# Fuzzy Functions, Relations, and Fuzzy Transforms: Theoretical Aspects and Applications to Fuzzy Systems

Guest Editors: Salvatore Sessa, Ferdinando Di Martino, and Irina G. Perfilieva

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# Editorial Board

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*Editorial*

# Fuzzy Functions, Relations, and Fuzzy Transforms: Theoretical Aspects and Applications to Fuzzy Systems

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Fuzzy functions, fuzzy relations, and fuzzy transforms are important for their applications to fuzzy systems. In this special issue, all terms are understood in their very general sense. We have received many papers, but the selection limited them to the actual version, which deals with essentially the following topics: fuzzy optimizations with fuzzy functions and relations (including also clustering algorithms), applications of fuzzy relations/transforms to decision making and approximation reasoning, and applications of fuzzy relations/transforms to data mining.

The contents of the papers can be resumed in the following way.

(i) In the paper of M. Shaverdi et al., the goal is to construct an approach based on multiple criteria decision making (MCDM) and balanced scorecard (BSC) for evaluating performance of three nongovernmental Iranian's banks via 21 indexes. The Fuzzy Analytic hierarchy process (FAHP) is adopted for calculating the related weights of each index and three MCDM analytical tools (TOPSIS, VIKOR, ELECTRE) to rank the banking performance.

(ii) In the paper of T. Yamamoto et al., a linear fuzzy clustering model based on the fuzzy C-medoid (FCMdd) concept is proposed. Strictly speaking, a fuzzy C-Means-like iterative algorithm is performed, and, in several numerical experiments, some suitable pre-imputation strategies properly select representative medoids of each cluster.

(iii) In the paper of S. Sachdeva at al., the problem to consume electricity more efficiently in developed and developing countries is performed. Since developed countries do not want to waste electricity and developing countries cannot waste electricity, it becomes important the concept of "Load Forecasting." By moving from daily to hourly basis of load forecasting, the correlated error increases. To reduce this error, a fuzzy method combined with an artificial network (ANN) and an orthogonal frequency division multiplexing (OFDM) transmission is used, obtaining a considerable reduction of 2-3% error.

(iv) In the paper of M. J. Hossain at al., a simplified fuzzy logic-based speed control scheme of an interior permanent magnet synchronous motor (IPMSM) is presented. A simplified fuzzy speed controller (FLC) for the IPMSM drive is incorporated in the drive system to maintain high performance standards. The efficacy of the proposed controller is verified by simulation at various dynamic operating conditions, and it is found to be robust and efficient.

(v) In the paper of F. Di Martino and S. Sessa, a system of fuzzy relation equations (SFRE) with the max-min composition for solving a problem of spatial analysis is integrated in a geographical information systems (GIS) tool. A precise geographical area is studied and divided in homogeneous subzones with respect to the parameters involved, and an expert settles the system of SFRE with the values of the impact coefficients considered as inputs. The best solutions of this system (considered as outputs) and the related results are associated to each subzone. Among others, an index which evaluates the reliability of these results is also given.

(vi) In the paper of I. Yusuf at al., a genetic algorithm (GA) is given for the design and implementation of a fuzzy logic controllers (FLC) for incubating eggs. It is determined the membership function of the FLC which makes the process as the fastest possible one as well.

(vii) In the paper of M. Hourali and G. A. Montazer, a novel approach for fuzzy ontology generation with two uncertainty degrees is presented. Indeed, the authors combine two uncertain models and propose a new ontology with two degrees of uncertainty, based on the concept expression and relation expression. The generated fuzzy ontology is implemented for expansion of the initial user's queries in the domain's concepts (software maintenance engineering (SME)). Experimental results show that the proposed model has better overall retrieval performance with respect to the keyword-based retrieval systems.

(viii) In the first paper of I. Perfilieva and V. Kreinovich, we underline that the original motivation of the concept of fuzzy transform comes from fuzzy modelling, but it is purely a mathematical transformation, and, hence, it is to be interpreted also in traditional (nonfuzzy) sense. Specifically, the authors show that the probabilistic interpretation of fuzzy modeling by Sanchez et al. (2002) can be modified into a natural probabilistic explanation of the fuzzy transform formulas involved.

(ix) In the paper of M. Yasud, a fuzzy clustering method which combines the deterministic annealing (DA) approach with an entropy is presented. In particular, by maximizing the Shannon entropy, the fuzzy entropy, or the Tsallis entropy within the framework of the fuzzy C-means (FCM) algorithm, the author obtains membership functions which are very similar to the statistical mechanical distribution functions.

(x) In the second paper of I. Perfilieva and V. Kreinovich, the authors discuss about large-scale (averaged) value of the predicted quantities. As example, they point out the impossibility to predict the exact future temperature at different spatial locations, but it is reasonably to predict average temperature over a region. Traditionally speaking, the resulting procedure is based on many differential equations, and, hence, it is very time-consuming. The authors show that similar quality large-scale prediction results can be obtained by applying an appropriate fuzzy transform and using averaged inputs to solve the corresponding discretized differential equations.

We hope that these topics would be captured and significantly pushed forward by interested readers. We hope that this issue will be regarded as the first of many our future scientific initiatives.

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*Research Article*

# FGA Temperature Control for Incubating Egg

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This paper investigates the use of genetic algorithms (GA) in the design and implementation of fuzzy logic controllers (FLC) for incubating egg. What is the best to determine the membership function is the first question that has been tackled. Thus it is important to select the accurate membership functions, but these methods possess one common weakness where conventional FLC use membership function generated by human operators. The membership function selection process is done with trial and error, and it runs step by step which takes too long in solving the problem. This paper develops a system that may help users to determine the membership function of FLC using the GA optimization for the fastest processing in solving the problems. The data collection is based on the simulation results, and the results refer to the transient response specification which is maximum overshoot. From the results presented, we will get a better and exact result; the value of overshot is decreasing from 1.2800 for FLC without GA to 1.0081 with GA (FGA).

## 1. Introduction

Clearly automatic control has played an important role in the advance of engineering and science. In addition to its extreme importance in robotic systems, and at time, automatic control has become an important and integral part of modern manufacturing and industrial processes. Automatic control is essential in such industrial operations as controlling pressure, temperature, humidity, viscosity, and flow in the process industries.

While modern control theory [1] has been easy to practice, fuzzy logic controllers (FLC) have been rapidly gaining popularity among practicing engineers. This increase of popularity can be attributed to the fact that fuzzy logic provides a powerful vehicle that allows engineers to incorporate human reasoning in the control algorithm.

In our daily life from the production lines in manufacturing plants, medical equipment, and agriculture to the consumer products such as washing machine and air conditioner, FLC can be applied. As for an example, the controller temperature set for incubating egg. Four factors are of major importance in egg incubation artificially: temperature, humidity, ventilation, and turning. Of these factors, temperature is the most critical. If the temperatures are not accurately controlled, the incubated egg will not be uniform. One of the problems with the incubate-egg systems occurs in the design of the temperature controllers. The temperature and relative humidity of incubator should be set at the specific value as proposed by Du et al. [2]. And it is very easy to overheat the eggs in incubators and difficult to maintain proper humidity.

Preferably, these controllers are designed with a high sensitivity to disturbance signals. However, when a change in a temperature set point occurs, there is a danger in saturating the zone temperature controllers as the magnitude of the temperature set point changes are generally greater than the magnitude of disturbances. Hence, the sensitivity of the controller to disturbance signals must be reduced to prevent

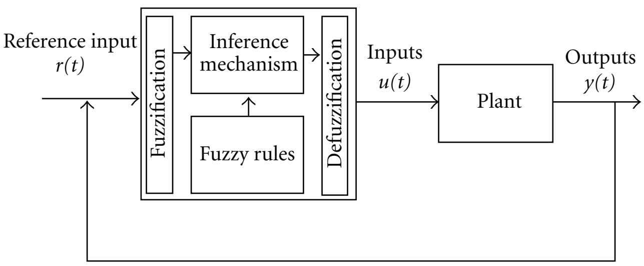



Figure 1: Fuzzy controller architecture.

saturation of the controllers to set point changes. Thus it is important to select the accurate membership functions for temperature setting an incubate egg systems.

Taking the above explanation, we propose to use control system based on FLC. The important part in FLC is during the process in selecting the membership function. The membership function of a fuzzy set is a generalization of the indicator function in classical sets. In fuzzy logic, it represents the degree of an extension of valuation.

Conventional FLC used membership function generated by human operators, who have been manually designing the membership function of FLC. To satisfy such requirements including one common weakness where the membership function selection process is done with trial and error, it runs step by step, which is too long in completing the problem.

A new approach for optimum coding of fuzzy controllers is using GA. GA is used to determine membership function especially designed in situations. We use GA to tune the membership function for terms of each fuzzy variable.

## 2. Fuzzy Logic Control

Fuzzy control provides a formal methodology for representing, manipulating, and implementing a human's heuristic knowledge about how to control a system. The fuzzy logic controller block diagram is given in Figure 1, it shows a fuzzy controller integrated in a closed-loop control system. The *plant output* is denoted by $y(t)$, the *plant input* is denoted by $u(t)$, and the *reference input* to the *fuzzy controller* is denoted by $r(t)$.

Basically, we can view the fuzzy controller as an artificial decision maker that operates in a closed-loop system in real-time. In gathered plant output data $y(t)$, compare it to the reference input $r(t)$, and then decide what the plant input $u(t)$ should be to ensure that the performance objectives will be met [3].

Performance of various control system can be analyzed by concentrating time response. The time response of a control system consists of two parts: the transient response and the steady-state response. The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In this research, specifying the transient-response characteristics of a control system to a unit-step input it is common to specify the maximum (percent) overshoot. The maximum overshoot ($M_p$) is the maximum peak value of the response curve measured from unity.

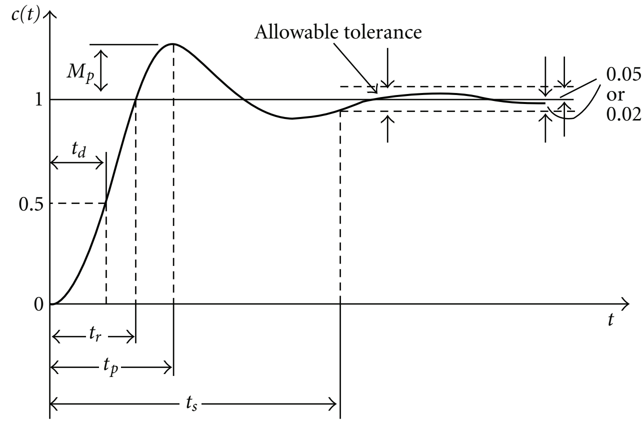



Figure 2: Transient and steady-state response.

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system [1, 3, 4]. These specifications are defined in what follows and are shown graphically in Figure 2.

## 3. Genetic Algorithm

The GA borrow ideas and attempt to simulate Darwin's theory on natural selection and Mandel's work in genetics on inheritance. The usual form of genetic algorithms was described by Goldberg [5]. Genetic algorithms are stochastic search techniques based on the mechanism of natural selection and natural genetics. Genetic algorithms, differing from conventional search techniques, start with an initial set of random solutions called "population." Each individual in the population is called a "chromosome," representing a solution to the problem at hand. For three variable problems hence, chromosomes will arrange three genes.

The chromosomes evolve through successive iterations, called generations. During each generation, the chromosomes are evaluated, using some measures of fitness. To create the next generation, new chromosomes, called offspring, are formed by crossover and mutation operator. A new generation is formed by selecting and rejecting. Fitter chromosomes have higher probabilities of being selected. After several generations, the algorithms converge to the best chromosome, which hopefully represents the optimal solution of the problem.

## 4. Design and Implementation

Basically, genetic algorithms (GA) have had a great measure of success in search and optimization problems. In this research, the GA are used to improve the performance of the fuzzy controller. Considering that the main attribute of the GA is its ability to solve the topological structure of an unknown system, then the problem of determining the fuzzy membership functions can also fall into this category.

For obtaining final (tuned) membership function by using GA, some functional mapping of the system will be given. Parameters of the initial membership function

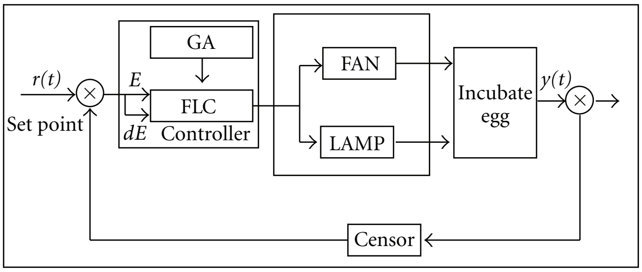



Figure 3: Block diagram of incubate egg by fuzzy genetic algorithms.

are then generated and coded as real numbers that are concatenated to make one long string to represent the whole parameter set of the membership function. A fitness function is then used to evaluate the fitness value of each set of membership function. Then the reproduction, crossover, and mutation operators are applied to obtain the optimal population (membership function), or more precisely, the final tuned value describes the membership function which is proposed.

Having now learnt the complicated procedures of designing FLC, a practical realization of this system is not easy to determine the *membership function* in FLC.

The conceptual idea is to have an automatic and intelligent scheme to tune the *fuzzy membership functions* of the closed-loop control for incubate egg, as indicated in Figure 3.

*4.1. Data Structures.* The most important data structures in GA are those that represent genes and chromosomes. Most researchers represent a chromosome as a string that is a binary code of a set of genes, but in our case, the real numbers code will be used; *real numbers code* is a more natural representation than *binary code* [6]. In our case, each *gene* corresponds to one *linguistic variable* whose definition is what the GA tries to evolve. In *fuzzy systems*, we represent the value of a *linguistic variable* by *membership function* and for our simulation design; there are three kinds of *linguistic variables*: the variable error signal as input-1 is parameter $X$, the rate of change in error as input-2 is parameter $Y$, and then the controller output is parameter $Z$, as shown in Figure 4.

For every variable, there are five shapes of *membership functions*; three are triangular and two trapezoidal ones. If *membership function* has triangular form, then it can be described by three parameters, a fixed number of *real number* is used to define each of the three parameters (which define completely the specific triangular membership function): if it is trapezoidal, it requires four parameters, as shown in Figure 5.

*4.2. Fitness Function.* An individual is evaluated, based on a certain function as the measurement performance. In the evolution of the nature, the highest valuable individual fitness will survive whereas the low valuable individual will die. The fitness calculation is a measure to know what the best particular solution to resolve the problem is.

The fitness function is the basis of the survival of the fittest premise of genetic algorithms. It is responsible for evaluating the parameter sets, and choosing which parameter sets are suitable. Since the fuzzy controller operates in a closed-loop specification, it can be analyzed by the maximum overshoot [1].

The *maximum overshoot* ($M_p$) is the maximum peak value of the response curve measured from the unity, and the amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

*4.3. Genetic Parameters.* An individual is evaluated. The decision to make in implementing a genetic algorithm is to set the values for the various parameters, such as population size, probability of crossover rate, and probability of mutation rate. These parameters typically interact with one another nonlinearly, so they cannot be optimized for all situations. There are no conclusive results on what is the best; most people use what has worked well in previously reported cases. In our case, we use population size of 5, 10, and 100 for comparing, while the probability of crossover rate is 0.1, 0.7, and 0.9 where the probability of mutation rate is 0.001, 0.05, and 0.1.

*4.4. Termination Conditions.* Genetic algorithms will typically run forever, until an appropriate termination condition is reached. For our research, the termination condition was the one that defines the maximum number of generations to be produced. When the generation number is completed by the GA, the new populations generating process is finished, and the best solution is the one among the individuals more adapted to the evaluation function.

## 5. Result and Analysis

As mentioned before, most important to implementing a genetic algorithm for improving the performance is how to set values of the various parameters, such as population size, the probability of crossover, and mutation rate. These parameters typically interact with each other.

1st experiment: combination of population size: (5, 0.7, 0.001), (10, 0.7, 0.001), and (100, 0.7, 0.001).

2nd experiment: combination of mutation rate: (10, 0.7, 0.001), (10, 0.7, 0.005), and (10, 0.7, 0.05).

3rd experiment: combination of crossover rate: (10, 0.1, 0.001), (10, 0.7, 0.001), and (10, 0.9, 0.001).

4th experiment: combination of crossover rate: (10, 0.1, 0.05), (10, 0.7, 0.05), and (10, 0.9, 0.05).

Referring to researches available beforehand [7, 8], it seems that (10, 0.7, 0.001) is the best combination rate, and therefore that value is chosen for the tests, although further tests show that it does not give even optimum results.

For the first step, we make a comparison of convergence rates for populations of 5, 10, and 100 individuals; the probability of crossover rate is 0.7, and the probability of mutation rate is 0.001. All the data can be shown in the

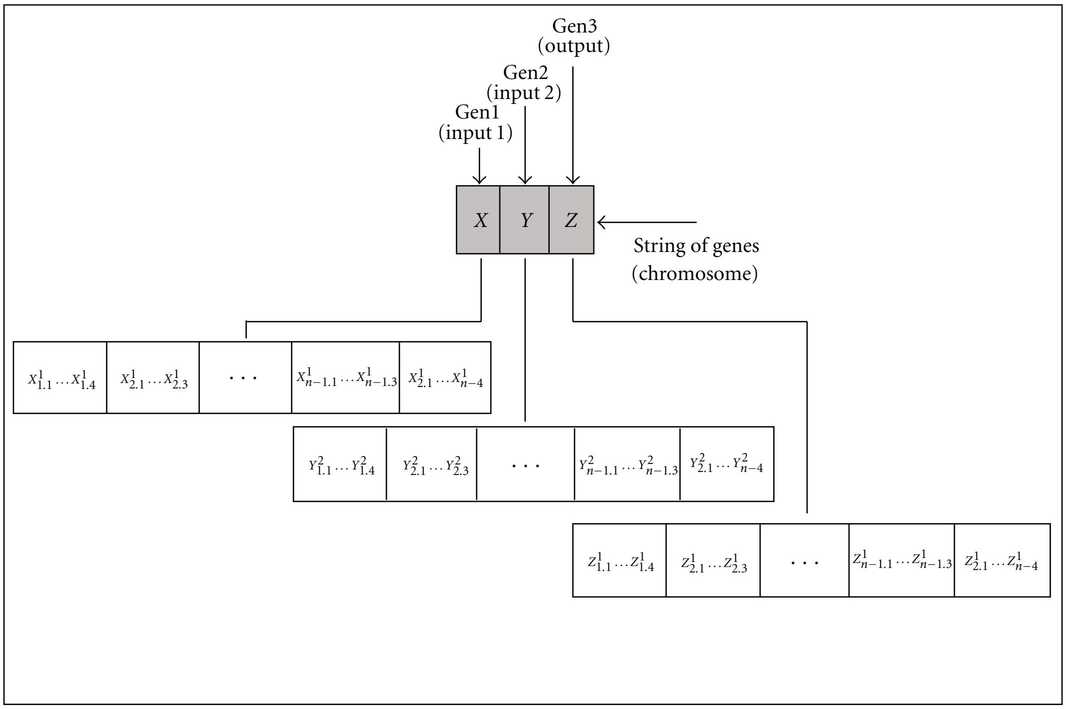



FIGURE 4: Structure of chromosome.

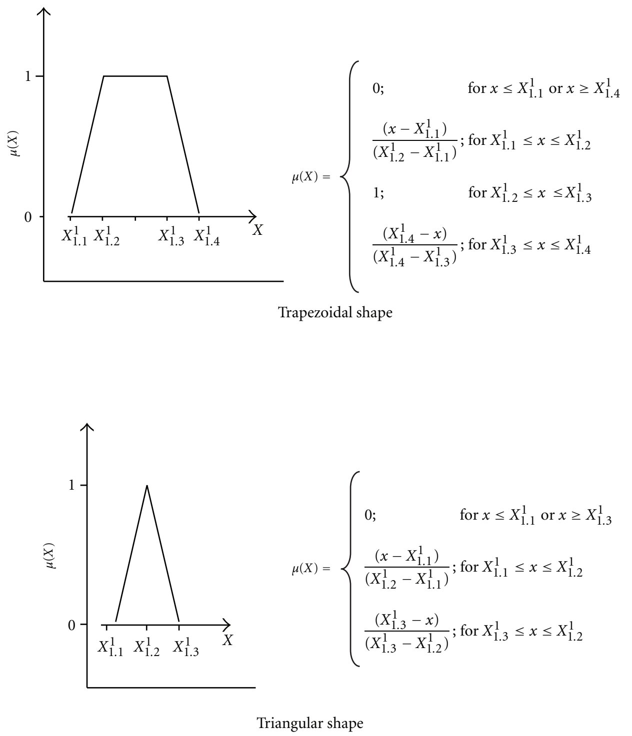



FIGURE 5: Mathematical characterization trapezoidal and triangular MF.

following graph in Figure 6(a). These results show that the performance of the GA for first generation with a population size of 5 and 10 are the same, that is, 0.0799. Then this result is increased for next generation till the final generation. When the population size is 5, the fitness is 0.0817 at the 25th generation while the fitness is 0.0855 at the 25th generation when the population size is 10.

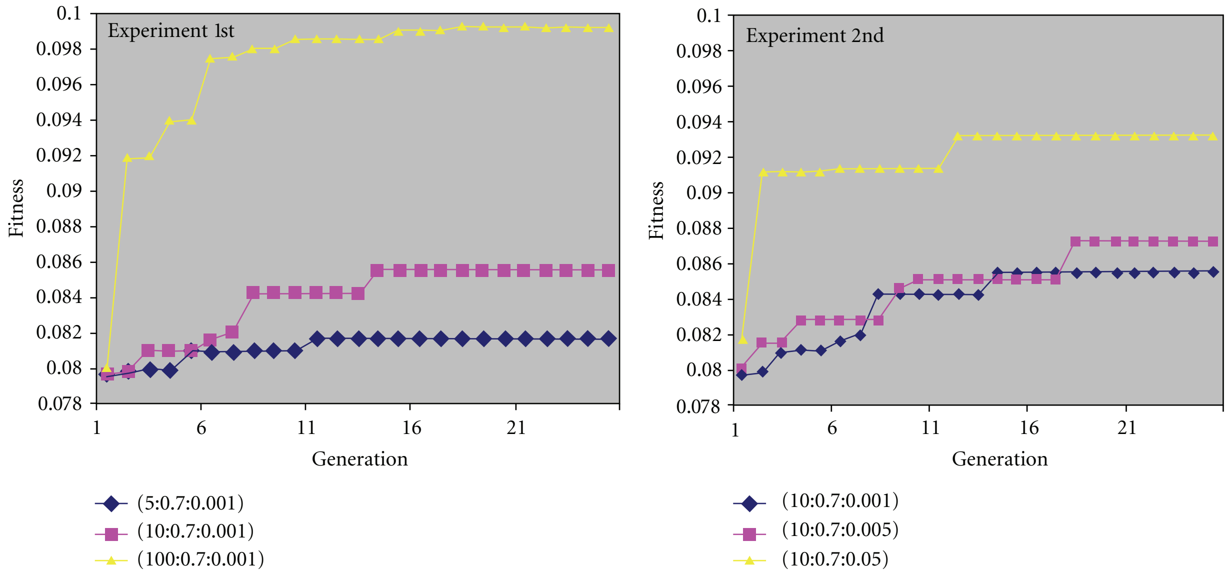
For the best result, the fitness is 0.0992 at 25th generation for the training data when the population size is 100, but very significantly, the consumed time is increased from 377 minutes when the population size is 10 to 985 minutes when the population size is 100. These results show that the performance of our GA is very sensitive to the population size.

From Figure 6(b), we can see the effect of a probability mutation rate on the fitness value. If the value of mutation rate is high, the fitness value gets better. The highest fitness value is 0.0932 for a probability mutation rate of 0.05. Secondly, it is 0.0872 (probability mutation rate = 0.005). The lowest value is 0.0855 (probability mutation rate = 0.001). For all these values the probability of crossover rate and the size of population are the same.

Figure 7(a) shows a comparison of convergence rates for three values of crossover rate 0.1, 0.7, and 0.9. In this case, the combination parameter (10, 0.9, 0.001) shows the best performance of GA, while the combination parameter (10, 0.1, 0.001) shows the lowest performance GA.

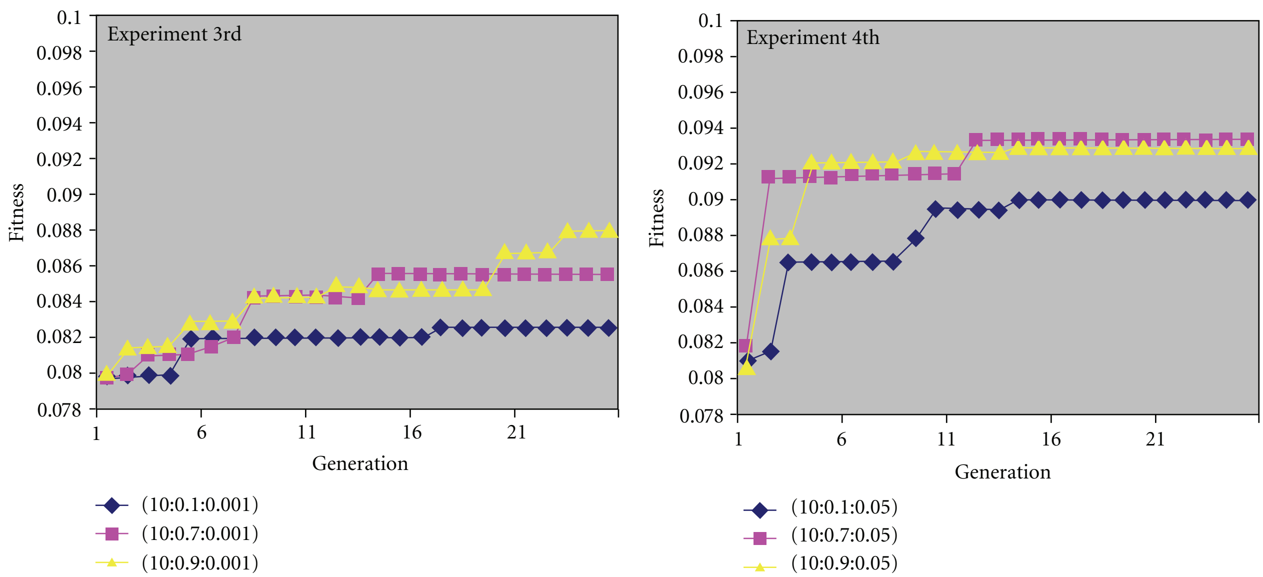
We can see the effect of a probability crossover rate on the fitness value. If the value of crossover rate is high, the fitness value gets better. The highest fitness value is 0.0880 for a probability crossover rate of 0.9. Secondly, it is 0.0855 (probability crossover rate = 0.7). The lowest value is 0.0825 (probability crossover rate = 0.1). For all these values the probability of mutation rate and the size of population are the same.

The Figure 7(b) shows a comparison of convergence rates for three values of crossover rate 0.1, 0.7, and 0.9. In this




(a) (b)

Figure 6: Comparison of convergent for (a) population size and (b) mutation rate.



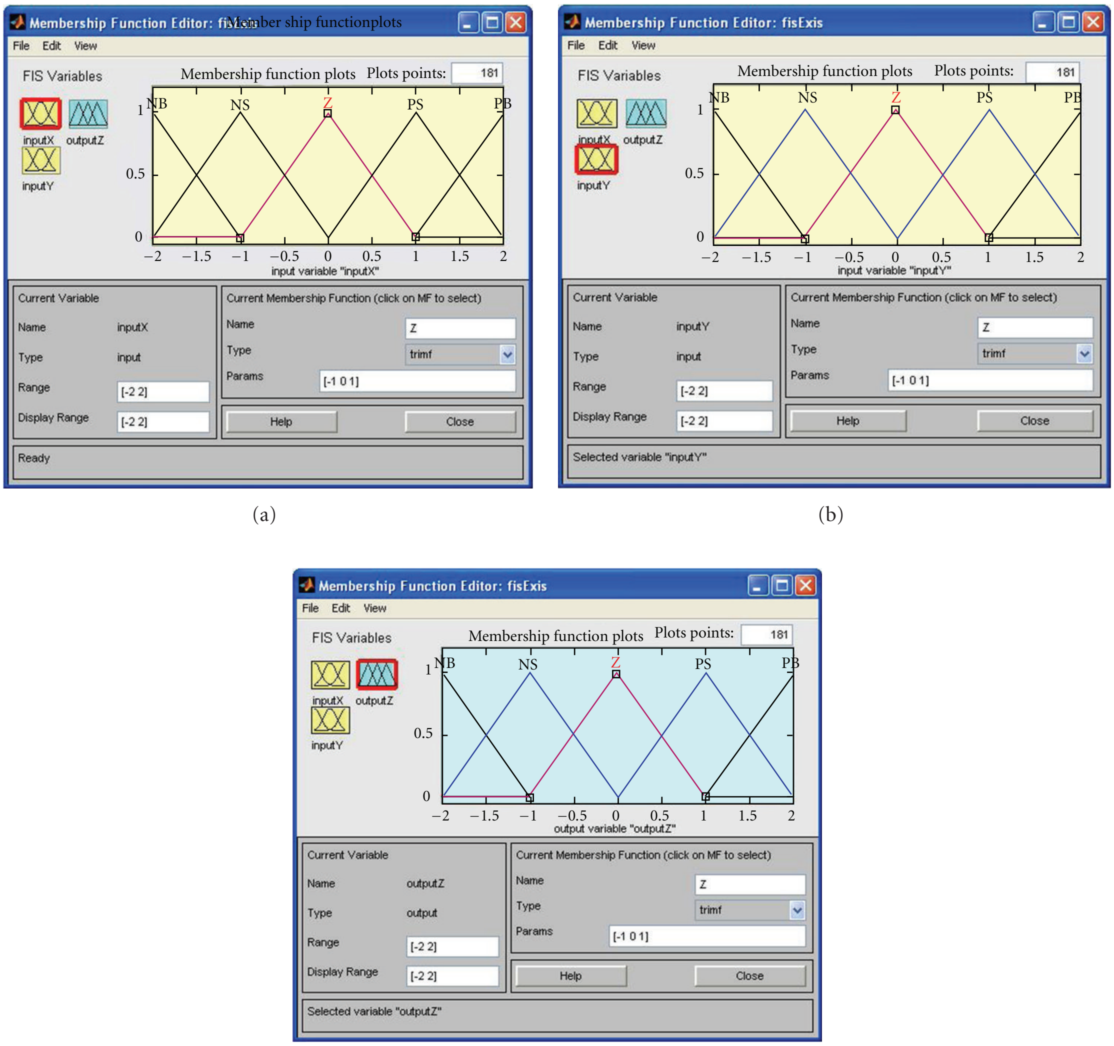

(a) (b)

Figure 7: Comparison of convergent for crossover rate while mutation rates set to (a) 0.001 and (b) 0.05.

case, the combination parameters (10, 0.7, 0.05) show the best performance of GA while the combination parameters (10, 0.1, 0.05) show the lowest performance GA.

The comparison between Figures 7(a) and 7(b) shows the fitness value only a little bit different for probability crossover rates are 0.7 and 0.9, when the probability mutation rate sets to 0.05. This situation shows that the values of probability crossover and mutation rate interact; both of them will affect each other. The determination of the probability crossover and mutation rate is more important. The interaction between crossover rate and mutation rate is significant; both of them will affect each other. The parameters settings vary from problem to problem.

From the results presented in this experiment, the system which we developed is very helpful to determine the membership function for the fastest processing in completing the problem. Figures 8(a), 8(b), and 8(c) show screen produced from system; the membership function exists (without GA) and will be used in initial population for the first chromosome, in the first population and generation (existing membership function, without GA). Compare with Figures 9(a), 9(b), and 9(c), GA were applied into fuzzy




(a) (b) (c)

Figure 8: Existing membership (without GA): (a) input-1, (b) input-2, and (c) output.

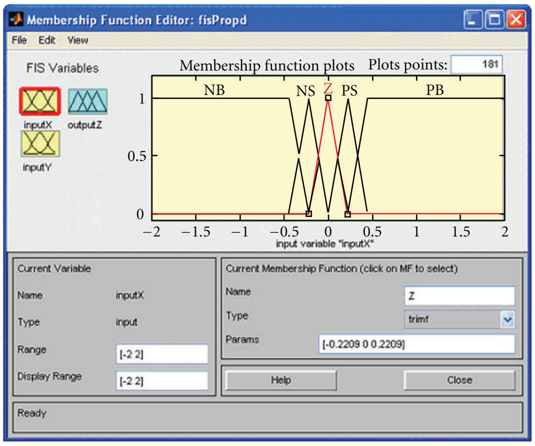
system to determine the membership function with GA/ FGA (proposed membership function).

During the execution of the GA, the fitness of each result is recorded. After evolution is complete, the evolved membership function is tested using the data and the results are compared with both FLC with and without GA. Generally speaking, after the membership function has been tuned with the GA, the improvement in performance of the FLC by using the GA is encouraged. We can compare fuzzy logic with and without genetic algorithms.

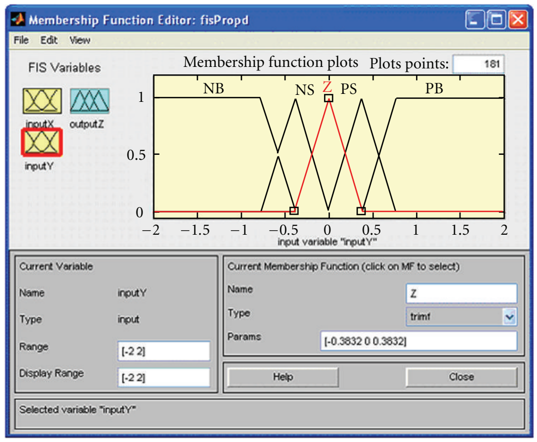
It is clear that GA are very promising in improving the performance of the FLC, to get more accurate in order to find the optimum result. From Figures 10 and 11, we can see that, after the execution of program and end of GA, the membership function is regulated automatically. We will get a better and exact result; the value of overshot is decreasing from 1.2800 for FLC without GA, to 1.0081 for FGA.

This is a research using *roulette wheel* or *fitness-based* method selection. But even each chromosome has a chance of selection directly proportional to its fitness but is done in a random manner, and then the effect of this situation causes the possibility of chromosomes with good fitness not joining as well as in the next generation because of not slipping away the selection. Therefore, we need the existence of a method of combination in the level of this selection that can maintain chromosomes with good fitness value so as these chromosomes can be maintained in the following generation.

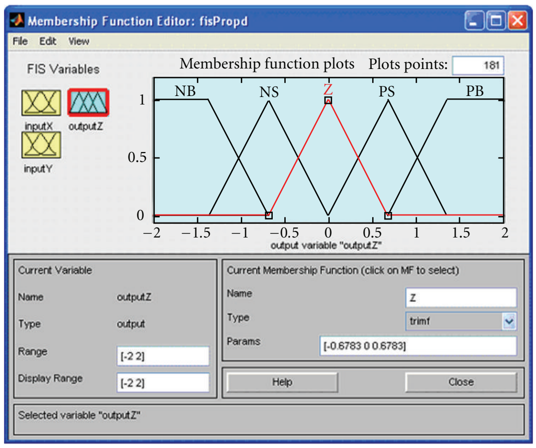
Our research proposes one method, by recording the value of fitness of each chromosome for every generation. Afterwards we will choose chromosome with the best fitness to enter the further generation directly. It was stated [5, 9, 10] that GA can be represented by a sequence of procedural steps for moving from one population of artificial "chromosomes" to a new population. GA uses "natural" selection and genetic-inspired techniques known as crossover and mutation. Nature has an ability to adapt and learn without being told what to do. In other words, nature finds good chromosomes blindly.




(a)




(b)




(c)

FIGURE 9: Existing membership (with GA/FGA): (a) input-1, (b) input-2, and (c) output.

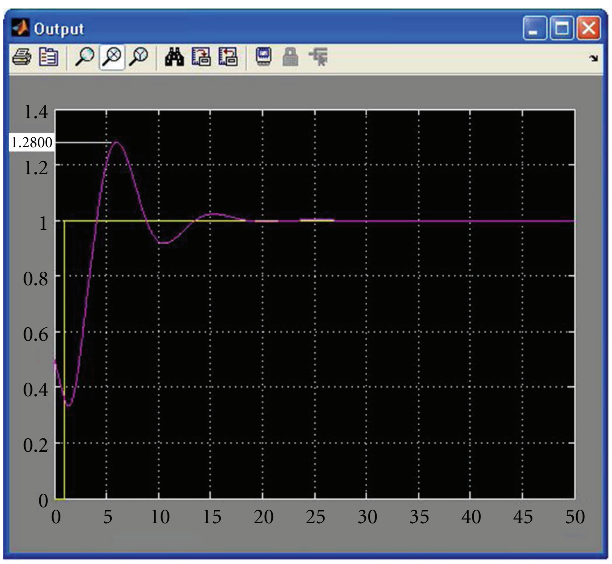



FIGURE 10: Response system for existing membership function (without GA).

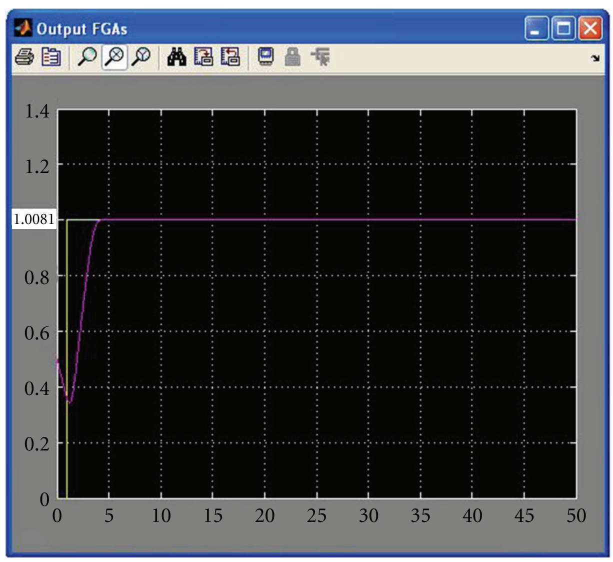



FIGURE 11: Response system for proposed membership function (using GA/FGA).

From our experiment, we found several matters which become important factors that influence performance of GA. They can be considered as follows.

(1) The selection procedure is used to choose suitable candidates for mating.

(2) The variable $\alpha$ for scaling factor in arithmetic crossover method.

(3) The value of $r$ and $b$ in dynamic or nonuniform mutation method.

## 6. Conclusion and Future Research

GA has been successfully applied to solve many optimization problems. In this research, GA are implemented to a system (programming language) for determining the *membership function* of FLC. By designing compact data structures for genes and chromosomes and an accurate fitness evaluation function, GA have been implemented which is very effective in finding more accurate membership functions for the fuzzy system. The data structures adopted are compact, and thus very convenient to manipulate by genetic operators.

From our experiment, we found that the *population size* was a significant factor to improve the performance of GA. Generally speaking, the larger *population size* will be better for performance of GA, but longer in processing time. A larger *population size* will be more diverse and thus will contain more *chromosomes*. A reasonable assumption held here is that, when more chromosomes are present, more good *chromosomes* will be present in the population. This is helpful to achieve a better solution.

GA need a longer time if probability of crossover and mutation rate is higher. So the interaction between *crossover rate* and *mutation rate* is significant; both of them will affect each other. The parameters settings vary from problem to problem.

In this research,

(i) combination parameter (10, 0.9, 0.05) will give the best value for solving our problem; the value of fitness is 0.0928 (maximum overshoot = 1.0776) and the processing time is 723 minutes.

(ii) Combination parameter (100, 0.7, 0.001) will give the best fitness value; the value of fitness is 0.0992 (maximum overshoot = 1.0081) and the processing time is 985 minutes.

The performance of GA can be further improved by using different combinations of selection strategies, *crossover* and *mutation methods*, and other genetic parameters such as population size, probability of crossover, and mutation rate.

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*Research Article*

# Deterministic Annealing Approach to Fuzzy C-Means Clustering Based on Entropy Maximization

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This paper is dealing with the fuzzy clustering method which combines the deterministic annealing (DA) approach with an entropy, especially the Shannon entropy and the Tsallis entropy. By maximizing the Shannon entropy, the fuzzy entropy, or the Tsallis entropy within the framework of the fuzzy c-means (FCM) method, membership functions similar to the statistical mechanical distribution functions are obtained. We examine characteristics of these entropy-based membership functions from the statistical mechanical point of view. After that, both the Shannon- and Tsallis-entropy-based FCMs are formulated as DA clustering using the very fast annealing (VFA) method as a cooling schedule. Experimental results indicate that the Tsallis-entropy-based FCM is stable with very fast deterministic annealing and suitable for this annealing process.

## 1. Introduction

Statistical mechanics investigates the macroscopic properties of a physical system consisting of several elements. Recently, research activities that attempt to apply statistical mechanical models or tools to information science have become popular.

The deterministic annealing (DA) method [1] is a deterministic variant of the simulated annealing (SA) method [2]. DA characterizes the minimization problem of the cost function as the minimization of the free energy, which depends on temperature and tracks its minimum while decreasing the temperature, and thus it can deterministically optimize the cost function at each temperature. Hence, DA is more efficient than SA, but does not guarantee a global optimal solution.

There exists a strong relationship between the membership functions of the fuzzy c-means (FCM) clustering [3] with the maximum entropy or entropy regularization methods [4, 5] and the statistical mechanical distribution function. That is, FCM regularized with the Shannon entropy gives a membership function similar to the Boltzmann (or Gibbs) distribution function [1, 4], and FCM regularized with the fuzzy entropy [6] gives a membership function similar to the Fermi-Dirac distribution function [7]. These membership functions are suitable for the annealing methods because they contain a parameter corresponding to the system temperature.

Tsallis [8] achieved nonextensive extension of the Boltzmann-Gibbs statistics. Tsallis postulated a generalization form of entropy with a generalization parameter $q$, which, in a limit of $q \rightarrow 1$, reaches the Shannon entropy. Later on, Ménard et al. [9] derived a membership function by regularizing FCM with the Tsallis entropy.

In this study, the membership function which takes the familiar form of the statistical mechanical distribution function is derived by maximizing the Shannon and fuzzy entropy within the framework of FCM. Similarly, the Tsallis entropy-based FCM membership function is derived [10, 11] by maximizing the Tsallis entropy. Then, the formulations of the free energy for these membership functions are calculated and examined from the statistical mechanical viewpoint.

On the other hand, there are some representative cooling schedules of the temperature for SA; for example, inversely proportional to a logarithmic function and inversely proportional to exponential function are well adopted. Rosen [12] proposed the more effective method for SA known as very fast annealing (VFA).

However, an applicability of VFA to DA is not known yet. In order to achieve good clustering by DA, a reliable annealing process is desirable. Therefore, by introducing VFA to DA, we formulate the Shannon- and Tsallis-entropy based FCMs as very fast DA clustering, to examine their reliabilities.

Experiments are performed on the numerical and iris data [13], and the obtained results indicate that Tsallis-entropy-based FCM clustering is suitable for very fast DA clustering because of its shape of the membership function.

## 2. Entropy Maximization Method

Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}(\mathbf{x}_k = (x_k^1, \ldots, x_k^p) \in R^p)$ be a data set in the $p$-dimensional real space, which should be divided into $c$ clusters. In addition, let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_c\}(\mathbf{v}_i = (v_i^1, \ldots, v_i^p))$ be the centers of clusters, and let $u_{ik} \in [0, 1]$ $(i = 1, \ldots, c;\ k = 1, \ldots, n)$ be the membership functions. Furthermore, let

$$J = \sum_{k=1}^{n} \sum_{i=1}^{c} u_{ik}^m d_{ik} \quad (m > 1) \tag{1}$$

be the objective function of FCM, where $d_{ik} = \|\mathbf{x}_k - \mathbf{v}_i\|^2$.

### *2.1. Shannon Entropy Maximization of FCM.*

First, we introduce the Shannon entropy into the FCM clustering. The Shannon entropy is given by

$$S = -\sum_{k=1}^{n} \sum_{i=1}^{c} u_{ik} \log u_{ik}. \tag{2}$$

Under the normalization constraint of

$$\sum_{i=1}^{c} u_{ik} = 1 \quad (\forall k), \tag{3}$$

and setting $m$ to 1, the fuzzy entropy functional is given by

$$\delta S - \sum_{k=1}^{n} \alpha_k \delta \left( \sum_{i=1}^{c} u_{ik} - 1 \right) - \beta \sum_{k=1}^{n} \sum_{i=1}^{c} \delta(u_{ik} d_{ik}), \tag{4}$$

where $\alpha_k$ and $\beta$ are the Lagrange multipliers and $\alpha_k$ must be determined so as to satisfy (3). The stationary condition for (4) leads to the following membership function

$$u_{ik} = \frac{e^{-\beta d_{ik}}}{\sum_{j=1}^{c} e^{-\beta d_{jk}}} \tag{5}$$

and the cluster centers

$$\mathbf{v}_i = \frac{\sum_{k=1}^{n} u_{ik} \mathbf{x}_k}{\sum_{k=1}^{n} u_{ik}}. \tag{6}$$

### *2.2. Fuzzy Entropy Maximization of FCM.*

We then introduce the fuzzy entropy into the FCM clustering.

The fuzzy entropy is given by

$$\hat{S} = -\sum_{k=1}^{n} \sum_{i=1}^{c} \{\hat{u}_{ik} \log \hat{u}_{ik} + (1 - \hat{u}_{ik}) \log(1 - \hat{u}_{ik})\}. \tag{7}$$

The fuzzy entropy functional is given by

$$\delta \hat{S} - \sum_{k=1}^{n} \alpha_k \delta \left( \sum_{i=1}^{c} \hat{u}_{ik} - 1 \right) - \beta \sum_{k=1}^{n} \sum_{i=1}^{c} \delta(\hat{u}_{ik} d_{ik}), \tag{8}$$

where $\alpha_k$ and $\beta$ are the Lagrange multipliers [14]. The stationary condition for (8) leads to the following membership function:

$$\hat{u}_{ik} = \frac{1}{e^{\alpha_k + \beta d_{ik}} + 1} \tag{9}$$

and the cluster centers

$$\mathbf{v}_i = \frac{\sum_{k=1}^{n} \hat{u}_{ik} \mathbf{x}_k}{\sum_{k=1}^{n} \hat{u}_{ik}}. \tag{10}$$

In (9), $\beta$ defines the extent of the distribution [7]. Equation (9) is formally normalized as

$$\hat{u}_{ik} = \frac{1}{e^{\alpha_k + \beta d_{ik}} + 1} \Big/ \sum_{j=1}^{c} \frac{1}{e^{\alpha_k + \beta d_{jk}} + 1}. \tag{11}$$

### *2.3. Tsallis Entropy Maximization of FCM.*

Let $\tilde{\mathbf{v}}_i$ and $\tilde{u}_{ik}$ be the centers of clusters and the membership functions, respectively.

The Tsallis entropy is defined as

$$\tilde{S} = -\frac{1}{q - 1} \left( \sum_{k=1}^{n} \sum_{i=1}^{c} \tilde{u}_{ik}^q - 1 \right), \tag{12}$$

where $q \in \mathbf{R}$ is any real number. The objective function is rewritten as

$$\tilde{U} = \sum_{k=1}^{n} \sum_{i=1}^{c} \tilde{u}_{ik}^q \tilde{d}_{ik}, \tag{13}$$

where $\tilde{d}_{ik} = \|\mathbf{x}_k - \tilde{\mathbf{v}}_i\|^2$.

Accordingly, the Tsallis entropy functional is given by

$$\delta \tilde{S} - \sum_{k=1}^{n} \alpha_k \delta \left( \sum_{i=1}^{c} \tilde{u}_{ik} - 1 \right) - \beta \sum_{k=1}^{n} \sum_{i=1}^{c} \delta \left( \tilde{u}_{ik}^q \tilde{d}_{ik} \right). \tag{14}$$

The stationary condition for (14) yields the following membership function:

$$\tilde{u}_{ik} = \frac{\left\{1 - \beta(1 - q) \tilde{d}_{ik}\right\}^{1/(1-q)}}{\tilde{Z}}, \tag{15}$$

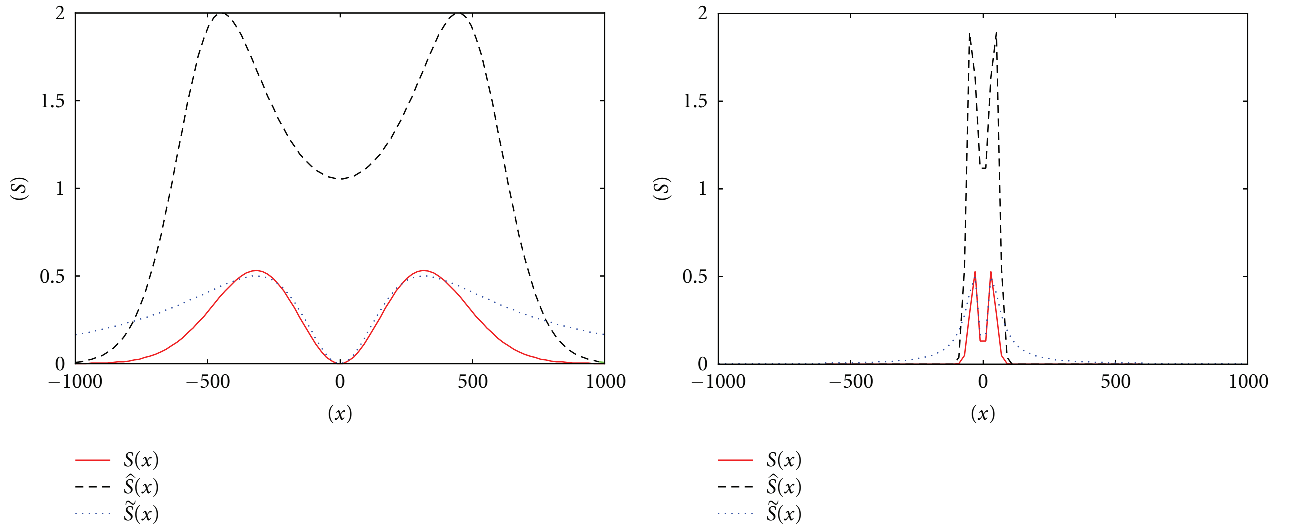
where

$$\tilde{Z} = \sum_{j=1}^{c} \left\{1 - \beta(1 - q) \tilde{d}_{jk}\right\}^{1/(1-q)}. \tag{16}$$

In this case, the cluster centers are given by

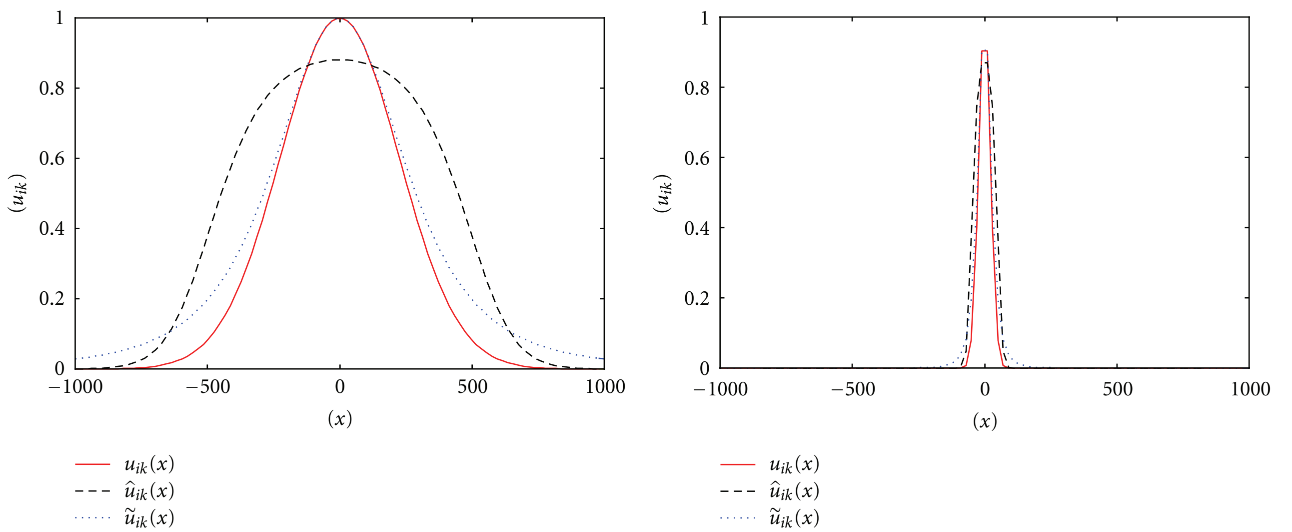
$$\tilde{\mathbf{v}}_i = \frac{\sum_{k=1}^{n} \tilde{u}_{ik}^q \mathbf{x}_k}{\sum_{k=1}^{n} \tilde{u}_{ik}^q}. \tag{17}$$

In the limit of $q \to 1$, the Tsallis entropy recovers the Shannon entropy [8] and $\tilde{u}_{ik}$ approaches $u_{ik}$ in (5).




