4 (*finite-time stability*). For a given time constant $T > 0$, system (2) ($u(t) = 0$, $w(t) = 0$) is said to be finite-time stable with respect to (c_1, c_2, T, H_i), if

$$\mathbb{E}\left\{ x_0^T H_i x_0 \right\} \le c_1 \Longrightarrow \mathbb{E}\left\{ x(t)^T H_i x(t) \right\} \le c_2, \quad \forall t \in [0, T], \tag{13}$$

where $0 < c_1 < c_2$, $H_i > 0$.

Definition 5 (*finite-time boundedness*). For a given time constant $T > 0$, system (2) ($u(t) = 0$) is said to be finite-time bounded with respect to (c_1, c_2, T, H_i, d), if the condition (13) holds, where $0 < c_1 < c_2$, $H_i > 0$.

Definition 6 (H_∞ *finite-time boundedness*). For a given time constant $T > 0$, system (2) ($u(t) = 0$) is said to be H_∞ finite-time bounded with respect to (c_1, c_2, T, H_i, d), if there exists a positive constant γ, such that the following two conditions are true:

(1) system (2) is finite-time bounded with respect to (c_1, c_2, T, H_i, d);

(2) under zero initial condition ($x(t_0) = 0, t_0 = 0$), for any external disturbance $w(t) \neq 0$ satisfying condition (10), the control output $z(t)$ of system (2) satisfies

$$\mathbb{E}\left\{ \int_0^T z^T(t)\, z(t)\, dt \right\} \le \gamma^2 \int_0^T w^T(t)\, w(t)\, dt. \tag{14}$$

Definition 7 (*finite-time* H_∞ *state feedback stabilization*). The system (2) is said to be finite-time H_∞ state feedback stabilizable with respect to (c_1, c_2, T, H_i, d), if there exist a positive constant γ and a state feedback controller in the form of (5), such that the closed-loop system (6) is H_∞ finite-time bounded.

Definition 8 (see [26]). In the Euclidean space $\{\mathbb{R}^n \times \mathscr{M} \times \mathbb{R}^+\}$, introduce the stochastic Lyapunov function for system (2) as $V(x(t), i)$, and the weak infinitesimal operator satisfies

$$\begin{aligned} &\mathscr{L}V(x(t), i) \\ &\quad = \lim_{\Delta_t \to 0} \frac{1}{\Delta_t} \left[\mathbb{E}\left\{ V\left(x(t+\Delta_t)\, r(t+\Delta_t) \right) \right\} - V(x(t), i) \right] \\ &\quad = \frac{\partial}{\partial t} V(x(t), i) + \frac{\partial}{\partial x} V(x(t), i)\, \dot{x}(t) + \sum_{j=1}^{N} \pi_{ij} V(x(t), j). \end{aligned} \tag{15}$$

Remark 9. It easily follows from (12) that $F_{11}^i \ge 0$, $F_{22}^i \ge 0$. So F_{11}^i and F_{22}^i can be decomposed as

$$F_{11}^i = \left(F_{11}^i\right)^{1/2} \left(F_{11}^i\right)^{1/2}, \qquad F_{22}^i = \left(F_{22}^i\right)^{1/2} \left(F_{22}^i\right)^{1/2}. \tag{16}$$

Remark 10. It is noticed that finite-time stability can be regarded as a particular case of finite-time boundedness by setting $w(t) = 0$. That is, finite-time boundedness implies finite-time stability, but the converse is not true.

Lemma 11 (see [27]). *Let T, M, F, and N be real matrices of appropriate dimensions with $F^T F \le I$; then for a positive scalar $\varepsilon > 0$, there holds:*

$$T + MFN + N^T F^T M^T \le T + \varepsilon M M^T + \varepsilon^{-1} N^T N. \tag{17}$$

The aim in this paper is to find a tractable solution to the problem of finite-time H_∞ state feedback stabilization.

3. Main Results

3.1. Finite-Time Boundedness Analysis. In this subsection, we will consider the problem of finite-time boundedness for the nominal system of nonlinear Markovian jump system (2) with $F(t, r_t) = 0$ for all $t \geq 0$; that is,

$$\begin{aligned} \dot{x}(t) &= A(r_t)x(t) + A_d(r_t)x(t-\tau) + B(r_t)u(t) \\ &\quad + G(r_t)w(t) + f(r_t, x(t), x(t-\tau)), \\ z(t) &= C(r_t)x(t) + C_d(r_t)x(t-\tau) + D(r_t)u(t) \\ &\quad + E(r_t)w(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau \quad 0]. \end{aligned} \tag{18}$$

Under the controller (5), the closed-loop system is

$$\begin{aligned} \dot{x}(t) &= (A(r_t) + B(r_t)K(r_t))x(t) \\ &\quad + (A_d(r_t) + B(r_t)K_d(r_t))x(t-\tau) \\ &\quad + G(r_t)w(t) + f(r_t, x(t), x(t-\tau)), \\ z(t) &= (C(r_t) + D(r_t)K(r_t))x(t) \\ &\quad + (C_d(r_t) + D(r_t)K_d(r_t))x(t-\tau) \\ &\quad + E(r_t)w(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau \quad 0]. \end{aligned} \tag{19}$$

Theorem 12. *Given $T > 0$, if there exist positive constants α and ε_{fi}, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{q\times q}$ and $S \in \mathbb{R}^{p\times p}$, and symmetric matrices $W_i \in \mathbb{R}^{n\times n}$, such that for all $i \in \mathscr{M}$*

$$\begin{bmatrix} \Lambda_{1i} & P_iA_{di} + \varepsilon_{fi}F_{12}^i & P_iG_i \\ * & -Q + \varepsilon_{fi}F_{22}^i & 0 \\ * & * & -\alpha S \end{bmatrix} < 0, \tag{20}$$

$$P_j - W_i \leq 0, \quad j \in L_{uk}^i, \ j \neq i, \tag{21}$$

$$P_j - W_i \geq 0, \quad j \in L_{uk}^i, \ j = i, \tag{22}$$

$$\frac{c_1\left[\lambda_{\max}(\widetilde{P}_i) + \tau\lambda_{\max}(\widetilde{Q}_i)\right] + d\lambda_{\max}(S)\left(1 - e^{-\alpha T}\right)}{\lambda_{\min}(\widetilde{P}_i)} < e^{-\alpha T}c_2, \tag{23}$$

then system (18) *($u = 0$) under partially known transition probabilities is finite-time bounded with respect to (c_1, c_2, T, H_i, d), where*

$$\begin{aligned} \Lambda_{1i} &= A_i^TP_i + P_iA_i + Q \\ &\quad + \sum_{j\in L_k^i}\pi_{ij}(P_j - W_i) + \varepsilon_{fi}^{-1}P_iP_i + \varepsilon_{fi}F_{11}^i - \alpha P_i, \\ \widetilde{P}_i &= H_i^{-1/2}P_iH_i^{-1/2}, \qquad \widetilde{Q}_i = H_i^{-1/2}QH_i^{-1/2}. \end{aligned} \tag{24}$$

Proof. For system (18) ($u = 0$), choose a Lyapunov function candidate

$$\begin{aligned} V(x(t), i) &= V_1(x(t), i) + V_2(x(t), i) \\ &= x(t)^TP_ix(t) + \int_{t-\tau}^{t} x^T(\xi)Qx(\xi)\,d\xi, \end{aligned} \tag{25}$$

where $P_i > 0$. Then by Definition 8, we get

$$\begin{aligned} \mathscr{L}V_1(x(t), i) &= x^T(t)\left[A_i^TP_i + P_iA_i + \sum_{j=1}^{N}\pi_{ij}P_j\right]x(t) \\ &\quad + x^T(t)P_iA_{di}x(t-\tau) + x^T(t)P_iG_iw(t) \\ &\quad + x^T(t)P_if_i + x^T(t-\tau)A_{di}^TP_ix(t) \\ &\quad + w^T(t)G_i^TP_ix(t) + f_i^TP_ix(t). \end{aligned} \tag{26}$$

Based on Lemma 11, there exist scalars ε_{fi} such that

$$\begin{aligned} &x^T(t)P_if_i + f_i^TP_ix(t) \\ &\quad \leq \varepsilon_{fi}f_i^Tf_i + \varepsilon_{fi}^{-1}x^T(t)P_iP_ix(t) \\ &\quad \leq \varepsilon_{fi}\left[x^T(t)F_{11}^ix(t) + x^T(t)F_{12}^ix(t-\tau)\right. \\ &\qquad \left. + x^T(t-\tau)F_{21}^ix(t) + x^T(t-\tau)F_{22}^ix(t-\tau)\right] \\ &\qquad + \varepsilon_{fi}^{-1}x^T(t)P_iP_ix(t). \end{aligned} \tag{27}$$

Substituting (27) into (26) yields

$$\begin{aligned} &\mathscr{L}V_1(x(t), i) \\ &\quad \leq x^T(t)\left[A_i^TP_i + P_iA_i + \varepsilon_{fi}^{-1}P_iP_i + \sum_{j=1}^{N}\pi_{ij}P_j + \varepsilon_{fi}F_{11}^i\right]x(t) \\ &\qquad + x^T(t)P_iG_iw(t) + x^T(t)\left[P_iA_{di} + \varepsilon_{fi}F_{12}^i\right]x(t-\tau) \\ &\qquad + x^T(t-\tau)\left[A_{di}^TP_i + \varepsilon_{fi}F_{21}^i\right]x(t) + w^T(t)G_i^TP_ix(t) \\ &\qquad + x^T(t-\tau)\varepsilon_{fi}F_{22}^ix(t-\tau). \end{aligned} \tag{28}$$

It is easy to obtain that

$$\mathscr{L}V_2(x(t), i) = x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau). \tag{29}$$

From (28) and (29), the following holds:

$$\begin{aligned}
&\mathscr{L}V(x(t),i)\\
&= \mathscr{L}V_1(x(t),i) + \mathscr{L}V_2(x(t),i)\\
&\le x^T(t)\Bigg[A_i^T P_i + P_i A_i + \varepsilon_{fi}^{-1} P_i P_i\\
&\qquad + \sum_{j=1}^{N} \pi_{ij} P_j + \varepsilon_{fi} F_{11}^i + Q\Bigg] x(t)\\
&\quad + x^T(t) P_i G_i w(t) + x^T(t)\left[P_i A_{di} + \varepsilon_{fi} F_{12}^i\right] x(t-\tau)\\
&\quad + w^T(t) G_i^T P_i x(t) + x^T(t-\tau)\left[A_{di}^T P_i + \varepsilon_{fi} F_{21}^i\right] x(t)\\
&\quad + x^T(t-\tau)\left[\varepsilon_{fi} F_{22}^i - Q\right] x(t-\tau).
\end{aligned} \tag{30}$$

Due to the fact that $\sum_{j=1}^{N} \pi_{ij} W_i = 0$ for arbitrary symmetric matrices W_i, (30) can be written as

$$\begin{aligned}
&\mathscr{L}V(x(t),i)\\
&\le x^T(t)\Bigg[A_i^T P_i + P_i A_i + \varepsilon_{fi}^{-1} P_i P_i\\
&\qquad + \sum_{j=1}^{N} \pi_{ij}\left(P_j - W_i\right) + \varepsilon_{fi} F_{11}^i + Q\Bigg] x(t)\\
&\quad + x^T(t) P_i G_i w(t) + x^T(t)\left[P_i A_{di} + \varepsilon_{fi} F_{12}^i\right] x(t-\tau)\\
&\quad + w^T(t) G_i^T P_i x(t) + x^T(t-\tau)\left[A_{di}^T P_i + \varepsilon_{fi} F_{21}^i\right] x(t)\\
&\quad + x^T(t-\tau)\left[\varepsilon_{fi} F_{22}^i - Q\right] x(t-\tau)\\
&= x^T(t)\Bigg[A_i^T P_i + P_i A_i + \varepsilon_{fi}^{-1} P_i P_i\\
&\qquad + \sum_{j\in L_k^i} \pi_{ij}\left(P_j - W_i\right) + \varepsilon_{fi} F_{11}^i + Q\Bigg] x(t)\\
&\quad + x^T(t) P_i G_i w(t) + x^T(t)\left[P_i A_{di} + \varepsilon_{fi} F_{12}^i\right] x(t-\tau)\\
&\quad + w^T(t) G_i^T P_i x(t) + x^T(t-\tau)\left[A_{di}^T P_i + \varepsilon_{fi} F_{21}^i\right] x(t)\\
&\quad + x^T(t-\tau)\left[\varepsilon_{fi} F_{22}^i - Q\right] x(t-\tau)\\
&\quad + x^T(t) \sum_{j\in L_{uk}^i} \pi_{ij}\left(P_j - W_i\right) x(t).
\end{aligned} \tag{31}$$

Noticing that $\pi_{ij} \ge 0$ for all $i \ne j$ and $\pi_{ii} = -\sum_{j=1, i\ne j}^{N} \pi_{ij} < 0$ for all $i \in \mathscr{M}$, if $i \in L_k^i$ (the elements of the diagonal are known), by inequalities (20) and (21), the following inequalities hold:

$$\begin{aligned}
\mathscr{L}V(x(t),i) &< \alpha x(t)^T P_i x(t) + \alpha w^T(t) S w(t)\\
&< \alpha x(t)^T P_i x(t) + \alpha \int_{t-\tau}^{t} x^T(\xi) Q x(\xi)\, d\xi\\
&\quad + \alpha w^T(t) S w(t)\\
&= \alpha V(x(t),i) + \alpha w^T(t) S w(t).
\end{aligned} \tag{32}$$

If $i \in L_{uk}^i$ (the elements of the diagonal are unknown), according to the inequalities (20)–(22), inequality (32) holds. Multiplying (32) by $e^{-\alpha t}$ yields

$$\mathscr{L}\left(e^{-\alpha t} V(x(t),i)\right) < \alpha e^{-\alpha t} w^T(t) S w(t). \tag{33}$$

Applying Dynkin's formula for (33), we obtain

$$e^{-\alpha t} V(x(t),i) - V(x_0,t_0) < \alpha \int_0^t e^{-\alpha s} w^T(s) S w(s)\, ds, \tag{34}$$

which shows

$$\begin{aligned}
V(x(t),i) &< e^{\alpha t} V(x_0,t_0) + \alpha e^{\alpha t} \int_0^t e^{-\alpha s} w^T(s) S w(s)\, ds\\
&< e^{\alpha t} V(x_0,t_0) + \alpha d \lambda_{\max}(S) e^{\alpha t} \int_0^t e^{-\alpha s} ds\\
&= e^{\alpha t}\left[V(x_0,t_0) + \alpha d \lambda_{\max}(S) \frac{1-e^{-\alpha t}}{\alpha}\right].
\end{aligned} \tag{35}$$

This together with $\widetilde{P}_i = H_i^{-1/2} P_i H_i^{-1/2}$ and $\widetilde{Q}_i = H_i^{-1/2} Q H_i^{-1/2}$ gives rise to

$$\begin{aligned}
V(x(t),i) < e^{\alpha t}\Big[&c_1\left(\lambda_{\max}\left(\widetilde{P}_i\right) + \tau \lambda_{\max}\left(\widetilde{Q}_i\right)\right)\\
&+ d\lambda_{\max}(S)\left(1 - e^{-\alpha t}\right)\Big].
\end{aligned} \tag{36}$$

Considering that

$$V(x(t),i) \ge x^T(t) P_i x(t) \ge \lambda_{\min}\left(\widetilde{P}_i\right) x^T(t) H_i x(t) \tag{37}$$

and combining (36) and (37), it follows that

$$\begin{aligned}
&\mathbb{E}\left\{x^T(t) H_i x(t)\right\}\\
&< \frac{e^{\alpha t}\left[c_1\left(\lambda_{\max}\left(\widetilde{P}_i\right) + \tau \lambda_{\max}\left(\widetilde{Q}_i\right)\right) + d\lambda_{\max}(S)\left(1 - e^{-\alpha t}\right)\right]}{\lambda_{\min}\left(\widetilde{P}_i\right)}\\
&< c_2.
\end{aligned} \tag{38}$$

Condition (38) implies that, for $t \in [0 \quad T]$, $\mathbb{E}\{x^T(t) H_i x(t)\} < c_2$.

The proof is complete. □

Corollary 13. *Given $T > 0$, if there exist positive constants α, ε_{fi}, and γ, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and $Q \in \mathbb{R}^{q\times q}$, and symmetric matrices $W_i \in \mathbb{R}^{n\times n}$, such that for all $i \in \mathscr{M}$*

$$\begin{bmatrix} \Lambda_{1i} & P_iA_{di} + \varepsilon_{fi}F_{12}^i & P_iG_i \\ * & -Q + \varepsilon_{fi}F_{22}^i & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{39}$$

$$P_j - W_i \le 0, \quad j \in L_{uk}^i, \ j \ne i, \tag{40}$$

$$P_j - W_i \ge 0, \quad j \in L_{uk}^i, \ j = i, \tag{41}$$

$$\begin{aligned} &c_1\left[\lambda_{\max}\left(\widetilde{P}_i\right) + \tau\lambda_{\max}\left(\widetilde{Q}_i\right)\right] + \frac{\gamma^2 d}{\alpha}\left(1 - e^{-\alpha T}\right) \\ &\quad < \lambda_{\min}\left(\widetilde{P}_i\right)e^{-\alpha T}c_2, \end{aligned} \tag{42}$$

then system (18) ($u = 0$) *under partially known transition probabilities is finite-time bounded with respect to (c_1, c_2, T, H_i, d), where*

$$\begin{aligned} \Lambda_{1i} &= A_i^T P_i + P_i A_i + Q \\ &\quad + \sum_{j\in L_k^i} \pi_{ij}\left(P_j - W_i\right) + \varepsilon_{fi}^{-1}P_iP_i + \varepsilon_{fi}F_{11}^i - \alpha P_i, \end{aligned} \tag{43}$$

$$\widetilde{P}_i = H_i^{-1/2}P_iH_i^{-1/2}, \qquad \widetilde{Q}_i = H_i^{-1/2}QH_i^{-1/2}.$$

3.2. Finite-Time H_∞ Performance Analysis. In this subsection, based on Corollary 13, some sufficient conditions will be provided ensuring the H_∞ finite-time boundedness of system (18) and the H_∞ finite-time stabilization of system (19).

Theorem 14. *Given $T > 0$ and $w(t)$ satisfying* (10), *system* (18) ($u = 0$) *under partially known transition probabilities is H_∞ finite-time bounded with respect to (c_1, c_2, T, H_i, d), if there exist positive constants α, ε_{fi}, and γ, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{q\times q}$, and symmetric matrices $W_i \in \mathbb{R}^{n\times n}$, such that for all $i \in \mathscr{M}$*

$$\begin{bmatrix} \Lambda_{1i} + C_i^TC_i & P_iA_{di} + \varepsilon_{fi}F_{12}^i + C_i^TC_{di} & P_iG_i + C_i^TE_i \\ * & -Q + \varepsilon_{fi}F_{22}^i + C_{di}^TC_{di} & C_{di}^TE_i \\ * & * & -\gamma^2 I + E_i^TE_i \end{bmatrix} < 0, \tag{44}$$

$$P_j - W_i \le 0, \quad j \in L_{uk}^i, \ j \ne i, \tag{45}$$

$$P_j - W_i \ge 0, \quad j \in L_{uk}^i, \ j = i, \tag{46}$$

$$\begin{aligned} &c_1\left[\lambda_{\max}\left(\widetilde{P}_i\right) + \tau\lambda_{\max}\left(\widetilde{Q}_i\right)\right] + \frac{\gamma^2 d}{\alpha}\left(1 - e^{-\alpha T}\right) \\ &\quad < \lambda_{\min}\left(\widetilde{P}_i\right)e^{-\alpha T}c_2, \end{aligned} \tag{47}$$

where

$$\begin{aligned} \Lambda_{1i} &= A_i^T P_i + P_i A_i + Q \\ &\quad + \sum_{j\in L_k^i} \pi_{ij}\left(P_j - W_i\right) + \varepsilon_{fi}^{-1}P_iP_i + \varepsilon_{fi}F_{11}^i - \alpha P_i, \end{aligned} \tag{48}$$

$$\widetilde{P}_i = H_i^{-1/2}P_iH_i^{-1/2}, \qquad \widetilde{Q}_i = H_i^{-1/2}QH_i^{-1/2}.$$

Proof. From (44), the following inequality holds:

$$\begin{aligned} &\begin{bmatrix} \Lambda_{1i} + C_i^TC_i & P_iA_{di} + \varepsilon_{fi}F_{12}^i + C_i^TC_{di} & P_iG_i + C_i^TE_i \\ * & -Q + \varepsilon_{fi}F_{22}^i + C_{di}^TC_{di} & C_{di}^TE_i \\ * & * & -\gamma^2 I + E_i^TE_i \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_{1i} & P_iA_{di} + \varepsilon_{fi}F_{12}^i & P_iG_i \\ * & -Q + \varepsilon_{fi}F_{22}^i & 0 \\ * & * & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_i^T \\ C_{di}^T \\ E_i^T \end{bmatrix}\begin{bmatrix} C_i & C_{di}^T & E_i \end{bmatrix} \\ &< 0. \end{aligned} \tag{49}$$

This together with (49) implies (39). Then based on (39)–(42), system (18) is finite-time bounded.

Then, let us prove that inequality (14) is satisfied for any external disturbance $w(t) \ne 0$ under zero initial condition. For system (18), choosing a Lyapunov function candidate (25), we have

$$\begin{aligned} &\mathscr{L}V(x(t), i) \\ &\quad \le x^T(t)\left[A_i^TP_i + P_iA_i + \varepsilon_{fi}^{-1}P_iP_i \right. \\ &\qquad \left. + \sum_{j\in L_k^i}\pi_{ij}\left(P_j - W_i\right) + \varepsilon_{fi}F_{11}^i + Q\right]x(t) \\ &\qquad + x^T(t)P_iG_iw(t) + x^T(t)\left[P_iA_{di} + \varepsilon_{fi}F_{12}^i\right]x(t-\tau) \\ &\qquad + w^T(t)G_i^TP_ix(t) + x^T(t-\tau)\left[A_{di}^TP_i + \varepsilon_{fi}F_{21}^i\right]x(t) \\ &\qquad + x^T(t-\tau)\left[\varepsilon_{fi}F_{22}^i - Q\right]x(t-\tau) \\ &\qquad + x^T(t)\sum_{j\in L_{uk}^i}\pi_{ij}\left(P_j - W_i\right)x(t), \end{aligned} \tag{50}$$

for any symmetric matrices W_i.

According to inequality (44), (45), and (46), we derive

$$\mathscr{L}V(x(t),i) < \alpha V(x(t),i) + \gamma^2 w^T(t)w(t) - z^T(t)z(t),$$
$$\mathscr{L}\left[e^{-\alpha t}V(x(t),i)\right] < e^{-\alpha t}\left[\gamma^2 w^T(t)w(t) - z^T(t)z(t)\right]. \tag{51}$$

Under zero initial condition, using Dynkin's formula yields

$$\begin{aligned} &e^{-\alpha t}V(x(t),i) \\ &\quad < \int_0^t e^{-\alpha s}\left[\gamma^2 w^T(s)w(s) - z^T(s)z(s)\right]ds, \\ &\mathbb{E}\int_0^t e^{-\alpha s}z^T(s)z(s)\,ds < \int_0^t e^{-\alpha s}\gamma^2 w^T(s)w(s)\,ds. \end{aligned} \tag{52}$$

Further, it implies that

$$\mathbb{E}\int_0^T z^T(s)z(s)\,ds < \gamma^2 e^{\alpha T}\int_0^t w^T(s)w(s)\,ds. \tag{53}$$

Therefore expression (14) holds with $\overline{\gamma} = \sqrt{e^{\alpha T}}\gamma$.

The proof is complete. □

Corollary 15. *Given $T > 0$ and $w(t)$ satisfying (10), system (19) under partially known transition probabilities is finite-time H_∞ state feedback stabilizable via a state feedback controller (5) with respect to (c_1, c_2, T, H_i, d), if there exist positive constants α, ε_{fi}, and γ, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{q\times q}$, and symmetric matrices $W_i \in \mathbb{R}^{n\times n}$, such that for all $i \in \mathscr{M}$*

$$\begin{bmatrix} \widetilde{\Lambda}_{1i} + \widetilde{C}_i^T\widetilde{C}_i & P_i\widetilde{A}_{di} + \varepsilon_{fi}F_{12}^i + \widetilde{C}_i^T\widetilde{C}_{di} & P_iG_i + \widetilde{C}_i^TE_i \\ * & -Q + \varepsilon_{fi}F_{22}^i + \widetilde{C}_{di}^T\widetilde{C}_{di} & \widetilde{C}_{di}^TE_i \\ * & * & -\gamma^2 I + E_i^TE_i \end{bmatrix} < 0, \tag{54}$$

$$P_j - W_i \le 0, \quad j \in L_{uk}^i,\ j \ne i, \tag{55}$$

$$P_j - W_i \ge 0, \quad j \in L_{uk}^i,\ j = i, \tag{56}$$

$$\begin{aligned} &c_1\left[\lambda_{\max}\left(\widetilde{P}_i\right) + \tau\lambda_{\max}\left(\widetilde{Q}_i\right)\right] + \frac{\gamma^2 d}{\alpha}\left(1 - e^{-\alpha T}\right) \\ &\quad < \lambda_{\min}\left(\widetilde{P}_i\right)e^{-\alpha T}c_2, \end{aligned} \tag{57}$$

where

$$\begin{aligned} \widetilde{\Lambda}_{1i} &= \widetilde{A}_i^TP_i + P_i\widetilde{A}_i + Q \\ &\quad + \sum_{j\in L_k^i}\pi_{ij}\left(P_j - W_i\right) + \varepsilon_{fi}^{-1}P_iP_i + \varepsilon_{fi}F_{11}^i - \alpha P_i, \\ \widetilde{A}_i &= A_i + B_iK_i, \qquad \widetilde{A}_{di} = A_{di} + B_iK_{di}, \\ \widetilde{C}_i^T &= C_i + D_iK_i, \qquad \widetilde{C}_{di} = C_{di} + D_iK_{di}, \\ \widetilde{P}_i &= H_i^{-1/2}P_iH_i^{-1/2}, \qquad \widetilde{Q}_i = H_i^{-1/2}QH_i^{-1/2}. \end{aligned} \tag{58}$$

It is clear that (54) is a nonlinear matrix inequality due to the existence of the nonlinear terms $K_i^TB_i^TP_i$, $P_iB_iK_i$, $K_{di}^TB_i^TP_i$, and $P_iB_iK_{di}$. In order to solve the desired controller K_i, we give the following result.

