Journal of Function Spaces

Nonlinear Operator Theory and Its Applications

Lead Guest Editor: Juan Martinez-Moreno
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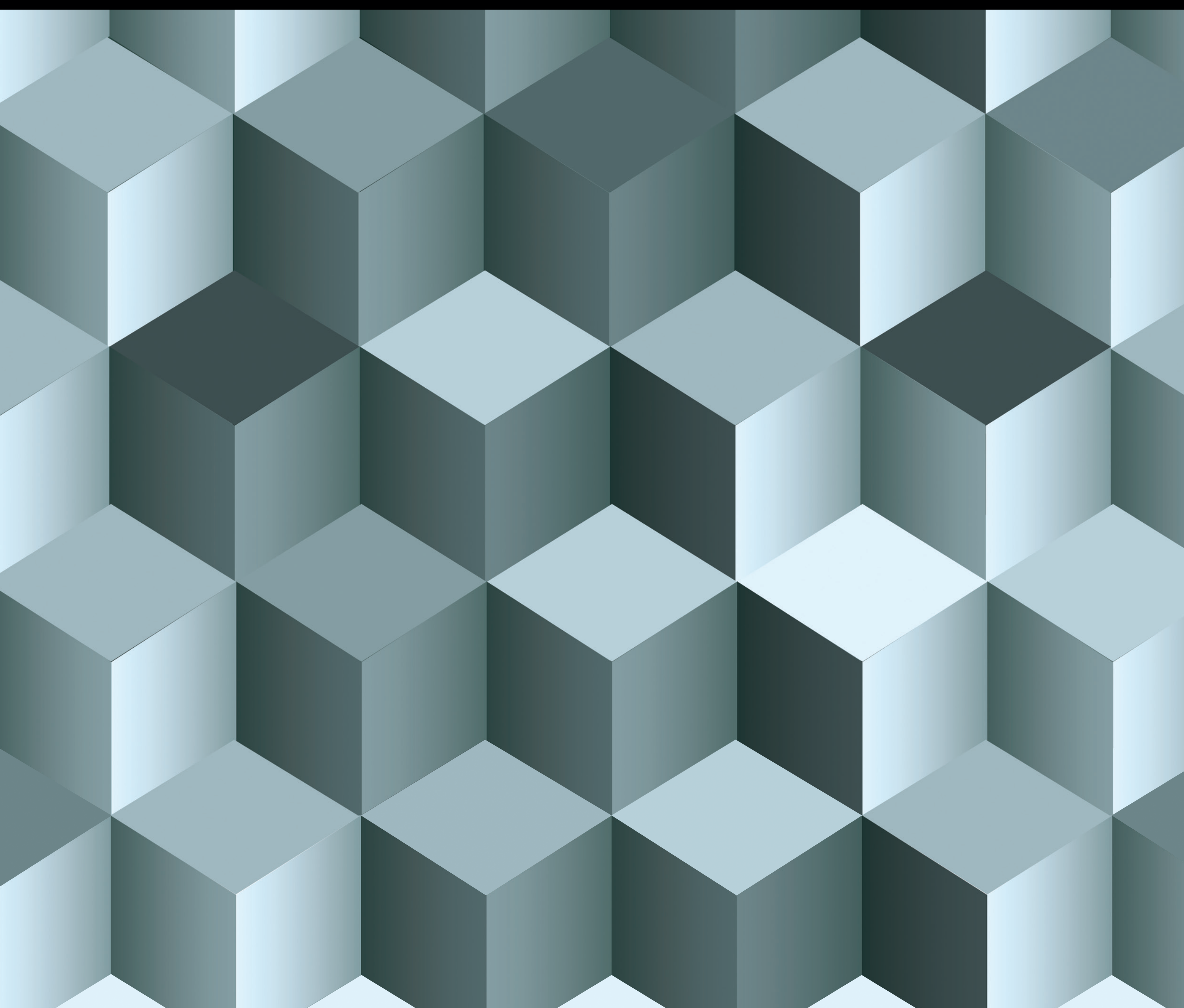

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Editorial Board

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 9713872, 2 pages
https://doi.org/10.1155/2018/9713872

Editorial

Nonlinear Operator Theory and Its Applications

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Received 7 August 2018; Accepted 7 August 2018; Published 2 September 2018

Nonlinear operator theory falls within the general area of nonlinear functional analysis, an area which has been of increasing research interest in recent years. Nonlinear operator theory applies to diverse nonlinear problems in many areas such as differential equations, nonlinear ergodic theory, game theory, optimization problems, control theory, variational inequality problems, equilibrium problems, and split feasibility problems.

This special issue reflects both the state-of-the-art theoretical research and important recent advances in applications.

Concerning this special issue, ten papers have been accepted and published with twenty-five different authors. Five manuscripts come from China with fourteen authors. Other papers come from Chile, Saudi Arabia, Turkey, Japan, and Poland.

The selected and published papers are the following items.

One paper proposes stochastic convex semidefinite programs (SCSDPs) to handle uncertain data in applications. For these models, S. Chen et al. design an efficient inexact stochastic approximation (SA) method and prove the convergence, complexity, and robust treatment of the algorithm and apply it for solving SCSDPs where the subproblem in each iteration is only solved approximately and show that it enjoys the similar iteration complexity as the exact counterpart if the subproblems are progressively solved to sufficient accuracy.

Another paper extends a number of existing results on b-metric spaces. For it, an existence and uniqueness of new contractive operators combining admissible and simulation functions are proved for complete b-metric spaces by A. S.S. Alharbi et al.

The Monge-Ampère equations are a type of important fully nonlinear elliptic equations. In the third paper, W. Shen establishes the global bifurcation results from the trivial solutions axis and from infinity for some Monge-Ampère equations and some applications are given.

The main aim of the fourth paper is to investigate the Mobius gyrovector spaces which are open balls centered at the origin in a real Hilbert space with the Mobius addition, the Mobius scalar multiplication, and the Poincaré metric introduced by Ungar. In particular, for an arbitrary point, K. Watanabe obtains the unique closest point in any closed gyrovector subspace, by using the ordinary orthogonal decomposition and shows that each element has the orthogonal gyroexpansion with respect to any orthogonal basis in a Mobius gyrovector space. Finally, a concrete procedure to calculate the gyrocoefficients of the orthogonal gyroexpansion is presented.

One of the papers studies a nonlocal fourth-order elliptic equation of Kirchhoff type with dependence on the gradient and Laplacian. Y. Ru et al. show that there exists a $b^* >0$ such that the problem has a nontrivial solution for some cases through an iterative method based on the mountain pass lemma and truncation method previously developed by Figuereido, Girard, and Matzeu.

A paper also studies fixed-point results in the setting of b-metric spaces. In this case, E. Karapinar et al. present generalized (α,ψ)-Meir-Keeler type contractions and, for them, establish a fixed-point result that improves, generalizes, and unifies many existing famous results in the corresponding literature. Two examples are presented to illustrate main results.

In another paper, by using two fixed-point theorems on cone, Q. Sun et al. discuss the existence results of positive solutions for a boundary value problem of fractional differential equation with integral boundary conditions.

The purpose of T. Xiong et al. in one of the papers is to introduce and study a class of new two-step viscosity iteration approximation methods for finding fixed points of set-valued nonexpansive mappings in ${\rm CAT(0)}$ Spaces. By means of some properties and characteristic to ${\rm CAT(0)}$ Spaces, and using Cauchy-Schwarz inequality and Xu's inequality, strong convergence theorems of the new two-step viscosity iterative process for set-valued nonexpansive and contraction operators in complete ${\rm CAT(0)}$ Spaces are provided.

Another paper's author, Tomonari Suzuki, by introducing the concept of $\sum$-semicompleteness in semimetric spaces, extends Caristi's fixed-point theorem to $\sum$-semicomplete semimetric spaces. Via this extension, $\sum$-semicompleteness is characterized and Banach contraction principle generalized.

In one of the papers, the existence and uniqueness of weak solutions for the boundary value problem modelling the stationary case of the bioconvective low problem are proved. The bioconvective model is a boundary value problem for a system of four equations: the nonlinear Stokes equation, the incompressibility equation, and two transport equations. The unknowns of the model are the velocity of the fluid, the pressure of the fluid, the local concentration of microorganisms, and the oxygen concentration. A. Coronel et al. derive some appropriate a priori estimates for the weak solution, which implies the existence, by application of Gossez theorem, and the uniqueness by standard methodology of comparison of two arbitrary solutions.

Conflicts of Interest

As Guest Editorial team of special issue named "Nonlinear Operator Theory and Its Applications" in Journal of Function Spaces, we declare that there are no conflicts of interest or private agreements with companies regarding our work for this special issue. We have no financial relationships through employment and consultancies, either stock ownership or honoraria, with industry.

Acknowledgments

We want to thank all the authors of these works, which provide a wide view of some of the most recent topics in the field. Also, we acknowledge with thanks the work done by the reviewers who collaborated to make this special issue possible. Our gratitude goes also to the editors of the journal for the support and help with the preparation of this special issue.

Juan Martinez-Moreno
Dhananjay Gopal
Vijay Gupta
Edixon Rojas
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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 1280241, 8 pages
https://doi.org/10.1155/2018/1280241

Research Article

Strong Convergence of New Two-Step Viscosity Iterative Approximation Methods for Set-Valued Nonexpansive Mappings in CAT(0) Spaces

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Received 3 December 2017; Accepted 24 May 2018; Published 2 July 2018

Academic Editor: Dhananjay Gopal

This paper is for the purpose of introducing and studying a class of new two-step viscosity iteration approximation methods for finding fixed points of set-valued nonexpansive mappings in CAT(0) spaces. By means of some properties and characteristic to CAT(0) space and using Cauchy-Schwarz inequality and Xu's inequality, strong convergence theorems of the new two-step viscosity iterative process for set-valued nonexpansive and contraction operators in complete CAT(0) spaces are provided. The results of this paper improve and extend the corresponding main theorems in the literature.

1. Introduction

In [1], the fixed point theory in CAT(0) spaces was first introduced and studied by Kirk. Further, Kirk [1] presented that each nonexpansive (single-valued) mapping on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. On the other hand, fixed point theory for set-valued mappings has been applied to applied sciences, game theory, and optimization theory. This promotes the rapid development of fixed point theory for single-valued (set-valued) operators in CAT(0) spaces, and it is natural and particularly meaningful to extensively study fixed point theory of set-valued operators. Particularly, some old relative works on Ishikawa iterations for multivalued mappings can be found in [2–4]. For more detail, we refer to [5–14] and the references therein.

Definition 1. Let $g : X \to X$ be a nonlinear operator on a metric space (X, d) and $G : E \to BC(X)$ be a set-valued operator, where $E \subset X$ is a nonempty subset and $BC(X)$ is the family of nonempty bounded closed subsets of X. Then

(i) g is said to be a *contraction*, if there exists a constant $\kappa \in [0, 1)$ such that

$$d(g(x), g(y)) \le \kappa d(x, y) \quad \forall x, y \in X. \tag{1}$$

Here, g is called *nonexpansive* when $\kappa = 1$ in (1).

(ii) G is said to be a nonexpansive, if

$$H(G(x), G(y)) \le d(x, y), \tag{2}$$

where $H(\cdot, \cdot)$ is Hausdorff distance on $BC(X)$, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b))\right\}, \quad \forall A, B \in BC(X). \tag{3}$$

Recently, Shi and Chen [5] first considered the following Moudafi's viscosity iteration for a nonexpansive mapping $g : E \to E$ with $\emptyset \ne Fix(g) = \{x \mid x = g(x)\}$ and a contraction mapping $f : E \to E$ in CAT(0) space X:

$$x_\alpha = \alpha f(x_\alpha) \oplus (1-\alpha) g(x_\alpha), \tag{4}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) g(x_n), \quad n \ge 1, \tag{5}$$

where $\alpha, \alpha_n \in (0, 1)$ and x_1 is an any given element in a nonempty closed convex subset $E \subseteq X$. $x_\alpha \in E$ is called unique fixed point of contraction $x \mapsto \alpha f(x) \oplus (1-\alpha) g(x)$. Shi and Chen [5] showed that $\{x_\alpha\}$ defined by (4) converges strongly to $\tilde{x} \in Fix(g)$ as $\alpha \to 0^+$, where $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ in CAT(0) space (X, d) satisfies the following property $\mathscr{P}$: for all $x, u, y_1, y_2 \in X$,

$$\begin{aligned} d(x, m_1) d(x, y_1) &\leq d(x, m_2) d(x, y_2) \\ &\quad + d(x, u) d(y_1, y_2), \end{aligned} \tag{6}$$

that is, an extra condition on the geometry of CAT(0) spaces is requested, where $m_i = P_{[x,y_i]} u$ for $i = 1, 2$. Further, the authors also found that the sequence $\{x_n\}$ generated by (5) converges strongly to $\tilde{x} \in Fix(g)$ under some suitable conditions about $\{\alpha_n\}$. Afterwards, based on the concept of quasilinearization introduced by Berg and Nikolaev [15], Wangkeeree and Preechasilp [6] explored strong convergence results of (4) and (5) in CAT(0) spaces without the property $\mathscr{P}$ and presented that the iterative processes (4) and (5) converges strongly to $\tilde{x} \in Fix(g)$ such that $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ is the unique solution of the following variational inequality:

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{y \tilde{x}} \right\rangle \geq 0, \quad \forall y \in Fix(g). \tag{7}$$

In [16], Panyanak and Suantai extended (4) and (5) to T being a set-valued nonexpansive mapping from E to $BC(X)$. That is, for each $\alpha \in (0, 1)$, let a set-valued contraction G_α on E be defined by

$$G_\alpha(x) = \alpha f(x) \oplus (1 - \alpha) Tx, \quad \forall x \in E. \tag{8}$$

By Nadler's [17] theorem, it is easy to know that there exists $x_\alpha \in E$ such that x_α is a fixed point of G_α, which does not have to be unique, and

$$x_\alpha \in \alpha f(x_\alpha) \oplus (1 - \alpha) Tx_\alpha, \tag{9}$$

i.e., for each x_α, there exists $y_\alpha \in Tx_\alpha$ such that

$$x_\alpha = \alpha f(x_\alpha) \oplus (1 - \alpha) y_\alpha. \tag{10}$$

Further, when the contraction constant coefficient of f is $k \in [0, 1/2)$ and $\{\alpha_n\} \subset (0, 1/(2-k))$ satisfying some suitable conditions, Panyanak and Suantai [16] proved strong convergence of one-step viscosity approximation iteration defined by (10) or the following iterative process in CAT(0) spaces:

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) y_n, \quad y_n \in M(x_n), \tag{11}$$

and $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, where M is a set-valued nonexpansive operator from E to $C(E)$, the family of nonempty compact subsets of E, $f : E \to E$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Moreover, Chang et al. [7] affirmatively answered the open question proposed by Panyanak and Suantai [16, Question 3.6]: "If $k \in [0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfying the same conditions, does $\{x_n\}$ converge to $\tilde{x} = P_{F(M)} f(\tilde{x})$", where $F(M)$ denotes the set of all fixed points of M.

On the other hand, Piatek [18] introduced and studied the following two-step viscosity iteration in complete CAT(0) spaces with the nice projection property $\mathscr{N}$:

$$\begin{aligned} y_n &= \alpha_n f(x_n) \oplus (1 - \alpha_n) g(x_n), \\ x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n) y_n, \quad \forall n \geq 1, \end{aligned} \tag{12}$$

where $x_1 \in E$ is an given element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$ satisfying some suitable conditions and the contraction coefficient of f is $k \in [0, 1/2)$.

Based on the ideas of Wangkeeree and Preechasilp [6] and Piatek [18] intensively, Kaewkhao et al. [19] omit the nice projection property $\mathscr{N}$. We note that the two-step viscosity iteration (12) is also considered and studied by Chang et al. [8] when the property $\mathscr{N}$ is not satisfied and $k \in [0, 1)$, which is due to the open questions in [19], where the property $\mathscr{N}$ depends on whether its metric projection onto a geodesic segment preserves points on each geodesic segment, that is, for every geodesic segment $\chi \subset X$ and $x, y \in X$, $m \in [x, y]$ implies $P_\chi m \in [P_\chi x, P_\chi y]$, where P_χ denotes the metric projection from X onto χ. For more works on the convergence analysis of (viscosity) iteration approximation method for (split) fixed point problems, one can refer to [20–27].

Motivated and inspired mainly by Panyanak and Suantai [16] and Piatek [18] and so on, we consider the following two-step viscosity iteration for set-valued nonexpansive operator $T : E \to C(E)$:

$$\begin{aligned} x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n) y_n, \\ y_n &= \alpha_n f(x_n) \oplus (1 - \alpha_n) z_n, \quad \forall n \geq 1, \end{aligned} \tag{13}$$

where E is a nonempty closed convex subset of complete CAT(0) space (X, d), $x_1 \in E$ is an given element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$, $f : E \to E$ is a contraction mapping, and $z_n \in T(x_n)$ satisfying $d(z_n, z_{n+1}) \leq d(x_n, x_{n+1})$ for any $n \in \mathbb{N}$.

By using the method due to Chang et al. [7, 8], the purpose of this paper is to prove some strong convergence theorems of the viscosity iteration procedure (13) in complete CAT(0) spaces. Hence, the results of Chang et al. [7, 8] and many others in the literature can be special cases of main results in this paper.

2. Preliminaries

Throughout of this paper, let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map $\xi : \mathbb{R} \supseteq [0, l] \to X$ such that $\xi(0) = x, \xi(l) = y$, and $d(\xi(s), \xi(t)) = |s - t|$ for each $s, t \in [0, l]$. In particular, ξ is a isometry and $d(x, y) = l$. The image of ξ is called a geodesic segment (or metric) joining x and y if unique is bespoke by $[x, y]$. The space (X, d) is called a geodesic space when every two points in X are joined by a geodesic, and X is called uniquely geodesic if there is exactly one geodesic joining x and y for any $x, y \in X$. A subset E of X is said to be convex if E includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(p, q, r)$ in a geodesic space (X, d) consists of three points p, q, r in X (the vertices of $\triangle$) and a choice of three geodesic segments $[p, q]$, $[q, r]$, $[r, p]$ (the edge of $\triangle$) joining them. A comparison triangle for geodesic triangle $\triangle(p, q, r)$ in X is a triangle $\overline{\triangle}(\overline{p}, \overline{q}, \overline{r})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$\begin{aligned} d_{\mathbb{R}^2}(\overline{p}, \overline{q}) &= d(p, q), \\ d_{\mathbb{R}^2}(\overline{q}, \overline{r}) &= d(q, r), \\ d_{\mathbb{R}^2}(\overline{r}, \overline{p}) &= d(r, p). \end{aligned} \tag{14}$$

A point $\overline{u} \in [\overline{p}, \overline{q}]$ is said to be a comparison point for $u \in [p, q]$ if $d(p, u) = d_{\mathbb{R}^2}(\overline{p}, \overline{u})$. Similarly, we can give the definitions of comparison points on $[\overline{q}, \overline{r}]$ and $[\overline{r}, \overline{p}]$.

Definition 2. Suppose that $\triangle$ is a geodesic triangle in (X, d) and $\overline{\triangle}$ is a comparison triangle for $\triangle$. A geodesic space is said to be a CAT(0) space, if all geodesic triangles of appropriate size satisfy the following comparison axiom (i.e., CAT(0) inequality):

$$d(u, v) \le d_{\mathbb{R}^2}(\overline{u}, \overline{v}), \quad \forall u, v \in \triangle, \overline{u}, \overline{v} \in \overline{\triangle}. \tag{15}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [28]). For other equivalent definitions and basic properties of CAT(0) spaces, one can refer to [29]. It is well known that every CAT(0) space is uniquely geodesic and any complete, simply connected Riemannina manifold having nonpositive sectional curvature is a CAT(0) space. Other examples for CAT(0) spaces include pre-Hilbert spaces [29], $\mathbb{R}$−trees [9], Euclidean buildings [30], and complex Hilbert ball with a hyperbolic metric [31] as special case.

Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d). By Proposition 2.4 of [29], it follows that, for all $x \in X$, there exists a unique point $x_0 \in E$ such that

$$d(x, x_0) = \inf\{d(x, y) : y \in E\}. \tag{16}$$

Here, x_0 is said to be *unique nearest point* of x in E.

Assume that (X, d) is a CAT(0) space. For all $x, y \in X$ and $t \in [0, 1]$, by Lemma 2.1 of Phompongsa and Panyanak [10], there exists a unique point $z \in [x, y]$ such that

$$\begin{aligned} d(x, z) &= (1 - t)\, d(x, y) \\ \text{and } d(y, z) &= t d(x, y). \end{aligned} \tag{17}$$

We shall denote by $tx \oplus (1 - t)y$ the unique point z satisfying (17). Now, we give some results about CAT(0) spaces for the proof of our main results.

Lemma 3 ([1, 10]). *Let (X, d) be a* CAT(0) *space. Then for each $x, y, z \in X$ and $\alpha \in [0, 1]$,*

(i) $d(\alpha x \oplus (1 - \alpha)y, z) \le \alpha d(x, z) + (1 - \alpha)d(y, z)$.

(ii) $d^2(\alpha x \oplus (1 - \alpha)y, z) \le \alpha d^2(x, z) + (1 - \alpha)d^2(y, z) - \alpha(1 - \alpha)d^2(x, y)$.

(iii) $d(\alpha x \oplus (1 - \alpha)z, \alpha y \oplus (1 - \alpha)z) \le \alpha d(x, y)$.

Lemma 4 ([11]). *Suppose that (X, d) is a* CAT(0) *space. Then for all $x, y \in X$ and $\alpha, \beta \in [0, 1]$,*

$$d(\alpha x \oplus (1 - \alpha) y, \beta x \oplus (1 - \beta) y) \le |\alpha - \beta|\, d(x, y). \tag{18}$$

Lemma 5 ([12]). *Assume that $\{x_n\}$ and $\{y_n\}$ are two bounded sequences in a* CAT(0) *space (X, d) and $\{\beta_n\}$ is a sequence in $[0, 1]$ with $0 < \liminf_n \beta_n \le \limsup_n \beta_n < 1$. If*

$$\begin{aligned} & x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) y_n, \quad \forall n \in \mathbb{N}, \\ & \limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0, \end{aligned} \tag{19}$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 6 ([32]). *Suppose that nonnegative real numbers sequence $\{u_n\}$ is defined by*

$$u_{n+1} \le (1 - \alpha_n) u_n + \alpha_n \beta_n, \quad \forall n \ge 1, \tag{20}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\beta_n\} \subset \mathbb{R}$ are two sequences satisfying

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \beta_n \le 0$ or $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} \{u_n\} = 0$.

Lemma 7 ([13]). *Assume that E is a closed convex subset of a complete* CAT(0) *space (X, d). If a set-valued nonexpansive operator $T : E \to BC(X)$ satisfies endpoint condition $\mathbb{C}$, i.e., $F(T) \ne \emptyset$ and $T(x) = \{x\}$ for any $x \in F(T)$ (see [33]), then $F(T)$ is closed and convex.*

In [15], Berg and Nikolaev introduced the concept of quasilinearization. Now we denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$, which is a vector. Define the quasilinearization by a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ as follows:

$$\begin{aligned} & \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \\ & \quad = \frac{1}{2} \left[d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right], \\ & \qquad \forall a, b, c, d \in X. \end{aligned} \tag{21}$$

One can easily know that

$$\begin{aligned} \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle &= \left\langle \overrightarrow{cd}, \overrightarrow{ab} \right\rangle, \\ \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle &= -\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle, \\ \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{ad}, \overrightarrow{bc} \right\rangle &= \left\langle \overrightarrow{ac}, \overrightarrow{bd} \right\rangle, \\ \left\langle \overrightarrow{ax}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{xb}, \overrightarrow{cd} \right\rangle &= \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \end{aligned} \tag{22}$$

for every $a, b, c, d, x \in X$. We say that a geodesic metric space (X, d) satisfies the Cauchy-Schwarz inequality if

$$\left| \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \right| \le d(a, b)\, d(c, d), \quad \forall a, b, c, d \in X. \tag{23}$$

From [15, Corollary 3], it is known that a geodesic space (X, d) is a CAT(0) space if and only if X satisfies the Cauchy-Schwarz inequality. Further, we give the following other properties of quasilinearization.

Lemma 8 ([14]). *Assume that E is a nonempty closed convex subset of a complete* CAT(0) *space (X, d). Then for $u \in X$ and $x \in E$,*

$$x = P_E u \quad \text{if and only if} \left\langle \overrightarrow{xu}, \overrightarrow{yx} \right\rangle \geq 0, \ \forall y \in E. \tag{24}$$

Lemma 9 ([6]). *For two points u and v in a* CAT(0) *space (X, d) and any $\alpha \in [0, 1]$, letting $u_\alpha = \alpha u \oplus (1 - \alpha)v$, then for all $x, y \in X$, the following results hold:*

(i) $\langle \overrightarrow{u_\alpha x}, \overrightarrow{u_\alpha y} \rangle \leq \alpha \langle \overrightarrow{ux}, \overrightarrow{u_\alpha y} \rangle + (1 - \alpha) \langle \overrightarrow{vx}, \overrightarrow{u_\alpha y} \rangle$;

(ii) $\langle \overrightarrow{u_\alpha x}, \overrightarrow{uy} \rangle \leq \alpha \langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1 - \alpha) \langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ and $\langle \overrightarrow{u_\alpha x}, \overrightarrow{vy} \rangle \leq \alpha \langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1 - \alpha) \langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Definition 10. A continuous linear functional μ is said to be Banach limit on ℓ_∞ if

$$\begin{gathered} \|\mu\| = \mu (1, 1, \cdots) = 1 \\ \text{and } \mu_n (u_n) = \mu_n (u_{n+1}), \quad \forall \{u_n\} \in \ell_\infty. \end{gathered} \tag{25}$$

Lemma 11 ([34]). *Suppose that, for real number α and all Banach limits μ, $(u_1, u_2, \cdots) \in \ell_\infty$ satisfies*

$$\begin{gathered} \mu_n (u_n) \leq \alpha, \\ \limsup_n (u_{n+1} - u_n) \leq 0. \end{gathered} \tag{26}$$

Then $\limsup_n u_n \leq \alpha$.

Lemma 12 ([16]). *Assume that (X, d) is a complete* CAT(0) *space, $E \subset X$ is a nonempty closed convex subset, $T : E \to C(E)$ is a set-valued nonexpansive operator satisfying endpoint condition $\mathbb{C}$, and $f : E \to E$ is a contraction with $k \in [0, 1)$. Then we have following results:*

(i) $\{x_\alpha\}$ generated by (10) converges strongly to $\tilde{x}$ as $\alpha \to 0^+$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$.

(ii) In addition, if $\{x_n\} \subset E$ is a bounded sequence such that $\lim_{n\to\infty} dist(x_n, T(x_n)) = 0$, where $dist(a, B)$ is the distance from a point $a \in X$ to the set $B \in C(X)$, then for all Banach limits μ,

$$d^2 (f (\tilde{x}), \tilde{x}) \leq \mu_n d^2 (f (\tilde{x}), x_n). \tag{27}$$

3. Main Results

Employing the preliminaries in the previous section, now we will study the strong convergence of the new two-step viscosity iteration (13) for set-valued nonexpansive operators in complete CAT(0) spaces.

Theorem 13. *Assume that (X, d) is a complete* CAT(0) *space, $E \subset X$ is a nonempty closed convex subset, $T : E \to C(E)$ is a set-valued nonexpansive operator satisfying endpoint condition $\mathbb{C}$, and $f : E \to E$ is contraction with $k \in [0, 1)$. If sequences $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ satisfy*

(L_1) $\lim_{n\to\infty} \alpha_n = 0$, (L_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (L_3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, then the sequence $\{x_n\}$ generated by (13) converges strongly to $\tilde{x}$, where

$$\begin{gathered} \tilde{x} = P_{F(T)} f (\tilde{x}), \\ \left\langle \overrightarrow{\tilde{x} f (\tilde{x})}, \overrightarrow{x \tilde{x}} \right\rangle \geq 0, \quad \forall x \in F (T). \end{gathered} \tag{28}$$

Proof. The proof shall be divided into the following four steps.

Step 1. We first prove that sequences $\{x_n\}$, $\{f(x_n)\}$, $\{y_n\}$, and $\{z_n\}$ are bounded. In fact, setting $p \in F(T)$, then from Lemma 3, we know

$$\begin{aligned} d (y_n, p) &\leq \alpha_n d (f (x_n), p) \\ &\quad + (1 - \alpha_n) dist (z_n, T (p)) \\ &\leq \alpha_n d (f (x_n), p) \\ &\quad + (1 - \alpha_n) H (T (x_n), T (p)) \\ &\leq \alpha_n d (f (x_n), p) + (1 - \alpha_n) d (x_n, p) \\ &\leq \alpha_n d (f (x_n), f (p)) + \alpha_n d (f (p), p) \\ &\quad + (1 - \alpha_n) d (x_n, p) \\ &\leq [1 - \alpha_n (1 - k)] d (x_n, p) \\ &\quad + \alpha_n d (f (p), p), \end{aligned} \tag{29}$$

and

$$\begin{aligned} d (x_{n+1}, p) &\leq \beta_n d (x_n, p) + (1 - \beta_n) d (y_n, p) \\ &\leq [1 - \alpha_n (1 - k) (1 - \beta_n)] d (x_n, p) \\ &\quad + \alpha_n (1 - k) (1 - \beta_n) \frac{d (f (p), p)}{1 - k} \\ &\leq \max \left\{ d (x_n, p), \frac{d (f (p), p)}{1 - k} \right\}. \end{aligned} \tag{30}$$

Thus, we obtain

$$d (x_n, p) \leq \max \left\{ d (x_1, p), \frac{d (f (p), p)}{1 - k} \right\}. \tag{31}$$

Hence, $\{x_n\}$ is bounded, so is $\{f(x_n)\}$. By (29), it is easy to know that $\{y_n\}$ is bounded. Since $d(z_n, p) \leq H(T(x_n, T(p)) \leq d(x_n, p)$, one can easily know that the sequence $\{z_n\}$ is also bounded.

Step 2. We present that $\lim_{n\to\infty} d(x_n, y_n) = 0$, $\lim_{n\to\infty} dist(x_n, T(x_n)) = 0$, $\lim_{n\to\infty} d(x_n, z_n) = 0$, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$,

$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$, and $\lim_{n\to\infty} dist(z_n, T(z_n)) = 0$. Indeed, by applying Lemmas 3 and 4, we have

$$\begin{aligned} d(y_n, y_{n+1}) &\le d(\alpha_n f(x_n) \\ &\oplus (1-\alpha_n) z_n, \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1}) z_{n+1}) \\ &\le d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \alpha_n f(x_n) \\ &\oplus (1-\alpha_n) z_{n+1}) + d(\alpha_n f(x_n) \\ &\oplus (1-\alpha_n) z_{n+1}, \alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}) \\ &+ d(\alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}, \alpha_{n+1} f(x_{n+1}) \\ &\oplus (1-\alpha_{n+1}) z_{n+1}) \le (1-\alpha_n) d(z_n, z_{n+1}) \\ &+ \alpha_n d(f(x_n), f(x_{n+1})) + |\alpha_n \\ &- \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \le [1 \\ &- \alpha_n (1-k)] d(x_n, x_{n+1}) + |\alpha_n \\ &- \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}), \end{aligned} \tag{32}$$

and so

$$\begin{aligned} &d(y_n, y_{n+1}) - d(x_n, x_{n+1}) \\ &\quad \le |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \\ &\quad - (1-k) \alpha_n d(x_n, x_{n+1}). \end{aligned} \tag{33}$$

From $\lim_{n\to\infty} \alpha_n = 0$ and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, and $\{z_n\}$, we know

$$\limsup_{n\to\infty} [d(y_{n+1}, y_n) - d(x_{n+1}, x_n)] \le 0. \tag{34}$$

It follows from Lemma 5 that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{35}$$

Thus,

$$\begin{aligned} dist(x_n, T(x_n)) &\le d(x_n, z_n) \\ &\le d(x_n, y_n) + \alpha_n d(f(x_n), z_n) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{36}$$

By (36), now we know that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{37}$$

Moreover,

$$d(x_n, x_{n+1}) = (1-\beta_n) d(x_n, y_n) \longrightarrow 0 \tag{38}$$

and

$$d(z_n, z_{n+1}) \le d(x_n, x_{n+1}) \longrightarrow 0 \tag{39}$$

as $n \to \infty$. By (36) and (37), we get

$$\begin{aligned} dist(z_n, T(z_n)) &\le d(z_n, x_n) + dist(x_n, T(x_n)) \\ &+ H(T(x_n), T(z_n)) \\ &\le 2d(x_n, z_n) + dist(x_n, T(x_n)) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{40}$$

Step 3. Now, we show that

$$\limsup_{n\to\infty} \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right) \le 0, \tag{41}$$

with $\tilde{x} = P_{F(T)} f(\tilde{x})$ satisfying

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \ge 0, \quad \forall x \in F(T). \tag{42}$$

Above all, since $T(x)$ is compact for any $x \in E$, then $T(x) \in BC(X)$. It follows from Lemma 7 that $F(T)$ is closed and convex, which implies that $P_{F(T)} u$ is well defined for any $u \in X$. By Lemma 12 (i), we know that $\{x_\alpha\}$ generated by (10) converges strongly to $\tilde{x} = P_{F(T)} f(\tilde{x})$ as $\alpha \to 0^+$. Then by Lemma 8, we know that $\tilde{x}$ is the unique solution of the following variational inequality:

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \ge 0, \quad \forall x \in F(T). \tag{43}$$

Next, since $\{z_n\}$ is bounded and $\lim_{n\to\infty} dist(z_n, T(z_n)) = 0$, it follows from Lemma 12 (ii) that for all Banach limits μ,

$$d^2(f(\tilde{x}), \tilde{x}) \le \mu_n d^2(f(\tilde{x}), z_n), \tag{44}$$

and so

$$\mu_n \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right) \le 0. \tag{45}$$

Further, $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$ implies that

$$\begin{aligned} &\limsup_{n\to\infty} \left[\left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_{n+1})\right)\right. \\ &\quad \left. - \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right)\right] = 0. \end{aligned} \tag{46}$$

By Lemma 11, we have

$$\limsup_{n\to\infty} \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right) \le 0. \tag{47}$$

Step 4. $\lim_{n\to\infty} x_n = \tilde{x}$ will be verified. In fact, by Lemma 3 and (13), now we know

$$\begin{aligned} d^2(x_{n+1}, \tilde{x}) &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}) \\ &- \beta_n (1-\beta_n) d^2(x_n, y_n) \\ &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}), \end{aligned} \tag{48}$$

and

$$\begin{aligned} d^2(y_n,\tilde{x}) &\le \alpha_n d^2(f(x_n),\tilde{x}) + (1-\alpha_n) d^2(z_n,\tilde{x}) \\ &\quad - \alpha_n (1-\alpha_n) d^2(f(x_n),z_n) \\ &\le (1-\alpha_n) H^2(T(x_n),T(\tilde{x})) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n) \\ &\quad + \alpha_n \left(d^2(f(x_n),\tilde{x}) - d^2(f(x_n),z_n)\right) \\ &\le (1-\alpha_n) d^2(x_n,\tilde{x}) + \alpha_n^2 d^2(f(x_n),z_n) \\ &\quad + \alpha_n \left(d^2(f(x_n),\tilde{x}) - d^2(f(x_n),z_n)\right). \end{aligned} \tag{49}$$

It follows from (21), Cauchy-Schwarz inequality, and Lemma 9 that

$$\begin{aligned} &\alpha_n \left(d^2(f(x_n),\tilde{x}) - d^2(f(x_n),z_n)\right) \\ &\quad \le 2\alpha_n \Big(d(f(x_n),f(\tilde{x})) d(z_n,\tilde{x}) \\ &\quad + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_n\tilde{x}} \right\rangle - d^2(z_n,\tilde{x})\Big) \\ &\quad \le 2\alpha_n \Big(kd(x_n,\tilde{x}) d(z_n,\tilde{x}) + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_n\tilde{x}} \right\rangle \\ &\quad - d^2(z_n,\tilde{x})\Big) \le \alpha_n k \left(d^2(x_n,\tilde{x}) + d^2(z_n,\tilde{x})\right) \\ &\quad + 2\alpha_n \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_n\tilde{x}} \right\rangle - 2\alpha_n d^2(z_n,\tilde{x}) \\ &\quad \le \alpha_n k d^2(x_n,\tilde{x}) + \alpha_n \Big(d^2(f(\tilde{x}),\tilde{x}) \\ &\quad - d^2(f(\tilde{x}),z_n)\Big). \end{aligned} \tag{50}$$

From (50) and (49), we know

$$\begin{aligned} d^2(y_n,\tilde{x}) &\le (1-\alpha_n(1-k)) d^2(x_n,\tilde{x}) \\ &\quad + \alpha_n \left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n). \end{aligned} \tag{51}$$

Combining (51) and (48), we get

$$\begin{aligned} &d^2(x_{n+1},\tilde{x}) \le \beta_n d^2(x_n,\tilde{x}) + (1-\beta_n) \\ &\quad \cdot \Big((1-\alpha_n(1-k)) d^2(x_n,\tilde{x}) \\ &\quad + \alpha_n \left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n)\Big) \le (1 \\ &\quad - (1-k)\alpha_n(1-\beta_n)) d^2(x_n,\tilde{x}) + \alpha_n (1 \\ &\quad - \beta_n) \left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) + (1-\beta_n) \\ &\quad \cdot \alpha_n^2 d^2(f(x_n),z_n) \le (1 \\ &\quad - (1-k)\alpha_n(1-\beta_n)) d^2(x_n,\tilde{x}) + \alpha_n (1 \\ &\quad - \beta_n) \left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n), \end{aligned} \tag{52}$$

i.e.,

$$u_{n+1} \le \left(1-\alpha'_n\right) u_n + \alpha'_n \beta'_n, \quad \forall n \ge 1, \tag{53}$$

where $u_n = d^2(x_n,\tilde{x})$, $\alpha'_n = (1-k)\alpha_n(1-\beta_n)$, and

$$\begin{aligned} &\beta'_n \\ &\quad = \frac{(1-\beta_n)\left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) + \alpha_n d^2(f(x_n),z_n)}{(1-k)(1-\beta_n)}. \end{aligned} \tag{54}$$

Thus, from the conditions (L_1)-(L_3) and the inequality (41), it follows that $\alpha'_n \in (0,1)$, and

$$\begin{gathered} \sum_{n=1}^{\infty} \alpha'_n = \infty, \\ \limsup_{n\to\infty} \beta'_n \le 0. \end{gathered} \tag{55}$$

Hence, it follows from Lemma 6 that $u_n \to 0$. This implies that the proof is completed. □

If $T \equiv g$ is a nonexpansive single-valued operator with $Fix(g) \ne \emptyset$, then from Theorem 13, one can easy to obtain the following result.

Corollary 14. *Suppose that f, E, and (X,d) are the same as in Theorem 13, and the conditions (L_1)-(L_3) in Theorem 13 are satisfied. If $g : E \to E$ is a nonexpansive single-valued operator with $Fix(g) \ne \emptyset$, then the sequence $\{x_n\}$ generated by (12) converges strongly to $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ with*

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{y\tilde{x}} \right\rangle \ge 0, \quad \forall y \in Fix(g). \tag{56}$$

Remark 15. Corollary 14 is the corresponding results of Theorem 3.1 in [8].

If $f \equiv I$, the identity operator, then by Theorem 13, now we directly have the following theorem.

Theorem 16. *Assume that T, E, and (X,d) are the same as in Theorem 13, and the conditions (L_1)-(L_3) in Theorem 13 hold. Then for any given $u, x_1 \in E$, sequence $\{x_n\}$ generated by*

$$\begin{gathered} y_n = \alpha_n u \oplus (1-\alpha_n) z_n, \\ d(z_n, z_{n+1}) \le d(x_n, x_{n+1}), \\ z_n \in T(x_n), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \quad \forall n \ge 1, \end{gathered} \tag{57}$$

converges strongly to the unique nearest point $\widetilde{x}$ of u in $F(T)$, i.e., $\widetilde{x} = P_{F(T)}u$, where $\widetilde{x}$ also satisfies

$$\left\langle \overrightarrow{\widetilde{x}u}, \overrightarrow{v\widetilde{x}} \right\rangle \geq 0, \quad \forall v \in F(T). \tag{58}$$

Remark 17. Theorems 13 and 16 also extend and improve the corresponding results of Chang et al. [7], Piatek [18], Kaewkhao et al. [19], Panyanak and Suantai [16], and many others in the literature.

4. Concluding Remarks

The purpose of this paper is to introduce and study the following new two-step viscosity iterative approximation for finding fixed points of a set-valued nonlinear mapping $G : D \to C(D)$ and a contraction mapping $g : D \to D$:

$$\begin{aligned} u_{n+1} &= \beta_n u_n \oplus (1-\beta_n) v_n, \\ v_n &= \alpha_n g(u_n) \oplus (1-\alpha_n) w_n, \quad \forall n \geq 1, \end{aligned} \tag{59}$$

where D is a nonempty closed convex subset of a metric space $\mathbb{E}$, $u_1 \in D$ is an any given element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, and $w_n \in G(u_n)$ satisfying $d(w_n, w_{n+1}) \leq d(u_n, u_{n+1})$ for any $n \in \mathbb{N}$.

By using the method due to Chang et al. [7, 8], Cauchy-Schwarz inequality, and Xu's inequality, we exposed strong convergence theorems of the new two-step viscosity iteration approximation (59) in complete CAT(0) spaces. The main theorems of this paper extend and improve the corresponding results of Chang et al. [7, 8], Piatek [18], Kaewkhao et al. [19], Panyanak and Suantai [16], and many others in the literature.

However, when g is a set-value contraction operator or is also nonexpansive in (59), whether can our main results be obtained? Furthermore, can our results be obtained when the iterations (13) (i.e., (10)), (12), and (57) become three-step iterations as in [35] or operator T is total asymptotically nonexpansive single-valued (set-valued) operator? These are still **open questions** to be worth further studying.

Conflicts of Interest

The authors declare that there are not any conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was partially supported by the Scientific Research Project of Sichuan University of Science & Engineering (2017RCL54) and the Scientific Research Fund of Sichuan Provincial Education Department (16ZA0256).

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 4051812, 5 pages
https://doi.org/10.1155/2018/4051812

Research Article

A Result on the Existence and Uniqueness of Stationary Solutions for a Bioconvective Flow Model

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Received 8 December 2017; Accepted 25 March 2018; Published 30 April 2018

Academic Editor: Juan Martinez-Moreno

In this note, we prove the existence and uniqueness of weak solutions for the boundary value problem modelling the stationary case of the bioconvective flow problem. The bioconvective model is a boundary value problem for a system of four equations: the nonlinear Stokes equation, the incompressibility equation, and two transport equations. The unknowns of the model are the velocity of the fluid, the pressure of the fluid, the local concentration of microorganisms, and the oxygen concentration. We derive some appropriate a priori estimates for the weak solution, which implies the existence, by application of Gossez theorem, and the uniqueness by standard methodology of comparison of two arbitrary solutions.

1. Introduction

Bioconvection is an important process in the biological treatment and in the life of some microorganisms. In a broad sense, biconvection originates from the concentration of upward swimming microorganisms in a culture fluid. It is well known that, under some physical assumptions, the process can be described by mathematical models which are called bioconvective flow models. The first model of this kind was derived by Moribe [1] and independently by Levandowsky et al. [2] (see also [3] for the mathematical analysis). In that model the unknowns are the velocity of the fluid, the pressure of the fluid, and the local concentration of microorganisms. More recently, Tuval et al. [4] have introduced a new bioconvective flow model considering an additional unknown variable, the oxygen concentration. Some advances in mathematical analysis and some numerical results for this new model are presented in [5] and [6], respectively.

In this paper, we are interested in the existence and uniqueness of solutions for the stationary problem associated with the bioconvective system given in [4] when the physical domain is a three-dimensional chamber [6] (a parallelepiped). Thus, the stationary bioconvective flow problem to be analyzed is formulated as follows. Given the external force $\mathbf{F}$, the source functions f_n, f_c, and the dimensionless function r, find the velocity of the fluid $\mathbf{u} = (u_1, u_2, u_3)^t$, the fluid pressure p, the local concentration of bacteria n, and the local concentration of oxygen c satisfying the boundary value problem:

$$-S_c \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + S_c \nabla p = \gamma S_c n \mathbf{g} + \mathbf{F}, \quad \text{in } \Omega := \prod_{i=1}^{3} [0, L_i], \tag{1}$$

$$\operatorname{div}(\mathbf{u}) = 0, \quad \text{in } \Omega, \tag{2}$$

$$-\Delta n + (\mathbf{u}\cdot\nabla)\, n + \chi \operatorname{div}(n r(c) \nabla c) = f_n, \quad \text{in } \Omega, \tag{3}$$

$$-\delta \Delta c + (\mathbf{u}\cdot\nabla)\, c + \beta r(c)\, n = f_c, \quad \text{in } \Omega, \tag{4}$$

$$\nabla c \cdot \boldsymbol{\nu} = \nabla n \cdot \boldsymbol{\nu} = 0, \quad \mathbf{u} = 0, \quad \text{on } \partial\Omega_L \ (x_3 = 0), \tag{5}$$

$$\chi n r(c) \nabla c \cdot \boldsymbol{\nu} - \nabla n \cdot \boldsymbol{\nu} = 0, \quad \mathbf{u} = 0, \quad \text{on } \partial\Omega_U := \partial\Omega - \partial\Omega_L. \tag{6}$$

Here $\boldsymbol{\nu}$ is the unit external normal to $\partial\Omega$; $\mathbf{g} = (0,0,-g)$ is a gravitational field with constant acceleration g; and S_c, γ, α, δ, and β are some physical parameters defined as follows:

$$S_c = \frac{\eta}{D_n \rho},$$
$$\gamma = \frac{V_b n_r (\rho_b - \rho) L^3}{\eta D_n},$$
$$\chi = \frac{\overline{\chi} c_{\text{air}}}{D_n}, \quad (7)$$
$$\delta = \frac{D_c}{D_n},$$
$$\beta = \frac{k n_r L^2}{c_{\text{air}} D_n},$$

with η being the fluid viscosity, D_n the diffusion constant for bacteria, D_c the diffusion constant for oxygen, ρ the fluid density, ρ_b the bacterial density, $V_b > 0$ the bacterial volume, n_r a characteristic cell density, L a characteristic length, $\overline{\chi}$ the chemotactic sensitivity, c_{air} the oxygen concentration above the fluid, and k the oxygen consumption rate.

We consider the standard notation of the Lebesgue and Sobolev spaces which are used in the analysis of Navier-Stokes and related equations of fluid mechanics; see [7–11] for details and specific definitions. In particular, we use the following rather common spaces notation:

$$H^m(\Omega) = W^{m,2}(\Omega),$$
$$\widetilde{H}^1(\Omega) = \left\{ f \in H^1(\Omega) : \int_\Omega f \, d\mathbf{x} = 0 \right\},$$
$$H_0^1(\Omega) = \overline{\mathbf{C}_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}, \quad (8)$$
$$\mathbf{C}_{0,\sigma}^\infty(\Omega) = \left\{ \mathbf{v} \in \left(C_0^\infty(\Omega)\right)^3 : \operatorname{div}(\mathbf{v}) = 0 \right\},$$
$$\mathbf{V} = \overline{\mathbf{C}_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{H_0^1(\Omega)}},$$

where $\overline{A}^{\|\cdot\|_B}$ denotes the completion of A in B. Also, we consider the notation for the applications $a_0 : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $b_0 : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, and $b : \mathbf{V} \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, which are defined as

$$a_0(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}),$$
$$a(\phi, \psi) = (\nabla \phi, \nabla \psi),$$
$$b_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}), \quad (9)$$
$$b(\mathbf{u}, \phi, \psi) = (\mathbf{u} \cdot \nabla \phi, \psi),$$

where $(\cdot,\cdot)$ is the standard inner product in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$. It is well known that a_0 and a are bilinear coercive forms and b_0 and b are well defined trilinear forms with the following properties:

$$b_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b_0(\mathbf{u}, \mathbf{w}, \mathbf{v}),$$
$$b(\mathbf{u}, \phi, \psi) = -b(\mathbf{u}, \psi, \phi),$$
$$b_0(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad (10)$$
$$b(\mathbf{u}, \phi, \phi) = 0,$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ and $\psi, \phi \in H^1(\Omega)$. Moreover, we need to introduce some notation related to some useful Sobolev inequalities and estimates for b and b_0. There exist $C_{\text{poi}} > 0$, $C_{\text{tr}} > 0$, and C_1 depending only on Ω such that

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \le C_{\text{poi}} \|\mathbf{u}\|_{\mathbf{V}},$$
$$\|c\|_{L^2(\Omega)} \le C_{\text{poi}} \|c\|_{\widetilde{H}^1(\Omega)},$$
$$\|\varphi\|_{L^1(\partial\Omega)} \le C_{\text{tr}} \|\varphi\|_{W^{1,1}(\Omega)}, \quad (11)$$
$$\left|b_0(\mathbf{u}, \mathbf{v}, \mathbf{w})\right| \le C_1 \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}},$$
$$|b(\mathbf{u}, c, n)| \le C_1 \|\mathbf{u}\|_{\mathbf{V}} \|c\|_{\widetilde{H}^1(\Omega)} \|n\|_{\widetilde{H}^1(\Omega)},$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, $c, n \in \widetilde{H}^1(\Omega)$, and $\varphi \in W^{1,1}(\Omega)$. For details on Poincaré and trace inequalities, we refer to [8] and for the estimates of b_0 and b consult [11].