(a) $T = 10^5$ (b) $T = 10^3$

Figure 1: The plots of the entropy functions $S$, $\overline{S}$ and $\widetilde{S}$ at (a) high and (b) low temperature ($n = 1$, $c = 2$, $q = 1.5$, $\alpha_k = -2$).




(a) $T = 10^5$ (b) $T = 10^3$

Figure 2: The plots of the membership functions $u_{ik}$, $\overline{u}_i k$ and $\widetilde{u}_{ik}$ at (a) high and (b) low temperature ($n = 1$, $c = 2$, $q = 1.5$, $\alpha_k = -2$).

## 3. Statistical Mechanical Interpretation of Entropy-Based FCM

*3.1. Shannon-Entropy-Based FCM Statistics.* In the Shannon-entropy-based FCM, the sum of the states (the partition function) for the grand canonical ensemble of fuzzy clustering can be written as

$$Z = \prod_{k=1}^{n} \sum_{i=1}^{c} e^{-\beta d_{ik}}. \tag{18}$$

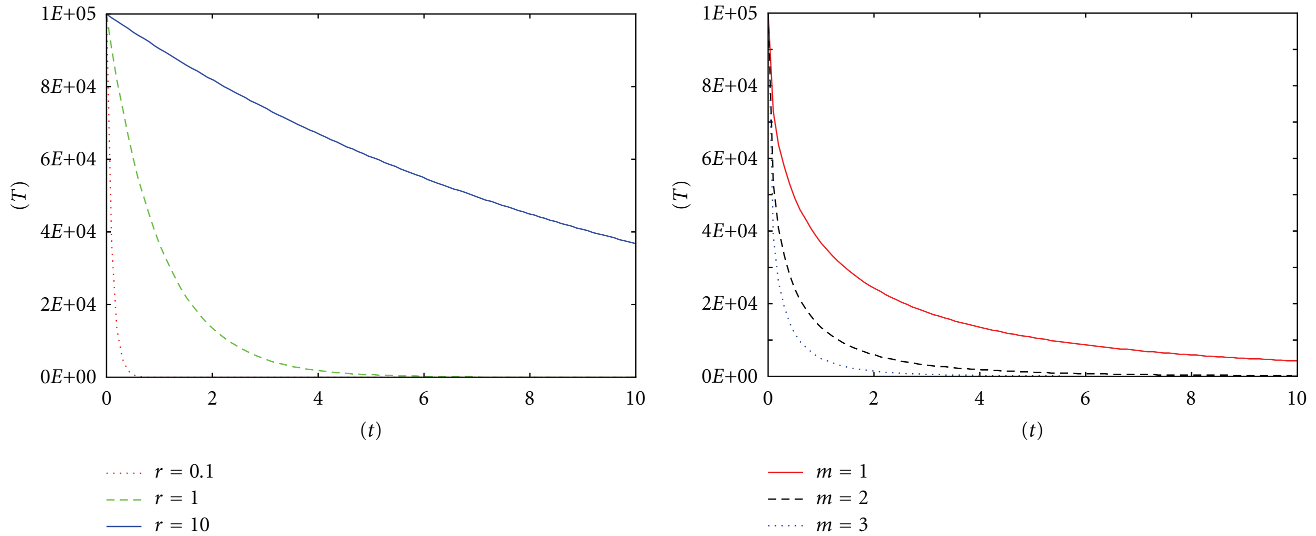
By substituting (18) for $F = -(1/\beta)(\log Z)$ [15], the free energy becomes

$$F = -\frac{1}{\beta} \sum_{k=1}^{n} \log \left\{ \sum_{i=1}^{c} e^{-\beta d_{ik}} \right\}. \tag{19}$$

Stable thermal equilibrium requires a minimization of the free energy. By formulating deterministic annealing as a minimization of the free energy, $\partial F / \partial \mathbf{v}_i = 0$ yields

$$\mathbf{v}_i = \frac{\sum_{k=1}^{n} u_{ik} \mathbf{x}_k}{\sum_{k=1}^{n} u_{ik}}. \tag{20}$$

This cluster center is the same as that in (6).




(a) Inversely proportional to an exponential function

(b) Very fast annealing

Figure 3: The plots of the cooling functions of (a) proportional to an exponential ($r = 0.1, 1, 10$) and (b) very fast annealing methods ($m = 1.0, 2.0, 3.0$).

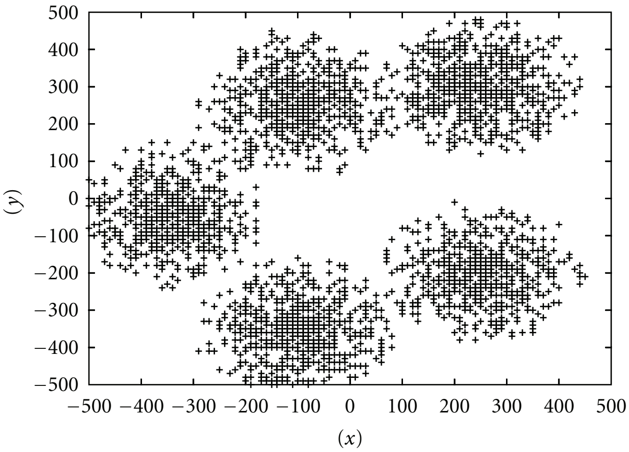



Figure 4: The numerical data.

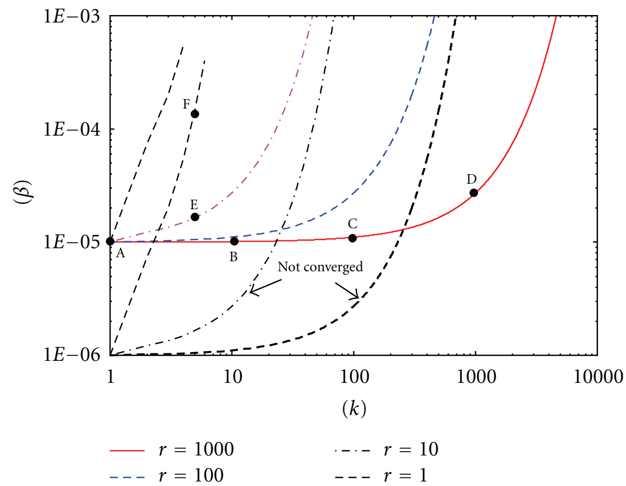



Figure 5: The inversely exponential cooling schedule of DA. The temperature decreases from $T_{\text{high}}$. (Inverse of temperature $\beta$ increases from $1.0 \times 10^{-5}$ or $1.0 \times 10^{-6}$ to $1.0 \times 10^{-3}$) The curves are parameterized by the temperature reduction rate $r$.

*3.2. Fuzzy-Entropy-Based FCM Statistics.* In the fuzzy-entropy based FCM, by analogy with statistical mechanics, the grand partition function for the grand canonical ensemble of fuzzy clustering can be written as

$$\hat{\Xi} = \prod_{k=1}^{n} \prod_{i=1}^{c} \left(1 + e^{-\alpha_k - \beta d_{ik}}\right), \tag{21}$$

because data can belong to any cluster. By substituting (21) for $\hat{F} = -(1/\beta)(\log \hat{\Xi} - \alpha_k \partial \log \hat{\Xi} / \partial \alpha_k)$ [15], the free energy becomes

$$\hat{F} = -\frac{1}{\beta} \sum_{k=1}^{n} \left\{ \sum_{i=1}^{c} \log\left(1 + e^{-\alpha_k - \beta d_{ik}}\right) + \alpha_k \right\}. \tag{22}$$

It should be noted that $J_{m=1} - T\hat{S}$, the Legendre transform of the fuzzy entropy, gives the same form for the free energy.

*3.3. The Tsallis-Entropy-Based FCM Statistics.* On the other hand, $\tilde{U}$ and $\tilde{S}$ satisfy

$$\tilde{S} - \beta \tilde{U} = \sum_{k=1}^{n} \frac{\tilde{Z}^{1-q} - 1}{1 - q}, \tag{23}$$

which leads to

$$\frac{\partial \tilde{S}}{\partial \tilde{U}} = \beta. \tag{24}$$

Equation (24) makes it possible to regard $\beta^{-1}$ as an artificial system temperature $T$ [15]. Then, the free energy can be

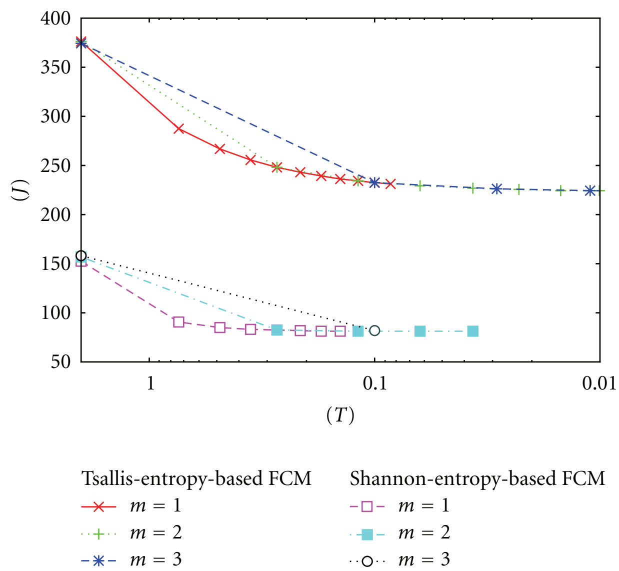



FIGURE 6: Reduction of the objective values for iris data with decreasing the temperature by VFA. The curves are parameterized by the temperature reduction rate $m$.

defined as

$$\tilde{F} = \tilde{U} - T\tilde{S} = -\frac{1}{\beta}\sum_{k=1}^{n}\frac{\tilde{Z}^{1-q} - 1}{1-q}. \tag{25}$$

$\tilde{U}$ can be derived from $\tilde{F}$ as

$$\tilde{U} = -\frac{\partial}{\partial\beta}\sum_{k=1}^{n}\frac{\tilde{Z}^{1-q} - 1}{1-q}. \tag{26}$$

$\partial\tilde{F}/\partial\tilde{\mathbf{v}}_i = 0$ also gives

$$\tilde{\mathbf{v}}_i = \frac{\sum_{k=1}^{n}\tilde{u}_{ik}^{q}\mathbf{x}_k}{\sum_{k=1}^{n}\tilde{u}_{ik}^{q}}. \tag{27}$$

## 4. Effects of Annealing Temperature

*4.1. Dependency of Shapes of Membership Functions on Temperature.* By reducing the temperature according to the annealing schedule, the deterministic annealing method achieves thermal equilibrium which minimizes the free energy. At absolute zero, the particle system settles down to the ground state, that is, the state of minimum energy. Figure 1 shows the forms of the entropy functions $S$, $\hat{S}$, and $\tilde{S}$. Figure 2 shows the forms of the membership functions $u_{ik}$, $\hat{u}_{ik}$, and $\tilde{u}_{ik}$.

In the deterministic annealing method, cluster distribution which minimizes the free energy is searched at the given temperature. At high temperature, the membership functions are widely distributed and clusters to which a data belongs are fuzzy. In case of $\tilde{u}_{ik}$ with $q = 2$, the width of the distribution is roughly proportional to $\beta^{-0.5}$. At the limit of low temperature, on the other hand, fuzzy clustering reaches hard clustering. The relationship $F = U - TS$ suggests that the higher temperature causes the larger entropy state, that is, chaotic state. This increase of the entropy is the result of the extent of the membership function.

In Figure 2, it can be seen that $\hat{u}_{ik}$ has a flat peak, though both $u_{ik}$ and $\tilde{u}_{ik}$ have Gaussian forms. Also, it can be found that $\tilde{u}_{ik}$ has a more gentle base slope than $u_{ik}$.

### 4.2. Cooling Schedule

*4.2.1. Representative Annealing Methods.* In SA, the temperature decreases according to a cooling schedule. The representative cooling schedules [16] for SA are

(i) proportional to an exponential function

$$T = T_{\text{high}} r^{t}, \tag{28}$$

where $T_{\text{high}}$ is a sufficiently high initial temperature, $r$ is a parameter which defines a temperature reduction speed, and $t$ is the number of iterations,

(ii) inversely linear function

$$T = \frac{T_{\text{high}}}{t}, \tag{29}$$

(iii) inversely proportional to a logarithmic function

$$T = \frac{T_{\text{high}}}{\ln(t)}, \tag{30}$$

(iv) inversely proportional to exponential function

$$T = \frac{T_{\text{high}}}{e^{r^{-1}t}}. \tag{31}$$

*4.2.2. Very Fast Annealing.* Rosen proposed another inversely proportional to exponential function known as very fast annealing (VFA).

In VFA, $T$ is given by

$$T = T_{\text{high}} e^{-mt^{(1/D)}}, \tag{32}$$

where $m$ is a temperature reduction parameter and $D$ is a dimension of a state space. Equations (31) and (32) are compared in Figure 3. It is observed that VFA initially decreases a temperature extremely.

In Section 6, we apply VFA as a cooling schedule of entropy based FCM clustering using DA.

## 5. Fuzzy C-Means as Clustering Algorithm Using Very Fast Annealing DA

The very fast deterministic annealing algorithm for the Tsallis-entropy-based FCM is given as follows.

(1) Set the number of clusters $c$, the highest temperature $T_{\text{high}}$, the temperature reduction rate $m$, and the threshold of convergence test $\delta_1$ and $\delta_2$;

(2) generate initial clusters at random positions and assign each data point to the nearest cluster. Set current temperature $T$ to $T_{\text{high}}$;

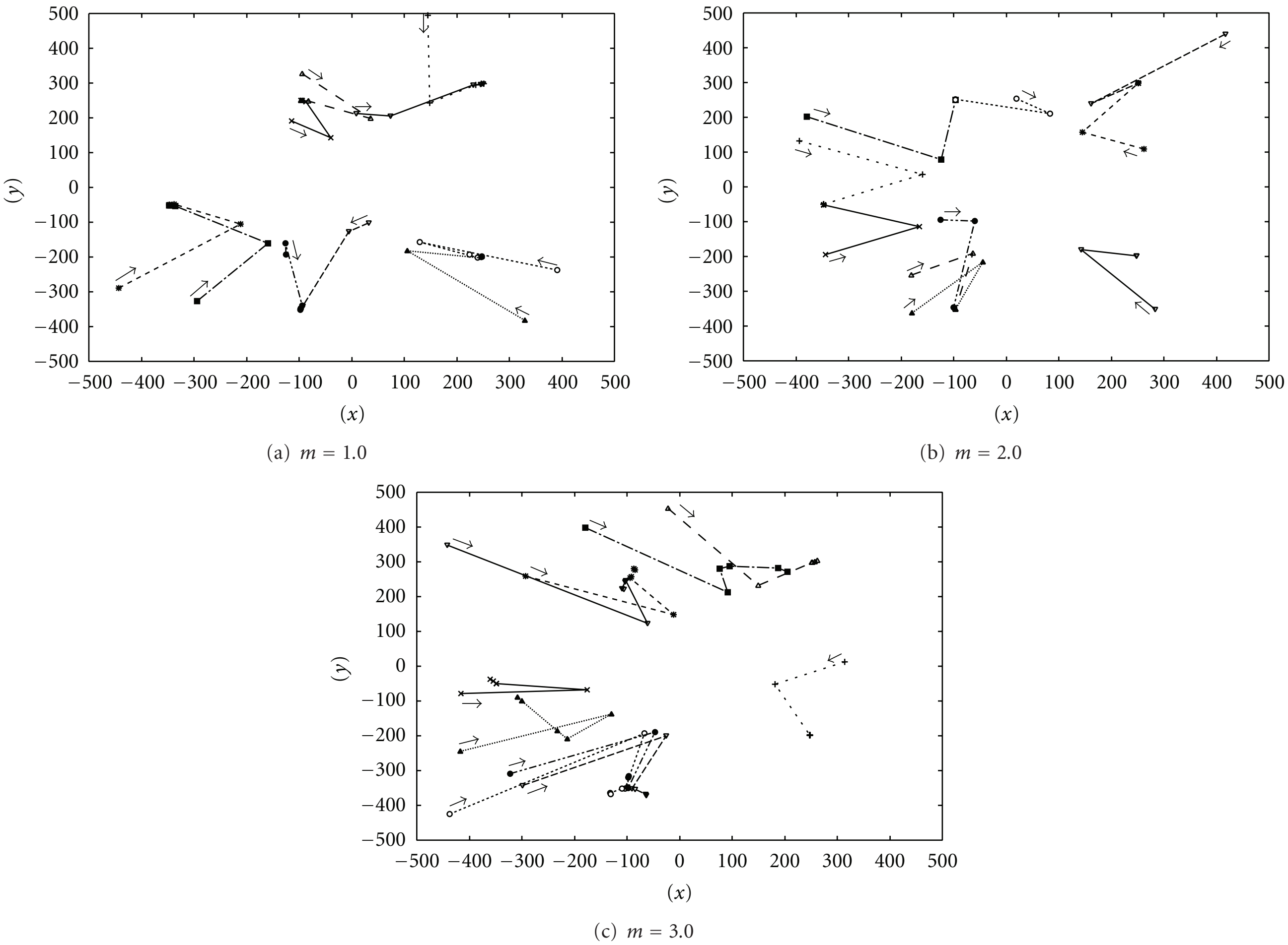



FIGURE 7: The shifts of cluster centers of Shannon entropy based clustering with decreasing the temperature by VFA ($m = 1.0, 2.0, 3.0$).

(3) calculate $\tilde{u}_{ik}$ by (15);

(4) calculate cluster centers by (17);

(5) compare the difference between the current centers and those obtained at the previous iteration $\hat{\mathbf{v}}_i$. If the convergence condition If $\max_{1\le i\le c}\|\mathbf{v}_i - \hat{\mathbf{v}}_i\| < \delta_1$ is satisfied, then go to (6), otherwise go back to (3);

(6) if $\max_{1\le i\le c}\|\mathbf{v}_i - \hat{\mathbf{v}}_i\| < \delta_2$ is satisfied, then stop. Otherwise decrease the temperature with (32) and go back to (3).

In case of Shannon-entropy-based FCM, (15) is replaced by (5) and (17) is replaced by (6), respectively.

## 6. Experiments

*6.1. Experiment 1.* In experiment 1, we generated five randomly placed clusters composed of 2,000 data points shown in Figure 4. We set $c$ to be 10, $\delta_1$ to be 50, and $\delta_2$ to be 2 (measured by the scale of Figure 4). We also set $T_{\text{high}} = 1.0\times10^6 (\beta = 1.0\times10^{-6})$ or $T_{\text{high}} = 1.0\times10^5 (\beta = 1.0\times10^{-5})$.

First, we have applied the inversely exponential scheduling method to the Tsallis-entropy-based FCM clustering. The cooling schedule is illustrated in Figure 5. The changes of $\beta$ are parameterized by the temperature reduction rate $r$: from 1 to 1000.

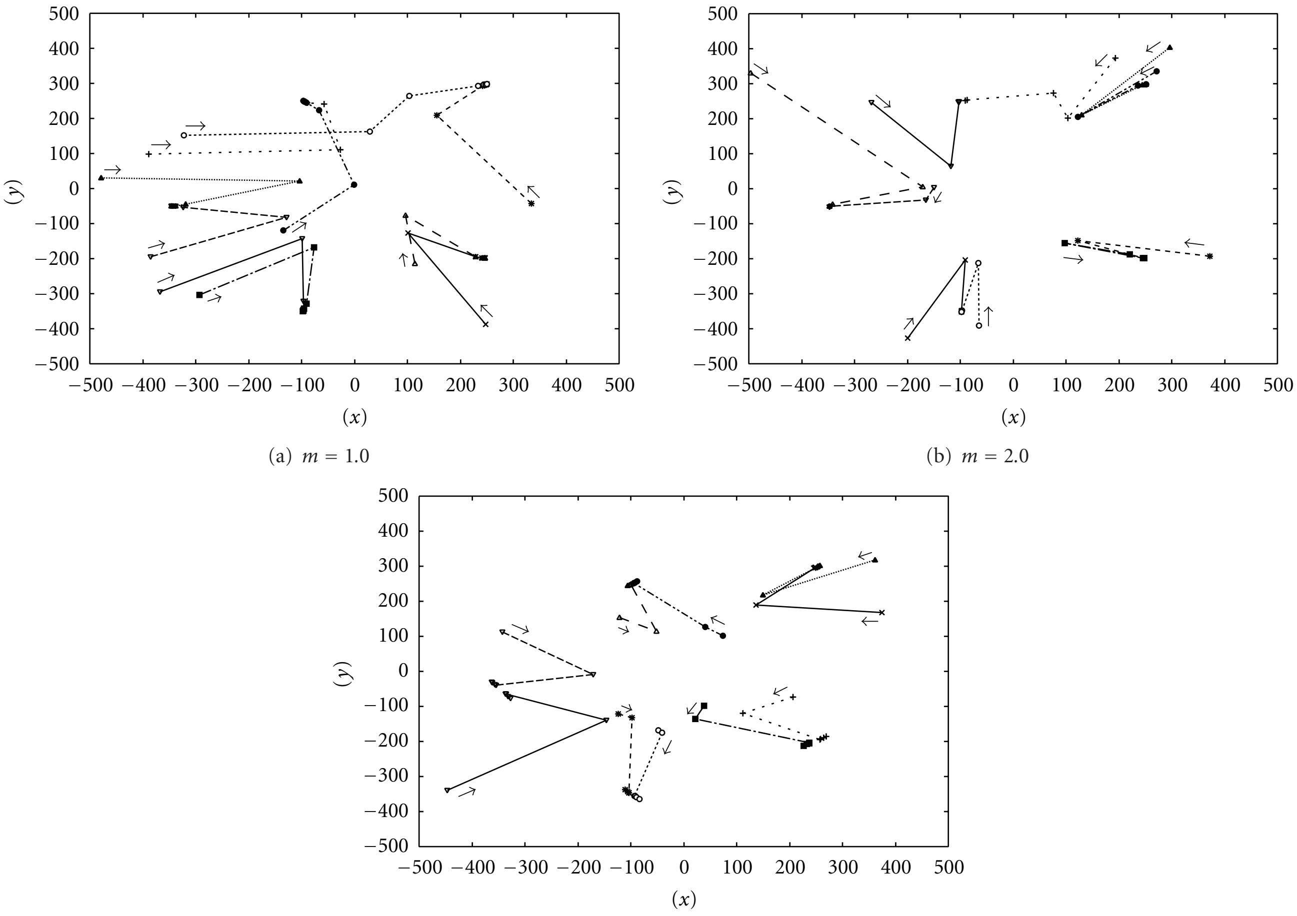
At the higher levels of $T$ (Figure 5 ($A$)), clusters are created near the center of gravity of data because $\beta$ is comparatively small and the membership function extends over the whole data area and is extremely uniform. As $T$ is lowered from Figure 5 ($B$) to ($C$), the width of the membership functions becomes narrower; that is, the Tsallis entropy decreases, and the associations become less fuzzy. And finally, the desired result is obtained.

In case of $r = 10$ or $r = 1$ (Figure 5 ($E$) or ($F$)), it is observed that $\tilde{u}_{ik}$ and $\tilde{\mathbf{v}}_i$ converge more rapidly. In case of $T_{\text{high}} = 1.0 \times 10^6 (\beta = 1.0 \times 10^{-6})$, however, the initial distribution of $\tilde{u}_{ik}$ becomes too wide and the algorithm is not converged with $r = 100$ and $r = 10$ (indicated by "not converged" in Figure 5). Thus, it is important to set $T_{\text{high}}$ and $r$ values properly.

To examine the effectiveness of VFA as a cooling schedule of DA, we made numerical experiments of the Shannon- and Tsallis-entropy-based FCM clustering.

The shifts of cluster centers with decreasing temperature are illustrated in Figures 7 and 8.

Initially, clusters are located randomly. At the higher levels of $T$, clusters move to near the center of gravity of




(a) $m = 1.0$

(b) $m = 2.0$

(c) $m = 3.0$

Figure 8: The shifts of cluster centers of Tsallis entropy based clustering with decreasing the temperature by VFA ($m = 1.0, 2.0, 3.0$, $q = 1.5$).

data because $\beta$ is comparatively small and the membership function extends over the whole data area and is extremely uniform.

As $T$ is lowered, the width of the membership functions becomes narrower and the associations of data become less fuzzy. In this process, in the Shannon-entropy-based FCM clustering, the clusters move to their nearest local data distribution centers. However, in the Tsallis-entropy-based FCM clustering, clusters can move a long distance to optimal positions because of their gentle base slopes.

Figures 9 and 10 illustrate the three-dimensional plots of $u_{ik}$ and $\tilde{u}_{ik}$ in the progress of very fast DA clustering.

At the higher temperature, roughness of $\tilde{u}_{ik}$ is smaller than that of $u_{ik}$. After that, the shapes of both membership functions do not change greatly, because VFA reduces the temperature extremely only at the early annealing stage.

Consequently, because the Tsallis-entropy-based FCM has gentle slope in the region far from the origin, clusters can move long distance to optimal positions stably and the temperature can be reduced rapidly. This feature makes it possible to use VFA as a cooling schedule of DA for the Tsallis-entropy-based FCM. On the other hand, final cluster positions obtained by the Shannon-entropy-based FCM tend to depend on their initial positions.

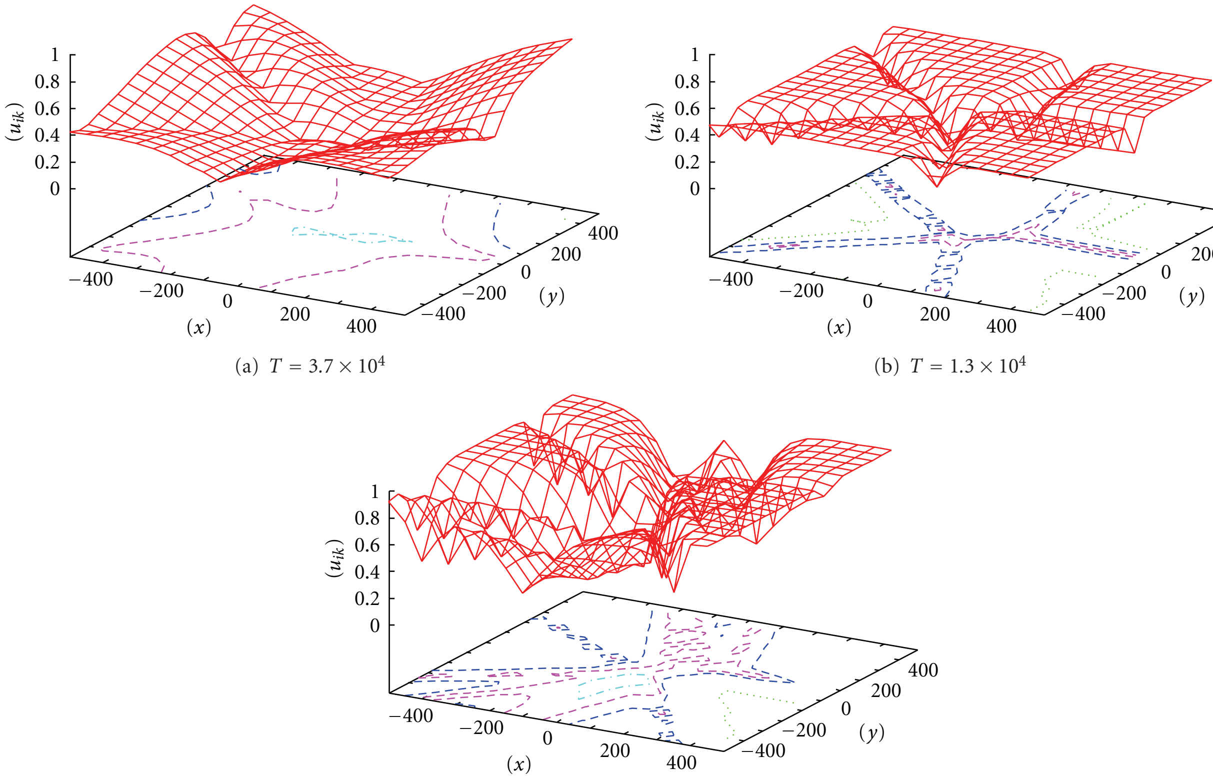
Table 1: Comparison of minimum, maximum, and average values of misclassified iris data (100 trials).

| Tsallis-entropy-based FCM | | | | Shannon-entropy-based FCM | | | |
|---|---|---|---|---|---|---|---|
| m | Min. | Max. | Ave. | m | Min. | Max. | Ave. |
| 1.0 | 14 | 14 | 14.00 | 1.0 | 15 | 16 | 15.01 |
| 2.0 | 14 | 15 | 14.01 | 2.0 | 11 | 13 | 12.97 |
| 3.0 | 14 | 15 | 14.68 | 3.0 | 11 | 14 | 13.59 |

*6.2. Experiment 2.* In experiment 2, the iris data set [13] consisting of 150 four-dimensional vectors of iris flowers are used. Three clusters of flowers detected are Versicolor, Virginia, and Setosa. Each cluster consists of 50 vectors.

The Shannon- and Tsallis-entropy-based FCM with DA are examined. VFA is used as a cooling schedule of DA. We set the parameters as follows: $c = 3$, $T_{\text{high}} = 2$, $\delta_1 = 0.1$, $\delta_2 = 0.01$, and $q = 1.5$.

The minimum, maximum, and average values of misclassified data of 100 trials are summarized in Table 1. The Shannon-entropy-based FCM gives slightly better results than the Tsallis-entropy-based FCM. However, it is found that the Tsallis-entropy-based FCM gives the best results

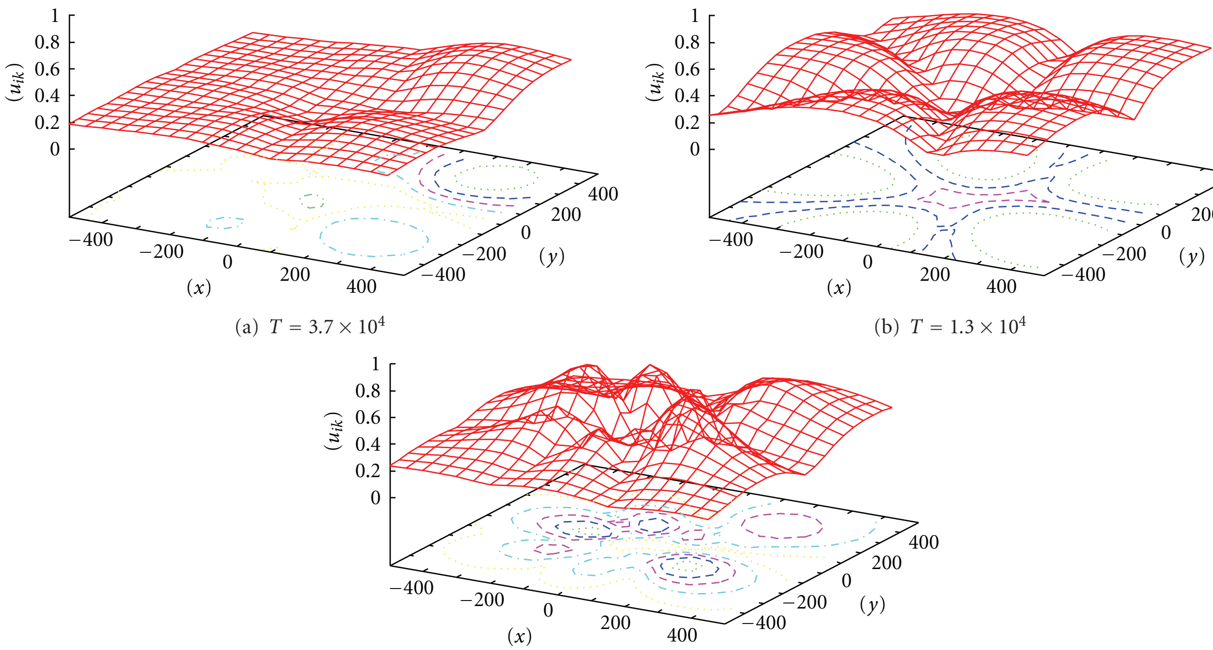



(a) $T = 3.7 \times 10^4$

(b) $T = 1.3 \times 10^4$

(c) $T = 4.9 \times 10^3$

FIGURE 9: The changes of the landscape of $u_{ik}$ with decreasing the temperature.




(a) $T = 3.7 \times 10^4$

(b) $T = 1.3 \times 10^4$

(c) $T = 4.9 \times 10^3$

FIGURE 10: The changes of the landscape of $\widetilde{u}_{ik}$ with decreasing the temperature ($q = 1.5$).

when the temperature reduction rate $m = 1.0$ or 2.0, though the best results for the Shannon-entropy-based FCM are obtained only when $m = 2.0$. Furthermore, variances of the Tsallis-entropy-based FCM are smaller than those of the Shannon-entropy-based FCM. These features indicate that a wide range of $m$ values are applicable to Tsallis-entropy-based FCM.

Figure 6 shows the reduction of the objective values of the Tsallis- and Shannon-entropy-based FCM with decreasing the temperature by VFA. The Shannon-entropy-based FCM does not converge properly when $T = 0.023$ for $m = 2.0$ and $T = 0.029$ for $m = 3.0$. That is, with larger $m$ values, the Shannon-entropy-based FCM becomes unstable.

## 7. Conclusion

By maximizing the Tsallis-entropy, the membership function of the Tsallis-entropy-based FCM is formulated. It has a more gentle base slope in the region far from the origin than that of the Shannon- and fuzzy-entropy-based FCMs. This feature allows clusters to move long distance and the temperature can be reduced rapidly in the Tsallis-entropy-based FCM.

Next, the deterministic annealing (DA) method using very fast annealing (VFA) as its cooling schedule is applied to the Tsallis-entropy-based FCM. VFA initially decreases a temperature extremely, and experimental results showed that the Tsallis-entropy-based FCM was suitable for DA combined with VFA.

Our future works include convergence and computational time test under various conditions (temperatures and parameters, especially $q$-value) of the Tsallis-entropy-based FCM using very fast deterministic annealing. They also include experiments and examinations of its applications.

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*Research Article*

# A Comparative Study on TIBA Imputation Methods in FCMdd-Based Linear Clustering with Relational Data

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Relational fuzzy clustering has been developed for extracting intrinsic cluster structures of relational data and was extended to a linear fuzzy clustering model based on Fuzzy $c$-Medoids (FCMdd) concept, in which Fuzzy $c$-Means-(FCM-) like iterative algorithm was performed by defining linear cluster prototypes using two representative medoids for each line prototype. In this paper, the FCMdd-type linear clustering model is further modified in order to handle incomplete data including missing values, and the applicability of several imputation methods is compared. In several numerical experiments, it is demonstrated that some pre-imputation strategies contribute to properly selecting representative medoids of each cluster.

## 1. Introduction

Relational fuzzy clustering is a relational extension of fuzzy clustering for revealing cluster structures buried in relational data. Relational Fuzzy $c$-Means (RFCM) [1] extended the Fuzzy $c$-Means (FCM) [2] clustering criterion with mutual dissimilarity measures instead of object-type observation in FCM. Although FCM and other variants of $k$-Means [3] use the clustering criterion of the distance between a data point and a cluster prototype, RFCM defines the clustering criterion by using mutual dissimilarities only. When the dissimilarities among objects are measured by squared Euclidean distances, the RFCM criterion is equivalent to the centroid-less formulation of the FCM criterion. Using other dissimilarity measures; however, the RFCM criterion has no clear connection with distances between data points and prototypes. In $k$-Medoids [4], cluster prototypes are selected from data points, and the clustering criterion coincides with one of mutual dissimilar degree among objects. So, $k$-Medoids can be directly extended to relational data analysis even if taking an average cannot be done in non-Euclidean space. Fuzzy $c$-Medoids (FCMdd) [5] is a fuzzy extension of $k$-Medoids and can deal with various dissimilarity measures.

Linear fuzzy clustering models [6, 7] extract linear substructures by modifying the point prototypes of FCM into lines, planes, and linear varieties. Because the subspace learning model in each cluster can be identified with fuzzy principal component analysis (fuzzy PCA) [8], they are often regarded as a kind of local principal component analysis (local PCA) [9]. This paper studies the FCMdd-based linear clustering model [10], which can reveal local linear substructures buried in relational data. In [10], Haga et al. defined each prototypical line by using two representative medoids and demonstrated that the clustering modal can be applied to Euclidean relational data. The FCMdd-type linear clustering model was further modified for dealing with non-Euclidean relational data [11, 12], in which data transformation, called $\beta$-spread transformation, was performed before applying the clustering algorithm in a similar manner to Non-Euclidean-type Relational Fuzzy (NERF) $c$-Means [13].

In this paper, a comparative study on the applicability of $\beta$-spread transformation is performed in FCMdd-based linear clustering of incomplete relational data. Hathaway and Bezdek [14] proposed several methods for imputing (predicting and substituting) missing elements of incomplete relational data and showed that imputation errors can be revised by $\beta$-spread transformation in NERF $c$-Means. This paper demonstrates that the performance of FCMdd-type linear fuzzy clustering for incomplete relational data can also be improved by $\beta$-spread transformation through several comparative experiments including an example of document clustering.

The remaining part of this paper is organized as follows. In Section 2, linear clustering and relational clustering are briefly reviewed. Section 3 introduces FCMdd-type linear clustering model and applies several imputation methods called TIBA. Comparative results are shown in Section 4, and conclusions are given in Section 5.

## 2. Linear Clustering and Relational Clustering

*2.1. FCM-Type Linear Clustering.* Assume that we have $m$-dimensional observations of $n$ patterns $\mathbf{x}_i = (x_{i1}, \ldots, x_{im})^\top$, $i = 1, \ldots, n$. With the goal of partitioning the $n$ patterns into $C$ clusters, the objective function for FCM-type clustering is defined as

$$L_{fcm} = \sum_{c=1}^{C} \sum_{i=1}^{n} u_{ci}^{\theta} D_{ci}. \tag{1}$$

$u_{ci}$ is the fuzzy membership degree of pattern $i$ to cluster $c$, and $\theta$ is the fuzzification parameter. The larger the $\theta$, the fuzzier the membership assignment. $D_{ci}$ is the clustering criterion which measures the deviation between pattern $i$ and the prototype of cluster $c$. In the original FCM clustering [2], cluster prototypes are given by the centroid vectors $\mathbf{b}_c$, and $D_{ci}$ is the squared Euclidean distance as follows:

$$D_{ci} = \|\mathbf{x}_i - \mathbf{b}_c\|^2. \tag{2}$$

The FCM model is reduced to the hard (nonfuzzy) $k$-Means model [3] when $\theta = 1$, in which cluster memberships are given by the nearest prototype principle.

Besides point-type prototypes $\mathbf{b}_c$ in FCM, Fuzzy $c$-Lines (FCL) [6] for extracting linear clusters used linear prototypes defined as

$$\mathrm{Line}_c(\mathbf{b}_c, \mathbf{a}_c) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{b}_c + t\mathbf{a}_c;\ t \in R\}, \tag{3}$$

where $\mathbf{a}_c$ is the basis vector of the principal subspace, and $\mathbf{b}_c$ is the centroid, which the linear prototype passes through. The clustering criterion is calculated as

$$D_{ci} = \|\mathbf{x}_i - \mathbf{b}_c\|^2 - \left|\mathbf{a}_c^\top(\mathbf{x}_i - \mathbf{b}_c)\right|^2. \tag{4}$$

The updating rules for membership $u_{ci}$ and the cluster center $\mathbf{b}_c$ are derived as

$$u_{ci} = \left[\sum_{l=1}^{C} \left(\frac{D_{ci}}{D_{li}}\right)^{1/(\theta-1)}\right]^{-1}, \tag{5}$$

$$\mathbf{b}_c = \frac{\sum_{i=1}^{n} u_{ci}^{\theta}\mathbf{x}_i}{\sum_{i=1}^{n} u_{ci}^{\theta}}. \tag{6}$$

The basis vectors $\mathbf{a}_c$ are the principal eigenvectors of the generalized fuzzy scatter matrices:

$$S_{\mathrm{fc}} = \sum_{i=1}^{n} u_{ci}^{\theta}(\mathbf{x}_i - \mathbf{b}_c)(\mathbf{x}_i - \mathbf{b}_c)^\top. \tag{7}$$

This linear clustering model has close relation with local PCA [9]. Indeed, when we consider only a single cluster ($C = 1$), the FCL clustering model is equivalent to the conventional PCA and the basis vector $\mathbf{a}_c$ is reduced to the principal component vector. In this sense, FCL is a type of local PCA, which simultaneously performs membership estimation (local fuzzy group extraction) and fuzzy PCA [8] in each local fuzzy group considering the fuzzy membership degree of $u_{ci}^{\theta}$. The prototypical line $\mathrm{Line}_c$ can be identified with principal subspace spanned by fuzzy principal component vector $\mathbf{a}_c$ from the local PCA view point.

When $\theta = 1$, the FCL model is also reduced to the hard (nonfuzzy) local PCA model [15, 16], in which cluster memberships are given by the nearest prototype principle.

*2.2. FCM-Type Relational Clustering.* RFCM [1] is the relational extension of FCM. When we have relational data composed of mutual relations among patterns $D = \{d_{ij}^2\}$, the FCM-type objective function is redefined as

$$L_{rfcm} = \sum_{c=1}^{C} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{u_{ci}^{\theta} u_{cj}^{\theta} d_{ij}^2}{2\sum_{t=1}^{n} u_{ct}^{\theta}}. \tag{8}$$

$d_{ij}$ can be any type of dissimilarity between patterns $i$ and $j$ but is assumed to be Euclidean-like one in RFCM. Indeed, this model is equivalent to FCM only when $d_{ij}^2$ is the squared Euclidean distance, and the clustering model derives only poor results if the relational information is highly non-Euclidean.

In order to modify RFCM for handling non-Euclidean distance metrics, Hathaway and Bezdek [13] considered NERF, which includes the following $\beta$-spread transformation:

$$D_\beta = D + \beta \times (M - I), \tag{9}$$

where $\beta$ is added to off-diagonal elements of non-Euclidean relational data $D$. $I$ is a unit matrix, and $\beta$ is a suitably chosen scalar. $M$ is a matrix whose elements are all 1. Hathaway and Bezdek discussed that $D_\beta$ is Euclidean if $PD_\beta P$ with $P = I - (1/n)M$ is negative semidefinite; that is, $\beta$ is greater than or equal to the largest eigenvalue of $PDP$. By the way, the basic RFCM iteration can be continued when clustering criteria are all nonnegative. In NERF, $\beta$ is gradually increased from 0 to a certain value by considering the negative elements of clustering criteria.

## 3. FCMdd-Type Linear Clustering and TIBA Imputation

*3.1. FCMdd-Type Linear Clustering.* Assume that $d_{ij}$ is the mutual Euclidean distance such that

$$d_{ii} = 0, \quad d_{ij} \geq 0, \qquad d_{ij} = d_{ji}, \qquad i, j = 0, \ldots, n. \tag{10}$$

FCMdd [5] is a fuzzy extension of $k$-medoids [4], which performs an FCM-like clustering by selecting $\mathbf{b}_c$ from patterns $\mathbf{x}_i$, $i = 1, \ldots, n$. The representative objects are called "medoids" and are given by solving combinatorial optimization problems. Haga et al. [10] applied the idea to linear fuzzy clustering, in which each linear prototype is spanned by two representative medoids $\mathbf{x}_{c_1}$ and $\mathbf{x}_{c_2}$ as

$$\text{Line}_c(\mathbf{x}_{c_1}, \mathbf{x}_{c_2}) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{x}_{c_1} + t(\mathbf{x}_{c_2} - \mathbf{x}_{c_1});\ t \in R\}. \quad (11)$$

The squared Euclidean distance between object $i$ and the prototypical line $\text{Line}_c$ is given as

$$D_{ci} = d_{i,c_1}^2 - \frac{\left(d_{i,c_1}^2 - d_{i,c_2}^2 + d_{c_1,c_2}^2\right)^2}{4d_{c_1,c_2}^2}. \quad (12)$$

With fixed fuzzy memberships $u_{ci}$, the optimal medoids are derived by the following combinatorial optimization problem:

$$(c_1, c_2) = \arg \min_{\substack{(k_1,k_2) \\ 1 \le k_1,k_2 \le n \\ k_1 \neq k_2}} \sum_{i=1}^{n} u_{ci}^{\theta} D_{ci}. \quad (13)$$

The optimal medoid set of $(c_1, c_2)$ is searched by enumerating all pairs of objects. In order to reduce the computational cost, a simplified medoid search process was also proposed, in which medoids are selected from a subset $X_c$ of objects:

$$(c_1, c_2) = \arg \min_{\substack{(k_1,k_2) \\ \mathbf{x}_{k_1}, \mathbf{x}_{k_2} \in X_c \\ k_1 \neq k_2}} \sum_{i=1}^{n} u_{ci}^{\theta} D_{ci}, \quad (14)$$

where $X_c = \{\mathbf{x}_i : u_{ci} > M_{\min}\}$.

This linear fuzzy clustering model was also extended to the 2D prototype case by spanning 2D prototypical planes using three medoids [10].

Although non-Euclidean relational data may bring negative values for the clustering criteria of (12), from the practical view point, we have no trouble in operating the conventional FCMdd-type linear clustering algorithm if all clustering criteria are not negative.

Yamamoto et al. [11] proposed a procedure for $\beta$-spread transformation so as to avoid negative criterion values in FCMdd-type linear clustering. Because a negative criterion value implies a non-Euclidean situation, relational data should be revised so that the criterion value is always nonnegative. In the previous research [12], it was shown that the clustering criterion $D_{ci}$ is always nonnegative if triangle inequality ($d_{c_1,c_2} \le d_{i,c_1} + d_{i,c_2}$) is satisfied. Then, $\beta$-spread transformation should be performed so that the following triangle inequality is satisfied for all objects:

$$d_{c_1,c_2} + \beta \le d_{i,c_1} + \beta + d_{i,c_2} + \beta. \quad (15)$$

A plausible value of $\Delta\beta$ in an iteration step is obtained as

$$\Delta\beta = \max\left\{\max_i \{d_{c_1,c_2} - d_{i,c_1} - d_{i,c_2} - \beta\}, 0\right\}. \quad (16)$$

Here, $\Delta\beta$ is positive when some $D_{ci}$ are negative, while $\Delta\beta$ is zero when all $D_{ci}$ are nonnegative. Then, $\beta$ is monotonically increasing.

A sample procedure including the automated $\beta$-spread transformation can be summarized as follows:

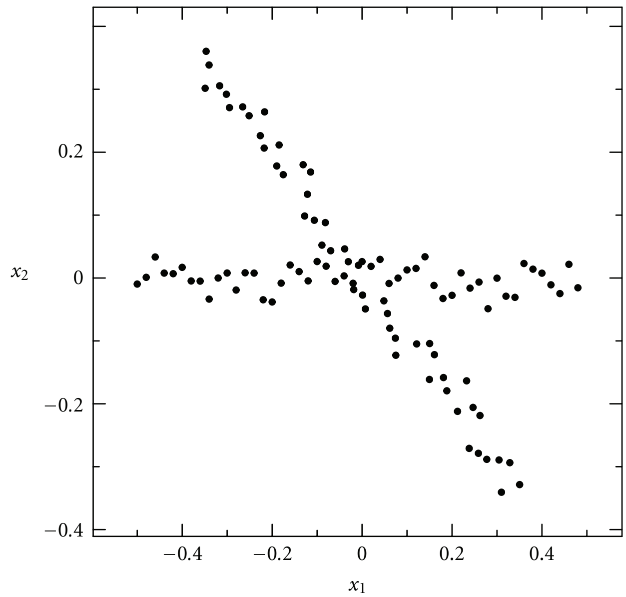



Figure 1: 2D plots of artificial data set.

*Step 1.* Set $\beta = 0$. Randomly initialize the prototypical medoids (two representative objects) of each cluster.

*Step 2.* Calculate the clustering criteria $D_{ci}$ by (12).

*Step 3.* If there is at least one object that has $D_{ci} < 0$, update $\beta = \beta + \Delta\beta$ by (16).

*Step 4.* Update fuzzy memberships by (5).

*Step 5.* Search medoids in each cluster.

*Step 6.* Repeat Steps 2–5 until a certain stopping criterion is satisfied.

In Step 6, such a stopping criterion as $\min |u_{ci}^{(\text{new})} - u_{ci}^{(\text{old})}| < \varepsilon$ is used where $\varepsilon$ is a small positive value.