Theorem 16. *Given $T > 0$, system (18) under partially known transition probabilities is finite-time H_∞ state feedback stabilizable via a state feedback controller with respect to (c_1, c_2, T, H_i, d), if there exist positive scalars α, γ, ε_{fi}, λ_1, and λ_2, symmetric positive definite matrices $X_i \in \mathbb{R}^{n\times n}$, symmetric matrices $\mathscr{W}_i \in \mathbb{R}^{n\times n}$, and matrices $Y_i \in \mathbb{R}^{m\times n}$ and $K_{di} \in \mathbb{R}^{n\times m}$ such that for all $i \in \mathscr{M}$*

$$\begin{bmatrix} \Pi_{11i}^1 & \Pi_{12i} & G_i & \Pi_{14i} & I & \varepsilon_{fi}X_i\left(F_{11}^i\right)^{1/2} & S_{1i}(x) \\ * & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0 & 0 \\ * & * & -\gamma^2 I & E_i^T & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_{fi}I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{fi}I & 0 \\ * & * & * & * & * & * & -M_{1i}(x) \end{bmatrix} < 0, \quad i \in L_k^i, \tag{59}$$

$$\begin{bmatrix} \Pi_{11i}^2 & \Pi_{12i} & G_i & \Pi_{14i} & I & \varepsilon_{fi}X_i\left(F_{11}^i\right)^{1/2} & S_{2i}(x) \\ * & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0 & 0 \\ * & * & -\gamma^2 I & E_i^T & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_{fi}I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{fi}I & 0 \\ * & * & * & * & * & * & -M_{2i}(x) \end{bmatrix} < 0, \quad i \in L_{uk}^i, \tag{60}$$

$$\begin{bmatrix} -\mathscr{W}_i & X_i \\ * & -X_j \end{bmatrix} < 0, \quad j \in L_{uk}^i,\ j \ne i, \tag{61}$$

$$X_j - \mathscr{W}_j > 0, \quad j \in L_{uk}^i,\ j = i, \tag{62}$$

$$\begin{bmatrix} -e^{-\alpha T}c_2 + c_1\tau\lambda_2 + \dfrac{\gamma^2 d}{\alpha}\left(1 - e^{-\alpha T}\right) & \sqrt{c_1} \\ \sqrt{c_1} & -\lambda_1 \end{bmatrix} < 0, \tag{63}$$

$$\lambda_1 H_i^{-1} < X_i < H_i^{-1}, \qquad 0 < Q < \lambda_2 H_i, \tag{64}$$

where

$$\begin{aligned} \Pi_{11i}^1 &= X_iA_i^T + A_iX_i + Y_i^TB_i^T + B_iY_i + Q_i \\ &\quad - \sum_{j\in L_k^i}\pi_{ij}\mathscr{W}_i + \pi_{ii}X_i - \alpha X_i, \\ \Pi_{11i}^2 &= X_iA_i^T + A_iX_i + Y_i^TB_i^T + B_iY_i + Q_i \\ &\quad - \sum_{j\in L_k^i}\pi_{ij}\mathscr{W}_i - \alpha X_i, \\ \Pi_{12i} &= A_{di} + B_iK_{di} + \varepsilon_{fi}X_iF_{12}^i, \\ \Pi_{22i} &= -Q + \varepsilon_{fi}F_{22}^i, \\ \Pi_{14i} &= X_iC_i^T + Y_i^TD_i^T, \\ \Pi_{24i} &= C_{di}^T + K_{di}^TD_i^T, \end{aligned}$$

$$
\begin{aligned}
&S_{1i}(x)\\
&= \left[\sqrt{\pi_{ik_1^i}}X_i, \ldots, \sqrt{\pi_{ik_{r-1}^i}}X_i, \sqrt{\pi_{ik_{r+1}^i}}X_i, \ldots, \sqrt{\pi_{ik_m^i}}X_i\right],\\
&M_{1i}(x) = \operatorname{diag}\left\{X_{k_1^i}, \ldots, X_{k_{r-1}^i}, X_{k_{r+1}^i}, \ldots, X_{k_m^i}\right\},\\
&S_{2i}(x) = \left[\sqrt{\pi_{ik_1^i}}X_i, \ldots, \sqrt{\pi_{ik_m^i}}X_i\right],\\
&M_{2i}(x) = \operatorname{diag}\left\{X_{k_1^i}, \ldots, X_{k_m^i}\right\},
\end{aligned}
\tag{65}
$$

with $k_1^i, k_2^i, \ldots k_m^i$ described in (9) *and $k_r^i = i$. Moreover, the finite-time H_∞ state feedback controller gains in* (5) *are given by $K_i = Y_i X_i^{-1}$.*

Proof. It is clear that system (18) is finite-time H_∞ state feedback stabilizable if the conditions (54)–(57) are satisfied. Notice that inequality (54) is equivalent to the following condition:

$$
\Sigma_i = \begin{bmatrix}
\widetilde{\Lambda}_{1i} & P_i\widetilde{A}_{di} + \varepsilon_{fi}F_{12}^i & P_iG_i & \widetilde{C}_i^T & P_i & \varepsilon_{fi}\left(F_{11}^i\right)^{1/2}\\
* & -Q + \varepsilon_{fi}F_{22}^i & 0 & \widetilde{C}_{di}^T & 0 & 0\\
* & * & -\gamma^2 I & E_i^T & 0 & 0\\
* & * & * & -I & 0 & 0\\
* & * & * & * & -\varepsilon_{fi}I & 0\\
* & * & * & * & * & -\varepsilon_{fi}I
\end{bmatrix}
< 0.
\tag{66}
$$

Pre- and postmultiplying inequality (66) by block diagonal matrix diag $\{P_i^{-1} \quad I \quad I \quad I \quad I \quad I\}$, letting $X_i = P_i^{-1}$, $Y_i = K_iX_i$, and $\mathscr{W}_i = P_i^{-1}W_iP_i^{-1}$, we have

$$
\begin{bmatrix}
\Xi_{1i} & \Pi_{12i} & G_i & \Pi_{14i} & I & \varepsilon_{fi}X_i\left(F_{11}^i\right)^{1/2}\\
* & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0\\
* & * & -\gamma^2 I & E_i^T & 0 & 0\\
* & * & * & -I & 0 & 0\\
* & * & * & * & -\varepsilon_{fi}I & 0\\
* & * & * & * & * & -\varepsilon_{fi}I
\end{bmatrix} < 0,
\tag{67}
$$

where

$$
\begin{aligned}
\Xi_{1i} &= X_iA_i^T + A_iX_i + Y_i^TB_i^T + B_iY_i + Q_i\\
&\quad + \sum_{j\in L_k^i}\pi_{ij}X_iX_j^{-1}X_i - \sum_{j\in L_k^i}\pi_{ij}\mathscr{W}_i - \alpha X_i.
\end{aligned}
\tag{68}
$$

Since $\pi_{ii} < 0$, $\forall i \in \mathscr{M}$, inequality (67) is discussed in the following two cases.

Case 1. When $i \in L_k^i$, the left side of (67) becomes

$$
\begin{bmatrix}
\Xi_{2i} & \Pi_{12i} & G_i & \Pi_{14i} & I & \varepsilon_{fi}X_i\left(F_{11}^i\right)^{1/2}\\
* & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0\\
* & * & -\gamma^2 I & E_i^T & 0 & 0\\
* & * & * & -I & 0 & 0\\
* & * & * & * & -\varepsilon_{fi}I & 0\\
* & * & * & * & * & -\varepsilon_{fi}I
\end{bmatrix}
+ \begin{bmatrix}
\sum_{j\in L_k^i, j\neq i}\pi_{ij}X_iX_j^{-1}X_i & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} < 0,
\tag{69}
$$

where

$$
\begin{aligned}
\Xi_{2i} &= X_iA_i^T + A_iX_i + Y_i^TB_i^T + B_iY_i + Q_i\\
&\quad - \sum_{j\in L_k^i}\pi_{ij}\mathscr{W}_i - \alpha X_i.
\end{aligned}
\tag{70}
$$

Applying Schur complement lemma to (69), then (59) easily follows.

Case 2. When $i \in L_{uk}^i$, the inequality (69) turns into

$$
\begin{bmatrix}
\Xi_{3i} & \Pi_{12i} & G_i & \Pi_{14i} & I & \varepsilon_{fi}X_i\left(F_{11}^i\right)^{1/2}\\
* & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0\\
* & * & -\gamma^2 I & E_i^T & 0 & 0\\
* & * & * & -I & 0 & 0\\
* & * & * & * & -\varepsilon_{fi}I & 0\\
* & * & * & * & * & -\varepsilon_{fi}I
\end{bmatrix}
+ \begin{bmatrix}
\sum_{j\in L_k^i, j\neq i}\pi_{ij}X_iX_j^{-1}X_i & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} < 0,
\tag{71}
$$

where

$$
\Xi_{3i} = X_iA_i^T + A_iX_i + Y_i^TB_i^T + B_iY_i + Q_i - \sum_{j\in L_k^i}\pi_{ij}\mathscr{W}_i.
\tag{72}
$$

Similar to the proving process of the case one, we can prove that (60) is true.

Pre- and postmultiplying inequalities (55) and (56) by P_i^{-1}, respectively, and letting $X_i = P_i^{-1}$, $Y_i = K_iX_i$,

and $\mathscr{W}_i = P_i^{-1} W_i P_i^{-1}$, we have

$$X_i X_j^{-1} X_i - R_i < 0, \quad j \in L_{uk}^i, \ j \neq i, \tag{73}$$

$$X_j - R_j > 0, \quad j \in L_{uk}^i, \ j = i. \tag{74}$$

Inequality (73) is equivalent to LMI (61). Denoting $\widetilde{X}_i = \widetilde{P}_i^{-1} = H_i^{1/2} X_i H_i^{1/2}$ and taking $\lambda_{\max}(\widetilde{X}_i) = 1/\lambda_{\min}(\widetilde{P}_i)$ into consideration, we conclude that condition (57) holds. Hence, the following conditions

$$\lambda_1 < \lambda_{\min}\left(\widetilde{X}_i\right), \qquad \lambda_{\max}\left(\widetilde{X}_i\right) < 1, \qquad 0 < \lambda_{\min}(Q), \qquad \lambda_{\max}(Q) < \lambda_2, \tag{75}$$

guarantee that

$$\frac{c_1}{\lambda_1} + c_1 \tau \lambda_2 + \frac{\gamma^2 d}{\alpha}\left(1 - e^{-\alpha t}\right) < e^{-\alpha t} c_2. \tag{76}$$

It should be easily observed that condition (76) implies LMI (63) and (75) is equivalent to (64). Therefore if LMIs (59)–(64) hold, the closed-loop system (19) is H_∞ finite-time bounded, and then system (18) can be stabilized via the state feedback controller (5).

This completes the proof of Theorem 16. □

3.3. Robust Finite-Time H_∞ Control. In this subsection, a robust finite-time H_∞ state feedback controller is designed to guarantee the finite-time H_∞ state feedback stabilization of system (2).

Theorem 17. *Given $T > 0$, the problem of robust finite-time H_∞ state feedback stabilizable for system (2) under partly known transition probabilities is solvable, if there exist positive scalars α, γ, ε_{fi}, ε_{1i}, ε_{2i}, ε_{3i}, ε_{4i}, λ_1, and λ_2, symmetric positive definite matrices $X_i \in \mathbb{R}^{n\times n}$, symmetric matrices $\mathscr{W}_i \in \mathbb{R}^{n\times n}$, and matrices $Y_i \in \mathbb{R}^{m\times n}$ and $K_{di} \in \mathbb{R}^{n\times m}$ such that for all $i \in \mathscr{M}$*

$$\begin{bmatrix}
\widetilde{\Pi}_{11i}^1 & \Pi_{12i} & G_i & \Pi_{14i} & X_i N_{1i}^T & Y_i^T N_{2i}^T & 0 & 0 & I & \varepsilon_{fi} X_i \left(F_{11}^i\right)^{1/2} & S_{1i}(x) \\
* & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0 & N_{3i}^T & K_{di}^T N_{2i}^T & 0 & 0 & 0 \\
* & * & -\gamma^2 I & E_i^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_{1i} I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon_{2i} I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_{3i} I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon_{4i} I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_{fi} I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\varepsilon_{fi} I & 0 \\
* & * & * & * & * & * & * & * & * & * & -M_{1i}(x)
\end{bmatrix} < 0, \quad i \in L_k^i, \tag{77}$$

$$\begin{bmatrix}
\widetilde{\Pi}_{11i}^2 & \Pi_{12i} & G_i & \Pi_{14i} & X_i N_{1i}^T & Y_i^T N_{2i}^T & 0 & 0 & I & \varepsilon_{fi} X_i \left(F_{11}^i\right)^{1/2} & S_{2i}(x) \\
* & \Pi_{22i} & 0 & \Pi_{24i} & 0 & 0 & N_{3i}^T & K_{di}^T N_{2i}^T & 0 & 0 & 0 \\
* & * & -\gamma^2 I & E_i^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_{1i} I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon_{2i} I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_{3i} I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon_{4i} I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_{fi} I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\varepsilon_{fi} I & 0 \\
* & * & * & * & * & * & * & * & * & * & -M_{2i}(x)
\end{bmatrix} < 0, \quad i \in L_{uk}^i, \tag{78}$$

$$\begin{bmatrix} -\mathscr{W}_i & X_i \\ * & -X_j \end{bmatrix} < 0, \quad j \in L_{uk}^i, \ j \neq i, \tag{79}$$

$$X_j - \mathscr{W}_j > 0, \quad j \in L_{uk}^i, \ j = i, \tag{80}$$

$$\begin{bmatrix} -e^{-\alpha T} c_2 + c_1 \tau \lambda_2 + \dfrac{\gamma^2 d}{\alpha}\left(1 - e^{-\alpha T}\right) & \sqrt{c_1} \\ \sqrt{c_1} & -\lambda_1 \end{bmatrix} < 0, \tag{81}$$

$$\lambda_1 H_i^{-1} < X_i < H_i^{-1}, \qquad 0 < Q < \lambda_2 H_i, \tag{82}$$

where

$$\begin{aligned}\widetilde{\Pi}_{11i}^1 &= X_i A_i^T + A_i X_i + Y_i^T B_i^T + B_i Y_i + Q_i \\ &\quad - \sum_{j\in L_k^i} \pi_{ij} \mathscr{W}_i + \varepsilon_{1i} M_{1i} M_{1i}^T + \varepsilon_{2i} M_{2i} M_{2i}^T \\ &\quad + \varepsilon_{3i} M_{3i} M_{3i}^T + \varepsilon_{4i} M_{2i} M_{2i}^T + \pi_{ii} X_i - \alpha X_i,\end{aligned}$$

$$\begin{aligned}\widetilde{\Pi}_{11i}^2 &= X_i A_i^T + A_i X_i + Y_i^T B_i^T + B_i Y_i + Q_i \\ &\quad - \sum_{j\in L_k^i} \pi_{ij} \mathscr{W}_i + \varepsilon_{1i} M_{1i} M_{1i}^T + \varepsilon_{2i} M_{2i} M_{2i}^T \\ &\quad + \varepsilon_{3i} M_{3i} M_{3i}^T + \varepsilon_{4i} M_{2i} M_{2i}^T - \alpha X_i,\end{aligned}$$

$$\begin{aligned}\Pi_{12i} &= A_{di} + B_i K_{di} + \varepsilon_{fi} X_i F_{12}^i, \\ \Pi_{22i} &= -Q + \varepsilon_{fi} F_{22}^i, \\ \Pi_{14i} &= X_i C_i^T + Y_i^T D_i^T, \\ \Pi_{24i} &= C_{di}^T + K_{di}^T D_i^T,\end{aligned}$$

$$\begin{aligned}&S_{1i}(x) \\ &= \left[\sqrt{\pi_{ik_1^i}} X_i, \ldots, \sqrt{\pi_{ik_{r-1}^i}} X_i, \sqrt{\pi_{ik_{r+1}^i}} X_i, \ldots, \sqrt{\pi_{ik_m^i}} X_i\right], \\ &M_{1i}(x) = \operatorname{diag}\left\{X_{k_1^i}, \ldots, X_{k_{r-1}^i}, X_{k_{r+1}^i}, \ldots, X_{k_m^i}\right\}, \\ &S_{2i}(x) = \left[\sqrt{\pi_{ik_1^i}} X_i, \ldots, \sqrt{\pi_{ik_m^i}} X_i\right], \\ &M_{2i}(x) = \operatorname{diag}\left\{X_{k_1^i}, \ldots, X_{k_m^i}\right\},\end{aligned} \tag{83}$$

with $k_1^i, k_2^i, \ldots k_m^i$ described in (9) *and $k_r^i = i$. Moreover, the finite-time H_∞ state feedback controller gains in* (5) *are given by $K_i = Y_i X_i^{-1}$.*

Proof. In (59) and (60), replacing A_i, A_{di}, and B_i with $(A_i + \Delta A_i)$, $(A_{di} + \Delta A_{di})$, and $(B_i + \Delta B_i)$, respectively, the following conditions are obtained:

$$\begin{aligned}\overline{\Pi}_{11i}^1 &= X_i A_i^T + X_i \Delta A_i^T + A_i X_i + \Delta A_i X_i + Y_i^T B_i^T + Y_i^T \Delta B_i^T \\ &\quad + B_i Y_i + \Delta B_i Y_i - \sum_{j\in L_k^i} \pi_{ij} R_i + \pi_{ii} X_i,\end{aligned}$$

$$\begin{aligned}\overline{\Pi}_{21i}^1 &= X_i A_i^T + X_i \Delta A_i^T + A_i X_i + \Delta A_i X_i + Y_i^T B_i^T \\ &\quad + Y_i^T \Delta B_i^T + B_i Y_i + \Delta B_i Y_i - \sum_{j\in L_k^i} \pi_{ij} R_i,\end{aligned}$$

$$\overline{\Pi}_{12i} = A_{di} + \Delta A_{di} + B_i K_{di} + \Delta B_i K_{di} + \varepsilon_{fi} X_i F_{12}^i. \tag{84}$$

Based on Lemma 11, there exist scalars ε_{1i}, ε_{2i}, ε_{3i}, and ε_{4i} such that

$$\begin{aligned}X_i \Delta A_i^T + \Delta A_i X_i &= X_i N_{1i}^T F_i^T(t) M_{1i}^T + M_{1i} F_i(t) N_{1i} X_i \\ &\le \varepsilon_{1i} M_{1i} M_{1i}^T + \varepsilon_{1i}^{-1} X_i N_{1i}^T N_{1i} X_i, \\ Y_i^T \Delta B_i^T + \Delta B_i Y_i &= Y_i^T N_{2i}^T F_i^T(t) M_{2i}^T + M_{2i} F_i(t) N_{2i} Y_i \\ &\le \varepsilon_{2i} M_{2i} M_{2i}^T + \varepsilon_{2i}^{-1} Y_i^T N_{2i}^T N_{2i} Y_i.\end{aligned}$$

$$\begin{aligned}&\begin{bmatrix} 0 & \Delta A_{di} & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & M_{3i} F_i(t) N_{3i} & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} M_{3i} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_i \begin{bmatrix} 0 & N_{3i} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ N_{3i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_i \begin{bmatrix} M_{3i}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\le \varepsilon_{3i} \begin{bmatrix} M_{3i} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} M_{3i}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad + \varepsilon_{3i}^{-1} \begin{bmatrix} 0 \\ N_{3i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & N_{3i} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\end{aligned}$$

$$
\begin{bmatrix}
0 & \Delta B_i K_{di} & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & M_{2i}F_i(t)N_{2i}K_{di} & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix} M_{2i} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_i \begin{bmatrix} 0 & N_{2i}K_{di} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ K_{di}^T N_{2i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_i \begin{bmatrix} M_{2i}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\leq \varepsilon_{4i} \begin{bmatrix} M_{2i} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} M_{2i}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
+ \varepsilon_{4i}^{-1} \begin{bmatrix} 0 \\ N_{3i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & N_{3i} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{85}
$$

Applying Schur complement lemma to (85), (77) can be obtained. Similar to the above proving process, we can prove that (78) holds. Therefore, if LMIs (77)–(82) hold, the closed-loop system (6) is robust H_∞ finite-time bounded, and further system (18) can be stabilized via the state feedback controller (5).

The proof is complete. □

Remark 18. It should be pointed out that the conditions in Theorems 16 and 17 are not strict linear matrix inequalities such as conditions (20), (39), (44), (54), (59), (60), (77), and (78), due to the product of unknown scalars and matrices. An efficient way to solve this problem is to choose the appropriate values of the unknown scalars and then solve a set of LMIs for the fixed values of these parameters. For example, if α, ε_{fi} are fixed, then conditions (59) and (60) of Theorem 16 can be converted to LMIs conditions.

4. Numerical Examples

This section considers the following four-mode uncertain nonlinear Markovian jump systems with time delay as follows.

Mode 1

$$
A_1 = \begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -0.2 & 0.3 \\ 0.1 & -0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
$$

$$
G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad C_{d1} = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix},
$$

$$
D_1 = E_1 = 0.1, \quad M_{11} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
$$

$$
N_{11} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},
$$

$$
M_{21} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad N_{21} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},
$$

$$
M_{31} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad N_{31} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}. \tag{86}
$$

Mode 2

$$
A_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & -0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},
$$

$$
G_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix},
$$

$$
C_{d2} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \quad D_2 = E_2 = 0.2, \quad M_{12} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},
$$

$$
N_{12} = \begin{bmatrix} 0.2 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad N_{22} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix},
$$

$$
M_{32} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.03 \end{bmatrix}, \quad N_{32} = \begin{bmatrix} 0.02 & 0.03 \\ 0 & 0.02 \end{bmatrix}. \tag{87}
$$

Mode 3

$$
A_3 = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \quad A_{d3} = \begin{bmatrix} 0.1 & -0.3 \\ -0.2 & 0.3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},
$$

$$
G_3 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 3 \end{bmatrix},
$$

Table 1

Case I	1	2	3	4	Case II	1	2	3	4
1	−1.2	0.3	0.5	0.4	1	?	0.3	?	0.4
2	0.2	−1	0.3	0.5	2	?	−1	0.3	?
3	0.8	0.1	−1.3	0.4	3	0.8	?	−1.3	?
4	0.2	0.1	0.5	−0.8	4	0.2	?	?	?
Case III	**1**	**2**	**3**	**4**	**Case VI**	**1**	**2**	**3**	**4**
1	−1.2	?	0.5	?	1	?	?	?	?
2	0.2	?	?	0.5	2	?	?	?	?
3	?	0.1	?	0.4	3	?	?	?	?
4	?	0.1	0.5	−0.8	4	?	?	?	?

