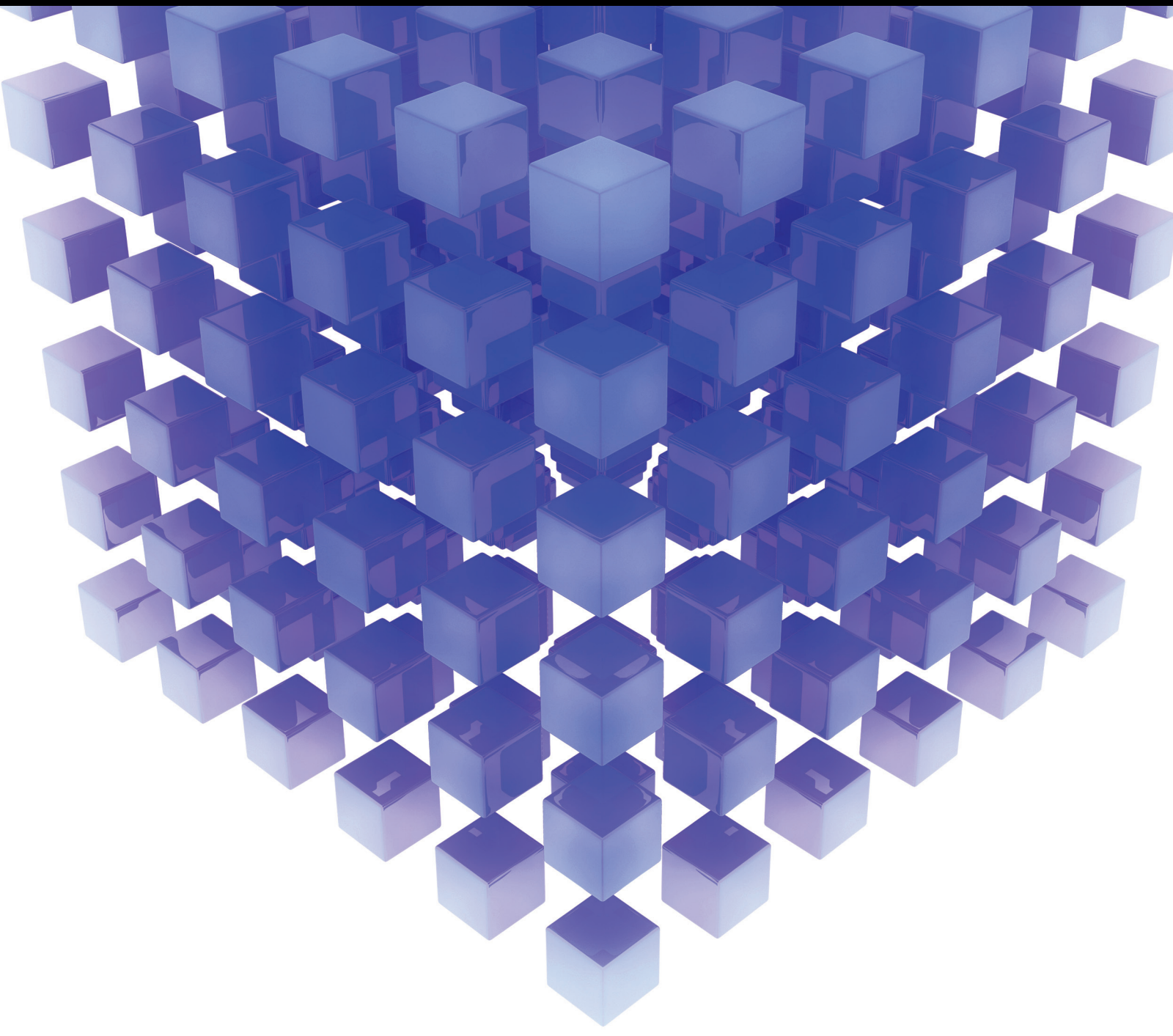
Mathematical Problems in Engineering

Time-Delay Systems and Their Applications in Engineering 2014

Guest Editors: Ligang Wu, Hak-Keung Lam, Yuxin Zhao, and Zhan Shu



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Editorial

Time-Delay Systems and Their Applications in Engineering 2014

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It is well known that, together with the increasing expectations of system dynamic performances, the delay information in practical systems or processes should not be neglected in the modelling procedure. Time delay is a very common phenomenon in the engineering world. In fact, many industrial processes include after effect phenomena in their inner dynamics. And actuators, sensors, and field networks that are involved in feedback loops usually introduce unavoidable delays. The theoretical research about delay related systems should be very important, and it is surely a rewarding topic for researchers. In the past few decades, there has been an increasing interest in time-delayed systems from control scientists and engineers. On one hand, time delays are inherent in various engineering systems and often deteriorate the main system performances. Those delays can cause instabilities of original systems and make it harder for the system analysis, controller design or state estimation. To tackle these disadvantages, special attention must be paid and necessary compensation should be applied in the design process. On the other hand, the nice properties of time delays may also bring a new vision for our practical design. For example, if dealt with properly, the delay elements can really be treated as the oscillation generator or chatting compensator.

In this special issue, most papers are concerned with the stability analysis and control problems of time-delay systems. Among them, five papers investigate the simplified or improved stability criteria of time-delayed systems; seven papers consider the extended stability analysis problems of the complex systems with time delay. Based on the existing control theory, three papers discuss the output-feedback control strategies for the stochastic high-order nonlinear systems and complex interconnected time-delay systems; one paper concerns the asymptotic tracking control for a class of nonlinear systems with unknown failures of hysteretic actuators; two papers cover the optimal control problem of the nonlinear systems with time delay and the calculus variations with time delays; four papers focus on the robust $\mathscr{H}_\infty$ or sliding model control schemes of time-delayed dynamic systems; one paper proposes a novel fractional-order PID controller for integrated pressurized water reactor; and the adaptive and impulsive control concepts are also involved in two papers. What is more, there are five papers concerning the observer, filter, monitor, and estimation problems of time-delayed dynamic systems or processes. One paper studies the identification problem of convex function on Riemannian manifold.

Stability is the most important problem for time-delay systems. It is the basic criterion for most of time-delayed control systems. This problem has been intensively studied during the past few years. "Modified Schur-Cohn Criterion for Stability of Delayed Systems" by J. I. Mulero-Martnez gives a modified Schur-Cohn criterion for time-delay linear time-invariant systems. "New Results on Stability and Stabilization of Markovian Jump Systems with Time Delay" by H. Xia et al. deals with the problem of stochastic stability and stabilization for a class of linear Markovian jumping linear systems with discrete time-varying delay. "Improved Results on Robust Stability for Systems with Interval Time-Varying Delays and Nonlinear Perturbations" by X. Zhou et al. investigates delay-dependent robust stability criteria for systems with interval time-varying delays and nonlinear perturbations. "A Delay Decomposition Approach to the

Stability Analysis of Singular Systems with Interval Time-Varying Delay" by J. Jiao and R. Zhang investigates delay-dependent stability problem for singular systems with interval time-varying delay. "Stability Analysis and $\mathscr{H}_\infty$ Output Tracking Control for Linear Systems with Time-Varying Delays" by K. H. Kim et al. studies the problem of stability analysis and $\mathscr{H}_\infty$ output tracking control for linear systems with time-varying delays. "Fixed Points and Stability of a Class of Integrodifferential Equations" by D. Pi studies a class of integrodifferential functional differential equations with variable delay. "Robust Stability Analysis of Nonlinear Fractional-Order Time-Variant Systems" by C. Chen and Y. Xie presents a stability theorem for a class of nonlinear fractional-order time-variant systems with fractional-order α $(0 \leq \alpha < 1)$ by using the Gronwall-Bellman lemma. "Stability and Hopf Bifurcation of a Predator-Prey Model with Distributed Delays and Competition Term" by L.-Z. Zheng considers a class of predator-prey systems with distributed delays and competition term. "On Less Conservative Stability Criteria for Neural Networks with Time-Varying Delays Utilizing Wirtinger-Based Integral Inequality" by O. M. Kwon et al. investigates the problem of stability analysis for neural networks with time-varying delays. "Simplified Stability Criteria for Delayed Neutral Systems" by X. Zhang et al. simplifies several existing asymptotic stability criteria in the form of linear matrix inequalities. "Stability Analysis of Fractional-Order Nonlinear Systems with Delay" by Y. Wang and T. Li studies the stability analysis of fractional-order nonlinear systems with delay. "A Less Conservative Stability Criterion for Delayed Stochastic Genetic Regulatory Networks" by T. Yu et al. concerns the problem of stability analysis for delayed stochastic genetic regulatory networks.

In order to obtain the desired system performance, some feasible controllers must be applied to regulate the systems with time delays. "Robust $\mathscr{H}_\infty$ Control of Neutral System with Time-Delay for Dynamic Positioning Ships" by D. Zhao et al. proposes the less conservative result for the neutral system with state-derivative feedback by the delay decomposition approach and linear matrix inequality. "Discrete-Time Sliding Mode Control for Uncertain Networked System Subject to Time Delay" by S. C. Garcia et al. deals with uncertain systems with networked sliding mode control, subject to time delay. "Adaptive Output Feedback Sliding Mode Control for Complex Interconnected Time-Delay Systems" by H. V. Van et al. extends the decentralized output-feedback sliding mode control scheme to stabilize a class of complex interconnected time-delay systems. "Asymptotic Tracking Control for a Class of Nonlinear Systems with Unknown Failures of Hysteretic Actuators" by C. Hou et al. proposes an adaptive failure compensation controller for a class of nonlinear systems preceded by hysteretic actuators. "A Novel Fractional-Order PID Controller for Integrated Pressurized Water Reactor Based on Wavelet Kernel Neural Network Algorithm" by Y. Zhao et al. presents a novel wavelet kernel neural network with wavelet kernel function. "Quasipolynomial Approach to Simultaneous Robust Control of Time-Delay Systems" by N. Semenič et al. investigates a control law for retarded time-delay systems, concerning infinite closed-loop spectrum assignment. "Research on Adaptive Neural Network Control System Based on Nonlinear U-Model with Time-Varying Delay" by F. Xu et al. proposes a new approach, namely, U-model approach, to solve the problems of analysis and synthesis for nonlinear systems. "Optimal Control with Time Delays via the Penalty Method" by M. Benharrat and D. F. M. Torres proves necessary optimality conditions of Euler-Lagrange type for a problem of the calculus of variations with time delays, where the delay in the unknown function is different from the delay in its derivative. "Robust Moving Horizon $\mathscr{H}_\infty$ Control of Discrete Time-Delayed Systems with Interval Time-Varying Delays" by F. Y. Tascikaraoglu et al. considers the design of a delay-dependent type moving horizon state-feedback control for a class of linear discrete-time systems subject to time-varying state delays, norm bounded uncertainties, and disturbances with bounded energies. "Time-Delayed Impulsive Control of Chaotic System Based on T-S Fuzzy Model" by C. Hu and H. Jiang is concerned with the time-delayed impulsive control and synchronization of general chaotic system based on T-S fuzzy model. "Optimal Control of a Nonlinear Time-Delay System in Batch Fermentation Process" by Y. Yu considers an optimal control problem involving a nonlinear time-delay system to get a high yield of products in batch process. "Output-Feedback Control for a Class of Stochastic High-Order Feedforward Nonlinear Systems with Delay" by Q. Ma et al. investigates the problem of global output-feedback stabilization for a class of stochastic high-order time-delay feedforward nonlinear systems with different power orders. "Adaptive Fuzzy Output-Feedback Method Applied to Fin Control for Time-Delay Ship Roll Stabilization" by R. Bai considers the ship roll stabilization by fin control system and investigates the adaptive fuzzy output-feedback control.

Some specific problems about observer, estimation, monitoring, filtering, and identification of time-delay systems and industrial processes are very important. These topics have been analyzed separately. "Observer Design for One-Sided Lipschitz Nonlinear Systems Subject to Measurement Delays" by S. Ahmad et al. presents a novel nonlinear observer-design approach to one-sided Lipschitz nonlinear systems in the presence of output delays. "Estimation of Nonlinear Functions of State Vector for Linear Systems with Time-Delays and Uncertainties" by I. Y. Song et al. focuses on estimation of a nonlinear function of state vector in discrete-time linear systems with time delays and model uncertainties. "$\mathscr{H}_\infty$ Filtering for Discrete Markov Jump Singular Systems with Mode-Dependent Time Delay Based on T-S Fuzzy Model" by C. Gong and Y. Zeng investigates the $\mathscr{H}_\infty$ filtering problem of discrete singular Markov jump systems with mode-dependent time delay based on T-S fuzzy model. "Monitoring of Nonlinear Time-Delay Processes Based on Adaptive Method and Moving Window" by Y. Fan et al. proposes a new adaptive KPCA algorithm for monitoring nonlinear time-delay process. "Experimental Evaluation of FSM-Based Testing Cost for Time-Delay Systems" by Z. Wei et al. presents a timer embedded FSM model to design a comparative strategy for assessing the coverage criteria and test suites generation methods for time-delay systems. "$\mathscr{H}_\infty$ Filtering for Networked Systems with Bounded Measurement Missing" by H. Yang et al. is concerned with the $\mathscr{H}_\infty$ filtering

problem for networked systems with bounded measurement missing. "The Identification of Convex Function on Riemannian Manifold" by L. Zou et al. presents the identification of convex function on Riemannian manifold using the Penot generalized directional derivative and the Clarke generalized gradient.

Many practical applications related with time delay are also concerned with. "Approximate Design of Optimal Disturbance Rejection for Discrete-Time Systems with Multiple Delayed Inputs: Application to a Jacket-Type Offshore Structure" by S.-Y. Han et al. is concerned with the problem of optimal disturbance rejection for a class of discrete-time systems with multiple delayed inputs. "The Impact of Awareness Programs with Recruitment and Delay on the Spread of an Epidemic" by L. Zuo et al. discusses the impact of awareness programs on the spread and control of infectious diseases in a given region. "The Application of Time-Delay-Dependent $\mathscr{H}_\infty$ Control in the Transmission Effect of Monetary Policy on Real Estate Market" by H. Guo et al. uses a time-delay-dependent $\mathscr{H}_\infty$ control method to analyze the transmission effect of monetary policy on the real estate market. "Modelling and Simulation of Volume Controlled Mechanical Ventilation System" by Y. Shi et al. derives a new mathematical model of the ventilation system. "Robust Predictive Control of Water Level in an Experimental Pilot Plant with Uncertain Input Delay" by F. Q. Rossi and R. K. H. Galvão is concerned with the predictive control of water level in an experimental pilot plant, in the presence of input constraints and uncertain time delays. "Study of Robust $\mathscr{H}_\infty$ Filtering Application in Loosely Coupled INS/GPS System" by L. Zhao et al. gives an illustration of minimum mean square Kalman filter in integration GPS/INS system and a contrast with traditional Kalman filter. "Modeling and Bifurcation Research of a Worm Propagation Dynamical System with Time Delay" by Y. Yao et al. adopts both vaccination and quarantine strategy to control the internet worm propagation. "Global Existence and Uniqueness of Solutions for a Free Boundary Problem Modeling the Growth of Tumors with a Necrotic Core and a Time Delay in Process of Proliferation" by S. Xu and M. Huang studies a mathematical model for the growth of necrotic tumors with time delays in proliferation.

For both the theoretical and practical impact, time-delay systems have been an enduring theme in the study of differential equations, stochastic processes, statistics theory, and systems theory. These research topics have broad applications to a number of areas such as mechanical, electrical, and chemical engineering. And the theoretical research results about time-delayed systems have played an important role in solving these delay related problems. Recently, some open problems about time delay have also emerged, such as using the delayed inputs to stabilize and regulate the original system, adaptive identification of delays in practical problems, taking advantage of the stochastic properties of time delay, and considering the delay information for observers. Needless to say, these open problems are both challenging and rewarding for theoretical researchers and practical engineers. And many articles in this special issue have been concerned with these most recent theoretical developments and practical applications.

Of course, the selected topics and papers are not a comprehensive representation of the area of this special issue. Nonetheless, they represent the rich and many-faceted knowledge and we hope the readers will find them well readable and useful.

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Research Article

Approximate Design of Optimal Disturbance Rejection for Discrete-Time Systems with Multiple Delayed Inputs: Application to a Jacket-Type Offshore Structure

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The study is concerned with problem of optimal disturbance rejection for a class of discrete-time systems with multiple delayed inputs. In order to avoid the two-point boundary value (TPBV) problem with items of time-delay and time-advance caused by multiple delayed inputs, the discrete-time system with multiple delayed inputs is transformed into a delay-free system by introducing a variable transformation, and the original performance index is reformulated as a corresponding form without the explicit appearance of time-delay items. Then, the approximate optimal disturbance rejection controller (AODRC) is derived from Riccati equation and Stein equation based on the reduced system and reformulated performance index, which is combined with feedback item of system state, feedforward item of disturbances, and items of delayed inputs. Also, the existence and uniqueness of AODRC are proved, and the stability of the closed-loop system is analysed. Finally, numerical examples of disturbance rejection for jacket-type offshore structure and pure mathematical model are illustrated to validate the feasibility and effectiveness of the proposed approach.

1. Introduction

In practice, time-delays are inherent in various engineering systems. As an efficient control algorithm for the improvement of the efficiency, reliability, and scalability of the dynamic systems, time-delays must be taken into consideration in process of modeling and controller design from engineering and control scientists. It is well known that the actuators and transmission of the communication information are with the electrical and electromagnetic characteristics. Therefore, delayed inputs are more general pervasive on dynamic modelling of engineering systems. Especially, with the rise of networked control systems (NCSs) and communication systems [1, 2], multiple delayed inputs are mainly caused by the online data acquisition, filtering, calculating controller, and transmitting the control force signals from a computer to the actuator in engineering systems. Recently, many methods have been proposed to solve the problem for the time-delay discrete-time systems. For example, sliding mode control (SMC) is a widely used technique to handle matched disturbances by designing sliding surface and reaching motion controller for the time-delay process [3–5]. Smith predictor has the potential capability to achieve an improvement of set-point response and disturbance response [6, 7]; observer based output feedback control is very powerful for systems with delayed input by using the predictor feedback while the state vectors are measurable or estimated online [8, 9]; H_∞ control has been intensively applied among the stability analysis, stabilization control, and robustness control for discrete-time systems with multiple delayed inputs [10, 11].

Disturbance rejection control has been viewed as a typical issue since the origins of control theory, and many researchers have been paying more attention to this field both academia and engineers. Many physical systems are subjected to persistent disturbances, such as offshore jacket platforms subjected to the waves [12, 13], helicopters subjected to the

winds [14], and vehicle suspension systems subjected to the road roughness [15, 16]. The external disturbances are often a source of poor performance of underlying control systems. The active disturbance rejection control has been viewed as an effective solution to solve the above problem. However, finding an explicit form of optimal control law for time-delay systems under disturbances remains difficult [17]. Therefore, some modification or improvement has been done in the conventional control schemes. For example, active disturbance rejection control has been shown to be an effective tool in dealing with real world problems of dynamic uncertainties, disturbances, and nonlinearities [18–20]; a robust H_∞ guaranteed cost control law is designed for uncertain Markovian jump systems with distributed delays and input delays in [21]; by using an inertial actuator for suppression of multiple unknown and/or time-varying vibrations, an active vibration control system was proposed in [22]; a disturbance estimator design method was proposed to design the control scheme for open-loop and unstable processes with time delay [23]. An overview of antidisturbance control for engineering systems with multiple disturbances can be found in [24].

For discrete-time systems with multiple delayed input under persistent disturbances, the traditional optimal control problem with respect to quadratic performance index will induce (TPBV) problem with time-delay and time-advance items. Although there have been many methods to solve the control problem for discrete-time systems with delayed input, it is difficult to find an optimal solution for this problem. On the one hand, augmented method could convert a delay system into free-delay ones. However, with the multiple delays or large delays, the dimension and stability of the delay-free system cannot be ensured; even the computing work would increase exponentially. On the other hand, virtually most of the studies on optimal control for discrete-time system with multiple input delays consider only approximate optimal control by using an iterative solution. For example, iterative learning algorithm was proposed to ensure the control performance for nonlinear discrete-time systems with multiple input delays in [25]; [26] proposed an optimal controller by using only measured input/output data from a class of linear discrete-time systems with multiple delays by using adaptive dynamic programming; a successive approximation approach (SAA) was proposed in [27] to design a suboptimal control for discrete linear systems with time delay, in which an iterative procedure is designed to solve the TPBV problem with time-delay items and nonlinear items; after that, based on SAA and augmented method, [28] designed an approximate tracking controller for discrete-time systems with multiple state and input delays.

The motivation of this work is to design an AODRC for discrete-time systems with multiply delayed inputs under persistent disturbances with respect to a typical quadric performance index. In order to deal with multiple delayed inputs, a variable transformation is proposed so that the original system is transformed into delay-free ones, and the quadratic performance index of the original system is simplified without the explicit appearance of time-delay items. Then, the AODRC is designed based on the maximum principle by solving Riccati and Stein equations, which is composed of a feedback item of system state, a feedforward item of disturbance state, and some control memory terms. In term of design of controller, compared to the previous work [25, 27, 28], the proposed AODRC is analytical solution so that the computation cost and time are reduced. In terms of control performance, by analysing the simulation results for an offshore platform and pure mathematical example with different cases in delayed inputs, the proposed AODRC can not only reduce the influence from multiple delayed inputs for underlying systems but also trade off among rejection effect opposed to the multiple delayed inputs, persistent disturbances, and energy consumption.

The rest of the paper is organized as follows. The disturbance rejection problem for discrete-time systems with multiple delayed inputs is formulated in Section 2. In Section 3, the original problem is reformulated as an equivalent one for delay-free transformed system with respect to a transformed performance index without the explicit appearance of time delay. Then, the proposed AODRC is presented in Section 4, in which its existence and uniqueness are proved, and the stability of system with multiple delayed inputs is analysed. In Section 5, the effectiveness and implement of the proposed AODRC are evaluated by applying it to numerical examples of pure mathematical example and a jacket-type offshore platform for different cases with delayed inputs. Finally, our findings are concluded in Section 6.

2. Problem Statement

Consider the discrete-time systems with multiple delayed inputs under persistent disturbances described by

$$\begin{aligned} x(k+1) &= Ax(k) + \sum_{i=1}^{N} B_i u(k-h_i) + Dv(k), \\ y(k) &= Cx(k), \quad k = 0, 1, 2, \ldots, \\ x(0) &= x_0, \\ u(k) &= 0, \quad k < 0, \end{aligned} \tag{1}$$

where $x \in R^n$ denotes the state vector; $u \in R^m$ is the control input; $v \in R^p$ denotes the persistent disturbances signal; $h_i > 0$ $(i = 1, 2, \ldots, N)$ are positive multiple delayed inputs, respectively; $y \in R^r$ is the output vector; N is the account of multiple delayed inputs. A, B_i $(i = 1, 2, \ldots, N)$, C, and D are constant matrices of appropriate dimensions.

Assumption 1. The pair (A, B_1) is completely controllable and the pair (A, C) is completely observable, where C is defined by $Q = C^T C$.

The dynamic characteristics of the persistent disturbances vector $v(k)$ are described by the following exosystem:

$$\begin{aligned} w(k+1) &= Gw(k) \\ v(k) &= Ew(k), \end{aligned} \tag{2}$$

where $w \in R^q$, $(p \le q \le r)$ is the disturbance state, whose initial condition is unknown. G and E are constant matrices

of appropriate dimensions with full rank. It is assumed that the pair (E, G) is completely observable. Also, the eigenvalues of the G satisfy

$$\left|\lambda_i(G)\right| \le 1, \quad i = 1, 2, \ldots, q. \tag{3}$$

Then, exosystem (2) of the persistent disturbances is stable but may not be asymptotically stable. It should be noted that the expression of exosystem (2) could describe the various forms of disturbances with known or unknown dynamic characteristics, such as the sinusoidal disturbances, periodic disturbances, step disturbances, and random signals.

The performance index could be depending on the dynamic characteristics of persistent disturbances. When the persistent disturbances are asymptotically stable, the traditional quadratic performance index could be chosen as

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[y^T(k)\,Qy(k) + u^T(k)\,Ru(k)\right]. \tag{4}$$

On the other hand, the persistent disturbances are stable but may not be asymptotically stable; the effect from persistent disturbance vector $v(k)$, the steady state of the state vector $x(k)$, and the control input $u(k)$ will not converge to zero synchronously. Therefore, the following infinite-time average performance index is chosen:

$$J = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}\left[y^T(k)\,Qy(k) + u^T(k)\,Ru(k)\right], \tag{5}$$

where $Q \in R^{n\times n}$ is positive semidefinite and $R \in R^{m\times m}$ is positive definite.

The aim of the optimal disturbance rejection is to find an optimal control law $u^*(k)$ for the discrete-time systems described by (1) and (2) with respect to the performance index (4) or (5) that make the performance index (4) or (5) obtain the minimum value.

3. Reformulation of Disturbance Rejection Problem

For discrete-time delay systems with quadratic performance index, the optimal control problem is difficult to be solved analytically caused by the TPBV problem with both time-delay and time-advance terms. In this section, a vector transformation is introduced to deal with the multiple delayed inputs, and the performance index is reformulated as the equivalent ones without the explicit appearance of time delay.

3.1. Delay-Free Transformation of Control System. By proposing the following variable transformation

$$\tilde{x}(k) = x(k) + \sum_{i=1}^{N}\sum_{j=k-h_i}^{k-1} A^{k-h_i-j-1}B_i u(j) \tag{6}$$

system (1) with multiple delayed input can be transformed into the following form:

$$\begin{aligned}\tilde{x}(k+1) &= A\tilde{x}(k) + \tilde{B}u(k) + Dv(k),\\ y(k) &= C\left(\tilde{x}(k) - \sum_{i=1}^{N}\sum_{j=k-h_i}^{k-1} A^{k-h_i-j-1}u(j)\right),\end{aligned} \tag{7}$$

where $\tilde{x}(k) \in R^n$ is the state vector of the delay-free system (7) and $\tilde{B} = \sum_{i=1}^{N} A^{-h_i}B_i$. Note that the transformed system (7) is in the form of a delay-free system, and the pair $(A, \tilde{B})$ is completely controllable if and only if (A, B_1) is completely controllable [29].

3.2. Simplification of Performance Index. Through the vector transformation (6) introduced, the original system with multiple delayed inputs (1) could be transformed into system (7) without the explicit appearance of time delay. In order to evaluate the effectiveness of the control law more exactly, it is necessary to transform the performance index (4) or (5) into the equivalent form for the transformed system (7).

By using (6) and (7), the quadratic performance index (4) could be transformed into the following form:

$$\begin{aligned}J = \sum_{k=0}^{\infty}\Big\{&\tilde{x}^T(k)\,C^TQC\tilde{x}(k) + u^T(k)\,Ru(k)\\ &- 2U_1^TC^TQC\tilde{x}(k) + U_1^TC^TQCU_1\Big\},\end{aligned} \tag{8}$$

where $U_1 = \sum_{i=1}^{N}\sum_{j=k-h_i}^{k-1} A^{k-h_i-j-1}u(j)$.

Based on (2), (6), (7), and (9), one gets

$$\begin{aligned}&\sum_{k=0}^{\infty}\tilde{x}^T(k)\,C^TQCU_1\\ &\quad= \sum_{k=0}^{\infty}\left\{\tilde{x}^T(k)\,C_1u(k) + w^T(k)\,C_2u(k)\right.\\ &\qquad+\sum_{t=1}^{N}\sum_{i=1}^{h_t}\sum_{j=1}^{h_t}\left(\left(CA^{-j}B_iu(k+i-1)\right)^T\right.\\ &\qquad\left.\left.\times\, QCA^{i-j-1}B_i\right)u(k)\right\},\\ &\sum_{k=0}^{\infty}U_1^TC^TQCU_1\\ &\quad= \sum_{k=0}^{\infty}\left\{u^T(k)\sum_{i=1}^{N}\sum_{j=1}^{h_i}\left(\left(CA^{-j}B_i\right)^TQCA^{-j}B_i\right)u(k)\right.\\ &\qquad+2\sum_{t=1}^{N}\sum_{i=1}^{h_t}\sum_{j=1}^{h_t}\left(\left(CA^{-j}B_iu(k+i-1)\right)^T\right.\\ &\qquad\left.\left.\times\, QCA^{i-j-1}B_i\right)u(k)\right\},\end{aligned} \tag{9}$$

where

$$C_1 = \sum_{i=1}^{N}\left(\left(\sum_{j=1}^{h_i}\left(CA^j\right)^T QACA^{j-1-h_i}B_i\right)\right),$$
$$C_2 = \sum_{i=1}^{N}\sum_{j=1}^{h_i}\sum_{t=0}^{h_i-j}\left(\left(CA^t DEG^{j-1}\right)^T QCA^{t+j-h_i-1}B_i\right). \tag{10}$$

Then, the quadratic performance index (4) could be reformed as the following form:

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{\tilde{x}^T(k)C^TQC\tilde{x}(k) + u^T(k)\tilde{R}u(k) - 2\tilde{x}^T(k)C_1u(k) - 2w^T(k)C_2u(k)\right\}, \tag{11}$$

where $\tilde{R} = R + \sum_{i=1}^{N}\sum_{j=1}^{h_i}((CA^{-j}B_i)^TQCA^{-j}B_i)$.

Actually, the above transformations are suitable for the infinite-time average quadratic performance index (5). The reformed performance index could be described as

$$J = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}\left\{\tilde{x}^T(k)C^TQC\tilde{x}(k) + u^T(k)\tilde{R}u(k) - 2\tilde{x}^T(k)C_1u(k) - 2w^T(k)C_2u(k)\right\}. \tag{12}$$

Then, the original optimal disturbance rejection is reformulated as an equivalent optimal regulation problem that designs an approximate control law $u^*(k)$ to make the quadratic performance index (11) or (12) obtain the minimum value subject to the discrete-time system (2) and (7) such that the negative effects from persistent disturbances and multiple delayed inputs are eliminated.

4. Approximate Design of Optimal Disturbance Rejection Controller

In this section, the proposed AODRC is presented in detail, and its existence and uniqueness are proved. In order to state the proposed control law clearly, the following matrices are defined:

$$A_1 = A + \tilde{B}\tilde{R}^{-1}C_1^T, \qquad A_2 = DE + \tilde{B}\tilde{R}^{-1}C_2^T,$$
$$Q_1 = C^TQC - C_1\tilde{R}^{-1}C_1^T, \qquad Q_2 = C_1\tilde{R}^{-1}C_2^T. \tag{13}$$

Theorem 2. *Consider the optimal disturbance rejection control problem for the discrete-time systems given by* (1) *and* (2) *with respect to the quadratic performance indexes* (4) *or* (5). *The approximate optimal disturbance rejection controller uniquely exists and can be formulated as*

$$u^*(k) = \tilde{R}^{-1}\left\{\left[C_1^T - \tilde{B}^TA_1^{-T}(P_1 - Q_1)\right] \times\left[x(k) + \sum_{i=1}^{N}\sum_{j=k-h_i}^{k-1}A^{k-h_i-j-1}Bu(j)\right] + \left(C_2^T - \tilde{B}^TA_1^{-T}(P_2 + Q_2)\right)w(k)\right\}, \tag{14}$$

where P_1 is the unique positive definite solution of the Riccati matrix equation

$$P_1 = Q_1 + A_1^TP_1\left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^TP_1\right)^{-1}A_1 \tag{15}$$

and P_2 is the unique solution of the Stein matrix equation

$$P_2 + Q_2 = A_1^TP_1\left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^TP_1\right)^{-1}A_2 + A_1^T\left[I - A_1^TP_1\left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^TP_1\right)^{-1}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\right]P_2G. \tag{16}$$

Proof. Define the Hamiltonian function for the optimal disturbance rejection problem as

$$H(\bullet) = \frac{1}{2}\left\{\tilde{x}^T(k)C^TQC\tilde{x}(k) + u^T(k)\tilde{R}u(k) - 2\tilde{x}^T(k)C_1u(k) - 2w^T(k)C_2u(k)\right\} + \lambda^T(k+1)\left[A\tilde{x}(k) + \tilde{B}u(k) + Dv(k)\right]. \tag{17}$$

Applying the maximum principle condition $\partial H/\partial u = 0$ to this specific Hamiltonian function (17) yields

$$u(k) = \tilde{R}^{-1}\left[\tilde{B}^T\lambda(k+1) - C_1^T\tilde{x}(k) - C_2^Tw(k)\right], \tag{18}$$

where $\lambda(k)$ is the solution of the following TPBV problem:

$$\tilde{x}(k+1) = A_1\tilde{x}(k) + A_2w(k) - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\lambda(k+1),$$
$$Q_1\tilde{x}(k) - Q_2w(k) + A_1^T\lambda(k+1) = \lambda(k),$$
$$\tilde{x}(0) = x(0),$$
$$\lambda(\infty) = 0. \tag{19}$$

To solve TPBV problem (19), let

$$\lambda(k) = P_1\tilde{x}(k) + P_2w(k). \tag{20}$$

Rearranging (19), one gets

$$\lambda(k+1) = A_1^{-T}\left[(P_1 - Q_1)\tilde{x}(k) + (P_2 + Q_2)w(k)\right],$$
$$x(k+1) = \left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^TP_1\right)^{-1} \times\left\{A_1x(k) + \left(A_2 - \tilde{B}\tilde{R}^{-1}\tilde{B}^TP_2G\right)w(k)\right\}. \tag{21}$$

Substituting the first formula of (21) into (18), the approximate optimal control law (14) is obtained. Note that

$$\begin{aligned}\lambda(k) &= Q_1\tilde{x}(k) - Q_2 w(k) + A_1^T\lambda(k+1)\\ &= \left[Q_1 + A_1^T P_1\left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^T P_1\right)^{-1} A_1\right]\tilde{x}(k)\\ &\quad + \left\{A_1^T\left[I - A_1^T P_1\left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^T P_1\right)^{-1}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\right]P_2 G\right.\\ &\quad \left. + A_1^T P_1\left(I + \tilde{B}\tilde{R}^{-1}\tilde{B}^T P_1\right)^{-1} A_2 - Q_2\right\} w(k).\end{aligned} \tag{22}$$

By comparing the parameters of (20) and (22), the Riccati equation (15) and Stein equation (16) can be obtained.

In the following, the existence and uniqueness of the proposed controller will be proved. Because P_1 and P_2 are unknown in the parameters of proposed control law (14), the existence and uniqueness of (14) are equivalent to that of P_1 and P_2. For Riccati matrix equation (15), because of the completely controllable and observable pairs (A, B_1) and (A, C), the matrix P_1 is existent and the unique solution of Riccati matrix equation (15). It should be noted that the first formula of (19) is equivalent to (7); based on the completely controllable pair $(A, \tilde{B})$, one gets

$$\left|\lambda_i\left\{\left[I + \tilde{B}\tilde{R}^{-1}\tilde{B}^T P_1\right]^T A_1\right\}\right| < 1, \quad i = 1, 2, \ldots, n. \tag{23}$$

Note (3); one gets

$$\begin{aligned}&\left|\lambda_i\left\{\left[I + \tilde{B}\tilde{R}^{-1}\tilde{B}^T P_1\right]^T A_1\right\}\right|\left|\lambda_j(G)\right| < 1,\\ &\qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, q.\end{aligned} \tag{24}$$

Then, P_2 is existent and the unique solution of Stein matrix equation (16). Therefore, the proposed approximate controller is existent and unique. □

According to [29], (6), and (14), one gets

$$\begin{aligned}\|x(k)\| \le \|\tilde{x}(k)\| &+ \sum_{i=1}^{N}\left[h_i \max_{1\le s\le h_i}\left\|A^{k-j-1}\right\|\right.\\ &\times \left\|B\tilde{R}^{-1}\left[\tilde{B}^T A_1^{-T}(P_1 - Q_1) - C_1^T\right]\right\|\\ &\left.\times \|\tilde{x}(k-s)\| \max_{1\le s\le h_i}\right]\\ &+ \sum_{i=1}^{N}\left[h_i \max_{1\le s\le h_i}\left\|A^{k-j-1}\right\|\|w(k-s)\|\right.\\ &\left.\times \left\|B\tilde{R}^{-1}\left[\tilde{B}^T A_1^{-T}(P_2 + Q_2) - C_2^T\right]\right\|\right].\end{aligned} \tag{25}$$

Because $\|\tilde{x}(k)\|$ and $\|w(k)\|$ are bounded, $\|x(k)\|$ is bounded. Then, the proposed control law (14) is a stabilizing control law for discrete-time system (1) with multiple delayed inputs. The proof is completed.

Table 1: Parameters of installed AMD device.

Name of parameter	Variable	Value	Unit
Mass	m_2	11855	kg
Natural frequency	ω_2	2.33	rad/s
Structural damping ratio	ξ_2	9.32	Percent

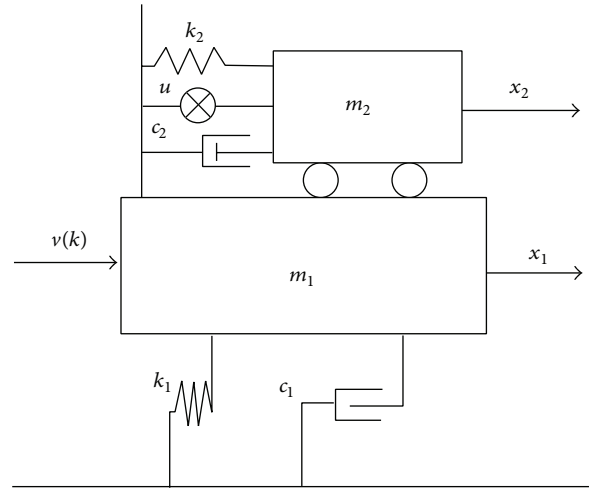

Figure 1: The sketch of offshore platform with AMD device.

5. Simulation

In order to demonstrate the effectiveness and feasibility of the proposed AODRC, numerous simulations are undertaken in this section. In case 1, the vibration control problem for offshore steel jacket platforms with single delayed input is simulated by using the proposed AODRC. In case 2, the disturbance rejection problem for pure mathematical example with two delayed inputs is considered.

5.1. Vibration Control for Offshore Jacket Platforms with Single Delayed Input. The jacket type platforms play an import role in the oil exploration and drilling operations. However, the jacket type platforms subject to external disturbances persistently, such as wave force, wind, and earthquake. In order to ensure the safety and production efficiency, the vibration caused by external disturbances could be rejected effectively, especially for wave force. In this subsection, the vibration control problem for offshore jacket platforms with single input delay is solved by the proposed control law.

Consider the jacket platform located in Bohai Bay, the sketch of offshore platform with an AMD device is shown in Figure 1. The structural parameters of offshore platform and installed AMD device on the deck of the jacket type platform are listed in Tables 1 and 2 [30], respectively.

Based on the JONSWAP spectrum, the wave can be described as

$$S_\eta(\omega) = \left[\frac{5H_s^2}{16\omega_0}\right]\left(\frac{\omega_0}{\omega}\right)^5 \exp\left[-1.25\left(\frac{\omega_0}{\omega}\right)^{-4}\right]\gamma^\beta, \tag{26}$$

Table 2: Parameters of offshore platform.

Name of parameter	Variable	Value	Unit
Equivalent characteristic diameter of legs	D	1.7	m
First modal mass	m_1	2371100	kg
Natural frequency	ω_1	2.20	rad/s
Structural damping ratio	ξ_1	4	Percent
Shape function of first mode	$\varphi(s)$	$1-\cos\left(\frac{\pi s}{2L}\right)\quad 0\le s\le L$	

where $H_s = 4\,\text{m}$ is the significant wave height, ω is the wave frequency, $\gamma = 3.3$ is the peakedness parameter, $\omega_0 = 0.87\,\text{rad/s}$ is the peak frequency, $\beta = \exp[-(\omega-\omega_0)^2/(2\sigma^2\omega_0^2)]$, in which σ is the shape parameter, $\sigma = 0.07\,(\omega \le \omega_0)$, and $\sigma = 0.09\,(\omega > \omega_0)$. Then, based on [30], the wave force can be formulated as an exosystem described by (2) with the water depth $d = 13.2\,\text{m}$, the drag coefficient $C_d = 1.2$, and the inertial coefficient $C_m = 2.0$. The external disturbances from irregular wave force are displayed in Figure 2.

By setting the sampling period $T = 0.1\,\text{s}$, $Q = \text{diag}(100\ 0\ 100\ 0)$, and $R = 1$, the discrete-time model of the offshore structure could be established with the same expression of (1) based on the previous investigations on the subject, which is described as

$$A = \begin{bmatrix} 0.9790 & 0.0002 & 0.0989 & 0.0000 \\ 0.0205 & 0.9789 & 0.0047 & 0.0952 \\ -0.4182 & 0.0041 & 0.9701 & 0.0010 \\ 0.3954 & -0.4152 & 0.0996 & 0.8993 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.0006 \\ 0.0621 \\ -0.0120 \\ 1.2205 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.0006 \\ 0.0000 \\ 0.0126 \\ 0.0006 \end{bmatrix}, \tag{27}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

According to the previous investigations on the subject, the range of delayed input is in the interval $(0, 1)\,\text{s}$. Therefore, to demonstrate the effectiveness of the proposed AODRC, the comparison results between the open-loop system, the feedback control, and the AODRC with single delayed input $h = 6$ are compared, in which feedback control law is just considered the feedback item in AODRC. Then, the curves of displacement of offshore structure, velocity of offshore structure, and control law are presented in Figures 3, 4, and 5, respectively.

In order to show the comparison results among the simulated case clearly, the values of the quadratic performance index with different delayed input are compared in Table 3. Note that the percentage number given in the parentheses indicates the reduced amount of the closed-loop response relative to the open-loop case. For instance, the value of performance index is reduced by 18.40% when delayed input $h = 2$ less than that of open-loop one, using feedback control, while the value of performance index is reduced by 67.23% less than that of open-loop one by using proposed AODRC.

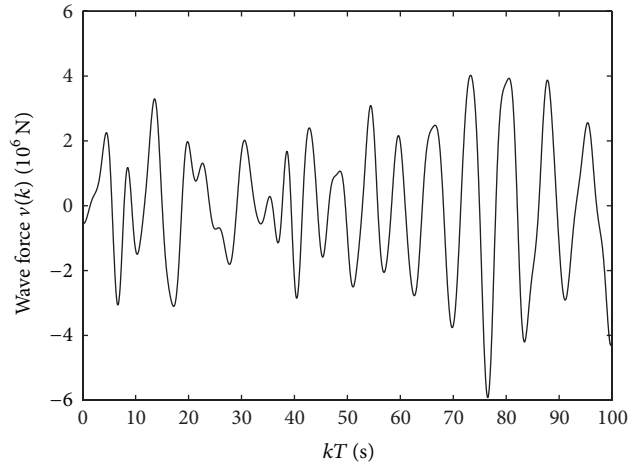

Figure 2: The external disturbances of wave force.

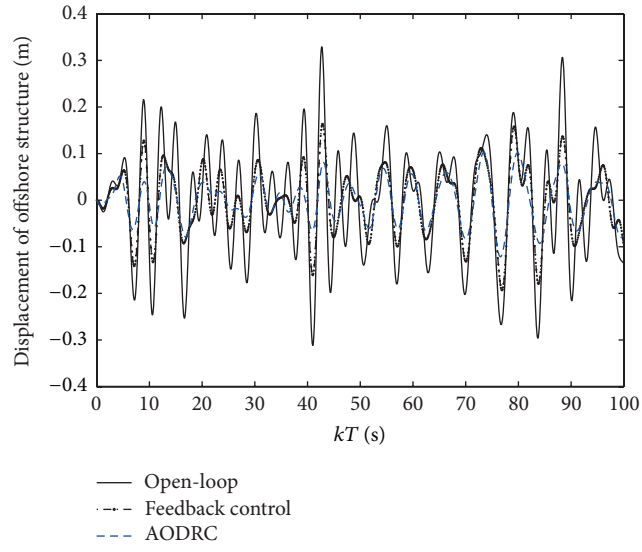

Figure 3: Curves of the displacement of offshore structure.

Table 3: Comparison of values of performance index with different delayed input.

	$h = 2$	$h = 4$	$h = 6$	$h = 8$
Open-loop	5.0953	5.0953	5.0953	5.0953
Feedback Control	4.1578 (18.40%)	3.5232 (30.85%)	3.0731 (39.69%)	3.1232 (38.70%)
AODRC	1.6695 (67.23%)	1.6875 (66.88%)	1.7076 (66.49%)	1.7366 (65.92%)

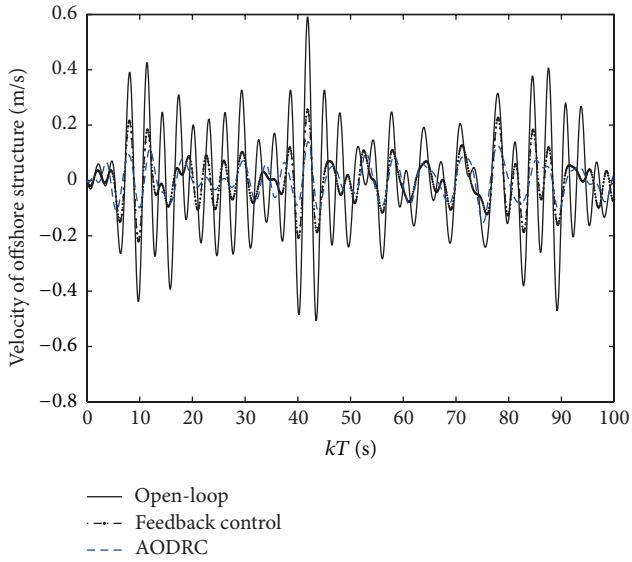

Figure 4: Curves of the velocity of offshore structure.

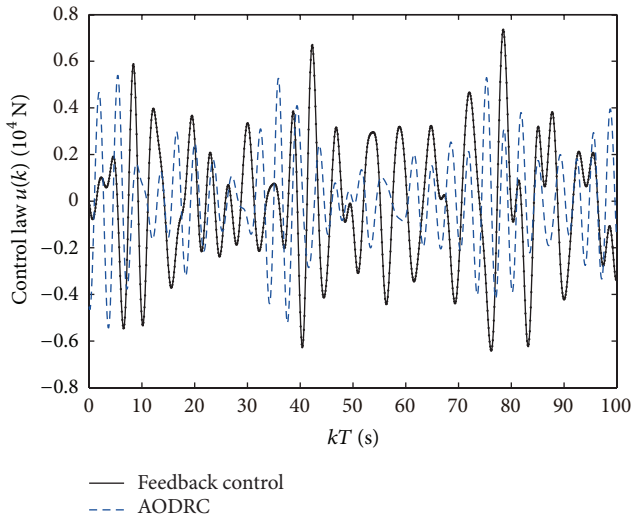

Figure 5: Curves of the control law.

In order to present the control results more intuitive, the maximum displacement, the maximum acceleration of offshore structure, and the maximum control law with different input delay are shown in Table 4. From Table 4, it can be seen clearly that the maximum displacement, the maximum acceleration of offshore structure, and the values of maximum control law with different input delay are close. The results show that the proposed control law could effectively eliminate the influences from different input delay and wave forces with high value.

It can be seen from Figures 3–5 and Tables 3-4 that the disturbances of wave force for offshore structure is reduced significantly by using the proposed AODRC. With the increase of delayed input, the proposed AODRC is still able to effectively control the displacement and velocity of offshore structure. What's more, the quadratic performance index is reduced significantly by using the proposed AODRC compared by the feedback control and open-loop ones. Therefore, we can conclude that the proposed AODRC is efficient to reject the disturbances from wave force for offshore structure, as well as being with lower consumption.

5.2. Disturbance Rejection for Pure Mathematical Example with Two Delayed Inputs. In order to verify the elasticity of the proposed AODRC, the discrete-time system with delayed inputs h_1 and h_2 under sinusoidal disturbances is considered. The matrices of (1) are chosen as

$$A = \begin{bmatrix} 1 & 0.4 \\ -0.2 & 0.9 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, \quad (28)$$
$$B_2 = \begin{bmatrix} 0.1 \\ 0.15 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$x(0) = \begin{bmatrix} 2 & 0 \end{bmatrix}^T.$$

The external disturbance $v(t)$ is chosen as sinusoidal disturbance; the matrices of (2) are given by

$$G = \begin{bmatrix} 0.4\sqrt{5} & 0.4 \\ -0.5 & 0.4\sqrt{5} \end{bmatrix},$$
$$E = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, \quad (29)$$
$$v(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T.$$

The curves of sinusoidal disturbances with (29) are shown in Figure 6. Also, the quadratic performance index is chosen as (5) with $Q = 1$, $R = 2$.

Then, the responses of system states x_1 and x_2 for different cases (open-loop, feedback control only, and proposed AODRC) with delayed inputs $h_1 = 1$ and $h_2 = 4$ are

Table 4: The maximum displacement, the maximum acceleration of offshore structure, and the maximum control law.

	$h = 2$	$h = 4$	$h = 6$	$h = 8$
Maximum displacement (m)	0.126	0.130	0.146	0.150
Maximum acceleration (m^2/s)	0.171	0.175	0.178	0.181
Maximum control input (10^4 N)	0.520	0.523	0.582	0.588

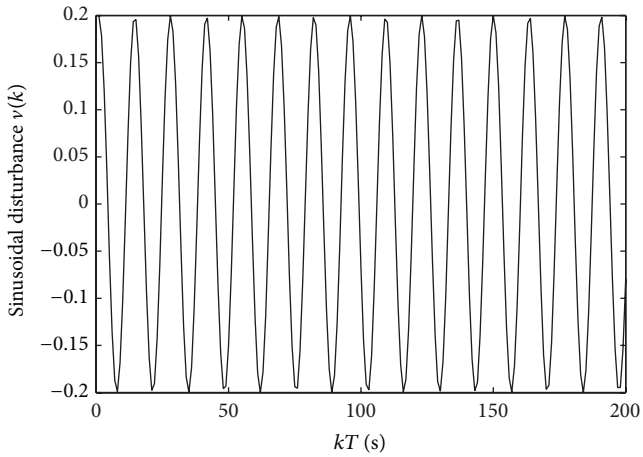

Figure 6: Curves of the sinusoidal disturbance $v(k)$.

simulated and shown in Figures 7 and 8. The curves of the proposed AODRC for different cases (the proposed AODRC and feedback control only) are shown in Figure 9. The values of quadratic performance index (5) are shown in Table 3 with two delayed inputs ($h_1 = 1, h_2 = 2$; $h_1 = 1, h_2 = 4$; $h_1 = 2, h_2 = 4$). Figures 7 and 8 and Table 5 show that the proposed AODRC is able to effectively ensure the system states in lower values and reject the disturbances from external sinusoidal input.

From Figures 1–9 and Tables 1–5, the simulation results show that the proposed AODRC for the offshore platform and pure mathematical example are effective to attenuate the vibration amplitudes of the system state, and the quadratic performance index of the controlled system under AODRC is smaller than those under the open-loop and feedback control only. Moreover, the proposed AODRC is more scalable for different cases with multiple delayed inputs.

6. Conclusion

This paper has presented an interesting approach to solve the disturbance rejection problem for discrete-time systems with multiple delayed inputs under persistent external disturbances. A discrete-time system with multiple input delays is transformed into a non-delayed system in form, and the quadratic performance index has been transformed into a relevant format without the explicit appearance of time delay. Then, the AODRC is deployed to eliminate the negative effects from multiple delayed inputs and persistent external disturbances. On the other hand, the disturbance rejection problem for jacket-type offshore structure is illustrated to validate the feasibility and effectiveness of the proposed approach.

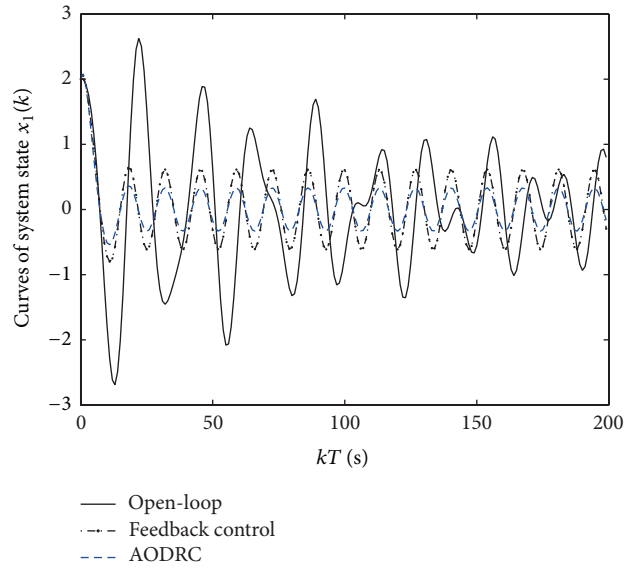

Figure 7: Curves of system state $x_1(k)$.

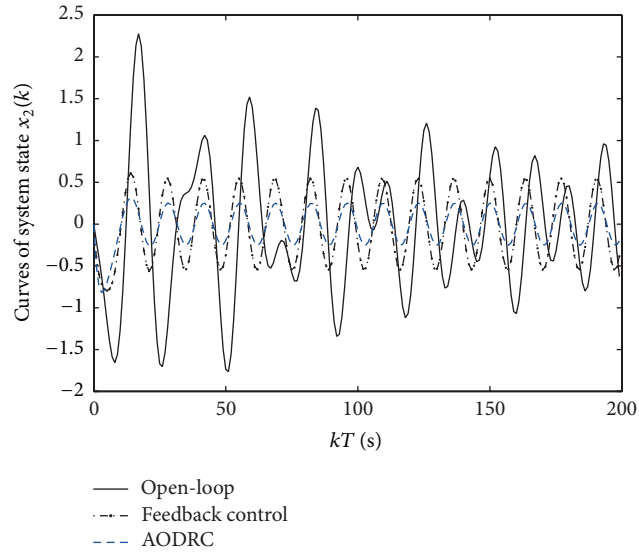

Figure 8: Curves of system state $x_2(k)$.

Table 5: Comparison of values of performance index with two delayed inputs.

	$h_1 = 1, h_2 = 2$	$h_1 = 1, h_2 = 4$	$h_1 = 2, h_2 = 4$
Open-loop	0.3422	0.3422	0.3422
Feedback Control	0.0836 (75.57%)	0.1055 (69.17%)	0.1035 (69.75%)
AODRC	0.0372 (89.13%)	0.0506 (85.21%)	0.0901 (66.49%)

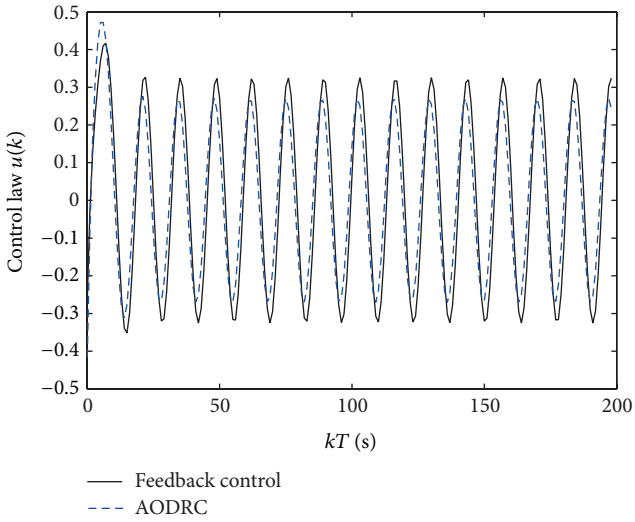

Figure 9: Curves of the control law.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Adaptive Output Feedback Sliding Mode Control for Complex Interconnected Time-Delay Systems

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We extend the decentralized output feedback sliding mode control (SMC) scheme to stabilize a class of complex interconnected time-delay systems. First, sufficient conditions in terms of linear matrix inequalities are derived such that the equivalent reduced-order system in the sliding mode is asymptotically stable. Second, based on a new lemma, a decentralized adaptive sliding mode controller is designed to guarantee the finite time reachability of the system states by using output feedback only. The advantage of the proposed method is that two major assumptions, which are required in most existing SMC approaches, are both released. These assumptions are (1) disturbances are bounded by a known function of outputs and (2) the sliding matrix satisfies a matrix equation that guarantees the sliding mode. Finally, a numerical example is used to demonstrate the efficacy of the method.

1. Introduction

Advancement in the field of engineering has led to increasingly complex large-scale systems [1]. In addition, time-delay systems often feature in real-world problems, for example, chemical processes, biological systems, economic systems, and hydraulic/pneumatic systems. Time delay commonly leads to a degradation and/or instability in system performance (e.g., [2, 3]). The stability of interconnected time-delay systems has therefore been the focus of much research, which has achieved useful results [4–8]. However, the solutions proposed by previous studies necessarily require that all state variables are available for measurements.

In many practical systems, the state variables are not accessible for direct measurement or the number of measuring devices is limited. Recently, various control approaches have been employed to overcome the above obstacles. In [9–11], based on the assumption that each isolated subsystem is of triangular form and includes internal dynamics, a class of decentralized stabilizing dynamic output feedback controller was proposed for interconnected time-delay systems. In [12], based on two adaptive neural networks, a class of decentralized stabilizing output feedback controllers was proposed for a class of uncertain nonlinear interconnected time-delay systems with immeasurable states and triangular structures. In [13], based on adaptive fuzzy control theory, a decentralized robust output feedback controller was proposed for a class of strict-feedback nonlinear interconnected time-delay systems. In [14], a new adaptive robust state observer was designed for a class of uncertain interconnected systems with multiple time-varying delays. By including fuzzy logic systems and fuzzy state observer, the authors of [15] presented an adaptive decentralized fuzzy output feedback control for interconnected systems when system states cannot be measured. The work in [16] investigated the issue of robust and reliable decentralized H_∞ tracking control for fuzzy interconnected time-delay systems. In [1], based on Lyapunov stability theory and the corresponding linear matrix inequalities (LMI), the design of a dynamic output feedback controller was proposed for uncertain interconnected systems of neutral type. The authors of [17] proposed two new stability criteria of the synchronization state for interconnected time-delay systems. The above work obtained important results related to decentralized control using only output variables. However, it should be noted that most of the existing results for interconnected time-delay

systems can only be obtained when the systems conform to a special structure [9–13]. The approaches proposed by [14–17] cannot be applied for interconnected time-delay systems with mismatched parameter uncertainties in the state matrix of each isolated subsystem. Therefore, it is important to develop a decentralized adaptive output feedback sliding mode control (SMC) law to stabilize interconnected time-delay systems with a more general structure.

Sliding mode control is a robust fast-response control strategy that has been successfully applied to a wide variety of practical engineering systems [2, 3, 18]. Generally speaking, SMC is attained by applying a discontinuous control law to drive state trajectories onto a sliding surface and force them to remain on it thereafter (this process is called reaching phase), and then to keep the state trajectories moving along the surface towards the origin with the desired performance (such motion is called sliding mode) [2, 3, 18]. Earlier work on decentralized adaptive SMC mainly focused on interconnected systems or nonlinear systems that satisfy the matching condition [19–22]. If the matching condition is not satisfied, then the mismatched uncertainty will affect the dynamics of the system in sliding mode. Thus, system behavior in sliding mode is not invariant to mismatched uncertainty. Many techniques, such as [23–25], have been applied to deal with mismatched uncertainties in sliding mode. The authors of [23] proposed a decentralized SMC law for a class of mismatched uncertain interconnected systems by using two sets of switching surfaces. In [24], a decentralized dynamic output feedback based on a linear controller was proposed for the same systems. In [25], by using a multiple-sliding surface, a new control scheme was presented for a class of decentralized multi-input perturbed systems. However, time delays are not included in the above approaches [23–25]. The existence of delay usually leads to a degradation and/or instability in system performance [2, 3]. In the limited available literature, results on applying sliding mode techniques to interconnected time-delay systems are very few [2, 3, 18]. A decentralized model reference adaptive control scheme was proposed for interconnected time-delay systems in [18]. An interconnected time-delayed system with dead-zone input via SMC in which all system state variables are available for feedback was considered in [2]. The authors of [3] investigated the global decentralized stabilization of a class of interconnected time-delay systems with known and uncertain interconnections. Their proposed approach uses only output variables. Based on Lyapunov stability theory, they designed a composite sliding surface and analyzed the stability of the associated sliding motion. As a result, the stability of interconnected time-delay systems is assured under certain conditions, the most important of which are that the disturbances must be bounded by a known function of outputs and that the sliding matrix must satisfy a matrix equation in order to guarantee sliding mode. However, in practical cases, these assumptions are difficult to achieve. Therefore, it would be worthwhile to design a decentralized adaptive output feedback SMC scheme for complex interconnected time-delay systems with a more general structure in which two of the above limitations are eliminated. To the best of our knowledge, no decentralized adaptive output feedback SMC scheme has so far been proposed for interconnected time-delay systems with unknown disturbance, mismatched parameter uncertainties in the state matrix, and mismatched interconnections and without the measurements of the states.

In this technical note, we extend the concept of decentralized output feedback sliding mode controller, introduced by Yan et al. in [3], for the aim of stabilizing complex interconnected time-delay systems. The main contributions of this paper are as follows.

(i) The interconnected time-delay systems investigated in this study include mismatched parameter uncertainties in the state matrix, mismatched interconnections, and unknown disturbance. Therefore, we consider a more general structure than the one considered in [2, 3, 18–25].

(ii) This approach uses the output information completely in the sliding surface and controller design. Therefore, conservatism is reduced and robustness is enhanced.

(iii) The two major limitations in [3] are both eliminated (disturbances must be bounded by a known function of outputs and the sliding matrix must satisfy a matrix equation in order to guarantee sliding mode). Hence, the proposed method can be applied to a wider class of interconnected time-delay systems.

Notation. The notation used throughout this paper is fairly standard. X^T denotes the transpose of matrix X. I_n and $0_{n\times m}$ are used to denote the $n\times n$ identity matrix and the $n\times m$ zero matrix, respectively. The subscripts n and $n\times m$ are omitted where the dimension is irrelevant or can be determined from the context. $\|x\|$ stands for the Euclidean norm of vector x and $\|A\|$ stands for the matrix induced norm of the matrix A. The expression $A > 0$ means that A is a symmetric positive definite. R^n denotes the n-dimensional Euclidean space. For the sake of simplicity, sometimes function $x_i(t)$ is denoted by x_i.

2. Problem Formulations and Preliminaries

We consider a class of interconnected time-delay systems that is decomposed into L subsystems. The state space representation of each subsystem is described as follows:

$$\begin{aligned}\dot{x}_i &= \left(A_i + \Delta A_i\right) x_i + B_i \left(u_i + G_i\left(t, x_i, x_{id_i}\right)\right) \\ &\quad + \sum_{\substack{j=1 \\ j\neq i}}^{L} \left[H_{ij} + \Delta H_{ij}\left(t, x_j, x_{jd_j}\right)\right] x_{jd_j}, \\ y_i &= C_i x_i,\end{aligned} \tag{1}$$

where $x_i \in R^{n_i}$, $u_i \in R^{m_i}$, and $y_i \in R^{p_i}$ with $m_i < p_i < n_i$ are the state variables, inputs, and outputs of the ith subsystem, respectively. The triplet (A_i, B_i, C_i) and H_{ij} represent known constant matrices of appropriate dimensions. The notations $x_{id_i} := x_i(t - d_i)$ and $y_{id_i} := y_i(t - d_i)$ represent delayed states and delayed outputs, respectively. The symbol $d_i := d_i(t)$ is

the time-varying delay, which is assumed to be known and is bounded by $\overline{d}_i$ for all d_i where $\overline{d}_i > 0$ is constant. The initial conditions are given by $x_i(t) = \chi_i(t)$ $(t \in [-\overline{d}_i, 0])$, where $\chi_i(t)$ are continuous in $[-\overline{d}_i, 0]$ for $i = 1, 2, 3, \ldots, L$. The matrices $\Delta A_i x_i$ and $\Delta H_{ij}(t, x_j, x_{jd_j})$ represent mismatched parameter uncertainties in the state matrix and mismatched uncertain interconnections with $\text{rank}[B_i : \Delta A_i : \Delta H_{ij}] > \text{rank}(B_i) = m_i$. The matrix $B_i G_i(t, x_i, x_{id_i})$ is the disturbance input. In this paper, only output variables y_i are assumed to be available for measurements.

For system (1), the following basic assumptions are made for each subsystem in this paper.

Assumption 1. All the pairs (A_i, B_i) are completely controllable.

Assumption 2. The matrices B_i and C_i are full rank and $\text{rank}(C_i B_i) = m_i$.

Assumption 3. The exogenous disturbance $G_i(t, x_i, x_{id_i})$ is assumed to be bounded and to satisfy the following condition:

$$\left\| G_i\left(t, x_i, x_{id_i}\right) \right\| \le c_i + b_i \left\| x_i \right\|, \tag{2}$$

where b_i and c_i are unknown bounds which are not easily obtained due to the complicated structure of the uncertainties in practical control systems.

Assumption 4. The mismatched parameter uncertainties in the state matrix of each isolated subsystem are satisfied as $\Delta A_i = D_i \Delta F_i(x_i(t), t) E_i$, where $\Delta F_i(x_i(t), t)$ is unknown but bounded as $\|\Delta F_i(x_i(t), t)\| \le 1$ and D_i, E_i are known matrices of appropriate dimensions.

Assumption 5. The mismatched uncertain interconnections are given as $\Delta H_{ij} = D_{ij} \Delta F_{ij}(t, x_j, x_{jd_j}) E_{ij}$, where $\Delta F_{ij}(t, x_j, x_{jd_j})$ is unknown but bounded as $\|\Delta F_{ij}(t, x_j, x_{jd_j})\| \le 1$ and D_{ij}, E_{ij} are any nonzero matrices of appropriate dimensions.

Remark 1. Assumption $\text{rank}(C_i B_i) = m_i$ is a limitation on the triplet (A_i, B_i, C_i) and has been utilized in most existing output feedback SMCs, for example [3, 26, 27]. This assumption guarantees the existence of the output sliding surface. Assumptions 4 and 5 were used in [6, 27].

Remark 2. There are two major assumptions in [3].

(i) The exogenous disturbances are bounded by a known function of outputs y_i. That is $\|G_i(t, x_i, x_{id_i})\| \le g_i(t, y_i, y_{id_i})$, where $g_i(t, y_i, y_{id_i})$ is known. This condition is quite restrictive.

(ii) The sliding matrix F_i satisfies $\Gamma_i C_i = F_i C_i A_i$ to guarantee sliding condition $S_i(x_i) = F_i y_i = 0$. This limitation is really quite strong.

In this paper, a decentralized adaptive output feedback SMC scheme is proposed for complex interconnected time-delay systems where the two above limitations are eliminated.

For later use, we will need the following lemma.

Lemma 3 (see [3, 26]). *Consider the following interconnected system:*

$$\begin{aligned} \dot{x}_i &= A_{ii} x_i + B_i u_i + \sum_{\substack{j=1 \\ j \ne i}}^{L} A_{ij} x_j, \\ y_i &= C_i x_i, \end{aligned} \tag{3}$$

where $x_i \in R^{n_i}$, $u_i \in R^{m_i}$, and $y_i \in R^{p_i}$ are the state variables, inputs, and outputs of the ith subsystem, respectively. Under assumption $\text{rank}(C_i B_i) = m_i$, it follows from [3, 26] that there exists a coordinate transformation $x_i \to z_i = T_i x_i$ such that the interconnected system (3) has the following regular form:

$$\begin{aligned} \dot{z}_i &= \begin{bmatrix} A_{ii1} & A_{ii2} \\ A_{ii3} & A_{ii4} \end{bmatrix} z_i + \sum_{\substack{j=1 \\ j \ne i}}^{L} \begin{bmatrix} A_{ij1} & A_{ij2} \\ A_{ij3} & A_{ij4} \end{bmatrix} z_j + \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} u_i, \\ y_i &= \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \end{aligned} \tag{4}$$

where $T_i A_{ii} T_i^{-1} = \left[\begin{smallmatrix} A_{ii1} & A_{ii2} \\ A_{ii3} & A_{ii4} \end{smallmatrix}\right]$, $T_i A_{ij} T_j^{-1} = \left[\begin{smallmatrix} A_{ij1} & A_{ij2} \\ A_{ij3} & A_{ij4} \end{smallmatrix}\right]$, and $T_i B_i = \left[\begin{smallmatrix} 0 \\ B_{i2} \end{smallmatrix}\right]$, $C_i T_i^{-1} = \begin{bmatrix} 0 & C_{i2} \end{bmatrix}$. The matrices $B_{i2} \in R^{m_i \times m_i}$ and $C_{i2} \in R^{p_i \times p_i}$ are nonsingular and A_{ii1} is stable.

3. Sliding Mode Control Design for Complex Interconnected Time-Delay Systems

In this section, we design a new decentralized adaptive output feedback SMC scheme for the system (1). There are three steps involved in the design of our decentralized adaptive output feedback SMC scheme. In the first step, a proper sliding function is constructed such that the sliding surface is designed to be dependent on output variables only. In the second step, we derive sufficient conditions in terms of LMI for the existence of a sliding surface guaranteeing asymptotic stability of the sliding mode dynamic. In the final step, based on a new Lemma, we design a decentralized adaptive output feedback sliding mode controller, which assures that the system states reach the sliding surface in finite time and stay on it thereafter.

3.1. Sliding Surface Design. Let us first design a sliding surface, which depends on only output variables. Since $\text{rank}(C_i B_i = m_i)$, it follows from Lemma 3 that there exists a coordinate transformation $z_i = T_i x_i$ such that the system (1) has the following regular form:

$$\begin{aligned} \dot{z}_i = &\left(\begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} + \begin{bmatrix} D_{i1} \\ D_{i2} \end{bmatrix} \Delta F_i \begin{bmatrix} E_{i1} & E_{i2} \end{bmatrix} \right) z_i \\ &+ \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} \left[u_i + G_i\left(t, T_i^{-1} z_i, T_i^{-1} z_{id_i}\right) \right] \end{aligned}$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{L}\left(\begin{bmatrix} H_{ij1} & H_{ij2}\\ H_{ij3} & H_{ij4}\end{bmatrix}+\begin{bmatrix} D_{ij1}\\ D_{ij2}\end{bmatrix}\Delta F_{ij}\begin{bmatrix} E_{ij1} & E_{ij2}\end{bmatrix}\right)z_{jd_j},$$

$$y_i=\begin{bmatrix}0 & C_{i2}\end{bmatrix}z_i, \tag{5}$$

where $T_i=\begin{bmatrix} T_{i1}\\ T_{i2}\end{bmatrix}$, $T_i^{-1}=\begin{bmatrix} W_{i1} & W_{i2}\end{bmatrix}$, $T_iA_iT_i^{-1}=\begin{bmatrix} A_{i1} & A_{i2}\\ A_{i3} & A_{i4}\end{bmatrix}$, $T_iH_{ij}T_j^{-1}=\begin{bmatrix} H_{ij1} & H_{ij2}\\ H_{ij3} & H_{ij4}\end{bmatrix}$, $T_iD_i\Delta F_iE_iT_i^{-1}=\begin{bmatrix} D_{i1}\\ D_{i2}\end{bmatrix}\Delta F_i\begin{bmatrix} E_{i1} & E_{i2}\end{bmatrix}$, $T_iD_{ij}\Delta F_{ij}E_{ij}T_j^{-1}=\begin{bmatrix} D_{ij1}\\ D_{ij2}\end{bmatrix}\Delta F_{ij}\begin{bmatrix} E_{ij1} & E_{ij2}\end{bmatrix}$, and $T_iB_i=\begin{bmatrix} 0\\ B_{i2}\end{bmatrix}$, $C_iT_i^{-1}=\begin{bmatrix}0 & C_{i2}\end{bmatrix}$. The matrices $B_{i2}\in R^{m_i\times m_i}$ and $C_{i2}\in R^{p_i\times p_i}$ are non-singular and A_{i1} is stable.

Letting $z_i=\begin{bmatrix} z_{i1}\\ z_{i2}\end{bmatrix}$, where $z_{i1}\in R^{n_i-m_i}$ and $z_{i2}\in R^{m_i}$, the first equation of (5) can be rewritten as

$$\begin{aligned}\dot z_{i1}&=\left(A_{i1}+D_{i1}\Delta F_iE_{i1}\right)z_{i1}+\left(A_{i2}+D_{i1}\Delta F_iE_{i2}\right)z_{i2}\\&\quad+\sum_{\substack{j=1\\ j\neq i}}^{L}\Big[\left(H_{ij1}+D_{ij1}\Delta F_{ij}E_{ij1}\right)z_{j1d_j}\\&\qquad+\left(H_{ij2}+D_{ij1}\Delta F_{ij}E_{ij2}\right)z_{j2d_j}\Big],\end{aligned} \tag{6}$$

$$\begin{aligned}\dot z_{i2}&=\left(A_{i3}+D_{i2}\Delta F_iE_{i1}\right)z_{i1}+\left(A_{i4}+D_{i2}\Delta F_iE_{i2}\right)z_{i2}\\&\quad+B_{i2}\left[u_i+G_i\left(t,T_i^{-1}z_i,T_i^{-1}z_{id_i}\right)\right]\\&\quad+\sum_{\substack{j=1\\ j\neq i}}^{L}\Big[\left(H_{ij3}+D_{ij2}\Delta F_{ij}E_{ij1}\right)z_{j1d_j}\\&\qquad+\left(H_{ij4}+D_{ij2}\Delta F_{ij}E_{ij2}\right)z_{j2d_j}\Big].\end{aligned} \tag{7}$$

Obviously, the system (6) represents the sliding-motion dynamic of the system (5), and, hence, the corresponding sliding surface can be chosen as follows:

$$\sigma_i\left(x_i\right)=K_iC_{i2}^{-1}y_i=0, \tag{8}$$

where $K_i=\begin{bmatrix} F_{i1} & F_{i2}\end{bmatrix}=\begin{bmatrix}0_{m_i\times(p_i-m_i)} & F_{i2}\end{bmatrix}$, $F_{i2}=\Xi_iP_i\Xi_i^T$, the matrix $P_i\in R^{(n_i-m_i)\times(n_i-m_i)}$ is defined later, and the matrix $\Xi_i\in R^{m_i\times(n_i-m_i)}$ is selected such that F_{i2} is nonsingular. Then, by using the second equation of (5), we have

$$\begin{aligned}\sigma_i\left(x_i\right)&=K_iC_{i2}^{-1}y_i=K_iC_{i2}^{-1}\begin{bmatrix}0 & C_{i2}\end{bmatrix}z_i\\&=K_i\begin{bmatrix} N_i & 0_{(p_i-m_i)\times m_i}\\ 0_{m_i\times(n_i-m_i)} & I_{m_i}\end{bmatrix}z=\begin{bmatrix}F_{i1}N_i & F_{i2}\end{bmatrix}z_i\\&=F_{i2}z_{i2}=0,\end{aligned} \tag{9}$$

where $N_i=[0_{(p_i-m_i)\times(n_i-p_i)}\quad I_{(p_i-m_i)}]$. In addition, the Newton-Leibniz formula is defined as

$$z_{i2d_i}=z_{i2}\left(t-d_i\right)=z_{i2}\left(t\right)-\int_{t-d_i}^{t}\dot z_{i2}\left(s\right)ds. \tag{10}$$

Therefore, in sliding modes $\sigma_i(x_i)=0$ and $\dot\sigma_i(x_i)=0$, we have $z_{i2}=0$ and $z_{j2d_j}=0$. Then, from the structure of systems (6)-(7), the sliding mode dynamics of the system (1) associated with the sliding surface (8) is described by

$$\dot z_{i1}=\left(A_{i1}+D_{i1}\Delta F_iE_{i1}\right)z_{i1}+\sum_{\substack{j=1\\ j\neq i}}^{L}\left(H_{ij1}+D_{ij1}\Delta F_{ij}E_{ij1}\right)z_{j1d_j}. \tag{11}$$

3.2. Asymptotically Stable Conditions by LMI Theory. Now we are in position to derive sufficient conditions in terms of linear matrix inequalities (LMI) such that the dynamics of the system (11) in the sliding surface (8) is asymptotically stable. Let us begin with considering the following LMI:

$$\begin{bmatrix}\Psi_i & P_iD_{i1} & E_{i1}^T\\ D_{i1}^TP_i & -\varphi_iI_{m_i} & 0\\ E_{i1} & 0 & -\varphi_i^{-1}I_{m_i}\end{bmatrix}<0,\quad i=1,2,\ldots,L, \tag{12}$$

where $\Psi_i=A_{i1}^TP_i+P_iA_{i1}+((L-1)/\varepsilon_i)P_i+\sum_{j=1,j\neq i}^{L}(q\varepsilon_jH_{ji1}^TP_jH_{ji1}+\overline{\varphi}_i^{-1}P_iD_{ij1}D_{ij1}^TP_i+\widehat{q}\overline{\varphi}_jE_{ji1}^TE_{ji1})$, $P_i\in R^{(n_i-m_i)\times(n_i-m_i)}$ is any positive matrix, and L is the number of subsystems and the scalars $q>1$, $\widehat{q}>1$, $\varphi_i>0$, $\varepsilon_i>0$, $\overline{\varphi}_i>0$, $i=1,2,\ldots,L$. Then, we can establish the following theorem.

Theorem 4. *Suppose that LMI (12) has solution $P_i>0$ and the scalars $q>1$, $\widehat{q}>1$, $\varphi_i>0$, $\varepsilon_i>0$, $\overline{\varphi}_i>0$, $i=1,2,\ldots,L$. Suppose also that the SMC law is*

$$u_i\left(t\right)=-\left(F_{i2}B_{i2}\right)^{-1}\left(\kappa_i\eta_i\left(t\right)+\overline{\kappa}_i\left\|y_i\right\|+\widehat{\kappa}_i\left\|y_{id_i}\right\|+\zeta_i\left(t\right)+\alpha_i\right)\frac{\sigma_i}{\left\|\sigma_i\right\|},\quad i=1,2,\ldots,L, \tag{13}$$

where $\kappa_i=\|F_{i2}\|(\|A_{i3}\|+\|D_{i2}\|\|E_{i1}\|)+\sum_{j=1,j\neq i}^{L}\beta_i\|F_{j2}\|(\|H_{ji3}\|+\|D_{ji2}\|\|E_{ji1}\|)$, $\overline{\kappa}_i=\|F_{i2}\|(\|A_{i4}\|+\|D_{i2}\|\|E_{i2}\|)\|F_{i2}^{-1}\|\|K_iC_{i2}^{-1}\|$, $\widehat{\kappa}_i=\sum_{j=1,j\neq i}^{L}\|F_{j2}\|(\|H_{ji4}\|+\|D_{ji2}\|\|E_{ji2}\|)\|F_{i2}^{-1}\|\|K_iC_{i2}^{-1}\|$, and the scalars $\alpha_i>0$, $\beta_i>1$, and the time functions $\zeta_i(t)$ and $\eta_i(t)$ will be designed later. The sliding surface is given by (8). Then, the dynamics of system (11) restricted to the sliding surface $\sigma_i(x_i)=0$ is asymptotically stable.

Before proofing Theorem 4, we recall the following lemmas.

Lemma 5 (see [27]). *Let X, Y, and F be real matrices of suitable dimension with $F^TF\leq I$; then, for any scalar $\varphi>0$, the following matrix inequality holds:*

$$XFY+Y^TF^TX^T\leq\varphi^{-1}XX^T+\varphi Y^TY. \tag{14}$$

Lemma 6 (see [28]). *The linear matrix inequality:*

$$\begin{bmatrix}Q\left(x\right) & \Pi\left(x\right)\\ \Pi\left(x\right)^T & R\left(x\right)\end{bmatrix}>0, \tag{15}$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $\Pi(x)$ depend affinely on x, is equivalent to $R(x) > 0$, $Q(x) - \Pi(x)R(x)^{-1}\Pi(x)^T > 0$.

Lemma 7. *Assume that $x \in R^n$, $y \in R^n$, $N \in R^{n\times n}$, and N is a positive definite matrix. Then, the inequality*

$$x^T Ny + y^T Nx \le \frac{1}{\varepsilon} x^T Nx + \varepsilon y^T Ny \tag{16}$$

holds for all $\varepsilon > 0$.

Proof of Lemma 7. For any $n \times n$ matrix $N > 0$, $N^{1/2}$ is well defined and $N^{1/2} > 0$. Let vector

$$\vartheta = \sqrt{\frac{1}{\varepsilon}} N^{1/2} x - \sqrt{\varepsilon} N^{1/2} y. \tag{17}$$

Then, we have

$$\begin{aligned} \vartheta^T \vartheta &= \left(\sqrt{\frac{1}{\varepsilon}} N^{1/2} x - \sqrt{\varepsilon} N^{1/2} y \right)^T \left(\sqrt{\frac{1}{\varepsilon}} N^{1/2} x - \sqrt{\varepsilon} N^{1/2} y \right) \\ &= \frac{1}{\varepsilon} x^T Nx - x^T Ny - y^T Nx + \varepsilon y^T Ny. \end{aligned} \tag{18}$$

Since $\vartheta^T \vartheta \ge 0$, it is obvious that

$$x^T Ny + y^T Nx \le \frac{1}{\varepsilon} x^T Nx + \varepsilon y^T Ny. \tag{19}$$

The proof is completed. □

Proof of Theorem 4. Now we are going to prove that the system (11) is asymptotically stable. Let us first consider the following positive definition function:

$$V = \sum_{i=1}^{L} z_{i1}^T P_i z_{i1}, \tag{20}$$

where the matrix $P_i \in R^{(n_i - m_i)\times(n_i - m_i)}$ is defined in (12). Then, the time derivative of V along the state trajectories of system (11) is given by

$$\begin{aligned} \dot{V} = &\sum_{i=1}^{L} z_{i1}^T \left(A_{i1}^T P_i + P_i A_{i1} + P_i D_{i1} \Delta F_i E_{i1} \right. \\ &\left. + E_{i1}^T \Delta F_i^T D_{i1}^T P_i \right) z_{i1} \\ &+ \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \left(z_{j1d_j}^T H_{ij1}^T P_i z_{i1} + z_{i1}^T P_i H_{ij1} z_{j1d_j} \right. \\ &+ z_{i1}^T P_i D_{ij1} \Delta F_{ij} E_{ij1} z_{j1d_j} \\ &\left. + z_{j1d_j}^T E_{ij1}^T \Delta F_{ij}^T D_{ij1}^T P_i z_{i1} \right). \end{aligned} \tag{21}$$

Applying Lemma 5 to (21) yields

$$\begin{aligned} \dot{V} \le &\sum_{i=1}^{L} z_{i1}^T \left(A_{i1}^T P_i + P_i A_{i1} + \varphi_i^{-1} P_i D_{i1} D_{i1}^T P_i \right. \\ &\left. + \varphi_i E_{i1}^T E_{i1} \right) z_{i1} \\ &+ \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \left(z_{j1d_j}^T H_{ij1}^T P_i z_{i1} + z_{i1}^T P_i H_{ij1} z_{j1d_j} \right. \\ &+ \overline{\varphi}_i^{-1} z_{i1}^T P_i D_{ij1} D_{ij1}^T P_i z_{i1} \\ &\left. + \overline{\varphi}_i z_{j1d_j}^T E_{ij1}^T E_{ij1} z_{j1d_j} \right), \end{aligned} \tag{22}$$

where the scalars $\varphi_i > 0$ and $\overline{\varphi}_i > 0$. By Lemma 7, it follows that for any $\varepsilon_i > 0$

$$\begin{aligned} &\sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \left(z_{i1}^T P_i H_{ij1} z_{j1d_j} + z_{j1d_j}^T H_{ij1}^T P_i z_{i1} \right) \\ &\le \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \left(\frac{1}{\varepsilon_i} z_{i1}^T P_i z_{i1} + \varepsilon_i z_{j1d_j}^T H_{ij1}^T P_i H_{ij1} z_{j1d_j} \right). \end{aligned} \tag{23}$$

From (22) and (23), it is obvious that

$$\begin{aligned} \dot{V} \le &\sum_{i=1}^{L} z_{i1}^T \left(A_{i1}^T P_i + P_i A_{i1} + \varphi_i^{-1} P_i D_{i1} D_{i1}^T P_i \right. \\ &\left. + \varphi_i E_{i1}^T E_{i1} \right) z_{i1} \\ &+ \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \left(\varepsilon_i z_{j1d_j}^T H_{ij1}^T P_i H_{ij1} z_{j1d_j} + \frac{1}{\varepsilon_i} z_{i1}^T P_i z_{i1} \right. \\ &+ \overline{\varphi}_i z_{j1d_j}^T E_{ij1}^T E_{ij1} z_{j1d_j} \\ &\left. + \overline{\varphi}_i^{-1} z_{i1}^T P_i D_{ij1} D_{ij1}^T P_i z_{i1} \right). \end{aligned} \tag{24}$$

Then, by using (24) and properties

$$\begin{aligned} &\sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \varepsilon_i z_{j1d_j}^T H_{ij1}^T P_i H_{ij1} z_{j1d_j} \\ &\quad = \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \varepsilon_j z_{i1d_i}^T H_{ji1}^T P_j H_{ji1} z_{i1d_i}, \\ &\sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \overline{\varphi}_i z_{j1d_j}^T E_{ij1}^T E_{ij1} z_{j1d_j} \\ &\quad = \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\ne i}}^{L} \overline{\varphi}_j z_{i1d_i}^T E_{ji1}^T E_{ji1} z_{i1d_i}, \end{aligned} \tag{25}$$

it generates

$$\begin{aligned}\dot{V} \le \sum_{i=1}^{L} z_{i1}^{T} \Big(& A_{i1}^{T}P_i + P_iA_{i1} + \varphi_i^{-1}P_iD_{i1}D_{i1}^{T}P_i \\ & + \varphi_i E_{i1}^{T}E_{i1}\Big) z_{i1} \\ + \sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L} \Big(& \varepsilon_j z_{i1d_i}^{T} H_{ji1}^{T}P_jH_{ji1}z_{i1d_i} \\ & + \frac{1}{\varepsilon_i} z_{i1}^{T}P_iz_{i1} + \overline{\varphi}_j z_{i1d_i}^{T}E_{ji1}^{T}E_{ji1}z_{i1d_i} \\ & + \overline{\varphi}_i^{-1} z_{i1}^{T}P_iD_{ij1}D_{ij1}^{T}P_iz_{i1}\Big). \end{aligned} \tag{26}$$

According to Assumption 5, E_{ij} is a free-choice matrix. Therefore, we can easily select matrix E_{ij} such that the matrix $E_{ji1}^{T}E_{ji1}$ is semipositive definite. Since the z_{i1} for $i = 1, 2, \ldots, L$ are independent of each other, then, from equation (31) of paper [3], the following is true:

$$V\left(z_{11d_1}, z_{21d_2}, z_{31d_3}, \ldots, z_{n1d_n}\right) \le qV\left(z_{11}, z_{21}, z_{31}, \ldots, z_{n1}\right) \tag{27}$$

for $q > 1$, and is equivalent to

$$\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L} \varepsilon_j z_{i1d_i}^{T}H_{ji1}^{T}P_jH_{ji1}z_{i1d_i} \le q\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L} \varepsilon_j z_{i1}^{T}H_{ji1}^{T}P_jH_{ji1}z_{i1} \tag{28}$$

which implies that

$$\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L} \overline{\varphi}_j z_{i1d_i}^{T}E_{ji1}^{T}E_{ji1}z_{i1d_i} \le \widehat{q}\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L} \overline{\varphi}_j z_{i1}^{T}E_{ji1}^{T}E_{ji1}z_{i1}, \tag{29}$$

where the scalar $\widehat{q} > 1$. Thus, from (26), (28), and (29), we achieve

$$\begin{aligned}\dot{V} \le \sum_{i=1}^{L} z_{i1}^{T} \Bigg[& A_{i1}^{T}P_i + P_iA_{i1} + \varphi_i^{-1}P_iD_{i1}D_{i1}^{T}P_i \\ & + \varphi_i E_{i1}^{T}E_{i1} + \frac{L-1}{\varepsilon_i}P_i \\ & + \sum_{\substack{j=1\\ j\neq i}}^{L} \Big(q\varepsilon_j H_{ji1}^{T}P_jH_{ji1} + \overline{\varphi}_i^{-1}P_iD_{ij1}D_{ij1}^{T}P_i \\ & + \widehat{q}\,\overline{\varphi}_j E_{ji1}^{T}E_{ji1}\Big) \Bigg] z_{i1}. \end{aligned} \tag{30}$$

In addition, by applying Lemma 6, LMI (12) is equivalent to the following inequality:

$$\begin{aligned} & A_{i1}^{T}P_i + P_iA_{i1} + \varphi_i E_{i1}^{T}E_{i1} \\ & + \frac{L-1}{\varepsilon_i}P_i + \varphi_i^{-1}P_iD_{i1}D_{i1}^{T}P_i \\ & + \sum_{\substack{j=1\\ j\neq i}}^{L} \Big(q\varepsilon_j H_{ji1}^{T}P_jH_{ji1} \\ & + \overline{\varphi}_i^{-1}P_iD_{ij1}D_{ij1}^{T}P_i + \widehat{q}\,\overline{\varphi}_j E_{ji1}^{T}E_{ji1}\Big) < 0. \end{aligned} \tag{31}$$

According to (30) and (31), it is easy to get

$$\dot{V} < 0. \tag{32}$$

The inequality (32) shows that LMI (12) holds, which further implies that the sliding motion (11) is asymptotically stable. □

Remark 8. Theorem 4 provides a new existence condition of the sliding surface in terms of strict LMI, which can be easily worked out using the LMI toolbox in Matlab.

Remark 9. Compared to recent LMI methods [1, 5–7], the proposed method offers less number of matrix variables in LMI equations making it easier to find a feasible solution.

In order to design a new decentralized adaptive output feedback sliding mode control scheme for complex interconnected time-delay system (1), we establish the following lemma.

Lemma 10. *Consider a class of interconnected time-delay systems that is decomposed into L subsystems*

$$\dot{\nu}_i = \left(A_{ii} + \Delta A_{ii}\right)\nu_i + \sum_{\substack{j=1\\ j\neq i}}^{L} A_{ij}\nu_{jd_j}, \tag{33}$$

where $\nu_i = \left[\begin{smallmatrix}\nu_{i1}\\ \nu_{i2}\end{smallmatrix}\right]$ *are the state variables of the ith subsystem with* $\nu_{i1} \in R^{n_i-m_i}$ *and* $\nu_{i2} \in R^{m_i}$. *The matrix* $A_{ii} = \left[\begin{smallmatrix}A_{ii1} & A_{ii2}\\ A_{ii3} & A_{ii4}\end{smallmatrix}\right]$ *is known matrices of appropriate dimensions. The matrices* $\Delta A_{ii} = \left[\begin{smallmatrix}\Delta A_{ii1} & \Delta A_{ii2}\\ \Delta A_{ii3} & \Delta A_{ii4}\end{smallmatrix}\right]$ *and* $A_{ij} = \left[\begin{smallmatrix}A_{ij1} & A_{ij2}\\ A_{ij3} & A_{ij4}\end{smallmatrix}\right]$ *are unknown matrices of appropriate dimensions. The notation* $\nu_{id_i} := \nu_i(t - d_i)$ *represents delayed states. The symbol* $d_i := d_i(t)$ *is the time-varying delay, which is assumed to be known and is bounded by* $\overline{d}_i$ *for all* d_i. *The initial conditions are given by* $\nu_i(t) = \chi_i(t)$ $(t \in [-\overline{d}_i, 0])$, *where* $\chi_i(t)$ *are continuous in* $[-\overline{d}_i, 0]$ *for* $i = 1, 2, 3, \ldots, L$. *If the matrix* A_{ii1} *is stable then* $\sum_{i=1}^{L} \|\nu_{i1}(t)\|$

is bounded by $\sum_{i=1}^{L} \phi_i(t)$ *for all time, where* $\phi_i(t)$ *is the solution of*

$$\dot{\phi}_i(t) = \widehat{k}_i \phi_i(t) + k_i \left[\left(\|A_{ii2}\| + \|\Delta A_{ii2}\| \right) \|v_{i2}\| + \sum_{\substack{j=1 \\ j \neq i}}^{L} \|A_{ji2}\| \left\| v_{i2d_i} \right\| \right], \quad i = 1, 2, \ldots, L \tag{34}$$

in which $\widehat{k}_i = k_i(\|\Delta A_{ii1}\| + \sum_{j=1, j\neq i}^{L} \beta_i \|A_{ji1}\|) + \lambda_i < 0$, $k_i > 0$. λ_i *is the maximum eigenvalue of the matrix* A_{ii1} *and the scalar* $\beta_i > 1$.