The main result of the paper is the existence and uniqueness of weak solutions for (1)–(6). Indeed, let us introduce some appropriate notation:

$$\Theta_1 := \frac{1 - C_{\text{tr}}}{1 - C_{\text{tr}} - 2\chi \|r\|_{L^1(\mathbb{R})} C_{\text{tr}} C_{\text{poi}}},$$
$$\Theta_2 := \frac{1 - C_{\text{tr}}}{1 - C_{\text{tr}} - C_{\text{tr}} C_{\text{poi}}}, \quad (12)$$

$$\Gamma_0 = \frac{|\Omega| \Theta_1 C_{\text{poi}}}{|\Omega| - \chi\beta\alpha_1 \|r\|^2_{L^\infty(\mathbb{R})} C^2_{\text{poi}} \Theta_1 \Theta_2} \left[\frac{\chi\alpha_1 \|r\|^2_{L^\infty(\mathbb{R})} \Theta_2}{\delta |\Omega|} \|f_c\|_{L^2(\Omega)} + \|f_n\|_{L^2(\Omega)} \right], \quad (13)$$

$$\Gamma_1 = \frac{\gamma S_c g C_{\text{poi}}}{S_c - C_1 C_{\text{poi}} \left(\gamma g \Gamma_0 + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\right)},$$
$$\Gamma_2 = \frac{1 - C_{\text{tr}}}{1 - 2\|r\|_{L^1(\mathbb{R})} \left(1 - C_{\text{tr}} + C_{\text{tr}} C_{\text{poi}}\right)}, \quad (14)$$

$$\Gamma_3 = \frac{1 - C_{\text{tr}}}{\delta \left(1 - C_{\text{tr}} - C_{\text{tr}} C_{\text{poi}}\right) - (C_1)^3 \|r\|_{\text{Lip}(\mathbb{R})} \Gamma_0}, \quad (15)$$

such that the result is precised as follows.

Theorem 1. *Let us consider that $f_c, f_b \in L^2(\Omega)$, $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\overline{n}$, the average of n on Ω, are given. Also consider notations (12)–(15). If we assume that the following assumptions,*

$$r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}),$$
$$1 - C_{tr} > C_{tr} C_{poi} \max\left\{2\chi \|r\|_{L^1(\mathbb{R})}, 1\right\}, \quad (16)$$
$$1 > \chi\beta\overline{n} \|r\|^2_{L^\infty(\mathbb{R})} C^2_{poi} \Theta_1 \Theta_2,$$

are satisfied, there is $(\mathbf{u}, p, n, c) \in \mathbf{V}\times H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)$ *satisfying (1)–(6). Moreover, if we consider that additionally* $r \in Lip(\mathbb{R})$ *and the following inequalities,*

$$S_c - C_1C_{poi}\left(\gamma g\Gamma_0 + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\right) > 0,$$
$$\delta\left(1 - C_{tr} - C_{tr}C_{poi}\right) - (C_1)^3 \|r\|_{L^1(\mathbb{R})}\Gamma_0 > 0, \tag{17}$$
$$C_1 \|r\|_{Lip(\mathbb{R})}\Gamma_0 < 1,$$

$$\Pi = \Gamma_1\Gamma_2 \left\{ C_1\Gamma_0 + \frac{\|r\|_{L^\infty(\mathbb{R})} C\Gamma_3\Theta_2C_{poi}}{\delta\left(1 - C_1\|r\|_{Lip(\mathbb{R})}\Gamma_0\right)}\left[\beta C_{poi}\|r\|_{L^\infty(\mathbb{R})}\Gamma_0 + \|f_c\|_{L^2(\Omega)}\right]\right\} < 1, \tag{18}$$

are satisfied, the weak solution is unique.

It should be noted that existence and uniqueness results are derived in [12, 13] for the bioconvection problem, when the concentration of oxygen is assumed to be constant. In the case of [12], the proof is based on the application of the Galerkin approximation and in [13] on the application of the Gossez theorem. Moreover, other related results are given in [3, 5]. In particular, in [5], a well detailed discussion of some particular models derived from (1)–(6) is given.

2. Proof of Theorem 1

2.1. Variational Formulation. By standard arguments, the variational formulation of (1)–(6) is given by

$$\begin{aligned}
&\text{Find } (\mathbf{u}, n, c) \in \mathbf{V}\times H^1(\Omega)\times H^1(\Omega) \text{ such that}\\
&S_ca_0(\mathbf{u},\mathbf{v}) + b_0(\mathbf{u},\mathbf{u},\mathbf{v}) = \gamma S_c(n\mathbf{g},\mathbf{v}) + (\mathbf{F},\mathbf{v}), \quad \forall \mathbf{v}\in\mathbf{V},\\
&a(n,\phi) + b(\mathbf{u},n,\phi) = \chi(nr(c)\nabla c, \nabla\phi) + (f_n,\phi), \quad \forall\phi\in H^1(\Omega),\\
&\delta a(c,\varphi) + b(\mathbf{u},c,\varphi) = -\beta(r(c)n,\varphi) + \delta\int_{\partial\Omega_U}\nabla c\cdot\boldsymbol{\nu}\varphi\,dS + (f_c,\varphi), \quad \forall\varphi\in H^1(\Omega).
\end{aligned} \tag{19}$$

We notice that if $f_c = f_n = 0$ and $\mathbf{u}_0$ is a solution of (1)-(2) with $n = 0$, we have that $(\mathbf{u}_0, 0, 0)$ is a solution of (19). However, $(\mathbf{u}_0, 0, 0)$ does not describe the bioconvective flow problem and we need to study the variational problem when the total local concentration of bacteria and the total local concentration of oxygen are some given strictly positive constants, that is, $\int_\Omega n_{\boldsymbol{\alpha}} d\mathbf{x} = \alpha_1 > 0$ and $\int_\Omega c_{\boldsymbol{\alpha}} d\mathbf{x} = \alpha_2 > 0$. Thus, by considering the change of variable $\widehat{n}_{\boldsymbol{\alpha}} = n_{\boldsymbol{\alpha}} - \alpha_1|\Omega|^{-1}$ and $\widehat{c}_{\boldsymbol{\alpha}} = c_{\boldsymbol{\alpha}} - \alpha_2|\Omega|^{-1}$, we can rewrite (19) as follows:

$$\text{Given } \boldsymbol{\alpha} = (\alpha_2, \alpha_2) \in \left]0,1\right]\times\left]0,1\right] \text{ find } (\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}}) \in \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega): \tag{20}$$

$$S_ca_0(\mathbf{u}_{\boldsymbol{\alpha}},\mathbf{v}) + b_0(\mathbf{u}_{\boldsymbol{\alpha}},\mathbf{u}_{\boldsymbol{\alpha}},\mathbf{v}) = \gamma S_c(\widehat{n}_{\boldsymbol{\alpha}}\mathbf{g},\mathbf{v}) + (\mathbf{F},\mathbf{v}), \tag{21}$$

$$a(\widehat{n}_{\boldsymbol{\alpha}},\phi) + b(\mathbf{u}_{\boldsymbol{\alpha}},\widehat{n}_{\boldsymbol{\alpha}},\phi) = \chi\left(\left(\widehat{n}_{\boldsymbol{\alpha}} + \frac{\alpha_1}{|\Omega|}\right)r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\phi\right) + (f_n,\phi), \tag{22}$$

$$\delta a(\widehat{c}_{\boldsymbol{\alpha}},\varphi) + b(\mathbf{u}_{\boldsymbol{\alpha}},\widehat{c}_{\boldsymbol{\alpha}},\varphi) = -\beta\left(r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\left(\widehat{n}_{\boldsymbol{\alpha}} + \frac{\alpha_1}{|\Omega|}\right),\varphi\right) + \delta\int_{\partial\Omega_U}\nabla\widehat{c}_{\boldsymbol{\alpha}}\cdot\boldsymbol{\nu}\varphi\,dS + (f_c,\varphi), \tag{23}$$

$$\forall(\mathbf{v},\phi,\varphi) \in \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega). \tag{24}$$

2.2. Some A Priori Estimates for $\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}$, *and* $\widehat{c}_{\boldsymbol{\alpha}}$

Proposition 2. *Consider that the assumptions for the existence result of Theorem 1 are satisfied. If we assume that* $(\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}})$ *is a solution of (20)–(24), then* $\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \le \Gamma_0$ *with* Γ_0 *defined on (13). Furthermore, the following estimates are valid:*

$$\begin{aligned}
\|\mathbf{u}_{\boldsymbol{\alpha}}\|_{\mathbf{V}} &\le C_{poi}\left(\gamma g\Gamma_0 + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\right),\\
\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} &\le \frac{\Theta_2C_{poi}}{\delta}\left[\beta C_{poi}\|r\|_{L^\infty(\mathbb{R})}\Gamma_0 + \|f_c\|_{L^2(\Omega)}\right].
\end{aligned} \tag{25}$$

Proof. In order to prove the estimates, we select the test functions $(\mathbf{v},\phi,\varphi) = (\mathbf{u}_{\boldsymbol{\alpha}}, \widehat{n}_{\boldsymbol{\alpha}}, \widehat{c}_{\boldsymbol{\alpha}})$ in (21)–(23). From (21) and (10), we deduce that

$$\|\mathbf{u}_{\boldsymbol{\alpha}}\|_{\mathbf{V}} \le \gamma gC_{\rm poi}^2\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} + (S_c)^{-1}C_{\rm poi}\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}. \tag{26}$$

Now, by the trace inequality and integration by parts, we have that

$$\begin{aligned}
\int_{\partial\Omega}\left|\nabla\widehat{n}_{\boldsymbol{\alpha}}\cdot\nu\widehat{n}_{\boldsymbol{\alpha}}\right|dS &\le C_{\rm tr}\left\|\widehat{n}_{\boldsymbol{\alpha}}\nabla\widehat{n}_{\boldsymbol{\alpha}}\cdot\nu\right\|_{W^{1,1}(\Omega)}\\
&\le C_{\rm tr}C_{\rm poi}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2 + C_{\rm tr}\int_{\partial\Omega}\left|\nabla\widehat{n}_{\boldsymbol{\alpha}}\cdot\nu\widehat{n}_{\boldsymbol{\alpha}}\right|dS,
\end{aligned} \tag{27}$$

which implies that

$$\int_{\partial\Omega}\left|\nabla\widehat{n}_{\boldsymbol{\alpha}}\cdot\nu\widehat{n}_{\boldsymbol{\alpha}}\right|dS \le \frac{C_{\rm tr}C_{\rm poi}}{1 - C_{\rm tr}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2. \tag{28}$$

Here, we have used the fact that $1 - C_{\mathrm{tr}} > 0$, as a consequence of the assumption (16). Then, by integration by parts we get the bound

$$\begin{aligned}
&\left(\widehat{n}_{\boldsymbol{\alpha}} r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\widehat{n}_{\boldsymbol{\alpha}}\right) \\
&\quad = \left(\nabla\left[\int_0^{\widehat{c}_{\boldsymbol{\alpha}}} r\left(m + \frac{\alpha_2}{|\Omega|}\right)dm\right], \nabla\left(\frac{\widehat{n}_{\boldsymbol{\alpha}}^2}{2}\right)\right) \\
&\quad = -\left(\int_0^{\widehat{c}_{\boldsymbol{\alpha}}} r\left(m + \frac{\alpha_2}{|\Omega|}\right)dm, \Delta\left(\frac{\widehat{n}_{\boldsymbol{\alpha}}}{2}\right)\right) \\
&\qquad + \int_{\partial\Omega}\left[\int_0^{\widehat{c}_{\boldsymbol{\alpha}}} r\left(m + \frac{\alpha_2}{|\Omega|}\right)dm\right]\nabla\left(\frac{\widehat{n}_{\boldsymbol{\alpha}}^2}{2}\right)\cdot\nu\, dS \\
&\quad \le 2\|r\|_{L^1(\mathbb{R})}\int_{\partial\Omega}\left|\widehat{n}_{\boldsymbol{\alpha}}\nabla\widehat{n}_{\boldsymbol{\alpha}}\cdot\nu\right| dS \\
&\quad \le \frac{2\|r\|_{L^1(\mathbb{R})} C_{\mathrm{tr}} C_{\mathrm{poi}}}{1 - C_{\mathrm{tr}}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2.
\end{aligned} \tag{29}$$

From (22), using the properties (10) and the inequality (29), we have that

$$\begin{aligned}
\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2 &= \chi\left(\widehat{n}_{\boldsymbol{\alpha}} r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\widehat{n}_{\boldsymbol{\alpha}}\right) \\
&\quad + \frac{\chi\alpha_1}{|\Omega|}\left(r\left(\widehat{c}_{\boldsymbol{\alpha}} + \frac{\alpha_2}{|\Omega|}\right)\nabla\widehat{c}_{\boldsymbol{\alpha}}, \nabla\widehat{n}_{\boldsymbol{\alpha}}\right) \\
&\quad + (f_n, \phi) \\
&\le \frac{2\chi\|r\|_{L^1(\mathbb{R})} C_{\mathrm{tr}} C_{\mathrm{poi}}}{1 - C_{\mathrm{tr}}}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}^2 \\
&\quad + \frac{\chi\alpha_1}{|\Omega|}\|r\|_{L^\infty(\mathbb{R})}\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \\
&\quad + C_{\mathrm{poi}}\|f_n\|_{L^2(\Omega)}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)},
\end{aligned} \tag{30}$$

or equivalently, we get the following estimate for $\widehat{n}_{\boldsymbol{\alpha}}$:

$$\begin{aligned}
&\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \\
&\quad \le \Theta_1\left[\frac{\chi\alpha_1}{|\Omega|}\|r\|_{L^\infty(\mathbb{R})}\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} + C_{\mathrm{poi}}\|f_n\|_{L^2(\Omega)}\right],
\end{aligned} \tag{31}$$

with Θ_1 being defined in (12). Similarly, from (23) and (28) with $\widehat{c}_{\boldsymbol{\alpha}}$ instead of $\widehat{n}_{\boldsymbol{\alpha}}$, we deduce that

$$\begin{aligned}
&\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \\
&\quad \le \frac{\Theta_2 C_{\mathrm{poi}}}{\delta}\left[\beta C_{\mathrm{poi}}\|r\|_{L^\infty(\mathbb{R})}\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} + \|f_c\|_{L^2(\Omega)}\right],
\end{aligned} \tag{32}$$

where Θ_2 is given in (12). Now, replacing the estimate (32) in (31) and applying (16), we deduce the existence of Γ_0 defined in (13) such that $\|\widehat{n}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)} \le \Gamma_0$. We notice that the second and third relation in (16) imply that $\Theta_i > 1$, $i = 1, 2$, and $|\Omega| > \chi\beta\alpha_1\|r\|_{L^\infty(\mathbb{R})}^2 C_{\mathrm{poi}}^2\Theta_1\Theta_2$, respectively, that is, $\Gamma > 0$ under (16). Moreover, from (26) and (31), we deduce the estimates given in (25) for $\|\mathbf{u}_{\boldsymbol{\alpha}}\|_{\mathbf{V}}$ and $\|\widehat{c}_{\boldsymbol{\alpha}}\|_{\widetilde{H}^1(\Omega)}$, concluding the proof of the Proposition. □

2.3. Proof of Theorem 1. To prove the existence, we can apply the Gossez theorem [9, 14]. Let us first define the mapping $G : \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega) \to (\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega))'$ by the relation

$$\begin{aligned}
&\langle\!\langle G(\mathbf{u}, n, c), (\mathbf{v}, \phi, \varphi)\rangle\!\rangle = \lambda_1\{S_c a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) \\
&\quad - \gamma S_c(n\mathbf{g}, \mathbf{v}) - (\mathbf{F}, \mathbf{v})\} + \lambda_2\Big\{a(n, \phi) + b(\mathbf{u}, n, \phi) \\
&\quad - \chi\left(\left(n + \frac{\alpha_1}{|\Omega|}\right) r\left(c + \frac{\alpha_2}{|\Omega|}\right)\nabla c, \nabla\phi\right) - (f_n, \phi)\Big\} \\
&\quad + \lambda_3\Big\{\delta a(c, \varphi) + b(\mathbf{u}, c, \varphi) \\
&\quad + \beta\left(r\left(c + \frac{\alpha_2}{|\Omega|}\right)\left(n + \frac{\alpha_1}{|\Omega|}\right), \varphi\right) - \delta\int_{\partial\Omega_U}\nabla c \\
&\quad \cdot\boldsymbol{\nu}\varphi\, dS - (f_c, \varphi)\Big\}, \\
&\qquad \forall(\mathbf{u}, n, c), (\mathbf{v}, \phi, \varphi) \in \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega),
\end{aligned} \tag{33}$$

with $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denoting the duality pairing between $\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ and $(\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega))'$ and λ_1, λ_2, and λ_3 are positive fixed constant. From (10), (12), and (29), we then have that

$$\begin{aligned}
&\langle\!\langle G(\mathbf{u}, n, c), (\mathbf{u}, n, c)\rangle\!\rangle \ge \Big\{\lambda_1 S_c\|\mathbf{u}\|_{\mathbf{V}}^2 \\
&\quad - \lambda_1\gamma S_c g\left(C_{\mathrm{poi}}\right)^2\|n\|_{\widetilde{H}^1(\Omega)}\|\mathbf{u}\|_{\mathbf{V}} + \frac{\lambda_2}{3\Theta_1}\|n\|_{\widetilde{H}^1(\Omega)}^2\Big\} \\
&\quad + \Big\{\frac{\lambda_2}{3\Theta_1}\|n\|_{\widetilde{H}^1(\Omega)}^2 \\
&\quad - \frac{\lambda_2\chi\alpha_1}{|\Omega|}\|r\|_{L^\infty(\mathbb{R})}\|c\|_{\widetilde{H}^1(\Omega)}\|n\|_{\widetilde{H}^1(\Omega)} \\
&\quad + \frac{\lambda_3\delta}{2\Theta_2}\|c\|_{\widetilde{H}^1(\Omega)}^2\Big\} + \Big\{\frac{\lambda_3\delta}{2\Theta_2}\|c\|_{\widetilde{H}^1(\Omega)}^2 \\
&\quad - \lambda_3\beta\left(C_{\mathrm{poi}}\right)^2\|r\|_{L^\infty(\mathbb{R})}\|c\|_{\widetilde{H}^1(\Omega)}\|n\|_{\widetilde{H}^1(\Omega)} \\
&\quad + \frac{\lambda_2}{3\Theta_1}\|n\|_{\widetilde{H}^1(\Omega)}^2\Big\} - C_{\mathrm{poi}}\big\{\lambda_1\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{u}\|_{\mathbf{V}} \\
&\quad + \lambda_2\|f_n\|_{L^2(\Omega)}\|n\|_{\widetilde{H}^1(\Omega)} + \lambda_3\|f_c\|_{L^2(\Omega)}\|c\|_{\widetilde{H}^1(\Omega)}\big\} \\
&\quad := \Upsilon_1 + \Upsilon_2 - \Upsilon_3.
\end{aligned} \tag{34}$$

Now, selecting λ_1, λ_2, λ_3 and r such that

$$\lambda_1 < \frac{4\lambda_2}{3\Theta_1\gamma^2 g^2 S_c\left(C_{\mathrm{poi}}\right)^4},$$

$$\lambda_2 < \frac{4\delta|\Omega|^2\lambda_3}{6\Theta_1\Theta_2\left(\chi\alpha_1\|r\|_{L^\infty(\mathbb{R})}\right)^2},$$

$$\lambda_3 < \frac{4\delta\lambda_2}{6\Theta_1\Theta_2\left(\beta\left(C_{\mathrm{poi}}\right)^2\|r\|_{L^\infty(\mathbb{R})}\right)^2}$$

$$r < \frac{\Upsilon_1 + \Upsilon_2}{C_{\mathrm{poi}}\left(\lambda_1\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \lambda_2\|f_n\|_{L^2(\Omega)} + \lambda_3\|f_c\|_{L^2(\Omega)}\right)}, \tag{35}$$

we can prove that $\langle\!\langle G(\mathbf{u}, n, c), (\mathbf{u}, n, c)\rangle\!\rangle$ is positive for all $(\mathbf{u}, n, c) \in \mathbf{V} \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega)$ such that $\|(\mathbf{u}, n, c)\|_{\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)} = r$. Moreover, we notice that it is straightforward to deduce that G is continuous between the weak topologies of $\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ and $(\mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega))'$. Thus, there is $(\mathbf{u}, n, c) \in \overline{B}_r(0) \subset \mathbf{V}\times\widetilde{H}^1(\Omega)\times\widetilde{H}^1(\Omega)$ such that $\langle\!\langle G(\mathbf{u}, n, c), (\mathbf{u}, n, c)\rangle\!\rangle = 0$, concluding the proof of existence.

To prove the uniqueness we consider that there are two solutions $(\mathbf{u}^i, n^i, c^i)$, $i = 1, 2$, satisfying (21)–(23). Then, subtracting, selecting the test functions $(\mathbf{v}, \phi, \varphi) = (\mathbf{u}^1 - \mathbf{u}^2, n^1 - n^2, c^1 - c^2)$, using (10), (16), (17), and applying Proposition 2, we get

$$\left\|\mathbf{u}^1 - \mathbf{u}^2\right\|_{\mathbf{V}} \le \Gamma_1 \left\|n^1 - n^2\right\|_{\widetilde{H}^1(\Omega)}, \tag{36}$$

$$\begin{aligned}\left\|n^1 - n^2\right\|_{\widetilde{H}^1(\Omega)} &\le \Gamma_2 \Big[C_1 \left\|\mathbf{u}^1 - \mathbf{u}^2\right\|_{\mathbf{V}} \left\|n^1\right\|_{\widetilde{H}^1(\Omega)} \\ &\quad + \|r\|_{L^\infty(\mathbb{R})} \left\|c^1 - c^2\right\|_{\widetilde{H}^1(\Omega)}\Big],\end{aligned} \tag{37}$$

$$\begin{aligned}\left\|c^1 - c^2\right\|_{\widetilde{H}^1(\Omega)} &\le C_1\Gamma_3 \Big[\left\|\mathbf{u}^1 - \mathbf{u}^2\right\|_{\mathbf{V}} \left\|c^2\right\|_{\widetilde{H}^1(\Omega)} \\ &\quad + \left(C_1\right)^2 \|r\|_{\mathrm{Lip}(\mathbb{R})} \left\|n^1\right\|_{\widetilde{H}^1(\Omega)} \left\|c^1 - c^2\right\|_{\widetilde{H}^1(\Omega)}\Big],\end{aligned} \tag{38}$$

with Γ_i being defined in (13)–(15). From (38), Proposition 2, and the first inequality in (18), we have that

$$\begin{aligned}&\left\|c^1 - c^2\right\|_{\widetilde{H}^1(\Omega)} \\ &\quad \le \frac{C_1\Gamma_3\Theta_2 C_{\mathrm{poi}}}{\delta\left(1 - \left(C_1\right)^2 \|r\|_{\mathrm{Lip}(\mathbb{R})}\Gamma_0\right)} \Big[\beta C_{\mathrm{poi}} \|r\|_{L^\infty(\mathbb{R})}\Gamma_0 \\ &\quad + \left\|f_c\right\|_{L^2(\Omega)}\Big] \left\|\mathbf{u}^1 - \mathbf{u}^2\right\|_{\mathbf{V}}.\end{aligned} \tag{39}$$

Then, replacing (39) in (37), using Proposition 2 to estimate $\|n^1\|_{\widetilde{H}^1(\Omega)}$, we obtain the bound $\|n^1 - n^2\|_{\widetilde{H}^1(\Omega)} \le \Pi(\Gamma_1)^{-1}\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{V}}$ with Π being defined in (18). Now, using this estimate in (36), we get that $\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{V}} \le \Pi\|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{V}}$. Thus using the fact that $\Pi \le 1$ we deduce that $\mathbf{u}^1 = \mathbf{u}^2$ on $\mathbf{V}$, which also implies that $n^1 = n^2$ and $c^1 = c^2$ on $\widetilde{H}^1(\Omega)$, concluding the uniqueness proof.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

The authors are supported by the project DIUBB 172409 GI/C at Universidad del Bío-Bío, Chile. Aníbal Coronel is supported by the projects DIUBB 183309 4/R and FAPEI at U. del Bío-Bío, Chile. Alex Tello and Ian Hess are supported by Conicyt-Chile through the grants program "Becas de Doctorado".

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 6461930, 6 pages
https://doi.org/10.1155/2018/6461930

Research Article

Positive Solutions for Boundary Value Problems of Fractional Differential Equation with Integral Boundary Conditions

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Received 29 November 2017; Accepted 22 February 2018; Published 22 April 2018

Academic Editor: Dhananjay Gopal

By using two fixed-point theorems on cone, we discuss the existence results of positive solutions for the following boundary value problem of fractional differential equation with integral boundary conditions: $D_{0^+}^{\alpha}x(t)+a(t)f(t,x(t))=0$, $t\in(0,1)$, $x(0)=x'(0)=0$, and $x(1)=\int_0^1 x(t)dA(t)$.

1. Introduction

Boundary value problem for fractional differential equation has aroused much attention in the past few years; many professors devoted themselves to the solvability of fractional differential equations, especially to the study of the existence of solutions for boundary value problems of fractional differential equation (see [1–28]). For example, Wang et al. [19] studied the existence of positive solutions for the following problem:

$$D_{0^+}^{\alpha}u(t)+\lambda f(t,u(t))=0,\quad t\in(0,1), \tag{1}$$

with the boundary conditions

$$u(0)=u'(0)=u(1)=0, \tag{2}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential operator of $2<\alpha\le 3$, λ is a positive parameter, and f may be singular at $t=0,1$ and may change sign. And Ma [14] discussed the positive solutions of

$$\begin{gathered} D_{0^+}^{\alpha}u(t)+a(t)f(t,u(t))=0,\quad t\in(0,1), \\ u(0)=u'(0)=0, \\ u(1)=\sum_{i=1}^{m}\beta_i u(\xi_i), \end{gathered} \tag{3}$$

where $m\ge 1$ is integer and $\xi_i,\beta_i>0$.

There have already been lots of books and papers involving the positive solutions for boundary value problems of fractional differential equation; however, only a few papers cover that for fractional differential equation boundary value problems with integral boundary conditions. Motivated by [14], we shall investigate the positive solutions of the following boundary value problem:

$$\begin{gathered} D_{0^+}^{\alpha}x(t)+a(t)f(t,x(t))=0,\quad t\in(0,1), \\ x(0)=x'(0)=0, \\ x(1)=\int_0^1 x(t)\,dA(t), \end{gathered} \tag{4}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential operator of $2<\alpha<3$, $A(t)$ is right continuous on $[0,1)$, left continuous at $t=1$, and nondecreasing on $[0,1]$ with $A(0)=0$, and

$\int_0^1 u(t)dA(t)$ denotes the Riemann-Stieltjes integrals of u with respect to A. And $a(t)$, $f(t, x(t))$ satisfies the following conditions:

(H1) $a \in L[0,1]$ is nonnegative and not identically zero on any compact subset of $(0,1)$, $\sigma = \int_0^1 t^{\alpha-1}dA(t) < 1$.

(H2) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

This paper consists of four sections. After the introduction, we recall some definitions, lemmas, and theorems in Section 2. And the main results of this paper are stated in Section 3. In the last section, we give two examples of the main results.

2. Preliminaries

Firstly, for convenience we recall some definitions, lemmas, and theorems.

Definition 1 (see [29, 30]). Let $f \in L^1(R^+)$ define the Riemann-Liouville fractional integral of order $\alpha > 0$ for f as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds, \tag{5}$$

where $\Gamma(\alpha)$ is Euler gamma function.

Definition 2 (see [29, 30]). Define the Riemann-Liouville fractional derivative of order $\alpha > 0$ for f as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad n = [\alpha]+1, \tag{6}$$

where f has absolutely continuous derivatives up to order $(n-1)$ on R^+.

Lemma 3. *Let $y \in C[0,1]$, $2 < \alpha \le 3$; then the boundary value problem*

$$\begin{gathered} D_{0^+}^{\alpha} x(t) + y(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \\ x(1) = \int_0^1 x(t)\, dA(t) \end{gathered} \tag{7}$$

has the unique solution $x(t) = \int_0^1 G_1(t,s)y(s)ds$, where

$$G_1(t,s) = G(t,s) + \frac{t^{\alpha-1}}{1-\sigma} \int_0^1 G(\tau,s)\, dA(\tau), \tag{8}$$

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{9}$$

Proof. The boundary value problem can be converted to an equivalent integral equation:

$$x(t) = -I_{0^+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \quad c_1, c_2, c_3 \in R. \tag{10}$$

Then the solution is

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t y(s)(t-s)^{\alpha-1} ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \tag{11}$$

It follows from the boundary conditions $x(0) = x'(0) = 0$ that $c_3 = c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 y(s)(1-s)^{\alpha-1} ds + \int_0^1 x(s)\, dA(s). \tag{12}$$

Thus we get

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t y(s)(t-s)^{\alpha-1} ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 y(s)(1-s)^{\alpha-1} ds \cdot t^{\alpha-1} \\ &\quad + \int_0^1 x(s)\, dA(s) \cdot t^{\alpha-1} \\ &= \int_0^1 G(t,s)\, y(s)\, ds + \int_0^1 x(s)\, dA(s) \cdot t^{\alpha-1}. \end{aligned} \tag{13}$$

Then we can obtain

$$\begin{aligned} \int_0^1 x(s)\, dA(s) &= \int_0^1 \left[\int_0^1 G(s,\tau)\, y(\tau)\, d\tau + \int_0^1 x(\tau)\, dA(\tau)\, s^{\alpha-1}\right] dA(s) \\ &= \int_0^1 \int_0^1 G(s,\tau)\, y(\tau)\, d\tau\, dA(s) \\ &\quad + \int_0^1 x(s)\, dA(s) \cdot \int_0^1 t^{\alpha-1} dA(t), \end{aligned} \tag{14}$$

which means

$$\int_0^1 x(s)\, dA(s) = \frac{\int_0^1 \int_0^1 G(s,\tau)\, y(\tau)\, d\tau\, dA(s)}{1 - \int_0^1 t^{\alpha-1} dA(t)}. \tag{15}$$

So

$$
\begin{aligned}
x(t) &= \int_0^1 G(t,s)\,y(s)\,ds \\
&\quad + \frac{\int_0^1\int_0^1 G(s,\tau)\,y(\tau)\,d\tau\,dA(s)}{1-\int_0^1 t^{\alpha-1}dA(s)}t^{\alpha-1} \\
&= \int_0^1 \left[G(t,s) + \frac{t^{\alpha-1}}{1-\sigma}\int_0^1 G(\tau,s)\,dA(\tau)\right] y(s)\,ds \\
&= \int_0^1 G_1(t,s)\,y(s)\,ds.
\end{aligned} \tag{16}
$$

□

Lemma 4 (see [20]). *$G(t,s)$ defined in (9) has the following properties:*

(i) $G(t,s) > 0$, $t,s \in (0,1)$.

(ii) $G(t,s) = G(1-s,1-t)$, $t,s \in [0,1]$.

(iii) $k(1-t)k(s) \le \Gamma(\alpha)G(t,s) \le (\alpha-1)k(s)$, $t,s \in [0,1]$ *where* $k(t) = t(1-t)^{\alpha-1}$.

Lemma 5. *If (H2) is satisfied, then $G_1(t,s)$ defined in (8) has the following properties:*

$$k(1-t)\,k(s) \le \Gamma(\alpha)\,G_1(t,s) \le Lk(s), \quad t,s \in [0,1], \tag{17}$$

where $L = (\alpha-1)(1+\int_0^1 dA(s)/(1-\sigma))$.

Proof. The proof can be easily accomplished by Lemma 4, so we omitted it. □

Theorem 6 (see [31]). *Let E be a Banach space and $P \subseteq E$ be a cone in E. Suppose that Ω_1, Ω_2 are two bounded open sets of E with $\overline{\Omega_1} \subset \Omega_2$. Assume that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tx\| \le \|x\|$ *for any* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$ *for any* $x \in P \cap \partial\Omega_2$ *or*

(ii) $\|Tx\| \ge \|x\|$ *for any* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$ *for any* $x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 7 (see [32]). *Let P be a cone of a Banach space E. $P_c = \{x \in P : \|x\| < c\}$. θ is a nonnegative continuous concave function on P, such that, for any $x \in \overline{P_c}$, $\theta(x) \le \|x\|$, and $P(\theta,b,d) = \{x \in P : \theta(x) \ge b,\ \|x\| \le d\}$. Assume that $T : \overline{P_c} \to \overline{P_c}$ is completely continuous, and there exist constants $a < b < d \le c$ such that*

(c1) $\{x \in P(\theta,b,d) : \theta(x) > b\} \ne \emptyset$ *and* $\theta(Tx) > b$ *for* $x \in P(\theta,b,d)$;

(c2) $\|Tx\| < a$ *for* $x \in \overline{P_a}$;

(c3) $\theta(Tx) > b$ *for any* $x \in P(\theta,b,c)$ *with* $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, x_3 with $\|x_1\| < a$, $b < \theta(x_2)$, $\|x_3\| > a$, and $\theta(x_3) < b$.

Let $E = C[0,1]$ be a Banach space with the maximum norm $\|x\| = \max_{0\le t\le 1}|x(t)|$; define the cone $P \subseteq E$ as

$$P = \{x \in E : x(t) \ge 0,\ t \in [0,1]\}. \tag{18}$$

Define a continuous operator $T : P \to E$ as

$$(Tx)(t) = \int_0^1 G_1(t,s)\,a(s)\,f(s,x(s))\,ds. \tag{19}$$

Lemma 8. *Assume that (H1) and (H2) hold; then $T : P \to P$ is a completely continuous operator.*

Proof. The lemma can be easily proven, so we omitted it. □

3. Main Results

We define the following notation: given $\delta \in (0,1/2)$, take

$$
\begin{aligned}
h &= \min_{\delta\le t\le 1-\delta} k(1-t) = \min_{\delta\le t\le 1-\delta}(1-t)\,t^{\alpha-1}, \\
M &= \frac{\Gamma(\alpha)}{L\int_0^1 k(s)\,a(s)\,ds}, \\
N &= \frac{\Gamma(\alpha)}{h\int_\delta^{1-\delta} k(s)\,a(s)\,ds}.
\end{aligned} \tag{20}
$$

Now we can obtain the following theorems.

Theorem 9. *Suppose that (H1) and (H2) are satisfied; there exist two positive constants $r_2 > r_1 > 0$ such that*

(H3) $f(t,x) \le Mr_2$, $(t,x) \in [0,1]\times[0,r_2]$;

(H4) $f(t,x) \ge Nr_1$, $(t,x) \in [0,1]\times[0,r_1]$.

Then boundary value problem (4) has at least one positive solution $x \in P$ such that $r_1 \le \|x\| \le r_2$.

Proof. The solution of boundary value problem (4) is equivalent to the fixed point of operator T. Let $\Omega_2 = \{x \in P : \|x\| < r_2\}$; when $x \in \partial\Omega_2$, for any $t \in [0,1]$ we have $0 \le x(t) \le r_2$. By Lemma 5 and (H3) we get

$$
\begin{aligned}
(Tx)(t) &= \int_0^1 G_1(t,s)\,a(s)\,f(s,x(s))\,ds \\
&\le \frac{LMr_2}{\Gamma(\alpha)}\int_0^1 k(s)\,a(s)\,ds = r_2 = \|x\|,
\end{aligned} \tag{21}
$$

which means when $x \in \partial\Omega_2$, $\|Tx\| \le \|x\|$.

Let $\Omega_1 = \{x \in P : \|x\| < r_1\}$; when $x \in \partial\Omega_1$, for any $t \in [0,1]$ we have $0 \le x(t) \le r_1$. By Lemma 5 and (H4) we get

$$\begin{aligned}(Tx)(t) &= \int_0^1 G_1(t,s)\,a(s)\,f(s,x(s))\,ds \\ &\ge k(1-t)\frac{Nr_1}{\Gamma(\alpha)}\int_0^1 k(s)\,a(s)\,ds \\ &\ge \min_{\delta\le t\le 1-\delta} k(1-t)\frac{Nr_1}{\Gamma(\alpha)}\int_\delta^{1-\delta} k(s)\,a(s)\,ds \\ &= r_1 = \|x\|,\end{aligned} \tag{22}$$

which means when $x \in \partial\Omega_1$, $\|Tx\| \ge \|x\|$.

It follows from Theorem 6 that we know that T has at least one fixed point in $(\overline{\Omega_2} \setminus \Omega_1)$, which means that the boundary value problem (4) has at least one solution. □

Theorem 10. *Suppose that (H1) and (H2) are satisfied; there exist four positive constants a, b, c, d with* $0 < a < b < (h/L)d < d < c$, *such that*

(H5) $f(t,x) < Ma$, $(t,x) \in [0,1] \times [0,a]$;

(H6) $f(t,x) > Nb$, $(t,x) \in [\delta, 1-\delta] \times [b,d]$;

(H7) $f(t,x) \le Mc$, $(t,x) \in [0,1] \times [0,c]$.

Then boundary value problem (4) has at least three positive solutions x_1, x_2, x_3, *such that*

$$\begin{aligned}&\max_{0\le t\le 1}|x_1(t)| < a, \\ &\qquad b < \min_{\delta\le t\le 1-\delta}|x_2(t)| < \max_{0\le t\le 1}|x_2(t)| \le c, \\ &\qquad a < \max_{0\le t\le 1}|x_3(t)| \le c, \\ &\min_{\delta\le t\le 1-\delta}|x_3(t)| < b.\end{aligned} \tag{23}$$

Proof. Define a nonnegative continuous concave function θ on P as

$$\theta(x) = \min_{\delta\le t\le 1-\delta} x(t). \tag{24}$$

If $x \in \overline{P_c} = \{x \in P : \|x\| \le c\}$, then $\|x\| \le c$; it follows from (H7) that $f(t,x) \le Mc$; hence

$$\begin{aligned}\|Tx\| &\le \frac{L}{\Gamma(\alpha)}\int_0^1 k(s)\,a(s)\,f(s,x(s))\,ds \\ &\le \frac{LMc}{\Gamma(\alpha)}\int_0^1 k(s)\,a(s)\,ds = c.\end{aligned} \tag{25}$$

Thus, $T(\overline{P_c}) \subseteq \overline{P_c}$. It follows from Lemma 8 that T is completely continuous. In the same way, let $x \in \overline{P_a}$; it follows from (H5) that $f(t,x) < Ma$ for any $t \in [0,1]$, which shows that condition (c2) of Theorem 7 is fulfilled.

Let $x(t) = (b+d)/2$; it is easy to know that $x \in P(\theta,b,d)$ and $\{x \in P(\theta,b,d) : \theta(x) > b\} \ne \emptyset$. If $x \in P(\theta,b,d)$, we have $b \le x(t) \le d$ for any $t \in [\delta, 1-\delta]$. We know $f(t,x(t)) > Nb$ for $\delta \le t \le 1-\delta$ by (H6). So we get

$$\begin{aligned}\theta(Tx) &= \min_{\delta\le t\le 1-\delta} Tx(t) \\ &= \min_{\delta\le t\le 1-\delta}\int_0^1 G_1(t,s)\,a(s)\,f(s,x(s))\,ds \\ &\ge \min_{\delta\le t\le 1-\delta}\int_0^1 \frac{k(1-t)\,k(s)}{\Gamma(\alpha)}a(s)\,f(s,x(s))\,ds \\ &> \frac{Nhb}{\Gamma(\alpha)}\int_\delta^{1-\delta} k(s)\,a(s)\,ds = b.\end{aligned} \tag{26}$$

So condition (c1) of Theorem 7 holds.

When $x \in P(\theta,b,c)$ with $\|Tx\| > d$, noting that

$$\|Tx\| \le \frac{L}{\Gamma(\alpha)}\int_0^1 k(s)\,a(s)\,f(s,x(s))\,ds, \tag{27}$$

thus

$$\begin{aligned}(Tx)(t) &\ge \frac{k(1-t)}{\Gamma(\alpha)}\int_0^1 k(s)\,a(s)\,f(s,x(s))\,ds \\ &\ge \frac{k(1-t)}{L}\|Tx\|,\end{aligned} \tag{28}$$

so we obtain

$$\begin{aligned}\theta(Tx) &= \min_{\delta\le t\le 1-\delta} Tx(t) \ge \frac{\min_{\delta\le t\le 1-\delta} k(1-t)}{L}\|Tx\| \\ &> \frac{h}{L}d > b.\end{aligned} \tag{29}$$

That is to say, (c3) is satisfied.

All conditions of Theorem 7 are satisfied, so T has at least three fixed points x_1, x_2, x_3, which means that the boundary value problem (4) has at least three positive solutions x_1, x_2, x_3, such that

$$\begin{aligned}&\max_{0\le t\le 1}|x_1(t)| < a, \\ &\qquad b < \min_{\delta\le t\le 1-\delta}|x_2(t)| < \max_{0\le t\le 1}|x_2(t)| \le c, \\ &\qquad a < \max_{0\le t\le 1}|x_3(t)| \le c, \\ &\min_{\delta\le t\le 1-\delta}|x_3(t)| < b.\end{aligned} \tag{30}$$

The proof of this theorem is finished. □

4. Some Examples

Now we present two examples to illustrate our main results.

Example 1. Let us see the following problem:

$$\begin{gathered}D_{0^+}^{5/2}x(t) + \frac{1}{4}\sin t + x + 3 = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \\ x(1) = \int_0^1 x(t)\,dt.\end{gathered} \tag{31}$$

Choose $\delta = 1/3$; we obtain that $L = 4$, $h = 2/\sqrt{243}$, $M = 105\sqrt{\pi}/64 \approx 2.91$, and $N = 76545\sqrt{\pi}/(704\sqrt{2} - 256) \approx 183.48$. Then for any $(t, x) \in [0, 1] \times [0, 2]$, we have $f(t, x) = (1/4)\sin t + x + 3 \leq 5.25 \leq Mr_2 \approx 5.82$, and for any $(t, x) \in [0, 1] \times [0, 0.01]$, we get $f(t, x) = (1/4)\sin t + x + 3 \geq 3 \geq Nr_1 \approx 1.83$.

Then the boundary value problem has at least one positive solution $x \in P$ such that $0.01 \leq \|x\| \leq 2$.

Example 2. We now study the following problem:

$$\begin{aligned} &D_{0^+}^{5/2} x(t) + f(t, x) = 0, \quad t \in (0, 1), \\ &x(0) = x'(0) = 0, \\ &x(1) = \int_0^1 x(t)\, dt, \end{aligned} \tag{32}$$

where

$$f(t, x) = \begin{cases} \dfrac{t}{3} + x^2, & x \leq 1; \\ 183 + \dfrac{t}{3} + x, & x > 1. \end{cases} \tag{33}$$

Choose $\delta = 1/3$; we have $M \approx 2.91$, $N \approx 183.48$, $L = 4$, and $h \approx 0.1283$. Let $a = 1/2$, $b = 1$, $c = 100$, and $d = 35$; then for any $(t, x) \in [0, 1] \times [0, 1/2]$, we have $f(t, x) = t/3 + x^2 \leq 0.59 < Ma \approx 1.46$, for any $(t, x) \in [1/3, 2/3] \times [1, 35]$, we have $f(t, x) = 183 + t/3 + x \geq 184.11 > Nb \approx 183.48$, and for any $(t, x) \in [0, 1] \times [0, 100]$, we have $f(t, x) = 183 + t/3 + x \leq 283.34 < Mc \approx 291$. Then by Theorem 10, we conclude that this boundary value problem has at least three positive solutions x_1, x_2, x_3, such that

$$\begin{aligned} &\max_{0 \leq t \leq 1} |x_1(t)| < \frac{1}{2}, \\ &1 < \min_{\delta \leq t \leq 1-\delta} |x_2(t)| < \max_{0 \leq t \leq 1} |x_2(t)| \leq 100, \\ &\frac{1}{2} < \max_{0 \leq t \leq 1} |x_3(t)| \leq 100, \\ &\min_{\delta \leq t \leq 1-\delta} |x_3(t)| < 1. \end{aligned} \tag{34}$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This project is supported by NNSF of China (11371221, 11571207), SDNSF (ZR2018MA011), and the Tai'shan Scholar Engineering Construction Fund of Shandong Province of China.

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 9435470, 7 pages
https://doi.org/10.1155/2018/9435470

Research Article

Characterization of $\sum$-Semicompleteness via Caristi's Fixed Point Theorem in Semimetric Spaces

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Received 7 December 2017; Accepted 2 January 2018; Published 1 March 2018

Academic Editor: Satish Shukla

Introducing the concept of $\sum$-semicompleteness in semimetric spaces, we extend Caristi's fixed point theorem to $\sum$-semicomplete semimetric spaces. Via this extension, we characterize $\sum$-semicompleteness. We also give generalizations of the Banach contraction principle.

1. Introduction

The following, very famous theorem is referred to as Caristi's fixed point theorem. See also [1–10] and references therein.

Theorem 1 (Theorem 1 in [11]). *Let (X, d) be a complete metric space and let T be a mapping on X. Let h be a lower semicontinuous function from X into $[0, \infty)$. Assume $h(Tx) + d(x, Tx) \le h(x)$ for all $x \in X$. Then T has a fixed point.*

In 1976, Kirk proved that Caristi's fixed point theorem characterizes the metric completeness.

Theorem 2 (Theorem 2 in [12]). *Let (X, d) be a metric space. Then the following are equivalent:*

(i) *X is complete.*

(ii) *Every mapping T on X has a fixed point provided there exists a lower semicontinuous function h from X into $[0, \infty)$ satisfying $h(Tx)+d(x, Tx) \le h(x)$ for all $x \in X$.*

Very recently, Theorem 1 was extended to semimetric spaces.

Theorem 3 (see [13]). *Let (X, d) be a $(\sum, \neq)$-complete semimetric space and let T be a mapping on X. Let h be a function from X into $(-\infty, +\infty]$ which is proper, bounded from below, and sequentially lower semicontinuous from above in the sense of Definition 6. Assume $h(Tx) + d(x, Tx) \le h(x)$ for all $x \in X$. Then T has a fixed point.*

Remark 4. See Definitions 5 and 6 for the definitions of $(\sum, \neq)$-completeness and others.

It is a very natural question of whether Theorem 3 characterizes $(\sum, \neq)$-completeness of the underlying space.

In this paper, we give a negative answer to this question (see Example 17). Motivated by this fact, we introduce the concept of $\sum$-semicompleteness and extend Theorem 3 to $\sum$-semicomplete semimetric spaces (see Corollary 12). And we characterize the $\sum$-semicompleteness via Corollary 12 (see Theorem 13). Also we give generalizations of the Banach contraction principle (see Section 4).

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ the sets of all positive integers, all rational numbers, and all real numbers, respectively. For an arbitrary set A, we also denote by $\#A$ the cardinal number of A.

In this section, we give some preliminaries.