Table 2

Case I	Completely known
Controller gains	$K_1 = \begin{bmatrix} -22.2335 & -19.0199 \end{bmatrix}$ $K_{d1} = \begin{bmatrix} -0.9097 & 0.9098 \end{bmatrix}$ $K_2 = \begin{bmatrix} -7.0199 & -2.3824 \end{bmatrix}$ $K_{d2} = \begin{bmatrix} -0.9490 & -0.4701 \end{bmatrix}$ $K_3 = \begin{bmatrix} -6.9528 & -9.8454 \end{bmatrix}$ $K_{d3} = \begin{bmatrix} 0.6466 & -0.3225 \end{bmatrix}$ $K_4 = \begin{bmatrix} -7.9573 & -2.4935 \end{bmatrix}$ $K_{d4} = \begin{bmatrix} -0.4998 & -0.2499 \end{bmatrix}$
Case II	Partially known
Controller gains	$K_1 = \begin{bmatrix} -22.5382 & -18.9685 \end{bmatrix}$ $K_{d1} = \begin{bmatrix} -0.8142 & 0.8142 \end{bmatrix}$ $K_2 = \begin{bmatrix} -7.9189 & -2.8820 \end{bmatrix}$ $K_{d2} = \begin{bmatrix} -0.9007 & -0.4391 \end{bmatrix}$ $K_3 = \begin{bmatrix} -6.9801 & -9.8455 \end{bmatrix}$ $K_{d3} = \begin{bmatrix} 0.6272 & -0.3112 \end{bmatrix}$ $K_4 = \begin{bmatrix} -8.1348 & -2.4949 \end{bmatrix}$ $K_{d4} = \begin{bmatrix} -0.4996 & -0.2498 \end{bmatrix}$
Case III	Partially known
Controller gains	$K_1 = \begin{bmatrix} -20.2412 & -18.1608 \end{bmatrix}$ $K_{d1} = \begin{bmatrix} -0.8925 & 0.9362 \end{bmatrix}$ $K_2 = \begin{bmatrix} -6.2413 & -2.7888 \end{bmatrix}$ $K_{d2} = \begin{bmatrix} -0.9283 & -0.5364 \end{bmatrix}$ $K_3 = \begin{bmatrix} -6.9091 & -9.8876 \end{bmatrix}$ $K_{d3} = \begin{bmatrix} 0.7272 & -0.2134 \end{bmatrix}$ $K_4 = \begin{bmatrix} -8.6329 & -2.4765 \end{bmatrix}$ $K_{d4} = \begin{bmatrix} -0.4998 & -0.2499 \end{bmatrix}$
Case VI	Completely unknown
Controller gains	$K_1 = \begin{bmatrix} -21.8153 & -18.7884 \end{bmatrix}$ $K_{d1} = \begin{bmatrix} -0.8143 & 0.8143 \end{bmatrix}$ $K_2 = \begin{bmatrix} -3.8757 & -0.3739 \end{bmatrix}$ $K_{d2} = \begin{bmatrix} -0.9008 & -0.4391 \end{bmatrix}$ $K_3 = \begin{bmatrix} -6.9153 & -9.8460 \end{bmatrix}$ $K_{d3} = \begin{bmatrix} 0.6272 & -0.3112 \end{bmatrix}$ $K_4 = \begin{bmatrix} -7.9143 & -2.4877 \end{bmatrix}$ $K_{d4} = \begin{bmatrix} -0.4996 & -0.2498 \end{bmatrix}$

$$C_{d3} = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}, \qquad D_3 = E_3 = 0.3,$$
$$M_{13} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad N_{13} = \begin{bmatrix} 0.2 & 0.3 \\ 0 & 0.5 \end{bmatrix},$$
$$M_{23} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad N_{23} = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix},$$
$$M_{33} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad N_{33} = \begin{bmatrix} 0.02 & 0.03 \\ 0 & 0.05 \end{bmatrix}. \tag{88}$$

Mode 4

$$A_4 = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}, \qquad A_{d4} = \begin{bmatrix} -0.1 & 0.3 \\ 0.2 & -0.1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$
$$G_4 = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, \qquad C_4 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$C_{d4} = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}, \qquad D_4 = E_4 = 0.4,$$
$$M_{14} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad N_{14} = \begin{bmatrix} 0.2 & 0.4 \\ 0 & 0.3 \end{bmatrix},$$
$$M_{24} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad N_{24} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix},$$
$$M_{34} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad N_{34} = \begin{bmatrix} 0.02 & 0.04 \\ 0 & 0.03 \end{bmatrix},$$
$$H_1 = H_2 = H_3 = H_4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad C_1 = 0.5,$$
$$C_2 = 4, \qquad d = 4, \qquad T = 1.2. \tag{89}$$

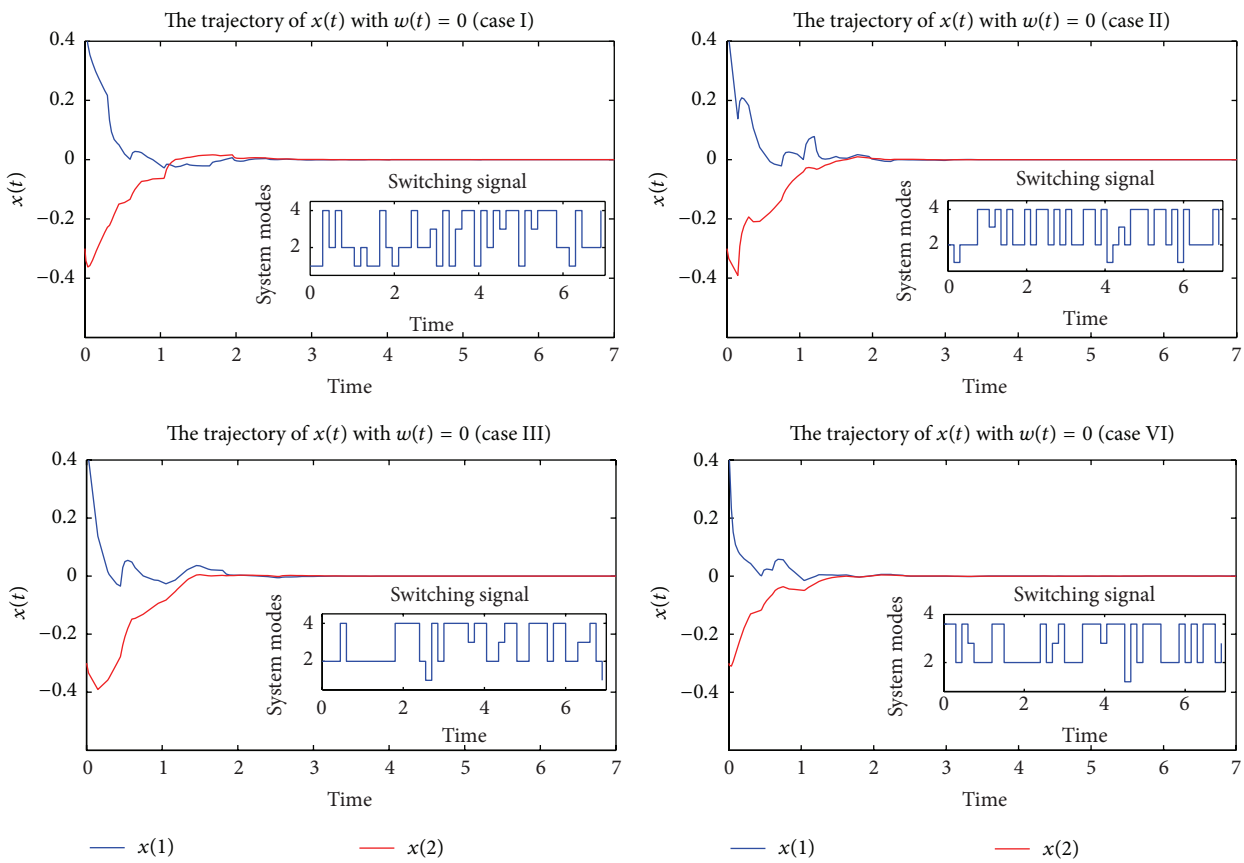

Figure 1: The trajectory of $x(t)$.

Choose $\tau = 1$, $\alpha = 0.5$, the exogenous disturbance $w(t) = [1/(5t+1) \quad 1/(t+1)]$, and the nonlinearities

$$f_1(x(t), x(t-\tau)) = \begin{bmatrix} 0.1\sin(x(t)) \\ 0.1\sin(x(t-\tau)) \end{bmatrix},$$

$$f_2(x(t), x(t-\tau)) = \begin{bmatrix} 0.1\sin(x(t-\tau)) \\ -0.15\sin(x(t)) \end{bmatrix},$$

$$f_3(x(t), x(t-\tau)) = \begin{bmatrix} 0.1\sin(x(t)) \\ 0.1\sin(x(t-\tau)) \end{bmatrix},$$

$$f_4(x(t), x(t-\tau)) = \begin{bmatrix} 0.1\sin(x(t-\tau)) \\ -0.15\sin(x(t)) \end{bmatrix},$$

$$F_{11}^1 = \begin{bmatrix} 1.1841 & 0.1704 \\ 0.1562 & 1.1370 \end{bmatrix}, \quad F_{22}^1 = \begin{bmatrix} 0.0606 & 0.1000 \\ 0.1000 & 0.3355 \end{bmatrix},$$

$$F_{11}^2 = \begin{bmatrix} 0.3299 & 0 \\ 0.7999 & 0.5000 \end{bmatrix}, \quad F_{22}^2 = \begin{bmatrix} 0.4000 & 0 \\ 0 & 0.2500 \end{bmatrix},$$

$$F_{11}^3 = \begin{bmatrix} 1.1841 & 0.1704 \\ 0.1562 & 1.1370 \end{bmatrix}, \quad F_{22}^3 = \begin{bmatrix} 0.0606 & 0.1000 \\ 0.1000 & 0.3355 \end{bmatrix},$$

$$F_{11}^4 = \begin{bmatrix} 0.3299 & 0 \\ 0.7999 & 0.5000 \end{bmatrix}, \quad F_{22}^4 = \begin{bmatrix} 0.4000 & 0 \\ 0 & 0.2500 \end{bmatrix},$$

$$F_{12}^1 = F_{12}^2 = F_{12}^3 = F_{12}^4 = 0. \tag{90}$$

The four cases for the transition probability matrix considered in Table 1.

Solving the LMIs (77)–(82) in Theorem 17, the robust finite-time H_∞ state feedback controller gains of K_i are given by Table 2.

Figures 1, 2, and 3 are presented. For every figure, the four different transition probability matrices cases are included, which can be better to demonstrate the effectiveness of the design method. Figure 1 depicts the trajectories of system state $x(t)$ and the corresponding switching signal. It can be seen that system (6) is robust finite-time stable, which implies that system (2) is robust finite-time H_∞ state feedback stabilizable via the designed state feedback controller (5). Figure 2 depicts the trajectories of system state $x(t)$ with $w(t) \neq 0$ and the corresponding switching signal. It can be seen that system (6) is robust finite-time bounded. The trajectory of the output $z(t)$ is described in Figure 3, which further shows the effectiveness of the designed controller (5).

5. Conclusions

In this paper, we have dealt with the problem of robust finite-time H_∞ control for a class of nonlinear Markovian jump systems with time delay under partially known transition probabilities. Based on the free-weighting matrices approach, all sufficient conditions have been firstly proposed to ensure

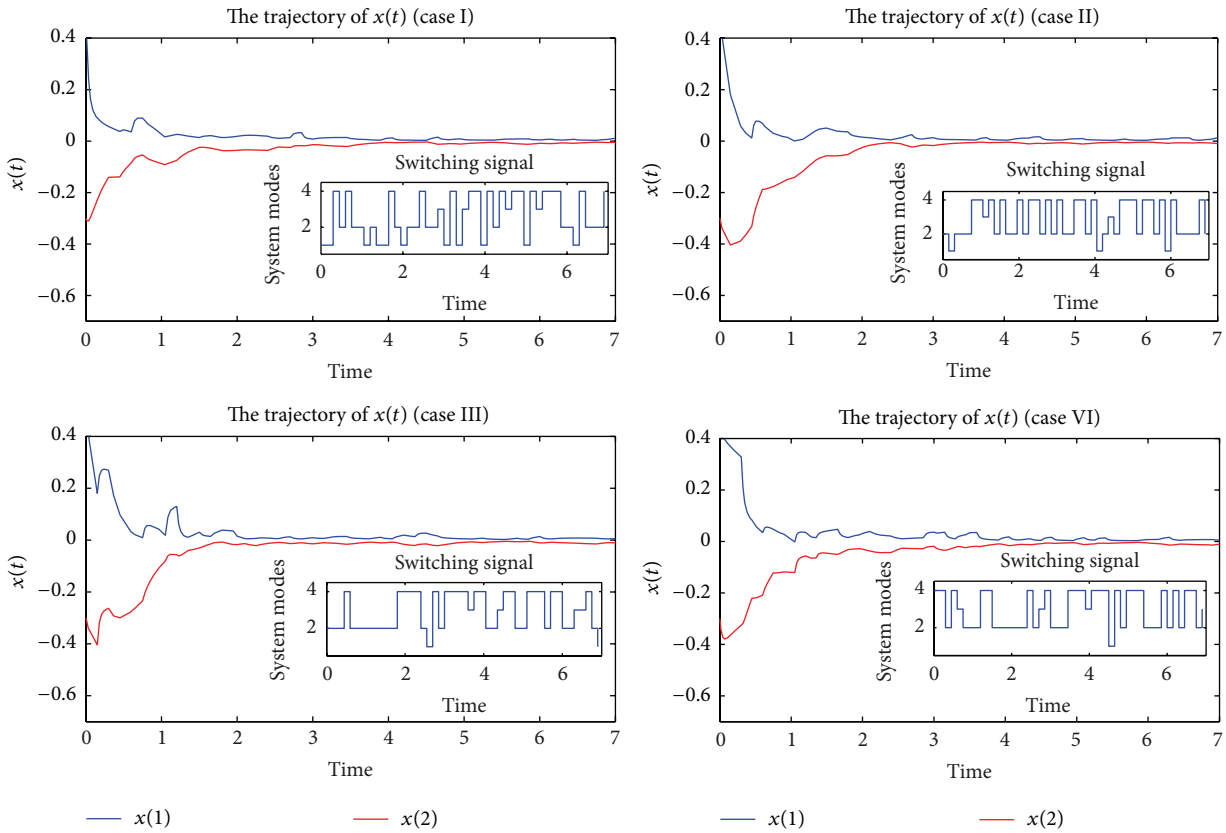

FIGURE 2: The trajectory of $x(t)$ with $w(t) \neq 0$.

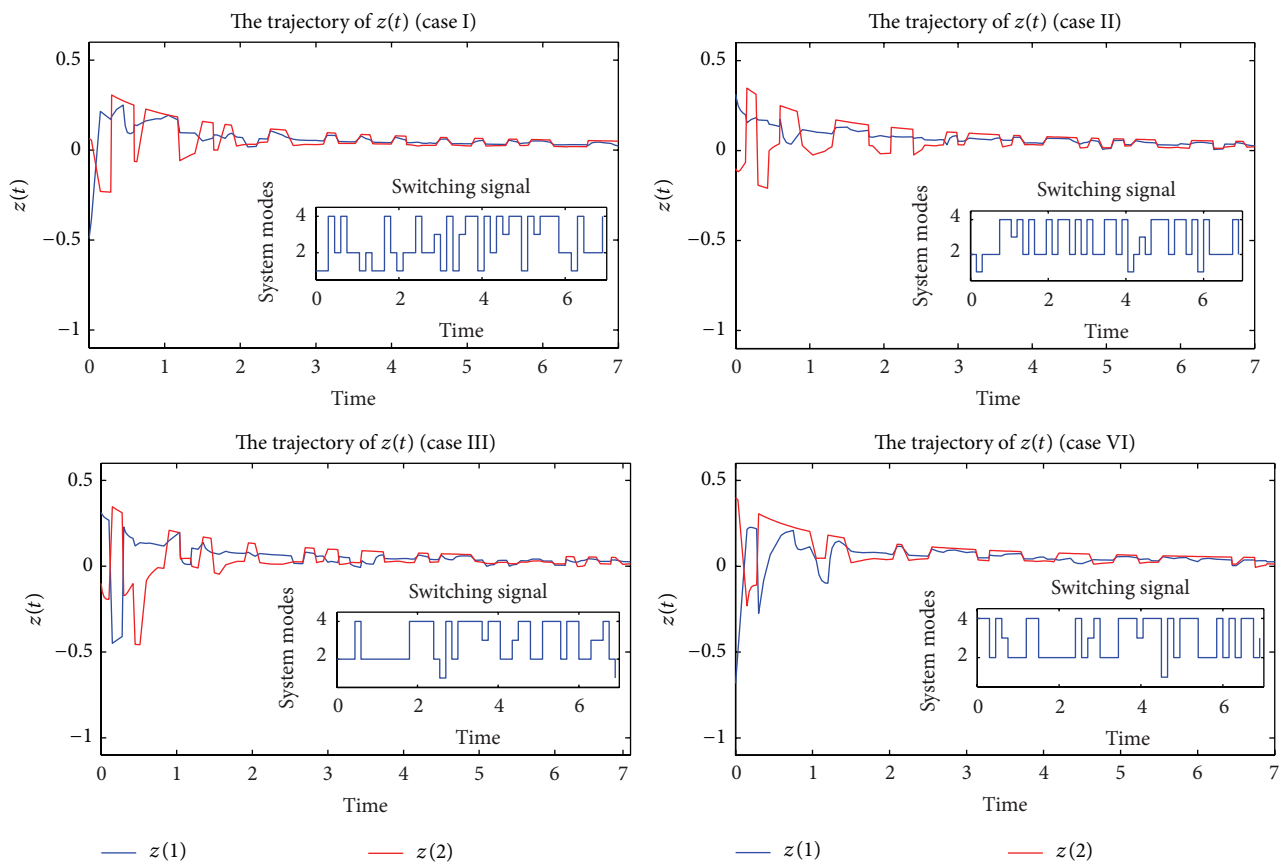

FIGURE 3: The trajectory of $z(t)$.

finite-time boundedness, H_∞ finite-time boundedness, and finite-time H_∞ state feedback stabilization for the given system. We have also designed a robust finite-time H_∞ state feedback controller, which guarantees the H_∞ finite-time boundedness of the closed-loop system. All the conditions have been presented in terms of strict linear matrix inequalities. Finally, a numerical example has been provided to demonstrate the effectiveness of all the results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Nonfragile H_∞ Control for Stochastic Systems with Markovian Jumping Parameters and Random Packet Losses

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This paper is concerned with the nonfragile H_∞ control problem for stochastic systems with Markovian jumping parameters and random packet losses. The communication between the physical plant and controller is assumed to be imperfect, where random packet losses phenomenon occurs in a random way. Such a phenomenon is represented by a stochastic variable satisfying the Bernoulli distribution. The purpose is to design a nonfragile controller such that the resulting closed-loop system is stochastically mean square stable with a guaranteed H_∞ performance level γ. By using the Lyapunov function approach, some sufficient conditions for the solvability of the previous problem are proposed in terms of linear matrix inequalities (LMIs), and a corresponding explicit parametrization of the desired controller is given. Finally, an example illustrating the effectiveness of the proposed approach is presented.

1. Introduction

During the past several decades, stochastic systems have been the main focus of research receiving much attention since realistic models of most engineering systems involve random exogenous disturbances [1, 2]. As a simple yet significant mathematical model, stochastic systems have come to play a key role in many branches of science and engineering [3]. For this reason, many fundamental issues have been extensively addressed for stochastic systems, and consequently fruitful results have been presented in the literature; see, for example, [2, 4, 5] and the references therein.

In addition to stochastic systems, there have been great efforts in the research of the modeling of dynamic systems subject to random abrupt changes in their parameters [6–8]. Such random abrupt changes may be caused by various factors, including the switching between economic scenarios, abrupt changes in the operation point for nonlinear plant, and actuator/sensor failure or repairs, to name just a few. Fortunately, Markov jump systems provide a natural framework for modeling these practical systems subject to random abrupt changes. Since the pioneering work on Markov jump systems was introduced in [9], considerable research results related to Markov jump systems or system with Markovian jumping parameters have been presented in terms of a variety of methods. For more details, we refer to the literature [10–17]. When Markovian jumping parameters appear in stochastic systems, many control issues have been studied recently by researchers. For example, robust stability and stabilization problems were investigated in [18], passivity-based control problem was addressed in [19], optimal control problems were studied in [12, 20–22], and the sliding-mode control problem was solved in [23].

It is worth noting that the controller design methods proposed in the previous literature require two critical assumptions. One is that the controller can be implemented exactly, and the other is that the communication between the physical plant and controller is always perfect. Such two assumptions, however, may not be unreasonable in practice. Firstly, in the implementation of a design controller, uncertainties or inaccuracies do occur because of round-off errors in numerical computation. Some existing control

synthesis methods have proven to be sensitive, or fragile, with respect to small perturbations in controller parameters. Therefore, it is an important question to design a controller, which guarantees that the controller is insensitive to some amount of errors with its gain, that is, the nonfragile or resilient control problem [24]. Secondly, a modern control system can hardly work without the help of the networks and the computers and their intercommunication. They bring a lot of advantages, but the existence of network-induced phenomena is unavoidable [25]. For instance, packet losses may occur due to the unreliability of the network links. These may limit the scope of the applications of the existing results related to stochastic systems with Markovian jumping parameters. The main purpose of this paper, therefore, is to shorten such a gap.

In this paper, we make the first attempt to deal with the nonfragile H_∞ control for stochastic systems with Markovian jumping parameters and random packet losses. The packet losses phenomena are assumed to exist in communication links between the physical plant and controller. Attention is focused on the design of a nonfragile controller such that the resulting closed-loop system is stochastically mean square stable, and meanwhile a prescribed H_∞ performance is satisfied. Sufficient conditions for the existence of such a controller are given in terms of LMIs. By solving a convex optimization problem, a desired nonfragile controller can be constructed based on the use of standard numerical algorithms [26].

Notation. Throughout this paper, for symmetric matrices P, the notation $P \geq 0$ (resp., $P > 0$) means that the matrix P is positive semidefinite (resp., positive definite); I and 0 represent the identity matrix and zero matrix with appropriate dimension. The notation M^T represents the transpose of the matrix M; $\operatorname{diag}\{\cdots\}$ stands for a block-diagonal matrix. In symmetric block matrices or complex matrix expressions, we employ an asterisk $(*)$ to represent a term that is induced by symmetry; $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathscr{P}$; $(\Omega, \mathscr{F}, \mathscr{P})$ is a probability space; Ω is the sample space, $\mathscr{F}$ is the σ-algebra of subsets of the sample space, and $\mathscr{P}$ is the probability measure on $\mathscr{F}$; $l_2[0,\infty)$ is the space of square-summable infinite vector sequences over $[0,\infty)$; $|\cdot|$ refers to the Euclidean vector norm; $\|\cdot\|_2$ stands for the usual $l_2[0,\infty)$ norm. Matrices, if not explicitly stated, are assumed to have compatible dimensions. Z^+ represents $\{0,1,2,\ldots\}$.

2. Problem Formulations

Consider the following discrete-time stochastic Markov jump system over a probability space $(\Omega, \mathscr{F}, \mathscr{P})$:

$$\begin{aligned}(\Sigma): x(k+1) &= A(\delta_k)x(k) + B_1(\delta_k)u(k) + C(\delta_k)v(k) \\ &\quad + [E(\delta_k)x(k) + F(\delta_k)v(k)]\omega(k), \\ z(k) &= D(\delta_k)x(k) + B_2(\delta_k)u(k) + G(\delta_k)v(k),\end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state vector; $u(k) \in \mathbb{R}^m$ is the controlled input; $z(k) \in \mathbb{R}^p$ is the controlled output; $v(k) \in \mathbb{R}^q$ is the exogenous disturbance input that belongs to $l_2[0,\infty)$. For each δ_k, $A(\sigma_k)$, $B_1(\sigma_k)$, $C(\sigma_k)$, $E(\sigma_k)$, $F(\sigma_k)$, $D(\sigma_k)$, $B_2(\sigma_k)$, and $G(\sigma_k)$ are real constant matrices with appropriate dimensions. $\omega(k)$ is a one-dimensional zero mean Gaussian white noise sequence on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with

$$\begin{aligned}&\mathbb{E}\{\omega(k)\} = 0; \qquad \mathbb{E}\left\{\omega^2(k)\right\} = 1; \\ &\mathbb{E}\{\omega(l)\omega(k)\} = 0, \quad l \neq k.\end{aligned} \tag{2}$$

In system (Σ), the system mode switching is governed by a discrete-time homogeneous Markov chain $\{\delta_k\}$ $(k \in Z^+)$, which takes values in a finite state space $\mathscr{S} = \{1,2,\ldots,\mathscr{N}\}$ with transition probability matrix $\Pi \triangleq \{\psi_{\alpha\beta}\}$ given by

$$\psi_{\alpha\beta} \triangleq \Pr\{\delta_{k+1} = \beta \mid \delta_k = \alpha\} \geq 0, \quad \forall \alpha, \beta \in \mathscr{S}, k \in Z^+, \tag{3}$$

with $0 \leq \psi_{\alpha\beta} \leq 1$, for any $\alpha, \beta \in \mathscr{S}$, and

$$\sum_{\beta=1}^{\mathscr{N}} \psi_{\alpha\beta} = 1, \quad \alpha \in \mathscr{S}. \tag{4}$$

In practice, it is usually of importance to require very accurate controllers to achieve given engineering specifications. However, the resulting closed-loop systems are sensitive to changes in controller gain. In this case, once there are some small perturbations in the controller parameters, the existence of these perturbations may cause a serious deterioration of system performance. Hence, it is imperative to consider the design of nonfragile controllers [24]. Consequently, in this paper, we are interested in designing the controller in the following form:

$$u_{oc} = (K_\alpha + \Delta K_\alpha(k))x_{ic}(k), \quad \alpha \in \mathscr{S}, \tag{5}$$

where $x_{ic}(k)$ is the input of the controller; u_{oc} is the output of the controller; K_α are the gain matrices of the controller, which will be determined; $\Delta K_\alpha(k)$ are real-valued unknown matrices denoting the additive gain variations as follows:

$$\Delta K_\alpha(k) = M_\alpha H_\alpha(k) N_\alpha, \quad \alpha \in \mathscr{S}, \tag{6}$$

where M_α and N_α, $\alpha \in \mathscr{S}$, are the known real constant matrices of appropriate dimensions and $H_\alpha(k)$ are unknown time-varying matrix functions, which satisfy the following constraint:

$$H_\alpha^T(k)H_\alpha(k) \leq I, \quad \alpha \in \mathscr{S}. \tag{7}$$

Remark 1. Normally, under the implicit assumption that the communication between the plant and the controller is perfect, one can readily get that the controlled input $u(k)$ is equivalent to the the output of controller u_{oc} and the measurement state of the plant $x(k)$ is also equivalent to the input of the controller $x_{ic}(k)$. As noted in the previous section, such an assumption is sometimes unpractical especially under networked environments because of the existence of the packet losses.