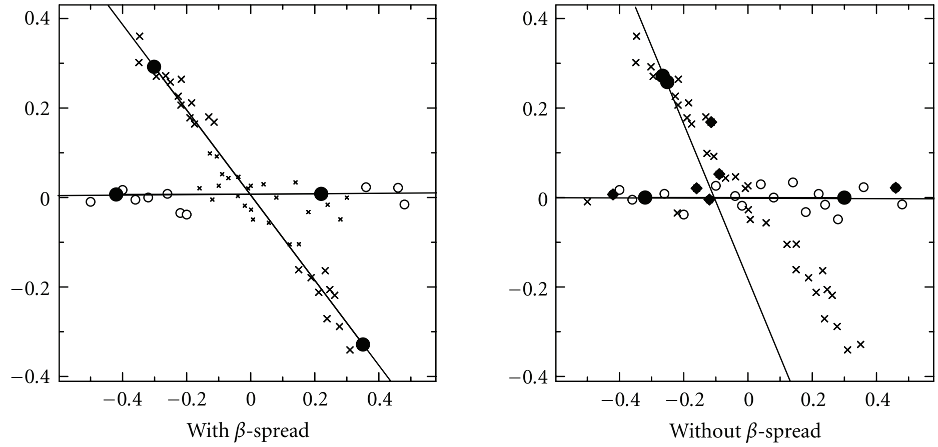
Although the proposed model is in the fuzzy clustering category, it is easily seen that a hard (non-fuzzy) version can be covered when $\theta = 1$, in which cluster memberships are given by the nearest prototype principle.

*3.2. Missing Value Imputation by TIBA.* Hathaway and Bezdek [14] demonstrated that the $\beta$-spread transformation is also useful for handling missing elements in relational data matrices. Although preimputation of missing elements may cause imputation errors and bring illegal effects in clustering process, $\beta$-spread transformation can decrease the illegal effects.

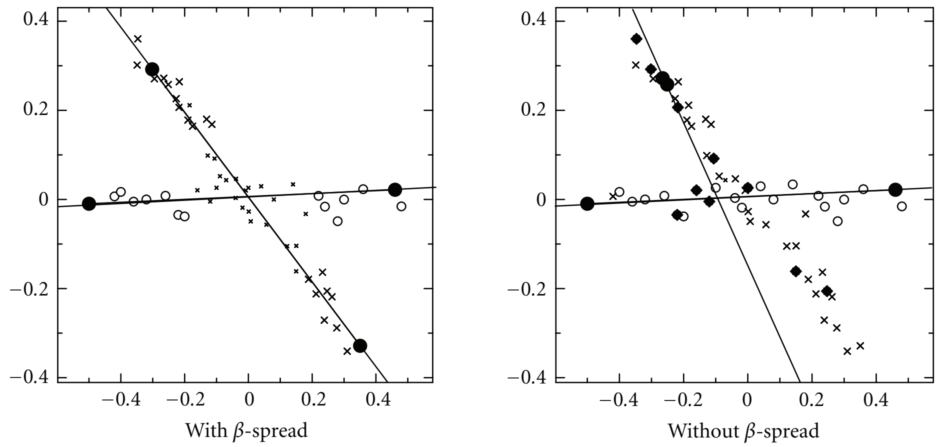
This paper considers the applicability of several imputation techniques in FCMdd-type linear clustering.

Hathaway and Bezdek [14] used three imputation techniques based on triangle inequality-based approximation (TIBA). The triangle inequality, which Euclidean relational data always satisfy, is represented as follows:

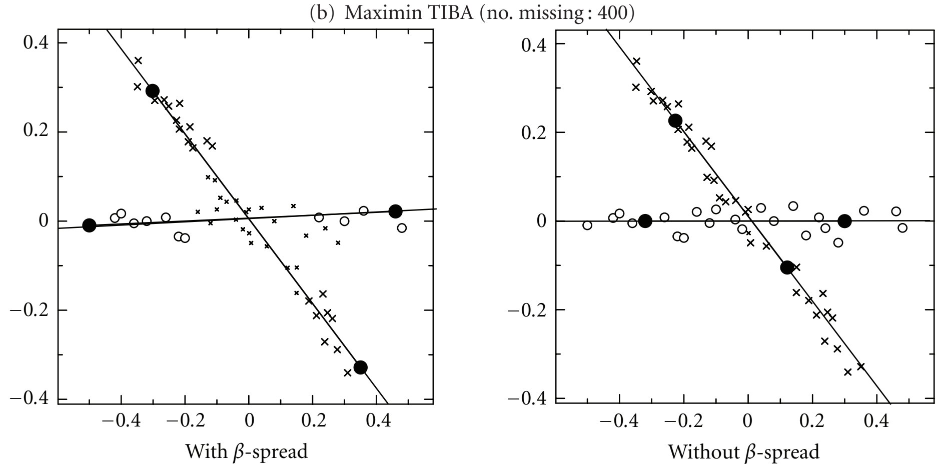
$$d_{ij} \le d_{ik} + d_{kj}. \quad (17)$$




(a) Minimax TIBA (no. missing : 500)




(b) Maximin TIBA (no. missing : 400)




(c) Average TIBA (no. missing : 500)

Figure 2: Comparison of cluster partitions from relational data imputed by three TIBAs, with (left)/without (right) $\beta$-spread transformation (Euclidean norm).

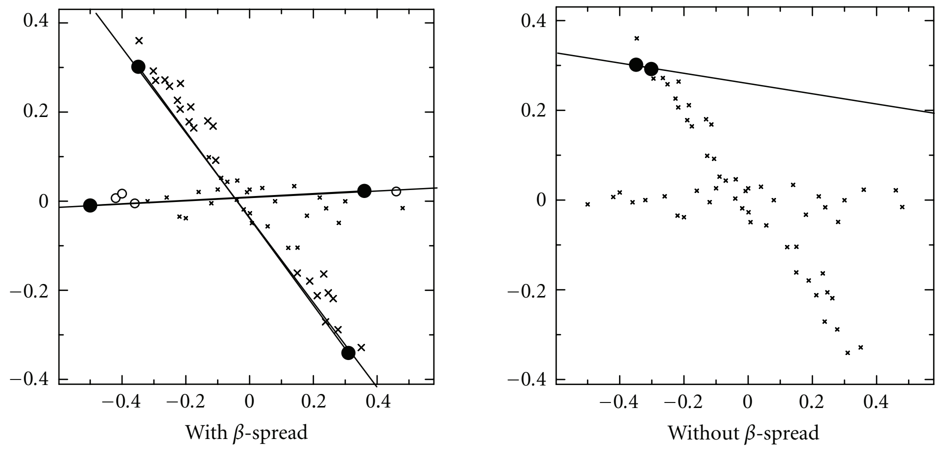
Assume that an element of relational matrix $\widetilde{d_{ij}}$ is missing and is to be preimputed before applying the clustering algorithm. Let $K_{ij}$ be the corresponding index set as

$$K_{ij} = \left\{ k \mid d_{ik} \quad \text{and} \quad d_{kj} \quad \text{observed} \right\}. \tag{18}$$

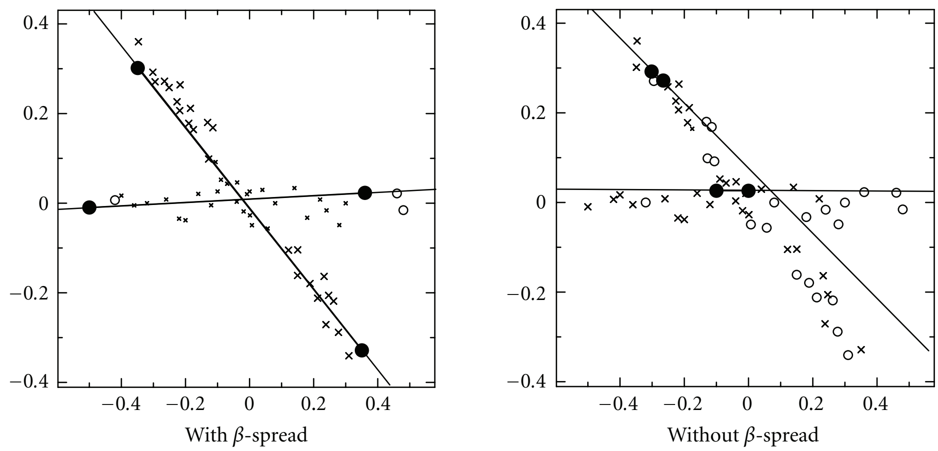
For each $k \in K_{ij}$, the triangle inequality (17) is given as the upper bound of $\widetilde{d_{ij}}$. Missing elements are replaced with the minimum upper bound of $\widetilde{d_{ij}}$:

$$\widetilde{d_{ij}} = \widetilde{d_{ji}} = \min_{k} \left\{ d_{ik} + d_{kj} \right\}, \tag{19}$$

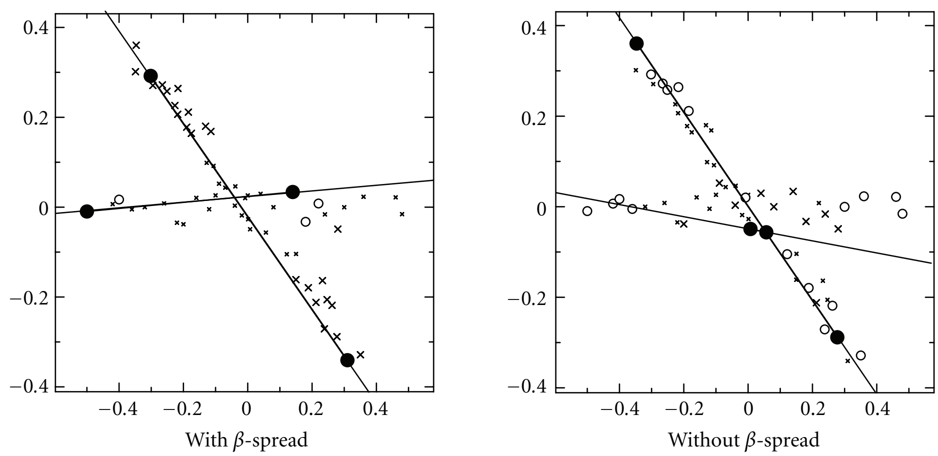
which is called minimax TIBA. By the way, $\widetilde{d_{ij}}$ is imputed by zero value if $K_{ij}$ is empty.




(a) Minimax TIBA (no. missing : 1000)




(b) Maximin TIBA (no. missing : 700)




(c) Average TIBA (no. missing : 1100)

Figure 3: Comparison of cluster partition from relational data imputed by three TIBAs, with (left)/without (right) $\beta$-spread transformation ($L_1$ norm).

The triangle inequality is also represented as follows:

$$
\begin{aligned}
d_{ik} &\leq \widetilde{d_{ij}} + d_{jk}, \\
d_{jk} &\leq \widetilde{d_{ji}} + d_{ik},
\end{aligned}
\tag{20}
$$

and brings the following inequalities:

$$
\begin{aligned}
\widetilde{d_{ij}} &\geq d_{ik} - d_{jk}, \\
\widetilde{d_{ji}} &\geq d_{jk} - d_{ik}.
\end{aligned}
\tag{21}
$$

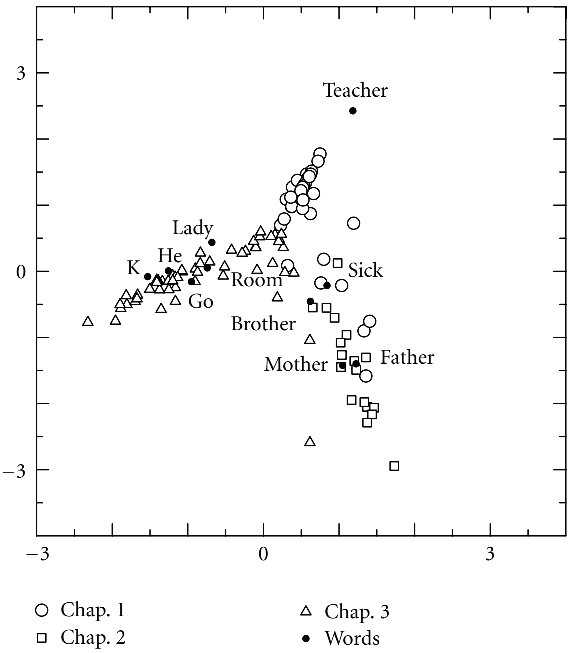



Figure 4: Document-keyword biplots [18].

So, the lower bound of $\widetilde{d_{ij}}$ is given as

$$\widetilde{d_{ij}} \geq \left| d_{ik} - d_{jk} \right|. \tag{22}$$

Missing elements are replaced with the maximum lower bound of $\widetilde{d_{ij}}$:

$$\widetilde{d_{ij}} = \widetilde{d_{ji}} = \max_k \left\{ \left| d_{ik} - d_{jk} \right| \right\}, \tag{23}$$

which is called maximin TIBA.

It is also possible to combine the previous two imputation values for predicting a reasonable estimation of missing values. The average values of minimax TIBA and maximin TIBA are used for imputing missing elements. This TIBA is called average TIBA.

These imputation techniques based on triangle inequalities can be easily applied to relational clustering problems. In the next section, these three imputation approaches are compared in FCMdd-type linear clustering tasks in conjunction with $\beta$-spread transformation.

## 4. Numerical Experiments

Two experimental results are shown in order to consider the applicability of the three TIBA imputation techniques in FCMdd-type linear clustering with $\beta$-spread transformation.

In previous researches, it has been shown that "soft" clustering models outperformed "hard" ones in local PCA tasks [15–17], and "fuzzy" models can be more useful than probabilistic ones [9]. Therefore, in this paper, the characteristics of the fuzzy version are investigated.

*4.1. Artificial Data Set.* An artificial relational data set composed of 60 patterns was generated from a 2D data set shown in Figure 1, in which patterns form two line-shaped clusters. It is obvious that the local linear structures cannot be extracted by the conventional point-prototype models such as FCM-like models and FCMdd. We made two relational data matrices. The first relational data matrix was generated by Euclidean norm, and the second one was generated by $L_1$ norm, which is non-Euclidean measure. The iterative algorithm was performed until the medoids became unchanged, and the model parameters were set as $(C, \theta) = (2, 2)$. In order to demonstrate the characteristics of the algorithm, the initial memberships were given in a supervised manner; that is, $(u_{1i}, u_{2i}) = (0.9, 0.1)$ for the first visual cluster and $(u_{1i}, u_{2i}) = (0.1, 0.9)$ for the second one.

In the previous research [12], it was demonstrated that the two linear substructures can be successfully revealed by the FCMdd-based linear clustering algorithm without $\beta$-spread transformation for Euclidean relational data while it can be done only with $\beta$-spread transformation for $L_1$ norm.

First, Euclidean incomplete relational data matrices were generated by removing a part of off-diagonal elements where $K_{ij}$ was not empty. In order to protect tridiagonal parts of relational data, the maximum number of missing elements was set as $n^2 - 3n + 2$.

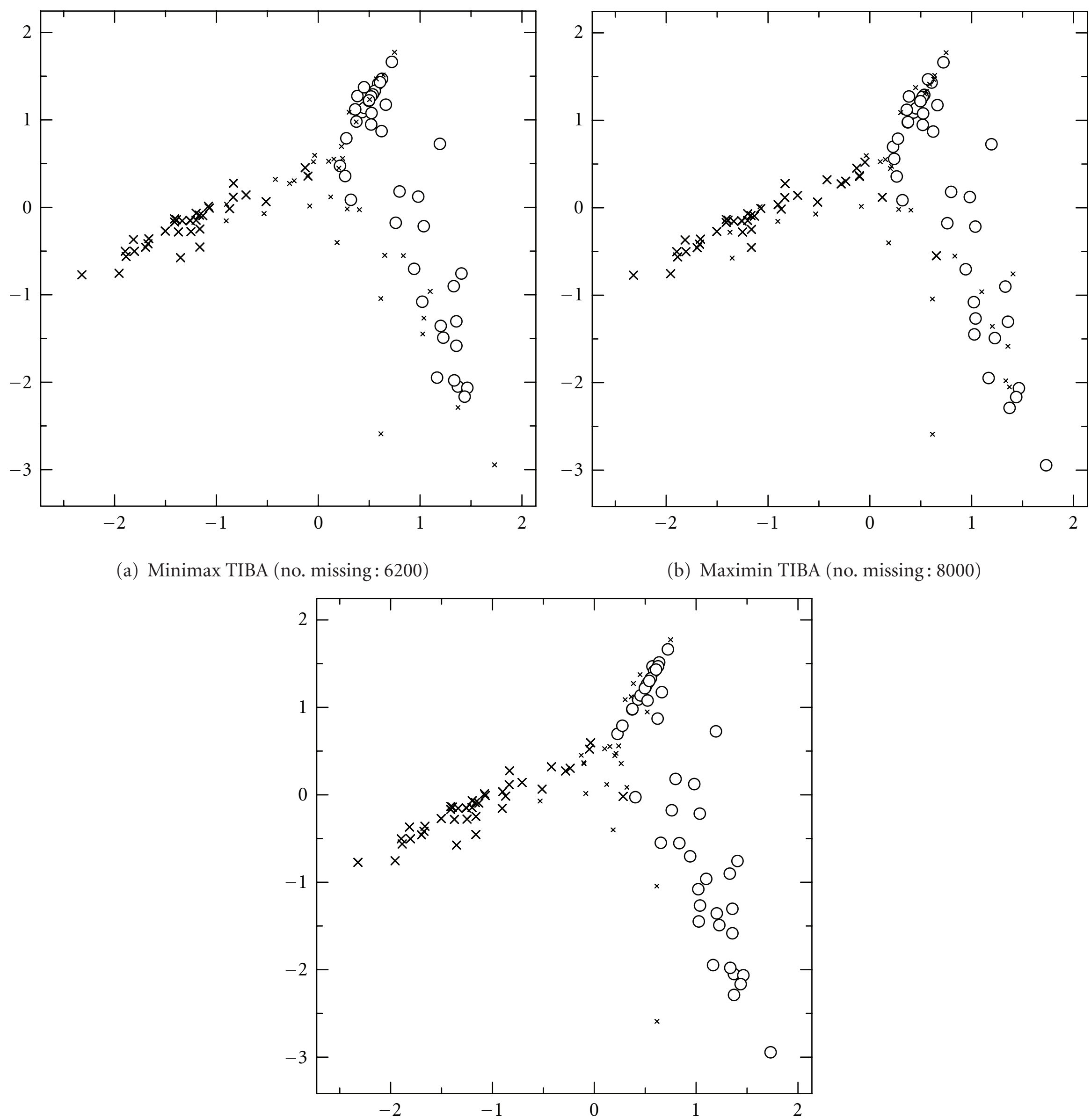
Clustering results are compared with those without $\beta$-spread transformation in Figure 2. Objects were partitioned into two clusters of circles and times, and smaller times mean that the patterns were shared almost equally by the two clusters. Medoids and prototypical lines are indicated by black circles and lines, respectively.

Each approximation method with $\beta$-spread transformation could estimate cluster medoids for capturing the two visual linear prototypes until the numbers of missing elements are less than about 30% although patterns having ambiguous memberships increased more than complete relational data. $\beta$-spread transformation performed on each approximation, minimax TIBA: $\beta = 0.04867$, maximin TIBA: $\beta = 0.044286$, average TIBA: $\beta = 0.051706$. Here, the maximum eigenvalues of PDP after imputation were, minimax TIBA: 0.053808, maximin TIBA: 0.1711612, average TIBA: 0.083694. So, the TIBA imputation brought a slightly non-Euclidean situation, and $\beta$-spread transformation successfully modified the data set.

On the other hand, without $\beta$-spread transformation, only average TIBA made it possible to extract linear substructures while minimax TIBA and maximin TIBA brought inappropriate results where some patterns depicted by black diamonds in Figure 2 had negative clustering criterion values.

These results imply that the FCMdd-type linear clustering can successfully extract linear substructures of incomplete Euclidean relational data using $\beta$-spread transformation although the three imputation techniques cause non-Euclidean relational matrices.

Second, FCMdd-type linear fuzzy clustering was applied to non-Euclidean relational data.




(a) Minimax TIBA (no. missing : 6200)

(b) Maximin TIBA (no. missing : 8000)

(c) Average TIBA (no. missing : 7200)

Figure 5: Comparison of cluster partition from incomplete "Kokoro" text data derived with Jaccard's coefficient imputed by three TIBAs.

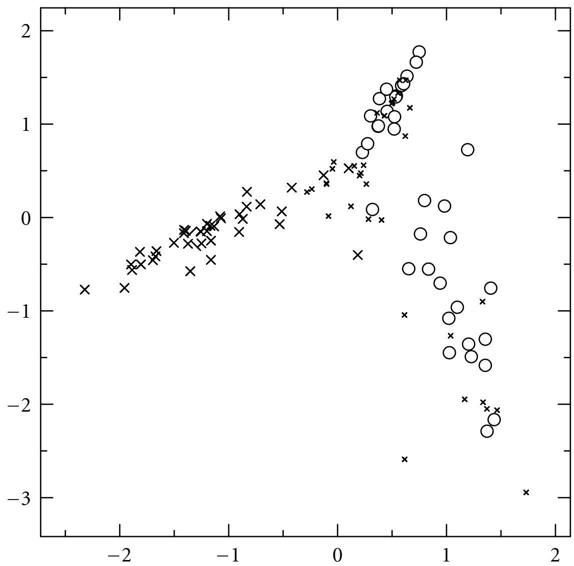
Incomplete relational data matrices were generated in the same manner with the Euclidean case. Clustering results are depicted in Figure 3.

With $\beta$-spread transformation, minimax TIBA and average TIBA could extract linear prototypes until the numbers of missing elements are less than about 60%, and maximin TIBA also could until about 40%. The parameters $\beta$ in $\beta$-spread transformation were minimax TIBA: $\beta = 0.23344$, maximin TIBA: $\beta = 0.137577$, average TIBA: $\beta = 0.165504$. The derived $\beta$ values are still smaller than the maximum eigenvalues of PDP, minimax TIBA: 0.655306, maximin TIBA: 0.427750, average TIBA: 0.619776.

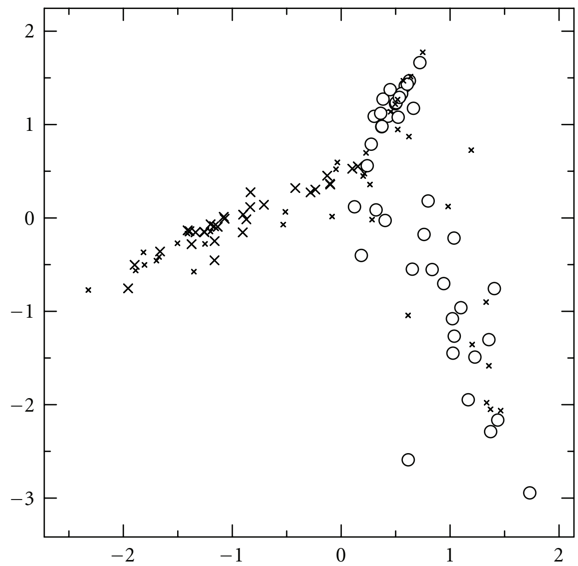
Without $\beta$-spread transformation; however, all the three TIBAs brought inappropriate partitions because many patterns had negative clustering criterion values.

In this way, $\beta$-spread transformation also works well in incomplete situations.

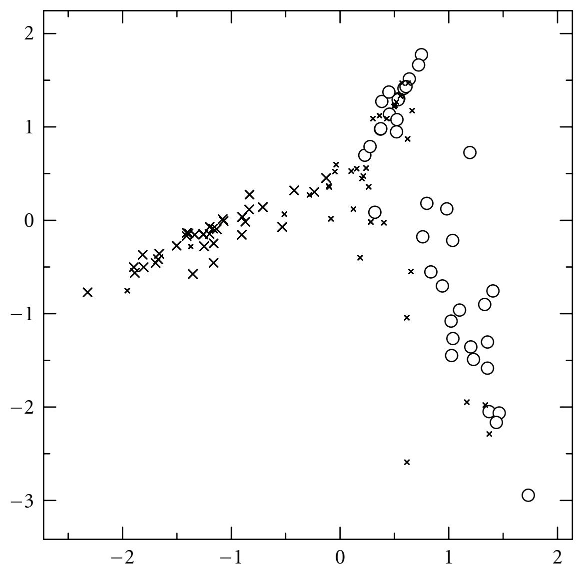
*4.2. Document Clustering.* In the second experiment, TIBA imputation methods are compared in a document classification task. A relational data set was generated using a famous Japanese novel "Kokoro" by Soseki Natsume. The novel is composed of 3 chapters (Sensei and I, My Parents and I, Sensei and His Testament), and the chapters include 36, 18, 56 sections, respectively. The text data (Japanese language) can be downloaded from Aozora Bunko (http://www.aozora.gr.jp/). The sections were used as individual text documents ($n = 110$), which should be partitioned without the chapter information. The text documents




(a) Minimax TIBA (no. missing : 5600)




(b) Maximin TIBA (no. missing : 7400)




(c) Average TIBA (no. missing : 6600)

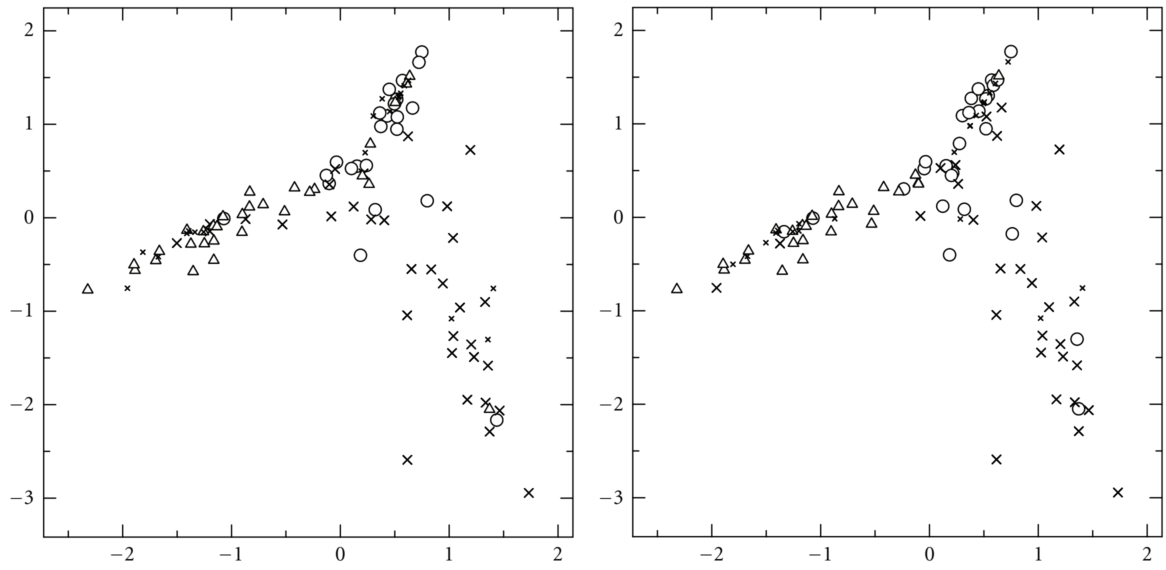
Figure 6: Comparison of cluster partition from incomplete “Kokoro” text data derived with Dice’s coefficient imputed by three TIBAs.

were preprocessed using “Chasen” morphological analysis system software (http://chasen.naist.jp/hiki/ChaSen/), which segments the Japanese text string into morphemes. Wada et al. [18] performed a PCA-based structural analysis with the 83 most frequently used substantives and verbs with their tf-idf weights and revealed that the chapter structure can be emphasized by using 10 meaningful keywords as is shown in Figure 4, which is 2D biplots of principal components. Chapters 2 and 3 form two linear clusters in 10D data space, and chapter 1 exists on their intersection. In this experiment, parameters were set as $C = 2$, $\theta = 2.0$ with the goal of revealing the two linear substructures.

Two relational data matrices were generated considering co-occurrence information of the 10 keywords. Jaccard coefficient and Dice coefficient are the similarity measures for asymmetric information on binary variables [19]. Assume that the cooccurrence information of keywords among two text documents are summarized in a $2 \times 2$ contingency table as shown in Table 1 where “1” means occurrence of the keyword.

Table 1: $2 \times 2$ contingency table for text documents.

| | keyword B | | |
|---|---|---|---|
| keyword A | 1 | 0 | Total |
| 1 | a | b | a + b |
| 0 | c | d | c + d |
| Total | a + c | b + d | |



(a) Jaccard (no. missing : 8000)

(b) Dice (no. missing : 7400)

Figure 7: Comparison of cluster partition of Fuzzy $c$-Medoids for incomplete "Kokoro" text data derived with Jaccard's and Dice's coefficients imputed by maximin TIBA.

Jaccard's coefficient is the similarity represented as

$$s_{ij} = \frac{a}{a+b+c}. \tag{24}$$

Dice's coefficient is also the similarity represented as

$$s_{ij} = \frac{2a}{2a+b+c}. \tag{25}$$

Because the linear clustering model uses distance (dissimilarity) measures, the similarity measures $s_{ij}$ were transformed into dissimilarity ones $d_{ij}$.

$$d_{ij} = \max_{k,l}\{s_{kl}\} - s_{ij}. \tag{26}$$

Before applying the FCMdd-based linear fuzzy clustering, randomly selected elements were withheld from the relational matrix with 11,772 elements and were imputed by the three TIBA methods. Then, the cluster partitions for Jaccard's index were derived as shown in Figure 5. Two clusters are depicted by circles and times, and small times mean ambiguous assignment. Documents were properly partitioned into two clusters considering linear substructures.

Minimax TIBA allowed with 50% missing values or fewer. Average TIBA tolerated 60% missing values or fewer. Maximin TIBA resulted in a good partition with 68% missing values or fewer. The parameters $\beta$ in $\beta$-spread transformation were given as minimax TIBA: $\beta = 0.960$, maximin TIBA: $\beta = 0.411$, average TIBA: $\beta = 0.435$. The derived $\beta$ values are still smaller than the maximum eigenvalues of PDP without missing elements, minimax TIBA: 5.662, maximin TIBA: 4.548, average TIBA: 2.427.

Clustering results for Dice coefficient are depicted in Figure 6. Our approach also extracted linear substructure from incomplete relational data of Dice coefficient. Minimax TIBA allowed with 48% missing values or fewer. Average TIBA tolerated 55% missing values or fewer. Maximin TIBA resulted in a good partition with 63% missing values or fewer. The parameters $\beta$ in $\beta$-spread transformation were given as, minimax TIBA: $\beta = 0.760$, maximin TIBA: $\beta = 0.383$, average TIBA: $\beta = 0.608$. The derived $\beta$ values are still smaller than the maximum eigenvalues of PDP, minimax TIBA: 6.169, maximin TIBA: 4.288, average TIBA: 2.219.

In the experiments, it was demonstrated that the TIBA imputation methods work well for incomplete non-Euclidean relational data in conjunction with $\beta$-spread transformation.

Finally, comparison with other methods is discussed. Although we have already many clustering algorithms, some of which are used in document clustering tasks [20], most of them are designed for finding groups composed of similar pattern from the view point of "point prototype" or "hierarchical aggregation". For example, Fuzzy $c$-Medoids (FCMdd) [5], which is a representative method of point-prototype models, can be applied to the relational data set of this subsection. Figure 7 shows the clustering results of finding three chapter structures of circles, times, and triangles. Small times mean ambiguous assignment as well. The conventional clustering methods are useful for finding such document groups considering mutual similarity among documents (or sometime keyword groups).

On the other hand, the proposed method is designed for a different purpose of finding "local linear structures" from the view point of local PCA, which is useful for cluster-wise information summarization such as local feature map construction. In this sense, the proposed method has different future application area from the conventional clustering tools.

## 5. Conclusion

This paper compared the applicability of TIBA imputation methods and $\beta$-spread transformation for handling incomplete relational data in FCMdd-type linear clustering. In numerical experiments, three imputation techniques of minimax TIBA, maximin TIBA, and average TIBA were compared using two data sets. The experimental results indicated that $\beta$-spread transformation still works well for incomplete data in conjunction with $\beta$-spread transformation. All the three TIBAs are useful for imputing incomplete non-Euclidean relational data.

From the view point of local PCA concept, the proposed method can be used for local information summarization or local feature map construction where data structures are visually summarized in low-dimensional space in conjunction with data clustering. The application is remained in future works. Another potential future work is an extension to the case of multidimensional prototype models, which is useful for constructing 2D feature map.

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*Research Article*

# An Intelligent Information Retrieval Approach Based on Two Degrees of Uncertainty Fuzzy Ontology

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In spite of the voluminous studies in the field of intelligent retrieval systems, effective retrieving of information has been remained an important unsolved problem. Implementations of different conceptual knowledge in the information retrieval process such as ontology have been considered as a solution to enhance the quality of results. Furthermore, the conceptual formalism supported by typical ontology may not be sufficient to represent uncertainty information due to the lack of clear-cut boundaries between concepts of the domains. To tackle this type of problems, one possible solution is to insert fuzzy logic into ontology construction process. In this article, a novel approach for fuzzy ontology generation with two uncertainty degrees is proposed. Hence, by implementing linguistic variables, uncertainty level in domain's concepts (Software Maintenance Engineering (SME) domain) has been modeled, and ontology relations have been modeled by fuzzy theory consequently. Then, we combined these uncertain models and proposed a new ontology with two degrees of uncertainty both in concept expression and relation expression. The generated fuzzy ontology was implemented for expansion of initial user's queries in SME domain. Experimental results showed that the proposed model has better overall retrieval performance comparing to keyword-based or crisp ontology-based retrieval systems.

## 1. Introduction

The process of searching specific information among a large number of information items is known as information retrieval (IR). Users of IR Systems expect to find the most relevant items to a certain query. The computing parameters such as recall and precision are used for effectiveness appraisal of these systems [1]. Generally, an information retrieval system does not present an ideal behavior. Users often receive large result sets, and they have to spend a considerable time to find these items which are really relevant to their initial queries. Indeed, this kind of searching information will neglect relevant documents that do not contain the index terms which are specified in the user's queries. Working with specific domain knowledge, the mentioned problem can be tackled by incorporating a knowledge base such as an ontology which builds the relationships between index terms and existing information retrieval systems [2].

One of the motivations of the semantic web is the implementation of ontologies to overcome the limitations of keyword-based search [3]. Ontology is a conceptualization of a domain into a human understandable, machine-readable format consisting of entities, attributes, relationship, and axioms [4]. It is used as a standard knowledge representation for the semantic web [5]. An ontology-based information retrieval system is constructed on specified domain knowledge. Applying knowledge structures for filtering and searching the user's relevant needed information is the objective of inserting an ontology to information retrieval systems. If the searched information is covered under the concept of the user's knowledge domain, using ontology will increase the probability of relevancy [6]. However, the conceptual formalism supported by a typical ontology may not be sufficient to represent uncertainty, commonly found in many application domains due to the lack of clear-cut boundaries between concepts of the domains. One possible solution to confront the uncertain and vague information is inserting the fuzzy logic to ontology constructing process [7]. Fuzzy set theory, among computational intelligence techniques, is a promising approach to improve the effectiveness of information retrieval systems [8]. It deals with uncertainty that may be present in document and query representations as well as in their relationships. It has already been used for indexing, clustering, and recommendation [2]. The fuzzy ontology is based on modification of an existing crisp ontology. The modification process is entirely incremental and the conversion process to a fuzzy ontology adds membership values to the currently existing relations, and may also add new entries to the ontology [9].

Several researchers have considered the usage of fuzzy ontologies to model the uncertainty in the domain knowledge [10], where Lee et al. proposed a fuzzy ontology for news summarization. In their paper, the fuzzy inference mechanism generated the membership degrees for each fuzzy concept of the fuzzy ontology [11]. They also proposed a novel type-2 fuzzy ontology and applied it to diet assessment by combining the type-2 fuzzy sets and the ontology model to propose a type-2 fuzzy ontology. Moreover, they used the type-2 fuzzy ontology to diet assessment domain to propose a fuzzy diet assessment agent for people with the average levels of physical activity [12]. A Type-2 Fuzzy Ontology (T2FO) is a knowledge representation model for describing the domain knowledge with uncertainty. It is an extension of the domain ontology and contains six layers, including a domain layer, a category layer, a fuzzy concept layer, a fuzzy variable layer, a fuzzy set layer, and a Type-2 fuzzy set (T2FS) layer. The concepts and relations of the T2FO are constructed by fuzzy variables, fuzzy sets, and T2FSs [12]. In a similar work, Lee et al. developed a type-2 fuzzy ontology and used it for personal diabetic-diet recommendation [13]. Quan et al. presented an automatic fuzzy ontology generation for semantic help-desk support for supporting customer services utilizing the semantic web technologies. It has been focused on an automatic generation approach, known as fuzzy formal concept analysis (FFCA) for fuzzy machine service ontology that can also deal with uncertainty data [14].

One application of fuzzy ontologies is in query expansion task. The main aim of query expansion is to add new meaningful terms to the initial query. Query expansion technology can improve the efficiency of the search engine by adding other terms which are closely related to the original query terms and disambiguate the user query [9]. Bahri et al. addressed fuzzy ontology implementation and query answering on databases. They propose a language to define fuzzy ontology schema and to query fuzzy ontology databases and an inferential engine to infer fuzzy concept instances and their membership degrees [15]. Pan et al. proposed a framework of fuzzy query languages for fuzzy ontologies and present ed query answering algorithms for query languages over fuzzy DL-Lite ontologies [16].

In this paper, a new method for ontology generation is proposed based on fuzzy theory with two degrees of uncertainty. Considering two uncertainty degrees in concept expression and relation expression and combining uncertain models to generate a new fuzzy ontology is the main contribution of this work. In this model, linguistic variables are used and membership degree of concepts to a certain domain and similarly membership degree of relations to concepts in SME domain are modeled as well. Then, these uncertain models are combined and a new ontology with two uncertainty degrees, both in concept expression and relation expression, is proposed. Finally, performance appraisal of information retrieval system based on proposed query expansion algorithm is measured in SME domain by comparing the lack of ontology, the crisp ontology, and the fuzzy ontology situations.

The rest of this paper is organized as follows: Section 2 reviews related work to the subject of paper. In Section 3, basic concepts of fuzzy set theory are explained. Fuzzification of the modification process ontology is depicted in Section 4. In Section 6, information retrieval system performance appraisal is discussed, and finally Section 7 concludes the paper.

## 2. Related Work

Fuzzy logic systems (FLSs) have been credited with providing an adequate methodology for designing robust systems that are able to deliver a satisfactory performance when contending with the uncertainty, noise, and imprecision attributed to real-world environments and applications [10]. As a result, FLSs have been used in wide range of applications including fuzzy ontologies.

Because the aim of this paper is proposing an information retrieval approach based on fuzzy ontology, in this section some fuzzy information retrieval models which are used as a tool for improving the retrieval performance are presented.

Parry has implemented fuzzy ontology for information retrieval which is focused on medical documents retrieval. This ontology has fuzzy values in its relations [17]. Zhai et al. presented a fuzzy ontology for semantic information retrieval in e-commerce domain, and a semantic query expansion method is used for this purpose. Their framework includes three parts: concepts, properties of concepts, and values of properties in which property value can be either standard data types or linguistic values of fuzzy concepts [18]. They also implemented fuzzy ontology for semantic information retrieval in supply chain management, traffic information retrieval, and intelligent transportation systems fields [19–21].

Leite and Ricarte presented a framework to encode a geographic knowledge base composed of multiple-related ontologies whose relationships were expressed as fuzzy relations. This knowledge organization was used in a fuzzy method to expand the user initial query [22].

An ontology-based spatial query expansion method was proposed by Fu et al., which considered a geographical ontology to expand geographic terms. Various factors are taken into account to support intelligent expansion of a spatial query, including types of spatial terms as encoded in the geographical ontology, types of nonspatial terms as encoded in the domain ontology, as well as the semantics of the spatial relationships and their context of use [23]. Bratsas et al. used a fuzzy query expansion and a fuzzy thesaurus to solve the Medical Computational Problem (MCP). In the experiments, the system was capable of retrieving the same MCP for distinct descriptions of the same problem. The system uses a unique fuzzy thesaurus for query expansion [24].

Ogawa et al. used the keyword connection matrix model which the knowledge about the relevant keywords or terms is encoded as a single fuzzy relation that expresses the degree of similarity between the terms. The information retrieval process uses the fuzzy keyword connection matrix to find similarities between the query terms as a way to improve query results [25].

Pereira et al. presented the fuzzy relational ontological model in information search systems that considers knowledge base as a fuzzy ontology with concepts representing the categories and the keywords of a domain. When the user enters a query, composed of concepts, the system performs its expansion and may add new concepts based on the ontology knowledge. After expansion, the similarity between the query and the documents is calculated by fuzzy operations [26].

The fuzzy rough set-based web query expansion method was represented by Cock and Cornelis, which used the tight upper approximation in fuzzy rough set theory to find terms to be added to the query. The knowledge base is a thesaurus that consists of a term-term relation. In this approach, a term Y will only be added to a query represented by the fuzzy set A if all the terms that are related to Y are also related to at least one keyword of the query. The proposed technique results were promising when the query had ambiguous terms [27].

Calegari and Sanchez proposed a fuzzy ontology-approach to improve semantic information retrieval and introduced an information retrieval algorithm that allows to derive a unique path among the entities involved in the query in order to obtain maxima semantic associations in the knowledge domain [28]. Bahri et al. implemented fuzzy ontology for query answering on databases. They proposed a language to define fuzzy ontology schema and to query fuzzy ontology databases and an inferential engine to infer fuzzy concept instances and their membership degrees [29].

By reviewing the studies in fuzzy ontology implementation, it can be observed that the proposed approaches do not have adequate ability in uncertainty representation of concepts and have not considered natural semantic relations between ontology concepts. These weaknesses may cause some problems in fuzzy semantic information retrieval. Since concepts belong with a specific membership degree to a certain domain and similarly relations belong with a certain membership degree to concepts, this paper proposes a fuzzy ontology with two degrees of uncertainty, both in concept expression and relation expression.

## 3. Fuzzy Membership Function and Linguistic Variables

The special structure of fuzzy numbers makes calculations very time-consuming and sophisticated. Generally for facilitating calculations and practical usage, particular fuzzy numbers are used. In this paper, the experts' opinions are described by linguistic variables which have been expressed in trapezoidal fuzzy numbers. In order to determine the relevancy of ontology elements to related specific domain (concept's membership degree to main domain and relation's membership degree to concepts), seven linguistic variables have been used as: "not relevant", "very low relevant", "low relevant", "medium relevant", "high relevant", "very high relevant", and "fully relevant". Figure 1 presents these linguistic variables and their corresponding trapezoidal fuzzy numbers.

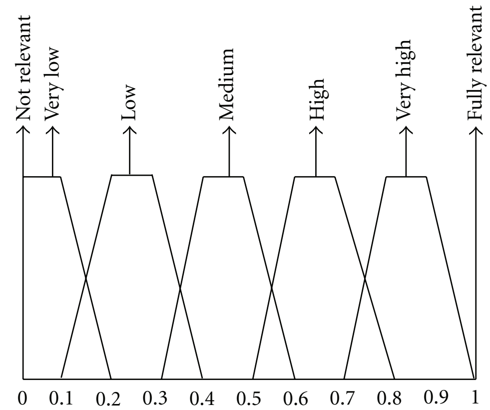


Figure 1: The membership function of linguistic variable.

## 4. Fuzzification the "Modification Process Ontology"

Software maintenance happens in a relatively disorganized way and naturally leads to the deterioration of software systems' structure. Lacking a complete knowledge of all the implementation details, apply modifications that will result in a loss of structure, which in turn makes the systems more difficult to understand fully and, therefore, to maintain [30]. To break this vicious circle, we aim at developing a knowledge management approach for software maintenance domain. This approach will be modeled as fuzzy ontology.

The initial modification process ontology, which has been proposed by Dias, is presented in Figure 2. This ontology organizes concepts from the modification request (and its causes) to the maintenance activities in SME domain [24]. In this study, the existing uncertainty in concepts and relations of the modification process ontology is modeled in two phases. Firstly, an uncertainty degree for describing the concepts of ontology is obtained using linguistic variables. Secondly, the relations of existing ontology have been fuzzified, and finally, a new ontology with a unique membership degree for each relation is proposed by implementing fuzzy composition. Assume that $U$ and $V$ are two collections of objects. An arbitrary fuzzy set $R$, defined in the Cartesian product $V \times W$, will be called a fuzzy relation in the space $U \times V$. $R$ is thus a function defined in the space $U \times V$, which takes values from the interval $[0, 1]$. Let $R_1$ be a fuzzy relation in $U \times V$ and $R_2$ a fuzzy relation in $V \times W$. For all

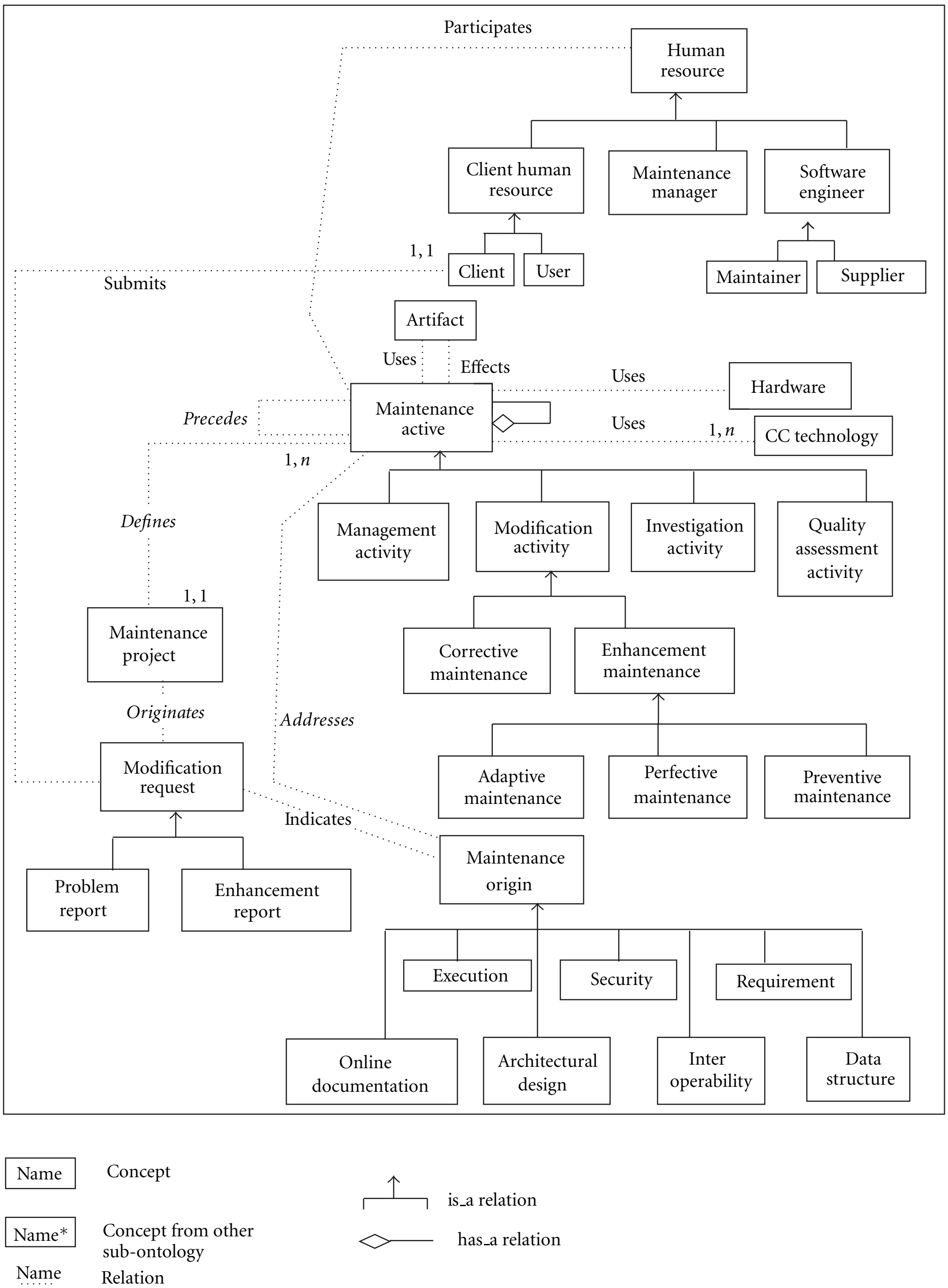


Figure 2: The modification process ontology.

$(u, w)$ in $U \times W$, max-min, max-product, and max-average compositions are defined as follows [31]:

$$\text{Max-Min}(R_1, R_2)(u, w) = \text{Max}(\text{Min}(R_1(u, v), R_2(v, w))) \text{ over all } v \text{ in the set } V, \tag{1}$$

$$\text{Max-Product}(R_1, R_2)(u, w) = \text{Max}(R_1(u, v) \times R_2(v, w)) \text{ over all } v \text{ in the set } V, \tag{2}$$

$$\text{Max-Average}(R_1, R_2)(u, w) = \left(\frac{1}{2}\right) \text{Max}\{(R_1(u, v) + R_2(v, w))\} \text{ over all } v \text{ in the set } V. \tag{3}$$

Table 1: Concept's fuzzification results of the modification process ontology.

| Row | Concepts | Average of expert's opinions | Defuzzification value |
|---|---|---|---|
| 1 | Human resource | (0.80,0.87,0.93,1) | 0.9 |
| 2 | Client human resource | (0.7,0.8,0.9,1) | 0.85 |
| 3 | Maintenance manager | (0.7,0.8,0.9,1) | 0.85 |
| 4 | Software engineer | (0.9,0.93,0.97,1) | 0.95 |
| 5 | Client | (0.7,0.8,0.9,1) | 0.85 |
| 6 | User | (0.63,0.73,0.83,0.93) | 0.78 |
| 7 | Maintainer | (0.9,0.93,0.97,1) | 0.95 |
| 8 | Supplier | (0.63,0.73,0.83,0.93) | 0.78 |
| 9 | Artifact | (0.43,0.53,0.63,0.73) | 0.58 |
| 10 | Maintenance activity | (0.63,0.73,0.83,0.93) | 0.78 |
| 11 | Hardware | (0.63,0.73,0.83,0.93) | 0.78 |
| 12 | CC technology | (0.37,0.47,0.57,0.87) | 0.52 |
| 13 | Management activity | (0.80,0.87,0.93,1) | 0.9 |
| 14 | Modification activity | (0.80,0.87,0.95,1) | 0.92 |
| 15 | Investigation activity | (0.80,0.87,0.93,1) | 0.9 |
| 16 | Quality assurance activity | (0.80,0.87,0.93,1) | 0.9 |
| 17 | Maintenance project | (0.63,0.73,0.83,0.93) | 0.78 |
| 18 | Corrective maintenance | (0.80,0.87,0.93,1) | 0.9 |
| 19 | Enhancement maintenance | (0.80,0.87,0.93,1) | 0.9 |
| 20 | Modification request | (0.80,0.87,0.93,1) | 0.9 |
| 21 | Adaptive maintenance | (0.7,0.8,0.93,1) | 0.87 |
| 22 | Perfective maintenance | (0.7,0.8,0.9,1) | 0.85 |
| 23 | Preventive maintenance | (0.7,0.8,0.9,1) | 0.85 |
| 24 | Problem report | (0.7,0.8,0.9,1) | 0.85 |
| 25 | Enhancement report | (0.7,0.8,0.9,1) | 0.85 |
| 26 | Maintenance origin | (0.80,0.87,0.93,1) | 0.9 |
| 27 | Execution | (0.7,0.8,0.9,1) | 0.85 |
| 28 | Security | (0.7,0.8,0.9,1) | 0.85 |
| 29 | Requirement | (0.7,0.8,0.9,1) | 0.85 |
| 30 | Online documentation | (0.7,0.8,0.9,1) | 0.85 |
| 31 | Architectural design | (0.7,0.8,0.9,1) | 0.85 |
| 32 | Interoperability | (0.7,0.8,0.9,1) | 0.85 |
| 33 | Data structure | (0.7,0.8,0.9,1) | 0.85 |

*4.1. Concepts' Fuzzification of the "Modification Process Ontology".* Although fuzzification of ontology's relations has been focused in almost all the revised papers, concept's fuzzification has been considered rarely [2, 4, 9, 11, 18–21]. So, this section considers fuzzification of ontology concepts particularly. The supportive idea is that existing concepts in ontology are not related completely to the considered SME domain, so each concept has a specific membership degree to the specific domain. For fuzzification of the concepts' ontology, a special questionnaire was designed and related experts were asked to determine the relevancy degree of the modification process ontology's concepts to the domain with following linguistic values: "not relevant", "very low relevant", "low relevant", "medium relevant", "high relevant", "very high relevant", and "fully relevant". These values were fuzzified by the membership function shown in Figure 1. In order to reach a unique membership degree for each concept of the ontology, trapezoidal fuzzy variable defuzzifier has been used. The defuzzification value of a trapezoidal fuzzy number $\widetilde{A} = [a_1, a_2, a_3, a_4]$, $a_1 \le a_2 \le a_3 \le a_4$, is defined as [32]

$$D = \frac{(a_1 + a_2 + a_3 + a_4)}{4}. \tag{4}$$

The results have been presented in Table 1.