Proof of Lemma 10. We are now in the position to prove Lemma 10. From (33), it is obvious that

$$\dot{v}_{i1}(t) = \left(A_{ii1} + \Delta A_{ii1}\right) v_{i1} + \left(A_{ii2} + \Delta A_{ii2}\right) v_{i2} + \sum_{\substack{j=1 \\ j \neq i}}^{L} \left(A_{ij1} v_{j1d_j} + A_{ij2} v_{j2d_j} \right). \tag{35}$$

From system (35), we have

$$v_{i1}(t) = \exp\left(A_{ii1}\right) v_{i1}(0) + \int_0^t \exp\left(A_{ii1}(t-\tau)\right) \times \left[\Delta A_{ii1} v_{i1} + \left(A_{ii2} + \Delta A_{ii2}\right) v_{i2} + \sum_{\substack{j=1 \\ j \neq i}}^{L} \left(A_{ij1} v_{j1d_j} + A_{ij2} v_{j2d_j} \right) \right] d\tau. \tag{36}$$

According to (36), we obtain

$$\begin{aligned} \|v_{i1}(t)\| \leq{}& \|\exp\left(A_{ii1}t\right)\| \|v_{i1}(0)\| \\ &+ \int_0^t \|\exp\left(A_{ii1}(t-\tau)\right)\| \left(\|A_{ii2}\| + \|\Delta A_{ii2}\|\right) \times \|v_{i2}\| \, d\tau \\ &+ \int_0^t \|\exp\left(A_{ii1}(t-\tau)\right)\| \left[\|\Delta A_{ii1}\| \|v_{i1}\| + \sum_{\substack{j=1 \\ j \neq i}}^{L} \|A_{ij1}\| \left\| v_{j1d_j} \right\| + \sum_{\substack{j=1 \\ j \neq i}}^{L} \|A_{ij2}\| \left\| v_{j2d_j} \right\| \right] d\tau. \end{aligned} \tag{37}$$

The stable matrix A_{ii1} implies that $\|\exp(A_{ii1}t)\| \leq k_i \exp(\lambda_i t)$ for some scalars $k_i > 0, i = 1, 2, \ldots, L$. Therefore, the above inequality can be rewritten as

$$\begin{aligned} \|v_{i1}(t)\| \exp\left(-\lambda_i t\right) \leq{}& k_i \|v_{i1}(0)\| + \int_0^t k_i \exp\left(-\lambda_i \tau\right) \left(\|A_{ii2}\| + \|\Delta A_{ii2}\|\right) \times \|v_{i2}\| \, d\tau \\ &+ \int_0^t k_i \exp\left(-\lambda_i \tau\right) \times \left[\|\Delta A_{ii1}\| \|v_{i1}(t)\| + \sum_{\substack{j=1 \\ j \neq i}}^{L} \|A_{ij1}\| \left\| v_{j1d_j} \right\| + \sum_{\substack{j=1 \\ j \neq i}}^{L} \|A_{ij2}\| \left\| v_{j2d_j} \right\| \right] d\tau. \end{aligned} \tag{38}$$

Let $s_i(t)$ be the right side term of the inequality (38)

$$\begin{aligned} s_i(t) = k_i \|v_{i1}(0)\| &+ \int_0^t k_i \exp\left(-\lambda_i \tau\right) \left(\|A_{ii2}\| + \|\Delta A_{ii2}\|\right) \|v_{i2}\| \, d\tau \\ &+ \int_0^t k_i \exp\left(-\lambda_i \tau\right) \left[\|\Delta A_{ii1}\| \|v_{i1}\| \right. \end{aligned}$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ij1}\right\|\left\|v_{j1d_j}\right\| +\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ij2}\right\|\left\|v_{j2d_j}\right\|\Bigg] d\tau. \tag{39}$$

Then, by taking the time derivative of $s_i(t)$, we can get that

$$\begin{aligned}\frac{d}{dt}s_i(t) &= k_i\exp(-\lambda_i t)\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\| \\ &\quad + k_i\exp(-\lambda_i t)\Bigg[\left\|\Delta A_{ii1}\right\|\left\|v_{i1}\right\| \\ &\quad +\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ij1}\right\|\left\|v_{j1d_j}\right\| \\ &\quad +\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ij2}\right\|\left\|v_{j2d_j}\right\|\Bigg].\end{aligned} \tag{40}$$

For the above equation, we multiply the term $(1/k_i)\exp(\lambda_i t)$ on both sides; then

$$\begin{aligned}\frac{1}{k_i}\exp(\lambda_i t)\frac{d}{dt}s_i(t) &= \left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\| \\ &\quad +\left\|\Delta A_{ii1}\right\|\left\|v_{i1}\right\|+\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ij1}\right\|\left\|v_{j1d_j}\right\| \\ &\quad +\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ij2}\right\|\left\|v_{j2d_j}\right\|.\end{aligned} \tag{41}$$

Then, by taking the summation of both sides of the above equation, we have

$$\begin{aligned}&\sum_{i=1}^{L}\frac{1}{k_i}\exp(\lambda_i t)\frac{d}{dt}s_i(t) \\ &\quad=\sum_{i=1}^{L}\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\|+\sum_{i=1}^{L}\left\|\Delta A_{ii1}\right\|\left\|v_{i1}\right\| \\ &\quad\quad+\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji2}\right\|\left\|v_{i2d_i}\right\|+\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji1}\right\|\left\|v_{i1d_i}\right\|.\end{aligned} \tag{42}$$

Since the v_{i1} for $i=1,2,\ldots,L$ are independent of each other, then, from equation (32) of paper [3], it is clear that

$$\left\|v_{i1d_i}\right\|\leq\beta_i\left\|v_{i1}\right\|,\quad i=1,2,\ldots,L \tag{43}$$

for some scalars $\beta_i>1, i=1,2,\ldots,L$. Then, by substituting (43) into (42), we achieve

$$\begin{aligned}&\sum_{i=1}^{L}\frac{1}{k_i}\exp(\lambda_i t)\frac{d}{dt}s_i(t) \\ &\quad=\sum_{i=1}^{L}\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\|+\sum_{i=1}^{L}\left\|\Delta A_{ii1}\right\|\left\|v_{i1}\right\| \\ &\quad\quad+\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji2}\right\|\left\|v_{i2d_i}\right\|+\sum_{i=1}^{L}\sum_{\substack{j=1\\ j\neq i}}^{L}\beta_i\left\|A_{ji1}\right\|\left\|v_{i1}\right\|.\end{aligned} \tag{44}$$

For the above equation, we multiply the term $k_i\exp(-\lambda_i t)$ to both sides. Since $\|v_{i1}\|\exp(-\lambda_i t)\leq s_i(t)$, one can get that

$$\begin{aligned}\sum_{i=1}^{L}\frac{d}{dt}s_i(t) &\leq \sum_{i=1}^{L}k_i\exp(-\lambda_i t) \\ &\quad\times\left[\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\|+\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji2}\right\|\left\|v_{i2d_i}\right\|\right] \\ &\quad+\sum_{i=1}^{L}\overline{k}_i s_i(t),\end{aligned} \tag{45}$$

where $\overline{k}_i = k_i\left(\|\Delta A_{ii1}\|+\sum_{j=1,j\neq i}^{L}\beta_i\|A_{ji1}\|\right)$. For the above inequality, we multiply the term $\exp(-\overline{k}_i t)$ to both sides, then

$$\begin{aligned}&\sum_{i=1}^{L}\frac{d}{dt}\left[s_i(t)\exp\left(-\overline{k}_i t\right)\right] \\ &\quad\leq\sum_{i=1}^{L}k_i\exp(-\lambda_i t) \\ &\quad\quad\times\left[\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\|+\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji2}\right\|\left\|v_{i2d_i}\right\|\right] \\ &\quad\quad\times\exp\left(-\overline{k}_i t\right).\end{aligned} \tag{46}$$

Since $\|v_{i1}\|\exp(-\lambda_i t)\leq s_i(t)$, integrating the above inequality on both sides, we obtain

$$\begin{aligned}&\sum_{i=1}^{L}\left\|v_{i1}\right\| \\ &\quad\leq\sum_{i=1}^{L}k_i\left\|v_{i1}(0)\right\|\exp\left(\left(\overline{k}_i+\lambda_i\right)t\right)\end{aligned}$$

$$+\sum_{i=1}^{L}\left\{\int_0^t k_i\exp\left(-\lambda_i\tau\right)\left[\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\|+\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji2}\right\|\left\|v_{i2d_i}\right\|\right]\times\exp\left(-\overline{k}_i\tau\right)d\tau\right\}\exp\left(\overline{k}_i t\right)\exp\left(\lambda_i t\right)$$

$$=\sum_{i=1}^{L}\left\{\phi_i\left(0\right)\exp\left(\left(\overline{k}_i+\lambda_i\right)t\right)+\int_0^t k_i\exp\left[\left(\overline{k}_i+\lambda_i\right)\left(t-\tau\right)\right]\times\left[\left(\left\|A_{ii2}\right\|+\left\|\Delta A_{ii2}\right\|\right)\left\|v_{i2}\right\|+\sum_{\substack{j=1\\ j\neq i}}^{L}\left\|A_{ji2}\right\|\left\|v_{i2d_i}\right\|\right]d\tau\right\}$$

$$=\sum_{i=1}^{L}\phi_i\left(t\right)\quad\text{if }\phi_i\left(0\right)\geq k_i\left\|v_{i1}\left(0\right)\right\|, \tag{47}$$

where the time function $\phi_i(t)$ satisfies (34). Hence, we can see that $\sum_{i=1}^{L}\phi_i(t)\geq\sum_{i=1}^{L}\|v_{i1}\|$ for all time, if $\phi_i(0)$ is sufficiently large. □

Remark 11. It is obvious that the time function $\phi_i(t)$ is dependent on only state variable v_{i2}. Therefore, we can replace state variable v_{i1} by a function of state variable v_{i2} in controller design. This feature is very useful in controller design using only output variables.

3.3. Decentralized Adaptive Output Feedback Sliding Mode Controller Design. Now, we are in the position to prove that the state trajectories of system (1) reach sliding surface (8) in finite time and stay on it thereafter. In order to satisfy the above aims, the modified decentralized adaptive output feedback sliding mode controller is selected to be

$$u_i\left(t\right)=-\left(F_{i2}B_{i2}\right)^{-1}\left(\kappa_i\eta_i\left(t\right)+\overline{\kappa}_i\left\|y_i\right\|+\widehat{\kappa}_i\left\|y_{id_i}\right\|+\zeta_i\left(t\right)+\alpha_i\right)\frac{\sigma_i}{\left\|\sigma_i\right\|},\quad i=1,2,\ldots,L, \tag{48}$$

where $\kappa_i=\|F_{i2}\|(\|A_{i3}\|+\|D_{i2}\|\|E_{i1}\|)+\sum_{j=1,j\neq i}^{L}\beta_i\|F_{j2}\|(\|H_{ji3}\|+\|D_{ji2}\|\|E_{ji1}\|)$, $\overline{\kappa}_i=\|F_{i2}\|(\|A_{i4}\|+\|D_{i2}\|\|E_{i2}\|)\|F_{i2}^{-1}\|\|K_iC_{i2}^{-1}\|$, $\widehat{\kappa}_i=\sum_{j=1,j\neq i}^{L}\|F_{j2}\|(\|H_{ji4}\|+\|D_{ji2}\|\|E_{ji2}\|)\|F_{i2}^{-1}\|\|K_iC_{i2}^{-1}\|$, and the scalars $\alpha_i>0$ and $\beta_i>1$. The adaptive law is defined as

$$\zeta_i\left(t\right)\geq\widehat{b}_i\left\|F_{i2}\right\|\left\|B_{i2}\right\|\times\left(\left\|W_{i1}\right\|\eta_i\left(t\right)+\left\|W_{i2}\right\|\left\|F_{i2}^{-1}\right\|\left\|K_iC_{i2}^{-1}\right\|\left\|y_i\right\|\right)+\left\|F_{i2}\right\|\left\|B_{i2}\right\|\widehat{c}_i+\breve{q}_i\frac{c_i^2}{4}, \tag{49}$$

where $\widehat{b}_i$ and $\widehat{c}_i$ are the solution of the following equations:

$$\dot{\widehat{b}}_i=\overline{q}_i\left\|F_{i2}\right\|\left\|B_{i2}\right\|\times\left(\left\|W_{i1}\right\|\eta_i\left(t\right)+\left\|W_{i2}\right\|\left\|F_{i2}^{-1}\right\|\left\|K_iC_{i2}^{-1}\right\|\left\|y_i\right\|\right),\qquad \dot{\widehat{c}}_i=\widetilde{q}_i\left(-\breve{q}_i\widehat{c}_i+\left\|F_{i2}\right\|\left\|B_{i2}\right\|\right) \tag{50}$$

in which $\left[W_{i1}\ \ W_{i2}\right]=T_i^{-1}$ and the scalars $\overline{q}_i>0$, $\widetilde{q}_i>0$, and $\breve{q}_i>0$. The time function $\eta_i(t)$ will be designed later. It should be pointed out that controller (48) uses only output variables.

Now let us discuss the reaching conditions in the following theorem.

Theorem 12. *Suppose that LMI (12) has solution $P_i>0$ and the scalars $q>1$, $\widehat{q}>1$, $\varphi_i>0$, $\varepsilon_i>0$, $\overline{\varphi}_i>0$, $i=1,2,\ldots,L$. Consider the closed loop of system (1) with the above decentralized adaptive output feedback sliding mode controller (48) where the sliding surface is given by (8). Then, the state trajectories of system (1) reach the sliding surface in finite time and stay on it thereafter.*

Proof of Theorem 12. We consider the following positive definite function:

$$V=\sum_{i=1}^{L}\left(\left\|\sigma_i\right\|+\frac{0.5}{\overline{q}_i}\widetilde{b}_i^2+\frac{0.5}{\widetilde{q}_i}\widetilde{c}_i^2\right), \tag{51}$$

where $\widetilde{b}_i(t)=b_i-\widehat{b}_i(t)$ and $\widetilde{c}_i(t)=c_i-\widehat{c}_i(t)$. Then, the time derivative of V along the trajectories of (9) is given by

$$\dot{V}=\sum_{i=1}^{L}\left(\frac{\sigma_i^T}{\left\|\sigma_i\right\|}F_{i2}\dot{z}_{i2}-\frac{1}{\overline{q}_i}\widetilde{b}_i\dot{\widehat{b}}_i-\frac{1}{\widetilde{q}_i}\widetilde{c}_i\dot{\widehat{c}}_i\right). \tag{52}$$

Substituting (7) into (52), we have

$$\dot{V}=\sum_{i=1}^{L}\frac{\sigma_i^T}{\left\|\sigma_i\right\|}F_{i2}\left[\left(A_{i3}+D_{i2}\Delta F_iE_{i1}\right)z_{i1}+\left(A_{i4}+D_{i2}\Delta F_iE_{i2}\right)z_{i2}\right]$$

$$+\sum_{i=1}^{L}\frac{\sigma_i^T}{\|\sigma_i\|}F_{i2}B_{i2}\left(u_i+G_i\left(t,T_i^{-1}z_i,T_i^{-1}z_{id_i}\right)\right)$$
$$-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{b}_i\dot{\widehat{b}}_i-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{c}_i\dot{\widehat{c}}_i$$
$$+\sum_{i=1}^{L}\sum_{\substack{j=1\\j\neq i}}^{L}\frac{\sigma_i^T}{\|\sigma_i\|}F_{i2}\left[\left(H_{ij3}+D_{ij2}\Delta F_{ij}E_{ij1}\right)z_{j1d_j}\right.$$
$$\left.+\left(H_{ij4}+D_{ij2}\Delta F_{ij}E_{ij2}\right)z_{j2d_j}\right]. \tag{53}$$

From (53), properties $\|AB\| \leq \|A\|\|B\|$ and $\|\Delta F_i\| \leq 1$, $\|\Delta F_{ij}\| \leq 1$ generate

$$\begin{aligned}\dot{V}\leq{}&\sum_{i=1}^{L}\|F_{i2}\|\left[\left(\|A_{i3}\|+\|D_{i2}\|\,\|E_{i1}\|\right)\|z_{i1}\|\right.\\&\left.+\left(\|A_{i4}\|+\|D_{i2}\|\,\|E_{i2}\|\right)\|z_{i2}\|\right]\\&+\sum_{i=1}^{L}\sum_{\substack{j=1\\j\neq i}}^{L}\|F_{i2}\|\left[\left(\left\|H_{ij3}\right\|+\left\|D_{ij2}\right\|\left\|E_{ij1}\right\|\right)\left\|z_{j1d_j}\right\|\right.\\&\left.+\left(\left\|H_{ij4}\right\|+\left\|D_{ij2}\right\|\left\|E_{ij2}\right\|\right)\left\|z_{j2d_j}\right\|\right]\\&+\sum_{i=1}^{L}\|F_{i2}\|\,\|B_{i2}\|\,\|G_i\|+\sum_{i=1}^{L}\frac{\sigma_i^T}{\|\sigma_i\|}F_{i2}B_{i2}u_i\\&-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{b}_i\dot{\widehat{b}}_i-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{c}_i\dot{\widehat{c}}_i.\end{aligned} \tag{54}$$

Since $\|G_i\| \leq c_i + b_i\|x_i\|$ and $x_i = W_{i1}z_{i1} + W_{i2}z_{i2}$, where $[W_{i1}\ \ W_{i2}] = T_i^{-1}$, we obtain

$$\begin{aligned}\dot{V}\leq{}&\sum_{i=1}^{L}\|F_{i2}\|\left[\left(\|A_{i3}\|+\|D_{i2}\|\,\|E_{i1}\|\right)\|z_{i1}\|\right.\\&\left.+\left(\|A_{i4}\|+\|D_{i2}\|\,\|E_{i2}\|\right)\|z_{i2}\|\right]\\&+\sum_{i=1}^{L}\frac{\sigma_i^T}{\|\sigma_i\|}F_{i2}B_{i2}u_i\\&+\sum_{i=1}^{L}\sum_{\substack{j=1\\j\neq i}}^{L}\|F_{i2}\|\left[\left(\left\|H_{ij3}\right\|+\left\|D_{ij2}\right\|\left\|E_{ij1}\right\|\right)\left\|z_{j1d_j}\right\|\right.\\&\left.+\left(\left\|H_{ij4}\right\|+\left\|D_{ij2}\right\|\left\|E_{ij2}\right\|\right)\left\|z_{j2d_j}\right\|\right]\\&+\sum_{i=1}^{L}b_i\|F_{i2}\|\,\|B_{i2}\|\left(\|W_{i1}\|\,\|z_{i1}\|+\|W_{i2}\|\,\|z_{i2}\|\right)\\&+\sum_{i=1}^{L}\|F_{i2}\|\,\|B_{i2}\|\,c_i-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{b}_i\dot{\widehat{b}}_i-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{c}_i\dot{\widehat{c}}_i.\end{aligned} \tag{55}$$

The facts $\sum_{i=1}^{L}\sum_{j=1,j\neq i}^{L}\|F_{i2}\|(\|H_{ij3}\| + \|D_{ij2}\|\|E_{ij1}\|)\|z_{j1d_j}\| = \sum_{i=1}^{L}\sum_{j=1,j\neq i}^{L}\|F_{j2}\|(\|H_{ji3}\| + \|D_{ji2}\|\|E_{ji1}\|)\|z_{i1d_i}\|$ and $\sum_{i=1}^{L}\sum_{j=1,j\neq i}^{L}\|F_{i2}\|(\|H_{ij4}\| + \|D_{ij2}\|\|E_{ij2}\|)\|z_{j2d_j}\| = \sum_{i=1}^{L}\sum_{j=1,j\neq i}^{L}\|F_{j2}\|(\|H_{ji4}\| + \|D_{ji2}\|\|E_{ji2}\|)\|z_{i2d_i}\|$ imply that

$$\begin{aligned}\dot{V}\leq{}&\sum_{i=1}^{L}\|F_{i2}\|\left[\left(\|A_{i3}\|+\|D_{i2}\|\,\|E_{i1}\|\right)\|z_{i1}\|\right.\\&\left.+\left(\|A_{i4}\|+\|D_{i2}\|\,\|E_{i2}\|\right)\|z_{i2}\|\right]\\&+\sum_{i=1}^{L}\frac{\sigma_i^T}{\|\sigma_i\|}F_{i2}B_{i2}u_i\\&+\sum_{i=1}^{L}\sum_{\substack{j=1\\j\neq i}}^{L}\|F_{j2}\|\left[\left(\left\|H_{ji3}\right\|+\left\|D_{ji2}\right\|\left\|E_{ji1}\right\|\right)\left\|z_{i1d_i}\right\|\right.\\&\left.+\left(\left\|H_{ji4}\right\|+\left\|D_{ji2}\right\|\left\|E_{ji2}\right\|\right)\left\|z_{i2d_i}\right\|\right]\\&+\sum_{i=1}^{L}b_i\|F_{i2}\|\,\|B_{i2}\|\left(\|W_{i1}\|\,\|z_{i1}\|+\|W_{i2}\|\,\|z_{i2}\|\right)\\&+\sum_{i=1}^{L}\|F_{i2}\|\,\|B_{i2}\|\,c_i-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{b}_i\dot{\widehat{b}}_i-\sum_{i=1}^{L}\frac{1}{\overline{q}_i}\widetilde{c}_i\dot{\widehat{c}}_i.\end{aligned} \tag{56}$$

Equation (9) implies that

$$\begin{aligned}\|z_{i2}\|&=\left\|F_{i2}^{-1}\right\|\left\|K_iC_{i2}^{-1}\right\|\|y_i\|,\\\left\|z_{i2d_i}\right\|&=\left\|F_{i2}^{-1}\right\|\left\|K_iC_{i2}^{-1}\right\|\left\|y_{id_i}\right\|.\end{aligned} \tag{57}$$

In addition, let $v_{i1} = z_{i1}$, $v_{i2} = z_{i2}$, $v_{j1d_j} = z_{j1d_j}$, $v_{j2d_j} = z_{j2d_j}$, $A_{ii1} = A_{i1}$, $\Delta A_{ii1} = D_{i1}\Delta F_iE_{i1}$, $A_{ii2} = A_{i2}$, $\Delta A_{ii2} = D_{i1}\Delta F_iE_{i2}$, $A_{ij1} = (H_{ij1} + D_{ij1}\Delta F_{ij}E_{ij1})$, $A_{ij2} = (H_{ij2} + D_{ij1}\Delta F_{ij}E_{ij2})$, and $\phi_i(t) = \eta_i(t)$. Then, by applying Lemma 10 to the system (6), we obtain

$$\sum_{i=1}^{L}\|z_{i1}\| \leq \sum_{i=1}^{L}\eta_i(t), \tag{58}$$

where $\eta_i(t)$ is the solution of

$$\dot{\eta}_i(t)=\widehat{k}_i\eta_i(t)+k_i\left[\left(\|A_{i2}\|+\|D_{i1}\Delta F_iE_{i2}\|\right)\|z_{i2}\|+\sum_{\substack{j=1\\j\neq i}}^{L}\left\|H_{ji2}+D_{ji1}\Delta F_{ji}E_{ji2}\right\|\left\|z_{i2d_i}\right\|\right] \tag{59}$$

in which $\widehat{k}_i = (\overline{k}_i + \lambda_i) < 0$ and $\overline{k}_i = k_i(\|D_{i1}\Delta F_iE_{i1}\| + \sum_{j=1,j\neq i}^{L}\beta_i\|H_{ji1} + D_{ji1}\Delta F_{ji}E_{ji1}\|)$. λ_i is the maximum eigenvalue of the matrix A_{i1} and the scalars $k_i > 0$, $\beta_i > 1$.

From (57) and $\|\Delta F_i\| \leq 1$, $\|\Delta F_{ji}\| \leq 1$, (59) can be rewritten as

$$\begin{aligned}\dot{\eta}_i(t) = {} & \widehat{k}_i \eta_i(t) \\ & + k_i \Bigg[\left(\|A_{i2}\| + \|D_{i1}\| \|E_{i2}\|\right) \left\|F_{i2}^{-1}\right\| \left\|K_i C_{i2}^{-1}\right\| \|y_i\| \\ & + \sum_{\substack{j=1 \\ j\neq i}}^{L} \left(\|H_{ji2}\| + \|D_{ji1}\| \|E_{ji2}\|\right) \left\|F_{i2}^{-1}\right\| \\ & \times \left\|K_i C_{i2}^{-1}\right\| \left\|y_{id_i}\right\| \Bigg],\end{aligned} \tag{60}$$

where $\widehat{k}_i = (\overline{k}_i + \lambda_i) < 0$ and $\overline{k}_i = k_i(\|D_{i1}\|\|E_{i1}\| + \sum_{j=1,j\neq i}^{L} \beta_i(\|H_{ji1}\| + \|D_{ji1}\|\|E_{ji1}\|))$. By (56), (57), and (58), we have

$$\begin{aligned}\dot{V} \leq {} & \sum_{i=1}^{L} \|F_{i2}\| \Big[\left(\|A_{i3}\| + \|D_{i2}\| \|E_{i1}\|\right) \eta_i \\ & + \left(\|A_{i4}\| + \|D_{i2}\| \|E_{i2}\|\right) \left\|F_{i2}^{-1}\right\| \left\|K_i C_{i2}^{-1}\right\| \|y_i\| \Big] \\ & + \sum_{i=1}^{L} \sum_{\substack{j=1 \\ j\neq i}}^{L} \|F_{j2}\| \Big[\beta_i \left(\|H_{ji3}\| + \|D_{ji2}\| \|E_{ji1}\|\right) \eta_i \\ & + \left(\|H_{ji4}\| + \|D_{ji2}\| \|E_{ji2}\|\right) \\ & \times \left\|F_{i2}^{-1}\right\| \left\|K_i C_{i2}^{-1}\right\| \left\|y_{id_i}\right\| \Big] \\ & + \sum_{i=1}^{L} b_i \|F_{i2}\| \|B_{i2}\| \left(\|W_{i1}\| \eta_i + \|W_{i2}\| \left\|F_{i2}^{-1}\right\| \left\|K_i C_{i2}^{-1}\right\| \|y_i\|\right) \\ & + \sum_{i=1}^{L} \|F_{i2}\| \|B_{i2}\| c_i + \sum_{i=1}^{L} \frac{\sigma_i^T}{\|\sigma_i\|} F_{i2} B_{i2} u_i \\ & - \sum_{i=1}^{L} \frac{1}{\overline{q}_i} \widetilde{b}_i \dot{\widehat{b}}_i - \sum_{i=1}^{L} \frac{1}{\overline{q}_i} \widetilde{c}_i \dot{\widehat{c}}_i.\end{aligned} \tag{61}$$

By substituting the controller (48) into (61), it is clear that

$$\begin{aligned}\dot{V} \leq {} & \sum_{i=1}^{L} b_i \|F_{i2}\| \|B_{i2}\| \left(\|W_{i1}\| \eta_i + \|W_{i2}\| \left\|F_{i2}^{-1}\right\| \left\|K_i C_{i2}^{-1}\right\| \|y_i\|\right) \\ & - \sum_{i=1}^{L} \zeta_i - \sum_{i=1}^{L} \alpha_i + \sum_{i=1}^{L} \|F_{i2}\| \|B_{i2}\| c_i \\ & - \sum_{i=1}^{L} \frac{1}{\overline{q}_i} \widetilde{b}_i \dot{\widehat{b}}_i - \sum_{i=1}^{L} \frac{1}{\overline{q}_i} \widetilde{c}_i \dot{\widehat{c}}_i.\end{aligned} \tag{62}$$

Considering (50) and (62), the above inequality can be rewritten as

$$\begin{aligned}\dot{V} \leq {} & \sum_{i=1}^{L} \widehat{b}_i \|F_{i2}\| \|B_{i2}\| \left(\|W_{i1}\| \eta_i + \|W_{i2}\| \left\|F_{i2}^{-1}\right\| \left\|K_i C_{i2}^{-1}\right\| \|y_i\|\right) \\ & - \sum_{i=1}^{L} \zeta_i - \sum_{i=1}^{L} \alpha_i + \sum_{i=1}^{L} \|F_{i2}\| \|B_{i2}\| \widehat{c}_i \\ & + \sum_{i=1}^{L} \breve{q}_i \left[-\left(\widehat{c}_i - \frac{c_i}{2}\right)^2 + \frac{c_i^2}{4} \right].\end{aligned} \tag{63}$$

By applying (49) to (63), we achieve

$$\dot{V} \leq -\sum_{i=1}^{L} \alpha_i - \sum_{i=1}^{L} \breve{q}_i \left(\widehat{c}_i - \frac{c_i}{2}\right)^2 < 0. \tag{64}$$

The above inequality implies that the state trajectories of system (1) reach the sliding surface $\sigma_i(x_i) = 0$ in finite time and stay on it thereafter. □

Remark 13. From sliding mode control theory, Theorems 4 and 12 together show that the sliding surface (8) with the decentralised adaptive output feedback SMC law (48) guarantee that (1) at any initial value the state trajectories will reach the sliding surface in finite time and stay on it thereafter; and (2) the system (1) in sliding mode is asymptotically stable.

Remark 14. The SMC scheme is often discontinuous which causes "chattering" in the sliding mode. This chattering is highly undesirable because it may excite high-frequency unmodelled plant dynamics. The most common approach to reduce the chattering is to replace the discontinuous function $\sigma_i/\|\sigma_i\|$ by a continuous approximation such as $\sigma_i/(\|\sigma_i\| + \mu_i)$, where μ_i is a positive constant [29]. This approach guarantees not asymptotic stability but ultimate boundedness of system trajectories within a neighborhood of the origin depending on μ_i.

Remark 15. The proposed controller and sliding surface use only output variables, while the bounds of disturbances are unknown. Therefore, this approach is very useful and more realistic, since it can be implemented in many practical systems.

4. Numerical Example

To verify the effectiveness of the proposed decentralized adaptive output feedback SMC law, our method has been applied to interconnected time-delay systems composed of two third-order subsystems, which is modified from [3].

The first subsystem's dynamics is given as

$$\begin{aligned}\dot{x}_1 = {} & \left(A_1 + \Delta A_1\right) x_1 + B_1 \left(u_1 + G_1\left(x_1, x_{1d_1}, t\right)\right) \\ & + \left(H_{12} + \Delta H_{12}\right) x_{2d_2}, \\ y_1 = {} & C_1 x_1,\end{aligned} \tag{65}$$

where $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} \in R^3$, $u_1 \in R^1$, $y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} \in R^2$, $A_1 = \begin{bmatrix} -8 & 0 & 1 \\ 0 & -8 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $H_{12} = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.02 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$. The mismatched parameter uncertainties in the state matrix of the first subsystem are $\Delta A_1 = D_1 \Delta F_1 E_1$ with $D_1 = \begin{bmatrix} -0.02 & 0.1 & 0.02 \end{bmatrix}^T$, $E_1 = \begin{bmatrix} 0.1 & 0 & 0.1 \end{bmatrix}$, and $\Delta F_1 = 0.4\sin^2(x_{11}^2 + t \times x_{12} + x_{13} + t \times x_{11}x_{12})$. The mismatched uncertain interconnections with the second subsystem are $\Delta H_{12} = D_{12}\Delta F_{12}E_{12}$ with $D_{12} = \begin{bmatrix} 0.1 & 0.02 & 0.1 \end{bmatrix}^T$, $\Delta F_{12} = 0.2\sin^2(x_{23d_2} + x_{23}x_{22d_2} + t \times x_{21}x_{22})$, and $E_{12} = \begin{bmatrix} 0.1 & 0.02 & 0.1 \end{bmatrix}$. The exogenous disturbance in the first subsystem is $\|G_1(x_1, x_{1d_1}, t)\| \le c_1 + b_1\|x_1\|$, where b_1 and c_1 can be selected by any positive value.

The second subsystem's dynamics is given as

$$\begin{aligned} \dot{x}_2 &= \left(A_2 + \Delta A_2\right) x_2 + B_2 \left(u_2 + G_2\left(x_2, x_{2d_2}, t\right)\right) \\ &\quad + \left(H_{21} + \Delta H_{21}\right) x_{1d_1}, \\ y_2 &= C_2 x_2, \end{aligned} \tag{66}$$

where $x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} \in R^3$, $u_2 \in R^1$, $y_2 = \begin{bmatrix} y_{21} \\ y_{22} \end{bmatrix} \in R^2$, $A_2 = \begin{bmatrix} -6 & 0 & 1 \\ 0 & -6 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $H_{21} = \begin{bmatrix} 0.1 & 0.02 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0.02 & 0.1 & 0.02 \end{bmatrix}$. The mismatched parameter uncertainties in the state matrix of the second subsystem are $\Delta A_2 = D_2 \Delta F_2 E_2$ with $D_2 = \begin{bmatrix} 0.1 & 0.02 & -0.1 \end{bmatrix}^T$, $E_2 = \begin{bmatrix} 0.1 & 0.1 & 0.03 \end{bmatrix}$, and $\Delta F_2 = 0.45\sin(x_{21} + x_{23}^2 + t \times x_{22} + x_{21}x_{22})$. The mismatched uncertain interconnection with the first subsystem is $\Delta H_{21} = D_{21}\Delta F_{21}E_{21}$ with $D_{21} = \begin{bmatrix} 0.1 & 0.1 & -0.1 \end{bmatrix}^T$, $E_{21} = \begin{bmatrix} 0.02 & 0.02 & 0.1 \end{bmatrix}$, and $\Delta F_{21} = 0.5\sin^3(x_{11d_1} + t \times x_{12d_1} + x_{11}x_{12}x_{13})$. The exogenous disturbance in the second subsystem is $\|G_2(x_2, x_{2d_2}, t)\| \le c_2 + b_2\|x_2\|$ where b_2 and c_2 can be selected by any positive value.

For this work, the following parameters are given as follows: $\varphi_1 = 0.6$, $\varphi_2 = 3.9$, $\overline{\varphi}_1 = 0.08$, $\overline{\varphi}_2 = 0.05$, $\varepsilon_1 = 2$, $\varepsilon_2 = 3$, $q = 100$, $\widehat{q} = 100$, $\breve{q}_1 = 4$, $\breve{q}_2 = 3$, $\overline{q}_1 = \overline{q}_2 = \widetilde{q}_1 = \widetilde{q}_2 = 1$, $\beta_1 = 80$, $\beta_2 = 150$, $k_1 = 1.002$, $k_2 = 1.01$, $b_1 = 0.8$, $b_2 = 0.1$, $c_1 = 0.3$, $c_2 = 0.5$, $\alpha_1 = 0.05$, $\alpha_2 = 0.06$. According to the algorithm given in [3], the coordinate transformation matrices for the first subsystem and the second subsystem are $T_1 = T_2 = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. By solving LMI (12), it is easy to verify that conditions in Theorem 4 are satisfied with positive matrices $P_1 = \begin{bmatrix} 0.2104 & -0.0017 \\ -0.0017 & 0.2305 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0.2669 & -0.0266 \\ -0.0266 & 0.2517 \end{bmatrix}$. The matrices Ξ_1 and Ξ_2 are selected to be $\Xi_1 = \begin{bmatrix} 0.2236 & 0.6708 \end{bmatrix}$ and $\Xi_2 = \begin{bmatrix} 0.8 & -0.4 \end{bmatrix}$. From (8), the sliding surface for the first subsystem and the second subsystem are $\sigma_1 = \begin{bmatrix} 0 & 0 & -0.1137 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix}^T = 0$ and $\sigma_2 = \begin{bmatrix} 0 & 0 & -0.2281 \end{bmatrix} \begin{bmatrix} x_{21} & x_{22} & x_{23} \end{bmatrix}^T = 0$. Theorem 4 showed that the sliding motion associated with the sliding surfaces σ_1 and σ_2 is globally asymptotically stable. The time functions $\eta_1(t)$ and $\eta_2(t)$ are the solution of $\dot{\eta}_1(t) = -7.954\eta_1(t) + 2.015\|y_1\| + 0.22\|y_{1d_1}\|$ and $\dot{\eta}_2(t) = -5.796\eta_2(t) + 2.024\|y_2\| + 0.215\|y_{2d_2}\|$, respectively. From Theorem 12, the decentralized adaptive output feedback sliding mode controller for the first subsystem and the second subsystem are

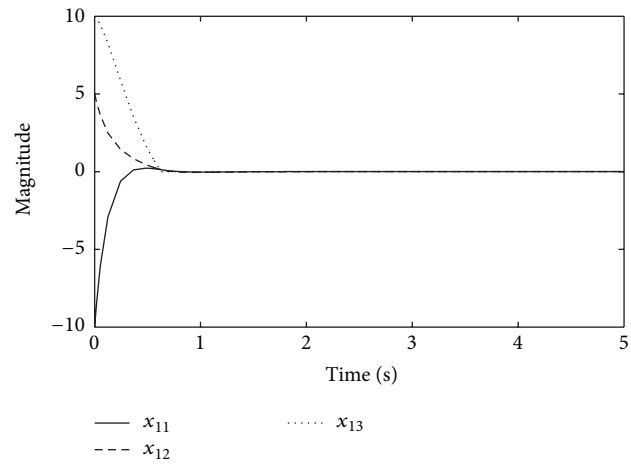

Figure 1: Time responses of states x_{11} (solid), x_{12} (dashed), and x_{13} (dotted).

$$\begin{aligned} u_1(t) &= \left(\zeta_1 + 0.05 + 1.0862\eta_1(t) + 0.00028\|y_1\| \right. \\ &\quad \left. + 0.0068\|y_{1d_1}\|\right) \frac{\sigma_1}{\|\sigma_1\|}, \end{aligned} \tag{67}$$

$$\begin{aligned} u_2(t) &= \left(\zeta_2 + 0.06 + 1.1048\eta_2(t) + 0.000684\|y_2\| \right. \\ &\quad \left. + 0.0125\|y_{2d_2}\|\right) \frac{\sigma_2}{\|\sigma_2\|}, \end{aligned} \tag{68}$$

where $\zeta_1 \ge 0.113\widehat{b}_1(\eta_1(t) + \|y_1\|) + 0.1137\widehat{c}_1 + 0.022$, $\dot{\widehat{b}}_1 = 0.113(\eta_1(t) + \|y_1\|)$, $\dot{\widehat{c}}_1 = -4\widehat{c}_1 + 0.113$, $\zeta_2 \ge 0.228\widehat{b}_2(\eta_2(t) + \|y_2\|) + 0.2281\widehat{c}_2 + 0.0625$, $\dot{\widehat{b}}_2 = 0.228(\eta_2(t) + \|y_2\|)$, and $\dot{\widehat{c}}_2 = -4\widehat{c}_2 + 0.2281$. Figures 5 and 6 imply that the chattering occurs in control input. In order to eliminate chattering phenomenon, the discontinuous controllers (67) and (68) are replaced by the following continuous approximations:

$$\begin{aligned} u_1(t) &= \left(\zeta_1 + 0.05 + 1.0862\eta_1(t) + 0.00028\|y_1\| \right. \\ &\quad \left. + 0.0068\|y_{1d_1}\|\right) \frac{\sigma_1}{\|\sigma_1\| + 0.0001}, \end{aligned} \tag{69}$$

$$\begin{aligned} u_2(t) &= \left(\zeta_2 + 0.06 + 1.1048\eta_2(t) + 0.000684\|y_2\| \right. \\ &\quad \left. + 0.0125\|y_{2d_2}\|\right) \frac{\sigma_2}{\|\sigma_2\| + 0.0001}. \end{aligned} \tag{70}$$

From Figures 7 and 8, we can see that the chattering is eliminated.

The time-delays chosen for the first subsystem and the second subsystem are $d_1(t) = 2 - \sin(t)$ and $d_2(t) = 1 - 0.5\cos(t)$. The initial conditions for two subsystems are selected to be $\chi_1(t) = \begin{bmatrix} -10 & 5 & 10 \end{bmatrix}^T$ and $\chi_2(t) = \begin{bmatrix} 10 & -8 & -10 \end{bmatrix}^T$, respectively. By Figures 1, 2, 3, 4, 5, 6, 7, and 8 it is clearly seen that the proposed controller is effective in

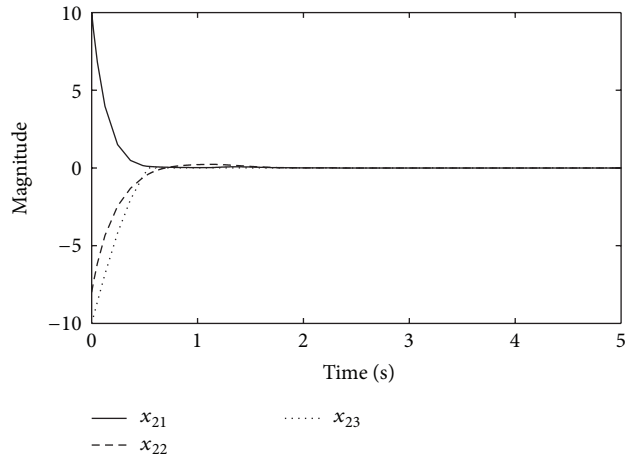

Figure 2: Time responses of states x_{21} (solid), x_{22} (dashed), and x_{23} (dotted).

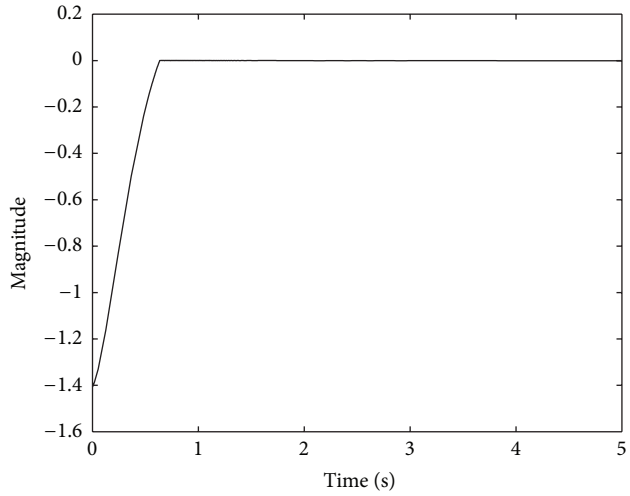

Figure 3: Time responses of sliding function σ_1.

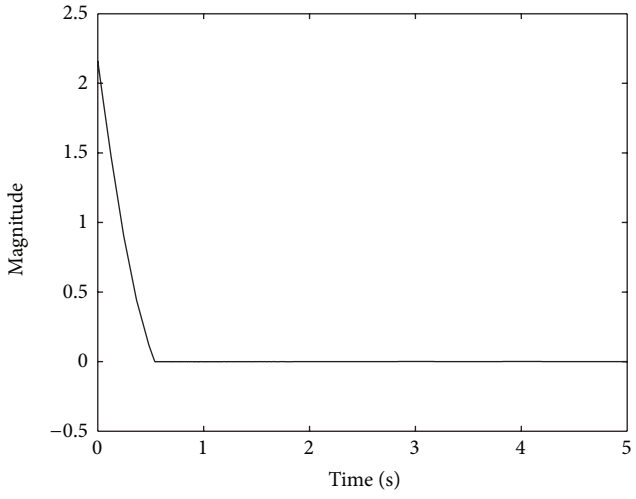

Figure 4: Time responses of sliding function σ_2.

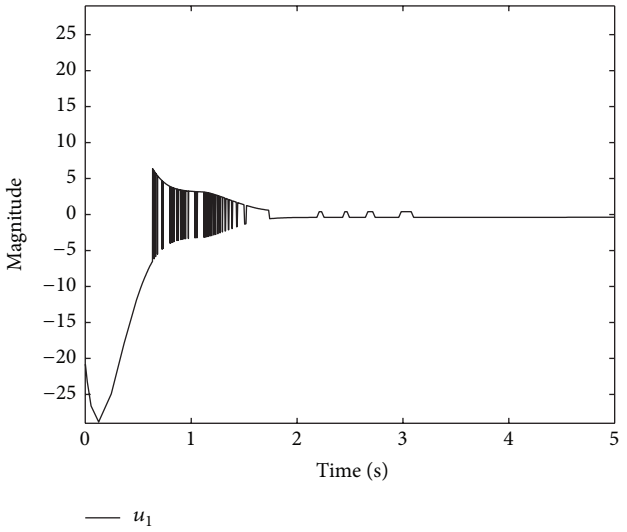

Figure 5: Time responses of discontinuous control input u_1 (67).

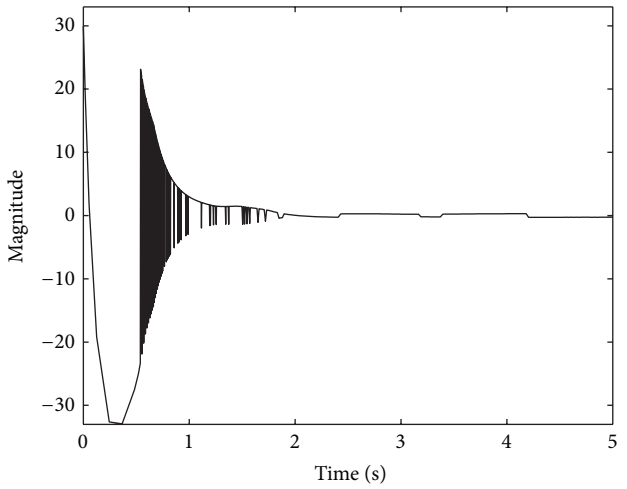

Figure 6: Time responses of discontinuous control input u_2 (68).

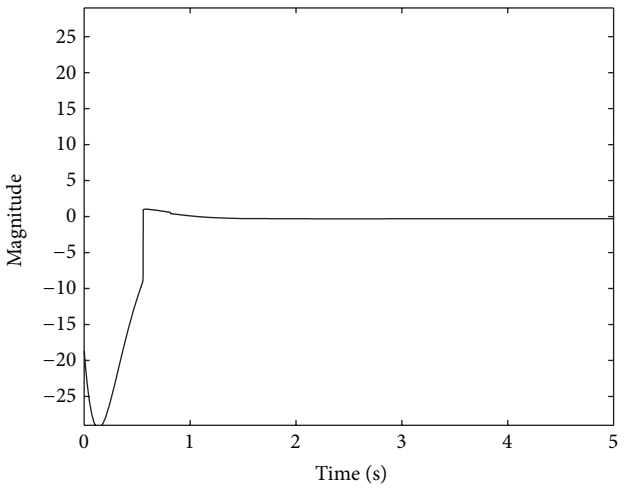

Figure 7: Time responses of continuous control input u_1 (69).

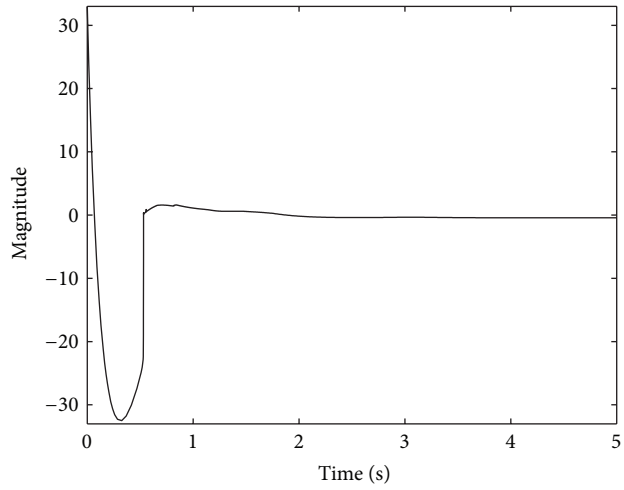

Figure 8: Time responses of continuous control input u_2 (70).

dealing with matched and mismatched uncertainties and the system has a good performance.

5. Conclusion

In this paper, a decentralized adaptive SMC law is proposed to stabilize complex interconnected time-delay systems with unknown disturbance, mismatched parameter uncertainties in the state matrix, and mismatched interconnections. Furthermore, in these systems, the system states are unavailable and no estimated states are required. This is a new problem in the application of SMC to interconnected time-delay systems. By establishing a new lemma, the two major limitations of SMC approaches for interconnected time-delay systems in [3] have been removed. We have shown that the new sliding mode controller guarantees the reachability of the system states in a finite time period, and moreover the dynamics of the reduced-order complex interconnected time-delay system in sliding mode is asymptotically stable under certain conditions.

Conflict of Interests

The authors declare that they have no conflict of interests regarding to the publication of this paper.

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Research Article

Estimation of Nonlinear Functions of State Vector for Linear Systems with Time-Delays and Uncertainties

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This paper focuses on estimation of a nonlinear function of state vector (NFS) in discrete-time linear systems with time-delays and model uncertainties. The NFS represents a multivariate nonlinear function of state variables, which can indicate useful information of a target system for control. The optimal nonlinear estimator of an NFS (in mean square sense) represents a function of the receding horizon estimate and its error covariance. The proposed receding horizon filter represents the standard Kalman filter with time-delays and special initial horizon conditions described by the Lyapunov-like equations. In general case to calculate an optimal estimator of an NFS we propose using the unscented transformation. Important class of polynomial NFS is considered in detail. In the case of polynomial NFS an optimal estimator has a closed-form computational procedure. The subsequent application of the proposed receding horizon filter and nonlinear estimator to a linear stochastic system with time-delays and uncertainties demonstrates their effectiveness.

1. Introduction

The Kalman filtering for dynamic systems with time-delays and its variations are well-known state estimation techniques in wide use in a variety of applications such as navigation, target tracking, vehicle state estimation, communications engineering, air traffic control, biomedical and chemical processing, and many other areas [1–6]. Ignorance of the effect of time- delays could cause unpredictable and unsatisfactory system performance with traditional Kalman filters [3, 6, 7]. These applications have motivated researchers to study the control and filtering problem for systems with time-delays.

However, some applications require the estimation of not only a state vector but also a nonlinear function of the state (NFS), which expresses practical and worthwhile information for control systems. For instance, in a mechanical application, such measurands include displacement (linear and angular), which can be interpreted as a quadratic form of a state vector, mass (weight, load, and density), and force (absolute, relative, static, torque, and pressure) [8]. In particular, prediction of future measurands depicted by an NFS can be helpful in several applications, such as system control and target tracking. It is well known that, for dynamical systems in which the stability issue depends explicitly on the time-delay and model uncertainty, in order to overcome this practical issue, it is necessary to study the estimation of NFS in dynamical systems with time-delay and model uncertainty. Most authors have not focused on estimation of NFS but have considered state estimation (filtering) only. To the best of our knowledge, there are no effective methods for estimation of an NFS in a dynamic system with time-delay and model uncertainty in the literature.

The Kalman filters are very difficult to be applied to dynamic system with uncertainty. The standard Kalman filter estimates the state of system if it fully fits the system model; that is, there are no model uncertainties. Otherwise, its algorithm may become unstable [9] and the estimate diverges [10] due to the limited knowledge of the system's

dynamic model and unestimated measurement noise. Thus, it is not robust against modeling uncertainty and disturbances [1]. For estimation problems where we attempt to achieve robustness against temporary uncertainty, numerous strategies have been suggested, rigorously investigated, and implemented over the past few decades. For this reason, in terms of structural adaptation, there are a number of methods currently available for such systems [11, 12]. The structure adaptation filters are based on the Bayesian approach in which the unknown parameter is assumed to be random with a prior known probability [13–15]. Next approach relative to the robust filtering problem with respect to various filtering performance criteria, such as H-infinity filtering and energy-to-peak for linear continuous and discrete-time-delay systems [2, 6, 16, 17].

In this paper, we are concerned with usage of the receding horizon strategy for designing robust receding horizon (finite-memory) filter/estimator for time-delay stochastic systems with model uncertainties. In this way the information obtained over the most recent time interval is only used [15, 18, 19]. As a result, the receding horizon filters are more robust against model uncertainties and numerical errors than standard Kalman filters, which utilize all measurements [20, 21]. Based on aforementioned literature, and to the best of the authors' knowledge, there are no existing results for the receding horizon estimation of NFS for linear systems with time-delays.

Motivated by the above problems, in this paper, we focus on estimating an arbitrary nonlinear function of the state in a linear system with time-delay and model uncertainty. The main contribution of the paper is derivation of receding horizon estimators based on the crucial Lyapunov-like equations for receding horizon initial conditions.

This paper is organized as follows. Section 2 presents a statement of the estimation problem for NFS within the linear system with time-delays. The optimal estimator for NFS is proposed. In Section 3, we present the receding horizon filter/estimator for discrete-time linear system with time-delays. Here, the exact recursive equations for determining receding horizon initial conditions (mean and covariance) are derived and discussed. In Section 4, we present several practical examples of NFS. In Section 5, we derive effective closed-form computational procedure for polynomial NFS. In Section 6, the unscented transformation is introduced, and its application for estimation of NFS is proposed. In Section 7, the efficiency of the proposed receding horizon estimators is studied on real dynamical model with time-delays. Finally, a brief conclusion is given in Section 8.

2. Problem Statement

We first consider a discrete-time linear system described by stochastic recursive equation with time-delays:

$$x(k+1) = \sum_{h=0}^{M} F_h(k-h)\, x(k-h) + w(k), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $x(k) \in \mathfrak{R}^n$ is an unknown state, $F_h(k)$, $h = 0, 1, \ldots, M$, are $n \times n$ time-varying matrices, $x(-s) \sim \mathbf{N}(\overline{x}_0, P_0)$, $s = 0, 1, \ldots, M$, are initial conditions, $w(k) \in \mathfrak{R}^n$ is a zero-mean white Gaussian noise with covariance $\mathrm{Cov}\{w(k), w(s)\} = Q(k)\delta_{ks}$, and δ_{ks} is the Kronecker function.

The discrete measurement $y(k) \in \mathfrak{R}^m$ is also described by recursive equation with time- delays:

$$y(k) = \sum_{d=0}^{L} H_d(k-d)\, x(k-d) + v(k), \tag{2}$$

where $H_d(k) \in \mathfrak{R}^{m\times n}$ is the measurement matrix, $d = 0, 1, \ldots, L$, and $v(k) \in \mathfrak{R}^m$ is a zero-mean white Gaussian noise with covariance $\mathrm{Cov}\{v(k), v(s)\} = R(k)\delta_{ks}$.

We also assume that the initial states $x(-s)$, $s = 0, 1, \ldots, M$, system noise $w(k)$, and measurement error $v(k)$ are mutually uncorrelated; that is,

$$\begin{aligned} \mathrm{Cov}\{x(-s), w(k)\} &= 0, \\ \mathrm{Cov}\{x(-s), v(k)\} &= 0, \\ \mathrm{Cov}\{w(k), v(k)\} &= 0, \\ s &= 0, 1, \ldots, M. \end{aligned} \tag{3}$$

A problem associated with such system (1), (2) is that of estimation of the nonlinear function of state (NFS) variables

$$z(k) = f[x(k)] : \mathfrak{R}^n \longrightarrow \mathfrak{R} \tag{4}$$

from the overall noisy sensor measurements

$$Y^k = \{y(s) : s = 1, \ldots, k\}. \tag{5}$$

Typical examples of such an NFS may be an arbitrary quadratic form $f[x(k)] = x(k)^T \Omega x(k)$, representing an energy-like function of an object or square distance $f[x(k)] = d^2(x(k), x^0(k))$ between the current $x(k)$ and nominal $x^0(k)$ states, respectively.

We propose an optimal estimation algorithm for NFS and its performance in the subsequent sections.

The optimal mean square estimate of the NFS $z(k) = f[x(k)]$ based on the overall sensor measurements (5) represents a conditional mean:

$$\widehat{z}(k) = \mathbf{E}\left[z(k) \mid Y^k\right] = \int f[x(k)]\, p\left(x(k) \mid Y^k\right) dx(k), \tag{6}$$

where $p(x(k) \mid Y^k)$ is a conditional probability density function (pdf).

To calculate a conditional pdf $p(x(k) \mid Y^k)$ we need to solve the filtering problem consisting in calculation of good estimates $\widehat{x}(k \mid k)$ of the state of a stochastic dynamical system based on noisy partial measurements (5).

Next we propose the new receding horizon filtering algorithm for discrete-time linear systems with time-delays and uncertainties (1), (2) determining the conditional pdf $p(x(k) \mid Y^k)$ in (6).

3. Receding Horizon Filter and Nonlinear Estimator

At first assume that the system model (1), (2) does not contain any uncertainties.

3.1. Kalman Filtering with Time-Delays for Model without Uncertainties. According to [22, 23] the optimal mean square estimate of state $\widehat{x}(k \mid k)$ based on all current measurements (5) represents conditional mean $\widehat{x}(k \mid k) = \mathbf{E}[x(k) \mid Y^k]$ which is described by the Kalman filter equations with time-delays (KFTD):

$$\begin{aligned}
&\widehat{x}(k-m \mid k) \\
&\quad = \widehat{x}(k-m \mid k-1) \\
&\qquad + G_m(k)\left[y(k) - \sum_{d=0}^{L} H_d(k-d)\,\widehat{x}(k-d \mid k-1)\right], \\
&\qquad\qquad m = 1, 2, \ldots, \overline{M}, \quad \overline{M} = \max\{M, L\}, \\
&\widehat{x}(k \mid k-1) = \sum_{h=0}^{M} F_h(k-h-1)\,\widehat{x}(k-h-1 \mid k-1), \\
&\widehat{x}(k \mid k) \\
&\quad = \widehat{x}(k \mid k-1) \\
&\qquad + G_0(k)\left[y(k) - \sum_{d=0}^{L} H_d(k-d)\,\widehat{x}(k-d \mid k-1)\right],
\end{aligned} \tag{7}$$

where Kalman gains $G_m(k)$, $m = 0, 1, \ldots, \overline{M}$, and error covariances

$$\begin{aligned}
P(k_1, k_2 \mid k) &= \operatorname{Cov}\{e(k_1 \mid k), e(k_2 \mid k)\}, \\
e(k_1 \mid k) &= x(k) - \widehat{x}(k_1 \mid k), \\
&\qquad k_1, k_2 \le k
\end{aligned} \tag{8}$$

are given by

$$\begin{aligned}
&G_m(k) \\
&\quad = \sum_{d=0}^{L} P(k-m, k-d \mid k-1)\, H_d^T(k-d) \\
&\qquad \times \left[R(k) + \sum_{d_1, d_2=0}^{L} H_{d_1}(k-d_1)\, P(k-d_1, k-d_2 \mid k-1)\right. \\
&\qquad\qquad \left. \times\, H_{d_2}^T(k-d_2)\right]^{-1}, \\
&P(k-h_1, k-h_2 \mid k) \\
&\quad = P(k-h_1, k-h_2 \mid k-1) \\
&\qquad - G_{h_1}(k) \sum_{d=0}^{L} H_d(k-d)\, P(k-d, k-h_2 \mid k-1), \\
&P(k+1, k+1 \mid k) \\
&\quad = \sum_{h_1, h_2=0}^{M} F_{h_1}(k-h_1)\, P(k-h_1, k-h_2 \mid k)\, F_{h_2}^T(k-h_2) \\
&\qquad + Q(k).
\end{aligned} \tag{9}$$

To run KFTD equations (7)–(9) we need to know the initial conditions for estimates and error covariances, which represent a priori mean and covariance; that is,

$$\begin{aligned}
\widehat{x}(-h \mid 0) &= \overline{x}_0, \\
P(-h, -j) &= P(-h, -j \mid 0) = P_0, \\
&\qquad h, j = 0, 1, 2, \ldots, \overline{M}.
\end{aligned} \tag{10}$$

In practice any system model contains uncertainties; therefore, to achieve robustness against temporary uncertainty, numerous strategies have been proposed. In the paper we use the effective receding horizon (finite-memory) strategy using only the measurements $y(k)$ obtained over the most recent time interval (receding horizon) [10, 20, 21].

3.2. Receding Horizon Filtering with Time-Delays for Model with Uncertainties. According to the receding horizon strategy the optimal (in mean square sense) estimate of the unknown state $x(k)$ based on the overall receding horizon sensor measurements,

$$Y_{k-\Delta}^{k} = \{y(s) : s = k-\Delta, k-\Delta+1, \ldots, k\}, \tag{11}$$

with horizon time interval Δ, represents the conditional mean,

$$\widehat{x}(k \mid k) = \mathbf{E}\left[x(k) \mid Y_{k-\Delta}^{k}\right]. \tag{12}$$

Using KFTD's equations (7)–(9) we propose their receding horizon version for estimation of the state $x(k)$ using receding horizon measurements $Y_{k-\Delta}^{k}$ on the interval $s \in [k-\Delta, k]$. We obtain

$$\begin{aligned}
&\widehat{x}(s-m \mid s) \\
&\quad = \widehat{x}(s-m \mid s-1) \\
&\qquad + G_m(s)\left[y(s) - \sum_{d=0}^{L} H_d(s-d)\,\widehat{x}(s-d \mid s-1)\right], \\
&\qquad\qquad s = k-\Delta, k-\Delta+1, \ldots, k; \\
&\qquad\qquad m = 1, 2, \ldots, \overline{M}, \quad \overline{M} = \max\{M, L\}, \\
&\widehat{x}(s \mid s-1) = \sum_{h=0}^{M} F_h(s-h-1)\,\widehat{x}(s-h-1 \mid s-1),
\end{aligned}$$

$$\widehat{x}(s \mid s) = \widehat{x}(s \mid s-1) + G_0(s)\left[y(s) - \sum_{d=0}^{L} H_d(s-d)\,\widehat{x}(s-d \mid s-1)\right], \tag{13}$$

where the receding horizon filter gains $G_m(k)$, $m = 0, 1, \ldots, \overline{M}$, and error autocovariances

$$P(s_1, s_2 \mid s) = \mathrm{Cov}\{e(s_1 \mid s), e(s_2 \mid s)\}, \quad e(s_1 \mid s) = x(s) - \widehat{x}(s_1 \mid s), \quad s_1, s_2 \le s \tag{14}$$

are described by

$$\begin{aligned} G_m(s) &= \sum_{d=0}^{L} P(s-m, s-d \mid s-1)\, H_d^T(s-d) \\ &\quad \times \left[R(s) + \sum_{d_1,d_2=0}^{L} H_{d_1}(s-d_1)\, P(s-d_1, s-d_2 \mid s-1) \times H_{d_2}^T(s-d_2)\right]^{-1}, \\ P(s-h_1, s-h_2 \mid s) &= P(s-h_1, s-h_2 \mid s-1) - G_{h_1}(s) \sum_{d=0}^{L} H_d(s-d)\, P(s-d, s-h_2 \mid s-1), \\ P(s+1, s+1 \mid s) &= \sum_{h_1,h_2=0}^{M} F_{h_1}(s-h_1)\, P(s-h_1, s-h_2 \mid s)\, F_{h_2}^T(s-h_2) + Q(s). \end{aligned} \tag{15}$$

In contrast to KFTD (7)–(9) the receding horizon filter (RHF) (13)–(15) needs to initialize $(M+1)$ horizon initial conditions at $s = k - \Delta$ which represent unconditional means and covariances of the state $x(k)$; that is,

$$\begin{aligned} m\left(k-\Delta-\overline{M}+1\right) &\equiv \widehat{x}\left(k-\Delta-\overline{M}+1 \mid k-\Delta\right) = \mathbf{E}\left\{x\left(k-\Delta-\overline{M}+1\right)\right\}, \\ m\left(k-\Delta-\overline{M}+2\right) &\equiv \widehat{x}\left(k-\Delta-\overline{M}+2 \mid k-\Delta\right) = \mathbf{E}\left\{x\left(k-\Delta-\overline{M}+2\right)\right\}, \\ &\vdots \\ m(k-\Delta+1) &\equiv \widehat{x}(k-\Delta+1 \mid k-\Delta) = \mathbf{E}\{x(k-\Delta+1)\}, \end{aligned} \tag{16}$$

$$P(h_1, h_2) \equiv P(h_1, h_2 \mid k-\Delta) = \mathrm{Cov}\{x(h_1), x(h_2)\}, \quad h_1, h_2 = k-\Delta-\overline{M}+1, \ldots, k-\Delta+1. \tag{17}$$

We have the following.

Theorem 1. *The horizon initial means* (16) *are described by*

$$m(t+1) = \sum_{h=0}^{M} F_h(t-h)\, m(t-h), \quad t = 0, 1, \ldots, k-\Delta+1, \quad m(0) = m(-1) = \cdots = m(-M) = \overline{x}_0. \tag{18}$$

Theorem 2. *The horizon initial covariances* (17) *satisfy Lyapunov-like recursive equations*

$$\begin{aligned} P(t+1, t+1) &= \sum_{h_1,h_2=0}^{M} F_{h_1}(t-h_1)\, P(t-h_1, t-h_2)\, F_{h_2}^T(t-h_2) + Q(t), \quad t = 0, 1, \ldots, k-\Delta+1, \\ P(t-h_1+1, t-h_2+1) &= \sum_{\ell_1=0}^{M} F_{\ell_1}(t-h_1-\ell_1)\, P(t-h_1-\ell_1) + Q(t-h_1)\,\delta_{t-h_1,t-h_2}, \quad k-h_1 \ge k-h_2, \end{aligned} \tag{19a}$$

$$P(t-h_1, t-h_2) = P(t-h_2, t-h_1)^T, \quad t-h_1 < t-h_2, \quad P(-s_1, s_2) = P_0, \quad s_1, s_2 = 1, 2, \ldots, M. \tag{19b}$$

Derivation of Lyapunov-like equations for mean and covariances (18), (19a), and (19b) is given in the appendix.

Remark 3. Original initial conditions (10) for the KFTD (7)–(9) at different time instants $s = 0, -1, \ldots, -\overline{M}$ are identical in contrast to new horizon initial conditions (18), (19a), and (19b) which are more realistic for practice.

Remark 4 (real-time implementation of RHF). The RHF equations (13)–(19b) can be divided into two parts as follows.

Part 1 (offline equations). We may note that the RHF gains $G_m(s)$ and error autocovariances $P(s_1, s_2 \mid s)$ can be *precomputed*, since they do not depend on the receding horizon measurements $Y_{k-\Delta}^{k}$, but only on the system matrices $F_h(k-h)$, $H_d(k-d)$, noise statistics $Q(k)$, $R(k)$, and horizon initial conditions (18), (19a), and (19b) which are the part of system model (1) and (2).

Part 2 (online equations). Thus, once the measurement schedule has been settled, the real-time implementation of

the RHF requires only the computation of the RHF estimates $\widehat{x}(s-m \mid s)$, $s=k-\Delta, k-\Delta+1,\ldots,k$, using only current measurements $y(k)$.

3.3. Conditional PDF and Optimal Receding Horizon Estimator. From (6) the optimal mean square estimate of the NCF $z(k)=f[x(k)]$ based on the receding horizon sensor measurements (11) also represents a conditional mean:

$$\widehat{z}(k)=\mathbf{E}\left[z(k)\mid Y_{k-\Delta}^{k}\right]=\int f[x(k)]\,p\left(x(k)\mid Y_{k-\Delta}^{k}\right)dx(k), \tag{20}$$

where $p(x(k)\mid Y_{k-\Delta}^{k})=\mathbf{N}(\widehat{x}(k\mid k),P(k\mid k))$ is a Gaussian probability density function with conditional mean $\widehat{x}(k\mid k)=\mathbf{E}[x(k)\mid Y_{k-\Delta}^{k}]$ and covariance $P(k\mid k)$ determined by RHF equations (13) and (15).

Thus, the estimate in (20) represents the *optimal receding horizon estimator* (RHE):

$$\widehat{z}(k)=F(\widehat{x}(k\mid k),P(k\mid k)), \tag{21}$$

which depends on the receding horizon estimate $\widehat{x}(k\mid k)$ and its error covariance $P(k\mid k)$.

Further, we consider several examples of application of the general nonlinear estimator (20).

4. Examples of Optimal Nonlinear Estimator

4.1. Estimation of Distance. Consider an NFS representing a distance between an unknown scalar position $x(k)$ and given point $a(k)$. In this case an NFS becomes $z(k)=|x(k)-a(k)|$. Using (20) the best estimate of the distance between an unknown position $x(k)$ and moving point $a(k)$, $k=0,1,\ldots$, is

$$\begin{aligned}\widehat{z}(k)&=\int_{-\infty}^{\infty}|x(k)-a(k)|\,\mathbf{N}(\widehat{x}(k\mid k),P(k\mid k))\,dx(k)\\&=|\widehat{x}(k\mid k)-a(k)|\,\Phi\left(\frac{|\widehat{x}(k\mid k)-a(k)|}{\sqrt{P(k\mid k)}}\right)\\&\quad+\sqrt{2P(k\mid k)}\exp\left[-\frac{(\widehat{x}(k\mid k)-a(k))^{2}}{2P(k\mid k)}\right],\\\Phi(u)&=\frac{1}{\sqrt{2\pi}}\int_{0}^{u}e^{-t^{2}/2}dt,\end{aligned} \tag{22}$$

where $\widehat{x}(k\mid k)$ and $P(k\mid k)$ are determined by (13) and (15) (or (7) and (9)), respectively, and $\Phi(u)$ is the error function.

In the particular case with $a(k)=0$, the estimate of an unknown modulus $z(k)=|x(k)|$ takes the form

$$\begin{aligned}\widehat{z}(k)&=\int_{-\infty}^{\infty}|x(k)|\,\mathbf{N}(\widehat{x}(k\mid k),P(k\mid k))\,dx(k)\\&=|\widehat{x}(k\mid k)|\,\Phi\left(\frac{|\widehat{x}(k\mid k)|}{\sqrt{P(k\mid k)}}\right)\\&\quad+\sqrt{2P(k\mid k)}\exp\left[-\frac{\widehat{x}(k\mid k)^{2}}{2P(k\mid k)}\right].\end{aligned} \tag{23}$$

4.2. Quadratic Function of State Vector. Consider an arbitrary quadratic function of a state vector $x(k)\in\mathfrak{R}^{n}$,

$$z(k)=f[x(k)]=x(k)^{T}\Omega x(k)+a(k)^{T}x(k),\quad \Omega^{T}=\Omega. \tag{24}$$

Then optimal nonlinear estimate (20) can be explicitly calculated in terms of a state estimate $\widehat{x}(k\mid k)$ and its error covariance $P(k\mid k)$. Using the formula [24]

$$\begin{aligned}\mathbf{E}\left(x^{T}\Omega x\right)&=\operatorname{tr}\left[\Omega\left(P+mm^{T}\right)\right],\\m&=\mathbf{E}(x),\ P=\operatorname{cov}(x,x),\end{aligned} \tag{25}$$

we obtain an optimal estimate for the quadratic function

$$\begin{aligned}\widehat{z}(k)=&\ \operatorname{tr}\left[\Omega\left(P(k\mid k)+\widehat{x}(k\mid k)\,\widehat{x}(k\mid k)^{T}\right)\right]\\&+a(k)^{T}\widehat{x}(k\mid k).\end{aligned} \tag{26}$$

In (25) and (26), $\operatorname{tr}(A)$ is the trace of a matrix A.

In the particular case a quadratic function represents a power of a signal $z(k)=x(k)^{2}$. Using (26) at $\Omega=1$, $a(k)=0$, we obtain

$$\widehat{z}(k\mid k)=P(k\mid k)+\widehat{x}(k)^{2}. \tag{27}$$

4.3. Estimation of Sine and Cosine Functions. In this case an NCF becomes

$$\begin{aligned}z^{(s)}(k)&=\sin(x(k)),\\z^{(c)}(k)&=\cos(x(k)),\end{aligned} \tag{28}$$

where $x(k)$ is an unknown angle.

Then the best estimates of the trigonometrical functions of an unknown angle are

$$\begin{aligned}\widehat{z}^{(s)}(k\mid k)&=\int_{-\infty}^{\infty}\sin(x(k))\,\mathbf{N}(\widehat{x}(k\mid k),P(k\mid k))\,dx(k)\\&=e^{-P(k|k)/2}\sin(\widehat{x}(k\mid k)),\\\widehat{z}^{(c)}(k\mid k)&=\int_{-\infty}^{\infty}\cos(x(k))\,\mathbf{N}(\widehat{x}(k\mid k),P(k\mid k))\,dx(k)\\&=e^{-P(k|k)/2}\cos(\widehat{x}(k\mid k)).\end{aligned} \tag{29}$$

5. Multivariate Polynomial Nonlinear Function

Here we consider a special NFS representing an arbitrary multivariate polynomial function of the form

$$\begin{aligned}&z(k)\\&=f[x(k)]\\&=\sum_{0\le\ell_1+\cdots+\ell_n\le n}A_{\ell_1\ell_2,\ldots,\ell_n}x_1(k)^{\ell_1}x_2(k)^{\ell_2},\ldots,x_n(k)^{\ell_n},\\&\qquad\qquad\ell_1,\ldots,\ell_n\ge 0.\end{aligned} \tag{30}$$

In this case, the algorithm for calculation of the optimal estimator $\widehat{z}(k) = \mathbf{E}[f(x(k)) \mid Y_{k-\Delta}^{k}]$ has a closed form because conditional expectation

$$\begin{aligned}\widehat{z}(k) &= \mathbf{E}\left[f(x(k)) \mid Y_{k-\Delta}^{k}\right] \\ &= \sum_{0\le \ell_1+\cdots+\ell_n\le n} A_{\ell_1\ell_2,\ldots,\ell_n}\mathbf{E}\left(x_1^{\ell_1}x_2^{\ell_2},\ldots,x_n^{\ell_n} \mid Y_{k-\Delta}^{k}\right)\end{aligned} \tag{31}$$

depends on high-order moments $m_{\ell_1\ell_2,\ldots,\ell_n} \equiv \mathbf{E}(x_1^{\ell_1}x_2^{\ell_2},\ldots,x_n^{\ell_n} \mid Y_{k-\Delta}^{k})$ of a multivariate Gaussian distribution, which can be explicitly calculated in terms of first- and second-order conditional moments $(\hat{x}(k \mid k), P(k \mid k))$ [24–26]. For example,

$$\begin{aligned}z(k) &= A_0 + \sum_{i=1}^{N} A_i x(k)^i, \quad x(k) \in \mathfrak{R}, \\ \widehat{z}(k) &= A_0 + \sum_{i=1}^{N} A_i m_i(k), \quad m_i(k) = \mathbf{E}\left[x(k)^i \mid Y_{k-\Delta}^{k}\right],\end{aligned} \tag{32}$$

where

$$\begin{aligned}m_1(k) &= \hat{x}(k \mid k), \\ m_2(k) &= \hat{x}(k \mid k)^2 + P(k \mid k), \\ m_3(k) &= \hat{x}(k \mid k)^3 + 3\hat{x}(k \mid k)P(k \mid k), \\ m_4(k) &= \hat{x}(k \mid k)^4 + 6\hat{x}(k \mid k)^2 P(k \mid k) + 3P(k \mid k)^2, \\ m_5(k) &= \hat{x}(k \mid k)^5 + 10\hat{x}(k \mid k)^3 P(k \mid k) \\ &\quad + 15\hat{x}(k \mid k)P(k \mid k)^2, \\ m_6(k) &= \hat{x}(k \mid k)^6 + 15\hat{x}(k \mid k)^4 P(k \mid k) \\ &\quad + 45\hat{x}(k \mid k)^2 P(k \mid k)^2 + 15P(k \mid k)^3, \\ &\vdots\end{aligned} \tag{33}$$

Here $\hat{x}(k \mid k)$ and $P(k \mid k)$ are determined by the RHF equations.

For general NFS the unscented transformation serves as a useful tool for their calculations.

Remark 5. For typical NFS $z(k) = f[x(k)]$ the integral (20) for normal distribution is known; that is, it can be explicitly expressed in terms of a receding horizon state estimate and its error covariance (see examples in Sections 4 and 5). But for general NFS the integral (20) can be calculated only approximately, for example, using the unscented transformation.

6. General Nonlinear Function and Unscented Transformation

The unscented transformation (UT) was first proposed by Julier and Uhlmann in light of the intuition that it is significantly easier to approximate the statistics of a transformed random variable, for example, the mean and covariance [27, 28]. During the last decade, the UT has become a powerful approach for designing new filtering and control algorithms for nonlinear dynamic models [29, 30]. Following this approach, the procedure to calculate the best estimate of an NFS (conditional mean)

$$\widehat{z}(k) = \mathbf{E}\left[f(x(k)) \mid Y_{k-\Delta}^{k}\right] \tag{34}$$

using the UT can be summarized as follows.

Generate the sigma points $\{X_s(k)\}_{s=0}^{2n}$ with corresponding weights $\{W_s\}_{s=0}^{2n}$:

$$\begin{aligned}X_0(k) &= \hat{x}(k \mid k), \qquad W_0 = \frac{\ell}{n+\ell}. \\ X_s(k) &= \hat{x}(k \mid k) + \left[\sqrt{(n+l)P(k \mid k)}\right]_s, \\ W_s &= \frac{1}{2(n+\ell)}, \\ X_{s+n}(k) &= \hat{x}(k \mid k) - \left[\sqrt{(n+l)P(k \mid k)}\right]_s, \\ W_{s+n} &= \frac{1}{2(n+\ell)}, \\ s &= 1, 2, \ldots, n,\end{aligned} \tag{35}$$

where $[\sqrt{P(k \mid k)}]_s$ is the sth column of the matrix square root of $P(k \mid k)$ and ℓ is the scaling parameter influencing the spreading of the points in the state space and thus the accuracy of the approximation [31]. Propagate each of these sigma points through a nonlinear function as

$$\xi_s(k) = f\left(X_s(k)\right), \quad s = 0, 1, \ldots, 2n \tag{36}$$

and the resulting best estimate of the NFS is given as

$$\widehat{z}(k) = \sum_{s=0}^{2n} W_s \xi_s(k). \tag{37}$$

Thus, the estimate (37) is approximately represented by known function of the receding horizon estimate $\hat{x}(k \mid k)$ and error covariance $P(k \mid k)$.

Application of the obtained results is illustrated by example of wind tunnel system with model uncertainties.

7. Numerical Example: The Wind Tunnel System

A comparative experimental analysis of the proposed RHF (11)–(19b) and KFTD (7)–(10) is considered on example of nonlinear estimation of a kinetic energy of the wind tunnel system with two model time-invariant uncertainties δ_1 and δ_2. In this case, we numerically evaluate and compare three mean square errors (MSEs) of the RHF, KFTD, and nonlinear estimator for kinetic energy.

A discretization of the state differential equations and the design based on a discrete-time model has been applied to the high-speed closed-air unit wind tunnel model by Armstrong

and Tripp [32]. The state vector $x(k) \in \Re^3$ consists of the state variables $x_1(k)$, $x_2(k)$, and $x_3(k)$, representing derivatives from a chosen equilibrium point of the following quantities: x_1 = Mach number, x_2 = actuator position guide vane angle in a driving fan, and x_3 = actuator rate. Then the discretized system model with a sampling period of $T = 0.5$ (sec) is given by

$$x(k+1) = \begin{bmatrix} 0.9032 + \delta_1 & 0 & 0 \\ 0 & 0.6245 & 0.0701 \\ 0 & -2.5247 & -0.0488 \end{bmatrix} x(k) + \begin{bmatrix} 0 & -0.0011 + \delta_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(k-1) + w(k), \tag{38}$$

where $\delta_1(k) = \delta_1$ and $\delta_2(k) = \delta_2$ are unknown time-invariant model parameters and a wind tunnel system with the initial mean and covariance is $\overline{x}(0) = \mathbf{E}[x(0)] = [3 \;\; 28 \;\; 10]^T$ and $P(0) = \text{diag}[1 \;\; 1 \;\; 0 \;\; 0]$. The covariance of the system white Gaussian noise $w(k) \in \Re^3$ is subjected to $Q(k) = \text{diag}\{0.01^2 \;\; 0.01^2 \;\; 0.01^2\}$.

The first coordinate $x_1(k)$, related to Mach number of the wind tunnel system, is observable through a measurement model. We have

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + v(k), \tag{39}$$

where $v(k) \in \Re^2$ is zero-mean white Gaussian measurement noise with covariance $R(k) = \text{diag}[0.5 \;\; 0.5]$. We assume that the true values of the model uncertainties δ_1 and δ_2 on the given *uncertainty interval* (UI) $T_{\text{UI}} = [80, 200]$ are

$$\begin{aligned} \delta_1(k) &= \delta_1 = 0.5, \\ \delta_2(k) &= \delta_2 = 0.1, \\ k &\in T_{\text{UI}} = [80, 200] \end{aligned} \tag{40}$$

and without it, that is,

$$\begin{aligned} \delta_1(k) &= \delta_2(k) = 0, \\ k &\in T_{\text{UI}} = [80, 200]. \end{aligned} \tag{41}$$

The horizon length Δ of the RHF (13)–(19b) and RHE (20) takes three values as $\Delta = 3, 5$, and 10.

The total kinetic energy of an actuator representing the NFS, $z(k) = f[x(k)]$, can be expressed as the sum of the translational kinetic energy of the center of mass (E^t) and the rotational kinetic energy about the center of mass (E^r). The kinetic energies can be expressed in the following quadratic form (see Section 4.2):

$$\begin{aligned} E^r &= \frac{I\omega^2}{2} = \frac{I\dot{x}_2^2}{2}, \qquad E^t = \frac{mv^2}{2} = \frac{mx_3^2}{2}, \\ \text{Total Energy} &= z(k) = E^r + E^t = \frac{I\dot{x}_2^2 + mx_3^2}{2}, \end{aligned} \tag{42}$$

where I is rotational inertia, ω is angular velocity, m is mass, and v is linear velocity. Concrete numerical values for $I = 0.136\,\text{kgm}^2$, $\omega = 55$ rad/s, $m = 7.39$ kg, and $v = 15$ m/s.

7.1. Simulation Results with Parametric Uncertainty. We assume that time-invariant uncertain model parameters $\delta_i(k) \equiv \delta_i$ take the form (40). We specially focus on comparing the MSEs of the Mach number of the wind tunnel system $x_1(k)$ that directly contain the uncertainties δ_i in (38) and the total kinetic energy $z(k)$ in (42).

Our point of interest is the behavior of the nonlinear RHE $\widehat{z}(k)$ using proposed RHF $\widehat{x}(k \mid k)$ around the uncertainty interval $T_{\text{UI}} = [80, 200]$. Here we describe the results of simulations of two filters (KFTD, RHF) and RHE $\widehat{z}(k)$ with three different receding horizon lengths Δ in terms of MSEs such that

$$\begin{aligned} P_1^{\text{KFTD}}(k \mid k) &= \mathbf{E}\left[x_1(k) - \widehat{x}_1^{\text{KFTD}}(k \mid k)\right]^2, \\ P_{1,\Delta}^{\text{RHF}}(k \mid k) &= \mathbf{E}\left[x_1(k) - \widehat{x}_{1,\Delta}^{\text{RHF}}(k \mid k)\right]^2, \\ P_z^{\text{KFTD}}(k \mid k) &= \mathbf{E}\left[z(k) - \widehat{z}^{\text{KFTD}}(k \mid k)\right]^2, \\ P_{z,\Delta}^{\text{RHF}}(k \mid k) &= \mathbf{E}\left[z(k) - \widehat{z}_{\Delta}^{\text{RHE}}(k \mid k)\right]^2, \\ &\text{for } \Delta = 3, 5, 10. \end{aligned} \tag{43}$$

In Figures 1 and 2, we observe that, within the uncertainty interval $T_{\text{UI}} = [80, 200]$, the time histories of the MSEs for the KFTD are notably larger than the other RHFs. However, the KFTD performs slightly worse than the RHF with the biggest horizon length of $\Delta = 10$. Also, the RHF/RHE with the small horizon length $\Delta = 3$ is more accurate than the RHF/RHE with the big horizon lengths $\Delta = 5$ and $\Delta = 10$. Therefore,

$$\begin{aligned} &P_{1,\Delta=3}^{\text{RHF}}(k \mid k) < P_{1,\Delta=5}^{\text{RHF}}(k \mid k) < P_{1,\Delta=10}^{\text{RHF}}(k \mid k) < P_1^{\text{KFTD}}(k \mid k), \\ &P_{z,\Delta=3}^{\text{RHF}}(k \mid k) < P_{z,\Delta=5}^{\text{RHF}}(k \mid k) < P_{z,\Delta=10}^{\text{RHF}}(k \mid k) < P_z^{\text{KFTD}}(k \mid k) \\ &\qquad \text{for } k \in T_{UI}. \end{aligned} \tag{44}$$

The reason for the presence of such a robustness property (44) is to compensate for the given uncertainties $\delta_i(k)$, as the horizon length Δ for sensor (memory of RHF) should be minimal. In this case, it is equal, as $\Delta = 3$.

The comparison shows us that performance of the RHF/RHE is quite good and it can be widely used in different areas of applications: industrial, military, space, inertial navigation, and others [33].

8. Conclusion

In this paper we propose a new RHF for discrete-time linear system with time-delays and model uncertainties with application to estimation of nonlinear functions of state vector. We have derived the Lyapunov-like recursive equations for horizon initial mean and covariance of a system state with an arbitrary number of time-delays and verified the effectiveness

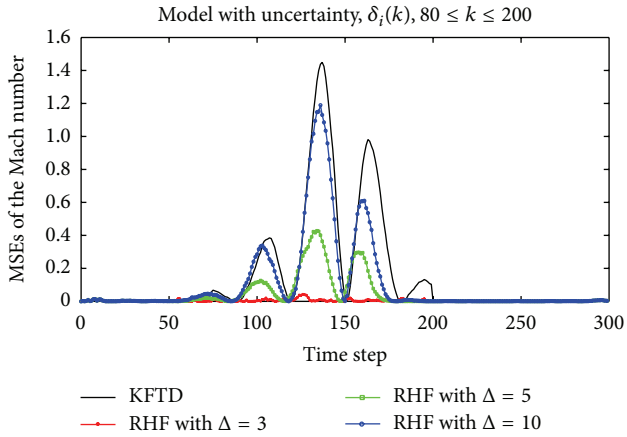

FIGURE 1: MSEs of the Mach number x_1 comparison among KFTD and three RHFs with different horizon lengths: $\Delta = 3$ (red line); $\Delta = 5$ (green line); $\Delta = 10$ (blue line); KFTD (black line). The x-axis represents time and y-axis represents MSEs.

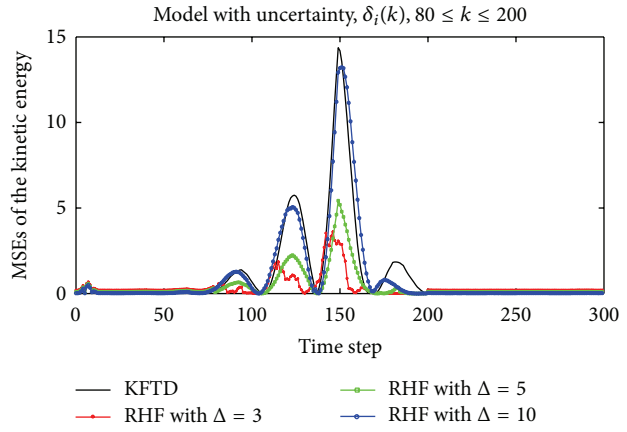

FIGURE 2: Comparison of MSEs of the total kinetic energy z using KFTD and three RHFs with horizon lengths: $\Delta = 3$ (red line); $\Delta = 5$ (green line); $\Delta = 10$ (blue line); KFTD (black line). The x-axis represents time and y-axis represents MSEs.

of the proposed filter RHF through numerical simulation and comparison with KFTD.

In order to estimate an arbitrary NCF in uncertain environment, a new RHE is proposed. Two important classes of an NCF (quadratic, polynomial) are considered in detail, and effective computational procedures based on formulas for high-order moments of a Gaussian distribution and UT transformation are proposed.

In verifying we implemented example for real dynamic system with time-delays and parametric model uncertainty $\delta(k)$. Through the implementations, the robustness of the RHF/RHE in terms of MSEs is demonstrated and the proposed receding horizon estimators have a potential to be used in a wide range of practical applications.

During the last decades, there has been extensive interest in the study of a class of physical systems modeled by hybrid system dynamics known as Markovian jump systems including multiple delays [34], mode-dependent delays [35], singular time-delays [36], and time-variant delays [37]. As a generalization of the obtained results for linear stochastic systems with time-delays and uncertainties we would like to point out that it is possible to extend the main results to Markovian jump systems with time-delays and complex dynamical systems with prevalent network induced phenomena such as missing measurements, sensor delays, multiple fading measurements, and signal quantization.

Appendix

Derivation of Equation for Horizon Initial Mean (18). Taking expectation on both sides of (1) and using $\mathbf{E}[w(t)] = 0$ we immediately obtain recursive equation (18) for mean $m(t) = \mathbf{E}[x(t)]$.

Derivation of Equation For Horizon Initial Covariance (19a). Subtracting (18) from (1) we obtain time propagation of the centered state:

$$\tilde{x}(t+1) = \sum_{h=0}^{M} F_h(t-h)\,\tilde{x}(t-h) + w(t), \quad t = 0, 1, 2, \ldots. \tag{A.1}$$

Next we have

$$\begin{aligned}
&\tilde{x}(t+1)\,\tilde{x}(t+1)^T \\
&= \sum_{h_1,h_2=0}^{M} F_{h_1}(t-h_1)\,\tilde{x}(t-h_1)\,\tilde{x}(t-h_2)^T F_{h_2}^T(t-h_2) \\
&\quad + w(t)\,w(t)^T + \sum_{h_1=0}^{M} F_{h_1}(t-h_1)\,\tilde{x}(t-h_1)\,w(t)^T \\
&\quad + \sum_{h_2=0}^{M} w(t)\,\tilde{x}(t-h_2)^T F_{h_2}^T(t-h_2).
\end{aligned} \tag{A.2}$$

Taking expectation on both sides of (A.2) and using the fact that current noise $w(t)$ does not depend on current and past states $\tilde{x}(t-h_1)$, $\tilde{x}(t-h_2)$ we obtain recursive equation for covariances (19a) and (19b),

$$\begin{aligned}
&P(t+1, t+1) \\
&= \sum_{h_1,h_2=0}^{M} F_{h_1}(t-h_1)\,P(t-h_1, t-h_2)\,F_{h_2}^T(t-h_1) + Q(t).
\end{aligned} \tag{A.3}$$

Note that (A.3) contains autocovariance:

$$P(t-h_1, t-h_2) = \mathbf{E}\left[\tilde{x}(t-h_1)\,\tilde{x}(t-h_2)^T\right], \tag{A.4}$$

$$\begin{aligned}
\tilde{x}(t-h) &= x(t-h) - m(t-h), \\
h_1, h_2 &= 0, 1, \ldots, M.
\end{aligned} \tag{A.5}$$

Derivation of Equation for Autocovariance (19b). Using "symmetric" property of autocovariance $P(t-h_2, t-h_1) = P(t-h_1, t-h_2)^T$ and without loss of generality we can assume that $k-h_1 \geq k-h_2$. Substituting $t \to t-h_1$ in (A.1) we obtain

$$\tilde{x}(t-h_1+1) = \sum_{\ell_1=0}^{M} F_{l_1}(t-h_1-\ell_1)\,\tilde{x}(t-h_1-\ell_1) + w(t-h_1). \tag{A.6}$$

Multiplying both sides of (A.6) by $\tilde{x}(t-h_2+1)^T$ and using (A.5) we obtain

$$\tilde{x}(t-h_1+1)\,\tilde{x}(t-h_2+1)^T = \sum_{\ell_1=0}^{M} F_{\ell_1}(t-h_1-\ell_1)\,\tilde{x}(t-h_1-\ell_1)\,\tilde{x}(t-h_2+1)^T + w(t-h_1)\,\tilde{x}(t-h_2+1)^T, \tag{A.7}$$

$$P(t-h_1+1, t-h_2+1) = \sum_{\ell_1=0}^{M} F_{\ell_1}(t-h_1-\ell_1)\,P(t-h_1-\ell_1, t-h_2+1) + \mathbf{E}\left[w(t-h_1)\,\tilde{x}(t-h_2+1)^T\right], \quad h_1, h_2 = 0, 1, \ldots, M. \tag{A.8}$$

It remains to calculate expectation in (A.8); that is,

$$\mathbf{E}\left[w(t-h_1)\,\tilde{x}(t-h_2+1)^T\right] \quad \text{for } t-h_1 \geq t-h_2. \tag{A.9}$$

Calculating product $w(t-h_1)\tilde{x}(t-h_2+1)^T$ using (A.6) and after that taking expectation we get

$$\mathbf{E}\left[w(t-h_1)\,\tilde{x}(t-h_2+1)^T\right] = \sum_{\ell_2=0}^{M} \mathbf{E}\left[w(t-h_1)\,\tilde{x}(t-h_2-\ell_2)^T\right] F_{l_2}^T(t-h_2-\ell_2) + \mathbf{E}\left[w(t-h_1)\,w(t-h_2)^T\right]. \tag{A.10}$$

According to assumption $t-h_1 \geq t-h_2$, "future" noise $w(t-h_1)$ does not depend on current and past states $\tilde{x}(t-h_2-\ell_2)$; therefore $\mathbf{E}[w(t-h_1)\tilde{x}(t-h_2-\ell_2)^T] = 0$. Next using property of white noise we obtain

$$\mathbf{E}\left[w(t-h_1)\,w(t-h_2)^T\right] = Q(t-h_1)\,\delta_{t-h_1,t-h_2}. \tag{A.11}$$

Finally using (A.8), (A.10), and (A.11) we get equation for autocovariance (19b).