Therefore, in this paper, the packet losses phenomena are considered in communication links. As a result, one has $u(k) \neq u_{oc}$ and $x(k) \neq x_{ic}(k)$, and the relations between them are modeled by using a stochastic method as follows:

$$x_{ic}(k) = \rho_k x(k), \qquad u(k) = \sigma_k u_{oc}. \tag{8}$$

Here, $\{\rho_k\}$ and $\{\sigma_k\}$ are two independent Bernoulli processes. As shown in (8), $\{\rho_k\}$ models the unreliable communication link from the sensor to the controller and $\{\sigma_k\}$ models the unreliable communication link from the controller to the actuator. Inspired by [27], a natural assumption on $\{\rho_k\}$ and $\{\sigma_k\}$ can be made as follows:

$$\begin{aligned} &\operatorname{Prob}\{\rho_k = 1\} = \mathbb{E}\{\rho_k\} = \overline{\rho}, \qquad \operatorname{Prob}\{\rho_k = 0\} = 1 - \overline{\rho}, \\ &\operatorname{Prob}\{\sigma_k = 1\} = \mathbb{E}\{\sigma_k\} = \overline{\sigma}, \qquad \operatorname{Prob}\{\sigma_k = 0\} = 1 - \overline{\sigma}, \end{aligned} \tag{9}$$

where either $\overline{\rho}$ or $\overline{\sigma}$ is a known constant satisfying $\overline{\rho} \in [0,1]$ and $\overline{\sigma} \in [0,1]$. Clearly, for $\{\rho_k\}$, when $\overline{\rho} = 0$ (resp., $\overline{\rho} = 1$), it means that the communication link from the sensor to the controller fails (resp., successful transmission), and $\{\sigma_k\}$ also has a similar inference. Throughout this paper, we also assume that the sequences $\omega(k)$, $\{\delta_k\}$, $\{\rho_k\}$, and $\{\sigma_k\}$ are mutually independent. Clearly, one can get

$$u(k) = \sigma_k \rho_k \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k). \tag{10}$$

In order to address the considered problem, before presenting further results, let us introduce a new Bernoulli process $\{\varrho_k\}$ satisfying $\varrho_k \equiv \sigma_k \rho_k$. Then, simple computation yields

$$\begin{aligned} &\operatorname{Prob}\{\varrho_k = 1\} = \mathbb{E}\{\varrho_k\} = \overline{\varrho} = \overline{\rho\sigma}, \\ &\operatorname{Prob}\{\varrho_k = 0\} = 1 - \overline{\rho\sigma}, \end{aligned} \tag{11}$$

which implies that

$$\begin{aligned} u(k) &= \sigma_k \rho_k \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k) \\ &= \varrho_k \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k) \\ &= \overline{\varrho} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k) \\ &\quad + \left(\varrho_k - \overline{\varrho}\right) \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k). \end{aligned} \tag{12}$$

Under the control law (12), the resulting closed-loop system can be obtained as

$$\begin{aligned} \left(\overline{\Sigma}\right): x(k+1) &= \Omega_{1\alpha}(k) x(k) + \left(\varrho_k - \overline{\varrho}\right) \Omega_{2\alpha}(k) x(k) \\ &\quad + C_\alpha \upsilon(k) + \left[E_\alpha x(k) + F_\alpha \upsilon(k)\right] \omega(k), \\ z(k) &= \Omega_{3\alpha}(k) x(k) + \left(\varrho_k - \overline{\varrho}\right) \Omega_{4\alpha}(k) x(k) + G_\alpha \upsilon(k), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Omega_{1\alpha}(k) &= A_\alpha + \overline{\varrho} B_{1\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right), \\ \Omega_{2\alpha}(k) &= B_{1\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right), \\ \Omega_{3\alpha}(k) &= D_\alpha + \overline{\varrho} B_{2\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right), \\ \Omega_{4\alpha}(k) &= B_{2\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right). \end{aligned} \tag{14}$$

Before formulating the problem to be investigated, we first introduce the following definition for system $(\overline{\Sigma})$.

Definition 2. The closed-loop system in (13) with $\upsilon(k) \equiv 0$ is said to be stochastically mean square stable (SMSS) if there exists a $\kappa > 0$ such that

$$\mathscr{E}\{\|x(k)\|\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{15}$$

for any initial condition $\|x(0)\| < \kappa$.

Definition 3. System $(\overline{\Sigma})$ is said to be SMSS with a guaranteed H_∞ performance level γ, if system $(\overline{\Sigma})$ is SMSS according to Definition 2, and the prescribed disturbance attenuation level γ is made small in the feasibility of

$$\|z(k)\|_E \leq \gamma \|\upsilon(k)\|_2, \tag{16}$$

for all nonzero $\omega(k) \in l_2[0,\infty)$ under zero initial conditions, where

$$\|z(k)\|_E \triangleq \mathscr{E}\left\{\sqrt{\sum_{k=0}^{\infty} z^T(k) z(k)}\right\}. \tag{17}$$

Now, let us state the problems concerned in this paper, which are listed as follows.

Problem I. Consider the stochastic system $(\overline{\Sigma})$, suppose that the controller gain matrices K_α and the additive gain variations $\Delta K_\alpha(k)$ are given, and determine under what condition the system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ.

Problem II. Consider the system $(\overline{\Sigma})$, and design a nonfragile controller in the form of (5) such that the resulting closed-loop system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ, in spite of the presence of packet losses phenomena.

3. Main Results

In this section, we will give an LMI approach to solving the nonfragile H_∞ control problem formulated in the previous section. Before proceeding further, we shall introduce the following lemmas, which will be used in the proof of the main results.

Lemma 4 (see [28]). *Given constant matrices $X = X^T, Y$ and $Z = Z^T > 0$ of appropriate dimensions, then*

$$X + Y^T Z Y < 0, \tag{18}$$

if and only if

$$\begin{bmatrix} X & Y^T \\ Y & -Z^{-1} \end{bmatrix} < 0, \tag{19}$$

or, equivalently,

$$\begin{bmatrix} -Z^{-1} & Y \\ Y^T & X \end{bmatrix} < 0. \tag{20}$$

Lemma 5 (see [29]). *Let A, L, E, H, and P be real matrices of appropriate dimensions with $H^T H \le I$. Then one has*

(1) *for any scalar $\epsilon > 0$ and vectors $x, y \in \mathbb{R}^n$,*

$$2x^T LHEy \le \epsilon^{-1} x^T LL^T x + \epsilon y^T E^T Ey, \tag{21}$$

(2) *for any scalar $\epsilon > 0$, such that $P - \epsilon LL^T > 0$,*

$$(A + LHE)^T P^{-1} (A + LHE) \le A^T \left(P - \epsilon LL^T\right)^{-1} A + \epsilon^{-1} E^T E. \tag{22}$$

Now, we first establish the following H_∞ performance analysis criterion, which will play a key role in derivation of the solution to the nonfragile H_∞ control problem.

Theorem 6. *Let the controller parameters in the filtering error system $(\overline{\Sigma})$, scalars $\gamma > 0$, $\overline{\varrho} > 0$, be given. Then, system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ, if there exist positive matrices $P_\alpha > 0$, such that, for each $\alpha \in \mathcal{S}$,*

$$\begin{bmatrix} -P_\alpha & 0 & \Omega_{1\alpha}^T(k) & \sqrt{\overline{\varrho}(1-\overline{\varrho})}\Omega_{2\alpha}^T(k) & E_\alpha^T & \Omega_{3\alpha}^T(k) & \sqrt{\overline{\varrho}(1-\overline{\varrho})}\Omega_{4\alpha}^T(k) \\ * & -\gamma^2 I & C_\alpha^T & 0 & F_\alpha^T & G_\alpha^T & 0 \\ * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0 \\ * & * & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{23}$$

where $\mathcal{P}_\alpha \triangleq \sum_{\beta\in\mathcal{S}} \psi_{\alpha\beta} P_\beta$.

Proof. Firstly, we need to establish the stochastic mean square stability criterion of system $(\overline{\Sigma})$. For this purpose, we consider system $(\overline{\Sigma})$ with $v(k) \equiv 0$ and choose a stochastic Lyapunov function for system $(\overline{\Sigma})$ as follows:

$$V(x(k), k) = x^T(k) P_\alpha x(k), \tag{24}$$

where P_α are the positive matrices to be determined for each $\alpha \in \mathcal{S}$. Then, we have that, for each $\delta_k = \alpha \in \mathcal{S}$ and $\delta_{k+1} = \beta \in \mathcal{S}$,

$$\begin{aligned} &\mathcal{E}\{V(x(k+1), k+1) - V(x(k), k)\} \\ &\quad = \mathcal{E}\{[V(x(k+1), k+1) - V(x(k), k)] \mid \\ &\qquad (x(k), \delta_k = \alpha)\} \\ &\quad = \sum_{\beta\in\mathcal{S}} \Pr\{\delta_{k+1} = \beta \mid \delta_k = \alpha\} x^T(k+1) P_\beta x(k+1) \\ &\qquad - x^T(k) P_\alpha x(k) \\ &\quad = \mathcal{E}\left\{x^T(k+1) \mathcal{P}_\alpha x(k+1)\right\} - x^T(k) P_\alpha x(k) \\ &\quad = [\Omega_{1\alpha}(k) x(k)]^T \mathcal{P}_\alpha [\Omega_{1\alpha}(k) x(k)] \\ &\qquad + \overline{\varrho}(1-\overline{\varrho}) x^T(k) \Omega_{2\alpha}^T(k) \mathcal{P}_\alpha \Omega_{2\alpha}(k) x(k) \\ &\qquad + [E_\alpha x(k)]^T \mathcal{P}_\alpha [E_\alpha x(k)] - x^T(k) P_\alpha x(k). \end{aligned} \tag{25}$$

On the other hand, it can be deduced from (23) that

$$\begin{bmatrix} -P_\alpha & \Omega_{1\alpha}^T(k) & \sqrt{\overline{\varrho}(1-\overline{\varrho})}\Omega_{2\alpha}^T(k) & E_\alpha^T \\ * & -\mathcal{P}_\alpha^{-1} & 0 & 0 \\ * & * & -\mathcal{P}_\alpha^{-1} & 0 \\ * & * & * & -\mathcal{P}_\alpha^{-1} \end{bmatrix} < 0. \tag{26}$$

By applying the Schur complement formula (i.e., Lemma 4) to (26), for system $(\overline{\Sigma})$ with $v(k) \equiv 0$, one can readily obtain that

$$\mathcal{E}\{V(x(k+1), k+1) - V(x(k), k)\} < 0; \tag{27}$$

that is, system $(\overline{\Sigma})$ with $v(k) \equiv 0$ is SMSS according to [14, 16]. Next, we will show the H_∞ performance analysis of system $(\overline{\Sigma})$. To this end, we also obtain that each $\delta_k = \alpha \in \mathcal{S}$ and $\delta_{k+1} = \beta \in \mathcal{S}$,

$$\begin{aligned} &\mathcal{E}\{V(x(k+1), k+1) - V(x(k), k)\} \\ &\quad = [\Omega_{1\alpha}(k) x(k) + C_\alpha v(k)]^T \mathcal{P}_\alpha [\Omega_{1\alpha}(k) x(k) + C_\alpha v(k)] \\ &\qquad + \overline{\varrho}(1-\overline{\varrho}) x^T(k) \Omega_{2\alpha}^T(k) \mathcal{P}_\alpha \Omega_{2\alpha}(k) x(k) \\ &\qquad + [E_\alpha x(k) + F_\alpha v(k)]^T \mathcal{P}_\alpha [E_\alpha x(k) + F_\alpha v(k)] \\ &\qquad - x^T(k) P_\alpha x(k). \end{aligned} \tag{28}$$

Note that

$$\begin{aligned} &\mathcal{E}\left\{z^T(k) z(k) - \gamma^2 v^T(k) v(k)\right\} \\ &\quad = [\Omega_{3\alpha}(k) x(k) + G_\alpha v(k)]^T [\Omega_{3\alpha}(k) x(k) + G_\alpha v(k)] \\ &\qquad + \overline{\varrho}(1-\overline{\varrho}) x^T(k) \Omega_{4\alpha}^T(k) \Omega_{4\alpha}(k) x(k) - \gamma^2 v^T(k) v(k). \end{aligned} \tag{29}$$

It can be verified that

$$\begin{aligned}
&\mathscr{E}\left\{z^T(k)z(k)-\gamma^2 v^T(k)v(k)\right.\\
&\quad\left.+V(x(k+1),k+1)-V(x(k),k)\right\}\\
&=\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]^T\mathscr{P}_\alpha\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]\\
&\quad+\bar{\varrho}(1-\bar{\varrho})x^T(k)\Omega_{2\alpha}^T(k)\mathscr{P}_\alpha\Omega_{2\alpha}(k)x(k)\\
&\quad+\left[E_\alpha x(k)+F_\alpha v(k)\right]^T\mathscr{P}_\alpha\left[E_\alpha x(k)+F_\alpha v(k)\right]\\
&\quad-x^T(k)P_\alpha x(k)+\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]^T\\
&\quad\times\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]\\
&\quad+\bar{\varrho}(1-\bar{\varrho})x^T(k)\Omega_{4\alpha}^T(k)\Omega_{4\alpha}(k)x(k)-\gamma^2 v^T(k)v(k)\\
&=\begin{bmatrix}x(k)\\ v(k)\end{bmatrix}^T\begin{bmatrix}-P_\alpha & 0\\ 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}x(k)\\ v(k)\end{bmatrix}\\
&\quad+\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]^T\\
&\quad\times\mathscr{P}_\alpha\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]\\
&\quad+\bar{\varrho}(1-\bar{\varrho})x^T(k)\Omega_{2\alpha}^T(k)\mathscr{P}_\alpha\Omega_{2\alpha}(k)x(k)\\
&\quad+\left[E_\alpha x(k)+F_\alpha v(k)\right]^T\mathscr{P}_\alpha\left[E_\alpha x(k)+F_\alpha v(k)\right]\\
&\quad+\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]^T\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]\\
&\quad+\bar{\varrho}(1-\bar{\varrho})x^T(k)\Omega_{4\alpha}^T(k)\Omega_{4\alpha}(k)x(k).
\end{aligned}\tag{30}$$

Similar to the derivation of (27), we apply the Schur complement to (23) and get

$$\begin{aligned}
&\mathscr{E}\left\{z^T(k)z(k)-\gamma^2 v^T(k)v(k)\right.\\
&\quad\left.+V(x(k+1),k+1)-V(x(k),k)\right\}<0.
\end{aligned}\tag{31}$$

For $k=0,1,2,\ldots$, summing up both sides of (31) under zero initial condition and noticing $V(x(\infty),\infty)\geq 0$, it can be verified that

$$\mathscr{E}\left\{\sum_{k=0}^{\infty}z^T(k)z(k)\right\}\leq\gamma^2\sum_{k=0}^{\infty}v^T(k)v(k),\tag{32}$$

or, equivalently, condition (16) is satisfied. This completes the proof. □

In the following, we will present a solution to the nonfragile H_∞ controller design problem for system (Σ) based on Theorem 6. The following theorem proposes a sufficient condition for the existence of such a controller for system (Σ).

Theorem 7. *Consider system (Σ), let scalars $\gamma>0$, $\bar{\varrho}>0$ be given, and let matrices $J_{1\alpha}$, $J_{2\alpha}$, and $J_{3\alpha}$ be fixed. Then, there exists an admissible controller in the form of (5) such that the resulting closed-loop system $(\bar{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ, in spite of the presence of packet losses phenomena if there exist matrices $Q_\alpha>0, X$ such that the following LMIs hold for each $\alpha\in\mathcal{S}$:*

$$\begin{bmatrix}
-Q_\alpha & 0 & \Gamma_{1\alpha}^T & \sqrt{\bar{\varrho}(1-\bar{\varrho})}Y_\alpha^T B_{1\alpha}^T & X^T E_\alpha^T & \Gamma_{6\alpha}^T & \sqrt{\bar{\varrho}(1-\bar{\varrho})}Y_\alpha^T B_{2\alpha}^T & X^T N_\alpha^T\\
* & -\gamma^2 I & C_\alpha^T & 0 & F_\alpha^T & G_\alpha^T & 0 & 0\\
* & * & \Gamma_{2\alpha} & \Gamma_{3\alpha} & 0 & \Gamma_{7\alpha} & \Gamma_{10\alpha} & 0\\
* & * & * & \Gamma_{4\alpha} & 0 & \Gamma_{8\alpha} & \Gamma_{11\alpha} & 0\\
* & * & * & * & \Gamma_{5\alpha} & 0 & 0 & 0\\
* & * & * & * & * & \Gamma_{9\alpha} & \Gamma_{12\alpha} & 0\\
* & * & * & * & * & * & \Gamma_{13\alpha} & 0\\
* & * & * & * & * & * & * & -\varepsilon_\alpha I
\end{bmatrix}<0,\tag{33}$$

where

$$\begin{aligned}
\Gamma_{1\alpha}&=A_\alpha X+\bar{\varrho}B_{1\alpha}Y_\alpha,\\
\Gamma_{2\alpha}&=\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{1\alpha}Q_\beta J_{1\alpha}^T-XJ_{1\alpha}^T-J_{1\alpha}X^T\\
&\quad+\varepsilon_\alpha\bar{\varrho}^2B_{1\alpha}M_\alpha M_\alpha^T B_{1\alpha}^T,\\
\Gamma_{3\alpha}&=\varepsilon_\alpha\bar{\varrho}\sqrt{\bar{\varrho}(1-\bar{\varrho})}B_{1\alpha}M_\alpha M_\alpha^T B_{1\alpha}^T,\\
\Gamma_{4\alpha}&=\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{2\alpha}Q_\beta J_{2\alpha}^T-XJ_{2\alpha}^T-J_{2\alpha}X^T\\
&\quad+\varepsilon_\alpha\bar{\varrho}(1-\bar{\varrho})B_{1\alpha}M_\alpha M_\alpha^T B_{1\alpha}^T,\\
\Gamma_{5\alpha}&=\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{3\alpha}Q_\beta J_{3\alpha}^T-XJ_{3\alpha}^T-J_{3\alpha}X^T,\\
\Gamma_{6\alpha}&=D_\alpha X+\bar{\varrho}B_{2\alpha}Y_\alpha,\\
\Gamma_{7\alpha}&=\varepsilon_\alpha\bar{\varrho}^2B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{8\alpha}&=\varepsilon_\alpha\bar{\varrho}\sqrt{\bar{\varrho}(1-\bar{\varrho})}B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{9\alpha}&=\varepsilon_\alpha\bar{\varrho}^2B_{2\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T-I,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{10\alpha} &= \varepsilon_\alpha \bar{\varrho}\sqrt{\bar{\varrho}(1-\bar{\varrho})}B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{11\alpha} &= \varepsilon_\alpha \bar{\varrho}(1-\bar{\varrho})B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{12\alpha} &= \varepsilon_\alpha \bar{\varrho}\sqrt{\bar{\varrho}(1-\bar{\varrho})}B_{2\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{13\alpha} &= \varepsilon_\alpha \bar{\varrho}(1-\bar{\varrho})B_{2\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T - I.
\end{aligned} \tag{34}$$

In this case, a suitable nonfragile H_∞ controller in the form of (5) *is given by*

$$K_\alpha = Y_\alpha X^{-1}, \quad 1 \le \alpha \le \mathcal{N}. \tag{35}$$

Proof. Introduce the new variables $Q_\alpha = X^T P_\alpha X$; then one can find that

$$\mathcal{P}_\alpha \triangleq \sum_{\beta\in\mathcal{S}} \psi_{\alpha\beta}P_\beta = \sum_{\beta\in\mathcal{S}} \psi_{\alpha\beta}X^{-T}Q_\beta X^{-1}, \tag{36}$$

which implies that

$$-\mathcal{P}_\alpha^{-1} = -X\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right)^{-1}X^T. \tag{37}$$

Note that

$$\begin{aligned}
&\left(J_{l\alpha}\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right) - X\right)\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right)^{-1}\\
&\quad\times\left(J_{l\alpha}\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right) - X\right) \ge 0, \quad l = 1,2,3.
\end{aligned} \tag{38}$$

It follows from (37) that

$$-\mathcal{P}_\alpha^{-1} \le \sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{l\alpha}Q_\beta J_{l\alpha}^T - XJ_{l\alpha}^T - J_{l\alpha}X^T, \quad l = 1,2,3. \tag{39}$$

In view of Lemma 5, one can get that

$$\Xi_{1\alpha}^T H_\alpha(k)\Xi_{2\alpha} + \Xi_{2\alpha}^T H_\alpha^T(k)\Xi_{1\alpha} \le \varepsilon_\alpha \Xi_{1\alpha}^T\Xi_{1\alpha} + \varepsilon_\alpha^{-1}\Xi_{2\alpha}^T\Xi_{2\alpha}, \tag{40}$$

where

$$\begin{aligned}
\Xi_{1\alpha} &= \left[0 \;\; 0 \;\; \bar{\varrho}M_\alpha^T B_{1\alpha}^T \;\; \sqrt{\bar{\varrho}(1-\bar{\varrho})}M_\alpha^T B_{1\alpha}^T \;\; 0 \;\; \bar{\varrho}M_\alpha^T B_{2\alpha}^T \;\; \sqrt{\bar{\varrho}(1-\bar{\varrho})}M_\alpha^T B_{2\alpha}^T\right],\\
\Xi_{2\alpha} &= \left[N_\alpha X \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right].
\end{aligned} \tag{41}$$

Using Lemma 4 and combining (33), (39), and (40) result in

$$\begin{bmatrix}
-Q_\alpha & 0 & \tilde{\Gamma}_{1\alpha}^T & \sqrt{\bar{\varrho}(1-\bar{\varrho})}\tilde{\Gamma}_{2\alpha}^T & X^T E_\alpha^T & \tilde{\Gamma}_{3\alpha}^T & \sqrt{\bar{\varrho}(1-\bar{\varrho})}\tilde{\Gamma}_{4\alpha}^T\\
* & -\gamma^2 I & C_\alpha^T & 0 & F_\alpha^T & G_\alpha^T & 0\\
* & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0 & 0\\
* & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0\\
* & * & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0\\
* & * & * & * & * & -I & 0\\
* & * & * & * & * & * & -I
\end{bmatrix} < 0, \tag{42}$$

where

$$\begin{aligned}
\tilde{\Gamma}_{1\alpha} &= A_\alpha X + \bar{\varrho}B_{1\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right),\\
\tilde{\Gamma}_{2\alpha} &= B_{1\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right),\\
\tilde{\Gamma}_{3\alpha} &= D_\alpha X + \bar{\varrho}B_{2\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right),\\
\tilde{\Gamma}_{4\alpha} &= B_{2\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right).
\end{aligned} \tag{43}$$

Then, by pre- and postmultiplying (42) by $\mathrm{diag}\{X^{-T}, I, I, I, I, I, I\}$ and its transpose, one has that inequality (23) holds. Therefore, in light of Theorem 6, we can conclude that the resulting closed-loop system is SMSS with a guaranteed H_∞ performance level γ. This completes the proof. □

4. An Illustrative Example

In this section, an example is used to illustrate the effectiveness of the presented nonfragile controller design method. Consider the discrete-time stochastic Markov jump system

(Σ) over a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with two modes ($\alpha = 1, 2$) and the following parameters:

$$A_1 = \begin{bmatrix} 1.1 & 0.31 & 0 \\ 0 & 0.33 & 0.21 \\ 0 & 0 & -0.52 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0.2 \\ 0 \\ 0 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad B_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} 0.1 & 0.2 & -0.1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.8 & -0.38 & 0 \\ -0.2 & 0 & 0.21 \\ 0.1 & 0 & -0.55 \end{bmatrix},$$
$$B_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 \\ 0.12 \\ 0 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} -0.12 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$
$$B_{22} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \\ 0 & 0.2 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.2 \\ 0 \\ 0 \end{bmatrix},$$
$$M_2 = M_1, \qquad N_2 = N_1. \tag{44}$$

Here, our aim is to design a nonfragile controller in the form of (5) such that the resulting closed-loop system is SMSS with a guaranteed H_∞ performance level γ, in spite of the presence of packet losses phenomena. To this end, we suppose that $\overline{\varrho} = 0.7$, $\gamma = 1$, $J_{11} = J_{12} = \mathrm{diag}\{0.25, 0.25, 0.25\}$, $J_{21} = J_{22} = \mathrm{diag}\{0.5, 0.5, 0.5\}$, and $J_{31} = J_{32} = \mathrm{diag}\{0.05, 0.05, 0.05\}$, and choose the transition probability matrix Π as $\Pi = \left[\begin{smallmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{smallmatrix}\right]$. Then, by applying Theorem 7, one can get feasible solutions as follows:

$$Y_1 = \begin{bmatrix} -14.5256 & -10.2590 & -1.2361 \\ -1.9851 & -0.9283 & 0.6443 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} -6.8634 & 6.1819 & -0.9802 \\ 0.3373 & -0.0477 & 2.2476 \end{bmatrix},$$
$$X = \begin{bmatrix} 12.7563 & 2.1242 & 0.7457 \\ 1.8940 & 26.0177 & -1.2706 \\ 0.6125 & 0.1118 & 13.4984 \end{bmatrix}. \tag{45}$$

Thus, the desired controller gains K_α ($\alpha = 1, 2,$) can be given by

$$K_1 = \begin{bmatrix} -1.0905 & -0.3050 & -0.0600 \\ -0.1548 & -0.0233 & 0.0541 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.5797 & 0.2850 & -0.0138 \\ 0.0191 & -0.0041 & 0.1651 \end{bmatrix}. \tag{46}$$

5. Conclusions

In this paper, we have studied the problem of nonfragile H_∞ control for stochastic systems with Markovian jumping parameters and random packet losses. An LMI approach has been developed to design a nonfragile controller which ensures both the stochastic mean square stability and a prescribed H_∞ performance level for the resulting closed-loop systems in the presence of random packet losses. The proposed approach has been illustrated to be effective by an example. It should be pointed out that the states of the system are assumed to be precisely known, but this is difficult to achieve in practice [30–32]. Therefore, one of our further research topics is to develop nonfragile output feedback controller design methods for stochastic Markov jump systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Finite-Time Boundedness for a Class of Delayed Markovian Jumping Neural Networks with Partly Unknown Transition Probabilities

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This paper is concerned with the problem of finite-time boundedness for a class of delayed Markovian jumping neural networks with partly unknown transition probabilities. By introducing the appropriate stochastic Lyapunov-Krasovskii functional and the concept of stochastically finite-time stochastic boundedness for Markovian jumping neural networks, a new method is proposed to guarantee that the state trajectory remains in a bounded region of the state space over a prespecified finite-time interval. Finally, numerical examples are given to illustrate the effectiveness and reduced conservativeness of the proposed results.