Definition 5. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a semimetric space if the following hold:

(D1) $d(x, x) = 0$.

(D2) $d(x, y) = 0 \Rightarrow x = y$.

(D3) $d(x, y) = d(y, x)$ (symmetry).

Definition 6. Let (X, d) be a semimetric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. Let $\kappa \in \mathbb{N}$ and let h be a function from X into $(-\infty, +\infty]$.

(i) $\{x_n\}$ is said to *converge* to x if $\lim_n d(x_n, x) = 0$.

(ii) $\{x_n\}$ is said to be *Cauchy* if $\lim_n \sup\{d(x_n, x_m) : m > n\} = 0$.

(iii) $\{x_n\}$ is said to be $\sum$-*Cauchy* if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$.

(iv) $\{x_n\}$ is said to be $(\sum, \neq)$-*Cauchy* if x_n $(n \in \mathbb{N})$ are all different and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$.

(v) X is said to be *Hausdorff* if $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$ imply $x = y$.

(vi) X is said to be κ-*Hausdorff* if

$$\lim_{n\to\infty} D\left(x, u^{(1)}{}_n, \ldots, u^{(\kappa)}{}_n, y\right) = 0 \tag{1}$$

implies $x = y$, where

$$\begin{aligned} D&\left(x, u^{(1)}{}_n, \ldots, u^{(\kappa)}{}_n, y\right) \\ &= d\left(x, u^{(1)}{}_n\right) + d\left(u^{(1)}{}_n, u^{(2)}{}_n\right) + \cdots \\ &\quad + d\left(u^{(\kappa-1)}{}_n, u^{(\kappa)}{}_n\right) + d\left(u^{(\kappa)}{}_n, y\right). \end{aligned} \tag{2}$$

(vii) X is said to be *complete* if every Cauchy sequence converges.

(viii) X is said to be $\sum$-*complete* if every $\sum$-Cauchy sequence converges.

(ix) X is said to be $(\sum, \neq)$-*complete* if every $(\sum, \neq)$-Cauchy sequence converges.

(x) X is said to be *semicomplete* if every Cauchy sequence has a convergent subsequence.

(xi) X is said to be $\sum$-*semicomplete* if every $\sum$-Cauchy sequence has a convergent subsequence.

(xii) X is said to be $(\sum, \neq)$-*semicomplete* if every $(\sum, \neq)$-Cauchy sequence has a convergent subsequence.

(xiii) d is said to be *sequentially lower semicontinuous* if $d(x, y) \leq \liminf_n d(x_n, y_n)$ provided $\{x_n\}$ converges to x and $\{y_n\}$ converges to y.

(xiv) h is said to be *sequentially lower semicontinuous* if $h(x) \leq \liminf_n h(x_n)$ provided $\{x_n\}$ converges to x.

(xv) h is said to be *sequentially lower semicontinuous from above* if $h(x) \leq \lim_n h(x_n)$ provided $\{x_n\}$ converges to x and $\{h(x_n)\}$ is strictly decreasing.

(xvi) h is said to be *proper* if $\{x \in X : h(x) \in \mathbb{R}\} \neq \varnothing$.

Remark 7.

(i) The definitions of κ-Hausdorffness and $\sum$-semicompleteness are new.

(ii) It is obvious that X is Hausdorff $\Leftrightarrow$ X is 1-Hausdorff.

(iii) It is also obvious that X is λ-Hausdorff $\Rightarrow$ X is κ-Hausdorff provided $\kappa < \lambda$.

Proposition 8. *Let (X, d) be a semimetric space. Then the following implications hold:*

(I) *(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (vi).*

(II) *(i) $\Rightarrow$ (v) $\Rightarrow$ (vi).*

(i) *X is $\sum$-complete.*

(ii) *X is $(\sum, \neq)$-complete.*

(iii) *X is $\sum$-semicomplete.*

(iv) *X is $(\sum, \neq)$-semicomplete.*

(v) *X is complete.*

(vi) *X is semicomplete.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (iv), and (v) $\Rightarrow$ (vi) obviously hold. We have already proved (i) $\Rightarrow$ (v) in [13].

In order to prove (iv) $\Rightarrow$ (iii), we assume (iv). Let $\{x_n\}$ be a $\sum$-Cauchy sequence in X. We consider the following two cases:

(a) There exists $z \in X$ satisfying $\#\{n \in \mathbb{N} : x_n = z\} = \infty$.

(b) For any $x \in X$, $\#\{n \in \mathbb{N} : x_n = x\} < \infty$.

In the first case, some subsequence of $\{x_n\}$ converges to z. In the second case, we define a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ as follows: $f(1) = 1$. We assume that $f(n)$ is defined. Then we define $f(n+1)$ by

$$f(n+1) = \max\left\{k \in \mathbb{N} : x_k = x_{f(n)}\right\} + 1. \tag{3}$$

By induction, we have defined $\{f(n)\}$. We note that $x_{f(n)}$ $(n \in \mathbb{N})$ are all different. We also have

$$\sum_{n=1}^{\infty} d\left(x_{f(n)}, x_{f(n+1)}\right) \leq \sum_{n=1}^{\infty} d\left(x_n, x_{n+1}\right) < \infty. \tag{4}$$

Thus, $\{x_{f(n)}\}$ is $(\sum, \neq)$-Cauchy. From (iv), there exists a subsequence of $\{x_{f(n)}\}$ which converges. It is obvious that the subsequence is also one of subsequences of $\{x_n\}$. Therefore we have shown (iii).

Let us prove (iii) $\Rightarrow$ (vi). We assume (iii). Let $\{x_n\}$ be a Cauchy sequence in X. Choose a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying

$$\sup\left\{d\left(x_\ell, x_m\right) : m > \ell\right\} < 2^{-n} \tag{5}$$

for any $\ell, n \in \mathbb{N}$ with $\ell \geq f(n)$. Then $\{x_{f(n)}\}$ satisfies

$$\sum_{n=1}^{\infty} d\left(x_{f(n)}, x_{f(n+1)}\right) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty, \tag{6}$$

thus $\{x_{f(n)}\}$ is $\sum$-Cauchy. From (iii), there exist $z \in X$ and a subsequence $\{g(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying $\lim_n d(x_{f\circ g(n)}, z) = 0$. Since $\{x_{f\circ g(n)}\}$ is also a subsequence of $\{x_n\}$, we obtain (vi). □

Proposition 9. *Let (X, d) be a semimetric space. Assume that d is sequentially lower semicontinuous. Then X is 2-Hausdorff.*

Proof. Suppose

$$\lim_{n\to\infty}\left(d\left(x,u_n\right)+d\left(u_n,v_n\right)+d\left(v_n,y\right)\right)=0. \quad (7)$$

Then $\{u_n\}$ converges to x and $\{v_n\}$ converges to y. So we have

$$d\left(x,y\right)\le\liminf_{n\to\infty}d\left(u_n,v_n\right)=0. \quad (8)$$

Thus, we obtain $x=y$. □

Proposition 10. *Let (X,d) be a $\sum$-complete semimetric space. Then X is κ-Hausdorff for any $\kappa\in\mathbb{N}$.*

Proof. Suppose

$$\lim_{n\to\infty}D\left(x,u^{(1)}{}_n,\ldots,u^{(\kappa)}{}_n,y\right)=0. \quad (9)$$

Choose a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying

$$D\left(x,u^{(1)}{}_{f(n)},\ldots,u^{(\kappa)}{}_{f(n)},y\right)<2^{-n} \quad (10)$$

for $n\in\mathbb{N}$. Define a sequence $\{v_n\}$ in X as follows:

$$\begin{aligned}v_{(2\kappa+2)i+1}&=x,\\ v_{(2\kappa+2)i+j+1}&=u^{(j)}{}_{f(2i+1)},\\ v_{(2\kappa+2)i+\kappa+2}&=y,\\ v_{(2\kappa+2)i+2\kappa-j+3}&=u^{(j)}{}_{f(2i+2)}\end{aligned} \quad (11)$$

for $i\in\mathbb{N}\cup\{0\}$ and $j\in\{1,\ldots,\kappa\}$. That is, $\{v_n\}$ is as follows:

$$\begin{aligned}&x,u^{(1)}{}_{f(1)},\ldots,u^{(\kappa)}{}_{f(1)},y,u^{(\kappa)}{}_{f(2)},\ldots,u^{(1)}{}_{f(2)},\\ &x,u^{(1)}{}_{f(3)},\ldots,u^{(\kappa)}{}_{f(3)},y,u^{(\kappa)}{}_{f(4)},\ldots,u^{(1)}{}_{f(4)},x,\ldots.\end{aligned} \quad (12)$$

We have

$$\begin{aligned}\sum_{i=1}^{\infty}d\left(v_i,v_{i+1}\right)&=\sum_{i=0}^{\infty}D\left(v_{(\kappa+1)i+1},\ldots,v_{(\kappa+1)(i+1)+1}\right)\\ &<\sum_{i=1}^{\infty}2^{-i}=1<\infty.\end{aligned} \quad (13)$$

Thus, $\{v_n\}$ is $\sum$-Cauchy. Since X is $\sum$-complete, $\{v_n\}$ converges to some z. From the definition of $\{v_n\}$, we have $x=z=y$. Thus, X is κ-Hausdorff. □

3. Caristi's Theorem

In this section, we first prove a Kirk-Saliga type fixed point theorem [14] in $\sum$-semicomplete semimetric spaces. See also [13].

Let α be an ordinal number. We denote by α^+ and α^- the successor and the predecessor of α, respectively. α is said to be *isolated* if α^- exists. On the other hand, α is said to be *limit* if $\alpha\ne 0$ holds and α^- does not exist. For $\kappa\in\mathbb{N}$, we define $\alpha+\kappa$ by

$$\alpha+\kappa=\alpha^{\overbrace{+\cdots+}^{\kappa}}. \quad (14)$$

Theorem 11. *Let (X,d) be a $\sum$-semicomplete semimetric space and let T be a mapping on X. Let h be a function from X into $(-\infty,+\infty]$ which is proper and bounded from below. Assume that h is sequentially lower semicontinuous from above in the sense of Definition 6. Assume also that there exists $\kappa\in\mathbb{N}$ satisfying the following:*

(i) *$h(Tx)\le h(x)$ for all $x\in X$.*

(ii) *$h(T^\kappa x)+d(x,Tx)\le h(x)$ for all $x\in X$.*

Then T has a fixed point.

Proof. Define a function H form X into $(-\infty,+\infty]$ by

$$H\left(x\right)=\sum_{j=0}^{\kappa-1}h\left(T^jx\right), \quad (15)$$

where T^0 is the identity mapping on X. We have from (ii)

$$H\left(Tx\right)+d\left(x,Tx\right)\le H\left(x\right)\quad \text{for } x\in X. \quad (16)$$

Arguing by contradiction, we assume $Tx\ne x$ for any $x\in X$. Let Ω be the first uncountable ordinal number. Using transfinite induction, we will define a net $\{u_\alpha:\alpha\in\Omega\}$ satisfying the following:

$(P_1:\alpha)$ $h(u_\alpha)\le h(u_\beta)$ and $H(u_\alpha)<H(u_\beta)$ for any $\beta\in\Omega$ with $\beta<\alpha$.

$(P_2:\alpha)$ $h(u_\alpha)<h(u_\beta)$ for any $\beta\in\Omega$ with $\beta+\kappa\le\alpha$.

$(P_3:\alpha)$ For any $\varepsilon>0$ and for any $\beta\in\Omega$ with $\beta<\alpha$, there exists a finite sequence $(\gamma_0,\ldots,\gamma_n)\in\Omega^{n+1}$ satisfying

$$\begin{aligned}&\beta=\gamma_0<\gamma_1<\cdots<\gamma_n=\alpha,\\ &H\left(u_\alpha\right)+\sum_{j=0}^{n-1}d\left(u_{\gamma_j},u_{\gamma_{j+1}}\right)<H\left(u_\beta\right)+\varepsilon.\end{aligned} \quad (17)$$

Fix $u\in X$ with $h(u)<\infty$. It follows from (i) that $H(u)<\infty$ holds. Put $u_0=u$. Then $(P_1:0)$–$(P_3:0)$ obviously hold.

Fix $\alpha\in\Omega$ with $0<\alpha$ and assume that $(P_1:\beta)$–$(P_3:\beta)$ hold for $\beta<\alpha$. We consider the following two cases:

(a) α is isolated.

(b) α is limit.

In the first case, we put $u_\alpha=Tu_{\alpha^-}$. For any $\beta<\alpha$, since $\beta\le\alpha^-$ and $u_{\alpha^-}\ne u_\alpha$ hold, we have by $(P_1:\alpha^-)$, (i) and (16)

$$\begin{aligned}&h\left(u_\alpha\right)\le h\left(u_{\alpha^-}\right)\le h\left(u_\beta\right),\\ &H\left(u_\alpha\right)<H\left(u_\alpha\right)+d\left(u_{\alpha^-},u_\alpha\right)\le H\left(u_{\alpha^-}\right)\le H\left(u_\beta\right).\end{aligned} \quad (18)$$

Thus, we have shown $(P_1:\alpha)$. For $\beta\in\Omega$ with $\beta+\kappa\le\alpha$, we have by $(P_1:\alpha)$ and (ii)

$$\begin{aligned}h\left(u_\alpha\right)&\le h\left(u_{\beta+\kappa}\right)=h\left(T^\kappa u_\beta\right)\\ &<h\left(T^\kappa u_\beta\right)+d\left(u_\beta,Tu_\beta\right)\le h\left(u_\beta\right).\end{aligned} \quad (19)$$

Thus, we have shown $(P_2 : \alpha)$. Fix $\varepsilon > 0$ and $\beta \in \Omega$ with $\beta < \alpha$. In the case where $\beta = \alpha^-$, putting $\gamma_0 = \beta$ and $\gamma_1 = \alpha$, we have by (16)

$$H(u_\alpha) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) = H(Tu_\beta) + d(u_\beta, Tu_\beta) \le H(u_\beta) < H(u_\beta) + \varepsilon. \tag{20}$$

In the other case, where $\beta < \alpha^-$, from $(P_3 : \alpha^-)$, there exists a finite sequence $(\gamma_0, \dots, \gamma_n) \in \Omega^{n+1}$ satisfying

$$\beta = \gamma_0 < \gamma_1 < \cdots < \gamma_n = \alpha^-, \quad H(u_{\alpha^-}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) < H(u_\beta) + \varepsilon. \tag{21}$$

Putting $\gamma_{n+1} = \alpha$, we have by (16)

$$H(u_\alpha) + \sum_{j=0}^{n} d(u_{\gamma_j}, u_{\gamma_{j+1}}) \le H(u_{\alpha^-}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) < H(u_\beta) + \varepsilon. \tag{22}$$

Thus, we have shown $(P_3 : \alpha)$. Therefore we have defined u_α satisfying $(P_1 : \alpha)$–$(P_3 : \alpha)$ in the first case.

In the second case, let $\{\beta_n\}$ be a strictly increasing sequence in Ω converging to α; that is, the following hold:

(j) $\beta_n < \alpha$ for $n \in \mathbb{N}$.

(jj) For any $\beta < \alpha$, there exists $n \in \mathbb{N}$ satisfying $\beta < \beta_n$.

For any $n \in \mathbb{N}$, from $(P_3 : \beta_{n+1})$, we can choose a finite sequence $({\gamma^{(n)}}_0, \dots, {\gamma^{(n)}}_{\nu_n}) \in \Omega^{\nu_n+1}$ satisfying

$$\beta_n = {\gamma^{(n)}}_0 < {\gamma^{(n)}}_1 < \cdots < {\gamma^{(n)}}_{\nu_n} = \beta_{n+1}, \quad \sum_{j=0}^{\nu_n-1} d(u_{{\gamma^{(n)}}_j}, u_{{\gamma^{(n)}}_{j+1}}) < H(u_{\beta_n}) - H(u_{\beta_{n+1}}) + 2^{-n}. \tag{23}$$

Since h is bounded from below, H is also bounded from below. So we have

$$\sum_{n=1}^{\infty} \sum_{j=0}^{\nu_n-1} d(u_{{\gamma^{(n)}}_j}, u_{{\gamma^{(n)}}_{j+1}}) < H(u_{\beta_1}) - \lim_{n\to\infty} H(u_{\beta_n}) + 1 < \infty. \tag{24}$$

Since X is $\sum$-semicomplete, the sequence

$$(\beta_1 =)\, {\gamma^{(1)}}_0, \dots, {\gamma^{(1)}}_{\nu_1} (= \beta_2 = {\gamma^{(2)}}_0), {\gamma^{(2)}}_1, \dots, {\gamma^{(2)}}_{\nu_2} (= \beta_3 = {\gamma^{(3)}}_0), {\gamma^{(3)}}_1, \dots \tag{25}$$

has a subsequence $\{\delta_n\}$ such that $\{u_{\delta_n}\}$ converges to some $u_\alpha \in X$. We note that $\{\delta_n\}$ is strictly increasing and it converges to α. Taking a subsequence, without loss of generality, we may assume $\delta_n + \kappa \le \delta_{n+1}$ for $n \in \mathbb{N}$. We have by $(P_2 : \delta_{n+1})$

$$h(u_{\delta_{n+1}}) < h(u_{\delta_n}) \quad \text{for } n \in \mathbb{N}. \tag{26}$$

Thus, $\{h(u_{\delta_n})\}$ is strictly decreasing. Fix $\varepsilon > 0$ and $\beta \in \Omega$ with $\beta < \alpha$. We can choose $\nu \in \mathbb{N}$ satisfying

$$\beta < \delta_\nu, \quad d(u_{\delta_\nu}, u_\alpha) < \varepsilon. \tag{27}$$

Since h is sequentially lower semicontinuous from above, we have from (i)

$$\begin{aligned} h(u_\alpha) &\le \lim_{n\to\infty} h(u_{\delta_n}) < h(u_{\delta_\nu}) \le h(u_\beta), \\ H(u_\alpha) &\le \kappa h(u_\alpha) \le \kappa \lim_{n\to\infty} h(u_{\delta_n}) \\ &= \kappa \inf \{h(u_\gamma) : \gamma < \alpha\} = \lim_{n\to\infty} H(u_{\delta_n}) \\ &< H(u_{\delta_\nu}) < H(u_\beta). \end{aligned} \tag{28}$$

We have shown $(P_1 : \alpha)$ and $(P_2 : \alpha)$. We can choose a finite sequence $(\gamma_0, \dots, \gamma_n) \in \Omega^{n+1}$ satisfying

$$\beta = \gamma_0 < \gamma_1 < \cdots < \gamma_n = \delta_\nu, \quad H(u_{\delta_\nu}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) < H(u_\beta) + \varepsilon. \tag{29}$$

Putting $\gamma_{n+1} = \alpha$, we have by $(P_1 : \alpha)$

$$H(u_\alpha) + \sum_{j=0}^{n} d(u_{\gamma_j}, u_{\gamma_{j+1}}) < H(u_{\delta_\nu}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) + \varepsilon < H(u_\beta) + 2\varepsilon. \tag{30}$$

Thus we have defined u_α satisfying $(P_1 : \alpha)$–$(P_3 : \alpha)$ in the second case.

Therefore by transfinite induction, we have defined the net $\{u_\alpha : \alpha \in \Omega\}$ satisfying $(P_1 : \alpha)$–$(P_3 : \alpha)$ for any $\alpha \in \Omega$. Since the net $\{H(u_\alpha) : \alpha \in \Omega\}$ is strictly decreasing, we obtain

$$\#\mathbb{Q} = \#\mathbb{N} < \#\Omega \le \#\mathbb{Q}, \tag{31}$$

which implies a contradiction. Therefore there exists a fixed point of T. □

Using Theorem 11, we can generalize Theorem 3.

Corollary 12. *Let (X, d) be a $\sum$-semicomplete semimetric space and let T be a mapping on X. Let h be a function from X into $(-\infty, +\infty]$ which is proper and bounded from below. Assume that h is sequentially lower semicontinuous from above in the sense of Definition 6. Assume also*

$$h(Tx) + d(x, Tx) \le h(x) \tag{32}$$

for all $x \in X$. Then T has a fixed point.

Via Corollary 12, we characterize the $\sum$-semicompleteness of X.

Theorem 13. *Let (X, d) be a semimetric space. Then the following are equivalent:*

(i) *X is $\sum$-semicomplete.*

(ii) *Every mapping T on X has a fixed point provided there exists a function h from X into $[0, \infty]$ such that h is proper and sequentially lower semicontinuous and (32) holds for all $x \in X$.*

Proof. By Corollary 12, we obtain (i) $\Rightarrow$ (ii).

In order to prove (ii) $\Rightarrow$ (i), we will show $\neg$ (i) $\Rightarrow$ $\neg$ (ii). We assume that X is not $\sum$-semicomplete. Then by Proposition 8, X is not $(\sum, \neq)$-semicomplete. So there exists a sequence $\{x_n\}$ in X such that x_n $(n \in \mathbb{N})$ are all different, $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ holds, and there does not exist a subsequence which converges. Define a mapping T on X and a function h from X into $(0, \infty]$ by

$$Tx = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ for some } n \in \mathbb{N} \\ x_1 & \text{if } x \notin \{x_n : n \in \mathbb{N}\}, \end{cases}$$
$$h(x) = \begin{cases} \sum_{j=n}^{\infty} d(x_j, x_{j+1}) & \text{if } x = x_n \text{ for some } n \in \mathbb{N} \\ \infty & \text{if } x \notin \{x_n : n \in \mathbb{N}\}. \end{cases} \tag{33}$$

We note that T and h are well defined because x_n $(n \in \mathbb{N})$ are all different and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ holds. Then h is proper, (32) holds for any $x \in X$, and T does not have a fixed point. Let $\{y_n\}$ be a sequence in X converging to some $y \in X$. Arguing by contradiction, we assume

$$h(y) > \liminf_{n\to\infty} h(y_n). \tag{34}$$

Then from the definition of h, there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $y_{f(n)} \in \{x_m : m > \nu\}$ for any $n \in \mathbb{N}$, where we put $\nu \in \mathbb{N} \cup \{0\}$ by

$$\nu = \begin{cases} n & \text{if } y = x_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } y \notin \{x_n : n \in \mathbb{N}\}. \end{cases} \tag{35}$$

Define a function g from $\mathbb{N}$ into $\{m \in \mathbb{N} : m > \nu\}$ by $y_{f(n)} = x_{g(n)}$. We consider the following two cases:

(a) $\limsup_n g(n) = \infty$.

(b) $\mu := \limsup_n g(n) < \infty$.

In the first case, $\{x_n\}$ has a subsequence converging to y. This is a contradiction. In the second case, we have

$$\begin{aligned} \infty &= \#\{n \in \mathbb{N} : g(n) = \mu\} = \#\left\{n \in \mathbb{N} : y_{f(n)} = x_\mu\right\} \\ &\leq \#\left\{n \in \mathbb{N} : y_n = x_\mu\right\}, \end{aligned} \tag{36}$$

which implies $y = x_\mu$. This is also a contradiction. Therefore we obtain $h(y) \leq \liminf_n h(y_n)$. Thus, h is sequentially lower semicontinuous. □

4. Banach's Theorem

The author has extended the Banach contraction principle [15, 16] to semicomplete semimetric spaces. Such a result will be published somewhere else. See also [17]. In this section, we give other generalizations.

Theorem 14. *Let (X, d) be a 2-Hausdorff $\sum$-semicomplete semimetric space. Let T be a contraction on X. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Proof. Let $r \in [0, 1)$ satisfy

$$d(Tx, Ty) \leq r d(x, y) \quad \text{for } x, y \in X. \tag{37}$$

Fix $u \in X$. Then we have

$$\sum_{n=1}^{\infty} d\left(T^n u, T^{n+1} u\right) \leq \sum_{n=1}^{\infty} r^n d(u, Tu) < \infty. \tag{38}$$

Since X is $\sum$-semicomplete, there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $\{T^{f(n)} u\}$ converges to some $z \in X$. We have

$$\lim_{n\to\infty} d\left(T^{f(n)+1} u, Tz\right) \leq \lim_{n\to\infty} r d\left(T^{f(n)} u, z\right) = 0. \tag{39}$$

Thus, $\{T^{f(n)+1} u\}$ converges to Tz. So we have

$$\lim_{n\to\infty} D\left(z, T^{f(n)} u, T^{f(n)+1} u, Tz\right) = 0. \tag{40}$$

From 2-Hausdorffness of X, we obtain $Tz = z$.

For any $x \in X$, we have

$$\begin{aligned} \lim_{n\to\infty} d(T^n x, z) &= \lim_{n\to\infty} d(T^n x, T^n z) \leq \lim_{n\to\infty} r^n d(x, z) \\ &= 0. \end{aligned} \tag{41}$$

The uniqueness of the fixed point follows from (41). □

Theorem 15. *Let (X, d) be a $(\sum, \neq)$-complete Hausdorff semimetric space. Let T be a contraction on X. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Proof. Let $r \in [0, 1)$ satisfy (37). Fix $u \in X$. We consider the following two cases:

(a) There exists $\nu \in \mathbb{N}$ satisfying $T^{\nu+1} u = T^\nu u$.

(b) $T^n u \neq T^{n+1} u$ for any $n \in \mathbb{N}$.

In the first case, $T^\nu u$ is a fixed point of T. In the second case, we have

$$\begin{aligned} d\left(T^{n+1} u, T^{n+2} u\right) &\leq r d\left(T^n u, T^{n+1} u\right) \\ &< d\left(T^n u, T^{n+1} u\right) \end{aligned} \tag{42}$$

for any $n \in \mathbb{N}$. Hence $T^n u$ $(n \in \mathbb{N})$ are all different. We also have

$$\sum_{n=1}^{\infty} d\left(T^n u, T^{n+1} u\right) \leq \sum_{n=1}^{\infty} r^n d(u, Tu) < \infty. \tag{43}$$

Since X is $(\sum, \neq)$-complete, $\{T^n u\}$ converges to some $z \in X$. We have

$$\lim_{n\to\infty} d\left(T^{n+1}u, Tz\right) \leq \lim_{n\to\infty} rd\left(T^n u, z\right) = 0. \tag{44}$$

Thus, $\{T^{n+1}u\}$ converges to Tz. Since X is Hausdorff, we obtain $Tz = z$. Therefore in both cases, there exists a fixed point of T. As in the proof of Theorem 14, we can prove the remainder. □

By Propositions 8–10 and Theorems 14 and 15, we obtain the following corollary.

Corollary 16. *Let (X, d) be a semimetric space. Assume that either of the following holds:*

(i) *X is $\sum$-complete.*

(ii) *X is $\sum$-semicomplete and d is sequentially lower and semicontinuous.*

Let T be a contraction on X. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

5. Example

We finally give an example which tells that Theorem 3 does not characterize $(\sum, \neq)$-completeness of the underlying space.

Example 17. Put $X = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$. Define a function d from $X \times X$ into $[0, \infty)$ as follows:

$$\begin{aligned} & d(x, x) = 0, \\ & d\left(2^{-n}, 2^{-(n+1)}\right) = d\left(2^{-(n+1)}, 2^{-n}\right) = 2^{-(n+1)} \\ & \qquad \text{for } n \in \mathbb{N}, \\ & d\left(2^{-2n}, 0\right) = d\left(0, 2^{-2n}\right) = 2^{-2n} \quad \text{for } n \in \mathbb{N}, \\ & d(x, y) = 1 \quad \text{otherwise}. \end{aligned} \tag{45}$$

Then the following assertions holds:

(i) (X, d) is a semimetric space.

(ii) X is not $(\sum, \neq)$-semicomplete.

(iii) X is $\sum$-semicomplete.

(iv) Every mapping T on X has a fixed point provided there exists a function h from X into $[0, \infty]$ such that h is proper and sequentially lower semicontinuous and (32) holds for all $x \in X$.

Proof. (i) obviously holds. (iv) follows from (iii) and Corollary 12.

We will show (ii). It is obvious that $\{2^{-n}\}$ is a $(\sum, \neq)$-Cauchy sequence. However, we have $\limsup_n d(2^{-n}, x) = 1$ for all $x \in X$. Therefore we obtain (ii).

In order to prove (iii), we will show that X is $(\sum, \neq)$-semicomplete. Let $\{x_n\}$ be a $(\sum, \neq)$-Cauchy sequence in X. We can choose $\mu, \nu \in \mathbb{N}$ satisfying

$$\begin{aligned} & 0 \notin \{x_n : n \geq \nu\}, \\ & x_\nu = 2^{-\mu}. \end{aligned} \tag{46}$$

From the definition of d, we have $x_{\nu+j} = 2^{-(\mu+j)}$ for $j \in \mathbb{N}$. So $\{x_n\}$ has a subsequence converging to 0. We have shown that X is $(\sum, \neq)$-semicomplete. By Proposition 8, X is $\sum$-semicomplete. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The author is supported in part by JSPS KAKENHI Grant no. 16K05207 from Japan Society for the Promotion of Science.

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 3264620, 4 pages
https://doi.org/10.1155/2018/3264620

Research Article

(α, ψ)-Meir-Keeler Contraction Mappings in Generalized b-Metric Spaces

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Received 29 September 2017; Accepted 20 December 2017; Published 28 January 2018

Academic Editor: Satish Shukla

We present a fixed point theorem for generalized (α, ψ)-Meir-Keeler type contractions in the setting of generalized b-metric spaces. The presented results improve, generalize, and unify many existing famous results in the corresponding literature.

1. Introduction and Preliminaries

The idea of a b-metric has been introduced in the papers [1, 2]. Very recently, this idea was extended in [3] to a generalized b-metric space in the following manner.

Definition 1. Let X be a nonempty set and $s \geq 1$ be a fixed constant. A function $d : X \times X \to [0, \infty]$ is called a generalized b-metric space (in brief, gbms) if and only if for $x, y, z \in X$ the conditions are satisfied:

(d_1) $d(x, y) = 0$ if and only if $x = y$.

(d_2) $d(x, y) = d(y, x)$.

(d_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

A triple (X, d, s) is called a generalized b-metric space.

On the other hand, Meir and Keeler [4] have proved the following very general result on the existence of fixed points of Meir-Keeler contraction mappings in metric spaces.

Theorem 2 (see [4]). *Let (X, d) be a complete metric space and $f : X \to X$ satisfy the following condition:*

(d) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(fx, fy) < \varepsilon. \tag{1}$$

f has a unique fixed point ξ. Moreover, for any $x \in X$,

$$\lim_{n\to\infty} f^n x = \xi, \tag{2}$$

where $f^n x$ denotes the nth iteration of f at a point x.

This result has been generalized and extended in many directions; see [5–15]. Using some auxiliary functions, the main purpose of this paper is to extend and generalize this result on generalized b-metric spaces.

For the sake of explicitness, we recall some notations. The symbols $\mathbb{N}, \mathbb{R}$ denote the natural and real numbers, respectively. Furthermore, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{R}_0^+ := [0, \infty)$.

Berinde [16] characterized comparison functions to define the contraction mappings in the setting of b-metric spaces.

Definition 3. Let $s \geq 1$ be a real number. A function $\phi : [0, \infty) \to [0, \infty)$ is called a (c)-comparison function if

(1) ϕ is increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

Denote Ψ as the set of (c)-comparison functions. We will need the following essential properties in our further discussion.

Lemma 4 (see [16–18]). *For a (c)-comparison function $\varphi : [0,+\infty) \to [0,+\infty)$, the following statements hold:*

(1) *The series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in [0,+\infty)$.*

(2) *The function $b_s : [0,+\infty) \to [0,+\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t)$, $t \in [0,\infty)$, is increasing and continuous at 0.*

(3) *Each iterate φ^k of φ for $k \geq 1$ is also a (c)-comparison function.*

(4) *φ is continuous at 0.*

(5) *$\varphi(t) < t$ for any $t > 0$.*

Inspired by Popescu [19], we introduce the concept of generalized α-orbital admissible mappings.

Definition 5. Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty]$ be a function. We say that T is a generalized α-orbital admissible if

$$\begin{gathered}\alpha(x,Tx) \geq 1 \Longrightarrow \alpha\left(Tx,T^2x\right) \geq 1,\\ \alpha(x,Tx) < \infty \Longrightarrow \alpha\left(Tx,Tx^2\right) < \infty.\end{gathered} \tag{3}$$

Notice that each α-orbital admissible mapping [19] is generalized α-orbital admissible.

Based on the concept of generalized α-orbital admissibility, we are the first who establish a fixed point result for a Meir-Keeler type contraction in the setting of generalized b-metric spaces.

2. Main Results

We start with this definition.

Definition 6. For an arbitrary constant $s \geq 1$, let T be a self-mapping defined on a generalized b-metric space (X,d,s). Then T is called an (α,ψ)-Meir-Keeler contractive mapping if there exist two auxiliary mappings $\alpha : X \times X \to [0,\infty]$ and $\psi \in \Psi$ such that

$$\begin{gathered}\varepsilon \leq \psi(d(x,y)) < \varepsilon + \delta\\ \text{implies } \alpha(x,y)\,d(Tx,Ty) < \varepsilon, \quad \forall x,y \in X.\end{gathered} \tag{4}$$

Remark 7. For $x \neq y$ and $d(x,y) < \infty$ with $\alpha(x,y) < \infty$, from (4) we derive that

$$\alpha(x,y)\,d(Tx,Ty) < \psi(d(x,y)). \tag{5}$$

Our main result is as follows.

Theorem 8. *Let $s \geq 1$ be a fixed constant and (X,d,s) be a complete generalized b-metric space. Suppose that a self-mapping $T : X \to X$ is an (α,ψ)-Meir-Keeler type contraction. Assume also that*

(i) *T is generalized α-orbital admissible;*

(ii) *there exists $x \in X$ such that $1 \leq \alpha(x,Tx) < \infty$;*

(iii) *T is continuous.*

Then for such x, one of the following statements holds:

(A) *For every $n \in \mathbb{N}_0$,*

$$d\left(T^n x, T^{n+1}x\right) = \infty \tag{6}$$

$$\text{or } \alpha\left(T^n x, T^{n+1}x\right) = \infty. \tag{7}$$

(B) *There exists $k \in \mathbb{N}_0$ such that $d(T^k x, T^{k+1}x) < \infty$ and $\alpha(T^k x, T^{k+1}x) < \infty$. In this case, there exists $u \in X$ such that $Tu = u$.*

Proof. On account of assumption (ii), there exists $x \in X$ such that $\alpha(x,Tx) \geq 1$. We suppose that case (A) is not satisfied. Consequently, we have to examine case (B). Consequently, there exists $k \in \mathbb{N}_0$ such that $d(T^k x, T^{k+1}x) < \infty$ and $\alpha(T^k x, T^{k+1}x) < \infty$. If $T^k x = T^{k+1}x$, the proof is completed. Assume that $d(T^k x, T^{k+1}) > 0$. By property of ψ and Remark 7, we have

$$\begin{aligned}&\alpha\left(T^k x, T^{k+1}x\right) d\left(T^{k+1}x, T^{k+2}x\right)\\ &\quad < \psi\left(d\left(T^k x, T^{k+1}x\right)\right) < d\left(T^k x, T^{k+1}x\right) < \infty.\end{aligned} \tag{8}$$

Since T is a generalized α-orbital admissible mapping, by (ii), we derive that

$$1 \leq \alpha(x,Tx) < \infty \Longrightarrow 1 \leq \alpha\left(Tx, T^2x\right) < \infty. \tag{9}$$

Recursively, we obtain that

$$1 \leq \alpha\left(T^{k+n}x, T^{k+n+1}x\right) < \infty \quad \forall n \in \mathbb{N}_0. \tag{10}$$

Applying (10) in (8), we get

$$d\left(T^{k+1}x, T^{k+2}x\right) < \psi\left(d\left(T^k x, T^{k+1}x\right)\right) < \infty. \tag{11}$$

Thus

$$d\left(T^{k+n}x, T^{k+n+1}x\right) < \infty \quad \forall n \in \mathbb{N}_0. \tag{12}$$

Again, on account of (10) and (12) in (8), by induction, one gets

$$d\left(T^{k+n}x, T^{k+n+1}x\right) < \psi^n\left(d\left(T^k x, T^{k+1}x\right)\right). \tag{13}$$

Consequently, for $n, \nu \in \mathbb{N}_0$, by (13) we have

$$\begin{aligned}d\left(T^{k+n}x, T^{k+n+\nu}x\right) &\leq sd\left(T^{k+n}x, T^{k+n+1}x\right) + \cdots\\ &\quad + s^{\nu-1} d\left(T^{k+n+\nu-2}x, T^{k+n+\nu-1}x\right)\\ &\quad + s^{\nu} d\left(T^{k+n+\nu-1}x, T^{k+n+\nu}x\right)\\ &< s\psi^n\left(d\left(T^k x, T^{k+1}x\right)\right) + \cdots\\ &\quad + s^{\nu-1}\psi^{n+\nu-2}\left(d\left(T^k x, T^{k+1}x\right)\right)\\ &\quad + s^{\nu}\psi^{n+\nu-1}\left(d\left(T^k x, T^{k+1}x\right)\right)\end{aligned}$$

$$\begin{aligned}&\leq s\sum_{m=0}^{\infty}s^m\psi^{n+m}\left(d\left(T^kx,T^{k+1}x\right)\right)\\&\leq s\sum_{m=0}^{\infty}s^m\psi^{m}\left(d\left(T^kx,T^{k+1}x\right)\right).\end{aligned}\tag{14}$$

Finally,

$$d\left(T^{k+n}(x),T^{k+n+v}x\right)\leq s\sum_{m=0}^{\infty}s^m\psi^{m}\left(d\left(T^kx,T^{k+1}x\right)\right),\tag{15}$$

for all $n,v\in\mathbb{N}_0$. By (15) and the fact that $\psi\in\Psi$, it follows that $\{T^nx\}$ is a Cauchy sequence of elements of X.

Since X is complete, there exists $u\in X$ with

$$\lim_{n\to\infty}d\left(T^nx,u\right)=0.\tag{16}$$

Since T is continuous, we get

$$u=\lim_{n\to\infty}T^{n+1}x=T\left(\lim_{n\to\infty}T^nx\right)=Tu,\tag{17}$$

and u is a fixed point of T, which ends the proof. □

Definition 9. Let $s\geq1$ be a fixed constant. We say that a generalized b-metric space (X,d,s) is regular if $\{x_n\}$ is a sequence in X such that $1\leq\alpha(x_n,x_{n+1})<\infty$ for all n and $x_n\to x\in X$ as $n\to\infty$; then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $1\leq\alpha(x_{n(k)},x)<\infty$ and $0<d(x_{n(k)},x)<\infty$ for all k.

Theorem 10. *Let $s\geq1$ be a fixed constant and (X,d,s) be a complete generalized b-metric space. Suppose that a self-mapping $T:X\to X$ is an (α,ψ)-Meir-Keeler type contraction. Assume also that*

(i) *T is a generalized α-orbital admissible mapping;*

(ii) *there exists $x\in X$ such that $1\leq\alpha(x,Tx)<\infty$;*

(iii) *(X,d,s) is regular.*

Then for such x, one of the following statements holds:

(A) *For every $n\in\mathbb{N}_0$,*

$$d\left(T^nx,T^{n+1}x\right)=\infty\tag{18}$$

$$\text{or }\alpha\left(T^nx,T^{n+1}x\right)=\infty.\tag{19}$$

(B) *There exists $k\in\mathbb{N}_0$ such that $d(T^kx,T^{k+1}x)<\infty$ and $\alpha(T^kx,T^{k+1}x)<\infty$. In this case, there exists $u\in X$ such that $Tu=u$.*

Proof. In case (B), following the proof of Theorem 8, we know that the sequence $\{T^n(x)\}$ converges to some $u\in X$. By Definition 9 and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $1\leq\alpha(T^{n(k)}x,u)<\infty$ and $0<d(T^{n(k)}x,u)<\infty$ for all k. Applying (5) for all k, we get that

$$\begin{aligned}d\left(T^{n(k)+1}x,Tu\right)&=d\left(T\left(T^{n(k)}x\right),Tu\right)\\&\leq\alpha\left(T^{n(k)}x,u\right)d\left(T\left(T^{n(k)}x\right),Tu\right)\\&<\psi\left(d\left(T^{n(k)}x,u\right)\right).\end{aligned}\tag{20}$$

Letting $k\to\infty$ in the above equality, we get $d(u,Tu)=0$; that is, $u=Tu$. □

For the uniqueness of a fixed point of an (α,ψ)-Meir-Keeler type contraction mapping T in $Y:=\{t\in X:d(T^kx,t)<\infty\}$, we shall consider the following condition:

(U) For all $x,y\in\mathrm{Fix}(T)$, we have $1\leq\alpha(x,y)<\infty$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 11. *By adding condition (U) to the hypotheses of Theorem 8 (resp., Theorem 10), T has at most one fixed point in $Y:=\{t\in X:d(T^kx,t)<\infty\}$.*

Proof. Let T be an (α,ψ)-Meir-Keeler type contraction. Owing to Theorem 8 (resp., Theorem 10), T has a fixed point $u\in X$.

Now, we shall show that T has at most one fixed point in Y. We argue by contradiction. For this, assume that there exist two distinct fixed points u_1 and u_2 of T, where $u_1,u_2\in Y$; that is,

$$d\left(T^kx,u_i\right)<\infty\quad\forall i=1,2.\tag{21}$$

We deduce

$$d\left(u_1,u_2\right)\leq sd\left(u_1,T^kx\right)+sd\left(T^kx,u_2\right)<\infty.\tag{22}$$

By condition (U), $1\leq\alpha(u_1,u_2)<\infty$ and since $0<d(u_1,u_2)<\infty$, in view of (5), one writes

$$\begin{aligned}d\left(u_1,u_2\right)&=d\left(Tu_1,Tu_2\right)\leq\alpha\left(u_1,u_2\right)d\left(Tu_1,Tu_2\right)\\&<\psi\left(d\left(u_1,u_2\right)\right)<d\left(u_1,u_2\right),\end{aligned}\tag{23}$$

which is a contradiction, so $u_1=u_2$. This completes the proof. □

3. Consequences

3.1. Meir-Keeler Contraction Mappings in gbms. In this section, we present our main result. By letting $\alpha(x,y)=1$ and $\phi(t)=t/2s$, we get the following result.

Theorem 12. *Let (X,d,s) be a generalized complete b-metric space and $T:X\to X$ satisfy the following: given $\varepsilon>0$, there exists $\delta>0$ such that*

$$\varepsilon\leq d\left(x,y\right)<\varepsilon+\delta\text{ implies }d\left(Tx,Ty\right)<\frac{\varepsilon}{2s},\quad x,y\in X.\tag{24}$$

Let $x\in X$. Then one of the following alternatives holds:

(A) For every $n\in\mathbb{N}_0$ ($\mathbb{N}_0$ being the set of all nonnegative integers),

$$d\left(T^nx,T^{n+1}x\right)=\infty.\tag{25}$$

(B) There exists $k\in\mathbb{N}_0$ such that $d(T^kx,T^{k+1}x)<\infty$.

In case (B), we assert the following:

(i) The sequence $\{T^m x\}$ is Cauchy in X.

(ii) There exists a point $u \in X$ such that $Tu = u$ and $\lim_{n\to\infty} d(T^n x, u) = 0$.

(iii) u is the unique fixed point of T in $B := \{t \in X : d(T^k x, t) < \infty\}$.

(iv) For every $t \in B$,

$$\lim_{n\to\infty} d\left(T^n t, u\right) = 0. \quad (26)$$

Remark 13. Unfortunately, if (X, d) is a metric space, we do not get the result of Meir-Keeler [4].

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 3742575, 12 pages
https://doi.org/10.1155/2018/3742575

Research Article

Inexact SA Method for Constrained Stochastic Convex SDP and Application in Chinese Stock Market

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Received 4 August 2017; Revised 17 November 2017; Accepted 13 December 2017; Published 23 January 2018

Academic Editor: Dhananjay Gopal

We propose stochastic convex semidefinite programs (SCSDPs) to handle uncertain data in applications. For these models, we design an efficient inexact stochastic approximation (SA) method and prove the convergence, complexity, and robust treatment of the algorithm. We apply the inexact method for solving SCSDPs where the subproblem in each iteration is only solved approximately and show that it enjoys the similar iteration complexity as the exact counterpart if the subproblems are progressively solved to sufficient accuracy. Numerical experiments show that the method we proposed was effective for uncertain problem.

1. Introduction

In this paper, we propose a class of optimization problems called stochastic convex semidefinite programs (SCSDPs):

$$\begin{aligned} \min \quad & f(X) = \mathbb{E}[F(X,\xi)] \\ \text{s.t.} \quad & \mathscr{A}(X) = b, \\ & X \succeq 0,\ X \in S^n, \end{aligned} \tag{1}$$

where F is a smooth convex function for every realization of ξ on S^n_+, ξ is a random matrix whose probability distribution P is supported on set $\Omega \in R^{r\times r}$, $\mathscr{A} : S^n \to R^m$ is a linear map, $b \in R^m$ and $X \succeq 0$ mean that $X \in S^n_+$, and S^n is the space of $n \times n$ real symmetric endowed with the standard trace inner product $\langle\cdot,\cdot\rangle$ and Frobenius norm $\|\cdot\|_F$. S^n_+ is the set of positive semidefinite matrices in S^n.

SCSDPs may be viewed as an extension of the following stochastic models:

(1) Stochastic (Linear) Semidefinite Programs (SLSDPs) ([1–5])

$$\begin{aligned} \min \quad & \langle C, X\rangle + E[Q(X,w)] \\ \text{s.t.} \quad & \langle A_i, X\rangle = b_i, \quad i = 1,\ldots,m_1, \\ & X \succeq 0, \end{aligned} \tag{2}$$

where $Q(X,w)$ is the minimum of the problem

$$\begin{aligned} \min \quad & \langle D(w), Y\rangle \\ \text{s.t.} \quad & \langle T_j, X\rangle + \langle W_j, Y\rangle = d_j(w), \quad i = 1,\ldots,m_2, \\ & Y \succeq 0, \end{aligned} \tag{3}$$

where $\langle A, B\rangle = \operatorname{trace}(A^T B)$ denotes the Frobenius inner product between A and B.

(2) Stochastic Convex Quadratic Semidefinite Programs (SCQSDPs)

$$\begin{aligned} \min \quad & \frac{1}{2}\langle X, \mathbb{E}[\mathcal{Q}(X,\xi)]\rangle + \langle C, X\rangle \\ \text{s.t.} \quad & \mathscr{A}(X) = b, \\ & X \succeq 0, \end{aligned} \tag{4}$$

where $\mathcal{Q}(\cdot,\xi) : S^n \times \Omega \to S^n$ is given self-adjoint positive semidefinite linear operator for every $\xi \in \Omega$ and $C \in S^n$.

(3) Stochastic Nearest Correlation Matrix. Now we consider an interesting example from finance industry. In stock research, sample correlation matrices (a symmetric positive semidefinite matrix with unit diagonal) constructed from vectors of

stock returns are used for predictive purposes. Unfortunately, on any day when an observation is made data is rarely available for all the stocks of interest. Higham [6] proposed a method that is to compute the sample correlations of pairs of stocks using data drawn only from the days on which both stocks have data available. Compute the nearest correlation matrix and use it for the subsequent stock analysis. This is a statistical application that motivates nearest correlation matrix problem

$$\begin{aligned} \min \quad & \frac{1}{2} \|\mathscr{L}(X-U)\|^2 \\ \text{s.t.} \quad & \operatorname{diag}(X) = e \\ & X \succeq 0, \end{aligned} \tag{5}$$

where $e \in R^n$ is a vector of all ones and $U \in S^n$ is given.

Furthermore, with the particularity of Chinese stock market (in Chinese stock market which is different from stock markets in other countries, stock price rise or fall during a day does not exceed ten percent), we could more accurately estimate the information of stock price in the future. We consider computing expected correlations of pairs of stocks returns for better response random factors of the stock market. In order to justify the subsequent stock analysis, it is desired to compute the nearest expectation correlation matrix and to use that matrix in the computation. We use this matrix to predict the correlation of these stocks in the future. The problem we consider is an important special case of (4)

$$\begin{aligned} \min \quad & \frac{1}{2} E\left[\|\mathscr{L}(X-U(\xi))\|^2\right] \\ \text{s.t.} \quad & \operatorname{diag}(X) = e \\ & X \succeq 0, \end{aligned} \tag{6}$$

where $U(\xi)$ is a stochastic matrix from $(\mathscr{F}, \Omega, P) \to S^n$.