*4.2. Relations' Fuzzification of the "Modification Process Ontology".* Resembling the mentioned process in Section 4.1, another questionnaire was designed for fuzzification of ontology relations in Figure 2. The experts were asked to determine the relevancy degree of each relation to the existed concepts with these values: "not relevant", "very low relevant", "low relevant", "medium relevant", "high relevant", "very high relevant", and "fully relevant". These linguistic variables were fuzzified with the membership function which has been shown in Figure 1. Table 2 shows the results of

Table 2: Relation's fuzzification results of the modification process ontology.

| Row | Ontology relations | Average of expert's opinions | Defuzzification value |
|---|---|---|---|
| 1 | participates | (0.7,0.8,0.9,1) | 0.85 |
| 2 | is a | (1,1,1,1) | 1.00 |
| 3 | is a | (1,1,1,1) | 1.00 |
| 4 | is a | (1,1,1,1) | 1.00 |
| 5 | is a | (1,1,1,1) | 1.00 |
| 6 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 7 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 8 | is a | (0.7,0.8,0.9,1) | 0.85 |
| 9 | submits | (0.9,0.93,0.97,1) | 0.95 |
| 10 | uses | (0.9,0.93,0.97,1) | 0.95 |
| 11 | affects | (0.9,0.93,0.97,1) | 0.95 |
| 12 | precedes | (0.9,0.93,0.97,1) | 0.95 |
| 13 | has a | (1,1,1,1) | 1.00 |
| 14 | uses | (0.7,0.8,0.9,1) | 0.85 |
| 15 | uses | (0.7,0.8,0.9,1) | 0.85 |
| 16 | defines | (0.9,0.93,0.97,1) | 0.95 |
| 17 | addresses | (0.7,0.8,0.9,1) | 0.85 |
| 18 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 19 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 20 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 21 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 22 | originates | (0.7,0.8,0.9,1) | 0.85 |
| 23 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 24 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 25 | is a | (0.7,0.8,0.9,1) | 0.85 |
| 26 | is a | (0.7,0.8,0.9,1) | 0.85 |
| 27 | is a | (0.7,0.8,0.9,1) | 0.85 |
| 28 | is a | (0.7,0.8,0.9,1) | 0.85 |
| 29 | is a | (0.7,0.8,0.9,1) | 0.85 |
| 30 | indicates | (0.63,0.73,0.83,0.93) | 0.78 |
| 31 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 32 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 33 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 34 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 35 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 36 | is a | (0.9,0.93,0.97,1) | 0.95 |
| 37 | is a | (0.9,0.93,0.97,1) | 0.95 |

modeling the ontology relations with trapezoidal numbers. In order to reach a unique membership degree for each relation of the ontology, the trapezoidal fuzzy variable defuzzifier (4) has been used. Column 2 of Table 2 shows the ontology relations which have been presented in Figure 3.

*4.3. Proposing the New Combined Model Based on Fuzzy Set Theory.* The final step of this phase is the combination of concepts and relations' membership degrees and proposing a unique membership degree for ontology relations in the SME domain. Because the relevancy degree of the ontology relations to the concepts has been fuzzified and the concepts of the SME domain are fuzzified as well, the membership degree of relations in the modification process ontology can be calculated by using fuzzy compositions. Implementing the proposed fuzzy ontology shows better results for max-product composition (2) in comparison to other fuzzy compositions, therefore this fuzzy composition type is used. The final fuzzy ontology has been shown in Figure 3.

## 5. The Query Expansion Algorithm

In this section, topics including the proposed query expansion algorithm, the algorithm in a pseudocode form and computational complexity of the crisp, and fuzzy ontology query expansion algorithms are discussed.

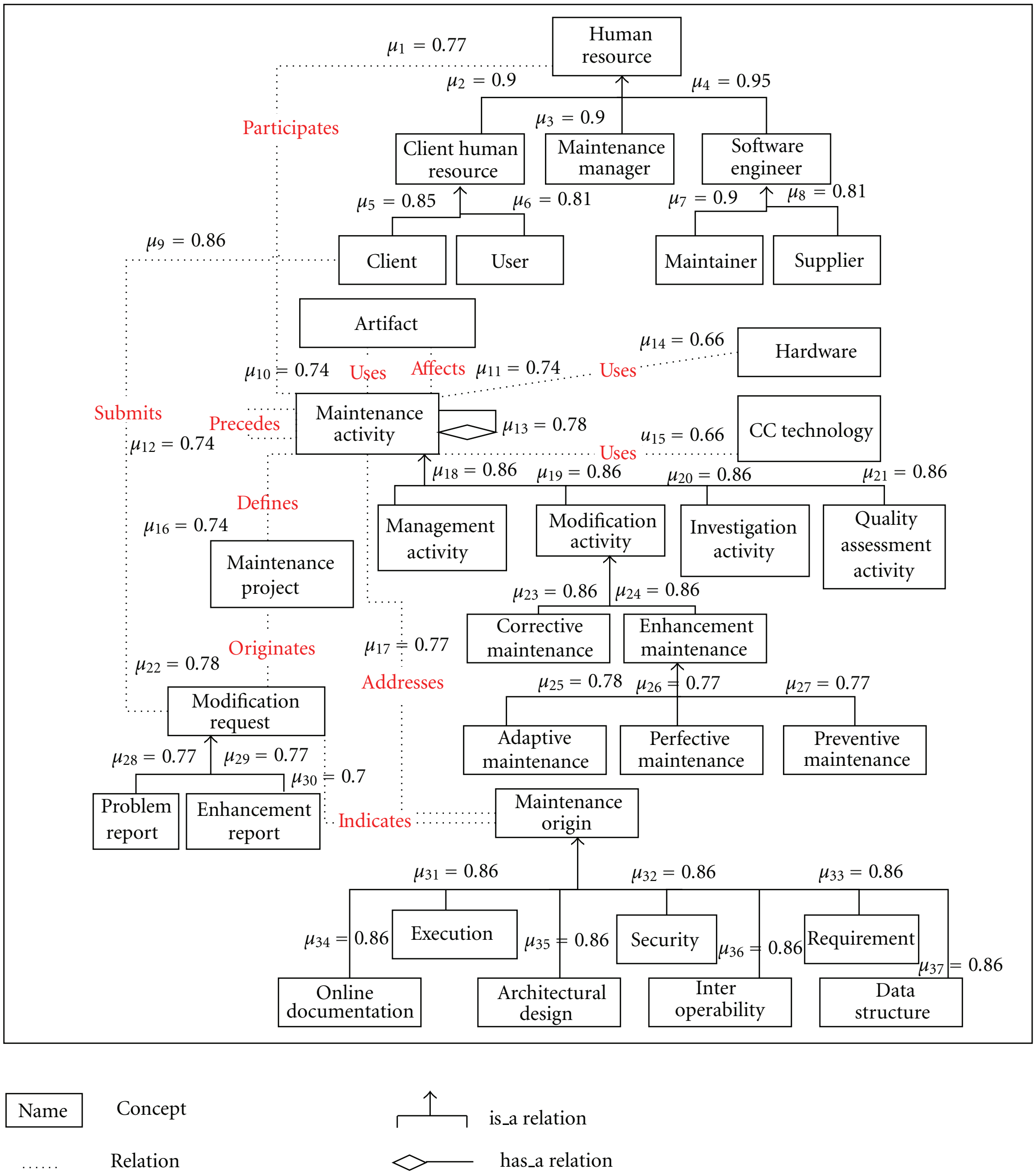


Figure 3: The final fuzzified modification process ontology.

*5.1. The Proposed Query Expansion Algorithm Based on the Ontology.* In this algorithm, it is assumed that entered user query terms are existed in the modification process ontology. If the query terms were not in the related ontology of a specific domain, the most common approach is adding synonym terms based on a general dictionary (commonly English comprehensive ontology WordNet). General dictionaries do not consider any specific domain, hence a satisfied accuracy will not be obtained [33]. Therefore, in the proposed query expansion algorithm, those situations are considered that the entered query terms exist in the modification process ontology. Different approaches are used for query expansion based on the existence or the lack of relations between query terms in the ontology. These approaches are explained in the following.

*(a) Query Expansion Based on Crisp Ontology.* With this supposition that query terms exist in the modification process ontology and these terms have or do not have a semantic relation with each other, two situations are possible.

*(i) Query terms exist in the modification process ontology and have semantic relation.* The associated relations of terms in an ontology are extracted (it is possible that one term is father or child of another term). For each father term, all its father terms (generalized terms) are extracted, and for each child term, all its child terms (specialized terms) are extracted as well and then these extracted terms are entered in the expanded query, consequently. For example, consider query = ("modification activity", "enhancement maintenance"). These terms have "modification activity" > "enhancement

```
//situation (a.i)
for (){
get(query);
if (query words exist in modification process ontology && have relation together){
    if (the query word is parent of other words)
      add(other parent of this word to the query terms);
    else if (the query word is child of other words)
      add(other child of this word to the query terms);
}//end if
}//end for
* * * * * * * * * * * * * * * * * * * * * * * * * * *
//situation (a.ii)
for (){
get(query);
    if (query words exist in modification process ontology && have no relation together){
    add(all parents and children of this words to query);
}//end for
* * * * * * * * * * * * * * * * * * * * * * * * * * *
//situation (b.i)
for (){
get(query);
  if (query words exist in modification process ontology && have relation together){
    if (the query word is parent of other words && their membership degree >=threshold limit)
      add(other parent of this word to the query terms);
  else if (the query word is child of other words && their membership degree >=threshold limit)
      add(other child of this word to the query terms);
}//end if
}//end for
* * * * * * * * * * * * * * * * * * * * * * * * * * *
//situation (b.ii)
for (){
get(query);
  if (query words exist in modification process ontology && have no relation together && their
  membership degree >=threshold limit){
    add(all parents and children of this words to query);
}//end for
* * * * * * * * * * * * * * * * * * * * * * * * * * *
```

Pseudocode 1

maintenance" relation in the ontology; thus, instead of the term "modification activity", generalized terms such as "maintenance activity" will be entered in the expanded query. Similarly, instead of the term "enhancement maintenance", specialized terms such as "adaptive maintenance", "perfective maintenance", and "preventive maintenance" are added in the expanded query.

*(ii) Query terms exist in the modification process ontology and do not have semantic relations.* In this situation, all related father and child terms will be entered in the expanded query. As a case in point, consider query = ("corrective maintenance", "maintenance project"). Because there is no relation between these query terms in the ontology, all of their father and child terms are added in the expanded query; so, instead of the term "corrective maintenance", the terms "modification activity", and instead of the term "maintenance project", the terms "maintenance activity" and "modification request" will be added in expanded query.

*(b) Query Expansion Based on Fuzzy Ontology.* This status is very similar to status (a) with the difference that terms added in the query have a membership degree equal or greater than a certain threshold limit. As is discussed in the next section, the maximum average precision belongs to threshold limit 0.78, so this limit is considered. For example, consider query = ("modification activity", "enhancement maintenance"). These terms have "modification activity" > "enhancement maintenance" relation in the ontology and obey the situation (i); thus, instead of term the "modification activity", generalized terms with a membership degree equal or greater than 0.78 such as "maintenance activity" will be entered in the expanded query. Similarly, instead of the term "enhancement maintenance", specialized terms with membership degree equal or greater than 0.78 such as "adaptive maintenance" are added in the expanded query.

As a more illustration, consider query = ("corrective maintenance", "maintenance project"). Because there is no relation between these query terms in the ontology, all of their father and child terms are added in the expanded query based on the situation (ii); so, instead of term "corrective maintenance", the term "modification activity" and instead of term "maintenance project" the term "modification request" which have a membership degree equal or greater than 0.78 will be added in expanded query.

*5.2. Pseudocode Form of the Algorithm.* The pseudocode form of the algorithm based on crisp and fuzzy ontology is as shown in Pseudocode 1.

*5.3. Computational Time of the Crisp and Fuzzy Ontology Query Expansion Algorithm.* Computational time of the crisp and fuzzy ontology query expansion algorithm in worth case of both situation (a) and (b) of our pseudocode will be $O(n)$, because in the worth case the user enters all of terms of the crisp and fuzzy ontology in the query and in normal case is $O(\text{constant})$.

## 6. Performance Evaluation of Information Retrieval System

The two most frequent and basic measures for information retrieval effectiveness are precision and recall [34, 35]. So, we used these measures for the ontology performance evaluation. Precision and recall are defined in terms of a set of retrieved documents (e.g., the list of documents produced by a web search engine for a query) and a set of relevant documents (e.g., the list of all documents on the internet that are relevant for a certain topic),

$$\begin{aligned} \text{Precision} &= \frac{\#(\text{relevant items retrieved})}{\#(\text{retrieved items})}, \\ &= P(\text{relevant} \mid \text{retrieved}) \\ \text{Recall} &= \frac{\#(\text{relevant items retrieved})}{\#(\text{relevant items})} \\ &= P(\text{retrieved} \mid \text{relevant}). \end{aligned} \tag{5}$$

The average precision plots at each standard recall level across all queries and evaluates overall system performance on a document/query corpus [36, 37].

For performance evaluation of information retrieval system in local searches, the Google Desktop search engine has been implemented, and a lot of related documents to the Software Maintenance Engineering domain have been entered in. For appropriate threshold limit determining in described situation (b), various thresholds from 0 to 0.9 have been considered. Furthermore, both unique and multiple query terms were entered in the search engine. Figure 4 shows the average precision of fuzzy ontology for different threshold limits. As it is obvious in this figure, the maximum average precision belongs to threshold limit 0.78.

As a result, the threshold limit 0.78 has been used for the situation (b) in the proposed query expansion algorithm. In Figure 5, the average precision for different levels of the recall values has been depicted for the three situations: the lack of ontology situation, the crisp ontology query expansion situation, and the fuzzy ontology query expansion situation. These situations are selected to evaluate the fuzzy ontology performance rather than the lack of ontology situation and the crisp ontology query expansion situation. As shown in Figure 5, the fuzzy query expansion situation has the best average precision in retrieving relevant documents.

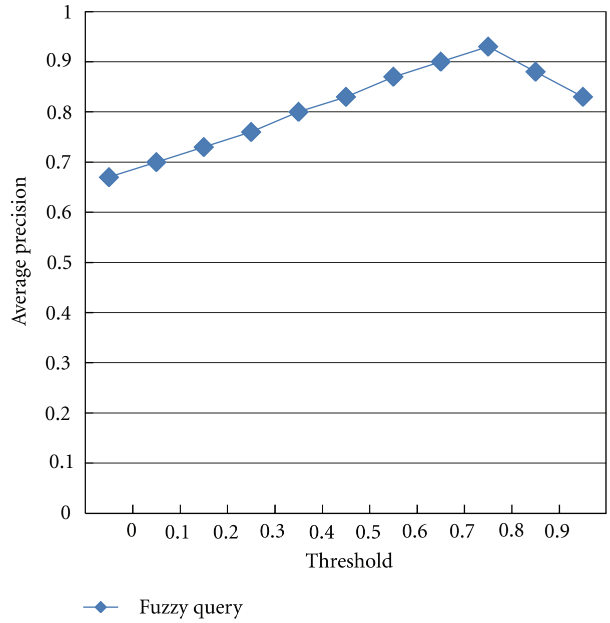


Figure 4: The average precision plot for different fuzzy query's threshold limits.

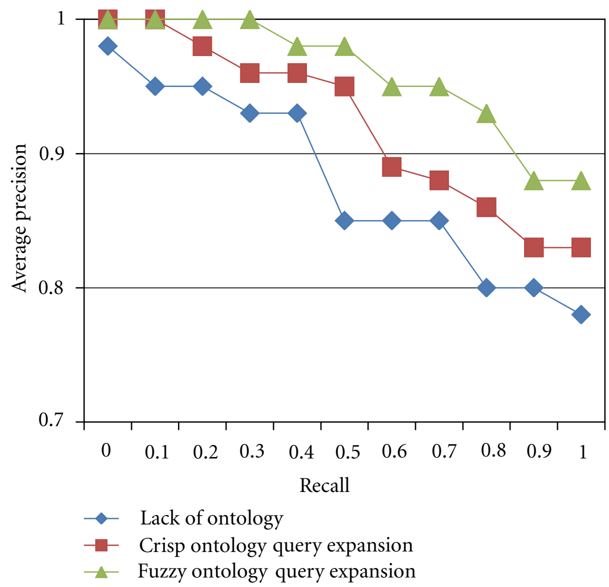


Figure 5: The average precision plot for three situations: lack of ontology, crisp ontology query expansion and fuzzy ontology query expansion.

## 7. Conclusion

In this paper we proposed a new approach for effective retrieving of information by implementing fuzzy ontology generation technique with two uncertainty degrees. Taking fuzzy ontology can tackle the uncertainty of relations in comparison to taking crisp ontology and make it possible to find uncertain information in a specific domain. For performance evaluation of information retrieval system, we considered three different situations (two situation with ontology existence (crisp or fuzzy) and one without any ontology). The empirical results of using proposed query expansion algorithm showed that crisp ontology's average precision increased 5% in comparison to take no ontology for expansion job and fuzzy ontology's average precision increased 3% in comparison to use crisp ontology query expansion situation. Furthermore, fuzzy ontology's average precision increased 8% in comparison to take no ontology for expansion. These results showed that the quality of information retrieval system using fuzzy ontology query expansion method improves remarkably.

In the future, we will implement fuzzy theory and neural network methods to build fuzzy ontology from unstructured data automatically. Also, we will compare the results with the proposed approach in this paper.

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*Research Article*

# Spatial Analysis and Fuzzy Relation Equations

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We implement an algorithm that uses a system of fuzzy relation equations (SFRE) with the max-min composition for solving a problem of spatial analysis. We integrate this algorithm in a Geographical Information System (GIS) tool, and the geographical area under study is divided in homogeneous subzones (with respect to the parameters involved) to which we apply our process to determine the symptoms after that an expert sets the SFRE with the values of the impact coefficients. We find that the best solutions and the related results are associated to each subzone. Among others, we define an index to evaluate the reliability of the results.

## 1. Introduction

A Geographical Information System (GIS) is used as a support decision system for problems in a spatial domain; in many cases, we use a GIS to analyze spatial distribution of data, spatial relations, the impact of event data on spatial areas; simple examples of this analysis are the creation of thematic maps, the geoprocessing operators, the buffer analysis, and so forth. Often the expert analyzes spatial data in a decision making process with the help of a GIS which involves integration of images, spatial layers, attributes information and an inference mechanism based on these attributes. The diversity and the inhomogeneity between the individual layers of spatial information and the inaccuracy of the results can lead to uncertain decisions, so that one needs the use of fuzzy inference calculus to handle these uncertain information. Many authors [1–5] propose models to solve spatial problems based on fuzzy relational calculus. In this paper, we propose an inferential method to solve spatial problems based on an algorithm for the resolution of a system of fuzzy relation equations (shortly, SFRE) given in [6] (cf. also [7, 8]) and applied in [9] to solve industrial application problems. Here we integrate this algorithm in the context of a GIS architecture. Usually an SFRE with max-min composition is read as

$$\begin{gathered}
(a_{11} \wedge x_1) \vee \cdots \vee (a_{1n} \wedge x_n) = b_1, \\
(a_{21} \wedge x_1) \vee \cdots \vee (a_{2n} \wedge x_n) = b_2, \\
\vdots \\
(a_{m1} \wedge x_1) \vee \cdots \vee (a_{mn} \wedge x_n) = b_m.
\end{gathered} \tag{1}$$

The system (1) is said consistent if it has solutions. In his pioneering paper [10], the author determines the greatest solution in case of max-min composition. After these results, many researchers have found algorithms which determine minimal solutions of max-min fuzzy relation equations (cf. e.g., [11–18]). In [6, 7] a method is described for the consistence of the system (1), and moreover it calculates the complete set of the solutions. This method is schematized in Figure 1 and described below.

(i) Input extraction: the input data are extracted and stored in the dataset.

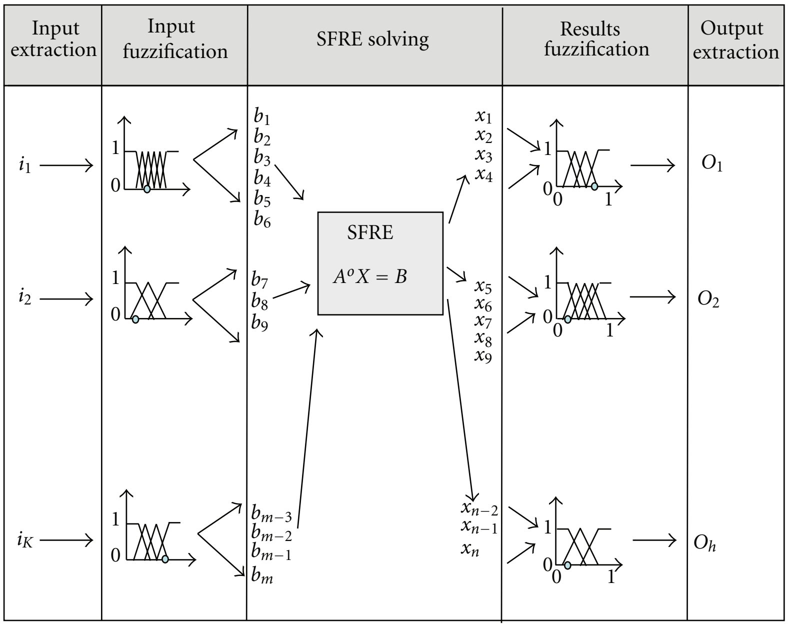



Figure 1: Resolution process of an SFRE.

(ii) The input variable is fuzzified. A fuzzy partition of the input domain is created; the corresponding membership degree of every input data is assigned to each fuzzy set.

(iii) The membership degrees of each fuzzy set determine the coefficients $\{b_1, \ldots, b_m\}$ of (1). The values of the coefficients $a_{ij}$ are set by the expert and the whole set of solutions $(x_1, \ldots, x_n)$ of (1) is determined as well.

(iv) A fuzzy partition of the domain $[0, 1]$ is created for the output variables $o_1, \ldots, o_k$; every fuzzy set of the partition corresponds to a determined value $x_j$.

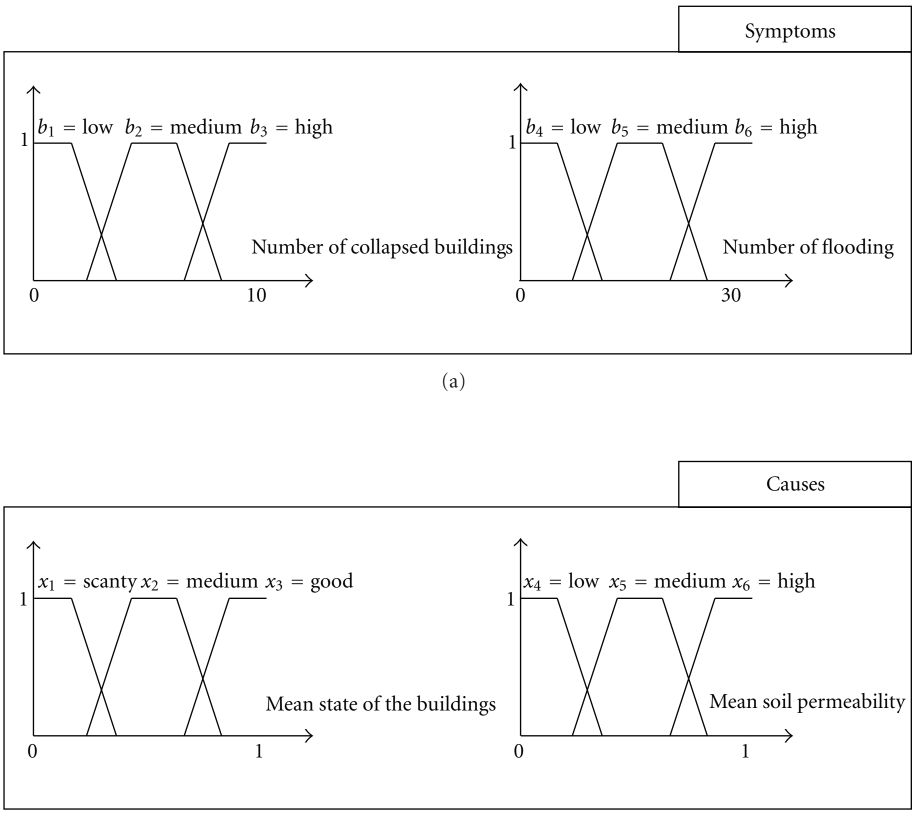
(v) The output data $o_1, \ldots, o_k$ are extracted. A partition of fuzzy sets corresponds to each output variable $o_j$ $(j = 1, \ldots, k)$; in this phase the linguistic label of the most appropriate fuzzy set is assigned to the output variable $o_j$.

This process has been applied to a real spatial problem in which the input data vary for each subzone of the geographical area. We have the same input data, and the expert applies the same SFRE (1) on each subzone. The expert starts from a valuation of input data, and he uses linguistic labels for the determination of the output results for each subzone. The input data are the facts or symptoms; the parameters to be determined are the causes. For example, let us consider a planning problem. A city planner needs to determine in each subzone the mean state of buildings $(x_1)$ and the mean soil permeability $(x_2)$, knowing the number of collapsed building in the last year $(b_1)$ and the number of flooding in the last year $(b_2)$. In Figure 2, we suppose to create for each symptom's and cause's variable domain a fuzzy partition of three fuzzy sets (generally, one is faced with trapezoidal or triangular fuzzy number, this last one is denoted in the sequel shortly with the acronym TFN). The expert creates the SFRE (1) for each subzone by setting the impact matrix A, whose entries $a_{ij}$ $(i = 1, \ldots, n$ and $j = 1, \ldots, m)$ represent the impact of the $j$th cause $x_j$ to the production of the $i$th symptom $b_i$, where the value of $b_i$ is the membership degree in the corresponding fuzzy set and let $B = [b_1, \ldots, b_m]$. In another subzone the input data vector $B$ and the matrix $A$ can vary. For example, we consider the equation:

$$(0.8 \wedge x_1) \vee (0.2 \wedge x_2) \vee (0.0 \wedge x_3) \vee (0.8 \wedge x_4) \vee (0.3 \wedge x_5) \vee (0.0 \wedge x_6) = b_3 = 0.9. \tag{2}$$

The expert sets for the symptom $b_3$ = "collapsed building in the last year = high" = 0.9, an impact 0.8 of the variable "mean state of buildings = scanty", an impact 0.2 of the variable "mean state of buildings = medium", an impact 0.0 of the variable "mean state of buildings = high", an impact 0.8 of the variable "mean soil permeability = low", an impact 0.3 of the variable "mean soil permeability = medium", or an impact 0.0 of the variable "mean soil permeability = high".

We can determine the maximal interval solutions of (1). Each maximal interval solution is an interval whose extremes are the values taken from a minimal solution and from the greatest solution. Every value $x_i$ belongs to this interval. If the SFRE (1) is inconsistent, it is possible to determine the rows for which no solution is permitted. If the expert decides to exclude the row for which no solution is permitted, he considers that the symptom $b_i$ (for that row) is not relevant to its analysis, and it is not taken into account. Otherwise, the expert can modify the setting of the coefficients of the matrix $A$ to verify if the new system has some solution. In general, the SFRE (1) has T




(a)

(b)

Figure 2: Examples of trapezoidal fuzzy numbers used for symptoms and causes.

maximal interval solutions $X_{\max(1)}, \ldots, X_{\max(T)}$. In order to describe the extraction process of the solutions, let $X_{\max(t)}$, $t \in \{1, \ldots, T\}$, be a maximal interval solution given below, where $X^{\text{low}}$ is a minimal solution and $X^{\text{gr}}$ is the greatest solution. Our aim is to assign the linguistic label of the most appropriate fuzzy sets corresponding to the unknown $\{x_{j_1}, x_{j_1}, \ldots, x_{j_s}\}$ related to an output variable $o_s$, $s = 1, \ldots, k$. For example, assume that the three fuzzy sets $x_1, x_2, x_3$ (resp., $x_4$, $x_5$, $x_6$) are related to $o_1$ (resp., $o_2$) and are represented from the TFNs given in Table 1, where $\text{INF}(j)$, $\text{MEAN}(j)$, and $\text{SUP}(j)$ are the three fundamental values of the generic TFN $x_j$, $j = j_1, \ldots, j_s$. We can write their membership functions $\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_h}$ as follows:

$$\mu_{j_1} = \begin{cases} 1, & \text{if } \text{INF}(j_1) \le x \le \text{MEAN}(j_1), \\ \dfrac{\text{SUP}(j_1) - x}{\text{SUP}(j_1) - \text{MEAN}(j_1)}, & \text{if } \text{MEAN}(j_1) < x \le \text{SUP}(j_1), \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

$$\mu_j = \begin{cases} \dfrac{x - \text{INF}(j)}{\text{MEAN}(j) - \text{INF}(j)}, & \text{if } \text{INF}(j) \le x \le \text{MEAN}(j), \\ \dfrac{\text{SUP}(j) - x}{\text{SUP}(j) - \text{MEAN}(j)}, & \text{if } \text{MEAN}(j) < x \le \text{SUP}(j), \\ 0, & \text{otherwise}, \end{cases} \quad j \in \{j_2, \ldots, j_{s-1}\}, \tag{4}$$

$$\mu_{j_s} = \begin{cases} \dfrac{x - \text{INF}(j_s)}{\text{MEAN}(j_s) - \text{INF}(j_s)}, & \text{if } \text{INF}(j_s) \le x \le \text{MEAN}(j_s), \\ 1, & \text{if } \text{MEAN}(j_s) < x \le \text{SUP}(j_s), \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

If $X\text{Min}_t(j)$ (resp., $X\text{Max}_t(j)$) is the min (resp., max) value of every interval corresponding to the unknown $x_j$, we can calculate the arithmetical mean value $X\text{Mean}_t(j)$ of

Table 1: TFNs values for the fuzzy sets.

| Unknown | INF($j$) | MEAN($j$) | SUP($j$) |
|---|---|---|---|
| $x_1$ | 0.0 | 0.2 | 0.4 |
| $x_2$ | 0.3 | 0.5 | 0.7 |
| $x_3$ | 0.6 | 0.8 | 1.0 |
| $x_4$ | 0.0 | 0.2 | 0.4 |
| $x_5$ | 0.3 | 0.5 | 0.7 |
| $x_6$ | 0.6 | 0.8 | 0.1 |

the $j$th component of the above maximal interval solution $X_{\max(t)}$ as

$$X\mathrm{Mean}_t(j) = \frac{X\mathrm{Min}_t(j) + X\mathrm{Max}_t(j)}{2}, \tag{6}$$

and we get the vector column $X\mathrm{Mean}_t = [X\mathrm{Mean}_t(1), \ldots, X\mathrm{Mean}_t(n)]^{-1}$ (cf. Table 2). The value given from $\max\{X\mathrm{Mean}_t(j_1), \ldots, X\mathrm{Mean}_t(j_s)\}$ obtained for the unknowns $x_{j_1}, \ldots, x_{j_s}$ corresponding to the output variable $o_s$, is the linguistic label of the fuzzy set assigned to $o_s$ and it is denoted by $\mathrm{score}_t\ (o_s)$, defined also as reliability of $o_s$ in the interval solution $t$. In our example, we have that "$o_1$ = *mean state of buildings = scanty*" and "$o_2$ = *mean soil permeability = medium*", hence $\mathrm{score}_t(o_1) = 0.70$ and $\mathrm{score}_t(o_2) = 0.55$. For the output vector $O = [o_1, \ldots, o_k]$, we define the following reliability index in the interval solution $t$ as

$$\mathrm{Rel}_t(O) = \frac{1}{k} \cdot \sum_{s=1}^{k} \mathrm{score}_t(o_s) \tag{7}$$

and then as final reliability index of $O$, the number $\mathrm{Rel}(O) = \max\{\mathrm{Rel}_t(O) : t = 1, \ldots, T\}$.

In our example, we have $\mathrm{Rel}_t(O) = (0.7 + 0.55)/2 = 0.625$. Therefore, the higher the reliability of our solution, the closer the final reliability index $\mathrm{Rel}(O)$ to 1. In Section 2, we give an extended and articulated overview on how to determine the whole set of the solutions of an SFRE, and in Section 3 we show how the proposed algorithm is applied in spatial analysis. Section 4 contains the results of our simulation.

## 2. SFRE: An Extended Overview

In this paper, we investigate the solutions of the SFRE (1), which is abbreviated in the following known form:

$$A \circ X = B, \tag{8}$$

where $A = (a_{ij})$ is the matrix of coefficients, $X = (x_1, x_2, \ldots, x_n)^{-1}$ is the column vector of the unknowns, and $B = (b_1, b_2, \ldots, b_m)^{-1}$ is the column vector of the known terms, being $a_{ij}, x_j, b_i \in [0, 1]$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We have the following definitions and terminologies: the whole set of all solutions $X$ of the SFRE (8) is denoted by $\Omega$. If $\Omega \neq \emptyset$, then the SFRE (8) is called consistent, otherwise it is called inconsistent. A solution $\widehat{X} \in \Omega$ is called a minimal solution if $X \le \widehat{X}$ for some $X \in \Omega$ implies $X = \widehat{X}$, where "$\le$" is the partial order induced in $\Omega$ from the natural order of $[0, 1]$. If the minimal solution is unique, then it is the least (or minimum) solution of the SFRE (8). We also recall that the system (8) has the unique greatest (or maximum) solution $X^{\mathrm{gr}} = (x_1^{\mathrm{gr}}, x_2^{\mathrm{gr}}, \ldots, x_n^{\mathrm{gr}})^{-1}$ if $\Omega \neq \emptyset$ [10]. A matrix interval $X_{\mathrm{interval}}$ of the following type:

$$X_{\mathrm{interval}} = \begin{pmatrix} [a_1, b_1] \\ [a_2, b_2] \\ [\ldots, \ldots] \\ [a_n, b_n] \end{pmatrix}, \tag{9}$$

where $[a_j, b_j] \subseteq [0, 1]$ for each $j = 1, \ldots, n$, is called an interval solution of the SFRE (8) if every $X = (x_1, x_2, \ldots, x_n)^{-1}$ such that $x_j \in [a_j, b_j]$ for each $j = 1, \ldots, n$, belongs to $\Omega$. If $a_j$ is a membership value of a minimal solution and $b_j$ is a membership value of $X^{\mathrm{gr}}$ for each $j = 1, \ldots, n$, then $X_{\mathrm{interval}}$ is called a maximal interval solution of the SFRE (8), and it is denoted by $X_{\max(t)}$, where $t$ varies from 1 to the number of minimal solutions. The SFRE (8) is said to be in normal form if $b_1 \ge b_2 \ge \cdots \ge b_m$. The time computational complexity to reduce an SFRE in a normal form is polynomial [6, 8]. Now we consider the matrix $A^* = (a_{ij}^*)$ so defined:

$$a_{ij}^* = \begin{cases} 0, & \text{if } a_{ij} < b_i, \\ b_i, & \text{if } a_{ij} = b_i, \\ 1, & \text{if } a_{ij} > b_i, \end{cases} \tag{10}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The linguistic description of $a_{ij}^*$ as S-type coefficient (Smaller) if $a_{ij} < b_i$, E-type coefficient (Equal) if $a_{ij} = b_i$, and G-type coefficient (Greater) if $a_{ij} > b_i$ is often used. $A^*$ is called augmented matrix, and the system $A^* \circ X = B$ is said associated to the SFRE (8). Without loss of generality, from now on we suppose that the system (8) is in normal form. We also obtained the following definitions and results from [6, 8, 19, 20].

*Definition 1.* Let the SFRE (8) be consistent and $A_j^* = \{a_{1j}^*, \ldots, a_{mj}^*\}$. If $A^*(j)$ contains G-type coefficients and $k \in \{1, \ldots, m\}$ is the greatest index of row such that $a_{kj}^* = 1$, then the following coefficients in $A^*(j)$ are called selected:

(i) $a_{ij}^*$ for $i \in \{1, \ldots, k\}$ with $a_{ij}^* \ge b_i = b_k$,

(ii) $a_{ij}^*$ for $i \in \{k+1, \ldots, m\}$ with $a_{ij}^* = b_i$.

*Definition 2.* If $A^*(j)$ does not contain G-type coefficients, but it contain E-type coefficients and $r \in \{1, \ldots, m\}$ is the smallest index of row such that $a_{rj}^* = b_r$, then any $a_{ij}^* = b_i$ in $A^*(j)$ for $i \in \{r, \ldots, m\}$ is called selected.

**Theorem 3.** *Consider an SFRE* (8)*. Then the following occurs.*

(i) *The SFRE* (8) *is consistent if and only if there exist at least one selected coefficient for each ith equation, $i = 1, \ldots, m$.*

(ii) *The complexity time function for determining the consistency of the SFRE* (8) *is $O(m \cdot n)$.*

Table 2: TFNs mean values.

| Output variable | Unknown component | Linguistic label | $XMin_t(j)$ | $XMax_t(j)$ | $XMean_t(j)$ |
|---|---|---|---|---|---|
| | $x_1$ | scanty | 0.6 | 0.8 | 0.70 |
| $o_1$ | $x_2$ | medium | 0.2 | 0.4 | 0.30 |
| | $x_3$ | good | 0.0 | 0.1 | 0.05 |
| | $x_4$ | low | 0.3 | 0.5 | 0.40 |
| $o_2$ | $x_5$ | medium | 0.4 | 0.7 | 0.55 |
| | $x_6$ | good | 0.0 | 0.3 | 0.15 |

Consequently, when an SFRE (8) is inconsistent, the equations for which no element is a selected coefficient could not be satisfied simultaneously with the other equations having at least one selected coefficient. Furthermore, a vector IND = (IND(1), ..., IND($m$)) is defined by setting IND($i$) equal to the number of selected coefficients in the $i$th equation for each $i = l, \ldots, m$. If IND($i$) = 0, then all the coefficients in the $i$th equation are not selected and the system is inconsistent. The system is consistent if IND($i$) $\neq 0$ for each $i = l, \ldots, m$ and the product

$$\mathrm{PN2} = \prod_{i=1}^{m} \mathrm{IND}(i), \tag{11}$$

gives the upper bound of the number of the eventual minimal solutions.

**Theorem 4.** *Let the SFRE* (8) *be consistent. Then the following occurs.*

(i) *The SFRE has a unique greatest solution $X^{gr}$ with component $x_j^{gr} = b_k$ if the jth column $A^*(j)$ of $A^*$ contains selected G-type coefficients $a_{kj}^*$ and $x_j^{gr} = 1$ otherwise.*

(ii) *The complexity time function for computing $X^{gr}$ is $O(m \cdot n)$.*

A help matrix $H = (h_{ij})$, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$, is defined as follows:

$$h_{ij} = \begin{cases} b_i, & \text{if } a_{ij}^* \text{ is selected,} \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Let $|H_i|$ be the number of coefficients $h_{ij}$ in the $i$th equation of the SFRE (8). Then the number of potential minimal solutions cannot exceed the value

$$\mathrm{PN1} = \prod_{i=1}^{m} |H_i|, \tag{13}$$

where PN2 ≤ PN1.

*Definition 5.* Let $h_i = (h_{i1}, h_{i2}, \ldots, h_{in})$ and $h_k = (h_{k1}, h_{k2}, \ldots, h_{kn})$ be the $i$th and the $k$th rows of the help matrix $H$. If for each $j = 1, \ldots n$, $h_{ij} \neq 0$ implies both $h_{kj} \neq 0$ and $h_{kj} \leq h_{ij}$, then the $i$th row (resp., equation) is said dominant over the $k$th row in $H$ (resp., equation) or that the $k$th row (resp., equation) is said dominated by the $i$th row (resp., equation).

In other terms, if the $i$th equation is dominant over the $k$th equation in (8), then the $k$th equation is a redundant equation of the system. By using Definition 5, we can build a matrix of dimension $m \times n$, called dominance matrix, with components:

$$h_{ij}^* = \begin{cases} 0, & \text{if the } i\text{th equation is dominated by} \\ & \text{another equation,} \\ h_{ij}, & \text{otherwise.} \end{cases} \tag{14}$$

For each $i = 1, \ldots, m$, now we set $|H_i^*|$ as the number of coefficients $h_{ij}^* = b_i \neq 0$ in the $i$th row of the dominance matrix $H^*$. When this value is 0, we set $|H_i^*| = 1$. Then the number of potential minimal solutions of the SFRE cannot exceed the value

$$\mathrm{PN3} = \prod_{i=1}^{m} |H_i^*|, \tag{15}$$

where PN3 ≤ PN2 ≤ PN1. In [6, 8, 20], the authors use the symbol $\langle b_i/j \rangle$ to indicate the coefficients $h_{ij}^* = b_i \neq 0$. We have $h_{ij}^* \wedge x_j = b_i$ if $x_j \in [b_i, 1]$ and $x_j = b_i$ is the $j$th component of a minimal solution. A solution of the $i$th equation can be written as

$$H_i = \sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle. \tag{16}$$

In [6, 8] the concept of concatenation $W$ is introduced to determine all the components of the minimal solutions and it is given by

$$W = \prod_{i=1}^{m} H_i = \prod_{i=1}^{m} \left( \sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle \right). \tag{17}$$

The following properties hold:

(i) commutativity:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle = \left\langle \frac{b_{i_2}}{j_2} \right\rangle \left\langle \frac{b_{i_1}}{j_1} \right\rangle, \tag{18}$$

(ii) associativity:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left( \left\langle \frac{b_{i_2}}{j_2} \right\rangle \left\langle \frac{b_{i_3}}{j_3} \right\rangle \right) = \left( \left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle \right) \left\langle \frac{b_{i_3}}{j_3} \right\rangle, \tag{19}$$

(iii) distributivity with respect to the addition:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left( \left\langle \frac{b_{i_2}}{j_2} \right\rangle + \left\langle \frac{b_{i_3}}{j_3} \right\rangle \right) = \left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle + \left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_3}}{j_3} \right\rangle, \tag{20}$$

(iv) absorption for multiplication:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle = \begin{cases} \left\langle \dfrac{b_{i_1} \wedge b_{i_2}}{j} \right\rangle, & \text{if } j_1 = j_2 = j, \\ \text{unchanged}, & \text{otherwise}, \end{cases} \tag{21}$$

(v) absorption for addition:

$$\begin{aligned} &\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle \cdots \left\langle \frac{b_{i_m}}{j_n} \right\rangle + \left\langle \frac{b_{k_1}}{j_1} \right\rangle \left\langle \frac{b_{k_2}}{j_2} \right\rangle \cdots \left\langle \frac{b_{k_m}}{j_n} \right\rangle \\ &\quad = \begin{cases} \left\langle \dfrac{b_{i_1}}{j_1} \right\rangle \left\langle \dfrac{b_{i_2}}{j_2} \right\rangle \cdots \left\langle \dfrac{b_{i_m}}{j_n} \right\rangle, & \text{if } b_{i_h} = b_{k_h}, \\ & h \in \{1, \ldots, m\}, \\ \text{unchanged}, & \text{otherwise}. \end{cases} \end{aligned} \tag{22}$$

We can determine the minimal solutions $X^{\text{low}(t)} = (x_1^{\text{low}(t)}, x_2^{\text{low}(t)}, \ldots, x_n^{\text{low}(t)})^{-1}$, $t \in \{1, \ldots, \text{PN}(3)\}$, with components

$$x_j^{\text{low}(t)} = \begin{cases} b_{i_t}, & \text{if } b_{i_t} \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{23}$$

The above definitions shall be clarified in the following example of an SFRE with 4 equations and 6 unknown:

$$\begin{aligned} &(1.0 \wedge x_1) \vee (0.0 \wedge x_2) \vee (0.0 \wedge x_3) \\ &\quad \vee (0.9 \wedge x_4) \vee (0.2 \wedge x_5) \vee (0.0 \wedge x_6) = 0.1, \\ &(0.5 \wedge x_1) \vee (0.3 \wedge x_2) \vee (0.4 \wedge x_3) \\ &\quad \vee (0.5 \wedge x_4) \vee (0.3 \wedge x_5) \vee (0.4 \wedge x_6) = 0.3, \\ &(0.7 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.2 \wedge x_3) \\ &\quad \vee (0.7 \wedge x_4) \vee (0.4 \wedge x_5) \vee (0.2 \wedge x_6) = 0.3, \\ &(0.4 \wedge x_1) \vee (0.7 \wedge x_2) \vee (0.2 \wedge x_3) \\ &\quad \vee (0.4 \wedge x_4) \vee (0.7 \wedge x_5) \vee (0.2 \wedge x_6) = 0.3. \end{aligned} \tag{24}$$

We have

$$A = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.9 & 0.2 & 0.0 \\ 0.5 & 0.3 & 0.4 & 0.5 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.2 & 0.7 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.2 & 0.4 & 0.7 & 0.2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix}. \tag{25}$$

By using the normal form, we obtain that

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.5 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.2 & 0.7 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.2 & 0.4 & 0.7 & 0.2 \\ 1.0 & 0.0 & 0.0 & 0.9 & 0.2 & 0.0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.1 \end{pmatrix}. \tag{26}$$

Now we compute the matrix $A^*$ and the vector IND as follows:

$$A^* = \begin{pmatrix} 1.0 & 0.3 & 1.0 & 1.0 & 0.3 & 1.0 \\ 1.0 & 1.0 & 0.0 & 1.0 & 1.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 1.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 1.0 & 1.0 & 0.0 \end{pmatrix}, \qquad \text{IND} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}. \tag{27}$$

The SFRE is consistent because each component of IND is not null. The greatest solution is given by

$$X^{\text{gr}} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.3 \\ 0.1 \\ 0.1 \\ 0.3 \end{pmatrix}. \tag{28}$$

Now we calculate the help matrix $H$ and the dominant matrix $H^*$ as follows:

$$\begin{aligned} H &= \begin{pmatrix} 0.0 & 0.3 & 0.3 & 0.0 & 0.0 & 0.3 \\ 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.0 \end{pmatrix}, \\ H^* &= \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.0 \end{pmatrix}. \end{aligned} \tag{29}$$

Then we have $|H_1^*| = |H_2^*| = |H_3^*| = 1$, $|H_1^*| = 3$ and hence PN3 $= 3$. By using the properties (18)–(23), we have that

$$\begin{aligned} W &= \left\langle \frac{0.3}{2} \right\rangle \left( \left\langle \frac{0.1}{1} \right\rangle + \left\langle \frac{0.1}{4} \right\rangle + \left\langle \frac{0.1}{5} \right\rangle \right) \\ &= \left\langle \frac{0.1}{1} \right\rangle \left\langle \frac{0.3}{2} \right\rangle + \left\langle \frac{0.3}{2} \right\rangle \left\langle \frac{0.1}{4} \right\rangle \\ &\quad + \left\langle \frac{0.3}{2} \right\rangle \left\langle \frac{0.1}{5} \right\rangle. \end{aligned} \tag{30}$$

The three minimal solutions are given by

$$X^{\text{low}(1)} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}, \quad X^{\text{low}(2)} = \begin{pmatrix} 0.0 \\ 0.3 \\ 0.0 \\ 0.1 \\ 0.0 \\ 0.0 \end{pmatrix}, \quad X^{\text{low}(3)} = \begin{pmatrix} 0.0 \\ 0.3 \\ 0.0 \\ 0.0 \\ 0.1 \\ 0.0 \end{pmatrix}. \tag{31}$$

The three maximal interval solutions are given by

$$X_{\max(1)} = \begin{pmatrix} [0.1, 0.1] \\ [0.3, 0.3] \\ [0.0, 0.3] \\ [0.0, 0.1] \\ [0.0, 0.1] \\ [0.0, 0.3] \end{pmatrix}, \quad X_{\max(2)} = \begin{pmatrix} [0.0, 0.1] \\ [0.3, 0.3] \\ [0.0, 0.3] \\ [0.1, 0.1] \\ [0.0, 0.1] \\ [0.0, 0.3] \end{pmatrix}, \tag{32}$$

$$X_{\max(3)} = \begin{pmatrix} [0.0, 0.1] \\ [0.3, 0.3] \\ [0.0, 0.3] \\ [0.0, 0.1] \\ [0.1, 0.1] \\ [0.0, 0.3] \end{pmatrix}.$$

In order to determine if an SFRE is consistent, hence its greatest solution and minimal solutions, we have used the universal algorithm of [6, 8] based on the above concepts. For brevity of presentation, here we do not give this algorithm which has been implemented and tested under C++ language. The C++ library has been integrated in the ESRI ArcObject Library of the tool ArcGIS 9.3 for a problem of spatial analysis illustrated in Section 3.