1. Introduction

Over the past decades, delayed neural networks have been successfully applied in the pattern recognition, signal processing, image processing, and pattern recognition problems. However, these successful applications mostly rely on the dynamic behaviors of delayed neural networks and some of these applications are dependent on stability of the equilibria of neural networks. Up to now, there have been a large number of results related to dynamical behaviors of delayed neural networks [1–8].

On the one, in the past few decades, Markovian jump systems have gained special research attention. Such class of systems is a special class of stochastic hybrid systems, which may switch from one to another at the different time. Such as component failures, sudden environmental disturbance and abrupt variations of a nonlinear system [9–11]. Moreover, it is shown that such jumping can be decided by a Markovian chain [12]. For the linear Markovian jumping systems, many important issues have been devoted extensively such as stability, stabilization, control synthesis, and filter design [13–16]. In reality, however, it is worth mentioning that most of the gotten results are based on the implicit assumptions that the complete knowledge of transition probabilities is known. It is known that in most situations, the transition probabilities rate of Markovian jump systems and networks is not known; it is difficult to obtain all the transition probabilities. Therefore, it is of great importance to investigate the partly unknown transition probabilities. Very recently, the systems with partially unknown transition probabilities have been fully investigated and many important results have been obtained; for a recent survey on this topic and related questions, one can refer to [17–23]. However, it has been shown that the existing delay-dependent results are conservative.

On the other hand, the practical problems which described system stay as not exceeding a given threshold over finite-time interval are considered. Compared with classical Lyapunov stability, finite-time stability was studied to tackle the transient behavior of systems in the finite-time interval. Recently, the concept of finite-time stability has been revisited in the terms of linear matrix inequalities (LMIs); some results have been obtained to guarantee that system is finite-time stable and finite-time bounded [24–39]. To the best of our knowledge, the finite-time stability analysis for Markovian jumping neural networks with mode-dependent time-varying delays and partially known transition rates has not been tackled, and such a situation motivates our present study.

The main contribution of this paper lies in proposing a novel method for finite-time boundedness of delayed Markovian jumping neural networks with partly unknown transition probabilities. The considered system is more general than the systems with completely known or completely unknown transition probabilities, which can be regarded as two special cases of the one tackled here. In contrast to study on Markovian jumping neural networks with time delays, the knowledge of the unknown elements is not required in our method. By employing the appropriate Lyapunov-Krasovskii functional, the sufficient conditions are obtained to ensure that the system does not exceed a given threshold in a specified time interval. The finite-time bounded criteria can be tackled in the form of LMIs. Finally, numerical examples are given to demonstrate that the derived results are less conservative and more useful than some existent ones.

2. Preliminaries

Given a probability space (Ω, F, P) where Ω, F and P, respectively, represents the sample space, the algebra of events and the probability measure which defined on Ω. In this paper, we consider the following n-neuron Markovian jumping neural network over the space (Ω, F, P) described by

$$\begin{gathered}\dot{x}(t) = -A_{r_t}x(t) + B_{r_t}f(x(t)) + C_{r_t}f\left(x\left(t-\tau_{r_t}(t)\right)\right) + J \\ x(t) = \phi(t), \quad t \in [-\tau, 0),\end{gathered} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathsf{T}}$ represents the neural state vector of the system, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^{\mathsf{T}}$ is the nonlinear activation function with the initial condition $f(0) = 0$, $A_{r_t} = \operatorname{diag}\{a_1(r_t), a_2(r_t), \ldots, a_n(r_t)\}$ describes the rate with each neuron which would reset its potential to resting state in isolation, $B_{r_t} = [b_{ij}(r_t)]_{n\times n}$ and $C_{r_t} = [c_{ij}(r_t)]$ are the connection weight matrices and the delayed connection weight matrices, respectively, and $J = [J_1, J_2, \ldots, J_n]^{\mathsf{T}}$ denotes a constant external input vector. $\tau_{r_t}(t)$ are the time-varying delays which satisfy

$$\begin{gathered}0 \le \tau_{r_t}(t) \le \tau_{r_t}, \\ 0 \le \dot{\tau}_{r_t}(t) \le d_{r_t} \le 1,\end{gathered} \tag{2}$$

where τ_{r_t} and d_{r_t} are constant scalars and $\tau = \max_{r_t}\{\tau_{r_t}\}$, $d = \max_{r_t}\{d_{r_t}\}$.

Remark 1. This assumption is often employed to investigate the stability of neural networks. It is worth noting that if this assumption is not true, corresponding time-delays are not a continuous function belonging to a given interval; neither the lower nor upper bounds for time-varying delays are available. Therefore, it may lead to more conservativeness.

Let the random form process $\{r_t, t \ge 0\}$ be the Markovian stochastic process taking values on the finite set $\mathcal{N} = \{1, 2, \ldots, N\}$ with transition rate matrix $\Omega = \{\mu_{ij}\}$, $i, j \in \mathcal{N}$; namely, for $r_t = i$, $r_{t+1} = j$, one has

$$\Pr\left(r_{t+h} = j \mid r_t = i\right) = \begin{cases}\mu_{ij}h + o(h), & \text{if } j \ne i \\ 1 + \mu_{ii}h + o(h), & \text{if } j = i,\end{cases} \tag{3}$$

where $h > 0$, $\lim_{h\to 0}(o(h)/h) = 0$, and $\mu \ge 0$ $(i, j \in \mathcal{N}, j \ne i)$, denote switching rate from mode i at time t to mode j at time $t + h$. For all $i \in \mathcal{N}$, $\mu_{ii} = -\sum_{j=1, j\ne i}\mu_{ij}$. Moreover, the Markovian process transition matrix Ω is defined as follows:

$$\Omega = \begin{bmatrix}\mu_{11} & \mu_{12} & \cdots & \mu_{1N} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{N1} & \mu_{N2} & \cdots & \mu_{NN}\end{bmatrix}. \tag{4}$$

Moreover, the transition rates of jumping process in this paper are considered to be partly accessed; that is, some elements in matrix Ω are unknown. Therefore, the transition rates matrix Ω which is Markovian jump system (1) may be as follows:

$$\Omega = \begin{bmatrix}\mu_{11} & ? & \cdots & \mu_{1N} \\ ? & \mu_{22} & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \cdots & \mu_{NN}\end{bmatrix}, \tag{5}$$

where ? represents the inaccessible elements. For notational clarity, for all $i \in \mathcal{N}$, we denote $\mathcal{N} = \mathcal{N}^i_{\mathcal{K}} + \mathcal{N}^i_{\mathcal{UK}}$ and we denote that

$$\begin{gathered}\mathcal{N}^i_{\mathcal{K}} \equiv \left\{j : \mu_{ij} \text{ is known}\right\}, \\ \mathcal{N}^i_{\mathcal{UK}} \equiv \left\{j : \mu_{ij} \text{ is unknown}\right\}.\end{gathered} \tag{6}$$

Moreover, if $\mathcal{N}^i_{\mathcal{K}} \ne \emptyset$, $\mathcal{N}^i_{\mathcal{K}}$ and $\mathcal{N}^i_{\mathcal{UK}}$ can be further described, respectively, as

$$\begin{gathered}\mathcal{N}^i_{\mathcal{K}} = \left\{\mathcal{K}^i_1, \mathcal{K}^i_2, \ldots, \mathcal{K}^i_m\right\}, \\ \mathcal{N}^i_{\mathcal{UK}} = \left\{\mathcal{UK}^i_1, \mathcal{UK}^i_2, \ldots, \mathcal{UK}^i_{N-m}\right\},\end{gathered} \tag{7}$$

where $\mathcal{N}^i_m \in \mathbb{Z}^+$ represents the mth known element with the index $\mathcal{N}^i_m$ in the ith row of matrix Ω. $\mathcal{UN}^i_{N-m} \in \mathbb{Z}^+$ represents the $N - m$th unknown element with the index $\mathcal{UN}^i_{Nm}$ in the ith row of matrix Ω.

Set $\mathcal{N}$ contains N modes of system (1) and, for $r_t = i \in \mathcal{N}$, the system matrices of the ith mode are denoted by A_i, B_i, and C_i, which are considered to be real known with appropriate dimensions.

Remark 2. The Markovian jump process $\{r_t, t \ge 0\}$ in the literature is always assumed μ_{ij} ether to be completely known $(\mathcal{N}^i_{\mathcal{K}})$ or completely unknown $(\mathcal{N}^i_{\mathcal{UK}})$. Therefore, our transition probabilities matrix considered in this paper is more general than the Markovian jump systems and therefore covers the existing ones.

Assumption 3. The neuron state-based nonlinear function $f(x(t))$ considered in Markovian jump system (1) is bounded and satisfies

$$0 \le \frac{f_s(\varsigma_1) - f_s(\varsigma_2)}{\varsigma_1 - \varsigma_2} \le \gamma_s, \quad s = 1, 2, \ldots, n \tag{8}$$

for all $\varsigma_1, \varsigma_2 \in \mathcal{R}$, with γ_s being known real constants with $s = 1, 2, \ldots, n$.

It should be noted that by using the Brouwer fixed-point theorem, there should exist at least the one equilibrium point for system (1). Assuming that $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^\mathsf{T}$ is the equilibrium point of (1) and using the transformation $z(\cdot) = x(\cdot) - x^*$, system (1) can be converted to the following system:

$$\dot{z}(t) = -A_{r_t} z(t) + B_{r_t} g(z(t)) + C_{r_t} g\left(z\left(t - \tau_{r_t}(t)\right)\right), \tag{9}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^\mathsf{T}$, $g(z(\cdot)) = [g_1(z_1(x(t))), g_2(x(t)), \ldots, g_n(x(t))]^\mathsf{T}$, and $g_i(z_i(z_i(\cdot))) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*)$, $i = 1, 2, \ldots, n$. According to Assumption 3, one can obtain that

$$0 \le \frac{g_i(z_i(t))}{z_i(t)} \le \gamma_i, \quad g_i(0) = 0,\ i = 1, 2, \ldots, n. \tag{10}$$

Definition 4 (see [33]). The nominal time-delayed Markovian jumping neural networks (1) are said to be stochastically finite-time bounded with respect to (c_1, c_2, T), if

$$\mathbb{E}\|x(t_1)\|^2 \le c_1 \Longrightarrow \mathbb{E}\|x(t_2)\|^2 \le c_2, \quad t_1 \in [-\tau, 0], \quad t_2 \in [0, T]. \tag{11}$$

Definition 5 (see [34]). Let $V(x_t, r_t)$ be a stochastic positive functional and define its weak infinitesimal operator as

$$\begin{aligned} &\pounds V(x_t, r_t = i) \\ &\quad = \lim_{\Delta \to 0} \frac{1}{\Delta} \left[\mathbb{E}\left\{V(x_{t+\Delta}, r_{t+\Delta}) \mid x_t, r_t = i\right\} - V(x_t, r_t = i)\right]. \end{aligned} \tag{12}$$

3. Finite-Time H_∞ Performance Analysis

In this section, one method would be employed to analyze the finite-time stability of Markovian jump systems with partial information on transition probabilities.

Theorem 6. *Given a time constant $T > 0$, the delayed Markovian jumping neural networks* (1) *are stochastically finite-time bounded with respect to (c_1, c_2, T), if there exist a positive constant $\eta > 0$, mode-dependent symmetric positive-definite matrices $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $W_1 > 0$, $W_2 > 0$ $(i \in \mathcal{N})$, a set of symmetric matrices S_υ $(\upsilon = 1, 2, \ldots, N)$, any appropriately dimensioned matrices M_i, N_i $(i \in \mathcal{N})$, Γ_s, and scalars λ_l $(l = 1, 2, \ldots, 6)$ such that the following matrix inequalities hold:*

$$\begin{gathered} \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} Q_{1j} - \left(1 + \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij}\right) W_1 + \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} Q_{1i} < 0, \\ Q_{1j} - W_1 + Q_{1i} < 0, \quad j \in \mathcal{N}_{\mathcal{UK}}^i,\ j \ne i, \\ Q_{1j} - W_1 + Q_{1i} < 0, \quad j \in \mathcal{N}_{\mathcal{UK}}^i,\ j = i, \\ \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} Q_{2j} - \left(1 + \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij}\right) W_2 + \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} Q_{2i} < 0, \\ Q_{2j} - W_2 + Q_{2i} < 0, \quad j \in \mathcal{N}_{\mathcal{UK}}^i,\ j \ne i, \\ Q_{2j} - W_2 + Q_{2i} < 0, \quad j \in \mathcal{N}_{\mathcal{UK}}^i,\ j = i, \end{gathered}$$

$$\begin{aligned} \Sigma_i = {} & e_1 \left(1 + \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij}\right) \left(-P_i A_i - A_i^\mathsf{T} P_i\right) e_1^\mathsf{T} \\ & + e_1 \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} P_j e_1^\mathsf{T} + 2 e_1 P_i B_i e_3^\mathsf{T} + 2 e_1 P_i C_i e_4 + e_1 Q_{1i} e_1^\mathsf{T} \\ & - \left(1 - d_i - \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} \tau_j\right) e_2 Q_{1i} e_2^\mathsf{T} + e_3 Q_{2i} e_3^\mathsf{T} \\ & - (1 - d_i) e_4 Q_{2i} e_4^\mathsf{T} + \sum_{j=1}^{N} \mu_{ij} \tau_j e_4 Q_{2i} e_4^\mathsf{T} \\ & + \tau e_1 W_1 e_1^\mathsf{T} + \tau e_3 W_2 e_3^\mathsf{T} + e_1 \Gamma_s M_i \Gamma_s e_1^\mathsf{T} \\ & - e_3 M_i e_3^\mathsf{T} + e_2 \Gamma_s N_i \Gamma_s e_2^\mathsf{T} - e_4 N_i e_4^\mathsf{T} \\ & - e_1 \sum_{j \in \mathcal{N}_{\mathcal{K}}^i} \mu_{ij} S_\upsilon e_1^\mathsf{T} < 0, \\ & e_1 \left(-P_i A_i - A_i^\mathsf{T} P_i + P_j - S_\upsilon\right) e_1^\mathsf{T} + e_2 \tau_j Q_{2i} e_2^\mathsf{T} < 0, \\ & \qquad j \in \mathcal{N}_{\mathcal{UK}}^i, \quad j \ne i, \\ & e_1 \left(-P_i A_i - A_i^\mathsf{T} P_i + P_j - S_\upsilon\right) e_1^\mathsf{T} + e_2 \tau_j Q_{2i} e_2^\mathsf{T} > 0, \\ & \qquad j \in \mathcal{N}_{\mathcal{UK}}^i, \quad j = i, \end{aligned} \tag{13}$$

$$c_1 e^{\eta T} \left(\lambda_2 + \tau \lambda_3 + \tau \overline{\gamma}_s^2 \lambda_4 + \tau^2 \lambda_5 + \tau^2 \overline{\gamma}_s^2 \lambda_6\right) < \lambda_1 c_2, \tag{14}$$

where

$$\begin{gathered} \lambda_1 = \max_{i \in \mathcal{N}} \lambda_{\min}(P_i), \qquad \lambda_2 = \max_{i \in \mathcal{N}} \lambda_{\max}(P_i), \\ \lambda_3 = \max_{i \in \mathcal{N}} \lambda_{\max}(Q_{1i}), \qquad \lambda_4 = \max_{i \in \mathcal{N}} \lambda_{\max}(Q_{2i}), \\ \lambda_5 = \lambda_{\max}(W_1), \qquad \lambda_6 = \lambda_{\max}(W_2), \qquad \overline{\gamma}_s = \max_s (\gamma_s). \end{gathered} \tag{15}$$

Proof. We consider the following the stochastic Lyapunov-Krasovskii functional:

$$V(z_t, r_t) = \sum_{l=1}^{4} V_l(z_t, r_t), \tag{16}$$

where

$$\begin{aligned} V_1(z_t, r_t) &= z^\mathsf{T}(t) P_{r_t} z(t), \\ V_2(z_t, r_t) &= \int_{t - \tau_{r_t}(t)}^{t} z^\mathsf{T}(s) Q_{1r_t} z(s)\, ds, \\ V_3(z_t, r_t) &= \int_{t - \tau_{r_t}(t)}^{t} g^\mathsf{T}(z(s)) Q_{2r_t} g(z(s))\, ds, \\ V_4(z_t, r_t) &= \int_{-\tau}^{0} \int_{t+\theta}^{t} z^\mathsf{T}(s) W_1 z(s)\, ds\, d\theta \\ &\quad + \int_{-\tau}^{0} \int_{t+\theta}^{t} g^\mathsf{T}(z(s)) W_2 g(z(s))\, ds\, d\theta \end{aligned} \tag{17}$$

with P_i, Q_{1i}, Q_{2i}, $(i = 1, 2, \ldots, N)$, W_1, and W_2 being positive definite matrices and

$$\sum_{j=1}^{N} \mu_{ij} Q_{1j} < W_1, \tag{18}$$

$$\sum_{j=1}^{N} \mu_{ij} Q_{2j} < W_2. \tag{19}$$

For notational simplicity, let

$$\xi(t) = \left[z^{\mathsf{T}}(t), z^{\mathsf{T}}(t - \tau_i(t)), g^{\mathsf{T}}(z(t)), g^{\mathsf{T}}(z(t - \tau_i(t)))\right]^{\mathsf{T}},$$
$$e_s = \left[\underbrace{0, \ldots, 0,}_{s-1} I, \underbrace{0, \ldots, 0}_{4-s}\right]^{\mathsf{T}}, \quad s = 1, \ldots, 4. \tag{20}$$

Let £ be the infinitesimal generator of random process $\{z_t, t \geq 0\}$; then for each $r_t = i$, $i \in \mathcal{N}$, we can obtain that

$$\begin{aligned}
\pounds V_1(z_t, i) &= 2z^{\mathsf{T}}(t) P_i \dot{z}(t) + z^{\mathsf{T}}(t) \sum_{j=1}^{N} \mu_{ij} P_j z(t) \\
&= \xi^{\mathsf{T}}(t) e_1 \left(-P_i A_i - A_i^{\mathsf{T}} P_i + \sum_{j=1}^{N} \mu_{ij} P_j\right) e_1^{\mathsf{T}} \xi(t) \\
&\quad + 2\xi^{\mathsf{T}}(t) e_1 P_i B_i e_3^{\mathsf{T}} \xi(t) + 2\xi^{\mathsf{T}}(t) e_1 P_i C_i e_4 \xi(t), \\
\pounds V_2(z_t, i) &= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \mathbb{E} \\
&\quad \times \left\{\left[\int_{t+\Delta-\tau_{r_{t+\Delta}}(t+\Delta)}^{t+\Delta} z^{\mathsf{T}}(s) Q_{1r_{t+\Delta}} z(s)\, ds \mid r_t = i\right]\right. \\
&\quad \left. - \int_{t-\tau_i(t)}^{t} z^{\mathsf{T}}(s) Q_{1i} z(s)\, ds\right\} \\
&= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{\int_{t+\Delta-\tau_i(t+\Delta)-\sum_{j=1}^{N}(\mu_{ij}\Delta+o(\Delta))\tau_j(t+\Delta)}^{t+\Delta} z^{\mathsf{T}}(s)\right. \\
&\quad \times \left[Q_{1i} + \sum_{j=1}^{N}\left(\mu_{ij}\Delta + o(\Delta)\right)\right] z(s)\, ds \\
&\quad \left. - \int_{t-\tau_i(t)}^{t} z^{\mathsf{T}}(s) Q_{1i} z(s)\, ds\right\} \\
&= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{\int_{t+\Delta-\tau_i(t+\Delta)-\sum_{j=1}^{N}(\mu_{ij}\Delta+o(\Delta))\tau_j(t+\Delta)}^{t+\Delta}\right. \\
&\quad \times z^{\mathsf{T}}(s) Q_{1i} z(s)\, ds \\
&\quad \left. - \int_{t-\tau_i(t)}^{t} z^{\mathsf{T}}(s) Q_{1i} z(s)\, ds\right\} \\
&\quad + \lim_{\Delta \to 0^+} \frac{1}{\Delta} \int_{t+\Delta-\tau_i(t+\Delta)-\sum_{j=1}^{N}(\mu_{ij}\Delta+o(\Delta))\tau_j(t+\Delta)}^{t+\Delta} z^{\mathsf{T}}(s) \\
&\quad \times \sum_{j=1}^{N}\left(\mu_{ij}\Delta + o(\Delta)\right) \\
&\quad \times Q_{1j} z(s)\, ds \\
&= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \int_{t}^{t+\Delta} z^{\mathsf{T}}(s) Q_{1i} z(s)\, ds \\
&\quad + \lim_{\Delta \to 0^+} \frac{1}{\Delta} \int_{t+\Delta-\tau_i(t+\Delta)-\sum_{j=1}^{N}(\mu_{ij}\Delta+o(\Delta))\tau_j(t+\Delta)}^{t+\Delta} \\
&\quad \times z^{\mathsf{T}}(s) \sum_{j=1}^{N}\left(\mu_{ij}\Delta + o(\Delta)\right) \\
&\quad \times Q_{1j} z(s)\, ds \\
&= \xi^{\mathsf{T}}(t) e_1 Q_{1i} e_1^{\mathsf{T}} \xi(t) - \left(1 - \dot{\tau}_i(t) - \sum_{j=1}^{N} \mu_{ij} \tau_j(t)\right) \\
&\quad \times \xi^{\mathsf{T}}(t) e_2 Q_{1i} e_2^{\mathsf{T}} \xi(t) \\
&\quad + \int_{t-\tau_i(t)}^{t} z^{\mathsf{T}}(s) \left(\sum_{j=1}^{N} \mu_{ij} Q_{1j}\right) z(s)\, ds \\
&\leq \xi^{\mathsf{T}}(t) e_1 Q_{1i} e_1^{\mathsf{T}} \xi(t) - \left(1 - d_i - \sum_{j=1}^{N} \mu_{ij} \tau_j(t)\right) \\
&\quad \times \xi^{\mathsf{T}}(t) e_2 Q_{1i} e_2^{\mathsf{T}} \xi(t) \\
&\quad + \int_{t-\tau_i(t)}^{t} z^{\mathsf{T}}(s) \left(\sum_{j=1}^{N} \mu_{ij} Q_{1j}\right) z(s)\, ds.
\end{aligned} \tag{21}$$

Similar to the process above, it yields

$$\begin{aligned}
\pounds V_3(z_t, i) &\leq \xi^{\mathsf{T}}(t) e_3 Q_{2i} e_3^{\mathsf{T}} \xi(t) - (1 - d_i) \xi^{\mathsf{T}}(t) e_4 Q_{2i} e_4^{\mathsf{T}} \xi(t) \\
&\quad + \sum_{j=1}^{N} \mu_{ij} \tau_j(t) \xi^{\mathsf{T}}(t) e_4 Q_{2i} e_4^{\mathsf{T}} \xi(t) \\
&\quad + \int_{t-\tau_i(t)}^{t} g^{\mathsf{T}}(z(s)) \left(\sum_{j=1}^{N} \mu_{ij} Q_{2i}\right) g(z(s))\, ds, \\
\pounds V_4(z_t, i) &= \tau \xi^{\mathsf{T}}(t) e_1 W_1 e_1^{\mathsf{T}} \xi(t) - \int_{t-\tau}^{t} z^{\mathsf{T}}(s) W_1 z(s)\, ds \\
&\quad + \tau \xi^{\mathsf{T}}(t) e_3 W_2 e_3^{\mathsf{T}} \xi(t) \\
&\quad - \int_{t-\tau}^{t} g^{\mathsf{T}}(z(s)) W_2 g(z(s))\, ds.
\end{aligned} \tag{22}$$