Alternatively, SCSDPs may be viewed as an extension of the following deterministic convex semidefinite programs ([7–11]):

$$\begin{aligned} \min \quad & F(X) \\ \text{s.t.} \quad & \mathscr{A}(X) = b, \\ & X \succeq 0, \ X \in S^n, \end{aligned} \tag{7}$$

where F is a smooth convex function on S^n_+ and $\mathscr{A} : S^n \to R^m$ is a linear map and $b \in R^m$ and $X \succeq 0$ mean that $X \in S^n_+$.

There are several methods available for solving determinate deterministic convex semidefinite programs and their special cases, which include the accelerated proximal gradient method [8], the alternating projection method [6], the quasi-Newton method [12], the inexact semismooth Newton-CG method [13], and the inexact interior-point method [14]. However, all the methods mentioned above may not be extended to efficiently solve the SCSDPs (1) because it might not be easy to evaluate $E[f(X,\xi)]$ in the following occasions:

(a) ξ is a random vector with a known probability distribution, but calculations of the expected value involve multidimensional integration, which is computationally expensive if not impossible.

(b) The function $f(X,\xi)$ is known, but the distribution of ξ is unknown and the information on ξ can only be obtained using past data or sampling.

(c) $E[f(X,\xi)]$ is not observable and it must be approximately evaluated through simulation.

Under these circumstances, the existing numerical methods for deterministic convex semidefinite programs are not applicable to SCSDPs and new methods are needed. On the other hand, Ariyawansa and Zhu [1] and Mehrotra and Gozevin [4] apply barrier decomposition algorithms and interior-point methods for solving stochastic (linear) semidefinite programs. However, these methods may not be extended for solving the convex problems.

The main purpose of this paper is to design an efficient algorithm to solve the general problems (1) including all the special cases mentioned above. The algorithm we propose here is based on the classical SA algorithm whose subproblem must be solved exactly to generate the next iterate point. The SA algorithm approach originates from the pioneering work of Robbins and Monro and is discussed in numerous publications [15–18]. In this paper, we design an inexact SA method which overcomes the limitation just mentioned. Specifically, in our inexact SA method, the subproblem is only solved approximately and we allow the error in solving subproblem to be deterministic or stochastic. From the theoretical point of view we analyze the convergence, complexity, and robust treatment of the algorithm. We also give the numerical results for illustrating the effectiveness of our method.

The rest of the paper is organized as follows. In Section 2, we focus on theory of the inexact SA method to solve the more general model-stochastic convex matrix program over abstract set Θ, and the proof of convergence is given whether the error is determinate or stochastic. In Section 3, we provide two error estimations of the algorithm. In Section 4, we apply the theoretical results proved before for solving SCSDPs (1). Numerical experiments will be showed at last.

2. Algorithm and Convergence Analysis of General Convex Stochastic Matrix Optimization

For more generality, we first consider the following stochastic convex optimization problem:

$$\min_{X \in \Theta} \quad \{f(X) = \mathbb{E}[F(X,\xi)]\}, \tag{8}$$

where $\Theta \subset R^{n\times n}$ is a nonempty closed convex set, ξ is a random matrix whose probability distribution P is supported on set $\Omega \subset R^{n\times n}$, and F is a smooth convex stochastic function on $\Theta \times \Omega$. We assume that the expectation

$$\mathbb{E}[F(X,\xi)] = \int_\Omega F(X,\xi)\, dP(\xi) \tag{9}$$

is well defined and finite valued for every $X \in \Theta$. We also assume that the expected value function $f(\cdot)$ is continuous and convex on Θ. If for every $\xi \in \Omega$ the function $F(\cdot,\xi)$ is convex on X, then it follows that $f(\cdot)$ is convex. With these assumptions, (8) becomes a convex programming problem.

Let f be any convex finite at Θ. A matrix X^* is called ε-subgradient of f at X (where $\varepsilon > 0$) if

$$f(X) \geq (f(X) - \varepsilon) + \langle X^*, Z - X \rangle, \quad \forall Z \in \Theta. \tag{10}$$

The set of all such ε-subgradients is denoted by $\partial_\varepsilon f(X)$.

Next we make the following assumptions (also in [19–21]) which will be used to analyze the convergence of algorithms in this paper:

(A1) It is possible to generate an independent identically distributed i.i.d sample $\xi_1, \xi_2, \ldots$ of realizations of random vector ξ.

(A2) There is an oracle, which, for a given input point $(X,\xi) \in \Theta\times\Omega$, returns inexact stochastic subgradient-a vector $G(X,\xi)$ such that $g(X) = \mathbb{E}[G(X,\xi)]$ is well defined and is a ε-subgradient of $f(\cdot)$ at X, that is, $g(X) \in \partial_\varepsilon f(X)$.

(A3) There is a positive number M such that

$$\mathbb{E}\left[\|G(X,\xi)\|_F^2\right] \leq M^2, \quad \forall X \in \Theta. \tag{11}$$

Remark 1. Assumption (A2) is different from the assumption used in [19]. We only need to get a ε-subgradient which is more likely got in practice.

Definition 2. The *convex hull* of set S (denoted by $\mathrm{co}(S)$) is the smallest convex polygon that contains all the points of S.

Throughout the paper, we use the following notations. $\|X\|_F = \sqrt{\sum_{i,j} X_{ij}^2}$ denotes the Frobenius norm, where X_{ij}^2 means the element at the ith row and jth column of X. $\langle X,Y\rangle = \mathrm{Trace}(X^TY)$ denotes standard trace inner product. By Π_Θ, we denote the metric projection operator onto the set X: that is, $\Pi_\Theta(X) = \arg\min_{X'\in\Theta}\|X - X'\|_F$. Note that Π_Θ is a nonexpanding operator, that is,

$$\|\Pi_{\Theta'} - \Pi_\Theta\|_F \leq \|X' - X\|_F, \quad \forall X, X' \in R^n. \tag{12}$$

The notation $[a]$ stands for the largest integer less than or equal to $a \in R$. By $\xi_{[t]} = (\xi_1, \ldots, \xi_t)$, we denote the history of the process $\xi_1, \ldots, \xi_t$ up to time t. Unless stated otherwise, all relations between random variables are supposed to hold almost surely.

In the following, we discuss theory of the inexact SA approach to the minimization problem (8).

The inexact SA algorithm solves (8) by mimicking the subgradient descent method

$$X_{j+1} = \Pi_\Theta\left(X_j - \gamma_j f'(X_j)\right), \tag{13}$$

where $f'(X) \in \partial f(X)$. These lead to an executable algorithm below.

Algorithm 3.

Step 1. Give the initial point $X_1 \in \Theta$ and set $j = 1$, iterate the following steps.

Step 2. Choose suitable step sizes $\gamma_j > 0$ and computational error e_j.

Step 3. Find an approximation solution of

$$X_{j+1} = \Pi_\Theta\left(X_j - \gamma_j G(X_j, \xi_j) - \gamma_j e_j\right). \tag{14}$$

Set $j := j + 1$; go back to Step 2.

In order to prove the convergence of Algorithm 3 we introduce the following lemma.

Lemma 4 (see [22]). *Let $\mathcal{F}_k$ be an increasing sequence of σ-algebras and let $\{t_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ be nonnegative random variables adapted to $\mathcal{F}_k$. If $\sum_{i=1}^\infty \alpha_k < \infty$, $\sum_{i=1}^\infty \beta_k < \infty$, it holds almost surely that*

$$t_{k+1} \leq (1 + \alpha_k) t_k + \beta_k - \rho_k, \tag{15}$$

and then t_k is convergent almost surely and $\Sigma\rho_k < \infty$ almost surely.

We now state the main convergence theorem.

Theorem 5. *Suppose that the stochastic optimization problem (8) has an optimal solution X^* and $e_j \to 0$. Let $\gamma_j = \theta/j$, where $\theta > 0$, and $X_j \subset \Theta$ be a sequence generated by (14); then X_j converges to X^*.*

Proof. Note that the iterate $X_j = X_j(\xi_{[j-1]})$ is a function of the history $\xi_{[j-1]} = (\xi_1, \ldots, \xi_{j-1})$ of the generated random process and hence is random.

Denote

$$\begin{aligned} D_j &= \frac{1}{2}\left\|X_j - X^*\right\|_F^2, \\ d_j &= \mathbb{E}\left[D_j\right] = \frac{1}{2}\mathbb{E}\left[\left\|X_j - X^*\right\|_F^2\right]. \end{aligned} \tag{16}$$

Since $X^* \in \Theta$ and hence $\Pi_\Theta(X^*) = X^*$, we can write

$$\begin{aligned} D_{j+1} &= \frac{1}{2}\left\|\Pi_\Theta\left(X_j - \gamma_j G(X_j,\xi_j) - \gamma_j e_j\right) - X^*\right\|_F^2 \\ &= \frac{1}{2}\left\|\Pi_\Theta\left(X_j - \gamma_j G(X_j,\xi_j) - \gamma_j e_j\right) - \Pi_\Theta(X^*)\right\|_F^2 \\ &\leq \frac{1}{2}\left\|X_j - \gamma_j G(X_j,\xi_j) - \gamma_j e_j - X^*\right\|_F^2 \\ &= D_j + \frac{1}{2}\gamma_j^2\left\|G(X_j,\xi_j) + e_j\right\|_F^2 \\ &\quad - \gamma_j\left\langle X_j - X^*, G(X_j,\xi_j) + e_j\right\rangle, \end{aligned} \tag{17}$$

where the second inequality is due to nonexpansionary of projection operator.

Since X_j is independent of ξ_j, we have

$$\begin{aligned}\mathbb{E}&\left[\left\langle X_j - X^*, G\left(X_j, \xi_j\right)\right\rangle\right] \\ &= \mathbb{E}\left\{\mathbb{E}\left[\left\langle X_j - X^*, G\left(X_j, \xi_j\right)\right\rangle \mid \xi_{[j-1]}\right]\right\} \\ &= \mathbb{E}\left\{\left\langle X_j - X^*, \mathbb{E}\left[G\left(X_j, \xi_j\right) \mid \xi_{[j-1]}\right]\right\rangle\right\} \\ &= \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right)\right\rangle\right].\end{aligned} \tag{18}$$

Since assumption (A3) and $e_j \rightarrow 0$, there is a positive number M such that

$$\begin{aligned}D_{j+1} \leq D_j &+ \frac{1}{2}\gamma_j^2 M^2 \\ &- \gamma_j \left\langle X_j - X^*, G\left(X_j, \xi_j\right) + e_j\right\rangle.\end{aligned} \tag{19}$$

Then, by taking expectation of both sides of (17) and using (18), we obtain

$$d_{j+1} \leq d_j - \gamma_j \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + e_j\right\rangle\right] + \frac{1}{2}\gamma_j^2 M^2. \tag{20}$$

For $\gamma_j = \theta/j$, we known that $\sum \theta^2/j^2 < \infty$. By Lemma 4, we know that $d_j = (1/2)\mathbb{E}[\|X_j - X^*\|_F^2]$ is convergent almost surely and

$$\sum_{j=1}^{\infty} \gamma_j \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + e_j\right\rangle\right] < \infty. \tag{21}$$

Suppose that $X_j \rightarrow \overline{X} \neq X^*$ and by the convexity of $f(x)$, we have that $\mathbb{E}[\langle X_j - X^*, g(X_j)\rangle] \geq \mathbb{E}[f(x_j) - f(X^*)] > 0$, when $j \rightarrow \infty$. For $\sum_{j=1}^{\infty} \gamma_j = \sum_{j=1}^{\infty} \theta/j = \infty$, we know that

$$\sum_{j=1}^{\infty} \gamma_j \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + e_j\right\rangle\right] = \infty. \tag{22}$$

It is in contradiction with (21); then we have $\mathbb{E}[\|X_j - X^*\|^2] \rightarrow 0$ almost surely. □

Sometimes, the error at each iteration j is relevant to the random variable ξ_j. In this environment, we consider stochastic inexact SA algorithm.

Algorithm 6.

Step 1. Give the initial point $X_1 \in \Theta$ and set $k = 1$; iterate the following steps.

Step 2. Choose suitable step sizes $\gamma_j > 0$ and computational stochastic error $e_j(\xi_j)$.

Step 3. Find an approximation solution of

$$X_{j+1} = \Pi_{\Theta}\left(X_j - \gamma_j G\left(X_j, \xi_j\right) - \gamma_j e_j\left(\xi_j\right)\right). \tag{23}$$

Set $j := j + 1$; go back to Step 2.

Theorem 7. *Suppose that the stochastic optimization problem (8) has an optimal solution X^*. We also supposed that stochastic error $e_j(\xi_j) \rightarrow 0$ and there is a positive number N such that $e_j(\xi_j) \leq N$. Let $\gamma_j = \theta/j$, where $\theta > 0$, and $X_j \subset \Theta$ be a sequence generated by (23); then X_j converges to X^*.*

Proof. Like the proof of Theorem 5, we obtain

$$\begin{aligned}d_{j+1} \leq d_j &- \gamma_j \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + \mathbb{E}\left[e_j\left(\xi_j\right)\right]\right\rangle\right] \\ &+ \frac{1}{2}\gamma_j^2 (M + N)^2.\end{aligned} \tag{24}$$

For $\gamma_j = \theta/j$, we known that $\sum \theta^2/j^2 < \infty$. By Lemma 4, we know that $d_j = (1/2)\mathbb{E}[\|X_j - X^*\|_F^2]$ is convergent almost surely and

$$\sum_{j=1}^{\infty} \gamma_j \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + \mathbb{E}\left[e_j\left(\xi_j\right)\right]\right\rangle\right] < \infty. \tag{25}$$

Suppose that $X_j \rightarrow \overline{X} \neq X^*$ and by the convexity of $f(x)$, we have that $\mathbb{E}[\langle X_j - X^*, g(X_j)\rangle] \geq \mathbb{E}[f(x_j) - f(X^*)] > 0$, when $j \rightarrow \infty$. For $\sum_{j=1}^{\infty} \gamma_j = \sum_{j=1}^{\infty} \theta/j = \infty$, we know that

$$\sum_{j=1}^{\infty} \gamma_j \mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + E\left[e_j\left(\xi_j\right)\right]\right\rangle\right] = \infty. \tag{26}$$

It is in contradiction with (25); then we have $\mathbb{E}[\|X_j - X^*\|_F] \rightarrow 0$ almost surely. □

3. Error Estimation

Suppose further that the expectation function $f(X)$ is differentiable and strongly convex on Θ; that is, there is constant $c > 0$ such that

$$\begin{aligned}\left\langle X' - X, \nabla f\left(X'\right) - \nabla f(X)\right\rangle &\geq c\left\|X' - X\right\|_F^2, \\ &\forall X', X \in \Theta.\end{aligned} \tag{27}$$

Next we will discuss the complexity of the above algorithms; we take the second algorithm, for instance.

Theorem 8. *Suppose that $f(X)$ is differentiable and strongly convex, $\nabla f(X)$ is Lipschitz continuous, and the conditions of Theorem 7 are satisfied; we have that*

$$\mathbb{E}\left[f\left(X_j\right) - f\left(X^*\right)\right] \leq \frac{(1/2)LQ(\theta)}{j}, \tag{28}$$

where

$$\begin{aligned}&Q(\theta) \\ &\quad = \max\left\{\theta^2 (M + N)^2 (2c\theta - 1)^{-1}, \left\|X_1 - X^*\right\|_F^2\right\}.\end{aligned} \tag{29}$$

Proof. Note that strong convexity of $f(X)$ implies that the minimizer X^* is unique. As optimality of X^* is unique, we have that

$$\left\langle X - X^*, \nabla f\left(X^*\right)\right\rangle \geq 0 \quad \forall X \in \Theta, \tag{30}$$

which together with (27) implies that $\langle X - X^*, \nabla f(X)\rangle \geq c\|X' - X_*\|_F^2$. In turn, it follows that $\langle X - X^*, g + \mathbb{E}[e_j(\xi_j)]\rangle \geq c\|X' - X_*\|_F^2$ for all $X \in \Theta$ and $g(X_j) + \mathbb{E}[e_j(\xi_j)] \in \partial f(X_j)$ and hence

$$\begin{aligned} &\mathbb{E}\left[\left\langle X_j - X^*, g\left(X_j\right) + \mathbb{E}\left[e_j\left(\xi_j\right)\right]\right\rangle\right] \\ &\quad \geq c\mathbb{E}\left[\left\|X_j - X^*\right\|_F^2\right] = 2cd_j. \end{aligned} \tag{31}$$

Therefore, it follows from (24) that

$$d_{j+1} \leq d_j\left(1 - 2c\gamma_j\right) + \frac{1}{2}\gamma_j^2 (M+N)^2. \tag{32}$$

For some constant $\theta > 1/(2c)$, by (32), we have

$$d_{j+1} \leq d_j\left(1 - \frac{2c\theta}{j}\right) + \frac{(1/2)\,\theta^2}{j^2 (M+N)^2}. \tag{33}$$

Next we will prove that

$$\mathbb{E}\left[\left\|X_j - X^*\right\|_F^2\right] = 2d_j \leq \frac{Q(\theta)}{j}, \tag{34}$$

where

$$\begin{aligned} &Q(\theta) \\ &\quad = \max\left\{\theta^2 (M+N)^2 (2c\theta - 1)^{-1}, \left\|X_1 - X^*\right\|_F^2\right\}. \end{aligned} \tag{35}$$

Using mathematical induction, for $j = 1$, we have

$$\mathbb{E}\left[\left\|X_1 - X^*\right\|_F^2\right] = 2d_1 \leq Q(\theta), \tag{36}$$

which holds naturally. For

$$\begin{aligned} 2d_{j+1} &\leq \frac{Q(\theta)}{j}\left(1 - 2c\frac{\theta}{j}\right) + \frac{\theta^2}{j^2}(M+N)^2 \\ &= \frac{\theta^2 (M+N)^2 (j - 2c\theta) + \theta^2 (M+N)^2 (2c\theta - 1)}{j^2 (2c\theta - 1)} \\ &= \frac{\theta^2 (M+N)^2 (j-1)}{j^2 (2c\theta - 1)}, \end{aligned} \tag{$*$}$$

$$\frac{Q(\theta)}{j+1} = \frac{\theta^2 (M+N)^2}{(2c\theta - 1)(j+1)}, \tag{$**$}$$

we know that $(*)/(**) < 1$; then we have $\mathbb{E}[\|X_j - X^*\|_F] = 2d_j \leq Q(\theta)/j$.

For $\nabla f(X)$ is Lipschitz continuous, there is constant $L > 0$ such that

$$\left\|\nabla f\left(X'\right) - \nabla f(X)\right\|_F \leq L\left\|X' - X\right\|_F, \quad \forall X',\ X \in \Theta, \tag{37}$$

and hence

$$\begin{aligned} &\mathbb{E}\left[f\left(X_j\right) - f\left(X^*\right)\right] \\ &\quad \leq \mathbb{E}\left[-\nabla f\left(X_j\right)\left(X_* - X_j\right) + \nabla f\left(X^*\right)\left(X_* - X_j\right)\right] \\ &\quad = \mathbb{E}\left[\left(\nabla f\left(X^*\right) - \nabla f\left(X_j\right)\right)\left(X_* - X_j\right)\right] \\ &\quad \leq L\mathbb{E}\left[\left\|X_j - X^*\right\|_F^2\right] \leq \frac{LQ(\theta)}{j}. \end{aligned} \tag{38}$$

□

When $\theta > 1/2c$, it follows from (38) that, after j iterations, the expected error in terms of the objective value is of order $O(j^{-1})$. But that the result is highly sensitive to a priori information on strongly convex factor c. In order to make the SA method applicable to general convex objectives rather than to strongly convex ones, one should replace the classical step sizes $\gamma_j = \theta/j$, which can be too small to ensure a reasonable rate of convergence. We take appropriate averages of the search points X_j rather than these points themselves. This processing method go back to [20, 23] and we call it "robust treatment." We will show the effectiveness of the above technique in the following theorem.

Theorem 9. *Suppose that f is convex; let $\nu_t \geq 0$ and $\sum_{t=i}^{j} \nu_t = 1$. Consider the points $\widetilde{X}_i^j = \sum_{t=i}^{j} \nu_t X_t$, and let*

$$D_\Theta = \max_{X \in \mathrm{co}\{X_k\}} \left\|X_k - X_1\right\|_F. \tag{39}$$

We know that

$$\mathbb{E}\left[f\left(\widetilde{X}_i^j\right) - f\left(X^*\right)\right] \leq \frac{D_\Theta^2\left(5(M+N) + \sqrt{N}\right)}{2n\gamma}. \tag{40}$$

Proof. Due to (24) and assumption (A3), we know that

$$\begin{aligned} \gamma_t \mathbb{E}\left[\left(f\left(X_t\right) - f\left(X^*\right)\right)\right] &\leq \gamma_t\left\langle X_t - X^*, G\left(X_t, \xi_j\right)\right\rangle \\ &\leq d_t - d_{t+1} + \frac{1}{2}\gamma_t^2 (M+N)^2 \\ &\quad - \gamma_t\left\langle \mathbb{E}\left[X_t - X^*\right], \mathbb{E}\left[e_t\left(\xi_t\right)\right]\right\rangle. \end{aligned} \tag{41}$$

It follows that whenever $1 \leq i \leq j$, we have

$$\begin{aligned} &\sum_{t=i}^{j} \gamma_t \mathbb{E}\left[f\left(X_t\right) - f\left(X^*\right)\right] \\ &\quad \leq \sum_{t=i}^{j}\left[d_t - d_{t+1}\right] + \frac{1}{2}(M+N)^2 \sum_{t=i}^{j} \gamma_t^2 \\ &\qquad - \sum_{t=i}^{j} \gamma_t \left\langle \mathbb{E}\left[X_t - X^*\right], \mathbb{E}\left[e_t\left(\xi_t\right)\right]\right\rangle \\ &\quad \leq d_i + \frac{1}{2}(M+N)^2 \sum_{t=i}^{j} \gamma_t^2 \\ &\qquad - \sum_{t=i}^{j} \gamma_t \left\langle \mathbb{E}\left[X_t - X^*\right], \mathbb{E}\left[e_t\left(\xi_t\right)\right]\right\rangle, \end{aligned} \tag{42}$$

and hence, setting $\nu_t = \gamma_t / \sum_{t=i}^{j} \gamma_t$,

$$\begin{aligned}\mathbb{E}\left[\sum_{t=i}^{j}\nu_t f(X_t)-f(X^*)\right] &= \mathbb{E}\left[\sum_{t=i}^{j}\frac{\gamma_t}{\sum_{t=i}^{j}\gamma_t}f(X_t)-\sum_{t=i}^{j}\frac{\gamma_t}{\sum_{t=i}^{j}\gamma_t}f(X^*)\right]=\mathbb{E}\left[\sum_{t=i}^{j}\frac{\gamma_t}{\sum_{t=i}^{j}\gamma_t}\left[f(X_t)-f(X^*)\right]\right]\\ &= \frac{1}{\sum_{t=i}^{j}\gamma_t}\sum_{t=i}^{j}\gamma_t\mathbb{E}\left[f(X_t)-f(X^*)\right]\\ &\le \frac{d_i+(1/2)(M+N)^2\sum_{t=i}^{j}\gamma_t^2-\sum_{t=i}^{j}\gamma_t\left\langle\mathbb{E}\left[X_t-X^*\right],\mathbb{E}\left[e_t(\xi_t)\right]\right\rangle}{\sum_{t=i}^{j}\gamma_t}.\end{aligned}\tag{43}$$

By convexity of Θ, we have $\widetilde{X}_i^j\in X$, and by convexity of f, we have $f(\widetilde{X}_i^j)\le\sum_{t=i}^{j}\nu_t f(X_t)$. Thus, by (43) and in view of $d_1\le D_\Theta^2$ and $d_i\le 4D_\Theta^2$, $i>1$, we get

$$\begin{aligned}&\mathbb{E}\left[f\left(\widetilde{X}_1^j\right)-f(X^*)\right]\\ &\quad\le\frac{D_\Theta^2+(M+N)^2\sum_{t=i}^{j}\gamma_t^2+\sum_{t=i}^{j}\gamma_t D_\Theta N}{2\sum_{t=i}^{j}\gamma_t},\\ &\mathbb{E}\left[f\left(\widetilde{X}_i^j\right)-f(X^*)\right]\\ &\quad\le\frac{4D_\Theta^2+(M+N)^2\sum_{t=i}^{j}\gamma_t^2+\sum_{t=i}^{j}\gamma_t D_\Theta N}{2\sum_{t=i}^{j}\gamma_t}.\end{aligned}\tag{44}$$

Now we can develop step size policies along with the associated efficiency estimates based on (44). Assume that the number n of iterations of the method is fixed in advance and that $\gamma_t=\gamma$. Then (44) becomes

$$\begin{aligned}&\mathbb{E}\left[f\left(\widetilde{X}_1^j\right)-f(X^*)\right]\\ &\quad\le\frac{D_\Theta^2+(M+N)^2 n\gamma^2+n\gamma D_\Theta N}{2n\gamma},\end{aligned}\tag{45}$$

$$\begin{aligned}&\mathbb{E}\left[f\left(\widetilde{X}_i^j\right)-f(X^*)\right]\\ &\quad\le\frac{4D_\Theta^2+(M+N)^2 n\gamma^2+n\gamma D_\Theta N}{2n\gamma}.\end{aligned}\tag{46}$$

Minimizing the right-hand side of (45) over $\gamma>0$, we arrive at the constant step size policy

$$\gamma=\frac{D_\Theta}{(M+N)\sqrt{n}},\quad t=1,\ldots,N.\tag{47}$$

Along with the associated efficiency estimate and $O(\mathbb{E}[e_t(\xi_t)])<O(1/N)$, we have

$$\mathbb{E}\left[f\left(\widetilde{X}_i^j\right)-f(X^*)\right]\le\frac{D_\Theta^2\left(5(M+N)+\sqrt{N}\right)}{2n\gamma}.\tag{48}$$

□

With constant step size policy (47) in the above theorem, after n iterations, the expected error is of order $O(n^{1/2})$. This is worse than the rate for the inexact SA algorithm. However, the error bounds (48) are guaranteed independently of any smoothness and/or strong convexity assumptions on f.

4. Specialization to the Case Where $\Theta=\{\mathscr{A}(X)=b,X\succeq 0\}$

To illustrate the advantage of inexact SA method over classical SA method, we apply our method to SCSDPs in this section. Problem (1) can be expressed in the form (8) with $\Theta=\{\mathscr{A}(X)=b,\ X\succeq 0,\ X\in S^n\}$. Subproblem (14) then becomes the following constrained minimization problem:

$$\begin{aligned}\min\quad&\frac{1}{2}\left\|X-X_j+\gamma_j G\left(X_j,\xi_j\right)+\gamma_j e_j\right\|^2\\ \text{s.t.}\quad&\mathscr{A}(X)=b,\\ &X\succeq 0.\end{aligned}\tag{49}$$

The KKT corresponding to problem (49) is written as

$$\begin{aligned}&X_{j+1}-X_j+\gamma_j G\left(X_j,\xi_j\right)-\mathscr{A}^*\mu_{j+1}-\lambda_{j+1}+\gamma_j e_j\\ &\quad=0,\\ &\mathscr{A}\left(X_{j+1}\right)-b=0,\\ &\left\langle X_{j+1},\lambda_{j+1}\right\rangle=\varepsilon_{j+1}\approx 0,\quad X_{j+1}\succeq 0,\ \lambda_{j+1}\succeq 0.\end{aligned}\tag{50}$$

We can get the algorithm for problem (49).

Algorithm 10.

Step 1. Given the initial point $X_1\in\Theta$ and setting $j=1$, iterate the following steps.

Step 2. Choose suitable step sizes $\gamma_j>0$ and computational error e_j.

Step 3. Solve the following equation:

$$\begin{aligned}X_{j+1}-X_j+\gamma_j G\left(X_j,\xi_j\right)-\mathscr{A}^*\mu_{j+1}-\lambda_{j+1}+\gamma_j e_j=0,\\ \mathscr{A}\left(X_{j+1}\right)-b=0,\end{aligned}$$

Table 1: Objective function is random matrix.

	Radius	r_P	r_D	CPU time (s)
(1)	0.5	$5.4987e-014$	$3.8803e-009$	0.2260
(2)	1	$5.1761e-014$	$9.8586e-009$	0.2868
(3)	1.5	$3.4534e-015$	$5.4413e-009$	0.5979
(4)	2	$7.9802e-016$	$1.8327e-009$	0.7178
(5)	2.5	$4.5433e-016$	$6.1692e-009$	0.8770
(6)	3	$8.3721e-015$	$3.4310e-009$	0.9087
(7)	3.5	$2.6872e-016$	$3.1458e-009$	1.1569
(8)	4	$4.5937e-016$	$9.1487e-009$	1.2725
(9)	4.5	$8.6907e-016$	$8.1782e-009$	1.3046
(10)	5	$6.6707e-016$	$6.1892e-009$	1.3068
(11)	6	$5.6907e-016$	$3.1882e-009$	1.5898

$$\begin{aligned} \left\langle X_{j+1}, \lambda_{j+1} \right\rangle &= 0, \\ X_{j+1} \succeq 0, \ \lambda_{j+1} &\geq 0. \end{aligned} \tag{51}$$

Set $j := j+1$; go back to Step 2.

5. Application and Numerical Results

In order to assess from a practical point of view the inexact SA method, we code the Algorithm in MATLAB and ran it on several subcategories of SDP. We implement our algorithm in MATLAB 2009a and use a computer with one 2.20 GHz processor and 2.00 GB RAM.

5.1. Numerical Results for Random Matrix. In our first example, we take a simple case of stochastic convex SDP

$$\begin{aligned} \min \quad & f(X) = \left\| X - \mathbb{E}\left[A(\xi)\right] \right\|_F \\ \text{s.t.} \quad & \operatorname{diag}(X) = e, \\ & X \succeq 0, \ X \in S^4, \end{aligned} \tag{52}$$

where $f : S_4^+ \times \Theta \to R$ is a convex function for every ξ and

$$A(\xi) = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \circ \xi, \tag{53}$$

where "$\circ$" means each element of A multiplied by random variable ξ. We let ξ mean 1 which is uniformly distributed whose interval radius is mutative from 0.5 to 4. Using inexact SA method to solve this problem, we list the results in Table 1.

Table 1 shows that the inexact SA method for solving stochastic convex SDP is effective. The abscissa of Figure 1 represent the interval radius of ξ and ordinate of Figure 1 represents norm difference of the optimal value between ξ which is a random variable and $\xi \equiv 1$. This means that the variance of random variable may impact approximation; however, this influence does not exceed the tolerance range.

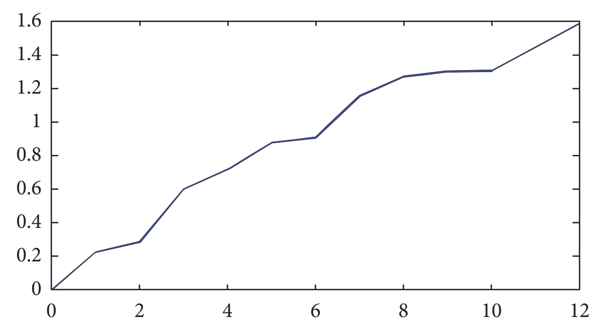

Figure 1: Discrete map.

5.2. Numerical Results for SCQSDPs. In this example, we tested the inexact SA method on the linearly constrained SCQSDPs:

$$\min \left\{ \frac{1}{2} E\left[\left\langle x, \xi \mathcal{Q} x \right\rangle\right] + \langle c, x \rangle : \mathcal{A}(x) = b, \ x \succeq 0 \right\}, \tag{54}$$

where $\xi \sim U[0.5, 1.5]$ is random variable, $\mathcal{Q}$, and $\mathcal{A}$ is given as follows:

$$\begin{aligned} \mathcal{Q}(x) &= \frac{1}{2}(Bx + xB), \quad \text{where } B \succ 0, \\ \mathcal{A}(x) &= \operatorname{diag}(x) = \left[x_{11}, x_{22}, \ldots, x_{nn}\right]^T. \end{aligned} \tag{55}$$

Then b and c can be generated at random. The MATLAB code for generating the matrix b is given by $b = \text{rand}(n,1)$; $b = b. * b$. We generated B as follows:

$$\text{randvec} = 1 + 9 * \text{rand}(n, 1);$$

$$\text{tmp} = \text{randn}\left(n, \text{ceil}\left(\frac{n}{4}\right)\right)$$

Table 2: Objective function is stochastic quadratic.

	n	r_P	r_D	CPU time (s)	Iter	Pobj
(1)	100	$9.2734e-014$	$1.0529e-007$	17.46	18	$6.8577e+002$
(2)	200	$7.1942e-014$	$2.4666e-007$	35.11	17	$1.6196e+003$
(3)	300	$5.2093e-014$	$5.1427e-007$	90.36	16	$2.3798e+003$
(4)	500	$5.6795e-014$	$1.3859e-007$	230.89	16	$4.9247e+003$
(5)	800	$4.5533e-014$	$1.1092e-007$	756.32	20	$8.5645e+004$
(6)	1000	$4.3823e-014$	$7.4480e-007$	1757.73	14	$1.1458e+004$
(7)	1200	$3.6972e-014$	$1.1656e-007$	3939.16	17	$4.9487e+004$
(8)	1500	$3.6907e-014$	$1.1882e-007$	7500.49	18	$5.7514e+005$

$$B = \operatorname{diag}(\text{randvec}) + \frac{\left(\text{tmp} * \text{tmp}'\right)}{n};$$
$$B = \frac{\left(B + B'\right)}{2}. \tag{56}$$

We show the elementary numerical results in Table 2.

Our numerical results are reported in Table 2, where n, r_P, r_D, iter, and pobj stand for, respectively, the matrix dimension, the primal infeasibility, the dual infeasibility, the number of outer iterations, and the primal objective value. For each instance, our algorithm can solve the convex QSDP problem efficiently with very good primal infeasibility. Experimental result shows that the exact SA algorithm can achieve a high degree of detecting reliability.

5.3. Predict Correlation Coefficient in the Chinese Stock Market. In this example, we discuss how to use our inexact SA method to predict correlation coefficient in the Chinese stock market. This is the main motivation of this paper. Chinese stock market is different from other countries, and the rise or fall in a day is not more than ten percent. This feature could make us estimate some index more accurately.

In this paper, our main concern is the correlation coefficient of stocks. Correlation coefficient is often useful to know if two stocks tend to move together. For a diversified portfolio, you would want stocks that are not closely related. It helps to measure the closeness of the returns. Usually, we use the determinate correlation coefficient based on historical data. This principle which we mentioned in the previous paragraph makes us discover that we can use stochastic factor to estimate the correlation coefficient in the future.

According to the difference of the *total market capitalization* (= total equity × stock prices) of company, we can divide stocks into two parts called small-cap stocks and big-cap stocks.

(1) Small-Cap Stocks. They refer to stocks with a relatively small market capitalization. The definition of small cap can vary among brokerages, but generally it is a company with a market capitalization of less than ¥0.1 *billion*.

Table 3 is the closing price of ten stocks of small-cap from 2017-6-3 to 2017-7-3, where CP is logogram of closing price. The second line of Table 3 is code of stocks.

We know that the change of the closing price in 2017.7.4 will not exceed ten percent and seriously up or down is small probability event. According to this and model (6), we can forecast the correlation coefficient from 2017.6.4 to 2017.7.4. We list the results in Table 4.

Now we give the true correlation coefficient of the stocks from 2017.6.4 to 2017.7.4 for comparison.

From Tables 4 and 5, we know that the correlation matrix we calculated is very close to the true correlation matrix. This mean that our method for predicting correlation matrix is effective.

(2) Big-Cap Stocks. In China, it is a term used by the investment community to refer to companies with a market capitalization value of more than ¥0.1 *billion*. Large cap is an abbreviation of the term "large market capitalization." Market capitalization is calculated by multiplying the number of a company's shares outstanding by its stock price per share.

Table 6 is the closing price of ten stocks of big-cap from 2017-6-3 to 2017-7-3.

Like in the small-cap stocks, we calculate the correlation matrix as shown in Table 7.

Now we also give the true correlation coefficient of the stocks from 2017.6.4 to 2017.7.4 for comparison.

From Tables 7 and 8, we know that the correlation matrix we calculated is even more close to the true correlation matrix compared to the case in small-cap stocks. This means that our method for predicting correlation matrix is more effective in the company whose total market capitalization is huge.

6. Conclusion

In this paper, we propose stochastic convex semidefinite programs (SCSDPs) to handle data uncertainty in financial problem. An efficient inexact stochastic approximation method is designed. We proved the convergence, complexity, and robust treatment of the algorithm. Numerical experiments show that the method we proposed was effective for SCSDP and also for its special cases. We also numerically demonstrated that the method is more effective in big-cap stocks.

Table 3: Closing price of small-cap stocks.

CP	002001	002003	002004	002005	002006	002007	002008	002009	002010	002011
6.3	18.140	9.550	14.500	11.330	8.760	26.000	13.500	9.620	9.380	12.380
6.4	18.260	9.250	13.870	10.940	9.430	25.430	13.250	9.600	9.440	12.270
6.5	18.300	9.240	14.010	11.310	9.090	25.300	13.360	9.680	9.600	12.160
6.6	17.680	9.070	13.990	10.950	8.520	25.800	12.770	9.500	9.560	11.930
6.7	17.130	8.950	13.890	10.950	8.070	25.680	12.250	9.240	9.530	11.720
6.13	16.820	8.750	13.370	10.950	8.270	24.500	12.170	8.940	9.380	11.400
6.14	17.060	8.800	13.630	11.460	8.220	24.450	12.790	9.120	9.620	11.860
6.17	17.370	8.770	14.090	11.570	8.280	24.260	12.710	9.300	9.620	11.780
6.18	17.150	8.810	14.450	11.360	8.230	24.800	12.770	9.260	9.860	11.540
6.19	17.280	8.810	14.300	11.160	8.110	25.350	12.990	9.120	9.590	11.500
6.20	16.750	8.620	13.960	10.620	7.750	24.580	12.300	8.650	9.200	10.740
6.21	16.670	8.480	13.990	10.350	7.900	24.560	11.950	8.730	8.760	10.330
6.24	15.640	8.200	13.500	9.3200	7.110	22.900	10.850	8.210	8.350	9.900
6.25	15.160	8.200	13.740	9.190	7.240	23.010	11.290	8.500	8.430	10.310
6.26	15.080	8.180	14.190	9.960	7.310	23.400	11.470	8.770	8.380	10.590
6.27	14.890	8.180	13.850	9.650	7.020	22.890	11.340	8.620	8.000	10.280
6.28	14.740	8.200	14.090	9.300	6.960	22.580	11.190	9.030	8.090	10.140
7.1	14.650	8.250	14.150	9.600	7.050	22.750	11.500	9.490	8.160	10.360
7.2	15.120	8.240	13.920	10.150	7.130	23.320	11.740	9.540	8.200	10.800
7.3	14.750	8.130	14.040	10.230	7.400	23.700	11.570	9.590	8.000	10.680

Table 4: Prediction correlation matrix.

CC	002001	002003	002004	002005	002006	002007	002008	002009	002010	002011
002001	1	0.9597	0.0690	0.8547	0.9343	0.9042	0.9003	0.4002	0.9128	0.8682
002003	0.9597	1	0.0681	0.8080	0.9129	0.9003	0.9110	0.6429	0.8139	0.9120
002004	0.0690	0.0681	1	0.2535	0.1418	0.2719	0.2700	0.4429	0.1797	0.1379
002005	0.8547	0.8080	0.2535	1	0.8080	0.8139	0.8896	0.5691	0.9100	0.8741
002006	0.9343	0.9129	0.1418	0.8080	1	0.8517	0.8964	0.5837	0.8255	0.8888
002007	0.9042	0.9003	0.2719	0.8139	0.8517	1	0.8624	0.5235	0.8381	0.8430
002008	0.9003	0.9110	0.2700	0.8896	0.8964	0.8624	1	0.6828	0.8517	0.9168
002009	0.4002	0.6429	0.4429	0.5691	0.5837	0.5235	0.6828	1	0.4691	0.7216
002010	0.9128	0.8139	0.1797	0.9100	0.8255	0.8381	0.8517	0.4691	1	0.8430
002011	0.8682	0.9120	0.1379	0.8741	0.8886	0.8430	0.9168	0.7216	0.8430	1

Table 5: True correlation matrix.

CC	002001	002003	002004	002005	002006	002007	002008	002009	002010	002011
002001	1	0.966	0.016	0.852	0.948	0.905	0.917	0.342	0.928	0.871
002003	0.966	1	0.043	0.833	0.957	0.908	0.931	0.493	0.908	0.929
002004	0.016	0.043	1	0.172	0.016	0.133	0.236	0.388	0.106	0.09
002005	0.852	0.833	0.172	1	0.821	0.832	0.916	0.512	0.913	0.899
002006	0.948	0.957	0.016	0.821	1	0.868	0.923	0.496	0.848	0.911
002007	0.905	0.908	0.133	0.832	0.868	1	0.87	0.437	0.864	0.849
002008	0.917	0.931	0.236	0.916	0.923	0.87	1	0.583	0.898	0.935
002009	0.342	0.493	0.388	0.512	0.496	0.437	0.583	1	0.339	0.647
002010	0.928	0.908	0.106	0.913	0.848	0.864	0.898	0.339	1	0.875
002011	0.871	0.929	0.09	0.899	0.911	0.849	0.935	0.647	0.875	1

Table 6: Closing price of big-cap stocks.

CP	002001	002003	002004	002005	002006	002007	002008	002009	002010	002011
6.3	9.840	4.790	10.71	10.38	4.81	6.740	12.990	9.280	13.770	12.350
6.4	9.740	4.730	10.70	10.28	4.77	6.730	12.740	9.160	13.750	12.180
6.5	9.650	4.770	10.59	10.10	4.76	6.710	12.880	9.060	13.360	12.150
6.6	9.450	4.700	10.39	10.00	4.71	6.710	12.720	9.040	13.180	11.760
6.7	9.350	4.560	10.26	9.990	4.66	6.600	12.120	8.840	13.170	11.520
6.13	9.020	4.300	9.990	9.880	4.53	6.390	11.350	8.340	12.320	11.000
6.14	9.020	4.500	10.06	9.960	4.55	6.290	11.340	8.310	12.210	11.070
6.17	9.010	4.620	10.03	9.910	4.51	6.350	11.120	8.160	12.100	10.970
6.18	9.070	4.590	10.10	9.950	4.33	6.350	11.220	8.140	12.200	11.120
6.19	8.890	4.560	9.960	9.750	4.28	4.680	11.150	8.110	11.980	10.940
6.20	8.420	4.340	9.520	9.290	4.17	4.500	10.820	7.940	11.470	10.610
6.21	8.280	4.370	9.480	9.450	4.15	4.450	11.050	7.730	11.710	10.590
6.24	7.520	4.060	8.690	8.510	4.05	4.210	10.090	7.480	10.930	9.5300
6.25	7.800	3.960	8.690	8.440	4.03	4.190	10.000	7.540	10.920	9.3000
6.26	7.770	3.930	8.580	8.300	3.92	4.100	9.9900	7.490	10.660	9.3100
6.27	7.880	4.230	8.660	8.210	3.92	4.110	9.8400	7.450	10.920	9.3300
6.28	8.280	4.100	9.020	8.570	3.93	4.180	10.130	7.510	11.600	9.9100
7.1	8.170	4.030	8.920	8.590	3.91	4.270	10.180	7.420	11.380	9.9900
7.2	8.150	3.990	8.880	8.480	3.92	4.240	10.190	7.280	11.210	9.8700
7.3	8.040	3.940	8.680	8.440	3.94	4.250	9.9700	7.060	11.350	9.6600

Table 7: Prediction correlation matrix.

CC	600000	600010	600015	600016	600019	600028	600030	600031	600036	600048
600000	1	0.9119	0.9776	0.9497	0.9388	0.9656	0.9447	0.9427	0.9766	0.9736
600010	0.9119	1	0.9477	0.9318	0.9209	0.9298	0.9020	0.8960	0.9030	0.9318
600015	0.9776	0.9477	1	0.9825	0.9646	0.9835	0.9467	0.9408	0.9716	0.9825
600016	0.9497	0.9318	0.9825	1	0.9437	0.9666	0.9050	0.8940	0.9547	0.9597
600019	0.9388	0.9209	0.9646	0.9437	1	0.9766	0.9587	0.9716	0.9358	0.9527
600028	0.9656	0.9298	0.9835	0.9666	0.9766	1	0.9597	0.9557	0.9567	0.9756
600030	0.9447	0.9020	0.9467	0.9050	0.9587	0.9597	1	0.9776	0.9408	0.9696
600031	0.9427	0.8960	0.9408	0.8940	0.9716	0.9557	0.9776	1	0.9318	0.9487
600036	0.9766	0.9030	0.9716	0.9547	0.9358	0.9567	0.9408	0.9318	1	0.9746
600048	0.9736	0.9318	0.9825	0.9597	0.9527	0.9756	0.9696	0.9487	0.9746	1

Table 8: True correlation matrix.

CC	600000	600010	600015	600016	600019	600028	600030	600031	600036	600048
600000	1	0.911	0.979	0.952	0.937	0.97	0.943	0.932	0.979	0.976
600010	0.911	1	0.951	0.932	0.92	0.925	0.901	0.9	0.887	0.93
600015	0.979	0.951	1	0.987	0.968	0.985	0.951	0.944	0.968	0.988
600016	0.952	0.932	0.987	1	0.946	0.97	0.907	0.893	0.955	0.965
600019	0.937	0.92	0.968	0.946	1	0.981	0.962	0.972	0.929	0.954
600028	0.97	0.925	0.985	0.97	0.981	1	0.965	0.952	0.96	0.982
600030	0.943	0.901	0.951	0.907	0.962	0.965	1	0.975	0.934	0.973
600031	0.932	0.9	0.944	0.893	0.972	0.952	0.975	1	0.909	0.942
600036	0.979	0.887	0.968	0.955	0.929	0.96	0.934	0.909	1	0.973
600048	0.976	0.93	0.988	0.965	0.954	0.982	0.973	0.942	0.973	1

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The work is partially supported by the Natural Science Foundation of China, Grants 11626051, 11626052, 11701061, and 11501074.

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 9269458, 7 pages
https://doi.org/10.1155/2018/9269458

Research Article

Global Bifurcation from Intervals for the Monge-Ampère Equations and Its Applications

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Received 8 August 2017; Accepted 20 November 2017; Published 9 January 2018

Academic Editor: Vijay Gupta

We shall establish the global bifurcation results from the trivial solutions axis or from infinity for the Monge-Ampère equations: $\det(D^2u) = \lambda m(x)(-u)^N + m(x) f_1(x, -u, -u', \lambda) + f_2(x, -u, -u', \lambda)$, in B, $u(x) = 0$, on ∂B, where $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ is the Hessian matrix of u, where B is the unit open ball of $\mathbb{R}^N$, $m \in C(\overline{B}, [0, +\infty))$ is a radially symmetric weighted function and $m(r) := m(|x|) \not\equiv 0$ on any subinterval of $[0, 1]$, λ is a positive parameter, and the nonlinear term $f_1, f_2 \in C(\overline{B} \times (\mathbb{R}^+)^3, \mathbb{R}^+)$, but f_1 is not necessarily differentiable at the origin and infinity with respect to u, where $\mathbb{R}^+ = [0, +\infty)$. Some applications are given to the Monge-Ampère equations and we use global bifurcation techniques to prove our main results.

1. Introduction

The Monge-Ampère equations are a type of important fully nonlinear elliptic equations [1–3]. Existence and regularity results of the Monge-Ampère equations can be found in [4–8] and the references therein.

We first consider the following real Monge-Ampère equations:

$$\begin{aligned} &\det\left(D^2u\right) = \lambda m(x)(-u)^N + F\left(x, -u, -u', \lambda\right), \quad \text{in } B, \\ &u(x) = 0, \quad \text{on } \partial B, \end{aligned} \tag{1}$$

where $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ is the Hessian matrix of u, B is the unit ball of $\mathbb{R}^N$, $m(x)$ is a weighted function, λ is a positive parameter, and $F \in C(\overline{B} \times (\mathbb{R}^+)^3, R^+)$.