## 3. SFRE in Spatial Analysis

We consider a specific area of study on the geographical map on which we have a spatial data set of "causes" and we want to analyze the possible "symptoms". We divide this area in P subzones (see, e.g, Figure 3), where a subzone is an area in which the same symptoms are derived by input data or facts, and the impact of a symptom on a cause is the same one as well. It is important to note that even if two subzones have the same input data, they can have different impact degrees of symptoms on the causes. For example, the cause that measures the occurrence of floods may be due to different degrees of importance to the presence of low porous soils or to areas subjected to continuous rains. Afterwards the area of study is divided in homogeneous subzones, hence the expert creates a fuzzy partition for the domain of each input variable and, for each subzone, he determines the values of the symptoms $b_i$, as the membership degrees of the corresponding fuzzy sets (cf. input fuzzification process of Figure 1). For each subzone, then the expert sets the most significant equations and the values $a_{ij}$ of impact of the $j$th cause to the $i$th symptom creating the SFRE (1). After the determination of the set of maximal interval solutions by using the algorithm of Section 2, the expert for each interval solution calculates, for each unknown $x_j$, the mean interval solution $X\text{Max}_{M,t}(j)$ with (6). The linguistic label $\text{Rel}_t(o_s)$ is assigned to the output variable $o_s$. Then he calculates the reliability index $\text{Rel}_t(O)$, given from formula (7), associated to this maximal interval solution $t$. After the iteration of this step, the expert determines the reliability index (7) for each maximal interval solution, by choosing the output vector $O$ for which $\text{Rel}(O)$ assumes the maximum value. Iterating the process for all the subzones, the expert can show the thematic map of each output variable. We schematize the whole process in Figure 4.

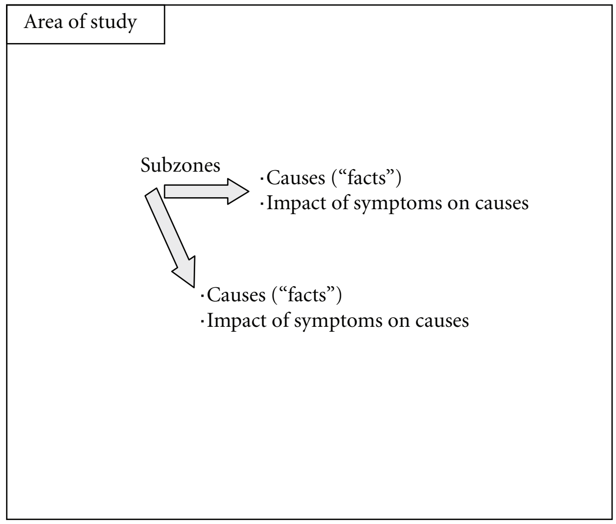



Figure 3: Subdivision in homogeneous subzones.

We suppose to subdivide the area of study in P subzones. The steps of the process are described below.

(i) In the spatial dataset, we associate $k$ facts $i_1, \ldots, i_h$ to every subzone.

(ii) For each input fact, a fuzzy partition in $m_f$ fuzzy sets is created for every $f = 1, \ldots, h$. To each fuzzy set, the expert associates a linguistic label. After the fuzzification process, the expert determines the $m$ most significant equations, where $m \le m_1 + m_2 + \cdots + m_k$. The input vector $B = [b_1, \ldots, b_m]$ is set, where each component $b_i$ ($i = 1, \ldots, m$) is the membership degree to the $i$th fuzzy set of the corresponding input fact. To create the fuzzy partitions, we use TFNs (cf. formulae (3), (4), (5)). The expert sets the impact of the $m$ symptoms to the $n$ causes by defining the impact matrix $A$ with entries $a_{ij}$ with $i = 1, \ldots, m$, $j = 1, \ldots, n$.

(iii) An SFRE (1) with $m$ equations and $n$ unknowns is created. We use the algorithm from [8] to determine

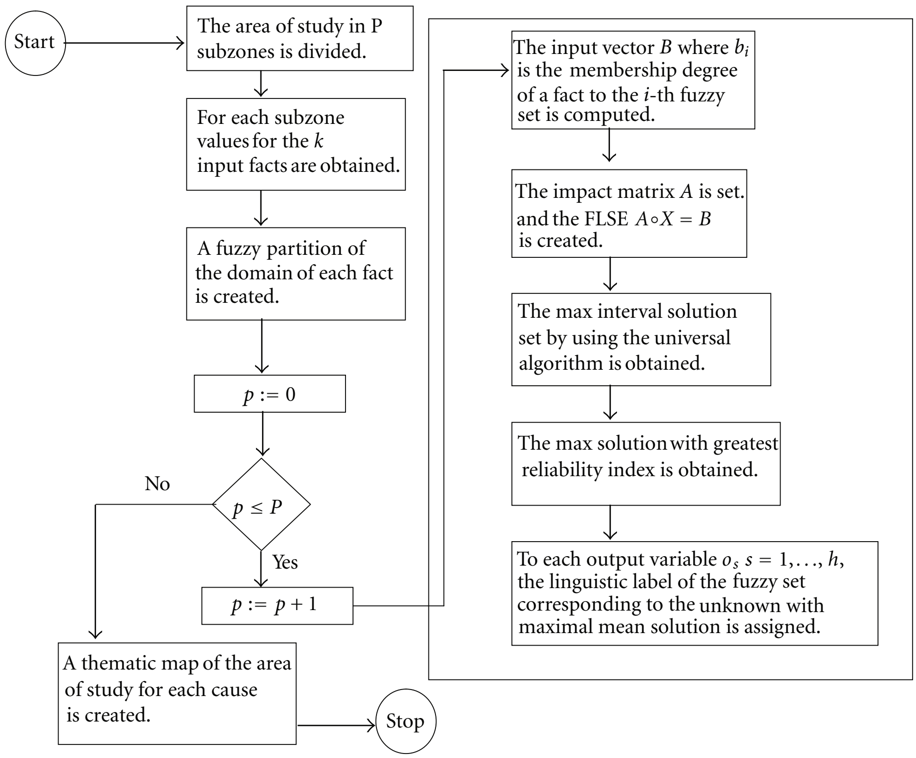



FIGURE 4: Flux diagram of the resolution problem.

all the solutions of (1). Thus we determine $T$ maximal interval solutions.

(iv) $\max \mathrm{Rel}_t := 0$// (the maximal reliability is initialized to 0).

(v) For each maximal interval solution $X_{\max,t}$, with $t = 1, \ldots, T$, we define the vector column $X\mathrm{Mean}_t$ via formula (6).

(vi) $\mathrm{Rel}_t := 0$.

(vii) For each output variable $o_s$, with $s = 1, \ldots, k$, if $x_{j_1}, \ldots, x_{j_s}$ are the unknown associated to $o_s$, let $\mathrm{score}_t(s) = \max\{X\mathrm{Mean}_t(j_1), \ldots, X\mathrm{Mean}_t(j_s)\}$.

(viii) $\mathrm{Rel}_t := \mathrm{Rel}_t + \mathrm{score}_t(\mathrm{o}_s)$.

(ix) Next $s$.

(x) $\mathrm{Rel}_t := \mathrm{Rel}_t/k$// (the reliability index is calculated via formula (7)).

(xi) If $\mathrm{Rel}_t > \max\mathrm{Rel}_t$, then the linguistic label of the fuzzy set corresponding to the unknown with maximum mean solution is assigned to the output vector $O = [o_1, \ldots, o_k]$.

(xii) Next $t$ with $t = 1, \ldots, T$.

(xiii) Next $p$ with $p = 1, \ldots, P$.

At the end of the process, the user can create a thematic map of a specific output variable over the area of study and also a thematic map of the reliability index value obtained for the output variable. If the SFRE related to a specific subzone is inconsistent, the expert can decide whether or not eliminate rows to find solutions: in the first case, he decides that the symptoms associated to the rows that make the system inconsistent are not considered and eliminates them, so reducing the number of the equations. In the second case, he decides that the correspondent output variable for this subzone remains unknown and it is classified as unknown on the map.

## 4. Simulation Results

Here we show the results of an experiment in which we apply our method to census statistical data agglomerated on four districts of the east zone of Naples (Italy) (Figure 5). We use the year 2000 census data provided by the ISTAT (Istituto Nazionale di Statistica). These data contain information on population, buildings, housing, family, employment work for each census zone of Naples. Every district is considered as a subzone with homogeneous input data given in Table 4.

In this experiment, we consider the following four output variables: "$o_1$ = *Economic prosperity*" (wealth and prosperity of citizens), "$o_2$ = *Transition into the job*" (ease of finding work), "$o_3$ = *Social Environment*" (cultural levels of citizens), and "$o_4$ = *Housing development*" (presence of building and residential dwellings of new construction). For each variable,

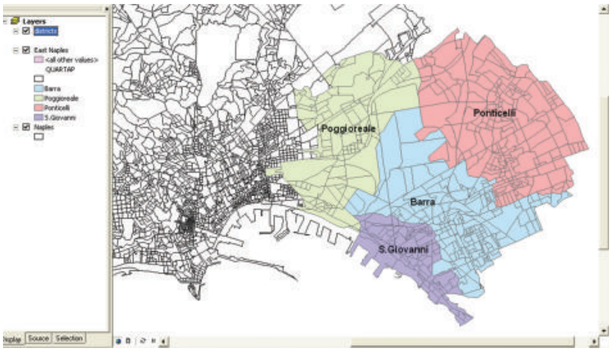



Figure 5: Area of study: four districts at east of Naples (Italy).

Table 3: Values of the TFNs low, mean, high.

| Output | Low | | | Mean | | | High | | |
|---|---|---|---|---|---|---|---|---|---|
| | INF | MEAN | SUP | INF | MEAN | SUP | INF | MEAN | SUP |
| $o_1$ | 0.0 | 0.3 | 0.5 | 0.3 | 0.5 | 0.8 | 0.5 | 0.8 | 1.0 |
| $o_2$ | 0.0 | 0.3 | 0.5 | 0.3 | 0.5 | 0.8 | 0.5 | 0.8 | 1.0 |
| $o_3$ | 0.0 | 0.3 | 0.5 | 0.3 | 0.5 | 0.8 | 0.5 | 0.8 | 1.0 |
| $o_4$ | 0.0 | 0.3 | 0.5 | 0.3 | 0.5 | 0.8 | 0.5 | 0.8 | 1.0 |

Table 4: Input data obtained for the four subzones.

| District | $i_1$ | $i_2$ | $i_3$ | $i_4$ | $i_5$ | $i_6$ | $i_7$ |
|---|---|---|---|---|---|---|---|
| Barra | 0.604 | 0.227 | 0.039 | 0.032 | 0.111 | 0.424 | 0.067 |
| Poggioreale | 0.664 | 0.297 | 0.060 | 0.051 | 0.086 | 0.338 | 0.149 |
| Ponticelli | 0.609 | 0.253 | 0.039 | 0.042 | 0.156 | 0.372 | 0.159 |
| S. Giovanni | 0.576 | 0.244 | 0.041 | 0.031 | 0.054 | 0.353 | 0.097 |

we create a fuzzy partition composed by three TFNs called "low", "mean", and "high" presented in Table 3.

Moreover, we consider the following seven input parameters: $i_1$ = percentage of people employed = number of people employed/total work force, $i_2$ = percentage of women employed = number of women employed/number of people employed, $i_3$ = percentage of entrepreneurs and professionals = number of entrepreneurs and professionals/number of people employed, $i_4$ = percentage of residents graduated = numbers of residents graduated/number of residents with age > 6 years, $i_5$ = percentage of new residential buildings = number of residential buildings built since 1982/total number of residential buildings, $i_6$ = percentage of residential dwellings owned = number of residential dwellings owned/total number of residential dwellings, and $i_7$ = percentage of residential dwellings with central heating system = number of residential dwellings with central heating system/total number of residential dwellings. In Table 4, we show these input data for the four subzones.

For the fuzzification process of the input data, the expert indicates a fuzzy partition for each input domain formed from three TFNs labeled "low", "mean", and "high", whose values are reported in Table 5. In Tables 6 and 7, we show the values obtained for the 21 symptoms $b_1, \ldots, b_{21}$; moreover, we report the input variable and the linguistic label of the correspondent TFN for each symptom $b_i$. In order to form the SFRE (1) in each subzone, the expert defines the equations by setting the impact values $a_{ij}$ by basing over the most significant symptoms.

Now we illustrate this procedure for each subzone.

*4.1. Subzone "Barra".* The expert chooses the significant symptoms $b_2$, $b_4$, $b_5$, $b_7$, $b_{10}$, $b_{11}$, $b_{15}$, $b_{17}$, $b_{18}$, $b_{19}$, by obtaining an SFRE (1) with $m = 10$ equations and $n = 12$ unknowns (Table 8).

The matrix $A$ of the impact values $a_{ij}$ has dimensions $10 \times 12$ and the vector $B$ of the symptoms $b_i$ has dimension $10 \times 1$ and both are given below. The SFRE (1) is inconsistent and eliminating the rows for which the value $\mathrm{IND}(j) = 0$, we obtain four maximal interval solutions $X_{\max(t)}$ $(t = 1, \ldots, 4)$ and we calculate the vector column $X\mathrm{Mean}_t$ on each maximal interval solution. Hence we associate to the output variable $o_s$ $(s = 1, \ldots, 4)$, the linguistic label of the fuzzy set with the higher value calculated with formula (6) obtained for the corresponding unknowns $x_{j_1}, \ldots, x_{j_s}$ and given in Table 8. For determining the reliability of our solutions, we use the index given by formula (7). We obtain that $\mathrm{Rel}_t(o_1) = \mathrm{Rel}_t(o_2) = \mathrm{Rel}_t(o_3) = \mathrm{Rel}_t(o_4) = 0.6025$ for $t = 1, \ldots, 4$ and hence $\mathrm{Rel}(O) = \max\{\mathrm{Rel}_t(O) : t = 1, \ldots, 4\} = 0.6025$ where $O = \{o_1, \ldots o_4\}$. We note that the same final set of linguistic labels associated to the output variables $o_1$ = "high", $o_2$ = "mean", $o_3$ = "low", and $o_4$ = "low" is obtained as well. The relevant quantities are given below.

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.3 & 0.5 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 & 0.6 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.2 & 0.0 & 0.8 & 0.3 & 0.1 & 0.8 & 0.2 & 0.2 & 0.3 & 0.0 & 0.0 \\ 0.5 & 0.3 & 0.1 & 0.6 & 0.4 & 0.1 & 0.6 & 0.4 & 0.1 & 0.1 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.2 & 0.1 & 0.4 & 0.1 & 0.2 & 0.5 & 0.1 & 0.3 & 0.7 & 0.3 \\ 0.1 & 0.4 & 0.4 & 0.1 & 0.4 & 0.4 & 0.1 & 0.5 & 0.5 & 0.2 & 0.4 & 0.5 \\ 0.5 & 0.2 & 0.0 & 0.4 & 0.3 & 0.0 & 0.4 & 0.3 & 0.0 & 1.0 & 0.1 & 0.0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.98 \\ 0.36 \\ 0.63 \\ 1.00 \\ 0.40 \\ 0.60 \\ 0.10 \\ 0.59 \\ 0.41 \\ 1.00 \end{pmatrix}, \tag{33}$$

Table 5: TFNs values for the input domains.

| Input variable | Low | | | Mean | | | High | | |
|---|---|---|---|---|---|---|---|---|---|
| | INF | MEAN | SUP | INF | MEAN | SUP | INF | MEAN | SUP |
| $i_1$ | 0.00 | 0.40 | 0.60 | 0.40 | 0.60 | 0.80 | 0.60 | 0.80 | 1.00 |
| $i_2$ | 0.00 | 0.10 | 0.30 | 0.10 | 0.30 | 0.40 | 0.30 | 0.50 | 1.00 |
| $i_3$ | 0.00 | 0.04 | 0.06 | 0.04 | 0.06 | 0.10 | 0.07 | 0.20 | 1.00 |
| $i_4$ | 0.00 | 0.02 | 0.04 | 0.02 | 0.04 | 0.07 | 0.04 | 0.07 | 1.00 |
| $i_5$ | 0.00 | 0.05 | 0.08 | 0.05 | 0.08 | 0.10 | 0.08 | 0.10 | 1.00 |
| $i_6$ | 0.00 | 0.10 | 0.30 | 0.10 | 0.30 | 0.60 | 0.30 | 0.60 | 1.00 |
| $i_7$ | 0.00 | 0.10 | 0.30 | 0.10 | 0.30 | 0.50 | 0.30 | 0.50 | 1.00 |

Table 6: TFNs for the symptoms $b_1 \div b_{12}$.

| Subzone | $b_1$: $i_1$ = low | $b_2$: $i_1$= mean | $b_3$: $i_1$ = high | $b_4$: $i_2$ = low | $b_5$: $i_2$ = mean | $b_6$: $i_2$ = high | $b_7$: $i_3$ = low | $b_8$: $i_3$ = mean | $b_9$: $i_3$ = high | $b_{10}$: $i_4$ = low | $b_{11}$: $i_4$ = mean | $b_{12}$: $i_4$ = high |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Barra | 0.00 | 0.98 | 0.02 | 0.36 | 0.63 | 0.00 | 1.00 | 0.00 | 0.00 | 0.40 | 0.60 | 0.00 |
| Poggioreale | 0.00 | 0.93 | 0.07 | 0.01 | 0.99 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.63 | 0.37 |
| Ponticelli | 0.00 | 0.91 | 0.05 | 0.23 | 0.76 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.93 | 0.07 |
| S. Giovanni | 0.12 | 0.88 | 0.00 | 0.28 | 0.72 | 0.00 | 0.95 | 0.05 | 0.00 | 0.45 | 0.55 | 0.00 |

$$
X_{\max(1)} = \begin{pmatrix} [0.40, 0.40] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix}, \quad
X_{\max(2)} = \begin{pmatrix} [0.40, 0.40] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix},
$$

$$
X_{\max(3)} = \begin{pmatrix} [0.40, 0.40] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix}, \quad
X_{\max(4)} = \begin{pmatrix} [0.40, 0.40] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix},
$$

$$
X\mathrm{Mean}_1 = \begin{pmatrix} 0.40 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}, \quad
X\mathrm{Mean}_2 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix},
$$

$$
X\mathrm{Mean}_3 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.18 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}, \quad
X\mathrm{Mean}_4 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.05 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}. \tag{34}
$$

*4.2. Subzone "Poggioreale".* The expert chooses the significant symptoms $b_2$, $b_5$, $b_8$, $b_{11}$, $b_{12}$, $b_{14}$, $b_{15}$, $b_{17}$, $b_{18}$, $b_{19}$, $b_{20}$, by obtaining an SFRE (1) with $m = 11$ equations and $n = 12$ unknowns (Table 9). The matrix $A$ of the impact values $a_{ij}$ has dimension $11 \times 12$ and the vector $B$ of the symptoms $b_i$ has dimension $11 \times 1$ which are given below. The SFRE (1)

Table 7: TFNs for the symptoms $b_{13} \div b_{21}$.

| Subzone | $b_{13}$: $i_5$ = low | $b_{14}$: $i_5$ = mean | $b_{15}$: $i_5$ =high | $b_{16}$: $i_6$ = low | $b_{17}$: $i_6$ = mean | $b_{18}$: $i_6$ = high | $b_{19}$: $i_7$ = low | $b_{20}$: $i_7$ = mean | $b_{21}$: $i_7$ = high |
|---|---|---|---|---|---|---|---|---|---|
| Barra | 0.00 | 0.00 | 0.10 | 0.00 | 0.59 | 0.41 | 1.00 | 0.00 | 0.00 |
| Poggioreale | 0.00 | 0.70 | 0.30 | 0.00 | 0.87 | 0.13 | 0.75 | 0.25 | 0.00 |
| Ponticelli | 0.00 | 0.00 | 1.00 | 0.00 | 0.76 | 0.24 | 0.70 | 0.30 | 0.00 |
| S. Giovanni | 0.87 | 0.13 | 0.00 | 0.00 | 0.82 | 0.18 | 1.00 | 0.00 | 0.00 |

Table 8: Final linguistic labels for the output variables in the district Barra.

| Output variable | $\text{Score}_1(o_s)$ | $\text{Score}_2(o_s)$ | $\text{Score}_3(o_s)$ | $\text{Score}_4(o_s)$ |
|---|---|---|---|---|
| $o_1$ | high | high | high | high |
| $o_2$ | mean | mean | mean | mean |
| $o_3$ | low | low | low | low |
| $o_4$ | low | low | low | low |

is inconsistent and eliminating the rows for which the value $\text{IND}(j) = 0$, we obtain 12 maximal interval solutions $X_{\max(t)}$ $(t = 1, \dots, 12)$, and we calculate the vector column $X\text{Mean}_t$ on each maximal interval solution. The relevant quantities are given below

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 0.9 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.0 & 0.0 & 0.1 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.6 & 0.3 & 0.4 & 0.6 & 0.2 & 0.4 & 0.6 & 0.0 & 0.1 & 0.2 \\ 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 \\ 0.0 & 0.1 & 0.5 & 0.1 & 0.2 & 0.5 & 0.1 & 0.2 & 0.5 & 0.0 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.0 & 0.8 & 0.5 & 0.3 & 0.5 & 0.3 & 0.1 & 0.7 & 0.3 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.3 & 0.6 & 0.2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.93 \\ 0.99 \\ 1.0 \\ 0.63 \\ 0.37 \\ 0.7 \\ 0.3 \\ 0.87 \\ 0.13 \\ 0.75 \\ 0.25 \end{pmatrix}, \tag{35}$$

$$X_{\max(1)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.13, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X_{\max(2)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.13, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix},$$

$$X_{\max(3)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X_{\max(4)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix},$$

$$X_{\max(5)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X_{\max(6)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix},$$

Table 9: Final linguistic labels for the output variables in the district Poggioreale.

| Output | $\text{Score}_1$ $(o_s)$ | $\text{Score}_2$ $(o_s)$ | $\text{Score}_3$ $(o_s)$ | $\text{Score}_4$ $(o_s)$ | $\text{Score}_5$ $(o_s)$ | $\text{Score}_6$ $(o_s)$ | $\text{Score}_7$ $(o_s)$ | $\text{Score}_8$ $(o_s)$ | $\text{Score}_9$ $(o_s)$ | $\text{Score}_{10}$ $(o_s)$ | $\text{Score}_{11}$ $(o_s)$ | $\text{Score}_{12}$ $(o_s)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $o_1$ | low | low | low | high | low | low | low | high | low | low | low | high |
| $o_2$ | low | low | low | mean | low | low | low | mean | low | low | low | mean |
| $o_3$ | low | low | low | low | low | low | low | low | low | low | low | low |
| $o_4$ | low | mean | low | mean | low | mean | low | mean | low | mean | low | mean |

$$X_{\max(7)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X_{\max(8)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X\text{Mean}_1 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.13 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.05 \end{pmatrix}, \quad X\text{Mean}_2 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.13 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.065 \end{pmatrix},$$

$$X_{\max(9)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.13, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X_{\max(10)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.13, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X\text{Mean}_3 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.13 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.065 \end{pmatrix}, \quad X\text{Mean}_4 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.13 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.065 \end{pmatrix},$$

$$X_{\max(11)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.13, 0.13] \end{pmatrix}, \quad X_{\max(12)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.13, 0.13] \end{pmatrix}, \quad X\text{Mean}_5 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.13 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.05 \end{pmatrix}, \quad X\text{Mean}_6 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.13 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.05 \end{pmatrix},$$

$$
XMean_7 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.13 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.065 \end{pmatrix}, \quad XMean_8 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.13 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.065 \end{pmatrix},
$$

$$
XMean_9 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.13 \\ 0.25 \\ 0.125 \\ 0.05 \end{pmatrix}, \quad XMean_{10} = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.13 \\ 0.125 \\ 0.25 \\ 0.05 \end{pmatrix},
$$

$$
XMean_{11} = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.13 \end{pmatrix}, \quad XMean_{12} = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.13 \end{pmatrix}. \tag{36}
$$

For determining the reliability of our solutions, we use the index given by formula (7). We obtain $\mathrm{Rel}(O_k) = 0.4675$ for $k = 1, \ldots, 12$. Then we obtain two final sets of linguistic labels associated to the output variables: $o_1$ = "low", $o_2$ = "low", $o_3$ = "low", $o_4$ = "low", and $o_1$ = "low", $o_2$ = "low", $o_3$ = "low", $o_4$ = "mean", with a same reliability index value 0.4675. The expert prefers to choose the second solution: $o_1$ = "low", $o_2$ = "low", $o_3$ = "low", $o_4$ = "mean" because he considers that in the last two years in this district the presence of building and residential dwellings of new construction has increased although marginally. We obtain four final thematic maps shown in Figures 6, 7, 8, 9 for the output variable $o_1$, $o_2$, $o_3$, $o_4$, respectively.

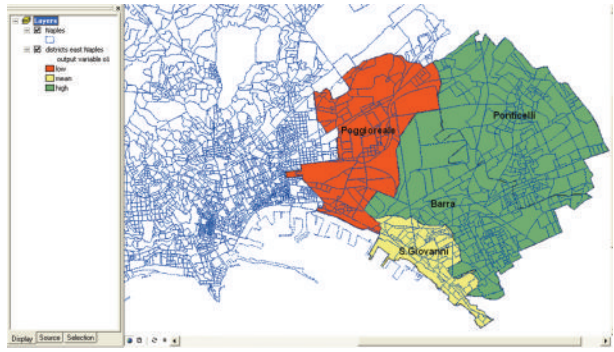



Figure 6: Thematic map for output variable $o_1$ (*Economic prosperity*).

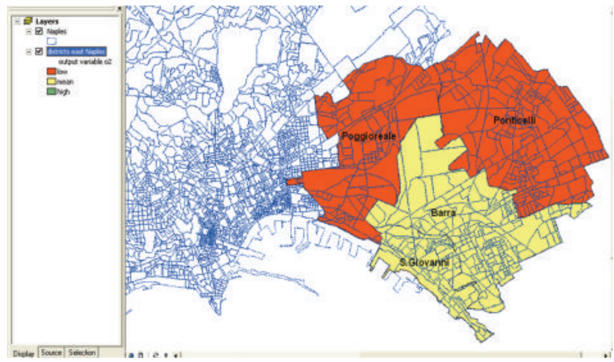



Figure 7: Thematic map of the output variable $o_2$ (*Transition into the job*).

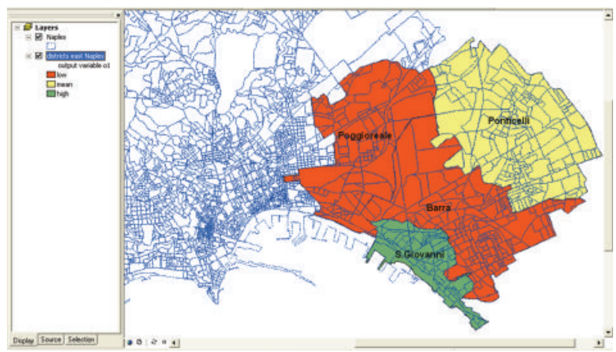



Figure 8: Thematic map for the output variable $o_3$ (*Social Environment*).

The results show that there was no housing development in the four districts in the last 10 years, and there is difficulty

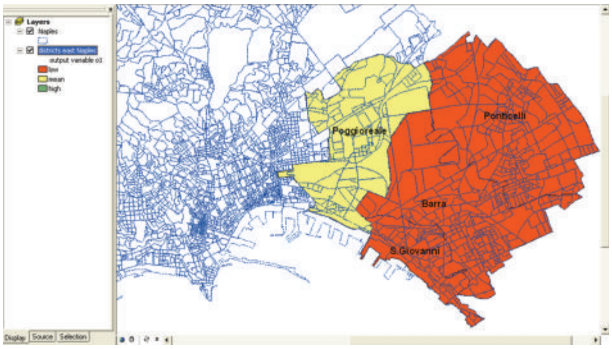



Figure 9: Thematic map for the output variable $o_4$ (*Housing development*).

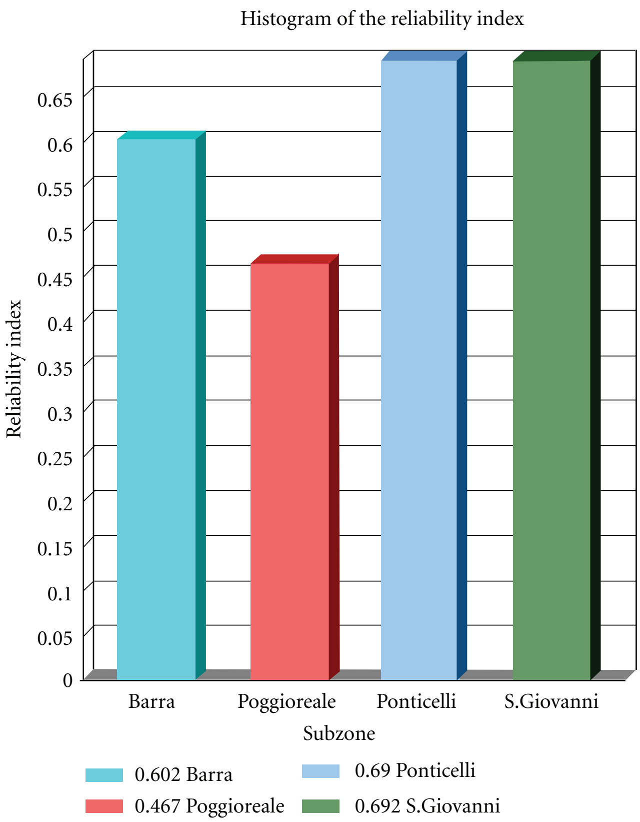



Figure 10: Histogram of the reliability index Rel($O$) for the four subzones.

in finding job positions. In Figure 10, we show the histogram of the reliability index Rel($O$) for each subzone, where $O = [o_1, o_2, o_3, o_4]$.

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*Research Article*

# Towards an (Even More) Natural Probabilistic Interpretation of Fuzzy Transforms (and of Fuzzy Modeling)

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In many practical applications, it turns out to be useful to use the notion of *fuzzy transform:* once we have functions $A_1(x) \geq 0, \ldots, A_n \geq 0$, with $\sum_{i=1}^{n} A_i(x) = 1$, we can then represent each function $f(x)$ by the coefficients $F_i = (\int f(x) \cdot A_i(x)dx)/(\int A_i(x)dx)$. Once we know the coefficients $F_i$, we can (approximately) reconstruct the original function $f(x)$ as $\sum_{i=1}^{n} F_i \cdot A_i(x)$. The original motivation for this transformation came from fuzzy modeling, but the transformation itself is a purely mathematical transformation. Thus, the empirical successes of this transformation suggest that this transformation can be also interpreted in more traditional (nonfuzzy) mathematics as well. Such an interpretation is presented in this paper. Specifically, we show that the 2002 probabilistic interpretation of fuzzy modeling by Sánchez et al. can be modified into a natural probabilistic explanation of fuzzy transform formulas.

## 1. Introduction: Fuzzy Transform and the Need for Its Probabilistic Interpretation

*1.1. Fuzzy Transform: A Definition.* The notion of a fuzzy transform (*F-transform*, for short) turned out to be very useful in many application areas such as image compression and solving differential equations under initial uncertainty; see, for example, [1, 2] and references therein.

Generally speaking, the F-transform of function $f$ is a vector with weighted local mean values of $f$ as components. The first step in the definition of the F-transform of $f : X \to \mathbb{R}$ is a selection of a *fuzzy partition* of universal set $X$ (e.g, a bounded interval $[a, b]$ on $\mathbb{R}$) by a finite set of *basic functions*

$$A_1(x) \geq 0, \ldots, A_n(x) \geq 0, \quad (1)$$

which are continuous and satisfy the condition $\sum_{i=1}^{n} A_i(x) = 1$. Basic functions are called *membership functions* of respective fuzzy sets, or, alternatively, *granules*, information pieces, etc. Their choice reflects the type of uncertainty which is related to the knowledge of $x$.

Once the basic functions are selected, we define the F-transform of a continuous function $f : X \to \mathbb{R}$ as a vector $(F_1, \ldots, F_n)$, where

$$F_i \stackrel{\text{def}}{=} \frac{\int f(x) \cdot A_i(x)dx}{\int A_i(x)dx}. \quad (2)$$

F-transform satisfies the following properties [1, 2]:

(i) $y = F_i$ minimizes $\int_a^b (f(x) - y)^2 A_i(x)dx$,

(ii) for a twice continuously differentiable function $f$, $F_i = f(x_i) + O(h_i^2)$, where $h_i$ is the length of the support of $A_i$.

F-transform is used in applications as a "skeleton model" of $f$. This model provides a compressed image if $f$ is an image [3], values of a trend if $f$ is a time series [4], a numeric model if $f$ is used in numeric computations (integration, differentiation) [5], etc.

Once we know the F-transform components $F_i$, we can (approximately) reconstruct the original function $f$ as

$$\overline{f}(x) = \sum_{i=1}^{n} F_i \cdot A_i(x). \quad (3)$$

In [1], the formula (3) is called the *F-transform inversion formula*. The formula (3) represents a continuous function that approximates $f$. Under certain reasonable conditions, a sequence of functions represented by (3) uniformly converges to $f$ (see [1] for more details).

*Example 1.* Let us give an example of the F-transform of $x^2$ on the domain $[0,1]$ with respect to $A_1,\dots,A_5$. For simplicity, we assume that basic functions $A_1,\dots,A_5$ are of triangular shape and constitute a uniform fuzzy partition of $[0,1]$. Their analytical representation is as follows:

$$A_1(x) = \begin{cases} 1-4x, & \text{if } x \in [0,0.25], \\ 0, & \text{otherwise,} \end{cases}$$
$$A_2(x) = \begin{cases} 4x, & \text{if } x \in [0,0.25], \\ 2-4x, & \text{if } x \in [0.25,0.5], \\ 0, & \text{otherwise,} \end{cases}$$
$$A_3(x) = \begin{cases} 4x-1, & \text{if } x \in [0.25,0.5], \\ 3-4x, & \text{if } x \in [0.5,0.75], \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$
$$A_4(x) = \begin{cases} 4x-2, & \text{if } x \in [0.5,0.75], \\ 4-4x, & \text{if } x \in [0.75,1.0], \\ 0, & \text{otherwise,} \end{cases}$$
$$A_5(x) = \begin{cases} 4x-3, & \text{if } x \in [0.75,1.0], \\ 0, & \text{otherwise.} \end{cases}$$

By (2), the values of the components $F_1,\dots,F_5$ of the F-transform are

$$\begin{aligned} F_1 &= \frac{1}{0.125} \cdot \int_0^{0.25} x^2 \cdot (1-4x)dx \approx 0.01, \\ F_2 &= \frac{1}{0.25} \cdot \left(\int_0^{0.25} x^2 \cdot 4x\,dx + \int_{0.25}^{0.5} x^2 \cdot (2-4x)dx\right) \\ &\approx 0.08, \\ F_3 &= \frac{1}{0.25} \cdot \left(\int_{0.25}^{0.5} x^2 \cdot (4x-1)dx + \int_{0.5}^{0.75} x^2 \cdot (3-4x)dx\right) \\ &\approx 0.26, \\ F_4 &= \frac{1}{0.25} \cdot \left(\int_{0.5}^{0.75} x^2 \cdot (4x-2)dx + \int_{0.75}^{1} x^2 \cdot (4-4x)dx\right) \\ &\approx 0.58, \\ F_5 &= \frac{1}{0.125} \cdot \int_{0.75}^{1} x^2 \cdot (4x-3)dx \\ &\approx 0.85. \end{aligned} \tag{5}$$

Figure 1 provides a graphical representation of the basic functions $A_1,\dots,A_5$, of the function $f(x) = x^2$, of its F-transform components $F_1,\dots,F_5$, and of the inverse F-transform $\overline{f}(x)$ of $x^2$.

*1.2. F-Transform: Original Motivation.* The original motivation for F-transform came from fuzzy modeling [1, 2]. For example, in the situation corresponding to the inverse F-transform, we have $n$ rules

$$\begin{gathered} \text{If } x \text{ is } A_1 \text{ then } y = F_1, \\ \vdots \\ \text{If } x \text{ is } A_n \text{ then } y = F_n. \end{gathered} \tag{6}$$

These rules are Takagi-Sugeno (TSK) rules with singleton (constant) right-hand sides. For TSK rules, the value corresponding to a given input $x$ is $\overline{f}(x) = (\sum_{i=1}^{n} F_i \cdot A_i(x))/(\sum_{i=1}^{n} A_i(x))$. Since $\sum_{i=1}^{n} A_i(x) = 1$, we get formula (3).

The purpose was to show that this type of modeling can be as useful in applications as more traditional techniques such as Fourier transform and wavelet transform. Moreover, F-transform has a potential advantage over Fourier and wavelet transforms: in contrast to the purely mathematical basic functions used in Fourier and wavelet transforms, the basic functions $A_i$ in a fuzzy partition usually come from natural language terms like "low" or "high" (for a detailed description of fuzzy modeling, see, e.g., [6, 7]).

Just like any other tool of applied mathematics, F-transform is not a panacea. It is more successful in some problems, and in other problems, it is less successful. It is therefore desirable to combine F-transform with other mathematical tools, so as to combine relative advantages of different techniques. For combining F-transform with other mathematical tools, it is desirable to come up with a purely mathematical (nonfuzzy) interpretation for this transform.

In particular, since most mathematical data processing tools are based on probability and statistics, it is desirable to come up with a probabilistic interpretation for F-transform.

*1.3. The Known Probabilistic Interpretation of Fuzzy Modeling Leads to a Probabilistic Interpretation of F-Transform.* We have mentioned that F-transform was originally designed as a particular case of fuzzy modeling. A seminal paper [8] provided a reasonable probabilistic model for a particular case of fuzzy modeling. Specifically, this paper shows that if we use piecewise constant probability density functions for describing the output, then we get a particular case of a fuzzy model—the case when we use product for "and" and sum

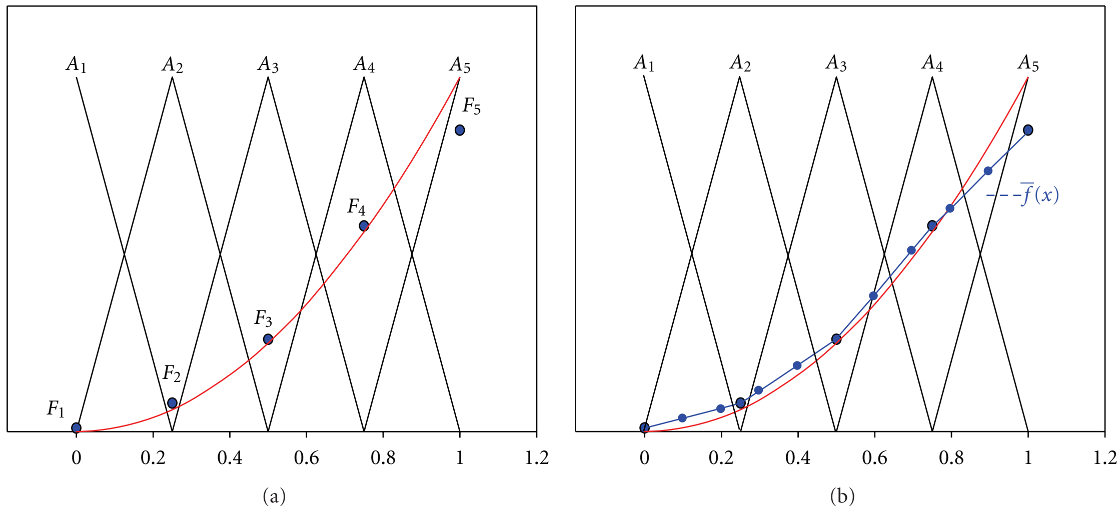


Figure 1: Function $x^2$ on $[0, 1]$ and its F-transform components $F_1, \ldots, F_5$ with respect to $A_1, \ldots, A_5$ (a). Function $x^2$ on $[0, 1]$ and its inverse F-transform $\overline{f}$ (b).

for "or." Since F-transform corresponds to exactly this type of fuzzy modeling, we thus get a probabilistic model for F-transform as well.

*1.4. What We Do in This Paper.* In this paper, we show that a modification of the probabilistic interpretation from [8] enables us to justify formulas of F-transform without making any additional assumptions about the probability distributions. In mathematical terms, this modification consists of using Bayes formulas—and making assumptions about *prior* distributions (a natural way to describe prior knowledge in statistics) instead of making assumptions about the *actual* distributions.

Thus, we get an even more natural probabilistic interpretation of F-transform. Specifically

(i) the paper [8] shows, in effect, that *there exists* a reasonable probabilistic interpretation of the F-transform formulas;

(ii) however, in principle, this interpretation leaves the possibility that *there exist other* equally reasonable assumptions about the probability distributions can lead to different formulas;

(iii) in our modified interpretation, we show that the basic probabilistic setting *uniquely determines* the F-transform formulas—without the need to make any assumptions about the probability distributions.

We also show that a similar modification can be applied to the probabilistic interpretation of general fuzzy modeling formulas.

*Comment 1.* From the mathematical viewpoint, the resulting formulas are very similar to the formulas from [8] (with the exception of the Bayes formula step). However, in our opinion, this mathematically *minor* modification leads to a *major* change in interpretation: now, to probabilistic researchers, F-transform is

(i) not just a possible model, corresponding to one of the possible reasonable choices of probability distributions,

(ii) but the model uniquely emerging from the natural probabilistic setting.

Similar conclusion can be made about the probabilistic interpretation of more general fuzzy models. In other words, our minor modification uncovers an even deeper fundamental meaning of the probabilistic interpretation originally proposed in [8].

## 2. A Natural Practical Problem that Leads to F-Transform

*2.1. Physical Setting: General Discussion.* Let us assume that we have a physical process that is characterized by two quantities $x$ and $z$, and we know that these quantities are related by a functional dependence $z = f(x)$.

In the ideal situation of complete knowledge,

(i) we know the exact value of $x$,

(ii) we have the exact description of the function $f$.

In this case, we can get the corresponding *exact* value $z = f(x)$ of the second quantity.

In practice, we know the value $x$ with uncertainty, that is, several different values of $x$ are consistent with our knowledge. We must therefore provide a reasonable *estimate* for $z$. Finding such an estimate will be the *first problem* with which we will be dealing. In this first problem, we assume that the function $f$ is known *exactly*.

If this function has to be determined *empirically*, then we will transform the empirical (often, partial) knowledge about $f$ into a reasonable estimate for this function. This will be the *second problem* with which we will be dealing in this section.

*2.2. First Problem: Estimating the Value $f(x)$ for an Imprecisely Known $x$.* If we only know one piece of information $X_i$ about $x$, what is the reasonable estimate for $z = f(x)$?

*2.3. Second Problem: Estimating the Function $z = f(x)$ Based on Partial Information about the Dependence between $x$ and $z$.* Assume that for every information piece $X_i$, $1 \le i \le n$, we have the corresponding measured value $F_i$ of $z$. Since we know only $n$ numerical characteristics $F_i$ of the unknown function $f$, we cannot exactly reconstruct this function. Instead, we need to provide a good estimate for each value $f(x)$ of this function.

## 3. A Natural Probabilistic Problem that Leads to the Probabilistic Interpretation of F-Transform

*3.1. Uncertainty in $x$: A General Probabilistic Description.* Assume that we have a *model* of the estimation procedure, that enables us, given the actual value $x$, to compute the probability $P(X_i \mid x) \ge 0$ of this procedure resulting in $X_i$—under the condition that the actual (unknown) value of the estimated quantity is $x$.

To simplify formulas, we denote

$$A_i(x) \stackrel{\text{def}}{=} P(X_i \mid x). \tag{7}$$

Since for every $x$, we must have exactly one of the $n$ possible outcomes, we thus conclude that the probabilities $P(X_1 \mid x)$, $\ldots$, $P(X_n \mid x)$ of different estimation results must add up to one, that is, we must have

$$P(X_1 \mid x) + \cdots + P(X_n \mid x) = 1. \tag{8}$$

In the above simplified notation, this formula takes the form

$$A_1(x) + \cdots + A_n(x) = 1. \tag{9}$$

*3.2. First Problem: Estimating the Value $f(x)$ for an Imprecisely Known $x$.* Let us consider the first problem. In practice, we do not know the exact value of the quantity $x$. Instead, we only have one of the information pieces $X_i$, $1 \le i \le n$. Under the assumption that we know $X_i$, what is the reasonable estimate for $z = f(x)$?

In terms of probability theory, we would like to find the conditional expected value $F_i \stackrel{\text{def}}{=} E[z \mid X_i] = E[f(x) \mid X_i]$ of $z = f(x)$ under the condition $X_i$.

By definition, this expected value is equal to

$$F_i = E[f(x) \mid X_i] = \int f(x) \cdot P(x \mid X_i)\, dx. \tag{10}$$

Thus, to compute this expected value, we must know the probabilities $P(x \mid X_i)$. Instead, we know the probabilities $P(X_i \mid x)$.

In general, the problem of reconstructing probabilities $P(H_x \mid X_i)$ of different hypotheses $H_x$ based on the observation $X_i$ from conditional probabilities $P(X_i \mid H_x)$ of this observation under different hypotheses $H_x$ is well known in probability theory; it is solved by applying the Bayes theorem. The continuous version of this theorem is

$$P(H_x \mid X_i) = \frac{P(X_i \mid H_x) \cdot P(H_x)}{\int P\left(X_i \mid H_y\right) \cdot P(y)\, dy}, \tag{11}$$

in which $P(H_x)$ is a prior probability of the hypothesis $H_x$ (strictly speaking, $P(H_x \mid X_i)$ and $P(H_x)$ are probability densities).

In our case, different hypotheses $H_x$ correspond to different possible values $x$ of the quantity of interest. Thus, (11) takes the form

$$P(x \mid X_i) = \frac{P(X_i \mid x) \cdot P(x)}{\int P(X_i \mid y) \cdot P(y)\, dy}. \tag{12}$$

Since there is no a priori reason to prefer one value of $x$ to the other, it is reasonable to assume that all the values $x$ are equally probable, that is, that all prior values $P(x)$ are equal to each other: $P(x) = P_0$.

Substituting $P(x) = P_0$ into the formula (12) and dividing both the numerator and the denominator by the common factor $P_0$, we get the expression

$$P(x \mid X_i) = \frac{P(X_i \mid x)}{\int P(X_i \mid y)\, dy}. \tag{13}$$

Substituting this expression into formula (10) (and renaming the variable in the denominator), we get

$$F_i = E[f(x) \mid X_i] = \frac{\int f(x) \cdot P(X_i \mid x) dx}{\int P(X_i \mid x) dx}. \tag{14}$$

In terms of the simplified notation (7), we thus get

$$F_i = E[f(x) \mid X_i] = \frac{\int f(x) \cdot A_i(x) dx}{\int A_i(x) dx}, \tag{15}$$

that is, exactly the formula (2) corresponding to F-transform.

*3.3. Second Problem: Estimating the Function $z = f(x)$ Based on Partial Information about the Dependence between $x$ and $z$.* In some practical situations, we do not know the exact expression for the function $f(x)$. Instead, we must estimate $f(x)$ from the empirical data, that is, from the previous results of simultaneous measuring $x$ and $z$.

In each such measurement, the only information that we get about $x$ is one of the values $X_1, \ldots, X_n$. For each case when the information about $x$ is $X_i$, we have one or several values $z$.

Ideally, we should have a large number of values $z$ corresponding to each $x$-measurement result $X_i$. Based on these values $z$, we should then be able to reconstruct the conditional distribution of $z$ under the condition of $X_i$. Based on these conditional distributions, we should be able to reconstruct the values $f(x)$ for all $x$.

In practice, however, we have only a few values $z$ corresponding to each $x$-measurement result $X_i$. In this case, at best, instead of the entire conditional probability distribution, we can only reconstruct a single parameter—the conditional mean $F_i = E[z \mid X_i]$. Since we only know $n$ characteristics $F_i$ of the unknown function $f(x)$, we cannot exactly reconstruct this function. Instead, we need to describe a good estimates for each value $f(x)$ of this function.

Similarly to the first problem, we take the mean as a reasonable estimate. Thus, in the above practical setting, the problem of estimating the function $f(x)$ takes the following form:

(i) for every $i$, we know the conditional mean $F_i = E[f(x) \mid X_i]$;

(ii) based on these conditional means, for every $x$, we want to estimate the mean value $\overline{f}(x) \stackrel{\text{def}}{=} E[z \mid x]$.

For this problem, the formula of full probability leads to the following result:

$$E[z \mid x] = \sum_{i=1}^{n} E[z \mid X_i] \cdot P(X_i \mid x). \tag{16}$$

By using the notations $\overline{f}(x)$ for $E[z \mid x]$, $F_i$ for $E[z \mid X_i]$, and $A_i(x)$ for $P(X_i \mid x)$, we can transform the formula (16) into the form

$$\overline{f}(x) = \sum_{i=1}^{n} F_i \cdot A_i(x), \tag{17}$$

that is, exactly the F-transform inversion formula (3).

*3.4. Conclusion.* The above (minor) modification of a probability model from [8] uniquely determined both basic formulas (2) and (3) related to F-transform.

*3.5. Relation with the Random Set Interpretation of Fuzzy Sets.* It is worth mentioning that the probabilistic interpretation from [8] is related to the random set interpretation of fuzzy sets (see, e.g., [9]).

In this interpretation, the meaning of an imprecise (fuzzy) term like "small" is based on the following idea. The fact that the term is imprecise means that for the same value $x$, some people will say that this value is small, while other people will say that this value is not small. To take this imprecision into account, we can store, for each person, a set of all the values that this person considers small.

Since there is no prior reason to prefer the opinion of one of these folks, we consider their opinions equally reasonable. We can then take the ratio $\mu_{\text{small}}(x)$ of people who consider $x$ to be small as a reasonable measure of smallness (this is actually one of the standard ways to construct a membership function corresponding to a certain term).

We can describe this ratio in probabilistic terms if we assume that all the persons are equally probable. In these terms, the value $\mu_{\text{small}}(x)$ can be interpreted as the probability $P(\text{small} \mid x)$ that a randomly selected person would consider $x$ to be small.

This interpretation of the membership function $A_i(x)$ as the conditional probability $P(X_i \mid x)$ is exactly what we used in our probabilistic interpretation of F-transform.

*3.6. Terminological Comment.* For completeness, let us explain why the above interpretation is called the random sets interpretation.

For crisp (well-defined) properties, each property can be described by the set of all the values that satisfy this property.

For each imprecise property like "small," instead of a *single* set describing all the values that satisfy this property, we have *several* sets describing the opinions of several persons. We consider the opinions of all these persons to be equally valid, so each of $N$ persons has the exact same probability $1/N$ of being correct. In this case, we have different sets, each occurring with probability $1/N$.

In mathematical terms, we can describe this situation by saying that we have a probability distribution on the class of all possible sets. In probability theory, such a distribution is called a *random set*—similarly to the fact that a probability distribution on the class of all possible numbers is called a *random number*.

## 4. Discussion

Let us discuss what the consequences of the above results for the meaning and usage of F-transforms are(the authors are greatly thankful to the anonymous referees who proposed the main ideas of this discussion). To start this discussion, let us recall why F-transforms were proposed in the first place.

*4.1. Need for F-Transforms and the Resulting Main Advantage of F-Transforms: Reminder.* One of the main objectives of F-transform is to approximate general functions by functions from a selected finite-parametric family. This is a well-known mathematical problem, and many successful techniques have been developed for solving this problem. For example, we can expand the original function by a polynomial, and then use the first few terms in this expansion as the desired approximation. We can also use transforms such as Fourier transform or wavelet transform, and keep only the first few terms in the corresponding expansion as the desired approximation.