This completes the derivation Lyapunov-like equations for receding horizon mean and covariances.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Discrete-Time Sliding Mode Control for Uncertain Networked System Subject to Time Delay

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We deal with uncertain systems with networked sliding mode control, subject to time delay. To minimize the degenerative effects of the time delay, a simpler format of state predictor is proposed in the control law. Some ultimate bounded stability analyses and stabilization conditions are provided for the uncertain time delay system with proposed discrete-time sliding mode control strategy. A numerical example is presented to corroborate the analyses.

1. Introduction

Networked control system (NCS) is a very convenient strategy of control for industrial plants, where actuators and machines are located in rustic setting [1]. In those environments, a digital device, which computes the control signals, is susceptible to degeneration. Many other advantages, such as simple installation, great flexibility, and low cost, make the NCS a strategy widely used and researched [2]. However, despite these advantages, NCS implies the need of data acquisition, conversion of analog signal in digital form, and processing the data to generate the control signal. These steps may cause time delays of several sampling periods, which tend to deteriorate the system performance [3]. Many control strategies that are robust with respect to parametric variations and nonlinearities of the plant exhibit great sensitivity when delays are present, losing all their features of robustness. Due to this and other problems and high control utilization, NCS has significantly attracted research communities with important results for numerous control methods such as fuzzy control, neural control, adaptive control, sliding mode control, optimal control techniques, and many more techniques [4–11].

In particular, the strategy sliding mode control (SMC) [12, 13] is more sensitive to time delay. The main advantage of SMC is its robustness with respect to matched uncertainties [14]. By using a high speed switching control law in order to take the states trajectory to a sliding surface, if the states used are delayed, the control law may not direct the states to this surface, which can also generate performance loss or even lead system to instability. The damage caused by delays to sliding mode control motivates several studies with important results, including SMC performed by NCS [15–18].

In this paper we deal with uncertain system with discrete-time sliding mode control (DSMC) performed with NCS subject to time delay. To minimize the degenerative effects of delay, a discrete-time state predictor in simplified form used in conjunction with DSMC is proposed. Some ultimate bounded stability analyses and stabilization conditions are provided for the uncertain time delay system with proposed discrete-time sliding mode control strategy. A numerical example is presented to corroborate the results shown in the analyses.

2. Systems with Time Delay due to NCS

Consider the following uncertain discrete-time system:

$$x_{k+1} = \Phi x_k + \Gamma v_k + f_k, \tag{1}$$

where $x_k \in \Re^n$ is the state vector, $v_k \in \Re^m$ is the control vector, and $f_k \in \Re^n$ is the bounded uncertainties vector. The

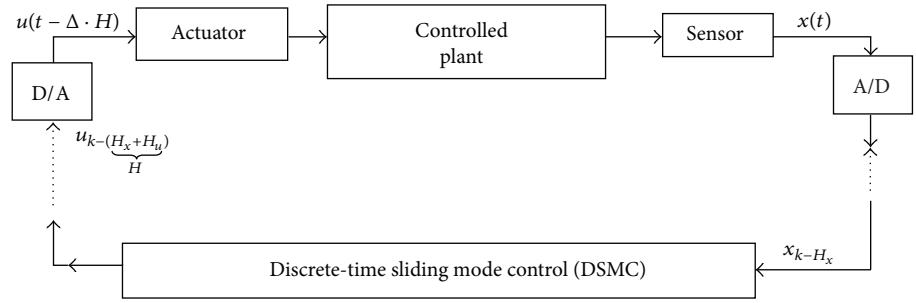

Figure 1: NCS representation in direct structure.

matrices $\Phi \in \Re^{nxn}$ and $\Gamma \in \Re^{nxm}$ are constants with nominal values of the plant, which is controllable.

The discrete-time law is given by

$$v(t) = v_k = Fx_k; \quad k\Delta \le t < (k+1)\Delta, \tag{2}$$

where Δ is the sampling period and $F \in \Re^{mxn}$ is the state feedback gain matrix, whose design can be done through various methods. For the design of matrix F, in this paper, we analyze the stability robustness of the technique discrete-time sliding mode control (DSMC) performed by NCS.

As represented in Figure 1, when the control of the system is performed by a network in direct structure, the presence of delay due to the time required for analog and digital conversion is very common for receiving, processing, and transmission of data.

When the delay is directly related to the time required for state vector signal transmission, we have

$$v_k = Fx_{k-H_x}, \tag{3}$$

where H_x is the number of periods of delayed sampling state vector. Thus, we have the equivalence to a system with delayed control signal as

$$x_{k+1} = \Phi x_k + \Gamma u_{k-H_x} + f_k, \tag{4}$$

where $u_{k-H_x} = Fx_{k-H_x}$.

When the delay is due to the time required for control signal transmission we have

$$x_{k+1} = \Phi x_k + \Gamma u_{k-H_u} + f_k, \tag{5}$$

where H_u is the number of sampling periods of the delayed control signal.

Usually these time delays occur simultaneously in the NCS so that

$$u_{k-H} = Fx_{k-H}, \tag{6}$$

where $H = H_x + H_u$. So, we have the following system:

$$x_{k+1} = \Phi x_k + \Gamma u_{k-H} + f_k. \tag{7}$$

If the sampling time is Δ, the control time delay in the continuous time system is $\Delta \cdot H$. It is known that time delays in control signals degrade system performance [6–9]. Specifically, sliding mode control is more sensitive to these types of failures. In the next section, the use of a simpler format of state predictor to minimize the damaging effects of the time delay is proposed.

3. State Predictor

For the system (7) the following control law is proposed:

$$u_k = F\widehat{x}_k, \tag{8}$$

where the predictive state vector $\widehat{x}_k$ is an estimate of actual state vector at sample time $k+H$, that is, x_{k+H}.

In Xia et al. [19], the authors present a predictor of states for discrete-time systems with control time delay given by

$$\widehat{x}_k = \Phi^H x_k + \sum_{i=-H+1}^{0} \Phi^{-i}\Gamma u(k-1+i). \tag{9}$$

By (9), we observe that to obtain the predicted states, several past samples of the control signal are needed and, for each sampling period, many calculations are required. We therefore propose predictors of states that require neither sampling control signal nor calculations that cause large processing time.

Consider the system (7) with the control law (8) and assume uncertainties are zero; that is, $f_k = 0$. So, $\widehat{x}_k = x_{k+H}$ and $u_{k-H} = Fx_k$. Then,

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma \underbrace{Fx_{(k+H)-H}}_{u_{k-H}} = (\Phi + \Gamma F)\, x_k; x_{k+2} \\ &= (\Phi + \Gamma F)\, x_{k+1} = (\Phi + \Gamma F)(\Phi + \Gamma F)\, x_k \\ &= (\Phi + \Gamma F)^2 x_k, \end{aligned} \tag{10}$$

and for the sample time $k+H$ we have

$$x_{k+H} = \underbrace{(\Phi + \Gamma F)}_{\Psi}^{H} x_k. \tag{11}$$

By (11), we define the predicted state vector as

$$\widehat{x}_k = \Psi^{(\widehat{H})} x_k, \tag{12}$$

where $\Psi = (\Phi + \Gamma F)$ is a stable matrix and $\widehat{H}$ is an estimate value of the actual delay H. Note that

$$x_k = \Psi^{(-\widehat{H})} \widehat{x}_k. \tag{13}$$

It should be noted that, in the presence of uncertainties, that is, $f_k \neq 0$ and $H \neq \widehat{H}$, the predictor (12) is not accurate.

The following proposition deals with the dynamics of the proposed predictor (12).

Proposition 1. *Consider the uncertain time delay system* (7) *with the control law* (8) *and the state predictor* (12). *Then the dynamics of predictive states are given by*

$$\widehat{x}_{k+1} = \overline{\Phi}\widehat{x}_k + \overline{\Gamma}\overline{u}_k + \Psi^{(\widehat{H})}\widetilde{e}_k, \tag{14}$$

where $\overline{\Phi} = \Psi^{(\widehat{H})}\Phi\Psi^{(-\widehat{H})}$, $\overline{\Gamma} = \Psi^{(\widehat{H})}\Gamma$, $\overline{u}_k = F\Psi^{(-\widehat{H})}\widehat{x}_k = \overline{F}\widehat{x}_k$, $\widetilde{e}_k = f_k - \Gamma F e_k$, *and* $e_k = x_k - \widehat{x}_{k-\widetilde{H}}$.

Proof. Due to uncertainties, we have $\widehat{x}_{k-H} \neq x_k$. We define the error vector as

$$\begin{aligned} e_k &= x_k - \widehat{x}_{k-H}, \\ \widehat{x}_{k-H} &= x_k - e_k. \end{aligned} \tag{15}$$

In (12), at sampling period $k+1$, it follows that

$$\widehat{x}_{k+1} = \Psi^{(\widehat{H})}x_{k+1}. \tag{16}$$

Substituting (7) into (16), we have

$$\widehat{x}_{k+1} = \Psi^{(\widehat{H})}\left(\Phi x_k + \Gamma u_{k-H} + f_k\right). \tag{17}$$

Using (8) and (13) into (17) we obtain

$$\widehat{x}_{k+1} = \Psi^{(\widehat{H})}\left(\Phi\underbrace{\Psi^{(-\widehat{H})}\widehat{x}_k}_{x_k} + \Gamma\underbrace{F\widehat{x}_{k-H}}_{u_{k-H}} + f_k\right). \tag{18}$$

Substituting (15) into (18) and rearranging, we get

$$\widehat{x}_{k+1} = \overline{\Phi}\widehat{x}_k + \overline{\Gamma}\overline{u}_k + \Psi^{(\widehat{H})}\widetilde{e}_k, \tag{19}$$

where $\overline{\Phi} = \Psi^{(\widehat{H})}\Phi\Psi^{(-\widehat{H})}$, $\overline{\Gamma} = \Psi^{(\widehat{H})}\Gamma$, $\overline{u}_k = F\Psi^{(-\widehat{H})}\widehat{x}_k = \overline{F}\widehat{x}_k$, and $\widetilde{e}_k = f_k - \Gamma F e_k$. □

Note 1. The predictive state vector $\widehat{x}_k$ of (19) has the same closed-loop dynamics of the actual state vector x_k. It can be seen by substituting (8) and (12) into the system (7):

$$x_{k+1} = \Phi x_k + \Gamma F\widehat{x}_{k-H} + f_k. \tag{20}$$

Considering (15) and the matrix $\Psi = (\Phi + \Gamma F)$, we obtain the closed-loop dynamics of the actual state vector

$$x_{k+1} = \underbrace{(\Phi + \Gamma F)}_{\Omega_{MF}=\Psi}x_k + \widetilde{e}_k. \tag{21}$$

Also, by (19), the closed-loop dynamics of the predictive state vector $\widehat{x}_k$ are

$$\widehat{x}_{k+1} = \underbrace{\left[\Psi^{(\widehat{H})}\underbrace{(\Phi + \Gamma F)}_{\Psi}\Psi^{(-\widehat{H})}\right]}_{\overline{\Omega}_{MF}=\Psi}\widehat{x}_k + \Psi^{(\widehat{H})}\widetilde{e}_k. \tag{22}$$

Comparing (21) with (22) it can be noted that the matrices $\overline{\Omega}_{MF}$ and Ω_{MF} are both equal to Ψ. Thus, we conclude that the dynamics of the predictive state vector $\widehat{x}_k$ are the same as actual state vector x_k.

Note 2. In (21) and (22) it can be seen that, with the control (8) and state predictor (12), the influence of uncertainties in the dynamics of the system increases with the presence of the time delay due to the prediction error $\widetilde{e}_k = f_k - \Gamma F e_k$. According to control strategy adopted for the calculation of gain matrix F, there will be greater or lesser influence on system performance.

In the next section, the stability robustness will be analyzed specifically for discrete-time sliding mode control (DSMC).

4. Discrete-Time Sliding Mode Control (DSMC) via NCS with Delay

In this work, the proposed DSMC law can be of the form

$$u(t) = u_k = u_k^{\text{eq}} + u_k^N, \quad k\Delta \le t < (k+1)\Delta, \tag{23}$$

where $u_k^{\text{eq}} \in \mathfrak{R}^m$ is the equivalent control vector, which establishes the system dynamics in sliding mode, and $u_k^N \in \mathfrak{R}^m$ is the control signal vector that takes the trajectory of states to the sliding surface. In sliding mode, the system becomes less sensitive to certain classes of disturbances and uncertainty [14].

4.1. Design of DSMC with Predictive States. The equivalent control u_k^{eq} is primarily designed. Later, the design of u_k^N is performed.

A linear discrete-time sliding surface is defined as

$$S_k = Gx_k, \tag{24}$$

where the gain matrix $G \in \mathfrak{R}^{mxn}$ is designed such that the sliding trajectory has desired dynamics. The equivalent control is obtained with the following sliding condition:

$$S_{k+1} = S_k. \tag{25}$$

Disregarding the uncertainties and substituting (7) and (24) into (25) we have

$$\begin{aligned} G\Phi x_k + G\Gamma u_{k-H}^{\text{eq}} &= Gx_k, \\ u_{k-H}^{\text{eq}} &= -(G\Gamma)^{-1}G(\Phi - I)x_k. \end{aligned} \tag{26}$$

Thus, for the sample period k we have

$$u_k^{\text{eq}} = F_e x_{k+H}. \tag{27}$$

Using the predictor $x_{k+H} = \widehat{x}_k$ defined in (12), we obtain

$$u_k^{\text{eq}} = F_e\widehat{x}_k, \tag{28}$$

where $F_e = -(G\Gamma)^{-1}G(\Phi - I)eG\Gamma$ is a nonsingular matrix.

Now, the control law u_k^N is designed. Consider the following Lyapunov function:

$$V_k = S_k^T S_k. \tag{29}$$

The control law u_k^N should provide to the system the condition of attractiveness to the sliding mode. For this to occur, the following condition of attractiveness must be satisfied:

$$V_{k+1} < V_k. \tag{30}$$

By (29), it follows that (30) becomes

$$S_{k+1}^T S_{k+1} < S_k^T S_k. \tag{31}$$

Defining [20],

$$\Delta S_{k+1} = S_{k+1} - S_k. \tag{32}$$

Disregarding the uncertainties and substituting (7) and (24) into (32), we have

$$\Delta S_{k+1} = G\Phi x_k + G\Gamma u_{k-H}^{\text{eq}} + G\Gamma u_{k-H}^N - Gx_k. \tag{33}$$

By replacing u_{k-H}^{eq}, (26), into (33) comes

$$\Delta S_{k+1} = G\Gamma u_{k-H}^N. \tag{34}$$

Replacing $S_{k+1} = S_k + \Delta S_{k+1}$ in condition (31) we find

$$\left\| S_k + G\Gamma u_{k-H}^N \right\| < \left\| S_k \right\|, \tag{35}$$

where $\|S_k\|$ is the Euclidian norm of vector $S_k \in \Re^m$.

A control law that satisfies the attractiveness condition (35) is defined as

$$u_k^N = F_n \widehat{x}_k, \tag{36}$$

where $\widehat{x}_k = x_{k+H}$, $F_n = -(G\Gamma)^{-1}\beta G$, and $\beta \in \Re^{mxm}$ is a diagonal matrix with $0 < \beta_{ii} < 1$, $i = 1, \ldots, m$. Thus, the control law $u_k = u_k^{\text{eq}} + u_k^N$ becomes

$$u_k = F\widehat{x}_k, \tag{37}$$

with $F = F_e + F_n$.

4.2. Robustness Analysis. This section analyzes the stability robustness of the system (7) in the following proposition.

Proposition 2. *Consider the uncertain time delay system* (7)*, with control law* (37) *and* (12)*. The attractiveness condition* (30) *is satisfied if* $\|Gx_k\| > \|G\widetilde{e}_k\|$*, with* $\widetilde{e}_k = f_k - \Gamma F(x_k - \widehat{x}_{k-H})$.

Proof. Taking into account the uncertainties, that is, $f_k \neq 0$ and $H \neq \widehat{H}$, (32) becomes

$$\Delta S_{k+1} = G\left(\Phi x_k + \Gamma u_{k-H} + f_k\right) - Gx_k. \tag{38}$$

Substituting (37), (28), and (36) into (38), one comes to

$$\Delta S_{k+1} = -\beta G x_k + G\widetilde{e}_k, \tag{39}$$

where $\widetilde{e}_k = f_k - \Gamma F e_k$. Substituting (39) into (30), using (32) and considering $\beta_{ii} = \overline{\beta} \in \Re^+$, $i = 1, \ldots, m$, we have

$$2\left(1-\overline{\beta}\right) S_k{}^T G\widetilde{e}_k < \left[1 - \left(1-\overline{\beta}\right)^2\right] \left\| S_k \right\|^2 - \left\| G\widetilde{e}_k \right\|^2. \tag{40}$$

If $0 < \overline{\beta} < 1.0$, so $2 > 2(1-\overline{\beta}) > 0$. Thus, it is possible to rearrange condition (40) as follows:

$$S_k{}^T G\widetilde{e}_k < \varsigma \left\| S_k \right\|^2 - \overline{\varsigma} \left\| G\widetilde{e}_k \right\|^2, \tag{41}$$

with $\varsigma = (1-(1-\overline{\beta})^2)/2(1-\overline{\beta})$ and $\overline{\varsigma} = 1/2(1-\overline{\beta})$. Considering that $0 < \overline{\beta} < 1,0$, so $(\overline{\varsigma}/\varsigma) > 1,0$. With this, it can be seen, by (41), that the condition that satisfies the cases where $S_k{}^T G\widetilde{e}_k > 0$ and $S_k{}^T G\widetilde{e}_k < 0$ is the following:

$$\frac{\left\| S_k \right\|}{\left\| G\widetilde{e}_k \right\|} > \sqrt{\frac{\overline{\varsigma}}{\varsigma}} > 1,0. \tag{42}$$

Therefore, to satisfy (41) the norm $\|S_k\| = \|Gx_k\|$ must be greater than $\|G\widetilde{e}_k\|$. □

Note 3. From Proposition 2 it follows that, when the uncertain time delay system (7) is in steady state, its states remain in the vicinity of the ideal equilibrium point (origin). This fact implies the concept "ultimate bounded stability" [21].

Proposition 3. *Consider the uncertain time delay system* (7) *with the control law* (37) *and that attractiveness condition* (30) *is satisfied. Also, the actual number of delayed samples H is unknown and the uncertainties are bounded; that is,* $\|f_k\| \leq d_{\max} < \infty$ *for all values of k. So, in steady state the norm* $\|\widetilde{e}_k\|$ *will be also bounded; that is,* $\lim_{k\to\infty} \|\widetilde{e}_k\| = \overline{b}_{\max} < \infty$*, with* $\widetilde{e}_k = f_k - \Gamma F(x_k - \widehat{x}_{k-H})$.

Proof. Consider the system (7) with $f_k \neq 0$ and the control law (37) with predictive state vector (12). So

$$x_{k+1} = \Phi x_k + \Gamma F \widehat{x}_{k-H} + f_k. \tag{43}$$

With $\widehat{x}_{k-H} = x_k - e_k$ we have

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma F\left(x_k - e_k\right) + f_k \\ &= \left(\Phi + \Gamma F\right) x_k + \left(f_k - \Gamma F e_k\right). \end{aligned} \tag{44}$$

For $\Psi = \Phi + \Gamma F$ and $\widetilde{e}_k = f_k - \Gamma F e_k$ we have

$$\begin{aligned} x_{k+1} &= \Psi x_k + \widetilde{e}_k; \\ x_{k+2} &= \Psi x_{k+1} + \widetilde{e}_{k+1} = \Psi\underbrace{\left(\Psi x_k + \widetilde{e}_k\right)}_{x_{k+1}} + \widetilde{e}_{k+1} \\ &= \Psi^2 x_k + \Psi\widetilde{e}_k + \widetilde{e}_{k+1}; \\ x_{k+3} &= \Psi^2 x_{k+1} + \Psi\widetilde{e}_{k+1} + \widetilde{e}_{k+2} \\ &= \Psi^2\underbrace{\left(\Psi x_k + \widetilde{e}_k\right)}_{x_{k+1}} + \Psi\widetilde{e}_{k+1} + \widetilde{e}_{k+2} \\ &= \Psi^3 x_k + \Psi^2\widetilde{e}_k + \Psi\widetilde{e}_{k+1} + \widetilde{e}_{k+2}; \end{aligned} \tag{45}$$

and for the sample time $k + H$ we have

$$\begin{aligned} x_{k+H} = \Psi^H x_k + \Psi^{(H-1)}\widetilde{e}_k + \Psi^{(H-2)}\widetilde{e}_{k+1} \\ + \Psi^{(H-3)}\widetilde{e}_{k+2} + \cdots + \widetilde{e}_{k+(H-1)}. \end{aligned} \tag{46}$$

Thus, by (46), the predictive state vector $\overline{x}_k = x_{k+H}$ is accurately obtained by the equation

$$\overline{x}_k = \Psi^{(H)} x_k + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left(\widetilde{e}_{k+i}\right). \tag{47}$$

The actual state vector is obtained as $x_k = \overline{x}_{k-H}$. So by (47) we have

$$\overline{x}_{k-H} = \Psi^{(H)} x_{k-H} + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left(\widetilde{e}_{k-H+i}\right). \tag{48}$$

We defined the predictor (not exact) as $\widehat{x}_k = \Psi^{\widehat{H}} x_k$ and the error is given by $e_k = x_k - \widehat{x}_{k-H}$. Then, with (48) and considering that $\widetilde{e}_k = f_k - \Gamma F e_k$, we have

$$\begin{aligned} e_k = &\sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left[f_{k-H+i} - (\Gamma F)\left(e_{k-H+i}\right)\right] \\ &+ \Psi_f x_{k-H}, \end{aligned} \tag{49}$$

where $\Psi_f = \Psi^{(H)} - \Psi^{(\widehat{H})}$. For the sampling period $k+N$ (49) becomes

$$\begin{aligned} &e_{k+N} + \left[\sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} (\Gamma F)\left(e_{k-H+i+N}\right)\right] \\ &= \left[\sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left(f_{k-H+i+N}\right)\right] + \Psi_f \left(x_{k-H+N}\right). \end{aligned} \tag{50}$$

It is easy to deduce that $x_{k-H+1} = \Psi x_{k-H} + \widetilde{e}_{k-H}$. So for sampling period $k-H+N$ we have

$$\begin{aligned} x_{k-H+1} &= \Psi x_{k-H} + \widetilde{e}_{k-H}; \\ x_{k-H+2} &= \Psi \underbrace{x_{k-H+1}}_{\Psi x_{k-H} + \widetilde{e}_{k-H}} + \widetilde{e}_{k-H+1} \\ &= \Psi^2 x_{k-H} + \Psi \widetilde{e}_{k-H} + \widetilde{e}_{k-H+1}; \\ &\vdots \\ x_{k-H+N} &= \Psi^{(N)} x_{k-H} + \sum_{i=0}^{N-1} \Psi^{-[i-(N-1)]} \left(\widetilde{e}_{k-H+i}\right). \end{aligned} \tag{51}$$

Thus, by (51), (50) can be rewritten as

$$\begin{aligned} &e_{k+N} + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} (\Gamma F)\left(e_{k+N-H+i}\right) \\ &= \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left(f_{k+N-H+i}\right) + \left(\Psi_f \Psi^{(N)}\right)\left(x_{k-H}\right) \\ &\quad + \Psi_f \left[\sum_{i=0}^{N-1} \Psi^{-[i-(N-1)]} \underbrace{\left(\widetilde{e}_{k-H+i}\right)}_{\left[\left(f_{k-H+i}\right)-(\Gamma F)\left(e_{k-H+i}\right)\right]}\right] e_{k+N} \\ &\quad + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} (\Gamma F)\left(e_{k+N-H+i}\right) \\ &= \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left(f_{k+N-H+i}\right) \\ &\quad + \left(\Psi_f \Psi^{(N)}\right)\left(x_{k-H}\right) + \Psi_f \sum_{i=0}^{N-1} \Psi^{-[i-(N-1)]} \left(f_{k-H+i}\right) \\ &\quad - \Psi_f \sum_{i=0}^{N-1} \Psi^{-[i-(N-1)]} \left[(\Gamma F)\left(e_{k-H+i}\right)\right] e_{k+N} \\ &\quad + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} (\Gamma F)\left(e_{k+N-H+i}\right) \\ &\quad + \Psi_f \sum_{i=0}^{N-1} \Psi^{-[i-(N-1)]} \left[(\Gamma F)\left(e_{k-H+i}\right)\right] \\ &= \left(\Psi_f \Psi^{(N)}\right)\left(x_{k-H}\right) + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} \left(f_{k+N-H+i}\right) \\ &\quad + \Psi_f \sum_{i=0}^{N-1} \Psi^{-[i-(N-1)]} \left(f_{k-H+i}\right). \end{aligned} \tag{52}$$

The norm of f_k is bounded. Suppose for simplicity, without loss of generality, that this vector is constant, $f_k = f$, such that $\|f\| = d_{\max}$. So, (52) becomes

$$\begin{aligned} &e_{k+N} + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]} (\Gamma F)\left(e_{k-H+i+N}\right) \\ &\quad + \Psi_f \sum_{j=0}^{N-1} \Psi^{-[j-(N-1)]} (\Gamma F)\left(e_{k-H+j}\right) \\ &= \left(\Psi_f \Psi^{(N)}\right)\left(x_{k-H}\right) \\ &\quad + \left(\Psi_f \sum_{j=0}^{N-1} \Psi^{-[j-(N-1)]} + \sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]}\right)(f). \end{aligned} \tag{53}$$

The closed-loop matrix Ψ is designed to be stable, that is, all its eigenvalues are inside in the unit circle. Then in the second member of (53) we have

$$\lim_{k,N\to\infty}\left\{\left(\Psi_f\sum_{j=0}^{N-1}\Psi^{-[i-(N-1)]}+\sum_{i=0}^{H-1}\Psi^{-[i-(H-1)]}\right)\right\}=\Upsilon,$$

$$\|\Upsilon\|<\infty,$$

$$\lim_{k,N\to\infty}\left\{\left(\Psi_f\Psi^{(N)}\right)\right\}=0. \tag{54}$$

So for (53) to be satisfied, in the first member of this equation, $\lim_{k\to\infty}e_k$ must be also a constant. Thus, it can be concluded that

$$\lim_{k,N\to\infty}\|e_k\|=b_{\max}<\infty,$$
$$\lim_{k,N\to\infty}\|\widetilde{e}_k\|=\overline{b}_{\max}<\infty. \tag{55}$$

□

Note 4. With $f_k = f$, by Proposition 3, we have that $\lim_{k\to\infty}e_k = e$ and $\|e\| = b_{\max} < \infty$. For stability, the attractiveness condition $\|x_k\| > \|\widetilde{e}_k\| = \|(f_k - \Gamma F e_k)\|$ is maintained until the system enters in the steady state, that is, until the state trajectory enters the neighborhood of the origin. The value of norm $\|e\| = b_{\max}$ is established, as can be seen in (53) and (54), by

$$\left(\Psi_f\sum_{j=0}^{N-1}\Psi^{-[j-(N-1)]}+\sum_{i=0}^{H-1}\Psi^{-[i-(H-1)]}\right)(f). \tag{56}$$

Then, the higher the value of H, the higher the value of $\|(f_k - \Gamma F e_k)\|$. This imposes a limit on the maximum number of delayed samples $H_{\max}$, in which it is possible to satisfy the attractiveness condition of sliding mode control.

Proposition 4. *Consider the uncertainties with bounded norm; that is,* $\|f_k\| \le d_{\max} < \infty$, *for all* k *and* $\lim_{k\to\infty}\|\widetilde{e}_k\| = \overline{b}_{\max} < \infty$, *where* $\widetilde{e}_k = f_k - \Gamma F(x_k - \widehat{x}_{k-H})$. *If the attractiveness condition* (30) *is satisfied, the system* (7) *will be uniformly ultimately bounded with control law* (37), (8), *and* (12).

Proof. This proof is a version of result and terminology presented in [19]. Consider a linear transformation for the uncertain time delay system (7) of the form

$$Tx_{k+1} = \begin{bmatrix} z_{1_{k+1}} \\ z_{2_{k+1}} \end{bmatrix} = \begin{bmatrix} \overline{\Phi}_{11} & \overline{\Phi}_{12} \\ \overline{\Phi}_{21} & \overline{\Phi}_{22} \end{bmatrix}\begin{bmatrix} z_{1_k} \\ z_{2_k} \end{bmatrix} + \begin{bmatrix} \overline{0} \\ \overline{\Gamma}_2 \end{bmatrix} u_{k-H} + \begin{bmatrix} \overline{f}_{1_k} \\ \overline{f}_{2_k} \end{bmatrix}, \tag{57}$$

where $z_{1_k} \in \Re^{n-m}$, $z_{2_k} \in \Re^m$, $\overline{f}_{1_k} \in \Re^{n-m}$ is the unmatched uncertainty vector; $\overline{f}_{2_k} \in \Re^m$ is the matched uncertainty vector; $\overline{\Phi}_{11}$, $\overline{\Phi}_{12}$, $\overline{\Phi}_{21}$,$\overline{\Phi}_{22}$, and $\overline{\Gamma}_2$ are constant matrices with appropriate dimensions; and $\overline{0} \in \Re^{(n-m)x(m)}$ is the zero matrix. Note that, because the uncertainties f_k has bounded norm, so $\|\overline{f}_{1_k}\| \le f_1 < \infty$ and $\|\overline{f}_{2_k}\| \le f_2 < \infty$ for all k.

Now, let us use the following Lyapunov function:

$$\overline{V}_k = z_{1_k}{}^T(P)\, z_{1_k} + S_k{}^T S_k, \tag{58}$$

where $0 < P = P^T \in \Re^{(n-m)x(n-m)}$. So, the stability condition is

$$\overline{V}_{k+1} - \overline{V}_k < 0. \tag{59}$$

The sliding surface is

$$S_k = GT^{-1}z_k = \overline{G}z_k = \begin{bmatrix} \overline{G}_1 & \overline{G}_2 \end{bmatrix}\begin{bmatrix} z_{1_k} \\ z_{2_k} \end{bmatrix}. \tag{60}$$

So,

$$z_{2_k} = \left(\overline{G}_2\right)^{-1}S_k - \left(\overline{G}_2\right)^{-1}\overline{G}_1 z_{1_k}. \tag{61}$$

Substituting (61) into (57) we have

$$z_{1_{k+1}} = \overline{\overline{\Phi}}_{11}z_{1_k} + \overline{\overline{\Phi}}_{12}S_k + \overline{f}_{1_k}, \tag{62}$$

where $\overline{\overline{\Phi}}_{11} = \overline{\Phi}_{11} - \overline{\Phi}_{12}(\overline{G}_2)^{-1}\overline{G}_1$ and $\overline{\overline{\Phi}}_{12} = \overline{\Phi}_{12}(\overline{G}_2)^{-1}$.

Also,

$$\begin{aligned} S_{k+1} &= Gx_{k+1}, \\ S_{k+1} &= G\Phi x_k + G\Gamma u_{k-H} + Gf_k, \\ u_{k-H} &= F\underbrace{\widehat{x}_{k-H}}_{x_k - e_k}, \\ S_{k+1} &= G\Phi x_k - G\Gamma\underbrace{(G\Gamma)^{-1}\left[G(\Phi - I) + \overline{\beta}G\right]}_{-F}x_k \\ &\quad + G\underbrace{(f_k - \Gamma F e_k)}_{\widetilde{e}_k}, \\ S_{k+1} &= \underbrace{Gx_k}_{S_k} - \overline{\beta}\underbrace{Gx_k}_{S_k} + G\widetilde{e}_k, \\ S_{k+1} &= \left(1-\overline{\beta}\right)\left(S_k + \overline{\overline{G}}\widetilde{e}_k\right), \end{aligned} \tag{63}$$

with $\overline{\overline{G}} = G/(1-\overline{\beta})$, $0 < \overline{\beta} < 1.0$.

Substituting (62) and (63) into (58)

$$\begin{aligned}
\overline{V}_{k+1} &= z_{1_{k+1}}{}^{T}[P]\, z_{1_{k+1}} + S_{k+1}{}^{T} S_{k+1} \\
&= \left(\overline{\overline{\Phi}}_{11} z_{1_k} + \overline{\overline{\Phi}}_{12} S_k + \overline{f}_{1_k}\right)^{T} \\
&\quad \times (P)\left(\overline{\overline{\Phi}}_{11} z_{1_k} + \overline{\overline{\Phi}}_{12} S_k + \overline{f}_{1_k}\right) \\
&\quad + \left(1-\overline{\beta}\right)^2 \left(S_k + \overline{\overline{G}}\widetilde{e}_k\right)^{T} \left(S_k + \overline{\overline{G}}\widetilde{e}_k\right) \\
&= z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{11} z_{1_k} + z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12} S_k \\
&\quad + z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{f}_{1_k} \\
&\quad + S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{11} z_{1_k} + S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12} S_k \\
&\quad + S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{f}_{1_k} \\
&\quad + \overline{f}_{1_k}^{T} P \overline{\overline{\Phi}}_{11} z_{1_k} + \overline{f}_{1_k}^{T} P \overline{\overline{\Phi}}_{12} S_k + \overline{f}_{1_k}^{T} P \overline{f}_{1_k} \\
&\quad + \left(1-\overline{\beta}\right)^2 \left(S_k{}^{T} S_k + 2 S_k{}^{T} \overline{\overline{G}}\widetilde{e}_k + f_k{}^{T} \overline{\overline{G}}^{T} \overline{\overline{G}}\widetilde{e}_k\right),
\end{aligned}$$

$$\begin{aligned}
\overline{V}_{k+1} &= z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{11} z_{1_k} + 2 z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12} S_k \\
&\quad + 2 z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{f}_{1_k} + S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12} S_k \\
&\quad + 2 S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{f}_{1_k} + \overline{f}_{1_k}^{T} P \overline{f}_{1_k} + \left(1-\overline{\beta}\right)^2 S_k{}^{T} S_k \\
&\quad + 2\left(1-\overline{\beta}\right)^2 S_k{}^{T} \overline{\overline{G}}\widetilde{e}_k + \left(1-\overline{\beta}\right)^2 \widetilde{e}_k^{T} \overline{\overline{G}}^{T} \overline{\overline{G}}\widetilde{e}_k,
\end{aligned}$$

$$\begin{aligned}
\overline{V}_{k+1} - \overline{V}_k &= z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{11} z_{1_k} - z_{1_k}^{T} P z_{1_k} \\
&\quad + 2 z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12} S_k + 2 z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{f}_{1_k} \\
&\quad + S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12} S_k + 2 S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{f}_{1_k} \\
&\quad + \overline{f}_{1_k}^{T} P \overline{f}_{1_k} + \left(1-\overline{\beta}\right)^2 S_k{}^{T} S_k - S_k{}^{T} S_k \\
&\quad + 2\left(1-\overline{\beta}\right)^2 S_k{}^{T} \overline{\overline{G}}\widetilde{e}_k + \left(1-\overline{\beta}\right)^2 \widetilde{e}_k^{T} \overline{\overline{G}}^{T} \overline{\overline{G}}\widetilde{e}_k,
\end{aligned}$$

$$\begin{aligned}
\overline{V}_{k+1} - \overline{V}_k &= z_{1_k}^{T} \underbrace{\left(\overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{11} - P\right)}_{-Q} z_{1_k} + 2 z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12} S_k \\
&\quad + 2 z_{1_k}^{T} \overline{\overline{\Phi}}_{11}{}^{T} P \overline{f}_{1_k} + S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12} S_k \\
&\quad + 2 S_k{}^{T} \overline{\overline{\Phi}}_{12}{}^{T} P \overline{f}_{1_k} + \overline{f}_{1_k}^{T} P \overline{f}_{1_k} \\
&\quad + \left[\left(1-\overline{\beta}\right)^2 - 1\right] S_k{}^{T} S_k \\
&\quad + 2\left(1-\overline{\beta}\right)^2 S_k{}^{T} \overline{\overline{G}}\widetilde{e}_k + \left(1-\overline{\beta}\right)^2 \widetilde{e}_k^{T} \overline{\overline{G}}^{T} \overline{\overline{G}}\widetilde{e}_k,
\end{aligned}$$

$$\begin{aligned}
\overline{V}_{k+1} - \overline{V}_k &\le -Q_{\min}\left\|z_{1_k}\right\|^2 + 2\left\|z_{1_k}\right\| \left\|\overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12}\right\| \|S_k\| \\
&\quad + 2\left\|z_{1_k}\right\| \left\|\overline{\overline{\Phi}}_{11}{}^{T} P\right\| \left\|\overline{f}_{1_k}\right\| \\
&\quad + \|S_k\| \left\|\overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12}\right\| \|S_k\| \\
&\quad + 2\|S_k\| \left\|\overline{\overline{\Phi}}_{12}{}^{T} P\right\| \left\|\overline{f}_{1_k}\right\| + \underbrace{\|P\|}\left\|\overline{f}_{1_k}\right\|^2 \\
&\quad + \left|\left(1-\overline{\beta}\right)^2 - 1\right| \|S_k\|^2 \\
&\quad + 2\left(1-\overline{\beta}\right)^2 \|S_k\| \left\|\overline{\overline{G}}\right\| \|\widetilde{e}_k\| + \left(1-\overline{\beta}\right)^2 \left\|\overline{\overline{G}}\right\|^2 \|\widetilde{e}_k\|^2,
\end{aligned}$$

$$\begin{aligned}
\overline{V}_{k+1} - \overline{V}_k &\le -Q_{\min}\left\|z_{1_k}\right\|^2 + 2\delta \left\|z_{1_k}\right\| \left\|\overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12}\right\| \\
&\quad + 2 f_1 \left\|z_{1_k}\right\| \left[\overline{\overline{\Phi}}_{11}{}^{T} P\right] + \delta^2 \left\|\overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12}\right\| \\
&\quad + 2\delta f_1 \left\|\overline{\overline{\Phi}}_{12}{}^{T} P\right\| + {f_1}^2 P + \left|\left(1-\overline{\beta}\right)^2 - 1\right| \delta^2 \\
&\quad + 2\delta\left(\overline{b}_{\max}\right)\left(1-\overline{\beta}\right)^2 \left\|\overline{\overline{G}}\right\| + \left(1-\overline{\beta}\right)^2 \left(\overline{b}_{\max}\right)^2 \left\|\overline{\overline{G}}\right\|^2,
\end{aligned} \tag{64}$$

where $Q_{\min}$ denote the minimum singular value of $Q = -(\overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{11} - P)$. Note that $\lim_{k\to\infty}\|\widetilde{e}_k\| = \overline{b}_{\max} < \infty$ and because the attractiveness condition (30) is satisfied we have $\|S_k\| \le \delta < \infty$ for all k.

After some algebraic manipulation, (64) becomes

$$\overline{V}_{k+1} - \overline{V}_k \le -Q_{\min}\left(\left\|z_{1_k}\right\| - \frac{\delta_1}{Q_{\min}}\right)^2 + \frac{{\delta_1}^2}{Q_{\min}} + \delta_2, \tag{65}$$

where

$$\begin{aligned}
\delta_1 &= \delta \left\|\overline{\overline{\Phi}}_{11}{}^{T} P \overline{\overline{\Phi}}_{12}\right\| + f_1 \left\|\overline{\overline{\Phi}}_{11}{}^{T} P\right\|, \\
\delta_2 &= \delta^2 \left\|\overline{\overline{\Phi}}_{12}{}^{T} P \overline{\overline{\Phi}}_{12}\right\| + 2\delta f_1 \left\|\overline{\overline{\Phi}}_{12}{}^{T} P\right\| + {f_1}^2 \|P\| \\
&\quad + \left|\left(1-\overline{\beta}\right)^2 - 1\right| \delta^2 + 2\delta\left(\overline{b}_{\max}\right)\left(1-\overline{\beta}\right)^2 \left\|G_\beta\right\| \\
&\quad + \left(1-\overline{\beta}\right)^2 \left(\overline{b}_{\max}\right)^2 \left\|G_\beta\right\|^2.
\end{aligned} \tag{66}$$

Therefore [19],

$$\overline{V}_{k+1} - \overline{V}_k < 0 \quad \text{if } \left\| z_{1_k} \right\| > \frac{\delta_1 + \sqrt{\delta_1{}^2 + Q_{\min}\delta_2}}{Q_{\min}} = r; \quad (67)$$

that is, the trajectory of z_{1_k} will enter into ball with center at the origin and radius r and will converge in finite time to the quasisliding mode band δ. At steady state it stays in the neighborhood of the origin. □

Proposition 5. *Consider the uncertain system without delay of the form*

$$x_{k+1} = \Phi x_k + \Gamma \upsilon_k + f_k, \quad (68)$$

in which the sliding mode control law does not use predictors, that is,

$$\upsilon_k = F x_k, \quad (69)$$

where $F = F_e + F_n$, $F_e = -(G\Gamma)^{-1}G(\Phi - I)$, *and* $F_n = -(G\Gamma)^{-1}\overline{\beta}G$, $0 < \overline{\beta} < 1.0$. *So the attractiveness condition* (30) *is satisfied if* $\|Gx_k\| > \|Gf_k\|$.

Proof. Taking into account the uncertainties f_k, (32) becomes

$$\Delta S_{k+1} = G\left(\Phi x_k + \Gamma \upsilon_k + f_k\right) - Gx_k. \quad (70)$$

Substituting (69) into (70),

$$\Delta S_{k+1} = -\beta G x_k + G f_k. \quad (71)$$

In this case, there is no need for using predictor, and the proof is similar to the proof of Proposition 2, (40)–(42), with $\widetilde{e}_k = f_k$. □

Note 5. Because $\|Gf_k\| < \|G\widetilde{e}_k\|$ and Propositions 2,3, and 5, it can be concluded that the use of proposed control law (37) with state predictor (12) stabilizes the delayed system, however, increases the limits around the ideal equilibrium point. It means that uncertainties affect more the performance of the system when time delay is present.

In the following section, the presented propositions and notes are validated through a numerical example.

5. Numerical Example: System of Second Order

In this section, a system of order 2 to validate the propositions and notes presented in the previous section is utilized. Three cases are simulated and graphical results are presented through phase plans.

5.1. System Model and Design of Controllers. The uncertain model is used as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_1(t) + \left[2 + \Delta_{22}\text{sen}(\pi t)\right] x_2(t) \\ &\quad + u(t-\tau) + \Delta_u \text{sen}(\pi t), \end{aligned} \quad (72)$$

where $x_1(t)$ and $x_2(t)$ are the states, $\Delta_{22} = 4.0$ is the amplitude of the parametric uncertainty of the plant, and $\Delta_u = 1.0$ is the amplitude of the disturbance in the control input. The time delay control input is $\tau = H\Delta$, $\Delta = 2$ milliseconds is the sampling period, and H is the actual number of delayed samples. It is considered that only a bounded range of H is known. The estimated time delay $\widehat{\tau}$ of the predictors used in the control design is 40 milliseconds; that is, the number of estimated delayed samples $\widehat{H}$ is 20. The initial states are $x_1(0) = 0.2$ and $x_2(0) = 0.2$. Note that this model without uncertainties has open-loop eigenvalues equal to "−0.4142" and "+2.4142"; thus, it represents an unstable plant.

The control law is generated through feedback of the states using the strategy DSMC, as follows.

(i) DSMC-P1 Controller:

$$u_k = F\widehat{x}_k. \quad (73)$$

The predicted state vector $\widehat{x}_k$ is obtained by means of (9). The matrix $F = F_e + F_n$ is the same as the one of DSMC-P2 described as follows:

(ii) DSMC-P2 Controller:

$$u_k = F\widehat{x}_k. \quad (74)$$

The predicted states vector $\widehat{x}_k$ is obtained by (12), $\widehat{x}_k = \Psi^{(\widehat{H})}x_k$. The matrix $F = F_e + F_n$ is obtained according to (28), (36), and (37). The sliding surface is given by $S_k = Gx_k$ and it is designed such that the sliding pole is the equivalent discrete of "−2", that is, "0.9960". So

$$\begin{aligned} G &= \begin{bmatrix} 996.0077 & 498.0050 \end{bmatrix}, \\ F_e &= \begin{bmatrix} -1.0000 & -3.9930 \end{bmatrix}, \\ F_n &= \begin{bmatrix} -49.8004 & -24.9002 \end{bmatrix}, \\ \Psi^{(\widehat{H})} &= (\Phi + \Gamma F)^{(20)} = \begin{bmatrix} 0.9709 & 0.0239 \\ -1.2248 & 0.3107 \end{bmatrix}. \end{aligned} \quad (75)$$

For $N = 4,000$ sampling periods, that is, 8.0 seconds,

$$\Psi^{(4,000)} = 1.0e-006 \begin{bmatrix} 0.1221 & 0.0048 \\ -0.2441 & -0.0095 \end{bmatrix}; \quad (76)$$

for $N = 10,000$ sampling periods, that is, 20.0 seconds,

$$\begin{aligned} \Psi^{(10,000)} &= 1.0e-017 \begin{bmatrix} 0.4608 & 0.0180 \\ -0.9216 & -0.0359 \end{bmatrix}, \\ \lim_{N\to\infty} \left\{\Psi^{(N)}\right\} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (77)$$

(iii) DSMC Controller:

$$\upsilon_k = F x_k, \quad (78)$$

where x_k is the actual state vector. The matrix $F = F_e + F_n$ is the same as the ones of DSMC-P2 and DSMC-P1.

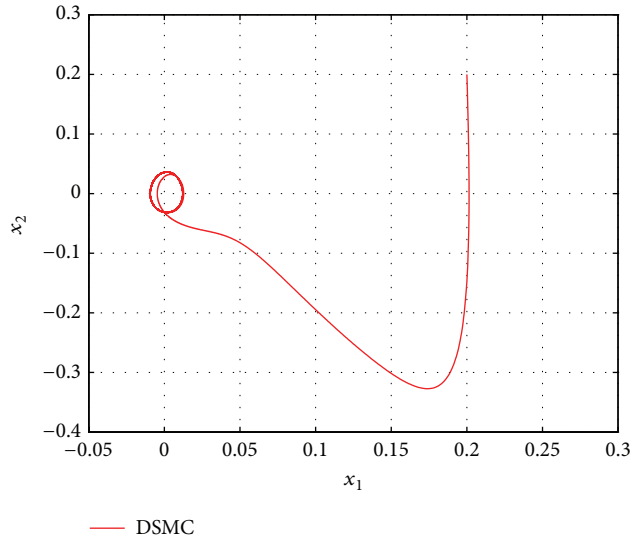

Figure 2: Phase plane for uncertain system without time delay, controlled by DSMC (without predictor).

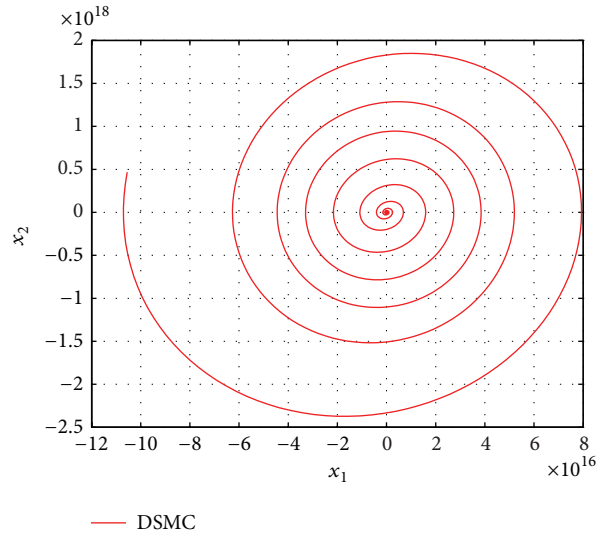

Figure 3: Phase plane for uncertain system with time delay, controlled by DSMC (without predictor).

5.2. Simulations and Analysis of Results. The computational simulations were performed using Matlab/Simulink software. Three cases are discussed as shown below.

Case (i). Uncertain system without time delay, controlled by DSMC.

In this case the uncertain system has no time delay, that is, $H = 0$. The used controller is DSMC. The result is shown in Figure 2, where it can be noted that the controller has stabilized the system, taking the steady state trajectory in the neighborhood of the origin. Under the point of view of disturbances rejection, the system with DSMC controller had a good performance.

Case (ii). Uncertain system with time delay, controlled by DSMC (without predictor).

In this case the actual value H is different at each simulation time interval as follows:

$$T_1 = \{0 \le t \le 20\,\text{s}\}: \quad H = 0;$$

$$T_2 = \{20\,\text{s} < t \le 40\,\text{s}\}: \quad H = \widehat{H}; \tag{79}$$

$$T_3 = \{40\,\text{s} < t \le 60\,\text{s}\}: \quad H = 1.5\widehat{H}.$$

DSMC without state predictor is used, that is, with the control law $\upsilon_k = Fx_k$. Figure 3 shows the result. It may be noted that the system became unstable.

Case (iii). Uncertain system with time delay, controlled by DSMC-P1 or DSMC-P2.

Figure 4 shows the result for the system with the same time delay in the intervals T_1, T_2, and T_3, but now the plant is controlled by DSMC-P1 or DSMC-P2, that is, with predictor (9) or (12), respectively. Now the system is stable. This shows

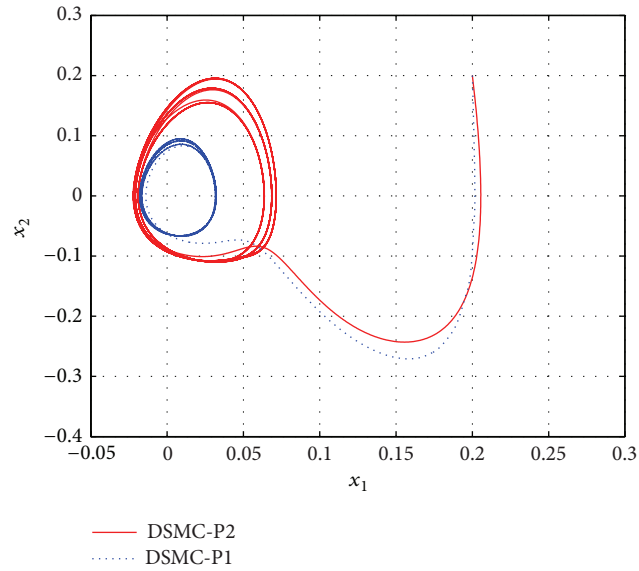

Figure 4: Phase plane for uncertain system with time delay, controlled by DSMC-P1 or DSMC-P2.

that this control strategy requires state predictor when time delay is present.

For each of the controllers, it can be seen that there are three different trajectories. From the smallest to the largest area, they represent the state trajectories for each time interval T_1, T_2, and T_3, respectively. The part "$(\sum_{i=0}^{H-1} \Psi^{-[i-(H-1)]})(f)$" of (53) increases as the delay is increased, implying a steady state error norm, $\lim_{k\to\infty}\|e_k\| = b_{\max}$, also higher at each interval T_1, T_2, and T_3. These results are in accordance with the Propositions 2, 3, and 4 and also with Notes 3 and 4.

The best performance was obtained using the DSMC-P1 controller, where the predictor is given by (9). However this predictor requires several past samples of the control signal, and, for each sampling period, many calculations are required. On the other hand, the DSMC-P2 controller also proved to be efficient. The main advantage is that it requires no previous samples of the control signal, and its computation, at each period, is very simple and quick.

It can be seen, by comparing Figures 2 and 4 that the delayed system controlled by DSMC-P1 or DSMC-P2 is stable, but it is more sensitive to disturbance. This fact is in accordance with Propositions 2 to 5 and Note 4.

6. Conclusion

This work approached uncertain systems with networked discrete-time sliding mode control (DSMC), subject to time delay. To minimize the degenerative effects of time delay, a simpler format of state predictor is used in the control law. The used state predictor has the advantage of fast computation, without the necessity of control signal sampling. The analyses and results from simulations showed the effectiveness of the proposed strategy with regard to the stabilization of uncertain time delay system, even in the presence of uncertainties and delays. The simulation results also confirm the analyses concerning the influence of the uncertainties in the networked control system performance with proposed DSMC strategy.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Asymptotic Tracking Control for a Class of Nonlinear Systems with Unknown Failures of Hysteretic Actuators

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An adaptive failure compensation controller for a class of nonlinear systems preceded by hysteretic actuators is proposed in this paper. Three types of high-gain functions are constructed to counteract the effects of the hysteresis, bounded modeling errors, and bounded disturbances. It is shown that the proposed controller not only ensures bounded signals and asymptotic tracking but also avoids possible chattering, despite the presence of unknown hysteretic actuator failures. Simulation results verify the desired failure compensation performance.

1. Introduction

The hysteresis phenomenon occurs in all the smart material-based actuators and sensors. With such nonlinearity in control systems, it may lead to undesirable inaccuracies or oscillations. On the other hand, actuator failures seem inevitable in practical systems. Thus, failure compensation of hysteretic actuators is an important and challenging problem. Adaptive failure compensation has received great attention in recent years [1–12]. However, available results based on adaptive approaches to address hysteretic actuator failures are very limited [13–16].

To address such a challenging problem, it is important to find an appropriate model for the hysteresis. As mentioned in [17], several models were proposed such as Duhen model, Preisach model, Prandtl-Ishlinskii hysteresis operator, Bouc-Wen differential model [17, 18]. The Bouc-Wen differential model is one of the most widely accepted models of the hysteresis. Actually, it can be shown that the hysteresis model presented in [13–16] is a special case of the Bouc-Wen hysteresis model.

In [13], an adaptive failure compensation scheme for a class of nonlinear systems was studied, where control gains are constants. To avoid possible chattering, the sign $(\cdot)$ functions were involved in the backstepping controller. When the control gains are nonlinear functions of system states, the effects of the actuator hysteresis can no longer be assumed bounded as in [13]. How to handle such effects is a challenging issue especially when there are possible actuator failures. In [14, 15], the idea is to separate such effects into two parts by applying Young's inequality. It is noted that the control methods in [14, 15] are very complicated and the tracking error cannot asymptotically converge to zero but to the so-called predefined bound. Furthermore, the existence of the estimators increases the order of the closed-loop system.

In this paper, we develop a backstepping [19, 20] adaptive compensation controller for a class of nonlinear systems preceded by hysteretic actuators described by Bouc-Wen model. Three types of high-gain functions are incorporated into the controller to counteract the effects of the hysteresis, bounded modeling errors, and bounded disturbances, respectively. In our design, the sign $(\cdot)$ function and a priori knowledge on the bounds of control gains are not required. Besides showing the stability of the closed-loop system, the tracking error is also ensured to achieve zero asymptotically.

The rest of the paper is outlined as follows. In Section 2, the control problem is formulated. In Section 3, a robust adaptive compensation scheme with high-gain functions is proposed. In Section 4, the stability analysis is presented. Simulation results are presented to show the proposed scheme is effective in Section 5. Finally, this paper is concluded in Section 6.

2. Problem Formulation

Consider a class of nonlinear systems in the following form [15]:

$$\begin{aligned}
&\dot{x}_i = x_{i+1} + a^T \varphi_i(\overline{x}_i) \quad (i = 1, 2, \ldots, \rho - 1)\\
&\quad \vdots\\
&\dot{x}_\rho = \varphi_0(x, \xi) + a^T \varphi_\rho(\overline{x}_\rho) + \sum_{j=1}^{m} b_j \beta_j(x) u_j + d(t) + \eta(x, t)\\
&\dot{\xi} = \psi(x, \xi) + a^T \phi(x, \xi),\\
&y = x_1,
\end{aligned} \tag{1}$$

where ρ is the relative degree of the system; $u_j \in R$, $j = 1, 2, \ldots, m$ are the inputs whose actuators may fail during system operation; $\overline{x}_i = [x_1, x_2, \ldots, x_i]^T$ $(i = 1, \ldots, n-1)$, $x = [x_1, x_2, \ldots, x_n]^T$ are the state vectors; $a = [a_1, \ldots, a_q]^T$ are unknown constant parameters; $b_j \in R$ $(j = 1, 2, \ldots, m)$ are unknown constant parameters with known signs; $\psi_0 \in R$, $\varphi_j \in R^\rho$ $(j = 1, \ldots, \rho)$, $\beta_j(x) \neq 0 \in R$ $(j = 1, \ldots, m)$, $\psi \in R^{n-\rho}$, and $\phi \in R^{(n-\rho)\times\rho}$ are known smooth functions; $|d(t)| \leq \overline{D}$ denotes bounded disturbance; and $\eta(x, t)$ is an unknown nonlinear function representing system modeling errors. There exists a known function $\delta(x, t)$ such that $|\eta(x, t)| \leq \delta(x, t)$.

The hysteresis nonlinearity can be described by Bouc-Wen model [17, 18]. Consider

$$\begin{aligned}
v_i &= \mu_i k_i u_{ci} + (1 - \mu_i) k_i \zeta_i = \mu_{i1} u_{ci} + \mu_{i2} \zeta_i\\
\dot{\zeta}_i &= \dot{u}_{ci} - \chi_{i1} |\dot{u}_{ci}| \, |\zeta_i|^{L_i - 1} - \chi_{i2} \dot{u}_{ci} |\zeta_i|^{L_i},
\end{aligned} \tag{2}$$

where $0 < \mu_i < 1$ $(i = 1, \ldots, m)$ are weighting parameters, k_i $(i = 1, \ldots, m)$ are stiffness coefficients, $\mu_{i1} = \mu_i k_i$, $\mu_{i2} = (1 - \mu_i)\, k_i$ $(i = 1, \ldots, m)$ are constants; u_{ci} is the input of the ith $(i = 1, \ldots, m)$ actuator, χ_{i1}, χ_{i2} $(i = 1, \ldots, m)$ describe the shape and amplitude of the ith hysteresis, respectively, L_i governs the smoothness of the transition from initial slope to the slope of the asymptote, and $\chi_{i1} \geq |\chi_{i2}|$, $L_i \geq 1$ $(i = 1, 2 \ldots, m)$. By Lemma 1 in [17], ζ_i $(i = 1, 2 \ldots, m)$ are bounded.

The actuator failure can be modeled as [8, 9, 13–15]

$$u_i = \rho_i \mu_{i1} v_i + d_i + u_{ki}, \quad \forall t \geq t_{iF}, \ (\rho_i u_{ki} = 0, i = 1, \ldots, m), \tag{3}$$

where $\rho_i \in [0, 1]$, u_{ki}, and t_{iF} are all unknown constants and $d_i = \rho_i \mu_{i2} \zeta_i$ $(i = 1, 2 \ldots, m)$ are bounded. For different values of ρ_i, three types of failures are included:

(1) $\rho_i = 1$, where the actuator works normally; namely, $u_i = u_{ci}$, which is regarded as a failure-free actuator;

(2) $0 < \rho_i < 1$; it implies $u_i = \rho_i u_{ci}$; the ith actuator is called partial loss of effectiveness (PLOE);

(3) $\rho_i = 0$; it indicates $u_i = u_{ki}$; the ith actuator is called total loss of effectiveness (TLOE).

Remark 1. The values of ρ_i can change only from $\rho_i = 1$ to some values with $0 \leq \rho_i < 1$. This means that possible changes from normal to any one of the failure cases are unidirectional. The uniqueness of t_{iF} indicates that a failure occurs only once on the ith actuator.

Substituting (2), (3) into (1), we have

$$\begin{aligned}
&\dot{x}_i = x_{i+1} + a^T \varphi_i(\overline{x}_i) \quad (i = 1, 2, \ldots, \rho - 1).\\
&\quad \vdots\\
&\dot{x}_\rho = \varphi_0(x, \xi) + a^T \varphi_\rho(\overline{x}_\rho)\\
&\qquad + \sum_{j=1}^{m} b_j \beta_j(x) \left(\rho_j \mu_{j1} u_{cj} + u_{kj} + d_j\right)\\
&\qquad + d(t) + \eta(x, t)\\
&\dot{\xi} = \psi(x, \xi) + a^T \phi(x, \xi)\\
&y = x_1.
\end{aligned} \tag{4}$$

To derive a suitable adaptive control scheme, the following assumptions are made.

Assumption 2. When TOLE type of actuator failures up to $m-1$, the remaining actuators can still achieve a desired control objective.

Remark 3. Note that all actuators are allowed to have partial loss of effectiveness simultaneously.

Assumption 4. The zero dynamics of $\dot{\xi} = \psi(x, \xi) + a^T \phi(x, \xi)$ is input to state stable with respect to x as its input.

Let $T_0 = 0$. Suppose that there are p_k $(0 \leq p_k \leq m)$ actuators failing at time instants t_k, $k = 1, 2, \ldots, q$, and $t_0 < t_1 < \cdots t_q < \infty$. In other words, all actuators work normally in time interval $[t_0, t_1)$ and no new failure will occur after time t_q. Let the set Q_{jT} denote the actuators of total failure in interval $[t_j, t_{j+1})$ and use the set $\overline{Q}_{jT}$ to represent other normal actuators. It can be concluded that $Q_{jT} \cup \overline{Q}_{jT} = \{1, 2, \ldots, m\}$.

Our objective is to design a control law $u_{ci}(t)$ for the nonlinear systems with p unknown actuator failures, when p changes at time instants t_k, $k = 1, 2, \ldots, q$, such that the output $y(t)$ asymptotically tracks a given reference signal $y_r(t)$ with up to ρth order derivatives bounded and that all closed-loop signals are bounded.

3. Adaptive Compensation Control Schemes

The backstepping technique [19, 20] is applied to derive an adaptive actuator failure compensation controller. The following change of coordinates is required:

$$\begin{aligned} z_1 &= x_1 - y_r \\ z_i &= x_i - y_r^{(i-1)} - \alpha_{i-1} \quad (i = 2, \ldots, \rho), \end{aligned} \tag{5}$$

where z_1 is the tracking error and α_ρ is the ρth stabilizing function. To illustrate the backstepping procedures, only the last step of the design is elaborated in details.

Step i. Consider $i = 1, \ldots, \rho - 1$; the ith stabilizing function α_i, the ith regressor ω_i, and the ith tuning function τ_i are chosen as

$$\begin{aligned} \alpha_1 &= -c_1 z_1 - \varphi_1(x_1)^T \widehat{a} \\ \alpha_i &= -z_{i-1} - c_i z_i - \omega_i^T \widehat{a} \\ &\quad + \sum_{k=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) \\ &\quad + \frac{\partial \alpha_{i-1}}{\partial \widehat{a}} \Gamma \tau_i + \sum_{k=2}^{i-1} \left(\frac{\partial \alpha_{k-1}}{\partial \widehat{a}} \Gamma \omega_i z_k \right), \quad (i = 2, \ldots, \rho - 1) \\ \omega_i &= \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k \quad (i = 1, \ldots, \rho - 1) \\ \tau_i &= \tau_{i-1} + \omega_i z_i \quad (i = 1, \ldots, \rho - 1), \end{aligned} \tag{6}$$

where c_i $(i = 1, \ldots, \rho - 1)$ are positive design parameters and $\widehat{a} \in R^q$ is estimator of the unknown vector a.

Step ρ. From (6), the derivative of z_ρ is

$$\begin{aligned} \dot{z}_\rho &= \varphi_0(x, \xi) + \varphi_\rho(\overline{x}_\rho)^T a \\ &\quad + \sum_{i=1}^{m} b_i \beta_i(x) \left(\rho_i \mu_{i1} u_{ci} + u_{ki} + d_i \right) + \eta(x, t) \\ &\quad + d(t) - \sum_{k=1}^{\rho-1} \left(\frac{\partial \alpha_{\rho-1}}{\partial x_k} x_{k+1} + \varphi_k^T a \right) \\ &\quad - \frac{\partial \alpha_{\rho-1}}{\partial \widehat{a}} \dot{\widehat{a}} - \sum_{k=1}^{\rho-1} \left(\frac{\partial \alpha_{\rho-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) - y_r^{(\rho)}. \end{aligned} \tag{7}$$

If the effects of hysteresis are treated as disturbances, it should be noted that the disturbances in (7) can be classified into three types: (1) $d(t)$ is bounded by an unknown constant D; (2) $\eta(x, t)$ is bounded by a known function; (3) $\sum_{j=1}^{m} b_j \beta_j(x) d_i$ cannot be bounded by any known function but $\beta_j(x)$ $(j = 1, \ldots, m)$ are known and $b_j d_j$ $(j = 1, 2 \ldots, m)$ are bounded. According to their different characteristics, three high-gain functions will be proposed to counteract the effects of the disturbances. At the final step, the stabilizing function α_ρ is given by

$$\begin{aligned} \alpha_\rho &= -\varphi_0 - \omega_\rho^T \widehat{a} - z_{\rho-1} - c_\rho z_\rho + y_r^{(\rho)} \\ &\quad + \sum_{k=1}^{\rho-1} \left(\frac{\partial \alpha_{\rho-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{\rho-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) \\ &\quad + \frac{\partial \alpha_{\rho-1}}{\partial \widehat{a}} \Gamma \tau_\rho - e^{f(t)} z_\rho - e^{f(t)} \delta(x)^2 z_\rho - e^{f(t)} \sum_{k=1}^{m} \beta_i^2(x) z_\rho, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \omega_\rho &= \varphi_\rho - \sum_{k=1}^{\rho-1} \frac{\partial \alpha_{\rho-1}}{\partial x_k} \varphi_k \\ \tau_\rho &= \tau_{\rho-1} + \omega_\rho z_\rho. \end{aligned} \tag{9}$$

$f(t)$ is bounded and $\lim_{t \to \infty} \int_0^t e^{-f(\tau)} d\tau$ exists, and c_ρ is a positive design parameter.

Let

$$\omega = \left[\alpha_\rho, \beta_1, \ldots, \beta_m \right]^T. \tag{10}$$

The control law and parameter update laws are obtained as follows

$$u_{ci} = \operatorname{sign}(b_i) \frac{1}{\beta_i(x)} \widehat{k}^T \omega, \quad (i = 1, \ldots, m) \tag{11}$$

$$\dot{\widehat{a}} = \Gamma \tau_\rho \tag{12}$$

$$\dot{\widehat{k}} = -\Gamma_k \omega z_\rho, \tag{13}$$

where $\widehat{k} = [\widehat{k}_1, \widehat{k}_2^T]^T$, $\widehat{k}_2 = [\widehat{k}_{21}, \widehat{k}_{22}^T, \ldots, \widehat{k}_{2m}^T]^T$, $\widehat{k}$ and $\widehat{a}$ are the estimates of k and a, respectively, and Γ, Γ_k are positive definite matrices chosen by users. If k is a desired constant vector which can be chosen to satisfy

$$\sum_{i=1, i \notin \overline{Q}_{jT}}^{m} |b_i| \rho_i k^T \omega = \alpha_\rho - \sum_{j \in \overline{Q}_{jT}} b_j \beta_j u_{kj}, \tag{14}$$

this gives

$$k_1 = \frac{1}{\sum_{i=1, i \notin \overline{Q}_{jT}}^{m} |b_i| \rho_i}, \qquad k_{2,j} = -\frac{b_j u_{kj}}{\sum_{i=1, i \notin \overline{Q}_{jT}}^{m} |b_i| \rho_i}. \tag{15}$$

Substituting (11)–(13) and (8) into (7), we have

$$\begin{aligned} \dot{z}_\rho &= -z_{\rho-1} - c_\rho z_\rho + \omega_\rho^T \widetilde{a} \\ &\quad + \sum_{i \in \overline{Q}_{jT}} |b_i| \rho_i \mu_{i1} \widetilde{k}^T \omega - \left(e^{f(t)} z_\rho - d(t) \right) \\ &\quad - \left(e^{f(t)} \delta(x)^2 z_\rho - \eta(x, t) \right) \\ &\quad - \left(e^{f(t)} \sum_{k=1}^{m} \beta_k^2(x) z_\rho - \sum_{k=1}^{m} \beta_k(x) b_k d_k \right). \end{aligned} \tag{16}$$

4. Stability Analysis

To prepare for the stability analysis, we rewrite the error system as

$$\dot{z} = A_z(z,t)\, z + W_\theta(z,t)^T \tilde{a} + D(t)\, e_\rho, \tag{17}$$

where the system matrices $A_z(z,t)$, $W_\theta(z,t)$, $D(t)$, and e_ρ are given by

$$\begin{aligned}
&A_z(z,t) \\
&= \begin{bmatrix}
-c_1 & 1 & 0 & \cdots & \cdots & 0 \\
1 & -c_2 & 1+\sigma_{23} & \sigma_{24} & \cdots & \sigma_{2\rho} \\
\vdots & -1-\sigma_{23} & \ddots & \ddots & \ddots & \vdots \\
\vdots & -\sigma_{24} & \ddots & \ddots & \ddots & \sigma_{\rho-2\rho} \\
\vdots & \vdots & \ddots & \ddots & \ddots & 1+\sigma_{\rho-1,\rho} \\
0 & -\sigma_{2\rho} & \cdots & -\sigma_{\rho-2,\rho} & -1-\sigma_{\rho-1,\rho} & -c_\rho
\end{bmatrix} \\
&W_\varepsilon(z,t) = \left[\omega_1 \omega_2, \ldots, \omega_\rho\right]^T \in R^\rho. \\
&W_\theta(z,t) = W_\varepsilon(z,t)\, \omega^T - \widehat{\rho}\left(\dot{y}_r + \overline{\alpha}_1\right) e_1 e_1^T \in R^{\rho\times\rho} \\
&D(t) = \sum_{i\in\overline{Q}_{jT}} |b_i|\, \rho_i \mu_{i1} \widetilde{k}^T \omega - \left(e^{f(t)} z_\rho - d(t)\right) \\
&\quad - \left(e^{f(t)} \delta(x)^2 z_\rho - \eta(x,t)\right) \\
&e_\rho = [0,\ldots,0,1]^T \in R^\rho \\
&\sigma_{jk} = -\frac{\partial \alpha_{j-1}}{\partial \widehat{a}} \Gamma \omega_k.
\end{aligned} \tag{18}$$

The closed-loop system has the following desired properties.

Theorem 5. *With the ISS of the zero dynamics of system* (1) *and m hysteretic actuators modeled in* (2) *with possible unknown failures by* (3)*, the controller* (11) *with the adaptive laws* (12) *and* (13) *ensures the boundedness of the closed-loop signals and the asymptotic output tracking:* $\lim_{t\to\infty}(y - y_r) \to 0$.

Proof. For each time interval (t_k, t_{k+1}), $k = 0, 1, 2, \ldots, q$, we have a Lyapunov function V in the following form:

$$V = \frac{1}{2}\sum_{i=1}^{\rho} z_i^2 + \frac{1}{2}\widetilde{a}^T \Gamma \widetilde{a} + \sum_{i\in\overline{Q}_{jT}} \frac{\rho_i \mu_{i1} |b_i|}{2} \widetilde{k}_j^T \Gamma_k \widetilde{k}_j. \tag{19}$$

Taking the derivative of (19) yields

$$\dot{V} = \sum_{i=1}^{\rho} z_i \dot{z}_i - \widetilde{a}^T \Gamma \dot{\widehat{a}} - \sum_{i\in\overline{Q}_{jT}} \rho_i \mu_{i1} |b_i|\, \widetilde{k}_j^T \Gamma_k \dot{\widehat{k}}_j. \tag{20}$$

Substituting (11)–(13) and (8) into (7), we have

$$\begin{aligned}
\dot{V} = &-\sum_{i=1}^{\rho} c_i z_i^2 - \left(e^{f(t)} z_\rho^2 - d(t)\, z_\rho\right) \\
&- \left(e^{f(t)} \delta(x)^2 z_\rho^2 - \eta(x,t)\, z_\rho\right) \\
&- \left(e^{f(t)} \sum_{k=1}^{m} \beta_k^2(x)\, z_\rho^2 - \sum_{k=1}^{m} \beta_k(x)\, b_k d_k z_\rho\right).
\end{aligned} \tag{21}$$

From $|d(t)| \le \overline{D}$, $|\eta(x,t)| \le \delta(x,t)$, and bounded d_i ($i = 1, 2, \ldots, m$), we get

$$\begin{aligned}
\dot{V} &\le -\sum_{i=1}^{\rho} c_i z_i^2 - \left(e^{f(t)} z_\rho^2 - \left|\overline{D}\right| \left|z_\rho\right|\right) \\
&\quad - \left(e^{f(t)} \delta(x)^2 z_\rho^2 - |\delta(x)| \left|z_\rho\right|\right) \\
&\quad - \left(e^{f(t)} \sum_{k=1}^{m} \beta_k^2(x)\, z_\rho^2 - \sum_{k=1}^{m} |\beta_k(x)| \left|b_k d_k\right| \left|z_\rho\right|\right) \\
&\le -\sum_{i=1}^{\rho} c_i z_i^2 + \frac{\overline{D}^2}{4} e^{-f(t)} + \frac{1}{4} e^{-f(t)} + \frac{1}{4}\sum_{k=1}^{m} \left|b_k^2 d_k^2\right| e^{-f(t)} \\
&= -\sum_{i=1}^{\rho} c_i z_i^2 + \frac{\sum_{k=1}^{m} \left|b_k^2 d_k^2\right| + \overline{D}^2 + 1}{4} e^{-f(t)} \\
&= -\sum_{i=1}^{\rho} c_i z_i^2 + D_0 e^{-f(t)},
\end{aligned} \tag{22}$$

where

$$D_0 = \frac{\sum_{k=1}^{m} \left|b_k^2 d_k^2\right| + \overline{D}^2 + 1}{4}. \tag{23}$$

From (22), we conclude that $V(t) \in L_\infty, t \in [t_0, t_1)$, so that z_i ($i = 1, \ldots, \rho$), $\dot{z}_i$ ($i = 1, \ldots, \rho$), and $\widetilde{a}, \widetilde{k}$ are bounded for $t \in [t_0, t_1)$. It follows from (8) and (10) that α_ρ and ω are bounded. Therefore, all closed-loop signals are bounded for (t_0, t_1). In order to prove the asymptotic tracking, considering the last time interval (t_q, ∞). From (22), we can obtain

$$c_i \int_0^t z_i^2(\tau)\, d\tau \le V(0) - V(t) + D_0 \int_0^t e^{-f(\tau)} d\tau. \tag{24}$$

Because of the boundedness of $D_0 \int_0^t e^{-f(\tau)} d\tau$, we have $z_i \in L_2$, ($i = 1, \ldots, \rho$). By Barbalat's lemma, $\lim_{t\to\infty}(z_i(t)) \to 0$ ($i = 1, \ldots, \rho$) can be obtained. This completes the proof. □

5. Simulations

We consider a second order nonlinear [15] system with two inputs described as

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1^T(x_1)\, \theta \\
\dot{x}_2 &= \varphi_2^T(x)\, \theta + \sum_{i=1}^{2} b_i \beta_i(x)\, u_i + \eta(x,t),
\end{aligned} \tag{25}$$

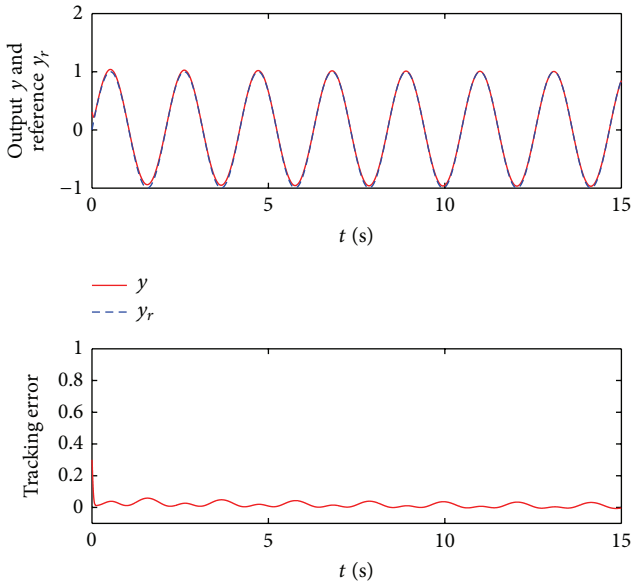

FIGURE 1: Output y, reference y_r, and error $e(t)$.

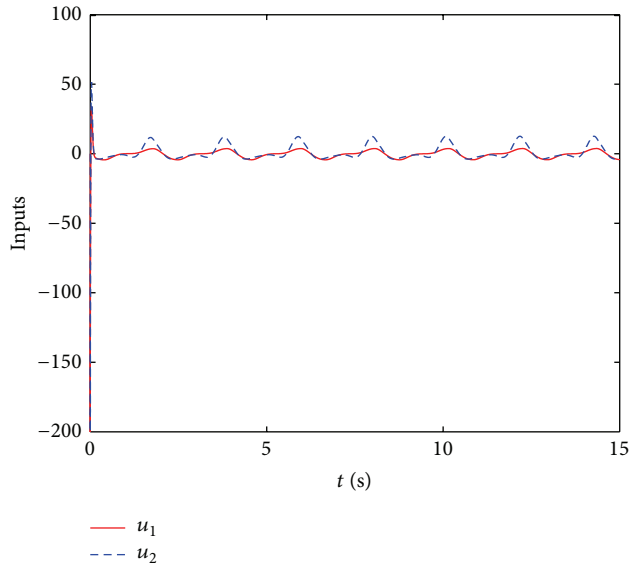

FIGURE 2: u_1 and u_2.

where $\varphi_1(x_1) = 1 + x_1^2$, $\varphi_2(x) = 1 + \sin(x_1)$, $\beta_1(x) = 1.9 + 0.1 \sin x_1$, $\beta_2(x) = e^{x_1}$, $\theta = 2$, $b_1 = b_2 = 1$, b_1, b_2 are unknown constants, u_1, u_2 are the outputs of two hysteretic actuators, x_1, x_2 are the states, and $\eta(x) = 0.1x_1^2(1 + \sin t)(1 + \sin x_2)$ is bounded by $\delta(x) = 0.4x_1^2$. The reference signal is set as $y_r(t) = \sin(3t)$. The backlash-like hysteresis is described by (2) with parameters $\mu_{11} = \mu_{21} = \mu_{12} = \mu_{22} = 1$, $\chi_{11} = \chi_{21} = 1$, $\chi_{12} = \chi_{22} = 0$, and $L_1 = L_2 = 1$. The high-gain function is chosen as $f(t) = e^{2.5 \arctan(t)}$. For simulation, we consider three actuator failure cases.

Case 1. There are no actuator failures.

By Theorem 5, we can obtain the actual control law and the update laws. The initial conditions are set as follows:

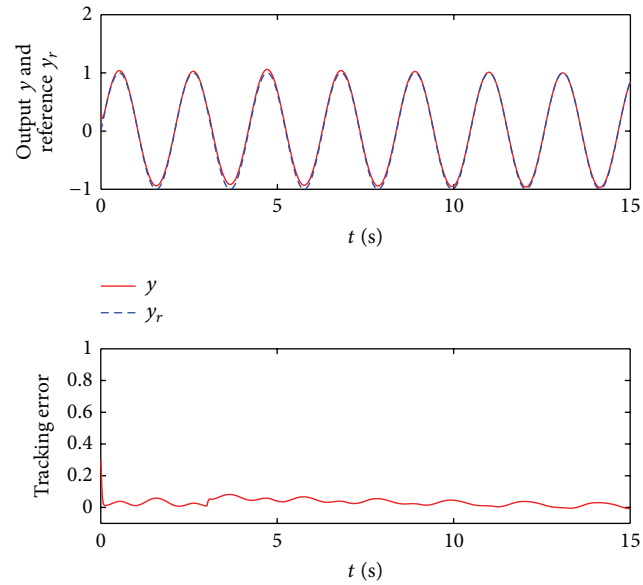

FIGURE 3: Output y, reference y_r, and error $e(t)$.

$$\begin{aligned} \widehat{a} &= 1.5, \qquad \Gamma = 0.0005, \\ \widehat{k} &= [0.4, 0, 0]^T, \qquad \Gamma_k = 0.0001 * I_3 \\ x_0 &= [0.3, 0]^T, \qquad c_1 = 60, \qquad c_2 = 60, \end{aligned} \tag{26}$$

where I_3 is the 3rd order identity matrix.

The simulation results including output $y(t)$, reference output $y_r(t)$, and tracking error $e(t)$ are shown in Figure 1; the actuators outputs u_1, u_2 are shown in Figure 2. The system responses are as expected. At the beginning, there is a transient response in tracking errors. But, as time goes on, the tracking errors become smaller and ultimately vanish. The proposed controller guarantees that asymptotic tracking is achieved.

Case 2. Actuator u_1 is stuck at $u_1 = 45$ from $t = 3$ s, thus undergoing a TLOE type of failure. By Theorem 5, we can obtain the actual control law and the update laws. The initial conditions are set as follows.

The other parameters are the same as those in Case 1.

The simulation results including output $y(t)$, reference output $y_r(t)$, and tracking error $e(t)$ are shown in Figure 3; the actuators outputs u_1, u_2 are shown in Figure 4. The system responses are as expected. When one of the actuators fails, there is a transient response in tracking errors. But, as time goes on, the tracking errors become smaller and ultimately vanish. The proposed controller guarantees that asymptotic tracking is achieved.

Case 3. Actuator u_2 is stuck at $u_2 = 40$ from $t = 3$ s and actuator u_1 loses 60% from $t = 6$ s. Thus, u_1 undergoes a PLOE type of failure while u_2 is a TLOE type of failure.

The other parameters are the same as those in Case 1.

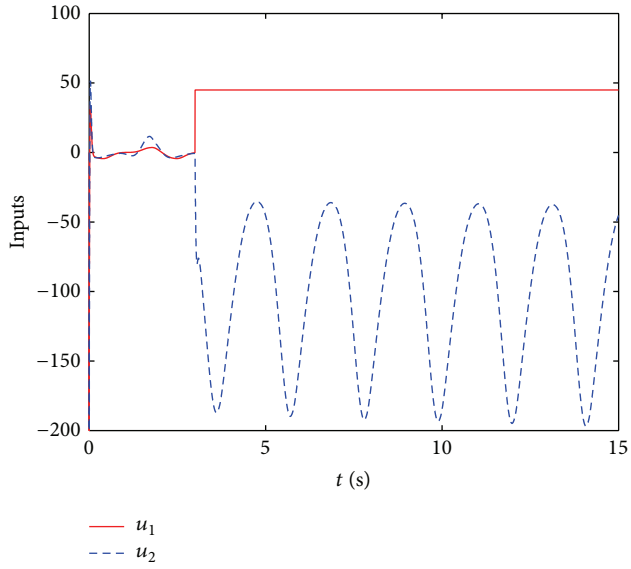

Figure 4: u_1 and u_2.

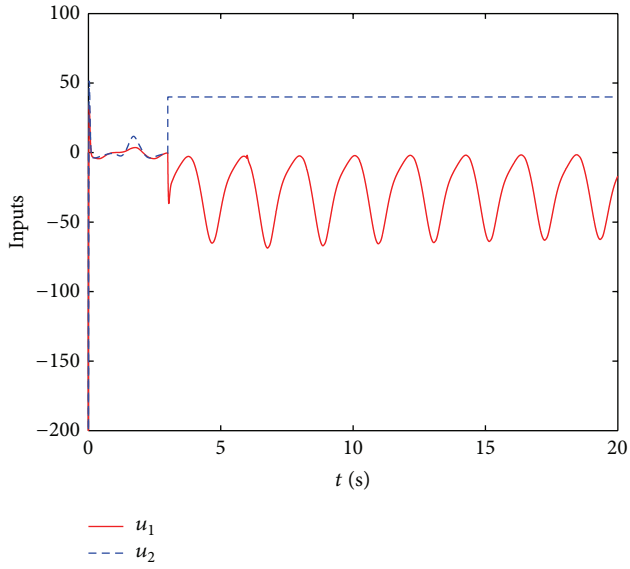

Figure 6: u_1 and u_2.

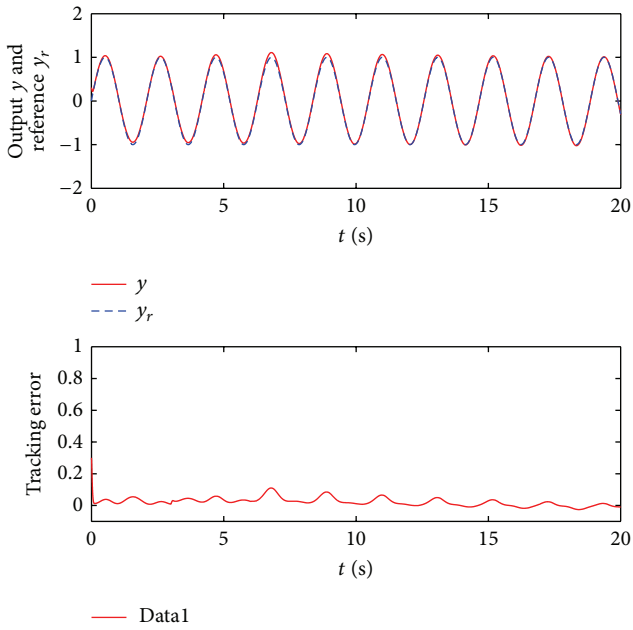

Figure 5: Output y, reference y_r, and error $e(t)$.

The simulation results including output $y(t)$, reference output $y_r(t)$, and tracking error $e(t)$ are shown in Figure 5; the actuators outputs u_1, u_2 are shown in Figure 6. The system responses are as expected. When the actuators fail, there is a transient response in tracking errors. But, as time goes on, the tracking errors become smaller and ultimately vanish. The proposed controller guarantees that asymptotic tracking is achieved.

6. Conclusions

This paper presents an adaptive failure compensation controller for a class of uncertain nonlinear systems dominated by the hysteresis actuator nonlinearity. We propose three types of high-gain functions to deal with the unknown bounded disturbances, unknown modeling errors, and unknown actuator failures. It has been shown that the tracking errors can converge to zero asymptotically while all the closed-loop signals remain bounded. Furthermore, the proposed scheme can avoid possible chattering. Simulation results illustrate the effectiveness of our proposed scheme.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Impact of Awareness Programs with Recruitment and Delay on the Spread of an Epidemic

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A compartment epidemic model with delay is given to discuss the impact of awareness programs on the spread and control of infectious diseases in a given region. It is assumed that there is a constant recruitment rate in the cumulative density of awareness programs, and further it is assumed that awareness programs can influence the susceptible to a limited extent. The system exhibits two equilibria: the disease-free equilibrium is stable if the basic reproduction number is less than unity for any delay and the unique endemic equilibrium exhibits Hopf-bifurcation under certain conditions. Numerical simulations prove the results of analysis and the significance of awareness programs in preventing and controlling the diseases, by investigating the relationship between the proportion of the infective and the dissemination rate and the implementation rate, respectively.

1. Introduction

Plenty of evidence shows that awareness programs, which can influence the susceptible to a limited extent due to some objective factors, play an important role in the spread and control of infectious diseases. For example, during the outbreak of SARS, H1N1 influenza pandemic, and HIV epidemic, public media had massive reports on the number of the infections and deaths per day, which had a great impact on the diseases control [1–4]. That is due to the fact that the spread of diseases is often accompanied by a rise in awareness of those in the social vicinity of infected individuals and a subsequent change in behavior, such as keeping social distancing, wearing protective masks, and vaccination [5, 6]. Such reactions can manifest themselves in lower susceptibility as people try to prevent themselves from catching the disease, but also in lower infectivity because of self-imposed quarantine or better hygiene, shorter durations of infectiousness, or longer immunity [7]. And once the infective are cured, they will be aware of the disease [8].

Recently some scholars used mathematical models to discuss the impact of awareness programs on the diseases spreading and controlling in a given region [6, 8–11]. Joshi et al. [9] formulated a model to investigate the effect on the HIV epidemic in Uganda and compared their model with three types of the susceptible to a standard SIR model and then pointed out that the awareness programs in Uganda are successful in combating diseases. Li and Cui [1] analyzed a SIS epidemic model incorporating media coverage under constant and pulse vaccination; then they obtained the exact periodic infection-free solution which is globally asymptotically stable under some conditions. In order to better describe populations mixed condition, some authors studied infectious diseases models on different networks [12–14]. In [12], the authors introduced a constant to represent the density level of media coverage from other regions. Funk et al. [14] formulated and analyzed a mathematical model in a host population; then they put forward that awareness programs can result in a lower size of the outbreak but do not affect the epidemic threshold, and if the behavioral response is treated as a local effect arising in the proximity of an outbreak, it can completely stop a disease from spreading, although only if the infection rate is below a threshold.

Some scholars consider some other factors in awareness program models. Liu and Wang et al. took into account the random perturbation [15, 16]. In [16], the authors extended a deterministic SIRS epidemic model to a stochastic differential equation, and then they discussed the exponential p-stability and global stability of unique positive solution. Some scholars

focused their attention at the contact rate, and most of them assumed that awareness programs will aid in modifying the contact rate between the susceptible and the infective [17–29]. Liu and Cui [17] used the contact rate $(\beta_1 - \beta_2 I/(m+I))$ and found that both of the two equilibria are asymptotically stable. Mirsa et al. [20] used the constant k with the contact rate $M/(k+M)$ to limit the effect of awareness programs on the susceptible and sought out the conditions that Hopf-bifurcation occurs. Tchuenche and Bauch [23] used an exponentially decreasing function $e^{-M(t)}$ to affect the force of infection; then numerical results showed the potential short-term beneficial effect of awareness programs. Pang and Cui [26] used $\beta(I) = \mu_1 - \mu_2 f(I)$ to show the contact rate after awareness programs alerts and found that though it is not a determined fact to eradicate the infection of the diseases, the effective awareness programs can postpone the arrival of the infection peak. Elenbaas et al. [28] introduced a segmented function to describe the media impact e^{-mI_c} when they formulate an epidemic model. A Filippov epidemic model was proposed to describe the real characteristics of media impact on the spread of infectious diseases by incorporating a piecewise continuous transmission rate $\beta e^{(-\alpha \epsilon I)} SI$ in [29]. Mathematical and bifurcation analyses with regard to the local and global stability of equilibria and local sliding bifurcations are performed.