Kutev [9] and Delano [10] treated the existence of convex radial solutions of problem (1) with $m \equiv 1$, $F = 0$ and $\lambda m(-u)^N + F = \lambda \exp f(|x|, u, |\nabla u|)$, respectively. Caffarelli et al. [11] have investigated problem (1) in general domains of $\mathbb{R}^N$.

In [9, 12], the authors have shown that problem (1) can reduce to the following boundary value problem:

$$\begin{aligned} \left(\left(u'\right)^N\right)' &= \lambda N r^{N-1} m(r)(-u)^N \\ &\quad + N r^{N-1} F\left(r, -u, -u', \lambda\right), \quad r \in (0,1), \\ u'(0) &= u(1) = 0, \end{aligned} \tag{2}$$

where $r = |x|$ with $x \in B$. By a solution of problem (2), we understand that it is a function which belongs to $C^2[0,1]$ and satisfies (2). It has been known that any negative solution of problem (2) is strictly convex in $(0, 1)$. Wang [13] and Hu and Wang [12] ($m \equiv 0$; $F = f(-u)$) also considered the existence of strictly convex solutions for problem (2) by using fixed index theorem. Lions [14] proved the existence of the first eigenvalue λ_1 of problem (1) with $\lambda m \equiv \lambda^N$, $F = 0$ via constructive proof.

By global bifurcation theorem, Dai and Ma [15] and Dai [16] studied the Monge-Ampère equations (1) with $\lambda m(x)(-u)^N + F$ equal $\lambda^N((-u)^N + g(-u))$ and $\lambda^N m(x)((-u)^N + g(-u))$, respectively, where $g : [0, +\infty) \to [0, +\infty)$ satisfies $\lim_{s\to 0^+} g(s)/s^N = 0$.

(H0) $m(x) \in C(\overline{B})$ is radially symmetric and $m(r) \geq 0$, $m(r) \not\equiv 0$ on any subinterval of $[0, 1]$, where $r = |x|$ with $x \in \overline{B}$.

However, the nonlinearities of the above papers are differentiable at the origin. In 1977, Berestycki [17] established an interval bifurcation theorem for the problems involving nondifferentiable nonlinearity. The main difficulties when dealing with this problem lie in the bifurcation results of [15, 16] which cannot be applied directly to obtain our results. Recently, Ma and Dai [18] and Dai and Ma [19, 20] also considered similar problems to [17].

Motivated by the above papers, we shall consider problem (1), where $F = m(x)f_1 + f_2$, with $f_1, f_2 \in C(\overline{B} \times (\mathbb{R}^+)^3, \mathbb{R}^+)$ being radially symmetric with respect to x, and $\mathbb{R}^+ = [0, +\infty)$.

It is clear that the radial solutions of (1) are equivalent to the solutions of problem (2), where m satisfies (H0) and f_1 and f_2 satisfy the following conditions.

(H1) $f_1 \in C([0,1] \times (\mathbb{R}^+)^3)$ and there exist $a_0, a^0 \in (0,\infty)$ such that

$$a_0 \le \liminf_{s\to 0^+} \frac{f_1(r,s,y,\lambda)}{s^N} < \limsup_{s\to 0^+} \frac{f_1(r,s,y,\lambda)}{s^N} \le a^0 \quad (3)$$

uniformly for $r \in [0,1]$, $0 < y \le 1$, and for all $\lambda \in \mathbb{R}^+$.

(H2) $f_1 \in C([0,1] \times (\mathbb{R}^+)^3)$ and there exist $a_\infty, a^\infty \in (0,\infty)$ such that

$$a_\infty \le \liminf_{s\to +\infty} \frac{f_1(r,s,y,\lambda)}{s^N} < \limsup_{s\to +\infty} \frac{f_1(r,s,y,\lambda)}{s^N} \le a^\infty \quad (4)$$

uniformly for $r \in [0,1]$, $y \ge C$ for some positive constant C large enough, and for all $\lambda \in \mathbb{R}^+$.

(H3) $f_2(r,s,y,\lambda) = o(s^N + y^N)$ near $(s,y) = (0,0)$ uniformly for $r \in (0,1)$ and λ on bounded sets.

(H4) $f_2(r,s,y,\lambda) = o(s^N + y^N)$ near $(s,y) = (+\infty,+\infty)$ uniformly for $r \in (0,1)$ and λ on bounded sets.

Under assumptions (H1) and (H3), we shall establish interval bifurcation of (2) from the trivial solutions axis by Rabinowitz [21]. Moreover, by the global bifurcation from infinity of Rabinowitz [22], we shall also establish a result involving global bifurcation of (2) from infinity with conditions (H2) and (H4).

Following the above theory (see Theorems 3 and 6), we shall investigate the existence of radial solutions for the following problem:

$$\begin{gathered} \det\left(D^2 u\right) = \gamma h(x) F(-u), \quad \text{in } B, \\ u(x) = 0, \quad \text{on } \partial B, \end{gathered} \quad (5)$$

where $F \in C(\mathbb{R}^+, \mathbb{R}^+)$.

It is clear that the radial solutions of (5) are equivalent to the solutions of the following problem:

$$\begin{gathered} \left(\left(u'\right)^N\right)' = \gamma N r^{N-1} h(r) F(-u), \quad r \in (0,1), \\ u'(0) = u(1) = 0, \end{gathered} \quad (6)$$

where γ is a positive parameter and h and $F = f_1 + f_2$ with $f_1, f_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy the following.

(A0) $h(x) \in C(\overline{B})$ is radially symmetric and $h(r) \ge 0$, $h(r) \not\equiv 0$ on any subinterval of $[0,1]$, where $r = |x|$ with $x \in \overline{B}$.

(A1) $f_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exist $a_0, a^0, a_\infty, a^\infty \in (0,\infty)$ such that

$$\begin{gathered} a_0 \le \liminf_{s\to 0^+} \frac{f_1(s)}{s^N} < \limsup_{s\to 0^+} \frac{f_1(s)}{s^N} \le a^0, \\ a_\infty \le \liminf_{s\to +\infty} \frac{f_1(s)}{s^N} < \limsup_{s\to +\infty} \frac{f_1(s)}{s^N} \le a^\infty. \end{gathered} \quad (7)$$

(A2) $f_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $f_2(s) > 0$ for $s \in (0,\infty)$ and there exist $f_2^0, f_2^\infty \in (0,\infty)$ such that

$$\begin{gathered} f_2^0 = \lim_{s\to 0^+} \frac{f_2(s)}{s^N}, \\ f_2^\infty = \lim_{s\to +\infty} \frac{f_2(s)}{s^N}. \end{gathered} \quad (8)$$

The rest of this paper is arranged as follows. In Section 2, we establish the bifurcation results which bifurcate from the trivial solutions axis and from infinity for problem (2), respectively. In Section 3, on the basis of the interval bifurcation result (Theorems 3 and 6), we give the intervals for the parameter γ which ensure existence of single or multiple strictly convex solutions for problem (6) under the assumptions of (A1)-(A2).

2. Global Interval Bifurcation

Let $Y = C[0,1]$ with the norm $\|u\|_\infty = \max_{r\in[0,1]}|u(r)|$. Let $E := \{u(r) \in C^1(0,1) \mid u'(0) = u(1) = 0\}$ with the usual norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$. Let $P^+ = \{u \in E : u(r) > 0,\ r \in (0,1)\}$. Set $K^+ = \mathbb{R} \times P^+$ under the product topology.

Now, we consider the following eigenvalue problem:

$$\begin{gathered} \left(\left(-v'\right)^N\right)' = \lambda N r^{N-1} m(r) v^N, \quad \text{in } r \in (0,1), \\ u'(0) = u(1) = 0. \end{gathered} \quad (9)$$

By [16, (4.2) of Section 4, p.11], the same proof as in Theorem 1.1 of [14], we can show that problem (9) possesses the first eigenvalue λ_1 which is positive, simple, and unique, and the corresponding eigenfunctions are positive in $(0,1)$ and concave on $[0,1]$.

By Sections 3-4 in [16], with a simple transformation $v = -u$, problem (2) can be equivalently written as

$$\begin{gathered} \left(\left(-v'\right)^N\right)' = \lambda N r^{N-1} m(r) v^N + N r^{N-1} F\left(r, v, v', \lambda\right), \\ r \in (0,1), \\ v'(0) = v(1) = 0, \end{gathered} \quad (10)$$

where m satisfies (H0) when $F \in C([0,1]\times(\mathbb{R}^+)^3, \mathbb{R}^+)$ satisfies (H2). According to Rabinowitz [21], using the same method to prove [16, Theorems 4.1 and 4.2] with obvious changes, we may get the following global bifurcation result.

Lemma 1 ([16, Theorem 4.2]). *Assume that (H0) holds and F satisfies (H3). Then $(\lambda_1, 0)$ is the unique bifurcation point* *of problem (10) and there exists an unbounded bifurcation continuum $C \subseteq (K^+ \cup \{(\lambda_1, 0)\})$ of solutions to problem (10) emanating from $(\lambda_1, 0)$.*

Hence, $F = m(r)f_1 + f_2$, where f_1, f_2 satisfy conditions (H1) and (H3), respectively. Let $\mathcal{S}$ denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions (λ, v) of (10) with $v \in P^+$ under the assumptions of (H0), (H1), and (H3). By an argument similar to that of [16, Lemma 4.1] with obvious changes, we can show that the following existence and uniqueness theorem is valid for problem (10).

Lemma 2 ([16, Lemma 4.1]). *If (λ, v) is a solution of (10) under the assumptions of (H0), (H1), and (H3) and v has a double zero, then $v \equiv 0$.*

The first main result for (10) is the following theorem.

Theorem 3. *Let (H0), (H1), and (H3) hold. Let $I_1^0 = [\lambda_1 - a^0, \lambda_1 - a_0]$. The component $\mathscr{C}$ of $\mathcal{S} \cup (I_1^0 \times \{0\})$ contains $I_1^0 \times \{0\}$, such that*

(i) *$\mathscr{C} \subset (K^+ \cup (I_1^0 \times \{0\}))$;*

(ii) *$\mathscr{C}$ is unbounded.*

To prove Theorem 3, we introduce the following auxiliary approximate problem:

$$\begin{aligned} \left(\left(-v'\right)^N\right)' &= \lambda N r^{N-1} m(r) v^N \\ &\quad + N r^{N-1} m(r) f_1\left(r, v|v|^{\epsilon}, v', \lambda\right) \\ &\quad + N r^{N-1} f_2\left(r, v, v', \lambda\right), \quad r \in (0,1), \\ v'(0) &= v(1) = 0. \end{aligned} \tag{11}$$

To prove Theorem 3, the next lemma will play a key role.

Lemma 4. *Let ϵ_n, $0 < \epsilon_n < 1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n, v_n) \in K^+$ such that (λ_n, v_n) is a nontrivial solution of problem (11) corresponding to $\epsilon = \epsilon_n$ and (λ_n, v_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times E$, then $\lambda \in I_1^0$.*

Proof. Let $w_n = v_n / \|v_n\|$; then, w_n satisfies the problem

$$\begin{aligned} \left(\left(-w_n'\right)^N\right)' &= \lambda N r^{N-1} m(r) w_n^N + N r^{N-1} m(r) f_1^n(r) \\ &\quad + N r^{N-1} f_2^n(r), \quad r \in (0,1), \\ w_n'(0) &= w_n(1) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} f_1^n(r) &= \frac{f_1\left(r, v_n(r)\left|v_n(r)\right|^{\epsilon}, v_n'(r), \lambda\right)}{\left\|v_n\right\|^N}, \\ f_2^n(r) &= \frac{f_2\left(r, v_n(r), v_n'(r), \lambda\right)}{\left\|v_n\right\|^N}. \end{aligned} \tag{13}$$

Let $\overline{f_2}(r, v, \lambda) = \max_{0 \le |s| \le v} |f_2(r, s, \lambda)|$ for all $r \in (0,1)$ and λ on bounded sets, and then $\overline{f_2}$ is nondecreasing with respect to v and $\lim_{v \to 0^+}(\overline{f_2}(r, v, \lambda)/v^N) = 0$ uniformly for $r \in [0,1]$ and λ on bounded sets. By an argument similar to that of [16, (4.7)] with obvious changes, we can show that $f_2^n(r) \to 0$ as $\|v\| \to 0$ uniformly for $r \in (0,1)$ and λ on bounded sets. By (H1), we have that w_n is convergent in E. Without loss of generality, we may assume that $w_n \to w$ in E with $\|w\| = 1$. Obviously, we have $w \in P^+$.

Now, we deduce the boundedness of λ. Let $\psi \in P^+$ be an eigenfunction of problem (9) with $F = 0$ corresponding to λ_1.

Similar to (4.12) in Lemma 4.5 of [16], by some simple calculations, we have that

$$\int_0^1 \left(\frac{w_n^{N+1}\left(-\psi'\right)^N}{\psi^N} - w_n\left(-w_n'\right)^N \right) = A + B, \tag{14}$$

where

$$\begin{aligned} A &= \int_0^1 \left[\left(\lambda_1 - \lambda_n\right) m(r) - \frac{f_1^n(r)}{w_n^N} - \frac{f_2^n(r)}{w_n^N} \right] \\ &\quad \cdot r^{N-1} N w_n^{N+1} dr, \\ B &= \left[\left(-w_n'\right)^{N+1} + N \left(\frac{-w_n \psi'}{\psi} \right)^{N+1} + (N+1) \right. \\ &\quad \left. \cdot w_n^N w_n' \left(\frac{-\psi'}{\psi} \right)^N \right]. \end{aligned} \tag{15}$$

Similar to the proof of (4.12) in Lemma 4.5 of [16], we can show that the left-hand side of (14) equals 0. The Young's inequality implies that $B \ge 0$. It follows that

$$\begin{aligned} &\int_0^1 \left[\left(\lambda_1 - \lambda_n\right) m(r) - \frac{f_1^n(r)}{w_n^N} - \frac{f_2^n(r)}{w_n^N} \right] \\ &\quad \cdot r^{N-1} N w_n^{N+1} dr \le 0. \end{aligned} \tag{16}$$

Similarly, we can also show that

$$\begin{aligned} &\int_0^1 \left[\left(\lambda_n - \lambda_1\right) m(r) + \frac{f_1^n(r)}{w_n^N} + \frac{f_2^n(r)}{w_n^N} \right] \\ &\quad \cdot r^{N-1} N \psi^{N+1} dr \le 0. \end{aligned} \tag{17}$$

Similar to [17, Lemma 1], we can easily show that

$$\begin{aligned} a_0 \int_0^1 m N r^{N-1} w^{N+1} dr &\le \int_0^1 \frac{f_1^n(r)}{w_n^N} m N r^{N-1} w_n^{N+1} dr \\ &\le a^0 \int_0^1 m N r^{N-1} w^{N+1} dr \end{aligned} \tag{18}$$

for n large enough.

It follows from (16) and (18) that

$$\int_0^1 \left(\lambda_1 - \lambda - a^0\right) m(r) N r^{N-1} w^{N+1} \le 0, \tag{19}$$

which implies $\lambda \ge \lambda_1 - a^0$.

Similarly, it follows from (17) and (18) that $\lambda \le \lambda_1 - a_0$.

Therefore, we have that $\lambda \in I_1^0$. ☐

Proof of Theorem 3. We divide the rest of the proofs into two steps.

Step 1. Using a similar method to prove [20, Theorem 1.1] with obvious changes, we may prove that $\mathscr{C} \subset (K^+ \cup (I_1^0 \times \{0\}))$.

Step 2. We prove that $\mathscr{C}$ is unbounded. Suppose on the contrary that $\mathscr{C}$ is bounded. Using a similar method to prove [20, Theorem 1.1] with obvious changes, we can find a neighborhood $\mathscr{O}$ of $\mathscr{C}$ such that $\partial\mathscr{O} \cap \mathscr{S} = \emptyset$.

In order to complete the proof of this theorem, we consider problem (11). For $\epsilon > 0$, it is easy to show that the nonlinear term $f(r, v|v|^\epsilon, \lambda) + g(r, v, \lambda)$ satisfies condition (H3). Let

$$\mathscr{S}_\epsilon = \overline{\{(\lambda, v) : (\lambda, v) \text{ satisfies } (11),\ v \not\equiv 0\}}^{\mathbb{R}\times E}. \tag{20}$$

By Lemma 1, there exists an unbounded continuum $\mathscr{C}_\epsilon$ of $\mathscr{S}_\epsilon$ bifurcating from $(\lambda_1, 0)$ such that

$$\mathscr{C}_\epsilon \subset (K^+ \cup \{(\lambda_1, 0)\}). \tag{21}$$

So there exists $(\lambda_\epsilon, v_\epsilon) \in \mathscr{C}_\epsilon \cap \partial\mathscr{O}$ for all $\epsilon > 0$. Since $\mathscr{O}$ is bounded in $\mathbb{R} \times P^+$, (11) shows that $(\lambda_\epsilon, v_\epsilon)$ is bounded in $\mathbb{R} \times C^2$ independently of ϵ. By the compactness of G_N, one can find a sequence $\epsilon_n \to 0$ such that $(\lambda_{\epsilon_n}, v_{\epsilon_n})$ converges to a solution (λ, v) of (11). So, $v \in \overline{P^+}$. If $v \in \partial P^+$, then from Lemma 2 it follows that $v \equiv 0$. By Lemma 4, $\lambda \in I_1^0$, which contradicts the definition of $\mathscr{O}$. On the other hand, if $v \in P^+$, then $(\lambda, v) \in \mathscr{S} \cap \partial\mathscr{O}$ which contradicts $\mathscr{S} \cap \partial\mathscr{O} = \emptyset$. □

From Theorem 3 and its proof, we can easily get a corollary.

Corollary 5. *Let (H0), (H1), and (H3) hold. There exists a subcontinuum $\mathscr{D}$ of solutions of (10) in $\mathbb{R} \times E$, bifurcating from $I_1^0 \times \{0\}$, such that*

(i) *$\mathscr{D} \subset (K^+ \cup (I_1^0 \times \{0\}))$;*

(ii) *$\mathscr{D}$ is unbounded.*

We add the points $\{(\lambda, \infty) \mid \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$. Let $\mathscr{T}$ denote the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions (v, λ) of (10) under conditions (H2) and (H4) with $v \in P^+$. Let S_N denote the spectral set of problem (9). Let $\overline{I}_\infty = [\overline{\lambda} - a^\infty, \overline{\lambda} - a_\infty]$, where $\overline{\lambda} \in S_N \backslash \{\lambda_1\}$.

According to Rabinowitz [22], our second main result for (10) is the following theorem.

Theorem 6. *Let (H0), (H2), and (H4) hold. Let $I_1^\infty = [\lambda_1 - a^\infty, \lambda_1 - a_\infty]$. There exists a connected component $\mathscr{D}$ of $\mathscr{T} \cup (I_1^\infty \times \{\infty\})$, containing $I_1^\infty \times \{\infty\}$. Moreover, if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap (\bigcup_{\overline{\lambda} \in S_N \backslash \{\lambda_1\}} (\overline{I}_\infty \cup I_1^\infty)) = I_1^\infty$ and $\mathscr{M}$ is a neighborhood of $I_1^\infty \times \{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

(1^o) *$\mathscr{D} - \mathscr{M}$ is bounded in $\mathbb{R} \times E$ in which case $\mathscr{D} - \mathscr{M}$ meets $\mathscr{R} = \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$;*

(2^o) *$\mathscr{D} - \mathscr{M}$ is unbounded.*

If (2^o) occurs and $\mathscr{D} - \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D} - \mathscr{M}$ meets $\overline{I}_\infty \times \{\infty\}$. Moreover, there exists a neighborhood $\mathscr{N} \subset \mathscr{M}$ of $I_1^\infty \times \{\infty\}$ such that $(\mathscr{D} \cap \mathscr{N}) \subset (K^+ \cup (I_1^\infty \times \{\infty\}))$.

Proof. We use a similar method to prove [18, Theorem 2.2] with obvious changes, but we give a rough sketch of the proof for readers' convenience. If $(\lambda, v) \in \mathscr{T}$ with $\|v\| \neq 0$, dividing (10) by $\|v\|^2$ and setting $w = v/\|v\|^2$ yield

$$\begin{aligned} \left(\left(-w'\right)^N\right)' &= \lambda N r^{N-1} m(r)\, w^N \\ &\quad + \frac{N r^{N-1} F(r, v, v', \lambda)}{\|v\|^{2N}}, \quad r \in (0,1), \\ w'(0) &= w(1) = 0. \end{aligned} \tag{22}$$

Define

$$\widetilde{h}(r, w, w', \lambda) = \begin{cases} \|w\|^{2N} h\left(r, \dfrac{w}{\|w\|^2}, \dfrac{w'}{\|w\|^2}, \lambda\right), & \text{if } w \neq 0, \\ 0, & \text{if } w = 0. \end{cases} \tag{23}$$

Clearly, (22) is equivalent to

$$\begin{aligned} \left(\left(-w'\right)^N\right)' &= \lambda N r^{N-1} m(r)\, w^N \\ &\quad + N r^{N-1} m(r) \widetilde{f_1}(r, w, w', \lambda) \\ &\quad + N r^{N-1} \widetilde{f_2}(r, w, w', \lambda), \quad r \in (0,1), \\ w'(0) &= w(1) = 0. \end{aligned} \tag{24}$$

It is obvious that $(\lambda, 0)$ is always the solution of (24). By simple computation, we can show that assumptions (H2) and (H4) imply that $\widetilde{f_1}, \widetilde{f_2}$ satisfy (H1) and (H3). Now, applying Theorem 3 to problem (24), we have that the component $\mathscr{C}$ of $\mathscr{S} \cup (I_1^0 \times \{0\})$, containing $I_1^0 \times \{0\}$, is unbounded and lies in $K^+ \cup (I_1^0 \times \{0\})$. Under the inversion $w \to w/\|w\|^2 = v, \mathscr{C} \to \mathscr{D}$ satisfying problem (10). By an argument similar to that of [18, Theorem 2.3], we can prove the existence of a neighborhood $\mathscr{N} \subset \mathscr{M}$ of $I_1^\infty \times \{\infty\}$ such that $(\mathscr{D} \cap \mathscr{N}) \subset (K^+ \cup (I_1^\infty \times \{\infty\}))$. □

3. Applications

In this section, we shall investigate the existence and multiplicity of convex solutions of problem (6). With a simple transformation $v = -u$, problem (6) can be written as

$$\begin{aligned} \left(\left(-v'\right)^N\right)' &= \gamma N r^{N-1} h(r) F(v), \quad r \in (0,1), \\ v'(0) &= v(1) = 0. \end{aligned} \tag{25}$$

By [16], in order to prove our main results, we need the following Sturm type comparison result.

Lemma 7 (see [16, Lemma 4.6]). *Let $b_i(r) \in C(0,1)$, $i = 1,2$, such that $b_2(r) \geq b_1(r)$ for $r \in (0,1)$ and the inequality is strict on some subset of positive measure in $(0,1)$. Also, let v_1 and v_2 be solutions of the following differential equations:*

$$\begin{gathered} \left(\left(-v'\right)^N\right)' = b_i(r) v^N, \quad r \in (0,1), \ i = 1,2, \\ v'(0) = v(1) = 0, \end{gathered} \tag{26}$$

respectively. If $v_1 \neq 0$ in $(0,1)$, then v_2 has at least one zero in $(0,1)$.

The main results of this section are the following theorems.

Theorem 8. *Let (A0), (A1), and (A2) hold. If $f_2^0 > -a_0$ and $f_2^\infty > -a_\infty$, for either*

$$\frac{\lambda_1}{f_2^0 + a_0} < \gamma < \frac{\lambda_1}{f_2^\infty + a^\infty} \tag{27}$$

or

$$\frac{\lambda_1}{f_2^\infty + a_\infty} < \gamma < \frac{\lambda_1}{f_2^0 + a^0}, \tag{28}$$

then problem (6) has at least one solution u such that it is negative and strictly convex in $(0,1)$.

Theorem 9. *Let (A0), (A1), and (A2) hold. If $f_2^0 > -a_0$ and $-a^\infty \leq f_2^\infty \leq -a_\infty$, for*

$$\frac{\lambda_1}{f_2^0 + a_0} < \gamma < \frac{\lambda_1}{f_2^\infty + a^\infty}, \tag{29}$$

then problem (6) has at least one solution u such that it is negative and strictly convex in $(0,1)$.

Theorem 10. *Let (A0), (A1), and (A2) hold. If $g_0 > -a_0$ and $f_2^\infty \leq -a^\infty$, for*

$$\gamma > \frac{\lambda_1}{f_2^0 + a_0}, \tag{30}$$

then problem (6) has at least one solution u such that it is negative and strictly convex in $(0,1)$.

Theorem 11. *Let (A0), (A1), and (A2) hold. If $-a^0 < f_2^0 \leq a_0$ and $f_2^\infty > -a_\infty$, for*

$$\frac{\lambda_1}{f_2^\infty + a_\infty} < \gamma < \frac{\lambda_1}{f_2^0 + a^0}, \tag{31}$$

then problem (6) has at least one solution u such that it is negative and strictly convex in $(0,1)$.

Theorem 12. *Let (A0), (A1), and (A2) hold. If $-a^0 \leq g_0$ and $f_2^\infty > -a_\infty$, for*

$$\gamma > \frac{\lambda_1}{f_2^\infty + a_\infty}, \tag{32}$$

then problem (6) has at least one solution u such that it is negative and strictly convex in $(0,1)$.

Remark 13. Note that if $f \equiv 0$, then Theorem 8 is equivalent to Theorem 4.1 of [15] or Theorem 5.1 of [16]. In this sense, Theorem 8 is also a generalization of Theorem 4.1 of [15] or Theorem 5.1 of [16].

Proof of Theorem 8. It suffices to prove that problem (25) has at least one solution v such that it is positive and strictly concave in $(0,1)$.

Firstly, we study the bifurcation phenomena for the following eigenvalue problem:

$$\begin{gathered} \left(\left(-v'\right)^N\right)' = \lambda\gamma N r^{N-1} h(r) f_2(v) \\ + \gamma N r^{N-1} h(r) f_1(v), \quad r \in (0,1), \\ v'(0) = v(1) = 0, \end{gathered} \tag{33}$$

where $\lambda > 0$ is a parameter.

Let $\zeta \in C(\mathbb{R}^+, \mathbb{R}^+)$ be such that

$$f_2(s) = f_2^0 s^N + \zeta(s) \tag{34}$$

with $\lim_{s\to 0^+} \zeta(s)/s^N = 0$. Let $\overline{\zeta}(v) = \max_{0\leq|s|\leq v} |\zeta(s)|$; then, $\overline{\zeta}(v)$ is nondecreasing and

$$\lim_{s\to 0^+} \frac{\overline{\zeta}(s)}{s^N} = 0. \tag{35}$$

Further, it follows from (35) that

$$\frac{|\zeta(v)|}{\|v\|^N} \leq \frac{\overline{\zeta}(|v|)}{\|v\|^N} \leq \frac{\overline{\zeta}(\|r\|_\infty)}{\|v\|^N} \leq \frac{\overline{\zeta}(\|r\|)}{\|v\|^N} \quad \text{as } \|v\| \longrightarrow 0. \tag{36}$$

Hence, (33), (34), and (36) imply that conditions (H1) and (H3) hold. Moreover, we have that $I_1^0 = [\lambda_1/\gamma f_2^0 - a^0/f_2^0, \lambda_1/\gamma f_2^0 - a_0/f_2^0]$.

By Theorem 3, there is a distinct unbounded component $\mathscr{D}_0$ of $\mathscr{S} \cup (I_1^0 \times \{0\})$, containing $I_1^0 \times \{0\}$ and lying in $(K^+ \cup (I_1^0 \times \{0\}))$.

Let $\xi \in C(\mathbb{R}^+, \mathbb{R}^+)$ be such that

$$f_2(s) = f_2^\infty s^N + \xi(s) \tag{37}$$

with $\lim_{s\to+\infty} \xi(s)/s^N = 0$. Let $\overline{v} = \max_{0\leq|s|\leq v} |\xi(s)|$, and then $\overline{\xi}$ is nondecreasing.

Define

$$\widetilde{v} = \max_{v/2\leq|s|\leq v} |\xi(s)|. \tag{38}$$

Then, we can see that

$$\begin{gathered} \lim_{s\to+\infty} \frac{\widetilde{\xi}(s)}{s^N} = 0, \\ \overline{v} \leq \frac{\overline{v}}{2} + \widetilde{v}. \end{gathered} \tag{39}$$

It is not difficult to verify that $\overline{v}/s$ is bounded in $\mathbb{R}^+$. From this fact and (37), it follows that

$$\limsup_{s\to+\infty}\frac{\overline{\xi}(v)}{v^N}\leq\limsup_{s\to+\infty}\frac{\overline{\xi}(v/2)}{v^N}=\limsup_{s\to+\infty}\frac{\overline{\xi}(v)}{v^N},\qquad(40)$$

where $t=u/2$. So, we have

$$\lim_{s\to+\infty}\frac{\overline{\xi}(s)}{s^N}=0.\qquad(41)$$

Further, it follows from (41) that

$$\frac{|\xi(v)|}{\|v\|^N}\leq\frac{\overline{\xi}(|v|)}{\|v\|^N}\leq\frac{\overline{\xi}(\|v\|_\infty)}{\|v\|^N}\leq\frac{\overline{\xi}(\|v\|)}{\|v\|^N}\quad\text{as }\|v\|\longrightarrow+\infty.\qquad(42)$$

Hence, (33), (37), and (42) imply that conditions (H2) and (H4) hold. Moreover, we have that $I_1^\infty=[\lambda_1/\gamma f_2^\infty-a^\infty/f_2^\infty,\lambda_1/\gamma f_2^\infty-a_\infty/f_2^\infty]$.

By Theorem 6, there is a distinct unbounded component $\mathscr{D}_\infty$ of $\mathscr{T}\cup(I_1^\infty\times\{\infty\})$, which satisfies the alternates of Theorem 6. Moreover, there exists a neighborhood $\mathscr{N}\subset\mathscr{M}$ of $I_1^\infty\times\{\infty\}$ such that $(\mathscr{D}_\infty\cap\mathscr{N})\subset(K^+\cup(I_1^\infty\times\{\infty\}))$.

We claim that $\mathscr{D}_0=\mathscr{D}_\infty$. It suffices to show that $\mathscr{D}_\infty$ meets some point $(\lambda_*,0)$ of $\mathscr{R}$. In fact, if this occurs, we can show that $\lambda_*\in I_1^0$. Suppose on the contrary that $\lambda_*\notin I_1^0$; hence, $\lambda_*\in\overline{I}_0$. So, $(\mathscr{D}_\infty\cap\mathscr{N})\subset\mathscr{D}_\infty\subset\overline{\mathscr{D}}_0\subset((\mathbb{R}\times\overline{P})\cup(\overline{I}_0\times\{0\}))$, noting $(\mathscr{D}_\infty\cap\mathscr{N})\cap(\mathbb{R}\times\{0\})=\emptyset$, which contradicts $(\mathscr{D}_\infty\cap\mathscr{N})\subset(K^+\cup(I_\infty\times\{\infty\}))$, where $\overline{\mathscr{T}}_0$ denotes the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions (λ,v) of (25) under conditions (H1) and (H3) with $v\in\overline{P}$, where $\overline{P}=\{v\mid(\overline{\lambda},v)\in(S_N\backslash\{\lambda_1\})\times E\}$, $\overline{I}_0=[\overline{\lambda}/\gamma f_2^0-a^0/f_2^0,\overline{\lambda}/\gamma f_2^0-a_0/f_2^0]$, where $\overline{\lambda}\in S_N\backslash\{\lambda_1\}$. $\overline{\mathscr{D}}_0$ is a connected component of $\overline{\mathscr{T}}_0\cup(\overline{I}_0\times\{0\})$, containing $\overline{I}_0\times\{0\}$.

Hence, $\lambda_*\in I_1^0$, and it follows that $\mathscr{D}_0=\mathscr{D}_\infty$.

Next, we shall show that (2°) of Theorem 6 does not occur. Suppose on the contrary that (2°) of Theorem 6 occurs; then, we shall deduce a contradiction. Firstly, we show that $\mathscr{D}_\infty-\mathscr{M}$ has a bounded projection on $\mathbb{R}$. We may claim $\mathscr{D}_\infty\subset K^+$. If $(\mathscr{D}_\infty-(\mathscr{D}_\infty\cap\mathscr{N}))\not\subset K^+$, then there exists $(\mu,v)\in(\mathscr{D}_\infty-(\mathscr{D}_\infty\cap\mathscr{N}))\cap(\mathbb{R}\times\partial P^+)$. Since $v\in\partial P^+$, by Lemma 4, $v\equiv0$; that is, (1°) of Theorem 6 occurs, which is a contradiction.

On the contrary, we suppose that $(\mu_n,v_n)\in\mathscr{D}_\infty-\mathscr{M}$ such that

$$\lim_{n\to\infty}\mu_n=\infty.\qquad(43)$$

It follows that

$$\begin{aligned}\left(\left(-v_n'\right)^N\right)'&=\mu_n^N\gamma Nr^{N-1}h(r)f_2(v_n)\\&\quad+\gamma Nr^{N-1}h(r)f_1(v_n),\quad r\in(0,1),\\v_n'(0)&=v_n(1)=0.\end{aligned}\qquad(44)$$

In view of (A1) and (A2), we have that $\lim_{n\to\infty}(\mu_n^NNr^{N-1}a(r)(f_2(v_n)/v_n^N)+Nr^{N-1}a(r)(f_1(v_n)/v_n^N))=\infty$ for any $r\in(0,1)$. By the Sturm comparison (Lemma 7), we get that v_n has at least one zero in $(0,1)$ for n large enough, and this contradicts the fact that v_n does not change its sign in $(0,1)$.

In the following, we show that the case of $\mathscr{D}_\infty-\mathscr{M}$ meeting $\overline{I}_\infty\times\{\infty\}$ is impossible. Assume on the contrary that $\mathscr{D}_\infty-\mathscr{M}$ meets $\overline{I}_\infty\times\{\infty\}$. So, there exists a neighborhood $\widetilde{\mathscr{N}}\subset\widetilde{\mathscr{M}}$ of $\overline{I}_\infty$ such that $(\mathscr{D}_\infty-\mathscr{M})\cap(\widetilde{\mathscr{N}}\backslash(\overline{I}_\infty\times\{\infty\}))\subset(\mathbb{R}\times\overline{P})$, where $\widetilde{\mathscr{M}}$ is a neighborhood of $\overline{I}_\infty\times\{\infty\}$ which satisfies the assumptions of Theorem 6, which contradicts $\mathscr{D}_\infty\subset K^+$, where $\overline{P}=\{v\mid(\overline{\lambda},v)\in(S_N\backslash\{\lambda_1\})\times E\}$ and $\overline{I}_\infty=[\overline{\lambda}/\gamma f_2^\infty-a^\infty/f_2^\infty,\overline{\lambda}/\gamma f_2^\infty-a_\infty/f_2^\infty]$, where $\overline{\lambda}\in S_N\backslash\{\lambda_1\}$.

For simplicity, we write $\mathscr{D}=\mathscr{D}_0=\mathscr{D}_\infty$. It is clear that any solution of (33) of the form $(1,v)$ yields a solution v of (25).

By (27), we obtain

$$\frac{\lambda_1^N}{\gamma f_2^0}-\frac{a_0}{f_2^0}<1<\frac{\lambda_1^N}{\gamma f_2^\infty}-\frac{a^\infty}{f_2^\infty}.\qquad(45)$$

By (28), we have

$$\frac{\lambda_1}{\gamma f_2^\infty}-\frac{a_\infty}{f_2^\infty}<1<\frac{\lambda_1}{\gamma f_2^0}-\frac{a^0}{f_2^0}.\qquad(46)$$

From $I_1^0=[\lambda_1/\gamma f_2^0-a^0/f_2^0,\lambda_1/\gamma f_2^0-a_0/f_2^0]$ and $I_1^\infty=[\lambda_1/\gamma f_2^\infty-a^\infty/f_2^\infty,\lambda_1/\gamma f_2^\infty-a_\infty/f_2^\infty]$, it follows that the subsets $I_1^0\times E$ and $I_1^\infty\times E$ of $\mathbb{R}\times E$ can be separated by the hyperplane $\{1\}\times E$. Furthermore, we have $\mathscr{D}$ crossing the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$. □

Proof of Theorems 9 and 10. The proof is similar to that of Theorem 8. In this case, it follows that (45) holds and (46) is impossible. □

Proof of Theorems 11 and 12. The proof is similar to that of Theorem 8. In this case, it follows that (46) holds and (45) is impossible. □

Remark 14. Note that if $-a^0<f_2^0\leq-a_0$ and $f_2^\infty\leq-a_\infty$ or $f_2^0\leq-a^0$ and $f_2^\infty\leq-a_\infty$, (45) and (46) are impossible, and it follows that the subsets $I_1^0\times E$ and $I_1^\infty\times E$ of $\mathbb{R}\times E$ cannot be separated by the hyperplane $\{1\}\times E$. In this case, we cannot give a suitable interval of γ in which there exist nodal solutions for (25). It would be interesting to have more information about this case.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The author was supported by the NSFC (no. 11561038) and Gansu Provincial National Science Foundation of China (no. 145RJZA087).

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Hindawi
Journal of Function Spaces
Volume 2018, Article ID 9857038, 7 pages
https://doi.org/10.1155/2018/9857038

Research Article

On Fourth-Order Elliptic Equations of Kirchhoff Type with Dependence on the Gradient and the Laplacian

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Received 26 September 2017; Accepted 20 November 2017; Published 2 January 2018

Academic Editor: Dhananjay Gopal

We consider a nonlocal fourth-order elliptic equation of Kirchhoff type with dependence on the gradient and Laplacian $\Delta^2 u - (a + b\int_\Omega |\nabla u|^2 dx)\Delta u = f(x, u, \nabla u, \Delta u)$, in Ω, $u = 0$, $\Delta u = 0$, on $\partial\Omega$, where a, b are positive constants. We will show that there exists $b^* > 0$ such that the problem has a nontrivial solution for $0 < b < b^*$ through an iterative method based on the mountain pass lemma and truncation method developed by De Figueiredo et al., 2004.

1. Introduction

This paper concerns with the existence of solutions of the fourth-order Kirchhoff type elliptic equation as follows:

$$\begin{aligned} &\Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u, \nabla u, \Delta u), \quad \text{in } \Omega, \\ &u = 0, \\ &\Delta u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded and smooth domain in R^N $(N \geq 5)$, a, b are positive constants, and $f : \Omega \times R \times R^N \times R \to R$ is locally Lipschitz continuous.

The fourth-order elliptic equation

$$\begin{aligned} &\Delta^2 u - a\Delta u = f(x, u), \quad \text{in } \Omega, \\ &u = 0, \\ &\Delta u = 0, \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

arises in the study of traveling waves in suspension bridges, which has been extensively investigated in recent years, such as [1–6]. To our attention, some authors paid more attention to a more general biharmonic elliptic problem

$$\begin{aligned} &\Delta^2 u + q\Delta u + \alpha(x) u = f(x, u, \Delta u, \nabla u), \quad \text{in } \Omega, \\ &u(x) = 0, \\ &\Delta u(x) = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

For this problem, due to the presence of Δu and ∇u in f, it is not variational. To overcome this difficulty, in [5], Wang deals with this problem via the upper and lower solutions and monotone iterative methods; in [7], the authors apply a technique developed by De Figueiredo et al. [8, 9] in studying a second-order elliptic problem involving the gradient, which "freezes" the gradient, and use truncation on the nonlinearity f. Thus the new problem becomes variational and an iterative scheme of the mountain pass "approximated" solutions is built.

In addition, the nonlocal fourth-order equation

$$\Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u), \quad \text{in } \Omega,$$

$$u = 0,$$

$$\Delta u = 0, \quad \text{on } \partial\Omega \tag{4}$$

has also been studied by many authors. We refer the readers to [10–20]. Particularly, Wang et al. studied the following fourth-order equation of Kirchhoff type equation

$$\begin{aligned} \Delta^2 u - \lambda\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u &= f(x,u), \quad \text{in } \Omega, \\ u &= 0, \\ \Delta u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where λ is a positive parameter. The authors showed that there exists λ^* such that the fourth-order elliptic equation has a nontrivial solution for $0 < \lambda < \lambda^*$ by using the mountain pass iterative techniques and the truncation method.

Motivated by these works, to study problem (1), we combine the famous mountain pass lemma with a technique developed by De Figueiredo et al. [8], which "freezes" the gradient and the Laplacian variable and use truncation on the nonlinearity of f. For convenience, we recall a definition and restate the mountain pass theorem.

Definition 1. Let X be a real Banach space and $I : X \to R$ a C^1-functional. A sequence $\{u_n\}$ in E is a (PS)-sequence for I if $I(u_n) \to C$ for some constant $C \geq 0$ as $n \to \infty$, while $\langle I'(u_n), u_n\rangle \to 0$ as $n \to \infty$. We say that the functional I satisfies the (PS)-condition if any (PS)-sequence for I has a convergent subsequence.

Theorem A (mountain pass lemma). *Let E be a real Banach space; $I \in C^1(E,R)$ satisfying (PS)-condition. Suppose the following:*

(1) *There exist $\rho > 0$, $\alpha > 0$ such that*

$$I|_{\partial B_\rho} \geq I(0) + \alpha, \tag{6}$$

where $B_\rho = \{u \in E \mid \|u\| \leq \rho\}$.

(2) *There is $e \in E$ and $\|e\| > \rho$ such that*

$$I(e) \leq I(0). \tag{7}$$

Then $I(u)$ has a critical value c which can be characterized as

$$C = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} I(u), \tag{8}$$

where $\Gamma = \{\gamma \in C([0,1],E) \mid \gamma(0) = 0,\ \gamma(1) = e\}$.

2. The Main Result

Theorem 2. *Assume that the function f satisfies the following conditions:*

(f_1) *$f : \Omega\times R\times R^N\times R \to R$ is locally Lipschitz continuous, and there exist $d_1 > 0$, $1 < p < (N+4)/(N-4)$ which satisfy $r := r_1 + r_2 < 2$ such that*

$$\left|f(x,t,\xi_1,\xi_2)\right| \leq d_1\left(1+|t|^p\right)\left(1+\left|\xi_1\right|\right)^{r_1}\left(1+\left|\xi_2\right|\right)^{r_2} \tag{9}$$

for all $(x,t,\xi_1,\xi_2) \in \Omega\times R^{N+2}$.

(f_2) $\lim_{t\to 0}(f(x,t,\xi_1,\xi_2)/t) = 0$ *uniformly with respect to $x \in \Omega$, $\xi_1 \in R^N$ and $\xi_2 \in R$.*

(f_3) *There exist $\Theta > 2$ and $t_1 > 0$ such that*

$$\begin{aligned} 0 < \Theta F(x,t,\xi_1,\xi_2) &\leq t f(x,t,\xi_1,\xi_2), \\ &\forall |t| \geq t_1,\ x\in\Omega,\ (\xi_1,\xi_2) \in R^{N+1}, \end{aligned} \tag{10}$$

where $F(x,t,\xi_1,\xi_2) = \int_0^t f(x,s,\xi_1,\xi_2)ds$.

(f_4) *There exist positive constants $\rho_i > 0$ $(i = 1,2,3)$ depending on a, b, Θ, and d_1 such that L_{ρ_i} $(i = 1,2,3)$ satisfy*

$$\left(\tau_1^2 L_{\rho_1} + \tau_1\tau_2 L_{\rho_2} + \tau_1\tau_3 L_{\rho_3}\right) < \min\{1,a\}, \tag{11}$$

where

$$\begin{aligned} L_{\rho_1} &= \sup\left\{\frac{\left|f(x,t',\xi_1,\xi_2) - f(x,t'',\xi_1,\xi_2)\right|}{|t'-t''|} : x \in \Omega,\ |t'|,|t''| \leq \rho_1,\ |\xi_1| \leq \rho_2,\ |\xi_2| \leq \rho_3\right\}, \\ L_{\rho_2} &= \sup\left\{\frac{\left|f(x,t,\xi_1',\xi_2) - f(x,t,\xi_1'',\xi_2)\right|}{|\xi_1'-\xi_1''|} : x \in \Omega,\ |t| \leq \rho_1,\ |\xi_1'|,|\xi_1''| \leq \rho_2,\ |\xi_2| \leq \rho_3\right\}, \\ L_{\rho_3} &= \sup\left\{\frac{\left|f(x,t,\xi_1,\xi_2') - f(x,t,\xi_1,\xi_2'')\right|}{|\xi_2'-\xi_2''|} : x \in \Omega,\ |t| \leq \rho_1,\ |\xi_1| \leq \rho_2,\ |\xi_2'|,|\xi_2''| \leq \rho_3\right\}, \end{aligned} \tag{12}$$

and τ_i $(i = 1,2,3)$ are the optimal constants of the following inequalities:

$$\begin{aligned} \left(\int_\Omega |u|^2\,dx\right)^{1/2} &\leq \tau_1\|u\|, \\ \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2} &\leq \tau_2\|u\|, \\ \left(\int_\Omega |\Delta u|^2\,dx\right)^{1/2} &\leq \tau_3\|u\|, \end{aligned} \tag{13}$$

where $\|\cdot\|$ is the norm of the Hilbert space $\mathbf{X} = \mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)$ defined by

$$\|u\|^2 = \int_\Omega |\Delta u|^2\,dx + \int_\Omega |\nabla u|^2\,dx. \tag{14}$$

Then there exists $b^* > 0$ such that (1) has at least a nontrivial solution for $0 < b < b^*$.

For each $\omega \in \mathbf{X}$ and $\mathbf{R} > 0$, we study the following "truncate" and "freezed" problem

$$\begin{gathered}\Delta^2 u_\omega^{\mathbf{R}} - \Phi_{\mathbf{R}}(\nabla\omega)\Delta u_\omega^{\mathbf{R}} = f_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right), \quad \text{in } \Omega, \\ u_\omega^{\mathbf{R}} = 0, \\ \Delta u_\omega^{\mathbf{R}} = 0, \\ \text{on } \partial\Omega,\end{gathered} \tag{15}$$

where

$$\begin{gathered}f_{\mathbf{R}}(x,t,\nabla\omega,\Delta\omega) = f\left(x,t,\nabla\omega\varphi_{\mathbf{R}}(\nabla\omega), \Delta\omega\varphi_{\mathbf{R}}(\Delta\omega)\right), \\ \Phi_{\mathbf{R}}(\nabla\omega) = a + b\int_\Omega \left|\nabla\omega\varphi_{\mathbf{R}}(\nabla\omega)\right|^2 dx,\end{gathered} \tag{16}$$

$\varphi_{\mathbf{R}} \in C^1(R,R)$ satisfies $|\varphi_{\mathbf{R}}| \le 1$, and

$$\varphi_{\mathbf{R}}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le \mathbf{R}, \\ 0, & \text{if } |\xi| \ge \mathbf{R}+1. \end{cases} \tag{17}$$

The associated functional $J_\omega^{\mathbf{R}} : \mathbf{X} \to R$ is defined as

$$\begin{aligned}J_\omega^{\mathbf{R}}\left(u_\omega^{\mathbf{R}}\right) = {}& \frac{1}{2}\int_\Omega \left|\Delta u_\omega^{\mathbf{R}}\right|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_\omega^{\mathbf{R}}\right|^2 dx \\ & - \int_\Omega F_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right) dx,\end{aligned} \tag{18}$$

where

$$F_{\mathbf{R}}(x,t,\nabla\omega,\Delta\omega) = \int_0^t f_{\mathbf{R}}(x,s,\nabla\omega,\Delta\omega)\, ds. \tag{19}$$

Lemma 3. *Let $\mathbf{R} > 0$ and $\omega \in \mathbf{X}$ be fixed. Then*

(1) *there exist constants $\rho > 0$, $\alpha > 0$ such that $J_\omega^{\mathbf{R}}(v)|_{\partial B_\rho} \ge \alpha$ with $B_\rho = \{v \in \mathbf{X} : \|v\| \le \rho\}$;*

(2) *for fixed v with $\|v\| = 1$, $J_\omega^{\mathbf{R}}(tv) \to -\infty$ as $t \to +\infty$.*

Proof. On one hand, by (f_2), for any $\varepsilon > 0$, there exists a constant $\sigma > 0$ such that, for $|t| < \sigma$, one has

$$F_{\mathbf{R}}(x,t,\nabla\omega,\Delta\omega) \le \frac{1}{2}\varepsilon t^2. \tag{20}$$

On the other hand, if $|t| > \sigma$, from (f_1) it follows that there exists $C_1 > 0$ such that

$$F_{\mathbf{R}}(x,t,\nabla\omega,\Delta\omega) \le C_1 |t|^{p+1}(\mathbf{R}+2)^r. \tag{21}$$

Then, from (20), (21), and the Sobolev inequality, we have

$$\begin{aligned}J_\omega^{\mathbf{R}}(v) = {}& \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega |\nabla v|^2 dx \\ & - \int_\Omega F_{\mathbf{R}}(x, v\nabla\omega, \Delta\omega)\, dx \\ \ge {}& \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega |\nabla v|^2 dx \\ & - \frac{\epsilon}{2}\int_\Omega |v|^2 dx - C_1(\mathbf{R}+2)^r \int_\Omega |v|^{p+1} dx \\ \ge {}& \left(\frac{\min\{1,a\}}{2} - \frac{\epsilon C}{2}\right)\|v\|^2 - C(\mathbf{R}+2)^r \|v\|^{p+1}\end{aligned} \tag{22}$$

for some positive constant C. Therefore, for sufficiently small $\epsilon > 0$, we can choose $\rho > 0$ and $\alpha > 0$ such that the first result of Lemma 3 holds.