From (18) and (19), we obtain that

$$\begin{aligned}
&\int_{t-\tau_i(t)}^{t} z^{\mathsf{T}}(s) \left(\sum_{j=1}^{N} \mu_{ij} Q_{1j}\right) z(s)\, ds \\
&\quad \leq \int_{t-\tau}^{t} z^{\mathsf{T}}(s) \left(\sum_{j=1}^{N} \mu_{ij} Q_{1j}\right) z(s)\, ds \\
&\quad \leq \int_{t-\tau}^{t} z^{\mathsf{T}}(s) W_1 z(s)\, ds,
\end{aligned}$$

$$\begin{aligned}
&\int_{t-\tau_i(t)}^{t} g^{\mathsf{T}}(z(s))\left(\sum_{j=1}^{N}\mu_{ij}Q_{2j}\right)g(z(s))\,ds \\
&\quad\le \int_{t-\tau}^{t} g^{\mathsf{T}}(z(s))\left(\sum_{j=1}^{N}\mu_{ij}Q_{2j}\right)g(z(s))\,ds \\
&\quad\le \int_{t-\tau}^{t} g^{\mathsf{T}}(z(s))\,W_2 g(z(s))\,ds.
\end{aligned} \tag{23}$$

Also, it results from (10) that for any appropriately dimensioned matrices M_i, N_i, $(i = 1,2,\ldots,N)$, one can obtain

$$\begin{aligned}
0 &\le \xi^{\mathsf{T}}(t)\,e_1\Gamma_s M_i\Gamma_s e_1^{\mathsf{T}}\xi(t) - \xi^{\mathsf{T}}(t)\,e_3 M_i e_3^{\mathsf{T}}\xi(t), \\
0 &\le \xi^{\mathsf{T}}(t)\,e_2\Gamma_s N_i\Gamma_s e_2^{\mathsf{T}}\xi(t) - \xi^{\mathsf{T}}(t)\,e_4 N_i e_4^{\mathsf{T}}\xi(t).
\end{aligned} \tag{24}$$

From (16)–(24), we have

$$\pounds V(z_t,i) \le \xi^{\mathsf{T}}(t)\,\Xi_i\xi(t), \tag{25}$$

where

$$\begin{aligned}
\Xi_i = {} & e_1\left(-P_iA_i - A_i^{\mathsf{T}}P_i + \sum_{j=1}^{N}\mu_{ij}P_j\right)e_1^{\mathsf{T}} \\
& + 2e_1P_iB_ie_3^{\mathsf{T}} + 2e_1P_iC_ie_4 + e_1Q_{1i}e_1^{\mathsf{T}} \\
& - \left(1 - d_i - \sum_{j=1}^{N}\mu_{ij}\tau_j\right)e_2Q_{1i}e_2^{\mathsf{T}} \\
& + e_3Q_{2i}e_3^{\mathsf{T}} - (1-d_i)\,e_4Q_{2i}e_4^{\mathsf{T}} \\
& + \sum_{j=1}^{N}\mu_{ij}\tau_je_4Q_{2i}e_4^{\mathsf{T}} + \tau e_1W_1e_1^{\mathsf{T}} \\
& + \tau e_3W_2e_3^{\mathsf{T}} + e_1\Gamma_sM_i\Gamma_se_1^{\mathsf{T}} - e_3M_ie_3^{\mathsf{T}} \\
& + e_2\Gamma_sN_i\Gamma_se_2^{\mathsf{T}} - e_4N_ie_4^{\mathsf{T}}.
\end{aligned} \tag{26}$$

By the fact that $\sum_{j\in\mathcal{N}}\mu_{ij} = 0$, we can rewrite Ξ_i as

$$\begin{aligned}
\Xi_i = {} & e_1\left(-P_iA_i - A_i^{\mathsf{T}}P_i + \sum_{j=1}^{N}\mu_{ij}P_j\right)e_1^{\mathsf{T}} \\
& + 2e_1P_iB_ie_3^{\mathsf{T}} + 2e_1P_iC_ie_4 + e_1Q_{1i}e_1^{\mathsf{T}} \\
& - \left(1 - d_i - \sum_{j=1}^{N}\mu_{ij}\tau_j\right)e_2Q_{1i}e_2^{\mathsf{T}} \\
& + e_3Q_{2i}e_3^{\mathsf{T}} - (1-d_i)\,e_4Q_{2i}e_4^{\mathsf{T}} \\
& + \sum_{j=1}^{N}\mu_{ij}\tau_je_4Q_{2i}e_4^{\mathsf{T}} + \tau e_1W_1e_1^{\mathsf{T}} \\
& + \tau e_3W_2e_3^{\mathsf{T}} + e_1\Gamma_sM_i\Gamma_se_1^{\mathsf{T}} - e_3M_ie_3^{\mathsf{T}} \\
& + e_2\Gamma_sN_i\Gamma_se_2^{\mathsf{T}} - e_4N_ie_4^{\mathsf{T}} \\
& - e_1\sum_{j=1}^{N}\mu_{ij}\left(P_iA_i + A_i^{\mathsf{T}}P_i + S_v\right)e_1^{\mathsf{T}}.
\end{aligned} \tag{27}$$

Thus, from (6), we have

$$\begin{aligned}
\Xi_i = {} & e_1\left(1 + \sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}\right)\left(-P_iA_i - A_i^{\mathsf{T}}P_i\right)e_1^{\mathsf{T}} \\
& + e_1\sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}P_je_1^{\mathsf{T}} + 2e_1P_iB_ie_3^{\mathsf{T}} \\
& + 2e_1P_iC_ie_4 + e_1Q_{1i}e_1^{\mathsf{T}} \\
& - \left(1 - d_i - \sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}\tau_j\right)e_2Q_{1i}e_2^{\mathsf{T}} + e_3Q_{2i}e_3^{\mathsf{T}} \\
& - (1-d_i)\,e_4Q_{2i}e_4^{\mathsf{T}} + \sum_{j=1}^{N}\mu_{ij}\tau_je_4Q_{2i}e_4^{\mathsf{T}} \\
& + \tau e_1W_1e_1^{\mathsf{T}} + \tau e_3W_2e_3^{\mathsf{T}} + e_1\Gamma_sM_i\Gamma_se_1^{\mathsf{T}} \\
& - e_3M_ie_3^{\mathsf{T}} + e_2\Gamma_sN_i\Gamma_se_2^{\mathsf{T}} - e_4N_ie_4^{\mathsf{T}} \\
& - e_1\sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}S_ve_1^{\mathsf{T}} \\
& + \sum_{j\in\mathcal{N}_{\mathcal{UK}}^i}\mu_{ij}\left[e_1\left(-P_iA_i - A_i^{\mathsf{T}}P_i + P_j - S_v\right)e_1^{\mathsf{T}}\right. \\
& \qquad \left. + e_2\tau_jQ_{2i}e_2^{\mathsf{T}}\right].
\end{aligned} \tag{28}$$

Then, for $j \in \mathcal{N}_{\mathcal{UK}}^i$ and if $i \in \mathcal{N}_{\mathcal{K}}^i$, $\Xi_i < 0$ can be guaranteed. On the other hand, for $j \in \mathcal{N}_{\mathcal{UK}}^i$ and if $i \notin \mathcal{N}_{\mathcal{K}}^i$, Ξ_i can be further expressed as

$$\begin{aligned}
\Xi_i = {} & e_1\left(1 + \sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}\right)\left(-P_iA_i - A_i^{\mathsf{T}}P_i\right)e_1^{\mathsf{T}} \\
& + e_1\sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}P_je_1^{\mathsf{T}} + 2e_1P_iB_ie_3^{\mathsf{T}} \\
& + 2e_1P_iC_ie_4 + e_1Q_{1i}e_1^{\mathsf{T}} \\
& - \left(1 - d_i - \sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}\tau_j\right)e_2Q_{1i}e_2^{\mathsf{T}} + e_3Q_{2i}e_3^{\mathsf{T}} \\
& - (1-d_i)\,e_4Q_{2i}e_4^{\mathsf{T}} + \sum_{j=1}^{N}\mu_{ij}\tau_je_4Q_{2i}e_4^{\mathsf{T}} \\
& + \tau e_1W_1e_1^{\mathsf{T}} + \tau e_3W_2e_3^{\mathsf{T}} + e_1\Gamma_sM_i\Gamma_se_1^{\mathsf{T}} \\
& - e_3M_ie_3^{\mathsf{T}} + e_2\Gamma_sN_i\Gamma_se_2^{\mathsf{T}} - e_4N_ie_4^{\mathsf{T}} \\
& - e_1\sum_{j\in\mathcal{N}_{\mathcal{K}}^i}\mu_{ij}S_ve_1^{\mathsf{T}} \\
& + \sum_{j\in\mathcal{N}_{\mathcal{UK}}^i, j\ne i}\mu_{ij}\left[e_1\left(-P_iA_i - A_i^{\mathsf{T}}P_i + P_j - S_v\right)e_1^{\mathsf{T}}\right. \\
& \qquad \left. + e_2\tau_jQ_{2i}e_2^{\mathsf{T}}\right] + \mu_{ii} \\
& \qquad \times\left[e_1\left(-P_iA_i - A_i^{\mathsf{T}}P_i + P_j - S_v\right)e_1^{\mathsf{T}}\right. \\
& \qquad \left. + e_2\tau_jQ_{2i}e_2^{\mathsf{T}}\right].
\end{aligned} \tag{29}$$

Similarly, (18) and (19) can be rewritten, respectively, as

$$\begin{aligned}
&\left\{\sum_{j\in\mathcal{N}^i_{\mathcal{K}}}\mu_{ij}Q_{1j}-\left(1+\sum_{j\in\mathcal{N}^i_{\mathcal{K}}}\mu_{ij}\right)W_1+\sum_{j\in\mathcal{N}^i_{\mathcal{K}}}\mu_{ij}Q_{1i}\right\}\\
&\quad+\sum_{j\in\mathcal{N}^i_{\mathcal{UK}},j\neq i}\mu_{ij}\left[Q_{1j}-W_1+Q_{1i}\right]\\
&\quad+\mu_{ii}\left[Q_{1j}-W_1+Q_{1i}\right]<0,\\
&\left\{\sum_{j\in\mathcal{N}^i_{\mathcal{K}}}\mu_{ij}Q_{2j}-\left(1+\sum_{j\in\mathcal{N}^i_{\mathcal{K}}}\mu_{ij}\right)W_2+\sum_{j\in\mathcal{N}^i_{\mathcal{K}}}\mu_{ij}Q_{2i}\right\}\\
&\quad+\sum_{j\in\mathcal{N}^i_{\mathcal{UK}},j\neq i}\mu_{ij}\left[Q_{2j}-W_2+Q_{2i}\right]\\
&\quad+\mu_{ii}\left[Q_{2j}-W_2+Q_{2i}\right]<0.
\end{aligned}\tag{30}$$

It is well known that $\mu_{ii}=-\sum_{j=1,j\neq i}^{N}\mu_{ij}<0$; according to (6), one can also obtain

$$\pounds V(z_t,i)<0.\tag{31}$$

On the other hand, from (32) and the needed constant $\eta>0$, it yields that

$$\mathbb{E}\{\pounds V(z_t,r_t)\}<\eta\mathbb{E}\{V(z_t,r_t)\},\tag{32}$$

from which we can easily get that

$$e^{-\eta t}\mathbb{E}\{V(z_t,r_t)\}<\mathbb{E}\{V(z_0,r_0)\}.\tag{33}$$

Note that $0\le t\le T$; we can obtain the following inequality:

$$\begin{aligned}
\mathbb{E}\{V(z_t,r_t)\}&<e^{\eta t}\mathbb{E}\{V(x_0,r_0)\}\\
&=e^{\eta t}\left[z^{\mathrm{T}}(0)P_{r_t}z(0)+\int_{-\tau_{r_t}(t)}z^{\mathrm{T}}(s)Q_{1r_t}z(s)\,ds\right.\\
&\qquad+\int_{-\tau_{r_t}(t)}g^{\mathrm{T}}(z(s))Q_{2r_t}g(z(s))\,ds\\
&\qquad+\int_{-\tau}^{0}\int_{\theta}^{0}z^{\mathrm{T}}(s)W_1z(s)\,ds\\
&\qquad\left.+\int_{-\tau}^{0}\int_{\theta}^{0}g^{\mathrm{T}}(z(s))W_1g(z(s))\,ds\right]\\
&<e^{\alpha t}\left[\max_{i\in\mathcal{N}}\lambda_{\max}(P_i)+\tau\max_{i\in\mathcal{N}}\lambda_{\max}(Q_{1i})\right.\\
&\qquad+\tau\overline{\gamma}_s^2\max_{i\in\mathcal{N}}\lambda_{\max}(Q_{2i})\\
&\qquad\left.+\tau^2\lambda_{\max}(W_1)+\tau^2\overline{\gamma}_s^2\lambda_{\max}(W_2)\right]\\
&\quad\times\sup_{-\tau\le s\le 0}\{x^{\mathrm{T}}(s)x(s)\}\\
&\le c_1e^{\eta T}\left(\lambda_2+\tau\lambda_3+\tau\overline{\gamma}_s^2\lambda_4+\tau^2\lambda_5+\tau^2\overline{\gamma}_s^2\lambda_6\right).
\end{aligned}\tag{34}$$

On the other hand, from (16), we can get

$$\mathbb{E}\{z^{\mathrm{T}}(t)P_iz(t)\}\ge\max_{i\in\mathcal{N}}\lambda_{\min}(P_i)\mathbb{E}\|z(t)\|^2.\tag{35}$$

Then, we can obtain

$$\mathbb{E}\|z(t)\|^2<\frac{c_1e^{\eta T}\left(\lambda_2+\tau\lambda_3+\tau\overline{\gamma}_s^2\lambda_4+\tau^2\lambda_5+\tau^2\overline{\gamma}_s^2\lambda_6\right)}{\lambda_1}.\tag{36}$$

By condition (14), we can obtain

$$\mathbb{E}\|z(t)\|^2<c_2.\tag{37}$$

By Definition 4, we conclude that Markovian jump system (1) is stochastically finite-time bounded with respect to (c_1,c_2,T). □

Remark 7. In this paper, it is in contrast with existing results for delay-dependent Markovian jump systems with partly unknown transition probabilities, and another different method is presented to tackle the unknown elements in the transition matrix. Compared with [33], some slack matrix variables S_v are introduced in this paper based on the probability identity $\sum_{j=1}^{N}\mu_{ij}=0$, which leads to less conservativeness than [33].

Remark 8. Theorem 6 develops a finite-time bounded criterion of Markovian jumping neural networks with time-varying delays and partially known transition rates. Theorem 6 makes full use of the information of the subsystems' upper bounds of the time-varying delays, which also brings us the less conservativeness.

Remark 9. In our paper, $\tau_i(t)$ and $\dot{\tau}_i(t)$ may indicate the different upper bounds during various time-delay intervals which satisfies condition (2), respectively. However, in existing work, for example, [17], $\tau_i(t)$ and $\dot{\tau}_i(t)$ are always extended to $\tau_i(t)\le\tau=\max\{\tau_i,i\in\mathcal{N}\}$ and $0\le\dot{\tau}_i(t)\le d=\max\{d_i,i\in\mathcal{N}\}$, respectively, which may inevitably lead to the conservativeness. Therefore, in order to reduce the conservatism, the cases above are taken into account by employing the stochastic Lyapunov-Krasovskii functional (16).

4. Illustrative Example

Example 1. Consider a class of delayed Markovian jumping neural networks (9) with two operation modes in [33]:

$$\begin{aligned}
&A_1=\begin{bmatrix}2&0\\0&1\end{bmatrix},\quad A_2=\begin{bmatrix}3&0\\0&2\end{bmatrix},\quad B_1=\begin{bmatrix}0.5&1\\-0.2&0.5\end{bmatrix},\\
&B_2=\begin{bmatrix}1.1&1\\-0.2&0.1\end{bmatrix},\quad C_1=\begin{bmatrix}0.9&0.1\\-0.1&0.1\end{bmatrix},\\
&C_2=\begin{bmatrix}0.3&-0.8\\0.1&0.2\end{bmatrix},\quad \Gamma_s=I_2.
\end{aligned}\tag{38}$$

The mode switching is governed by a Markov chain that has the following transition rate matrix:

$$\Omega=\begin{bmatrix}-0.5&0.5\\0.3&-0.3\end{bmatrix}.\tag{39}$$

In this paper, let the initial values for $c_1 = 0.25$, $T = 2$, $\eta = 1$, and time-varying delay be $\tau_1(t) = \tau_2(t) = 0.2 \times |\cos t|$, which means that $\tau = 0.2$ and $d = 0.2$. Through Theorem 6 and optimization over value c_2, it yields that delayed Markovian jumping neural networks (9) are finite-time bounded with respect to (c_1, c_2, T) with minimal $c_2 = 5.0312$ while minimal c_2 in [33] is 5.4296, which shows the less conservative result in this paper.

Example 2. Consider a class of delayed Markovian jumping neural networks (9) with partially known transition rates and operation modes described as follows:

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2.3 & 0 \\ 0 & 2.5 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0.6 \\ 0.1 & 0.3 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.4 \end{bmatrix}, \quad \Gamma_s = I_2. \tag{40}$$

The three cases of the transition rates matrices are considered as

$$\text{Case I: } \Omega = \begin{bmatrix} -0.8 & 0.3 & 0.5 \\ 0.1 & -0.8 & 0.7 \\ 0.7 & 0.4 & -1.1 \end{bmatrix},$$
$$\text{Case II: } \Omega = \begin{bmatrix} -0.8 & ? & ? \\ 0.1 & -0.8 & 0.7 \\ 0.7 & 0.4 & -1.1 \end{bmatrix}, \tag{41}$$
$$\text{Case III: } \Omega = \begin{bmatrix} -0.8 & ? & ? \\ ? & -0.8 & ? \\ 0.7 & 0.4 & -1.1 \end{bmatrix}.$$

With the same mode switching rates, initial values and time-varying delays, through Theorem 6 and optimization over value c_2, it yields that in Case I, $c_2 = 4.8124$; in Case II, $c_2 = 4.6121$; in Case III, $c_2 = 4.5372$. Therefore, the delayed Markovian jumping neural networks (9) are finite-time bounded with respect to (c_1, c_2, T).

Remark 10. The accessibility of the jumping process $\{r_t, t \geq 0\}$ in the existing literature is commonly assumed to be completely accessible or completely unaccessible. Note that the transition probabilities are still viewed as accessible in this paper. Therefore, the transition probabilities matrix considered in this paper is more general assumption than Markovian jump systems.

5. Conclusions

Unlike most existing research results focusing on Lyapunov stability property of Markovian jump system, our paper investigated finite-time stability which concerns the boundedness of state during the delayed Markovian jump interval. In this paper, we have examined the problems of finite-time boundedness for a class of delayed Markovian jumping neural networks with partly unknown transition probabilities. Based on the analysis result, the static state feedback finite-time boundedness is given. Although the derived result is not in LMIs form, we can turn it into LMIs feasibility problem by fixing some parameters. At last, numerical examples are also given to demonstrate the effectiveness of the proposed approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Cascading Dynamics of Heterogenous Scale-Free Networks with Recovery Mechanism

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In network security, how to use efficient response methods against cascading failures of complex networks is very important. In this paper, concerned with the highest-load attack (HL) and random attack (RA) on one edge, we define five kinds of weighting strategies to assign the external resources for recovering the edges from cascading failures in heterogeneous scale-free (SF) networks. The influence of external resources, the tolerance parameter, and the different weighting strategies on SF networks against cascading failures is investigated carefully. We find that, under HL attack, the fourth kind of weighting method can more effectively improve the integral robustness of SF networks, simultaneously control the spreading velocity, and control the outburst of cascading failures in SF networks than other methods. Moreover, the third method is optimal if we only knew the local structure of SF networks and the uniform assignment is the worst. The simulations of the real-world autonomous system in, Internet have also supported our findings. The results are useful for using efficient response strategy against the emergent accidents and controlling the cascading failures in the real-world networks.

1. Introduction

The robustness properties of complex networks subject to either random breakdown or intentional attacks have attracted considerable interest [1, 2], due to the blackouts in US power grids [3, 4], the large-scale congestion in the Internet [5], and the electrical blackout in Italy [6]. These accidents have threatened the network safety and resulted in enormous loss in economy.

As a result, many issues have been investigated carefully, including the robustness of the topological structure of networks [7–11], the description of cascading phenomenon and transition [12], the protection strategies against cascade [13–18], the cost of attack and defense [19, 20], and the reliability metrics of networks [21, 22]. In addition, the vulnerability of the real-world networks has become an important topic in the design of engineering safety [23, 24]. The cascading failure in power systems [25, 26] and the attacks in computer networks [27] have attracted more consideration. Some researches focus on the stability analysis for the uncertain systems [28–30] and the analysis of cyberphysical networking systems [31]. Especially, the robustness and cascading failures in interdependent networks [32–35] have become a hot topic for the past few years. Also, traffic bound [36], traffic delay [37], and the control systems of heavy inputs and delay systems [38, 39] are considered carefully.

The cascading failures [35, 40], which originate very locally but often result in a global collapse, have become one of the hottest topics in network safety. On the one hand, by characterizing the load on nodes, the considerable cascading models under the attacks on nodes have been presented. The conditions of the global cascade are explored [40, 41], where every node is assumed to have the same capacity [41]. The influence of the removal of nodes on reducing the efficiency of networks is investigated [42]. The cascading failures of the North American power grid under the loss of nodes [43]

and the cascading failures induced by flux fluctuations [44] are also probed. On the other hand, the cascading dynamics induced by the edge-based attacks have also been probed. These researches focus on the cascading model by assigning the load and adopting a local load redistribution to edges [45, 46] and the model that the overloaded edges break down with some probability [47]. The size of cascade and the cost of investment under the removal of targeted edges [48] and the cascade by adopting the Ohw' and Kirchhoff conservation law [49] are also probed.

However, once the cascading failures emerged, the important question is concerned with the efficient response to disasters. In real-world networks, there always exists some emergency mechanism or "recovery mechanism" that can be regarded as the coming external resources (e.g., manpower, number of vehicles) into networks, which recovers the overloaded components to the normal state. For example, police could deal with the emergent accidents or chaos in roads and the technical experts could handle the breakdown and repair the components damaged in technical networks, such as the electric power system and the Internet network. These accidents or breakdown can be caused by the natural events (earthquakes, floods, or extreme weather) or the intentional attacks. Such recovery mechanism can effectively counteract the overload and relieve the stress, which could make the edges or nodes from "overloaded" to "congestion" (the midstate) and maybe to "normal." Yet this recovery mechanism has not been considered in previous works. Therefore, it is important to investigate the influence of the recovery mechanism on increasing the robustness of the networks against cascading failures, especially for the network safety. We argue that probing this question will give us important implications in using efficient strategy to deal with the disasters happening in real-world networks.

In this paper, induced by highest-load attack (HL) and random attack (RA) on one edge, we study the cascading dynamics of the heterogeneous scale-free (SF) network with recovery mechanism that is represented by the external resources τ entering into SF network. Our novel model defines four kinds of weighing strategies to assign the external resource to the edges for recovering the networks from cascading failures. The influence of τ, the tolerance parameter α, and the different weighting strategies on improving the robustness against cascading failures in SF networks is investigated. We find that, firstly, under intentional attack, the fourth weighing method can more effectively decrease the number of avalanched edges, reduce the spreading speed of cascading failures, and control the outburst of cascading failures in SF networks than other methods. Secondly, as the most efficient strategy under intentional attack, the fourth weighting method needs to compute the betweenness centrality of nodes, which implies that the topological structure of SF networks is needed. Therefore, the third weighting method will be optimal if we only knew the local structure of network (namely, the degree of nodes). On the other hand, as an example in real-world networks, the simulation of the autonomous system in the Internet with scale-free characteristics also shows the same results of SF network model. It means that the simulation of real-world networks supports our findings.

The rest of this paper is organized as follows. Section 2 develops the novel model of cascading dynamics with recovery mechanism under edge-based attack, in which the external resource is assigned to the links according to the weight of links in SF network. In Section 3, we describe four kinds of weighting strategies to measure the weight of the links in SF networks. In Section 4, we compare the influence of four kinds of weighting strategies on the robustness of SF network against cascading failures and analyze the results of our simulations. Section 5 summarizes the most important findings and offers the future research.

2. The Cascading Dynamics with Recovery Mechanism

In this section, we focus on the development of cascading model on the weighted scale-free network subject to random and intentional attack on one edge.

Since many real-world networks have been observed to have a typical power-law degree distribution $P(k) \propto k^{-\gamma}$ (γ is the scale exponent), the vulnerability and the robustness of such scale-free networks (SF) under attacks have been an important problem in studying the cascading failures of complex networks [10, 40, 45–47, 50].

Therefore, in this paper, we focus on the cascading dynamics of the Barabási-Albert scale-free network model generated according to the rule of growth and preferential attachment [50]. On the other hand, The large-scale congestion in the Internet has drawn attention to the robustness of the autonomous system (AS) [5]. Therefore, as an example in the real-world networks, considering that the autonomous system (AS) formed by the graph of routers comprising the Internet from the BGP (Border Gateway Protocol) logs has been observed to show power-law degree distribution [51], we also focus on the autonomous system (AS) defined as AS1470 which has 1470 nodes and 3997 edges and the mean degree $\langle k \rangle \approx 4.26$. Here, we define the adjacent matrix of network considered as $A = (a_{ij})_{N\times N}$, where $a_{ij} = 1$ if the node i links to the node j; otherwise $a_{ij} = 0$. We denote w_{ij} as the weight of the edge e_{ij} in network.

Generally, the development of cascading model is based on the following three factors: the definition of the original load, the correlation between the original load and the capacity, and the dynamical redistribution of load after the attacks. Similarly, the cascading dynamics in this paper is modeled as follows.