All existing approximation techniques take a function $f(x)$ and approximate this function. In situations in which the only information that we have about the desired dependence $y = f(x)$ are the values of $y$ measured for several values of $x$, this is the only thing we can do. However, in practice, we often have additional expert knowledge about the dependence $y = f(x)$. It is therefore desirable to take this understanding into account when we approximate a function.

The expert knowledge is often imprecise (fuzzy), that is, formulated in terms of imprecise expert rules. A natural way to describe imprecise rules is to use fuzzy logic and fuzzy modeling, and, as we have shown, the fuzzy modeling approach naturally leads to F-transforms.

The ability to take into account expert knowledge is thus the main advantage of F-transforms, the main reason why F-transform has led to many successful applications.

*4.2. The Probabilistic Interpretation of F-Transform Leads to an Additional Advantage of F-Transform in Comparison with Other Approximation Techniques.* The above probabilistic interpretation of F-transforms shows that each component $F_i$ of an F-transform can be interpreted as a mean value $E[f(x) \mid X_i]$ of the approximating function $f(x)$ under the condition that the unknown value $x$ is consistent with the measurement result $X_i$. It is well known that in probability theory, the mean value can be alternatively described as the value $z$ that minimizes the mean square difference between this value and the actual value $f(x)$, that is, that minimizes the expression $E[(f(x)-z)^2 \mid X_i]$. Thus, the above relation provides an additional advantage of F-transforms in comparison with other approximation tools:

(i) F-transforms not only reflect expert knowledge,

(ii) F-transforms also provide a solution which is *optimal* (in a well-defined reasonable sense).

*4.3. Gauging the Accuracy of the Resulting Approximation.* We have shown that each component $z = F_i$ of the F-transform provides the approximation to $f(x)$ which is the most accurate. The next natural question is how accurate is it? In other words, what is the corresponding mean square difference $\sigma^2 = E[(f(x) - z)^2 \mid X_i]$? It turns out that the answer to this question can also be provided in terms of F-transforms.

Namely, as it is known, for $z = F_i = E[f(x) \mid X_i]$, we have

$$\begin{aligned}\sigma^2 &= E\Big[(f(x) - z)^2 \mid X_i\Big] \\ &= E[f^2(x) \mid X_i] - \Big(E\Big[(f(x) - z)^2 \mid X_i\Big]\Big)^2.\end{aligned} \tag{18}$$

That is, $\sigma^2 = E[f^2(x) \mid X_i]-F_i^2$. The expression $E[f^2(x) \mid X_i]$ can also be described in terms of F-transforms. Indeed, our result about the relation between F-transform and conditional expected value applies to all possible functions, including the square $f^2(x)$ of the original function $f(x)$. Thus, each value $E[f^2(x) \mid X_i]$ is equal to the $i$th component $S_i$ of the F-transform of this square.

So, we arrive at the following conclusion. If we only know that $x$ is consistent with the measurement result $X_i$, then

(i) a reasonable approximation for $f(x)$ is the value $F_i$: the $i$th component of the F-transform,

(ii) the root mean square accuracy $\sigma$ of this approximation is determined by the formula $\sigma^2 = S_i - F_i^2$, where $S_i$ is the $i$th component of the F-transform of the function $f^2(x)$.

Similarly, for the second problem—reconstructing $f(x)$ when we only know finitely many values corresponding to different $i$—the mean square accuracy of the corresponding approximation of the actual (unknown) function $f(x)$ by its inverse F-transform $\overline{f}(x)$ is equal to

$$\sigma^2(x) \stackrel{\text{def}}{=} E\Big[\Big(f(x) - \overline{f}(x)\Big)^2 \mid x\Big] = E[f^2(x) \mid x] - \Big(\overline{f}(x)\Big)^2. \tag{19}$$

The first term $E[f^2(x) \mid x]$ in this difference is equal to the inverse F-transform

$$\overline{f^2}(x) = \sum_{i=1}^{n} S_i \cdot A_i(x), \tag{20}$$

where the values $S_1, \dots, S_n$ form an F-transform of the squared function $f^2(x)$.

Thus, we arrive at the following conclusion:

(i) If we only know the values $F_1, \dots, F_n$ of the F-transform of the actual (unknown) dependence $f(x)$, then, as a reasonable approximation to $f(x)$, we can take the inverse F-transform $\overline{f}(x) = \sum_{i=1}^{n} F_i \cdot A_i(x)$.

(ii) If, in addition to the values $F_i$, we also know the F-transform $S_1, \dots, S_n$ of the square $f^2(x)$, then we can estimate the root means square accuracy $\sigma(x) = \sqrt{E[(f(x) - \overline{f}(x))^2 \mid x]}$ by using the formula

$$\sigma^2(x) = \overline{f^2}(x) - \Big(\overline{f}(x)\Big)^2, \tag{21}$$

where $\overline{f^2}(x) = \sum_{i=1}^{n} S_i \cdot A_i(x)$ is the inverse F-transform of the squared function.

## 5. A Similar Modification of a Probabilistic Interpretation Is Possible for Mamdani-Style Fuzzy Modeling (and Fuzzy Control)

*5.1. From F-Transform to Fuzzy Modeling.* Let us show that the above modification of a probabilistic interpretation from [8] can be extended from F-transform to a more general case of Mamdani-type fuzzy modeling and fuzzy control.

*Comment 2.* In this section, we concentrate on Mamdani's approach since F-transform can be viewed as a particular case of this approach, and since for Mamdani's approach, a probabilistic interpretation is possible [8]. Please note that while Mamdani's approach was historically the first, at present, there are many different approaches to fuzzy modeling and fuzzy control; we mention some of them in this chapter, but there are many others; see, for example, [10–12]. How to best interpret these other approaches in probabilistic terms—and whether such an interpretation is at all possible—is an interesting open question.

For example, an interesting question is how to interpret type-2 approaches to fuzzy modeling and fuzzy control; see, for example, [13–16]; maybe via interval-valued probabilities?

*5.2. Mamdani's Approach to Fuzzy Modeling and Fuzzy Control: A Brief Reminder.* In Mamdani's approach, we start with rules like

"if $x$ is small, then $u$ should be medium",

and then use membership functions for "small" and "medium" to transform these rules into an exact control strategy.

In general, we have rules

"if $x$ has a property $A_i$ then $u$ has the property $B_i$" ($1 \leq i \leq n$),

with known membership functions $A_i(x)$ and $B_i(u)$ for the corresponding properties. Mamdani's methodology is based on saying that for each input $x$, the value $u$ is a reasonable value of control if and only if one of the above $n$ rules is applicable, that is,

(i) either the first rule is applicable, that is, $x$ satisfies the property $A_1$ and $u$ satisfies the property $B_1$,

(ii) or the second rule is applicable, that is, $x$ satisfies the property $A_2$ and $u$ satisfies the property $B_2$,

(iii) $\cdots$

(iv) or the $n$th rule is applicable, that is, $x$ satisfies the property $A_n$ and $u$ satisfies the property $B_n$.

Once we select functions $f_{\&}(a,b)$ and $f_{\vee}(a,b)$ to represent "and" and "or" (these functions are called *t-norm* and *t-conorm*), we can thus describe the degree of our belief $\mu_x(u)$ that $u$ is reasonable (for a given input $x$) as

$$\mu_x(u) = f_{\vee}\left(f_{\&}(A_1(x), B_1(u)), \ldots, f_{\&}(A_n(x), B_n(u))\right). \tag{22}$$

In particular, if we select $f_{\&}(a,b) = a \cdot b$ and $f_{\vee}(a,b) = \min(a+b, 1)$ (and if the added values do not go beyond 1), we get

$$\mu_x(u) = \sum_{i=1}^{n} A_i(x) \cdot B_i(u). \tag{23}$$

Once we know this membership function, we can find the appropriate value of $u$ by using the so-called *centroid defuzzification*:

$$\overline{u}(x) = \frac{\int u \cdot \mu_x(u)du}{\int \mu_x(u)du}. \tag{24}$$

*5.3. A Natural Probabilistic Analog of Mamdani's Approach to Fuzzy Modeling.* In [8], it was shown that in a probabilistic setting, we get formulas similar to Mamdani rules corresponding to $f_{\&}(a,b) = a \cdot b$ and $f_{\vee}(a,b) = \min(a+b,1)$—if we assume a uniform distribution on the outputs. Let us show that by using Bayes formula, we can avoid this additional assumption, and thus, make the resulting probabilistic analog of Mamdani's fuzzy modeling even more natural.

Similarly to the above probabilistic interpretation of F-transform, let us assume that we have possible pieces of information $X_1, \ldots, X_n$ about the quantity $x$, and that for each piece of information, we also know the corresponding probability $P(X_i \mid x)$ which we will be denoted by $A_i(x)$.

Similarly, let us assume that we have possible pieces of information $U_1, \ldots, U_m$ about $u$, and we know the corresponding probabilities $P(U_i \mid u)$ which we will denote by $B_i(u)$.

We know that $u$ depends on $x$, but we do not know the exact dependence. Instead, for each information $X_i$ about $x$, we know the corresponding information $U_j$ about the corresponding $u$.

Since we did not select any specific order for the information $U_i$, we can select the value corresponding to $X_1$ as $U_1$, the value corresponding to $X_2$ by $U_2$, etc. Under this selection, the available information simply means that if $x$ is described by the piece of information $X_i$, then the corresponding $u$ is described by the piece of information $U_i$.

Our objective is, given these rules and given a new value $x$, to find a good estimate for the appropriate $u$.

Due to the formula of full probability, the conditional probability density $P(u \mid x)$ of $u$ under the condition $x$ has the form

$$P(u \mid x) = \sum_{i=1}^{n} P(u \mid U_i) \cdot P(X_i \mid x). \tag{25}$$

We know the probabilities $P(X_i \mid x) = A_i(x)$. The probability densities $P(u \mid U_i)$ can be determined by using the Bayes theorem—similarly to the F-transform case—as

$$P(u \mid U_i) = \frac{P(U_i \mid u)}{\int P(U_i \mid y)dy}, \tag{26}$$

that is, in terms of the values $B_i(u)$, as

$$P(u \mid U_i) = \frac{B_i(u)}{\int B_i(y)dy}. \tag{27}$$

Substituting the formula (27) and the expression (7) into the formula (25) (and changing the multiplication order), we get the formula

$$P(u \mid x) = \sum_{i=1}^{n} A_i(x) \cdot \frac{B_i(u)}{\int B_i(y)dy}. \tag{28}$$

Once we know these probabilities, we can produce the mean $\overline{u}$ as a reasonable estimate for $u$:

$$\overline{u}(x) = \frac{\int u \cdot P(u \mid x)du}{\int P(u \mid x)du}. \tag{29}$$

These are exactly the formulas derived in [8] from the additional assumption of a piecewise constant output distribution. Thus, our (minor) modification of [8] indeed uniquely determines the corresponding probabilistic analog of Mamdani's formulas.

*5.4. In Mamdani-Type Setting, Fuzzy and Probabilistic Formulas Are, in General, Different.* It is worth mentioning that

(i) while in F-transform, the probabilistic and fuzzy derivations lead to exactly the same formulas,

(ii) in the general fuzzy modeling case, as mentioned in [8], the formulas are somewhat different:

(a) the formula (29) is exactly the same as (24), with $P(u \mid x)$ instead of $\mu_x(u)$;

(b) the formula (28) is slightly different from Mamdani's formula (23)—by the integral in the denominator.

*5.5. Cases when Fuzzy and Probabilistic Formulas Coincide.* For F-transform (and, more generally, in all the cases when the value $\int B_i(y)dy$ is the same for all $i$), this additional denominator simply divides all the values $P(u \mid x)$ by the constant. This constant appears both in the numerator and in the denominator of the formula (28) and thus, it does not affect the resulting value $\overline{u}(x)$.

Another case when the fuzzy and probabilistic formulas coincide is the case of the Takagi-Sugeno (TSK) approach; see, for example, [10]. This equivalence is, in effect, proven in [8]. In the TSK approach, rules have the type

"if $x$ has a property $A_i$ then $u = f_i(x)$" $(1 \le i \le n)$,

for known functions $f_i(x)$. In the probabilistic setting, we assume that under a piece of information $U_i$, we must take $u = f_i(x)$. Thus, for a given input $x$, we select $f_i(x)$ with probability $P(X_i \mid \mathrm{x}) = A_i(x)$, where $\sum_{i=1}^n A_i(x) = 1$. The resulting mean $\overline{u}(x)$ is thus equal to $\sum_{i=1}^n A_i(x) \cdot f_i(x)$. For the case when $\sum_{i=1}^n A_i(x) = 1$, this is exactly the TSK formula.

*5.6. Comparison between Fuzzy and Probabilistic Modeling.* For Mamdani-type situations when fuzzy and probabilistic formulas are different, the comparison of the corresponding probabilistic and fuzzy rules is done, in detail, in [8].

Let us add three more situations to this comparison, situations that are naturally related to our modified derivation.

*5.7. Case when Probabilistic Control Is Better.* When the values $\int B_i(y)dy$ are different, probabilistic control and fuzzy control lead, in general, to a different value $\overline{u}$. We will show, on an example originally proposed by R. Yager, that in this case, the result of the probabilistic control is closer to common sense that the result of Mamdani's control.

Indeed, let us consider the situation in which we have two rules:

(i) the first rule is a more general rule saying that if $x$ is small, then $u$ should be small;

(ii) the second rule is a very specific rule, saying that if $x$ is very close to 0.11, then $u$ should be very close to 0.15.

Intuitively, if we have a value $x$ for which a very specific rule is applicable, for example, the value $x = 0.11$, then this specific rule should have a priority over the general rule. However, since the width of the membership function $B_2(u)$ is small, the corresponding term in (23) will practically not affect the resulting estimate (24).

In contrast, in the probabilistic control, the effect of $B_2(u)$ is normalized by, crudely speaking, the total width of the corresponding function $B_2(u)$. Thus, even the most specific rules will have—as desired—the significant influence on the result (29).

*Comment 3.* It should be mentioned that the problem with specific rules occurs only in Mamdani's approach to fuzzy control. In the alternative *logical* approach, this problem does not appear; see, for example, [17].

*5.8. Another Case When Probabilistic Control Is Better.* The probabilistic interpretation enables us to naturally consider more general situations in which the rules are themselves probabilistic, that is, when, for each $i$ and $j$, we know the conditional *probability* $P(U_i \mid X_j)$ that if $x$ has the property $X_j$, then $u$ has the property $U_i$.

In other words, instead of the original rules

"if $x$ has the property $X_i$, then $u$ has the property $U_i$, "

we now have rules

"if $x$ has the property $X_j$, then $u$ has the property $U_i$ with probability $P(U_i \mid X_j)$.″

Indeed, in this case, due to the formula of full probability, the conditional probability density $P(u \mid x)$ of $z$ under the condition $x$ has the form

$$P(u \mid x) = \sum_{i=1}^{n} \sum_{j=1}^{n} P(u \mid U_i) \cdot P\Big(U_i \mid X_j\Big) \cdot P\Big(X_j \mid x\Big). \tag{30}$$

Here, we know the original probabilities $P(U_i \mid X_j)$ and the probabilities $P(X_i \mid x) = A_i(x)$. The probability densities $P(u \mid U_i)$ can be determined by using the Bayes theorem as an expression (27). Substituting the formula (27) and the expression $P(X_i \mid x) = A_i(x)$ into the formula (30) (and changing the multiplication order), we get the formula

$$P(u \mid x) = \sum_{i=1}^{n} \sum_{j=1}^{n} P\Big(U_i \mid X_j\Big) \cdot A_j(x) \cdot \frac{B_i(u)}{\int B_i(y)\, dy}. \tag{31}$$

Once we know these probabilities, we can produce the mean $\overline{u}$ by using the formula (29).

*5.9. In Some Cases, Fuzzy Control Is Better.* We have shown that in some situations, probabilistic control is better than the original Mamdani's fuzzy control. However, in other situations, the fuzzy control is better. Let us give two examples.

*5.10. Case when Mamdani's Formulas Are Better.* The above probabilistic formulas only work for the case when $\sum_{i=1}^n A_i(x) = 1$, that is, in the probabilistic terms, when the properties $A_i$ are mutually exclusive. In practice, we may have nonexclusive properties, in which case we may have $\sum_{i=1}^n A_i(x) > 1$.

It is not clear how to handle this situation within the probabilistic approach. However, such situations are not a problem if we apply fuzzy control: its formulas are applicable no matter whether we satisfy the requirement $\sum_{i=1}^n A_i(x) = 1$ or not.

*Other Cases when Mamdani's Formulas Are Better.* The probabilistic interpretation is only possible when we use multiplication and addition as "and" and "or" operations $f_\&$ and $f_\vee$.

Fuzzy control does not necessarily have to use these operations, it can use different t-norms and t-conorms. It is an empirical fact that in many control situations, the use of t-norm different from the product and of the t-conorm different from the sum leads to a much better quality control—for example, a more stable or a smoother one.

In [18], we have formulated the problem of selecting the t-norm and the t-conorm as a precise optimization problem, and for several objective functions like smoothness or stability, we gave an explicit analytical solutions to these optimization problem—specifically, we described the selection that leads to the optimal values of smoothness or stability. In many of these case, the optimal selection is indeed different from the probabilistic case of product and sum. Thus, fuzzy control methodology indeed leads to a better quality control.

## 6. Conclusion

The fuzzy transform (F-transform) techniques have been lately shown to be very successful in various applications, including applications where until recently, only more traditional tools like Fourier transform or wavelet transform have been applied. In many other applications, however, the traditional tools have a clear advantage. It is therefore desirable to combine F-transform with the more traditional tools, so as to combine the relative advantages of both techniques. To make this combination easier, it is desirable to interpret F-transform in traditional mathematical terms.

In this paper, we describe a modification of a probabilistic interpretation described in [8]. In this modification, the corresponding probabilistic model uniquely leads to the formulas of the F-transform. A similar modification is described in a more general situation of fuzzy modeling.

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*Research Article*

# Why Fuzzy Transform Is Efficient in Large-Scale Prediction Problems: A Theoretical Explanation

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In many practical situations like weather prediction, we are interested in large-scale (averaged) value of the predicted quantities. For example, it is impossible to predict the exact future temperature at different spatial locations, but we can reasonably well predict average temperature over a region. Traditionally, to obtain such large-scale predictions, we first perform a detailed integration of the corresponding differential equation and then average the resulting detailed solution. This procedure is often very time-consuming, since we need to process all the details of the original data. In our previous papers, we have shown that similar quality large-scale prediction results can be obtained if, instead, we apply a much faster procedure—first average the inputs (by applying an appropriate fuzzy transform) and then use these averaged inputs to solve the corresponding (discretization of the) differential equation. In this paper, we provide a general theoretical explanation of why our semiheuristic method works, that is, why fuzzy transforms are efficient in large-scale predictions.

## 1. Formulation of the Problem

*1.1. Predictions Are Needed.* One of the main objectives of science is to predict the future values of the physical quantities. For example, it is desirable to predict tomorrow's weather, the weather for several days ahead, and so forth. For a spreading flu epidemic, it is desirable to predict how this epidemic will spread if we do not introduce any restrictions on travel-and how this spread will change if such restrictions are introduced.

*1.2. Detailed Predictions Are Often Impossible.* Of course, ideally, it is desirable to have predictions which are as detailed as possible. For example, ideally, we would like to know the exact value of tomorrow's temperature and wind speed at all possible spatial locations within a given region—or to predict exactly where the epidemics will spread and exactly how many people will fall ill if we do not introduce any travel restrictions.

However, in many practical situations, such a detailed prediction is impossible. In some of these situations, prediction is potentially possible, but it requires such a large amount of computations that even on the fastest modern computers, the computations finish long after the future event (that we are trying to predict) has already occurred.

*1.3. Large-Scale Predictions Are Usually Sufficient.* In many practical situations in which we cannot predict the *exact* values of the future quantities, it is often sufficient to predict the *average* values of the future quantities, averaged over certain areas.

For example, from the practical viewpoint, even though we cannot predict the exact value of tomorrow's temperature at all possible spatial locations, it would be beneficial to predict the *average* temperature over a given small geographic region. Similarly, for an epidemic, even though we are unable to predict where exactly it will spread and how many people will fall ill in different small towns, it is very beneficial to be able to predict how many people *on average* will get ill in the region.

For predicting time series, for example, financial time series formed by the prices of different stocks at different moments of time, though it is impossible to predict the exact values of the future prices, it is desirable to at least be able to predict the *trends*, that is, the prices averaged over a certain time period.

*Comment.* For clarity and simplicity, in the following text, we will describe the case when both the input $x(t)$ and the output $y(t)$ depend only on time $t$. The exact same formulas can also be applied if we have a spatial dependence; in this case, $t$ and $s$ are the corresponding spatial points.

*1.4. Towards a Precise Mathematical Description of Quantities Predicted by Large-Scale Prediction.* Instead of predicting the values $y(t)$ for different moments of time $t$, we predict the weighted averages $\overline{y}(t)$, that is, the average of the values $y(s)$ for the values $s$ which are close to $t$.

It is reasonable to assume that for different moments $t$ we use the same averaging, that is, the weight with which the value $y(s)$ contributes to $\overline{y}(t)$ depends only on the difference $t - s$ and not on the absolute values of $t$ or $s$. Under this assumption, the general formula for the weighted average takes the form

$$\overline{y}(t) = \int w(t-s) \cdot y(s)ds, \tag{1}$$

where all the weights are nonnegative and for each $t$, the total weight of all the values $y(s)$ is equal to 1:

$$\int w(s)ds = 1. \tag{2}$$

*1.5. An Example and a Useful Equivalent Reformulation of Averaging.* A natural example of such averaging is a *Gaussian averaging*, where we use Gaussian weights:

$$w(s) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{s^2}{2\sigma^2}\right). \tag{3}$$

It is often convenient to represent this Gaussian weight function as

$$w(s) = \text{const} \cdot W(s), \tag{4}$$

where the new weight function $W(s)$ is described by a simpler formula

$$W(s) = \exp\left(-\frac{s^2}{2\sigma^2}\right). \tag{5}$$

This new weight function satisfies the property and

$$\max_s W(s) = 1. \tag{6}$$

*1.6. Large-Scale Quantities and Fuzzy Transform.* A similar representation is often useful for other weight functions as well. In general, once we know this new weight function $W(s)$, we can use the normalized condition (2) to find that

$$w(s) = \frac{W(s)}{\int W(t)dt}. \tag{7}$$

Thus, in terms of the new weight function $W(s)$, the weighted average (1) takes the form

$$\overline{y}(t) = \frac{\int W(t-s) \cdot y(s)ds}{\int W(s)ds}. \tag{8}$$

Expression (8) is a particular case of the expression of a *fuzzy transform* [1–3] which is, in general, defined as

$$Y = \frac{\int A(s) \cdot y(s)ds}{\int A(s)ds} \tag{9}$$

for some function $A(s) \geq 0$ for which $\max_s A(s) = 1$. For a special uniform case [2, 3], we have several functions $A(s)$ of the form $A_n(s) = W(t_n - s)$, where $W(s)$ is a given function. The corresponding values $Y_n$ of the fuzzy transforms are then equal to

$$Y_n = \frac{\int A_n(s) \cdot y(s)ds}{\int A_n(s)ds} = \frac{\int W(t_n - s) \cdot y(s)ds}{\int W(s)ds}, \tag{10}$$

that is, coincide with the values $\overline{y}(t_n)$ corresponding to different points $t_n$.

Thus, from the mathematical viewpoint, the weighted averages are simply the values of the fuzzy transform.

*1.7. Typical Prediction Procedure: Solving a Differential Equation.* Most relations in physics are described by differential equations. In particular, the relation between the observed signals $x(t)$ and the predicted values $y(t)$ can also be described by a differential equation.

*1.8. Traditional Procedure for Large-Scale Predictions.* Since prediction usually means solving a known differential equation, a usual procedure for large-scale predictions is as follows:

(i) first, we use the known values $x(t)$ to solve the differential equations and get the values $y(t)$;

(ii) then, we apply the weighted average procedure (8) to the resulting values $y(t)$ and get the desired large-scale predictions $\overline{y}(t)$.

*1.9. Drawbacks of the Traditional Procedure.* The main drawback of the traditional procedure is that we spend a lot of computation time to get a detailed solution $y(t)$—but at the end, we only return a few values corresponding to large-scale predictions.

For example, in weather prediction, we spend hours of computer time on high-performance supercomputers to solve a complex system of differential equations with thousand of variables and then only use the large-scale weighted average of this solution.

*1.10. Natural Idea.* We are only interested in *large-scale* predictions, that is, only in the weighted *averages* of the result $y(t)$ of solving the differential equation, averages that ignore the fine structure of the solution $y(t)$. So why not start with the averaged values of the input $x(t)$, that is, why not ignore the fine structure of $x(t)$ from the very beginning and thus, save computation time.

In other words,

(1) traditionally, we first *integrate* the differential equation and then *average* the solution;

(2) what we propose is that we first *average* and only then *integrate*; in this manner, we will need fewer values to integrate and, thus, less computation time.

*1.11. Empirically, This Idea Seems to Work.* For several differential equations, we implemented the above idea of how to speed up computations. Specifically,

(i) instead of the original input $x(t)$, we use the fuzzy transform values $X_1, \ldots, X_n$,

(ii) then we use the values $X_i$ in the discretized version of the original differential equation, then

(iii) we use the results $Y_1, \ldots, Y_n$ of this solution as an estimate for the desired large-scale averages (= fuzzy transform of $y(t)$).

Surprisingly, we got a very good approximation to the values $Y_i$ computed based on the detailed $y(t)$ [2–12].

*1.12. What We Do in This Paper.* In this paper, we provide a theoretical explanation for the empirical success of the fuzzy-transform-based methods of speeding up computations.

This explanation makes us confident that this fuzzy transform technique can be successfully used in other large-scale prediction problems as well.

## 2. Theoretical Explanation

*2.1. Linearization.* Usually, the effect of each input value $x(t)$ on the prediction results is small. In this sense, we can say that the inputs are relatively small. Thus, we can use the standard technique of dealing with dependence on small value:

(1) extend the dependence of $y(t)$ on $x(s)$ in Taylor series,

(2) ignore quadratic and higher order terms, and thus

(3) keep only linear terms in this dependence.

In this case, we get the following dependence:

$$y(t) = y_0(t) + \int y_1(t,s) \cdot x(s)ds, \tag{11}$$

for some functions $y_0(t)$ and $y_1(t,s)$.

*2.2. Shift-Invariance.* We are interested in systematic predictions, predictions that need to be repeated again and again. In these predictions, there is no fixed moment of time: if we start with the same input repeated later (i.e., shifted in time, from $x(t)$ to $x_{\text{new}}(t) = x(t - t_0)$), we get the same result (similarly shifted) $y_{\text{new}}(t) = y(t - t_0)$.

For the formula (11), this shift-invariance means that

(1) first, we must have $y_0(t) = y_0(t - t_0)$ for all $t$ and $t_0$; in particular, for $t_0 = t$, we conclude that $y_0(t) = y_0(0)$, that is, $y_0$ should not depend on time at all: $y_0(t) = y_0$;

(2) second, we must have $y_1(t,s) = y_1(t - t_0, s - t_0)$ for all $t$, $s$, and $t_0$; in particular, for $t_0 - s$, we conclude that $y_1(t,s) = y_1(t - s, 0)$ and that the function $y_1(t,s)$ should only depend on the difference $t - s$.

Thus, we arrive at the following dependence:

$$y(t) = y_0 + \int y_1(t - s) \cdot x(s)ds. \tag{12}$$

*2.3. Main Result: Formulation.* In the traditional approach, we first find the detailed output (12) and then average it by applying the averaging

$$\overline{y}(t) = \frac{\int W(t-s) \cdot y(s)ds}{\int W(s)ds}. \tag{13}$$

An alternative approach is to first apply the same averaging to the original signal $x(t)$, resulting in

$$\overline{x}(t) = \frac{\int W(t-s) \cdot x(s)ds}{\int W(s)ds}, \tag{14}$$

and try use this averaged signal $\overline{x}(t)$ as the input to the corresponding dynamical systems (i.e., in effect, to transformation (12)):

$$\overline{y}_f(t) = y_0 + \int y_1(t-s) \cdot \overline{x}(s)ds. \tag{15}$$

Our claim is that these two approaches always lead to the same result, that is,

$$\overline{y}_f(t) = \overline{y}(t) \tag{16}$$

for all moments of time $t$.

*Proof.* In terms of the normalized weight function (7), the original signal has the form

$$\overline{y}(t) = \int w(t-s) \cdot y(s)ds, \tag{17}$$

where $y(s)$ is determined by formula (12). Substituting the expression

$$y(s) = y_0 + \int y_1(\mathrm{s} - u) \cdot x(u)du \tag{18}$$

into formula (17), we conclude that

$$\overline{y}(t) = y_0 + \int w(t-s) \cdot y_1(s-u) \cdot x(u)dsdu, \tag{19}$$

that is,

$$\overline{y}(t) = y_0 + \int \overline{w}(t,u) \cdot x(u)du, \tag{20}$$

where

$$\overline{w}(t,u) \stackrel{\text{def}}{=} \int w(t-s) \cdot y_1(s-u)ds. \tag{21}$$

Similarly, in terms of the normalized weight function $w(t)$, we have

$$\overline{x}(t) = \int w(t-s) \cdot x(s)ds. \tag{22}$$

Substituting the corresponding formula

$$\overline{x}(s) = \int w(s-u) \cdot x(u)du \tag{23}$$

into expression (15) for $\overline{y}_f(t)$, we conclude that

$$\overline{y}_f(t) = y_0 + \int y_1(t-s) \cdot w(s-u) \cdot x(u)dsdu, \tag{24}$$

that is,

$$\overline{y}_f(t) = y_0 + \int \overline{w}_f(t,u) \cdot x(u)du, \tag{25}$$

where

$$\overline{w}_f(t,u) \stackrel{\text{def}}{=} \int y_1(t-s) \cdot w(s-u)ds. \tag{26}$$

In view of formulas (20) and (25), to prove that the values $\overline{y}(t)$ and $\overline{y}_f(t)$ always coincide, it is sufficient to prove that the corresponding functions $\overline{w}(t,u)$ and $\overline{w}_f(t,u)$ coincide for all $t$ and $u$. These functions are defined by expressions (21) and (26).

To prove that these expressions coincide, let us try to transform them into each other. In expression (26), we take the value of the normalized weight function $w(t)$ at the point $s-u$. In contrast, in expression (21), we use the value $w(t-s)$ for the corresponding auxiliary variable $s$. To transform expression (26) into the form (21), let us introduce a new auxiliary variable $v$ for which $s-u = t-v$. From this formula, we conclude that $s = t+u-v$, hence $t-s$ takes the form $t-(t+u-v) = v-u$. Thus, in terms of the new variable $v$, the integrated expression in (26) takes the form

$$\begin{aligned} y_1(t-s) \cdot w(s-u) &= y_1(v-u) \cdot w(t-v) \\ &= w(t-v) \cdot y_1(v-u). \end{aligned} \tag{27}$$

Hence, the integrals of these two expressions must also coincide:

$$\int y_1(t-s) \cdot w(s-u)ds = \int w(t-v) \cdot y_1(v-u)dv. \tag{28}$$

The right-hand side of this equality is exactly the expression (21)—the only difference is that we use a different name for the integration variable ($v$ instead of $s$). Thus, the functions $\overline{w}(t,u)$ and $\overline{w}_f(t,u)$ indeed coincide—and, hence, $\overline{y}_f(t) = \overline{y}(t)$.

The equality is proven. □

*Comment.* In the ideal case, when quadratic terms can be completely ignored and there is no dependence on absolute time, the new method leads to *exact* same large-scale predictions as the traditional one. In practice, if we take into account that

(i) the quadratic terms are small but non-zero, and that

(ii) there may be an underlying trend-like dependence on absolute time (like global warming in weather prediction),

we end up with *approximate* equality between the traditional and fuzzy-transform-based predictions—and this approximate equality is what we observed in our experiments [2–12].

Since large-scale predictions are approximate anyway, this approximate equality means that, in terms of accuracy, the new predictions are, in effect, as good as the traditional ones. Since the new predictions are much faster to compute, they have a clear practical advantage.

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*Research Article*

# Combining Fuzzy MCDM with BSC Approach in Performance Evaluation of Iranian Private Banking Sector

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The objective of this study is to construct an approach based on multiple criteria decision making (MCDM) and balanced scorecard (BSC) for evaluating performance for three nongovernmental Iranian's banks. Following the literature relating to banking performance and BSC concepts, experts and managers select 21 indexes for evaluation. Furthermore, fuzzy analytic hierarchy process (FAHP) calculated the relative weights of each chosen index in order to tolerate vagueness and ambiguity of information, and three MCDM analytical tools (TOPSIS, VIKOR, and ELECTRE) were adopted to rank the banking performance. The results indicate that a customer "$C$" has the most significant BSC perspectives and the customer satisfaction "$C_1$" is the most major index in banking sector. This proposed fuzzy MCDM method combined with BSC approach is a comprehensive and up-to-date model that can be a useful and effective assessment tool.

## 1. Introduction

In today's competitive world, only the companies that observe the needs of their customers and provide them with satisfaction, can compete with others and gain benefit. This requirement causes the loyalty of customers to products and services of that organization or company. Also, bank as an organization that plays an important role in the economical development and blooming of the country must do some robust steps to assess the needs of its customers in the services which provide for them. To galvanize competition among banks, more emphasis is necessary to be put on the internal operational performance. This equals a need for finding an effective way to evaluate the performance of the entire organization to its objectives. The model presented aims at evaluating the general performance of banking which is a key factor in banking studies. Many researches are faced with financial factors to evaluate banking performance.

Recently, considerations and attentions to nonfinancial factors are increasing. Many indexes in current studies are repetitive and in some cases they were not identified. This paper not only considers new and practical indexes but also determines significant ranking for these indexes. We proposed a new and comprehensive model that can be a useful and effective assessment tool for business enterprise.

Nowadays many organizations employ numerous theories and methods of evaluation for assessment in their departments. These approaches consist of ratio analysis, total production analysis, regression analysis, Delphi analysis, balanced scorecard (BSC), analytic hierarchy process (AHP), data envelopment analysis (DEA), and so forth. Each method has its own principles, prospects and flaws [1]. Managers select this kind of method depending on the level and type of organization. Anyway, all successful organizations have common characteristics as specific outlook, positive actions and effective performance assessment. BSC is the expanded tool for assessing the performance, which acceptably programs and controls the organization to attain its goals [2–4]. BSC breaks down the conventional financial restrictions and assesses the performance of the organization from four perspectives: financial, customer, internal business process, and learning and growth [5]. To get to the best possible result, it is necessary to adapt the banking relations to the customer needs [6]. Therefore, experts and managers use a BSC framework for expanding the evaluative guidelines of the banking performance [7–9].

Bellman and Zadeh [10] developed the theory of decision behaviour in a fuzzy environment. Scientists developed many models and employed them in such different fields as control engineering, artificial intelligence, management science, and multi criteria decision making (MCDM). The concept of integrating the fuzzy logic and MCDM leads to the Fuzzy MCDM. Many academic and practical works have been done in applying MCDM to assessing criteria and selecting options [11–19]. With respect to other methods, BSC criterion is more concrete and comprehensive. In this research, the fuzzy MCDM approach is based on the above four perspectives of the BSC to create a model for assessing the performance of banking organizations. Our aims are fourfold: (1) specificity of the performance indexes in banks to create a hierarchical framework for performance assessment. (2) Usage of the Fuzzy AHP to find out the fuzzy value of indexes based on mental judgment. (3) Usage of TOPSIS, VIKOR, and ELECTRE for rating the performance of three banks. (4) Suggestions based on the outcome of the study for assessing performance. The structure of the paper is as follows: In Section 1 we introduce the concepts of performance assessment and BSC. in Section 2 we review the literature and in Section 3 we employ framework for performance assessment in analytic methods and we use the Fuzzy MCDM for assessing the banking performance. In Section 4 we explain the practical usage for assessing the banking performance consisting of the BSC hierarchical framework of assessment indexes and to present data analysis and discussion. Section 5 is conclusive.

## 2. Performance Evaluation and BSC

In this section, the paper briefly reviews some relevant concepts as the performance evaluation index and the BSC.

*2.1. Defining Performance Evaluation.* DuPont and General Motors designed the first systems of evaluating trade based on financial index in 1920s. Both companies had decentralized their unit benefits and synchronically considered their capital return to evaluate the financial performance of their units. Ralph Cordinerz, the senior executive director of General Electric, unveiled his dissatisfaction with such kinds of systems in 1951. This action ameliorated the general replication and the equilibrium between the short-term and long-term goals along with the profitability, market share, efficiency, and the workers' satisfaction [20]. There is a great deal of literature on performance assessment and successful performance management [21]. The traditional rating of performance was based on rigid factors as financial return, capital return, and benefit return. However, this method of performance rating cannot precisely define companies for applying the aforementioned approach [21]. Assessing the performance of a bank can be both uncertain and contradictory [9]. Previous studies on performance evaluation have examined the scale economy and employed the traditional statistical methods such as correlation analysis [22], translog cost function [23, 24], loglinear models, or the Data Envelopment Analysis (DEA) method [25, 26].

*2.2. Performance Evaluation Index.* The experts have defined the performance measurement as a control system in any company which investigates its daily operations and assesses to what extent the company has reached its goals. The company must specify a set of indexes that reflect its performance. These indexes can be quantitative or nonquantitative. For example, we can consider an index like the lead time as a quantitative (or financial) index, during the customer satisfaction is a qualitative (or nonfinancial) index. During performing BSC, directors have difficulty specifying strategies and selecting scales.

In the initial step of performing BSC, it is important to collect the possible ideas regarding the performance via interviewing business managers and discussing their vision, mission, and strategy. Meyer and Markiewicz [27] enclosed the vital factors of successful banking performance under categories: profitability, efficiency, human resources management, risk management, quality of services, capital management, and competition status.

Collier [7] employed the models of structural equations using such factors as quality process errors, employee turnover rate, labour productivity, on-time delivery, and the unit cost to analyze the process of bank performance. Arshadi and Lawrence [22] have used the multidimensional indexes such as profitability, the price of banking services, and the share of loan market. Most of previous studies have focused on the customer and what bank and what general services they choose. According to the relevant literature, the criteria upon which the customer evaluates and chooses the bank consists of price, ease of access, customer services, place, credibility and reliability, modern facilities, interest rates, working hours, incentive suggestions, the domain of product, and the policy of service price [28–32]. In their latest research, Devlin and Gerrard [32] tried to reveal the relative importance of different criteria for selecting banks using the quantitative statistical methodology. They provided the customers' choosing criteria and multiple banking and created the point by point comparison of the relative importance of the selection criteria vital to the selection of the primary and secondary banks.

*2.3. Balanced Scorecard (BSC).* Norton (the management director of Nolan Norton Institute) and Robert Kaplan (the Harvard University professor) suggested BSC in 1992. BSC specifies the organizational performance from the following four perspectives: financial, customer, internal business process, and learning and growth, which are in relation with the four following subordinates: accounting and financial, marketing, value chain, And human resources. We can use these financial and nonfinancial criteria to help directors and employees align with the outlooks of organizations from four perspectives. BSC makes directors ready for the applications of the necessary tools for a successful competitive strive [8].

In reality, current organizations accept the introduction of the BSC by Kaplan and Norton and the integration of financial and nonfinancial scales in specifying the performance of both profiting and nonprofiting organizations [33, 34]. If they recognize the most appropriate scale for their need, they can save both time and money. Delicate

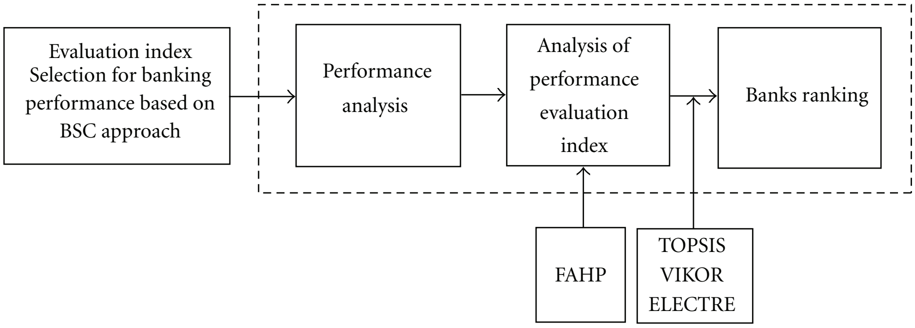


Figure 1: Performance evaluation framework of the research.

nonfinancial scales such as customer relations may be deemed more than half of the properties of a company. The important principle of the BSC is achieving success via the key nonfinancial factors after succeeding in key financial factors. These standards enable companies to manage their performance effectively and prognosticate their profits [6].

BSC is a general tool employed by many trades for assessing the performance in the different aspects. The advantages of using the BSC for banks are numerous. Some examples are the following: (1) usage of the BSC approach as a framework for the strategic assessment and development banks; (2) usage of the BSC approach for developing strategic goals and performance factors; (3) the approach is a way for specifying and supervising the performance of key stimulants that may lead to successful application of the bank's strategy; (4) the BSC approach is an effective tool to ensure the constant betterment of the bank's system and process [35]. Davis and Albright [3] presented an empirical analysis that investigated the effect of BSC on the financial performance of banks. Kim and Davidson [36] employed the BSC framework to assess the commercial performance of the Information Technology expenses in banking industry using the $t$-test and regression. Kuo and Chen [37] employed the four perspectives of the BSC via the fuzzy Delphi method to create performance assessment markers for stimulating the service industries. Leung et al. [38] suggested the custom-built performance assessment model using AHP and ANP to establish the BSC.

A major part of the literature relevant to financial industries have employed and utilized the BSC for performance evaluation [3, 39]. Much of these studies however, point to the need of creating an effective mechanism based on the relative weight for choosing assessment standards. Thus, this research develops a banking performance assessment model not only for investigating the relative importance among factors, but also for examining the delicate flaws to eliminate to reach the desired level.

## 3. Research Framework and Analytic Methods

Figure 1 explains the analytic structure of this paper. The performance analysis is based on the selective assessment standard. First, we have employed FAHP approach to calculate the relative weight of the performance assessment indexes. Then, we have utilized the multistandard analytic tools of decision making consisting of TOPSIS, VIKOR and ELECTRE based on these weights to rate and ameliorate the banking performance and specify the best options.

Table 1: Membership function of linguistic scales.

| Linguistic scale | Triangular fuzzy scale | Triangular fuzzy reciprocal scale |
|---|---|---|
| Just equal | (1, 1, 1) | (1, 1, 1) |
| Equally important | (1, 1, 3) | (1/3, 1, 1) |
| Weakly important | (1, 3, 5) | (1/5, 1/3, 1) |
| Strongly important | (3, 5, 7) | (1/7, 1/5, 1/3) |
| Very strongly important | (5, 7, 9) | (1/9, 1/7, 1/5) |
| Absolutely important | (7, 9, 9) | (1/9, 1/9, 1/7) |

*3.1. Linguistic Variables at Fuzzy Set.* According to Zadeh [40], a conventional quantification of reasonable expression in situations that are complex or hard to define is really difficult; in these cases the notion of a linguistic variable is vital. A linguistic variable defined as a variable whose values are words or sentences in a natural or artificial language. Here, we use five basic linguistic terms, for comparing the best plan evaluation criteria as "absolutely important," "very strongly important," "essentially important," "weakly important," and "equally important" according to a fuzzy five-level scale [11]. The membership function of a linguistic term is defined by Mon et al. [41] and displayed in Table 1.

We use linguistic variables to assess the performance value of alternative for each criterion as "very good," "good," "fair," "poor," and "very poor". A TFN fuzzy partition (see Figure 3 e.g.) is used to indicate the membership function of the expression values (Figure 2).

*3.2. Fuzzy AHP.* We summarize the procedure for determining the criteria weights in the FAHP method.

*Step 1.* Construct the pairwise comparison matrices among all the elements/criteria in the dimensions of the hierarchy system through expert questionnaire. Assign linguistic terms

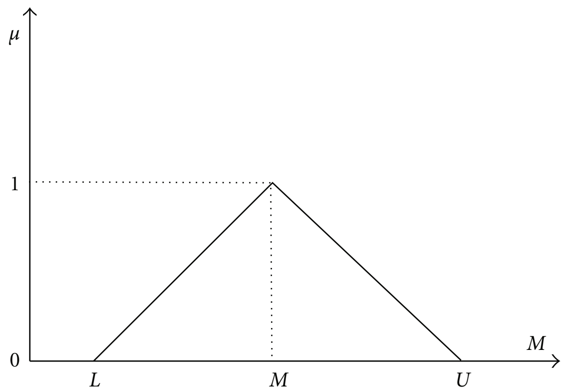


Figure 2: Hierarchical framework of research model based on BSC.

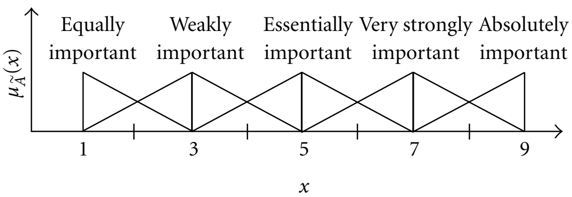


Figure 3: Membership function of a triangular fuzzy number.

by TFN to the pairwise comparisons by asking each expert its viewpoint elements/criteria, such as

$$\widetilde{A} = \begin{bmatrix} 1 & \widetilde{a}_{12} & \cdots & \widetilde{a}_{1n} \\ \widetilde{a}_{21} & 1 & \cdots & \widetilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{n1} & \widetilde{a}_{n2} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & \widetilde{a}_{12} & \cdots & \widetilde{a}_{1n} \\ 1/\widetilde{a}_{21} & 1 & \cdots & \widetilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/\widetilde{a}_{n1} & 1/\widetilde{a}_{n2} & \cdots & 1 \end{bmatrix}. \quad (1)$$

*Step 2.* Compute the fuzzy geometric and compute the fuzzy weights of each criterion by normalization as Buckley [42] described as follows:

$$\begin{aligned} \widetilde{r} &= \left(\widetilde{a}_{j1} \otimes \widetilde{a}_{j2} \otimes \cdots \otimes \widetilde{a}_{in}\right)^{1/n}, \\ \widetilde{w}_i &= \widetilde{r}_i(\widetilde{r}_1 \otimes \cdots \otimes \widetilde{r}_n)^{-1}, \end{aligned} \quad (2)$$

where $\widetilde{a}_{in}$ is fuzzy comparison value of criterion$i$ to criterion $n$, thus, $\widetilde{r}_i$ is geometric mean of fuzzy comparison value of criterion $i$ to each criterion, $\widetilde{w}_i$ is the fuzzy weight of the $i$th criterion, can be indicated by a TFN, $\widetilde{w}_i = (Lw_i, Mw_i, Uw_i)$. Here $Lw_i$, $Mw_i$, and $Uw_i$ stand for the lower, middle, and upper values of the fuzzy weight of the $i$th criterion.

Bellman and Zadeh [10] were the first to propose the decision making problem in fuzzy environments and they announced the initiation of FMCDM. This analysis method has been widely applied to deal with DM problems involving multiple criteria evaluation/selection of alternatives in various fields. The practical applications reported in the literatures: weapon system evaluation [41], optimizating the design process of truck components [42]. These studies have shown the advantages obtained by handling unquantifiable/qualitative criteria and by applying this method obtained quite reliable results. According to a variety of subjective judgments among the experts' viewpoints, this research employs the overall valuation of fuzzy judgments to acquire more reasonable assessment. The following are the methods and procedures of the FMCDM theory.

*(1) Alternatives Measurement.* used the linguistic variables measurement to demonstrate the criteria performance (effect-values) by expressions such as "very good," "good," "fair," "poor," "very poor," which used the experts to express their subjective judgments, and a TFN within the scale range of 0–100 for each linguistic variable. Assume $\widetilde{E}_{ij}^k$ to indicate the fuzzy performance value of evaluator $k$ towards alternative $i$ under criterion $j$ and indicate all of the evaluation criteria. Since the perception of each experts differs according to the evaluator's experience and knowledge, and the definitions of the linguistic variables vary as well, this study uses the concept of average value to integrate the fuzzy judgment values of $m$ evaluators, that is,

$$\widetilde{E}_{ij} = \left(\frac{1}{m}\right) \otimes \left(\widetilde{E}_{ij}^1 \oplus \widetilde{E}_{ij}^2 \oplus \cdots \oplus \widetilde{E}_{ij}^m\right). \quad (3)$$

The sign $\otimes$ denotes fuzzy multiplication, the sign $\oplus$ indicates fuzzy addition, $\widetilde{E}_{ij}$ shows the average fuzzy number of the judgment of the decision makers, which a triangular fuzzy number has demonstrated as $\widetilde{E}_{ij}^k = (L\widetilde{E}_{ij}^k, M\widetilde{E}_{ij}^k, U\widetilde{E}_{ij}^k)$. The method put forward by Buckley [42] has computed the end-point values $LE_{ij}$, $ME_{ij}$, and $UE_{ij}$, that is,

$$LE_{ij} = \frac{\sum LE_{ij}^k}{m}, \quad ME_{ij} = \frac{\sum ME_{ij}^k}{m}, \quad UE_{ij} = \frac{\sum UE_{ij}^k}{m}. \quad (4)$$

*(2) Fuzzy Synthetic Decision.* the calculation of fuzzy numbers has integrated the weights of each criterion of fuzzy performance values and was found at the fuzzy performance value of the basic evaluation. Regarding each criterion, FAHP has derived weight $\widetilde{w}_j$; the criteria weight vector $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_j, \ldots, \widetilde{w}_n)^t$ can be acquired, whereas the fuzzy performance value of each alternative under $n$ criteria has attained the fuzzy performance matrix $\widetilde{E}$ of each of the alternatives, that is, $\widetilde{E} = (\widetilde{E}_{ij})$.The vector $\widetilde{w}$ and fuzzy matrix $\widetilde{E}$ deduced the final fuzzy synthetic decision, and the obtained result is the fuzzy synthetic decision matrix $\widetilde{R}$, that is,

$$\widetilde{R} = \widetilde{E} \odot \widetilde{w}. \quad (5)$$

The sign "$\odot$" indicates the computation of the fuzzy numbers, consisting of fuzzy addition and fuzzy multiplication. Since the calculation of fuzzy multiplication is somehow complicated, the result usually represents and the approximate fuzzy number $\widetilde{R}_i$, as $\widetilde{R}_i = (LR_i, MR_i, UR_i)$, where $LR_i$,

$MR_i$, and $UR_i$, respectively, are the lower, middle, and upper synthetic performance values of the alternative $i$, that is,

$$LR_i = \sum_{j=1}^{n} LE_{ij} \times Lw_j,$$
$$MR_i = \sum_{j=1}^{n} Mw_j \times ME_{ij}, \quad (6)$$
$$UR_i = \sum_{j=1}^{n} Uw_j \times UE_{ij}.$$

*(3) Ranking the Fuzzy Number.* the result of the fuzzy synthetic decision obtained by each alternative is a fuzzy number. The procedure of defuzzification is to locate the Best Nonfuzzy Performance value (BNP) [12]. Methods utilized in such defuzzified fuzzy ranking usually include mean of maximal (MOM), center of area (COA), and $\alpha$-cut. The application of COA method is a simple and practical way to avoid the need to prefer some evaluators. Therefore, we utilize this method. The following formula calculates the BNP value of the fuzzy number $\widetilde{R}_i$:

$$\mathrm{BNP}_i = \left( \frac{[(UR_i - LR_i) + (MR_i - LR_i)]}{3} \right) + LR_i, \quad \forall i. \quad (7)$$

According to the value of the derived BNP for each alternative, the ranking of each of the alternatives and criteria can then proceed.