Now, we show that (f_3) implies that there exist $a_2, a_3 > 0$ such that

$$\begin{gathered}F_{\mathbf{R}}(x,t,\xi_1,\xi_2) \ge a_2|t|^\theta - a_3, \\ \forall x \in \Omega,\ t \in R,\ \xi_1 \in R^N,\ \xi_2 \in R.\end{gathered} \tag{23}$$

In fact, from (f_3), we have $f(x,t,\xi_1,\xi_2)/F(x,t,\xi_1,\xi_2) \ge \theta/t$, for any $|t| \ge t_1$. Being integral from t_1 to t, we get

$$\ln F(x,t,\xi_1,\xi_2) - \ln F(x,t_1,\xi_1,\xi_2) \ge \theta(\ln t - \ln t_1); \tag{24}$$

namely,

$$F(x,t,\xi_1,\xi_2) \ge \frac{F(x,t_1,\xi_1,\xi_2)}{t_1^\theta}|t|^\theta, \quad \forall |t| \ge t_1. \tag{25}$$

Then

$$\begin{aligned}F_{\mathbf{R}}(x,t,\xi_1,\xi_2) &= F\left(x,t,\xi_1\varphi_{\mathbf{R}}(\xi_1), \xi_2\varphi_{\mathbf{R}}(\xi_2)\right) \\ &\ge \frac{|t|^\theta}{t_1^\theta} F\left(x,t_1,\xi_1\varphi_{\mathbf{R}}(\xi_1), \xi_2\varphi_{\mathbf{R}}(\xi_2)\right).\end{aligned} \tag{26}$$

Let

$$\begin{aligned}a_2 &= \frac{1}{t_1^\theta}\min_{x\in\Omega,\xi_1\in R^N,\xi_2\in R} F\left(x,t_1,\xi_1\varphi_{\mathbf{R}}(\xi_1), \xi_2\varphi_{\mathbf{R}}(\xi_2)\right) \\ &= \frac{1}{t_1^\theta}\min_{x\in\Omega,|\xi_1|\le\mathbf{R}+1,|\xi_2|\le\mathbf{R}+1} F(x,t_1,\xi_1,\xi_2) > 0, \\ a_3 &= \max_{x\in\Omega,|t|\le t_1,|\xi_1|\le\mathbf{R}+1,|\xi_2|\le\mathbf{R}+1} F(x,t_1,\xi_1,\xi_2) > 0,\end{aligned} \tag{27}$$

and then inequality (23) holds.

Taking an arbitrary $v \in \mathbf{X}$ with $\|v\| = 1$, then from (23), we get

$$\begin{aligned}J_\omega^{\mathbf{R}}(tv) = {}& \frac{1}{2}\int_\Omega |\Delta tv|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega |\nabla tv|^2 dx \\ & - \int_\Omega F_{\mathbf{R}}(x,tv,\nabla\omega,\Delta\omega)\, dx \\ \le {}& \frac{1}{2}\int_\Omega |\Delta tv|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla v)\int_\Omega |\nabla tv|^2 dx \\ & - a_2 t^\Theta \int_\Omega |v|^\Theta dx + a_3|\Omega| \\ \le {}& \frac{\max\{1,\Phi_{\mathbf{R}}(\nabla\omega)\}}{2} t^2\|v\|^2 - a_2 t^\Theta \int_\Omega |v|^\Theta dx \\ & + a_3|\Omega| \longrightarrow -\infty, \quad \text{as } t \longrightarrow +\infty,\end{aligned} \tag{28}$$

which implies that the second result of Lemma 3 holds. □

Lemma 4. *Let $\mathbf{R} > 0$ and $\omega \in \mathbf{X}$ be fixed. Then the functional $J_\omega^{\mathbf{R}}(\cdot)$ satisfies the (PS)-condition.*

Proof. Let $\{u_n\} \subset \mathbf{X}$ be a (PS)-sequence; namely,

$$J_\omega^{\mathbf{R}}(u_n) \longrightarrow C, \quad \left\langle J_\omega^{\prime\mathbf{R}}(u_n), u_n\right\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{29}$$

From the standard processes, we only need to prove that $\{u_n\}$ is bounded in $\mathbf{X}$. On a contradiction, suppose that $\|u_n\| \to +\infty$; then, from (f_3), we obtain

$$\begin{aligned} &J_\omega^{\mathbf{R}}(u_n) - \frac{1}{\Theta}\left\langle J_\omega^{\prime\mathbf{R}}(u_n), u_n\right\rangle = \frac{1}{2}\int_\Omega \left|\Delta u_n\right|^2 dx + \frac{1}{2} \cdot \Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_n\right|^2 dx \\ &- \int_\Omega F_{\mathbf{R}}(x, u_n, \nabla\omega, \Delta\omega)\, dx - \frac{1}{\Theta}\int_\Omega \left|\Delta u_n\right|^2 dx - \frac{1}{\Theta}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_n\right|^2 dx + \frac{1}{\Theta} \\ &\cdot \int_\Omega f_{\mathbf{R}}(x, u_n, \nabla\omega, \Delta\omega)\, u_n dx = \left(\frac{1}{2} - \frac{1}{\Theta}\right) \cdot \left[\int_\Omega \left|\Delta u_n\right|^2 dx + \Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_n\right|^2 dx\right] \\ &+ \int_\Omega \left[\frac{1}{\Theta} f_{\mathbf{R}}(x, u_n, \nabla\omega, \Delta\omega)\, u_n - F_{\mathbf{R}}(x, u_n, \nabla\omega, \Delta\omega)\right] dx \ge \left(\frac{1}{2} - \frac{1}{\Theta}\right) \\ &\cdot \min\{1, a\}\|u\|^2 + \int_\Omega \left[\frac{1}{\Theta} f_{\mathbf{R}}(x, u_n, \nabla\omega, \Delta\omega)\, u_n - F_{\mathbf{R}}(x, u_n, \nabla\omega, \Delta\omega)\right] dx \ge \left(\frac{1}{2} - \frac{1}{\Theta}\right) \\ &\cdot \min\{1, a\}\|u\|^2. \end{aligned} \tag{30}$$

On the other hand, from (29) we know that

$$J_\omega^{\mathbf{R}}(u_n) - \frac{1}{\Theta}\left\langle J_\omega^{\prime\mathbf{R}}(u_n), u_n\right\rangle \le C + C\left\|u_n\right\|. \tag{31}$$

Then, from the above inequalities, we get

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\min\{1, a\}\left\|u_n\right\|^2 \le C + C\left\|u_n\right\|, \tag{32}$$

which contradicts with $\|u_n\| \to +\infty$. Therefore the sequence $\{u_n\}$ is bounded in $\mathbf{X}$. □

Lemma 5. *For any $\mathbf{R} > 0$ and $\omega \in \mathbf{X}$, problem (15) has a nontrivial weak solution.*

Proof. By Theorem A, Lemmas 3, and 4, the result holds. □

Lemma 6. *Let $\mathbf{R} > 0$ be fixed. Then there exist positive constants ν_1 and $\nu_2 := \nu_2(\mathbf{R})$, independent of ω, such that*

$$\nu_1 \le \left\|u_\omega^{\mathbf{R}}\right\| \le \nu_2 \tag{33}$$

for every solution $u_\omega^{\mathbf{R}}$ obtained in Lemma 5.

Proof. Firstly, since $J_\omega^{\mathbf{R}}(u_\omega^{\mathbf{R}}) \le \max_{t\ge0} J_\omega^{\mathbf{R}}(tv)$, from (23) it follows that

$$\begin{aligned} J_\omega^{\mathbf{R}}(tv) &\le \frac{\max\{1, \Phi_{\mathbf{R}}\}}{2} t^2\|v\|^2 - a_2 t^\Theta\int_\Omega |v|^\Theta dx + a_3|\Omega| \\ &\le \frac{\max\left\{1, [a + b(\mathbf{R}+1)]^2|\Omega|\right\}}{2} t^2 - a_2 t^\Theta\int_\Omega |v|^\Theta dx + a_3|\Omega|. \end{aligned} \tag{34}$$

As $\Theta > 2$, we can get a $C > 0$ such that $J_\omega^{\mathbf{R}}(u_\omega^{\mathbf{R}}) \le C$; that is,

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left|\Delta u_\omega^{\mathbf{R}}\right|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_\omega^{\mathbf{R}}\right|^2 dx \\ &\quad \le C + \int_\Omega F_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right) dx. \end{aligned} \tag{35}$$

Define $D = \{x \in \Omega : |u_\omega^{\mathbf{R}}| > t_1\}$, where t_1 is defined in (f_3). Then we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left|\Delta u_\omega^{\mathbf{R}}\right|^2 dx + \frac{1}{2}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_\omega^{\mathbf{R}}\right|^2 dx \\ &\le C + \int_\Omega F_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right) dx \\ &= C + \int_{\Omega\setminus D} F_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right) dx + \int_D F_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right) dx \\ &\le C + a_1\left(t_1 + \frac{|t_1|^{p+1}}{p+1}\right)(\mathbf{R}+2)^r|\Omega\setminus D| \\ &\quad + \frac{1}{\Theta}\int_\Omega \left|\Delta u_\omega^{\mathbf{R}}\right|^2 dx + \frac{1}{\Theta}\Phi_{\mathbf{R}}(\nabla\omega)\int_\Omega \left|\nabla u_\omega^{\mathbf{R}}\right|^2 dx. \end{aligned} \tag{36}$$

Furthermore, we have

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{\Theta}\right)\min\{1, a\}\left\|u_\omega^{\mathbf{R}}\right\|^2 \\ &\le C + a_1\left(t_1 + \frac{|t_1|^{p+1}}{p+1}\right)(\mathbf{R}+2)^r|\Omega\setminus D| \\ &\le C(\mathbf{R}+2)^r, \end{aligned} \tag{37}$$

where C is independence of b, $\mathbf{R} > 0$, and $\omega \in \mathbf{H}$. Therefore, $\|u_\omega^{\mathbf{R}}\| \le \nu_2$, for some $\nu_2 := \nu_2(\mathbf{R}) > 0$.

Secondly, from (f_1) and (f_2), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f_{\mathbf{R}}\left(x, u_\omega^{\mathbf{R}}, \nabla\omega, \Delta\omega\right) \le \epsilon\left|u_\omega^{\mathbf{R}}\right| + C_\epsilon(\mathbf{R}+2)^r\left|u_\omega^{\mathbf{R}}\right|^p. \tag{38}$$

Since $\langle J_\omega^{\prime\mathbf{R}}(u_\omega^{\mathbf{R}}), u_\omega^{\mathbf{R}}\rangle = 0$, it is easy to obtain that

$$\min\{1, a\}\left\|u_\omega^{\mathbf{R}}\right\|^2 \le C_2\epsilon\left\|u_\omega^{\mathbf{R}}\right\|^2 + C_3 C_\epsilon\left\|u_\omega^{\mathbf{R}}\right\|^{p+1} \tag{39}$$

for some constants $C_2, C_3 \geq 0$. Therefore, there exists $\nu_1 > 0$ such that $\|u_\omega^{\mathbf{R}}\| \geq \nu_1$. $\square$

Lemma 7 (see [7]). *Let* $\mathbf{R}$ *be fixed, and choose* $\omega \in C^{4,\alpha}(\overline{\Omega})$ *for* $\alpha \in (0,1)$. *If* $u_\omega^{\mathbf{R}} \in \mathbf{X}$ *is a weak solution of problem (15), then* $u_\omega^{\mathbf{R}} \in C^{4,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$, *and* $\Delta(u_\omega^{\mathbf{R}})(x) = 0$ *if* $x \in \partial\Omega$.

Lemma 8. *There exist three constants* $\eta_i > 0$ $(i = 1,2,3)$, *independent of* b, ω, *and* $\mathbf{R}$, *such that*

$$\begin{aligned} \left\|u_\omega^{\mathbf{R}}\right\|_{C^0} &\leq \eta_1 (\mathbf{R}+2)^{r/2}, \\ \left\|\nabla u_\omega^{\mathbf{R}}\right\|_{C^0} &\leq \eta_2 (\mathbf{R}+2)^{r/2}, \\ \left\|\Delta u_\omega^{\mathbf{R}}\right\|_{C^0} &\leq \eta_3 (\mathbf{R}+2)^{r/2}. \end{aligned} \tag{40}$$

In addition, there exists $\overline{\mathbf{R}}$ *such that*

$$\eta_i \left(\overline{\mathbf{R}}+2\right)^{r/2} \leq \overline{\mathbf{R}} \quad (i = 1,2,3). \tag{41}$$

Proof. From (37) and the proof of Lemma 6, there exists $C > 0$, independent of $R > 0$ and $\omega \in \mathbf{X}$, such that

$$\left\|u_\omega^{\mathbf{R}}\right\| \leq C (\mathbf{R}+2)^{r/2}. \tag{42}$$

Then by Lemma 7 and the Sobolev embedding theorem, the inequalities in the lemma are as follows. In addition, since $r/2 < 1$ and $\lim_{s\to\infty}((s+2)^{r/2}/s) = 0$, there exists a sufficiently large $\overline{\mathbf{R}} > 0$ such that $\theta_i(\overline{\mathbf{R}}+2)^{r/2} \leq \overline{\mathbf{R}}$. $\square$

Now let $u_n^{\mathbf{R}}$ $(n = 1,2,\ldots)$ be the weak solution of the following problem:

$$\begin{aligned} \Delta^2 u_n^{\mathbf{R}} - \Phi_{\mathbf{R}}\left(\nabla u_{n-1}^{\mathbf{R}}\right) \Delta u_n^{\mathbf{R}} &= f_{\mathbf{R}}\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_{n-1}^{\mathbf{R}}\right), \\ &\qquad \text{in } \Omega \\ u_n^{\mathbf{R}} &= 0, \\ \Delta u_n^{\mathbf{R}} &= 0, \\ &\qquad \text{on } \partial\Omega \end{aligned} \tag{43}$$

with $\omega = u_{n-1}^{\mathbf{R}}$, where $u_{n-1}^{\mathbf{R}}$ was found in Lemma 5 and $\mathbf{R} = \overline{\mathbf{R}}$ obtained in Lemma 8. From Lemmas 6–8, we have $u_n^{\mathbf{R}} \in C^4(\overline{\Omega})$ satisfying $\|u_n^{\mathbf{R}}\| \geq \nu_1$ and

$$\left\|u_n^{\mathbf{R}}\right\|_{C^0}, \left\|\nabla u_n^{\mathbf{R}}\right\|_{C^0}, \left\|\Delta u_n^{\mathbf{R}}\right\|_{C^0} \leq \overline{\mathbf{R}}. \tag{44}$$

Thus

$$\begin{aligned} \Phi_{\mathbf{R}}\left(\nabla u_{n-1}^{\mathbf{R}}\right) &= a + b\int_\Omega \left|\nabla u_{n-1}^{\mathbf{R}} \varphi_{\mathbf{R}}\left(\nabla u_{n-1}^{\mathbf{R}}\right)\right|^2 dx \\ &= a + b\int_\Omega \left|\nabla u_{n-1}^{\mathbf{R}}\right|^2 dx, \\ f_{\mathbf{R}}\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_{n-1}^{\mathbf{R}}\right) &= f\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_{n-1}^{\mathbf{R}}\right). \end{aligned} \tag{45}$$

Lemma 9. *Assume that* (f_4) *holds. Let*

$$\begin{aligned} \rho_1 &= \inf\left\{\sigma_1 : \left\|u_n^{\mathbf{R}}\right\|_{C^0} \leq \sigma_1,\ \forall n \in N^+\right\} > 0, \\ \rho_2 &= \inf\left\{\sigma_2 : \left\|\nabla u_n^{\mathbf{R}}\right\|_{C^0} \leq \sigma_2,\ \forall n \in N^+\right\} > 0, \\ \rho_3 &= \inf\left\{\sigma_3 : \left\|\Delta u_n^{\mathbf{R}}\right\|_{C^0} \leq \sigma_3,\ \forall n \in N^+\right\} > 0. \end{aligned} \tag{46}$$

Then $\{u_n^{\mathbf{R}}\}$ *strongly converges in* $\mathbf{X}$.

Proof. Let $u_{n+1}^{\mathbf{R}}$ and $u_n^{\mathbf{R}}$ be the weak solutions of (43) with $\omega = u_n^{\mathbf{R}}$ and $\omega = u_{n-1}^{\mathbf{R}}$, respectively. Then, multiplying (43) by $(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}})$, we obtain

$$\begin{aligned} \min\{1,a\}\left\|u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right\|^2 &\leq \int_\Omega \left[f\left(x, u_{n+1}^{\mathbf{R}}, \nabla u_n^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right)\right. \\ &\quad \left.- f\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_{n-1}^{\mathbf{R}}\right)\right]\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad + b\left(\int_\Omega \left|\nabla u_n^{\mathbf{R}}\right|^2 dx\right)\int_\Omega \Delta u_{n+1}^{\mathbf{R}}\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad - b\left(\int_\Omega \left|\nabla u_{n-1}^{\mathbf{R}}\right|^2 dx\right)\int_\Omega \Delta u_n^{\mathbf{R}}\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx. \end{aligned} \tag{47}$$

Furthermore, by (f_1), (f_4), and the Hölder inequality, we have

$$\begin{aligned} &\int_\Omega \left[f\left(x, u_{n+1}^{\mathbf{R}}, \nabla u_n^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right) - f\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_{n-1}^{\mathbf{R}}\right)\right] \\ &\quad \cdot \left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad = \int_\Omega \left[f\left(x, u_{n+1}^{\mathbf{R}}, \nabla u_n^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right)\right. \\ &\quad \left.- f\left(x, u_n^{\mathbf{R}}, \nabla u_n^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right)\right]\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad + \int_\Omega \left[f\left(x, u_n^{\mathbf{R}}, \nabla u_n^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right)\right. \\ &\quad \left.- f\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right)\right]\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad + \int_\Omega \left[f\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_n^{\mathbf{R}}\right)\right. \\ &\quad \left.- f\left(x, u_n^{\mathbf{R}}, \nabla u_{n-1}^{\mathbf{R}}, \Delta u_{n-1}^{\mathbf{R}}\right)\right]\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad \leq \tau_1^2 L_{\rho_1}\left\|u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right\|^2 + \tau_1\tau_2 L_{\rho_2}\left\|u_n^{\mathbf{R}} - u_{n-1}^{\mathbf{R}}\right\| \\ &\quad \cdot \left\|u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right\| + \tau_1\tau_3 L_{\rho_3}\left\|u_n^{\mathbf{R}} - u_{n-1}^{\mathbf{R}}\right\| \cdot \left\|u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right\|, \\ &b\left(\int_\Omega \left|\nabla u_n^{\mathbf{R}}\right|^2 dx\right)\int_\Omega \Delta u_{n+1}^{\mathbf{R}}\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad - b\left(\int_\Omega \left|\nabla u_{n-1}^{\mathbf{R}}\right|^2 dx\right)\int_\Omega \Delta u_n^{\mathbf{R}}\left(u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right) dx \\ &\quad \leq b\rho_3 |\Omega|^{1/2}\left\|u_{n+1}^{\mathbf{R}} - u_n^{\mathbf{R}}\right\|_{L^2} \cdot \int_\Omega \left(\left|\nabla u_n^{\mathbf{R}}\right| + \left|\nabla u_{n-1}^{\mathbf{R}}\right|\right) \\ &\quad \cdot \left(\nabla\left(u_n^{\mathbf{R}} - u_{n-1}^{\mathbf{R}}\right)\right) dx \leq 2b\rho_2\rho_3 |\Omega|\, \tau_1\tau_2 \left\|u_{n+1}^{\mathbf{R}}\right. \\ &\quad \left.- u_n^{\mathbf{R}}\right\| \cdot \left\|u_n^{\mathbf{R}} - u_{n-1}^{\mathbf{R}}\right\|. \end{aligned} \tag{48}$$

Hence, by (47) and (48), we get

$$\begin{aligned}\min\{1,a\}&\left\|u_{n+1}^{\mathbf{R}}-u_n^{\mathbf{R}}\right\|^2\\ &\le \tau_1^2 L_{\rho_1}\left\|u_{n+1}^{\mathbf{R}}-u_n^{\mathbf{R}}\right\|^2+\tau_1\tau_2 L_{\rho_2}\left\|u_n^{\mathbf{R}}-u_{n-1}^{\mathbf{R}}\right\|\\ &\quad\cdot\left\|u_{n+1}^{\mathbf{R}}-u_n^{\mathbf{R}}\right\|+\tau_1\tau_3 L_{\rho_3}\left\|u_n^{\mathbf{R}}-u_{n-1}^{\mathbf{R}}\right\|\\ &\quad\cdot\left\|u_{n+1}^{\mathbf{R}}-u_n^{\mathbf{R}}\right\|+2b\rho_2\rho_3\tau_1\tau_2|\Omega|\left\|u_{n+1}^{\mathbf{R}}-u_n^{\mathbf{R}}\right\|\\ &\quad\cdot\left\|u_n^{\mathbf{R}}-u_{n-1}^{\mathbf{R}}\right\|;\end{aligned}\tag{49}$$

namely,

$$\begin{aligned}&\left\|u_{n+1}^{\mathbf{R}}-u_n^{\mathbf{R}}\right\|\\ &\quad\le\frac{\tau_1\tau_2L_{\rho_2}+\tau_1\tau_3L_{\rho_3}+2b\rho_2\rho_3\tau_1\tau_2|\Omega|}{\min\{1,a\}-L_{\rho_1}\tau_1^2}\left\|u_n^{\mathbf{R}}-u_{n-1}^{\mathbf{R}}\right\|.\end{aligned}\tag{50}$$

Now, choosing $b^* = (\min\{1,a\} - L_{\rho_1}\tau_1^2 - L_{\rho_2}\tau_1\tau_2 - L_{\rho_3}\tau_1\tau_3)/2\rho_2\rho_3\tau_1\tau_2|\Omega|$, then

$$\frac{\tau_1\tau_2L_{\rho_2}+\tau_1\tau_3L_{\rho_3}+2b\rho_2\rho_3\tau_1\tau_2|\Omega|}{\min\{1,a\}-L_{\rho_1}\tau_1^2}<1,\qquad \text{for } 0<b<b^*.\tag{51}$$

Therefore, $\{u_n^{\mathbf{R}}\}$ converges strongly in $\mathbf{X}$. □

Proof of Theorem 2. Firstly, from Lemma 6, we get $\|u_n^{\mathbf{R}}\|\ge\nu_1>0$, and $\|u_n^{\mathbf{R}}\|_{C^0}$, $\|\nabla u_n^{\mathbf{R}}\|_{C^0}$, $\|\Delta u_n^{\mathbf{R}}\|_{C^0}$ are uniformly bounded. Secondly, set $v_n=\Delta u_n^{\mathbf{R}}$; then

$$\begin{aligned}\Delta v_n&=h(x)\\ &=f\left(x,u_n^{\mathbf{R}},\nabla u_{n-1}^{\mathbf{R}},\Delta u_{n-1}^{\mathbf{R}}\right)\\ &\quad+\left(a+b\int_\Omega\left|\nabla u_{n-1}^{\mathbf{R}}\right|^2dx\right)\Delta u_n^{\mathbf{R}}.\end{aligned}\tag{52}$$

Since $\|h\|_{C^\beta}\le C$, for some positive constant C, by the Schauder theorem, there exists a constant $C_0>0$ such that $\|v_n\|_{C^{2,\beta}}\le C_0$; that is, $\|u_n^{\mathbf{R}}\|_{C^{4,\beta}}\le C_0$. Furthermore, by the Arzela-Ascoli theorem and Lemma 9, the sequence $\{u_n^{\mathbf{R}}\}$ satisfies

$$\frac{\partial^j}{\partial x_i^j}u_n^{\mathbf{R}}(x)\longrightarrow\frac{\partial^j}{\partial x_i^j}u^{\mathbf{R}}(x),\quad\text{as } n\longrightarrow\infty\tag{53}$$

uniformly in $\overline{\Omega}$ for $j=0,1,2,3,4$ and $i=1,\dots,N$. Finally, passing to the limit in (43), we obtain that $u^{\mathbf{R}}(x)$ is a classical solution of (1). □

Example 10. Consider the following problem:

$$\begin{aligned}\Delta^2u-\left(a+b\int_\Omega|\nabla u|^2dx\right)\Delta u&=f(x,u,\nabla u,\Delta u),\quad\text{in }\Omega,\\ u&=0,\\ \Delta u&=0,\quad\text{on }\partial\Omega,\end{aligned}\tag{54}$$

where $f(x,t,\xi_1,\xi_2)=\alpha(x)|t|t(1+|\xi_1|)^{1/2}(1+|\xi_2|)^{3/4}+\beta(x)t^3$ and $\alpha(x)$ and $\beta(x)$ are positive and continuous functions. It is easy to verify that $f(x,t,\xi_1,\xi_2)$ satisfies all the conditions of (f_1)–(f_4).

3. Conclusion

The paper considers a class of fourth-order elliptic equations of Kirchhoff type with dependence on the gradient and Laplacian. The existence of a nontrivial solution of (1) is established when we choose appropriate b^* such that $0<b<b^*$. The paper generalized the conclusions in [7, 14] and weakened the condition in [7]. In the following research work, we will also consider problem (1), but we just truncate the right side of the equation, and the left of the equation remains the same.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

Acknowledgments

Yunhai Wang was supported by the projects of Guizhou Provincial Science and Technology Fund (QKH-JICHU [2017], Grant no. 1408) and the doctoral starting up foundation of Guizhou Institute of Technology (Grant no. XJGC20150408). Fanglei Wang was supported by NNSF of China (no. 11501165) and the Fundamental Research Funds for the Central Universities (2015B19414).

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Hindawi
Journal of Function Spaces
Volume 2017, Article ID 1518254, 13 pages
https://doi.org/10.1155/2017/1518254

Research Article

Orthogonal Gyroexpansion in Möbius Gyrovector Spaces

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Received 12 August 2017; Accepted 17 September 2017; Published 13 December 2017

Academic Editor: Juan Martinez-Moreno

We investigate the Möbius gyrovector spaces which are open balls centered at the origin in a real Hilbert space with the Möbius addition, the Möbius scalar multiplication, and the Poincaré metric introduced by Ungar. In particular, for an arbitrary point, we can easily obtain the unique closest point in any closed gyrovector subspace, by using the ordinary orthogonal decomposition. Further, we show that each element has the orthogonal gyroexpansion with respect to any orthogonal basis in a Möbius gyrovector space, which is similar to each element in a Hilbert space having the orthogonal expansion with respect to any orthonormal basis. Moreover, we present a concrete procedure to calculate the gyrocoefficients of the orthogonal gyroexpansion.

1. Introduction

A. Ungar initiated study on gyrogroups and gyrovector spaces (cf. [1]). Gyrovector spaces are generalized vector spaces, with which they share important analogies, just as gyrogroups are analogous to groups. The first known gyrogroup was the ball of Euclidean space $\mathbb{R}^3$ endowed with Einstein's velocity addition associated with the special theory of relativity. Another example of a gyrogroup is the open unit disc in the complex plain endowed with the Möbius addition. Ungar extended these gyroadditions to the ball of an arbitrary real inner product space, introduced a common gyroscalar multiplication, and observed that the ball endowed with gyrooperations are gyrovector spaces (cf. [2, 3]). He describes that gyrovector spaces provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry. In particular, Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry, and similarly, Einstein gyrovector spaces form the setting for the Beltrami-Klein ball model. Readers may consult [4, 5] and the references therein for general information about gyrogroups and gyrovector spaces.

Gyrooperations are generally not commutative, associative, or distributive. Thus the theory of gyrovector spaces falls within the general area of nonlinear functional analysis. They are enjoying algebraic rules such as left and right gyroassociative, gyrocommutative, scalar distributive, and scalar associative laws, so there exist rich symmetrical structures which we should clarify precisely. Many elementary problems are still unsolved. We refer to [6–10] as examples of recent papers for gyrovector spaces, their generalizations, and related matters.

In [8], Abe and the author of the present article showed that any finitely generated gyrovector subspace in the Möbius gyrovector space coincides with the intersection of the linear subspace generated by the same generators and the Möbius ball. As an application, they presented a notion of orthogonal gyrodecomposition and clarified the relation to the ordinary orthogonal decomposition.

The importance of the orthogonal expansion of each vector with respect to an orthonormal basis in a Hilbert space cannot be overemphasized in both theory and application of functional analysis. In this paper we will introduce a concept of orthogonal gyroexpansion of each element with respect to an orthogonal basis in a Möbius gyrovector space and reveal analogies that it shares with its classical counterpart. Such problems seem to be quite fundamental and important for developing pure and applied mathematics, since one of the virtues of gyrovector spaces is that they have properties which are fully analogous to vector space properties. Moreover, the gyrocoefficients of the orthogonal gyroexpansion can be concretely calculated by a procedure that is given here.

The paper is organized as follows. Section 2 is the preliminaries. In Section 3, we introduce a notion of gyrolinear independency for finite sets in a gyrovector space and show that it coincides with the notion of the linear independency. In Section 4, we give a notion of orthogonal gyroexpansions with respect to a complete orthogonal sequence in the Möbius gyrovector space, and we present an explicit procedure to obtain the orthogonal gyroexpansions.

2. Preliminaries

Let us briefly recall the definitions of two models of gyrovector spaces, that is, the Möbius and Einstein gyrovector spaces. For precise definitions and basic results of gyrocommutative gyrogroups and gyrovector spaces, see [4].

Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be a real inner product space with a binary operation $+$ and a positive definite inner product $\cdot$ and let $\mathbb{V}_s$ be the ball

$$\mathbb{V}_s = \{\mathbf{a} \in \mathbb{V} : \|\mathbf{a}\| < s\} \tag{1}$$

for any fixed $s > 0$.

Definition 1 (see [4, Definitions 3.40 and 6.83]). The Möbius addition $\oplus_{\mathrm{M}}$ and the Möbius scalar multiplication $\otimes_{\mathrm{M}}$ are given by the equations

$$\begin{aligned} &\mathbf{a} \oplus_{\mathrm{M}} \mathbf{b} \\ &= \frac{\left(1 + \left(2/s^2\right) \mathbf{a} \cdot \mathbf{b} + \left(1/s^2\right) \|\mathbf{b}\|^2\right) \mathbf{a} + \left(1 - \left(1/s^2\right) \|\mathbf{a}\|^2\right) \mathbf{b}}{1 + \left(2/s^2\right) \mathbf{a} \cdot \mathbf{b} + \left(1/s^4\right) \|\mathbf{a}\|^2 \|\mathbf{b}\|^2} \\ &r \otimes_{\mathrm{M}} \mathbf{a} = s \tanh\left(r \tanh^{-1} \frac{\|\mathbf{a}\|}{s}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &\qquad (\text{if } \mathbf{a} \neq \mathbf{0}), \quad r \otimes_{\mathrm{M}} \mathbf{0} = \mathbf{0} \end{aligned} \tag{2}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$. The addition $\oplus_{\mathrm{M}}$ and scalar multiplication $\otimes_{\mathrm{M}}$ for the set $\|\mathbb{V}_s\| = \{\pm\|\mathbf{a}\|;\ \mathbf{a} \in \mathbb{V}_s\}$ in the axiom (VV) of gyrovector space are defined by the equations

$$\begin{aligned} a \oplus_{\mathrm{M}} b &= \frac{a + b}{1 + \left(1/s^2\right) ab} \\ r \otimes_{\mathrm{M}} a &= s \tanh\left(r \tanh^{-1} \frac{a}{s}\right) \end{aligned} \tag{3}$$

for any $a, b \in \|\mathbb{V}_s\|$, $r \in \mathbb{R}$.

We simply denote $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}$ by $\oplus, \otimes$, respectively. If several kinds of operations appear in a formula simultaneously, we always give priority by the following order: (i) ordinary scalar multiplication; (ii) gyroscalar multiplication $\otimes$; (iii) gyroaddition $\oplus$; that is,

$$\begin{aligned} r_1 \otimes w_1 \mathbf{a}_1 \oplus r_2 \otimes w_2 \mathbf{a}_2 = &\left\{r_1 \otimes \left(w_1 \mathbf{a}_1\right)\right\} \\ &\oplus \left\{r_2 \otimes \left(w_2 \mathbf{a}_2\right)\right\}, \end{aligned} \tag{4}$$

and the parentheses are omitted in such cases. In general, we note that gyroaddition does not distribute with (both ordinary and gyro) scalar multiplications:

$$\begin{aligned} t (\mathbf{a} \oplus \mathbf{b}) &\neq t\mathbf{a} \oplus t\mathbf{b}, \\ r \otimes (\mathbf{a} \oplus \mathbf{b}) &\neq r \otimes \mathbf{a} \oplus r \otimes \mathbf{b}. \end{aligned} \tag{5}$$

In the limit of large s, $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole space $\mathbb{V}$. The next proposition suggests that each result in linear analysis can be restored from the counterpart in gyrolinear analysis.

Proposition 2 (see [4, p. 78]). *The Möbius addition (resp., Möbius scalar multiplication) reduces to the vector addition (resp., scalar multiplication) as $s \to \infty$; that is,*

$$\begin{aligned} \mathbf{a} \oplus \mathbf{b} &\longrightarrow \mathbf{a} + \mathbf{b} \quad (s \longrightarrow \infty) \\ r \otimes \mathbf{a} &\longrightarrow r\mathbf{a} \quad (s \longrightarrow \infty). \end{aligned} \tag{6}$$

Definition 3 (see [4, Definitions 3.45 and 6.86]). The Einstein addition $\oplus_{\mathrm{E}}$ and the Einstein scalar multiplication $\otimes_{\mathrm{E}}$ are given by the equations

$$\begin{aligned} &\mathbf{a} \oplus_{\mathrm{E}} \mathbf{b} = \frac{1}{1 + (\mathbf{a} \cdot \mathbf{b})/s^2} \left\{\mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}} \mathbf{b} + \frac{1}{s^2} \frac{\gamma_{\mathbf{a}}}{1 + \gamma_{\mathbf{a}}} (\mathbf{a} \cdot \mathbf{b}) \mathbf{a}\right\} \\ &r \otimes_{\mathrm{E}} \mathbf{a} = s \tanh\left(r \tanh^{-1} \frac{\|\mathbf{a}\|}{s}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &\qquad (\text{if } \mathbf{a} \neq \mathbf{0}), \quad r \otimes_{\mathrm{E}} \mathbf{0} = \mathbf{0} \end{aligned} \tag{7}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$, where $\gamma_{\mathbf{a}} = 1/\sqrt{1 - \|\mathbf{a}\|^2/s^2}$.

Note that each of the Einstein scalar multiplication and the operations on the set $\|\mathbb{V}_s\|$ is identical to the corresponding operation for the Möbius gyrovector spaces.

Definition 4 (see [4, Definition 6.88]). An isomorphism from a gyrovector space $(G_1, \oplus_1, \otimes_1)$ to a gyrovector space $(G_2, \oplus_2, \otimes_2)$ is a bijective map $\phi : G_1 \to G_2$ that preserves gyrooperations and keeps the inner product of normalized elements invariant; that is,

$$\begin{aligned} \phi\left(\mathbf{a} \oplus_1 \mathbf{b}\right) &= \phi(\mathbf{a}) \oplus_2 \phi(\mathbf{b}) \\ \phi\left(r \otimes_1 \mathbf{a}\right) &= r \otimes_2 \phi(\mathbf{a}) \\ \frac{\phi(\mathbf{a})}{\|\phi(\mathbf{a})\|} \cdot \frac{\phi(\mathbf{b})}{\|\phi(\mathbf{b})\|} &= \frac{\mathbf{a}}{\|\mathbf{a}\|} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|} \quad (\text{if } \mathbf{a}, \mathbf{b} \neq \mathbf{0}) \end{aligned} \tag{8}$$

for any $\mathbf{a}, \mathbf{b} \in G_1$, $r \in \mathbb{R}$.

Theorem 5 (see [4, Table 6.1]). *Let $\phi_{\mathrm{EM}} : \mathbb{V}_s \to \mathbb{V}_s$ be the map defined by the equation*

$$\phi_{\mathrm{EM}}(\mathbf{a}) = 2 \otimes \mathbf{a} \tag{9}$$

for any $\mathbf{a} \in \mathbb{V}_s$. Then ϕ_{EM} is an isomorphism from the Möbius gyrovector space to the Einstein gyrovector space.

Thus, most of results established for the Möbius gyrovector spaces in the sequel can be transformed to corresponding results for the Einstein gyrovector spaces by the isomorphism stated above.

3. Gyrolinear Independency

We begin with consideration of a counterpart in a gyrovector space to the notion of linearly independent sets in a linear space.

Definition 6. A finite subset $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{V}_s$ is gyrolinearly independent if, for any permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ and for any order of gyroaddition, the following implication holds:

$$r_{i_1} \otimes \mathbf{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \mathbf{a}_{i_n} = \mathbf{0} \Longrightarrow \\ r_1 = \cdots = r_n = 0. \tag{10}$$

Example 7. Let $\mathbb{V} = \mathbb{R}^2$ with the Euclidean inner product and $s = 1$. If we identify $\mathbb{V}_1$ with the open unit disc $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ in the complex plain $\mathbb{C}$ by $(x, y) = x + iy$, then it is well-known that the Möbius addition reduces to

$$a \oplus b = \frac{a + b}{1 + \overline{a}b} \tag{11}$$

for any $a, b \in \mathbb{D}$ (cf. [4, (3.127)]). If we take

$$a = \frac{i}{2}, \quad b = -\frac{2}{5} - \frac{2}{5}i, \quad c = \frac{1}{2}, \tag{12}$$

then

$$a \oplus (b \oplus c) = 0, \quad (a \oplus b) \oplus c = \frac{4 + 16i}{53 - 8i}. \tag{13}$$

This means that $\{a, b, c\}$ is not gyrolinearly independent, and if we put

$$r_1 = \frac{\tanh^{-1}\left(\left(-33 + \sqrt{689}\right)/20\right)}{\tanh^{-1}(1/2)}, \quad r_2 = \frac{\tanh^{-1}\left(\left(17 - \sqrt{689}\right)/20\right)}{\tanh^{-1}(1/2)}, \tag{14}$$

then it is readily checked that

$$\begin{aligned} r_1 \otimes c \oplus r_2 \otimes a &= \frac{-33 + \sqrt{689}}{20} \oplus \frac{17 - \sqrt{689}}{20} i \\ &= \frac{\left(\left(-33 + \sqrt{689}\right)/20\right) + \left(\left(17 - \sqrt{689}\right)/20\right) i}{1 + \left(\left(-33 + \sqrt{689}\right)/20\right) \cdot \left(\left(17 - \sqrt{689}\right)/20\right) i} \\ &= b. \end{aligned} \tag{15}$$

It is immediate to see the following lemma by the fact that $1 \otimes \mathbf{a} = \mathbf{a}$ and $0 \otimes \mathbf{a} = \mathbf{0}$ and Definition 6. We omit the proof.

Lemma 8. *Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{V}_s$ be gyrolinearly independent. Then*

(i) *each element is nonzero;*

(ii) *any subset is also gyrolinearly independent.*

Lemma 9. *Suppose that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent in $\mathbb{V}_s$ and*

$$r_1 \otimes \mathbf{a}_1 \oplus r_2 \otimes \mathbf{a}_2 = \lambda_1 \otimes \mathbf{a}_1 \oplus \lambda_2 \otimes \mathbf{a}_2 \tag{16}$$

Then one has $r_1 = \lambda_1$ and $r_2 = \lambda_2$.

Proof. Without loss of generality, we may assume that $s = 1$, $r_j \neq 0$, and $\lambda_j \neq 0$. If we put

$$\begin{aligned} \alpha &= \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} \cdot \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}, \\ c_j &= \tanh\left(r_j \tanh^{-1}\|\mathbf{a}_j\|\right), \\ d_j &= \tanh\left(\lambda_j \tanh^{-1}\|\mathbf{a}_j\|\right) \end{aligned} \tag{17}$$

for $j = 1, 2$, then, from the definitions of $\oplus, \otimes$, it follows that $-1 \leq \alpha \leq 1$, $0 < |c_j|, |d_j| < 1$, and

$$\begin{aligned} r_1 \otimes \mathbf{a}_1 \oplus r_2 \otimes \mathbf{a}_2 &= \lambda_1 \otimes \mathbf{a}_1 \oplus \lambda_2 \otimes \mathbf{a}_2 \\ &= t_1 \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + t_2 \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}, \end{aligned} \tag{18}$$

where we put

$$\begin{aligned} t_1 &= \frac{\left(1 + 2c_1c_2\alpha + c_2^2\right)c_1}{1 + 2c_1c_2\alpha + c_1^2c_2^2} = \frac{\left(1 + 2d_1d_2\alpha + d_2^2\right)d_1}{1 + 2d_1d_2\alpha + d_1^2d_2^2} \\ t_2 &= \frac{\left(1 - c_1^2\right)c_2}{1 + 2c_1c_2\alpha + c_1^2c_2^2} = \frac{\left(1 - d_1^2\right)d_2}{1 + 2d_1d_2\alpha + d_1^2d_2^2}. \end{aligned} \tag{19}$$

This means that (c_1, c_2) and (d_1, d_2) are solutions to the system of equations

$$\begin{aligned} x^2y^2 + \left(\gamma x^2 + 2\alpha x - \gamma\right) y + 1 &= 0 \\ xy^2 + \left((2\alpha + \beta) x^2 - \beta\right) y + x &= 0, \end{aligned} \tag{20}$$

where we put $\beta = t_1/t_2$ and $\gamma = 1/t_2$. Then, we have $\beta \neq 0$ and $1 + \beta(2\alpha + \beta) < \gamma^2$ by [8, Lemma 2.2]. So we can apply [8, Theorem 2.4] to obtain that $c_j = d_j$, which yields that $r_j = \lambda_j$ for $j = 1, 2$. This completes the proof. □

Theorem 10. *Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a linearly independent set in $\mathbb{V}_s$. Suppose that two gyrolinear combinations $r_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus r_n \otimes \mathbf{a}_n$, $\lambda_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus \lambda_n \otimes \mathbf{a}_n$ are given the same order of gyroaddition and*

$$r_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus r_n \otimes \mathbf{a}_n = \lambda_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus \lambda_n \otimes \mathbf{a}_n \tag{21}$$

Then one has $r_j = \lambda_j$ $(j = 1, \ldots, n)$.

Proof. Without loss of generality, we may assume that $s = 1$. Assume that the theorem is valid up to n. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\}$ be a linearly independent set in $\mathbb{V}_1$ and let the following formula

$$\begin{aligned} &\left(r_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus r_m \otimes \mathbf{a}_m\right) \\ &\quad \oplus \left(r_{m+1} \otimes \mathbf{a}_{m+1} \oplus \cdots \oplus r_{n+1} \otimes \mathbf{a}_{n+1}\right) \\ &= \left(\lambda_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus \lambda_m \otimes \mathbf{a}_m\right) \\ &\quad \oplus \left(\lambda_{m+1} \otimes \mathbf{a}_{m+1} \oplus \cdots \oplus \lambda_{n+1} \otimes \mathbf{a}_{n+1}\right) \end{aligned} \tag{22}$$

show the latest gyroadditions. Put

$$\begin{aligned}\mathbf{a} &= r_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus r_m \otimes \mathbf{a}_m \\ \mathbf{b} &= r_{m+1} \otimes \mathbf{a}_{m+1} \oplus \cdots \oplus r_{n+1} \otimes \mathbf{a}_{n+1} \\ \mathbf{a}' &= \lambda_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus \lambda_m \otimes \mathbf{a}_m \\ \mathbf{b}' &= \lambda_{m+1} \otimes \mathbf{a}_{m+1} \oplus \cdots \oplus \lambda_{n+1} \otimes \mathbf{a}_{n+1}.\end{aligned} \tag{23}$$

Then $\mathbf{a}, \mathbf{a}'$ (resp., $\mathbf{b}, \mathbf{b}'$) belong to the linear span of $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ (resp., $\{\mathbf{a}_{m+1}, \ldots, \mathbf{a}_{n+1}\}$). If $\mathbf{a} = \mathbf{0}$, then we have $\mathbf{b} = \mathbf{a}' \oplus \mathbf{b}'$. By [8, Theorem 3.3], we can express $\mathbf{b}, \mathbf{a}', \mathbf{b}'$ of the form

$$\begin{aligned}\mathbf{b} &= t_{m+1} \frac{\mathbf{a}_{m+1}}{\|\mathbf{a}_{m+1}\|} + \cdots + t_{n+1} \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|} \\ \mathbf{a}' &= t_1' \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + t_m' \frac{\mathbf{a}_m}{\|\mathbf{a}_m\|} \\ \mathbf{b}' &= t_{m+1}' \frac{\mathbf{a}_{m+1}}{\|\mathbf{a}_{m+1}\|} + \cdots + t_{n+1}' \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|}.\end{aligned} \tag{24}$$

By the definition of $\oplus$, it follows that

$$\begin{aligned}&t_{m+1} \frac{\mathbf{a}_{m+1}}{\|\mathbf{a}_{m+1}\|} + \cdots + t_{n+1} \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|} \\ &\quad = c_1 \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + c_{n+1} \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|},\end{aligned} \tag{25}$$

where we put

$$c_j = \begin{cases} \dfrac{\left(1 + 2\mathbf{a}' \cdot \mathbf{b}' + \|\mathbf{b}'\|^2\right) t_j'}{1 + 2\mathbf{a}' \cdot \mathbf{b}' + \|\mathbf{a}'\|^2 \|\mathbf{b}'\|^2} & (j = 1, \ldots, m) \\ \dfrac{\left(1 - \|\mathbf{a}'\|^2\right) t_j'}{1 + 2\mathbf{a}' \cdot \mathbf{b}' + \|\mathbf{a}'\|^2 \|\mathbf{b}'\|^2} & (j = m+1, \ldots, n+1). \end{cases} \tag{26}$$

Since $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\}$ is linearly independent, we have $c_1 = \cdots = c_m = 0$, which implies that $t_1' = \cdots = t_m' = 0$; that is, $\mathbf{a}' = \mathbf{0}$. By the assumption of our induction, it follows that $r_j = \lambda_j$ for all j.