(1) The original load on the edge e_{ij}: in many physical network structures, the physical flows (data packets or energy) are always forwarded along the edges according to the shortest path routing strategy. For a given pair of nodes (m, n), the flows are transmitted along the shortest paths connecting them; maybe there exist some shortest paths through the edge e_{ij}. Therefore, it is natural to define the total number of shortest paths passing through e_{ij} between any pair of nodes in a network as the load on e_{ij}. Naturally, for our weighted SF network, the load $L_{ij}(t)$ on the edge e_{ij} at time t is defined

as the number of the shortest paths through it ($t = 0$ means the initial load $L_{ij}(0)$ before attack). Now we assume that the original load on e_{ij} is $L_{ij}(0)$.

(2) The capacity C_{ij} of the edge e_{ij}: we suppose that C_{ij} is the maximum load that an edge e_{ij} could handle and is proportional to the initial load $L_{ij}(0)$; that is,

$$C_{ij} = (1 + \alpha) L_{ij}(0), \quad \forall ij, \tag{1}$$

where $\alpha \geq 0$ is the tolerance parameter. The higher α means that the edge has the higher capacity and the higher ability against failures. Also, it is rational in designing the real-world networks including power grids and the Internet, because the capacity of the links in these networks is always limited by the cost.

In most of the previous models, there were only two states assigned to a node or edge: *normal* or *overloaded*; besides the node or edge would break down (i.e., overloaded) once the load on them exceeded their capacity. However, in real-world networks, there exists some emergency mechanism that will handle the congestion state, relieve the pressure on them, and thus reduce the probability of the overload. For example, in transportation networks, the external resources (such as manpower or vehicles) will come to deal with the emergent events and recover the road from the "congested or overloaded" road to the "normal" state. Therefore, we assign a recovery rate τ_{ij} to every edge e_{ij} and assume that the threshold C^*_{ij} is the upper bound load on e_{ij} in normal state. Naturally, we define

$$\tau_{ij} = \frac{1 + \left(w_{ij} / \sum_{1 \leq i < j \leq N} w_{ij}\right) \cdot \tau}{10}, \tag{2}$$

$$C^*_{ij} = \left(1 + \alpha \cdot \tau_{ij}\right) L_{ij}(0), \quad \forall ij, \tag{3}$$

where w_{ij} is the weight of the edge e_{ij} and τ is an adjustable parameter which represents the external resources entering into the network. Here we assume $\tau \geq 1$. When developing (2) and (3), we required the following.

(i) We hope that the external resources τ enter into the network according to the importance of the edge e_{ij} that is measured by the normalized weight $w_{ij} / \sum_{1 \leq i < j \leq N} w_{ij}$. The recovery rate τ_{ij} should increase monotonically with the increasing τ. For some τ, the bigger w_{ij}, the more external resources are assigned to the edge e_{ij}, and then the recovery rate τ_{ij} can be closer to the upper bound $(1 + \tau)/10$.

(ii) We can control the parameter τ to adjust the recovery rate τ_{ij}. When $\tau = 0$, there is no external resource and $\tau_{ij} = 0.1$ is the initial recovery rate.

(iii) We have $C^*_{ij} \propto \tau_{ij} \propto \tau$. The bigger τ is, the higher τ_{ij} is, and then the closer C^*_{ij} is to C_{ij}. It implies that, when the more external resources entering into the network are assigned to the edges, the more easily the links are recovered from the abnormal to normal state. Namely, the external resources have only positive effect on the edge e_{ij}.

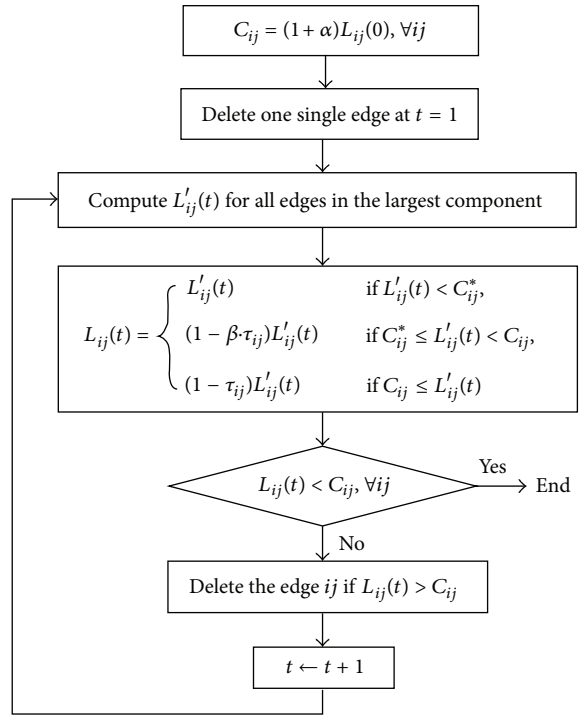

Figure 1: The evolving procedure of cascading failures in networks with the external resource τ.

We can find that such definition is rational in the actual situations and highlights the protection of the important edges. Of course, we can choose other functions of (2) and (3) satisfying these conditions.

(3) The redistribution of load: when a few edges break down, at some time t, we assume the temporary load on the edge e_{ij} as $L'_{ij}(t)$ after the redistribution of load. Then, the edge e_{ij} will get a number of external resources according to (2) once the load on e_{ij} exceeds the threshold C^*_{ij}. It means that the recovery rate τ_{ij} will work according to the degree of $L'_{ij}(t)$ exceeding the threshold C^*_{ij}. Finally, the true load $L_{ij}(t)$ on e_{ij} becomes

$$L_{ij}(t) = \begin{cases} L'_{ij}(t) & \text{if } L'_{ij}(t) < C^*_{ij}, \\ \left(1 - \beta \cdot \tau_{ij}\right) L'_{ij}(t) & \text{if } C^*_{ij} \leq L'_{ij}(t) < C_{ij}, \\ \left(1 - \tau_{ij}\right) L'_{ij}(t) & \text{if } C_{ij} \leq L'_{ij}(t), \end{cases} \tag{4}$$

where $\beta = (L'_{ij}(t) - C^*_{ij})/(C_{ij} - C^*_{ij})$. In fact, in (4), the final load $L_{ij}(t)$ indicates the three states of edge e_{ij}: *normal* (if $L_{ij}(t) < C^*_{ij}$); *congestion* (if $C^*_{ij} \leq L_{ij}(t) < C_{ij}$); *overloaded* (if $L_{ij}(t) \geq C_{ij}$). $C^*_{ij} \leq L_{ij}(t) \leq C_{ij}$ means that the edge deals with the load busily and still works; $L_{ij}(t) \geq C_{ij}$ implies that the edge e_{ij} cannot handle the too high a load even with the recovery mechanism, and as a result, the edge fails. Thus, a larger τ_{ij} leads to the stronger ability to handle the load on the edge, and finally the network will have the stronger robustness, which is consistent with the actual situations in

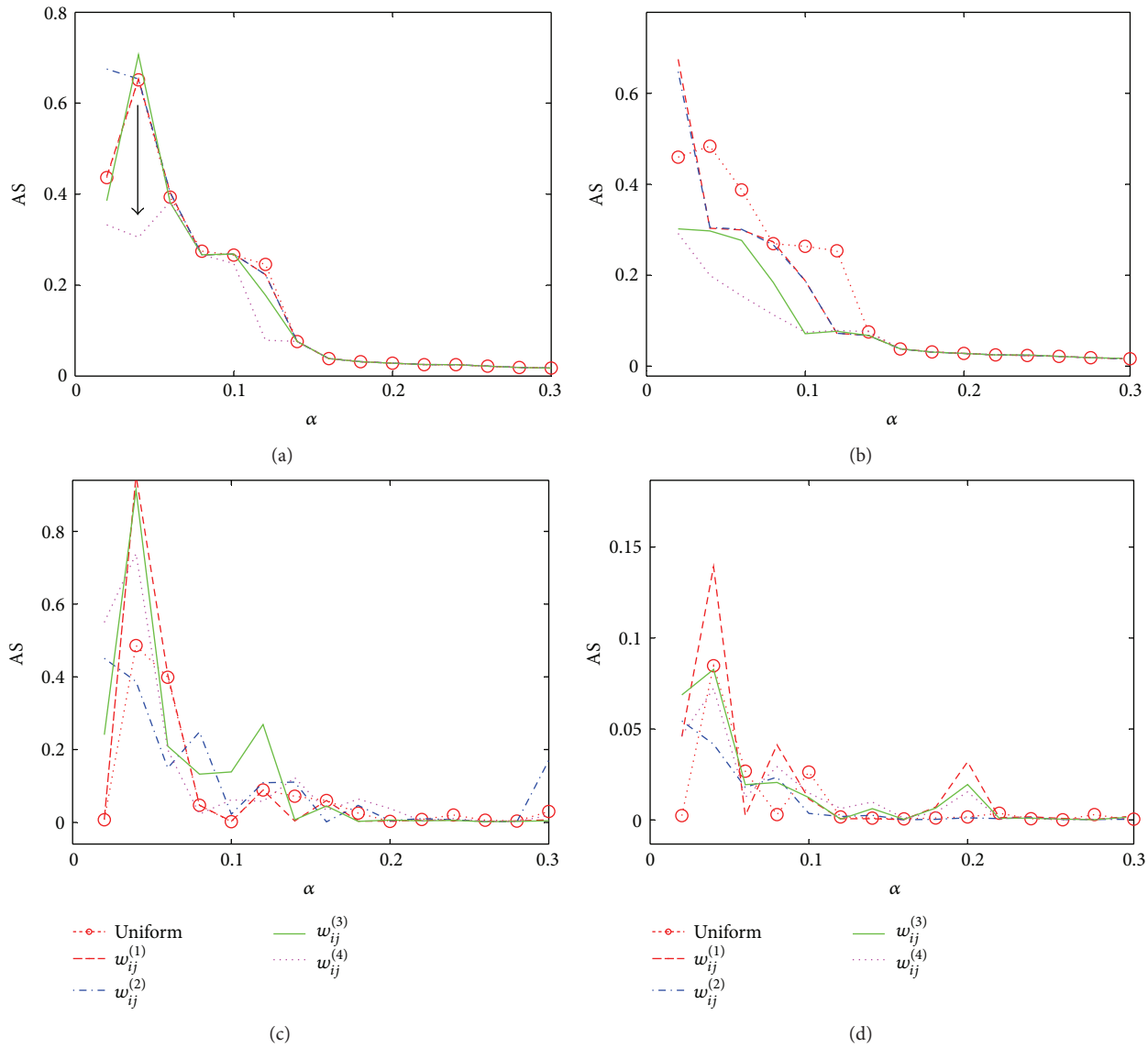

Figure 2: For SF network, after the cascade stops, the avalanche size AS as a function of α for different strategies with (a) $\tau = 20$ under HL attack, (b) $\tau = 100$ under HL attack, (c) $\tau = 20$ under RA attack, and (d) $\tau = 100$ under RA attack, respectively. Here (c) and (d) are averaged over 20 runs.

many real-world networks. Generally, the more important the edges are, the higher investment and the force are on them.

In (2), the external resource τ is assigned to the edge e_{ij} according to the weight of e_{ij}, so that (2) highlights the protection of the important edges in SF network. However, the external resources are limited and the higher τ represents the higher cost for protection. Naturally, it is needed to measure how important the edge is in order to find the efficient response strategy against disasters. This will be discussed in the following sections.

3. The Weighting Strategy

In the description of network characterization, the centrality is significant for measuring the importance of an element (node or edge) in studying cascading failures, which can be used to measure the topological position of an element in network. In this part, we will introduce four kinds of weighting methods to measure the centrality of an edge e_{ij}, which is regarded as the weight w_{ij} of e_{ij} and can reflect the importance of e_{ij} in network.

(1) The weighting strategy $w_{ij}^{(1)}$: in many real-world networks, the flows are forwarded along the edge according to the shortest path routing strategy. Thus, the edge betweenness centrality is always used to measure the centrality of the edge [52, 53], which is defined as

$$B_{ij} = \sum_{a \neq b} \frac{\sigma_{ab}\left(e_{ij}\right)}{\sigma_{ab}}, \tag{5}$$

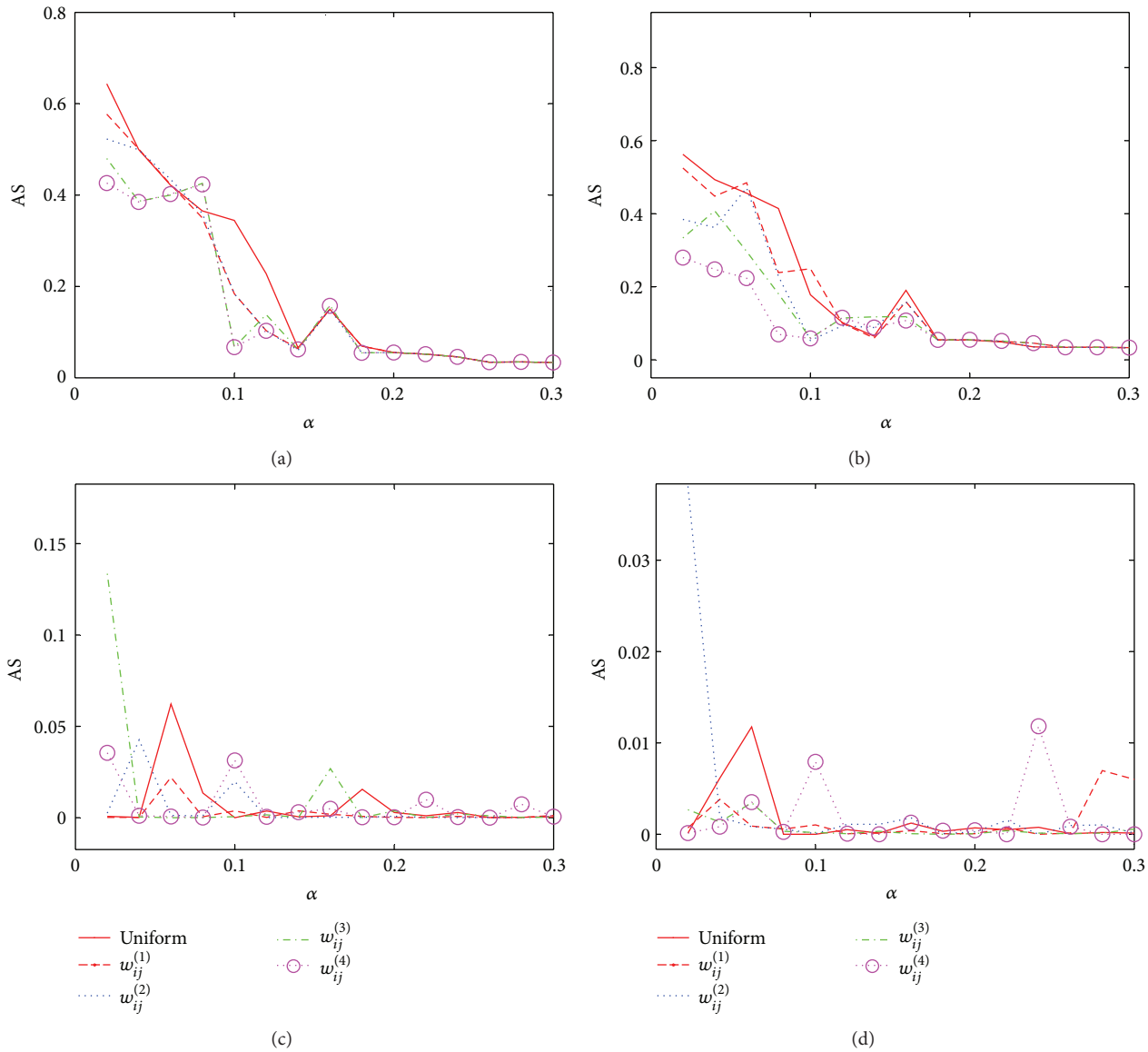

FIGURE 3: For autonomous system network, after the cascade stops, the avalanche size AS as a function of α for different strategies with (a) $\tau = 20$ under HL attack, (b) $\tau = 100$ under HL attack, (c) $\tau = 20$ under RA attack, and (d) $\tau = 100$ under RA attack, respectively. Here (c) and (d) are averaged over 20 runs.

where $\sigma_{ab}(e_{ij})$ is the number of the shortest paths between the nodes a and b passing through the edge e_{ij}. Then, we define the weight of the edge e_{ij} as

$$w_{ij}^{(1)} = B_{ij}. \tag{6}$$

(2) The weighting strategy $w_{ij}^{(2)}$: however, in real networks, the edge centrality is always related to some intrinsic quality of the end node of the edge. For example, in traffic networks, the design of the highway or the airlines always depends on the population or the economic development conditions (like GDP) among cities. These intrinsic characteristics can be seen as the quality of the node (city). The lines or roads connected to the nodes (city) with high quality always have high edge betweenness centrality, which have not been considered in the previous models yet.

Thus we define a novel edge betweenness centrality of e_{ij} as

$$B'_{ij} = \sum_{a \neq b} \frac{\sum_{k \in P_{ab}(e_{ij})} w_k}{\sum_{k \in P_{ab}} w_k}, \tag{7}$$

where P_{ab} is the set of all shortest paths between the nodes a and b, $P_{ab}(e_{ij})$ is the shortest paths between a and b passing through the edge e_{ij}, and w_k is the intrinsic quality of node k. (Here we choose the degree of node k as w_k; of course, one can choose other rational values.) Note that the definition of B'_{ij} incorporates the intrinsic characteristics of nodes with the network structure, which can better reflect the weight

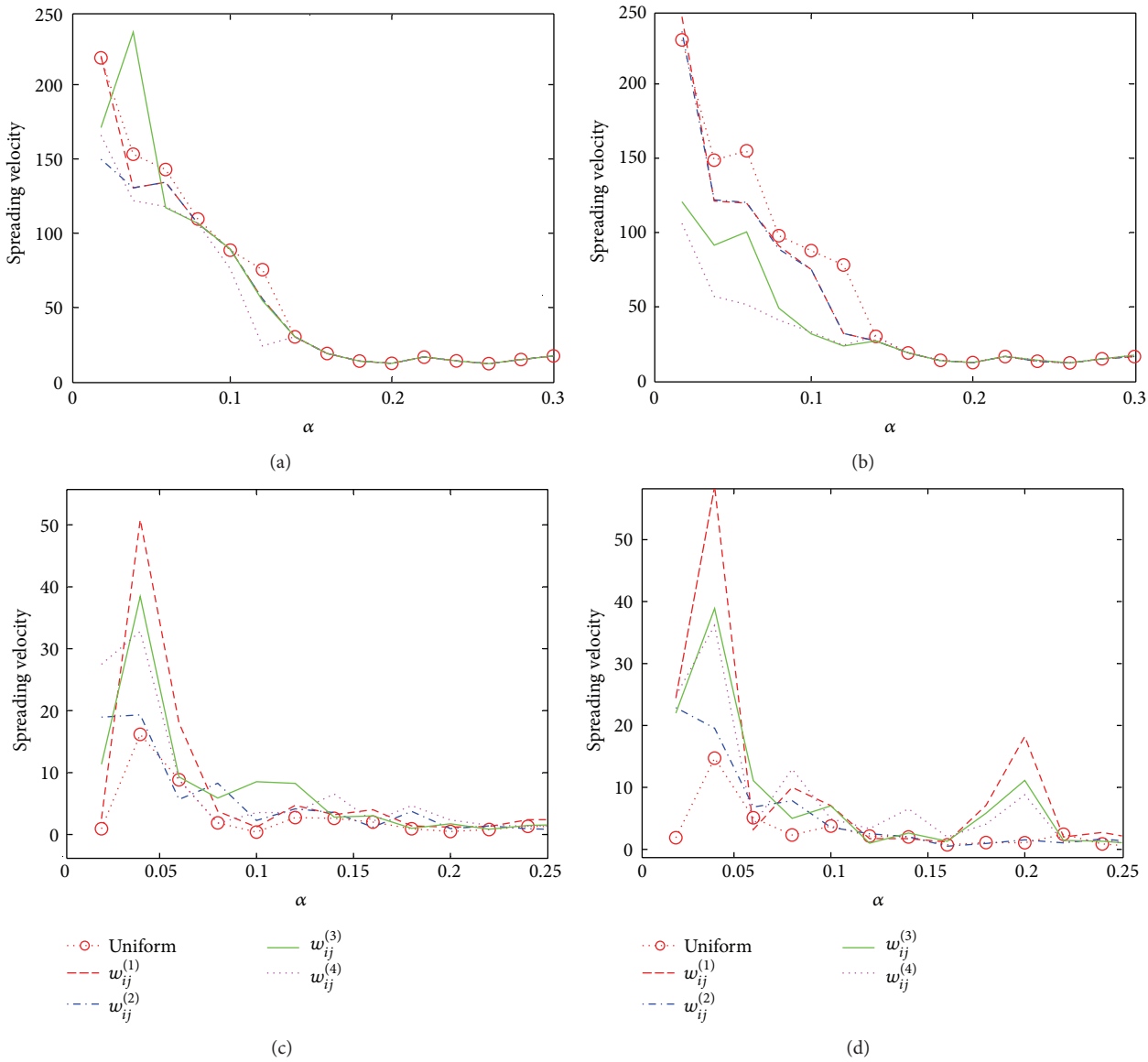

FIGURE 4: After the cascade stops, the spreading velocity V of failures in SF network as a function of α for different strategies with (a) $\tau = 20$ under HL attack, (b) $\tau = 100$ under HL attack, (c) $\tau = 20$ under RA attack, and (d) $\tau = 100$ under RA attack, respectively. Here (c) and (d) are averaged over 20 runs.

importance of edges in actual situations. Specially, (7) will degenerate into the definition in (5) if every node has a uniform intrinsic quality ($w_k = 1$). Now we assume the weight of the edge e_{ij} as

$$w_{ij}^{(2)} = B'_{ij}. \tag{8}$$

(3) The weighting strategy $w_{ij}^{(3)}$: another centrality measure of the edge e_{ij} is the product of the nodes degree of the end node i and j, which has been used to measure the weight of the edge e_{ij} [45, 46]; that is,

$$w_{ij}^{(3)} = \left(k_i k_j\right)^{\theta}, \tag{9}$$

where k_i and k_j are the degrees of nodes i and j, respectively. Here we assume $\theta = 1$.

(4) The weighting strategy $w_{ij}^{(4)}$: usually, the link is also important when the end of a link is important;this is in accordance with the real-world networks [45–47]. Moreover, the importance of one end i of a link e_{ij} can be measured by the node betweenness centrality [51, 53]; that is,

$$B_i = \sum_{a \neq b} \frac{\sigma_{ab}(i)}{\sigma_{ab}}, \tag{10}$$

where $\sigma_{ab}(i)$ is the number of the shortest paths between the nodes a and b passing through the node i. This motivated the introduction of another weight measure for an edge. Therefore, we assume that the weight of the edge e_{ij} depends

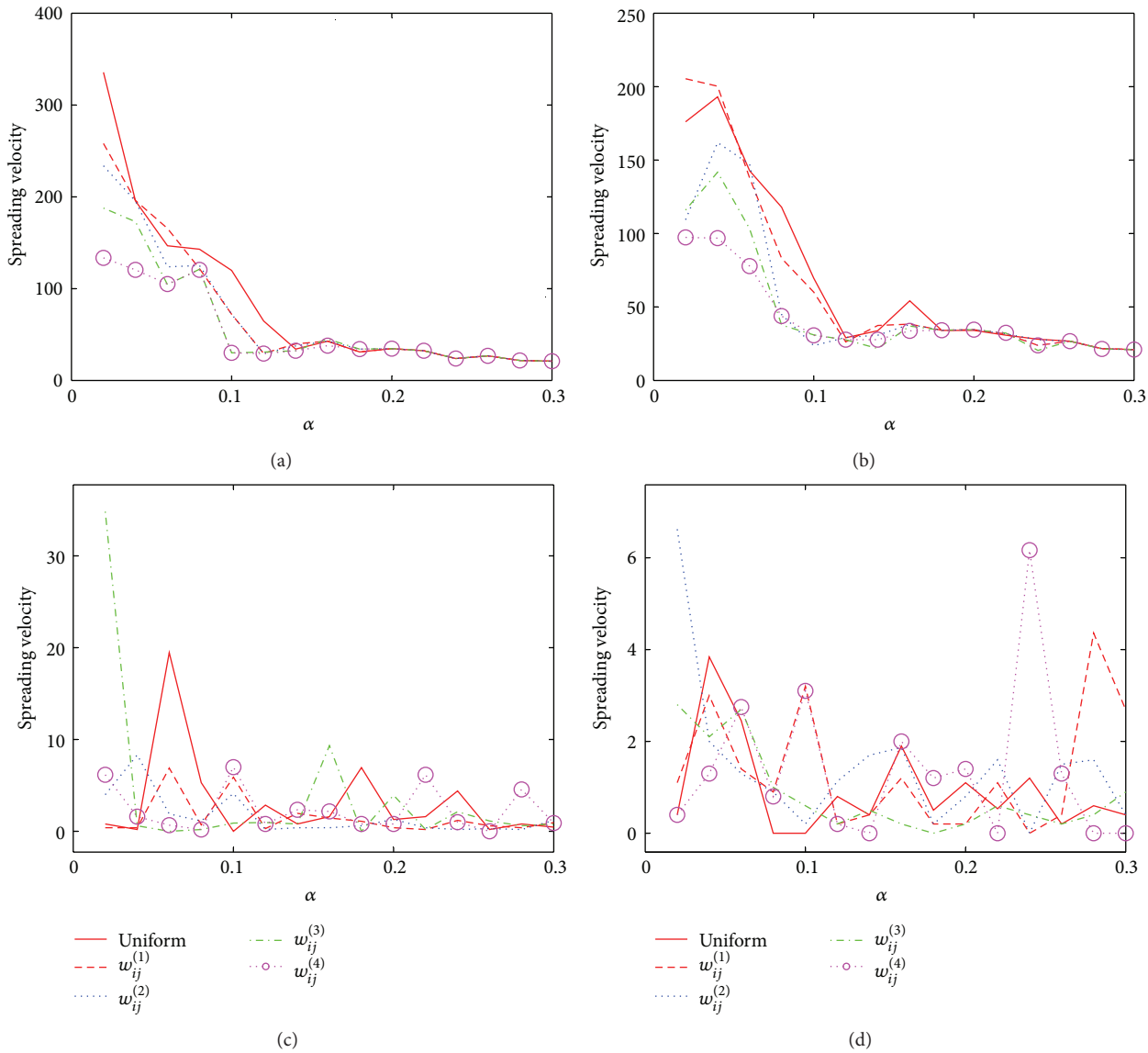

FIGURE 5: After the cascade stops, the spreading velocity V of failures in autonomous system network as a function of α for different strategies with (a) $\tau = 20$ under HL attack, (b) $\tau = 100$ under HL attack, (c) $\tau = 20$ under RA attack, and (d) $\tau = 100$ under RA attack, respectively. Here (c) and (d) are averaged over 20 runs.

on the product of betweenness centrality of the end nodes i and j, which is defined as

$$w_{ij}^{(4)} = \left(B_i B_j\right)^{\theta}. \tag{11}$$

Here we assume $\theta = 1$.