But what we regard as unreasonable is that most of the articles assume that the cured infective become unaware of the disease. In fact people have a certain consciousness about the disease once they get sick. Therefore we propose a delayed mathematical model for predicting the future course of any epidemic by considering some of the infect join the aware susceptible after recovery. We hold the opinion that awareness programs can influence the susceptible to a limited extent for some objective factors and then consider the interaction between the susceptible and awareness programs as Holling type-II functional response. In addition we make a constant to represent the density level of media coverage from other regions with the disease because other regions can also effect the region that we consider. In fact the results about global stability of other delayed systems could be further utilized for other related problems [30–32].

The rest of this paper is organized as follows. In the next section, a mathematical model with delay has been proposed to capture the dynamics of the effect of awareness programs. It is assumed that diseases spread due to the contact between the susceptible and the infective only. Then we analyze the conditions of the stability of equilibria and the existence of Hopf-bifurcation in Section 3. Furthermore, in Section 4 we perform some numerical examples to validate the analysis in Section 3 and then introduce the importance of the dissemination rate, implementation rate, and the delay in disease control. In Section 5 we discuss the above contents.

2. Mathematical Model and Equilibrium Analysis

In the region under consideration the rate of immigration of the susceptible is b. It is assumed that the disease spreads due to the direct contact between the susceptible and the infective only and due to awareness programs the susceptible avoid being in contact with the infective and form a different class with a proportion λ, named the aware susceptible. So the total population is divided into three classes: the susceptible, the aware susceptible, and the infective, the proportions of which at time t in the total population are $X(t)$, $X_m(t)$, and $Y(t)$. It is assumed that the aware susceptible may lose awareness with passage of time and become susceptible with the proportion of λ_0 again. The infective can be cured with a proportion ν and a fraction q of recovered people will become aware and join the aware susceptible class whereas the remaining fraction $p(p+q=1)$ will join the susceptible.

Consider that the cumulative density of awareness programs driven by media in that region at time t is $M(t)$, which is related to the infective. We make the constant μ represent the executed rate of awareness programs. As the time passes, some campaigns lose their impact on people and lead to the diminution of the awareness programs, so we introduce μ_0 to denote the rate of their depletion. Moreover m_0 represents the density level of media coverage on the disease from other regions. Using the fact that there is a limited extent of the awareness programs influence on the susceptible due to some objective factors, we introduce k to limit the effect and consider the interaction between them as Holling type-II functional response. It is plausible that the policy makers need some time to gather the cases of the infective generally; then we introduce τ and consider that the cumulative density of awareness programs at time t being executed will be in accordance with the infected cases reported at time $t-\tau$ $(\tau > 0)$. Keeping the above facts in mind, the dynamics of model is governed by the following system of nonlinear delay differential equations:

$$\begin{aligned}
X'(t) &= b - \beta X(t) Y(t) + \lambda_0 X_m(t) \\
&\quad - \lambda X(t) \frac{M(t)}{1 + kM(t)} + p\nu Y(t) - dX(t), \\
X_m'(t) &= \lambda X(t) \frac{M(t)}{1 + kM(t)} \\
&\quad - \lambda_0 X_m(t) + q\nu Y(t) - dX_m(t), \\
Y'(t) &= \beta X(t) Y(t) - \nu Y(t) - dY(t), \\
M'(t) &= m_0 + \mu Y(t-\tau) - \mu_0 M(t).
\end{aligned} \tag{1}$$

Here $X(0) = X_0 > 0$, $X_m(0) = X_{m0} \geq 0$, and $Y(\theta) = Y_0 \geq 0$ for $\theta \in [-\tau, 0]$ and $M(0) = M_0 \geq 0$.

In the above model, the constants β and $d(d=b)$, respectively, represent the contact rate of the unaware susceptible with the infective and the natural death rate. All the constants in the system are assumed to be positive.

Using the fact that $X(t) + Y(t) + X_m(t) = 1$, $p + q = 1$ system (1) is reduced to the following system:

$$\begin{aligned} X_m'(t) &= \lambda\left(1 - X_m(t) - Y(t)\right)\frac{M(t)}{1 + kM(t)} \\ &\quad - \left(\lambda_0 + d\right)X_m(t) + q\nu Y(t), \\ Y'(t) &= \beta\left(1 - X_m(t) - Y(t)\right)Y(t) - (\nu + d)Y(t), \\ M'(t) &= m_0 + \mu Y(t - \tau) - \mu_0 M(t). \end{aligned} \tag{2}$$

Now it is sufficient to study system (2) in detail rather than system (1).

For the analysis of system (2), we need the region of attraction which is given by the set $\Omega = \{(X_m, Y, M) \in \Re_+^3 : 0 \le X_m, Y \le 1, 0 \le M \le (m_0 + \mu)/\mu_0\}$, and it attracts all solutions initiating in the interior of the positive orthant.

The system (2) has two equilibria.

(i) Disease-free equilibrium $E_0(\lambda m_0/(\lambda m_0 + (\lambda_0 + d)(\mu_0 + km_0)), 0, m_0/\mu_0)$.

(ii) Endemic equilibrium $E_*(X_m^*, Y^*, M^*)$.

Define the basic reproduction number $R_0 = \beta/(\nu + d)$. The existence of equilibrium E_0 is trivial; then we prove the existence of E_* in detail. When $R_0 > 1$, in the equilibrium E_* the values of X_m^*, M^* are

$$\begin{aligned} X_m^* &= 1 - \frac{\nu + d}{\beta} - Y^*, \\ M^* &= \frac{m_0}{u_0} + \frac{\mu}{\mu_0}Y^*, \end{aligned} \tag{3}$$

with Y^* satisfying the equation

$$A_1 Y^{*2} + A_2 Y^* + A_3 = 0, \tag{4}$$

where

$$\begin{aligned} A_1 &= \mu k\beta\left(\lambda_0 + d + q\nu\right), \\ A_2 &= \mu k\left(\lambda_0 + d\right)(\nu + d - \beta) + \lambda\mu(\nu + d) \\ &\quad + \beta\left(\lambda_0 + d + q\nu\right)\left(\mu_0 + km_0\right), \\ A_3 &= \lambda m_0(\nu + d) + \left(\lambda_0 + d\right)(\nu + d - \beta)\left(\mu_0 + km_0\right). \end{aligned} \tag{5}$$

Solving (4) we get

$$Y^* = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}. \tag{6}$$

We obtain $A_1 > 0$, and when $\beta\lambda < k(\lambda_0 + d)(\beta - \nu - d)$,

$$\begin{aligned} A_3 &= \lambda m_0(\nu + d) + \left(\lambda_0 + d\right)(\nu + d - \beta)\left(\mu_0 + km_0\right) \\ &= \beta\lambda m_0\left(\frac{1}{R_0} - 1\right) + \beta\lambda m_0 \\ &\quad + \beta\left(\lambda_0 + d\right)\left(\mu_0 + km_0\right)\left(\frac{1}{R_0} - 1\right) \\ &= \beta\lambda m_0\left(\frac{1}{R_0} - 1\right) + \beta\mu_0\left(\lambda_0 + d\right)\left(\frac{1}{R_0} - 1\right) \\ &\quad + \beta m_0\left[\lambda - k\left(\lambda_0 + d\right)\left(1 - \frac{1}{R_0}\right)\right] < 0, \end{aligned} \tag{7}$$

then we get $Y^* = (-A_2 + \sqrt{A_2^2 - 4A_1A_3})/2A_1$ for $Y^* > 0$.

Remark 1. From the expression of Y^*, it is easy to note that $dY^*/d\lambda < 0$ and $dY^*/d\mu < 0$, which shows that the equilibrium proportion of the infective decreases as the rate of dissemination of awareness among the susceptible and the implementation rate of awareness programs increases.

3. Stability Analysis

In this section we present the local stability of E_0, E_* and explore the conditions of Hopf-bifurcation by taking delay τ as a bifurcation parameter.

3.1. Stability of Equilibria without Delay ($\tau = 0$)

Theorem 2. *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$.*

Proof. The Jacobian matrix corresponding to the system (2) when $\tau = 0$ is given below:

$$J = \begin{bmatrix} -\dfrac{\lambda M}{1 + kM} - \lambda_0 - d & -\dfrac{\lambda M}{1 + kM} + \nu q & \lambda\left(1 - Y - X_m\right)\dfrac{1}{(1 + kM)^2} \\ -\beta Y & \beta\left(1 - X_m\right) - (\nu + d) - 2\beta Y & 0 \\ 0 & \mu & -\mu_0 \end{bmatrix}. \tag{8}$$

The characteristic equation at E_0 is of the form

$$\begin{aligned}&(\eta+\mu_0)\left[\eta+\frac{\lambda m_0}{km_0+\mu_0}+\lambda_0+d\right]\\&\times\left[\eta+\frac{\beta\lambda m_0}{\lambda m_0+(\lambda_0+d)(\mu_0+km_0)}\right.\\&\left.+\beta\left(\frac{1}{R_0}-1\right)\right]=0,\end{aligned}\tag{9}$$

where η is the eigenvalue. We get

$$\begin{aligned}&\eta_1=-\mu_0<0,\\&\eta_2=-\left(\frac{\lambda m_0}{km_0+\mu_0}+\lambda_0+d\right)<0,\\&\eta_3=-\beta\left[\frac{\lambda m_0}{\lambda m_0+(\lambda_0+d)(\mu_0+km_0)}+\left(\frac{1}{R_0}-1\right)\right],\end{aligned}\tag{10}$$

so $\eta_3<0$ when $R_0<1$. Thus E_0 is locally asymptotically stable if $R_0<1$. □

The form of characteristic equation at E_* is

$$\eta^3+Q_1\eta^2+Q_2\eta+(Q_3+Q_4)=0,\tag{11}$$

where

$$\begin{aligned}&Q_1=B_1+B_3+\mu_0,\\&Q_2=\mu_0(B_1+B_3)+B_1B_3-\beta Y^*B_2,\\&Q_3=\mu_0B_1B_3-\beta\mu_0Y^*B_2,\\&Q_4=-\beta\mu Y^*B_4,\\&B_1=\lambda M^*\frac{1}{1+kM^*}+\lambda_0+d,\\&B_2=\lambda M^*\frac{1}{1+kM^*}-\nu q,\\&B_3=\beta(X_m^*-1)+2\beta Y^*+(\nu+d),\\&B_4=\lambda(X_m^*+Y^*-1)\frac{1}{(1+kM^*)^2}.\end{aligned}\tag{12}$$

Theorem 3. *When the endemic equilibrium E_* exists, it is locally asymptotically stable provided $m_0(\lambda_0+d+q\nu)>\mu\mu_0$.*

Proof. For the characteristic equation

$$\eta^3+Q_1\eta^2+Q_2\eta+(Q_3+Q_4)=0,\tag{13}$$

it is easy to show

$$\begin{aligned}Q_1&=\lambda M^*\frac{1}{1+kM^*}+\lambda_0+d\\&\quad+\beta(X_m^*-1)+2\beta Y^*+(\nu+d)+\mu_0\\&=\lambda M^*\frac{1}{1+kM^*}+\lambda_0+d+\beta(X_m^*-1)\\&\quad+2\beta Y^*+\beta(1-X_m^*-Y^*)+\mu_0\\&=\lambda M^*\frac{1}{1+kM^*}+\lambda_0+d+\beta Y^*+\mu_0>0,\\Q_2&=\mu_0(B_1+B_3)+\left(\lambda M^*\frac{1}{1+kM^*}+\lambda_0+d\right)\\&\quad\times[\beta(X_m^*-1)+2\beta Y^*+(\nu+d)]\\&\quad-\beta Y^*\left(\lambda M^*\frac{1}{1+kM^*}-\nu q\right)\\&=\mu_0(B_1+B_3)+\beta Y^*\left(\frac{\lambda M^*}{1+kM^*}+\lambda_0+d\right)\\&\quad-\beta Y^*\left(\frac{\lambda M^*}{1+kM^*}-\nu q\right)\\&=\mu_0(B_1+B_3)+(\lambda_0+\nu q+d)\beta Y^*>0,\\Q_3+Q_4&=\mu_0B_1B_3-\beta Y^*(\mu B_4+\mu_0B_2)\\&=\mu_0\beta Y^*\left(\lambda M^*\frac{1}{1+kM^*}+\lambda_0+d\right)\\&\quad-\beta Y^*\left[\mu\lambda(X_m^*+Y^*-1)\frac{1}{(1+kM^*)^2}\right.\\&\quad\left.+\mu_0\left(\lambda M^*\frac{1}{1+kM^*}-\nu q\right)\right]\\&=\beta Y^*\mu\lambda(1-X_m^*-Y^*)\frac{1}{(1+kM^*)^2}\\&\quad+\mu_0\beta Y^*(\lambda_0+d)+\nu q\mu_0\beta Y^*>0.\end{aligned}\tag{14}$$

And

$$\begin{aligned}&Q_1Q_2-(Q_3+Q_4)\\&=(B_1+B_3)(\mu_0B_1+\mu_0B_3+B_1B_3-\beta Y^*B_2)\\&\quad+\mu_0^2(B_1+B_3)-Q_4\end{aligned}$$

$$= \mu_0 (B_1 + B_3)^2 + \mu_0^2 (B_1 + B_3) + B_3 \left[(B_1 + B_3)(B_1 - B_2) + \mu B_4 \right], \tag{15}$$

where

$$\begin{aligned}
&(B_1 + B_3)(B_1 - B_2) + \mu B_4 \\
&= \left(\frac{\lambda M^*}{1 + kM^*} + \lambda_0 + d + \beta Y^* \right)(\lambda_0 + d + q\nu) \\
&\quad - \mu\lambda \left(1 - X_m^* - Y^*\right) \frac{1}{(1 + kM^*)^2} \\
&= \frac{\lambda}{1 + kM^*} \left[M^* (\lambda_0 + d + q\nu) - \frac{\mu}{1 + kM^*} \right] \\
&\quad + (\lambda_0 + d + \beta Y^*)(\lambda_0 + d + q\nu) + \frac{\mu\lambda \left(X_m^* + Y^*\right)}{1 + kM^*} \\
&= \frac{\lambda}{(1 + kM^*)^2} \left[M^* (\lambda_0 + d + q\nu) + k(M^*)^2 \right. \\
&\qquad \left. \times (\lambda_0 + d + q\nu) - \mu \right] \\
&\quad + (\lambda_0 + d + \beta Y^*)(\lambda_0 + d + q\nu) \\
&\quad + \mu\lambda \left(X_m^* + Y^*\right) \frac{1}{1 + kM^*} \\
&= \frac{\lambda}{(1 + kM^*)^2} \left[\frac{m_0 (\lambda_0 + d + q\nu)}{\mu_0} \right. \\
&\qquad + \frac{\mu Y^* (\lambda_0 + d + q\nu)}{\mu_0} \\
&\qquad \left. + kM^{*2} (\lambda_0 + d + q\nu) - \mu \right] \\
&\quad + (\lambda_0 + d + \beta Y^*)(\lambda_0 + d + q\nu) \\
&\quad + \mu\lambda \left(X_m^* + Y^*\right) \frac{1}{1 + kM^*} > 0,
\end{aligned} \tag{16}$$

so $Q_1 Q_2 - (Q_3 + Q_4) > 0$. According to Hurwitz criterion, we can claim that all the eigenvalues will be either negative or having negative real part. Thus, the equilibrium E_* is locally asymptotically stable. □

3.2. Stability of Equilibria with Delay ($\tau > 0$)

Theorem 4. *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$.*

Similar to the proof of Theorem 2, this proof is omitted.

We linearize system (2) about E_* to study the stability of the endemic equilibrium and get

$$\frac{du}{dt} = M_1 u(t) + M_2 u(t - \tau), \tag{17}$$

where

$$\begin{aligned}
u(t) &= \left[x(t) \quad y(t) \quad z(t) \right]^T, \\
M_1 &= \begin{bmatrix} -\dfrac{\lambda M^*}{1 + kM^*} - \lambda_0 - d & -\dfrac{\lambda M^*}{1 + kM^*} + \nu q & \lambda \left(1 - Y^* - X_m^*\right) \dfrac{1}{(1 + kM^*)^2} \\ -\beta Y^* & \beta \left(1 - X_m^*\right) - (\nu + d) - 2\beta Y^* & 0 \\ 0 & 0 & -\mu_0 \end{bmatrix}, \\
M_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu & 0 \end{bmatrix}.
\end{aligned} \tag{18}$$

In the above, x, y, and z are small perturbations around E_*.

Then the form of the characteristic equation of the system is

$$\eta^3 + Q_1 \eta^2 + Q_2 \eta + Q_3 = Q_4 e^{-\eta\tau}, \tag{19}$$

where η is the eigenvalue.

To show the Hopf-bifurcation, we need to show that (19) has a pair of purely imaginary roots. For this, substituting $\eta = i\omega$ ($\omega > 0$) into (19) and separating real and imaginary parts, we get the following transcendental equations:

$$\begin{aligned}
Q_2 \omega - \omega^3 &= Q_4 \sin(\omega\tau), \\
Q_1 \omega^2 - Q_3 &= Q_4 \cos(\omega\tau).
\end{aligned} \tag{20}$$

Squaring and adding the above equations and substituting $\psi = \omega^2$, we get

$$h(\psi) = \psi^3 + P_1 \psi^2 + P_2 \psi + P_3 = 0, \tag{21}$$

where $P_1 = Q_1^2 - 2Q_2$, $P_2 = Q_2^2 - 2Q_1 Q_3$, and $P_3 = Q_3^2 - Q_4^2$.

If the coefficients in $h(\psi)$ satisfy the conditions of Routh-Hurwitz criterion, then E_* is locally asymptotically stable for all delays $\tau > 0$, provided it is stable in absence of delay. In the following we consider that the values of P_i $(i = 1, 2, 3, 4)$ do not satisfy the Routh-Hurwitz criterion.

Lemma 5. *If* $B_2 > 0$, $\lambda_0 + d + q\nu < \mu\lambda(\nu + d)(1/\beta(1 + kM^*)^2)$, (21) *will have at least one positive root* ψ_0.

Proof. For $h(\psi) = \psi^3 + P_1\psi^2 + P_2\psi + P_3$, it is obvious that

$$\begin{aligned}
P_1 &= (B_1 + B_3 + \mu_0)^2 \\
&\quad - 2\left[(B_1 + B_3)\mu_0 + B_1B_3 - \beta B_2 Y^*\right] \\
&= B_1^2 + B_3^2 + \mu_0^2 + 2\beta B_2 Y^* > 0, \\
P_2 &= \left[(B_1 + B_3)\mu_0 + B_1B_3 - \beta B_2 Y^*\right]^2 \\
&\quad - 2(B_1 + B_3 + \mu_0)(B_1B_3\mu_0 - \mu_0\beta B_2 Y^*) \\
&= B_1^2\mu_0^2 + B_3^2\mu_0^2 + (B_1B_3)^2 \\
&\quad + (B_2B_3)^2 - 2B_1B_2B_3^2 + 2B_2B_3\mu_0^2 \\
&= \left(B_1^2 + B_3^2\right)\mu_0^2 + (B_1B_3 - B_2B_3)^2 \\
&\quad + 2\mu_0^2 B_2B_3 > 0, \\
P_3 &= (Q_3 + Q_4)(Q_3 - Q_4) \\
&= B_3(\mu_0B_1 - \mu_0B_2 + \mu B_4)(Q_3 + Q_4) \\
&= B_3(Q_3 + Q_4) \\
&\quad \times \left[\mu_0(\lambda_0 + d + q\nu)\right. \\
&\qquad \left. - \mu\lambda(1 - X_m^* - Y^*)\frac{1}{(1 + kM^*)^2}\right] \\
&< B_3(Q_3 + Q_4)\left[(\lambda_0 + d + q\nu)\right. \\
&\qquad \left. - \mu\lambda(\nu + d)\frac{1}{\beta(1 + kM^*)^2}\right] < 0.
\end{aligned} \tag{22}$$

Then we have $h(0) < 0$ and $h(+\infty) \to +\infty$; thus (21) has at least one positive root ψ_0. □

Denote $\omega_0 = \sqrt{\psi_0}$; then (21) has a pair of purely imaginary roots $(\pm i\omega_0)$.

Now we turn to the bifurcation analysis. We use the delay τ as the bifurcation parameter. We view the solutions of (21) as a function of the bifurcation parameter τ, and let $\eta(\tau) = \gamma(\tau) + i\omega(\tau)$ be the eigenvalue of (21) such that, for the initial value of the bifurcation parameter τ_0, we have $\gamma(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$ $(\omega_0 > 0)$. To establish the Hopf bifurcation at $\tau = \tau_0$, we need to show that $d\,\mathrm{Re}\,\eta(\tau)/d\tau|_{\tau=\tau_0} > 0$.

Lemma 6. *One has the following transversality condition:*

$$\left.\frac{d\,\mathrm{Re}\,\eta(\tau)}{d\tau}\right|_{\tau=\tau_0} > 0. \tag{23}$$

Proof. Differentiating with respect to τ from (21), we get

$$\left(\frac{d\eta}{d\tau}\right)^{-1} = \frac{d\tau}{d\eta} = \frac{3\eta^2 + 2Q_1\eta + Q_2 - Q_4\tau e^{-\eta\tau}}{Q_4\eta e^{-\eta\tau}}. \tag{24}$$

So

$$\begin{aligned}
&\mathrm{Sign}\left\{\mathrm{Re}\,\frac{d(\eta)}{d(\tau)}\right\}_{\tau=\tau_0} \\
&= \mathrm{Sign}\left\{\mathrm{Re}\,\frac{d(\tau)}{d(\eta)}\right\}_{\eta=i\omega_0} \\
&= \mathrm{Sign}\left\{\mathrm{Re}\left[\frac{3\eta^2 + 2Q_1\eta + Q_2 - Q_4\tau e^{-\eta\tau}}{Q_4\eta e^{-\eta\tau}}\right]_{\eta=i\omega_0}\right\} \\
&= \mathrm{Sign}\left\{\mathrm{Re}\left[\frac{3\eta^2 + 2Q_1\eta + Q_2}{Q_4\eta e^{-\eta\tau}}\right]_{\eta=i\omega_0}\right. \\
&\qquad \left. - \mathrm{Re}\left[\frac{\tau}{\eta}\right]_{\eta=i\omega_0}\right\} \\
&= \mathrm{Sign}\left\{\mathrm{Re}\left[-\frac{3\eta^2 + 2Q_1\eta + Q_2}{\eta^4 + Q_1\eta^3 + Q_2\eta^2 + Q_3\eta}\right]_{\eta=i\omega_0}\right\} \\
&= \mathrm{Sign}\left\{\mathrm{Re}\left[-\frac{\left(Q_2 - 3\omega_0^2\right) + 2Q_1\omega_0 i}{\left(\omega_0^4 - Q_2\omega_0^2\right) + \left(Q_3\omega_0 - Q_1\omega_0^3\right)i}\right]\right\} \\
&= \mathrm{Sign}\left\{\frac{\omega_0^2\left[3\omega_0^4 + \left(2Q_1^2 - 4Q_2\right)\omega_0^2 + \left(Q_2^2 - 2Q_1Q_3\right)\right]}{Q_4^2\omega_0^4}\right\} \\
&= \mathrm{Sign}\left\{\frac{3\omega_0^4 + 2P_1\omega_0^2 + P_2}{Q_4^2}\right\} > 0.
\end{aligned} \tag{25}$$

This proves the lemma. Now we have the following theorem. □

Theorem 7. *The endemic equilibrium* E_* *of the system is locally asymptotically stable when* $\tau < \tau_0$ *and becomes unstable for* $\tau > \tau_0$ *provided*

$$\frac{\mu\mu_0}{m_0} < \lambda_0 + d + q\nu < \min\left\{\frac{\lambda M^*}{1 + kM^*}, \frac{\mu\lambda(\nu + d)}{\beta(1 + kM^*)^2}\right\}. \tag{26}$$

When $\tau = \tau_0$, *a Hopf-bifurcation occurs, leading a family of periodic solutions bifurcating from* E_* *as* τ *passes through the critical value* τ_0, *where*

$$\tau_0 = \frac{1}{\omega_0}\mathrm{arc\ cos}\,\frac{Q_1\omega_0^2 - Q_3}{Q_4}. \tag{27}$$

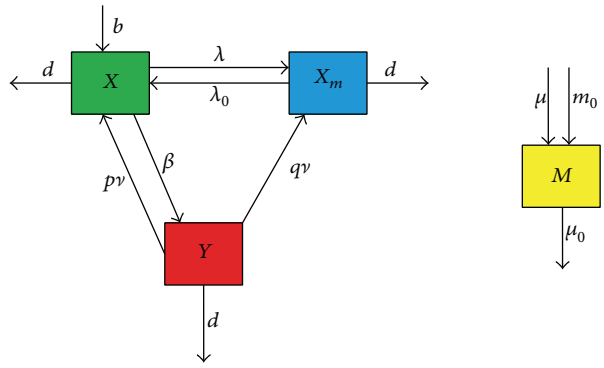

Figure 1: Schematic model flow diagram.

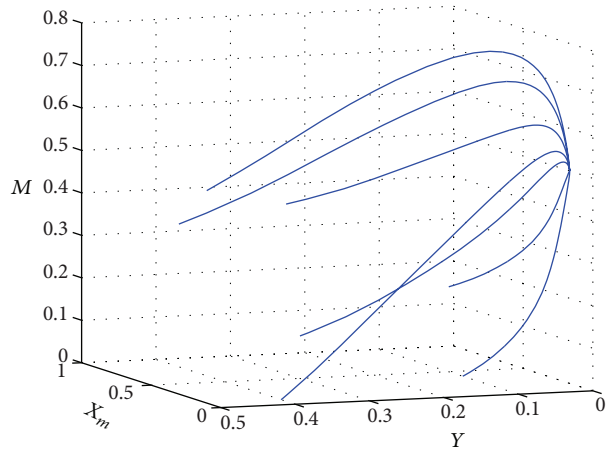

Figure 2: The stability of the disease-free equilibrium E_0 with $\tau = 10$.

4. Numerical Simulations and Results

To check the feasibility of our analysis in Section 3 about E_0, we present some numerical computations in this section using MatLab by choosing the following set of parameter values: $d = 0.002$, $\beta = 0.4$, $\mu_0 = 0.08$, $\mu = 0.2$, $\nu = 0.46$, $q = 0.6$, $\lambda = 0.2$, $\lambda_0 = 0.4$, $m_0 = 0.04$, $k = 0.5$, and $\tau = 10$ (see Figure 1), and then we get Figure 2 with different initial values, from which we can know that E_0 is stable and all the trajectories approach it. The numerical simulations support the analysis given.

Then we choose the following set of parameter values which satisfy the condition in Theorem 7: $d = 0.00004$, $\beta = 0.3$, $\mu_0 = 0.14$, $\mu = 0.35$, $\nu = 0.2$, $q = 0.15$, $\lambda = 0.019$, $\lambda_0 = 0.001$, $m_0 = 0.000005$, and $k = 0.0003$. The numerical value of τ_0 computed is found to be 30.12. When $\tau > 0$, we give different τ as follows: $\tau = 20$, $\tau = 25$, $\tau = 35$, and $\tau = 40$ (Figures 3 and 4). As shown in Figures 3 and 4 the variables approach their equilibria when τ is less than τ_0, whereas, as τ exceeds its critical value τ_0, all variables start showing oscillatory behavior. This indicates that, in the latter case, sometimes the infective will be high and sometimes will be low and it may be difficult to make the prediction regarding the size of epidemic. It is clear that τ plays a key role in the prevention and control of diseases.

In the following we let $\tau = 10$ and make λ, μ change from 0.05 to 0.8 and research the influence of the dissemination rate λ and the implementation rate μ on the infective $Y(t)$ separately. The variation of $Y(t)$ with respect to time t for different values of λ and μ is shown in Figure 5, and we discover all of the proportions of Y^* reductions as λ and μ increase, which proves the conclusions of the remark, but λ has greater influence on $Y(t)$ than μ. In addition the reason why $Y(t)$ has a similar trend in Figure 5 is that λ and μ have a similar trend with $M(t)$. There really is an effort here to make clear that λ and μ play a key role in the prevention and control of diseases.

A comparison between the oscillations in Y and M is presented in Figure 6. From this figure, we obtain that, as the infective increase, the awareness programs also start growing with some time lag due to time delay in the execution of awareness programs. As soon as the awareness programs reach a potentially high value, the infective start decreasing due to the execution of awareness programs in the infective. Awareness programs after certain time also start decreasing, which eventually results in the increase of the infective. Both

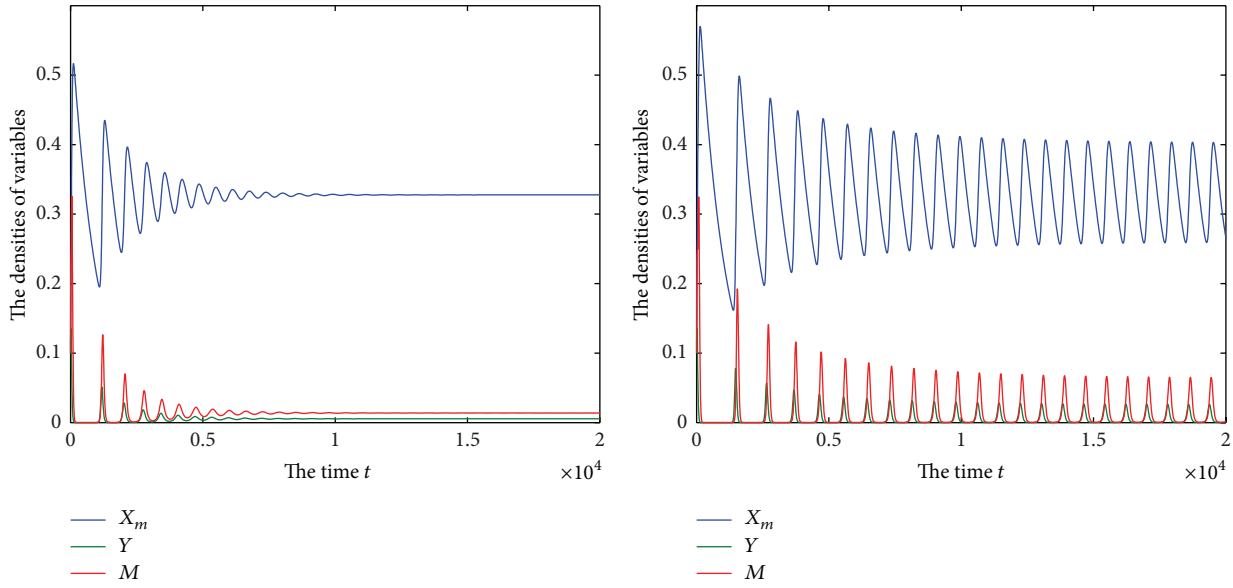

FIGURE 3: The stability of the endemic equilibria E_* with $\tau = 20$ and $\tau = 40$, respectively.

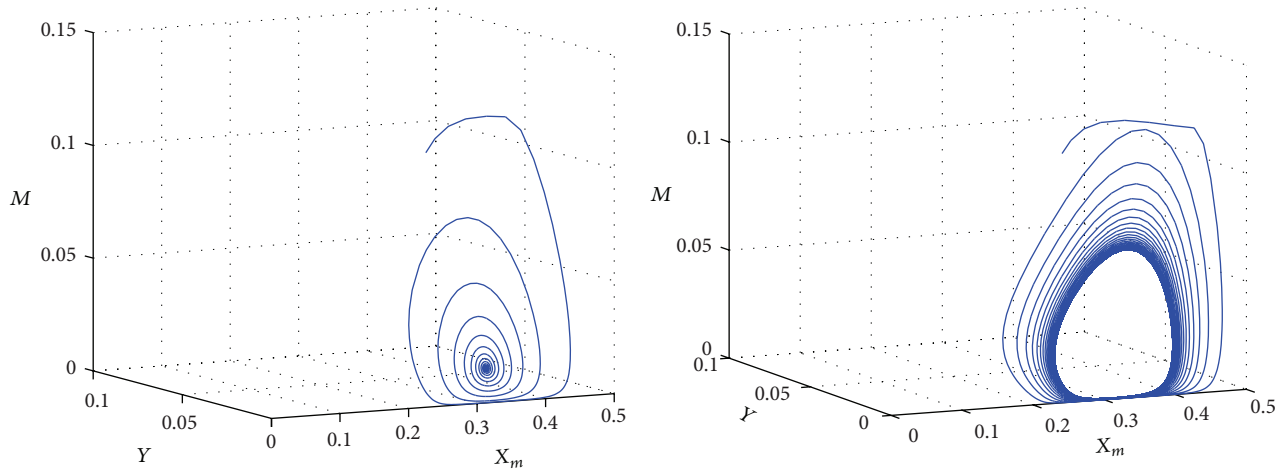

FIGURE 4: The stability of the endemic equilibria E_* with $\tau = 25$ and $\tau = 35$, respectively.

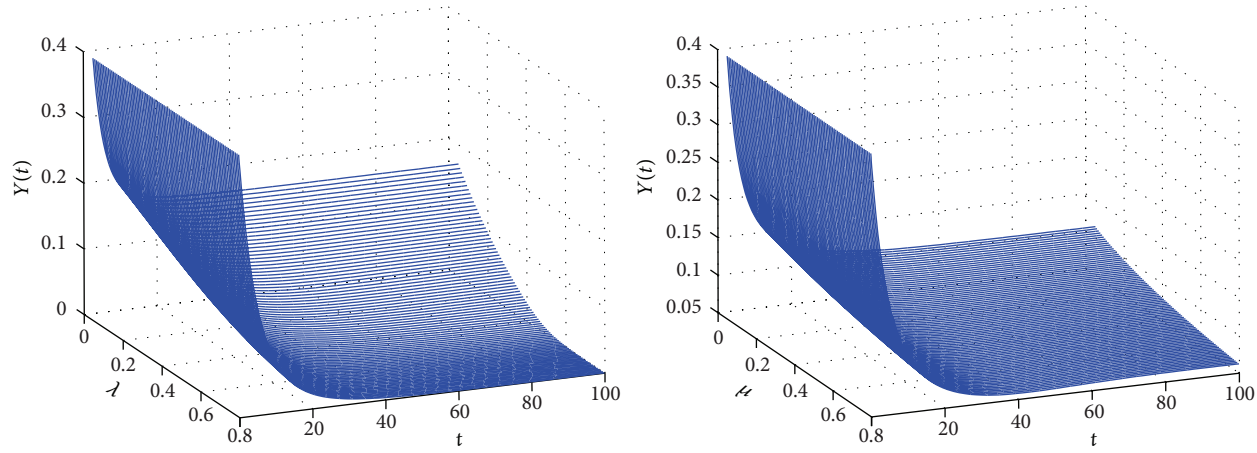

FIGURE 5: When $\tau = 10$, the stability of Y^* with variational λ and μ, respectively.

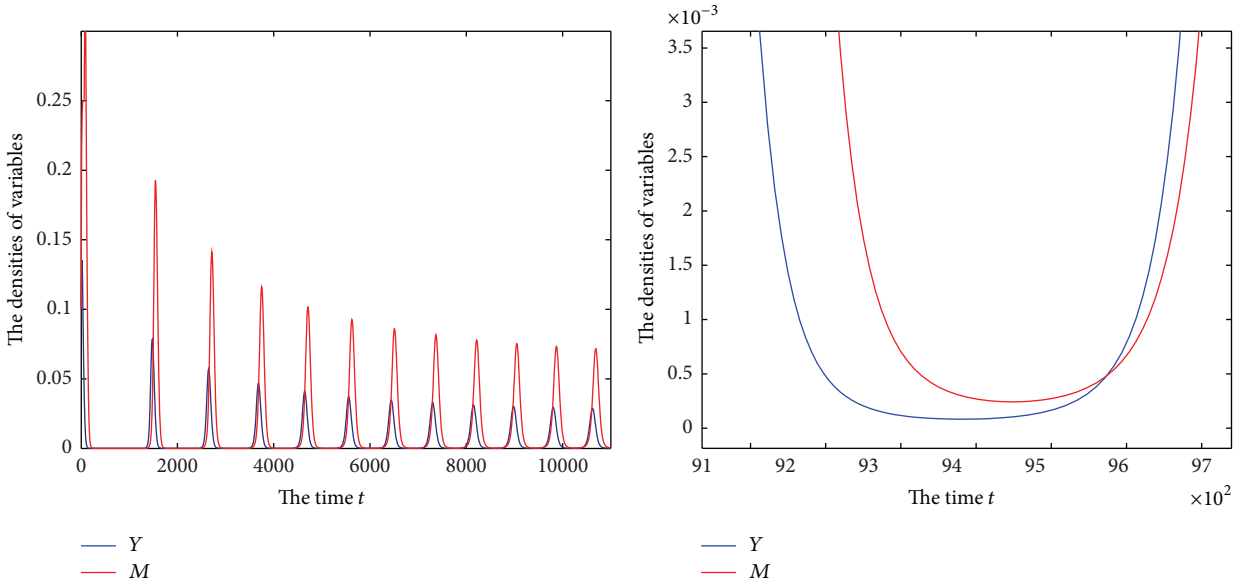

Figure 6: Comparison between variations of Y and M with time.

of the figures show the interplay between Y and M, which proves the existence of delay.

5. Discussion

In this paper, a nonlinear mathematical model with delay and awareness programs driven by media has been proposed and analyzed. It is assumed that pathogens are transmitted via direct contact between the susceptible and the infective. The model exhibits two equilibria, and the disease-free equilibrium has been shown to be stable for basic reproduction number $R_0 < 1$ when $\tau \geq 0$. For $R_0 > 1$, it leads to the existence and stability of an endemic equilibrium under some conditions in absence of time delay. But for $\tau > 0$ this equilibrium is locally asymptotically stable when the delay is suitably small; that is, $\tau < \tau_0$, while a loss of stability by a Hopf-bifurcation can occur as delay increases. Numerical simulations prove the stability of equilibria and show that the value of delay sustainable for the disease is 30.12. The numerical results suggest that if we want to reduce the proportion of the infective, we can increase the dissemination rate λ and implementation rate μ. At last we obtain the existence of τ from Figure 6. In short media is a key tool for influencing people behavior towards the disease to devise proper policies for controlling the epidemic.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publishing of this paper.

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Research Article

Robust H_∞ Control of Neutral System with Time-Delay for Dynamic Positioning Ships

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Due to the input time-delay existing in most thrust systems of the ships, the robust H_∞ controller is designed for the ship dynamic positioning (DP) system with time-delay. The input delay system is turned to a neutral time-delay system by a state-derivative control law. The less conservative result is derived for the neutral system with state-derivative feedback by the delay-decomposition approach and linear matrix inequality (LMI). Finally, the numerical simulations demonstrate the asymptotic stability and robustness of the controller and verify that the designed DP controller is effective in the varying environment disturbances of wind, waves, and ocean currents.

1. Introduction

The dynamic positioning (DP) systems of ships keep floating structures in fixed position or predetermined track for marine operation purpose exclusively by means of active thrusters [1]. Conventional DP control systems are designed based on linearization PID controllers in each degree of freedom combined with notch-filters. Subsequently linear optimal controller combined with Kalman filter theory was proposed [2, 3]. In recent years, the nonlinear controllers and estimators were employed to handle the inherent nonlinear characteristics of the vessel dynamics [4, 5]. In [6–8], Lyapunov methods and back-stepping technique were used to design a passive nonlinear observer to estimate the vessel velocity. Many scholars have studied the DP controllers but they did not consider the time-delay in the ship thrust system.

Time-delays as a source of instability and poor performance often appear in many dynamic systems, so it is very important to make a stability analysis of systems with time-delays. In vessel's DP control, the time-delays generally exist in thruster driver, including main propellers, tunnel thrusters, azimuth thrusters, and rudders. Compared with the rapidly changed instructions from the controller, the response time of thrusters, which generally lasts several seconds, is longer. Therefore, it is crucial to take account of the time-delay during the design of the DP controller. In the past, many useful approaches have been applied to guarantee the stabilization and the H_∞ performance of systems. Recently, more attention has been paid to the delay-dependent system which, in general, can be considered as less conservative than the delay-independent ones. By using the Lyapunov-Razumikhin functional approach or the Lyapunov-Krasovskii functional approach, several stability criteria have been proposed for the delay-dependent systems [9–18]. Based on a model transformation and Park's inequality, a delay-dependent stability criterion for the uncertain neutral systems with time-varying discrete delay was proposed in [10]. Reference [11] proposed a delay-dependent stability criterion for time-varying delay systems by means of an augmented model. In [12], the triple integral Lyapunov-Krasovskii functional approaches which utilized more information about the states and the delay states were proposed. Scholar Min Wu proposed the free weighting matrix method to avoid the conservatism of the transformation results, but it requires more possible decidable variables which make the computation quite complex [13, 14]. References [15–17]

proposed the delay-decomposition approach, that the delay interval is uniformly divided into N equidistant subintervals, and proposed a new delay-dependent stability criterion. The above researches for the stability of time-delays system are to increase the upper bound of the stability condition and reduce the conservatism of the algorithm. The H_∞ controllers are often utilized to stabilize the unstable systems or improve the system's performance [18]. Reference [19] proposed an H_∞ observer-based controller for the uncertain neutral time-delay systems. Shariati proposed an H_∞ controller for a linear system with time-delay, which used composite state-derivative feedback [20]. Reference [21] proposed a nonfragile controller for the discrete-time descriptor systems with multiple state delays. Reference [22] proposed a Lyapunov function with a piecewise analysis method for the semi-Markovian jump systems with mode-dependent delays. The state-derivative feedback can effectively improve the performance of the controller and it is generally used in practical systems such as the mechanical system and the ship DP system, where the accelerometers and the velocity sensors are used for measuring the system's motion. Moreover, the acceleration feedback can suppress varying disturbances, and the use of acceleration feedback in DP control problems has been discussed in [8].

In this paper, the robust H_∞ control problem for the neutral time-delay system has been considered. The less conservative results were derived for the neutral system with state-derivative feedback by a delay-decomposition approach and LMI method. The robust H_∞ controller is used in the ship DP control systems, and the designed closed-loop control law with speed and acceleration feedback enables DP controller to perform better and resist external environmental disturbances greatly.

This paper is organized as follows. In Section 2, the problem formulation for the DP system is discussed. In Section 3.1, the H_∞ performance analysis is proposed. In Section 3.2, the H_∞ state-derivative feedback controller is designed for the neutral time-delay system. Simulation results are provided in Section 4, which demonstrate the effectiveness and the advantages of the proposed controller under the ocean disturbances. Conclusions are given in Section 5.

2. Problem Formulation and Preliminaries

Here, the model of ship DP system with time-delay is established. The equations of motion of surface vessels can be described by the following model [1]:

$$\begin{aligned} M\dot{\nu} + D\nu &= u + w, \\ \dot{n} &= J(\psi)\nu, \\ M &= \begin{bmatrix} m - X_{\dot{u}} & 0 & 0 \\ 0 & m - Y_{\dot{v}} & mx_G - Y_{\dot{r}} \\ 0 & mx_G - Y_{\dot{r}} & I_z - N_{\dot{r}} \end{bmatrix}, \\ D &= \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -Y_r & -N_r \end{bmatrix}, \end{aligned} \tag{1}$$

where $n = [X \;\; Y \;\; \psi]$ is the position and orientation of the vessel with respect to an inertial reference coordinate system and $\nu = [u \;\; v \;\; r]$ is a vector of velocities given in the body-fixed coordinate system; $J(\psi)$ is the transformation matrix between the inertial and body-fixed coordinate frames; u is a control vector of forces and moments provided by the propulsion system. Generally, time-delay exists in the propulsion system, which means the input delay exists in the ship control system; w is the disturbance input of the system. The inertia matrix M includes hydrodynamic added inertia; D is a strictly positive damping matrix due to linear wave drift damping and laminar skin friction; m is the mass of the ship; I_z is the moment of inertia. $X_{\dot{u}}$, $Y_{\dot{v}}$, $N_{\dot{r}}$, and $Y_{\dot{r}}$ are the accessional mass caused by liquid forces, respectively.

The time-delays mainly existing in the ship's propulsion system are considered in this paper, and the formula (1) is transformed into the state-space model of the vessel with time-delay as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau(t)) + Ew(t), \\ z(t) &= Cx(t) + D_1 u(t - \tau(t)), \\ A &= \begin{bmatrix} 0 & I \\ -M^{-1}G & -M^{-1}D \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \\ E &= \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \end{aligned} \tag{2}$$

where $x(t) = [n^T, \nu^T]^T$ is the system state including the position, heading, and velocities of vessel; $u(t - \tau(t))$ is the system input and also is the control vector of forces and moments with time-delay provided by the propulsion system; $\tau(t)$ is the time-delay in the propulsion system; B is a constant matrix describing the actuator configuration; $w(t)$ is the disturbance input of the system caused by the environment disturbances from wind, currents, and waves; $z(t)$ is the controlled system output. The matrices C, D_1 are assumed to be known.

In ship DP system, a variety of sensors are installed to measure the state of the ship motion, such as global position system (GPS) for measuring the ship's position, attitude sensor for measuring the pitch, roll and heave, and so forth, and gyrocompass for measuring the ship's heading. Using the measured information for feedback control is a practical approach of the ship motion control. Velocity feedback in the DP controller can improve the performance of the control system. High-precision commercial inertial measurement units, which are used to measure the acceleration in the positioning control of vessels to suppress varying disturbances, can be easily interfaced with the existing control systems. Lindegaard used the acceleration feedback to increase the performance of DP systems [8]. Acceleration feedback (AFB) will provide a virtual mass in addition to the physical mass of the vessel. Therefore, the vessel becomes less sensitive to the environmental excitations.

Using the state-derivative feedback control law in the position control as well as the velocity and acceleration

feedback can suppress varying disturbances. The aim of this paper is to design the composite control law as follows:

$$u = K_1 x + K_2 \dot{x}, \tag{3}$$

where K_1 and K_2 are the controller gain to be designed; the design goal is to make the system (2) asymptotically stable and meet certain performance index, where $\dot{x}(t) = [\dot{n}^T, \dot{v}^T]^T$, $\dot{n}^T$ is the velocity vector of vessel, and $\dot{v}$ is the acceleration velocity of vessel. The velocity and acceleration feedback loop is established by measuring the velocity and acceleration in the DP control of vessels to suppress varying disturbances in the sea environment. By applying the feedback control law (3), the input time-delay system (2) is turned into a time-delay closed-loop system of the neutral type which is described as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BK_1 x(t - \tau(t)) \\ &\quad + BK_2 \dot{x}(t - \tau(t)) + Ew(t), \\ z(t) &= Cx(t) + D_1 K_1 x(t - \tau(t)) \\ &\quad + D_1 K_2 \dot{x}(t - \tau(t)). \end{aligned} \tag{4}$$

Neutral time-delay systems contain delays both in its state and its derivatives of the state. As it is shown in (4), both the coefficients of $x(t - \tau)$ and $\dot{x}(t - \tau)$ are dependent on the controller's parameters. Thus, our objective is to design a state feedback controller $u = K_1 x + K_2 \dot{x}$, in which for a given $\tau(t) > 0$, the following requirements are satisfied:

(1) the closed-loop system (4) with $w(t) = 0$ is asymptotically stable;

(2) under zero initial condition, the closed-loop system satisfies $\|z\|_2 < \gamma \|w\|_2$ for any nonzero $w \in L_2[0, \infty]$, where $\gamma > 0$ is a prescribed scalar.

In order to obtain the main results, we state the following well-known lemmas.

Lemma 1 (see [17]). *For any constant matrix $0 < R = R^T \in R^{n \times n}$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, then*

$$\begin{aligned} &\left(\int_0^\gamma \omega(s)\, ds \right)^T R \left(\int_0^\gamma \omega^T(s)\, ds \right) \\ &\quad \le \gamma^{-1} \int_0^\gamma \omega(s) R \omega^T(s)\, ds. \end{aligned} \tag{5}$$

Lemma 2 (see [18]). *For given matrices A_{11}, A_{12}, A_{21}, A_{22} with the appropriate dimensions, $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} < 0$ holds if and only if $A_{22} < 0$, $A_{11} - A_{12} A_{22}^{-1} A_{12}^T < 0$.*

3. Main Results

In this section, the delay-dependent stability criterion for the neutral system (4) will be presented by means of the Lyapunov-Krasovskii functional approach. In order to derive less conservative stability criterion for the system (4), we employ the delay-decomposition approach for the neutral time-delay system and divide $\tau(t)$ into several components by an integer N with the length of each component denoted by $\eta(t)$; that is,

$$\eta(t) = \frac{\tau(t)}{N}. \tag{6}$$

Therefore, $\eta(t)$ represents a partition of the time-varying delay $\tau(t)$, which satisfies

$$\begin{aligned} 0 &\le \eta(t) \le \eta, \\ 0 &\le \dot{\eta}(t) \le \nu. \end{aligned} \tag{7}$$

3.1. Robust Stabilization. In this section, we focus on the analysis of the H_∞ performance.

Theorem 3. *Given scalar $\tau \ge 0$, the closed-loop system (4) with $w(t) = 0$ is asymptotically stable for a continuous time-varying state delay $\tau(t)$ satisfying $0 \le \tau(t) \le \tau$, if there exist some positive definite symmetric matrices $P_1 > 0$, $S_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$ $M_1 > 0$, $M_2 > 0$, such that the following symmetric linear matrix inequalities hold:*

$$\begin{bmatrix} \Omega_{11} & BK_1 & BK_2 & P_1^{-1} Z_1 & 0 & P_1^{-1} A^T & P_1^{-1} A^T & P_1^{-1} A^T & P_1^{-1} A^T \\ * & \Omega_{22} & 0 & Z_2 & Z_3 & K_1^T B^T & K_1^T B^T & K_1^T B^T & K_1^T B^T \\ * & * & -S_2 & 0 & 0 & K_2^T B^T & K_2^T B^T & K_2^T B^T & K_2^T B^T \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -S_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\eta^2 Z_1 & 0 & 0 \\ * & * & * & * & * & * & * & -\eta^2 Z_2 & 0 \\ * & * & * & * & * & * & * & * & -\eta^2 Z_3 \end{bmatrix} < 0, \tag{8}$$

where

$$\begin{aligned} \Omega_{11} &= 2AP_1^{-1} + M_1 - M_2, \\ \Omega_{22} &= -(1 - 2\nu) Q_2 + (1 + 2\nu) Q_3 - Z_2 - Z_3, \\ \Omega_{44} &= -(1 - \nu) Q_1 + (1 + \nu) Q_2 - Z_1 - Z_2, \\ \Omega_{55} &= -(1 - 3\nu) Q_3 - Z_3. \end{aligned} \tag{9}$$

The asterisk represents the symmetric form of the matrix (8).

Proof. Choose a suitable Lyapunov-Krasovskii function candidate for the system (4) as the following form:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t). \tag{10}$$

Here,

$$\begin{aligned} V_1(t) &= x^T P_1 x(t), \\ V_2(t) &= \int_{t-\tau}^t \dot{x}^T(s) S_2 \dot{x}(s)\, ds, \\ V_3(t) &= \sum_{i=1}^N \int_{t-i\eta(t)}^{t-(i-1)\eta(t)} x^T(s) Q_i x(s)\, ds, \\ V_4(t) &= \sum_{i=1}^N \int_{-i\eta}^{-(i-1)\eta} \int_{t+\theta}^t \eta \dot{x}^T(s) Z_i \dot{x}(s)\, ds\, d\theta, \end{aligned} \tag{11}$$

where

$$\begin{aligned} P_1 &= P_1^T > 0, \\ S_2 &= S_2^T > 0, \\ Q_1 &= Q_1^T > 0, \\ Q_2 &= Q_2^T > 0, \\ Z_1 &= Z_1^T > 0, \\ Z_2 &= Z_2^T > 0, \\ Z_3 &= Z_3^T > 0. \end{aligned} \tag{12}$$

Now calculate the derivative of $V(t)$ along the state trajectory of the system (4). Employing Lemma 1, the following upper bound for $\dot{V}(t)$ can be obtained as follows:

$$\begin{aligned} \dot{V}_1(t) &= 2x^T(t) P_1 \dot{x}(t) \\ &= 2x^T(t) P_1 \left(Ax(t) + BK_1 x(t-\tau(t)) \right. \\ &\qquad \left. + BK_2 \dot{x}(t-\tau(t)) + E\omega(t)\right), \\ \dot{V}_2(t) &= \dot{x}^T(t) S_2 \dot{x}(t) - \dot{x}^T(t-\tau) S_2 \dot{x}(t-\tau), \\ \dot{V}_3(t) &= \sum_{i=1}^{N} \left[\left(1-(i-1)\dot{\eta}(t)\right) \times x^T\left(t-(i-1)\eta(t)\right) \right. \\ &\qquad \cdot Q_i x\left(t-(i-1)\eta(t)\right) \\ &\qquad \left. - \left(1-i\dot{\eta}(t)\right) x^T\left(t-i\eta(t)\right) Q_i x\left(t-i\eta(t)\right)\right] \\ &\le \sum_{i=1}^{N} \left[\left(1+(i-1)\nu\right) \times x^T\left(t-(i-1)\eta(t)\right) \right. \\ &\qquad \cdot Q_i x\left(t-(i-1)\eta(t)\right) \\ &\qquad \left. - (1-i\nu) x^T\left(t-i\eta(t)\right) Q_i x\left(t-i\eta(t)\right)\right], \\ \dot{V}_4(t) &= \eta^2 \dot{x}^T(t) \sum_{i=1}^{N} Z_i \dot{x}(t) - \sum_{i=1}^{N} \int_{t-i\eta}^{t-(i-1)\eta} \eta \dot{x}^T(\theta) Z_i \dot{x}(\theta)\, d\theta \\ &\le \eta^2 \dot{x}^T(t) \sum_{i=1}^{N} Z_i \dot{x}(t) \\ &\quad - \sum_{i=1}^{N} \left[\left(x\left(t-(i-1)\eta(t)\right) - x\left(t-i\eta(t)\right)\right)^T \right. \\ &\qquad \times Z_i \left(x\left(t-(i-1)\eta(t)\right) - x\left(t-i\eta(t)\right)\right) \\ &\qquad \left. - \left(1-i\dot{\eta}(t)\right) x^T\left(t-i\eta(t)\right) Q_i x\left(t-i\eta(t)\right)\right]. \end{aligned} \tag{13}$$

By simple derivations, get $N = 3$; the upper bound for $\dot{V}(t)$ is obtained as follows:

$$\begin{aligned} \dot{V}(t) &\le \dot{x}^T(t) S_2 \dot{x}(t) - \dot{x}^T(t-\tau) S_2 \dot{x}(t-\tau) \\ &\quad + 2x^T(t) P_1 \left(Ax(t) + BK_1 x(t-\tau(t)) \right. \\ &\qquad \left. + BK_2 \dot{x}(t-\tau(t)) + E\omega(t)\right) \\ &\quad + x^T(t) Q_1 x(t) \\ &\quad - (1-\nu) x^T\left(t-\eta(t)\right) Q_1 x\left(t-\eta(t)\right) \\ &\quad + (1+\nu) x^T\left(t-\eta(t)\right) Q_2 x\left(t-\eta(t)\right) \\ &\quad - (1-2\nu) x^T\left(t-2\eta(t)\right) Q_2 x\left(t-2\eta(t)\right) \\ &\quad + (1+2\nu) x^T\left(x-2\eta(t)\right) Q_3 x\left(t-2\eta(t)\right) \\ &\quad - (1-3\nu) x^T\left(t-3\eta(t)\right) Q_3 x\left(t-3\eta(t)\right) \\ &\quad + \eta^2 \dot{x}^T(t) \left(Z_1+Z_2+Z_3\right) \dot{x}(t) - x^T(t) Z_1 x(t) \\ &\quad + 2x^T(t) Z_1 x\left(t-\eta(t)\right) \\ &\quad - x^T\left(t-\eta(t)\right) Z_1 x\left(t-\eta(t)\right) \\ &\quad - x^T\left(t-\eta(t)\right) Z_2 x\left(t-\eta(t)\right) \\ &\quad + 2x^T\left(t-2\eta(t)\right) Z_2 x\left(t-\eta(t)\right) \\ &\quad - x^T\left(t-2\eta(t)\right) Z_2 x\left(t-2\eta(t)\right) \\ &\quad - x^T\left(t-2\eta(t)\right) Z_3 x\left(t-2\eta(t)\right) \\ &\quad + 2x^T\left(t-3\eta\right) Z_3 x\left(t-2\eta(t)\right) \\ &\quad - x^T\left(t-3\eta(t)\right) Z_3 x\left(t-3\eta(t)\right) \\ &\le \varsigma^T(t) \psi \varsigma(t), \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\varsigma^T(t) \\ &= \left[x^T(t), x^T(t-\tau), \dot{x}^T(t-\tau), x^T\left(t-\eta\right), x^T\left(t-3\eta\right)\right], \\ &\psi = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & Z_1 & 0 \\ * & \Sigma_{22} & \Sigma_{23} & Z_2 & Z_3 \\ * & * & \Sigma_{33} & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 \\ * & * & * & * & \Sigma_{55} \end{bmatrix}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Sigma_{11} &= 2P_1 A + Q_1 - Z_1 + A^T \theta A, \\ \Sigma_{12} &= P_1 B K_1 + A^T \theta B K_1, \\ \Sigma_{13} &= P_1 B K_2 + A^T \theta B K_2, \\ \Sigma_{22} &= -(1-2\nu) Q_2 + (1+2\nu) Q_3 - Z_2 - Z_3 \\ &\qquad + K_1^T B^T \theta B K_1, \\ \Sigma_{23} &= K_1^T B^T \theta B K_2, \\ \Sigma_{33} &= -S_2 + K_2^T B^T \theta B K_2, \end{aligned}$$

$$\Sigma_{44} = -(1-\nu)Q_1 + (1+\nu)Q_2 - Z_1 - Z_2,$$
$$\Sigma_{55} = -(1-3\nu)Q_3 - Z_3, \tag{16}$$

where $\theta = S_2 + \eta^2(Z_1 + Z_2 + Z_3)$.

Remark 1. The decomposition number N is a positive integer and N is not less than 2. The conservatism will be reduced gradually as the N is increasing, and the time-delay tends to the certain upper boundary. When N equals 2 or 3, the conservatism is small enough.

By using Schur complement, the following matrix inequality is the equivalence of the condition $\psi < 0$:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0, \tag{17}$$

where

$$\Xi_{11} = \begin{bmatrix} \Sigma_{11} & P_1BK_1 & P_1BK_2 & Z_1 & 0 \\ * & \Sigma_{22} & 0 & Z_2 & Z_3 \\ * & * & -S_2 & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 \\ * & * & * & * & \Sigma_{55} \end{bmatrix},$$
$$\Xi_{12} = \begin{bmatrix} \Delta & \Delta & \Delta & \Delta \end{bmatrix}, \tag{18}$$

where

$$\Sigma_{11} = 2P_1A + Q_1 - Z_1,$$
$$\Sigma_{22} = -(1-2\nu)Q_2 + (1+2\nu)Q_3 - Z_2 - Z_3,$$
$$\Sigma_{44} = -(1-\nu)Q_1 + (1+\nu)Q_2 - Z_1 - Z_2,$$
$$\Sigma_{55} = -(1-3\nu)Q_3 - Z_3,$$
$$\Delta = \begin{bmatrix} A & BK_1 & BK_2 & 0 & 0 \end{bmatrix}^T,$$
$$\Xi_{22} = \operatorname{diag}\left(-S_2 \quad -\eta^2 Z_1 \quad -\eta^2 Z_2 \quad -\eta^2 Z_3\right). \tag{19}$$

Premultiplying and postmultiplying the matrix Ξ in (17) by $\Phi^T$and Φ, where $\Phi = \operatorname{diag}(P_1^{-1}, I, I, I, I, I, I, I, I)$ and applying some change of variables, $M_1 = P_1^{-1}Q_1P_1^{-1}$, $M_2 = P_1^{-1}Z_1P_1^{-1}$, we obtain result (8); hence, we have $\dot{V}(t) < 0$, then system (4) with $w(t) = 0$ is asymptotically stable. This completes the proof. □

3.2. Robust H_∞ Control

Theorem 4. *Given scalar $\tau \geq 0$, the closed-loop system (4) is asymptotically stable and $\|T_{zw}\|_\infty < \gamma$ for a continuous time-varying state delay $\tau(t)$ satisfying $0 \leq \tau(t) \leq \tau$, if there exist positive definite symmetric matrices $P_1 > 0$, $S_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$ $M_1 > 0$, $M_2 > 0$. Moreover, the H_∞ composite state-derivative feedback control law is given by $u = K_1x + K_2\dot{x}$, such that the following symmetric linear matrix inequalities hold:*

$$\begin{bmatrix} \Omega_{11} & BK_1 & BK_2 & P_1^{-1}Z_1 & 0 & E & 0 & P_1^{-1}A^T & P_1^{-1}A^T & P_1^{-1}A^T & P_1^{-1}A^T & P_1^{-1}C^T \\ * & \Omega_{22} & 0 & Z_2 & Z_3 & 0 & 0 & K_1^TB^T & K_1^TB^T & K_1^TB^T & K_1^TB^T & K_1^TD_1{}^T \\ * & * & -S_2 & 0 & 0 & 0 & 0 & K_2^TB^T & K_2^TB^T & K_2^TB^T & K_2^TB^T & K_2^TD_1{}^T \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 & E^T & E^T & E^T & E^T & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -S_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\eta^2 Z_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\eta^2 Z_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\eta^2 Z_3 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{20}$$

where the asterisk represents the symmetric form of the matrix (20):

$$\Omega_{11} = 2AP_1^{-1} + M_1 - M_2,$$
$$\Omega_{22} = -(1-2\nu)Q_2 + (1+2\nu)Q_3 - Z_2 - Z_3,$$
$$\Omega_{44} = -(1-\nu)Q_1 + (1+\nu)Q_2 - Z_1 - Z_2,$$
$$\Omega_{55} = -(1-3\nu)Q_3 - Z_3. \tag{21}$$

Proof. Choose the same Lyapunov-Krasovskii function given in (10) for the system (4) as the following form. Assuming the zero initial condition, that is, $\phi(t) = 0$, $\forall t \in [-\tau, 0]$, we can get $V(q(t))|_{t=0} = 0$. For a prescribed $\gamma > 0$, take account of the performance index as follows:

$$J_{zd}(w) = \int_0^\infty \left(z^Tz - \gamma^2 d^Td\right) d\tau, \tag{22}$$

where

$$d(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}. \tag{23}$$

Therefore, the performance index (22) can be rewritten as

$$J_{zd}(w) = \int_0^\infty \left(z^T z - \gamma^2 w^T w - \gamma^2 \dot{w}^T \dot{w} \right) d\tau. \tag{24}$$

Then for the random nonzero external disturbances $w \in L_2[0, \infty]$, since $V(t)|_{t=0} = 0$ and $V(t)|_{t\to\infty} \geq 0$, we can obtain

$$\begin{aligned} J_{zd}(w) &= \int_0^\infty \left(z^T z - \gamma^2 w^T w - \gamma^2 \dot{w}^T \dot{w} + \dot{V}(t) \right) d\tau \\ &\quad + V(t)|_{t=0} - V(t)|_{t\to\infty} \\ &\leq \int_0^\infty \left(z^T z - \gamma^2 w^T w - \gamma^2 \dot{w}^T \dot{w} + \dot{V}(t) \right) d\tau. \end{aligned} \tag{25}$$

Introducing (4) and (14) into the above, we can obtain the following inequality:

$$J_{zd}(w) \leq \int_0^\infty \xi^T(t) \Sigma \xi(t)\, dt. \tag{26}$$

With defined

$$\begin{aligned} \xi^T(t) &= \left[x^T(t), x^T(t-\tau), \dot{x}^T(t-\tau), x^T(t-\eta), \right. \\ &\quad \left. x^T(t-3\eta), w(t), \dot{w}(t) \right], \\ \Sigma &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & Z_1 & 0 & \Sigma_{16} & 0 \\ * & \Sigma_{22} & \Sigma_{23} & Z_2 & Z_3 & \Sigma_{26} & 0 \\ * & * & \Sigma_{33} & 0 & 0 & \Sigma_{36} & 0 \\ * & * & * & \Sigma_{44} & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 \\ * & * & * & * & * & \Sigma_{66} & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Sigma_{11} &= 2P_1A + Q_1 - Z_1 + C^T C + A^T \theta A, \\ \Sigma_{12} &= P_1 B K_1 + C^T D_1 K_1 + A^T \theta B K_1, \\ \Sigma_{13} &= P_1 B K_2 + C^T D_1 K_2 + A^T \theta B K_2, \\ \Sigma_{16} &= P_1 E + A^T \theta E, \\ \Sigma_{22} &= -(1-2\nu) Q_2 + (1+2\nu) Q_3 \\ &\quad - Z_2 - Z_3 + K_1^T D_1^T D_1 K_1 + K_1^T B^T \theta B K_1 \\ \Sigma_{23} &= K_1^T D_1^T D_1 K_2 + K_1^T B^T \theta B K_2, \\ \Sigma_{26} &= K_1^T B^T \theta E, \\ \Sigma_{33} &= -S_2 + K_2^T D_1^T D_1 K_2 + K_2^T B^T \theta B K_2, \\ \Sigma_{36} &= K_2^T B^T \theta E \\ \Sigma_{44} &= -(1-\nu) Q_1 + (1+\nu) Q_2 - Z_1 - Z_2, \\ \Sigma_{55} &= -(1-3\nu) Q_3 - Z_3, \\ \Sigma_{66} &= -\gamma^2 + E^T \theta E, \end{aligned} \tag{28}$$

where $\theta = S_2 + \eta^2 (Z_1 + Z_2 + Z_3)$

By using Schur complement, the following matrix inequality (29) is the equivalent of the condition of $\Sigma < 0$:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0, \tag{29}$$

where

$$\begin{aligned} \Xi_{11} &= \begin{bmatrix} \Sigma_{11} & P_1 B K_1 & P_1 B K_2 & Z_1 & 0 & P_1 E & 0 \\ * & \Sigma_{22} & -\frac{1}{2} S_2 & Z_2 & Z_3 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix}, \\ \Xi_{12} &= \begin{bmatrix} \Delta_1 & \Delta_1 & \Delta_1 & \Delta_1 & \Delta_2 \end{bmatrix}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Sigma_{11} &= 2P_1A + Q_1 - Z_1, \\ \Sigma_{22} &= -(1-2\nu) Q_2 + (1+2\nu) Q_3 - Z_2 - Z_3, \\ \Sigma_{44} &= -(1-\nu) Q_1 + (1+\nu) Q_2 - Z_1 - Z_2, \\ \Sigma_{55} &= -(1-3\nu) Q_3 - Z_3, \\ \Delta_1 &= \begin{bmatrix} A & BK_1 & BK_2 & 0 & 0 & E & 0 \end{bmatrix}^T, \\ \Delta_2 &= \begin{bmatrix} C & D_1 K_1 & D_1 K_2 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\ \Xi_{22} &= \operatorname{diag}\left(-S_2 \quad -\eta^2 Z_1 \quad -\eta^2 Z_2 \quad -\eta^2 Z_3 \quad -I \right). \end{aligned} \tag{31}$$

Premultiplying and postmultiplying the matrix Ξ in (29) by $\operatorname{diag}(P_1^{-1}, I, I, I, I, I, I, I, I, I, I)$, and applying some change of variables, $M_1 = P_1^{-1} Q_1 P_1^{-1}$, $M_2 = P_1^{-1} Z_1 P_1^{-1}$, it is easy to know that the LMI (20) is satisfied, then

$$J_{zd} = \int_0^\infty \left(z^T z - \gamma^2 w^T w - \gamma^2 \dot{w}^T \dot{w} \right) d\tau \leq 0. \tag{32}$$

Therefore

$$\int_0^\infty \left(z^T z\right) d\tau \le \gamma^2 \int_0^\infty \left(d^T d\right) d\tau. \tag{33}$$

The proof is completed. □

4. Simulations

In order to validate the performance of the designed H_∞ controller, the certain DP ship simulations and the marine simulation environments are accomplished in MATLAB7.0. The main parameters of the ship are referenced to the Marine Systems Simulator (MSS) developed by the Norwegian university of Science and Technology. The ship is named as S-175. The ship's main parameters are set as follows. Length = 175 m, beam = 25.4 m, tonnage = $2.4009e + 4\,\text{m}^3$, and draft = 9.5 m. Then, the matrices M and D in (1) can be obtained as follows:

$$M = \begin{bmatrix} 2.6415e+07 & 0 & 0 \\ 0 & 33455476 & 19417351 \\ 0 & 14917351 & 6.5209e+10 \end{bmatrix},$$
$$D = \begin{bmatrix} 2.2204e+04 & 0 & 0 \\ 0 & 2.2204e+05 & -1.7746e+06 \\ 0 & -1.7746e+06 & 7.1506e+08 \end{bmatrix}. \tag{34}$$

The matrices A, B, and E can be calculated according to formula (2). The marine simulation environment includes wind, waves, and currents. The simulation environments are set as the following: the wave model used in the paper is PM spectrum. The wave's direction is 120 degree; the significant wave height is 5 m; and the peak frequency is 0.8 rad/s. The current direction is 90 degree, and the current speed is 0.2 m/s. The wind speed is 10 m/s; wind direction angle is 225 degree. The initial position of the vessel is $n_p = [0\ \ 0\ \ 0^\circ]^T$. The final desired position is $n_d = [10\ \ 10\ \ 10^\circ]^T$. When the γ is 0.5 and the maximum propeller delay τ is 3.07 sec, the gain matrix of the state feedback controller is calculated through LMI toolbox as follows:

$$K_1 = \begin{bmatrix} -1.094 & 0 & 0 & -72.071 & 0 & 0 \\ 0 & -41.886 & 196.778 & 0 & 75.3 & 1446 \\ 0 & 0.323 & -1.027 & 0 & -0.055 & 0.447 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.001 & 0 & 0 & 0.006 & 0 & 0 \\ 0 & -0.011 & 0.011 & 0 & -0.0075 & 0.1992 \\ 0 & -35.505 & 98.995 & 0 & 4.767 & -8.552 \end{bmatrix}. \tag{35}$$

We compare our results for neutral time-delay system in Theorem 4 with the results using the same state-derivative feedback in [20]. Table 1 shows the results of the upper bound of time-delay for different values of γ. It can be seen that the proposed H_∞ state-feedback controller in this paper provides larger delay bounds than the previous results as listed in Table 1.

Table 1: The upper bounds of τ for various values of γ.

γ	0.5	1.0	10.0
τ	1.56	3.61	5.65
τ (Theorem 4), $N = 2$	2.86	6.16	10.16
τ (Theorem 4), $N = 3$	3.07	6.27	10.28

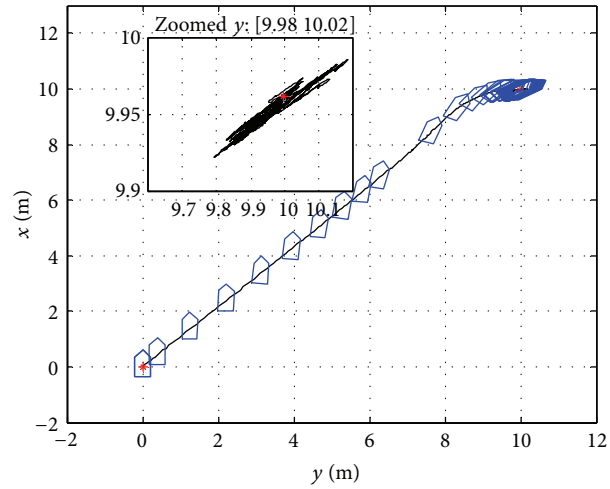

Figure 1: Position and heading of the vessel using H_∞ controller.

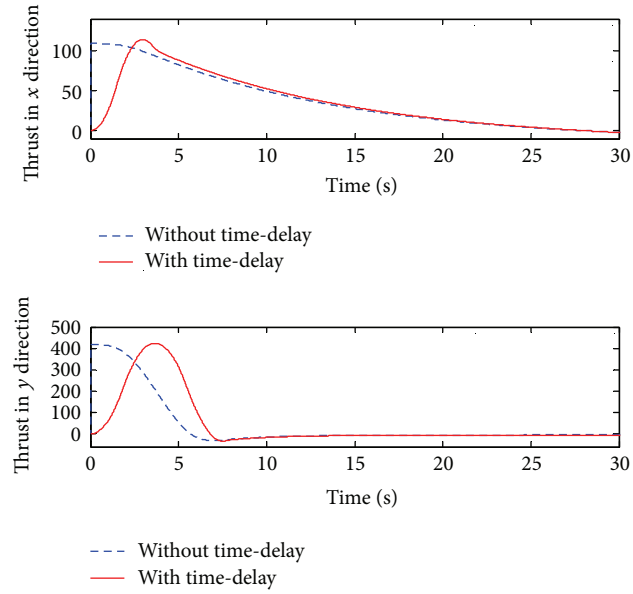

Figure 2: Comparisons of the thrust with and without time-delay.

The simulation results of the H_∞ state-feedback controller under the ocean environment are shown in Figures 1–6. The response curve of the ship's position is shown in Figure 1. The simulation results indicate that the ship can achieve the given position quickly with small overshoot. Under the action of wind, waves, and currents, the ship's position maintains given position of 0.5 m error range. Figure 2 shows the curves of the thrust without time-delay and with 3 sec of time-delay. Figures 3 and 4 show

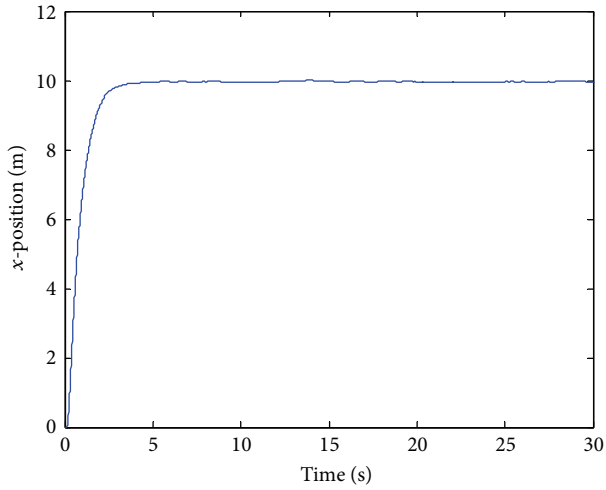

Figure 3: Position of the x direction of the vessel using H_∞ controller.

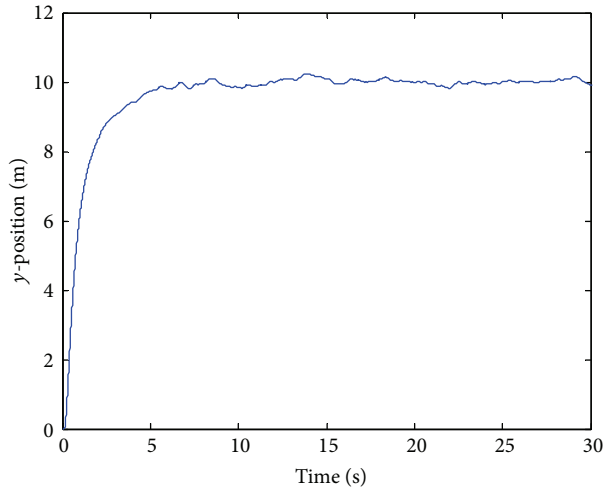

Figure 4: Position of the y direction of the vessel using H_∞ controller.

the position curves of the x direction and the y direction, respectively. Figures 5 and 6 show the ship's speed of the x and y directions, respectively. Figures 7 and 8 show the ship's acceleration of two directions. From the results, we can know that when delays exist in the propeller, the designed controller can make the ship stable and achieve the desired position even under the external disturbances (wind, waves, and currents).

In order to demonstrate that the proposed H_∞ controller has good delay stability, the simulation results are compared with the results of the PD controller. The simulations with disturbances of wind, waves, and currents are shown in Figures 9–12. Figure 9 shows the position response of the vessel using the H_∞ controller without time-delay. Figure 10 shows that the position response of the vessel using the H_∞ controller with 3 sec of time-delay. It can be seen that the controller has the delay stability under the same parameters sets of the controller with Figure 9. Figure 11 shows the position

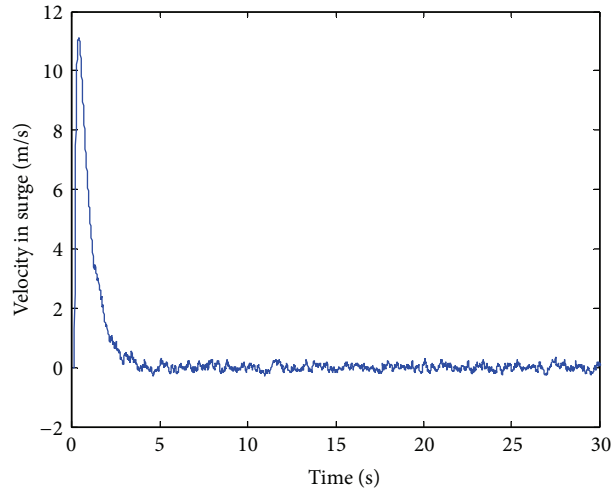

Figure 5: Velocity of the x direction of the vessel using H_∞ controller.

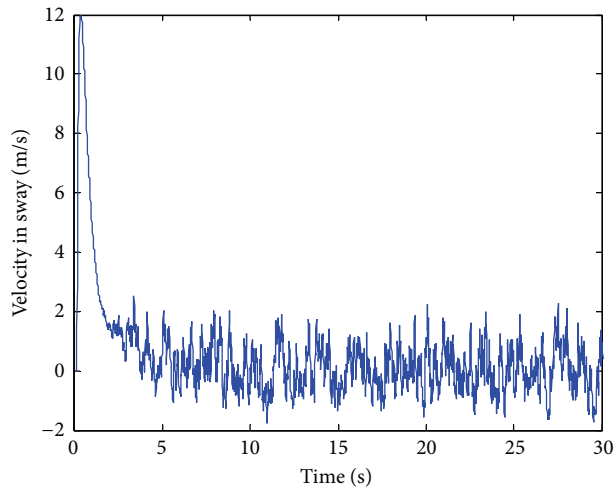

Figure 6: Velocity of the y direction of the vessel using H_∞ controller.

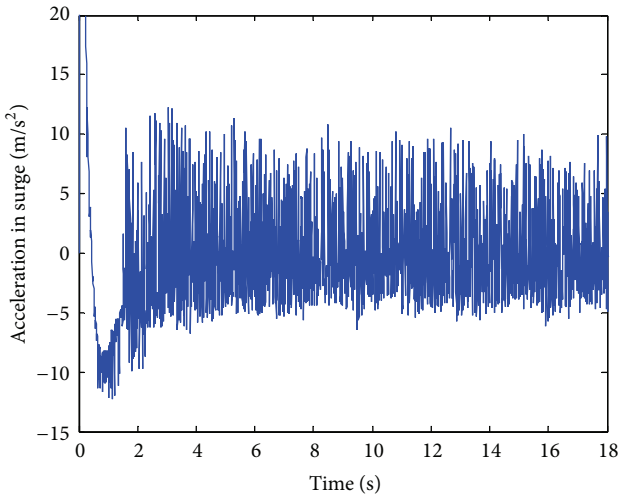

Figure 7: Acceleration of the x direction of the vessel using H_∞ controller.

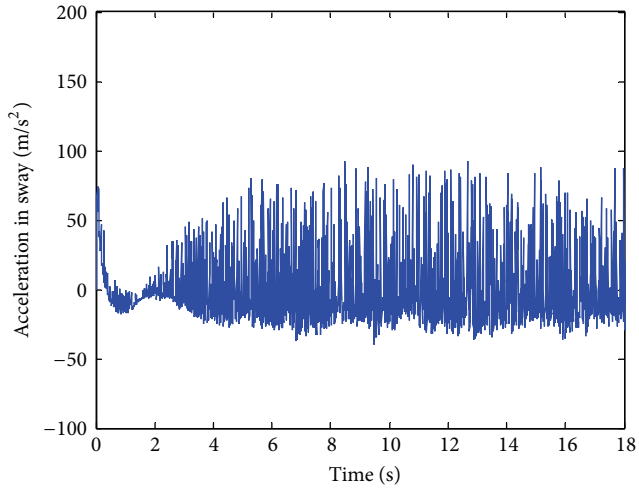

FIGURE 8: Acceleration of the y direction of the vessel using H_∞ controller.

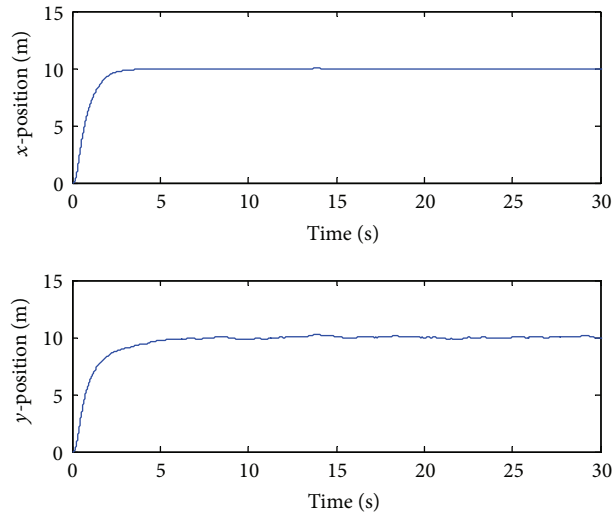

FIGURE 9: Positions of the vessel using H_∞ controller without time-delay.

response of the vessel using the PD controller without time-delay. And Figure 12 shows the position response of the vessel using the PD controller with time-delay under the same parameters sets of the controller with Figure 11. It can be seen when the time-delay is existing the PD controller is vibrating. We can see that the H_∞ controller has better delay stability than the PD controller. The parameters of the PD controller are listed as follows:

$$K_P = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 10 \\ 10 & 10 & 10 \end{bmatrix},$$
$$K_d = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}. \quad (36)$$

The results mentioned above can be seen as that the designed controller has not only the delay stability, but also the rejection capability for the external disturbance, that is, the control robustness.

5. Conclusions

In this paper, the robust H_∞ controller with time-delay was designed. The state-space model with time-delay for the ship motion was established. The designed closed-loop control law with speed and acceleration feedback enables DP controller to perform better and resist external environmental disturbances. The less conservative results were derived for the neutral system by adopting a delay-decomposition approach and LMI method. Some results for the ship DP systems were conducted to indicate that the delays exist in

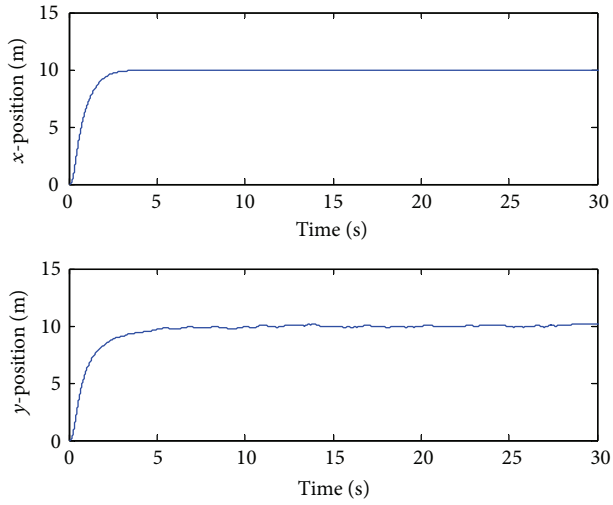

Figure 10: Positions of the vessel using H_∞ controller with time-delay.

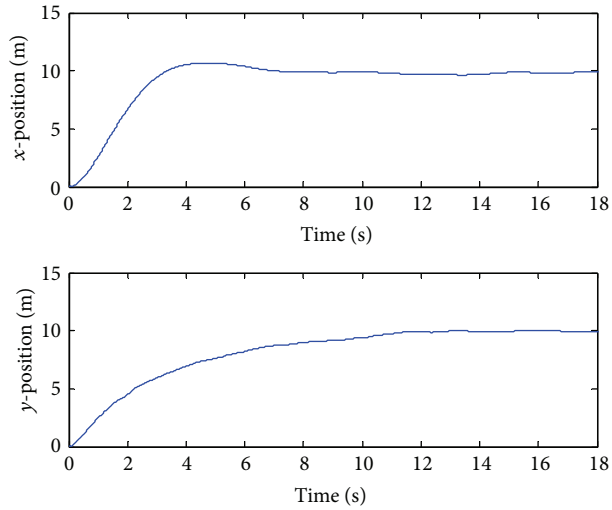

Figure 11: Positions of the vessel using the PD controller without time-delay.

the thruster system. The simulation results show that the designed controller can make the ship stable and achieve the desired position under the ocean environment (wind, waves, and currents), and the controller thereby makes the performance of the DP system better. We will study the observer-based H_∞ problem for uncertain neutral time-delay systems in our future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

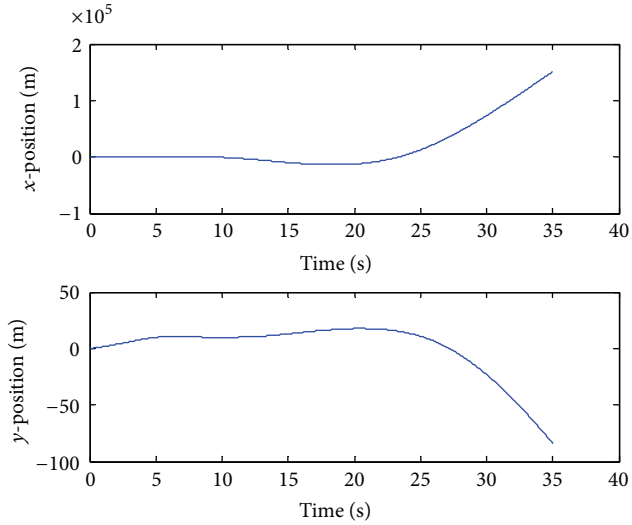

Figure 12: Positions of the vessel using the PD controller with time-delay.

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Research Article

Modified Schur-Cohn Criterion for Stability of Delayed Systems

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A modified Schur-Cohn criterion for time-delay linear time-invariant systems is derived. The classical Schur-Cohn criterion has two main drawbacks; namely, (i) the dimension of the Schur-Cohn matrix generates some round-off errors eventually resulting in a polynomial of s with erroneous coefficients and (ii) imaginary roots are very hard to detect when numerical errors creep in. In contrast to the classical Schur-Cohn criterion an alternative approach is proposed in this paper which is based on the application of triangular matrices over a polynomial ring in a similar way as in the Jury test of stability for discrete systems. The advantages of the proposed approach are that it halves the dimension of the polynomial and it only requires seeking real roots, making this modified criterion comparable to the Rekasius substitution criterion.

1. Introduction

Stability theory for linear time invariant systems with delays have long been in existence, but it still remains an active area of research, with applications in power system control, biology, human-in-the-loop control, neural networks, control in the context of automotive engines, haptic control, and synthetic biology, [1].

In this work we treat with linear time invariant time-delayed systems (LTI-TDS) with commensurate delays described by the state-space equation [2]:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - k\tau), \quad m \in \mathbb{Z}^+, \tag{1}$$

where $x \in \mathbb{R}^n$ stands for the state vector, $A_0, A_k \in \mathbb{R}^{n\times n}$ are given system matrices, and $\tau > 0$ is a delay time. For these systems the characteristic quasi-polynomial $p_c(s)$ possesses the form of

$$p_c(s,\tau) = \sum_{k=0}^{q} a_k(s)\, e^{-k\tau s}, \tag{2}$$

where q is the degree of commensuracy in the dynamics and $a_k \in \mathbb{R}[s]$ are polynomials of degree at most $n - k$.

It is well known that the analysis of the stability of LTI-TDS lies on the root continuity argument; that is, the location of the poles of p_c varies continuously with respect to delay. This means that any root crossing from the left half plane to the right half-plane will need to pass through the imaginary axis; as a result the computation of crossing frequencies is crucial when analyzing the stability and also for the plot of the root locus of p_c as τ varies [3–9].

There are five criteria in the literature to compute delay margins of general time-delayed systems: (i) Schur-Cohn (Hermite matrix formulation) [2, 10–12]; (ii) elimination of transcendental terms in the characteristic equation [13]; (iii) matrix pencil, Kronecker sum method [2, 11, 12, 14]; (iv) Kronecker multiplication and elementary transformation [15]; (v) Rekasius substitution, [4, 16, 17]. These criteria are based on the determination of all the imaginary roots of the characteristic equation and require numerical procedures which result in vast range of precisions in finding pure imaginary roots of polynomials. Some criteria show subtle inapplicabilities mainly from a numerical implementation viewpoint [18]. In particular methods (i), (iii), and (iv) lead to polynomials of degree $2q^2$, while (ii) requires finding the roots of a polynomial of degree $q2^q$. The Kronecker multiplication method generates the results with the smallest

number of significant digits among the above five methods. In the literature, the Rekasius substitution approach is the most attractive since the generated polynomial has degree q^2, much smaller than all the other criteria. Another advantage of this method is that only real roots are searched for, instead of complex ones, which is very crucial from a practical view point since complex roots are difficult to compute when numerical errors are dragged and propagated. For a total comparison of these methods, demonstrating their strengths and weakness, the reader is referred to [19].

This paper is focused on improving the Schur-Cohn procedure so as to obtain the advantageous properties of the Rekasius substitution criterion. Specifically, we modify the Schur-Cohn criterion to generate a real polynomial of degree q^2 whose real roots determine the pure imaginary roots of the characteristic equation. The original Schur-Cohn procedure computes the determinant of a partitioned matrix $A = \begin{pmatrix} \Lambda_1(s) & \Lambda_2(s) \\ \Lambda_2^*(s) & \Lambda_1^*(s) \end{pmatrix}$ where $\Lambda_1, \Lambda_2 \in \mathbb{C}[s]$ are appropriate $q \times q$ matrices over the ring of polynomials in the unknown s and Λ^* denotes the Hermitian (or conjugated complex) of Λ. The classical Schur-Cohn procedure for high dimensions ($2q^2 > 10$) becomes numerically unreliable due to repeated round-off errors in the determinant expansion procedure unless the operation is performed using very large number of significant digits. This particular point alone brings a weakness from the numerical deployment viewpoint, since the operation ultimately yields poor precision in determining the desired imaginary roots: because of the accumulated numerical errors the imaginary roots shape up in the form of $s = j\omega + \epsilon$, $\epsilon \ll 1$ and $j = \sqrt{-1}$, which leads the numerical error in real parts to be at least in the order of 10 magnitudes larger than the computational precision. Therefore, it is problematic to decide whether these roots, which are very close to the imaginary axis, are really imaginary roots or not. As an attempt to overcome the problems of the Schur-Cohn method some authors have proposed a modification based on the determination of the real eigenvalues of a constant matrix, [11]. By contrast, our modification stems from the applications of some transformations on A by using triangular matrices over the polynomial ring in a similar way as in the Jury test of stability for discrete systems [20].

The remainder of the paper is structured as follows: in Section 2, the basic idea of the modified stability criterion is explained and a numerical example is provided. Finally, a summary and an outlook are given in Section 3.