Similarly, we may assume that $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \neq \mathbf{0}$, so $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent. By the definition of $\oplus$, we can rewrite the equation

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{a}' \oplus \mathbf{b}' \tag{27}$$

as

$$t_1 \frac{\mathbf{a}}{\|\mathbf{a}\|} + t_2 \frac{\mathbf{b}}{\|\mathbf{b}\|} = t_1' \frac{\mathbf{a}'}{\|\mathbf{a}'\|} + t_2' \frac{\mathbf{b}'}{\|\mathbf{b}'\|}, \tag{28}$$

so we obtain that

$$\begin{aligned}t_1 \frac{\mathbf{a}}{\|\mathbf{a}\|} &= t_1' \frac{\mathbf{a}'}{\|\mathbf{a}'\|}, \\ t_2 \frac{\mathbf{b}}{\|\mathbf{b}\|} &= t_2' \frac{\mathbf{b}'}{\|\mathbf{b}'\|}.\end{aligned} \tag{29}$$

Therefore, (27) can be changed to the following equation:

$$1 \otimes \mathbf{a} \oplus 1 \otimes \mathbf{b} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{b}, \tag{30}$$

where

$$\begin{aligned}r_1 &= \frac{\tanh^{-1}\left(t_1 \|\mathbf{a}'\| / t_1'\right)}{\tanh^{-1} \|\mathbf{a}\|}, \\ r_2 &= \frac{\tanh^{-1}\left(t_2 \|\mathbf{b}'\| / t_2'\right)}{\tanh^{-1} \|\mathbf{b}\|}.\end{aligned} \tag{31}$$

By the previous lemma, we can conclude that $1 = r_1 = r_2$, which implies that $\mathbf{a} = \mathbf{a}'$, $\mathbf{b} = \mathbf{b}'$. Then, the assumption of our induction shows that $r_j = \lambda_j$ $(j = 1, \ldots, n+1)$. This completes the proof. □

Theorem 11. *For any finite subset in $\mathbb{V}_s$, two notions of linearly independent and gyrolinearly independent coincide.*

Proof. (⇒) It immediately follows from the previous theorem.

(⇐) We may assume that $s = 1$. Assume that the theorem is valid up to n, the number of elements of the finite set. Suppose that $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\} \subset \mathbb{V}_1$ is gyrolinearly independent and

$$t_1 \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + t_{n+1} \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|} = \mathbf{0}. \tag{32}$$

By Lemma 8(ii) and the assumption of our induction, it suffices to show that $t_{n+1} = 0$. On the contrary, assume that $t_{n+1} \neq 0$. Then, it is obvious that

$$t_1 \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + t_n \frac{\mathbf{a}_n}{\|\mathbf{a}_n\|} \neq \mathbf{0}. \tag{33}$$

Take a positive number M satisfying that

$$\begin{aligned}\frac{t_1}{M} \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + \frac{t_n}{M} \frac{\mathbf{a}_n}{\|\mathbf{a}_n\|} &\in \mathbb{V}_1, \\ \frac{t_{n+1}}{M} \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|} &\in \mathbb{V}_1.\end{aligned} \tag{34}$$

Thus we have

$$\|\mathbf{a}\| \frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{t_{n+1}}{M} \frac{\mathbf{a}_{n+1}}{\|\mathbf{a}_{n+1}\|} = \mathbf{0}, \tag{35}$$

where we put

$$\mathbf{a} = \frac{t_1}{M} \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + \frac{t_n}{M} \frac{\mathbf{a}_n}{\|\mathbf{a}_n\|}. \tag{36}$$

From [8, Theorem 2.1], we can rewrite (35) in the form of

$$r \otimes \mathbf{a} \oplus r_{n+1} \otimes \mathbf{a}_{n+1} = \mathbf{0}. \tag{37}$$

We can also rewrite $r \otimes \mathbf{a}$ in the form of

$$r \otimes \mathbf{a} = r_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus r_n \otimes \mathbf{a}_n \tag{38}$$

by using [8, Theorem 3.3], so we obtain the following equation:

$$(r_1 \otimes \mathbf{a}_1 \oplus \cdots \oplus r_n \otimes \mathbf{a}_n) \oplus r_{n+1} \otimes \mathbf{a}_{n+1} = \mathbf{0}. \tag{39}$$

Since $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\}$ is assumed to be gyrolinearly independent, we can conclude that $r_1 = \cdots = r_{n+1} = 0$, which implies that $t_{n+1} = 0$. This is a contradiction and completes the proof. □

Although the contents in the rest of this section are actually known and used repeatedly in [4], we give their proofs for the convenience of readers.

Lemma 12 (see also [10, Proposition 2.3]).

$$\|\mathbf{a} \oplus \mathbf{b}\|^2 = \frac{\|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2}{1 + (2/s^2)\,\mathbf{a} \cdot \mathbf{b} + (1/s^4)\,\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2} \tag{40}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$.

Proof. By using the definition of $\oplus$, one can easily calculate the inner product of $\mathbf{a} \oplus \mathbf{b}$ with itself to obtain

$$\begin{aligned} \|\mathbf{a} \oplus \mathbf{b}\|^2 &= \left(\frac{1}{1 + (2/s^2)\,\mathbf{a} \cdot \mathbf{b} + (1/s^4)\,\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2}\right)^2 \\ &\cdot \left\{\left(1 + \frac{2}{s^2}\mathbf{a} \cdot \mathbf{b} + \frac{1}{s^2}\|\mathbf{b}\|^2\right)^2 \|\mathbf{a}\|^2 \right. \\ &+ 2\left(1 + \frac{2}{s^2}\mathbf{a} \cdot \mathbf{b} + \frac{1}{s^2}\|\mathbf{b}\|^2\right)\left(1 - \frac{1}{s^2}\|\mathbf{a}\|^2\right)\mathbf{a} \cdot \mathbf{b} \\ &\left. + \left(1 - \frac{1}{s^2}\|\mathbf{a}\|^2\right)^2 \|\mathbf{b}\|^2\right\}. \end{aligned} \tag{41}$$

If we put $\mathbf{u} = \mathbf{a}/s$, $\mathbf{v} = \mathbf{b}/s$, then it is easy to factorize the second factor as

$$\begin{aligned} &\left(1 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2\right)^2 \|s\mathbf{u}\|^2 + 2\left(1 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2\right) \\ &\quad \cdot \left(1 - \|\mathbf{u}\|^2\right) s^2 \mathbf{u} \cdot \mathbf{v} + \left(1 - \|\mathbf{u}\|^2\right)^2 \|s\mathbf{v}\|^2 \\ &\quad = s^2\left(\|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2\right)\left(1 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\right) \\ &\quad = \left(\|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2\right) \\ &\quad \cdot \left(1 + \frac{2}{s^2}\mathbf{a} \cdot \mathbf{b} + \frac{1}{s^4}\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\right); \end{aligned} \tag{42}$$

hence we can conclude identity (40). □

Definition 13 (see [4, Definition 2.7, (2.1)]). Recall that the inverse element of $\mathbf{a}$ is denoted by $\ominus\mathbf{a}$ in a gyrogroup, and one uses the notation

$$\mathbf{a} \ominus \mathbf{b} = \mathbf{a} \oplus (\ominus\mathbf{b}) \tag{43}$$

as in group theory.

Lemma 14. *The following formulae hold:*

(i) $(\mathbf{a}/s) \oplus (\mathbf{b}/s) = (\mathbf{a} \oplus \mathbf{b})/s$.

(ii) $0 < 1 - (2/s^2)\mathbf{a} \cdot \mathbf{b} + (1/s^4)\|\mathbf{a}\|^2\|\mathbf{b}\|^2 < 2^2$.

(iii) $\|\mathbf{a} - \mathbf{b}\| \le 2\|\mathbf{a} \ominus \mathbf{b}\|$,

for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$, *where* $\oplus$ *in the left-hand side of (i) is in the space* $\mathbb{V}_1$.

Proof. (i) It immediately follows from the definition of $\oplus$.

(ii) By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 0 &< \left(1 - \frac{1}{s^2}\|\mathbf{a}\|\,\|\mathbf{b}\|\right)^2 \le 1 - \frac{2}{s^2}\mathbf{a} \cdot \mathbf{b} + \frac{1}{s^4}\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 \\ &\le 1 + \frac{2}{s^2}\|\mathbf{a}\|\,\|\mathbf{b}\| + \frac{1}{s^4}\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 \\ &= \left(1 + \frac{1}{s^2}\|\mathbf{a}\|\,\|\mathbf{b}\|\right)^2 < 2^2. \end{aligned} \tag{44}$$

(iii) From (ii) just established, identity (40) in Lemma 12, and the fact that $\ominus\mathbf{a} = -\mathbf{a}$ in $\mathbb{V}_s$, we have

$$\begin{aligned} \frac{\|\mathbf{a} - \mathbf{b}\|^2}{2^2} &\le \frac{\|\mathbf{a}\|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2}{1 - (2/s^2)\,\mathbf{a} \cdot \mathbf{b} + (1/s^4)\,\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2} \\ &= \|\mathbf{a} \ominus \mathbf{b}\|^2. \end{aligned} \tag{45}$$

This completes the proof. □

Lemma 15 (see [4, Theorem 8.33]). *Let* $\{\mathbf{c}_j\}_{j=1}^n$ *be an orthogonal set in* $\mathbb{V}_s$. *Then, for any permutation* $(i_1, \ldots, i_n)$ *of the numbers* $\{1, \ldots, n\}$ *and any order of gyroaddition for* $\mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_n}$, *the following equality holds:*

$$\frac{\left\|\mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_n}\right\|^2}{s} = \frac{\left\|\mathbf{c}_{i_1}\right\|^2}{s} \oplus \cdots \oplus \frac{\left\|\mathbf{c}_{i_n}\right\|^2}{s}. \tag{46}$$

Here, $\oplus$ *in the right-hand side are in the space* $\|\mathbb{V}_s\|$.

Proof. The previous lemma (i) shows that

$$\left\|\frac{\mathbf{a}}{s} \oplus \frac{\mathbf{b}}{s}\right\|^2 = \left\|\frac{\mathbf{a} \oplus \mathbf{b}}{s}\right\|^2 = \frac{\|\mathbf{a} \oplus \mathbf{b}\|^2}{s^2}. \tag{47}$$

for any $\mathbf{a}, \mathbf{b}$ in $\mathbb{V}_s$. On the other hand, if $\{\mathbf{a}, \mathbf{b}\}$ is orthogonal, then it follows from identity (40) in Lemma 12 that

$$\begin{aligned} \left\|\frac{\mathbf{a}}{s} \oplus \frac{\mathbf{b}}{s}\right\|^2 &= \frac{\|\mathbf{a}/s\|^2 + \|\mathbf{b}/s\|^2}{1 + \|\mathbf{a}/s\|^2\,\|\mathbf{b}/s\|^2} \\ &= \frac{1}{s} \cdot \frac{\|\mathbf{a}\|^2/s + \|\mathbf{b}\|^2/s}{1 + (1/s^2)\left(\|\mathbf{a}\|^2/s\right)\left(\|\mathbf{b}\|^2/s\right)} \\ &= \frac{1}{s}\left(\frac{\|\mathbf{a}\|^2}{s} \oplus \frac{\|\mathbf{b}\|^2}{s}\right). \end{aligned} \tag{48}$$

Thus the theorem holds for $n = 2$.

Assume that the theorem is valid up to n. Let $\{\mathbf{c}_j\}_{j=1}^{n+1}$ be an orthogonal set in $\mathbb{V}_s$ and let the following equation

$$\mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_{n+1}} = \left(\mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_m}\right) \oplus \left(\mathbf{c}_{i_{m+1}} \oplus \cdots \oplus \mathbf{c}_{i_{n+1}}\right) \tag{49}$$

show the latest gyroaddition $\oplus$. If we put

$$\mathbf{a} = \mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_m}, \quad \mathbf{b} = \mathbf{c}_{i_{m+1}} \oplus \cdots \oplus \mathbf{c}_{i_{n+1}}, \tag{50}$$

then $\{\mathbf{a}, \mathbf{b}\}$ is orthogonal. From the case of $n = 2$, it follows that

$$\frac{\left\|\mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_{n+1}}\right\|^2}{s} = \frac{\|\mathbf{a} \oplus \mathbf{b}\|^2}{s} = \frac{\|\mathbf{a}\|^2}{s} \oplus \frac{\|\mathbf{b}\|^2}{s} = \frac{\left\|\mathbf{c}_{i_1} \oplus \cdots \oplus \mathbf{c}_{i_m}\right\|^2}{s} \oplus \frac{\left\|\mathbf{c}_{i_{m+1}} \oplus \cdots \oplus \mathbf{c}_{i_{n+1}}\right\|^2}{s}. \tag{51}$$

Due to the assumption of our induction, we can conclude that

$$= \left(\frac{\left\|\mathbf{c}_{i_1}\right\|^2}{s} \oplus \cdots \oplus \frac{\left\|\mathbf{c}_{i_m}\right\|^2}{s}\right) \oplus \left(\frac{\left\|\mathbf{c}_{i_{m+1}}\right\|^2}{s} \oplus \cdots \oplus \frac{\left\|\mathbf{c}_{i_{n+1}}\right\|^2}{s}\right). \tag{52}$$

This completes the proof. □

4. The Poincaré Metric and Orthogonal Gyroexpansion in the Möbius Gyrovector Space

In this section, we give a notion of orthogonal gyroexpansions with respect to a complete orthogonal sequence in the Möbius gyrovector space, which is fully analogous to the notion of the orthogonal expansions with respect to a complete orthonormal sequence in a Hilbert space. It is an application of the orthogonal gyrodecomposition which was established in [8, Theorem 4.2], and we present an explicit procedure to obtain the orthogonal gyroexpansions in the Möbius gyrovector space.

Definition 16 (see [4, Definition 6.8, 6.17 (6.286) and (6.293)]). The Möbius gyrodistance function d on a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is defined by the equation

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} \ominus \mathbf{a}\|. \tag{53}$$

Moreover, the Poincaré distance function h on the ball $\mathbb{V}_s$ is introduced by the equation

$$h(\mathbf{a}, \mathbf{b}) = \tanh^{-1} \frac{d(\mathbf{a}, \mathbf{b})}{s} \tag{54}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$. Then h satisfies the triangle inequality [4, (6.294)], so that $(\mathbb{V}_s, h)$ is a metric space.

Remark 17. As pointed out in [4, 6.17, p.217], the distance function $h(\mathbf{a}, \mathbf{b})$ is the obvious generalization into the ball $\mathbb{V}_s$ of the well-known Poincaré distance function on the disc $\mathbb{D}$. There are a number of literatures dealing with relationship between hyperbolic geometry and Hilbert spaces. In particular, Goebel and Reich [11] introduced the hyperbolic metric ρ on the open unit ball of a complex Hilbert space, from a viewpoint of holomorphic function theory. They developed the study of the Hilbert ball, which leads to research on ρ-convexity, nonexpansive mappings, fixed point theorems, and so forth, and [11] is cited in many bibliography such as [12]. The definition of ρ is equivalent to

$$\rho(x, y) = \tanh^{-1}(1 - \sigma(x, y))^{1/2}, \tag{55}$$

where

$$\sigma(x, y) = \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{\left|1 - \langle x, y\rangle\right|^2} \tag{56}$$

for any elements x, y in the Hilbert ball. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then it is easy to see that h and ρ coincide with the Poincaré metric on $\mathbb{D}$. In general, however, h and ρ do not coincide for higher dimensional spaces. We clarify the relationship between h and ρ below.

Lemma 18. *Let $\mathbb{V}$ be a real inner product space. Then the norm of the Einstein addition of two elements is given by the equation*

$$\|\mathbf{a}\oplus_{\mathrm{E}}\mathbf{b}\|^2 = \frac{1}{\left(1 + (\mathbf{a}\cdot\mathbf{b})/s^2\right)^2}\left\{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}\cdot\mathbf{b} - \frac{1}{s^2}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 + \frac{1}{s^2}(\mathbf{a}\cdot\mathbf{b})^2\right\} \tag{57}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$.

Proof. At first, consider the case $s = 1$. From the definition of $\oplus_{\mathrm{E}}$, it is easy to calculate the inner product of $\mathbf{a}\oplus_{\mathrm{E}}\mathbf{b}$ with itself as follows:

$$\begin{aligned}(1 + \mathbf{a}\cdot\mathbf{b})^2\|\mathbf{a}\oplus_{\mathrm{E}}\mathbf{b}\|^2 &= \left\langle \mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}}\mathbf{b} + \frac{\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}, \mathbf{a} + \frac{1}{\gamma_{\mathbf{a}}}\mathbf{b} + \frac{\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}\right\rangle = \|\mathbf{a}\|^2 + \frac{1}{\gamma_{\mathbf{a}}^2}\|\mathbf{b}\|^2 \\ &\quad + \left(\frac{\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}\right)^2(\mathbf{a}\cdot\mathbf{b})^2\|\mathbf{a}\|^2 + \frac{2}{\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b}) + \frac{2\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})\|\mathbf{a}\|^2 + \frac{2}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2 + \left(1 - \|\mathbf{a}\|^2\right)\|\mathbf{b}\|^2 \\ &\quad + \left(\frac{1/\sqrt{1-\|\mathbf{a}\|^2}}{1 + 1/\sqrt{1-\|\mathbf{a}\|^2}}\right)^2(\mathbf{a}\cdot\mathbf{b})^2\|\mathbf{a}\|^2\end{aligned}$$

$$+2\sqrt{1-\|\mathbf{a}\|^2}(\mathbf{a}\cdot\mathbf{b})+\frac{2\cdot\left(1/\sqrt{1-\|\mathbf{a}\|^2}\right)}{1+1/\sqrt{1-\|\mathbf{a}\|^2}}(\mathbf{a}\cdot\mathbf{b})\cdot\|\mathbf{a}\|^2+\frac{2}{1+1/\sqrt{1-\|\mathbf{a}\|^2}}(\mathbf{a}\cdot\mathbf{b})^2=\|\mathbf{a}\|^2+\left(1-\|\mathbf{a}\|^2\right)\|\mathbf{b}\|^2+\left(\frac{1}{\sqrt{1-\|\mathbf{a}\|^2}+1}\right)^2(\mathbf{a}\cdot\mathbf{b})^2\|\mathbf{a}\|^2+2\sqrt{1-\|\mathbf{a}\|^2}(\mathbf{a}\cdot\mathbf{b})+\frac{2}{\sqrt{1-\|\mathbf{a}\|^2}+1}(\mathbf{a}\cdot\mathbf{b})\|\mathbf{a}\|^2+\frac{2\sqrt{1-\|\mathbf{a}\|^2}}{\sqrt{1-\|\mathbf{a}\|^2}+1}(\mathbf{a}\cdot\mathbf{b})^2=\|\mathbf{a}\|^2+\left(1-\|\mathbf{a}\|^2\right)\|\mathbf{b}\|^2+\left(\frac{1-\sqrt{1-\|\mathbf{a}\|^2}}{\|\mathbf{a}\|^2}\right)^2(\mathbf{a}\cdot\mathbf{b})^2\|\mathbf{a}\|^2+2\sqrt{1-\|\mathbf{a}\|^2}(\mathbf{a}\cdot\mathbf{b})+\frac{2\left(1-\sqrt{1-\|\mathbf{a}\|^2}\right)}{\|\mathbf{a}\|^2}(\mathbf{a}\cdot\mathbf{b})\cdot\|\mathbf{a}\|^2+\frac{2\sqrt{1-\|\mathbf{a}\|^2}\left(1-\sqrt{1-\|\mathbf{a}\|^2}\right)}{\|\mathbf{a}\|^2}(\mathbf{a}\cdot\mathbf{b})^2=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2+2\mathbf{a}\cdot\mathbf{b}-\|\mathbf{a}\|^2\|\mathbf{b}\|^2+(\mathbf{a}\cdot\mathbf{b})^2. \tag{58}$$

Thus the lemma holds for $s=1$. For general $s>0$, let $\mathbf{a},\mathbf{b}\in\mathbb{V}_s$. If we put $\mathbf{u}=\mathbf{a}/s$ and $\mathbf{v}=\mathbf{b}/s$, then it is immediate to see that

$$\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}=\frac{\mathbf{a}\oplus_{\mathrm{E}}\mathbf{b}}{s}, \tag{59}$$

and we can easily deduce identity (57) by applying the case $s=1$ to $\mathbf{u},\mathbf{v}$. This completes the proof. □

Theorem 19. *Let one use the notations ρ and σ in $\mathbb{V}_s$ by*

$$\rho(\mathbf{a},\mathbf{b})=\tanh^{-1}(1-\sigma(\mathbf{a},\mathbf{b}))^{1/2}, \tag{60}$$

where

$$\sigma(\mathbf{a},\mathbf{b})=\frac{\left(s^2-\|\mathbf{a}\|^2\right)\left(s^2-\|\mathbf{b}\|^2\right)}{\left(s^2-\mathbf{a}\cdot\mathbf{b}\right)^2} \tag{61}$$

for $\mathbf{a},\mathbf{b}\in\mathbb{V}_s$. Then the following identities hold:

(i) $\|\mathbf{a}\ominus_{\mathrm{E}}\mathbf{b}\|/s=(1-\sigma(\mathbf{a},\mathbf{b}))^{1/2}$.

(ii) $h_{\mathrm{E}}(\mathbf{a},\mathbf{b})=\tanh^{-1}(\|\mathbf{a}\ominus_{\mathrm{E}}\mathbf{b}\|/s)=\rho(\mathbf{a},\mathbf{b})$.

(iii) $2h(\mathbf{a},\mathbf{b})=2\tanh^{-1}(\|\mathbf{a}\ominus\mathbf{b}\|/s)=\rho(2\otimes\mathbf{a},2\otimes\mathbf{b})$.

Proof. (i) and (ii) immediately follow from the previous lemma. (iii) It is not difficult to see that we may assume $s=1$. By (ii) just established, it suffices to show that

$$2\otimes\|\mathbf{a}\ominus\mathbf{b}\|=\left\|2\otimes\mathbf{a}\ominus_{\mathrm{E}}2\otimes\mathbf{b}\right\|. \tag{62}$$

Note that

$$2\otimes a=\frac{(1+a)^2-(1-a)^2}{(1+a)^2+(1-a)^2}=\frac{2a}{1+a^2} \tag{63}$$

for real number $0\le a<1$. For any $\mathbf{a},\mathbf{b}\in\mathbb{V}_1$, if we put $a=\|\mathbf{a}\|$, $b=\|\mathbf{b}\|$, and $\alpha=(\mathbf{a}/\|\mathbf{a}\|)\cdot(\mathbf{b}/\|\mathbf{b}\|)$, then, by the definition or the axioms of gyrovector spaces, we have

$$\begin{gathered}\|2\otimes\mathbf{a}\|=2\otimes\|\mathbf{a}\|=\frac{2a}{1+a^2},\\(2\otimes\mathbf{a})\cdot(2\otimes\mathbf{b})=\frac{2a}{1+a^2}\cdot\frac{2b}{1+b^2}\cdot\alpha.\end{gathered} \tag{64}$$

By identity (40) in Lemma 12,

$$\begin{aligned}2\otimes\|\mathbf{a}\ominus\mathbf{b}\|&=2\otimes\sqrt{\frac{\|\mathbf{a}\|^2-2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2}{1-2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{a}\|^2\|\mathbf{b}\|^2}}\\&=\frac{2\sqrt{\left(a^2-2ab\alpha+b^2\right)/\left(1-2ab\alpha+a^2b^2\right)}}{1+\left(a^2-2ab\alpha+b^2\right)/\left(1-2ab\alpha+a^2b^2\right)}\\&=\frac{2\sqrt{\left(a^2-2ab\alpha+b^2\right)\left(1-2ab\alpha+a^2b^2\right)}}{\left(1-2ab\alpha+a^2b^2\right)+\left(a^2-2ab\alpha+b^2\right)}.\end{aligned} \tag{65}$$

On the other hand, identity (57) in the previous lemma shows that

$$\begin{aligned}\|2\otimes\mathbf{a}\ominus_{\mathrm{E}}2\otimes\mathbf{b}\|&=\sqrt{\frac{\|2\otimes\mathbf{a}\|^2+\|2\otimes\mathbf{b}\|^2-2(2\otimes\mathbf{a})\cdot(2\otimes\mathbf{b})-\|2\otimes\mathbf{a}\|^2\|2\otimes\mathbf{b}\|^2+\{(2\otimes\mathbf{a})\cdot(2\otimes\mathbf{b})\}^2}{\{1-(2\otimes\mathbf{a})\cdot(2\otimes\mathbf{b})\}^2}}\\&=\frac{\sqrt{\left(2a/\left(1+a^2\right)\right)^2+\left(2b/\left(1+b^2\right)\right)^2-2\cdot\left(2a/\left(1+a^2\right)\right)\cdot\left(2b/\left(1+b^2\right)\right)\cdot\alpha-\left(2a/\left(1+a^2\right)\right)^2\left(2b/\left(1+b^2\right)\right)^2+\left\{\left(2a/\left(1+a^2\right)\right)\cdot\left(2b/\left(1+b^2\right)\right)\cdot\alpha\right\}^2}}{1-\left(2a/\left(1+a^2\right)\right)\cdot\left(2b/\left(1+b^2\right)\right)\cdot\alpha}\\&=\frac{2\sqrt{\left(a^2-2ab\alpha+b^2\right)\left(1-2ab\alpha+a^2b^2\right)}}{\left(1+a^2\right)\left(1+b^2\right)-4ab\alpha}.\end{aligned} \tag{66}$$

This completes the proof. □

In the rest of the paper, we should concentrate to investigate the Möbius ball endowed with the Poincaré metric h introduced by Ungar. We can perform gyrolinear algebraic operations which behave quite well for orthogonal sequences in the Möbius gyrovector spaces, like as linear algebraic ones in Hilbert spaces.

Lemma 20. *For any sequence* $\{\mathbf{a}_n\}_{n=1}^{\infty}$ *and any element* $\mathbf{a}$ *in* $\mathbb{V}_s$,

(i) $h(\mathbf{a}_n, \mathbf{a}) \to 0\ (n \to \infty) \Leftrightarrow d(\mathbf{a}_n, \mathbf{a}) \to 0\ (n \to \infty)$,

(ii) $h(\mathbf{a}_n, \mathbf{a}_m) \to 0\ (n, m \to \infty) \Leftrightarrow d(\mathbf{a}_n, \mathbf{a}_m) \to 0\ (n, m \to \infty)$.

Proof. It is obvious, because both tanh and $\tanh^{-1}$ are uniformly continuous on a neighborhood of 0. □

Lemma 21. *For any fixed* $\mathbf{a} \in \mathbb{V}_s$, *the map* $\mathbb{V}_s \ni \mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{a} \in (-s^2, s^2)$ *is continuous, where one considers the metric h on both sets.*

Proof. Suppose that $h(\mathbf{x}_n, \mathbf{x}) \to 0$. Then, from the previous lemma and Lemma 14(iii), it follows that $\|\mathbf{x}_n - \mathbf{x}\| \le 2\|\mathbf{x}_n \ominus \mathbf{x}\| \to 0$. Therefore, we have

$$\left|\mathbf{x}_n \cdot \mathbf{a} \ominus \mathbf{x} \cdot \mathbf{a}\right| = \left|\frac{\mathbf{x}_n \cdot \mathbf{a} - \mathbf{x} \cdot \mathbf{a}}{1 - (1/s^4)(\mathbf{x}_n \cdot \mathbf{a})(\mathbf{x} \cdot \mathbf{a})}\right| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{67}$$

This implies that $h(\mathbf{x}_n \cdot \mathbf{a}, \mathbf{x} \cdot \mathbf{a}) \to 0$. □

We should make sure of two definitions here. One of them is quite usual; another is very natural.

For any nonempty subset A of $\mathbb{V}_s$, we denote $A^{\perp}$ as the orthogonal complement of A in $\mathbb{V}$; that is,

$$A^{\perp} = \{\mathbf{x} \in \mathbb{V}; \mathbf{x} \cdot \mathbf{a} = 0\ \forall \mathbf{a} \in A\}. \tag{68}$$

A nonempty subset M of $\mathbb{V}_s$ is a gyrovector subspace if M is closed under gyroaddition and gyroscalar multiplication; that is, $\mathbf{a}, \mathbf{b} \in M$ and $r \in \mathbb{R}$ imply that $\mathbf{a} \oplus \mathbf{b} \in M$ and $r \otimes \mathbf{a} \in M$.

Lemma 22. $A^{\perp} \cap \mathbb{V}_s$ *is an h-closed gyrovector subspace.*

Proof. From the definitions of $\oplus$ and $\otimes$, it is immediate to see that $A^{\perp} \cap \mathbb{V}_s$ forms a gyrovector subspace. Moreover, $A^{\perp} \cap \mathbb{V}_s$ is obviously h-closed by the previous lemma. □

Lemma 23. *If* $\mathbf{a}_n, \mathbf{a} \in \mathbb{V}_s$ *and* $\|\mathbf{a}_n - \mathbf{a}\| \to 0$, *then* $h(\mathbf{a}_n, \mathbf{a}) \to 0$.

Proof. It suffices to show that $d(\mathbf{a}_n, \mathbf{a}) \to 0$. By the assumption $\|\mathbf{a}\| < s$, we can obtain

$$\begin{aligned} d(\mathbf{a}_n, \mathbf{a})^2 &= \|\mathbf{a}_n \ominus \mathbf{a}\|^2 \\ &= \frac{\|\mathbf{a}_n\|^2 - 2\mathbf{a}_n \cdot \mathbf{a} + \|\mathbf{a}\|^2}{1 - (2/s^2)\,\mathbf{a}_n \cdot \mathbf{a} + (1/s^4)\|\mathbf{a}_n\|^2 \|\mathbf{a}\|^2} \\ &\longrightarrow 0 \quad (n \longrightarrow \infty), \end{aligned} \tag{69}$$

where we used identity (40) in Lemma 12. □

Proposition 24. *Let M be a gyrovector subspace of* $\mathbb{V}_s$. *Then the closure* $\overline{M}^h$ *with respect to the metric h is also a gyrovector subspace.*

Proof. Suppose that $\mathbf{a}, \mathbf{b} \in \overline{M}^h$. There exist sequences $\{\mathbf{a}_n\}, \{\mathbf{b}_n\} \subset M$ such that $h(\mathbf{a}_n, \mathbf{a}), h(\mathbf{b}_n, \mathbf{b}) \to 0$. By Lemmas 20 and 14(iii), we have $\|\mathbf{a}_n - \mathbf{a}\|, \|\mathbf{b}_n - \mathbf{b}\| \to 0$. From the definitions of $\oplus, \otimes$, it is easy to see that $\|(\mathbf{a}_n \oplus \mathbf{b}_n) - (\mathbf{a} \oplus \mathbf{b})\| \to 0$ and $\|r \otimes \mathbf{a}_n - r \otimes \mathbf{a}\| \to 0$. By Lemma 23, it follows that $h(\mathbf{a}_n \oplus \mathbf{b}_n, \mathbf{a} \oplus \mathbf{b}) \to 0$ and $h(r \otimes \mathbf{a}_n, r \otimes \mathbf{a}) \to 0$. Since $\mathbf{a}_n \oplus \mathbf{b}_n, r \otimes \mathbf{a}_n \in M$, we can conclude that $\mathbf{a} \oplus \mathbf{b}, r \otimes \mathbf{a} \in \overline{M}^h$. This completes the proof. □

Lemma 25. *Any finitely generated gyrovector subspace is h-closed.*

Proof. Let M be a gyrovector subspace generated by nonzero elements $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ in $\mathbb{V}_s$. For an arbitrary element $\mathbf{x} \in \overline{M}^h$, there exists a sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subset M$ such that $h(\mathbf{x}_k, \mathbf{x}) \to 0$. Then, from Lemmas 20(i) and 14(iii), it follows that $\|\mathbf{x}_k - \mathbf{x}\| \to 0$. By [8, Theorem 3.3], we have

$$M = \left\{t_1 \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} + \cdots + t_n \frac{\mathbf{a}_n}{\|\mathbf{a}_n\|}; t_1, \ldots, t_n \in \mathbb{R}\right\} \cap \mathbb{V}_s. \tag{70}$$

Since $\{t_1(\mathbf{a}_1/\|\mathbf{a}_1\|) + \cdots + t_n(\mathbf{a}_n/\|\mathbf{a}_n\|);\ t_1, \ldots, t_n \in \mathbb{R}\}$ is a finite dimensional linear subspace, it is closed with respect to the norm topology. Therefore $\mathbf{x} \in M$. This completes the proof. □

From now on, we assume that the carrier $\mathbb{V}$ of the Möbius gyrovector space $\mathbb{V}_s$ is complete as a metric space with respect to the norm induced by the inner product. Thus, $\mathbb{V}$ is a real Hilbert space.

Theorem 26. *Let* $\mathbb{V}$ *be a real Hilbert space. Then* $(\mathbb{V}_s, h)$ *is a complete metric space.*

Although this fact is well-known and it can be deduced by existing results and Theorem 19, we give a direct proof here in order to show how gyrovector space approach is fully analogous to vector space approach.

Proof. Without loss of generality, we may assume that $s = 1$. Suppose that $\{\mathbf{a}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{V}_1, h)$. From Lemmas 14(iii) and 20(ii), it follows that

$$\|\mathbf{a}_n - \mathbf{a}_m\| \le 2\|\mathbf{a}_n \ominus \mathbf{a}_m\| = 2d(\mathbf{a}_n, \mathbf{a}_m) \longrightarrow 0 \quad (n, m \longrightarrow \infty), \tag{71}$$

which implies that $\{\mathbf{a}_n\}$ is a Cauchy sequence with respect to the norm of $\mathbb{V}$. Hence there exists a unique element $\mathbf{a} \in \mathbb{V}$ such that $\|\mathbf{a}\| \le 1$, $\|\mathbf{a}_n - \mathbf{a}\| \to 0$. In order to show that $\|\mathbf{a}\| < 1$, on the contrary, we assume that $\|\mathbf{a}\| = 1$. By the assumption that $\{\mathbf{a}_n\}$ is a Cauchy sequence in $(\mathbb{V}_1, h)$, there exists a natural number m_0 such that

$$d(\mathbf{a}_{m+p}, \mathbf{a}_m)^2 < \frac{1}{2} \tag{72}$$

for any $m \geq m_0$ and any p. On the other hand, from identity (40) in Lemma 12, we have

$$d\left(\mathbf{a}_{m+p}, \mathbf{a}_m\right)^2 = \frac{\left\|\mathbf{a}_{m+p}\right\|^2 - 2\mathbf{a}_{m+p} \cdot \mathbf{a}_m + \left\|\mathbf{a}_m\right\|^2}{1 - 2\mathbf{a}_{m+p} \cdot \mathbf{a}_m + \left\|\mathbf{a}_{m+p}\right\|^2 \left\|\mathbf{a}_m\right\|^2}. \tag{73}$$

Now we fix $m \geq m_0$ and let $p \to \infty$. Then, from the fact that $\mathbf{a}_{m+p} \to \mathbf{a}$ and the assumption $\|\mathbf{a}\| = 1$, we can obtain

$$\frac{1}{2} \geq \frac{\|\mathbf{a}\|^2 - 2\mathbf{a} \cdot \mathbf{a}_m + \left\|\mathbf{a}_m\right\|^2}{1 - 2\mathbf{a} \cdot \mathbf{a}_m + \|\mathbf{a}\|^2 \left\|\mathbf{a}_m\right\|^2} = 1, \tag{74}$$

which is a contradiction. This implies that $\|\mathbf{a}\| < 1$. By Lemma 23, the proof is complete. □

Theorem 27. *Let $\mathbb{V}$ be a real Hilbert space. If A is a closed subset in $(\mathbb{V}_s, h)$, then A is relatively closed in $(\mathbb{V}_s, \|\cdot\|)$. Therefore, the orthogonal gyrodecomposition is applicable to h-closed gyrovector subspaces in the sense of [8, Theorem 4.2].*

Proof. Denote by $\overline{A}$ the closure of A with respect to the norm topology. It suffices to show that $A = \overline{A} \cap \mathbb{V}_s$. One of the inclusions ($\subset$) is trivial. If $\mathbf{x} \in \overline{A} \cap \mathbb{V}_s$, then there exists a sequence $\{\mathbf{x}_n\} \subset A$ such that $\|\mathbf{x}_n - \mathbf{x}\| \to 0$. So we can apply Lemma 23 to obtain that $h(\mathbf{x}_n, \mathbf{x}) \to 0$. Hence $\mathbf{x} \in A$ and this completes the proof. □

Theorem 28. *Let M be a gyrovector subspace of $\mathbb{V}_s$ and $\mathbf{x} \in \mathbb{V}_s$. Then one has*

$$\mathbf{x} \in M^{\perp} \Longleftrightarrow \|\mathbf{x} \ominus \mathbf{m}\| \geq \|\mathbf{x}\| \quad (\mathbf{m} \in M). \tag{75}$$

Proof. ($\Rightarrow$) Suppose that $\mathbf{x} \in M^{\perp} \cap \mathbb{V}_s, \mathbf{m} \in M$. By identity (40) in Lemma 12, we obtain

$$\|\mathbf{x} \ominus \mathbf{m}\|^2 = \frac{\|\mathbf{x}\|^2 + \|\mathbf{m}\|^2}{1 + \left(1/s^4\right)\|\mathbf{x}\|^2 \|\mathbf{m}\|^2} \geq \|\mathbf{x}\|^2. \tag{76}$$

($\Leftarrow$) Suppose that $\mathbf{m} \in M$. For an arbitrary positive real number t, take

$$r = t\frac{|\mathbf{x} \cdot \mathbf{m}|}{\mathbf{x} \cdot \mathbf{m}}. \tag{77}$$

Since $r \otimes \mathbf{m} \in M$, it follows from identity (40) in Lemma 12 that

$$\begin{aligned} \|\mathbf{x}\|^2 &\leq \|\mathbf{x} \ominus (r \otimes \mathbf{m})\|^2 \\ &= \frac{\|\mathbf{x}\|^2 - 2\mathbf{x} \cdot (r \otimes \mathbf{m}) + \|r \otimes \mathbf{m}\|^2}{1 - \left(2/s^2\right)\mathbf{x} \cdot (r \otimes \mathbf{m}) + \left(1/s^4\right)\|\mathbf{x}\|^2 \|r \otimes \mathbf{m}\|^2}. \end{aligned} \tag{78}$$

By the axiom (V7), we have $\|r \otimes \mathbf{m}\| = |r| \otimes \|\mathbf{m}\| = t \otimes \|\mathbf{m}\|$ and the inequality

$$\begin{aligned} &\|\mathbf{x}\|^2 \left\{1 - \frac{2}{s^2}\mathbf{x} \cdot (r \otimes \mathbf{m}) + \frac{1}{s^4}\|\mathbf{x}\|^2 (t \otimes \|\mathbf{m}\|)^2\right\} \\ &\quad \leq \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot (r \otimes \mathbf{m}) + (t \otimes \|\mathbf{m}\|)^2, \end{aligned} \tag{79}$$

which yields the following inequality:

$$2\mathbf{x} \cdot (r \otimes \mathbf{m}) \leq \left(1 + \frac{1}{s^2}\|\mathbf{x}\|^2\right)(t \otimes \|\mathbf{m}\|)^2; \tag{80}$$

namely,

$$\begin{aligned} &2s \tanh\left(r\tanh^{-1}\frac{\|\mathbf{m}\|}{s}\right)\frac{\mathbf{x} \cdot \mathbf{m}}{\|\mathbf{m}\|} \\ &\quad \leq \left(1 + \frac{1}{s^2}\|\mathbf{x}\|^2\right)\left(s \tanh\left(t\tanh^{-1}\frac{\|\mathbf{m}\|}{s}\right)\right)^2. \end{aligned} \tag{81}$$

Note that $\mathbf{x} \cdot \mathbf{m}$ and r have the same signature, so we have

$$\begin{aligned} &\tanh\left(r \tanh^{-1}\frac{\|\mathbf{m}\|}{s}\right)\mathbf{x} \cdot \mathbf{m} \\ &\quad = \tanh\left(t \tanh^{-1}\frac{\|\mathbf{m}\|}{s}\right)|\mathbf{x} \cdot \mathbf{m}|. \end{aligned} \tag{82}$$

Therefore, we can obtain the inequality

$$2\frac{|\mathbf{x} \cdot \mathbf{m}|}{\|\mathbf{m}\|} \leq \left(1 + \frac{1}{s^2}\|\mathbf{x}\|^2\right)s \tanh\left(t \tanh^{-1}\frac{\|\mathbf{m}\|}{s}\right). \tag{83}$$

Since $t > 0$ is arbitrary, we can let $t \to +0$ and conclude that $\mathbf{x} \cdot \mathbf{m} = 0$. This completes the proof. □

Lemma 29. *In a gyrocommutative gyrogroup, one has*

$$\mathbf{a} \ominus (\mathbf{b} \oplus \mathbf{c}) = \operatorname{gyr}[\mathbf{a}, \ominus\mathbf{b}]\{(\ominus\mathbf{b} \oplus \mathbf{a}) \ominus \mathbf{c}\}. \tag{84}$$

This lemma can be obtained if we put a, b and c as $\ominus\mathbf{b}, \mathbf{a}$ and $\ominus\mathbf{c}$, respectively, in [4, Theorem 3.9]. However, we give a proof for the convenience of readers by using gyroautomorphic inverse property $\ominus(\mathbf{a} \oplus \mathbf{b}) = \ominus\mathbf{a} \ominus \mathbf{b}$, left gyroassociative law (G3), gyrocommutativity (G6), and gyroautomorphism (G4).

Proof.

$$\begin{aligned} \mathbf{a} \ominus (\mathbf{b} \oplus \mathbf{c}) &= \mathbf{a} \oplus (\ominus\mathbf{b} \ominus \mathbf{c}) \\ &= (\mathbf{a} \oplus (\ominus\mathbf{b})) \oplus \operatorname{gyr}[\mathbf{a}, \ominus\mathbf{b}](\ominus\mathbf{c}) \\ &= \operatorname{gyr}[\mathbf{a}, \ominus\mathbf{b}](\ominus\mathbf{b} \oplus \mathbf{a}) \oplus \operatorname{gyr}[\mathbf{a}, \ominus\mathbf{b}](\ominus\mathbf{c}) \\ &= \operatorname{gyr}[\mathbf{a}, \ominus\mathbf{b}]\{(\ominus\mathbf{b} \oplus \mathbf{a}) \oplus (\ominus\mathbf{c})\}. \end{aligned} \tag{85}$$

□

Theorem 30. *Let M be an h-closed gyrovector subspace of $\mathbb{V}_s$ and $\mathbf{x} \in \mathbb{V}_s$.*

(i) *Let*

$$\mathbf{x} = \mathbf{y} \oplus \mathbf{z}, \quad \mathbf{y} \in M, \ \mathbf{z} \in M^{\perp} \cap \mathbb{V}_s \tag{86}$$

be the orthogonal gyrodecomposition of $\mathbf{x} \in \mathbb{V}_s$ by Theorem 27 and [8, Theorem 4.2]. Then $\mathbf{y}$ is the closest point to $\mathbf{x}$ in M. Thus $\mathbf{y}$ satisfies the identity

$$h(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{m} \in M} h(\mathbf{x}, \mathbf{m}). \tag{87}$$

(ii) *Conversely, let* $\mathbf{y}$ *be the closest point to* $\mathbf{x}$ *in* M*; namely,* $\mathbf{y}$ *is an element in* M *satisfying identity (87). Then*

$$\mathbf{x} = \mathbf{y} \oplus (\ominus \mathbf{y} \oplus \mathbf{x}) \quad (88)$$

is the orthogonal gyrodecomposition. Thus $\ominus\mathbf{y} \oplus \mathbf{x} \in M^{\perp} \cap \mathbb{V}_s$.

Proof. Note that $h(\mathbf{a}_1, \mathbf{a}_2) \le h(\mathbf{b}_1, \mathbf{b}_2)$ if and only if $\|\mathbf{a}_1 \ominus \mathbf{a}_2\| \le \|\mathbf{b}_1 \ominus \mathbf{b}_2\|$.

(i) Suppose that $\mathbf{x} = \mathbf{y} \oplus \mathbf{z}, \mathbf{y} \in M, \mathbf{z} \in M^{\perp} \cap \mathbb{V}_s$. For any $\mathbf{m} \in M$, we have

$$\begin{aligned} \mathbf{x} \ominus \mathbf{m} &= \mathbf{x} \ominus (\mathbf{y} \oplus (\ominus \mathbf{y} \oplus \mathbf{m})) \\ &= \operatorname{gyr}[\mathbf{x}, \ominus\mathbf{y}] \{(\ominus \mathbf{y} \oplus \mathbf{x}) \ominus (\ominus \mathbf{y} \oplus \mathbf{m})\} \\ &= \operatorname{gyr}[\mathbf{x}, \ominus\mathbf{y}] \{\mathbf{z} \ominus (\ominus \mathbf{y} \oplus \mathbf{m})\} \end{aligned} \quad (89)$$

by the previous lemma. From $\ominus\mathbf{y} \oplus \mathbf{x} = \mathbf{z} \in M^{\perp} \cap \mathbb{V}_s, \ominus\mathbf{y} \oplus \mathbf{m} \in M$ and Theorem 28, it follows that

$$\begin{aligned} \|\mathbf{x} \ominus \mathbf{y}\| &= \|\mathbf{z}\| \le \|\mathbf{z} \ominus (\ominus \mathbf{y} \oplus \mathbf{m})\| \\ &= \|\operatorname{gyr}[\mathbf{x}, \ominus\mathbf{y}] (\mathbf{z} \ominus (\ominus \mathbf{y} \oplus \mathbf{m}))\| = \|\mathbf{x} \ominus \mathbf{m}\|, \end{aligned} \quad (90)$$

because each gyroautomorphism preserves the norm. Since $\mathbf{m} \in M$ is arbitrary, we can conclude $\|\mathbf{x} \ominus \mathbf{y}\| \le \inf_{\mathbf{m} \in M} \|\mathbf{x} \ominus \mathbf{m}\|$, and the opposite inequality trivially holds. Thus $\mathbf{y}$ satisfies identity (87).

(ii) Put $\mathbf{z} = \ominus\mathbf{y} \oplus \mathbf{x}$. For any $\mathbf{m} \in M$, we have

$$\begin{aligned} \mathbf{x} \ominus (\mathbf{y} \oplus \mathbf{m}) &= \operatorname{gyr}[\mathbf{x}, \ominus\mathbf{y}] \{(\ominus \mathbf{y} \oplus \mathbf{x}) \ominus \mathbf{m}\} \\ &= \operatorname{gyr}[\mathbf{x}, \ominus\mathbf{y}] (\mathbf{z} \ominus \mathbf{m}) \end{aligned} \quad (91)$$

by the previous lemma. From $\mathbf{y} \oplus \mathbf{m} \in M$ and identity (87), it follows that

$$\begin{aligned} \|\mathbf{z}\| &= \|\ominus \mathbf{y} \oplus \mathbf{x}\| = \|\mathbf{x} \ominus \mathbf{y}\| \le \|\mathbf{x} \ominus (\mathbf{y} \oplus \mathbf{m})\| \\ &= \|\operatorname{gyr}[\mathbf{x}, \ominus\mathbf{y}] (\mathbf{z} \ominus \mathbf{m})\| = \|\mathbf{z} \ominus \mathbf{m}\|, \end{aligned} \quad (92)$$

because each gyroautomorphism preserves the norm. Thus we can apply Theorem 28 and obtain that $\mathbf{z} \in M^{\perp}$. This completes the proof. □

The following lemma plays a key role in our orthogonal gyroexpansion.