*3.3. TOPSIS.* TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) is one of the practical MADM techniques to solve real-world problems [43]. TOPSIS method was firstly suggested by Hwang and Yoon [44]. Based on this technique, the best alternative would be the one that has nearest distance to the positive ideal solution and farthest distance from the negative ideal solution [45]. The positive ideal solution has been defined as a solution that maximizes the benefit criteria and minimizes the cost criteria, whereas the negative ideal solution maximizes the cost criteria and minimizes the benefit criteria [46]. Briefly, all best values attainable of criteria have composed the positive ideal solution, whereas all worst values obtainable of criteria has made up the negative ideal solution [47]. This paper has used TOPSIS method for determining the final ranking of the private banking sector with following steps.

*Step 1.* Normalizing decision matrix via (10):

$$r_{ij} = \frac{w_{ij}}{\sqrt{\sum_{j=1}^{J} w_{ij}^2}}, \quad j = 1, 2, 3, \ldots, J. \quad (8)$$

*Step 2.* Forming weighted normalized decision matrix:

$$i = 1, 2, 3, \ldots, n, \quad v_{ij} = w_i \times r_{ij}, \qquad j = 1, 2, 3, \ldots, J. \quad (9)$$

*Step 3.* Calculating positive ideal solution (PIS), and negative ideal solution (NIS):

$$A^+ = \{v_1^+, v_2^+, \ldots, v_n^+\}, \quad (10)$$

$$A^- = \{v_1^-, v_2^-, \ldots, v_n^-\}. \quad (11)$$

*Step 4.* Computing the distance of each alternative from PIS and NIS:

$$d_i^+ = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^+\right)^2}, \quad j = 1, 2, \ldots, J, \quad (12)$$

$$d_i^- = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^-\right)^2}, \quad j = 1, 2, \ldots, J. \quad (13)$$

*Step 5.* Determining the closeness coefficient of each alternative:

$$CC_i = \frac{d_i^-}{d_i^+ + d_i^-}, \quad i = 1, 2, \ldots, J. \quad (14)$$

*Step 6.* Obtaining the ranking of alternatives by comparing $CC_i$ values.

*3.4. VIKOR.* Opricovic [48], Opricovic and Tzeng [49] developed VIKOR, the Serbian name: VlseKriterijumska Optimizacija I Kompromisno Resenje, means multicriteria optimization and compromise solution [50] that is a suitable tool to evaluate each alternative. Experts elaborated the VIKOR method for multicriteria optimization of the complicated systems [51]. This method concentrates on ranking and selecting from a set of alternatives and chooses compromise solutions for a problem with conflicting criteria, which can sustain the decision makers to obtain a best final decision. The concept of VIKOR is based on compromise (feasible) solution which is the closest to the ideal alternative, and a compromise means an agreement established by mutual concessions [52]. It introduces the multicriteria ranking index according to the special measure of "closeness" to the "ideal" solution [48].

According to Opricovic and Tzeng [51], the PLp-metric utilized has developed the multicriteria measure for compromise ranking as an aggregating function in a compromise programming method. The various $J$ alternatives have been described as $a_1, a_2, \ldots, a_j$. For alternative $a_j$, $f_{ij}$ denotes the rating of the $i$th aspect, for example, $f_{ij}$ is the value of $i$th criterion function for the alternative $a_j$; $n$ is the number of criteria. Development of the VIKOR method started with the following form of $L_p - metric$:

$$L_{pi} = \left\{ \sum_{j=1}^{n} \left[ \frac{\left(f_j^* - f_{ij}\right)}{\left(f_j^* - f_j^-\right)} \right]^p \right\}^{1/p}, \quad 1 \le p \le \infty; \ i = 1, 2, \ldots, m. \quad (15)$$

VIKOR method uses $L_{1,j}$ and $L_{\infty j}$ to formulate ranking measurement value. Also VIKOR interprets $L_{1,j}$ as "concordance"

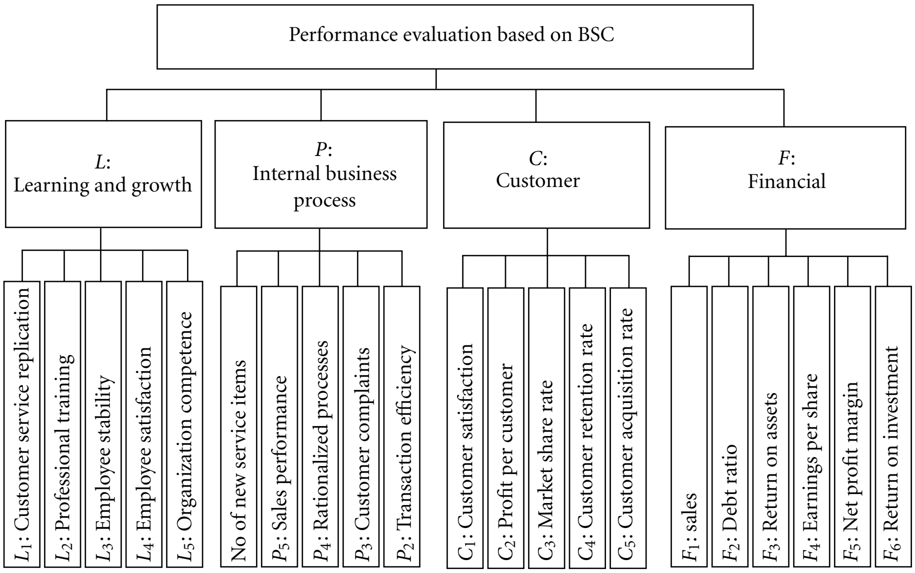


Figure 4: Membership function of linguistics variables for comparing two criteria.

and can contribute decision makers with information about the maximum group utility or "majority". Similarly, VIKOR interprets $L_{\infty j}$ as "discordance" and contributes decision makers with information about the minimum individual regret of the "opponent".

*3.5. ELECTRE.* The ELECTRE methodology is based on the concordance and discordance indices described as follows. First, the data of the decision matrix are defined, and the sums of the weights of all criteria equal to 1 are assumed. For an ordered pair of alternatives $(A_j, A_k)$, the concordance index $c_{jk}$ is the sum of all the weights for those criteria where the performance score of $A_j$ is least as high as that of $A_k$, that is.

$$c_{jk} = \sum_{i:a_{ij} \geq a_{ik}}^{n} W_i, \quad j,k = 1,2,\ldots,n, \; j \neq k. \tag{16}$$

Vividly, the concordance index lies between 0 and 1.

The calculation of the discordance index $d_{jk}$ is a little more complicated: $d_{jk} = 0$ if $a_{ij} > a_{ik}$, $i = 1,\ldots,m$, that is, the discordance index is zero if $A_j$ performs better than $A_k$ on all criteria. Otherwise,

$$d_{jk} = \max_{i=1,\ldots,m} \frac{a_{ik} - a_{ij}}{\max_{j=1,\ldots,n} a_{ij} - \max_{k=1,\ldots,m} a_{ik}}, \quad j,k = 1,\ldots,n, \; j \neq k, \tag{17}$$

that is, for each criterion where $A_k$ outperforms $A_j$, the ratio is computed between the difference in performance level between $A_k$ and $A_j$ and the maximum difference in score on the criterion concerned between any pair of alternatives. The maximum of these ratios (which must lie between 0 and 1) is the discordance index. A concordance threshold $c^*$ and discordance threshold $d^*$ are then defined such that $0 < d^* < c^* < 1$. Then, $A_j$ outranks $A_k$ if the $c_{jk} > c^*$ and $d_{jk} < d^*$, that is, the concordance index is above and the discordance index is below its threshold, respectively.

This outranking described as a partial ranking on the set of alternatives. The set of all alternatives that outrank at least one other alternative are considered and are themselves not outranked. This set contains the promising alternatives for this decision problem. Interactively changing the level thresholds has shown changing in the size of this set.

The ELECTRE I method is utilized to construct a partial ranking and determine on a set of promising alternatives. ELECTRE II is applied for ranking the alternatives. In ELECTRE III an outranking degree is implemented, representing an outranking creditability between two alternatives which makes this method more [53].

## 4. Assessing the Performance of the Private Banks of Iran

We employ four perspectives as a framework for assessing the standards of performance (Figure 4). Based on this framework, the research uses the FAHP to weight the fuzzy indexes and utilizes the analytic tools TOPSIS, VIKOR, and ELECTRE to assess the performance of the banks based and to reduce the distance among the three banks.

*4.1. BSC Performance Evaluation Criteria Framework.* We summarize 51 indexes of the banking performance assessment by using four perspectives of the BSC and reviewing the related literature. we used a questionnaire to homogenize the proportions of the assessment indexes. A committee consisting of 12 academicians and 12 experiential banking experts has selected 21 assessment indexes. Figure 4 presents the hierarchical framework of performance assessment index of the BSC (4 perspectives and 21 indexes) for banking and their titles. We grouped these 21 assessment indexes in 4 BSC perspectives. The financial perspective consistss of 6 indexes

Table 2: fuzzy weight of BSC performance indexes by FAHP.

| Criteria (dimension and index) | Local weights | Overall weights | BNP* | Standard BNP** | Rank |
|---|---|---|---|---|---|
| $F$ | (0.137,0.192,0.273) | | 0.20 | *0.26* | 2 |
| $F_1$ | (0.211,0.293,0.411) | (0.029,0.056,0.112) | 0.31 | 0.07 | 7 |
| $F_2$ | (0.164,0.222,0.310) | (0.029,0.056,0.112) | 0.23 | 0.04 | 15 |
| $F_3$ | (0.229,0.330,0.475) | (0.038,0.077,0.155) | 0.35 | 0.09 | 2 |
| $F_4$ | (0.235,0.333,0.471) | (0.030,0.063,0.129) | 0.35 | 0.07 | 4 |
| $F_5$ | (0.210,0.308,0.437) | (0.020,0.042,0.083) | 0.32 | 0.05 | 9 |
| $F_6$ | (0.229,0.312,0.424) | (0.030,0.063,0.129) | 0.32 | 0.07 | 6 |
| $C$ | (0.164,0.232,0.327) | | 0.24 | *0.32* | 1 |
| $C_1$ | (0.289,0.392,0.537) | (0.048,0.091,0.176) | 0.41 | 0.11 | 1 |
| $C_2$ | (0.202,0.278,0.376) | (0.033,0.065,0.123) | 0.29 | 0.07 | 5 |
| $C_3$ | (0.158,0.230,0.327) | (0.022,0.044,0.089) | 0.24 | 0.05 | 8 |
| $C_4$ | (0.261,0.355,0.483) | (0.034,0.067,0.132) | 0.37 | 0.08 | 3 |
| $C_5$ | (0.191,0.274,0.373) | (0.018,0.038,0.075) | 0.28 | 0.04 | 11 |
| $P$ | (0.010,0.136,0.198) | | 0.18 | *0.23* | 3 |
| $P_1$ | (0.200,0.279,0.379) | (0.019,0.039,0.076) | 0.29 | 0.05 | 10 |
| $P_2$ | (0.168,0.246,0.354) | (0.013,0.028,0.056) | 0.26 | 0.03 | 17 |
| $P_3$ | (0.136,0.190,0.267) | (0.013,0.026,0.051) | 0.20 | 0.03 | 18 |
| $P_4$ | (0.150,0.217,0.304) | (0.012,0.024,0.048) | 0.22 | 0.03 | 19 |
| $P_5$ | (0.089,0.126,0.174) | (0.012,0.024,0.047) | 0.13 | 0.03 | 20 |
| $L$ | (0.095,0.139,0.202) | | 0.15 | *0.19* | 4 |
| $L_1$ | (0.135,0.198,0.291) | (0.011,0.022,0.046) | 0.21 | 0.03 | 21 |
| $L_2$ | (0.134,0.192,0.274) | (0.018,0.037,0.075) | 0.20 | 0.04 | 12 |
| $L_3$ | (0.166,0.226,0.325) | (0.016,0.031,0.066) | 0.24 | 0.04 | 13 |
| $L_4$ | (0.166,0.226,0.325) | (0.015,0.031,0.060) | 0.23 | 0.04 | 16 |

* BNP (Best nonfuzzy performance) = $[(U/L) + (M/L)]/3 + L$.
**STD_BNP: standardized BNP.

($F_1 - F_6$), the customer perspective consist of 5 indexes ($C_1 - C_5$), the internal processes perspective has 5 indexes ($P_1 - P_5$), and the development and learning perspective has 5 indexes ($L_1 - L_5$) as well.

*4.2. The Weight of Assessment Criterion.* Based on the BSC hierarchical framework for the performance assessment index, we distributed the fuzzy AHP questionnaires which used Triangular Fuzzy Numbers (TNF) among the experts of the industry to achieve their expert opinion. Table 2 lists the relative importance (fuzzy weights) of each of the performance indexes by FAHP. The results reveal that the most important of the four perspectives is the customer perspective (0.315), and then is the financial perspective (0.263), after that is the internal processes (0.232), and the last is the development and learning (0.190). The five important assessment indexes consist of customer satisfaction ($C_1$), return on assets ($F_3$), customer retention rate ($C_4$), earning per share ($F_4$), and the profit per customer ($C_2$), while the least important indexes are the customer service replication ($L_1$), sales performance ($P_5$), and the rational processes ($P_4$).

*4.3. Ranking of the Banking Performance.* This research has taken three private Iranian banks (in abbreviation titles of S bank, P bank, and E bank) as an illustrative example and the experts have evaluated them based on the evaluation performance criterion. Although, there are differences of subjective judgments between experts' viewpoints and discretions, the overall employed evaluation of fuzzy judgments was combination of various mangers and experts opinions in order to achieve objective and comprehensive evaluation. We use the five linguistic variables, "very dissatisfied", "not satisfied", "fair", "satisfied," and "very satisfied" to measure the banking performance according to the evaluation criteria. A TFN presents each linguistic variable in a range of 0–100. Various experts have integrated the average fuzzy judgment values of each criteria of the three banks and Table 3 presents this findings.

Then, the final fuzzy synthetic judgment of the three banks is deduced from the fuzzy criteria weights (Table 2) and the fuzzy judgment values (Table 3). Consequently, based on FAHP, we employed the three MCDM analytical tools TOPSIS, VIKOR, and ELECTRE, respectively, to rank the banking performance. Referring to Table 2, the paper has integrated the BNP values to average fuzzy judgments of various experts. Table 4 contains these results.

The relative closeness (RCi) to the ideal solution and evaluation result by TOPSIS is presented in Table 5. The relative closeness values for three banks are E bank (RC = 0.57), P bank (RC = 0.515), and S bank (RC = 0.481).

Table 3: Average of fuzzy judgments value of experts.

| S bank | P bank | E bank | Indexes |
|---|---|---|---|
| 69.44 | 73.33 | 81 | $F_1$ |
| 50 | 70.8 | 62.15 | $F_2$ |
| 49.75 | 60.89 | 50.5 | $F_3$ |
| 61.25 | 45.9 | 78.63 | $F_4$ |
| 68.6 | 57.54 | 64.42 | $F_5$ |
| 60.33 | 70.35 | 73.67 | $F_6$ |
| 60.69 | 70.81 | 79.33 | $C_1$ |
| 64.6 | 70 | 68.63 | $C_2$ |
| 61.33 | 67.60 | 70.33 | $C_3$ |
| 59.75 | 63.63 | 71.60 | $C_4$ |
| 55.5 | 53.25 | 72.43 | $C_5$ |
| 67.83 | 61.53 | 74.6 | $P_1$ |
| 64.75 | 66.33 | 70.12 | $P_2$ |
| 62.63 | 60 | 59.41 | $P_3$ |
| 56.53 | 71.7 | 68.30 | $P_4$ |
| 75.33 | 69.34 | 52.13 | $P_5$ |
| 70.33 | 66.67 | 75 | $L_1$ |
| 71.94 | 59.66 | 80.81 | $L_2$ |
| 50 | 55.85 | 60.33 | $L_3$ |
| 53.30 | 79.44 | 72.50 | $L_4$ |
| 74.66 | 65.33 | 71.70 | $L_5$ |

Table 4: Indexes weight of three banks by FAHP.

| S bank | P bank | E bank | Indexes |
|---|---|---|---|
| 0.54 | 0.57 | 0.63 | $F_1$ |
| 0.47 | 0.66 | 0.58 | $F_2$ |
| 0.52 | 0.65 | 0.54 | $F_3$ |
| 0.56 | 0.42 | 0.76 | $F_4$ |
| 0.62 | 0.52 | 0.58 | $F_5$ |
| 0.49 | 0.75 | 0.60 | $F_6$ |
| 0.55 | 0.56 | 0.62 | $C_1$ |
| 0.55 | 0.60 | 0.58 | $C_2$ |
| 0.57 | 0.63 | 0.52 | $C_3$ |
| 0.56 | 0.53 | 0.63 | $C_4$ |
| 0.53 | 0.50 | 0.69 | $C_5$ |
| 0.57 | 0.52 | 0.63 | $P_1$ |
| 0.56 | 0.57 | 0.60 | $P_2$ |
| 0.60 | 0.57 | 0.57 | $P_3$ |
| 0.50 | 0.63 | 0.60 | $P_4$ |
| 0.57 | 0.60 | 0.45 | $P_5$ |
| 0.57 | 0.54 | 0.61 | $L_1$ |
| 0.58 | 0.48 | 0.65 | $L_2$ |
| 0.52 | 0.58 | 0.63 | $L_3$ |
| 0.45 | 0.66 | 0.60 | $L_4$ |
| 0.61 | 0.53 | 0.59 | $L_5$ |

These RC values mean that E bank has the smallest gap for achieving the desired level among the three banks, whereas the S bank has the largest gap. Similarly, we used VIKOR to rank the banking performance based on the fuzzy weights

Table 5: The separa

| Final ranking | CL | $d_i^-$ | $d_i^+$ | Banks |
|---|---|---|---|---|
| 1 | 0.57 | 0.49 | 0.73 | E bank |
| 2 | 0.51 | 0.52 | 0.49 | P bank |
| 3 | 0.48 | 0.53 | 0.49 | S bank |

Table 6: The performance matrix with the best value $x_j^+$ and worst value $x_j^-$ by VIKOR.

| Indexes | bank S | bank P | bank E | $x_i^-$ | $x_i^+$ |
|---|---|---|---|---|---|
| $F_1$ | 69.44 | 73.33 | 81.00 | 69.44 | 81.00 |
| $F_2$ | 50.00 | 70.80 | 62.15 | 50.00 | 70.80 |
| $F_3$ | 49.75 | 60.89 | 50.50 | 49.75 | 60.89 |
| $F_4$ | 61.25 | 45.90 | 78.63 | 45.90 | 78.63 |
| $F_5$ | 68.60 | 57.54 | 64.42 | 57.54 | 68.60 |
| $F_6$ | 60.33 | 70.35 | 73.67 | 60.33 | 73.67 |
| $C_1$ | 60.69 | 70.81 | 79.33 | 60.69 | 79.33 |
| $C_2$ | 64.60 | 70.00 | 68.63 | 64.60 | 70.00 |
| $C_3$ | 61.33 | 67.60 | 70.33 | 61.33 | 70.33 |
| $C_4$ | 59.75 | 63.63 | 71.60 | 59.75 | 71.60 |
| $C_5$ | 55.50 | 53.25 | 72.43 | 53.25 | 72.43 |
| $P_1$ | 67.83 | 61.53 | 74.60 | 61.53 | 74.60 |
| $P_2$ | 64.75 | 66.33 | 70.12 | 64.75 | 70.12 |
| $P_3$ | 62.63 | 60.00 | 59.41 | 59.41 | 62.63 |
| $P_4$ | 56.53 | 71.70 | 68.30 | 56.53 | 71.70 |
| $P_5$ | 75.33 | 69.34 | 52.13 | 52.13 | 75.33 |
| $L_1$ | 70.33 | 66.67 | 75.00 | 66.67 | 75.00 |
| $L_2$ | 71.94 | 59.66 | 80.81 | 59.66 | 80.81 |
| $L_3$ | 50.00 | 55.85 | 60.33 | 50.00 | 60.33 |
| $L_4$ | 53.50 | 79.44 | 72.50 | 53.50 | 79.44 |
| $L_5$ | 74.66 | 65.33 | 71.70 | 65.33 | 74.66 |

Table 7: The $Q_i$ values with $v = 0, 0.5, 1$ and preference order ranking by VIKOR.

| $Q_i$ ($v = 1$) | $Q_i$ ($v = 0.5$) | $Q_i$ ($v = 0$) | Banks |
|---|---|---|---|
| 0.42 (1) | 0.39 (1) | 0.36 (1) | E bank |
| 0.44 (2) | 0.43 (2) | 0.41 (2) | P bank |
| 0.49 (3) | 0.47 (3) | 0.43 (3) | S bank |

of the BSC performance evaluation indexes by FAHP as presented in Table 2.

Table 6 shows the performance matrix given with the best value $x_j^+$ (aspired levels) and the worst value $x_j^-$ (worst levels). Then, $Q_i$ values (with $v = 0, 0.5, 1$) are computed and Table 7 shows the preference ranking order. The performance ranking order of the three banks by VIKOR in basic way ($v = 0.5$) is E Bank ($Q_i = 0.398$) > P Bank ($Q_i = 0.43$) > S Bank ($Q_i = 0.471$). The results of two other ways for computing $Q_i$ (with $v = 0.5, 1$) are the same as the basic one ($v = 0.5$).

Ultimately, the final values and preference order ranking by these two MCDM methods (TOPSIS and VIKOR) are

Table 8: Summary of final preference order ranking by VIKOR and TOPSIS.

| VIKOR | TOPSIS | Banks |
|---|---|---|
| 0.39 (1) | 0.57 (1) | E bank |
| 0.43 (2) | 0.51 (2) | P bank |
| 0.47 (3) | 0.48 (3) | S bank |

summarized in Table 8, which indicates that the ranking results by TOPSIS and VIKOR are identical. However, the normalized value in the VIKOR method does not depend on the assessment unit of a criterion function, while the vector normalization value at TOPSIS, may depend on the evaluation unit. TOPSIS introduces two reference points but it does not consider the relative importance of distances from these two points. VIKOR uses linear normalization but TOPSIS method uses vector normalization [51].

The last method is the ELECTRE (Elimination ET Choix Traduisant la REalité).This method is an outranking relations approach and Roy has developed this method in Europe. An outranking relation allows to conclude that an alternative outranks an alternative b if there are enough arguments to confirm that it is at least as good as b, while there is no essential reason to refuse this statement. To develop the outranking relation, the decision maker, in collaboration with the decision analyst, must specify the weights of the evaluation criteria, as well as some technical parameters (preference, indifference, and veto thresholds). The definition of these parameters enables the examination of whether there is a sufficient majority of criteria for which is better than b (concordance) and if the unfavorable deviations for the rest of the criteria (discordance) are not too high. In this case it is possible to conclude that alternative outranks alternative b. Furthermore, through this modeling procedure it is possible to identify the cases where the performances of two alternatives on the evaluation criteria differ significantly, thus making impossible their comparison (incomparability). In first step in ELECTRE technique the indexes are normalized, as Table 9 shows as well and we utilized the weight of Tables 2 and 6.

A concordance and discordance index set is first defined for every pair of alternatives

$$\begin{aligned}
S_{12} &= \{F_4, F_5, C_5, P_1, P_3, P_5, L_1, L_2, L_5\},\\
S_{13} &= \{F_5, P_3, P_5, L_5\},\\
S_{21} &= \{F_1, F_2, F_3, F_6, C_1, C_2, C_3, C_4, P_4, L_3, L_4\},\\
S_{23} &= \{F_2, F_3, C_2, P_4, P_5, L_4\},\\
S_{31} &= \{F_1, F_2, F_3, F_4, F_6, C_1, C_2, C_3, C_4, C_5, P_1, P_2,\\
&\qquad P_4, L_1, L_2, L_3, L_4\},\\
S_{32} &= \{F_1, F_4, F_5, F_6, C_1, C_3, C_4, C_5, P_1, P_2,\\
&\qquad L_1, L_2, L_3, L_5\}.
\end{aligned} \tag{18}$$

Second, for each pair, the DM's weights for the corresponding concordance set are summed to arrive at a global concordance index

$$\begin{aligned}
I_{12} &= W_{F4}, W_{F5}, W_{C5}, W_{P1}, W_{P3}, W_{P5}, W_{L1}, W_{L2}, W_{L5}\\
&= 0.041 + 0.03 + 0.023 + 0.026 + 0.018\\
&\quad + 0.018 + 0.015 + 0.025 + 0.023 = 0.218,\\
I_{13} &= W_{F5}, W_{P3}, W_{P5}, W_{L5}\\
&= 0.03 + 0.018 + 0.018 + 0.023 = 0.089,\\
I_{21} &= W_{F1}, W_{F2}, W_{F3}, W_{F6}, W_{C1}, W_{C2},\\
&\quad W_{C3}, W_{C4}, W_{P4}, W_{L3}, W_{L4}\\
&= 0.037 + 0.024 + 0.059 + 0.04 + 0.061\\
&\quad + 0.044 + 0.031 + 0.018 + 0.023 = 0.402,\\
I_{23} &= W_{F2}, W_{F3}, W_{C2}, W_{P4}, W_{P5}, W_{L4}\\
&= 0.024 + 0.059 + 0.044 + 0.018\\
&\quad + 0.017 + 0.023 = 0.184,\\
I_{31} &= W_{F1}, W_{F2}, W_{F3}, W_{F4}, W_{F6}, W_{C1}, W_{C2}, W_{C3}, W_{C4},\\
&\quad W_{C5}, W_{P1}, W_{P2}, W_{P4}, W_{L1}, W_{L2}, W_{L3}, W_{L4}\\
&= 0.041 + 0.021 + 0.049 + 0.053 + 0.042\\
&\quad + 0.068 + 0.043 + 0.032 + 0.049 + 0.030\\
&\quad + 0.028 + 0.019 + 0.017 + 0.016\\
&\quad + 0.028 + 0.024 + 0.021 = 0.582,\\
I_{32} &= W_{F1}, W_{F4}, W_{F5}, W_{F6}, W_{C1}, W_{C3},\\
&\quad W_{C4}, W_{C5}, W_{P1}, W_{P2}, W_{L1}, W_{L2}, W_{L3}, W_{L5}\\
&= 0.041 + 0.053 + 0.028 + 0.042\\
&\quad + 0.068 + 0.032 + 0.049 + 0.030 + 0.028\\
&\quad + 0.019 + 0.016 + 0.028 + 0.024 + 0.022 = 0.481,
\end{aligned} \tag{19}$$

$$I_{kl} = \begin{pmatrix} - & 0.0218 & 0.089 \\ 0.402 & - & 0.184 \\ 0.582 & 0.481 & - \end{pmatrix}, \qquad \bar{I} = \frac{1.96}{6} = 0.326,$$
$$H = \begin{pmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 1 & - \end{pmatrix}, \tag{20}$$

$$\begin{aligned}
D_{12} &= \{F_1, F_2, F_3, F_6, C_1, C_2, C_3, C_4, P_4, L_3, L_4\},\\
D_{13} &= \{F_1, F_2, F_3, F_4, F_6, C_1, C_2, C_3, C_4, C_5,\\
&\qquad P_1, P_2, P_4, L_1, L_2, L_3, L_4\},\\
D_{21} &= \{F_4, F_5, C_5, P_1, P_3, P_5, L_1, L_2, L_5\},\\
D_{23} &= \{F_1, F_4, F_5, F_6, C_1, C_3, C_4, C_5,\\
&\qquad P_1, P_2, L_1, L_2, L_3, L_5\},\\
D_{31} &= \{F_5, P_3, P_5, L_5\},\\
D_{32} &= \{F_2, F_3, C_2, P_4, P_5, L_4\}.
\end{aligned} \tag{21}$$

Table 9: Normalized data of indexes based on ELECTRE method.

| Indexes | S bank | P bank | E bank | Weight | S bank | P bank | E bank |
|---|---|---|---|---|---|---|---|
| $F_1$ | 69.44 | 73.33 | 81.00 | 0.07 | 0.04 | 0.04 | 0.04 |
| $F_2$ | 50.00 | 70.80 | 62.15 | 0.04 | 0.02 | 0.02 | 0.02 |
| $F_3$ | 49.75 | 60.89 | 50.50 | 0.09 | 0.05 | 0.06 | 0.05 |
| $F_4$ | 61.25 | 45.90 | 78.63 | 0.07 | 0.04 | 0.03 | 0.05 |
| $F_5$ | 68.60 | 57.54 | 64.42 | 0.05 | 0.03 | 0.03 | 0.03 |
| $F_6$ | 60.33 | 70.35 | 73.67 | 0.07 | 0.04 | 0.04 | 0.04 |
| $C_1$ | 60.69 | 70.81 | 79.33 | 0.11 | 0.05 | 0.06 | 0.07 |
| $C_2$ | 64.60 | 70.00 | 68.63 | 0.07 | 0.04 | 0.04 | 0.04 |
| $C_3$ | 61.33 | 67.60 | 70.33 | 0.05 | 0.03 | 0.03 | 0.03 |
| $C_4$ | 59.75 | 63.63 | 71.60 | 0.08 | 0.04 | 0.04 | 0.05 |
| $C_5$ | 55.50 | 53.25 | 72.43 | 0.04 | 0.02 | 0.02 | 0.03 |
| $P_1$ | 67.83 | 61.53 | 74.60 | 0.05 | 0.03 | 0.02 | 0.03 |
| $P_2$ | 64.75 | 66.33 | 70.12 | 0.03 | 0.02 | 0.02 | 0.02 |
| $P_3$ | 62.63 | 60.00 | 59.41 | 0.03 | 0.02 | 0.02 | 0.02 |
| $P_4$ | 56.53 | 71.70 | 68.30 | 0.03 | 0.01 | 0.02 | 0.02 |
| $P_5$ | 75.33 | 69.34 | 52.13 | 0.03 | 0.02 | 0.02 | 0.01 |
| $L_1$ | 70.33 | 66.67 | 75.00 | 0.03 | 0.02 | 0.01 | 0.02 |
| $L_2$ | 71.94 | 59.66 | 80.81 | 0.04 | 0.03 | 0.02 | 0.03 |
| $L_3$ | 50.00 | 55.85 | 60.33 | 0.04 | 0.02 | 0.02 | 0.02 |
| $L_4$ | 53.30 | 79.44 | 72.50 | 0.04 | 0.02 | 0.02 | 0.02 |
| $L_5$ | 74.66 | 65.33 | 71.70 | 0.04 | 0.02 | 0.02 | 0.02 |

Similarly, a global discordance index for each pair of alternatives is defined

$$\begin{gathered} NI_{12} = 1, \quad NI_{13} = 1, \quad NI_{21} = 0.965, \\ NI_{23} = 1, \quad NI_{31} = 0.354, \quad NI_{32} = 0.453, \end{gathered} \tag{22}$$

$$\begin{gathered} NI_{kl} = \begin{pmatrix} - & 1 & 1 \\ 0.965 & - & 1 \\ 0.354 & 0.453 & - \end{pmatrix}, \\ \overline{N}I = \frac{4.807}{6} = 0.801, \\ H = \begin{pmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 1 & - \end{pmatrix}, \end{gathered} \tag{23}$$

$$F = \begin{pmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 1 & - \end{pmatrix} \longrightarrow A3 > A2 > A1.$$

Finally, summery ranking by ELECTRE has represented in Table 10. Regarding Table 10, the final values of the three banks calculated ELECTRE are nearly close to TOPISIS, VIKOR results.

Table 10: Summary ranking by ELECTRE.

| ELECTRE | Banks |
|---|---|
| (1) | E bank |
| (2) | P bank |
| (3) | S bank |

*4.4. Gap Analysis Regard to Three Methods.* Here we note that the final results that calculated TOPISIS, VIKOR, and ELECTRE are nearly close to each other. Based on literature of paper, TOPSIS and VIKOR are both based on relative closeness to ideal solution. While the VIKOR method introduces the ranking index based on the measure of closeness to ideal solution, TOPSIS method has chosen the alternative according to the "shortest distance" from the ideal solution and the "farthest distance" from the negative ideal solution. Totally, a comparative method between VIKOR and TOPSIS shows that they use different normalizations and introduce various aggregating functions for ranking the performance that may have different results. ELECTRE is the third method based on outranking relations approach. However the logic of VIKOR and TOPSIS is different from ELECTRE, but the result of all three methods is the same. In this case, the VIKOR and TOPSIS results are identical but VIKOR is a better method of assessment.

*4.5. Discussion.* We use a performance analysis on three banks by FMCDM approach based on the BSC perspectives. We adopt the FAHP and the three MCDM methods (TOPSIS, VIKOR and ELECTRE) in the performance, analysis for computing the fuzzy weights of the indexes, ranking the banking performance and improving the gaps between three banks. The results describe essential findings as follows. Here, the FAHP can not only involve the logical thought of human beings but also focuses on the relative importance of the evaluation criteria at the banking performance.

Based on Table 2 findings, the result of the FAHP analysis shows that the "Customer" dimension is the most significant dimension of the BSC and "customer satisfaction" is the most crucial evaluation index. Hence, in addition to paying attention to financial indexes like ROA and EPS, banks also must ensure that their customers remain loyal to them and develop new markets to attract new customers and sustain their retention. Therefore, based on the fuzzy weights of the evaluation criteria, the performance ranking order of the three banks using TOPSIS method is E Bank > P Bank > S Bank. The VIKOR ranking order result is the same as the TOPSIS and ELECTRE one. Furthermore, the result of VIKOR ranking is more precise and real in comparison to TOPSIS. When comparing the performance of S Bank with two other banks, as shown in Table 3, it is obvious that S Bank has the lowest performance value in the "Customer" dimension while customer is the most critical BSC factor based onto the experts and managers views. As far as mentioned before, S Bank has the poorest performance value in the "market share rate" index too. It means that recognizing market share as a major factor in that bank's growth strategy is vital.

Therefore, S Bank should develop new service items and technology and/or provide more improved promotions to attract new clients in order to remain in banking industry besides retaining existing customers. So, S bank must first focuses on customer satisfaction and then on financial return.

## 5. Conclusions

By summarizing, we can say (1) the experts have integrated and has selected 21 evaluation indexes as the proper banking performance according to BSC perspectives; (2) by applying the FAHP, the ranking order of banking performance is as follow: "*C*: Customer", "*F*: Finance", "*P*: Internal process," and "*L*: Learning and growth." The top five ranking of the evaluation indexes are, respectively, "$C_1$: Customer satisfaction", "$F_3$: Return on assets", "$C_4$: Customer retention rate," "$F_4$: Earnings per share", and "$C_2$: Profit per customer"; (3) ranking of the banking performance of the three banks by adopting the MCDM analytical tools as VIKOR, TOPSIS, and ELECTRE are E Bank, P Bank, and S Bank.

Thus we suggest that (1) although finding proper performance evaluation indexes to fit all experts and organization needs does not seem easy, we recommend tailoring the performance evaluation indexes to meet the organization's overall objective as well as individual goals. (2) We suggest to adopt the performance assessment indexes of the BSC perspectives which may not be mutually independent and other analytical methods like Analytic Network Process (ANP) to solve the interactive relationships among indexes. (3) We suggest to combine the other MCDM tools and several other techniques like Analytic Network Neuron (ANN) and FTOPSIS and compare their finding results. Fourthly, we suggest to utilize other performance evaluation techniques besides BSC like EFQM to achieve more suitable indexes. Finally we propose to investigate this approach in different industries such oil and gas, petrochemical, healthcare, and food for further validation.

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*Research Article*

# Simplified Fuzzy Control for Flux-Weakening Speed Control of IPMSM Drive

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This paper presents a simplified fuzzy logic-based speed control scheme of an interior permanent magnet synchronous motor (IPMSM) above the base speed using a flux-weakening method. In this work, nonlinear expressions of $d$-axis and $q$-axis currents of the IPMSM have been derived and subsequently incorporated in the control algorithm for the practical purpose in order to implement fuzzy-based flux-weakening strategy to operate the motor above the base speed. The fundamentals of fuzzy logic algorithms as related to motor control applications are also illustrated. A simplified fuzzy speed controller (FLC) for the IPMSM drive has been designed and incorporated in the drive system to maintain high performance standards. The efficacy of the proposed simplified FLC-based IPMSM drive is verified by simulation at various dynamic operating conditions. The simplified FLC is found to be robust and efficient. Laboratory test results of proportional integral (PI) controller-based IPMSM drive have been compared with the simulated results of fuzzy controller-based flux-weakening IPMSM drive system.

## 1. Introduction

In recent years, the IPMSM has become increasingly popular for its use in high performance drive (HPD) applications due to desirable features, such as high torque to current ratio, high power to weight ratio, high efficiency, low noise, and robust operation. The advantageous features of the IPMSM for modern drives application are well established [1, 2]. Fast and accurate speed response, quick recovery of speed from any disturbance, and insensitivity to parameter variations are some of the important criteria for HPD systems. These HPDs are used in robotics, rolling mills, machine tools, and so forth. The conventional proportional-integral (PI) and proportional-integral-derivative (PID) controllers have been widely utilized as speed controller in flux-weakening region in IPMSM drive [3]. However, difficulties in obtaining the exact $d$-$q$ axis reactance parameters of the IPMSM make the design approaches for these controllers cumbersome. Moreover, since the operation of the IPMSM is strongly affected by the rotor magnetic saliency, saturation, and armature reaction effects [2], conventional fixed-gain PI and PID controllers are very sensitive to parameter variations, along with step change of command speed, and load disturbance [4–6]. Therefore, an intelligent speed controller demands special attention for the IPMSM drives, which are used in HPD systems.

In an FLC, the system control parameters are adjusted by a fuzzy rule-based system, which is a logical model of the human behavior for process control. The advantages of FLC over the conventional controllers are as follows (1) the design of FLC does not need the exact mathematical model of the system; (2) the FLC is more robust than the conventional controllers; (3) it can handle nonlinear functions of any arbitrary complexity; (4) it is based on the linguistic control rules, which are also the basis of human logic. Many researches have been carried out with the fuzzy algorithms for ac (alternating current) motors drives [5–12]. These works are limited to the cases where the flux-weakening mode of operation is not considered. Researchers have focused their attention on the vector control of IPMSM drive by forcing the $d$-axis current equal to zero, which essentially linearizes the motor model for speed control up to the base speed [13, 14]. There exists a need to extend the fuzzy logic-based control to the practical case IPMSM drive in the flux-weakening region where the IPMSM operates above the base speed. Recently, researchers have turned their attention to the flux-weakening control of IPMSM drives [10–12].

The IPMSM drive operation with vector control scheme strategy is well established [15]. However, the vector control techniques used in IPMSM drives become complicated due to the nonlinearity of the developed torque for nonzero value of $d$-axis current. In real time, the electromagnetic torque and the flux producing $d$-axis current are nonlinear in nature and the generated or back emf (electromotive force) of an IPMSM is directly proportional to the rotor speed. As the rotor speed increases, the back emf increases in the linear fashion since excitation flux is constant due to the permanent magnets. Thus, to reach a desired speed, the terminal voltage must be increased to overcome the back emf. It is the real-time practice that the inverter should be capable of supplying the required voltage by PWM or any other suitable techniques up to the base or rated speed. For the drive operation above the base speed, an indirect flux control method of field-weakening method can be applied to the drive, so that the terminal voltage will remain constant after the base speed. This flux- or field-weakening strategy is very important from the limitation of IPMSM and inverter rating points of view, which optimizes the drive efficiency. However, owing to the permanent magnet construction of the rotor, nothing can be done on the rotor side from the control point of view. It is possible to weaken the field by controlling the stator current in such a way that a direct axis current component in the rotating frame axis can be generated which will oppose the main field produced by the permanent magnet.

The problem with the associated flux-weakening control technique is that its implementation in real time becomes complicated because there exists a complex nonlinear relationship between $d$-axis current and speed and also among $d$- and $q$-axis currents. Some researchers solved this problem by considering look-up table [3]. In this work, these nonlinearties are incorporated in the IPMSM drive system with a simpler expression of $d$-axis and $q$-axis current above the base speed using PI and fuzzy logic controller. The simplified expressions of $d$-axis and $q$-axis currents have been derived using curve fitting method and used in the simulation as well as real-time implementation. The objective of this paper is to present a simplified FLC-based speed controller for the IPMSM drive. The system is designed in such as way as to maintain the high performance drives employing a less complex algorithm of FLC, which reduces the computational burden and allows for real-time implementation above base speed. The proposed IPMSM drive system has been simulated using MATLAB/SIMULINK. Results of PI controller-based IPMSM drive have been compared with those obtained from the FLC-based IPMSM drive. The comparisons confirm the efficacy of the proposed system using FLC-based IPMSM drive system.

The organization of the paper is as follows: Section 2 provides the mathematical modeling of the IPMSM; flux weakening control algorithm is presented in Section 3; Section 4 describes the simplified fuzzy logic control design technique; Section 5 discusses the laboratory implementation of PI-based flux-weakening IPMSM drive; Section 6 presents the performance of the controller through a series of nonlinear simulation results. Concluding remarks and suggestions for future works are given in Section 7.

## 2. IPMSM Dynamics

The mathematical model of an IPMSM drive can be described by the following equations in a synchronously rotating rotor $d$-$q$ reference frame as:

$$\begin{bmatrix} v_d \\ v_q \end{bmatrix} = \begin{bmatrix} R + pL_d & -P\omega_r L_q \\ P\omega_r L_d & R + pL_q \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} + \begin{bmatrix} 0 \\ P\omega_r \psi_f \end{bmatrix}, \quad (1)$$

$$T_e = T_L + J_m p\omega_r + B_m \omega_r, \quad (2)$$

$$T_e = \frac{3P}{2}\left(\psi_f i_q + \left(L_d - L_q\right) i_d i_q\right), \quad (3)$$

where $v_d, v_q$ = $d$- and $q$-axis stator voltages; $i_d, i_q$ = $d$- and $q$-axis stator currents; $R$ = stator per phase resistance; $L_d, L_q$ = $d$- and $q$-axis stator inductances; $J_m$ = moment of inertia of the motor and load; $B_m$ = friction coefficient of the motor; $P$ = number of poles of the motor; $\omega_r$ = rotor speed in angular frequency; $p$ = differential operator $(=d/dt)$; $\psi_f$ = rotor magnetic flux linking the stator; $T_e, T_L$ = electromagnetic and load torques; $J_m$ = moment of inertia of the motor and load.

## 3. Flux-Weakening Algorithm

The steady-state voltage equations are derived from (1)

$$\begin{aligned} v_q &= R_s i_q + \omega_r L_d i_d + \omega_r \psi_f, \\ v_d &= -\omega_r L_q i_q + R_s i_d. \end{aligned} \quad (4)$$

For a limiting case of constant power of zero torque, the $q$-axis current is zero; therefore, the above (4) becomes

$$\begin{aligned} v_q &= \omega_r L_d i_d + \omega_r \psi_f, \\ v_d &= -R_s i_d. \end{aligned} \quad (5)$$

Considering

$$V_s^2 = v_d^2 + v_q^2, \quad (6)$$

one can find the maximum value of speed for the maximum available inverter (also stator) voltage $V_s$ from (5), and (6) as follows

$$\omega_{r\max} = \frac{\sqrt{V_s^2 - R_s^2 i_d}}{\psi_f + i_d L_d}. \quad (7)$$

The denominator of the above (7) must be positive giving condition of maximum stator current to be applied to counter the permanent magnet flux linkages as

$$i_{d\max} \le \left(-\frac{\psi_f}{L_d}\right). \quad (8)$$

Equations (7) and (8) are considered very important for real-time implementation because these expressions provide upper limiting values of speed and $d$-axis current for a given IPMSM. By considering stator resistance $R_s = 0$, voltage-limited ellipse equation can also be derived from (4), and (7) as

$$\left(\frac{i_d + \psi_f/L_d}{(V_s/\sqrt{2})/\omega_r L_d}\right)^2 + \left(\frac{i_q}{(V_s/\sqrt{2})/\omega_r L_q}\right)^2 = 1. \quad (9)$$

Plots of $q$-axis versus $d$-axis currents are shown in Figure 1 for speed range 200 rad/sec to 850 rad/sec (base speed 188 rad/sec) for a maximum inverter voltage of 250 volt.

Considering

$$I_s = \sqrt{i_d^2 + i_q^2}, \quad (10)$$

the expression of $d$-axis current can be derived from (5) as

$$i_d = \frac{-L_d\psi_f \pm L_q\sqrt{\left[\left\{\left(L_d^2 - L_q^2\right)\left(V_s/\omega_r L_q\right)^2 - I_s\right\} + \psi_f^2\right]}}{L_d^2 - L_q^2}. \quad (11)$$

With the IPMSM data given in the appendix, the expression of $i_d$ in (11) has been simplified for the real-time implementation using curve fitting method for a working operating range of speed of 188 rad/sec (base speed) to 276 rad/sec (doubling base speed) as

$$i_d = -0.000119\omega_r^2 - 0.080316\omega_r + 10.5269. \quad (12)$$

Using (10), the expression for $q$-axis current $i_q$ has been obtained which is also simplified using curve fitting method and is given as

$$i_q = -0.260375 i_d^2 - 0.244651 i_d + 3.4422727. \quad (13)$$

Equations (12) and (13) are the key equations used for the flux-weakening control of IPMSM. Block diagram in Figure 2 shows the control scheme of the motor drive.

Using (12), the command $d$-axis current $i_d^*$ is computed first, subsequently reference $d$-axis current $i_q^*$ is calculated using (13). The command torque is obtained from a PI and fuzzy type speed controller. An estimated torque is calculated using (3), (12), and (13) and compared with the command torque. As long as the command torque is greater than the estimated torque, (12), and (13) are used to compute the three phase reference currents with the vector rotator. If the command torque is less than estimated torque, reference $q$-axis current is calculated using the command torque rather than the estimated torque. The speed error is processed by the fuzzy controller to generate the torque-producing current component command $i_q^*(n)$. The complete design of fuzzy controller is shown in Section 4. The hysteresis current controller compares the reference three phase currents with actual currents and generates base signals for the transistorized inverters.

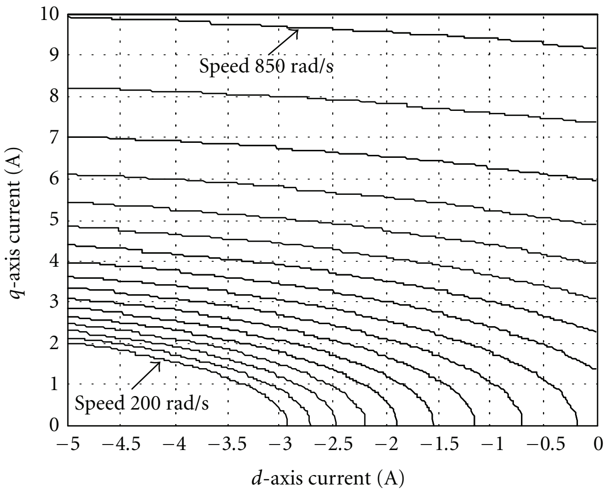



Figure 1: Voltage-limited ellipses of IPMSM for flux-weakening method of speed control.

## 4. Design of Simplified FLC for IPMSM

In this work, the fuzzy logic controller is based on the fuzzy set and fuzzy logic theory introduced by Zadeh, with the vector control techniques incorporated with the FLC to obtain the highest torque sensitivity of the IPMSM drive. The vector control technique is formulated within the $d$-$q$ synchronously rotating rotor reference frame. The complexity of the control arises due to the nonlinear nature of the torque expressed by (3). Moreover, $L_d$ and $L_q$ undergo significant variations at different steady-state and dynamic loading condition [2]. The dynamic model of the IPMSM may be rewritten from (1) to (3) as [16–18],

$$p i_q = \frac{\left(v_q - R i_q - K_b\omega_r\right)}{L_q}, \quad (14)$$

$$p\omega_r = \frac{(T_e - T_L - B_m\omega_r)}{J_m}, \quad (15)$$

where $K_b = p\Psi_f$. As the FLC can handle any nonlinearity, one can consider the load as unknown nonlinear mechanical characteristics. The load can be modeled using the following equation as [4]:

$$T_L = A\omega_r^2 + B\omega_r + C, \quad (16)$$

where $A$, $B$, and $C$ are arbitrary constants. To make the control task easier, the equation of an IPMSM expressed as a single input nd single output system by combining (15) and (16) in continuous time domain form as [16–18],

$$J_m\frac{d\omega_r}{dt} = T_e - (B_m + B)\omega_r - A\omega_r^2 - C. \quad (17)$$

A small incremental change $\Delta T_e$ of the electrical torque $T_e$ results in a corresponding change $\Delta\omega_r$ of the speed $\omega_r$, then (17) can be rewritten as

$$J_m\frac{d(\Delta\omega_r)}{dt} = \Delta T_e - (B_m + B)(\Delta\omega_r) - A(\Delta\omega_r^2). \quad (18)$$

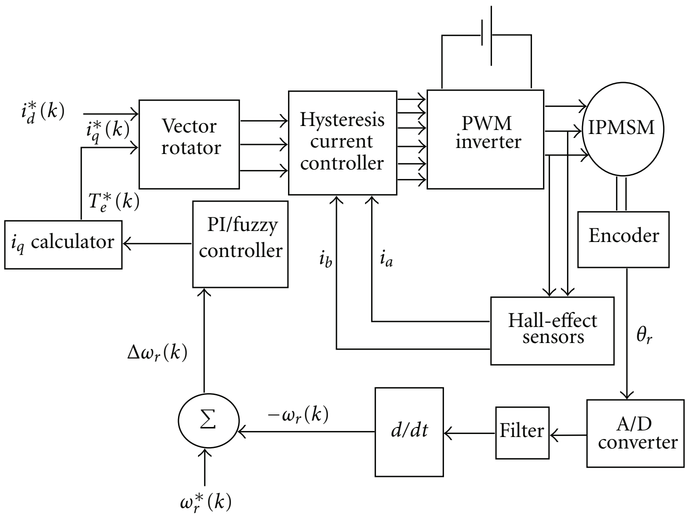



Figure 2: Block diagram of flux-weakening speed control of IPMSM drive.