2. Delayed Systems. Schur-Cohn Modified Method

The stability of the system (1) is determined by its characteristic quasi-polynomial in s [21]:

$$p_c\left(s; e^{-s\tau}, \ldots, e^{-ms\tau}\right) = \det\left(sI - A_0 - \sum_{k=1}^{m} A_k e^{-sk\tau}\right) = \sum_{k=0}^{q} a_k(s)\, e^{-k\tau s}, \tag{3}$$

$$a_0(s) = s^n + \sum_{i=0}^{n-1} a_{0i} s^i, \qquad a_k(s) = \sum_{i=0}^{n-1} a_{ki} s^i, \quad k = 1, \ldots, q. \tag{4}$$

The characteristic quasi-polynomial contains time delays of commensurate nature with degree q; that is, p_c depends on $e^{-k\tau s}$ for $k = 0, \ldots, q$, with $a_q(s) \neq 0$ and $\operatorname{rank}(A_k) \leq q \leq n = \operatorname{rank}(A_0)$. The starting-point of all the stability criteria is the determination of the roots of p_c on the imaginary axis. In fact, it is well known from the stability theory of LTI-TDS that the pure imaginary characteristic roots are the only possible transition points from stable to unstable behavior, and vice versa. We want to determine all the imaginary roots of $p_c(s; e^{-s\tau}, \ldots, e^{-ms\tau})$, and to this end it is convenient to introduce the variable $z = e^{-s\tau}$ and write p_c as a polynomial in two unknowns over the field of reals, $p_c(s, z) = \sum_{i=0}^{q} a_k(s) z^k \in \mathbb{R}[s, z]$.

We define the bivariate polynomial $r \in \mathbb{C}[\omega, z]$ as $r(\omega, z) = \sum_{i=0}^{q} b_i(\omega) z^i$ such that $b_i(\omega) = a_i(j\omega) \in \mathbb{C}[\omega]$. With every bivariate polynomial over the complex field, $r(\omega, z)$, we associate two triangular matrices over the polynomial ring $\mathbb{C}[\omega]$:

$$U_q(r(\omega)) = \begin{pmatrix} \overline{b}_q(\omega) & \overline{b}_{q-1}(\omega) & \overline{b}_{q-2}(\omega) & \cdots & \overline{b}_1(\omega) \\ 0 & \overline{b}_q(\omega) & \overline{b}_{q-1}(\omega) & \cdots & \overline{b}_2(\omega) \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \overline{b}_q(\omega) & \overline{b}_{q-1}(\omega) \\ 0 & \cdots & 0 & 0 & \overline{b}_q(\omega) \end{pmatrix},$$

$$L_q(r(\omega)) = \begin{pmatrix} 0 & \cdots & 0 & 0 & b_0(\omega) \\ 0 & \cdots & 0 & b_0(\omega) & b_1(\omega) \\ \vdots & & & & \\ 0 & b_0(\omega) & b_1(\omega) & \cdots & b_{q-2}(\omega) \\ b_0(\omega) & b_1(\omega) & b_2(\omega) & \cdots & b_{q-1}(\omega) \end{pmatrix}. \tag{5}$$

For the sake of simplicity we write U_q, L_q instead of $U_q(r(\omega))$, $L_q(r(\omega))$. The Schur-Cohn criterion solves the problem of computing the values of ω such that $p_c(j\omega; z, \ldots, z^q) = 0$ by multiplying $p_c(j\omega; z, \ldots, z^q)$ by z^{-k} for $k = 0, \ldots, q-1$, and $\overline{p_c(j\omega; z, \ldots, z^q)}$ by z^k for $k = 1, \ldots, q$. This leads to a system of $2q$ homogeneous equations in the unknowns z^k, $k = -q, \ldots, 0, \ldots, q-1$ which has the following matrix representation:

$$\begin{pmatrix} L_q & \overline{U}_q J \\ J U_q & J \overline{L}_q J \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6}$$

where J is the counteridentity matrix (i.e. an antidiagonal matrix with ones), $\mathbf{e}_1 = J l_{q-1}^T(z)$, $\mathbf{e}_2 = J l_{q-1}^T(z^{-1}) z^{-1}$, and

$l_{q-1}^T(z) = (1, z, z^2, \ldots, z^{q-1})$; the homogeneous system in (7) is the starting point of the Schur-Cohn method. If we make the change of unknowns $\mathbf{e}_i \mapsto J\mathbf{e}_i$ and we take into account the idempotence of J, (6) can be transformed into

$$\begin{pmatrix} L_q & \overline{U}_q \\ JU_q & J\overline{L}_q \end{pmatrix} \begin{pmatrix} J\mathbf{e}_1 \\ J\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7}$$

It is easily verified that $U_q^* \overline{L}_q$ is of the same structure as L_q, and in particular it is symmetric. Therefore $U_q^* \overline{L}_q = (U_q^* \overline{L}_q)^T = L_q^* \overline{U}_q$. As a consequence we obtain

$$\begin{aligned} &\begin{pmatrix} L_q^* & -U_q^* \\ O & I \end{pmatrix} \begin{pmatrix} I & O \\ O & J \end{pmatrix} \begin{pmatrix} L_q & \overline{U}_q \\ JU_q & J\overline{L}_q \end{pmatrix} \\ &= \begin{pmatrix} L_q^* & -U_q^* \\ O & I \end{pmatrix} \begin{pmatrix} L_q & \overline{U}_q \\ U_q & \overline{L}_q \end{pmatrix} \\ &= \begin{pmatrix} L_q^* L_q - U_q^* U_q & O \\ U_q & \overline{L}_q \end{pmatrix}. \end{aligned} \tag{8}$$

The homogeneous system (7) has a nontrivial solution so long as $\det \begin{pmatrix} L_q & \overline{U}_q \\ JU_q & J\overline{L}_q \end{pmatrix} = 0$. We use the fact that

$$\begin{aligned} &\det\left(\begin{pmatrix} L_q^* & -U_q^* \\ O & I \end{pmatrix} \begin{pmatrix} I & O \\ O & J \end{pmatrix}\right) \\ &= \det \begin{pmatrix} L_q^* & -U_q^* J \\ O & J \end{pmatrix} = \det\left(L_q^* J\right) = \det\left(\overline{L}_q\right)(-1)^q, \end{aligned} \tag{9}$$

and taking determinants in (8) yields

$$\begin{aligned} &\det\left(\overline{L}_q\right)(-1)^q \det \begin{pmatrix} L_q & \overline{U}_q \\ JU_q & J\overline{L}_q \end{pmatrix} \\ &= \det\left(\overline{L}_q\right) \det\left(L_q^* L_q - U_q^* U_q\right). \end{aligned} \tag{10}$$

As a result,

$$\det \begin{pmatrix} L_q & \overline{U}_q \\ JU_q & J\overline{L}_q \end{pmatrix} = (-1)^q \det\left(L_q^* L_q - U_q^* U_q\right). \tag{11}$$

Therefore, the problem has been reduced to find the pure imaginary roots of $\det(L_q^* L_q - U_q^* U_q)$ which is just the determinant of the Jury matrix $J_q(r(\omega))$ over $\mathbb{C}[\omega]$ defined as

$$J_q(r(\omega)) = \begin{pmatrix} L_q & U_q \\ \overline{U}_q & \overline{L}_q \end{pmatrix} \in \mathbb{C}^{2q \times 2q}[\omega]. \tag{12}$$

This can be seen by a similar argument as above in (8),

$$\begin{aligned} &\begin{pmatrix} L_q^* & -U_q^* \\ O & I \end{pmatrix} \begin{pmatrix} L_q & \overline{U}_q \\ U_q & \overline{L}_q \end{pmatrix} \\ &= \begin{pmatrix} L_q^* L_q - U_q^* U_q & L_q^* \overline{U}_q - U_q^* \overline{L}_q \\ U_q & \overline{L}_q \end{pmatrix} \\ &= \begin{pmatrix} L_q^* L_q - U_q^* U_q & O \\ U_q & \overline{L}_q \end{pmatrix}, \end{aligned} \tag{13}$$

and $\det(J_q(r(\omega))) = \det(L_q^* L_q - U_q^* U_q)$.

In the sequel we define the polynomial $\zeta(\omega) := \det(J_q(r(\omega)))$ and the set of zeros of a polynomial p is denoted as $\mathcal{Z}(p)$.

Proposition 1. *The real polynomial $\zeta(\omega)$ is even.*

Proof. We prove that $\zeta(\lambda) = 0$ if and only if $\zeta(-\lambda) = 0$. First of all note that $b_i(-\omega) = \overline{b_i(\omega)}$ and, then, $r(-\omega, z) = \overline{r}(\omega, z)$. As a consequence,

$$\begin{aligned} U_q(\overline{r}(\omega)) &= \overline{U}_q(r(\omega)) = U_q(r(-\omega)), \\ L_q(\overline{r}(\omega)) &= \overline{L}_q(r(\omega)) = L_q(r(-\omega)). \end{aligned} \tag{14}$$

Thus,

$$\begin{aligned} &L_q^*(r(-\omega)) L_q(r(-\omega)) - U_q^*(r(-\omega)) U_q(r(-\omega)) \\ &\quad = L_q^T(r(\omega)) \overline{L}_q(r(\omega)) - U_q^T(r(\omega)) \overline{U}_q(r(\omega)), \end{aligned} \tag{15}$$

and $\zeta(\lambda) = \zeta(-\lambda)$. □

From the above development we obtain the following criterion.

Criterion 1. Let $\omega_0 \in \mathbb{R}^+$ be a real root of $\zeta(\sqrt{s})$, then $\pm j\sqrt{\omega_0}$ is a pair of pure imaginary roots of the LTI-TDS system in (1).

Proof. It is immediate that $\pm j\omega$ is a pair of pure imaginary roots of $p_c(s; z, \ldots, z^q)$ if and only if ω is a positive real root of $p_c(j\omega; z, \ldots, z^q)$. According to the Schur-Cohn method the roots of $p_c(j\omega; z, \ldots, z^q)$ can be determined by searching for the real roots in the system (7). Additionally, it was shown above that the solution of (7) implies the computing of the roots of the even polynomial $\zeta(\omega)$; henceforth, if $\omega_0 \in \mathbb{R}^+$ is a root of $\zeta(\sqrt{s})$, $\pm\sqrt{\omega_0}$ is a pair of real roots of $\zeta(s)$ and $\pm j\omega_0$ is a pair of pure imaginary roots of $p_c(s; z, \ldots, z^q)$. □

Note that this criterion is advantageous since firstly we only need to compute real roots and secondly $\deg(\zeta(\sqrt{\cdot})) = q$. In the following the set of crossing frequencies is defined as

$$\Omega := \left\{\omega \in \mathbb{R}^+ : \omega^2 \in \mathcal{Z}\left(\zeta\left(\sqrt{\cdot}\right)\right) \cap \mathbb{R}^+\right\}. \tag{16}$$

Example 2. Let us consider the LTI-TDS system $\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau)$ where the system matrices are defined as

$$A_0 = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix}. \tag{17}$$

The quasi-polynomial characteristic is

$$\begin{aligned} &p_c\left(s; e^{-s\tau}, \ldots, e^{-ms\tau}\right) \\ &\quad = a_3(s) e^{-3\tau s} + a_2(s) e^{-2\tau s} + a_1(s) e^{-\tau s} + a_0(s) \end{aligned} \tag{18}$$

with

$$\begin{aligned} a_0(s) &= s^3 + 6s^2 + 45.5s + 111 \\ a_1(s) &= 0.9s^2 - 116.8s - 22.1 \\ a_2(s) &= 90.9s - 185.1 \\ a_3(s) &= 119.4. \end{aligned} \tag{19}$$

In the frequency domain ($s = j\omega$), we define the bivariate polynomial $r(\omega, z) = \sum_{i=0}^{3} a_i(j\omega)z^i$, and from (19) we build the triangular matrices $U_3(r)$ and $L_3(r)$ as follows:

$$\begin{aligned} U_3 &= \begin{pmatrix} 119.4 & -185.1 & -0.9\omega^2 - 22.1 \\ 0 & 119.4 & -185.1 \\ 0 & 0 & 119.4 \end{pmatrix} \\ &\quad + j\omega \begin{pmatrix} 0 & -90.9 & 116.8 \\ 0 & 0 & -90.9 \\ 0 & 0 & 0 \end{pmatrix} \\ L_3 &= \begin{pmatrix} 0 & 0 & 111.0 - 6.0\omega^2 \\ 0 & 111.0 - 6.0\omega^2 & -0.9\omega^2 - 22.1 \\ 111.0 - 6.0\omega^2 & -0.9\omega^2 - 22.1 & -185.1 \end{pmatrix} \\ &\quad + j\omega \begin{pmatrix} 0 & 0 & 45.5 - \omega^2 \\ 0 & 45.5 - \omega^2 & -116.8 \\ 45.5 - \omega^2 & -116.8 & 90.9 \end{pmatrix}. \end{aligned} \tag{20}$$

Then we compute the polynomial $\zeta(\sqrt{s})$ according to Criterion 1:

$$\begin{aligned} \zeta(\sqrt{s}) &= s^9 - 165.81s^8 - 18682s^7 + 2.4869 \times 10^5 s^6 \\ &\quad + 35185s^5 - 4.1197 \times 10^9 s^4 + 9.4110 \times 10^{10} s^3 \\ &\quad - 7.1137 \times 10^{11} s^2 + 1.8933 \times 10^{12} s \\ &\quad - 1.0144 \times 10^{12}, \end{aligned} \tag{21}$$

whose positive real roots are $\mathcal{Z}(\zeta(\sqrt{\cdot})) \cap \mathbb{R}^+ = \{8.4802, 4.4564, 9.2141, 0.70634, 240.35\}$ and then

$$\pm j\Omega = \{\pm j2.9121, \pm j2.111, \pm j3.0355, \pm j0.84044, \pm j15.503\} \tag{22}$$

is the set of pure imaginary roots of the delayed system. Note that from the fundamental theorem of algebra there are only a finite number of crossing frequencies and that $\mathcal{Z}(\zeta(\sqrt{\cdot})) \cap \mathbb{R}^+$ represents the set of squares of crossing frequencies.

The method presented above is the first step to test the stability of the characteristic quasi-polynomial $p_c(s, e^{-s\tau})$ and to compute the delay margin for LTI delay systems. It is common practice to make the following assumption on p_c.

Assumption 3. The characteristic quasi-polynomial p_c is stable at $\tau = 0$, that is, $\mathcal{Z}(p_c(\cdot; 0)) \subset \mathbb{C}^-$, where $\mathbb{C}^- := \{s \in \mathbb{C} : \mathrm{Re}\{s\} < 0\}$.

Recall that the delay margin $\overline{\tau}$ is the smallest deviation of τ from $\tau = 0$ such that the system becomes unstable:

$$\overline{\tau} := \min\left\{\tau \geq 0 : p_c\left(j\omega, e^{-j\omega\tau}\right) = 0 \text{ for some } \omega \in \mathbb{R}^+\right\}. \tag{23}$$

For any $\tau \in [0, \overline{\tau})$, the system is stable, and whenever $\tau = \infty$, the system is said to be stable independent of delay. For each finite delay τ we can associate a crossing frequency which represents the first contact or crossing of the roots of p_c from the stable region to the unstable one. The determination of the delay margin requires solving the bivariate polynomial $p_c \in \mathbb{C}[\omega, z]$ in two phases: (i) computing the positive real roots of $\zeta(\sqrt{s})$ so as to obtain the crossing frequencies $\omega \in \Omega$ of p_c, (ii) finding the roots of $p_c(j\omega) \in \mathbb{C}[z]$ on the boundary of the unit disk $\mathbb{D}$ (denoted as $\partial\mathbb{D}$), at each crossing frequency ω. A detailed analysis of the delay margin $\overline{\tau}$ can be found in [11] and in Theorem 2.11, pp. 59-60 in [2]. Under the assumption that p_c is stable at $\tau = 0$, the two-step stability method is summarized in Table 1.

Example 4. Taking into account the positive crossing frequencies $\omega_k \in \Omega$ computed for the LTI-TDS system in Example 2, we can derive the set of roots of $p_c(j\omega_k) \in \mathbb{C}[z]$ onto the unit circle $\partial\mathbb{D}$. Let $z_k \in \partial\mathbb{D} \cap \mathcal{Z}(p_c(j\omega_k))$, from the identity $z_k = e^{-j\omega_k\tau} = e^{j\arg(z_k)}$ it follows that $\tau_k = (2\pi - \arg(z_k))/\omega_k$ (with $\arg(z_k) \in [0, 2\pi]$) and the system can cross the stability boundary $\partial\mathbb{C} = j\mathbb{R}$ from the stable half-plane $\mathbb{C}^-$ to the unstable half-plane $\mathbb{C}^+ := \mathbb{C} \setminus (\mathbb{C}^- \cup \partial\mathbb{C})$ or vice versa at τ_k. If the root crosses $\partial\mathbb{C}$ from $\mathbb{C}^+$ to $\mathbb{C}^-$ at τ_k, there exists a $\tau < \tau_k$ such that makes the root to cross from $\mathbb{C}^-$ to $\mathbb{C}^+$. As a consequence the stability margin is defined as

$$\overline{\tau} = \min_{1 \leq k \leq q}\left\{\frac{2\pi - \arg(z_k)}{\omega_k} : \omega_k \in \Omega, z_k \in \partial\mathbb{D} \cap \mathcal{Z}\left(p_c\left(j\omega_k\right)\right)\right\}. \tag{24}$$

Under Assumption 3 the system is unstable at $\tau = \overline{\tau}$ and at least a root crosses $\partial\mathbb{C}$ from $\mathbb{C}^-$ to $\mathbb{C}^+$ at $\overline{\tau}$. If we choose $\tau \in [0, \overline{\tau})$, then $\tau\omega_k \neq \theta_k$ and $p_c(j\omega_k; e^{-j\omega\tau}) \neq 0$ for all $\omega \in \mathbb{R}^+$. Therefore, $p_c(s; e^{-s\tau})$ is stable for all $\tau \in [0, \overline{\tau})$. For each crossing frequency ω_k a delay τ_k is computed as shown in Table 2. The delay margin is then computed as the minimum value of these delays, that is, $\overline{\tau} = \min\{\tau_k : k = 1, \ldots, 5\} = 0.16230$.

The crossing direction of those roots onto the imaginary axis (either from stable to unstable complex plane or vice versa) as τ increases is a quantity called the root tendency (RT), [7, 8]. For a cross frequency ω_0 and a delay τ_0, the root tendency is defined as

$$(\mathrm{RT})_{\omega,\tau} = \mathrm{sgn}\left(\mathrm{Re}\left\{\left.\frac{\partial s}{\partial \tau}\right|_{(j\omega_0, \tau_0)}\right\}\right), \tag{25}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function. In the following lemma we propose a simple method to compute the root tendency based on the implicit function theorem.

Lemma 5 (root tendency). *Let p_c be a characteristic quasi-polynomial as defined in* (3) *and let τ_0 be the delay for*

Table 1: Stability analysis of the quasipolynomial p_c in terms of the delay margin $\overline{\tau}$.

Set of squares of crossing frequencies	Solutions of $p_c(\omega) \in \mathbb{C}[z]$ on $\partial\mathbb{D}$, $\omega \in \Omega$	Stability of the quasi-polynomial $p_c \in \mathbb{C}[\omega, z]$
$\mathcal{Z}(\zeta(\sqrt{\cdot})) \cap \mathbb{R}^+ = \varnothing$		
$\mathcal{Z}(\zeta(\sqrt{\cdot})) \cap \mathbb{R}^+ = (0)$		Stable, independent of delay ($\overline{\tau} = \infty$)
$\mathcal{Z}(\zeta(\sqrt{\cdot})) \cap \mathbb{R}^+ \neq \varnothing$, $\mathcal{Z}(q) \cap \mathbb{R}^+ \neq (0)$	$\mathcal{Z}(p_c(j\omega)) \cap \partial\mathbb{D} = \varnothing$	
	$\mathcal{Z}(p_c(j\omega)) \cap \partial\mathbb{D} \neq \varnothing$	Stable, dependent of delay ($\overline{\tau} < \infty$)

Table 2: Delays corresponding to the crossing frequencies of the LTI-TDS system of the Example 2.

Crossing frequency (ω_k)	Roots of $p_c(j\omega_k)$ on $\partial\mathbb{D}$ ($z_k = e^{-j\omega_k\tau}$)	Phase of $z_k(\theta_k)$	Delay (τ_k)
2.9121	$0.85692 - j0.51548$	5.7416	0.18597
2.111	$-0.26768 - j0.96352$	4.4414	0.87247
3.0355	$0.88110 - j0.47297$	5.7905	0.16230
0.84044	$0.97521 + j0.22129$	0.22314	7.2106
15.503	$-0.95537 + j0.29541$	2.8417	0.22199

Table 3: Root tendency for the crossing frequencies of the LTI-TDS system of the Example 2.

Crossing frequency (ω_k)	Delay (τ_k)	$\sum_{k=0}^{q}(a_k'(j\omega_0) - (k-1)\tau_0 a_k(j\omega_0))e^{-jk\omega_0\tau_0}$	$\sum_{k=1}^{q} k a_k(j\omega_0)e^{-jk\omega_0\tau_0}$	$(RT)_{\omega,\tau}$
2.9121	0.18597	$-48.520 + j30.258$	$74.136 - j58.701$	-1
2.111	0.87247	$-140.62 + j342.86$	$148.78 - j182.37$	$+1$
3.0355	0.16230	$-47.511 + j27.405$	$94.101 - j41.172$	$+1$
0.84044	7.2106	$872.31 - j692.90$	$-119.43 + j99.556$	-1
15.503	0.22199	$-902.35 - j919.93$	$1825.5 + j4475.4$	$+1$

which the system crosses the stability boundary at $j\omega_0$, for some $\omega_0 \in \Omega$, that is, $p_c(j\omega_0; \tau_0) = 0$. Let us assume that $(\partial p_c(s;\tau)/\partial s)|_{(j\omega_0,\tau_0)} = \sum_{k=0}^{q}(a_k'(j\omega_0) - (k-1)\tau a_k(\omega_0))$ $e^{-j\omega_0 k\tau_0} \neq 0$. Then, the root tendency of $j\omega_0$ is given by

$$
\begin{aligned}
&(\mathrm{RT})_{\omega_0,\tau_0} \\
&\quad = -\operatorname{sgn}\left(\operatorname{Im}\left(\left(\sum_{k=1}^{q} k a_k(j\omega_0)\, e^{-jk\omega_0\tau_0}\right)\right.\right. \\
&\qquad \times \left(\sum_{k=0}^{q}\left(a_k'(j\omega_0) - (k-1)\,\tau_0 a_k(j\omega_0)\right)\right. \\
&\qquad \left.\left.\left. \times e^{-jk\omega_0\tau_0}\right)^{-1}\right)\right).
\end{aligned}
\tag{26}
$$

Proof. In virtue of the implicit function theorem, there exist a $\delta > 0$, a neighborhood $U_0 \subset \mathbb{C}$ of $j\omega_0$, and a continuous function $s : (\tau_0 - \delta, \tau_0 + \delta) \to U_0$ such that $p_c(s(\tau); \tau) = 0$ for all $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$; this is also obvious from the continuity of the roots $s(\cdot)$ with respect to the delay τ. Furthermore, since p_c is differentiable and $(\partial p_c(s;\tau)/\partial s)|_{(j\omega_0,\tau_0)} \neq 0$, it is clear that

$$
\left.\frac{\partial s}{\partial \tau}\right|_{(j\omega_0,\tau_0)} = -\frac{\partial p_c(s;\tau)/\partial\tau|_{(j\omega_0,\tau_0)}}{\partial p_c(s;\tau)/\partial s|_{(j\omega_0,\tau_0)}},
\tag{27}
$$

where

$$
\begin{aligned}
&\left.\frac{\partial p_c(s;\tau)}{\partial s}\right|_{(j\omega_0,\tau_0)} \\
&\quad = \sum_{k=0}^{q}\left(a_k'(j\omega_0) - (k-1)\,\tau_0 a_k(j\omega_0)\right) e^{-jk\omega_0\tau_0} \\
&\qquad + p_c(j\omega_0;\tau_0), \\
&\left.\frac{\partial p_c(s;\tau)}{\partial \tau}\right|_{(j\omega_0,\tau_0)} = -j\omega_0 \sum_{k=1}^{q} k a_k(j\omega_0)\, e^{-jk\omega_0\tau_0}.
\end{aligned}
\tag{28}
$$

Owing to $p_c(j\omega_0; \tau_0) = 0$ and from the chain rule of derivation, the result in (26) follows immediately. □

Example 6. Following with Example 2 and using the root tendency lemma we can compute the crossing direction for each crossing frequency as shown in the fifth column in Table 3; RT $= -1$ indicates that the crossing is from $\mathbb{C}^+$ to $\mathbb{C}^-$ and RT = +1 implies a crossing from $\mathbb{C}^-$ to $\mathbb{C}^+$.

We finish this analysis by noting that it is not possible in general to separate $\zeta(\sqrt{s})$ with a factor containing exactly all the real roots of $\zeta(\sqrt{s})$ as stated in the following lemma.

Lemma 7. *Given a polynomial $p \in \mathbb{R}_n[s]$ there does not exist a general procedure of factorization of p in polynomials $p_R \in \mathbb{R}[s]$ and $p_C \in \mathbb{R}[s]$, $p(s) = p_R(s)p_C(s)$, such that p_R includes exactly all the real roots of p (and p_C all the pairs of complex conjugated roots of p not in $\mathbb{R}$).*

Proof. We prove this assertion by a counterexample: without loss of generality we assume that p is monic. Let us assume that p has only real roots which are ordered as $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, $m \leq n$. Always we can find a value $\delta \in \mathbb{R}$ such that $\alpha_1 < \delta < \alpha_2$. We make the change of variable $s \mapsto s + \delta$ resulting into a polynomial $p_\delta(s) = p(s + \delta)$. The roots of p_δ are located at $\alpha'_k = \alpha_k - \delta$ and it is easy to see that $\alpha'_1 < 0$ and $\alpha'_k > 0$ for $k = 2, \ldots, m$. Then we apply the map $s \mapsto s^2$ which produces a polynomial $q(s) = p_\delta(s^2)$ with a pair of pure conjugated complex roots at $\pm\mathbf{i}\sqrt{\delta - a_1}$ and pairs of symmetric real roots at $\pm\sqrt{a_k - \delta}$ for $k = 2, \ldots, m$. Under the hypothesis of separability, $q(s) = q_R(s)q_C(s)$ and, undoing in reverse order the changes of variables, $s \mapsto \sqrt{s} \mapsto s + \delta$, we obtain that $q_R(\sqrt{s - \delta}) = (s - \alpha_1)$ and $q_C(\sqrt{s - \delta}) = p(s)/(s - \alpha_1)$. Therefore, we might isolate roots of p by successively repeating the procedure onto q_C. This is a contradiction since from the Galois theory it is well known that it is not always possible to solve a polynomial equation by radicals. □

3. Conclusions

We overcome the drawbacks of the original Schur-Cohn criterion for delay systems, namely, the high degree of the polynomial that emerges from the Schur-Cohn matrix and the difficulty of detection of complex roots, especially pure imaginary roots, due to the propagation of numerical errors. In particular, we decrease the degree of the generated polynomial to a half and we reduce the problem numerically to that of seeking the real roots of this polynomial, not the complex ones. These properties are exactly those that make the Rekasius substitution criterion so attractive in the stability theory of delay systems. Therefore, from a practical viewpoint, the modified Schur-Cohn criterion is comparable to other widely used criteria in the literature. Our method is based on making transformations on the Schur-Cohn matrix via triangular matrices over a polynomial ring in one unknown. This leads to the determinant of the Jury matrix over a polynomial ring, which is proved to be an even polynomial. The pure imaginary roots of the Schur-Cohn polynomial are those of the even polynomial, and in virtue of the change of variable $\omega \mapsto \sqrt{\omega}$, we only need to seek real roots, which is more precise from a numerical viewpoint, while the degree of the polynomial is halved.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Observer Design for One-Sided Lipschitz Nonlinear Systems Subject to Measurement Delays

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This paper presents a novel nonlinear observer-design approach to one-sided Lipschitz nonlinear systems in the presence of output delays. The crux of the approach is to overcome the practical consequences of time delays, encountered due to distant sensor position and time lag in measurement, for estimation of physical and engineering nonlinear system states. A Lyapunov-Krasovskii functional is employed, the time derivative of which is solved using Jensen's inequality, one-sided Lipschitz condition, and quadratic inner-boundedness, and, accordingly, design conditions for delay-range-dependent nonlinear observer for delayed one-sided Lipschitz systems are derived. Further, novel solutions to the problems of delay-dependent observer synthesis of one-sided Lipschitz models and delay-range-dependent state estimation of linear and Lipschitz nonlinear systems are deduced from the present delay-range-dependent technique. An observer formulation methodology for retrieval of one-sided Lipschitz nonlinear-system states, which is robust against L_2 norm-bounded perturbations, is devised. The resultant design conditions, in contrast to the conventional procedures, can be solved via less conservative linear matrix inequality- (LMI-) based routines that succeed by virtue of additional LMI variables, meaningful transformations, and cone complementary linearization algorithm. Numerical examples are worked out to illustrate the effectiveness of the proposed observer-synthesis approach for delayed one-sided Lipschitz systems.

1. Introduction

State estimation using an observer is a methodology widely employed in physical, biomedical, and engineering fields owing to its multitude of applications in road-gradient and vehicle-mass estimation, coestimation for lithium-polymer battery cells, online monitoring of nonlinear bioprocesses, identification and analysis of vascular tumor growth, detection and reconstruction of sensor faults, robust control of stochastic systems under disturbances, and cylinder-pressure reconstruction [1–7]. Observer synthesis for nonlinear systems has received considerable attention within the control field over the past few decades, as it makes possible the application of state estimation to control design, energy system analysis, fault diagnosis, chaos-based secure communications, synchronization studies, and unknown input recovery [8–11]. Notable work in this regard has been concentrated on continuous-time systems, while a certain quantity of research has been devoted to discrete-time and time-delay dynamical models [12–15]. For the nonlinear systems, the observer design remains a challenging research problem: no generalized or widely applicable solution has yet been reported or even explored. Nonlinear observer design in the presence of time delays is thought-provoking, particularly as sensor technologies, conditioning units, and measurement systems often introduce unavoidable time delays that can sabotage the practicality of a monitoring or a control system [16–18].

In the literature, two broad and widely employed methodologies for observer design of nonlinear systems are nonlinear state transformation, for which the state-estimation error dynamics are transformed into linear ones [19, 20], and the direct method, based on the original system, by which the estimation error dynamics are obtained in nonlinear form [21–23]. Thus far, for the class of nonlinear systems observing the Lipschitz condition (see, e.g., [24] and references therein),

multiple observer designs based on the direct method have been presented to address the existence condition for full-order and reduced-order observers, robustness in observer design subject to disturbances, robust sensor fault reconstruction, and observer synthesis for discrete-time systems [25–28]. The Lipschitz condition in numerical analysis and mathematics, found to be conservative and region based, is now being replaced by a more spacious and less conservative one-sided Lipschitz condition. Observer-design schemes, based on characteristics of Lipschitz functions, ensure stability of state-estimation error only for small values of Lipschitz constants and result into infeasibility if the aforesaid constant is large. These facts escalate the demand of the one-sided Lipschitz constant for approximating an upper bound on nonlinear component of a dynamical system to accomplish viable estimation of the full state vector.

In recent years, several observer-design problems for one-sided Lipschitz nonlinear systems have been investigated [29–32]. For example, a state observer-design scheme for discrete-time systems with mathematical artifacts on the Lyapunov function for obtainment of simple linear matrix inequality (LMI) conditions for asymptotic stability of state-estimation error was carried out [29]. Full-order and reduced-order observer designs for one-sided Lipschitz systems using the Riccati equation approach demonstrating less conservatism than the Lipschitz counterpart also have been studied [30]. Further, a methodology applicable to monotonic and Lipschitz as well as one-sided Lipschitz nonlinearities was presented, by which the observer gain matrix is determined by solving LMIs [31]. In another approach, the analysis and deduction problem in a unified LMI framework, which provides the condition for existence of a nonlinear state observer, is addressed by incorporating the concept of quadratic inner-boundedness [32]. Still, however, for delayed one-sided Lipschitz nonlinear systems, asymptotic stability conditions to the observer-design dilemma remain elusive owing to the twofold involvement of time delays and one-sided Lipschitz dynamics. In this regard, the presence of output delays in practical systems, unavoidable due to the distant sensor position, digital processor computations, and measurement system processing, can result in oscillations, lags, and even instability, making the traditional observer-design approaches like [29–32] infeasible for state estimation. Moreover, the one-sided Lipschitz constant is either significantly smaller than or at most equal to the traditional Lipschitz constant, which fact facilitates a more suitable observer construction of nonlinear time-delay systems by reckoning the influence of nonlinear and delayed nonlinear parts. If observer-design techniques can be developed for delayed one-sided Lipschitz systems, these schemes can be effectively utilized or reformulated for monitoring and control of complex forms of engineering systems.

Time delays, varying in an interval [33] and appearing in state, input, and output variables as well as in state derivatives, are frequently encountered in engineering and physical systems [33–38]. In recent years, the stability analysis and control design problems were investigated for time-variant and time-invariant delayed systems (see, for instance, reference [39, 40]). Delay-dependent stability approach assuming that the time delay belongs to an interval from zero to a constant value provides less conservative results than does the conventional delay-independent scheme [41]. Nevertheless, because the Lyapunov function ignores the lower bound of the time delay, conservatism remains; therefore, the lower bound should be incorporated to establish less restricted results. Recently, delay-range-dependent techniques addressing the problem of conservatism have been developed (see, e.g., [33, 42–45]) for linear time-delay systems, based on various Lyapunov-Krasovskii (LK) approaches, Jensen's inequality, the free-weighting matrix, Newton Leibniz, and others; however, the field of delay-range-dependent observer synthesis for linear and, particularly, for nonlinear time-delay systems is still relatively immature. The beauty of such delay-range-dependent state-estimation methodologies lies in their applicability to systems with either large or small delays; therefore, it will be interesting to explore a state-of-the-art delay-range-dependent observer-design strategy for the one-sided Lipschitz nonlinear systems with measurement delays.

Motivated by the aforementioned linear delay-range-dependent approaches and one-sided Lipschitz nonlinear observer construction methodologies, the present study explores the problem of a nonlinear observer design for one-sided Lipschitz nonlinear systems under output-measurement and processing delays. By application of an LK functional, the time derivative of which is solved using Jensen's inequality, incorporating the one-sided Lipschitz condition and quadratic inner-boundedness, exploiting the standard matrix inequality procedures and regarding the nonzero lower bound of the output delay, a delay-range-dependent Luenberger-type observer-synthesis scheme ensuring asymptotic estimation of the states of delayed one-sided Lipschitz nonlinear plants was established. To the best of the authors' knowledge, the proposed observer-design scheme enabling time-delay one-sided Lipschitz nonlinear systems to overcome the practical limitation of sensors and measurement systems is herein introduced for the first time. The more general feature of the proposed one-sided Lipschitz observer-synthesis treatment relative to the Lipschitz one for the case of output delays is addressed by establishing a relation, and no such delay-range-dependent results for observer formulation using the Lipschitz condition have yet been effectively reported. Furthermore, a delay-dependent approach is deduced as a particular case of the present delay-range-dependent observer-design methodology, by taking the zero value of the lower bound on the time delay. A delay-rate-independent condition, owing to its importance in the field of full state vector estimation, is provided by application of the proposed LK functional treatment.

The scope of the resultant observer-design study for nonlinear interval time-delay systems is extended to fast time-varying delays, and the corresponding conditions are detailed. In addition, robust observer-synthesis scheme in the presence of output delays is provided against L_2 norm-bounded perturbations and disturbances to guarantee L_2 gain reduction from unwanted signals to the state-estimation error. Another contribution of the present work is the formulation of a less conservative LMI-based routine compared with the delay-range-dependent approaches like [46],

obtained by introducing additional LMI variables and meaningful transformations, solvable via a cone complementary linearization algorithm. Two numerical simulation examples are provided to illustrate the effectiveness of the proposed observer-design methodology for delayed one-sided Lipschitz nonlinear systems.

The remainder of the paper is organized as follows. In Section 2, the problem is formulated and some important concepts are introduced. Section 3 presents the main design conditions for the delay-range-dependence-oriented nonlinear dynamical observer strategies and, further, discusses their various modifications. Sections 4 and 5 provide simulations and concluding remarks, respectively.

Standard notation is used throughout the paper. The Euclidean norm of a vector x is shown by $\|x\|$, and the L_2 norm of the vector by $\|x\|_2 = \sqrt{\int_0^\infty \|x\|^2 dt}$. For vectors x and y of the same dimension, the notation $\langle x, y\rangle$ represents the inner product of the vectors. The mathematical quantity $\sup_{\|d\|_2\neq 0}(\|e\|_2/\|d\|_2)$ defines the L_2 gain for a system with an input vector d and an output vector e. A symmetric positive (or semipositive) definite matrix X is stated by matrix inequality $X > 0$ (or $X \geq 0$). The transpose of a matrix A is noted as A^T. The notation $\mathrm{diag}(x_1, x_2, \ldots, x_n)$ denotes a block diagonal matrix with entry x_i, for $i = 1, 2, \ldots, n$, at the corresponding diagonal entry.

2. System Description

Consider a class of one-sided Lipschitz nonlinear dynamical systems with time-varying output delays, given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x,u) + d, \\ y(t) &= Cx(t-\tau), \end{aligned} \tag{1}$$

where $x \in \mathfrak{R}^n$, $u \in \mathfrak{R}^m$, $y \in \mathfrak{R}^p$, and $d \in \mathfrak{R}^n$ are the state vector, the control input, the output, and the disturbance to the system, respectively. The linear constant matrices of the dynamical system are represented by A and C, and the nonlinear function is denoted by $f(x,u) \in \mathfrak{R}^n$. A continuous time-varying differentiable function τ refers to the time delay at the output, satisfying

$$\begin{aligned} 0 \leq h_1 \leq \tau \leq h_2, \\ \dot{\tau} \leq \mu. \end{aligned} \tag{2}$$

The function $f(x,u)$ belongs to the one-sided Lipschitz nonlinearities owing to the definition given below (see, for instance, [29–32]). Another concept employed for the observer design is quadratic inner-boundedness (see [30, 32, 47]).

Definition 1. A nonlinear function $f(x,u)$ is said to be one-sided Lipschitz in a region D enclosing the origin if there exists a scalar $\rho \in \mathfrak{R}$ such that the relation

$$\langle f(x,u) - f(\overline{x},u), x - \overline{x}\rangle \leq \rho\|x-\overline{x}\|^2, \tag{3}$$

holds $\forall x, \overline{x} \in D$, where ρ is the one-sided Lipschitz constant.

Definition 2. A nonlinear function $f(x,u)$ is said to satisfy the quadratic inner-boundedness condition in a defined region D, if there exist scalars $\beta, \alpha \in \mathfrak{R}$, such that

$$\begin{aligned} &(f(x,u) - f(\overline{x},u))^T (f(x,u) - f(\overline{x},u)) \\ &\quad \leq \beta\|x-\overline{x}\|^2 + \alpha\langle x-\overline{x}, f(x,u) - f(\overline{x},u)\rangle \end{aligned} \tag{4}$$

is satisfied for all $x, \overline{x} \in D$.

The one-sided Lipschitz and quadratic inner-boundedness conditions extrapolate the definitive Lipschitz theory to a more ecumenical category of nonlinear systems and have inbuilt advantages in observer synthesis. For a given function $f(x,u)$ satisfying (3)-(4), the Lipschitz condition holds, whereas the reverse is not true (see details in [30, 32, 47]). Further, the one-sided Lipschitz constant ρ and the quadratic inner-boundedness parameter β can be any real numbers, unlike the Lipschitz constant, which needs to be always positive.

The aim of the present study is to propose an observer-design methodology for a dynamic one-sided Lipschitz nonlinear system (1) subject to time-varying output delays belonging to an interval.

3. Observer Design

Consider a Luenberger-like observer for a delayed one-sided Lipschitz nonlinear system (1) formulated as

$$\begin{aligned} \dot{\widehat{x}}(t) &= A\widehat{x}(t) + f(\widehat{x},u) + L(y(t) - \widehat{y}(t)), \\ \widehat{y}(t) &= C\widehat{x}(t-\tau), \end{aligned} \tag{5}$$

where $L \in \mathfrak{R}^{n\times m}$ is the observer gain matrix. The state-estimation error is given by

$$e = x - \widehat{x}. \tag{6}$$

From (1) and (5)-(6), we have the error dynamics:

$$\dot{e}(t) = Ae(t) + f(x,u) - f(\widehat{x},u) - L(y(t) - \widehat{y}(t)) + d, \tag{7}$$

which reduce further to

$$\dot{e}(t) = Ae(t) + \Phi(x,\widehat{x},u) - LCe(t-\tau) + d, \tag{8}$$

by substitution of

$$\Phi(x,\widehat{x},u) = f(x,u) - f(\widehat{x},u). \tag{9}$$

Now, we provide an LMI-based sufficient condition to test the state-estimation ability of an observer (5) for a given observer gain matrix L. Note that a sophisticated guess of the observer gain matrix can be obtained using the traditional observer-design methodologies such as the standard high-gain-observer approach and techniques (see, e.g., [10, 29–32], etc.) that do not involve time delays.

Theorem 3. *Consider a one-sided Lipschitz nonlinear system* (1) *satisfying the time-delay bounds given by* (2)*, the one-sided Lipschitz condition* (3)*, and the quadratic inner-boundedness criterion* (4)*. Suppose there exist symmetric matrices* $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$, *and* $Z_j \in \mathfrak{R}^{n\times n}$ *for* $i = 1, 2, 3$, *and* $j = 1, 2$, *and scalars* ε_1 *and* ε_2, *such that the LMIs*

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \varepsilon_1 > 0, \quad \varepsilon_2 > 0, \quad \forall i = 1, 2, 3, \ j = 1, 2, \tag{10}$$

$$\begin{bmatrix} Y_1 + \rho\varepsilon_1 I + \beta\varepsilon_2 I & -PLC & Z_1 & 0 & P - \frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} & h_1 A^T Z_1 & h_{12} A^T Z_2 \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & -h_1 C^T L^T Z_1 & -h_{12} C^T L^T Z_2 \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & h_1 Z_1 & h_{12} Z_2 \\ * & * & * & * & -\varepsilon_2 I & h_1 Z_1 & h_{12} Z_2 \\ * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & -Z_2 \end{bmatrix} < 0 \tag{11}$$

are satisfied for a given matrix L, where

$$\begin{aligned} Y_1 &= PA + A^T P + \sum_{i=1}^{3} Q_i - Z_1, \\ \Lambda_1 &= (1-\mu) Q_3 + 2Z_2, \\ \Lambda_2 &= Q_1 + Z_1 + Z_2, \\ \Lambda_3 &= Q_2 + Z_2, \\ h_{12} &= h_2 - h_1. \end{aligned} \tag{12}$$

Then, there exists a Luenberger-type observer (5) *such that the state-estimation error e asymptotically converges to the origin.*

Proof. Define an LK functional candidate (see, for instance, [43]) as

$$\begin{aligned} V(e,t) = {} & e^T P e + \sum_{i=1}^{2} \int_{t-h_i}^{t} e^T(\alpha) Q_i e(\alpha)\, d\alpha \\ & + \int_{t-\tau}^{t} e^T(\alpha) Q_3 e(\alpha)\, d\alpha \\ & + \int_{-h_1}^{0} \int_{t+s}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha)\, d\alpha\, ds \\ & + \int_{-h_2}^{-h_1} \int_{t+s}^{t} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha\, ds. \end{aligned} \tag{13}$$

Acquiring the time derivative of (13) yields

$$\begin{aligned} \dot{V}(e,t) = {} & 2e^T P \dot{e} + \sum_{i=1}^{2} \left\{ e^T Q_i e - e^T(t-h_i) Q_i e(t-h_i) \right\} \\ & + e^T Q_3 e - (1-\mu) e^T(t-\tau) Q_3 e(t-\tau) \\ & + \dot{e}^T \left(h_1^2 Z_1 + h_{12}^2 Z_2 \right) \dot{e} - \int_{t-h_1}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha)\, d\alpha \\ & - \int_{t-h_2}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha. \end{aligned} \tag{14}$$

Employing (8) and (14) and rearranging the terms, the upper bound on $\dot{V}(e,t)$ is obtained as

$$\begin{aligned} \dot{V}(e,t) \le {} & 2e^T P \left(Ae + \Phi(x,\hat{x},u) + d - LCe(t-\tau) \right) \\ & + \sum_{i=1}^{3} e^T Q_i e - \sum_{i=1}^{2} e^T(t-h_i) Q_i e(t-h_i) \\ & - (1-\mu) e^T(t-\tau) Q_3 e(t-\tau) \\ & - \int_{t-h_1}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha)\, d\alpha \\ & + \left(Ae + \Phi(x,\hat{x},u) + d - LCe(t-\tau) \right)^T \\ & \times \left(h_1^2 Z_1 + h_{12}^2 Z_2 \right) \\ & \times \left(Ae + \Phi(x,\hat{x},u) + d - LCe(t-\tau) \right) \\ & - \int_{t-h_2}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha. \end{aligned} \tag{15}$$

Applying Jensen's inequality reveals

$$\begin{aligned} & - \int_{t-h_1}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha)\, d\alpha \\ & \le -\left(\int_{t-h_1}^{t} \dot{e}(\alpha)\, d\alpha \right)^T Z_1 \left(\int_{t-h_1}^{t} \dot{e}(\alpha)\, d\alpha \right) \\ & \le -\left(e(t) - e(t-h_1) \right)^T Z_1 \left(e(t) - e(t-h_1) \right). \end{aligned} \tag{16}$$

Similarly, we have

$$\begin{aligned} & - \int_{t-h_2}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha \\ & = - \int_{t-h_2}^{t-\tau} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha \\ & \quad - \int_{t-\tau}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha, \end{aligned}$$

$$
\begin{aligned}
&-\int_{t-h_2}^{t-h_1} h_{12}\dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha \\
&\quad \le -\left(\int_{t-h_2}^{t-\tau} \dot{e}(\alpha)\, d\alpha\right)^T Z_2 \left(\int_{t-h_2}^{t-\tau} \dot{e}(\alpha)\, d\alpha\right) \\
&\qquad -\left(\int_{t-\tau}^{t-h_1} \dot{e}(\alpha)\, d\alpha\right)^T Z_2 \left(\int_{t-\tau}^{t-h_1} \dot{e}(\alpha)\, d\alpha\right), \\
&-\int_{t-h_2}^{t-h_1} h_{12}\dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)\, d\alpha \\
&\quad \le -\left(e(t-\tau)-e(t-h_2)\right)^T Z_2 \left(e(t-\tau)-e(t-h_2)\right) \\
&\qquad -\left(e(t-h_1)-e(t-\tau)\right)^T Z_2 \left(e(t-h_1)-e(t-\tau)\right).
\end{aligned} \tag{17}
$$

Combining the results of (15)–(17), we have

$$
\begin{aligned}
&\dot{V}(e,t) \\
&\le e^T \left[PA + A^T P + \sum_{i=1}^{3} Q_i + A^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right) A - Z_1\right] e \\
&\quad - 2e^T \left[PLC + A^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right) LC\right] e(t-\tau) \\
&\quad + 2e^T Z_1 e(t-h_1) + e^T(t-\tau) \\
&\quad \times \left[-(1-\mu) Q_3 - 2Z_2 + C^T L^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right) LC\right] \\
&\quad \times e(t-\tau) + 2e^T(t-\tau) Z_2 e(t-h_1) + 2e^T(t-\tau) \\
&\quad \times Z_2 e(t-h_2) + e^T(t-h_1)\left(-Q_1 - Z_1 - Z_2\right) e(t-h_1) \\
&\quad + e^T(t-h_2)\left(-Q_2 - Z_2\right) e(t-h_2) \\
&\quad + 2e^T\left[P + A^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\right] \Phi(x,\hat{x},u) \\
&\quad + 2e^T\left[P + A^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\right] d + \Phi^T(x,\hat{x},u) \\
&\quad \times \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) \Phi(x,\hat{x},u) + 2\Phi^T(x,\hat{x},u) \\
&\quad \times \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) d + d^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right) d \\
&\quad - 2d^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right) LC e(t-\tau) \\
&\quad - 2e^T(t-\tau)\left(C^T L^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right)\right) \Phi(x,\hat{x},u).
\end{aligned} \tag{18}
$$

Under $d = 0$, (18) implies

$$
\dot{V}(e,t) \le \Psi_1^T \Upsilon_1 \Psi_1, \tag{19}
$$

where

$$
\Psi_1^T = \left[e^T \;\; e^T(t-\tau) \;\; e^T(t-h_1) \;\; e^T(t-h_2) \;\; \Phi^T(x,\hat{x},u)\right], \tag{20}
$$

$$
\Upsilon_1 = \begin{bmatrix}
Y_1 + A^T Y_4 A & -PLC - A^T Y_4 LC & Z_1 & 0 & Y_2 \\
* & -(1-\mu) Q_3 - 2Z_2 + Y_3 LC & Z_2 & Z_2 & -Y_3 \\
* & * & -Q_1 - Z_1 - Z_2 & 0 & 0 \\
* & * & * & -Q_2 - Z_2 & 0 \\
* & * & * & * & Y_4
\end{bmatrix} < 0, \tag{21}
$$

$$
\begin{aligned}
Y_2 &= P + A^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right), \\
Y_3 &= C^T L^T\left(h_1^2 Z_1 + h_{12}^2 Z_2\right), \\
Y_4 &= \left(h_1^2 Z_1 + h_{12}^2 Z_2\right).
\end{aligned} \tag{22}
$$

The one-sided Lipschitz condition given by (3) is equivalent to $\rho e^T e - e^T \Phi \ge 0$. For a positive scalar ε_1, the expression can be written as

$$
\Psi_1^T \begin{bmatrix}
\rho\varepsilon_1 I & 0 & 0 & 0 & \dfrac{-\varepsilon_1 I}{2} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\dfrac{-\varepsilon_1 I}{2} & 0 & 0 & 0 & 0
\end{bmatrix} \Psi_1 \ge 0. \tag{23}
$$

The quadratic inner-boundedness condition implies $\Phi^T \Phi \le \beta e^T e + \alpha e^T \Phi$, which for a positive scalar ε_2 results in

$$
\Psi_1^T \begin{bmatrix}
\beta\varepsilon_2 I & 0 & 0 & 0 & \dfrac{\alpha\varepsilon_2 I}{2} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\dfrac{\alpha\varepsilon_2 I}{2} & 0 & 0 & 0 & -\varepsilon_2 I
\end{bmatrix} \Psi_1 \ge 0. \tag{24}
$$

Merging (21), (23), and (24) using the S-procedure entails

$$\begin{bmatrix} \begin{pmatrix} Y_1 + A^T Y_4 A \\ +\rho\varepsilon_1 I + \beta\varepsilon_2 I \end{pmatrix} & -PLC - A^T Y_4 LC & Z_1 & 0 & Y_2 - \dfrac{\varepsilon_1 I}{2} + \dfrac{\alpha\varepsilon_2 I}{2} \\ * & -(1-\mu)Q_3 - 2Z_2 + Y_3 LC & Z_2 & Z_2 & -Y_3 \\ * & * & -Q_1 - Z_1 - Z_2 & 0 & 0 \\ * & * & * & -Q_1 - Z_2 & 0 \\ * & * & * & * & -\varepsilon_2 I + Y_4 \end{bmatrix} < 0. \tag{25}$$

Applying the Schur complement (see, e.g., [35]) to (25) produces (11), which implies that $\dot{V}(e,t) \leq \Psi_1^T \Upsilon_1 \Psi_1 < 0$; that is, the error e asymptotically converges to the origin. This finishes the proof of Theorem 3. □

Theorem 3 ensures state estimation by means of an observer for a given gain matrix L. If a guess for the observer gain matrix L is unobtainable, the following Theorem 4 provides a solution.

Theorem 4. *Consider a one-sided Lipschitz nonlinear system* (1) *satisfying the time-delay bounds given by* (2)*, the one-sided Lipschitz condition* (3)*, and the quadratic inner-boundedness criterion* (4)*. Suppose there exist symmetric matrices* $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$, *and* $Z_j \in \mathfrak{R}^{n\times n}$ *for* $i = 1, 2, 3$, *and* $j = 1, 2$, *matrix* $X \in \mathfrak{R}^{n\times m}$, *and scalars* ε_1 *and* ε_2, *such that the inequalities* (10) *and*

$$\begin{bmatrix} Y_1 + \rho\varepsilon_1 I + \beta\varepsilon_2 I & -XC & Z_1 & 0 & P - \dfrac{\varepsilon_1 I}{2} + \dfrac{\alpha\varepsilon_2 I}{2} & h_1 A^T P & h_{12} A^T P \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & -h_1 C^T X^T & -h_{12} C^T X^T \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & h_1 P & h_{12} P \\ * & * & * & * & -\varepsilon_2 I & h_1 P & h_{12} P \\ * & * & * & * & * & -T_1 & 0 \\ * & * & * & * & * & * & -T_2 \end{bmatrix} < 0 \tag{26}$$

are satisfied, where $T_1 = PZ_1^{-1}P$ *and* $T_2 = PZ_2^{-1}P$. *Then, there exists a Luenberger-type observer* (5) *such that the state-estimation error e asymptotically converges to the origin.*

Proof. Employing the congruence transform using $\mathrm{diag}(I, I, I, I, I, PZ_1^{-1}, PZ_2^{-1})$ to the inequality (11) and defining $X = PL$ and $T_i = PZ_i^{-1}P$ for $i = 1, 2$, we obtain (26). This completes the proof of Theorem 4. □

Remark 5. One-sided Lipschitz nonlinear observer designs for output-delay systems are presented in Theorems 3 and 4 by incorporating the quadratic inner-boundedness condition and the one-sided Lipschitz constraint. It should be emphasized that the proposed approach, in contrast to the approaches in [29–32] considering the delay free systems, fills the research gap in observer synthesis for one-sided Lipschitz nonlinear output-delay systems. It is also notable that the estimation results for the full state vector of the one-sided Lipschitz nonlinear systems, in any form of time delay, are lacking in the literature. The presented approach in the present study can be further unfolded to the state estimation of nonlinear dynamic systems under delays in the states.

Remark 6. The projected observer-design methodology provided in Theorems 3 and 4 can be applied to a broad category of nonlinear systems for two reasons. First, it leads to a more general feature of one-sided Lipschitz nonlinearity compared with the Lipschitz one that guarantees state estimation for a relatively larger category of nonlinear systems. Second, the proposed observer scheme has been inferred for time-varying time-delay systems that satisfy the interval $0 \leq h_1 \leq \tau \leq h_2$, which commonly is not addressed. Consequently, the presented delay-range-dependent observer-synthesis methodology is less conservative and more pragmatic than the traditional approaches such as [16, 17, 23–32, 34].

The proposed delay-range-dependent nonlinear observer-design scheme renders constructive linear results, as in a particular case $f(x,u) = Bu$ below (see Corollary 7). The novel delay-range-dependent Lipschitz observer-synthesis scheme can also be deduced from the one-sided Lipschitz case (see Corollary 8) by taking a specific form of the quadratic inner-boundedness constants given by $\beta = l^2$ and $\alpha = 0$, where l is the Lipschitz constant for $f(x,u)$.

Corollary 7. *Consider a linear system* (1) *with* $f(x,u) = Bu$ *(where B is a constant input matrix of an appropriate dimension) satisfying the time-delay bounds given by* (2)*. Suppose there exist symmetric matrices* $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$,

and $Z_j \in \mathfrak{R}^{n\times n}$ for $i = 1,2,3$, and $j = 1,2$, and a matrix $X \in \mathfrak{R}^{n\times m}$ such that the inequalities

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \forall i = 1,2,3, \ j = 1,2,$$
$$\begin{bmatrix} Y_1 & -XC & Z_1 & 0 & h_1A^TP & h_{12}A^TP \\ * & -\Lambda_1 & Z_2 & Z_2 & -h_1C^TX^T & -h_{12}C^TX^T \\ * & * & -\Lambda_2 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & 0 \\ * & * & * & * & -T_1 & 0 \\ * & * & * & * & * & -T_2 \end{bmatrix} < 0 \tag{27}$$

are satisfied. Then, there exists a Luenberger-type observer (5) *with observer gain matrix $L = P^{-1}X$ such that the state-estimation error e asymptotically converges to the origin.*

Remark 9. The outcomes of Corollaries 7 and 8 demonstrate that the derived results in Theorems 3 and 4 are applicable even for linear (see, e.g., [13]) and, particularly, for Lipschitz nonlinear systems, and such linear or Lipschitz nonlinear delay-range-dependent results for the case of measurement delays have not been fully explored so far. The deduced linear results introduce flexibility in the estimation of state vector under the influence of measurement or output delays. The introduced Lipschitz constant extracts the aftermath of the nonlinear term and facilitates the solution of the optimization problem without complexity.

are satisfied. Then, there exists a Luenberger-type observer (5) *for $f(\widehat{x},u) = Bu$ with observer gain matrix $L = P^{-1}X$ such that the state-estimation error e asymptotically converges to the origin.*

Corollary 8. *Consider a Lipschitz nonlinear system* (1) *satisfying the time-delay bounds given by* (2) *and the quadratic inner-boundedness criterion* (4) *for $\beta = l^2$ and $\alpha = 0$, where l is the Lipschitz constant for $f(x,u)$. Suppose there exist symmetric matrices $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$, and $Z_j \in \mathfrak{R}^{n\times n}$ for $i = 1,2,3$, and $j = 1,2$, matrix $X \in \mathfrak{R}^{n\times m}$, and a scalar ε_2, such that the inequalities* (27), *$\varepsilon_2 > 0$, and*

$$\begin{bmatrix} Y_1 + \varepsilon_2 l^2 I & -XC & Z_1 & 0 & P & h_1A^TP & h_{12}A^TP \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & -h_1C^TX^T & -h_{12}C^TX^T \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & h_1P & h_{12}P \\ * & * & * & * & -\varepsilon_2 I & h_1P & h_{12}P \\ * & * & * & * & * & -T_1 & 0 \\ * & * & * & * & * & * & -T_2 \end{bmatrix} < 0 \tag{28}$$

While $h_1 = 0$, Theorem 3 reduces to the following corollary, which provides the delay-dependent observer-design approach.

Corollary 10. *Consider a one-sided Lipschitz nonlinear system* (1) *satisfying the time-delay bounds given by* (2) *(with $h_1 = 0$, i.e., $0 \le \tau \le h_2$) and the one-sided Lipschitz condition* (3) *along with the quadratic inner-boundedness criterion* (4). *Suppose there exist symmetric matrices $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$, and $Z_j \in \mathfrak{R}^{n\times n}$ for $i = 2,3$, and $j = 2$, a matrix $X \in R^{n\times m}$, and scalars ε_1 and ε_2, such that the inequalities*

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \varepsilon_1 > 0, \quad \varepsilon_2 > 0, \quad \forall i = 2,3, \ j = 2,$$
$$\begin{bmatrix} \begin{pmatrix} PA + A^TP + Q_2 + Q_3 \\ -Z_2 + \rho\varepsilon_1 I + \beta\varepsilon_2 I \end{pmatrix} & -XC + Z_2 & 0 & P - \dfrac{\varepsilon_1 I}{2} + \dfrac{\alpha\varepsilon_2 I}{2} & h_2A^TP \\ * & -\Lambda_1 & Z_2 & 0 & -h_2C^TX^T \\ * & * & -\Lambda_3 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & h_2P \\ * & * & * & * & -T_2 \end{bmatrix} < 0, \tag{29}$$

are affirmed. Then, there exists a Luenberger-type observer (5) *with observer gain matrix $L = P^{-1}X$ such that the state-estimation error e asymptotically converges to the origin.*

Remark 11. Theorems 3 and 4 provide delay-range-dependent observer-design schemes for one-sided Lipschitz nonlinear systems such that the delay satisfies an interval $0 \le h_1 \le \tau \le h_2$; whereas, in Corollary 10, the traditional delay-dependent approach, assuming the lower bound as zero, is used to obtain interesting and novel delay-dependent results as a special category. Hence, the proposed derivation in Theorems 3 and 4, employing an advance delay-range-dependent concept, can be less conservative and more appropriate than the traditionalistic delay-dependent approaches. It is worth mentioning that an advanced delay-range-dependent rather than delay-dependent approach has been exploited in the present work to establish state-estimation strategies.

By taking $Q_3 = 0$, Theorem 4 reduces to the following delay-rate-independent approach for the fast time-varying delay case.

Corollary 12. *Consider a one-sided Lipschitz nonlinear system* (1) *satisfying the time-delay bound given by* (2) *with* $\dot{\tau} \geq 1$, *a one-sided Lipschitz condition* (3), *and the quadratic inner-boundedness criterion* (4). *Suppose there exist symmetric matrices* $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$, *and* $Z_j \in \mathfrak{R}^{n\times n}$ *for* $i = 1, 2$, *and* $j = 1, 2$, *a matrix* $X \in \mathfrak{R}^{n\times m}$, *and scalars* ε_1 *and* ε_2, *such that the inequalities*

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \varepsilon_1 > 0, \quad \varepsilon_2 > 0, \quad \forall i = 1, 2, \ j = 1, 2,$$

$$\begin{bmatrix} \begin{pmatrix} PA + A^T P + Q_1 + Q_2 \\ -Z_1 + \rho\varepsilon_1 I + \beta\varepsilon_2 I \end{pmatrix} & -XC & Z_1 & 0 & P - \dfrac{\varepsilon_1 I}{2} + \dfrac{\alpha\varepsilon_2 I}{2} & h_1 A^T P & h_{12} A^T P \\ * & -2Z_2 & Z_2 & Z_2 & 0 & -h_1 C^T X^T & -h_{12} C^T X^T \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & h_1 P & h_{12} P \\ * & * & * & * & -\varepsilon_2 I & h_1 P & h_{12} P \\ * & * & * & * & * & -T_1 & 0 \\ * & * & * & * & * & * & -T_2 \end{bmatrix} < 0, \tag{30}$$

are satisfied. Then, there exists a Luenberger-type observer (5) *with observer gain matrix* $L = P^{-1}X$ *such that the state-estimation error* e *asymptotically converges to the origin.*

Remark 13. Corollary 12 provides a delay-range-dependent stability criterion for the estimation error dynamics (8) under fast time-varying delays. If $\mu \geq 1$, Corollary 12, instead of Theorem 4, can be effectively utilized to design an observer; consequently, a delay-rate-independent transformation of the delay-range-dependent scheme of Theorem 4 for observer synthesis is operable. Such results as in Corollary 12 are applicable to the nonlinear systems with fast varying delays and are detailed to provide a plausible solution to the observer-design dilemma.

While designing an observer for a nonlinear system (1), a major issue emerged with unknown rapidly varying quantities such as external disturbances (for disturbance rejection see e.g., [48]), due to the fact that a small perturbation can cause the parametric estimates to drift towards infinity. Consequently, an observer for which the state-estimation error can diverge for a small-degree of perturbation is considered as fragile (see, e.g., [49]). Therefore, the scope of Theorem 4 is broadened in Theorem 14 to address the matter of robustness for observer formulation.

Theorem 14. *Consider a one-sided Lipschitz nonlinear system* (1) *satisfying the time-delay bounds given by* (2) *and a one-sided Lipschitz condition* (3) *along with the quadratic inner-boundedness criterion* (4). *Suppose there exist symmetric matrices* $P \in \mathfrak{R}^{n\times n}$, $Q_i \in \mathfrak{R}^{n\times n}$, *and* $Z_j \in \mathfrak{R}^{n\times n}$ *for* $i = 1, 2, 3$, *and* $j = 1, 2$, *a matrix* $X \in R^{n\times m}$, *and scalars* γ, ε_1, *and* ε_2, *such that the inequalities* (10), $\gamma > 0$, *and*

$$\begin{bmatrix} \begin{pmatrix} Y_1 + \rho\varepsilon_1 I \\ +\beta\varepsilon_2 I \end{pmatrix} & -XC & Z_1 & 0 & \begin{pmatrix} P - \dfrac{\varepsilon_1 I}{2} \\ +\dfrac{\alpha\varepsilon_2 I}{2} \end{pmatrix} & P & I & h_1 A^T P & h_{12} A^T P \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & 0 & 0 & -h_1 C^T X^T & -h_{12} C^T X^T \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma I & 0 & h_1 P & h_{12} P \\ * & * & * & * & * & * & -\gamma I & h_1 P & h_{12} P \\ * & * & * & * & * & * & * & -T_1 & 0 \\ * & * & * & * & * & * & * & * & -T_2 \end{bmatrix} < 0 \tag{31}$$

are satisfied. Then, there exists a Luenberger-type observer (5) *with observer gain matrix* $L = P^{-1}X$ *such that the state-estimation error* e *asymptotically converges to the origin, if* $d = 0$, *and, if* $d \neq 0$, *the* L_2 *gain of the state-estimation error* e *with regard to the disturbance* d *is bounded by* γ.

Proof. Consider the LK functional (13). Incorporating the inequality for L_2 gain reduction

$$J(e,t) = \dot{V}(e,t) + \gamma^{-1} e^T e - \gamma d^T d < 0 \tag{32}$$

yields

$$\int_0^t \dot{V}(e,t)\,dt + \int_0^t \left(\gamma^{-1}e^Te\right)dt - \int_0^t \left(\gamma d^Td\right)dt < 0,$$

$$V(e,t) - V(e,0) + \int_0^t \left(\gamma^{-1}e^Te\right)dt - \int_0^t \left(\gamma d^Td\right)dt < 0, \tag{33}$$

which further implies

$$\sqrt{\int_0^t \left(e^Te\right)dt} < \gamma\sqrt{\int_0^t \left(d^Td\right)dt}. \tag{34}$$

That is, $\|e\|_2 < \gamma\|d\|_2$. Considering the case when $d = 0$, inequality (32) shows that the estimation error asymptotically converges to zero. And if d belongs to L_2 bounded perturbations, (34) produces $\|e\|_2/\|d\|_2 < \gamma$ (see also [50, 51]). Working out the same procedure, under disturbance $d \neq 0$, as seen in the proof of Theorem 3, we have $J(e,t) \leq \Psi_2^T\Upsilon_2\Psi_2$, where

$$\Upsilon_2 = \begin{bmatrix} W & -PLC - A^TY_4LC & Z_1 & 0 & Y_2 - \frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} & Y_2 \\ * & -\Lambda_1 + Y_3LC & Z_2 & Z_2 & -Y_3 & -Y_3 \\ * & * & -\Lambda_2 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I + Y_4 & Y_4 \\ * & * & * & * & * & -\gamma I + Y_4 \end{bmatrix},$$

$$W = Y_1 + A^TY_4A + \gamma^{-1}I + \rho\varepsilon_1 I + \beta\varepsilon_2 I,$$

$$\Psi_2^T = \begin{bmatrix} e^T & e^T(t-\tau) & e^T(t-h_1) & e^T(t-h_2) & \Phi^T & d^T \end{bmatrix}. \tag{35}$$

Applying the Schur complement, employing congruence transformation using $\mathrm{diag}(I, I, I, I, I, I, PZ_1^{-1}, PZ_2^{-1})$, and defining $X = PL$ and $T_i = PZ_i^{-1}P$, for $i = 1, 2$, yield (31), which completes the proof. □

Remark 15. Theorem 14, in contrast to Theorem 4, ensures robustness of the observer design for one-sided Lipschitz nonlinear systems subject to time-varying interval time delays under L_2 norm-bounded disturbances. The results for delay-range-dependent observer design against disturbances are rare, and, in this regard, the proposed methodology for robust state estimation is utilitarian. Moreover, for one-sided Lipschitz dynamical processes, the proposed methodology can be appealing due to its effective utilization in the environment involving perturbations.

The constraints in Theorems 4 and 14 include nonlinear terms contained in $\varphi = \mathrm{diag}(-PZ_1^{-1}P, -PZ_2^{-1}P)$, which introduce difficulties in determining the observer gain matrix. The diagonal structure φ can be replaced by $\mathrm{diag}(-T_1, -T_2)$, where $T_1 = PZ_1^{-1}P$ and $T_2 = PZ_2^{-1}P$, to solve the constraints using cone complementary linearization technique [36, 52]. The original feasibility problem in Theorems 4 or 14 can be solved by optimizing min $\mathrm{Trace}\{Z_1S_1 + Z_2S_2 + T_1\overline{T}_1 + T_2\overline{T}_2 + P\overline{P}\}$ subject to

$$\begin{bmatrix} P & I \\ * & \overline{P} \end{bmatrix} > 0, \qquad \begin{bmatrix} Z_i & I \\ * & S_i \end{bmatrix} > 0, \tag{36}$$

$$\begin{bmatrix} T_i & I \\ * & \overline{T}_i \end{bmatrix} > 0, \tag{37}$$

in addition to the constraints in Theorems 4 or 14 as seen in [46], where S_i, $\overline{T}_i$, and $\overline{P}$ are employed to represent the inverses of Z_i, T_i, and P, respectively, for $i = 1, 2$. Since we have defined $T_i = PZ_i^{-1}P$, (37) further yields

$$\begin{bmatrix} PZ_i^{-1}P & I \\ * & \overline{T}_i \end{bmatrix} > 0, \quad i = 1, 2. \tag{38}$$

Applying congruence transform $\mathrm{diag}(\overline{P}, I)$ and substituting $S_i = Z_i^{-1}$, the inequality

$$\begin{bmatrix} S_i & \overline{P} \\ * & \overline{T}_i \end{bmatrix} > 0, \quad i = 1, 2, \tag{39}$$

is implicitly obtained. Hence, by including constraint (39) and using $T_i = PS_iP$ for $i = 1, 2$, a more appropriate optimization problem than [46] is obtained as

$$\begin{aligned} &\min && \mathrm{Trace}\left(\sum_{i=1}^{2}\left(Z_iS_i + 0.5PS_iP\overline{T}_i + 0.5T_i\overline{T}_i\right) + P\overline{P}\right), \\ &\text{subject to} && (36), (37), (39), \text{and inequalities in the} \\ &&& \text{Theorems 4 or 14.} \end{aligned} \tag{40}$$

Note that both conditions $T_i = \overline{T}_i^{-1}$ and $PZ_i^{-1}P = \overline{T}_i^{-1}$, for $i = 1, 2$, are given equal weight factors in the objective function of (40).

Remark 16. Recently, a control approach for TS fuzzy systems based on the cone complementary linearization algorithm was developed in [46]. A similar approach by solving min $\mathrm{Trace}\{Z_1S_1 + Z_2S_2 + T_1\overline{T}_1 + T_2\overline{T}_2 + P\overline{P}\}$ subject to (36), (37), and the inequalities in Theorems 4 or 14 can be used to determine observer gain matrix; however, it can lead to conservative results. The inclusion of (39) (or (38)) along with (37) and the application of the modified objective function in optimization (40) ensure $T_i = \overline{T}_i^{-1}$ and $PZ_i^{-1}P = \overline{T}_i^{-1}$ for $i = 1, 2$, which, in contrast to [46], enforces the additional condition $T_i = PZ_i^{-1}P$ required for obtainment of unique solutions to T_1 and T_2 matrices. Hence, constraint (39), defined in the present case, is mandatory in order to obtain a less conservative solution to the nonlinear optimization problem.

Remark 17. The objective function in (40) contains highly nonlinear terms, $PS_iP\overline{T}_i$ for $i = 1, 2$. To deal with this

problem, linearization of the trace function is employed to obtain

$$\begin{aligned}\psi\left(S_i, T_i, \overline{T}_i, P, \overline{P}, Z_i\right) &= \text{Trace}\Bigg(\sum_{i=1}^{2}\Big(Z_{io}S_i + Z_iS_{io} + 0.5 \\ &\quad \times \left(P_oS_{io}P_o\overline{T}_i + P_oS_{io}P\overline{T}_{io} + P_oS_iP_o\overline{T}_{io}\right. \\ &\quad \left.+PS_{io}P_o\overline{T}_{io} + T_i\overline{T}_{io} + T_{io}\overline{T}_i\right)\Big) \\ &\quad +P_o\overline{P} + P\overline{P}_o\Bigg),\end{aligned} \tag{41}$$

where the subscript o is used to represent the corresponding constant matrices of $S_i, T_i, \overline{T}_i, P, \overline{P}$, and Z_i, for $i = 1, 2$, appearing from the linearization process. By application of (41), optimization problem (40) can be solved via available LMI-tools and the cone complementary linearization algorithm (see details in [36, 52]) to obtain an appropriate solution to Theorems 4 or 14. The LMI-based proficiencies are preferred owing to their ability in addressing large scale, complex, multiconstraint, and multiobjective optimization problems.

Generally, cone complementary linearization algorithm based solutions for determining controller or observer gains require extra time and computations because of their iterative nature in contrast to the simple LMIs. It should be noted that the observer or controller design conditions like in Theorem 3 for delay-range-dependent systems are hard to convert into LMIs. Additionally, the present work provides an offline solution for the observer gain calculation, for which extra design computations will not affect either feasibility of the constraints or real-time state-estimation application.

4. Simulation Results

In this section, we verify the proposed nonlinear time-delay observer-design methodologies using two numerical examples.

Example 18. Consider the dynamics of a nonlinear system borrowed from [16] under delayed output, given by

$$\begin{aligned}\dot{x}_1 &= c_1x_2(t) - l_1x_1(t), \\ \dot{x}_2 &= c_2x_3(t) - l_2x_2(t), \\ \dot{x}_3 &= c_3x_1(t)\,x_2(t) + c_4\cos\left(x_2(t)\right) - l_3x_3(t) + c_3u(t), \\ y(t) &= x_1(t - \tau(t)).\end{aligned} \tag{42}$$

Using the parametric values mentioned in [16], we come up with the model as

$$\begin{aligned}f(x,u) &= \begin{bmatrix}0 & 2\cos\left(x_2(t)\right) + 8u(t) & 0\end{bmatrix}^T, \\ A &= \begin{bmatrix}-1 & 0.9 & 0 \\ 0 & -1 & 0.4 \\ 0.48 & 0.3 & -1\end{bmatrix}, \qquad C = \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix}^T.\end{aligned} \tag{43}$$

The output experiences a measurement delay $\tau(t)$ and the input function is $u(t) = \sin(0.35(t))$. For the state vector estimation in the presence of output delay, the gain vector for the observer is calculated as

$$L = \begin{bmatrix}-2.9070 \\ 3.2632 \\ -3.0164\end{bmatrix}, \tag{44}$$

from the given eigenvalues of $\overline{\lambda} = \{-0.030, -0.031, -0.032\}$. The parameters for quadratic inner-boundedness inequality and one-sided Lipschitz condition are selected as $\alpha = 1.5$, $\beta = -1$, and $\rho = 3$, respectively. A comparison of the computed upper bounds on delay, that is, h_2, for $h_1 = 0$, that ensures state estimation is listed in Table 1. It can be affirmed that the results of the proposed observer strategy are broader and applicable for a large range of delays in contrast to [16]. It is also notable that the proposed approach can also be employed for a specific interval of time delay with $h_1 \neq 0$.

Example 19. Consider the dynamics of a moving object in Cartesian coordinates (see [30, 32, 47]) described by

$$\begin{aligned}f(x,u) &= -\left(x_1^2 + x_2^2\right)\begin{bmatrix}x_1 \\ x_2\end{bmatrix}, \\ A &= \begin{bmatrix}1 & 1 \\ -1 & 1\end{bmatrix}, \qquad C = \begin{bmatrix}1 \\ 0\end{bmatrix}^T.\end{aligned} \tag{45}$$

The parameters of the one-sided Lipschitz condition and quadratic inner-boundedness inequality are

$$\rho = 0, \qquad \alpha = -100, \qquad \beta = -99. \tag{46}$$

In contrast to [30, 32, 47], it is assumed that the output of the system is subject to a measurement delay of $\tau = 0.12$ sec, and the output is available with $y(t) = x_1(t - \tau)$. The goal is to estimate the state vector in the presence of the output delay. By solving the optimization problem (40) for Theorem 14 with parameters $h_1 = 0.05$, $h_2 = 0.25$, and $\mu = 0$, we obtain the observer gain matrix as

$$L = \begin{bmatrix}0.564 \\ 0.080\end{bmatrix}, \tag{47}$$

for $\gamma = 450.191$. The results of the proposed methodology in the absence of disturbance are shown in Figures 1–3. It is evident in Figures 1 and 2 that the observer states are converging to the plant states, while, as depicted in Figure 3, the state-estimation errors are converging to zero.

To evaluate the observer performance against disturbances, we select

$$d = \begin{bmatrix}0.2\sin(12t) \\ 0.15\cos(8t)\end{bmatrix}. \tag{48}$$

Figure 4 plots the state-estimation errors against perturbations. It is notable that the estimation errors are converging in a region neighboring the origin, indicating the observer's robustness against time-varying disturbances.

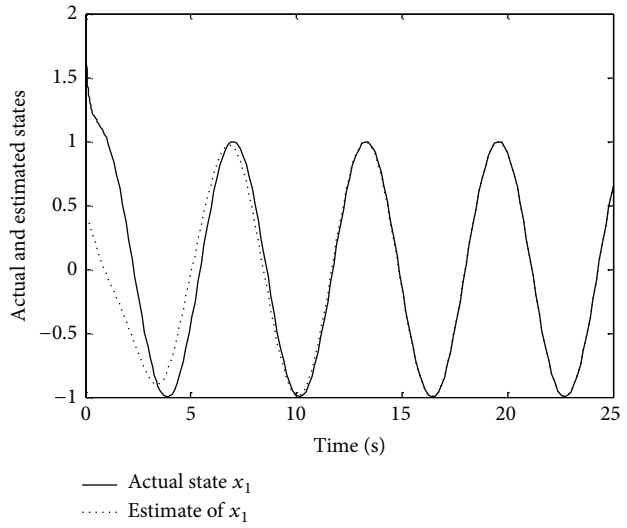

FIGURE 1: Estimation of the plant state x_1 under measurement delay.

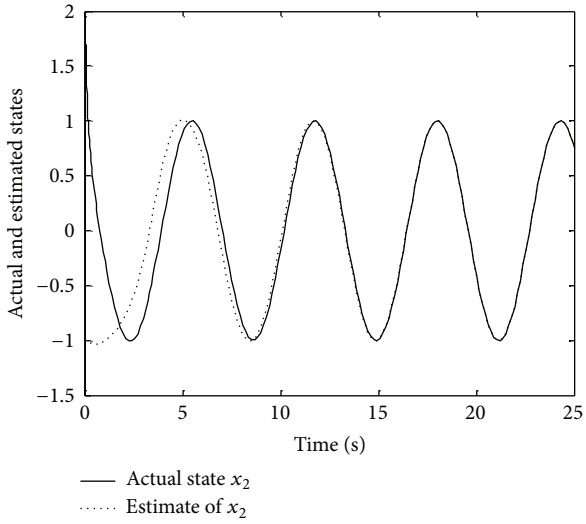

FIGURE 2: Estimation of the plant state x_2 under measurement delay.

TABLE 1: Allowable upper bound of h_2 for $h_1 = 0$ s, $\alpha = 1.5$, $\beta = -1$, and $\rho = 3$.

Methods	Observer-design approach in [16]	The proposed methodology in Theorem 3
h_2	16 sec	$\approx 10^{11}$ sec

Focusing on the conservatism of the traditional LMI approaches, the state observer methodologies [30, 32, 47] did not consider time delays; therefore, their applicability to time-delay systems is doubtful. For instance, in [32], the observer gain matrix for the delay free system (45) is found to be

$$L = \begin{bmatrix} 0.0079 \\ 0.2661 \end{bmatrix}. \quad (49)$$

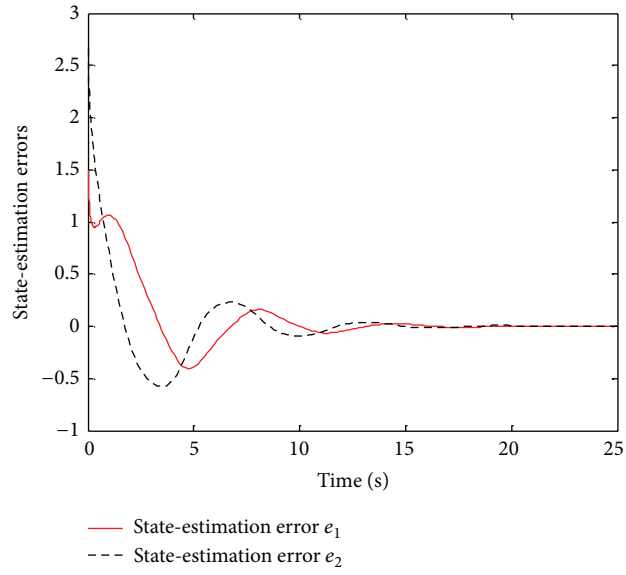

FIGURE 3: State-estimation errors converging to zero by application of the proposed observer strategy against output delay.

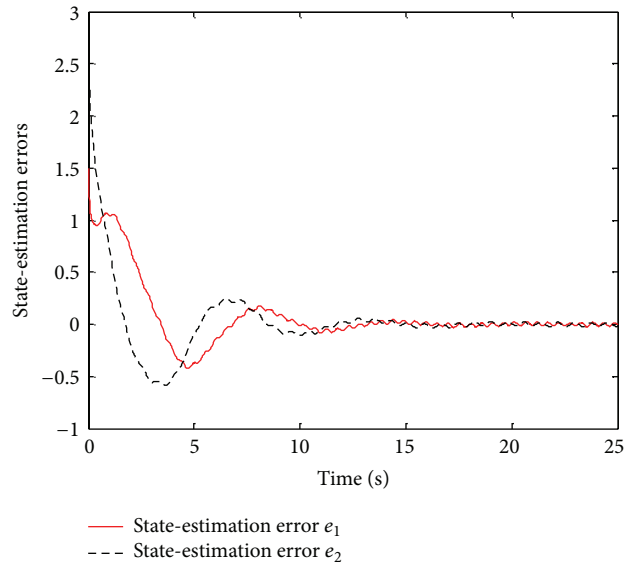

FIGURE 4: Robust state estimation using the proposed observer against disturbances and measurement delay.

With this observer gain, the LMI condition of Theorem 3 is found to be infeasible for a delay range between $h_1 = 0.05$ and $h_2 = 0.25$. Contrastingly, the proposed approach in Theorems 3–14 can be applied for observer synthesis of one-sided Lipschitz nonlinear systems subject to measurement delays. It should also be pointed out that the proposed observer gain in (47) cannot be applied for delay free systems because it does ensure stability of error dynamics; however, a suitable observer gain can also be calculated from the proposed approach in Theorems 3–14 by setting $h_1 = h_2 = 0$. To end, the proposed observer-design methodology can be

efficaciously applied to state estimation of one-sided Lipschitz nonlinear systems against output delays and perturbations.

5. Conclusion

A new approach to the nonlinear observer-design problem for nonlinear systems subject to delayed output measurements was presented in this paper. By application of Jensen's inequality, LK functional, LMI tools, and appropriate matrix transformations, the delay-range-dependent conditions, solvable via the cone complementary linearization algorithm for observer synthesis of one-sided Lipschitz nonlinear systems with time-varying output delays, were derived. An observer construction methodology for estimation of the states of dynamical nonlinear systems that is robust against disturbances was formulated by application of L_2 stability theory. The methodology is less conventional and more pragmatic than the traditional approaches, due to consideration of unavoidable output delays for observer design of one-sided Lipschitz nonlinear systems. The additional contributions of the present work are the incorporation of delay-range and time-varying delays, as well as the treatment of fast time-varying delays in the system dynamics and nonlinear terms in the design constraints for observer derivation. The resultant observer-synthesis approach can be applied to the estimation of the states of industrial nonlinear systems with output-interval time-varying delays and disturbances. Simulation results demonstrated the tractability and effectiveness of the projected one-sided Lipschitz observer-design schemes for nonlinear output-delay systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Quasipolynomial Approach to Simultaneous Robust Control of Time-Delay Systems

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A control law for retarded time-delay systems is considered, concerning infinite closed-loop spectrum assignment. An algebraic method for spectrum assignment is presented with a unique optimization algorithm for minimization of spectral abscissa and effective shaping of the chains of infinitely many closed-loop poles. Uncertainty of plant delays of a certain structure is considered in a sense of a robust simultaneous stabilization. Robust performance is achieved using mixed sensitivity design, which is incorporated into the addressed control law.

1. Introduction

Time-delay systems are an important and well established topic in modern control theory [1–3]. Its diverse applications, for instance, in dynamics of fluids, internal combustion, heating systems, regenerative chatter in metal cutting, and networked control systems, led to the development of various complex approaches to system and controller synthesis [4–8]. Infinite dimensional spectrum of such systems might cause difficulties in appropriate spectrum assignment using standard control laws, which consequently means that stabilization cannot be always achieved. Due to the complexity of the spectrum, much interest has been shown in the development of control laws based on the computation of rightmost poles, since such algorithms ensure results with high precision [9–12]. In principle, with a limited number of controller parameters only a few poles can be placed to desired positions in the complex plane [13]. As it has been shown in [8], some controller parameters might be used to directly assign a few dominant poles to an arbitrary position in the complex plane. The remaining controller parameters are then used to shift the chains of infinitely many system poles as far to the left of the dominant poles as possible. Another pole-placement-based technique has been introduced in [14] for retarded systems and in [15] for neutral systems. The method is based on continuous modifications applied to controller parameters in order to shift the rightmost or unstable poles to the left half plane in a quasicontinuous way, while monitoring other eigenvalues. A direct optimization approach [16, 17], used to minimize the spectral abscissa, has been introduced in [18].

Alternative eigenvalue based control laws might be considered, which incorporate different algebraic approaches [19]. General SISO time-delay plants treated in the form of a transfer function and its admissible coprime-inner/outer factorizations, derived by using numerical algorithms for computation of closed-loop poles, were discussed in [20]. In addition, an optimal $\mathcal{H}_\infty$ controller was designed.

In this paper, a control law for retarded time-delay systems is considered, which results in a controller in the form of a transfer function as a quotient of quasipolynomials. An individual quasipolynomial in the numerator and the denominator consists of several delayed terms that result in a closed-loop quasipolynomial of a certain structure of sequentially solvable polynomial Diophantine equations. The structure of the controller is examined and derived in the case of infinite solutions of individual polynomial equations. As has been

shown in [21] only a few such equations can be solved for an arbitrary polynomial, which suggests that the remainder of the closed-loop spectrum cannot be assigned accordingly. In [21], the unsuitable dynamics of the remaining spectrum has been algebraically eliminated by the specific structure imposed on the controller, which allows finite spectrum assignment. In our presented work, elimination of any part of the spectrum is not admissible and therefore we tackle the problem of infinite closed-loop spectrum assignment, which is indispensable especially in the case of uncertain delays. As soon as delays are perturbed, any delay compensation technique based on preestimated delays does not eliminate delayed terms out of the closed-loop completely, which results in an infinite closed-loop spectrum. We demonstrate that certain estimation technique of a predetermined region of poles [22] might be used in order to shape infinite closed-loop spectrum as well as infinite closed-loop spectrum with uncertain delays. We present an algebraic method for infinite closed-loop spectrum assignment, which reduces the number of parameters in the search routine for the appropriate stabile closed-loop spectrum. The algorithm for the search routine of the appropriately shaped infinite closed-loop spectrum is presented. It is shown that mixed sensitivity design might be incorporated into the addressed control law regarding uncertain time delays to obtain an optimal controller.

The paper is organized as follows. After some preliminaries, we derive different controller parameterizations concerning different types of free polynomials. Then, we propose an algebraic method for closed-loop spectrum assignment and present a spectrum shaping technique, where we show spectrum shaping with uncertain delays as well. After that, we give algorithm of a search routine for appropriate stabile closed-loop spectrum and present the mixed sensitivity design for robust controller synthesis. Finally, we demonstrate our results on an example and give final remarks in conclusions.

2. Preliminaries

Time-delay system of a retarded type, with internal or state and output discrete commensurate delays, is defined as in [22, 23]:

$$\dot{x}(t) = \overline{A}_0 x(t) + \sum_{i=1}^{\overline{q}} \overline{A}_i x(t - h_i\tau) + \overline{B}_0 u(t), \tag{1}$$

$$y(t) = \overline{C}_0 x(t) + \sum_{i=1}^{\overline{q}} \overline{C}_i x(t - h_i\tau) + \overline{D}_0 u(t), \tag{2}$$

where $\overline{A}_i \in \mathbb{R}^{n\times n}$, $\overline{B}, \overline{C}_i \in \mathbb{R}^n$, $i \in \mathbb{N}_1^{\overline{q}}$, are system defined parameters, $b(t), y(t) \in \mathbb{R}$ are input and output scalars, $x(t) \in \mathbb{R}^n$ is state vector, time delays $\{h_1\tau < \cdots < h_i\tau < \cdots < h_{\overline{q}}\tau\}$ are expressed as multiples of $\tau \in \mathbb{R}_+$, $h_i \in \mathbb{N}$, and initial condition is a segment of continuous functions $C([-h_i\tau, 0], \mathbb{R}^n)$ equipped with a supremum norm.

Definition 1. Let $Q_i(s)$ for $i \in \mathbb{Z}_0^q$ be polynomials with real coefficients and d_i are nonnegative real numbers in an ascending order. A function of the form

$$Q(s) = \sum_{i=0}^{q} Q_i(s) e^{-d_i s} \tag{3}$$

is a retarded quasipolynomial if $\deg Q_0 > \deg Q_i$, $i \in \mathbb{N}_1^q$.

System (1) can be represented with a SISO transfer function [19, 21]:

$$\begin{aligned} G(s) &= \frac{B(s)}{A(s)} \\ &= \sum_{i=0}^{\overline{q}} \overline{C}_i^T e^{-h_i\tau s} \left(sI - \sum_{i=0}^{\overline{q}} \overline{A}_i e^{-h_i\tau s} \right)^{-1} \overline{B}_0 + \overline{D}_0 u, \end{aligned} \tag{4}$$

where $A(s)$ and $B(s)$ are retarded quasipolynomials:

$$\begin{aligned} A(s) &= \det\left(sI - \sum_{i=0}^{\overline{q}} \overline{A}_i e^{-h_i\tau s} \right) = \sum_{i=0}^{n_h} A_i(s) e^{-i\tau s} \\ &= \sum_{j=0}^{n} A_j^*\left(e^{-\tau s}\right) s^j = \sum_{i=0}^{n_h} \sum_{j=0}^{n} a_{ij} s^j e^{-i\tau s}, \end{aligned} \tag{5}$$

with $i \in \mathbb{Z}_0^{n_h}$ and $j \in \mathbb{Z}_0^n$,

$$\begin{aligned} B(s) &= \sum_{i=0}^{\overline{q}} \overline{C}_i^T e^{-h_i\tau s} Adj\left(sI - \sum_{i=0}^{\overline{q}} \overline{A}_i e^{-h_i\tau s} \right) \overline{B}_0 + \overline{D}_0 A(s) \\ &= \sum_{i=0}^{m_h} B_i(s) e^{-i\tau s} = \sum_{j=0}^{m} B_j^*\left(e^{-\tau s}\right) s^j = \sum_{i=0}^{m_h} \sum_{j=0}^{m} b_{ij} s^j e^{-i\tau s}, \end{aligned} \tag{6}$$

with $i \in \mathbb{Z}_0^{m_h}$ and $j \in \mathbb{Z}_0^m$; $A^*(e^{-\tau s}) = \sum_{i=0}^{n_h} a_{ij} s^j e^{-i\tau s}$ and $B^*(e^{-\tau s}) = \sum_{i=0}^{m_h} b_{ij} s^j e^{-i\tau s}$ are alternative polynomials in variable $e^{-\tau s}$ and represent an equivalent description of quasipolynomials as Definition 1.