Lemma 31. *If* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is an orthogonal set in* $\mathbb{V}_s$*, then the associative law holds; that is,*

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}. \quad (93)$$

Proof. By [4, (3.147), (3.148)], the gyration in the Möbius gyrovector spaces $\mathbb{V}_s$ can be expressed by the equation

$$\operatorname{gyr}[\mathbf{u}, \mathbf{v}] \mathbf{w} = \mathbf{w} + 2 \frac{A\mathbf{u} + B\mathbf{v}}{D}, \quad (94)$$

where

$$\begin{aligned} A &= -\frac{1}{s^4} \mathbf{u} \cdot \mathbf{w} \|\mathbf{v}\|^2 + \frac{1}{s^2} \mathbf{v} \cdot \mathbf{w} + \frac{2}{s^4} (\mathbf{u} \cdot \mathbf{v}) (\mathbf{v} \cdot \mathbf{w}) \\ B &= -\frac{1}{s^4} \mathbf{v} \cdot \mathbf{w} \|\mathbf{u}\|^2 - \frac{1}{s^2} \mathbf{u} \cdot \mathbf{w} \\ D &= 1 + \frac{2}{s^2} \mathbf{u} \cdot \mathbf{v} + \frac{1}{s^4} \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \end{aligned} \quad (95)$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_s$. See also [10, Proposition 2.14] for a proof by hand calculation. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthogonal, then we have $A = B = 0$, so that $\operatorname{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$. This completes the proof. □

Definition 32. (i) Let $\{\mathbf{a}_n\}_n$ be a sequence in $\mathbb{V}_s$. One says that a series

$$(((\mathbf{a}_1 \oplus \mathbf{a}_2) \oplus \mathbf{a}_3) \oplus \cdots \oplus \mathbf{a}_n) \oplus \cdots \quad (96)$$

converges if there exists an element $\mathbf{x} \in \mathbb{V}_s$ such that $h(\mathbf{x}, \mathbf{x}_n) \to 0$ $(n \to \infty)$, where the sequence $\{\mathbf{x}_n\}_n$ is defined recursively by $\mathbf{x}_1 = \mathbf{a}_1$ and $\mathbf{x}_n = \mathbf{x}_{n-1} \oplus \mathbf{a}_n$. In this case, we say the series converges to $\mathbf{x}$ and denote

$$\mathbf{x} = (((\mathbf{a}_1 \oplus \mathbf{a}_2) \oplus \mathbf{a}_3) \oplus \cdots \oplus \mathbf{a}_n) \oplus \cdots. \quad (97)$$

(ii) Let $\{a_n\}_n$ be a sequence in $\mathbb{R}$ with $|a_n| < s$ for all n. We say that a series

$$\sum_{n=1}^{\infty}{}^{\oplus} a_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n \oplus \cdots \quad (98)$$

converges if there exists $x \in \mathbb{R}$ with $|x| < s$ such that $x_n \to x$, where the sequence $\{x_n\}_n$ is defined recursively by $x_1 = a_1$ and $x_n = x_{n-1} \oplus a_n$. In this case, we say the series converges to x and denote

$$x = \sum_{n=1}^{\infty}{}^{\oplus} a_n. \quad (99)$$

Theorem 33. *Let* $\{\mathbf{e}_n\}_{n=1}^{\infty}$ *be an orthonormal sequence in a real Hilbert space* $\mathbb{V}$*. Let* $\{w_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *such that* $0 < w_n < s$ *for all* n*. For any sequence* $\{r_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$*, the following are equivalent:*

(i) *The series*

$$r_1 \otimes w_1 \mathbf{e}_1 \oplus r_2 \otimes w_2 \mathbf{e}_2 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n \oplus \cdots \quad (100)$$

converges to an element $\mathbf{x} \in \mathbb{V}_s$.

(ii) *The series* $\sum_{n=1}^{\infty}{}^{\oplus} (r_n \otimes w_n)^2 / s$ *converges to* $x \in \mathbb{R}$ *with* $|x| < s$.

Note that parentheses are not necessary in the formula in (i) above by Lemma 31.

Proof. (i) ⇒ (ii). Put

$$\begin{aligned} \mathbf{x}_n &= r_1 \otimes w_1 \mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n \\ \mathbf{x} &= r_1 \otimes w_1 \mathbf{e}_1 \oplus r_2 \otimes w_2 \mathbf{e}_2 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n \oplus \cdots. \end{aligned} \quad (101)$$

From Lemma 15, it follows that

$$\frac{\|\mathbf{x}_n\|^2}{s} = \frac{\|r_1 \otimes w_1 \mathbf{e}_1\|^2}{s} \oplus \cdots \oplus \frac{\|r_n \otimes w_n \mathbf{e}_n\|^2}{s} = \frac{(r_1 \otimes w_1)^2}{s} \oplus \cdots \oplus \frac{(r_n \otimes w_n)^2}{s}. \quad (102)$$

By the assumption, we have $h(\mathbf{x}, \mathbf{x}_n) \to 0$. It follows from Lemma 20(i) and Lemma 14(iii) that $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$. Thus we have

$$\lim_{n\to\infty} \sum_{j=1}^{n} {}^{\oplus} \frac{(r_j \otimes w_j)^2}{s} = \frac{\|\mathbf{x}\|^2}{s} < s. \quad (103)$$

(ii) ⇒ (i). Suppose $x = \sum_{n=1}^{\infty} {}^{\oplus} (r_n \otimes w_n)^2/s < s$. We put

$$\mathbf{x}_n = r_1 \otimes w_1 \mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n, \quad c_n = \sum_{j=1}^{n} {}^{\oplus} \frac{(r_j \otimes w_j)^2}{s}. \quad (104)$$

By the assumption, for any $\varepsilon > 0$, there exists a number n_0 such that

$$n \geq n_0 \Longrightarrow 0 \leq x \ominus c_n < \varepsilon. \quad (105)$$

The last inequality implies that $(x - c_n)/(1 - (1/s^2)xc_n) < \varepsilon$. For $n_0 \leq m < n$, by Lemma 15,

$$\begin{aligned} d(\mathbf{x}_n, \mathbf{x}_m)^2 &= \|\ominus \mathbf{x}_m \oplus \mathbf{x}_n\|^2 \\ &= \|r_{m+1} \otimes w_{m+1} \mathbf{e}_{m+1} \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n\|^2 \\ &= c_n \ominus c_m = \frac{c_n - c_m}{1 - (1/s^2) c_n c_m} \\ &\leq \frac{x - c_m}{1 - (1/s^2) x c_m} < \varepsilon. \end{aligned} \quad (106)$$

Note that the strict inequality $x < s$ is crucial in the argument above. This implies that $\{\mathbf{x}_n\}_n$ is a Cauchy sequence with respect to the metric h by Lemma 20(ii). Since $(\mathbb{V}_s, h)$ is complete by Theorem 26, there exists a unique element $\mathbf{x} \in \mathbb{V}_s$ such that $h(\mathbf{x}, \mathbf{x}_n) \to 0$. This completes the proof. □

Example 34. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ defined by $a_n = 1/2n$. For $s = 1$, it is easy to see that

$$x_n = a_1 \oplus \cdots \oplus a_n = 1 - \frac{1}{n+1} \quad (n = 1, 2, \ldots). \quad (107)$$

Put $r_n = \tanh^{-1}(1/\sqrt{2n})/\tanh^{-1}(1/2)$. Then, we have $r_n \otimes (1/2) = \tanh(r_n \tanh^{-1}(1/2)) = 1/\sqrt{2n}$. It follows that

$$\sum_{j=1}^{n} {}^{\oplus} \left(r_j \otimes \frac{1}{2}\right)^2 = \sum_{j=1}^{n} {}^{\oplus} \frac{1}{2j} = 1 - \frac{1}{n+1}, \quad (108)$$

which does not converge to an element $x \in \mathbb{R}$ with $|x| < 1$. This example can be considered as a counterpart in the Möbius gyrovector space to the series $\sum_{n=1}^{\infty}(1/2n)$.

Theorem 35. *Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be a complete orthonormal sequence in a real Hilbert space $\mathbb{V}$. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $0 < w_n < s$ for all n. Then, for any $\mathbf{x} \in \mathbb{V}_s$, we have the orthogonal gyroexpansion*

$$\mathbf{x} = r_1 \otimes w_1 \mathbf{e}_1 \oplus r_2 \otimes w_2 \mathbf{e}_2 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n \oplus \cdots, \quad (109)$$

where the sequence of gyrocoefficients $\{r_n\}_{n=1}^{\infty}$ is determined by the following equations:

$$\begin{aligned} x_n &= \mathbf{x} \cdot \mathbf{e}_n, \\ \mathbf{x}_n^{(1)} &= \sum_{j=1}^{n} x_j \mathbf{e}_j, \\ \mathbf{x}_n^{(2)} &= \sum_{j=n+1}^{\infty} x_j \mathbf{e}_j \\ \mathbf{u}_j &= \mu_{j-1}^{(2)} \cdots \mu_1^{(2)} x_j \mathbf{e}_j \quad (j = 2, 3, \ldots) \\ \mathbf{u}_1 &= x_1 \mathbf{e}_1 = \mathbf{x}_1^{(1)} \\ \mathbf{v}_j &= \mu_{j-1}^{(2)} \cdots \mu_1^{(2)} \mathbf{x}_j^{(2)} \quad (j = 2, 3, \ldots) \\ \mathbf{v}_1 &= \mathbf{x}_1^{(2)} \\ \mu_j^{(1)} &= \frac{\|\mathbf{u}_j\|^2 + \|\mathbf{v}_j\|^2 + s^2 - \sqrt{\left(\|\mathbf{u}_j\|^2 + \|\mathbf{v}_j\|^2 + s^2\right)^2 - 4s^2 \|\mathbf{u}_j\|^2}}{2\|\mathbf{u}_j\|^2} \\ \mu_j^{(2)} &= \frac{\|\mathbf{u}_j\|^2 + \|\mathbf{v}_j\|^2 - s^2 + \sqrt{\left(\|\mathbf{u}_j\|^2 + \|\mathbf{v}_j\|^2 + s^2\right)^2 - 4s^2 \|\mathbf{u}_j\|^2}}{2\|\mathbf{v}_j\|^2} \\ r_j &= \frac{\tanh^{-1}\left(\mu_j^{(1)} \mu_{j-1}^{(2)} \cdots \mu_1^{(2)} x_j / s\right)}{\tanh^{-1}(w_j/s)} \end{aligned} \quad (110)$$

for all $j, n = 1, 2, \ldots$. If $x_j = 0$, then we do not define $\mu_j^{(1)}$ but define as $r_j = 0$ and continue the procedure. If $\mathbf{v}_n = \mathbf{0}$, then we do not define $\mu_n^{(2)}$ but define as $r_j = 0$ for all $j \geq n+1$ and finish the procedure.

Proof. It is not difficult to see that we may assume $s = 1$. It is obvious that the series $\sum_{n=1}^{\infty} x_n \mathbf{e}_n$ converges to $\mathbf{x}$ in the norm topology and that

$$\mathbf{x} = \mathbf{x}_n^{(1)} + \mathbf{x}_n^{(2)} \quad (111)$$

is the orthogonal decomposition with respect to the closed linear subspace generated by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Let

$$\mathbf{x} = \mathbf{y}_n \oplus \mathbf{z}_n, \quad \mathbf{y}_n \in M_n, \ \mathbf{z}_n \in M_n^{\perp} \cap \mathbb{V}_1 \quad (112)$$

be the orthogonal gyrodecomposition with respect to M_n by Theorem 27 and [8, Theorem 4.2], where M_n is an h-closed gyrovector subspace generated by $\{w_1\mathbf{e}_1, \ldots, w_n\mathbf{e}_n\}$. Then, $\mathbf{y}_n, \mathbf{z}_n$ are given by the equations

$$\mathbf{y}_n = \lambda_n^{(1)}\mathbf{x}_n^{(1)}, \qquad \mathbf{z}_n = \lambda_n^{(2)}\mathbf{x}_n^{(2)}, \tag{113}$$

where

$$\lambda_n^{(1)} = \frac{\|\mathbf{x}\|^2 + 1 - \sqrt{\left(\|\mathbf{x}\|^2 + 1\right)^2 - 4\left\|\mathbf{x}_n^{(1)}\right\|^2}}{2\left\|\mathbf{x}_n^{(1)}\right\|^2}, \qquad \lambda_n^{(2)} = \frac{\|\mathbf{x}\|^2 - 1 + \sqrt{\left(\|\mathbf{x}\|^2 + 1\right)^2 - 4\left\|\mathbf{x}_n^{(1)}\right\|^2}}{2\left\|\mathbf{x}_n^{(2)}\right\|^2}. \tag{114}$$

Since $\|\mathbf{x}_n^{(1)} - \mathbf{x}\| \longrightarrow 0$,

$$\lambda_n^{(1)} \longrightarrow \frac{\|\mathbf{x}\|^2 + 1 - \sqrt{\left(\|\mathbf{x}\|^2 + 1\right)^2 - 4\|\mathbf{x}\|^2}}{2\|\mathbf{x}\|^2} = 1, \tag{115}$$

so that we have

$$\mathbf{y}_n = \lambda_n^{(1)}\mathbf{x}_n^{(1)} \longrightarrow \mathbf{x} \tag{116}$$

in the norm topology, which implies that $h(\mathbf{y}_n, \mathbf{x}) \to 0$.

Next, we express $\mathbf{y}_n$ in the form of a gyrolinear combination

$$\mathbf{y}_n = r_1 \otimes w_1\mathbf{e}_1 \oplus r_2 \otimes w_2\mathbf{e}_2 \oplus \cdots \oplus r_n \otimes w_n\mathbf{e}_n \tag{117}$$

and present a concrete procedure to seek the gyrocoefficients r_n.

For $n = 1$, by using the above decomposition, we take $r_1 = \tanh^{-1}\lambda_1^{(1)}x_1/\tanh^{-1}w_1$. It follows that

$$\begin{aligned} r_1 \otimes w_1\mathbf{e}_1 &= \tanh\left(r_1\tanh^{-1}\|w_1\mathbf{e}_1\|\right)\frac{w_1\mathbf{e}_1}{\|w_1\mathbf{e}_1\|} \\ &= \lambda_1^{(1)}x_1\mathbf{e}_1 = \lambda_1^{(1)}\mathbf{x}_1^{(1)} = \mathbf{y}_1, \\ \mathbf{x} &= r_1 \otimes w_1\mathbf{e}_1 \oplus \mathbf{z}_1. \end{aligned} \tag{118}$$

Suppose that we proceed up to the n-th step and obtain the quantities, identities (110) for $j = 1, \ldots, n$, and

$$\mathbf{x} = r_1 \otimes w_1\mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n\mathbf{e}_n \oplus \mu_n^{(2)}\mathbf{v}_n. \tag{119}$$

Now,

$$\begin{aligned} \mu_n^{(2)}\mathbf{v}_n &= \mu_n^{(2)}\cdots\mu_1^{(2)}x_{n+1}\mathbf{e}_{n+1} + \mu_n^{(2)}\cdots\mu_1^{(2)}\sum_{j=n+2}^{\infty}x_j\mathbf{e}_j \\ &= \mathbf{u}_{n+1} + \mathbf{v}_{n+1} \end{aligned} \tag{120}$$

is the orthogonal decomposition with respect to the finite dimensional linear subspace generated by $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$. Let

$$\mu_n^{(2)}\mathbf{v}_n = \mathbf{y}'_{n+1} \oplus \mathbf{z}'_{n+1}, \qquad \mathbf{y}'_{n+1} \in M_{n+1},\ \mathbf{z}'_{n+1} \in M_{n+1}{}^{\perp} \cap \mathbb{V}_1 \tag{121}$$

be the orthogonal gyrodecomposition with respect to M_{n+1}. Then, $\mathbf{y}'_{n+1}, \mathbf{z}'_{n+1}$ are given by the equations

$$\mathbf{y}'_{n+1} = \mu_{n+1}^{(1)}\mathbf{u}_{n+1}, \qquad \mathbf{z}'_{n+1} = \mu_{n+1}^{(2)}\mathbf{v}_{n+1}, \tag{122}$$

where

$$\begin{aligned} \mu_{n+1}^{(1)} &= \frac{\|\mathbf{u}_{n+1}\|^2 + \|\mathbf{v}_{n+1}\|^2 + 1 - \sqrt{\left(\|\mathbf{u}_{n+1}\|^2 + \|\mathbf{v}_{n+1}\|^2 + 1\right)^2 - 4\|\mathbf{u}_{n+1}\|^2}}{2\|\mathbf{u}_{n+1}\|^2} \\ \mu_{n+1}^{(2)} &= \frac{\|\mathbf{u}_{n+1}\|^2 + \|\mathbf{v}_{n+1}\|^2 - 1 + \sqrt{\left(\|\mathbf{u}_{n+1}\|^2 + \|\mathbf{v}_{n+1}\|^2 + 1\right)^2 - 4\|\mathbf{u}_{n+1}\|^2}}{2\|\mathbf{v}_{n+1}\|^2}. \end{aligned} \tag{123}$$

By taking

$$r_{n+1} = \frac{\tanh^{-1}\left(\mu_{n+1}^{(1)}\mu_n^{(2)}\cdots\mu_1^{(2)}x_{n+1}\right)}{\tanh^{-1}w_{n+1}}, \tag{124}$$

we have

$$\begin{aligned} \mathbf{x} &= r_1 \otimes w_1\mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n\mathbf{e}_n \oplus \left(\mathbf{y}'_{n+1} \oplus \mathbf{z}'_{n+1}\right) \\ &= r_1 \otimes w_1\mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n\mathbf{e}_n \\ &\quad \oplus \left(\mu_{n+1}^{(1)}\mathbf{u}_{n+1} \oplus \mu_{n+1}^{(2)}\mathbf{v}_{n+1}\right) \\ &= r_1 \otimes w_1\mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n\mathbf{e}_n \oplus r_{n+1} \otimes w_{n+1}\mathbf{e}_{n+1} \\ &\quad \oplus \mu_{n+1}^{(2)}\mathbf{v}_{n+1}. \end{aligned} \tag{125}$$

Thus, we can inductively take a sequence $\{r_n\}_{n=1}^{\infty}$ by the procedure above.

Finally, from the uniqueness of the orthogonal gyrodecomposition with respect to the h-closed gyrovector subspace M_n, it follows that

$$\mathbf{y}_n = r_1 \otimes w_1 \mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n \tag{126}$$

and the series converges as follows:

$$\mathbf{x} = r_1 \otimes w_1 \mathbf{e}_1 \oplus \cdots \oplus r_n \otimes w_n \mathbf{e}_n \oplus \cdots . \tag{127}$$

This completes the proof. □

Theorem 36. *Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a real Hilbert space $\mathbb{V}$. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $0 < w_n < s$ for all n. Then the following are equivalent:*

(i) *$\{\mathbf{e}_n\}_{n=1}^{\infty}$ is complete.*

(ii) *The h-closed gyrovector subspace generated by $\{w_n\mathbf{e}_n\}_{n=1}^{\infty}$ coincides with $\mathbb{V}_s$.*

(iii) *$\|\mathbf{x}\|^2 = \sum_{n=1}^{\infty}{}^{\oplus}(r_n \otimes w_n)^2/s$ for all $\mathbf{x} \in \mathbb{V}_s$,*

where $\{r_n\}_{n=1}^{\infty}$ is the sequence determined by identities (110).

Proof. It is easy to deduce implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) from the previous theorem.

(iii) $\Rightarrow$ (i) Suppose that there exists an element $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x} \cdot \mathbf{e}_n = 0$ for all n. By multiplying nonzero scalar, we may assume that $\mathbf{x} \in \mathbb{V}_s$. Then we have $x_n = 0$, hence $r_n = 0$ for all n. Thus (iii) is violated.

(ii) $\Rightarrow$ (i) Suppose that there exists an element $\mathbf{x}$ such that $\mathbf{x} \cdot \mathbf{e}_n = 0$ for all n. We may assume that $\mathbf{x} \in \mathbb{V}_s$. Then $\{\mathbf{x}\}^{\perp} \cap \mathbb{V}_s$ is an h-closed gyrovector subspace by Lemma 22. Since it contains $w_n\mathbf{e}_n$ for all n, it coincides with $\mathbb{V}_s$ by the assumption. Therefore, we have $\mathbf{x} \in \{\mathbf{x}\}^{\perp} \cap \mathbb{V}_s$, which implies that $\mathbf{x} = \mathbf{0}$. This completes the proof. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The author would like to thank Professor Michio Seto for bringing information on literatures [11, 12].

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Hindawi
Journal of Function Spaces
Volume 2017, Article ID 2068163, 7 pages
https://doi.org/10.1155/2017/2068163

Research Article

On the Power of Simulation and Admissible Functions in Metric Fixed Point Theory

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Received 6 August 2017; Accepted 9 October 2017; Published 5 November 2017

Academic Editor: Edixon Rojas

We investigate the existence and uniqueness of certain operators which form a new contractive condition via the combining of the notions of admissible function and simulation function contained in the context of complete b-metric spaces. The given results not only unify but also generalize a number of existing results on the topic in the corresponding literature.

1. Introduction

The crucial notion of this research is the simulation function which is defined by Khojasteh et al. [1]. After that, Argoubi et al. [2] relaxed the conditions of the notion of simulation function a little bit to guarantee that the considered set is nonempty.

In this manuscript, we respond to the question, how do we guarantee the existence of fixed points of the new contraction defined by the help of the admissible function and the simulation function in the frame of complete b-metric spaces? The presented main theorem of the paper covers and unifies a huge number of published results on the topic in the related literature.

Definition 1 (see [2], cf. [1]). Let $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ be a mapping that satisfies the following inequality and the condition below:

($\mathcal{S}_1$) $\sigma(r, s) < s - r$ for each $r, s > 0$.

($\mathcal{S}_2$) if $\{r_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \sigma(r_n, s_n) < 0. \tag{1}$$

We shall use the letter $\mathcal{S}$ to indicate the class of all simulation functions $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$. It is obvious from the axiom ($\mathcal{S}_2$) that

$$\sigma(r, r) < 0 \quad \text{for every } r > 0. \tag{2}$$

Note that the condition $\sigma(0, 0) = 0$ in the original definition of the simulation function is removed in Definition 1. Indeed, this condition gives a contradiction when one takes $s = r$ in the first condition ($\mathcal{S}_1$). For further detail on the discussion, see, for example, [2].

Throughout the paper, we shall use $\mathbb{R}_0^+$ to represent nonnegative real numbers.

The following example [1, 3, 4] shall be helpful to illustrate the worth of the notion of simulation function.

Example 2. Suppose that Φ denotes the set of all continuous functions $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\phi(r) = 0$ if, and only if, $r = 0$. The following functions $\sigma_1 - \sigma_6$ form a simulation function.

(i) $\sigma_1(t, s) = \phi_1(s) - \phi_2(t)$ for each $t, s \in \mathbb{R}_0^+$ and $\phi_1, \phi_2 \in \Phi$, where $\phi_1(t) < t \leq \phi_2(t)$ for each $t > 0$.

(ii) Let $R, T : \mathbb{R}_0^+ \to (0, \infty)$ be two continuous functions with respect to each variable and the inequality $R(t,s) > T(t,s)$ holds for each $t, s > 0$. Then,

$$\sigma_2(t,s) = s - \frac{R(t,s)}{T(t,s)} t \quad \text{for each } t, s \in \mathbb{R}_0^+. \tag{3}$$

(iii) $\sigma_3(t,s) = s - \phi_3(s) - t$ for each $t, s \in [0, \infty)$.

(iv) For each $s, t \in \mathbb{R}_0^+$,

$$\sigma_4(t,s) = s\varphi(s) - t, \tag{4}$$

where $\varphi : \mathbb{R}_0^+ \to [0,1)$ with $\limsup_{t \to r^+} \varphi(t) < 1$ for each $r > 0$.

(v) For each $s, t \in \mathbb{R}_0^+$

$$\sigma_5(t,s) = \eta(s) - t, \tag{5}$$

where $\eta \in \Phi$ and it is upper semicontinuous.

(vi) For each $s, t \in [0, \infty)$,

$$\sigma_6(t,s) = s - \int_0^t \phi(u)\, du, \tag{6}$$

where ϕ is a self-mapping on $\mathbb{R}_0^+$ with the following properties:

(1) $\int_0^\varepsilon \phi(u) du$ exists,

(2) for every $\varepsilon > 0$, $\int_0^\varepsilon \phi(u) du > \varepsilon$.

For the further attracted simulation function examples see, for example, [1, 3, 4].

In 1993 Czerwik [5] proposed a more general frame for the notion of standard metric, so called a b-metric.

Definition 3. For $M \neq \emptyset$, let $d_b : M \times M \to \mathbb{R}_0^+$ be a function satisfying the following conditions:

(1) $d_b(p,q) = 0$ if and only if $p = q$.

(2) $d_b(p,q) = d_b(q,p)$ for each $p, q \in M$.

(3) $d_b(p,q) \leq s[d_b(p,r) + d_b(r,q)]$ for each $p, q, r \in M$, where $s \geq 1$.

Here, d_b is called a b-metric. Further, the triple (M, d_b, s) is called a b-metric space.

For the special case of $s = 1$, the notion of b-metric turns into the standard metric. Consequently, the notion of b-metric is more general than the standard metric.

For the sake of completeness, we recollect standard but interesting three examples of b-metric spaces; see, for example, [6, 7] and the related references therein.

Example 4. Let $M = \mathbb{R}$. For all $p, q \in M$, we define a function d_b as

$$d_b(p,q) = |p - q|^2. \tag{7}$$

Then, d_b is a b-metric on real numbers. The first two axioms are fulfilled in a straightforward way. The last axiom is satisfied for $s = 2$:

$$|p - q|^2 \leq 2\left[|p - r|^2 + |r - q|^2\right]. \tag{8}$$

Example 5. For a fixed $p \in (0,1)$, consider

$$M = l_p(\mathbb{R}) = \left\{ t = \{t_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |t_n|^p < \infty \right\}. \tag{9}$$

We introduce the corresponding distance functions as

$$d_b(t,s) = \left(\sum_{n=1}^{\infty} |t_n - s_n|^p \right)^{1/p} \quad \text{for each } t, s \in M. \tag{10}$$

Then, (M, d_b, s) forms a b-metric space with the constant $s = 2^{1/p}$.

Example 6. Suppose that E is a Banach space with the zero vector 0_E of E. Take P as a cone in E such that $\mathrm{int}(P) \neq \emptyset$ and further, $\preceq$ is partial ordering with respect to P. For a nonempty set M, we define a mapping $c : M \times M \to E$ as follows:

$(M1)$ $0 \preceq c(p,q)$ for each $p, q \in M$.

$(M2)$ $c(p,q) = 0$ if and only if $p = q$.

$(M3)$ $c(p,q) \preceq c(x,z) + c(z,y)$, for each $p, q \in M$.

$(M4)$ $c(p,q) = c(q,p)$ for each $p, q \in M$.

Then, the mapping c is called cone metric on M. Moreover, the pair (M, c) is said to be a cone metric space.

If a normal cone P in E is normal with the normality constant K, then, the mapping $D : M \times M \to \mathbb{R}_0^+$, defined by $D_b(x,y) = \|c(x,y)\|$, forms a b-metric space where the function $c : M \times M \to E$ is a cone metric. Moreover, the triple (M, D_b, s) forms a b-metric space with the constant $s := K \geq 1$.

Suppose that (M, d_b, s) is a b-metric space. A self-mapping T on M is said to be a *$\mathcal{S}$-contraction* with respect to σ [1], if the following inequality is fulfilled:

$$\sigma\left(d_b(Tp, Tq),\ d_b(p,q)\right) \geq 0 \quad \text{for each } p, q \in M,\ \sigma \in \mathcal{S}. \tag{11}$$

On account of (σ_2), we derive that

$$d_b(Tp, Tq) \neq d_b(p,q) \quad \text{for each distinct } p, q \in M. \tag{12}$$

Taking (12) into account, we find that T cannot be an isometry whenever T is a $\mathcal{S}$-contraction. Moreover, if T is a $\mathcal{S}$-contraction in the setting of b-metric space with a fixed point, then the desired fixed point is necessarily unique.

Theorem 7. *In a complete b-metric space, each $\mathcal{S}$-contraction has a unique fixed point.*

This theorem can be stated also as follows: each $\mathcal{S}$-contraction yields a Picard sequence that converges to a unique fixed point.

For a family $\Psi := \{\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+\}$, if the following two conditions are fulfilled,

(i) each function $\psi \in \Psi$ is nondecreasing;

(ii) there exist $a \in (0,1)$ and $k_0 \in \mathbb{N}$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that for any $t \in \mathbb{R}^+$ and for $k \geq k_0$ we have

$$\psi^{k+1}(t) \leq a\psi^{k}(t) + v_k. \quad (13)$$

Here, Ψ is called the class of (c)-comparison functions (see [8]). For a $\psi \in \Psi$ the notation ψ^n indicates the nth iteration of the function ψ. The following lemma is recollected from [8].

Lemma 8. *For a $\psi \in \Psi$, we have*

(i) *for each $t \in \mathbb{R}^+$, the sequence $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$;*

(ii) *for any $t \in \mathbb{R}^+$, the inequalities $\psi(t) < t$ are fulfilled;*

(iii) *each auxiliary function ψ is continuous at 0;*

(iv) *for any $t \in \mathbb{R}^+$, the series $\sum_{k=1}^{\infty} \psi^k(t)$ is convergent.*

Berinde [9] characterized (c)-comparison functions to use for the contraction mappings in the setting of b-metric spaces, as follows.

Definition 9. Fix a real number $s \geq 1$. An increasing function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be (b)-comparison if there exist a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$, $k_0 \in \mathbb{N}$, and $a \in (0,1)$ such that for any $t \geq 0$ and for $k \geq k_0$,

$$s^{k+1}\phi^{k+1}(t) \leq as^{k}\phi^{k}(t) + v_k. \quad (14)$$

Lemma 10 (see [10]). *Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a (b)-comparison function. Define a self-mapping b_s on $\mathbb{R}_0^+$ as $b_s = \sum_{k=0}^{\infty} s^k\phi^k(t)$:*

(1) *For any $t \in \mathbb{R}_0^+$, the series $\sum_{k=0}^{\infty} s^k\phi^k(t)$ is convergent.*

(2) *The function b_s is increasing and continuous at $t = 0$.*

Remark 11. It is obvious that each (b)-comparison function is a comparison function. Consequently, on account of Lemma 8, we deduce that any (b)-comparison function ψ satisfies the inequality $\psi(t) < t$.

Popescu [11] introduces the notion of the α-orbital admissible as follows.

Definition 12 (see [11]). Suppose that T is a self-mapping over a nonempty set M and $\alpha : M \times M \to \mathbb{R}_0^+$ is a function. The mapping T is called an α-orbital admissible if the following implication is provided:

$$\alpha(x, Tx) \geq 1 \Longrightarrow \alpha(Tx, T^2x) \geq 1. \quad (15)$$

We should mention the notion of the α-orbital admissible [11, 12] inspired from the notion of the α-orbital admissible notion defined in [13, 14].

In this paper, by combining the notion of the simulation function together with the admissible functions, we shall consider a new type contractive mapping in the frame of complete b-metric spaces. Accordingly, our results improve and extend the main results in [15] in twofold: first, we investigate the existence and uniqueness of a fixed point in b-metric spaces instead of standard metric space. Secondly, we extend the condition $\sigma(\alpha(p,q)d_b(Tp,Tq),(d_b(p,q)) \geq 0$ for each $p,q \in M$ by adding an auxiliary function ϕ into account. Consequently, we investigate the existence and uniqueness of a fixed point in the new extended condition $\sigma(\alpha(p,q)d_b(Tp,Tq),\phi(d_b(p,q))) \geq 0$ for each $p,q \in M$. We illustrate that the class of the new contractive mapping covers several well-known contractive mappings.

2. Main Results

We start this section by defining the $(\alpha - \phi)$-type $\mathcal{S}$-*contraction* which is a generalization of the notion of $\mathcal{S}$-*contraction*.

Definition 13. Let M be a nonempty set $s \geq 1$ and $\alpha : M \times M \to [0,\infty)$ be function. Suppose that T is a self-mapping defined over a b-metric space (M, d_b, s). The self-mapping T is called an $(\alpha - \phi)$-type $\mathcal{S}$-*contraction* with respect to σ if there are $\sigma \in \mathcal{S}$ and $\phi \in \Phi$ such that

$$\sigma(\alpha(p,q)d_b(Tp,Tq), \phi(d_b(p,q))) \geq 0 \quad \text{for each } p,q \in M. \quad (16)$$

Before stating our main theorem, we shall give lemmas that have a crucial role in the proof of the main result.

Lemma 14. *Let M be a nonempty set. Suppose that $\alpha : M \times M \to \mathbb{R}_0^+$ is a function and $T : M \to M$ is an α-orbital admissible mapping. If there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$ and $p_n = Tp_{n-1}$ for $n = 0, 1, \ldots$, then, we have*

$$\alpha(p_n, p_{n+1}) \geq 1, \quad \text{for each } n = 0, 1, \ldots. \quad (17)$$

Proof. On account of the assumptions of the theorem, there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$. Owing to the fact that T is α-orbital admissible, we find

$$\alpha(p_0, p_1) = \alpha(p_0, Tp_0) \geq 1 \Longrightarrow \alpha(Tp_0, Tp_1) = \alpha(p_1, p_2) \geq 1. \quad (18)$$

By iterating the above inequality, we derive that

$$\alpha(p_n, p_{n+1}) = \alpha(Tp_{n-1}, Tp_n) \geq 1, \quad \text{for each } n = 0, 1, \ldots. \quad (19)$$

□

Theorem 15. *Let M be a nonempty set, $s \geq 1$, and $\alpha : M \times M \to \mathbb{R}_0^+$ be a function. Suppose that a continuous self-mapping T over a complete b-metric space (M, d_b, s) is α-orbital admissible. Suppose also the mapping T forms an $(\alpha - \phi)$-type $\mathcal{S}$-contraction with respect to σ. If there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$, then there exists $p \in M$ such that $Tp = p$.*

Proof. Based on the assumption, there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$. We shall construct an iterative sequence $\{p_n\}$ in M by setting $p_{n+1} = Tp_n$ for each $n \geq 0$. By Lemma 14, we have (17); that is,

$$\alpha(p_n, p_{n+1}) \geq 1, \quad \text{for each } n = 0, 1, \ldots. \tag{20}$$

Taking (16) and (17) into account, for each $n \geq 1$, we derive that

$$\begin{aligned} 0 &\leq \sigma(\alpha(p_n, p_{n-1}) d_b(Tp_n, Tp_{n-1}), \phi(d_b(p_n, p_{n-1}))) \\ &= \sigma(\alpha(p_n, p_{n-1}) d_b(p_{n+1}, p_n), \phi(d_b(p_n, p_{n-1}))) \\ &< \phi(d_b(p_n, p_{n-1})) - \alpha(p_n, p_{n-1}) d_b(p_{n+1}, p_n). \end{aligned} \tag{21}$$

Accordingly, from (16) and (21) we conclude that

$$\begin{aligned} d_b(p_n, p_{n+1}) &\leq \alpha(p_n, p_{n-1}) d_b(p_n, p_{n+1}) \\ &\leq \phi(d_b(p_n, p_{n-1})) < d_b(p_n, p_{n-1}) \\ &\qquad \text{for each } n = 1, 2, \ldots. \end{aligned} \tag{22}$$

Hence, we conclude that the constructive sequence $\{d_b(p_n, p_{n-1})\}$ is bounded from below by zero, and moreover, it is nondecreasing. Hereby, there exists $\theta \geq 0$ such that $\lim_{n\to\infty} d_b(p_n, p_{n-1}) = \theta \geq 0$. We shall indicate that

$$\lim_{n\to\infty} d_b(p_n, p_{n-1}) = 0. \tag{23}$$

Suppose, on the contrary, that $\theta > 0$. On account of inequality (22), we find that

$$\begin{aligned} \lim_{n\to\infty} \alpha(p_n, p_{n-1}) d_b(p_n, p_{n+1}) &= \theta, \\ \lim_{n\to\infty} \phi(d_b(p_n, p_{n-1})) &= \theta. \end{aligned} \tag{24}$$

Taking $s_n = \alpha(p_n, p_{n-1}) d_b(p_n, p_{n+1})$ and $t_n = d_b(p_n, p_{n-1})$ together with the condition (σ_3), we derive that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \sigma(\alpha(p_n, \\ &\quad p_{n-1}) d_b(p_{n+1}, p_n), \phi(d_b(p_n, p_{n-1}))) \\ &< 0, \end{aligned} \tag{25}$$

a contradiction. Consequently, we have $\theta = 0$.

In the next step, we shall show that the constructive sequence $\{p_n\}$ is Cauchy. By iteration on the inequality (22), we derive that

$$d_b(p_{n+1}, p_n) \leq \psi^n(d_b(p_1, p_0)), \quad \text{for each } n \geq 1. \tag{26}$$

From (26) and using the triangular inequality, for each $p \geq 1$, we have

$$\begin{aligned} d_b(p_n, p_{n+p}) &\leq s \cdot d_b(p_n, p_{n+1}) + s^2 \cdot d_b(p_{n+1}, p_{n+2}) \\ &\quad + \cdots + s^{p-2} \cdot d_b(p_{n+p-3}, p_{n+p-2}) + s^{p-1} \\ &\quad \cdot d_b(p_{n+p-2}, p_{n+p-1}) + s^p \cdot d_b(p_{n+p-1}, p_{n+p}) \leq s \\ &\quad \cdot d_b(p_0, p_1) + s^2 \cdot d_b(p_0, p_1) + \cdots + s^{p-2} \\ &\quad \cdot d_b(p_0, p_1) + s^{p-1} \cdot d_b(p_0, p_1) + s^p \cdot d_b(p_0, p_1) \\ &= \frac{1}{s^n} \cdot \left[s^{n+1} \cdot d_b(p_0, p_1) + \cdots + s^{n+p-1}\right. \\ &\quad \left. \cdot d_b(p_0, p_1) + s^{n+p} \cdot d_b(p_0, p_1)\right] \leq \frac{1}{s^n} \cdot \left[s^{n+1}\right. \\ &\quad \left. \cdot d_b(p_0, p_1) + \cdots + s^{n+p} \cdot d_b(p_0, p_1)\right] = \frac{1}{s^n} \\ &\quad \cdot \sum_{i=n+1}^{n+p} s^i \cdot d_b(p_0, p_1) < \frac{1}{s^n} \cdot \sum_{i=n+1}^{\infty} s^i \cdot d_b(p_0, p_1). \end{aligned} \tag{27}$$

The precedent inequality is

$$\begin{aligned} d_b(p_n, p_{n+p}) &< \frac{1}{s^n} \cdot \sum_{i=n+1}^{\infty} s^i \cdot d_b(p_0, p_1) \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty, \end{aligned} \tag{28}$$

which yields that $\{p_n\}$ is a Cauchy sequence in (M, d_b, s). Since (M, d_b, s) is complete, there exists $p \in M$ such that

$$\lim_{n\to\infty} d_b(p_n, p) = 0. \tag{29}$$

Since T is continuous, we obtain from (29) that

$$\lim_{n\to\infty} d_b(p_{n+1}, Tp) = \lim_{n\to\infty} d_b(Tp_n, Tp) = 0. \tag{30}$$

Combining the uniqueness of the limit together with (29) amd (30), we find that p forms a fixed point of T; that is, $Tp = p$. □

The continuity condition can be relaxed in Theorem 15 by replacing a suitable condition like the given below.

Definition 16. Let $s \geq 1$. We say that a b-metric space (M, d_b, s) is *regular* if $\{p_n\}$ is a sequence in M such that $\alpha(p_n, p_{n+1}) \geq 1$ for each n and $p_n \to x \in M$ as $n \to \infty$; then there is a subsequence $\{p_{n(k)}\}$ of $\{p_n\}$ such that $\alpha(p_{n(k)}, x) \geq 1$ for each k.

By removing the continuity condition from the main result, Theorem 15 is possible. But, in this case, we should add the "regularity" condition which is mentioned in Definition 16.

Theorem 17. *Let M be a nonempty set, $s \geq 1$, and $\alpha : M \times M \to \mathbb{R}_0^+$ be a function. Suppose that (M, d_b, s) is regular and* *a self-mapping T on a complete b-metric space (M, d_b, s) is α-orbital admissible. Suppose also the mapping T forms an $(\alpha - \phi)$-type $\mathcal{S}$-contraction with respect to σ. If there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$, then there exists $p \in M$ such that $Tp = p$.*

Proof. By repeating the steps in the proof of Theorem 15, we guarantee that the sequence $\{p_n\}$ defined by $p_{n+1} = Tp_n$ for each $n \geq 0$ converges for some $p \in M$. From (17) and regularity of the metric, there exists a subsequence $\{p_{n(k)}\}$ of $\{p_n\}$ such that $\alpha(p_{n(k)}, p) \geq 1$ for each k. By implementing (16), for each k, we get that

$$\begin{aligned} 0 &\leq \sigma\left(\alpha\left(p_{n(k)}, p\right) d_b\left(Tp_{n(k)}, Tp\right), \phi\left(d_b\left(p_{n(k)}, p\right)\right)\right) \\ &= \sigma\left(\alpha\left(p_{n(k)}, p\right) d_b\left(p_{n(k)+1}, Tp\right), \phi\left(d_b\left(p_{n(k)}, p\right)\right)\right) \\ &< \phi\left(d_b\left(p_{n(k)}, p\right)\right) - \alpha\left(p_{n(k)}, p\right) d_b\left(p_{n(k)+1}, Tp\right), \end{aligned} \tag{31}$$

which leads to

$$\begin{aligned} d_b\left(p_{n(k)+1}, Tp\right) &= d_b\left(Tp_{n(k)}, Tp\right) \\ &\leq \alpha\left(p_{n(k)}, p\right) d_b\left(Tp_{n(k)}, Tp\right) \\ &\leq \phi\left(d_b\left(p_{n(k)}, p\right)\right) < d_b\left(p_{n(k)}, p\right). \end{aligned} \tag{32}$$

Taking $k \to \infty$ in inequality (32), we deduce that $d_b(u, Tp) = 0$; that is, $p = Tp$. □

Note that, in Theorems 15 and 17, we observe only the existence of the fixed point of the given operator. As a next step, we shall investigate the uniqueness of the obtained fixed point. Let $\mathrm{Fix}(T)$ represent the set of all fixed points of operator T. For this purpose, we need the following additional condition:

($\mathcal{U}$) $\alpha(p, q) \geq 1$ for each $p, q \in \mathrm{Fix}(T)$.

Theorem 18. *Under the assumption of additional condition ($\mathcal{U}$), the obtained fixed point p of the operator T defined in Theorem 15 (resp., Theorem 17) turns to be unique fixed point.*

Proof. Let T be an α-orbital admissible $\mathcal{S}$-*contraction* with respect to σ. Regarding Theorem 15 or Theorem 17, we guarantee the existence of a fixed point of the mapping T; namely, $p = Tp$. Suppose p is not the unique fixed point of T; thus, there exists q with $p \neq q$. So, we have $d_b(p, q) > 0$. Regarding the condition ($\mathcal{U}$), the definition of T yields that

$$\sigma\left(\alpha\left(p, q\right) d_b\left(Tp, Tq\right), \phi\left(d_b\left(p, q\right)\right)\right) \geq 0. \tag{33}$$

Due to definition of the auxiliary function σ, the inequality above implies that

$$\begin{aligned} d_b\left(p, q\right) &= d_b\left(Tp, Tq\right) \\ &\leq \alpha\left(p, q\right) d_b\left(Tp, Tq\right), \phi\left(d_b\left(p, q\right)\right) \\ &< d_b\left(p, q\right), \end{aligned} \tag{34}$$

which is a contradiction. Thus, p is the unique fixed point of T. □

3. Consequences

3.1. Consequences in the Setting of b-Metric Space. Consider a mapping $\phi_1 \in \phi$ related with an α-orbital admissible $\mathcal{S}$-*contraction* with respect to σ, namely T; that is, $\sigma_E(\alpha(p, q)d_b(Tp, Tq), \phi_1(d_b(p, q))$. For a function $\psi \in \Psi$, we set

$$\sigma_E(t, s) = \psi(s) - t \quad \text{for each } s, t \in \mathbb{R}_0^+. \tag{35}$$

It is straightforward that σ_{BW} is a simulation function. Combining the observations above, we have

$$\begin{aligned} &\sigma_E \alpha\left(p, q\right) d_b\left(Tp, Tq\right), \phi_1\left(d_b\left(p, q\right)\right) \\ &\quad = \psi\left(\phi_1\left(d_b\left(p, q\right)\right)\right) - \alpha\left(p, q\right) d_b\left(Tp, Tq\right) \\ &\qquad \text{for each } s, t \in \mathbb{R}_0^+, \end{aligned} \tag{36}$$

which is equivalent to

$$\begin{aligned} &\alpha\left(p, q\right) d_b\left(Tp, Tq\right) \leq \phi\left(d_b\left(p, q\right)\right), \\ &\qquad \text{for each } p, q \in M, \end{aligned} \tag{37}$$

where

$$\phi(t) := \psi\left(\phi_1(t)\right) \in \Phi. \tag{38}$$

Thus, the above sample of simulation function together with Theorem 18 yields the following result.

Theorem 19. *Let (M, d) be a complete b-metric space and let $T : M \to M$ be defined as*

$$\begin{aligned} &\alpha\left(p, q\right) d_b\left(Tp, Tq\right) \leq \phi\left(d_b\left(p, q\right)\right), \\ &\qquad \textit{for each } p, q \in M, \end{aligned} \tag{39}$$

where $\alpha : M \times M \to \mathbb{R}_0^+$ and $\phi \in \Phi$. Suppose that

(i) *T is α-orbital admissible;*

(ii) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$;*

(iii) *either T is continuous or (M, d) is regular.*

Then there exists $p \in M$ such that $Tp = p$. Moreover, if the condition ($\mathcal{U}$) is fulfilled, then we guarantee that the obtained fixed point p of T is unique.

Letting $\phi(t) = kt$ with $k \in [0, 1)$, Theorem 19 implies the following.

Theorem 20. *Let (M, d) be a complete b-metric space and let $T : M \to M$ be defined as*

$$\begin{aligned} &\alpha\left(p, q\right) d_b\left(Tp, Tq\right) \leq k d_b\left(p, q\right), \\ &\qquad \textit{for each } p, q \in M, \end{aligned} \tag{40}$$

where $k \in [0, 1)$, $\alpha : M \times M \to \mathbb{R}_0^+$, and $\phi \in \Phi$. Suppose that

(i) *T is α-orbital admissible;*

(ii) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$;*

(iii) *either T is continuous or (M, d) is regular.*

Then there exists $p \in M$ such that $Tp = p$. Moreover, if the condition $(\mathcal{U})$ is fulfilled, then we guarantee that the obtained fixed point p of T is unique.

By letting $\alpha(p,q) = 1$ in Theorems 19 and 20 we get the main results of Theorems 1 and 2 of Czerwik [5]. Notice that, in this case, conditions (i)–(iii) of Theorems 19 and 20 are fulfilled trivially.

3.2. Consequences in the Setting of Standard Metric Space. In this section, we consider the results in the setting of standard metric. Thus, we consider $s = 1$ throughout this section. We shall show that a number of existing fixed point results in the literature are the simple consequence of our main results. In particular, by taking Example 2 into consideration, we can list many well-known results as a consequence of our main results.

If $\psi \in \Psi$ and we define

$$\sigma_E(t,s) = \psi(s) - t \quad \text{for each } s,t \in \mathbb{R}_0^+, \tag{41}$$

then σ_{BW} is a simulation function (cf. Example 2 (v)).

First, we derive the very interesting recent results of Samet et al. [13] as a corollary of Theorem 18.

Theorem 21. *Theorems 2.1 and 2.2 in [13] are consequences of the following.*

Proof. Taking $\sigma_E(r,s) = \psi(s) - r$ for each $s,r \in [0,\infty)$ in Theorem 18, we derive that

$$\alpha(p,q)\, d_b(Tp,Tq) \le \psi(d_b(p,q)), \quad \text{for each } p,q \in M. \tag{42}$$

We skip the details. □

As is well known, the main theorem in [13] covers several fixed point results, including the pioneer fixed point theorem of Banach. Moreover, as is shown in [13, 16], several fixed point theorems in different settings (in the sense of partially ordered set, in the sense of cyclic mapping, etc.) can be concluded from Theorems 2.1 and 2.2 in [13] by setting $\alpha(p,q)$ in a proper way.

Notice also that one can express the main result of Khojasteh et al. [1] as a straight consequence of our main result.

Theorem 22. *Theorem 18 yields Theorem 7.*

Proof. It is sufficient to take $\alpha(p,q) = 1$ for each $p,q \in M$ in Theorem 21. □

It is obvious that all presented results in [1] follow from our main result.

4. Conclusion

It is very easy to see that one can list a further outcome of our main results by letting the mappings $\sigma, \alpha, \psi, \phi$ in a suitable way like in Example 2. More precisely, by following the techniques in [13, 16] one can easily derive a number of well-known fixed point results in the distinct settings (such as in the frame of *cyclic contraction* and in the setting of *partially ordered set endowed with a metric*). We prefer not to list all consequences due to our concerns on the length of the paper. This paper can be also considered as a continuation of the recent paper [17].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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