By replacing all the continuous quantities of (18) by their finite differences, the discrete time signal model of the simplified IPMSM with nonlinear load can be given as [16–18]

$$\Delta T_e(n) = \frac{-J_m}{t_s}\Delta e(n) + (B_m + B)\Delta\omega_r(n) + A\{\Delta\omega_r(n)\}^2. \tag{19}$$

Hence,

$$T_e(n) = \int_{\text{discrete}} \Delta T_e(n) = f(\Delta e(n), \Delta\omega_r(n)), \tag{20}$$

where $\Delta e(n) = \Delta\omega_r(n) - \Delta\omega_r(n-1)$ is the change of speed error, $\Delta\omega_r(n) = \omega_r^*(n) - \omega_r(n)$ is the present sample of speed error, $\Delta\omega_r(n-1)$ is the past sample of speed error, $\omega_r(n)$ is the present sample of actual speed, $\omega_r^*(n)$ is the present sample of command speed, $t_s$ is the sampling time interval, and $f$ denotes the nonlinear function. Thus, the purpose of using the FLC is to map the nonlinear functional relationship between electrical torque $T_e$ and rotor speed $\omega_r$. From this command torque $T_e$, (12) and (13) are used to calculate the necessary $q$- and $d$-axis currents to produce the rotor speed $\omega_r$. In real time, the motor position information and output of the simplified FLC in terms of the command $q$-axis and $d$-axis currents $i_q^*$ and $i_d^*$ are used to get the motor command phase current $i_a^*$, $i_b^*$, and $i_c^*$ by using Park's transformation.

The model of the IPMSM expressed by (20) defines the input and output linguistic variables for the FLC of the IPMSM drive. According to (20), the input of the proposed FLC is the present sample of speed error and the change of speed error, which is the difference between present and past sample of speed errors. However, it has been observed that the effect of the inclusion of the change of speed error on the motor speed response is negligible and does not produce an improvement in motor drive performance in measure with the necessary increase in computational burden as compared to when it is omitted. The omission of the $\Delta e(n)$ term produces an FLC-based drive with acceptably responsive and accurate tracking of the command speed. Thus, the input vectors of the FLC can be reduced to only $\Delta\omega_r(n)$, producing a much simplified FLC as compared to input vectors of $\Delta\omega_r(n)$ and $\Delta e(n)$ with the nonsimplified system. This simplification reduces computational burden and lowers the computer power required to implement the FLC scheme in real-time. Thus, this simplified FLC is a significant factor for real time implementation of the laboratory IPMSM drive system [17].

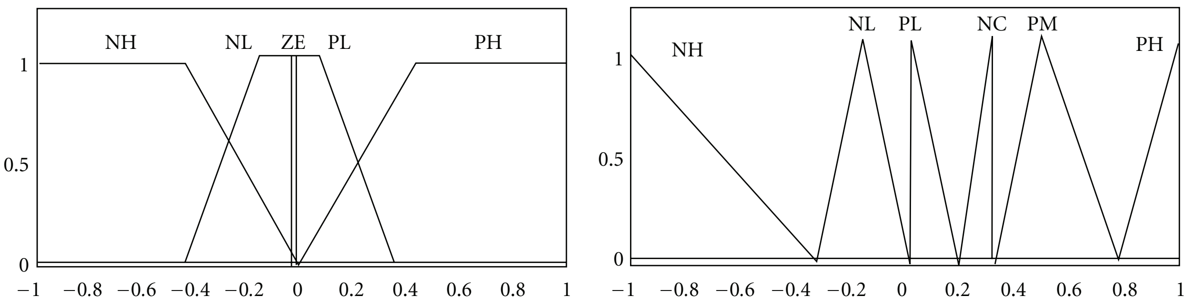
Table 1: Fuzzy rule-based matrix.

| $\Delta\omega_r$ | PH | PL | NL | NH | ZE |
|---|---|---|---|---|---|
| $T_e$ | PH | PL | NL | NH | NC |

The block diagram of the proposed FLC-based IPMSM drive incorporating field-weakening method is shown in Figure 2. Next, scaling factors $K_w$ and $K_i$ are chosen for fuzzification and obtaining the appropriate actual command current. The factor $K_w$ is chosen so that the normalized value of speed error $\Delta\omega_r$ remains within the limit of $\pm 1$. The factor $K_i$ is chosen so that the rated current $i$ can be produced by the controller for rated conditions. In this paper, the constants are taken as $K_w = \omega_r^*$ (command speed) and $K_i = 10$ in order to get the optimum drive performance. After selecting the scaling factors, the next step is to choose the membership functions of $\Delta\omega_r$ and $i_{qn}^*$, which form an important element of the FLC. The membership functions used for the input and output fuzzy sets are shown in Figure 3. The trapezoidal functions are used as membership functions for all the fuzzy sets except the fuzzy set ZE (zero) of the input vectors. The triangular membership functions are used for the fuzzy set ZE of the input vectors and all the fuzzy sets of the output vector. The trapezoidal and triangular functions are used to reduce the computation for online implementation. The rules used for the proposed FLC algorithm are as follows

(i) if $\Delta\omega_r$ is PH (positive high), $T_e$ is PH (positive high);

(ii) if $\Delta\omega_r$ is PL (positive low), $T_e$ is PM (positive medium);




(a) (b)

FIGURE 3: Member functions for (a) speed error $\Delta\omega_{rn}$ and (b) command torque $T_e^*$.

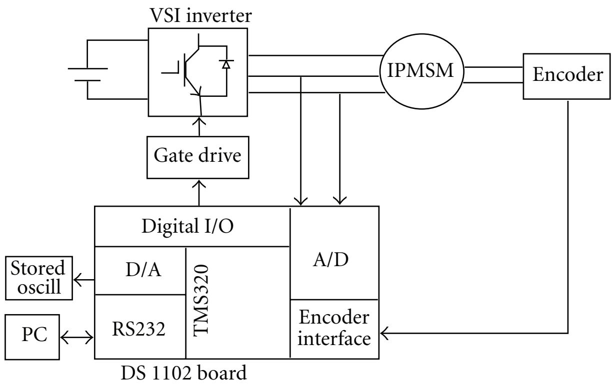



FIGURE 4: Hardware schematic for real-time implementation.

TABLE 2: Machine parameters.

| Motor rated power | 3-phase, 1 hp |
|---|---|
| Rated voltage | 250 V |
| Rated current | 3 A |
| Rated frequency | 60 Hz |
| Pole pair number ($P$) | 2 |
| $d$-axis inductance, $L_d$ | 42.44 mH |
| $q$-axis inductance, $L_q$ | 79.57 mH |
| Stator resistance, $R$ | 1.93 Ω |
| Motor inertia, $J_m$ | 0.003 kg m$^2$ |
| Friction coefficient, $B_m$ | 0.001 Nm/rad/sec |
| Magnetic flux constant, $\psi_f$ | 0.311 volts/rad/sec |

(iii) if $\Delta\omega_r$ is NL (negative low), $T_e$ is NL (negative low);

(iv) if $\Delta\omega_r$ is NH (negative high), $T_e$ is NH (negative high).

Mathematically, the trapezoidal membership function can be defined as

$$\text{Trapezoidal}: \; f(x; a, b, c, d) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \dfrac{d-x}{d-c} & c \leq x \leq d \\ 0 & x \geq d. \end{cases} \quad (21)$$

The triangular membership function can be obtained from the trapezoidal function by setting $b = c$. The rules used for the proposed FLC algorithm are shown in Table 1. Based on the rules, the fuzzy-rule-based matrix is shown above. For this study, Mamdani-type fuzzy inference is used [15]. The values of the constants, membership functions, fuzzy sets for the input/output variables, and the rules used in this study are selected by trial and error to obtain the optimum drive performance. In this study, the center of gravity defuzzification is used. The output function is given as

$$i = \frac{\sum_{k=1}^{N} i\mu_{C(k)}(i)}{\sum_{k=1}^{N} \mu_{C(k)}(i)_d}, \quad (22)$$

where $N$ is the total number of rules and $\mu_{C(k)}(i)$ denotes the output membership grade for the $k$th rule with the output subset $C$.

## 5. Laboratory Implementation of PI-Based Flux-Weakening IPMSM Rive

The complete IPMSM drive system has been implemented in the laboratory for a 1-hp laboratory IPMSM using DSPACE DSP (digital signal processing) controller board [8]. The complete hardware schematic for real-time implementation of the IPMSM drive is shown in Figure 4. Machine parameters are given in the Appendix. The DSP board is installed in a PC with uninterrupted communication capabilities through dual-port memory. The DSP has been supplemented by a set of on-board peripherals used in digital control systems, such as A/D (analog to digital), D/A (digital to analog) converters, and incremental encoder interfaces. The DS 1102 is also equipped with a TI TMS320P14, 16-bit microcontroller DSP that acts as a slave processor and is used for some special purposes.

In this work, slave processor is used for digital I/O configuration. The actual motor currents are measured by the Hall-effect sensors which have good frequency response and fed to the DSP board through A/D converter. As the motor neutral is isolated, only two-phase currents are fed back, and the other phase current is calculated from them. Three phase

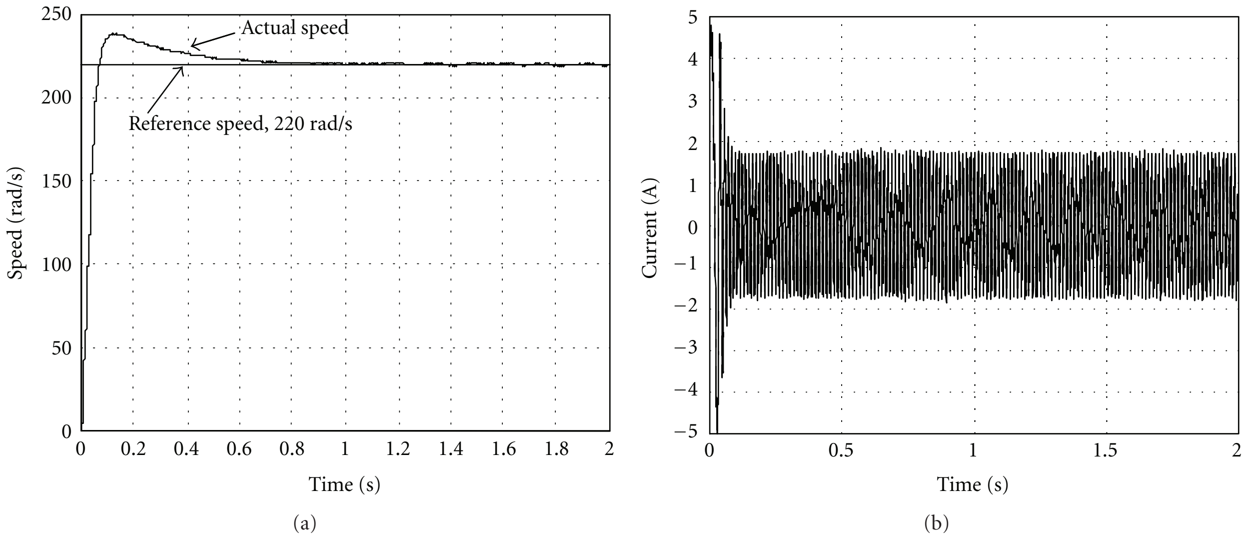



Figure 5: (a) Simulated speed of IPMSM drive for flux-weakening control with reference speed 220 rad/s using PI controller. (b) Simulated current of IPMSM drive for flux-weakening control with reference speed 220 rad/s using PI controller.

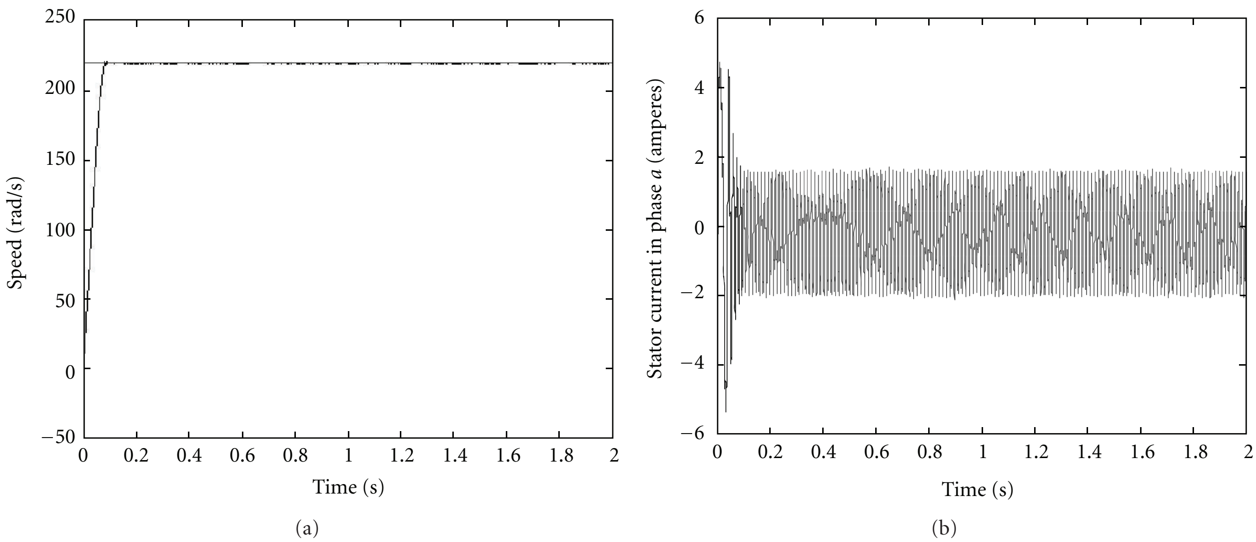
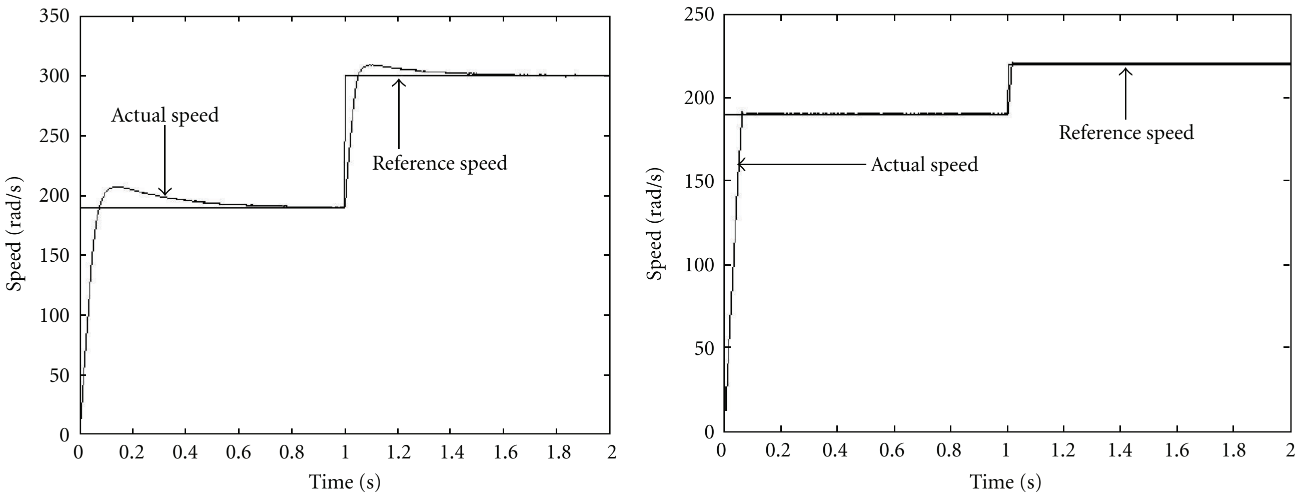



Figure 6: (a) Simulated speed of IPMSM drive for flux-weakening control with reference speed 220 rad/s using fuzzy controller. (b) Simulated current of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using fuzzy controller.

reference currents are generated utilizing reference $q$- and $d$-axis currents and rotor position angle obtained through encoder mounted on the shaft of the motor. Computed three phase reference currents are converted to upper and lower hysteresis by adding and subtracting a reselected band. Hysteresis currents are compared with actual motor currents, and PWM (pulse-width modulation) base drive signals are generated. All computations for generating reference currents and consequently base drive signals for the inverter are done by developing a program in ANSI C programming language. The program is compiled using Texas Instrument C compiler and downloaded to the DSP controller board. The sampling frequency for experimental implementation of the proposed drive is 10 kHz.

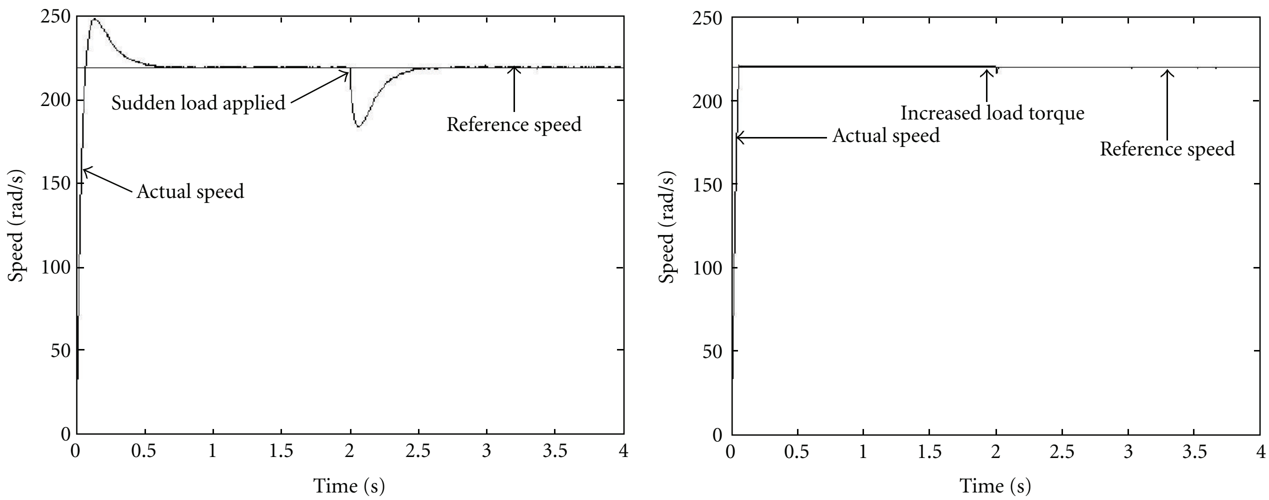
## 6. Results and Discussions

The performance of FLC-based flux-weakening control-based IPMSM drive has been evaluated by computer simulation. The speed and current responses are observed under different operating conditions such as various command speeds, sudden application of load, step change in command speed, and at different loading conditions. Some of the sample results are presented in this paper. Figures 5 and 6 show the simulated starting performance of the drive with PI- and FLC-based drive systems, respectively, for flux-weakening control-based IPMSM drive system with reference speed of 220 rad/sec at a load of 2 N-m. Although the PI controller is tuned to give an optimum response, the fuzzy




(a) (b)

Figure 7: (a) Simulated speed of IPMSM drive for flux-weakening control due to step change of reference speed using PI controller. (b) Simulated speed of IPMSM drive for flux-weakening control due to step change of reference speed using fuzzy controller.

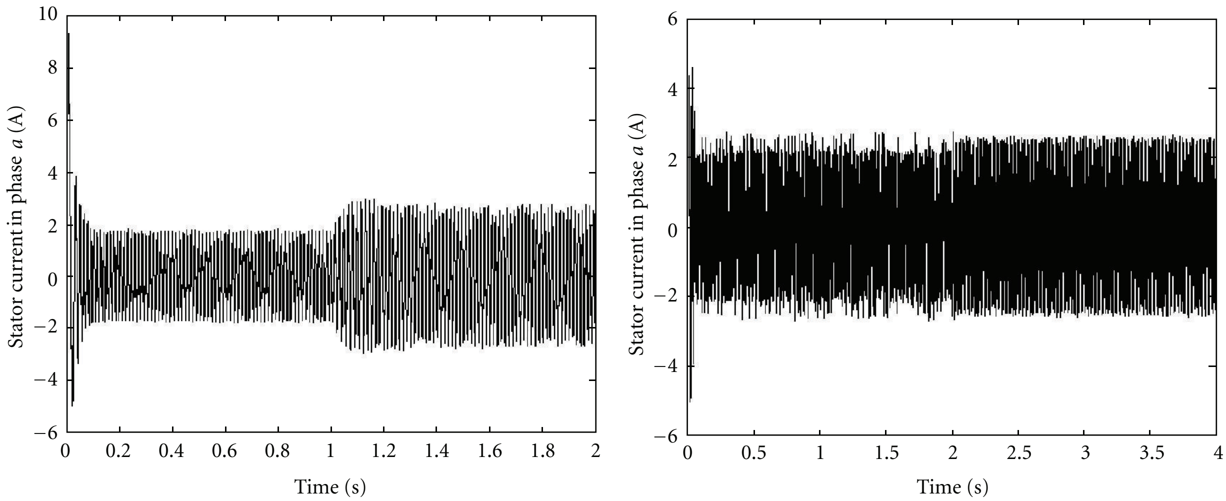



(a) (b)

Figure 8: (a) Simulated speed of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using PI controller with sudden load applied. (b) Simulated speed of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using fuzzy controller with sudden load applied.

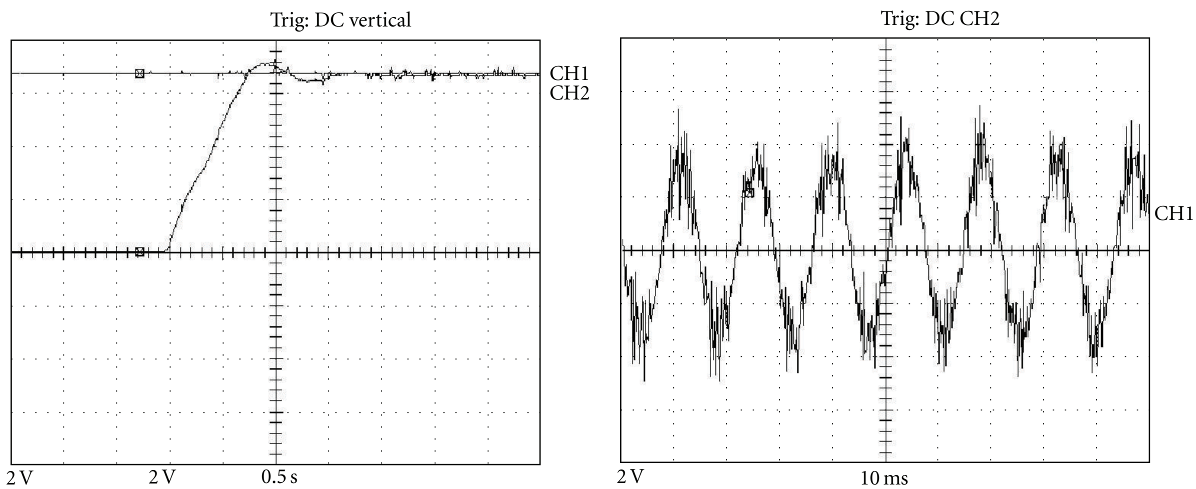
controller yielded better performances in terms of faster response time without any overshoot and lower starting current.

Figures 7(a) and 7(b) show speed responses of the drive system using PI and FLC, respectively, with a step change in the reference speed. It is evident from Figures 7(a) and 7(b) that the proposed FLC-based drive system can follow the command speed without any overshoot and steady-state error. Thus, the FLC-based drive system is not affected by the sudden change of command speed. So, a good tracking has been achieved for the FLC, whereas the PI-controller-based drive system is affected with sudden change in command speed. Figures 8(a) and 8(b) show speed responses of the drive system using PI and FLC, respectively, with a sudden change in loading torque. The motor was started with no load, and this value was increased to 2 N-m after two seconds causing a drop in motor speed. The PI took less than 0.5 second, and fuzzy logic controller took negligible time to respond to this change in torque for operating the motor at the command speed. Figures 9(a) and 9(b) show simulated current response, respectively, for PI, and fuzzy controller-based IPMSM drive system with sudden application of a load of 2 N-m. Figures 10(a) and 10(b) show experimental speed and steady-state current response, respectively, for PI-based flux-weakening control-based IPMSM drive system with reference speed at a load of 2 N-m.




(a) (b)

Figure 9: (a) Simulated current of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using fuzzy controller with sudden load applied. (b) Simulated current of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using fuzzy controller with sudden load applied.




(a) (b)

Figure 10: (a) Experimental speed of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using PI controller ($Y$-axis: 1 div = 65 rad/sec). (b) Experimental steady-state current response of IPMSM drive for flux-weakening control with reference speed 220 rad/sec using PI controller ($Y$-axis: 1 div = 2 A).

## 7. Conclusions

In this paper, a new approach for fuzzy logic-based algorithm of flux-weakening method has been applied for the speed control of IPMSM drive above the base speed. In this work, relatively simpler expressions of $d$- and $q$-axis currents have been derived and incorporated in the IPMSM drive system. Simplified fuzzy controller for the IPMSM has also been designed and implemented through simulation. The IPMSM drive system is efficient enough to operate in no load and loading condition. Derived equation of voltage-limited ellipse, which has been plotted in Figure 1, may dictate a new approach of flux-weakening method for an optimum value of stator current, which will provide better performance in terms of efficiency. From the obtained results, it is obvious that the FLC-based IPMSM drive has been found superior to the conventional PI-controller based-system.

## Appendix

See Table 2.

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*Research Article*

# Efficient Load Forecasting Optimized by Fuzzy Programming and OFDM Transmission

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Today, it is very important for developed and developing countries to consume electricity more efficiently. Though developed countries do not want to waste electricity and developing countries cannot waste electricity. This leads to the concept: load forecasting. This paper is written for the short-term load forecasting on daily basis, hourly, or half-hourly basis or real time load forecasting. But as we move from daily to hourly basis of load forecasting, the error of load forecasting increases. The analysis of this paper is done on previous year's load data records of an engineering college in India using the concept of fuzzy methods. The analysis has been done on Mamdani-type membership functions and OFDM (Orthogonal Frequency Division Multiplexing) transmission scheme. To reduce the error of load forecasting, fuzzy method has been used with Artificial Neural Network (ANN) and OFDM transmission is used to get data from outer world and send outputs to outer world accurately and quickly. The error has been reduced to a considerable level in the range of 2-3%. For further reducing the error, Orthogonal Frequency Division Multiplexing (OFDM) can be used with Reed-Solomon (RS) encoding. Further studies are going on with Fuzzy Regression methods to reduce the error more.

## 1. Introduction

Forecasting for future load demand requirement is the most important key for power system planning. The capacities of the generation, transmission, and distribution capacities are strictly dependant on the accurate energy and load forecasting for that system [1]. For transmission of data from outer world to load forecasting model and sending outputs from this model to outer world accurately, OFDM transmission can be used due to its less bit error rate [2]. The Energy Management System (EMS) demands accurate load forecasting and Short-Term Load Forecasting (STLF) gives better and accurate results [2, 3]. The short-term load forecasting is especially significant for economic load dispatch, load management scheduling, and optimum power flow with minimum transmission loss, fuel management, and contingency planning [3, 4]. The sources are limited and the costs for those are very high. Moreover, the advancements are going on in electrical and electronics technology, computer and control technology, which have led to a reasonable cause for the further development of techniques for load prediction for system operation. Load forecasting techniques can be divided into three categories: short-term forecasting—as hourly, daily, or weekly forecasting; mid-range forecasting—extends from a month to one year; and long-term forecasting—ranging from one year to ten years [5]. All these types of forecasting methods are useful for different types of systems and define the size of the system.

The method that has been stressed upon is short-term load forecasting. A number of methods and techniques have already been devised for prediction of load such as Artificial Neural Networks (ANNs), Fuzzy Logic, and Regression

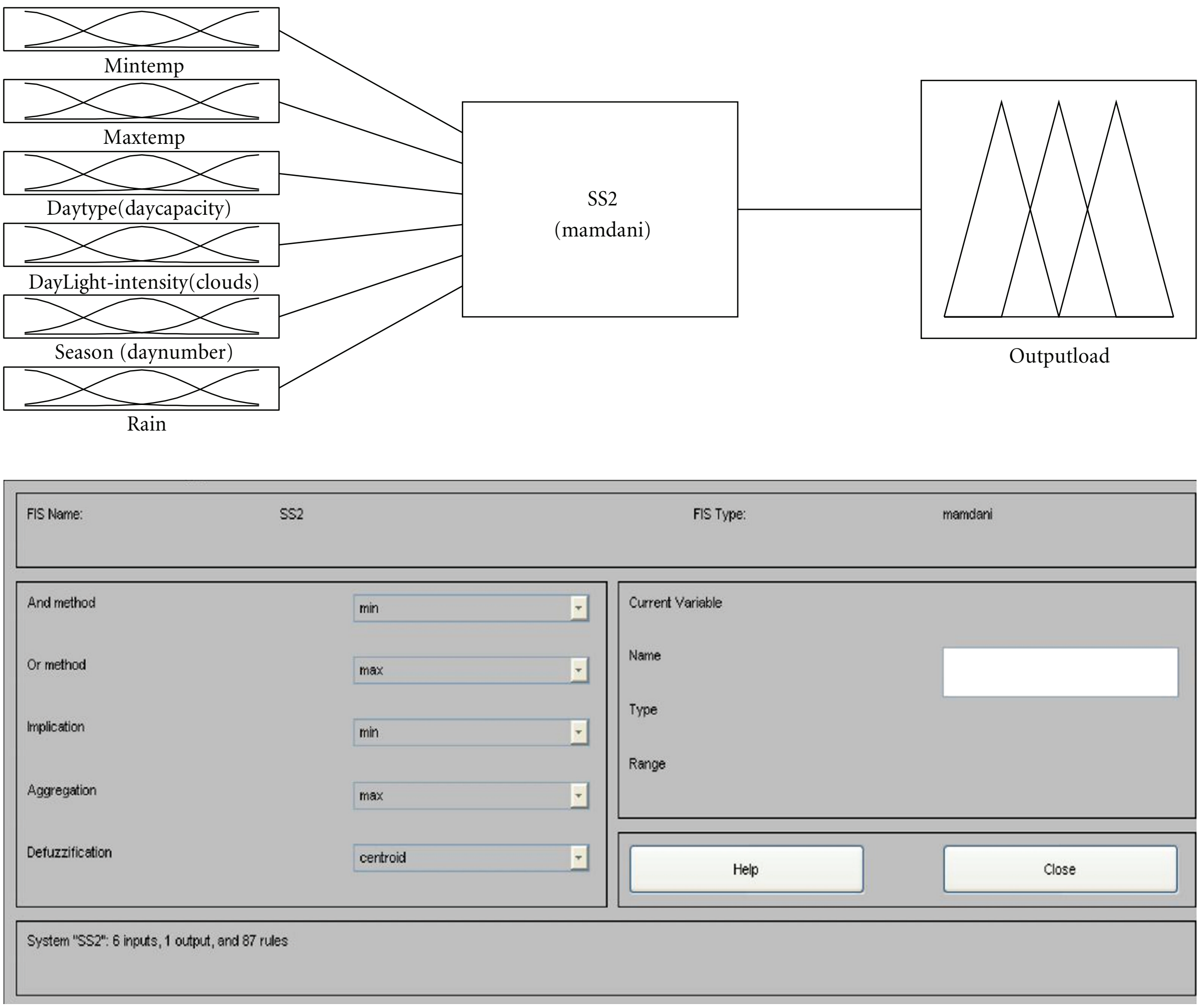


Figure 1: Flow and processing of data through fuzzy inference system.

Methods. Neural networks are having the properties of slow convergence time and poor ability to process a large number of variables at a time. Though, on other side, fuzzy logic gives a platform to represent and process data in linguistic terms, which makes the systems easily readable, understandable, and operateable [6, 7]. This is why the fuzzy logic has been used to deal with the input parameters information after detailed analysis of data and knowledge base (IF-THEN rules).

In fact, the load demand heavily depends on the number of factors such as weather, day type, and season. These factors actually decide the load to be forecasted depending on the conditions of these parameters on that day. The weather and seasons are factors which possess the nonlinear behavior with the load. One of the other important factors is day type. Day type generally means working day, week end, or a special day. It is important to extract a relation between electric load and the parameters affecting it. As accurate the parameters (weather, season, or day type) are judged, accurate will be the load forecasted for the day. For accuracy, OFDM and UWB systems can be used for calculating parameters.

## 2. The Work

In this study, a short-term load forecasting method using fuzzy logic has been developed, and a proposal to the advancement of the study with the use of artificial neural network (ANN) in different ways has been put up. A part of complete and generalized software using fuzzy logic has been tried to put into existence to forecast electrical load for domestic as well as commercial areas such as industries, institute, or residential colonies.

The system input parameters are day's minimum temperature, day's maximum temperature, season, day capacity, rain, and daylight intensity (Cloudy). Day's minimum temperature is a temperature when working hours start. All these parameters are put as input to fuzzy system, and the inputs are first of all scaled in the required value limits and fuzzified. Previous data (historical data or heuristic knowledge) which has already been stored in data base is used for inference. Rule base is designed to follow the heuristic knowledge according to the membership functions of various inputs. As in Figure 1, degree of membership for different input

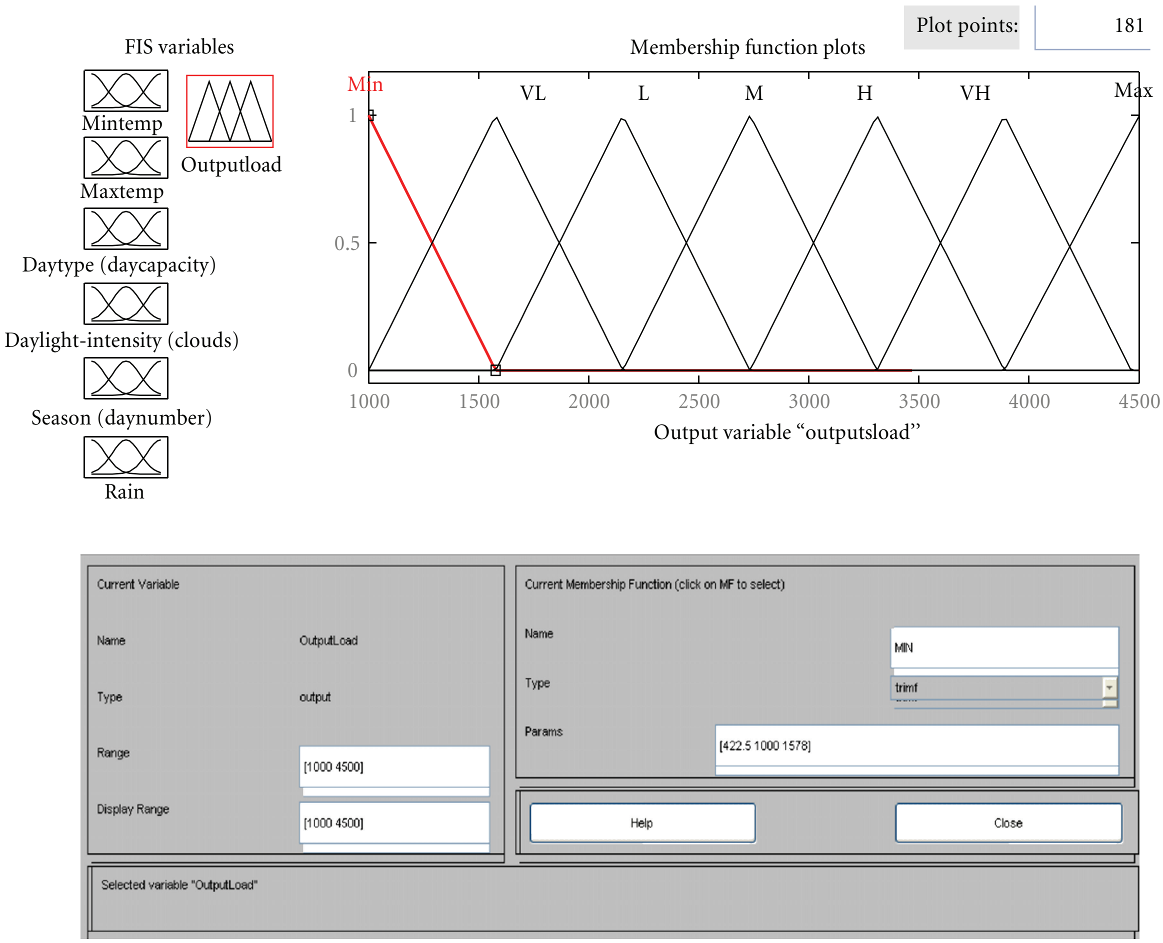


Figure 2: Membership functions of fuzzy inference system.

parameters is found out in the range of [0-1] and then defuzzified to get the crisp output which is then de-scaled to the required units and range [8, 9].

## 3. Historical Data and Key Factors

A good quality of historical data for input parameters for the last few years has been stored in data base management system (DBMS) for accurate load forecasting [6]. Short-term load forecasting mainly depends on the following conditions:

(i) day capacity,

(ii) weather conditions,

(iii) day temperature.

Though the day capacity can be defined as working day or non-working day (weekend or holiday). But as per this study, weekend and holiday are put in the same category when no work or negligible work is done. One more category as special day has been considered. This is the category when work is done after regular 8 working hours of the day (means if work is done for 9 Hrs. in a day shows one complete regular day and 1 Hr. of special day) or 9 Hrs. of special day depending on the type of work.

Overall working in an institute can be divided into two parts: Class (Theory and Tutorials) and Practical Labs and workshops. The day capacity is very much dependant on two factors:

(i) the type of work (either theory or practical),

(ii) day elongation.

So day capacity can be calculated as

$$\mathrm{DC} = \sum_{i}^{n} T_i \times D, \tag{1}$$

where $n$ is number of jobs done simultaneously in the same campus and DC is day capacity, $T_i$ is evaluation factor for the type of work, and $D$ is elongation of the day in (1).

Two main factors have been defined to decide weather conditions: cloudy and/or rainy weather. Cloudy weather gives an important effect of the daylight intensity, meaning that, the more the clouds, the lesser will be the daylight intensity and the more will be the consumption of electricity. These factors somehow are related to days minimum temperature and days maximum temperature.

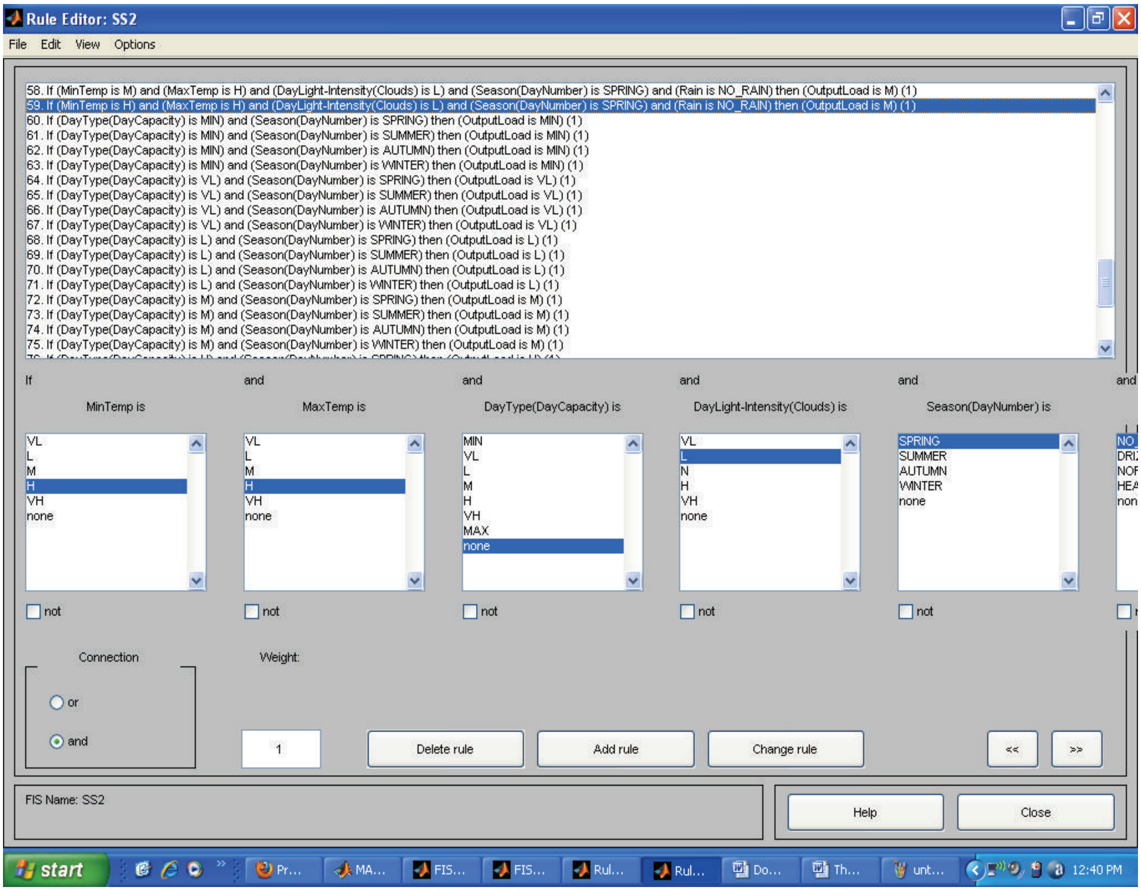


Figure 3: Rules of fuzzy system.

Actually, there can be a comparison between two working days with similar day capacity but different weather conditions; load consumed on both the days will be different. It can also happen that for two days, one is working and the other is nonworking with different weather conditions, the load consumed is the same [10, 11].

## 4. Load Forecasting

*4.1. Fuzzification.* Fuzzy linguistic variables are used to represent various inputs as well as output parameters as the member of fuzzy sets. A linguistic variable is used to define the value qualitatively by a linguistic term like any symbol serving its name and quantitatively by a corresponding membership function, that is, the meaning of a fuzzy set. In this work, we take example of temperature. We defined temperature as minimum temperature and maximum temperature which demonstrates the concept of linguistic variable. In order to express the fuzziness of information, this paper makes an arrangement of fuzzy subsets for different inputs and outputs in complete universe of discourse as membership functions [12, 13]. The relationship between several inputs and output may be nonlinear but linear membership functions have been used for simplicity and only the membership function for seasons is taken as ridge-shaped membership function such as gbell mf, gauss mf, and gauss2mf.

The day's minimum temperature and Maximum Temperature are represented as fuzzy subset [Very Low (VL), Low (L), Medium (M), High (H), Very High (VH)].

The linguistic variables of day capacity are represented as [Minimum (min), Very Low (VL), Low (L), Medium (M), High (H), Very High (VH), Maximum (max)].

The fuzzy subset for day capacity is [Very Low (VL), Low (L), Normal (N), High (H), Very High (VH)].

The season's fuzzy subset is given with the names of seasons as [Spring, Summer, Autumn, Winter].

The rain forecast has been given by fuzzy subset [No Rain, Drizzling, Normal Rain, Heavy Rain].

Similarly, the output factor load also has been assigned as fuzzy subset with membership functions [Minimum (min), very low (VL), Low (L), medium (M), High (H), Very High (VH), Maximum (max)].

*4.2. Fuzzy Rule Base.* This is the part of fuzzy system where heuristic knowledge is stored in terms of "IF-THEN Type" Rules. The rule base is used to send information to fuzzy inference system (FIS) to process through inference mechanism to numerically evaluate the information embedded in

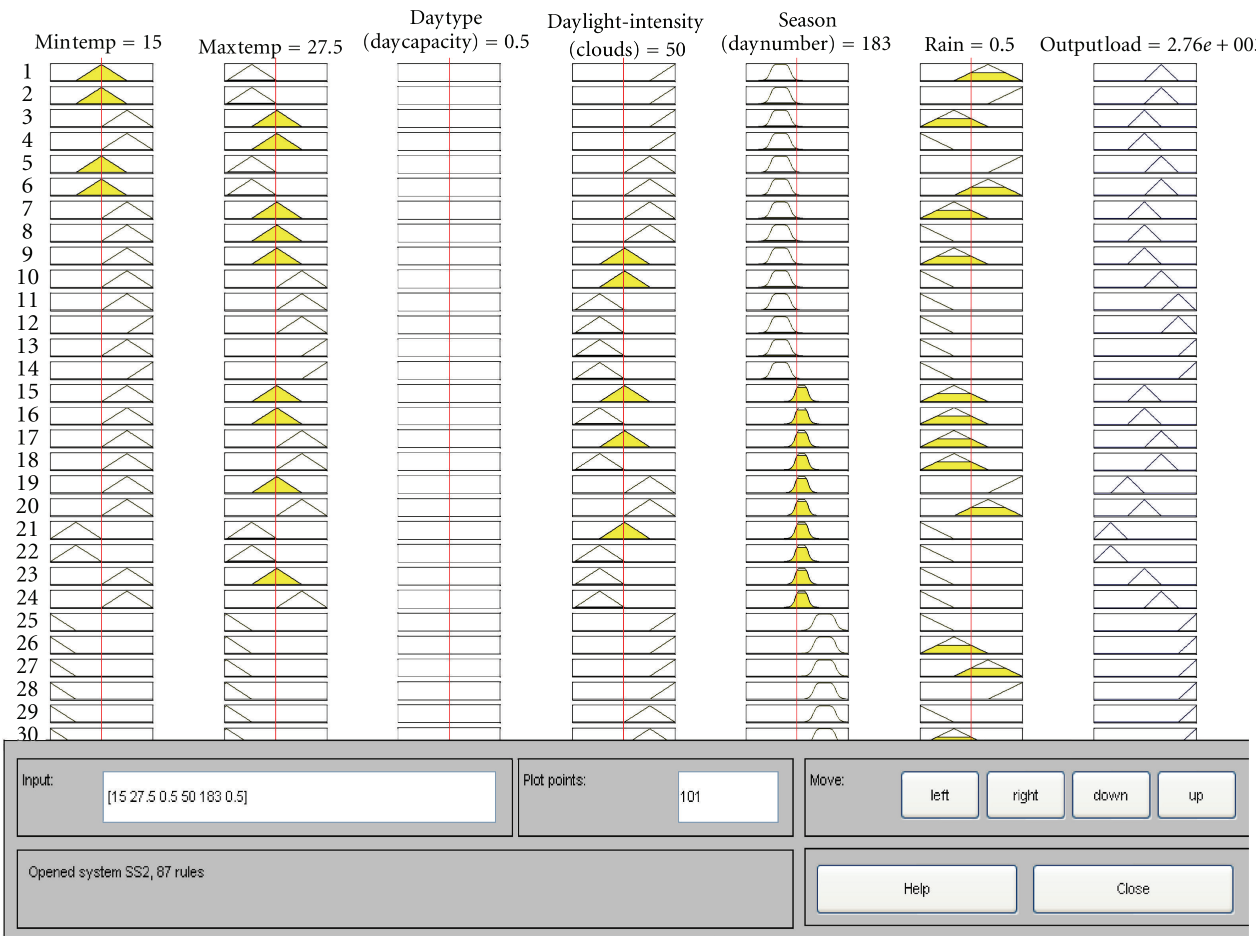


Figure 4: Rule Viewer of fuzzy system.

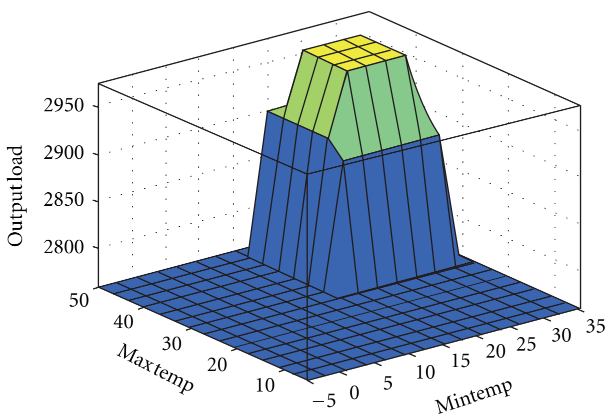


Figure 5: Surface viewer of fuzzy system.

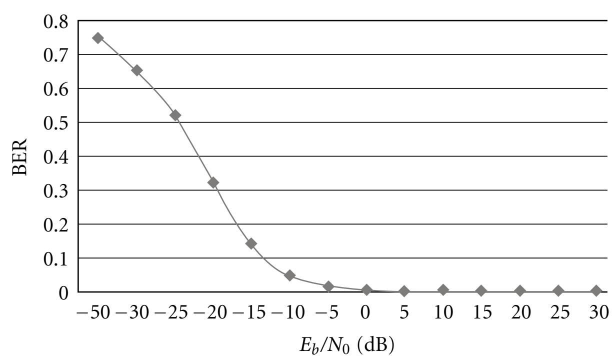


Figure 6: BER Plot of OFDM System with RS encoding and QPSK modulation scheme.

the fuzzy rule base to get the output. Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic. The mapping then provides a basis from which decisions can be made, or patterns discerned.

The different Rule Viewers of the Fuzzy Rule Base of the system is shown in Figure 4, though Figure 5 shows the Surface Viewer of the fuzzy optimized system.

## 5. Results

From the actual load, forecasted load and the % of error in the forecasted load for the data processed can be written as

$$\%\text{Error} = \frac{\text{AL} - \text{FL}}{\text{AL}} \times 100, \quad (2)$$

where AL is actual load and FL is forecasted load.

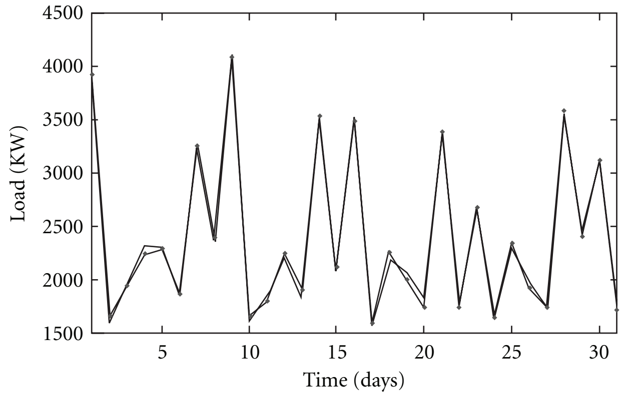


FIGURE 7: Comparison of actual load (–) and forecasted load.

The important factors for error are "inputs from outer world" and "output to outer world". Because if the transmission of data from outer world to fuzzy system is not accurate, the error rate of the system increases and the output data also comes with some error. So for transmission of data from outer world to fuzzy system for forecasting the load, OFDM can be used as a transmission scheme. Because OFDM system can handle number of users at same time with very less Bit Error Rate (BER) as shown in Figure 6.

So at overall the OFDM system minimizes the BER to 0.00436%, and it effects the overall load forecasting of a region. The forecasted load for the month of October has been shown for the reason that this is the mid time of a working session in an engineering college in India. Moreover, the change of season also takes place in this duration.

Both the results have been compared graphically, as in Figure 7, showing the minute variations in the actual and forecasted loads for the same session.

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