Considering control law

$$u(t) = \frac{P(s)}{R(s)} \left(y(t) - r(t) \right), \tag{7}$$

with $r(t)$ being any given bounded reference signal, $P(s)$ and $R(s)$ are retarded quasipolynomials of the form (3) and the controller structure

$$\begin{aligned} K(s) &= \frac{P(s)}{R(s)} = \frac{\sum_{i=0}^{l_h} P_i(s) e^{-i\tau s}}{\sum_{i=0}^{k_h} R_i(s) e^{-i\tau s}} \\ &= \frac{\sum_{j=0}^{l} P_j^*\left(e^{-\tau s}\right) s^j}{\sum_{j=0}^{k} R_j^*\left(e^{-\tau s}\right) s^j} \\ &= \frac{\sum_{i=0}^{l_h} \sum_{j=0}^{l} p_{ik} s^j e^{-i\tau s}}{\sum_{i=0}^{k_h} \sum_{j=0}^{k} r_{ik} s^j e^{-i\tau s}}. \end{aligned} \tag{8}$$

Assumption 1. We consider controller structure (8) with the same number of delays in the numerator and the denominator $k_h = l_h = \nu$ and $\deg R_0 = \deg P_0 > \deg R_i = \deg P_i$, $i \in \mathbb{N}_1^\nu$, with $R_0(s)$ being monic.

Assumption 2. Quasipolynomials $A(s)$ and $B(s)$ as well as polynomials $A_0(s)$ and $B_0(s)$ have no common zeroes. Complex Laplace argument (s) is hereafter omitted for clarity.

Controller (8) results in closed-loop characteristic quasipolynomial of the form

$$\begin{aligned} D &= \sum_{i=0}^{\nu+q} D_i e^{-i\tau s} = AR + BP \\ &= \sum_{i=0}^{\nu+q} \sum_{j=\mathrm{Max}(0,i-\nu)}^{\mathrm{Min}(i,q)} \left(A_j R_{i-j} + B_j P_{i-j}\right) e^{-i\tau s}, \end{aligned} \tag{9}$$

where $q = \max(n_h, m_h)$ and ν is as Assumption 1.

Regarding the closed-loop structure (9), which might be represented as a sum of several polynomial Diophantine equations with specific delay terms $e^{-i\tau s}$, the following result can be stated.

Proposition 2. *Let A, B, P, and R be quasipolynomials given by Definition 1 and Assumptions 1 and 2 hold. Then individual polynomial Diophantine equations in* (9) *for* $i \in \mathbb{Z}_0^\nu$,

$$A_0 R_i + B_0 P_i = D_i - \sum_{j=1}^{i} \left(A_j R_{i-j} + B_j P_{i-j}\right), \tag{10}$$

have solutions for arbitrary D_i*, where the equation for* $i = 0$ *has always infinite solutions (*$\deg R_0 > \deg A_0 - 1$*) and equations for* $i \in \mathbb{N}_1^\nu$ *have a unique (*$\deg R_i = \deg A_0 - 1$*) or infinite solutions (*$\deg R_i > \deg A_0 - 1$*), if the following is true:*

$$\begin{aligned} \deg R_0 = \deg P_0 &\le \deg R_i + \deg A_0 \\ &- \max\left(\max_j \left(\deg A_j\right), \max_k \left(\deg B_k\right)\right), \end{aligned} \tag{11}$$

$j \in \mathbb{N}_1^{n_h}$, $k \in \mathbb{N}_1^{m_h}$, *and* $d_{i \deg D_i} \neq a_{i \deg A_i} + b_{i \deg B_i} p_{0 \deg P_0}$ *in case of* $\deg A_0 + \deg R_i = \deg A_i + \deg R_0 = \deg B_i + \deg P_0$, $i \in \mathbb{N}_1^\nu$.

Proof. Polynomial equations (10) need to be solved sequentially since individual controller variables are present in several equations. Equations (10) for $i \in \mathbb{N}_1^\nu$ become

$$A_0 R_i + B_0 P_i = D_i - \cdots - A_i R_0 - B_i P_0. \tag{12}$$

For the expression on the left side, the following must hold:

$$\deg R_i \ge \deg A_0 - 1, \tag{13}$$

in order, (12) remains consistent polynomial equation. For the same reason, the degree of the expression on the right must not exceed the degree of the expression on the left; therefore, the following must hold:

$$\begin{aligned} &\max\left(\deg A_i, \deg B_i\right) + \deg R_0 \\ &\le \deg A_0 + \deg R_i = \deg D_i. \end{aligned} \tag{14}$$

From (14), (11) follows straightforwardly.

In the case of the equality in expression (14), selecting leading term of D_i according to $d_{i \deg D_i} = a_{i \deg A_i} + b_{i \deg A_i} p_{0 \deg P_0}$ results in the reduction of the degree of the expression on the right and therefore inconsistency of (12).

It can be shown that the first equation of (10) for $i = 0$ has always infinite solutions. The minimal allowed degree of R_i is $\deg R_i = \deg A_0 - 1$, $i \in \mathbb{N}_1^\nu$, and the maximal allowed degree of R_0 is determined by the equality of (11). If we join the two expressions, we can conclude that

$$\begin{aligned} \deg R_0 = 2\ \deg A_0 - \max\left(\max_j \left(\deg A_j\right), \max_k \left(\deg B_k\right)\right) \\ - 1 > \deg A_0 - 1, \end{aligned} \tag{15}$$

and consequently the first Diophantine equation always has infinite solutions. □

Remark 3. According to Proposition 2, only ν out of $\nu + q$ equations can be arbitrarily assigned as equations for $i \in \mathbb{N}_{\nu+1}^{\nu+q}$ consist of controller variables already assigned in equations $i \in \mathbb{Z}_0^\nu$, which suggests that infinite closed-loop spectrum assignment is not a straightforward task.

3. Controller Parameterization

For the parameterization of infinite number of controllers, we make the following assumption and give the result.

Assumption 3. To derive free parameters in all solvable Diophantine equations (10), the following must hold: $\deg R_i > \deg A_0 - 1$ and $\deg R_0 = 2\ \deg A_0 - \max_j(\deg A_j) > \deg A_0$, which we derive with the same procedure as (15) in the proof of Proposition 2.

Theorem 4. *Considering Assumptions 1, 2, and 3, the controller* (8) *can be expressed in the form*

$$K_\kappa = \left\{ \frac{\sum_{i=0}^{\nu} P_i e^{-i\tau s} + A_0 \sum_{i=0}^{\nu} \kappa_i e^{-i\tau s} + A_1 e^{-\tau s} \sum_{i=0}^{\nu-1} \kappa_i e^{-i\tau s} + \cdots + A_{n_h} e^{-n_h \tau s} \sum_{i=0}^{\nu-n_h} \kappa_i e^{-i\tau s}}{\sum_{i=0}^{\nu} R_i e^{-i\tau s} - B_0 \sum_{i=0}^{\nu} \kappa_i e^{-i\tau s} - B_1 e^{-\tau s} \sum_{i=0}^{\nu-1} \kappa_i e^{-i\tau s} - \cdots - B_{m_h} e^{-m_h \tau s} \sum_{i=0}^{\nu-m_h} \kappa_i e^{-i\tau s}} = \frac{P + \sum_{i=0}^{\nu} \sum_{k=0}^{\min(i,n_h)} A_k \kappa_{i-k} e^{-i\tau s}}{R - \sum_{i=0}^{\nu} \sum_{k=0}^{\min(i,m_h)} B_k \kappa_{i-k} e^{-i\tau s}} \right\}, \tag{16}$$

where κ_i are polynomials of degree $0 \leq \deg \kappa_i = \deg R_i - \deg A_0$, $i \in \mathbb{Z}_0^v$, which results in the following closed-loop characteristic quasipolynomial:

$$\begin{aligned} D &= A\left(R - \sum_{i=0}^{v}\sum_{k=0}^{\min(i,m_h)} B_k\kappa_{i-k}e^{-i\tau s}\right) \\ &\quad + B\left(P + \sum_{i=0}^{v}\sum_{k=0}^{\min(i,n_h)} A_k\kappa_{i-k}e^{-i\tau s}\right) \\ &= \sum_{i=0}^{v}\sum_{j=0}^{\min(i,q)} \left(A_jR_{i-j} + B_jP_{i-j}\right)e^{-i\tau s} \\ &\quad + \sum_{i=v+1}^{v+q}\sum_{j=i-v}^{\min(i,q)} \left(A_j\left(R_{i-j} - \sum_{k=0}^{\min(i-j,m_h)} B_k\kappa_{i-j-k}\right)\right. \\ &\qquad \left. + B_j\left(P_{i-j} + \sum_{k=0}^{\min(i-j,n_h)} A_k\kappa_{i-j-k}\right)\right)e^{-i\tau s}. \end{aligned} \tag{17}$$

The proof of Theorem 4 can be found in the appendix.

Remark 5. Polynomial equations (10) need to be solved sequentially for each change of a specific free parameter in an individual polynomial equation. The importance of the result of Theorem 4 lies in the fact that all the infinitely many solutions of individual polynomial equations are expressed using free polynomials κ_i. It is evident that as soon as controller variables P and R are derived it is not needed to solve polynomial equations again in case of a change of any parameter of free polynomials.

In the same way as individual Diophantine equations, the whole closed-loop characteristic quasipolynomial (9) can be treated as a single Diophantine equation. This leads to the following closed-loop [24] representation:

$$\begin{aligned} D &= A(R - B\lambda) + B(P + A\lambda) \\ &= AR + BP, \end{aligned} \tag{18}$$

where λ is a sum of polynomials λ_i as defined in (19) and a corresponding controller structure of the form

$$\begin{aligned} K_\lambda = \left\{\frac{P + A\lambda}{R - B\lambda}\right\} &= \left\{\frac{\sum_{i=0}^{v} P_ie^{-i\tau s} + \sum_{i=0}^{n_h} A_ie^{-i\tau s}\sum_{i=0}^{v}\lambda_ie^{-i\tau s}}{\sum_{i=0}^{v} R_ie^{-i\tau s} - \sum_{i=0}^{m_h} B_ie^{-i\tau s}\sum_{i=0}^{v}\lambda_ie^{-i\tau s}}\right. \\ &\left. = \frac{P + \sum_{i=0}^{v+n_h}\sum_{k=0}^{\min(i,n_h)} A_k\lambda_{i-k}e^{-i\tau s}}{R - \sum_{i=0}^{v+m_h}\sum_{k=0}^{\min(i,m_h)} B_k\lambda_{i-k}e^{-i\tau s}}\right\}, \end{aligned} \tag{19}$$

where λ_i are polynomials of degree $0 \leq \deg R_i - \deg A_0 \leq \deg \lambda_i \leq \deg R_i - \deg B_i - 1$, $i \in \mathbb{Z}_0^v$.

The two presented controller structures (16) and (19) consisting of different types of free polynomials might be joined in a unified controller structure K_γ.

Theorem 6. *Considering Assumptions 1, 2, and 3, unified controller structure*

$$K_\gamma = \left\{\frac{P + \sum_{i=0}^{v}\sum_{k=0}^{\min(i,n_h)} A_k\kappa_{i-k}e^{-i\tau s} + \sum_{i=0}^{v+n_h}\sum_{k=0}^{\min(i,n_h)} A_k\lambda_{i-k}e^{-i\tau s}}{R - \sum_{i=0}^{v}\sum_{k=0}^{\min(i,m_h)} B_k\kappa_{i-k}e^{-i\tau s} - \sum_{i=0}^{v+m_h}\sum_{k=0}^{\min(i,m_h)} B_k\lambda_{i-k}e^{-i\tau s}}\right\}, \tag{20}$$

where λ_i and κ_i are polynomials of degree as in (16) *and* (19)*, respectively, results in the closed-loop characteristic quasipolynomial* (17).

The proof of Theorem 6 can be found in the appendix.

4. Infinite Closed-Loop Spectrum Assignment

Due to the part of the closed-loop quasipolynomial that cannot be assigned arbitrarily, finite closed-loop spectrum assignment cannot be achieved without the cancelation or compensation of that part of the spectrum [21]. On the other hand, appropriate infinite closed-loop spectrum assignment might be achieved with the help of a numerical routine for reliably computing the rightmost poles of a closed-loop spectrum. The quasipolynomial of the form (9) might be determined with a direct search routine by continuously selecting controller variables and by closely observing the rightmost dominant poles of the closed-loop system by shifting the chains of infinitely many poles as far to the left of the dominant poles as possible. This might be achieved by minimizing the norm of the delayed terms in (9) [22]. Such search routine is computationally expensive since the number of controller parameters rapidly increases with the higher order of individual controller polynomials and the number of delayed controller polynomials. Another search routine could be constructed in a manner of continuously selecting closed-loop solution objectives of (9) represented as an overdetermined system of linear equations, where the individual closed-loop solution objective is designed by selecting values of the first equation representing a Hurwitzian polynomial and zeroing all other equations. Such solution objective will tend to minimize the effects of delayed parts of (9) by shifting the chains of infinitely many poles as far to the left as possible and consists of far less search parameters. The control objective can be expressed as an overdetermined system of linear equations by deriving the approximate solution of the expression $\min_x \|Sx - \overline{D}\|$, [25] using ordinary least squares method by

$$x = \left(S^TS\right)^{-1}S^T\overline{D}, \tag{21}$$

where $x = \left[\overline{R}_0 \ \overline{P}_0 \ \cdots \ \overline{R}_\nu \ \overline{P}_\nu\right]^T$, $x \in \mathbb{R}^{\sum_{i=0}^{\nu} 2 \deg R_i}$, $\overline{R}_i \in \mathbb{R}^{\deg R_i}$, and $\overline{P}_i \in \mathbb{R}^{\deg P_i}$, $i \in \mathbb{Z}_0^\nu$, are controller parameters (8), S is Sylvester matrix of (9), and $\overline{D}$ is the appropriately sized solution objective of the closed-loop quasipolynomial.

Through a close inspection of (9), we can conclude that, by selecting the sufficiently large number of delayed terms in (8), the solutions of the first few equations will never be included in the last few equations in (9), which do not have arbitrary solutions. Having that in mind, we can directly assign to the first few equations an arbitrary closed-loop polynomial D_i and derive solutions of the remaining equations according to (21), which is presented in the following result defining the sufficient structure of controller (8).

Proposition 7. *The solutions of the first jth polynomial equations in the sum* (9) *or* (17) *will not be part of the $\mathbb{N}_{\nu+1}^{\nu+q}$ equations if $\nu = q + j$, $j \in \mathbb{N}$. Therefore, assigning first jth equations arbitrarily and rearranging remaining equations $\mathbb{N}_{j+1}^{\nu+q}$ in* (9) *into* (21)*, where the solution objective $\overline{D}$ is appropriately sized vector of zeroes and*

$$x = \left[\overline{R}_{j+1} \ \overline{P}_{j+1} \ \cdots \ \overline{R}_\nu \ \overline{P}_\nu\right]^T, \quad x \in \mathbb{R}^{\sum_{i=j+1}^{\nu} 2 \deg R_i}, \quad (22)$$

will result in a consistent system of overdetermined equations.

Proof. According to Proposition 2, equations for $i \in \mathbb{Z}_0^\nu$ in the sum (9) have arbitrary solutions. Choosing $\nu = q + j$, the remaining equations, which do not have arbitrary solutions, become for $i \in \mathbb{N}_{q+j+1}^{2q+j}$ and consist of controller variables P_i and R_i, where $i \in \mathbb{N}_{j+1}^{q+j}$. Therefore, controller variables P_i and R_i, where $i \in \mathbb{Z}_0^j$, are not present in the remaining equations. So assigning and exchanging these variables with the values everywhere in (9) and rearranging remaining equations of (9) into (21) will result in a consistent system of overdetermined equations. □

Remark 8. The result of Proposition 7 can be directly applied onto a search routine by continuously selecting polynomials of the first or of the first few equations in the sum (9), where the first one is always Hurwitzian, deriving solutions of the first few equations by Proposition 2 and deriving solutions of the remaining equations by (21) and by closely inspecting closed-loop poles of (9). Hence, such search routine allows direct assignment of poles of the first nondelayed equation or of the first few equations and tends to minimize the effects of the remaining delayed equations.

5. Shaping of the Infinite Closed-Loop Spectrum

In order to appropriately shape the chains of infinitely many poles in the closed loop, by shifting them as far to the left from the dominant rightmost poles as possible, we can either compute sufficiently many rightmost poles, which is computationally expensive, or use the estimation technique of a predetermined region of poles presented in [9]. From the information of the real part of the rightmost pole $r = \max(\mathfrak{R}(s))$ and the distance from the real part of rightmost pole d on the real axes of the complex plane, we can specify a search criterion based on the predetermined region. This means that all the characteristic poles of the closed-loop quasipolynomial in the vertical strip of the complex plain $d < \mathfrak{R}(s) \leq r$ must belong to the predetermined region of a certain size. The estimation of the region with the belonging poles is given with the following result.

Proposition 9 (see [9]). *All characteristic roots of* (1) *satisfy $\mathfrak{R}(s) \geq \xi$, $\xi \in \mathbb{R}$, belonging to the set*

$$\Omega_\xi := \left\{\bigcup \sigma\left(\overline{A}_0 + \sum_{k=1}^{\overline{q}} \overline{A}_k z^{h_k}\right) : z \in \mathbb{C}, |z| \leq e^{-\xi\tau}, k \in \mathbb{N}_1^{\overline{q}}\right\} \bigcap \{s \in \mathbb{C} : \mathfrak{R}(s) \geq \xi\}. \quad (23)$$

The boundary of Ω_ξ is included in

$$\psi_\xi := \left(\bigcup_{\omega \in [0, 2\pi)} \sigma\left(\overline{A}_0 + \sum_{k=1}^{\overline{q}} \overline{A}_k e^{-\xi\tau h_k} e^{j\omega h_k}\right)\right) \bigcap \{s \in \mathbb{C} : \mathfrak{R}(s) \geq \xi\}. \quad (24)$$

For more information and proof see [26, Appendix A.1].

Remark 10. To derive the boundary ψ_0 of characteristic roots of (1) in the vertical stripe $d < \mathfrak{R}(s) \leq r$, we need to shift the origin of (1) by $-r+d$, which can be done by introducing new variable $s \to s - r + d$ in the characteristic matrix $\Delta(s)$:

$$\Delta(s) := sI - \overline{A}_0 - \sum_{k=1}^{\overline{q}} \overline{A}_k e^{-s\tau h_k}. \quad (25)$$

After shifting the origin, (24) must be executed in ph_q grid points over the interval $[0, 2\pi)$, because the function $\omega \to e^{j\omega h_q}$ has period $2\pi/h_q$. The value of the boundary ψ_0 might present a constraint of the search routine of the closed-loop quasipolynomial.

6. Shaping of the Infinite Closed-Loop Spectrum in Case of a Delay Mismatch

The structure of the delay uncertainty is in a form of multiplicative

$$\left\{h_1\widetilde{\tau} < \cdots < h_i\widetilde{\tau} < \cdots < h_q\widetilde{\tau} : \widetilde{\tau} = \tau(1 + \Delta\overline{\tau}), \ \overline{\tau} \in \mathbb{R}, \Delta \in [0, 1]\right\}, \quad (26)$$

or additive uncertainty

$$\left\{h_1\widetilde{\tau} < \cdots < h_i\widetilde{\tau} < \cdots < h_q\widetilde{\tau} : \widetilde{\tau} = \tau + \frac{\Delta\overline{\tau}}{h_i}, \ \overline{\tau} \in \mathbb{R}, \Delta \in [0, 1]\right\}, \quad (27)$$

which both introduce slightly more delayed terms in (9) as a result of multiplication between the delayed terms of the plant and the controller. Therefore, the closed-loop spectrum might change significantly as well. The same constraint as in Proposition 9 and Remark 10 might also be applied to the plant with an uncertain delay. It can be shown that, in the search routine, it is sufficient to consider only the boundary $\psi_{\xi,\overline{\tau}}$ as the boundaries $\psi_{\xi,\Delta\overline{\tau}}$ are included inside of $\psi_{\xi,\overline{\tau}}, \psi_{\xi,\Delta\overline{\tau}} \subseteq \psi_{\xi,\overline{\tau}}$. First we give an important property of time-delay systems.

Proposition 11 (see [22]). *If s is a characteristic root of the system* (1)*, then it satisfies*

$$|s| \leq \left\|\overline{A}_0\right\|_2 + \sum_{i=1}^{\overline{q}} \left\|\overline{A}_i\right\|_2 e^{-\Re(s)\tau h_i}. \tag{28}$$

Proof. The expression $\Delta(s) = 0$ (25) is equivalent to

$$s \in \sigma\left(\overline{A}_0 + \sum_{i=1}^{\overline{q}} \overline{A}_i e^{-s\tau h_i}\right). \tag{29}$$

Interpreting the argument of $\sigma(\cdot)$ as a matrix leads to

$$|s| \leq \left\|\overline{A}_0 + \sum_{i=1}^{\overline{q}} \overline{A}_i e^{-s\tau h_i}\right\|_2, \tag{30}$$

from which (28) follows straightforwardly. □

Using Proposition 11 and the result in Proposition 9, we can derive the following result.

Proposition 12. *For any positive $\Delta \in [0, 1]$, $\psi_{\xi,\Delta\overline{\tau}} \subseteq \psi_{\xi,\overline{\tau}}$ holds, where in* (28) *only the delay is perturbed according to* (26) *or* (27).

Proof. Regarding Proposition 11 and the multiplicative uncertainty (26) leads to

$$\begin{aligned} |s| &\leq \left\|\overline{D}_0\right\|_2 + \sum_{k=1}^{q} \left\|\overline{D}_k\right\|_2 e^{-\sigma\tau(1+\Delta\overline{\tau})h_k} \\ &\leq \left\|\overline{D}_0\right\|_2 + \sum_{k=1}^{q} \left\|\overline{D}_k\right\|_2 e^{-\sigma\tau(1+\overline{\tau})h_k}, \end{aligned} \tag{31}$$

and regarding the additive uncertainty (27) leads to

$$\begin{aligned} |s| &\leq \left\|\overline{D}_0\right\|_2 + \sum_{k=1}^{q} \left\|\overline{D}_k\right\|_2 e^{-\sigma(\tau h_k+\Delta\overline{\tau})} \\ &\leq \left\|\overline{D}_0\right\|_2 + \sum_{k=1}^{q} \left\|\overline{D}_k\right\|_2 e^{-\sigma(\tau h_k+\overline{\tau})}, \end{aligned} \tag{32}$$

where $\overline{D}_i$, $i \in \mathbb{Z}_0^q$, is the closed-loop quasipolynomial (9) in matrix form considering the delay uncertainty. Both expressions (31) and (32) are based on a simple comparison of exponential functions; namely, $e^{-\sigma\Delta\overline{\tau}} \leq e^{-\sigma\overline{\tau}}$ and $e^{-\sigma\tau\Delta\overline{\tau}h_k} \leq e^{-\sigma\tau\overline{\tau}h_k}$, respectively, $\sigma \in \mathbb{R}_-$. Since the area that contains all the characteristic roots in specific complex plane $\Re(s)$ of perturbed system is larger, it is sufficient to consider the boundary $\psi_{\xi,\overline{\tau}}$ of the largest set $\Omega_{\xi,\overline{\tau}}$. □

Finally, we can present the algorithm for the derivation of the infinite closed-loop spectrum, considering appropriate shaping of the chains of the infinitely many poles and constraints on the delay mismatch. The main objective of the algorithm is the minimization of function:

$$(K) \longrightarrow \alpha(K) + \psi_0(K) + \psi_{0,\overline{\tau}}(K), \tag{33}$$

where α is spectral abscissa:

$$\alpha := \sup\{\Re(s), s \text{ is characteristic root}\}. \tag{34}$$

Besides the spectral abscissa and the criterion based on Proposition 9, we might improve the objective function (34) by additional constraints, for instance, on the position of the dominant rightmost poles in the complex plane in the sense of damping, overshoot, or rise time [27].

As the spectral abscissa is a nonsmooth function, a gradient sampling algorithm presented in [16, 17] or differential evolution [28] might be used for the minimization of (34).

The important part of the algorithm is computation of only a few rightmost characteristic roots of (1). There are several numerical solutions [9–12] in the form of algorithms, with a distinct exception [9], which is based on an estimation of all roots in a predetermined region (24). Such an algorithm can be easily adapted in a way of computation of only a few rightmost roots with continuously shifting of the origin of the complex plane, in such a way that the predetermined region of all roots is as small as possible but large enough that it contains only a few rightmost roots (Remark 10). The shifting of the origin might be performed using a bisection algorithm. In this way, only a few rightmost roots are computed, which is much more efficient in the search for the appropriate closed-loop quasipolynomial, in contrast to the derivation of large number of roots, which is computationally very expensive.

Algorithm 13. We have the following.

(1) Choose $\nu = q + j$ as Proposition 7.

(2) Assign a Hurwitzian polynomial to first and any polynomial to the remaining jth equations according to the optimization method and derive a closed-loop spectrum (9) using Proposition 7.

(3) Derive the first few rightmost poles of (9).

(4) Derive ψ_0 spectrum of (9) using Proposition 9.

(5) Derive the first few rightmost poles of (9) considering delay uncertainty (26) or (27).

(6) Derive $\psi_{0,\overline{\tau}}$ spectrum of (9) using Proposition 9 and considering delay uncertainty (26) or (27).

(7) Compute the objective function (33) and derive improved values from minimization algorithm for step (2).

(8) Continue with step (2) until appropriate closed-loop spectrum is derived.

7. Robust Optimization via Mixed Sensitivity Design

Mixed sensitivity can be presented as an optimization problem of minimizing the $\mathscr{H}_\infty$ norm [24, 29] of the

$$\min_{\lambda} \left\| \begin{bmatrix} W_S(s)S(s,\lambda) \\ W_T(s)T(s,\lambda) \end{bmatrix} \right\|_\infty \leq \gamma, \quad \gamma \in \mathbb{R}_+, \tag{35}$$

where $W_S(s)$ and $W_T(s)$ are weighting filters in a form of frequency depended bounds, which characterize robust stability and robust performance; $S(s,\lambda)$ and $T(s,\lambda)$ are sensitivity and complementary sensitivity. Applying (19) to sensitivity and complementary sensitivity leads to

$$\begin{aligned} S(s,\lambda) &= \frac{A(R - B\lambda)}{AR + BP} \\ &= \frac{A\left(R - \sum_{i=0}^{\nu+m_h} \sum_{k=0}^{\min(i,m_h)} B_k \lambda_{i-k} e^{-i\tau s}\right)}{AR + BP}, \\ T(s,\lambda) &= \frac{B(P + A\lambda)}{AR + BP} \\ &= \frac{B\left(P + \sum_{i=0}^{\nu+n_h} \sum_{k=0}^{\min(i,n_h)} A_k \lambda_{i-k} e^{-i\tau s}\right)}{AR + BP}. \end{aligned} \tag{36}$$

When considering the delay uncertainty (26) or (27), structures of $S(s,\lambda)$ and $T(s,\lambda)$ are slightly altered. Therefore, the characteristic quasipolynomial depends on polynomials λ_i as well:

$$\begin{aligned} \widetilde{S}(s,\lambda) &= \frac{\widetilde{A}(R - B\lambda)}{\widetilde{A}(R - B\lambda) + \widetilde{B}(P + A\lambda)}, \\ \widetilde{T}(s,\lambda) &= \frac{\widetilde{B}(P + A\lambda)}{\widetilde{A}(R - B\lambda) + \widetilde{B}(P + A\lambda)}, \end{aligned} \tag{37}$$

where $\widetilde{A}$ and $\widetilde{B}$ are as in (4), only perturbed according to (26) or (27), respectively. By applying the uncertain $\widetilde{S}(s,\lambda)$ and $\widetilde{T}(s,\lambda)$ to the optimization procedure (35), we derive

$$\min_{\lambda} \left\| \begin{bmatrix} W_S(s)S(s,\lambda) \\ W_T(s)T(s,\lambda) \\ \widetilde{W}_S(s)\widetilde{S}(s,\lambda) \\ \widetilde{W}_T(s)\widetilde{T}(s,\lambda) \end{bmatrix} \right\|_\infty \leq \gamma, \quad \gamma \in \mathbb{R}_+, \tag{38}$$

with $\widetilde{W}_S(s)$ and $\widetilde{W}_T(s)$ being weighting filters characterizing sensitivity and complementary sensitivity of the plant with uncertain delays.

Optimal controller design might be performed by minimizing (38) and by closely observing right most poles of $\widetilde{T}(s,\lambda)$ in sense or infinite closed-loop spectrum shaping according to Algorithm 13.

8. Example: Level Control of a Chain of Evaporators

The dynamics of the level control of a chain of evaporators may be modeled by a delayed first order plus integrator transfer function [30, 31]:

$$F = \frac{-ke^{-\tau s}}{s(Ts + 1)}, \tag{39}$$

where parameters $\tau = 5s$, $k = -0.1$, and $T = 5s$. Time delay considered with appropriate uncertainty, for instance, using (26), $\widetilde{\tau} = \tau(1 + \Delta 2)$. Representing the transport delay as output delay and incorporating direct nondelayed output connections into the model (39) to represent the model in the form of (2), which might be achieved either by the observer [32, 33] or even by a classical smith predictor [31, 34], lead to the following model representation:

$$\widetilde{F} = \frac{-k\left(s + 1 + e^{-\widetilde{\tau} s}\right)}{s(Ts + 1)}. \tag{40}$$

The following controller was designed

$$K_\gamma = \frac{P_0 + P_5 e^{-5s} + A_0\lambda_0 + A_0\lambda_5 e^{-5s}}{R_0 + R_5 e^{-5s} - B_0\lambda_0 - B_5\lambda_0 e^{-5s} - B_0\lambda_5 e^{-5s}}, \tag{41}$$

where the number of delayed controller polynomials was set according to Proposition 7, $\nu = q + j = 5 + 0 = 5$. The degree of individual controller polynomials was set using the result of Proposition 2. Selecting $\deg P_5 = \deg R_5 = 1$ and $1 < \deg P = \deg R_0 = 2 \leq 3$. According to Theorem 6 and (19) $0 \leq \deg \lambda_0 = 0 \leq 1$ and $0 \leq \deg \lambda_5 = 0 \leq 0$. Applying Algorithm 13, where ψ_0 and $\psi_{0,\bar{\tau}}$ were computed according to Proposition 9 and Remark 10 for $d = 0.2$ (Figure 1). What is clear is that the higher the value d is, the larger the region $\psi_{0,\bar{\tau}}$ is. So from the numerical point of view, d should be set small. Optimal $\mathscr{H}_\infty$ controller (38) was designed for $W_S = \widetilde{W}_S = (0.5s + 0.01746)/(s + 0.002182)$ and $W_T = \widetilde{W}_T = (8s{+}4.583)/(s{+}9.165)$ (Figure 4) with the following controller parameters: $p_{02} = 1$, $p_{01} = -1.3884 \cdot 10^5$, $p_{00} = -5.1942 \cdot 10^5$, $p_{51} = -0.4432 \cdot 10^5$, $p_{50} = 2.3201 \cdot 10^5$, $r_{02} = 1$, $r_{01} = 0.012 \cdot 10^5$, $r_{00} = 3.4481 \cdot 10^5$, $r_{51} = -44.3231$, $r_{50} = -655.9617$, $\lambda_0 = -1000$, and $\lambda_5 = 0$. The step response of $\overline{T}(s,\lambda)$ is shown in Figure 2 in blue and of $T(s,\lambda)$ in red. Closed-loop poles of $T(s,\lambda)$ are shown in Figure 3 in blue and of $\overline{T}(s,\lambda)$, $\widetilde{\tau} = 3\tau$, and $\Delta = 1$ in green, respectively.

9. Conclusions

A unified controller structure for retarded time-delay systems composed of two different types of free polynomials was derived. An algebraic method for infinite closed-loop spectrum assignment was presented and a search algorithm for the appropriate infinite closed-loop spectrum was proposed. To shape the chains of infinitely many poles, a search criterion based on predetermined regions of poles was established and it was shown that the same technique was applicable for the closed-loop spectrum with uncertain time delays. Finally, optimal robust controller synthesis using mixed sensitivity approach was demonstrated in a control example.

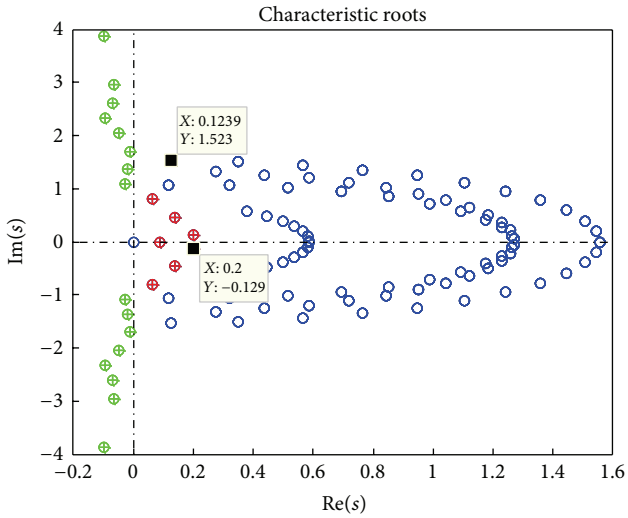


Figure 1: Poles of $\widetilde{T}(s,\lambda)$ shifted by $-r + 0.2$ in green and red, respectively; the spectrum $\psi_{0,\tilde{\tau}}$ is in blue.

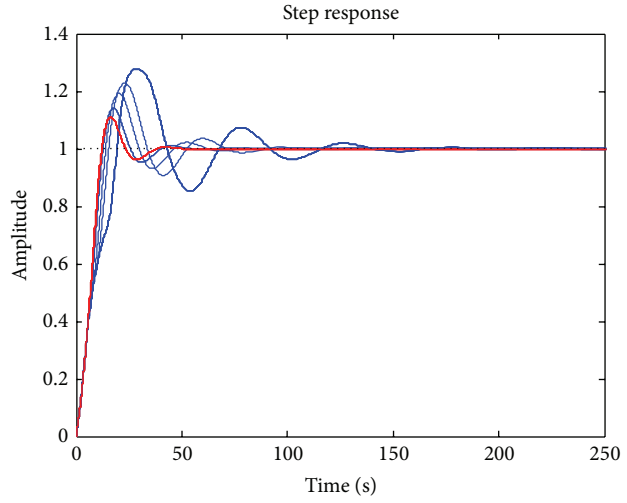


Figure 2: Step response of $\widetilde{T}(s,\lambda)$ in blue and of $T(s,\lambda)$ in red.

Appendix

Proof of Theorem 4. Considering Assumptions 1, 2, and 3, polynomial equations (10) have infinite solutions, where an individual polynomial equation in (10) can be represented in the form of an underdetermined system of linear equations with free parameters. The number of free parameters coincides with the number of parameters in κ_i, which represent an equivalent representation.

As shown in Proposition 2, if in the previous equation free parameters change, this leads to the change of the term on the right side of (10), which results in the change of the solutions in the current equation. Therefore, (10) need to be solved sequentially, but as Theorem 4, only once. After the first solution of (10), changes of free parameters relative to

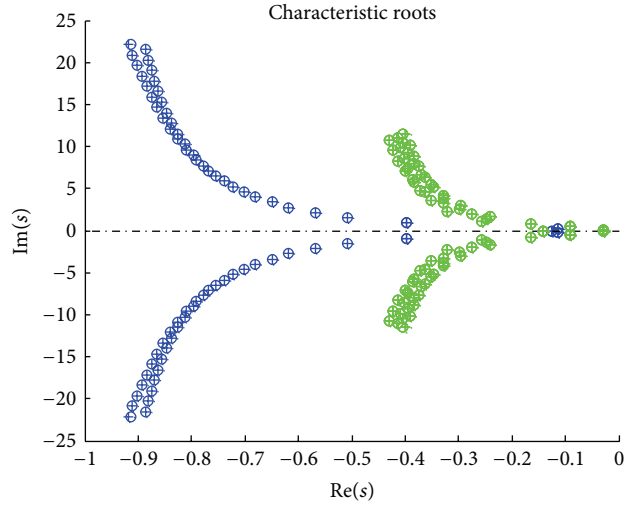


Figure 3: Closed-loop poles of $T(s,\lambda)$ in blue and of $\widetilde{T}(s,\lambda)$ in green.

the free parameters of the original solution can be expressed with polynomials κ_i:

$$A_0 R_{Ni} + B_0 P_{Ni} + \sum_{j=1}^{i} A_j R_{Ni-j} + B_j P_{Ni-j} = D_i. \tag{A.1}$$

Changing the free parameters in the first equation can be represented as a change of controller variables R_0 and P_0 to $R_{N0} = R_0 - B_0\kappa_0$ and $P_{N0} = P_0 + A_0\kappa_0$, which results in

$$A_0 (R_0 - B_0\kappa_0) + B_0 (P_0 + A_0\kappa_0) = D_i, \tag{A.2}$$

and the change of the solutions in the second equation

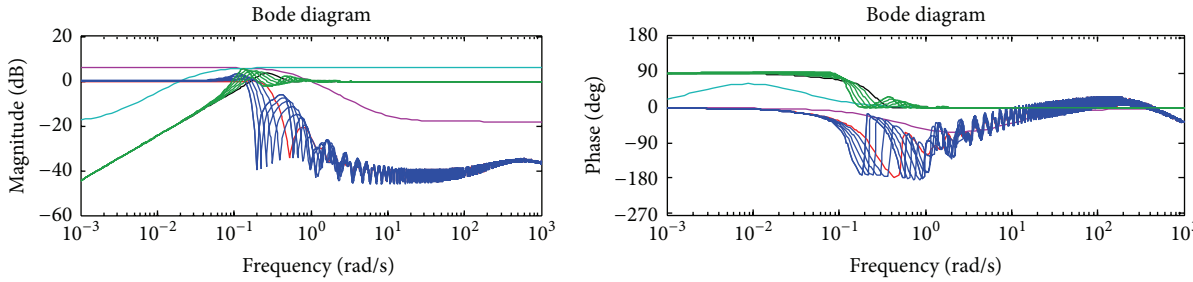
$$\begin{aligned} A_0 R'_{N1} + B_0 P'_{N1} = {} & D_1 - A_1 (R_0 - B_0\kappa_0) \\ & - B_1 (P_0 + A_0\kappa_0). \end{aligned} \tag{A.3}$$

By insertion of D_1 from (10) into (A.3), the amount of the change of R'_{N1} and P'_{N1} can be expressed in accordance with the change of R_{N0} and P_{N0}:

$$\begin{aligned} & A_0 R'_{N1} + B_0 P'_{N1} \\ & \quad = A_0 R_1 + B_0 P_1 + A_1 B_0\kappa_0 - B_1 A_0\kappa_0 \\ & \quad = A_0 (R_1 - B_1\kappa_0) + B_0 (P_1 + A_1\kappa_0). \end{aligned} \tag{A.4}$$

Considering the change of free parameters in the second equation same as in (A.2), we derive $R_{N1} = R'_{N1} - B_0\kappa_1 = R_1 - B_1\kappa_0 - B_0\kappa_1$ and $P_{N1} = P'_{N1} + A_0\kappa_1 = P_1 + A_1\kappa_0 + A_0\kappa_1$. When substituting R_{N1} and P_{N1} in (A.1), the second equation becomes

$$\begin{aligned} & A_0 (R_1 - B_1\kappa_0 - B_0\kappa_1) + B_0 (P_1 + A_1\kappa_0 + A_0\kappa_1) \\ & \quad = D_1 - A_1 (R_0 - B_0\kappa_0) - B_1 (P_0 + A_0\kappa_0). \end{aligned} \tag{A.5}$$



(a) (b)

FIGURE 4: Bode diagram of $1/W_S$ in pale blue, $1/W_T$ in purple, $T(s,\lambda)$ in red, $\widetilde{T}(s,\lambda)$ in blue, $S(s,\lambda)$ in black, and $\widetilde{S}(s,\lambda)$ in green.

Derivation of appropriate controller variables is executed sequentially. Generally we can write for the ith equation

$$\begin{aligned} &A_0R'_{N1} + B_0P'_{N1} \\ &\quad = D_i - \sum_{j=1}^{i} A_j \left(R_{i-j} - \sum_{k=0}^{\min(i-j,m_h)} B_k\kappa_{i-j-k} \right) \\ &\quad + B_j \left(P_{i-j} + \sum_{k=0}^{\min(i-j,n_h)} A_k\kappa_{i-j-k} \right). \end{aligned} \tag{A.6}$$

After replacing D_i in (A.6) with an equivalent representation from (10) and cancelation of individual terms, we derive

$$\begin{aligned} &A_0R'_{Ni} + B_0P'_{Ni} \\ &\quad = A_0R_i + B_0P_i + \sum_{j=1}^{i} \left(A_jR_{i-j} + B_jP_{i-j} \right) \\ &\quad - \sum_{j=1}^{i} A_j \left(R_{i-j} - \sum_{k=0}^{\min(i-j,m_h)} B_k\kappa_{i-j-k} \right) \\ &\quad - B_j \left(P_{i-j} + \sum_{k=0}^{\min(i-j,n_h)} A_k\kappa_{i-j-k} \right) \\ &\quad = A_0 \left(R_i - \sum_{k=1}^{\min(i,m_h)} B_k\kappa_{i-k} \right) \\ &\quad + B_0 \left(P_i + \sum_{k=1}^{\min(i,n_h)} A_k\kappa_{i-k} \right). \end{aligned} \tag{A.7}$$

After considering the influence of free parameters, we derive the final form of R_{Ni} and P_{Ni}:

$$\begin{aligned} A_0R_{Ni} + B_0P_{Ni} &= A_0 \left(R_i - \sum_{k=0}^{\min(i,m_h)} B_k\kappa_{i-k} \right) \\ &\quad + B_0 \left(P_i + \sum_{k=0}^{\min(i,n_h)} A_k\kappa_{i-k} \right). \end{aligned} \tag{A.8}$$

After replacing controller variables in (A.1) with (A.8), we derive (17), which completes the proof of Theorem 4. □

Proof of Theorem 6. The two presented types of free polynomials expressed in (16) and (19) might be joined in a unified controller structure K_γ (20). The proof relies on the fact that free polynomials λ cancel out in the closed loop (18) and therefore might be included into the controller structure (16) as well. The following closed-loop representation with the controller K_γ,

$$\begin{aligned} D &= A \left(R - \sum_{i=0}^{\nu} \sum_{k=0}^{\min(i,m_h)} B_k\kappa_{i-k}e^{-i\tau s} - B\lambda \right) \\ &\quad + B \left(P + \sum_{i=0}^{\nu} \sum_{k=0}^{\min(i,n_h)} A_k\kappa_{i-k}e^{-i\tau s} + A\lambda \right), \end{aligned} \tag{A.9}$$

leads to the same closed-loop representation as in (17). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Optimal Control with Time Delays via the Penalty Method

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We prove necessary optimality conditions of Euler-Lagrange type for a problem of the calculus of variations with time delays, where the delay in the unknown function is different from the delay in its derivative. Then, a more general optimal control problem with time delays is considered. Main result gives a convergence theorem, allowing us to obtain a solution to the delayed optimal control problem by considering a sequence of delayed problems of the calculus of variations.

1. Introduction

Over the past years, there has been an increasing interest in time delay problems of the calculus of variations and control [1–4]. Such interest is explained for their importance in control and engineering [5–8]. Indeed, time delays are inherent in various real systems, such as control systems and optimal control problems in engineering [9, 10].

In this paper we improve recent optimality conditions for time delay variational problems. In [11] necessary optimality conditions of Euler-Lagrange, DuBois-Reymond, and Noether type were obtained for problems of the calculus of variations with a time delay. The results of [11] were then extended to delayed variational problems with higher order derivatives in [2]. Here we model time delay variational problems in a more realistic way: while in [2, 11] the delay on functions and their derivatives (and control variables) is always the same, here we consider different delays for the functions and derivatives/controls.

The text is organized as follows. In Section 2 we formulate the delayed problem of the calculus of variations, where the delay in the unknown functions is different from the delay in their derivatives. The main result in this section is Theorem 4, which provides necessary optimality conditions of Euler-Lagrange type. Control strategies via an exterior penalty method are then investigated in Section 3. The idea is to replace the optimal control problem with time delays by a series of delayed problems of the calculus of variations. The main result gives a convergence theorem that allows us to obtain a solution to delayed optimal control problems with linear delayed control systems, by considering a sequence of variational problems with time delays of the type considered before in Section 2 (see Theorem 7). We end with Section 4 of conclusions.

2. Calculus of Variations with Time Delays

We consider the following fundamental problem of the calculus of variations with time delays, where the delay in the function we are looking for is different from the delay in its derivative:

$$\min \int_0^{\top} L\left(t, x(t), x\left(t-\tau_1\right), \dot{x}(t), \dot{x}\left(t-\tau_2\right)\right) dt \tag{1}$$

subject to

$$\begin{aligned} x(t) &= \theta_1(t), \quad t \in \left[-\tau_1, -\tau_2\right] =: I_1, \\ x(t) &= \theta_2(t), \quad t \in \left[-\tau_2, 0\right] =: I_2, \\ x(\top) &= \alpha, \end{aligned} \tag{2}$$

where $L : [0, \top] \times \mathbb{R}^{4N} \to \mathbb{R}$, $(t, a, \overline{a}, b, \overline{b}) \to L(t, a, \overline{a}, b, \overline{b})$, is the Lagrangian, $\top > 0$ is fixed in $\mathbb{R}$, τ_1 and τ_2 are two given positive real numbers such that $\tau_2 < \tau_1 < \top$, and $\theta_1(\cdot)$ and $\theta_2(\cdot)$ are given piecewise smooth functions. Let $I := [0, \top]$, $L^2(I, \mathbb{R}^N)$ be the Lebesgue space of measurable functions such that

$$\|x\|_{L^2} = \left(\int_0^\top \|x(t)\|_{\mathbb{R}^N}^2 dt \right)^{1/2} < \infty \tag{3}$$

and $H^1(I, \mathbb{R}^N)$ the Sobolev space of functions having their weak first derivative lying in $L^2(I, \mathbb{R}^N)$ and represented by

$$x(t) = x(\tau) + \int_\tau^t \dot{x}(s)\, ds \tag{4}$$

for all τ and t in I. We denote that

(i) $\mathscr{H}$ is the space of all functions $x : [-\tau_1, \top] \to \mathbb{R}^N$ such that $x_{/I_1} \in L^2(I_1, \mathbb{R}^N)$, $x_{/I_2} \in H^1(I_2, \mathbb{R}^N)$, and $x_{/I} \in H^1(I, \mathbb{R}^N)$, which is a Hilbert space with the norm

$$\|x\|_{\mathscr{H}} = \left(\left\|x_{/I_1}\right\|_{L^2(I_1, \mathbb{R}^N)}^2 + \left\|x_{/I_2}\right\|_{H^1(I_2, \mathbb{R}^N)}^2 + \left\|x_{/I}\right\|_{H^1\left(I, \mathbb{R}^N\right)}^2 \right)^{1/2}; \tag{5}$$

(ii) $D := \{x(\cdot) \in \mathscr{H} : x_{/I_1} = \theta_1, x_{/I_2} = \theta_2, \text{ and } x(\top) = \alpha\}$;

(iii) $J : \mathscr{H} \to \mathbb{R}$ is the functional

$$J(x(\cdot)) = \int_0^\top L\left(t, x(t), x\left(t - \tau_1\right), \dot{x}(t), \dot{x}\left(t - \tau_2\right)\right) dt. \tag{6}$$

Our problem (1) and (2) takes then the following form:

$$\min_{x(\cdot) \in D} J(x(\cdot)). \tag{7}$$

We make the following assumptions on the data of problem (7):

(A_1) Lagrangian L is a C^1 Carathéodory mapping; that is, it is of class C^1 in $(a, \overline{a}, b, \overline{b})$ for almost all $t \in [0, \top]$ and is measurable in t for every $(a, \overline{a}, b, \overline{b})$;

(A_2) there exist $\gamma_i(\cdot) \in L^2(I, \mathbb{R}^+)$, $i = 1, \ldots, 5$, such that a.e. in $t \in I$

$$\begin{aligned} \left|L\left(t, a, \overline{a}, b, \overline{b}\right)\right| &\le \gamma_1(t), \\ \left\|\partial_i L\left(t, a, \overline{a}, b, \overline{b}\right)\right\| &\le \gamma_i(t), \quad i = 2, \ldots, 5, \end{aligned} \tag{8}$$

where $\partial_i L$ is the partial derivative of L with respect to its ith argument.

Definition 1 (cone of tangents). Let Z be a normed space, $A \subset Z$, and $a \in \overline{A}$. The cone of tangents $T(A, a)$ is the set of all $z \in Z$ with the property that there is a sequence (a_n) in A converging strongly to a and a sequence of nonnegative numbers (α_n) such that $\alpha_n(a_n - a) \to z$.

Lemma 2. *The set D is an affine linear subspace of $\mathscr{H}$ and the cone of tangents $T(D, x(\cdot))$ is given by*

$$\begin{aligned} &T(D, x(\cdot)) \\ &\quad = \left\{ v(\cdot) \in \mathscr{H} : v(\cdot)_{/I_1} = 0, v(\cdot)_{/I_2} = 0,\ v(\top) = 0 \right\}. \end{aligned} \tag{9}$$

Proof. Let $v(\cdot) \in T(D, x(\cdot))$. Then there exist $(x_n(\cdot))_n \subset D$ and $\lambda_n \ge 0$ such that $x_n(\cdot) \to x(\cdot)$ in D implies that $\lambda_n(x_n(\cdot) - x(\cdot)) \to v(\cdot)$ in $\mathscr{H}$. Since $x_n(\cdot), x(\cdot) \in D$ for all n, we have

$$\begin{aligned} x_n(\tau) = x(\tau) &= \theta_1(\tau), \quad \tau \in \left[-\tau_1, -\tau_2\right], \\ x_n(\tau) = x(\tau) &= \theta_2(\tau), \quad \tau \in \left[-\tau_2, 0\right], \\ x_n(\top) &= x(\top) = \alpha. \end{aligned} \tag{10}$$

Hence,

$$\begin{aligned} \lambda_n\left(x_n(\tau) - x(\tau)\right) &= 0, \quad \tau \in \left[-\tau_1, -\tau_2\right], \\ \lambda_n\left(x_n(\tau) - x(\tau)\right) &= 0, \quad \tau \in \left[-\tau_2, 0\right], \\ \lambda_n\left(x_n(\top) - x(\top)\right) &= 0. \end{aligned} \tag{11}$$

Therefore, $v(\cdot) \in \mathscr{H}$ with $v(\tau) = 0$ for all $\tau \in [-\tau_1, -\tau_2]$, $v(\tau) = 0$ for almost all $\tau \in [-\tau_2, 0]$, and $v(\top) = 0$. Thus,

$$\begin{aligned} &T(D, x(\cdot)) \\ &\quad \subset \left\{ v(\cdot) \in \mathscr{H} : v(\cdot)_{/I_1} = 0, v(\cdot)_{/I_2} = 0,\ v(\top) = \alpha \right\} \\ &\quad = K. \end{aligned} \tag{12}$$

Conversely, let $v(\cdot) \in K$ for $x(\cdot) \in D$. Define $x_n(\cdot) = x(\cdot) + (1/n)v(\cdot)$. Then $n(x_n(\cdot) - x(\cdot)) = v(\cdot)$ with $v(\cdot) \in \mathscr{H}$. Hence, $v(\cdot) \in T(D, x(\cdot))$. □

For convenience, we introduce the operator $[\cdot]_{\tau_1}^{\tau_2}$ defined by

$$[x]_{\tau_1}^{\tau_2}(t) = \left(t, x(t), x\left(t - \tau_1\right), \dot{x}(t), \dot{x}\left(t - \tau_2\right)\right). \tag{13}$$

Proposition 3. *Under conditions (A_1) and (A_2), the mapping $J(\cdot)$ is Fréchet differentiable and*

$$\begin{aligned} J'(x(\cdot))(v(\cdot)) = \int_0^\top \Big(&\left\langle \partial_2 L[x]_{\tau_1}^{\tau_2}(t), v(t) \right\rangle \\ &+ \left\langle \partial_3 L[x]_{\tau_1}^{\tau_2}(t), v\left(t - \tau_1\right) \right\rangle \\ &+ \left\langle \partial_4 L[x]_{\tau_1}^{\tau_2}(t), \dot{v}(t) \right\rangle \\ &+ \left\langle \partial_5 L[x]_{\tau_1}^{\tau_2}(t), \dot{v}\left(t - \tau_2\right) \right\rangle \Big)\, dt. \end{aligned} \tag{14}$$

Proof. Let $v(\cdot) \in \mathscr{H}$. We have

$$\begin{aligned} J'(x(\cdot); v(\cdot)) &= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left(J(x(\cdot) + \lambda v(\cdot)) - J(x(\cdot))\right) \\ &= \lim_{\lambda \to 0^+} \int_0^\top \frac{1}{\lambda} \left[L[x + \lambda v]_{\tau_1}^{\tau_2}(t) - L[x]_{\tau_1}^{\tau_2}(t)\right] dt. \end{aligned} \tag{15}$$

Define

$$\Psi_\lambda(t) = \frac{1}{\lambda}\left[L[x+\lambda v]_{\tau_1}^{\tau_2}(t) - L[x]_{\tau_1}^{\tau_2}(t)\right] \tag{16}$$

and

$$\begin{aligned}\Psi(t) = {} & \left\langle \partial_2 L[x]_{\tau_1}^{\tau_2}(t), v(t)\right\rangle \\ & + \left\langle \partial_3 L[x]_{\tau_1}^{\tau_2}(t), v(t-\tau_1)\right\rangle \\ & + \left\langle \partial_4 L[x]_{\tau_1}^{\tau_2}(t), \dot v(t)\right\rangle \\ & + \left\langle \partial_5 L[x]_{\tau_1}^{\tau_2}(t), \dot v(t-\tau_2)\right\rangle.\end{aligned} \tag{17}$$

Then, $\Psi_\lambda(t) \to \Psi(t)$ as $\lambda \to 0^+$ for almost all $t \in [0,\top]$. On the other hand,

$$\left|\Psi_\lambda(t)\right| \le g(t) \tag{18}$$

a.e. in $t \in [0,\top]$ with

$$\begin{aligned}g(t) = \gamma(t)\,[& \|v(t)\|_{\mathbb{R}^N} + \left\|v(t-\tau_1)\right\|_{\mathbb{R}^N} \\ & + \|\dot v(t)\|_{\mathbb{R}^N} + \left\|v(t-\tau_2)\right\|_{\mathbb{R}^N}]\end{aligned} \tag{19}$$

a function not depending on λ, and

$$\left|\Psi_\lambda(t)\right| \le g(t) + 1 \tag{20}$$

for almost all $t \in [0,\top]$ and λ sufficiently small. Since $[0,\top]$ has finite measure, Lebesgue's theorem yields that

$$\int_0^\top \Psi_\lambda(t)\,dt \longrightarrow \int_0^\top \Psi(t)\,dt \tag{21}$$

as $\lambda \to 0^+$. Hence,

$$\begin{aligned}J'(x(\cdot))(v(\cdot)) = \int_0^\top \Big(& \left\langle \partial_2 L[x]_{\tau_1}^{\tau_2}(t), v(t)\right\rangle \\ & + \left\langle \partial_3 L[x]_{\tau_1}^{\tau_2}(t), v(t-\tau_1)\right\rangle \\ & + \left\langle \partial_4 L[x]_{\tau_1}^{\tau_2}(t), \dot v(t)\right\rangle \\ & + \left\langle \partial_5 L[x]_{\tau_1}^{\tau_2}(t), \dot v(t-\tau_2)\right\rangle\Big)\,dt.\end{aligned} \tag{22}$$

This is the directional derivative of J in the direction v. To finish the proof, we need to show that $J'(x(\cdot); v(\cdot))$ is linear and bounded in v and continuous in x. The linearity is obvious. We begin by proving that $J'(x(\cdot);\cdot)$ is bounded from $\mathscr{H}$ to $\mathbb{R}$:

$$\begin{aligned}\left|J'(x(\cdot); v(\cdot))\right| \le {} & \int_0^\top \left|\left\langle \partial_2 L[x]_{\tau_1}^{\tau_2}(t), v(t)\right\rangle\right| dt \\ & + \int_0^\top \left|\left\langle \partial_3 L[x]_{\tau_1}^{\tau_2}(t), v(t-\tau_1)\right\rangle\right| dt \\ & + \int_0^\top \left|\left\langle \partial_4 L[x]_{\tau_1}^{\tau_2}(t), \dot v(t)\right\rangle\right| dt \\ & + \int_0^\top \left|\left\langle \partial_5 L[x]_{\tau_1}^{\tau_2}(t), \dot v(t-\tau_2)\right\rangle\right| dt \\ \le {} & \int_0^\top \gamma_2(t)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_0^\top \gamma_3(t)\,\left\|v(t-\tau_1)\right\|_{\mathbb{R}^N} dt \\ & + \int_0^\top \gamma_4(t)\,\|\dot v(t)\|_{\mathbb{R}^N} dt \\ & + \int_0^\top \gamma_5(t)\,\left\|\dot v(t-\tau_2)\right\|_{\mathbb{R}^N} dt \\ \le {} & \int_0^\top \gamma_2(t)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_{-\tau_1}^{\top-\tau_1} \gamma_3(t+\tau_1)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_0^\top \gamma_4(t)\,\|\dot v(t)\|_{\mathbb{R}^N} dt \\ & + \int_{-\tau_2}^{\top-\tau_2} \gamma_5(t+\tau_2)\,\|\dot v(t)\|_{\mathbb{R}^N} dt \\ \le {} & \int_0^\top \gamma_2(t)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_{-\tau_1}^{-\tau_2} \gamma_3(t+\tau_1)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_{-\tau_2}^{0} \gamma_3(t+\tau_1)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_0^{\top-\tau_1} \gamma_3(t+\tau_1)\,\|v(t)\|_{\mathbb{R}^N} dt \\ & + \int_0^\top \gamma_4(t)\,\|\dot v(t)\|_{\mathbb{R}^N} dt \\ & + \int_{-\tau_2}^{0} \gamma_5(t+\tau_2)\,\|\dot v(t)\|_{\mathbb{R}^N} dt \\ \le {} & M\|v(\cdot)\|_{\mathscr{H}}.\end{aligned} \tag{23}$$

We still need to prove the continuity of $J'(\cdot)$. Let $x_n(\cdot) \to x(\cdot)$ in $\mathscr{H}$. Then,

$$\begin{aligned}
&\left|\left[J'\left(x_n(\cdot)\right) - J'\left(x(\cdot)\right)\right](v(\cdot))\right| \\
&\quad \le \int_0^{\top} \left|\left\langle \partial_2 L[x_n - x]_{\tau_1}^{\tau_2}(t), v(t)\right\rangle\right| dt \\
&\qquad + \int_0^{\top} \left|\left\langle \partial_3 L[x_n - x]_{\tau_1}^{\tau_2}(t), v(t-\tau_1)\right\rangle\right| dt \\
&\qquad + \int_0^{\top} \left|\left\langle \partial_4 L[x_n - x]_{\tau_1}^{\tau_2}(t), \dot{v}(t)\right\rangle\right| dt \\
&\qquad + \int_0^{\top} \left|\left\langle \partial_5 L[x_n - x]_{\tau_1}^{\tau_2}(t), \dot{v}(t-\tau_2)\right\rangle\right| dt \\
&\quad \le \int_0^{\top} \left\|\partial_2 L[x_n - x]_{\tau_1}^{\tau_2}(t)\right\| \|v(t)\|_{\mathbb{R}^N} dt \\
&\qquad + \int_0^{\top} \left\|\partial_3 L[x_n - x]_{\tau_1}^{\tau_2}(t)\right\| \|v(t-\tau_1)\|_{\mathbb{R}^N} dt \\
&\qquad + \int_0^{\top} \left\|\partial_4 L[x_n - x]_{\tau_1}^{\tau_2}(t)\right\| \|\dot{v}(t)\|_{\mathbb{R}^N} dt \\
&\qquad + \int_0^{\top} \left\|\partial_5 L[x_n - x]_{\tau_1}^{\tau_2}(t)\right\| \|\dot{v}(t-\tau_2)\|_{\mathbb{R}^N} dt \\
&\quad = \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\mathscr{I}_2 &\le \left\|\partial_2 L[x_n - x]_{\tau_1}^{\tau_2}(\cdot)\right\| \|v(t)\|_{L^2}, \\
\mathscr{I}_3 &\le \left\|\partial_3 L[x_n - x]_{\tau_1}^{\tau_2}(\cdot)\right\| \|v(t)\|_{L^2([-\tau_1,\top],\mathbb{R}^N)}, \\
\mathscr{I}_4 &\le \left\|\partial_4 L[x_n - x]_{\tau_1}^{\tau_2}(\cdot)\right\| \|\dot{v}(t)\|_{L^2}, \\
\mathscr{I}_5 &\le \left\|\partial_3 L[x_n - x]_{\tau_1}^{\tau_2}(\cdot)\right\| \|\dot{v}(t)\|_{L^2([-\tau_1,\top],\mathbb{R}^N)}.
\end{aligned} \tag{25}$$

On the other hand, $x_n(\cdot) \to x(\cdot)$ in $\mathscr{H}$. From Lebesgue's theorem, there exists $\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3 \subset \mathbb{N}$ such that $\mathbb{k}_1 \subset \mathbb{k}_2 \subset \mathbb{k}_3$ and

$$\begin{aligned}
&x_k(t) \longrightarrow x(t), \quad \text{a.e. } t \in [0,\top], \quad \forall k \in \mathbb{k}_1, \\
&\dot{x}_k(t) \longrightarrow \dot{x}(t), \quad \text{a.e. } t \in [0,\top], \quad \forall k \in \mathbb{k}_1, \\
&x_k(t) \longrightarrow x(t), \quad \text{a.e. } t \in [-\tau_2, 0], \quad \forall k \in \mathbb{k}_2, \\
&\dot{x}_k(t) \longrightarrow \dot{x}(t), \quad \text{a.e. } t \in [-\tau_2, 0], \quad \forall k \in \mathbb{k}_2, \\
&x_k(t) \longrightarrow x(t), \quad \text{a.e. } t \in [-\tau_1, -\tau_2], \quad \forall k \in \mathbb{k}_3.
\end{aligned} \tag{26}$$

Hence,

$$\begin{aligned}
&\dot{x}_k(t-\tau_2) \longrightarrow \dot{x}(t-\tau_2), \quad \text{a.e. } t \in [0,\tau_2], \quad \forall k \in \mathbb{k}_2, \\
&x_k(t-\tau_1) \longrightarrow x(t-\tau_1), \quad \text{a.e. } t \in [0,\tau_1], \quad \forall k \in \mathbb{k}_3.
\end{aligned} \tag{27}$$

Since $L(t,\cdot,\cdot,\cdot)$ is C^1-Carathéodory, assumption (A_2) assures from Lebesgue's theorem that

$$\left\|\partial_i L[x_n - x]_{\tau_1}^{\tau_2}(\cdot)\right\| \longrightarrow 0, \quad i = 2,3,4,5. \tag{28}$$

This implies that $\mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5 \to 0$. Then, $J'(x_k(\cdot)) \to J'(x(\cdot))$. The proof is complete. □

Theorem 4 (necessary optimality conditions of Euler-Lagrange type for problem (1) and (2)). *Under conditions* (A_1) *and* (A_2)*, if* $\overline{x}(\cdot)$ *is a minimizer to problem* (1) *and* (2)*, then* $\overline{x}(\cdot)$ *satisfies the following Euler-Lagrange equations with time delay:*

$$\begin{aligned}
&\frac{d}{dt}\left\{\partial_4 L[\overline{x}]_{\tau_1}^{\tau_2}(t) + \partial_5 L[\overline{x}]_{\tau_1}^{\tau_2}(t+\tau_2)\right\} \\
&\qquad = \partial_2 L[\overline{x}]_{\tau_1}^{\tau_2}(t) + \partial_3 L[\overline{x}]_{\tau_1}^{\tau_2}(t+\tau_1), \\
&\qquad\qquad \text{a.e. } t \in [0, \top - \tau_1], \\
&\frac{d}{dt}\left\{\partial_4 L[\overline{x}]_{\tau_1}^{\tau_2}(t) + \partial_5 L[\overline{x}]_{\tau_1}^{\tau_2}(t+\tau_2)\right\} = \partial_2 L[\overline{x}]_{\tau_1}^{\tau_2}(t), \\
&\qquad\qquad \text{a.e. } t \in]\top - \tau_1, \top - \tau_2], \\
&\frac{d}{dt}\partial_4 L[\overline{x}]_{\tau_1}^{\tau_2}(t) = \partial_2 L[\overline{x}]_{\tau_1}^{\tau_2}(t), \quad \text{a.e. } t \in]\top - \tau_2, \top].
\end{aligned} \tag{29}$$

Proof. If $\overline{x}(\cdot)$ is a minimizer to problem (1) and (2), then

$$J'(\overline{x}(\cdot))(v(\cdot)) = 0 \tag{30}$$

for all $v(\cdot) \in T(D, \overline{x}(\cdot))$; that is,

$$\begin{aligned}
&\int_0^{\top} \left(\left\langle p_2(t), v(t)\right\rangle + \left\langle p_3(t), v(t-\tau_1)\right\rangle \right. \\
&\qquad \left. + \left\langle p_4(t), \dot{v}(t)\right\rangle + \left\langle p_5(t), \dot{v}(t-\tau_2)\right\rangle\right) dt = 0
\end{aligned} \tag{31}$$

for all $v(\cdot) \in T(D, \overline{x}(\cdot))$ with $p_i(t) = \partial_i L[\overline{x}]_{\tau_1}^{\tau_2}(t)$, $i = 2,3,4,5$. Integration by parts yields

$$\begin{aligned}
&\int_0^{\top} \left\langle p_4(t), \dot{v}(t)\right\rangle dt = -\int_0^{\top} \left\langle \dot{p}_4(t), v(t)\right\rangle dt, \\
&\int_0^{\top} \left\langle p_5(t), v(t-\tau_2)\right\rangle dt = \left\langle p_5(\top), v(\top - \tau_2)\right\rangle \\
&\qquad + \int_0^{\top} \left\langle \dot{p}_5(\tau), v(\tau - \tau_2)\right\rangle d\tau.
\end{aligned} \tag{32}$$

By (31) and (32), we obtain that

$$\begin{aligned}
&\int_0^{\top} \left(\left\langle p_2(t), v(t)\right\rangle + \left\langle p_3(t), v(t-\tau_1)\right\rangle \right. \\
&\qquad \left. - \left\langle \dot{p}_4(t), v(t)\right\rangle + \left\langle \dot{p}_5(t), v(t-\tau_2)\right\rangle\right) dt \\
&\qquad + \left\langle p_5(t), v(\top - \tau_2)\right\rangle = 0
\end{aligned} \tag{33}$$

for all $v(\cdot) \in T(D, \overline{x}(\cdot))$. On the other hand,

$$\begin{aligned}
\int_0^{\top} \langle p_3(t), v(t-\tau_1)\rangle\, dt &= \int_{-\tau_1}^{\top-\tau_1} \langle p_3(\tau+\tau_1), v(\tau)\rangle\, d\tau \\
&= \int_{-\tau_1}^{0} \langle p_3(\tau+\tau_1), v(\tau)\rangle\, d\tau \\
&\quad + \int_{0}^{\top-\tau_1} \langle p_3(\tau+\tau_1), v(\tau)\rangle\, d\tau \\
&= \int_{0}^{\top-\tau_1} \langle p_3(\tau+\tau_1), v(\tau)\rangle\, d\tau, \\
\int_0^{\top} \langle \dot{p}_5(t), v(t-\tau_2)\rangle\, dt &= \int_{-\tau_2}^{\top-\tau_2} \langle \dot{p}_5(\tau+\tau_2), v(\tau)\rangle\, d\tau \\
&= \int_{-\tau_2}^{0} \langle \dot{p}_5(\tau+\tau_2)\, v(\tau)\rangle\, d\tau \\
&\quad + \int_{0}^{\top-\tau_2} \langle \dot{p}_5(\tau+\tau_2), v(\tau)\rangle\, d\tau \\
&= \int_{0}^{\top-\tau_2} \langle \dot{p}_5(\tau+\tau_2), v(\tau)\rangle\, d\tau.
\end{aligned} \tag{34}$$

Hence,

$$\begin{aligned}
&\int_0^{\top} \langle p_2(t) - \dot{p}_4(t), v(t)\rangle\, dt \\
&\quad + \int_0^{\top-\tau_1} \langle p_3(t+\tau_1), v(t)\rangle\, dt \\
&\quad - \int_0^{\top-\tau_2} \langle \dot{p}_5(t+\tau_2), v(t)\rangle\, dt \\
&\quad + \langle p_5(\top), v(\top-\tau_2)\rangle = 0
\end{aligned} \tag{35}$$

for all $v(\cdot) \in T(D, \overline{x}(\cdot))$. Put

$$\begin{aligned}
\overline{p}_3(t+\tau_1) &= \begin{cases} p_3(t+\tau_1) & \text{if } t \in [0, \top-\tau_1], \\ 0 & \text{if } t \in\,]\top-\tau_1, \top], \end{cases} \\
q(t+\tau_2) &= \begin{cases} \dot{p}_5(t+\tau_2) & \text{if } t \in [0, \top-\tau_2], \\ 0 & \text{if } t \in\,]\top-\tau_2, \top]. \end{cases}
\end{aligned} \tag{36}$$

Then,

$$\begin{aligned}
&\int_0^{\top} \langle p_2(t) - \dot{p}_4(t) + \overline{p}_3(t+\tau_1) - q(t+\tau_2), v(t)\rangle\, dt \\
&\quad + \langle p_5(\top), v(\top-\tau_2)\rangle = 0
\end{aligned} \tag{37}$$

for all $v(\cdot) \in T(D, \overline{x}(\cdot))$. In particular, for v such that $v(\tau) = 0$ for almost all $\tau \in [-\tau_1, 0]$ and $v(\tau) = 0$ for almost all $\tau \in [\top-\tau_2, 0]$, we have

$$p_2(t) - \dot{p}_4(t) + \overline{p}_3(t+\tau_1) - q(t+\tau_2) = 0 \quad \text{a.e. } t \in [0, \top] \tag{38}$$

or

$$\begin{aligned}
&\dot{p}_4(t) + \dot{p}_5(t+\tau_2) = p_2(t) + p_3(t+\tau_1) \\
&\qquad \text{a.e. } t \in [0, \top-\tau_1], \\
&\dot{p}_4(t) + \dot{p}_5(t+\tau_2) = p_2(t) \quad \text{a.e. } t \in\,]\top-\tau_1, \top-\tau_2], \\
&\dot{p}_4(t) = p_2(t) \quad \text{a.e. } t \in\,]\top-\tau_2, \top].
\end{aligned} \tag{39}$$

The proof is complete. □

3. Optimal Control with Time Delays

Now we prove existence of an optimal solution to more general problems of optimal control with time delays. The result is obtained via the exterior penalty method [12, 13] and Theorem 4. The optimal control problem with time delays is defined as follows:

$$\min \int_0^{\top} l(t, x(t), \dot{x}(t-\tau_2), u(t))\, dt \tag{40}$$

subject to

$$\begin{aligned}
&\dot{x}(t) = Ax(t-\tau_1) + Bu(t), \quad t \in [0, \top] =: I, \\
&x(t) = \theta_1(t), \quad t \in [-\tau_1, -\tau_2] =: I_1, \\
&x(t) = \theta_2(t), \quad t \in [-\tau_2, 0] =: I_2, \\
&x(\top) = \alpha,
\end{aligned} \tag{41}$$

where $x(\cdot) \in \mathscr{H}$, $u(\cdot) \in U_0 = \{u(\cdot) \in L^2([0, \top], U) : u(0) = 0\}$, A is an $N \times N$ matrix, B is an $N \times m$ matrix, and $l : [0, \top] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$, $(t, a, \overline{b}, c) \to l(t, a, \overline{b}, c)$. The final time $\top > 0$ is fixed in $\mathbb{R}$, τ_1 and τ_2 are two given positive real numbers such that $\tau_2 < \tau_1 < \top$, and, as before, $\theta_1(\cdot)$ and $\theta_2(\cdot)$ are given piecewise smooth functions. In the sequel, we denote by $\theta(\cdot)$ the function defined by $\theta(t) = \theta_1(t)$, $t \in I_1$, and $\theta(t) = \theta_2(t)$, $t \in I_2$. We make the following assumptions on the data of the problem.

(H_1) The mapping l is a C^1-Carathéodory mapping; that is, l is C^1 in $(a, \overline{b}, c)$ for almost all $t \in [0, T]$ and is measurable in t for every $(a, \overline{b}, c) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^m$.

(H_2) There exist $\gamma_i(\cdot) \in L^2(I, \mathbb{R}^+)$, $i = 1, \ldots, 5$, such that

$$\begin{aligned}
&\left| l(t, a, \overline{b}, c) \right| \le \gamma_1(t), \\
&\left\| \partial_i l(t, a, \overline{b}, c) \right\| \le \gamma_i(t) \\
&\qquad \text{a.e. } t \in I, \quad i = 2, \ldots, 4,
\end{aligned} \tag{42}$$

where $\partial_i l$ is the partial derivative of l with respect to its ith argument, $i = 2, \ldots, 4$.

(H_3) There exists $\rho > 0$ such that for almost all $t \in [0, T]$ and for all $(a, \overline{b}, c) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^m$

$$l(t, a, \overline{b}, c) \ge \rho \|c\|_{\mathbb{R}^m}. \tag{43}$$

(H_4) $l(t, a, \overline{b}, c)$ is convex in $(\overline{b}, c)$.

Using the exterior penalty function method, we consider the following sequence of unconstrained optimal control problems corresponding to (40) and (41):

$$\begin{aligned}
&\inf \int_0^{\top} l\left(t, x(t), \dot{x}\left(t-\tau_2\right), u(t)\right) dt \\
&\quad + \frac{c_n}{2} \int_0^{\top} \left\| \dot{x}(t) - Ax\left(t-\tau_1\right) - Bu(t) \right\|_{\mathbb{R}^N}^2 dt, \\
&\qquad x(t) = \theta_1(t) \quad \text{a.e. } t \in I_1, \\
&\qquad x(t) = \theta_2(t) \quad \text{a.e. } t \in I_2, \\
&\qquad x(\top) = \alpha, \\
&\qquad x(\cdot) \in \mathscr{H}, \quad u(\cdot) \in U_0,
\end{aligned} \tag{$\mathscr{P}_n$}$$

where $c_{n+1} \geq c_n$, $c_n \to \infty$. Denote

$$\begin{aligned}
&L_n\left(t, a, \overline{a}, b, \overline{b}, c\right) := l\left(t, a, \overline{b}, c\right) + \frac{c_n}{2} \left\| b - A\overline{a} + Bc \right\|_{\mathbb{R}^N}^2, \\
&J_n(x(\cdot), u(\cdot)) \\
&\quad := \int_0^{\top} L_n\left(t, x(t), x\left(t-\tau_1\right), \dot{x}(t), \dot{x}\left(t-\tau_2\right), u(t)\right) dt, \\
&\qquad D := \left\{ x(\cdot) \in \mathscr{H} : x(\top) = \alpha \right\}.
\end{aligned} \tag{44}$$

The sequence of unconstrained optimal control problems takes then the following form:

$$\begin{aligned}
&\inf\ J_n(x(\cdot), u(\cdot)), \\
&\qquad x(\cdot) \in D, \\
&\qquad u(\cdot) \in U_0,
\end{aligned} \tag{$\mathscr{P}_n$}$$

$n \in \mathbb{N}$.

Lemma 5. *The cone of tangents $T(U_0, u(\cdot))$ is given by*

$$T\left(U_0, u(\cdot)\right) = \left\{ w(\cdot) \in L^2\left(I, \mathbb{R}^m\right) : w(0) = 0 \right\}. \tag{45}$$

Proof. It is similar to the proof of Lemma 2. □

It is well known that the penalty function method is a very effective technique for solving constrained optimization problems via unconstrained ones. The main question is the convergence of the sequence of solutions of the unconstrained optimal control problems to the original/constrained problem. Before giving the convergence theorem, we begin with some preparatory results, which are a direct consequence of the necessary optimality conditions given by Theorem 4.

Proposition 6. *For every n, if $(x_n(\cdot), u_n(\cdot)) \in D \times U_0$ is an optimal solution to $(\mathscr{P}_n)$, then*

(i)

$$\begin{aligned}
&\frac{d}{dt}\phi_n(t) = A^*\phi_n\left(t+\tau_1\right) + \frac{1}{c_n}a_n(t) - \frac{1}{c_n}e_n(t) \\
&\qquad\qquad \text{a.e. } t \in \left[0, \top - \tau_1\right], \\
&\frac{d}{dt}\phi_n(t) = \frac{1}{c_n}a_n(t) - \frac{1}{c_n}e_n(t) \quad \text{a.e. } t \in \left]\top - \tau_1, \top - \tau_2\right], \\
&\frac{d}{dt}\phi_n(t) = \frac{1}{c_n}a_n(t) \quad \text{a.e. } t \in \left]\top - \tau_2, \top\right], \\
&B^*\phi_n(t) = \frac{1}{c_n}b_n(t) \quad \text{a.e. } t \in [0, \top],
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&\phi_n(t) = \dot{x}_n(t) - Ax_n\left(t-\tau_1\right) + Bu_n(t), \\
&a_n(t) = l_a'\left(t, x_n(t), \dot{x}_n\left(t-\tau_2\right), u_n(t)\right), \\
&e_n(t) = l_b'\left(t, x_n(t), \dot{x}_n\left(t-\tau_2\right), u_n(t)\right), \\
&b_n(t) = l_c'\left(t, x_n(t), \dot{x}_n\left(t-\tau_2\right), u_n(t)\right);
\end{aligned} \tag{47}$$

(ii) *there exists $M > 0$ such that $\|\phi_n(t)\|_X \leq M$ for all $t \in [0, \top]$ and all n sufficiently large.*

Proof. (i) Let $(x_n(\cdot), u_n(\cdot)) \in D \times U_0$ be an optimal solution to $(\mathscr{P}_n)$. Then, by Lemmas 2 and 5 and Theorem 4, we obtain the necessary conditions of item (i) for problem $(\mathscr{P}_n)$.

(ii) Since $u_n(0) = 0$ and $\dot{\theta}(0^+)$ exists, $\dot{x}_n(0)$ is defined and there exists $k > 0$ such that $\|\phi_n(0)\| \leq k$. By the first equation of item (i), we have

$$\begin{aligned}
\phi_n(t) &= \phi_n(0) + A^* \int_0^t \phi_n\left(\tau + \tau_1\right) d\tau \\
&\quad + \frac{1}{c_n} \int_0^t \left(a_n(\tau) + e_n(\tau)\right) d\tau, \\
&\qquad t \in \left[0, \top - \tau_1\right].
\end{aligned} \tag{48}$$

Consequently,

$$\begin{aligned}
\left\|\phi_n(t)\right\| &\leq \left\|\phi_n(0)\right\| + \left\|A^*\right\| \int_0^t \left\|\phi_n\left(\tau + \tau_1\right)\right\| d\tau + R_n \\
&\leq k + \alpha \int_0^t \left\|\phi_n\left(\tau + \tau_1\right)\right\| d\tau + R_n
\end{aligned} \tag{49}$$

for all $t \in [0, \top - \tau_1]$ with $k = \|\phi_n(0)\|$, $\alpha = \|A^*\|$, and $R_n = (1/c_n)(\|\gamma_2(\cdot)\|_{L^2} + \|\gamma_3(\cdot)\|_{L^2})$. By Gronwall's lemma, we obtain that

$$\left\|\phi_n(t)\right\| \leq \left(k + R_n\right) \exp\left(\alpha\left(\top - \tau_1\right)\right) \quad \forall t \in \left[0, \top - \tau_1\right]. \tag{50}$$

The second and third equalities of item (i) give

$$\left\|\phi_n(t)\right\| \le \left\|\phi_n\left(\top-\tau_1\right)\right\| + R_n \quad \forall t \in \left]\top-\tau_1, \top\right]. \tag{51}$$

Now, inequalities (50) and (51) imply that

$$\left\|\phi_n(t)\right\| \le M_n \quad \forall n,\ \forall t \in [0,\top] \tag{52}$$

with

$$M_n = k\exp\left(\alpha\left(\top-\tau_1\right)\right) + R_n\left(1+\exp\left(\alpha\left(\top-\tau_1\right)\right)\right). \tag{53}$$

Since $R_n \to 0$, there exists $M > 0$ such that

$$\left\|\phi_n(t)\right\|_{\mathbb{R}^N} \le M \tag{54}$$

for all $t \in [0,\top]$ and for all n large. □

We are now ready to prove the convergence theorem, which reads as follows.

Theorem 7 (penalty convergence theorem). *If hypotheses* (H_1)–(H_4) *hold and problem* (40) *and* (41) *has a finite value, then the sequence* $(x_n(\cdot), u_n(\cdot))_n$ *of solutions to* $(\mathscr{P}_n)$ *contains a subsequence* $(x_k(\cdot), u_k(\cdot))_k$ *such that*

(i) $x_k(\cdot) \to x(\cdot)$ *strongly in* $C(I,\mathbb{R}^N)$;

(ii) $u_k(\cdot) \to u(\cdot)$ *weakly in* $L^2(I,\mathbb{R}^m)$;

(iii) $\dot{x}_k(\cdot) \to \dot{x}(\cdot)$ *weakly in* $L^2(I,\mathbb{R}^N)$;

with $(x(\cdot), u(\cdot))$ *a solution to problem* (40) *and* (41).

Proof. Let $(x_n(\cdot), u_n(\cdot)) \in D \times U_0$ be an optimal solution to $(\mathscr{P}_n)$ for every n. By Proposition 6,

$$\begin{aligned}\left\|\dot{x}_n(t)\right\| &\le M + \|A\|\left\|x_n\left(t-\tau_1\right)\right\| + \|B\|\left\|u_n(t)\right\| \\ &\le M + \beta\left\|x_n\left(t-\tau_1\right)\right\| + \sigma\left\|u_n(t)\right\|.\end{aligned} \tag{55}$$

Because

$$\begin{aligned}x_n(t) &= \theta(0) + \int_0^t \dot{x}_n(\tau)\,d\tau \quad \forall t \in [0,\top], \\ x_n(t) &= \theta(t) \quad \text{a.e. } t \in \left[-\tau_1, 0\right],\end{aligned} \tag{56}$$

it follows that

$$\begin{aligned}\left\|x_n(t)\right\| &\le \|\theta(0)\| + \int_0^t \left\|\dot{x}_n(\tau)\right\| d\tau \\ &\le \|\theta(0)\| + M\top + \beta\int_0^t \left\|x_n(\tau-h)\right\| d\tau \\ &\quad + \sigma\int_0^t \left\|u_n(\tau)\right\| d\tau.\end{aligned} \tag{57}$$

On the other hand, if $\mathscr{M}$ denotes the finite value of (40) and (41), then

$$\int_0^\top l\left(t, x_n(t), u_n(t)\right) dt \le J_n\left(x_n(\cdot), u_n(\cdot)\right) \le \mathscr{M}. \tag{58}$$

By assumption (H_3), there exists $K > 0$ such that

$$\left\|u_n(\cdot)\right\|_{L^2} \le K. \tag{59}$$

Thus,

$$\begin{aligned}\left\|x_n(t)\right\| \le\ & \|\theta(0)\| + M\top + \sigma\top K \\ &+ \beta\int_{-\tau_1}^0 \|\theta(\tau)\|\,d\tau + \beta\int_0^{t-\tau_1} \left\|x_n(\tau)\right\| d\tau.\end{aligned} \tag{60}$$

By Gronwall's lemma, we obtain that

$$\left\|x_n(t)\right\| \le \psi \quad \text{for } n \text{ sufficiently large, } \forall t \in [0,\top], \tag{61}$$

where

$$\psi = \left(\|\theta(0)\| + M\top + \sigma\top K + \beta\tau_1\|\theta(\cdot)\|\right)\exp\left(\beta\left(\top-\tau_1\right)\right). \tag{62}$$

Similarly, for n sufficiently large,

$$\left\|\dot{x}_n(t)\right\| \le M + \beta\left\|x_n\left(t-\tau_1\right)\right\| + \sigma\left\|u_n(t)\right\|. \tag{63}$$

For all $t \in [0,h]$, we have

$$\left\|\dot{x}_n(t)\right\| \le M + \gamma(t) + \beta\left\|\theta\left(t-\tau_1\right)\right\| + \sigma\left\|u_n(t)\right\| = \omega(t). \tag{64}$$

Since $\omega(\cdot) \in L^2(I,\mathbb{R})$ and $(u_n(\cdot))_n$ is bounded in $L^2(I,\mathbb{R}^m)$, with $[0,h]$ of finite measure, there exists $\varrho > 0$ such that

$$\left\|\dot{x}_n(\cdot)\right\| \le \varrho \quad \text{in } L^2\left(\left[0,\tau_1\right],\mathbb{R}^N\right), \text{ for } n \text{ sufficiently large.} \tag{65}$$

For all $t \in [\tau_1,\top]$ we have

$$\left\|\dot{x}_n(t)\right\| \le M + \gamma(t) + \beta\psi + \sigma\left\|u_n(t)\right\|. \tag{66}$$

As before, we can assert that

$$\begin{aligned}&\exists\widehat{\varrho} > 0 : \left\|\dot{x}_n(\cdot)\right\| \le \widehat{\varrho} \\ &\text{in } L^2\left(\left[\tau_1,\top\right],\mathbb{R}^N\right) \text{ for } n \text{ sufficiently large.}\end{aligned} \tag{67}$$

By (65) and (67), there exists $\eta > 0$ such that

$$\left\|\dot{x}_n(\cdot)\right\| \le \eta \tag{68}$$

in $L^2([0,\top],\mathbb{R}^N)$ for n sufficiently large. Therefore, there exists a subsequence $(\dot{x}_k(\cdot))_k$ of $(\dot{x}_n(\cdot))_n$ converging to $\sigma(\cdot) \in L^2(I,\mathbb{R}^N)$. Since $x_n(t) = \theta(0) + \int_0^t \dot{x}_n(\tau)d\tau$ for all $t \in I$, by the use of (61), the sequence $(x_n(\cdot))_n$ is equi-bounded and equi-continuous (because $(\dot{x}_n(\cdot))_n$ is bounded in $L^2(I,\mathbb{R}^N)$). Ascoli's theorem implies that

$$x_k(\cdot) \longrightarrow x(\cdot) \quad \text{strongly in } C\left(I,\mathbb{R}^N\right). \tag{69}$$

Since

$$x_k(0) = \theta(0), \qquad \int_0^t \dot{x}_n(\tau)\,d\tau \longrightarrow \int_0^t \sigma(\tau)\,d\tau, \tag{70}$$

we obtain that $x(t) = \theta(0) + \int_0^t \sigma(\tau)d\tau$ and $\dot{x}(t) = \sigma(t)$ a.e. $t \in I$. The sequence $(u_n(\cdot))_n$ is bounded in $L^2(I, \mathbb{R}^m)$. Thus, there exists a subsequence $(u_k(\cdot))_k$ such that $u_k(\cdot) \rightharpoonup u(\cdot)$ weakly in $L^2(I, \mathbb{R}^N)$. To complete the proof, we show that $(x(\cdot), u(\cdot))$ is an optimal solution to (40)-(41). By Proposition 6, we have

$$B^* \phi_k(t) = \frac{1}{c_k} b_k(t) \quad \text{a.e. } t \in I. \tag{71}$$

Hence,

$$\int_0^t \left\| B^* \phi_k(\tau) \right\| d\tau = \frac{1}{c_k} \int_0^t \left\| b_k(\tau) \right\| d\tau \leq \frac{1}{c_k} M \tag{72}$$

with $M = \top \|\gamma_4(\cdot)\|_{L^2}$. We conclude that

$$\int_0^t B^* \phi_k(\tau)\, d\tau \longrightarrow 0 \quad \forall t \in I. \tag{73}$$

On the other hand,

$$\int_0^t B^* \phi_k(\tau)\, d\tau \longrightarrow \int_0^t B^* \phi(\tau)\, d\tau \quad \forall t \in I. \tag{74}$$

Consequently,

$$\int_0^t B^* \phi(\tau)\, d\tau = 0 \quad \forall t \in I. \tag{75}$$

This implies that

$$B^* \phi(t) = 0 \quad \forall t \in I. \tag{76}$$

Thus,

$$\begin{aligned} \phi(t) &= 0 \quad \forall t \in I, \\ \dot{x}(t) &= Ax\left(t - \tau_1\right) + Bu(t) \quad \forall t \in I \end{aligned} \tag{77}$$

and $x(t) = \theta(t)$ a.e. $t \in [-\tau_1, 0]$, $x(\top) = \alpha$. Then, $(x(\cdot), u(\cdot))$ is an admissible pair and

$$\int_I l\left(t, x(t), \dot{x}\left(t - \tau_2\right), u(t)\right) dt \geq \mathscr{M}. \tag{78}$$

On the other hand,

$$\int_I l\left(t, x_k(t), \dot{x}_k\left(t - \tau_2\right), u_k(t)\right) dt \leq \mathscr{M}. \tag{79}$$

Now the hypotheses (H_1), (H_2), and (H_4), together with Lebesgue's theorem, assert that

$$\int_I l\left(t, x(t), \dot{x}\left(t - \tau_2\right), u(t)\right) dt \leq \mathscr{M}; \tag{80}$$

that is,

$$\int_I l\left(t, x(t), \dot{x}\left(t - \tau_2\right), u(t)\right) dt = \mathscr{M}. \tag{81}$$

This implies that the pair $(x(\cdot), u(\cdot))$ is a solution to problem (40) and (41). □

4. Conclusion

New optimality conditions for problems of the calculus of variations and optimal control with time delays, where the delay in the unknown function differs from the delay in its derivative/control, were obtained. The proofs are first given in the simpler context of the delayed calculus of variations and then extended to delayed optimal control problems by using a penalty method. New results include a convergence theorem (see Theorem 7), which is of great practical interest because it allows us to obtain a solution to a delayed optimal control problem by considering a sequence of simpler problems of the calculus of variations. Previous results in the literature [2, 3, 11] consider the delay in the unknown function to be the same as the delay in its derivative. There is, however, no justification for the delays to be the same. In contrast with those results, here we consider the case of multiple time delays. Moreover, the procedure of our proofs is completely different from the case of one time delay only, which relies on the Lagrange multiplier method. Such approach introduces a new unknown function, the Lagrange multiplier, for which it is hard to set the interpolation space. Indeed, the Lagrange multiplier must be carefully selected in order to be possible to obtain an accurate solution. Otherwise, the resulting system of equations may become singular, in particular if the number of degrees of freedom is too large. Here we use a penalty method, which requires only the choice of one scalar parameter. Big values of this parameter are used in order to impose the boundary conditions in a proper manner. Furthermore, in our case the use of the penalty method replaces a constrained optimization problem (the delayed optimal control problem) by a sequence of unconstrained problems of the calculus of variations with time delay whose solutions converge to the solution of the original constrained problem. Similarly to [11], our results can be easily extended for controls with time delay.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Robust Moving Horizon $\mathcal{H}_\infty$ Control of Discrete Time-Delayed Systems with Interval Time-Varying Delays

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In this study, design of a delay-dependent type moving horizon state-feedback control (MHHC) is considered for a class of linear discrete-time system subject to time-varying state delays, norm-bounded uncertainties, and disturbances with bounded energies. The closed-loop robust stability and robust performance problems are considered to overcome the instability and poor disturbance rejection performance due to the existence of parametric uncertainties and time-delay appeared in the system dynamics. Utilizing a discrete-time Lyapunov-Krasovskii functional, some delay-dependent linear matrix inequality (LMI) based conditions are provided. It is shown that if one can find a feasible solution set for these LMI conditions iteratively at each step of run-time, then we can construct a control law which guarantees the closed-loop asymptotic stability, maximum disturbance rejection performance, and closed-loop dissipativity in view of the actuator limitations. Two numerical examples with simulations on a nominal and uncertain discrete-time, time-delayed systems, are presented at the end, in order to demonstrate the efficiency of the proposed method.