(5) The uniform strategy: finally, we should note that the SF network considered will become an unweighted network if every edge has the uniform weight (e.t., $w_{ij} = 1$). It means that every edge e_{ij} will get the uniform external resource according to (2). We defined this strategy as the uniform assignment strategy.

Now one can see that the external resource τ, the different weighting methods w_{ij}, and the tolerance parameter α would have great influence on the robustness of SF network subject to attacks on edges. This will be discussed in the following sections.

4. The Simulation and Analysis

In this paper, we mainly consider two kinds of attacks on one edge e_{ij}. (1) Highest-load attack (HL): we remove one edge with the highest initial load; (2) random attack (RA): we randomly choose one edge e_{ij} and then remove it. The attack originates from the removal of one edge and leads to the redistribution of load on other edges, and then some of them would fail as the load exceeds the capacity. This process is repeated until no edge fails, and at this moment, the cascade can be considered to be completed. Thus, the cascading process with the recovery mechanism τ under

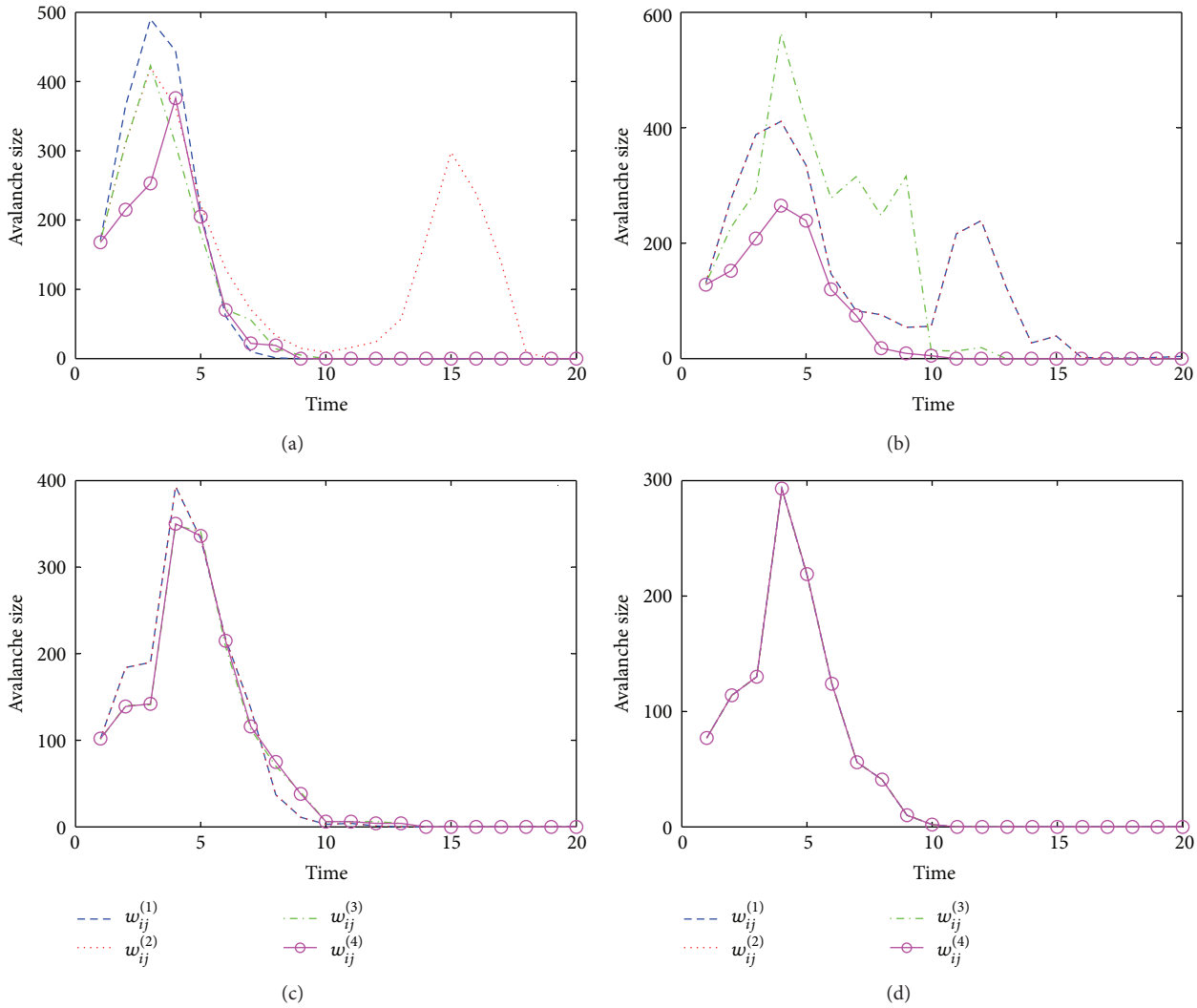

FIGURE 6: For SF network with $\tau = 20$ under HL attack, the avalanche size $N_{\text{ae}}(t)$ in each time step t as a function of t for (a) $\alpha = 0.02$, (b) $\alpha = 0.04$, (c) $\alpha = 0.06$, and (d) $\alpha = 0.08$, respectively.

edge-based attacks can be described in Figure 1. Now, in the following section, we will reveal the function of the recovery mechanism on the network robustness against cascading failures from three aspects: improving the integral robustness, controlling the spreading velocity of cascading failures, and controlling the burst of cascading failures.

4.1. Improving the Integral Robustness against Cascading Failures. Now, in the first part of this section, we focus on the function of the recovery mechanism on improving the robustness of the heterogeneous scale-free network (SF) against cascading failures, which is quantified by the following metrics: the avalanche size (AS) after cascade failures which is defined as follows:

$$\text{AS} = \frac{\sum_t N_{\text{ae}}(t)}{N_e - 1}, \tag{12}$$

where $N_{\text{ae}}(t)$ and N_e are the number of the avalanched edges at each time step t under attack and the total number of edges in initial networks, respectively. From (12), we can see that the metric AS can be regarded as a function of α and τ, and then AS could quantify the integral robustness of structure against cascading failures.

From Figures 2(a) and 2(b), it is clear that, for SF network model and the autonomous system network AS1470 subject to HL attack, as the external resources τ are assigned to the edges according to the weighting method $w_{ij}^{(4)}$, it could be better at decreasing the avalanche size (AS) thus improving the integrity of SF networks than other strategies. Especially, the effect is obvious for smaller tolerance parameter α ($\alpha < 0.2$) and more external resources ($\tau = 100$). For example, as $\alpha = 0.04$ and $\tau = 20$, the weighting method $w_{ij}^{(4)}$ could decrease the avalanche size AS from about 0.71 to 0.3 (see the arrow in Figure 2(a)). The simulations of the real-world networks (AS1470) have proved these findings (see Figures 3(a) and 3(b)). Moreover, as shown in Figures 2(b) and 3(b), the weighting method $w_{ij}^{(3)}$ is suboptimal and the uniform

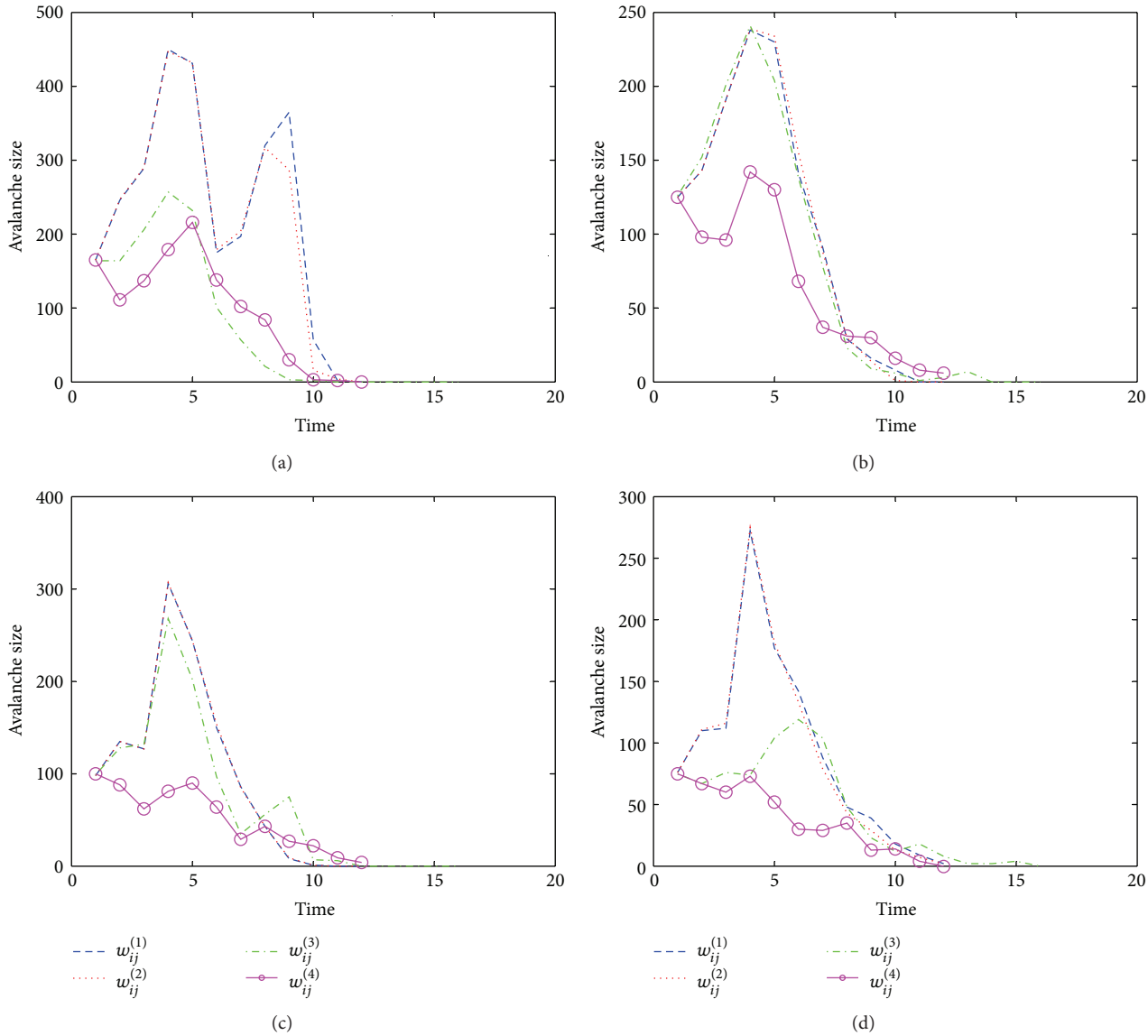

FIGURE 7: For SF network with $\tau = 100$ under HL attack, the avalanche size $N_{ae}(t)$ in each time step t as a function of t for (a) $\alpha = 0.02$, (b) $\alpha = 0.04$, (c) $\alpha = 0.06$, and (d) $\alpha = 0.08$, respectively.

method ($w_{ij} = 1$) is the worst. Although the weighting method $w_{ij}^{(4)}$ is optimal, it depends on the betweenness centrality of the nodes that needs to know the whole topological structure of SF network from (11). It implies that the third weighting strategy $w_{ij}^{(3)}$ is suggested if we only knew the local structure of networks, such as the degree of nodes.

On the other hand, as shown in Figures 2(c), 2(d), 3(c), and 3(d), under RA attack, the difference among the four kinds of weighting strategies is not clear if with fewer external resource (e.g., $\tau = 20$). But, as $\alpha \geq 0.1$, it seems that the second weighting strategy $w_{ij}^{(2)}$ and the uniform assignment strategy ($w_{ij} = 1$) are optimal if with sufficient external resource (e.g., $\tau = 100$).

4.2. Controlling the Spreading Velocity of Cascading Failures. In the second part of this section, to further measure how efficient the different weighting strategies are in response to the cascading failures in SF network, we will explore the spreading velocity of cascading failures, which is computed by V:

$$V = \frac{\sum_t N_{ae}(t)}{T}, \tag{13}$$

where $N_{ae}(t)$ is the number of the avalanched edges at each time step t under attacks and T is the evolving time step of cascading propagation in networks (see Figure 1).

As shown in Figures 4(a), 4(b), 5(a), and 5(b), under HL attack, the weighing method $w_{ij}^{(4)}$ can obviously reduce the spreading velocity of cascading failures in both the SF network model and AS1470 network, regardless of the quantity of external resources τ. Moreover, the third weighting method $w_{ij}^{(3)}$ is suboptimal if having more resources

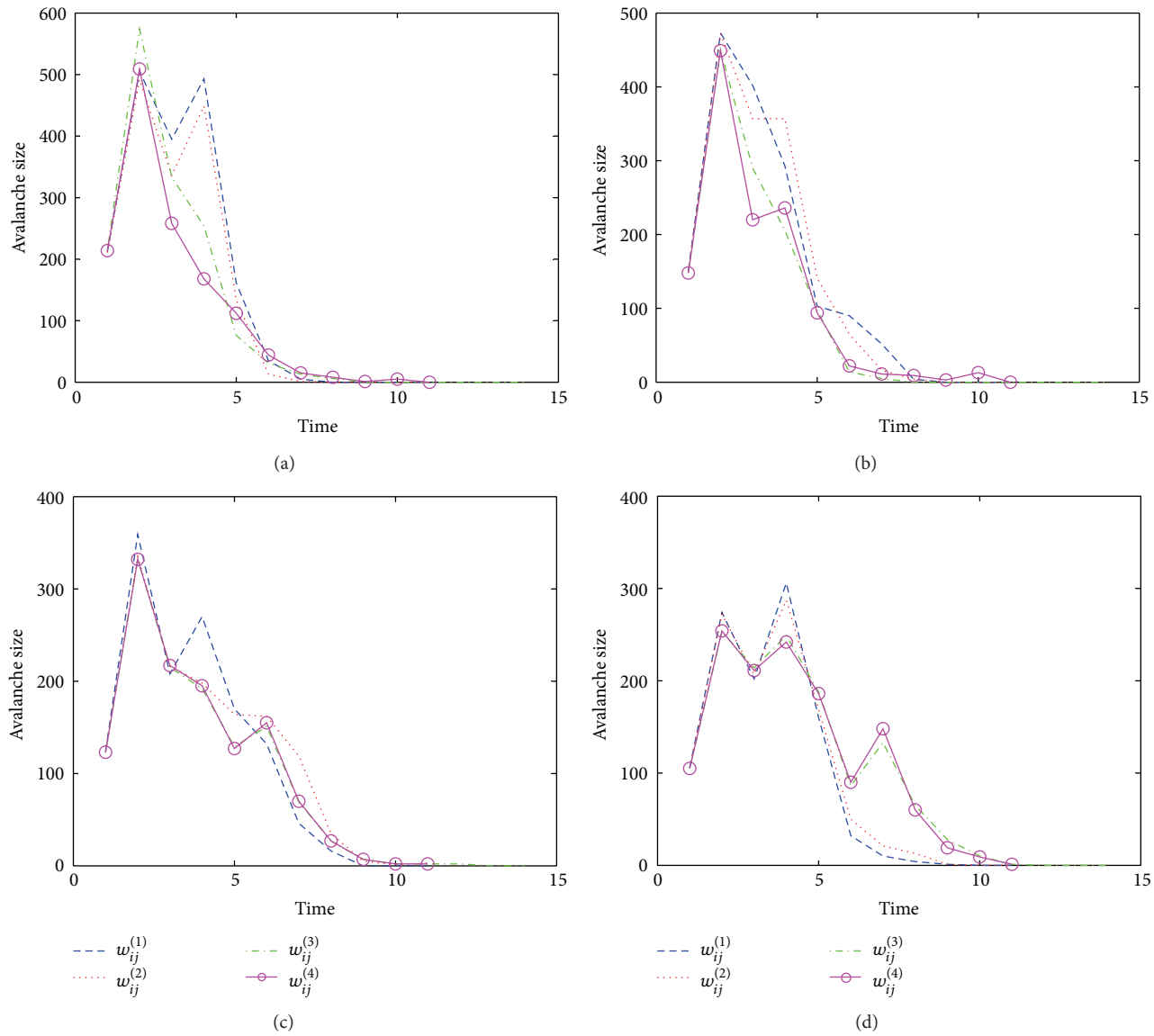

FIGURE 8: For autonomous system network with $\tau = 20$ under HL attack, the avalanche size $N_{ae}(t)$ in each time step t as a function of t for (a) $\alpha = 0.02$, (b) $\alpha = 0.04$, (c) $\alpha = 0.06$, and (d) $\alpha = 0.08$, respectively.

(e.g., $\tau = 100$). It reveals that, under HL attack, the external resource assigned to edges according to the method $w_{ij}^{(4)}$ can control the spreading speed V of cascading failures in heterogeneous scale-free networks more efficiently.

4.3. Controlling the Process of Cascading Failures. In the previous two parts of this section, the function of different weighting methods on improving the robustness of networks against cascading failures has been shown. However, another question that whether the weighting methods could control the outbreak of cascading failures also should be considered. In this part of this section, we focus on controlling the process of cascading failures in networks and plot the avalanche size $N_{ae}(t)$ in each time step t under HL attack to explore this question.

As shown in Figures 6 and 7, under HL attack with different tolerance parameters α ($\alpha = 0.02$, 0.04, 0.06, and 0.08), we can see that the weighting method $w_{ij}^{(4)}$ can more effectively control the outburst of cascading failures in SF network model than other methods. Especially, with more external resources ($\tau = 100$), the more obviously can $w_{ij}^{(4)}$ reduce the peak of cascading failures (see Figure 7). Moreover, the simulations of the autonomous system AS1470 also show the similar findings (see Figures 8 and 9).

5. Conclusion

In this paper, we study the cascading dynamics of heterogeneous scale-free (SF) network with the recovery mechanism subject to edge-based attack. The recovery mechanism is

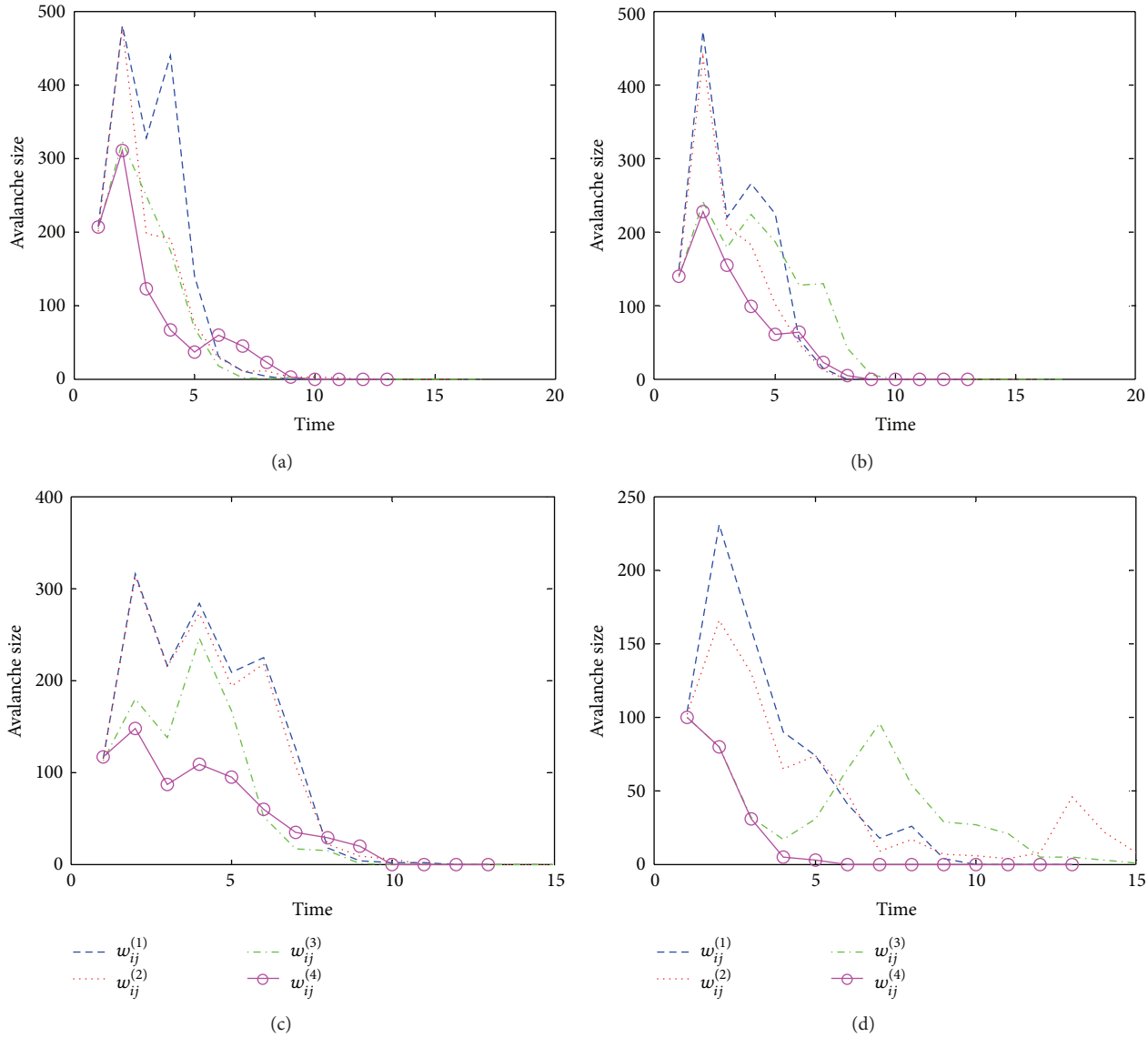

FIGURE 9: For autonomous system network with $\tau = 100$ under HL attack, the avalanche size $N_{ae}(t)$ in each time step t as a function of t for (a) $\alpha = 0.02$, (b) $\alpha = 0.04$, (c) $\alpha = 0.06$, and (d) $\alpha = 0.08$, respectively.

represented by the external resources τ that are distributed to the edge e_{ij} according to five kinds of weighting strategies: $w_{ij}^{(1)}$, $w_{ij}^{(2)}$, $w_{ij}^{(3)}$, $w_{ij}^{(4)}$, and the uniform strategy. We mainly investigate the influence of τ and different weighting strategies on the cascading dynamics of SF networks subject to intentional attack and random breakdown. On the whole, the main contributions of this paper are listed as follows.

(1) Under intentional attack, $w_{ij}^{(4)}$ is the most efficient response strategy against cascading failures in SF networks, which can obviously improve the integral robustness, simultaneously reduce the spreading speed, and control the outbreak of cascading failures in SF networks. Especially, the more external resources are, the more efficient $w_{ij}^{(4)}$ is. The uniform assignment strategy is the worst strategy.

(2) Although the method $w_{ij}^{(4)}$ is optimal, it needs to compute the betweenness centrality of node that depends on the whole structure of networks. Therefore, $w_{ij}^{(3)}$ will be optimal if we only knew the local structure of SF network (e.g., the degree of nodes). The simulations of autonomous system network have proved these results. However, the recent research [54] has shown that, the node betweenness centrality can be approximately estimated by using the local information of nodes in order to reduce the computational complexity in large networks. This implies that the weighting method $w_{ij}^{(4)}$ defined in this paper has great significance in the protection of actual scale-free networks.

(3) Under random breakdown, although the difference among the five kinds of weighting methods is not

clear in terms of the protection result against cascading effect, the uniform assignment strategy ($w_{ij} = 1$) can better decrease the spreading velocity of failures in SF network than other strategies.

The results remind us to take different actions on handling and controlling the emergent disasters in heterogeneous SF networks. Here we just highlight the protection of the important links. Our approach makes contributions to understanding the dynamics of disaster spreading and provides some possible countermeasures to control the disasters and finally to repair the system damaged.

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