1. Introduction

Time-delay systems have drawn a considerable amount of attention in the last few decades due to the fact that these systems represent the behaviour of processes more close to real world situations. The primary effects of delay in the behaviour of physical systems are on deterioration of performance and exhibition of an unstable respond. There are many sources of delay in a system which can be exemplified such as long transmission lines or intensive communication channels, approximations in the identification or modelling of real systems, and finite rate or capability of computing power for control and communication purposes in remote systems. In the literature, a great deal of studies have comprehensively investigated the time delayed systems in terms of their negative effects on the stability [1–3] and performance [4–6] of feedback systems.

$\mathcal{H}_\infty$ control mainly deals with the synthesis of feedback or feed-forward controller for dynamical systems to establish a stable behaviour or respond in a robust sense when the controlled system is subjected to an external disturbance effect. Particularly, in the last 2 decades, $\mathcal{H}_\infty$ control problem of time-delay systems received a considerable amount of interest, which might be constant or even time-varying. The main reason arises from the fact that the so-called bounded real lemma (BRL) signifying the $\mathcal{H}_\infty$ performance of a given plant or system devoid of any delay effect proves to yield a necessary condition while its counterpart given for the case of an existence of time-delay can afford to achieve only a sufficient condition. This observation implies that the $\mathcal{H}_\infty$ results concerning time-delay systems inherently involve potential conservativeness which lead the researchers in the time-delay community to construct further investigation to achieve better $\mathcal{H}_\infty$ performance. Therefore, for continuous time case, one can refer to [7–11] and the references therein for the $\mathcal{H}_\infty$ control of time-delay systems (TDSs) having zero lower delay bound and [10, 12–16] for the $\mathcal{H}_\infty$ control of TDSs with interval time-delays. For the discrete-time counterpart, the reader can refer to [17–19] and the references therein.

Moving horizon control (MHC), also called receding horizon control (RHC) or model predictive control (MPC)

in the literature, is a widely used method of control of industrial processes, especially having large number of variables and constraints, due to its superior capabilities in handling constraints on control and states, operating with less human intervention and reacting dynamically to system changes easily. The main approach in MPC is to solve a constraint optimization problem on-line at each time instant by utilizing the past and current feedback information and apply only the very recent control solution to the system. For a good survey and recent approaches on the concept, one can refer to [20–26] and the references therein. The general classification and areas of application of these systems have been comprehensively investigated in references [27, 28].

It is well known that uncertainties and time-delays cannot be avoided in real life, especially, in many processes such as chemical, biological, and network problems where the MPCs have been widely used. However, it is apparently seen from the literature that only very few results exist concerning the MPC of time-delay systems. Based on the *L-K* functional approach and the use of some relaxation matrices [29] has considered the problem of receding horizon control for state-delayed systems where the delay is fixed. Also, they have presented an eigenvalue search algorithm to check the closed-loop stability of the system. However their control strategy is delay-independent, therefore highly conservative. Reference [30] introduced a novel optimization method for MPC of uncertain time-delay systems having constant state delays and constraint input. But, again, their method does not depend on the size of delay. A robust one-step LMI based MPC scheme is developed for discrete time-delay systems having fixed delay with polytopic-type uncertainty in [31]. Their proposed MPC method is obtained by minimizing a new cost function that includes multiterminal weighting terms, subject to constraints on input.

The deficiency of MPC systems in disturbance attenuation has drawn the attention towards the moving horizon $\mathscr{H}_\infty$ control (MHHC) scheme in recent years, [26, 32, 33]. In this perspective, a Riccati-based MHHC algorithm was developed for a nominal time-delay system by [29, 34]. Then, a similar approach was proposed in [35] based on LMIs. However, all the abovementioned approaches are based on criteria which do not depend on the size of delay and therefore provide more conservative results. While taking the time-delay bounds into consideration, a delay-dependent MPC is proposed by [31] which guarantees only the closed-loop stability. Reference [35] has dealt with the receding horizon $\mathscr{H}_\infty$ control for constraint time-delay systems having fixed delays. However, their method is delay-independent as the ones indicated above. Finally, proposing a new cost function for a finite horizon dynamic game problem which includes two terminal weighting terms and a delay-independent LMI condition, [36] has studied the receding horizon $\mathscr{H}_\infty$ control problem for time-delay systems with fixed state delays.

Among these aforementioned work, it is apparently seen that only very few results exist concerning the MPC of TDSs which gives us a motivation to study this problem. Furthermore, to the best of author's knowledge, there does not exist any other reference in literature which deals with the robust $\mathscr{H}_\infty$ MPC of uncertain time-delay systems having time-varying delays in delay-dependent fashion. Therefore, combining these two sources of inspiration leads us to study that particular subject thoroughly.

In this paper, we investigate the design of a stabilizing $\mathscr{H}_\infty$ state-feedback controller for linear state-delayed interval TDS having norm-bounded uncertainties and constrained inputs. First, based on the selection of an *L-K* functional, a stabilizing delay-dependent $\mathscr{H}_\infty$ controller is introduced for nominal TDS having delay in the state, similar to the previous study of authors [37]. Then, the existing result is extended to cover TDS having norm-bounded uncertainties. Finally, the proposed approach is adapted to the so-called moving horizon scheme introduced in [26], to obtain a robust, delay-dependent moving horizon $\mathscr{H}_\infty$ control technique which ensures a dissipative closed-loop system. The proposed technique is practically implementable since it takes the input constraints into consideration and it does not employ any linearization techniques such as cone-complementary method which is widely used in delay-dependent control approaches such as those used in [11].

The rest of this paper is organized as follows: Section 2 states the problem formulation. Mathematical background is given in Section 3 in order to provide a priori knowledge about the methodology followed. Section 4 is devoted to the proposed method. Two different illustrative examples are presented in Section 5 to demonstrate the effectiveness of the approach. Finally, Section 6 concludes the paper.

Notation. R and R^n represent the set of real numbers and of n-dimensional real vectors, respectively. Identity matrix with an appropriate dimension is denoted by I. $X > 0$ ($\geq$, < 0) implies that X is a positive-definite (positive semidefinite, negative-definite) matrix. Likewise, $X > Y$ means that $X-Y$ is positive definite. $(\cdot^T)$ indicates transpose of a real matrix. To avoid repetition, "$*$" denotes off-diagonal block completion of a symmetric matrix. x_k and $x(k)$ are used interchangeably to simplify the notation especially in long mathematical expressions. Again, in order to make the notation easy, we drop the time-dependence of functions when it does not make any trouble. Finally, $\|x\|$ stands for the standard Euclidean 2-norm of the vector x. $\|x\|_{\max}$ is the absolute value of the entry of a vector x having the largest magnitude.

2. Problem Formulation

Let us consider a class of uncertain linear discrete-time, time-delay system of the form

$$\begin{aligned} x(k+1) &= A(k)\,x(k) + A_d(k)\,x(k-d(k)) \\ &\quad + B_w w(k) + B_u(k)\,u(k), \\ z(k) &= Cx(k) + C_d x(k-d(k)) + D_w w(k) + D_u u(k), \\ x(k) &= \phi(k), \quad k \in [-d_{\max}, 0] \end{aligned} \tag{1}$$

subject to control constraints

$$\left\|u_i(k)\right\|_{\max} \leq u_{i,\max}^2, \quad \forall k \geq 0,\ i = 1, 2, \ldots, m, \tag{2}$$

where k stands for the kth sample-time, $x(k) \in \mathrm{R}^n$ is the state vector, $u(k) \in \mathrm{R}^m$ represents the control input, $u_{i,\max}$

is a known magnitude bound on the control effort in the ith channel, $\phi(k)$ is a given initial condition sequence, and $w(k) \in \mathrm{R}^q$ is a disturbance signal in l_2, which satisfies

$$\sum_{i=0}^{\infty} \|w(i)\|^2 \leq \alpha^2. \tag{3}$$

$z(k) \in \mathrm{R}^p$ is the controlled output and $d(k) \in \mathrm{R}$ represents a time-varying delay which satisfies

$$d_{\min} \leq d(k) \leq d_{\max} \quad \forall k \geq 0, \tag{4}$$

where the nonnegative integers $d_{\min}$ and $d_{\max}$ stand for the lower and upper bounds of the delay $d(k)$, respectively, and all are assumed to be known. The time-varying delay $d(k)$ reduces to a constant delay d when $d_{\min} = d_{\max} = d$. On the other hand, the uncertain system matrices are assumed to be in the form of $A(k) = A + \Delta A(k)$, $A_d(k) = A_d + \Delta A_d(k)$, and $B_u(k) = B_u + \Delta B_u(k)$. Here, the unknown matrices $\Delta A(k)$, $\Delta A_d(k)$, and $\Delta B_u(k)$ are real-valued time-varying matrix functions representing the parameter uncertainties of the system and are in the form of

$$\left[\Delta A(k) \;\; \Delta A_d(k) \;\; \Delta B_u(k)\right] = GF(k)\left[E_1 \;\; E_2 \;\; E_3\right], \tag{5}$$

where G, E_1, E_2 are known time-invariant matrices and $F(k)$ is an unknown time-varying matrix function with Lebesgue measurable elements satisfying

$$F(k)^T F(k) \leq I, \quad \forall k \geq 0. \tag{6}$$

The objective of infinitive horizon MHHC problem is to obtain the state-feedback control law in the form of $u(k) = K_k x(k)$ at each instant k, so that the following conditions hold for the closed-loop system.

(I) The closed-loop system

$$\begin{aligned} x(k+1) &= A_{\mathrm{cl}}(k)\,x(k) + A_d(k)\,x(k - d(k)) + B_w w(k), \\ z(k) &= C_{\mathrm{cl}} x(k) + C_d x(k - d(k)) + D_w w(k), \\ x(k) &= 0, \quad k \in [-d_{\max}, 0], \end{aligned} \tag{7}$$

where $A_{\mathrm{cl}}(k) \triangleq A(k) + B_u(k)K_k$ and $C_{\mathrm{cl}} \triangleq C(k) + D_u(k)K_k$ is globally uniformly asymptotically stable under the conditions $w(k) = 0$ for all $k \geq 0$.

(II) Consider

$$\|z(k)\|^2 \leq \gamma^2 \|w(k)\|^2 \quad \forall k \geq 0 \tag{8}$$

and for all nonzero disturbance signal $w(k) \in l_2[0, \infty)$ and a given scalar $\gamma > 0$ under the condition $x(k) = 0$ for all $k \in [-d_{\max}, 0]$.

(III) Let us define two nested ellipsoids as follows:

$$\begin{aligned} \varepsilon_1(P, Q, r) &:= x \in \mathrm{R}^n : V(k) \leq r \quad \forall k \geq 0 \\ \varepsilon_2(P, Q, r, \alpha) &:= x \in \mathrm{R}^n : \gamma^2\alpha^2 + V(k) \leq r \quad \forall k \geq 0, \end{aligned} \tag{9}$$

where $V(k)$ is the Lyapunov function. If the initial state $x(0)$ satisfies $\gamma^2\alpha^2 + V(0) \leq r$, that is, $x(0)$ belongs to ε_2, then all perturbed state trajectories remain in the ellipsoid ε_1.

(IV) The constraint on the size of control effort (2) is satisfied for all $k \geq 0$.

In order to avoid complexity of the main result section, some related and required background material are provided next.

3. Preliminaries

We start with the definition of *dissipativity* which plays an important role during the implementation of moving horizon control.

Definition 1 (dissipativity). The system (7) with supply rate $s(w(k), z(k)) \triangleq \gamma^2\|w(k)\|^2 - \|z(k)\|^2$ is said to be dissipative if there exists a quadratic storage function $V(x(k))$ such that

$$V(k) + s(w(k), z(k)) \geq V(k+1) \quad \forall k \geq 0. \tag{10}$$

This implies that the change of storage function from step k to step $(k+1)$ is always less than the supplied rate to the system. Therefore, inequality (10) is named as dissipation inequality. If the dissipation inequality (10) holds true for any system trajectory of (1) and (4), then for any disturbance signal satisfying (3), the energy of the controlled output satisfies

$$\sum_{i=0}^{\infty} \|z(i)\|^2 \leq r, \tag{11}$$

where r is an upper bound for output energy and l_2 gain of the closed-loop system (7) from the disturbance $w(k)$ to the performance output $z(k)$ is bounded by γ.

4. Main Results

Taking $A(k) = A$, $A_d(k) = A_d$, and $B_u(k) = B_u$ in (1), the following theorem describes a criterion for constructing an $\mathscr{H}_\infty$ state-feedback controller in the form $u(k) = Kx(k)$, such that the closed-loop nominal system (7) with (4) is globally asymptotically stable.

Theorem 2. *Given the scalar constants $d_{\min}$, $d_{\max}$, and γ, the state-feedback control law $u(k) = LY^{-1}x(k)$ globally asymptotically stabilizes the time-delayed system* (1) *with* (4) *with an $\mathscr{H}_\infty$ disturbance attenuation level, γ, if there exist matrices $Y = Y^T > 0$, $W = W^T > 0$, and L with appropriate dimensions satisfying*

$$\begin{bmatrix} Y & * & * & * & * & * \\ 0 & W & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * \\ AY + B_u L & A_d W & B_w & Y & * & * \\ CY + D_u L & C_d W & D_w & 0 & I & * \\ Y & 0 & 0 & 0 & 0 & d_m^{-1} W \end{bmatrix} > 0, \tag{12}$$

where $d_m \triangleq d_{\max} - d_{\min} + 1$.

Proof. Let us choose a candidate L-K functional as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{13}$$

where

$$\begin{aligned} V_1(k) &= x^T(k) Px(k), \\ V_2(k) &= \sum_{s=k-d(k)}^{k-1} x^T(s) Qx(s), \\ V_3(k) &= \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=k+i-1}^{k-1} x^T(s) Qx(s). \end{aligned} \tag{14}$$

In view of the closed-loop system trajectory (7), one can define a first difference of the energy function $V(k)$ as follows:

$$\Delta V(k) \triangleq V(k+1) - V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k), \tag{15}$$

where

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1) Px(k+1) - x^T(k) Px(k) \\ &= x^T(k) \left(A_{\mathrm{cl}}^T PA_{\mathrm{cl}} - P\right) x(k) \\ &\quad + 2x^T(k) A_{\mathrm{cl}}^T PA_d x(k-d(k)) \\ &\quad + 2x^T(k) A_{\mathrm{cl}}^T PB_w w(k) \\ &\quad + x^T(k-d(k)) A_d^T PA_d x(k-d(k)) \\ &\quad + 2x^T(k-d(k)) A_d^T PB_w w(k) \\ &\quad + w^T(k) B_w^T PB_w w(k), \end{aligned} \tag{16}$$

$$\begin{aligned} \Delta V_2(k) &= \sum_{s=k-d(k+1)+1}^{k} x^T(s) Qx(s) - \sum_{s=k-d(k)}^{k-1} x^T(s) Qx(s) \\ &= x^T(k) Qx(k) - x^T(k-d(k)) Qx(k-d(k)) \\ &\quad + \sum_{s=k-d(k+1)+1}^{k-1} x^T(s) Qx(s) - \sum_{s=k-d(k)+1}^{k-1} x^T(s) Qx(s). \end{aligned} \tag{17}$$

Since

$$\begin{aligned} &\sum_{s=k-d(k+1)+1}^{k} x^T(s) Qx(s) \\ &= \sum_{s=k-d_{\min}+1}^{k} x^T(s) Qx(s) + \sum_{s=k-d(k+1)+1}^{k-d_{\min}} x^T(s) Qx(s) \\ &\le \sum_{s=k-d(k)+1}^{k} x^T(s) Qx(s) + \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^T(s) Qx(s) \end{aligned} \tag{18}$$

we can write

$$\begin{aligned} \Delta V_2(k) &\le x^T(k) Qx(k) - x^T(k-d(k)) Qx(k-d(k)) \\ &\quad + \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^T(s) Qx(s). \end{aligned} \tag{19}$$

Finally,

$$\begin{aligned} &\Delta V_3(k) \\ &= \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \left(\sum_{s=k+i}^{k} x^T(s) Qx(s) - \sum_{s=k+i-1}^{k-1} x^T(s) Qx(s) \right) \\ &= \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \left(x^T(k) Qx(k) - x^T(k+i-1) Qx(k+i-1) \right) \\ &= \left(d_{\max} - d_{\min}\right) x^T(k) Qx(k) - \sum_{i=k-d_{\max}+1}^{k-d_{\min}} x^T(i) Qx(i). \end{aligned} \tag{20}$$

Then, in the light of (16), (19), and (20), a bound on $\Delta V(k)$ can be obtained as follows:

$$\begin{aligned} \Delta V(k) &\le x^T(k) \left(A_{\mathrm{cl}}^T PA_{\mathrm{cl}} - P\right) x(k) \\ &\quad + 2x^T(k) A_{\mathrm{cl}}^T PA_d x(k-d(k)) \\ &\quad + 2x^T(k) A_{\mathrm{cl}}^T PB_w w(k) \\ &\quad + x^T(k-d(k)) A_d^T PA_d x(k-d(k)) \\ &\quad + 2x^T(k-d(k)) A_d^T PB_w w(k) \\ &\quad + w^T(k) B_w^T PB_w w(k) \\ &\quad + x^T(k) Qx(k) - x^T(k-d(k)) Qx(k-d(k)) \\ &\quad + \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^T(s) Qx(s) \\ &\qquad + \left(d_{\max} - d_{\min}\right) x^T(k) Qx(k) \\ &\quad - \sum_{s=k-d_{\max}+1}^{k-d_{\min}} x^T(s) Qx(s). \end{aligned} \tag{21}$$

Then, if we define the right-hand side of (21) with $\overline{\Delta} V(k)$, it is obvious that

$$\begin{aligned} &\Delta V(k) + z^T(k) z(k) - \gamma^2 w^T(k) w(k) \\ &\quad \le \overline{\Delta} V(k) + z^T(k) z(k) - \gamma^2 w^T(k) w(k). \end{aligned} \tag{22}$$

Therefore, if

$$\overline{\Delta} V(k) + z^T(k) z(k) - \gamma^2 w^T(k) w(k) < 0 \tag{23}$$

is satisfied for all $k \geq 0$, then the nominal closed-loop system (7) with (4) is guaranteed to be globally, uniformly, and asymptotically stable when $w(k) = 0$ for all $k \geq 0$, and we get $\|z\|_2^2 \leq \gamma^2 \|w\|_2^2$ for all $w(k) \in l_2[0, \infty)$. Equivalently, a straightforward application of the Schur complement formula on (23) allows us to write

$$\begin{bmatrix} d_m Q - P & * & * & * & * \\ 0 & -Q & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ A_{\mathrm{cl}} & A_d & B_w & -P^{-1} & * \\ C_{\mathrm{cl}} & C_d & D_w & 0 & -I \end{bmatrix} < 0, \tag{24}$$

where $d_m := d_{\max} - d_{\min} + 1$. Then, substituting $A_{\mathrm{cl}} = A + B_u K$ and $C_{\mathrm{cl}} = C + D_u K$ into (24), pre- and postmultiplying (24) by $\mathrm{diag}\{P^{-1}, Q^{-1}, I, I, I\}$, and applying the Schur complement which is followed by defining $Y := P^{-1}$, $W := Q^{-1}$ immediately lead to the LMI condition given in (12). This concludes our proof. □

Now, let us consider (1) with uncertainties. The following corollary considers the robust state-feedback $\mathscr{H}_\infty$ controller design problem for the uncertain linear time-delay system given in (1), (4) with (5).

Corollary 3. *The control law $u(k) = LY^{-1}x(k)$ robustly asymptotically stabilizes the uncertain system* (1) *and guarantees the $\mathscr{H}_\infty$ performance index, γ, for any time-varying delay, $d(k)$, satisfying* (4)*, if there exist matrices $Y = Y^T > 0$, $W = W^T > 0$, and L in appropriate dimensions and a scalar $\epsilon > 0$ which satisfy the following LMI condition:*

$$\begin{bmatrix} Y & * & * & * & * & * & * \\ 0 & W & * & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * & * \\ AY + B_u L & A_d W & B_w & Y - \epsilon G G^T & * & * & * \\ CY + D_u L & C_d W & D_w & 0 & I & * & * \\ Y & 0 & 0 & 0 & 0 & d_m^{-1} W & * \\ E_2 L + E_1 Y & E_3 W & 0 & 0 & 0 & 0 & \epsilon I \end{bmatrix} > 0. \tag{25}$$

Proof. Replacing A, A_d, and B_u with $A(k) = A + \Delta A$, $A_d(k) = A_d + \Delta A_d$, and $B_u(k) = B_u + \Delta B_u$ in (12) and utilizing the definitions given in (5) and (6) together with the definitions

$$\begin{aligned} \Pi &\triangleq \begin{bmatrix} E_2 L + E_1 Y & E_3 W & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \Theta &\triangleq \begin{bmatrix} 0 & 0 & 0 & G^T & 0 & 0 \end{bmatrix}^T \end{aligned} \tag{26}$$

yield

$$\begin{bmatrix} -Y & * & * & * & * & * \\ 0 & -W & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AY + B_u L & A_d W & B_w & -Y & * & * \\ CY + D_u L & C_d W & D_w & 0 & -I & * \\ Y & 0 & 0 & 0 & 0 & -d_m^{-1} W \end{bmatrix} + \Pi^T F(k) \Theta^T + \Theta F(k)^T \Pi < 0. \tag{27}$$

Note that for any matrices Σ_1 and Σ_2 of appropriate dimensions, there always exists an $\epsilon > 0$ such that

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 < \frac{1}{\epsilon} \Sigma_1^T \Sigma_1 + \epsilon \Sigma_2^T \Sigma_2. \tag{28}$$

Therefore

$$\Pi^T F(k) \Theta^T + \Theta F(k)^T \Pi \leq \frac{1}{\epsilon} \Pi^T \Pi + \epsilon \Theta \Theta^T. \tag{29}$$

Then, if

$$\begin{bmatrix} -Y & * & * & * & * & * \\ 0 & -W & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AY + B_u L & A_d W & B_w & -Y & * & * \\ CY + D_u L & C_d W & D_w & 0 & -I & * \\ Y & 0 & 0 & 0 & 0 & -d_m^{-1} W \end{bmatrix} + \frac{1}{\epsilon} \Pi^T \Pi + \epsilon \Theta \Theta^T < 0 \tag{30}$$

is satisfied, (27) is also satisfied. Finally, applying Schur complement formula to (30) yields (25). This completes the proof. □

Next we will introduce some LMI conditions which fulfill the requirement given in item (III).

Lemma 4. *Suppose that there exists a solution (γ, Y, W, X) that meets the stability condition* (25) *with input constraint* (2) *for system* (1) *and* (4)*, then state trajectory starting from $x(0)$ remains in $\varepsilon_1(P, r)$, if there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ for given positive scalar constants α and r, such that*

$$\begin{bmatrix} r-\gamma^2\alpha^2 & * & * & \cdots & * & \cdots & * & * & * \\ x(k) & Y & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ x(k-1) & 0 & d_m^{-1}W & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x(k-d_{\min}) & 0 & 0 & \cdots & d_m^{-1}W & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x(k-d_{\max}+2) & 0 & 0 & \cdots & 0 & \cdots & 3^{-1}W & 0 & 0 \\ x(k-d_{\max}+1) & 0 & 0 & \cdots & 0 & \cdots & 0 & 2^{-1}W & 0 \\ x(k-d_{\max}) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & W \end{bmatrix} \geq 0, \tag{31}$$

where $d_m = (d_{\max} - d_{\min} + 1)$.

Proof. The LMI condition (31) corresponds to the state constraint that is defined in (9). Utilizing the energy bound of disturbance given in (3), the dissipation inequality (10) implies

$$V(x(k)) + \sum_{i=0}^{k-1} \|z(i)\|^2 \leq V(x(0)) + \gamma^2\alpha^2 \quad \forall k \geq 0. \tag{32}$$

Therefore, the output energy is bounded as (11). As explained in [27],

(i) if $r > 0$, $x(k) \in \varepsilon_1(P, Q, r)$ for all $k \geq 0$;

(ii) if $r > \gamma^2\alpha^2$, $x(k) \in \varepsilon_1(P, Q, r)$ for $x(0) \in \varepsilon_2(P, r, \alpha)$.

Here, by choosing $r > \gamma^2\alpha^2$, the state trajectory starting from $x(0)$ meets $\sum_{i=0}^{\infty} \|z(i)\|^2 \leq r$ and $V(x(k)) \leq r$ if $V(x(k)) \leq r - \gamma^2\alpha^2$ with respect to (32) as mentioned in (8). Note that it is easy to see that the LMI condition (31) is congruent to

$$\begin{aligned} x^T(k) P x(k) &+ \sum_{s=k-d(k)}^{k-1} x^T(s) Q x(s) \\ &+ \sum_{i=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=k+i-1}^{k-1} x^T(s) Q x(s) \leq r - \gamma^2\alpha^2 \end{aligned} \tag{33}$$

in view of the variable changes $Y := P^{-1}$ and $W := Q^{-1}$ and the successive implementations of Schur complement formulae together with the bounding

$$\begin{aligned} \sum_{s=k-d(k)}^{k-1} x^T(s) Q x(s) \leq &\sum_{s=k-d_{\min}}^{k-1} x^T(s) Q x(s) \\ &+ \sum_{s=k-d_{\max}+1}^{k-d_{\min}-1} x^T(s) Q x(s), \end{aligned} \tag{34}$$

which has been used to eliminate the requirement on the knowledge of $d(k)$. □

On the other hand, we need to take the constrained input into consideration to fulfill the requirement given in item (IV) of the problem statement. Therefore, regarding the state trajectories belonging to ε_1, the control constraint condition given in (2) is satisfied as follows [38]:

$$\begin{aligned} \max_{k\geq 0} |u_i(k)|^2 &= \max_{k\geq 0} \left| \left(LY^{-1}\right)_i x(k) \right|^2 \\ &\leq \max_{x\in\varepsilon_1(P,r)} \left| \left(LY^{-1}\right)_i x \right|^2 \\ &\leq r \left\| \left(LY^{-1/2}\right)_i x \right\|^2 = r\left(LY^{-1}L^T\right)_{ii}. \end{aligned} \tag{35}$$

Hence, taking the Schur complement of (35) allows us to obtain the following LMI condition which is equivalent to the constraint (2):

$$\begin{bmatrix} \dfrac{1}{rX} & L \\ L^T & Y \end{bmatrix} \geq 0, \quad X_{ii} \leq u_{i,\max}^2 \tag{36}$$

for some $X = X^T > 0$.

Therefore, for given scalar constants $\alpha > 0$ and $r > 0$, the optimization problem can be stated as follows:

$$\begin{aligned} \min_{L,Y,W,X} \quad & \gamma^2 \\ \text{subject to} \quad & (22)\ (26),\ (31). \end{aligned} \tag{37}$$

Next, we will extend the $\mathscr{H}_\infty$ controller design approach to the moving horizon scheme. Exploiting the moving horizon strategy, one could solve the optimization problem given in (37) by utilizing the actual state measurements provided at each time instant k. Unfortunately, this simple implementation of the moving horizon strategy might fail to guarantee

the dissipation requirement given in the statement of the problem (see [39] for details). For that reason, in order to extend the idea of $\mathscr{H}_\infty$ control approach provided for the TDSs to the moving horizon platform, one needs to introduce some additional convex requirement so that the dissipativity of the closed-loop system is ensured. Next theorem provides an LMI condition for the optimization problem so that the closed-loop system is guaranteed to be dissipative.

Theorem 5. *Suppose that there exists an optimal solution* $(\gamma_0, Y_0, W_0, X_0)$ *that meets the LMIs* (25), (31), *and* (36) *for system* (1) *that is controlled with* $u = K_0 x$, *where* $K_0 = L_0 Y_0^{-1}$. *Then the dissipation inequality*

$$\begin{aligned}\sum_{i=0}^{k}\left(\|z(i)\|^2-\gamma^2\|w(i)\|^2\right)&\le x(0)^T P_0 x(0)+\sum_{s=-d_0}^{-1}x(s)^T Q_0 x(s)\\&\quad+\sum_{j=-d_{\min}+2}^{-d_{\max}+1}\sum_{s=j-1}^{-1}x(s)^T Q_0 x(s)\end{aligned}\tag{38}$$

also holds if the LMI condition

$$\begin{bmatrix}\Omega & * & * & \cdots & * & \cdots & * & *\\ x(k) & Y_k & 0 & \cdots & 0 & \cdots & 0 & 0\\ x(k-1) & 0 & d_m^{-1}W_k & \cdots & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots\\ x(k-d_{\min}) & 0 & 0 & \cdots & d_m^{-1}W_k & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots\\ x(k-d_{\max}+2) & 0 & 0 & \cdots & 0 & \cdots & 2^{-1}W_k & 0\\ x(k-d_{\max}+1) & 0 & 0 & \cdots & 0 & \cdots & 0 & W_k\end{bmatrix}\ge 0,\tag{39}$$

where $d_m = (d_{\max} - d_{\min} + 1)$ *and*

$$\begin{aligned}\Omega &= p_0 + q_0 - p_{k-1} - q_{k-1} + x^T(k)P_{k-1}x(k)\\&\quad+\sum_{s=k-d_{\max}}^{k-1}x^T(s)Q_{k-1}x(s)\\&\quad+\sum_{j=-d_{\max}+2}^{-d_{\min}+1}\sum_{s=j+k-1}^{k-1}x^T(s)Q_{k-1}x(s).\end{aligned}\tag{40}$$

with

$$\begin{aligned}p_k &= p_{k-1} - x(k)^T\left[P_{k-1}-P_k\right]x(k)\\q_k &= q_{k-1} - \sum_{s=k-d_{\min}}^{k-1}x(s)^T Q_{k-1}x(s) - \sum_{k-d_{\min}}^{k-1}x(s)^T Q_k x(s)\\&\quad-\sum_{j=-d_{\max+2}}^{-d_{\min}+1}\sum_{s=j+k-1}^{k-1}x(s)^T\left[Q_{k-1}-Q_k\right]x(s)\end{aligned}\tag{41}$$

holds true when Y_k *and* W_k *are replaced with* $Y_0 = Y_0^T$ *and* $W_0 = W_0^T$, *respectively.*

Proof. Assume there exists a feasible solution set which holds (39). Let us consider the predetermined Lyapunov-Krasovskii candidate functional for the ith energy level as follows:

$$\begin{aligned}V_i &= x^T(i)Px(i)+\sum_{s=i-d(i)}^{i-1}x^T(s)Qx(s)\\&\quad+\sum_{j=-d_{\max}+2}^{-d_{\min}+1}\sum_{s=i+j-1}^{i-1}x^T(s)Qx(s).\end{aligned}\tag{42}$$

By rearranging inequality (10) at each step i and assuming that the applied control at ith instant is kept constant up to the $(i+1)$th instant, one can write the dissipation inequality as

$$V_{i+1} + \|z(i)\|^2 \le \gamma^2\|w(i)\|^2 + V_i\tag{43}$$

for $i = 0, 1, \ldots, k$, where V_{i+1} is the energy of the closed-loop system attained by using the Lyapunov matrices obtained at the ith step. Then we get

$$\begin{aligned}\sum_{i=0}^{k}\left(\|z(i)\|^2-\gamma^2\|w(i)\|^2\right)&\le x(0)^T P_0 x(0) - x(k+1)^T P_k x(k+1)\\&\quad-\sum_{i=1}^{k}x(i)^T\left[P_{i-1}-P_i\right]x(i)+\sum_{s=-d_0}^{-1}x(s)^T Q_0 x(s)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{s=-d_{k+1}+k+1}^{k} x(s)^T Q_k x(s) \\
&- \sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \\
&+ \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j-1}^{-1} x(s)^T Q_0 x(s) \\
&- \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+k}^{k} x(s)^T Q_k x(s) \\
&- \sum_{i=1}^{k} \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s).
\end{aligned} \tag{44}
$$

Assuming that $P_k \geq 0$, $Q_k \geq 0$ for all $k \geq 0$, one can write

$$
\begin{aligned}
&\sum_{i=0}^{k} \left(\|z(i)\|^2 - \gamma^2 \|w(i)\|^2\right) \\
&\leq x(0)^T P_0 x(0) - \sum_{i=1}^{k} x(i)^T \left[P_{i-1} - P_i\right] x(i) \\
&\quad + \sum_{s=-d_0}^{-1} x(s)^T Q_0 x(s) \\
&\quad - \sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \\
&\quad + \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j-1}^{-1} x(s)^T Q_0 x(s) \\
&\quad - \sum_{i=1}^{k} \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s).
\end{aligned} \tag{45}
$$

Note that if

$$
\begin{aligned}
&\sum_{i=1}^{k} x(i)^T \left(P_{i-1} - P_i\right) x(i) + \sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \\
&\quad + \sum_{i=1}^{k} \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \geq 0
\end{aligned} \tag{46}
$$

then the closed-loop system behaves dissipatively. However, inequality (46) involves the term

$$
\sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \tag{47}
$$

which is dependent on the knowledge of $d(i)$. Therefore, if one can find a lower bound, B, such that

$$
\sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \geq B \geq 0 \tag{48}
$$

then we guarantee the dissipativity of the overall closed-loop system. Note that

$$
\begin{aligned}
&\sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \\
&\quad \geq \sum_{i=1}^{k} \sum_{s=i-d_{\min}}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s).
\end{aligned} \tag{49}
$$

Hence, we need to satisfy

$$
\begin{aligned}
&\sum_{i=1}^{k} x(i)^T \left(P_{i-1} - P_i\right) x(i) + \sum_{i=1}^{k} \sum_{s=i-d_{\min}}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \\
&\quad + \sum_{i=1}^{k} \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \geq 0.
\end{aligned} \tag{50}
$$

We infer from (41) that

$$
\begin{aligned}
p_{k-1} &= p_0 - \sum_{i=1}^{k-1} x(i)^T \left[P_{i-1} - P_i\right] x(i) \\
q_{k-1} &= q_0 - \sum_{i=1}^{k} \sum_{s=i-d_i}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s) \\
&\quad - \sum_{i=1}^{k} \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+i-1}^{i-1} x(s)^T \left[Q_{i-1} - Q_i\right] x(s).
\end{aligned} \tag{51}
$$

Applying Schur complement formula to LMI (39) yields

$$
\begin{aligned}
&p_0 - p_{k-1} + x(k)^T \left[P_{k-1} - P_k\right] x(k) + q_0 - q_{k-1} \\
&\quad + \sum_{s=k-d_{\min}}^{k-1} x(s)^T \left[Q_{k-1} - Q_k\right] x(s) \\
&\quad + \sum_{j=-d_{\max+2}}^{-d_{\min}+1} \sum_{s=j+k-1}^{k-1} x(s)^T \left[Q_{k-1} - Q_k\right] x(s) \geq 0.
\end{aligned} \tag{52}
$$

Then, using the definitions given in (51) and in (52) yields (50). This concludes the proof. □

Therefore, in view of (36) together with Corollary 3, Lemma 4, and Theorem 5, we present a new convex optimization problem as follows.

Theorem 6. *Given scalar constants $\alpha_0 > 0$ and $r_0 > 0$, at each step k, if one can find a common feasible solution set $\{\gamma_k, L_k, Y_k, W_k, X_k\}$ to the convex optimization problem*

$$
\begin{aligned}
&\min_{L,Y,W,X} && \gamma^2 \\
&\text{subject to} && (25), (31), (36), (39)
\end{aligned} \tag{53}
$$

then system (1) *controlled with state-feedback control law* $u(k) = K_k x(k)$, *where* $K_k = L_k Y_k^{-1}$, *has the following properties.*

(i) *The closed-loop system is* $\mathcal{H}_\infty$ *stable.*

(ii) *The* $\mathcal{H}_\infty$ *gain* γ *from disturbance input* w *to controlled output* z *is minimum.*

(iii) *The control input constraint is satisfied.*

(iv) *The closed-loop system is dissipative.*

The following receding-horizon $\mathcal{H}_\infty$ algorithm can be used to find the optimal control law for system (1) with (4) which repeats at each time instant k.

Algorithm 7.

Step 1. Defining the step size for γ as $d\gamma$; first, fix γ to a sufficiently large value and solve the optimization problem (37) for given α_0, r_0, $d_{\max}$, and $d_{\min}$. If there exists a feasible solution to the problem, set $\gamma = \gamma - d\gamma$ until infeasibility. Then, assign $\gamma_{\text{opt}} = \gamma + d\gamma$. Compute P_0, Q_0 and $K_0 = L_0 Y_0^{-1}$ and the initial dissipation level is $p(0) = x(0)^T P_0 x(0)$ and $q(0) = \sum_{s=-d_{\min}}^{-1} x^T(s) Q_0 x(s) + \sum_{s=-d_{\max}+2}^{-d_{\min}+1} \sum_{s=j-1}^{-1} x^T(s) Q_0 x(s)$. Go to Step 3.

Step 2. According to given α_0, r_0, $d_{\max}$, and $d_{\min}$ and obtained dissipation level with $p(k)$ and $q(k)$, solve the optimization problem (53). If there exists a feasible solution, set $\gamma(k) = \gamma(k) - d\gamma$, solve (53) until infeasibility. Then, assign $\gamma(k) \triangleq \gamma(k) + d\gamma$. If there is no possible solution increase r until a feasible solution is found and repeat the minimization procedure for $\gamma(k)$. Then, for $\gamma_{\min}$, compute P_k, Q_k and $K_k = L_k Y_k^{-1}$ and the dissipation level p_k and q_k. Go to Step 3.

Step 3. Apply the control signal $u(k) = K_k x(k)$ to the system. Change the step from k to $k + 1$ and continue with Step 2.

5. Numerical Examples

In this section we give two numerical examples to demonstrate the efficiency of the proposed method for time-varying state delayed systems with and without norm-bounded parametric uncertainties, respectively. Note that the results are obtained by using SeDuMi and YALMIP toolboxes operated under MATLAB.

Example 1. The following example is borrowed from [39] in which a nominal $\mathcal{H}_\infty$ control problem has been considered. Consider a state-delayed system (1) whose model parameters are given by

$$A = \begin{bmatrix} 0 & 1 \\ -0.14 & 0.9 \end{bmatrix} \qquad A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad B_w = B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \qquad C_d = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad D_w = \begin{bmatrix} 0 \end{bmatrix} \qquad D_u = \begin{bmatrix} 0.1 \end{bmatrix}$$

$$x(i) = \begin{bmatrix} 0 & -1.5 \end{bmatrix}^T \quad i = 0, -1, \ldots, -d_{\max}$$

$$\|u(k)\| \le 1, \quad \forall k \ge 0. \tag{54}$$

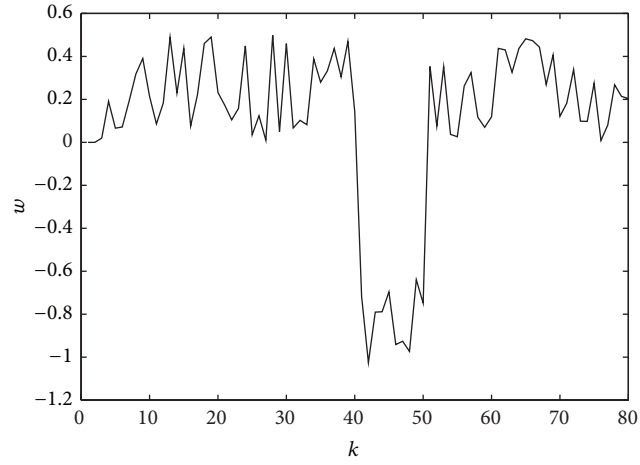

Figure 1: The time-history of the disturbance signal affecting the system.

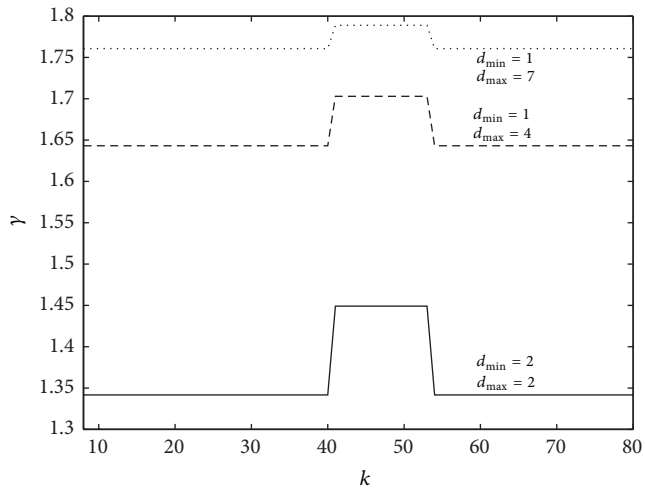

Figure 2: The variation of disturbance attenuation level, γ, for different delay sizes.

Choosing $r_0 = 0.17$ and using the energy-bounded disturbance ($\alpha = 0.3$), whose time-history is as shown in Figure 1, we compute $\gamma_{\text{opt}} = 1.09$ for $d_{\max} = d_{\min} = 2$ at the initial step of algorithm. Applying the proposed algorithm on the system, we obtain satisfactory results in γ minimization which also leads us to obtain a significant improvement in disturbance rejection performance. Figure 2 shows the variations of γ with respect to various delay bounds. It can be seen in the figure that the value of γ decreases while the bounds on the delay are tightened. Thus, the conservatism of the approach can be reduced by bounding the delay if the upper and lower bounds of delay are known. For different delay bounds, $d_{\max}$ and $d_{\min}$, the variation of the controlled output, z, and the control signal, u, are shown in Figures 3, 4, and 5, proving that the results are satisfactory and the control input constraint is satisfied for the system that involves time-varying delay.

Besides, the simulations are repeated for the same example with a new A_d matrix determined in a way that $A + A_d$

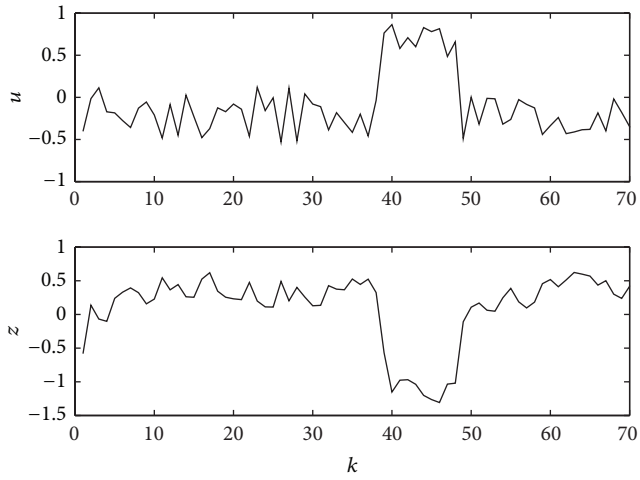

Figure 3: The time-history of the controlled output, z, and the control input, u, for $d_{\max} = d_{\min} = 2$.

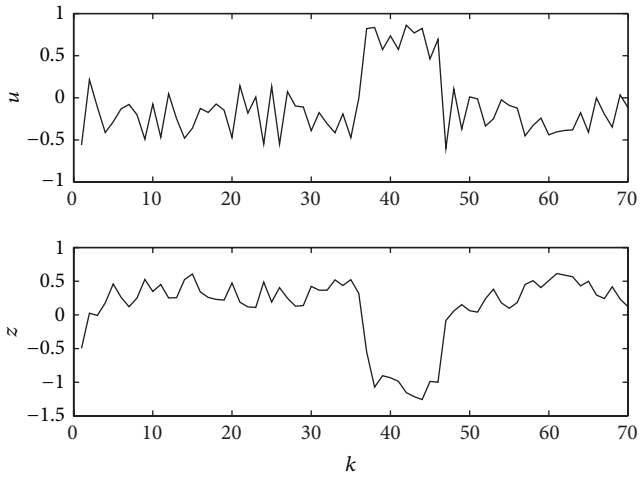

Figure 4: The time-history of the controlled output, z, and the control input, u, for $d_{\max} = 4$, $d_{\min} = 1$.

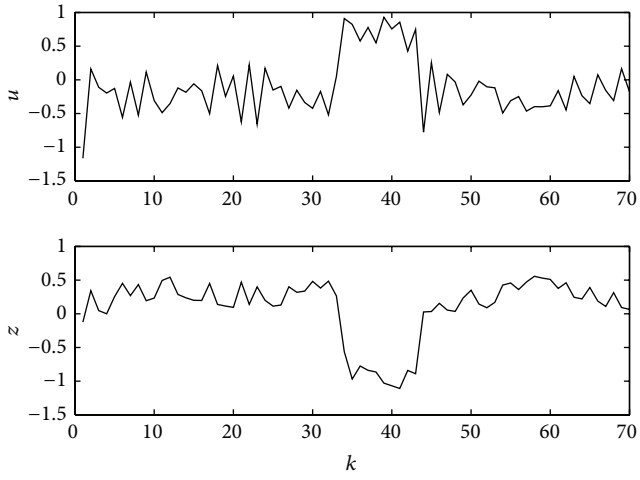

Figure 5: The time-history of the controlled output, z, and the control input, u, for $d_{\max} = 7$, $d_{\min} = 1$.

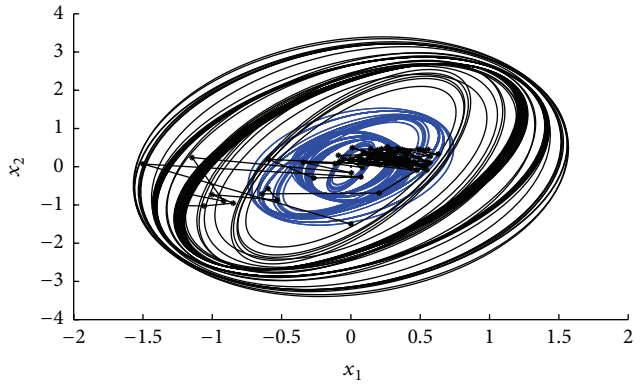

Figure 6: Ellipsoidal bounds on state trajectories, for $d_{\max} = 2$, $d_{\min} = 2$.

has a pole outside of the unit circle when the delay $d(k) = 0$. Consider the following:

$$A = \begin{bmatrix} 0 & 1 \\ -0.14 & 0.9 \end{bmatrix} \qquad A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}. \tag{55}$$

Hence, the effectiveness of the control algorithm is demonstrated also for unstable systems. The state trajectory that remains in the time-invariant ellipsoids is given in Figure 6.

Example 2. We utilize the example given in [40] in revised form as follows:

$$\begin{gathered} A = \begin{bmatrix} 0 & 0.1 \\ -0.14 & 0.9 \end{bmatrix} \qquad A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} \\ B_w = B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{0.1} \end{bmatrix} \qquad C_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ D_w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ x(i) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \quad i = 0, -1, \ldots, -d_{\max} \\ \|u(k)\| \le 1, \quad \forall k \ge 0, \end{gathered} \tag{56}$$

where $\|\Delta A\| = \|\Delta A_d\| = \|\Delta B_u\| \le 0.1$. The expansion of the proposed method to the time-delay systems having parametric uncertainties is shown in the example. The time-history of the disturbance applied to the system is as shown in Figure 1. We compute $\gamma_{\text{opt}} = 1.26$ for $d_{\max} = d_{\min} = 2$ and we choose $r_0 = 0.1$. During the simulations, the best variation of r is obtained at each step. Figure 7 shows the variations in γ with respect to different delay-bounds. For various delay bounds, $d_{\max}$ and $d_{\min}$, we obtain the time-history of performance outputs and the control efforts as shown in Figures 8, 9, and 10. Besides, Figure 11 shows the invariant ellipsoidal bounds on the state trajectory of the delayed system.

6. Conclusion

The MPC of a class of linear discrete-time, uncertain time-delayed systems having time-varying interval delays was

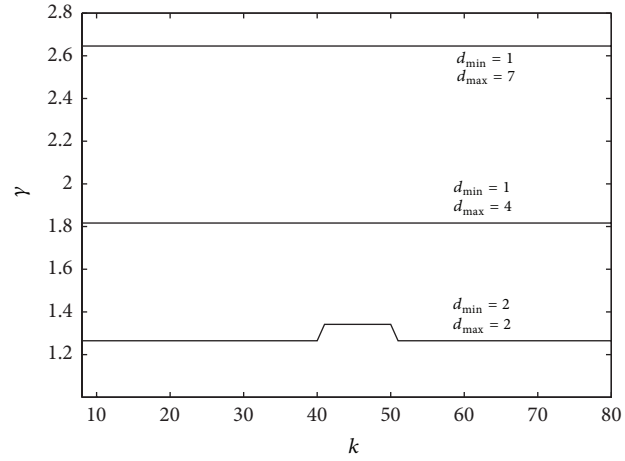

Figure 7: The variations of disturbance attenuation level, γ, for different delay sizes.

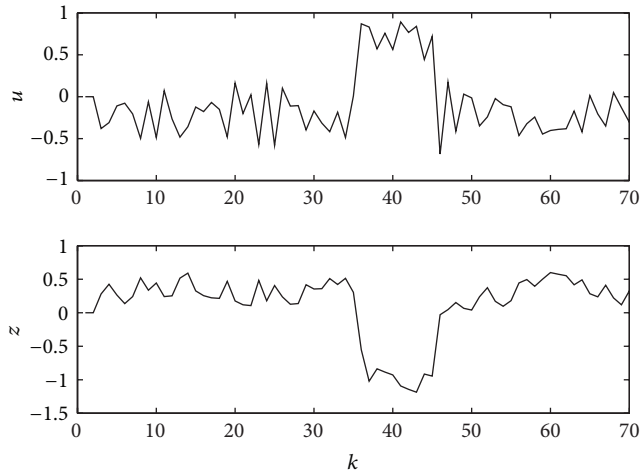

Figure 10: The time-history of the controlled output, z, and the control input, u, for $d_{\max} = 7$, $d_{\min} = 1$.

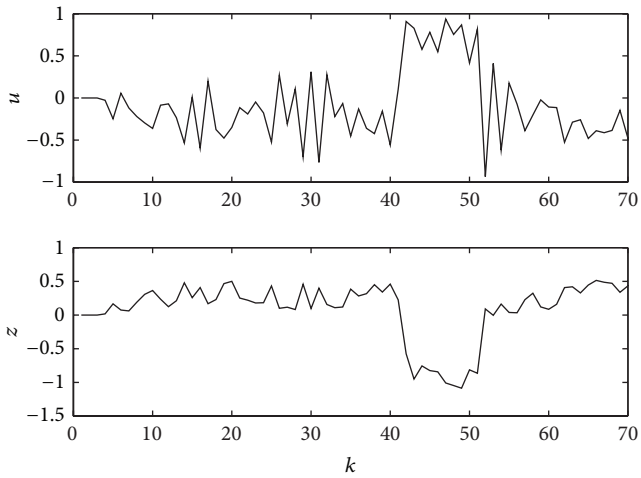

Figure 8: The time-history of the controlled output, z, and the control input, u, for $d_{\max} = 2$, $d_{\min} = 2$.

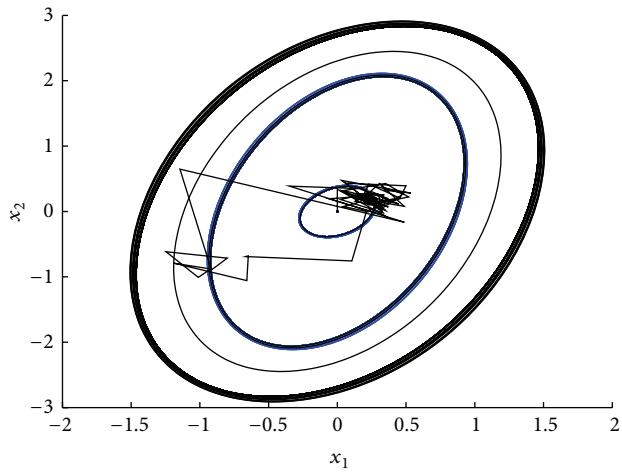

Figure 11: The ellipsoidal bounds on state trajectory of the uncertain system for $d_{\min} = d_{\max} = 2$.

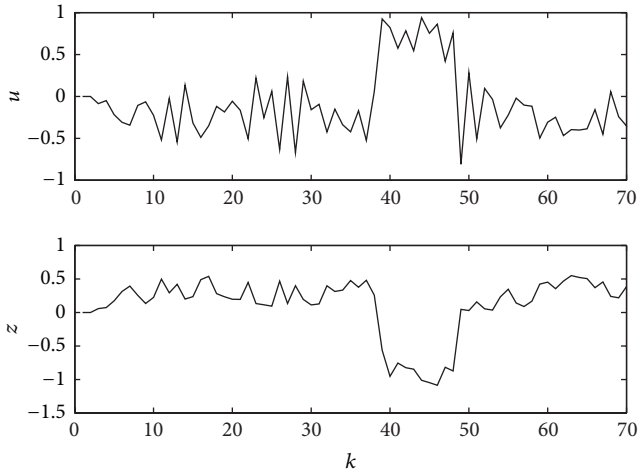

Figure 9: The time-history of the controlled output, z, and the control input, u, for $d_{\max} = 4$, $d_{\min} = 1$.

considered. Utilizing a standard discrete-time Lyapunov-Krasovskii functional, some delay-dependent, linear matrix inequality (LMI) based conditions which need to be solved iteratively in each step of run-time were provided. The provided LMI based conditions guarantee the closed-loop asymptotic stability, maximum disturbance attenuation performance, and closed-loop dissipativity in consideration of the physical limitations of the actuator. Two numerical examples which consist of nominal and uncertain system models were considered to demonstrate the applicability of the proposed approach. Both the numerical results and simulation results validated that the proposed approach of this note can be efficiently used for the control of discrete-time, time-delayed systems having uncertainties and physical control limitations. For a further study, the conservatism of the approach can be alleviated by using complete *LK* functionals and/or some delay-decomposition methods. However, it is probable that both strategies would impose noticeable

amount of additional computational loads to the controller which will prohibit the use of the proposed method in real-time. Also these types of methods cannot be solved by using convex optimization methods and would require the use of some linearisation techniques such as cone-complementary methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Time-Delayed Impulsive Control of Chaotic System Based on T-S Fuzzy Model

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This paper is concerned with the time-delayed impulsive control and synchronization of general chaotic system based on T-S fuzzy model. By utilizing impulsive control theory, time-delayed feedback control technique, and T-S fuzzy model, some useful and new conditions are derived to guarantee the stability and synchronization of the addressed chaotic system. Finally, some numerical simulations are given to illustrate the effectiveness of the derived results.

1. Introduction

Chaos may well be considered together with relativity and quantum mechanics as one of the three monumental discoveries of the twentieth century [1]. The term chaos associated with an interval map was first formally introduced into mathematics by Li and Yorke in 1975, where they established a simple criterion for chaos in one-dimensional difference equations, that is, the well-known "period three implies chaos." Initially regarded as a curiosity that interested only the mathematics community, it was later revamped when observed in meteorology, physics, chemistry, biology, and so on [2]. Over the past four decades, chaos has matured as a science and has given us deep insights into previously intractable and inherently nonlinear natural phenomena. There is, however, still no unified, universally accepted, and rigorous mathematical definition of chaos in the scientific literature to provide a fundamental basis for studying such exotic phenomena. Roughly speaking, chaotic behavior is a seemingly random behavior of a deterministic system that is characterized by sensitive dependence on initial conditions. From a practical point of view, a chaotic motion can be defined as a bounded invariant motion of a deterministic system, that is, not an equilibrium solution or a periodic solution or a quasiperiodic solution [2].

Although chaos is a very attractive subject for study, it was once believed to be neither predictable nor controllable due to its intrinsic topological complexity. However, recent research has shown that chaos can actually be useful under certain circumstances, such as enhanced mixing of chemical reactants. On the other hand, chaos should be weakened or completely suppressed when it is harmful. Therefore, it is very necessary and important to investigate chaos and chaos control.

Currently, many different techniques and methods based on conventional control theory have been extensively investigated to achieve chaos control [3–11]. For instance, the so-called OGY control method, introduced by Ott et al. [7], employs the classical feedback control idea, which might be understood as a kind of pole-placement method and is technically quite simple, but virtually it takes advantage of chaos itself in the sense of using its structural stability and its basic property of having a dense set of periodic orbits near a saddle. Such special properties are not available for nonchaotic systems. Therefore, such a control methodology was not first recommended by control theorists and engineers who usually, if not always, try to use brute force to regulate and stabilize unstable dynamics [1]. When brute force type of controls is not allowed, for example, in fragile and microscopic biological control systems such

as human brain and heart regulations, new control methods utilizing the extreme sensitivity of chaos to tiny variations are very desirable. This usually leads to some nonconventional approaches.

Fuzzy control as a nontraditional method of control, in recent years, has received much attention as a powerful tool for the nonlinear control. Compared with the traditional control paradigm, the advantages of the fuzzy control paradigm are twofold. Firstly, a precise mathematical model of the controlled system is not required. In addition, a satisfactory nonlinear controller can often be developed empirically without using complicated mathematics [1]. In view of its superiority, fuzzy controllers are often used for controlling conventional control methods as an extra layer of control in order to improve the effectiveness of the control.

On the other hand, the main drawback of fuzzy control systems is the lack of a systematic modeling and control design methodology. Particularly, stability analysis of fuzzy system is not easy, and parameter tuning is generally a time-consuming procedure, due to the nonlinear and multiparametric nature of fuzzy control systems [2]. To resolve these problems, a linear system can be adopted as the consequent part of a fuzzy rule, which leads to the so-called T-S model.

The T-S fuzzy dynamic model originates from Takagi and Sugeno. In this framework, a nonlinear dynamical system is first approximated by a T-S fuzzy model. In this type of fuzzy model, local dynamics in different state-space regions are represented by linear models. The overall model of the system is achieved by fuzzy blending of these linear models. The control design is carried out based on the fuzzy model. For each local linear model, a feedback control is designed. The resulting overall controller, which is nonlinear in general, is again a fuzzy blending of each individual controller [1, 2]. Here, it is noted that the defuzzified output of the constructed T-S fuzzy model is mathematically identical to that of the original nonlinear system. Clearly, the T-S fuzzy model provides a successful method to describe certain complex nonlinear systems using some local linear subsystems. In view of its convenience, T-S fuzzy control theory has been extensively applied to chaos control and synchronization in some literatures [12–21].

In addition, impulsive control method has been also widely used to stabilize and synchronize chaotic systems in recent years [13, 16, 18–27]. Impulsive control gives an efficient method to deal with systems which cannot endure continuous disturbance. Additionally, in synchronization process, the driven system receives the information about the active variables of the driving system only in the discrete times. This drastically reduces the amount of information transmitted from the driving system to driven system which makes this method more efficient and more useful in a great number of applications, such as in encrypt communications; this point is essential [18].

The delayed feedback control, as one of the important chaos control means, was introduced by Lithuanian physicist K. Pyragas in 1992 to stabilize an unstable periodic orbit of a nonlinear control system [1, 2, 9]. Based on this method, the control input is fed by the difference between the current state and the delayed state. Currently, the delayed feedback control method has been extensively applied to controlling chaos [8, 9, 12, 22, 28–34].

Although impulsive control, fuzzy control, and delayed feedback control are widely proposed in study of chaos and chaos control, to the best of our knowledge, there are few results on the stabilization and synchronization of chaotic system based on T-S fuzzy model by using the so-called time-delayed impulsive control which combine the superiorities of impulsive control and time-delayed feedback control. In addition, most previous researches have been restricted to linear impulsive control functions (e.g., see [18–20, 23, 25, 27]); few authors have considered the nonlinear impulsive control functions. Besides, the impulsive controllers are independent of time delays in most existing researches (see [13, 16, 18–21, 23–27]). Motivated by the above discussion, the objective of this paper is to develop a time-delayed impulsive controller for stabilization and synchronization of general chaotic systems based on a T-S fuzzy model. In this paper, the impulsive controller can be a nonlinear function. In addition, because of using time-delayed feedback control technique, the delayed impulsive controllers are derived from controlling chaotic system; this extends some previous impulsive control results to some extent.

The paper is organized as follows. In Section 2, the general T-S fuzzy control model of chaotic system is presented by applying T-S fuzzy theory. In Section 3, by combining the time-delayed feedback control technique and the impulsive control method, some sufficient conditions are obtained to guarantee that the origin of the controlled chaotic system is globally asymptotically stable. In Section 4, some time-delayed impulsive synchronization conditions are provided. In Section 5, the effectiveness and feasibility of the developed methods have been shown by some numerical simulations.

2. The T-S Fuzzy Control Model of Chaotic System

A T-S fuzzy model is described by a set of if-then rules, which characterize local relations of a nonlinear system in the state space. The main feature of a T-S model is to express the local dynamics of each fuzzy rule by a linear state-space system model, and the overall fuzzy system is modeled by fuzzy "blending" of these local linear system models through some suitable membership functions.

Consider the following chaotic system:

$$\dot{x}(t) = f(t, x(t)), \tag{1}$$

where $x \in R^n$ is the state variable and $f \in C[R^+ \times R^n, R^n]$ is a nonlinear vector-valued function satisfying $f(t, 0) \equiv 0$ for $t \in R^+$. We will construct a T-S fuzzy control model for (1) in the following form (see [1, 2]).

Rule i: if $z_1(t)$ is M_i^1, $z_2(t)$ is $M_i^2, \ldots, z_p(t)$ is M_i^p, then

$$\dot{x}(t) = A_i x(t) + u_i(t), \quad i = 1, 2, \ldots, r, \tag{2}$$

where the premise variables $z_1(t), \ldots, z_p(t)$ are proper state variables; each M_i^j and $u_i(t)$ denote the fuzzy set and the input

control signal, respectively; each A_i denotes a $n \times n$ constant matrix; r is the number of the fuzzy rules.

Using the singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier (see [1, 2]), the final output of the T-S fuzzy control system (2) for system (1) is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) \left[A_i x(t) + u_i(t)\right], \tag{3}$$

where $z(t) = (z_1(t), \ldots, z_p(t))$ and

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \qquad w_i(z(t)) = \prod_{j=1}^{p} M_i^j\left(z_j(t)\right), \tag{4}$$

in which $M_i^j(z_j(t))$ is the grade of membership of $z_j(t)$ in M_i^j, $w_i(z(t)) \geq 0$ $(i = 1, 2, \ldots, r)$, and $\sum_{i=1}^{r} w_i(z(t)) > 0$.

It is clear that

$$\sum_{i=1}^{r} h_i(z(t)) = 1, \quad h_i(z(t)) \geq 0, \ i = 1, 2, \ldots, r \tag{5}$$

for all t, where $h_i(z(t))$ can be regarded as the normalized weight of the if-then rules.

Remark 1. If the input control signal in system (2) is invalid, that is, $u_i(t) \equiv 0$ for any $t \in R^+$ and $i = 1, 2, \ldots, r$, then system (1) can be represented by the following T-S fuzzy model.

Rule i: if $z_1(t)$ is M_i^1, $z_2(t)$ is $M_i^2, \ldots, z_p(t)$ is M_i^p, then

$$\dot{x}(t) = A_i x(t), \quad i = 1, 2, \ldots, r. \tag{6}$$

Similar to the above discussion, the overall output of system (6) is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) A_i x(t), \tag{7}$$

where $z_i(t)$, M_i^j, A_i, and $h_i(z(t))$ are defined in (2) and (3), respectively. In this case, system (1) is mathematically identical to the defuzzified output of the constructed T-S fuzzy model (7).

Finally, for convenience and further study, we introduce the following lemmas.

Lemma 2 (see [29]). *For any two real column vectors ζ, η with the same dimensions and any positive constant θ, the following inequality holds:*

$$2\zeta^T \eta \leq \frac{1}{\theta}\zeta^T \zeta + \theta \eta^T \eta. \tag{8}$$

Lemma 3 (see [29]). *Assume that P is a $n \times n$ real symmetric matrix, and λ_1 and λ_2 are the smallest and the largest eigenvalues of P, respectively. Then for any $\xi \in R^n$, the following inequality holds:*

$$\lambda_1 \xi^T \xi \leq \xi^T P \xi \leq \lambda_2 \xi^T \xi. \tag{9}$$

3. Time-Delayed Fuzzy Impulsive Control

In this section, we will design a time-delayed impulsive controller for chaotic system (1) based on fuzzy model (2). In the following, we assume that the controller shares the same fuzzy sets with system (2). More precisely, the ith control rule is formulated in the following form.

Control rule i: if $z_1(t)$ is M_i^1, $z_2(t)$ is $M_i^2, \ldots, z_p(t)$ is M_i^p, then

$$u_i(t) = \sum_{k=1}^{\infty} F_i l_k(t) x(t) + \sum_{k=1}^{\infty} \delta(t - kT) I_{ik}(x(t), x(t - T)) \tag{10}$$

for $i = 1, 2, \ldots, r$, where $T > 0$ is the feedback time-delay, each F_i denotes $n \times n$ constant matrix, the switching function $l_k(\cdot)$ is defined by the following form:

$$l_k(t) = \begin{cases} 1, & (k-1)T < t \leq kT, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

$\delta(\cdot)$ is the Dirac delta function, each $I_{ik}(\cdot) \in C[R^n \times R^n, R^n]$ denotes the additive change of the state at time kT satisfying $I_{ik}(0, 0) = 0$, and the following condition is also satisfied.

(H) There exist constants $a_{ik} > 0$ and $b_{ik} \geq 0$ such that

$$\left\|u + I_{ik}(u, v)\right\|^2 \leq a_{ik}\|u\|^2 + b_{ik}\|v\|^2 \tag{12}$$

for any $u, v \in R^n$. Here and throughout this paper, $\|y(t)\| = [y^T(t) y(t)]^{1/2}$ is the Euclidean norm.

From (10), system (3) can be rewritten as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t)) \left[A_i x(t) + u_i(t)\right] \\ &= \sum_{i=1}^{r} h_i(z(t)) A_i x(t) \\ &\quad + \sum_{i=1}^{r} h_i(z(t)) \sum_{k=1}^{\infty} F_i l_k(t) x(t) \\ &\quad + \sum_{i=1}^{r} h_i(z(t)) \sum_{k=1}^{\infty} \delta(t - kT) I_{ik}(x(t), x(t - T)), \end{aligned} \tag{13}$$

which implies that

$$\begin{aligned} &x(kT + h) - x(kT) \\ &= \int_{kT}^{kT+h} \left\{ \sum_{i=1}^{r} h_i(z(t)) A_i x(t) \right. \\ &\qquad \left. + \sum_{i=1}^{r} h_i(z(t)) \sum_{k=1}^{\infty} F_i l_k(t) x(t) \right. \end{aligned}$$

$$+\sum_{i=1}^{r}h_i(z(t))\times\sum_{k=1}^{\infty}\delta(t-kT)\,I_{ik}(x(t),x(t-T))\Bigg\}dt, \tag{14}$$

where $h>0$ is sufficiently small. As $h\to 0^+$, by applying the properties of the Dirac delta function, we obtain

$$x(kT^+)-x(kT)=\sum_{i=1}^{r}h_i(z(kT))\,I_{ik}(x(kT),x((k-1)T)). \tag{15}$$

In this paper, we assume that $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T$ is left continuous at kT ($k\in Z^+$); that is, $x_i(kT)=\lim_{t\to kT^-}x_i(t)$. Thus, system (13) is rewritten as

$$\begin{aligned}\dot{x}(t)&=\sum_{i=1}^{r}h_i(z(t))(A_i+F_i)x(t),\quad t\neq kT,\\ \Delta x(t)&=\sum_{i=1}^{r}h_i(z(t))\,I_{ik}(x(t),x(t-T)),\quad t=kT,\ k\in Z^+.\end{aligned} \tag{16}$$

Remark 4. Evidently, the control input $u_i(t)$ is an n-dimensional vector function denoted by $(u_{i1}(t),u_{i2}(t),\ldots,u_{in}(t))^T$ and it will converge to zero if the origin is asymptotically stable. On the other hand, if controller (10) is invalid, that is, $u_i(t)\equiv 0$ for all $t\geq 0$, then it is easy to see that system (16) is equivalent to system (7) in this case. In addition, it is evident that $x(t)=0$ is an equilibrium of system (16).

In the following, we discuss the globally asymptotical stability of the origin of system (16). Firstly, for convenience, we introduce the following mathematical denotations.

Let P be a symmetric and positive definite matrix and $\lambda_1>0$ and $\lambda_2>0$ are the smallest and the largest eigenvalues of P, respectively.

β_i denotes the largest eigenvalue of $P(A_i+F_i)+(A_i+F_i)^TP$. $\alpha=\max_{1\leq i\leq r}\{\beta_i\}$, $\beta=\alpha/\lambda_1$, and

$$H_k=\frac{(\theta^2+1)\lambda_2}{2\theta\lambda_1}\sum_{i=1}^{r}h_i(z(kT))\,a_{ik},\quad k\in Z^+, \tag{17}$$

$$B_k=\frac{(\theta^2+1)\lambda_2}{2\theta\lambda_1}\sum_{i=1}^{r}h_i(z(kT))\,b_{ik},\quad k\in Z^+, \tag{18}$$

where $\theta>0$ is an arbitrary real number.

$G(t):R^+\to R^+$ denotes a differentiable at $t\neq kT$ and bounded function; that is, there exist $m>0$ and $M>0$ such that

$$m\leq G(t)\leq M,\quad \forall t\in R^+. \tag{19}$$

Furthermore, denote $\eta=M\lambda_2/m\lambda_1$ and

$$L_k=H_k+\frac{B_k}{\eta L_{k-1}}\exp(-\beta T)\frac{G((k-1)T^+)}{G(kT)} \tag{20}$$

for $k\in Z^+$, where $L_0=1$.

Remark 5. From $a_{ik}>0$ in (H), we know that $H_k>0$ for $k\in Z^+$, which implies that the definition of L_k is reasonable.

The following theorem is provided to guarantee that the origin of system (16) is asymptotically stable.

Theorem 6. *Under assumption (H), if there exists a constant $\theta>0$ such that one of the following conditions is satisfied:*

(1) *$\beta\geq 0$ and there exists a constant $\mu>1$ such that*

$$\beta T+\ln(\mu\eta L_k)+\ln\frac{G((k+1)T)}{G(kT^+)}\leq 0,\quad \forall k\in Z^+, \tag{21}$$

(2) *$\beta<0$ and there exists a constant ε satisfying $0\leq\varepsilon<-\beta$ such that*

$$\ln(\eta L_k)-\varepsilon T+\ln\frac{G((k+1)T)}{G(kT^+)}\leq 0,\quad \forall k\in Z^+, \tag{22}$$

then the controlled system (16) *is globally asymptotically stable at its origin.*

Proof. Choose a Lyapunov function as follows:

$$V(x(t))=x^T(t)Px(t). \tag{23}$$

Let $W(t)=G(t)V(x(t))$. For $t\neq kT$, the upper right derivative of $W(t)$ along the solution of system (16) is

$$\begin{aligned}D^+W(t)&=G(t)D^+V(x(t))+D^+G(t)V(x(t))\\&=G(t)\sum_{i=1}^{r}h_i(z(t))\,x^T(t)\\&\qquad\times\left[(A_i+F_i)^TP+P(A_i+F_i)\right]\\&\qquad\times x(t)+\dot{G}(t)V(x(t))\\&\leq G(t)\sum_{i=1}^{r}h_i(z(t))\,\beta_i x^T(t)x(t)+\dot{G}(t)V(x(t))\\&\leq\left(\beta+\frac{\dot{G}(t)}{G(t)}\right)G(t)V(x(t))\\&=\left(\beta+\frac{\dot{G}(t)}{G(t)}\right)W(t).\end{aligned} \tag{24}$$

Let $p(t)=\beta+(\dot{G}(t)/G(t))$; then we have

$$D^+W(t)\leq p(t)W(t),\quad t\in((k-1)T,kT],\ k\in Z^+, \tag{25}$$

which implies that

$$W(t) \le W\left((k-1)T^{+}\right) \exp\left[\int_{(k-1)T^{+}}^{t} p(s)\,ds\right]. \tag{26}$$

It follows that

$$\omega(t) \le \eta\omega\left((k-1)T^{+}\right) \exp\left[\int_{(k-1)T^{+}}^{t} p(s)\,ds\right] \tag{27}$$

for $t \in ((k-1)T, kT]$ and $k \in Z^{+}$, where $\omega(t) = x^{T}(t)x(t)$ and $\eta = M\lambda_2/m\lambda_1 \ge 1$.

For $t = kT (k \in Z^{+})$, we obtain

$$\begin{aligned}
&V\left(x\left(kT^{+}\right)\right)\\
&= V\left(x(kT) + \Delta x(kT)\right)\\
&= \Bigg[x(kT)\\
&\qquad + \sum_{i=1}^{r} h_i\left(z(kT)\right) I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\Bigg]^{T}\\
&\quad \times P\Bigg[x(kT)\\
&\qquad + \sum_{i=1}^{r} h_i\left(z(kT)\right) I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\Bigg]\\
&\le \lambda_2\Bigg[x(kT)\\
&\qquad + \sum_{i=1}^{r} h_i\left(z(kT)\right) I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\Bigg]^{T}\\
&\quad \times \Bigg[x(kT)\\
&\qquad + \sum_{i=1}^{r} h_i\left(z(kT)\right) I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\Bigg]\\
&= \lambda_2 \sum_{i=1}^{r} h_i\left(z(kT)\right)\\
&\qquad \times \left[x(kT) + I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\right]^{T}\\
&\quad \times \sum_{i=1}^{r} h_i\left(z(kT)\right)\\
&\qquad \times \left[x(kT) + I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\right]\\
&= \lambda_2 \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(kT)\right) h_j\left(z(kT)\right)\\
&\qquad \times \left[x(kT) + I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\right]^{T}\\
&\qquad \times \left[x(kT) + I_{jk}\left(x(kT), x\left((k-1)T\right)\right)\right]\\
&\le \lambda_2 \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(kT)\right) h_j\left(z(kT)\right)\\
&\qquad \times \left[\frac{\theta}{2}\left\|x(kT) + I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\right\|^{2}\right.\\
&\qquad \left. + \frac{1}{2\theta}\left\|x(kT) + I_{jk}\left(x(kT), x\left((k-1)T\right)\right)\right\|^{2}\right]\\
&= \frac{\theta\lambda_2}{2} \sum_{i=1}^{r} h_i\left(z(kT)\right)\\
&\qquad \times \left\|x(kT) + I_{ik}\left(x(kT), x\left((k-1)T\right)\right)\right\|^{2}\\
&\quad + \frac{\lambda_2}{2\theta} \sum_{j=1}^{r} h_j\left(z(kT)\right)\\
&\qquad \times \left\|x(kT) + I_{jk}\left(x(kT), x\left((k-1)T\right)\right)\right\|^{2}\\
&\le \lambda_2\left(\frac{\theta}{2} + \frac{1}{2\theta}\right) \sum_{i=1}^{r} h_i\left(z(kT)\right)\\
&\qquad \times \left[a_{ik}\|x(kT)\|^{2} + b_{ik}\|x\left((k-1)T\right)\|^{2}\right]\\
&= \frac{\left(\theta^{2}+1\right)\lambda_2}{2\theta} \sum_{i=1}^{r} h_i\left(z(kT)\right) a_{ik}\omega\left(x(kT)\right)\\
&\quad + \frac{\left(\theta^{2}+1\right)\lambda_2}{2\theta} \sum_{i=1}^{r} h_i\left(z(kT)\right) b_{ik}\omega\left(x\left((k-1)T\right)\right).
\end{aligned} \tag{28}$$

Hence, we have

$$\omega\left(kT^{+}\right) \le H_k\omega(kT) + B_k\omega\left((k-1)T\right). \tag{29}$$

From (27), we have

$$\omega(t) \le \eta\omega\left((k-1)T^{+}\right) \exp\left[\beta\left(t - (k-1)T\right)\right] \frac{G(t)}{G\left((k-1)T^{+}\right)} \tag{30}$$

for $t \in ((k-1)T, kT]$ and $k \in Z^{+}$.

Then for $t \in (0, T]$, we get

$$\omega(t) \le \omega(0)\,\eta \exp\left(\beta t\right) \frac{G(t)}{G\left(0^{+}\right)}, \tag{31}$$

which implies that

$$\omega(T) \le \omega(0)\,\eta \exp\left(\beta T\right) \frac{G(T)}{G\left(0^{+}\right)}. \tag{32}$$

This together with (29) implies that

$$\begin{aligned}
&\omega\left(T^{+}\right)\\
&\quad\le H_1\omega(T)+B_1\omega(0)\\
&\quad\le H_1\omega(0)\eta\exp\left(\beta T\right)\frac{G(T)}{G(0^{+})}+B_1\omega(0)\\
&\quad=\left[H_1+\frac{B_1}{\eta}\exp\left(-\beta T\right)\frac{G\left(0^{+}\right)}{G(T)}\right]\\
&\qquad\times\omega(0)\eta\exp\left(\beta T\right)\frac{G(T)}{G(0^{+})}\\
&\quad=L_1\omega(0)\eta\exp\left(\beta T\right)\frac{G(T)}{G(0^{+})}.
\end{aligned}\tag{33}$$

Similarly, for $t\in(T,2T]$, we obtain

$$\begin{aligned}
\omega(t)&\le\omega\left(T^{+}\right)\eta\exp\left[\beta(t-T)\right]\frac{G(t)}{G(T^{+})}\\
&\le L_1\omega(0)\eta^{2}\exp\left(\beta t\right)\frac{G(T)}{G(0^{+})}\times\frac{G(t)}{G(T^{+})},\\
\omega\left(2T^{+}\right)&\le H_2\omega(2T)+B_2\omega(T)\\
&\le H_2L_1\omega(0)\eta^{2}\exp\left(2\beta T\right)\frac{G(T)}{G(0^{+})}\\
&\quad\times\frac{G(2T)}{G(T^{+})}+B_2\omega(0)\eta\exp\left(\beta T\right)\frac{G(T)}{G(0^{+})}\\
&=L_1\left[H_2+\frac{B_2}{\eta L_1}\exp\left(-\beta T\right)\frac{G\left(T^{+}\right)}{G(2T)}\right]\\
&\quad\times\omega(0)\eta^{2}\exp\left(2\beta T\right)\frac{G(T)}{G(0^{+})}\times\frac{G(2T)}{G(T^{+})}\\
&=L_1\times L_2\omega(0)\eta^{2}\exp\left(2\beta T\right)\frac{G(T)}{G(0^{+})}\times\frac{G(2T)}{G(T^{+})}.
\end{aligned}\tag{34}$$

Further for $t\in(2T,3T]$, we have

$$\begin{aligned}
\omega(t)&\le\omega\left(2T^{+}\right)\eta\exp\left[\beta(t-2T)\right]\frac{G(t)}{G(2T^{+})}\\
&\le L_1\times L_2\omega(0)\eta^{3}\exp\left(\beta t\right)\frac{G(T)}{G(0^{+})}\\
&\quad\times\frac{G(2T)}{G(T^{+})}\times\frac{G(t)}{G(2T^{+})}.
\end{aligned}\tag{35}$$

Hence, generally, for $t\in((k-1)T,kT]$, we obtain

$$\begin{aligned}
\omega(t)&\le L_0\times L_1\times L_2\times\cdots\times L_{k-1}\omega(0)\eta^{k}\exp\left(\beta t\right)\\
&\quad\times\frac{G(T)}{G(0^{+})}\times\frac{G(2T)}{G(T^{+})}\times\cdots\times\frac{G(t)}{G\left((k-1)T^{+}\right)},
\end{aligned}\tag{36}$$

where $L_0=1$ and $k\in Z^{+}$.

In the following, we consider the following two cases.

Case 1. $\beta\ge0$; that is, condition (1) holds.

It follows from (21) that

$$L_k\eta\exp\left(\beta T\right)\frac{G((k+1)T)}{G(kT^{+})}\le\frac{1}{\mu},\quad\forall k\in Z^{+}.\tag{37}$$

This combined with (36) yields

$$\begin{aligned}
\omega(t)&\le L_1\times L_2\times\cdots\times L_{k-1}\omega(0)\eta^{k}\exp\left(k\beta T\right)\\
&\quad\times\frac{G(T)}{G(0^{+})}\times\frac{G(2T)}{G(T^{+})}\times\cdots\times\frac{G(t)}{G\left((k-1)T^{+}\right)}\\
&=\omega(0)\eta\exp\left(\beta T\right)\frac{G(t)G(T)}{G(0^{+})G(kT)}\\
&\quad\times L_1\eta\exp\left(\beta T\right)\frac{G(2T)}{G(T^{+})}\\
&\quad\times L_2\eta\exp\left(\beta T\right)\frac{G(3T)}{G(2T^{+})}\\
&\quad\times\cdots\times L_{k-1}\eta\exp\left(\beta T\right)\frac{G(kT)}{G\left((k-1)T^{+}\right)}\\
&\le\frac{\eta M^{2}}{m^{2}\mu^{k-1}}\omega(0)\exp\left(\beta T\right).
\end{aligned}\tag{38}$$

Note that $k\to\infty$ as $t\to\infty$; from $\mu>1$, we obtain $\lim_{t\to\infty}\omega(t)=0$.

Case 2. $\beta<0$; that is, condition (2) holds.

It follows from (22) that

$$\eta L_k\exp\left(-\varepsilon T\right)\frac{G((k+1)T)}{G(kT^{+})}\le1,\quad\forall k\in Z^{+}.\tag{39}$$

So, we have

$$\begin{aligned}
\omega(t)&\le L_1\times L_2\times\cdots\times L_{k-1}\omega(0)\eta^{k}\exp\left(-\varepsilon t\right)\\
&\quad\times\exp\left[\left(\beta+\varepsilon\right)t\right]\\
&\quad\times\frac{G(T)}{G(0^{+})}\times\frac{G(2T)}{G(T^{+})}\times\cdots\times\frac{G(t)}{G\left((k-1)T^{+}\right)}\\
&\le\omega(0)\eta\exp\left[\left(\beta+\varepsilon\right)t\right]\frac{G(t)G(T)}{G(0^{+})G(kT)}\\
&\quad\times L_1\eta\exp\left(-\varepsilon T\right)\frac{G(2T)}{G(T^{+})}\\
&\quad\times L_2\eta\exp\left(-\varepsilon T\right)\frac{G(3T)}{G(2T^{+})}\\
&\quad\times\cdots\times L_{k-1}\eta\exp\left(-\varepsilon T\right)\frac{G(kT)}{G\left((k-1)T^{+}\right)}\\
&\le\frac{M^{2}}{m^{2}}\omega(0)\eta\exp\left[\left(\beta+\varepsilon\right)t\right].
\end{aligned}\tag{40}$$

Hence, from $\varepsilon+\beta<0$, we obtain $\lim_{t\to\infty}\omega(t)=0$.

From the above discussion, we know that the trivial solution $x(t) \equiv 0$ of system (16) is asymptotically stable. This completes the proof of Theorem 6. □

Remark 7. If $\mu = 1$ or $\varepsilon = -\beta$ in Theorem 6, from the proved process of Theorem 6, we have

$$\omega(t) \le \frac{\eta M^2 Q}{m^2}\omega(0), \tag{41}$$

where $Q = \max\{1, \exp(\beta T)\}$. In this case, it is clear that the origin of system (16) is stable. In addition, under condition (2) in Theorem 6, it is evident that the origin of system (16) is globally exponentially stable.

Remark 8. If $H_k \equiv H$ for all $k \in Z^+$ in (18), inequality (21) can be replaced by the following inequality:

$$\beta T + \ln(\mu\eta\Lambda_k) + \ln\frac{G((k+1)T)}{G(kT^+)} \le 0 \quad \forall k \in Z^+, \quad (21')$$

where H is a constant and $\Lambda_k = H + (B_k/\eta H)\exp(-\beta T)(G((k-1)T^+)/G(kT))$. In fact, from the definition of L_k, we have $L_k \ge H$ for all $k \in Z^+$ in this case, which implies that $L_k \le \Lambda_k$ for all $k \in Z^+$. Hence, from (36), we obtain

$$\begin{aligned}\omega(t) \le{}& \Lambda_0 \times \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_{k-1}\eta^k \omega(0)\exp(\beta t)\\ &\times \frac{G(T)}{G(0^+)} \times \frac{G(2T)}{G(T^+)} \times \cdots \times \frac{G(t)}{G((k-1)T^+)},\end{aligned} \tag{42}$$

for $t \in ((k-1)T, kT]$, where $\Lambda_0 = 1$. Similar to the analytic technique of (36), we know that $\lim_{t\to\infty}\omega(t) = 0$ under condition $(21')$.

Similarly, in this case, inequality (22) can be replaced by the following inequality:

$$\ln(\eta\Lambda_k) - \varepsilon T + \ln\frac{G((k+1)T)}{G(kT^+)} \le 0, \quad \forall k \in Z^+. \quad (22')$$

Let $G(t)$ be a constant function and $P = I$ in Theorem 6; then $\eta = 1$ and $\beta = \alpha = \max_{1\le i\le r}\{\beta_i\}$, where I is the identity matrix and β_i denotes the largest eigenvalue of $(A_i + F_i) + (A_i + F_i)^T$. Besides, let $I_{ik}(x(t), x(t-T)) = I_k(x(t), x(t-T))$ for $i = 1, 2, \ldots, r$ and $k \in Z^+$. In this case, a_{ik} and b_{ik} are reduced to a_k and b_k, respectively. H_k, B_k, and L_k can be rewritten as

$$\begin{aligned}\overline{H}_k &= \frac{(\theta^2+1)}{2\theta}a_k, \quad k \in Z^+,\\ \overline{B}_k &= \frac{(\theta^2+1)}{2\theta}b_k, \quad k \in Z^+,\\ \overline{L}_k &= \overline{H}_k + \frac{\overline{B}_k}{\overline{L}_{k-1}}\exp(-\beta T), \quad k \in Z^+,\end{aligned} \tag{43}$$

where $\overline{L}_0 = 1$. Under those assumptions, the following results are directly obtained from Theorem 6.

Corollary 9. *Under assumption* (H)*, system* (16) *is asymptotically stable at its origin, if one of the following conditions is satisfied.*

(1) $\beta \ge 0$ *and there exists a constant* $\mu > 1$ *such that*

$$\beta T + \ln(\mu\overline{L}_k) \le 0 \quad \forall k \in Z^+. \tag{44}$$

(2) $\beta < 0$ *and there exists a constant* ε *satisfying* $0 \le \varepsilon < -\beta$ *such that*

$$\ln\overline{L}_k - \varepsilon T \le 0 \quad \forall k \in Z^+. \tag{45}$$

In the following, we consider the case where the impulsive controller is reduced to linearly impulsive controller; that is,

$$I_{ik}(x(t), x(t-T)) = K_{ik}x(t) + C_{ik}x(t-T) \tag{46}$$

for $i = 1, 2, \ldots, r$ and $k \in Z^+$ in Theorem 6, where each K_{ik} and C_{ik} are the $n \times n$ constant matrices; then system (16) can be rewritten as the following form:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))(A_i + F_i)x(t), \quad t \ne kT,\\ \Delta x(t) &= \sum_{i=1}^{r} h_i(z(t))\left[K_{ik}x(t) + C_{ik}x(t-T)\right],\\ &\qquad t = kT,\ k \in Z^+.\end{aligned} \tag{47}$$

In this case, we obtain

$$\begin{aligned}&\left\|u + I_{ik}(u, v)\right\|^2\\ &\quad = \left(u + K_{ik}u + C_{ik}v\right)^T\left(u + K_{ik}u + C_{ik}v\right)\\ &\quad = u^T\left(I + K_{ik}\right)^T\left(I + K_{ik}\right)u\\ &\qquad + 2u^T\left(I + K_{ik}\right)^T C_{ik}v + v^T C_{ik}^T C_{ik}v\\ &\quad \le 2u^T\left(I + K_{ik}\right)^T\left(I + K_{ik}\right)u + 2v^T C_{ik}^T C_{ik}v\\ &\quad \le \gamma_{ik}u^T u + \rho_{ik}v^T v,\end{aligned} \tag{48}$$

where $\gamma_{ik} \ge 2\lambda_{\max}[(I + K_{ik})^T(I + K_{ik})]$ and $\rho_{ik} \ge 2\lambda_{\max}(C_{ik}^T C_{ik})$; I is the identity matrix.

Let $P = I$ and

$$\begin{aligned}\mathbb{H}_k &= \sum_{i=1}^{r} h_i(z(kT))\gamma_{ik}, \quad k \in Z^+,\\ \mathbb{B}_k &= \sum_{i=1}^{r} h_i(z(kT))\rho_{ik}, \quad k \in Z^+.\end{aligned} \tag{49}$$

In addition, let $G(t)$ be a constant function and $\Delta_k = \mathbb{H}_k + (\mathbb{B}_k/\Delta_{k-1})\exp(-\beta T)$ for $k \in Z^+$, where $\Delta_0 = 1$. From Theorem 6, we have the following results.

Corollary 10. *System* (47) *is asymptotically stable at its origin, if one of the following conditions is satisfied.*

(1) $\beta \geq 0$ *and there exists a constant* $\mu > 1$ *such that*

$$\beta T + \ln\left(\mu \Delta_k\right) \leq 0, \quad \forall k \in Z^+. \tag{50}$$

(2) $\beta < 0$ *and there exists a constant* ε *satisfying* $0 \leq \varepsilon < -\beta$ *such that*

$$\ln \Delta_k - \varepsilon T \leq 0, \quad \forall k \in Z^+. \tag{51}$$

Choose $K_{ik} = K$ and $C_{ik} = C$ for all $i = 1, 2, \ldots, r$ and $k \in Z^+$ in system (47), where K and C are the $n \times n$ constant matrices, respectively. Let $\gamma \geq 2\lambda_{\max}[(I+K)^T(I+K)] > 0$ and $\rho \geq 2\lambda_{\max}(C^TC)$. Let $G(t)$ be a constant and $P = I$; then $\beta = \alpha = \max_{1\leq i\leq r}\{\beta_i\}$, where β_i denotes the largest eigenvalue of $(A_i+F_i)+(A_i+F_i)^T$. In addition, under those assumptions, $\eta = 1$, $\mathbb{H}_k = \gamma$, and $\mathbb{B}_k = \rho$ for all $k \in Z^+$. Denote $\Gamma = \gamma+(\rho/\gamma)\exp(-\beta T)$. The following results are directly obtained from Theorem 6 and Remark 8.

Corollary 11. *For system* (47)*, the following statements hold.*

(1) *If* $\beta \geq 0$ *and there exists a constant* $\mu > 1$ *such that*

$$\beta T + \ln\left(\mu \Gamma\right) \leq 0, \tag{52}$$

then the origin of system (47) *is asymptotically stable.*

(2) *If* $\beta < 0$ *and there exists a constant* ε *satisfying* $0 \leq \varepsilon < -\beta$ *such that*

$$\ln \Gamma - \varepsilon T \leq 0, \tag{53}$$

then the origin of system (47) *is exponentially stable.*

Remark 12. From Corollary 11, we can obtain an estimate of the upper bound of T when $\beta > 0$. In fact, it follows from Corollary 11 that

$$T \leq \frac{1}{\beta}\left[\ln\left(\gamma - \mu\rho\right) - \ln\left(\mu\gamma^2\right)\right]. \tag{54}$$

4. Time-Delayed Fuzzy Impulsive Synchronization

The synchronization of chaotic system based on T-S fuzzy model has been extensively studied in previous researches. For instance, [13, 15, 16] obtained some results on the synchronization of chaotic system by using fuzzy impulsive control technique and adaptive synchronization approach, respectively. But, to the best of our knowledge, there are few results on the synchronization of chaotic system based on T-S fuzzy model by applying time-delayed impulsive control method. In this section, we consider time-delayed impulsive synchronization of chaotic systems based on T-S fuzzy model. In order to deal with synchronization of chaotic systems, we need to design the control input so that the driven system achieves asymptotic synchronization with the driving system, provided that the two systems start from different initial conditions. The driving system is given by system (7), suppose that the driven system has the same premise variables as the driving system, and the output states of driving system are observable. The fuzzy rules of the driven system are presented by the following form.

Rule i: if $z_1(t)$ is M_i^1, $z_2(t)$ is $M_i^2, \ldots, z_p(t)$ is M_i^p, then

$$\dot{y}(t) = A_i y(t) + u_i(t), \tag{55}$$

where $z_j(t)$, M_i^j, and A_i are defined in system (2), and $y(t)$ is the state variable of the driven system; let $u_i(t)$ be the external control inputs represented by the following form:

$$\begin{aligned} u_i(t) = &\sum_{k=1}^{\infty} F_i l_k(t)\left[y(t) - x(t)\right] \\ &+ \sum_{k=1}^{\infty} \delta(t - kT) \\ &\times I_{ik}\left(y(t) - x(t), y(t-T) - x(t-T)\right), \end{aligned} \tag{56}$$

where $x(t)$ is the state variable of the driving system, $I_{ik}(\cdot)$ is defined in (10) satisfying assumption (H), and $I_{ik}(0,0) = 0$.

Using the same methods in Section 3, the final output of the driven system is represented as follows:

$$\begin{aligned} \dot{y}(t) &= \sum_{i=1}^{r} h_i(z(t)) A_i y(t) + \sum_{i=1}^{r} h_i(z(t)) F_i\left[y(t) - x(t)\right], \\ &\qquad t \neq kT, \\ \Delta y(t) &= \sum_{i=1}^{r} h_i(z(t)) I_{ik}\left(y(t) - x(t), y(t-T) - x(t-T)\right), \\ &\qquad t = kT, \end{aligned} \tag{57}$$

where $k \in Z^+$.

Let $e(t) = y(t) - x(t)$; then from (7) and (57), the error system can be expressed as the following form:

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{r} h_i(z(t))\left(A_i + F_i\right) e(t), \quad t \neq kT, \\ \Delta e(t) &= \sum_{i=1}^{r} h_i(z(t)) I_{ik}(e(t), e(t-T)), \\ &\qquad t = kT,\ k \in Z^+. \end{aligned} \tag{58}$$

We observe that the synchronization of driving system (7) and driven system (57) is converted to the asymptotical stability of the origin for system (58). On the other hand, it is evident that error system (58) is similar to system (16), and then similar to Theorem 6, we have the following statements, which guarantee the synchronization of driving-driven systems (7) and (57).

Theorem 13. *For driving-driven systems* (7) *and* (57)*, the following statements are true.*

(1) *Systems* (7) *and* (57) *are globally asymptotically synchronized, if* $\beta \geq 0$ *and there exists a constant* $\mu > 1$ *such that*

$$\beta T + \ln(\mu\eta L_k) + \ln\frac{G((k+1)T)}{G(kT^+)} \leq 0, \quad \forall k \in Z^+. \tag{59}$$

(2) *Systems* (7) *and* (57) *are globally exponentially synchronized, if* $\beta < 0$ *and there exists a constant* ε *satisfying* $0 \leq \varepsilon < -\beta$ *such that*

$$\ln(\eta L_k) - \varepsilon T + \ln\frac{G((k+1)T)}{G(kT^+)} \leq 0 \quad \forall k \in Z^+. \tag{60}$$

Theorem 13 can be proved by using a similar argument as the proof of Theorem 6, and we omit it here.

Remark 14. If the impulsive controllers I_{ik} reduce to the linear impulsive forms (46), similar to the above discussion in Section 3, it is easy to see that driving-driven systems (7) and (57) are globally asymptotically synchronized if the conditions of Corollaries 10 and 11 are satisfied, respectively.

Remark 15. Evidently, Theorems 6 and 13 are established based on the fact that the impulsive internals are equidistant and equivalent to the feedback time-delay T. For the generally impulsive sequences $\{t_k\}_{k=1}^{\infty}$ satisfying $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} \cdots$ and $\lim_{k\to\infty} t_k = \infty$, the corresponding results are not given because of the complexity of this case in this paper. Hence, an important and interesting problem is whether we can also obtain the similar results to guarantee the stability and synchronization of the addressed chaotic system based on the generally impulsive sequences.

5. Numerical Simulations

In this section, based on the results obtained in the previous sections, some numerical simulations are presented to show the effectiveness and feasibility of our results.

Consider the Rössler system described by the following form:

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 10x_2, \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - \frac{8}{3}x_3. \end{aligned} \tag{61}$$

Figure 1 shows that system (61) is chaotic. In the following, firstly, we consider time-delayed impulsive control of system (61) based on T-S fuzzy model.

Now, we can construct an exact T-S fuzzy model of system (61), which can be expressed as follows:

$$\begin{aligned} &\text{Rule 1: if } x_1 \text{ is about } M_1, \quad \text{then } \dot{x} = A_1x(t) + u_1(t), \\ &\text{Rule 2: if } x_1 \text{ is about } M_2, \quad \text{then } \dot{x} = A_2x(t) + u_2(t), \end{aligned} \tag{62}$$

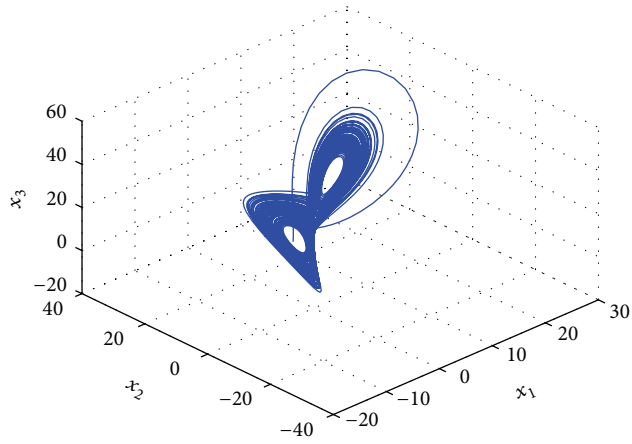

Figure 1: The chaotic behavior of system (61).

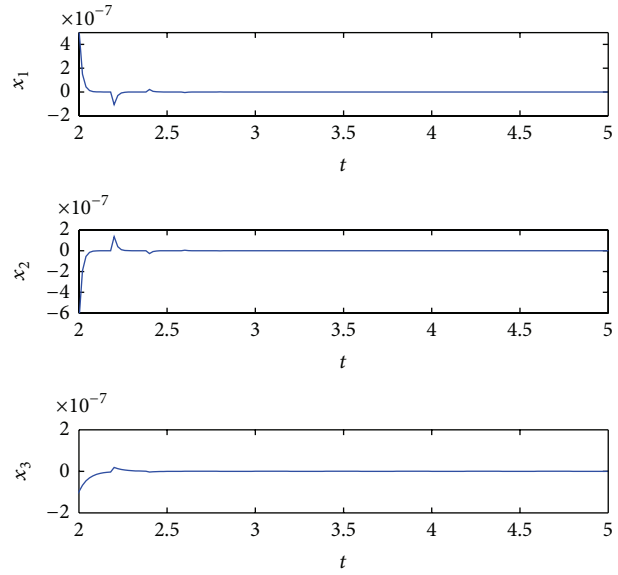

Figure 2: Time response of system (61).

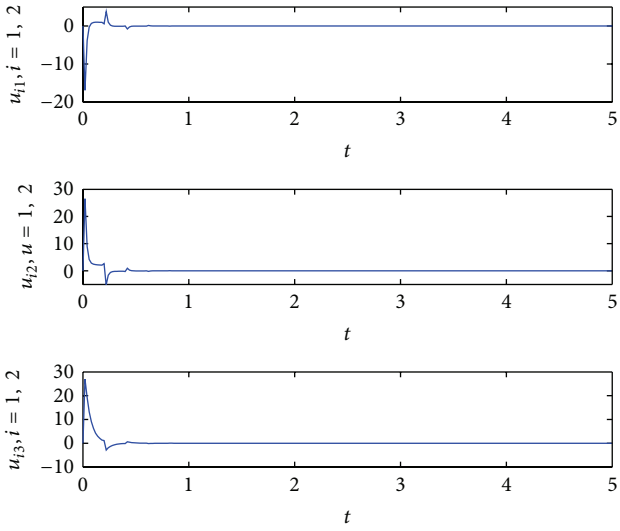

Figure 3: The plots of the control inputs.

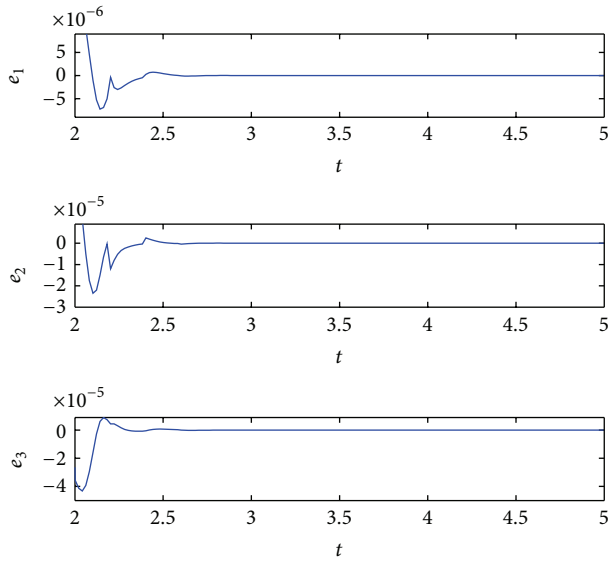

Figure 4: The plot of the synchronization errors.

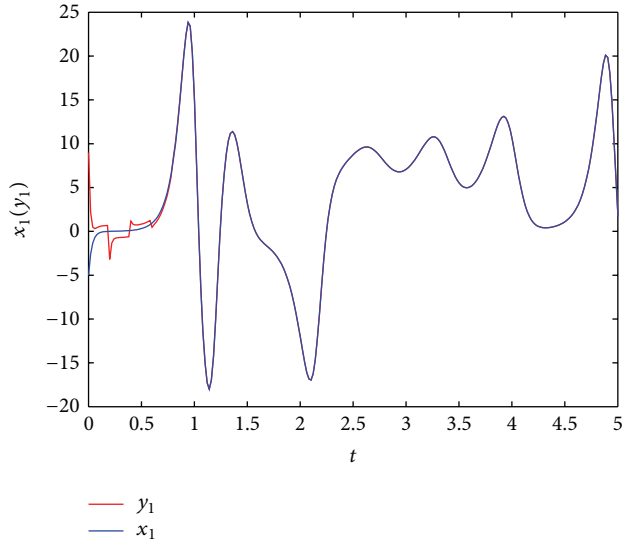

Figure 5: The curve of x_1 and y_1.

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$, $M_1 = -30$, $M_2 = 30$,

$$A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -30 \\ 0 & 30 & -\frac{8}{3} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 30 \\ 0 & -30 & -\frac{8}{3} \end{pmatrix}, \tag{63}$$

and the membership functions are $M_1(x_1) = (1/2) - (x_1/2d)$ and $M_2(x_1) = (1/2) + (x_1/2d)$ with $d = 30$.

From (10), the delayed impulsive controllers can be designed in the following form.

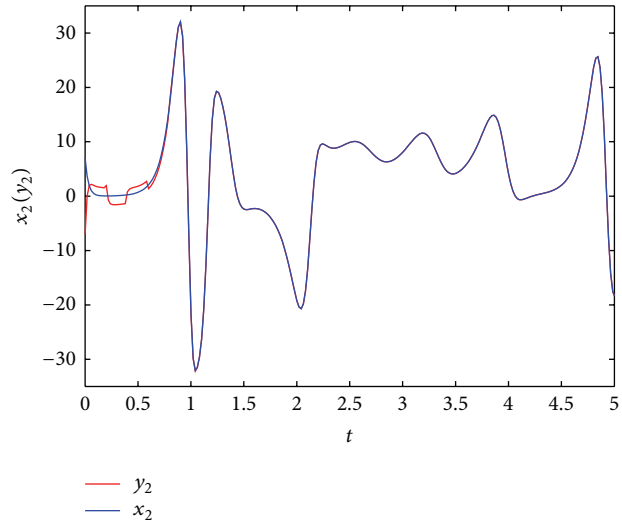

Figure 6: The curve of x_2 and y_2.

Control Rule 1: if x_1 is about M_1, then

$$u_1(t) = \sum_{k=1}^{\infty} F_1 l_k(t)\, x(t) + \sum_{k=1}^{\infty} \delta(t - kT) \left[K_1 x(t) + C_1 x(t - T)\right]. \tag{64}$$

Control Rule 2: if x_1 is about M_2, then

$$u_2(t) = \sum_{k=1}^{\infty} F_2 l_k(t)\, x(t) + \sum_{k=1}^{\infty} \delta(t - kT) \left[K_2 x(t) + C_2 x(t - T)\right]. \tag{65}$$

From system (16), the overall time-delayed impulsive control system of system (61) based on T-S fuzzy model can be rewritten as

$$\dot{x}(t) = \sum_{i=1}^{2} M_i(x_1(t)) (A_i + F_i) x(t), \quad t \neq kT,$$
$$\Delta x(t) = \sum_{i=1}^{2} M_i(x_1(t)) \left[K_i x(t) + C_i x(t - T)\right], \tag{66}$$
$$t = kT, \ k \in Z^+.$$

Choose $F_1 = F_2 = \mathrm{diag}(-13, -13, -13)$, $K_1 = K_2 = \mathrm{diag}(-0.6, -0.6, -0.6)$, and $C_1 = C_2 = \mathrm{diag}(-0.2, -0.2, -0.2)$; let $G(t)$ be a constant function and $P = I$. By simple computation, we obtain $\eta = 1$, $\beta = 2.0512$, $\gamma = 0.32$, and $\rho = 0.08$. Let $\mu = 1.3$; then from Remark 12, we have $0 \leq T \leq 0.2360$. Here, we choose $T = 0.2$. Hence, from Corollary 11, we know that the origin of system (61) is asymptotically stable. The simulation results of stabilizing chaotic system (61) for $T = 0.2$ are shown in Figure 2. Besides, it is evident that $u_1(t)$

and $u_2(t)$ are equivalent under the above parameters, which are simulated in Figure 3.

In the second simulation, we study time-delayed impulsive synchronization of the chaotic system (61) based on T-S fuzzy model in Figure 4.

The driving system is given in (62) with $u_1(t) = u_2(t) = 0$. Assume that the states of the delayed impulsive controlled driven system are $y(t) = (y_1(t), y_2(t), y_3(t))^T$, and the fuzzy rules of the driven system are represented by

$$\begin{aligned} &\text{Rule 1: if } x_1 \text{ is about } M_1, \quad \text{then } \dot{y} = A_1 y(t) + u_1(t), \\ &\text{Rule 2: if } x_1 \text{ is about } M_2, \quad \text{then } \dot{y} = A_2 y(t) + u_2(t), \end{aligned} \tag{67}$$

$$\begin{aligned} u_1(t) &= \sum_{k=1}^{\infty} F_1 l_k(t)\, e(t) \\ &\quad + \sum_{k=1}^{\infty} \delta(t - kT) \left[K_1 e(t) + C_1 e(t - T)\right], \end{aligned} \tag{68}$$

$$\begin{aligned} u_2(t) &= \sum_{k=1}^{\infty} F_2 l_k(t)\, e(t) \\ &\quad + \sum_{k=1}^{\infty} \delta(t - kT) \left[K_2 e(t) + C_2 e(t - T)\right], \end{aligned}$$

where $e(t) = y(t) - x(t)$, M_i and A_i are defined in (62), F_i, K_i, and C_i are chosen in (65), respectively, and the feedback delay $T = 0.2$.

Then from (58), we know that the error system can be expressed as follows:

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{2} M_i(x_1(t)) (A_i + F_i) e(t), \quad t \neq kT, \\ \Delta e(t) &= \sum_{i=1}^{2} M_i(x_1(t)) \left[K_i e(t) + C_i e(t - T)\right], \\ &\qquad t = kT, \ k \in Z^+. \end{aligned} \tag{69}$$

Let $G(t)$ be a constant function and $P = I$; from Remark 14 and the above discussion for the stability of system (61), we know that the origin of the error system (69) is asymptotically stable; that is, system (62) without the external input and system (67) are globally asymptotically synchronized. Figure 3 shows the simulation result of the stability of the origin for error system (69). The synchronization of driving system (62) with respect to initial value $x(0) = (-5, 7, -4)^T$ and of driven system (67) with initial value $y(0) = (9, -7, 10)^T$ is shown in Figures 5, 6, and 7. The control inputs are depicted in Figure 8.

6. Conclusion

In the paper, by using impulsive control method and the delayed feedback control mean, the stabilization and synchronization of chaotic system based on T-S fuzzy model are proposed; some useful and new conditions are obtained to ensure the stability and synchronization of the addressed chaotic system. In particular, some sufficient conditions which lead to the globally exponential stability of chaotic system are also obtained. It is noted that the impulsive functions can be nonlinear in this letter, which extend some previous results. Besides, in view of applying time-delayed feedback control technique, the delayed impulsive controllers are proposed in this paper. Finally, some numerical simulations are represented to show the effectiveness and feasibility of the developed methods.

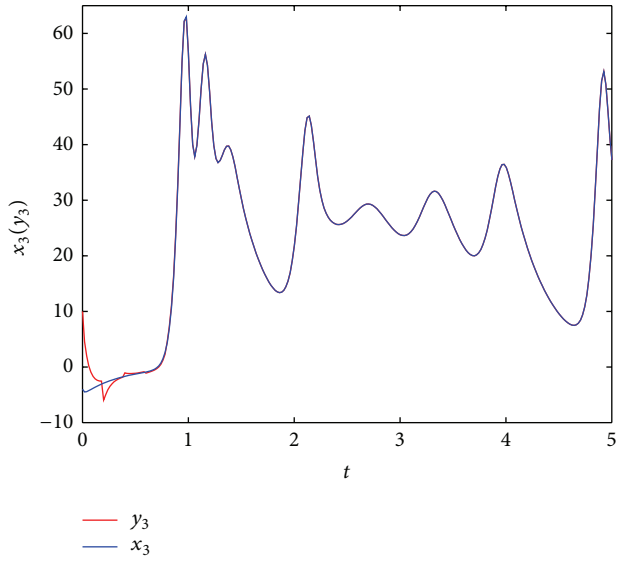

Figure 7: The curve of x_3 and y_3.

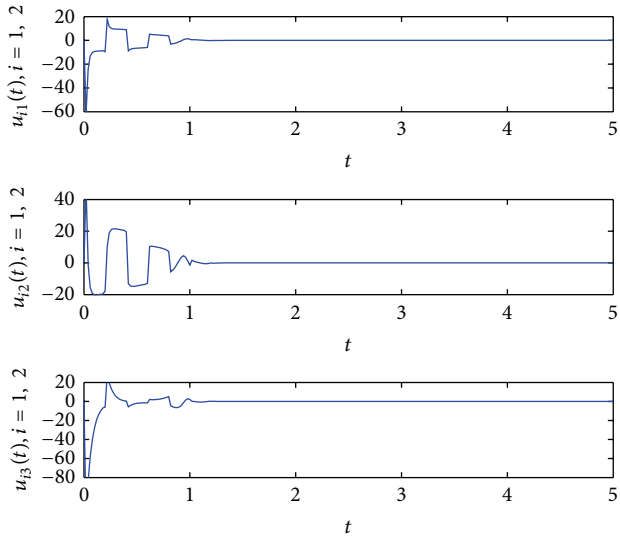

Figure 8: Time response of the control inputs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

H_∞ Filtering for Discrete Markov Jump Singular Systems with Mode-Dependent Time Delay Based on T-S Fuzzy Model

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This paper investigates the H_∞ filtering problem of discrete singular Markov jump systems (SMJSs) with mode-dependent time delay based on T-S fuzzy model. First, by Lyapunov-Krasovskii functional approach, a delay-dependent sufficient condition on H_∞-disturbance attenuation is presented, in which both stability and prescribed H_∞ performance are required to be achieved for the filtering-error systems. Then, based on the condition, the delay-dependent H_∞ filter design scheme for SMJSs with mode-dependent time delay based on T-S fuzzy model is developed in term of linear matrix inequality (LMI). Finally, an example is given to illustrate the effectiveness of the result.

1. Introduction

Recently, some efforts on filtering and control problem have been put on the Markov jump linear systems with Markovian jump time delays [1–7]. For example, [1] dealt with the robust stabilizability and H_∞ disturbance attenuation for a class of uncertain descriptor time-delay systems with jumping parameters. The transition of the jumping parameters in the systems is governed by a finite state Markov process. [2] addressed the problems of delay-dependent robust H_∞ control and filtering for Markovian jump linear systems with norm-bounded parameter uncertainties and time-varying delays. The robust H_∞ filtering problem for mode-dependent time-delay discrete Markov jump singular systems with parameter uncertainties is discussed in [3]. The energy-to-peak filtering problem of Markov jump systems with interval time-varying delay is investigated in [4]. The ℓ_2-ℓ_∞ fuzzy control problem was considered for nonlinear stochastic Markov jump systems with neutral time-delays in [5].

In practical applications, to research the nonlinear time-delay system, the scholars considered the T-S fuzzy time-delay model which is a kind of effective representation, and many analysis and synthesis methods for T-S fuzzy time-delay systems have been developed over the past years [8–17]. So, many scholars extended this model to Markov jump systems with time delays [18–28]. Reference [18] was concerned with an H_∞ control for a class of T-S fuzzy Markov jump system under unreliable communication links. Reference [19] concerned with the adaptive synchronization for T-S fuzzy neural networks with stochastic noises and Markovian jumping parameters. Reference [20] was concerned with the stability and stabilization problems for a class of nonlinear systems with Markovian jump parameters. The T-S fuzzy model was employed to represent the Markovian jump nonlinear systems with partly unknown transition probabilities. Reference [21] dealt with the delay-dependent asymptotic stability analysis problem for a class of fuzzy bidirectional associative memory neural networks with time-varying interval delays and Markovian jumping parameters by T-S fuzzy model. In [22], the asymptotic stability of fuzzy Markovian jumping genetic regulatory networks with time-varying delays by delay decomposition approach was investigated.

For singular systems, since the regularity, causality, and stability are needed to be considered, this class of systems is more complicated and difficult compared with standard state

space systems. The strict conditions of linear matrix inequalities (LMI) are not easy to obtain. Many references [29–31] discussed the control problems for SMJSs.

However, to the best of our knowledge, the H_∞ filtering problem of discrete SMJSs with mode-dependent time delay based on T-S fuzzy model has not been addressed, which is the focus of this paper.

This paper is organized as follows: In Section 2, the filter for SMJSs with mode-dependent time delay based on T-S fuzzy model is formulated. In Section 3 we give the sufficient condition to assure asymptotic stability and the H_∞ noise-attenuation level bound for the SMJSs filtering-error systems. Based on the condition in Section 3, we present a stable fuzzy filter in term of LMIs. Section 4 provides illustrative examples to demonstrate the effectiveness of the proposed method. Conclusions are given in Section 5.

Notations. The notations used throughout this paper are fairly standard. The superscript "T" stands for matrix transpose, and the notation $P > 0$ ($P \geq 0$) means that matrix P is real symmetric and positive (or being positive semi-definite). I and 0 are used to denote appropriate dimensions identity matrix and zero matrix, respectively. The notation $*$ in a symmetric always denotes the symmetric block in the matrix. The parameter diag$\{\cdots\}$ denotes a block-diagonal matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. System Descriptions and Preliminaries

Given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is the algebra of events and $\mathcal{P}$ is the probability measure define on $\mathcal{F}$. We consider a class of discrete-time uncertain nonlinear SMJSs with time-varying delay over space $(\Omega, \mathcal{F}, \mathcal{P})$. which can be represented by the following T-S model.

Plant Rule l. If s_{1k} is F_{l1} and s_{2k} is F_{l2} and...and s_{nk} is F_{ln} then

$$\begin{aligned} Ex_{k+1} &= A_l(r_k)x_k + A_{dl}(r_k)x_{k-d(r_k)} + B_l(r_k)\omega_k, \\ y_k &= C_l(r_k)x_k + C_{dl}(r_k)x_{k-d(r_k)} + D_{1l}(r_k)\omega_k, \\ z_k &= L_l(r_k)x_k + L_{dl}(r_k)x_{k-d(r_k)} + D_{2l}(r_k)\omega_k, \\ x_k &= \phi_k, \\ k &= -\overline{d}, \ldots, -1, 0, \quad l = 1, 2, \ldots, m, \end{aligned} \tag{1}$$

$$\tag{2}$$

where, $k \in Z$, $s_{1k}, s_{2k}, \ldots, s_{nk}$ are the premise variables that are measurable and each F_{lg} ($g = 1, 2, \ldots, n$) is fuzzy set. $x_k \in R^n$ is the state vector. y_k is the measurement. z_k is the signal to be estimated. ω_k is the disturbance input vector which belongs to $L_2 = \{\{a_k, k \in Z\} \mid \sum_{k=0}^{\infty} a_k^T a_k < \infty\}$. m is the number of IF-THEN rules, ϕ_k is a vector-valued initial function. $\{r_k, k \in Z\}$ is a Markov chain taking values in finite space $s = \{1, 2, \ldots, N\}$, with a transition probability from mode i at time k to mode j at time $k+1$ as

$$p_{ij} = \Pr\{r_{k+1} = j \mid r_k = i\}, \tag{3}$$

with $p_{ij} \geq 0$ for $i, j \in s$. $\sum_{j=1}^{N} p_{ij} = 1$, $d(r_k)$ is a positive integer, denoting the time delay of the system involved in node r_k and $0 < \underline{d}_i \leq d(i) \leq \overline{d}_i < \infty$. For each $i \in s$, the matrix $E \in R^{n\times n}$ is a known constant and singular, and rank$(E) = r < n$.

For each $i \in s$, $A_l(i)$, $A_{dl}(i)$, $B_l(i)$, $C_l(i)$, $C_{dl}(i)$, $D_{1l}(i)$, $L_l(i)$, $L_{1l}(i)$, and $D_{2l}(i)$ are known constant matrices with appropriate dimensions.

Remark 1. In this paper, we consider only the case $\underline{d} \leq d(i) \leq \overline{d} < \infty$, with $\underline{d} = \min\{\underline{d}_i, i \in s\}$, $\overline{d} = \max\{\overline{d}_i, i \in s\}$.

By using center-average defuzzifier, product inference, and singleton fuzzifier, the dynamic fuzzy model (1) can be expressed by the following global model:

$$\begin{aligned} &\sum_{l=1}^{m} u_l(s_k) Ex_{k+1} \\ &\quad = \sum_{l=1}^{m} u_l(s_k) A_l(r_k) x_k + \sum_{l=1}^{m} u_l(s_k) A_{dl}(r_k) x_{k-d(r_k)} \\ &\qquad + \sum_{l=1}^{m} u_l(s_k) B_l(r_k) \omega_k, \\ y_k &= \sum_{l=1}^{m} u_l(s_k) C_l(r_k) x_k + \sum_{l=1}^{m} u_l(s_k) C_{dl}(r_k) x_{k-d(r_k)} \\ &\qquad + \sum_{l=1}^{m} u_l(s_k) D_{1l}(r_k) \omega_k, \\ z_k &= \sum_{l=1}^{m} u_l(s_k) L_l(r_k) x_k + \sum_{l=1}^{m} u_l(s_k) L_{dl}(r_k) x_{k-d(r_k)} \\ &\qquad + \sum_{l=1}^{m} u_l(s_k) D_{2l}(r_k) \omega_k, \end{aligned} \tag{4}$$

with

$$u_l(s_k) = \frac{\alpha_l(s_k)}{\sum_{l=1}^{m} \alpha_l(s_k)}, \quad \alpha_l(s_k) = \prod_{g=1}^{n} F_{lg}(s_{gk}). \tag{5}$$

$F_{lg}(s_{gk})$ is the grade of membership of s_{gk} in F_{lg}, and it is assumed that $\alpha_l(s_k) \geq 0$, $l = 1, 2, \ldots, m$. $\sum_{l=1}^{m} \alpha_l(s_k) > 0$ for all k. Therefore, $u_l(s_k) \geq 0$ and $\sum_{l=1}^{m} u_l(s_k) = 1$ for all k.

The system (4) can be represented by the following equation:

$$\begin{aligned} Ex_{k+1} &= A(k, r_k)x_k + A_d(k, r_k)x_{k-d(r_k)} + B(k, r_k)\omega_k, \\ y_k &= C(k, r_k)x_k + C_d(k, r_k)x_{k-d(r_k)} + D_1(k, r_k)\omega_k, \\ z_k &= L(k, r_k)x_k + L_d(k, r_k)x_{k-d(r_k)} + D_2(k, r_k)\omega_k, \end{aligned} \tag{6}$$

with

$$\begin{aligned}
&\sum_{l=1}^{m} u_l(s_k) A_l(r_k) = A(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) A_{dl}(r_k) = A_d(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) B_l(r_k) = B(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) C_l(r_k) = C(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) C_{dl}(r_k) = C_d(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) D_{1l}(r_k) = D_1(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) L_l(r_k) = L(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) L_{dl}(r_k) = L_d(k, r_k),\\
&\sum_{l=1}^{m} u_l(s_k) D_{2l}(r_k) = D_2(k, r_k).
\end{aligned} \tag{7}$$

Definition 2. System $Ex_{k+1} = A(r_k)x_k$ or the pair $(E, A(r_k))$ is said to be

(1) regular if $\det(zE - A(r_k)) \not\equiv 0$ for any $r_k = i, i \in s$,

(2) causal if it is regular and $\text{degree}(\det(zE - A(r_k))) = \text{rank}(E)$ for any $r_k = i, i \in s$.

Definition 3. System (6) is said to be stochastically stable, if for every initial state (ϕ, r_0), the following condition $E\{\sum_{k=0}^{\infty} \|x_k(\phi, r_0)\|^2 \mid \phi, r_0\} < \infty$ is satisfied.

In this paper, employing the idea of PDC, we consider the filter based on T-S fuzzy-model with order $\hat{n}$ ($\hat{n} = n$ for full-order filter, and $1 \le \hat{n} \le n$ for reduced-order filter):

$$\begin{aligned}
\hat{E}\hat{x}_{k+1} &= \hat{A}(k, r_k)\hat{x}_k + \hat{B}(k, r_k) y_k,\\
\hat{z}_k &= \hat{C}(k, r_k)\hat{x}_k + \hat{D}(k, r_k) y_k,
\end{aligned} \tag{8}$$

where,

$$\begin{aligned}
\hat{A}(k, r_k) &= \sum_{l=1}^{m} u_l(s_k)\hat{A}_l(r_k),\\
\hat{B}(k, r_k) &= \sum_{l=1}^{m} u_l(s_k)\hat{B}_l(r_k),\\
\hat{C}(k, r_k) &= \sum_{l=1}^{m} u_l(s_k)\hat{C}_l(r_k).
\end{aligned} \tag{9}$$

For each $r_k = i, i \in s$, the matrices $\hat{E} \in R^{\hat{n}\times\hat{n}}$, $\hat{A}_l(r_k) \in R^{\hat{n}\times\hat{n}}$, $\hat{B}_l(r_k) \in R^{\hat{n}\times m}$, and $\hat{C}_l(r_k) \in R^{q\times\hat{n}}$ are to be calculate.

Remark 4. $\hat{n}$ may be either equal to n or less than n.

Let

$$\bar{x}_k = \left[x_k^T, \hat{x}_k^T\right], \qquad \bar{z}_k = z_k - \hat{z}_k. \tag{10}$$

Then, the filtering system from (6) and the filter (8) can be written as

$$\begin{aligned}
\bar{E}\bar{x}_{k+1} &= \bar{A}(k, r_k)\bar{x}_k + \bar{A}_d(k, r_k)\bar{x}_{k-d(r_k)} + \bar{B}(k, r_k)\omega_k,\\
\bar{z}_k &= \bar{L}(k, r_k)\bar{x}_k + \bar{L}_d(k, r_k)\bar{x}_{k-d(r_k)} + \bar{D}(k, r_k)\omega_k,
\end{aligned} \tag{11}$$

where, for each $r_k = i, i \in s$,

$$\begin{aligned}
\bar{E} &= \begin{bmatrix} E & 0\\ 0 & \hat{E}\end{bmatrix},\\
\bar{A}(k, i) &= \begin{bmatrix} A(k, i) & 0\\ \hat{B}(k, i)C(k, i) & \hat{A}(k, i)\end{bmatrix},\\
\bar{A}_d(k, i) &= \begin{bmatrix} A_d(k, i) & 0\\ \hat{B}(k, i)C_d(k, i) & 0\end{bmatrix},\\
\bar{B}(k, i) &= \begin{bmatrix} A_d(k, i)\\ \hat{B}(k, i)D_1(k, i)\end{bmatrix},\\
\bar{L}(k, i) &= \left[L(k, i) - \hat{D}(k, i)C(k, i), -\hat{C}(k, i)\right],\\
\bar{L}_d(k, i) &= \left[L_d(k, i) - \hat{D}(k, i), 0\right],\\
\bar{D}(k, i) &= D_2(k, i) - \hat{D}(k, i)D_1(k, i).
\end{aligned} \tag{12}$$

The H_∞ filtering problem is formulated as follows. Given system (1) and a scalar $\gamma > 0$, determine a filter in the form of (8) such that the filtering error system (11) is regular, causal, stochastically stable and satisfies the following H_∞ performance:

$$E\left\{\sum_{k=0}^{\infty} \bar{z}_k^T \bar{z}_k\right\} < \gamma^2 \sum_{k=0}^{\infty} \omega_k^T \omega_k. \tag{13}$$

For the convenience, we use the following notations:

$$\underline{p} = \min\left\{p_{ij}, i \in s\right\}, \qquad \rho = 1 + \left(1 - \underline{p}\right)\left(\overline{d} - \underline{d}\right). \tag{14}$$

Lemma 5 (see [32]). *Given matrices X, Y, and Z with appropriate dimensions, Y is symmetric and $Y > 0$. Then*

$$-Z^T Y^{-1} Z \le X^T Y X + X^T Z + Z^T X. \tag{15}$$

Remark 6. Without loss of generality, we assume that $0 < p_{ij} < 1$ and also $0 < \underline{p} < 1$.

3. Main Results

In this section, we proposed the sufficient conditions and design method for the H_∞ filter in term of LMIs. First of all,

we present a delay-dependent condition such that system (6) is regular, causal, and stochastically stable. Consider system (6) with $\omega_k = 0$

$$Ex_{k+1} = A(k, r_k)\, x_k + A_d(k, r_k)\, x_{k-d(r_k)}, \qquad x_k = \phi_k, \quad k = -\overline{d}, \ldots, -1, 0. \tag{16}$$

Theorem 7. *For a given scalar γ, the filtering error system* (11) *is regular, causal, and stochastically stable and satisfies* (13)*, if, for each mode $i \in s$, there exist symmetric matrices $X_i > 0$, $Z > 0$, $U > 0$, S_i and matrices N_{i1} and N_{i2} that satisfy the following set of coupled LMIs:*

$$\begin{bmatrix} \overline{\Phi}_{i11} & \overline{\Phi}_{i12} & \overline{\Phi}_{i13} & \overline{d}N_{i1} & \overline{L}^T(k,i) \\ * & \overline{\Phi}_{i22} & \overline{\Phi}_{i23} & \overline{d}N_{i2} & \overline{L}_d^T(k,i) \\ * & * & \overline{\Phi}_{i33} - \gamma^2 I & 0 & \overline{D}^T(k,i) \\ * & * & * & -\overline{d}Z & 0 \\ * & * & * & * & I \end{bmatrix}, \tag{17}$$

where,

$$\begin{aligned} \Phi_{i11} &= \overline{A}^T(k,i)\,\overline{X}_i\overline{A}(k,i) - \overline{A}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{A}(k,i) \\ &\quad - \overline{E}_k^T X_i \overline{E} + \rho U + N_{i1}\overline{E} + \overline{E}_k^T N_{il}^T \\ &\quad + \overline{d}\left(\overline{A}(k,i) - \overline{E}\right)^T Z\left(\overline{A}(k,i) - \overline{E}\right), \\ \Phi_{i12} &= \overline{A}^T(k,i)\,\overline{X}_i\overline{A}_d(k,i) - \overline{A}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{A}_d(k,i) \\ &\quad - N_{i1}\overline{E} + \overline{E}_k^T N_{i2}^T + \overline{d}\left(\overline{A}(k,i) - \overline{E}\right)^T Z\overline{A}_d(k,i), \\ \Phi_{i22} &= \overline{A}_d^T(k,i)\,\overline{X}_i\overline{A}_d(k,i) - \overline{A}_d^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{A}_d(k,i) \\ &\quad - U - N_{i2}\overline{E} - \overline{E}_k^T N_{i2}^T + \overline{d}\overline{A}_d^T(k,i)\,Z\overline{A}_d(k,i), \\ \overline{X}_i &= \sum_{j=1}^{N} p_{ij}X_j, \qquad \overline{S}_i = \sum_{j=1}^{N} p_{ij}S_j, \\ \overline{\Phi}_{i13} &= \overline{A}^T(k,i)\,\overline{X}_i\overline{B}(k,i) - \overline{A}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{B}(k,i) \\ &\quad + \overline{d}\left(\overline{A}(k,i) - \overline{E}\right)^T Z\overline{B}(k,i), \\ \overline{\Phi}_{i23} &= \overline{A}_d^T(k,i)\,\overline{X}_i\overline{B}(k,i) - \overline{A}_d^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{B}(k,i) \\ &\quad + \overline{d}\overline{A}_d(k,i)\,Z\overline{B}(k,i), \\ \overline{\Phi}_{i33} &= \overline{B}^T(k,i)\,\overline{X}_i\overline{B}(k,i) - \overline{B}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{B}(k,i) \\ &\quad + \overline{d}\overline{B}^T(k,i)\,Z\overline{B}(k,i). \end{aligned} \tag{18}$$

$\overline{E} = \mathrm{diag}\{E, \widehat{E}\}$, $\overline{R} = \mathrm{diag}\{R, \widehat{R}\}$, $\widehat{R} \in R^{\hat{n}\times\hat{n}}$ *is any constant matrix satisfying* $\widehat{R}\widehat{E} = 0$ *with* $\mathrm{rank}(\widehat{R}) = \hat{n} - \mathrm{rank}(\widehat{E})$.

Proof. The proof is omitted because it is similar to [3]. □

Obviously, the filter cannot be obtained from (17). In order to design the filter for system (1) and without loss of generality, we assume that $P = n + \hat{n}$.

Theorem 8. *Let $\gamma > 0$, ϵ_1, ϵ_2, ϵ_3, ϵ_4 and ϵ_5 be given scalars. There exists a filter in the form of* (8) *with $\widehat{E} = E$ such that the filtering error system* (11) *is regular, causal, and stochastically stable and satisfies* (13)*; if for each mode $i \in s$, there exist matrices $P_i > 0$, $S_i > 0$, $\overline{Z} > 0$, $U > 0$, G_i, N_{i1}, N_{i2}, $\widehat{A}(k,i)$, $\widehat{B}(k,i)$, $\widehat{C}(k,i)$, and $\widehat{D}(k,i)$ that satisfy the following coupled LMIs:*

$$\Phi_i = \begin{bmatrix} \Phi_{i11} & \Phi_{i12} & \Phi_{i13} & \overline{d}\epsilon_5 N_{i1} & \overline{L}^T(k,i) & \overline{A}^T(k,i) & \overline{d}\left(\overline{A}(k,i) - \overline{E}\right)^T & \epsilon_2 I & 0 \\ * & \Phi_{i22} & \Phi_{i23} & \overline{d}\epsilon_5 N_{i2} & \overline{L}_d^T(k,i) & \overline{A}_d^T(k,i) & \overline{d}\overline{A}_d^T(k,i) & \epsilon_3 I & 0 \\ * & * & \Phi_{i33} & 0 & \overline{D}^T(k,i) & \overline{B}^T(k,i) & \overline{d}\overline{B}^T(k,i) & \epsilon_4 I & 0 \\ * & * & * & -\overline{d}\overline{Z} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & G_i^T + G_i & 0 & 0 & \epsilon_1 W_i \\ * & * & * & * & * & 0 & \overline{d}\overline{Z} - 2\overline{d}\epsilon_5 I & 0 & 0 \\ * & * & * & * & * & 0 & 0 & -\overline{S}_i & 0 \\ * & * & * & * & * & 0 & 0 & 0 & -\overline{P} \end{bmatrix} < 0, \tag{19}$$

where

$$\begin{aligned} \Phi_{i11} &= \rho U + P_i + N_{i1}\overline{E} + \overline{E}^T N_{i1}^T + \epsilon_1\overline{E}^T + \epsilon_1\overline{E} \\ &\quad + \epsilon_2\overline{A}^T(k,i)\,\overline{R}^T + \epsilon_2\overline{R}\,\overline{A}(k,i), \\ \Phi_{i12} &= -N_{i1}\overline{E} + \overline{E}^T N_{i2} + \epsilon_3\overline{A}^T(k,i)\,\overline{R}^T + \epsilon_2\overline{R}\,\overline{A}_d(k,i), \\ \Phi_{i13} &= \epsilon_4\overline{A}^T(k,i)\,\overline{R}^T + \epsilon_2\overline{R}\,\overline{B}(k,i), \\ \Phi_{i23} &= \epsilon_4\overline{A}_d^T(k,i)\,\overline{R}^T + \epsilon_3\overline{R}\,\overline{B}(k,i), \end{aligned}$$

$$\Phi_{i22} = -U - N_{i2}\overline{E} - \overline{E}^T N_{i2}^T + \epsilon_3 \overline{A}_d^T(k,i)\,\overline{R}^T + \epsilon_3 \overline{R}\,\overline{A}_d(k,i),$$
$$\Phi_{i33} = -\gamma^2 I + \epsilon_4 \overline{B}^T(k,i)\,\overline{R}^T + \epsilon_4 \overline{R}\,\overline{B}(k,i),$$
$$W_i = \left[\sqrt{p_{i1}}G_i, \sqrt{p_{i2}}G_i, \ldots, \sqrt{p_{iN}}G_i\right],$$
$$\overline{P} = \operatorname{diag}\{P_1, P_2, \ldots, P_N\}. \tag{20}$$

$\overline{R} = \operatorname{diag}\{R, R\}$; $R \in R^{n\times n}$ *is any constant matrix satisfying* $RE = 0$ *with* $\operatorname{rank}(R) = n - r$.

Proof. Since $X_i > 0$, (17) is equivalent to

$$\begin{bmatrix}
\widetilde{\Phi}_{i11} & \widetilde{\Phi}_{i12} & \widetilde{\Phi}_{i13} & \overline{d}N_{i1} & \overline{L}^T(i) & \overline{A}^T(i) & \overline{d}\left(\overline{A}(i) - \overline{E}\right)^T \\
* & \widetilde{\Phi}_{i22} & \widetilde{\Phi}_{i23} & \overline{d}N_{i2} & \overline{L}_d^T(i) & \overline{A}_d^T(i) & \overline{d}\overline{A}_d^T(i) \\
* & * & \widetilde{\Phi}_{i33} & 0 & \overline{D}^T(i) & \overline{B}^T(i) & \overline{d}\overline{B}^T(i) \\
* & * & * & -\overline{dZ} & 0 & 0 & 0 \\
* & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & -\overline{X}_i^{-1} & 0 \\
* & * & * & * & * & 0 & -\overline{dZ}^{-1}
\end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned}
\widetilde{\Phi}_{i11} &= -\overline{A}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{A}(k,i) - \overline{E}^T X_i \overline{E} + \rho U \\
&\quad + N_{i1}\overline{E} + \overline{E}^T N_{i1}^T, \\
\widetilde{\Phi}_{i12} &= -\overline{A}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{A}_d(k,i) - N_{i1}\overline{E} + \overline{E}^T N_{i2}^T, \\
\widetilde{\Phi}_{i13} &= -\overline{A}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{B}(k,i), \\
\widetilde{\Phi}_{i22} &= -\overline{A}_d^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{A}_d(k,i) - U - N_{i2}\overline{E} - \overline{E}N_{i2}^T, \\
\widetilde{\Phi}_{i23} &= -\overline{A}_d^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{B}(k,i), \\
\widetilde{\Phi}_{i33} &= -\gamma^2 I + \overline{B}^T(k,i)\,\overline{R}^T\overline{S}_i\overline{R}\,\overline{B}(k,i);
\end{aligned} \tag{22}$$

then, based on Lemma 5 and $X_i > 0$, $S_i > 0$, $Z > 0$, for any matrix G_i with appropriate dimensions, and scalars ϵ_i, $i = 1, \ldots, 5$, the following inequalities hold:

$$\begin{aligned}
-\overline{X}_i^{-1} &\le G_i\overline{X}_i G_i^T + G_i^T + G_i, \\
-\overline{E}^T X_i \overline{E} &\le \epsilon_1 \overline{E}^T + \epsilon_1 \overline{E} + \epsilon_1^2 X_i^{-1}, \\
-Z^{-1} &\le \epsilon_5^2 Z - 2\epsilon_5 I, \\
&-\begin{bmatrix} \overline{A}^T(k,i) \\ \overline{A}_d^T(k,i) \\ \overline{B}^T(k,i) \end{bmatrix} \overline{R}^T\overline{S}_i\overline{R} \begin{bmatrix} \overline{A}^T(k,i) \\ \overline{A}_d^T(k,i) \\ \overline{B}^T(k,i) \end{bmatrix}^T \\
&\le \begin{bmatrix} \overline{A}^T(k,i) \\ \overline{A}_d^T(k,i) \\ \overline{B}^T(k,i) \end{bmatrix} \overline{R}^T \begin{bmatrix} \epsilon_2 I \\ \epsilon_3 I \\ \epsilon_4 I \end{bmatrix}^T + \begin{bmatrix} \epsilon_2 I \\ \epsilon_3 I \\ \epsilon_4 I \end{bmatrix} \overline{R} \begin{bmatrix} \overline{A}^T(k,i) \\ \overline{A}_d^T(k,i) \\ \overline{B}^T(k,i) \end{bmatrix}^T \\
&\quad + \begin{bmatrix} \epsilon_2 I \\ \epsilon_3 I \\ \epsilon_4 I \end{bmatrix} \overline{S}_i^{-1} \begin{bmatrix} \epsilon_2 I \\ \epsilon_3 I \\ \epsilon_4 I \end{bmatrix}^T.
\end{aligned} \tag{23}$$

Let $P_i = \epsilon_1^2 X_i^{-1}$, $\overline{Z} = \epsilon_5^2 Z$. According to the inequalities (23), and applying the Schur complement, it is obtained that if (19) holds, then (21) holds. The proof is completed. □

Then, we readily obtain the following theorem.

Theorem 9. *Let* $\gamma > 0$, ϵ_1, ϵ_2, ϵ_3, ϵ_4 *and* ϵ_5 *be given scalars. There exists a filter in the form of* (8) *with* $\widehat{E} = E$ *such that the filtering error system* (11) *is regular, causal, stochastically stable and satisfies* (13)*; if for each mode* $i \in s$*, there exist matrices* $P_i > 0$, $S_i > 0$, $\overline{Z} > 0$, $U > 0$, G_i, N_{i1}, N_{i2}, $\widehat{A}_l(i)$, $\widehat{B}_l(i)$, $\widehat{C}_l(i)$, $\widehat{D}_l(i)$ *that satisfy the following coupled LMIs:*

$$\begin{aligned}
&\Phi_{ll} < 0, \quad (1 = 1, 2, \ldots, m), \\
&\frac{\Phi_{lj} + \Phi_{jl}}{2} < 0, \quad \left(1 \le l < j \le m\right),
\end{aligned} \tag{24}$$

$$\Phi_{lj} = \begin{bmatrix}
\Phi_{ilj11} & \Phi_{ilj12} & \Phi_{ilj13} & \overline{d}\epsilon_5 N_{i1} & \overline{L}_{lj}^T(i) & \overline{A}_{lj}^T(i) & \overline{d}\left(\overline{A}_{lj}(i) - \overline{E}\right)^T & \epsilon_2 I & 0 \\
* & \Phi_{ilj22} & \Phi_{ilj23} & \overline{d}\epsilon_5 N_{i2} & \overline{L}_{dlj}^T(i) & \overline{A}_{dlj}^T(i) & \overline{d}\overline{A}_{dlj}^T(i) & \epsilon_3 I & 0 \\
* & * & \Phi_{ilj33} & 0 & \overline{D}_{lj}^T(i) & \overline{B}_{lj}^T(i) & \overline{d}\overline{B}_{lj}^T(i) & \epsilon_4 I & 0 \\
* & * & * & -\overline{dZ} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & G_i^T + G_i & 0 & 0 & \epsilon_1 W_i \\
* & * & * & * & * & 0 & \overline{dZ} - 2\overline{d}\epsilon_5 I & 0 & 0 \\
* & * & * & * & * & 0 & 0 & -\overline{S}_i & 0 \\
* & * & * & * & * & 0 & 0 & 0 & -\overline{P}
\end{bmatrix}, \tag{25}$$

where

$$\begin{aligned}
\Phi_{ilj11} &= \rho U + P_i + N_{i1}\overline{E} + \overline{E}^T N_{i1}^T + \epsilon_1 \overline{E}^T + \epsilon_1 \overline{E} \\
&\quad + \epsilon_2 \overline{A}_{lj}^T(i)\,\overline{R}^T + \epsilon_2 \overline{R}\,\overline{A}_{lj}(i), \\
\Phi_{ilj12} &= -N_{i1}\overline{E} + \overline{E}^T N_{i2}^T + \epsilon_3 \overline{A}_{lj}^T(i)\,\overline{R}^T + \epsilon_2 \overline{R}\,\overline{A}_{dlj}(i), \\
\Phi_{ilj13} &= \epsilon_4 \overline{A}_{lj}^T(i)\,\overline{R}^T + \epsilon_2 \overline{R}\,\overline{B}_{lj}(i), \\
\Phi_{i23} &= \epsilon_4 \overline{A}_{dlj}^T(i)\,\overline{R}^T + \epsilon_3 \overline{R}\,\overline{B}_{lj}(i), \\
\Phi_{ilj22} &= -U - N_{i2}\overline{E} - \overline{E}^T N_{i2}^T + \epsilon_3 \overline{A}_{dlj}^T(i)\,\overline{R}^T + \epsilon_3 \overline{R}\,\overline{A}_{dlj}(i), \\
\Phi_{ilj33} &= -\gamma^2 I + \epsilon_4 \overline{B}_{lj}^T(i)\,\overline{R}^T + \epsilon_4 \overline{R}\,\overline{B}_{lj}(i), \\
W_i &= \left[\sqrt{p_{i1}}G_i, \sqrt{p_{i2}}G_i, \ldots, \sqrt{p_{iN}}G_i\right], \\
\overline{P} &= \operatorname{diag}\{P_1, P_2, \ldots, P_N\}.
\end{aligned} \tag{26}$$

$\overline{R} = \operatorname{diag}\{R, R\}$; $R \in R^{n\times n}$ *is any constant matrix satisfying* $RE = 0$ *with* $\operatorname{rank}(R) = n - r$.

Proof. Noting that $u_j(s_k) \geq 0$, $\sum_{l=1}^m u_l(s_k) = 1$, $\sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) = 1$

$$\begin{aligned}
\overline{E} &= \begin{bmatrix} E & 0 \\ 0 & \widehat{E} \end{bmatrix} = \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \begin{bmatrix} E & 0 \\ 0 & \widehat{E} \end{bmatrix}, \\
\overline{A}(k,i) &= \begin{bmatrix} A(k,i) & 0 \\ \widehat{B}(k,i)\,C(k,i) & \widehat{A}(k,i) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{l=1}^m u_l(s_k) A_l(i) & 0 \\ \sum_{j=1}^m u_j(s_k) \widehat{B}_j(i) \sum_{l=1}^m u_l(s_k) C_l(i) & \sum_{l=1}^m u_l(s_k) \widehat{A}_j(i) \end{bmatrix} \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \begin{bmatrix} A_l(i) & 0 \\ \widehat{B}_j(i) C_l(i) & \widehat{A}_j(i) \end{bmatrix} \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \overline{A}_{lj}(i), \\
\overline{A}_d(k,i) &= \begin{bmatrix} A_d(k,i) & 0 \\ \widehat{B}(k,i)\,C_d(k,i) & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{l=1}^m u_l(s_k) A_{dl}(i) & 0 \\ \sum_{j=1}^m u_j(s_k) \widehat{B}_j(i) \sum_{l=1}^m u_l(s_k) C_{dl}(i) & 0 \end{bmatrix} \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \begin{bmatrix} A_{dl}(i) & 0 \\ \widehat{B}_j(i) C_{dl}(i) & 0 \end{bmatrix} \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \overline{A}_{dlj}(i), \\
\overline{B}(k,i) &= \begin{bmatrix} A_d(k,i) \\ \widehat{B}(k,i)\,D_1(k,i) \end{bmatrix} \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \begin{bmatrix} A_{dl}(i) \\ \widehat{B}_j(i) D_{1l}(i) \end{bmatrix} \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \overline{B}_{lj}(i), \\
\overline{L}(k,i) &= \left[L(k,i) - \widehat{D}(k,i)\,C(k,i), -\widehat{C}(k,i)\right] \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \left[L_l(i) - \widehat{D}_j(i)\,C_l(i), -\widehat{C}_j(i)\right] \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \overline{L}_{lj}(i), \\
\overline{L}_d(k,i) &= \left[L_d(k,i) - \widehat{D}(k,i)\,C_d(k,i), 0\right] \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \left[L_{dl}(i) - \widehat{D}_j(i)\,C_{dl}(i), 0\right] \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \overline{L}_{dlj}(i), \\
\overline{D}(k,i) &= \left[D_2(k,i) - \widehat{D}(k,i)\,D_1(k,i)\right] \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \left[D_{2l}(i) - \widehat{D}_j(i)\,D_{1l}(i)\right] \\
&= \sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \overline{D}_{lj}(i), \\
\sum_{l=1}^m u_l(s_k) \sum_{j=1}^m u_j(s_k) \Phi_{ij} &= \sum_{l=1}^m u_l(s_k) \Phi_{ll} \\
&\quad + \sum_{l<j}^m u_l(s_k)\,u_j(s_k) \left(\Phi_{lj} + \Phi_{jl}\right).
\end{aligned} \tag{27}$$

Therefore, condition (19) can guarantee that condition (24) holds. This completes the proof. □

4. Example

In this section, we give a numerical example to illustrate the use of the presented method. Consider the system of the form with two fuzzy rules and two modes.

The first rule parameters with 1st mode are as follows:

$$E = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1(1) = \begin{bmatrix} 1.2 & 0.2 \\ 1.2 & 2.2 \end{bmatrix},$$
$$A_{d1}(1) = \begin{bmatrix} 0.1 & -0.1 \\ 0.3 & -0.2 \end{bmatrix}, \quad B_1(1) = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix},$$
$$C_1(1) = \begin{bmatrix} -0.5 & 0.3 \end{bmatrix}, \quad (28)$$
$$C_{d1}(1) = \begin{bmatrix} 0.4 & 0.1 \end{bmatrix}, \quad D_{11}(1) = 0.1,$$
$$L_1(1) = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}, \quad L_{d1}(1) = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix},$$
$$D_{21}(1) = 0.2.$$

The first rule parameters with 2nd mode are as follows:

$$A_1(2) = \begin{bmatrix} -0.3 & 1 \\ -2 & 4 \end{bmatrix}, \quad A_{d1}(2) = \begin{bmatrix} 0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix},$$
$$B_1(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad C_1(2) = \begin{bmatrix} -0.4 & 0.6 \end{bmatrix},$$
$$C_{d1}(2) = \begin{bmatrix} 0.4 & -0.3 \end{bmatrix}, \quad D_{11}(2) = 0, \quad (29)$$
$$L_1(2) = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}, \quad L_{d1}(2) = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix},$$
$$D_{21}(2) = 0.2.$$

The second rule parameters with 1st mode are as follows:

$$E = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2(1) = \begin{bmatrix} 1.1 & 0.2 \\ 1.1 & 2.2 \end{bmatrix},$$
$$A_{d2}(1) = \begin{bmatrix} 0.2 & -0.1 \\ 0.2 & -0.2 \end{bmatrix}, \quad B_2(1) = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix},$$
$$C_2(1) = \begin{bmatrix} -0.4 & 0.3 \end{bmatrix}, \quad (30)$$
$$C_{d2}(1) = \begin{bmatrix} 0.5 & 0.2 \end{bmatrix}, \quad D_{12}(1) = 0.2,$$
$$L_2(1) = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix}, \quad L_{d2}(1) = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix},$$
$$D_{22}(1) = 0.3.$$

The second rule parameters with 2nd mode are as follows:

$$A_2(2) = \begin{bmatrix} -0.5 & 1 \\ -2 & 5 \end{bmatrix}, \quad A_{d2}(2) = \begin{bmatrix} 0.2 & 0 \\ -0.3 & -0.1 \end{bmatrix},$$
$$B_2(2) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad C_2(2) = \begin{bmatrix} -0.5 & 0.6 \end{bmatrix},$$
$$C_{d2}(2) = \begin{bmatrix} 0.5 & -0.3 \end{bmatrix}, \quad D_{12}(2) = 0.1, \quad (31)$$
$$L_2(2) = \begin{bmatrix} 0.3 & 0.5 \end{bmatrix}, \quad L_{d2}(2) = \begin{bmatrix} 0.4 & 0.4 \end{bmatrix},$$
$$D_{22}(2) = 0.1.$$

The time-delays $d(1) = 2, d(2) = 3$. The mode switching is governed by a Markov chain that has the following transition probability:

$$p_{11} = 0.9, \quad p_{12} = 0.1; \quad p_{21} = 0.2, \quad p_{22} = 0.8. \quad (32)$$

For the full-order H_∞ filter with $\widehat{E} = E$, we choose that $\gamma = 1$, and let $R = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 5 \end{smallmatrix}\right]$, $\overline{R} = \mathrm{diag}\{R, R\}$, $\epsilon_1 = -9$, $\epsilon_2 = -4$, $\epsilon_3 = 0.01$, $\epsilon_4 = -0.01$, $\epsilon_5 = 70$. Solving the LMIs (24). The solutions are as follows:

$$P_1 = \begin{bmatrix} 31.1 & 20.7 & 0 & 0 \\ 20.7 & 54.1 & 0 & 0 \\ 0 & 0 & 34 & 3.6 \\ 0 & 0 & 3.6 & 1764.1 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 27.9 & -212.1 & 0 & 0 \\ -12.1 & 91.5 & 0 & 0 \\ 0 & 0 & 31.3 & 2.7 \\ 0 & 0 & 2.7 & 1376.5 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} 1230.8 & -1.2 & 0 & 0 \\ -1.2 & 1230.2 & 0 & 0 \\ 0 & 0 & 1229.8 & 0 \\ 0 & 0 & 0 & 1228.4 \end{bmatrix},$$
$$S_2 = \begin{bmatrix} 1229.9 & 0.3 & 0 & 0 \\ 0.3 & 1228.8 & 0 & 0 \\ 0 & 0 & 1229.5 & 0 \\ 0 & 0 & 0 & 1228.4 \end{bmatrix},$$
$$Z = \begin{bmatrix} 74.0485 & -0.2410 & 0 & 0 \\ -0.2410 & 74.1101 & 0 & 0 \\ 0 & 0 & 73.9490 & -0.0228 \\ 0 & 0 & -0.0228 & 60.0563 \end{bmatrix},$$
$$U = \begin{bmatrix} 7.3966 & 2.5395 & 0 & 0 \\ 2.5395 & 12.1655 & 0 & 0 \\ 0 & 0 & 9.3139 & 1.3498 \\ 0 & 0 & 1.3498 & 675.6563 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} -0.3077 & -0.1549 & 0 & 0 \\ -0.1491 & -0.6274 & 0 & 0 \\ 0 & 0 & -0.4069 & -0.0388 \\ 0 & 0 & -0.0388 & -19.4679 \end{bmatrix},$$
$$G_2 = \begin{bmatrix} -0.3639 & -0.2226 & 0 & 0 \\ -0.2226 & -0.66165 & 0 & 0 \\ 0 & 0 & -0.4152 & -0.0422 \\ 0 & 0 & -0.0422 & -20.9454 \end{bmatrix},$$
$$N_{11} = \begin{bmatrix} -0.0150 & 0 & 0 & 0 \\ -0.0049 & 0 & 0 & 0 \\ 0 & 0 & -0.0150 & 0 \\ 0 & 0 & -0.0024 & 0 \end{bmatrix},$$
$$N_{12} = \begin{bmatrix} -0.0150 & 0 & 0 & 0 \\ -0.0050 & 0 & 0 & 0 \\ 0 & 0 & -0.0149 & -0.0422 \\ 0 & 0 & -0.0020 & 0 \end{bmatrix},$$
$$N_{21} = \begin{bmatrix} 0.0150 & 0 & 0 & 0 \\ 0.0049 & 0 & 0 & 0 \\ 0 & 0 & 0.0150 & 0 \\ 0 & 0 & 0.0022 & 0 \end{bmatrix},$$

$$N_{22} = \begin{bmatrix} 0.0150 & 0 & 0 & 0 \\ 0.0050 & 0 & 0 & 0 \\ 0 & 0 & 0.0150 & 0 \\ 0 & 0 & 0.0022 & 0 \end{bmatrix},$$

$$\widehat{A}_1(1) = \begin{bmatrix} 0.0494 & 0.2265 \\ -0.2274 & 104.9215 \end{bmatrix},$$

$$\widehat{B}_1(1) = \begin{bmatrix} 0.0463 \\ -0.2358 \end{bmatrix}, \qquad \widehat{C}_1(1) = \begin{bmatrix} 1.2250 & 0.4325 \end{bmatrix},$$

$$\widehat{D}_1(1) = 0.4822,$$

$$\widehat{A}_2(1) = \begin{bmatrix} 0.0463 & 0.2226 \\ -0.2358 & 106.6209 \end{bmatrix}, \qquad \widehat{B}_2(1) = \begin{bmatrix} -0.2546 \\ -0.5478 \end{bmatrix},$$

$$\widehat{C}_2(1) = \begin{bmatrix} 2.0125 & 0.5214 \end{bmatrix}, \qquad \widehat{D}_2(1) = -0.0200,$$

$$\widehat{A}_1(2) = \begin{bmatrix} 0.0487 & 0.1986 \\ -0.2641 & 91.3907 \end{bmatrix}, \qquad \widehat{B}_1(2) = \begin{bmatrix} 2.0125 \\ 0.6578 \end{bmatrix},$$

$$\widehat{C}_1(2) = \begin{bmatrix} 3.2156 & 0.6514 \end{bmatrix}, \qquad \widehat{D}_1(2) = 0.0125,$$

$$\widehat{A}_2(2) = \begin{bmatrix} 0.0487 & 0.1996 \\ -0.2641 & 91.5907 \end{bmatrix}, \qquad \widehat{B}_2(2) = \begin{bmatrix} 0.9854 \\ -1.5278 \end{bmatrix},$$

$$\widehat{C}_2(2) = \begin{bmatrix} 0.2145 & 0.8456 \end{bmatrix}, \qquad \widehat{D}_2(2) = 0.0178. \tag{33}$$

5. Conclusion

The problem on H_∞ filter design has been addressed for T-S fuzzy-model-based SMJSs. Based on the Lyapunov-Krasovskii functional approach, a sufficient condition for the existence of H_∞ filter, which stabilizes the T-S fuzzy-model-based filtering-error systems and guarantees a prescribed level on disturbance attenuation, has been obtained in terms of LMI form. The numerical example has shown the effectiveness of the proposed method. In addition, the ℓ_2-ℓ_∞ filtering problems for T-S fuzzy delayed systems are also challenging, which could be our further work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Research on Adaptive Neural Network Control System Based on Nonlinear U-Model with Time-Varying Delay

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U-model can approximate a large class of smooth nonlinear time-varying delay system to any accuracy by using time-varying delay parameters polynomial. This paper proposes a new approach, namely, U-model approach, to solving the problems of analysis and synthesis for nonlinear systems. Based on the idea of discrete-time U-model with time-varying delay, the identification algorithm of adaptive neural network is given for the nonlinear model. Then, the controller is designed by using the Newton-Raphson formula and the stability analysis is given for the closed-loop nonlinear systems. Finally, illustrative examples are given to show the validity and applicability of the obtained results.

1. Introduction

Since most physical systems and processes in the real world are nonlinear, researchers have been devoting their efforts to seeking an effective means of controlling nonlinear systems. Among the many developments, there are growing interests in the adaptive control of complex nonlinear systems. On the other hand, in control systems, time-varying delay is almost the inherent feature of general physical systems, which often degrades the system's performance and even causes instability. Therefore, time-varying delays have received great attention in recent years, and many researchers have studied various analytical techniques and developed many synthesis methods for time-delay systems. For instance, stability analysis is carried out in [1–7] and the controllers are designed in [8–15]. The time-varying delay being widespread in nonlinear control system, it is often difficult to establish accurate mathematical model because of its complex dynamic characteristics, so it is difficult to design a practical controller. For the nonlinear time-varying delay systems, it is a hot and difficult problem to design a controller with high control precision and good system stability in control theory. The difficulty in modeling the nonlinearities makes nonlinear time-varying delay systems much more difficult to control than linear systems, so a general high accuracy model becomes critical.

The main difficulty in nonlinear time-varying delay control system design lies in the lack of general control oriented modelling framework, and the model is very important in the design of nonlinear control system; at the same time, the model must be able to describe various time-varying delay systems sufficiently. Nonlinear U-model represents a large class of smooth nonlinear system by using the way of time-varying delay parameters polynomial and the nonlinear plants without any loss of original nonlinear characteristics, so the nonlinear time-varying delay system is represented by U-model. U-model constructs a general model for the controller design in the traditional nonlinear system and builds a bridge between the design method of the nonlinear dynamic systems and the linear control systems. In methodology, U-model provides a straightforward routine to design nonlinear control systems by using linear control system design techniques.

Because the identification process of the time-varying delays system is often required online, it is difficult to solve the identification problem by using traditional methods, and because the artificial neural network has strong ability of

identification of nonlinear multivariable systems, it is widely used in system identification. Zou and Wang [16] proposed an identification algorithm based on recurrent neural network, which selected NARMAX model to describe discrete nonlinear time-varying system, the parameters of NARMAX model were identified by regression neural network online. Pillonetto [17] presented a method of time-varying autoregressive system identification in the reproducing kernel Hilbert space. Hsiao [18] proposed an algorithm to identify time-varying systems by using maximum a posteriori estimation. Liu et al. [19] initiated a novel approach for simultaneously identifying the topological structure and unknown parameters of uncertain general complex networks with time delay. For the identification of time-varying delay systems, the precision of identification is not only related to identification algorithm, but also has direct relationship with the choice of system model, considering the U-model can be well represented for a class of discrete time-varying delay nonlinear systems, and has the characteristics of general and high precision, which, combined with artificial neural network, has strong ability of nonlinear multivariable system identification; this paper presents a method to identify the time-varying delays parameters of U-mode by neural network.

In the study of nonlinear controller design, Zhu and Guo [20] proposed a pole assignment controller design scheme for a class of nonlinear controlled plant with known structure and parameters, which used Diophantine equations corresponding polynomial to complete pole assignment based on U-model. Shafiq and Butt [21] proposed an internal model control method based on U-model, which is used to solve a class of known dynamic nonlinear plants, considering that the parameters are known, so U-model can be applied directly. Khan and Shafiq [22] put forward an adaptive inverse method based on the minimum variance (LMS) of U-model and applied to control the complex nonlinear industrial process system. The structural parameters of the nonlinear plant are unknown. Wu et al. [23] presented an adaptive stochastic nonlinear system control based on U-model, applying weighted iterative least square method to identify the unknown plant, by using control of pole assignment. Motivated by these earlier developments, in this paper, we propose a new approach for the control design of nonlinear time-varying delay systems with unknown structure and use neural network to identify the time-varying delay parameters of U-model. The adaptive controller is designed by Newton-Raphson formula. According to the identification results of neural network, the parameters of the Newton Raphson formula are adjusted online.

The remaining portion of the paper is organized as follows. Section 2 introduces U-model representation of nonlinear systems. The design of nonlinear time-varying delay control system is described in Section 3. Based on the results obtained in Section 3, illustrative examples are given in Section 4 to demonstrate the effectiveness of the theoretical results. Finally, some concluding remarks are given in Section 5.

2. U-Model Representation of Nonlinear Systems

In Zhu and Guo [20], U-model was firstly proposed to approximate a class of nonlinear time-varying delay model, which can be represented as

$$y(k) = \sum_{j=0}^{m} \alpha_j \left(k - d_j\right) u^j (k-1) + e(k), \tag{1}$$

where m is the degree of model input $u(k-1)$ and $\alpha_j(k-d_j)$ is a function of past inputs $u(k-d_1), \ldots, u(k-d_j)$ and outputs $y(k-1), \ldots, y(k-n)$, d_j is the delay parameter of $\alpha_j(k-d_j)$, and $e(k), \ldots, e(k-n)$ are errors. Adjusting the polynomial, it can be concluded that the control model with time-varying delay parameter can be expressed as the input power series.

From formula (1), it can be seen that the U-model has simple mathematical form; to apply linear control system design methodologies to the nonlinear model, a further transformation is applied as follows:

$$U(k) = y(k), \tag{2}$$

$$U(k) = \sum_{j=0}^{m} \alpha_j \left(k - d_j\right) u^j (k-1) + e(k). \tag{3}$$

Formula (3) is a mathematical expression of U-model.

In order to estimate time-varying delay parameter $\Theta(k)$ of the U-model, U-model vector representation is given as

$$y(k) = U(k) = \Theta(k)\, \Psi(k) + e(k), \tag{4}$$

where $\Theta(k) = [\alpha_0(k-d_0), \ldots, \alpha_j(k-d_j)]$ is the delay parameter and $\Psi(k) = [1, u(k-1), \ldots, u^j(k-1)]^T$.

U-model can be a good representation of a class of nonlinear time-varying delay plant. The transformation of U-model will not lose any characteristics of the original nonlinear model, compared with the other control model in Xu et al. [24]; representation of the U-model is more practical, and controller design methodologies derived from linear systems can be developed accordingly to design nonlinear discrete time systems.

3. Design of Nonlinear Time-Varying Delay Control System

3.1. Identification of Neural Network. Because neural network has the advantages of approximating any nonlinear system, it is often used for the identification of nonlinear systems. The time-varying delay parameters $\Theta(k) = [\alpha_0(k-d_0), \ldots, \alpha_j(k-d_j)]$ can be regarded as the weights of the neural network, through online adjustment of weights making the identification error meet the system requirements of precision, thus, obtaining time-varying delay parameters of the system; the time-varying delay parameters will be used as the input of the controller, and adjusting the output of the controller online, the structure of nonlinear neural network identification is shown in Figure 1.

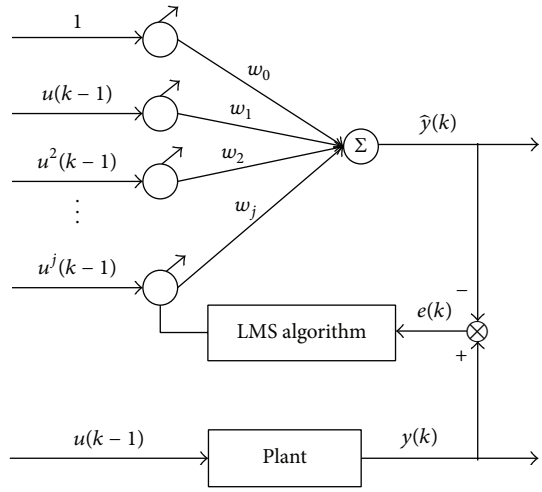

Figure 1: The structure of nonlinear neural network identification.

Widrow-Hoff learning rule (LMS algorithm) is used to adjust the weights of the neural network identification. Input vector $\Psi(k) = [1, u(k-1), \ldots, u^j(k-1)]^T$, the weights vector $\Theta(k) = [\alpha_0(k-d_0), \ldots, \alpha_j(k-d_j)]$, and the transfer function of the output node uses the linear function; the output of the neural network is

$$y(k) = U(k) = \Theta(k)\Psi(k) + e(k), \tag{5}$$

where the weights vector $\Theta(k) = W(k) = [w_0(k), w_1(k), \ldots, w_j(k)]$. In addition, in the process of identification, the error $e(k)$ can be identified as a part of the parameter $\alpha_0(k-d_0)$. In the process of weights adjustment, error signal is

$$e(k) = y(k) - W(k)\Psi(k). \tag{6}$$

The objective function is defined as

$$E(w) = \frac{1}{2}e^2(k). \tag{7}$$

According to formula (7), finding the partial derivative of $E(w)$, we obtain

$$\frac{\partial E}{\partial w} = -\Psi(k)e(k). \tag{8}$$

Therefore, according to the gradient descent method, the adjustment rules of the weights are as follows:

$$w(n+1) = w(n) + \eta\Psi(k)e(k), \tag{9}$$

where η is the learning rate.

In the identification of neural network, it has a great influence on convergence speed of the algorithm to select suitable learning rate and accuracy of the error, if the learning rate selected is too small, after a finite times adjustment, the weights may not reach the control precision required but reduce convergence speed of the algorithm. If the error precision selected is too small, it will lead to reduce the speed of convergence. For fast time-varying delay system, it has a great impact on the stability and control precision of the controller to select appropriate learning rate and accuracy of the error. In addition, when the highest order of $u(k-1)$ is higher than the recognition structure's, we can identify it as a part of $\alpha_0(k-d_0)$.

3.2. The Design of Adaptive Control System. After sampling of nonlinear time varying delay plant, it can be represented as U-model structure of discrete form, this paper uses neural network to identify the time-varying delay parameters of U-model. The controller adjusts the output according to the parameters obtained by the neural network identification. The adaptive controller is designed by adopting the Newton-Raphson formula and adjusts the parameters according to the results of identification online. The output of adaptive controller and the input of controlled plant are reciprocal processes in the system, when the controller and the controlled plant match exactly; namely, the output of the system always can track the input accurately. The structure diagram of adaptive control system is as shown in Figure 2: where $U(k)$ is the input of the system, $u(k-1)$ is the output of the controller, $y(k)$ is the output of the system, $\hat{y}(k)$ is the output of the neural network identification, and $e(k)$ is the system error.

The Newton-Raphson formula is used to solve the roots of equation in Zhu et al. [25]; in this paper, the Newton-Raphson formula is used to design controller to make the design of controller simplify, it is the most widely used method to solve the nonlinear equations; compared with other methods, this method has high convergence speed. General mathematical description of the Newton-Raphson formula is as follows:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{10}$$

where $f(x_k)$ is a nonlinear function, and the Newton-Raphson formula can be used to solve nonlinear equation (3) to obtain the root; the equation is described as follows:

$$\begin{aligned} u_{k+1}(k-1) &= u_k(k-1) - \frac{\phi[u_k(k-1)] - U(k)}{d\phi[u_k(k-1)]/du(k-1)} \\ &= u_k(k-1) - \frac{\sum_{j=0}^{M}\alpha_j(k-j)u_k^j(k-1) - U(k)}{d\left[\sum_{j=0}^{M}\alpha_j(k-j)u_k^j(k-1)\right]/du(k-1)}, \end{aligned} \tag{11}$$

where the subscript k is the times of iterative algorithms. In the Newton-Raphson formula, in order to ensure the real roots of nonlinear equations, we need to define the conditions in the algorithm: avoiding the denominator is zero and the iterative algorithm has faster convergence speed.

3.3. Stability Analysis. The stability is the basic requirement in the design of control system; in order to check the validity of the adaptive neural network control method proposed,

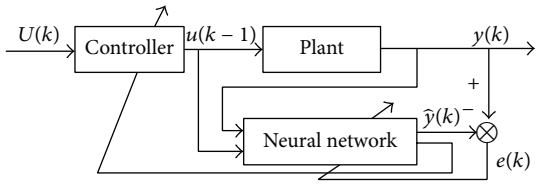

Figure 2: Adaptive control system structure diagram.

suppose the nonlinear time-varying delay plants are stable. The controller is designed by using the Newton-Raphson formula; in order to ensure that the Newton-Raphson formula always has a real root, the convergence condition needs to be limited.

If the U-model expression is such as $y(k) = \alpha_0(k - d_0) + \alpha_1(k - d_1)u(k - 1)$, the equation always has a real root; if the U-model expression is such as $y(k) = \alpha_0(k - d_0) + \alpha_1(k - d_1)u(k-1)+\alpha_2(k-d_2)u^2(k-1)$ and when the equation has no real root, in order to keep the equation having a real solution, make $\alpha_0(k-d_0) = U(k)$; ensure the equation always has a real root: $u(k - 1) = -\alpha_2(k - d_2)/\alpha_1(k - d_1)$.

For the high order equation $y(k) = \alpha_0(k - d_0) + \alpha_1(k - d_1)u(k - 1) + \cdots + \alpha_j(k - d_j)u^j(k - 1)$, it can always ensure that the equation has a real root by limiting the condition of solution of the equation. As long as there is a real root, namely, the controller is stable, the whole control system designed is stable.

4. The Simulation Research

There are four simulation examples of adaptive neural network control system based on time-varying delay nonlinear U-model; in the simulation examples, the parameters of neural network identification are given as follows: the maximum times of weights iteration is 1000, the learning rate is 0.01, the error precision is 0.0001, and the weight change is 0.0001. We choose initial weights $\Theta(k) = [\alpha_0(k - d_0)\alpha_1(k - d_1)] = [1 \ \ 1]$ in model 1 and $\Theta(k) = [\alpha_0(k - d_0)\alpha_1(k - d_1)\alpha_2(k - d_2)] = [1 \ \ 2 \ \ 1]$ in model 2. The parameters of the controller are given as follows: the maximum times of iterations is 100, the minimum change of iterations is 0.001; when the change of the iteration is less than the minimum change, stop the iterative algorithm, the final value for adaptive controller will be obtained.

Model 1. Choosing nonlinear time-varying delay system: $y(k) = \alpha_0(k - d_0) + \alpha_1(k - d_1)u(k - 1)$, where $\alpha_0(k - d_0) = 0.2y(k - 1)u(k - 2)$; $d_0 = 2$; $\alpha_1(k - d_1) = 0.8, d_1 = 0$. The nonlinear model of the plant is obtained: $y(k) = 0.2y(k - 1)u(k - 2) + 0.8u(k - 1)$; the system contains only the zero order and one order of $u(k - 1)$ can be regarded as a linear system. After choosing sine and square wave signal as the input signal, the tracking curve diagram of the system is as shown in Figures 3 and 4.

Model 2. Add two orders of $u(k - 1)$ to the nonlinear time-varying delay system in model 1; the expression of nonlinear

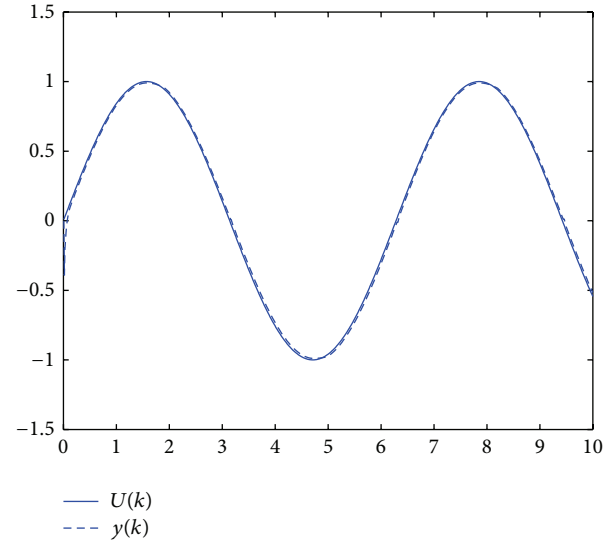

Figure 3: The response curve of sine signal.

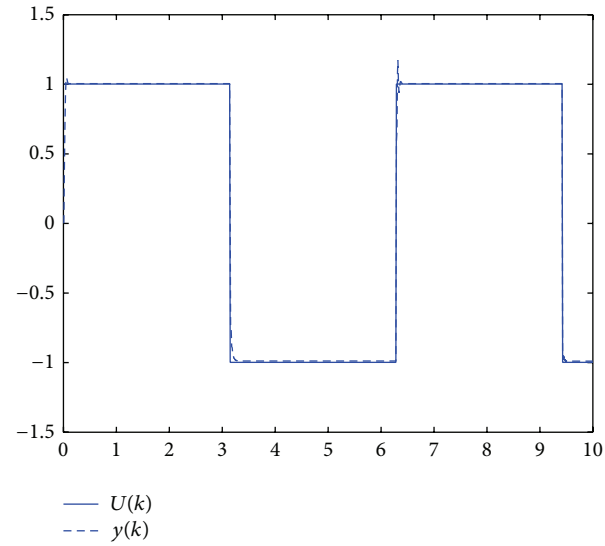

Figure 4: The response curve of square wave signal.

time-varying delay system is $y(k) = 0.2y(k - 1)u(k - 2) + 0.8u(k-1) + 0.02u^2(k-2)$; it contains a nonlinear term. After choosing sine and square wave signal as the input signal, the tracking curve diagram of the system is as shown in Figures 5 and 6.

Model 3. Choosing nonlinear time-varying delay system: $y(k) = \alpha_0(k-d_0)+\alpha_1(k-d_1)u(k-1)+\alpha_2(k-d_2)u^2(k-1)$, there are two delay coefficients in this system, where $\alpha_0(k - d_0) = 0.1y(k-1)u(k-d_0)$, $d_0 = 2$; $\alpha_1(k-d_1) = 0.8u(k-2)$, $d_1 = 2$; $\alpha_2(k-d_2) = -0.3y(k-1)$, $d_2 = 0$. The nonlinear model of the plant is obtained: $y(k) = 0.1y(k-1)u(k-2)+0.8u(k-2)u(k-1)-0.3y(k-1)u^2(k-1)$. After choosing sine and square wave

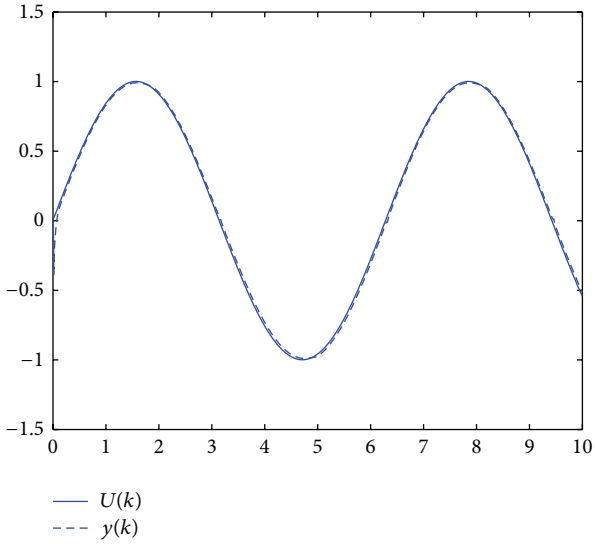

Figure 5: The response curve of sine signal.

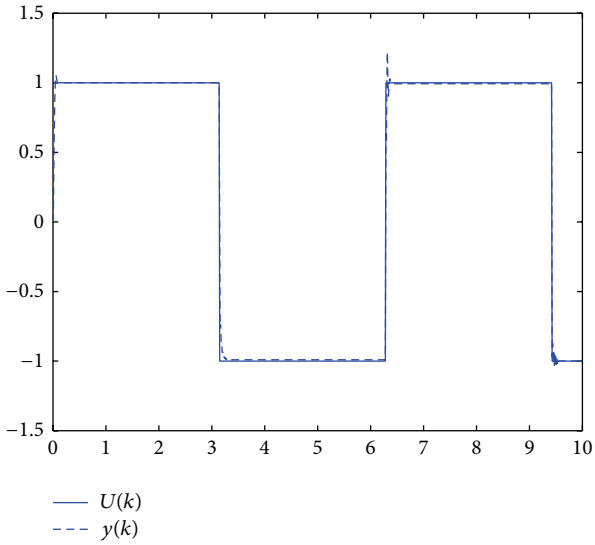

Figure 6: The response curve of square wave signal.

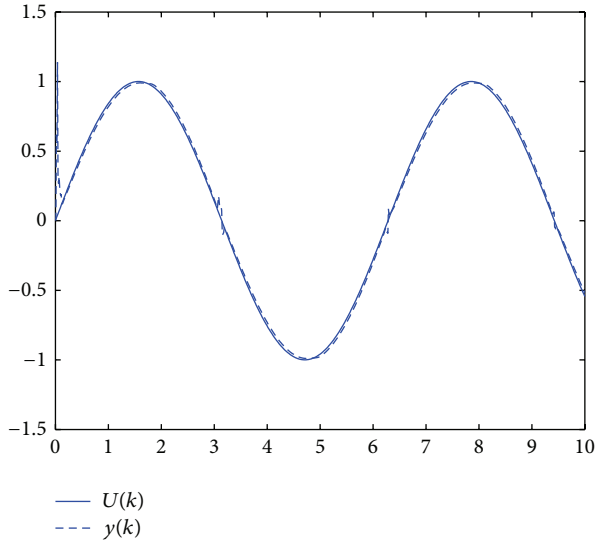

Figure 7: The response curve of sine signal.

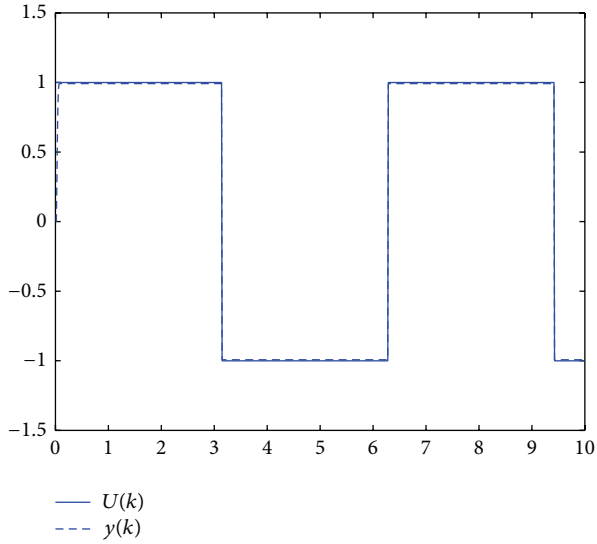

Figure 8: The response curve of square wave signal.

signal as the input signal, the tracking curve diagram of the system is as shown in Figures 7 and 8.

Model 4. Consider the following time-varying system: $y(k) = \alpha_0(k-d_0) + \alpha_1(k-d_1)u(k-1) + \alpha_2(k-d_2)u^2(k-1)$, where $\alpha_0(k-d_0) = 0.2y(k-1)u(k-2) + 0.5u^3(k-1)$, $d_0 = 2$; $\alpha_1(k-d_1) = -0.4y(k-2)$, $d_1 = 0$; $\alpha_2(k-d_2) = 0.5y(k-1)$, $d_2 = 0$. The nonlinear model of the plant is obtained: $y(k) = 0.2y(k-1)u(k-2) - 0.4y(k-2)u(k-1) + 0.5y(k-1)u^2(k-1) + 0.5u^3(k-1)$. After choosing sine and square wave signal as the input signal, the tracking curve diagram of the system is as shown in Figures 9 and 10.

It can be seen from the four simulation models above that U-model has a simple mathematical form; the identification method of adaptive neural network has high accuracy of identification, and it can be used to obtain precise parameters for the design of adaptive controller, so the adaptive controller designed has high control precision for nonlinear time-varying delay systems. The output of the system can track the input accurately, achieving the control of high precision for nonlinear time-varying delay systems.

5. Conclusions

The method of the nonlinear adaptive neural network control system based on U-model is presented in this paper; it has solved a class of nonlinear time-varying delay problems about the identification and the design of control system. U-model describes the nonlinear plant by the time-varying delay polynomial, which constructs a general and high accuracy nonlinear model and provides a precise mathematical model for the design of nonlinear time-varying delay systems.

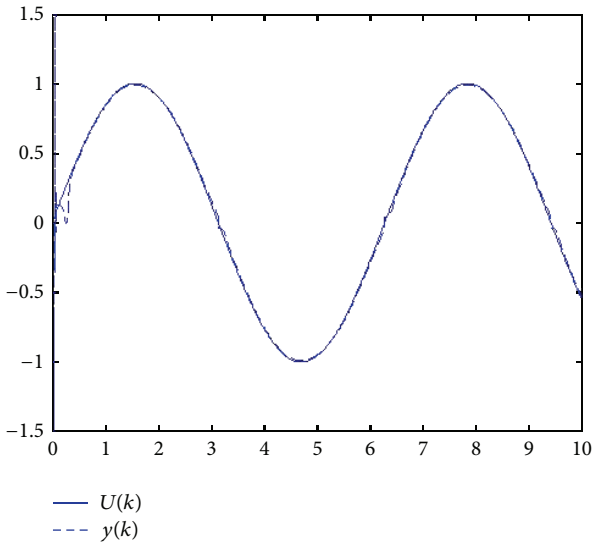

Figure 9: The response curve of sine signal.

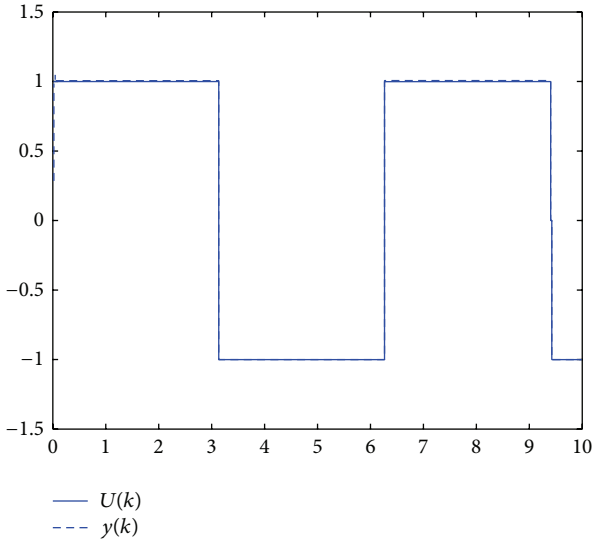

Figure 10: The response curve of square wave signal.

On this basis, the algorithm of adaptive neural network identification based on U-model is proposed, which uses a general time-varying delay nonlinear model to identify the nonlinear plant online and provides a theoretical basis for the establishment of unknown nonlinear time-varying delay plant model and the study of the identification methods of nonlinear time-varying delay systems. The adaptive neural network algorithm presented in this paper provides a new research direction for online identification of nonlinear time-varying delay system which used the Newton-Raphson formula to design the controller of nonlinear time-varying delay systems based on the identification of nonlinear U-model. Performance analysis of nonlinear time-varying delay control systems is given, and the stability of control system is proved by analysis of the real root.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Novel Fractional-Order PID Controller for Integrated Pressurized Water Reactor Based on Wavelet Kernel Neural Network Algorithm

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This paper presents a novel wavelet kernel neural network (WKNN) with wavelet kernel function. It is applicable in online learning with adaptive parameters and is applied on parameters tuning of fractional-order PID (FOPID) controller, which could handle time delay problem of the complex control system. Combining the wavelet function and the kernel function, the wavelet kernel function is adopted and validated the availability for neural network. Compared to the conservative wavelet neural network, the most innovative character of the WKNN is its rapid convergence and high precision in parameters updating process. Furthermore, the integrated pressurized water reactor (IPWR) system is established by RELAP5, and a novel control strategy combining WKNN and fuzzy logic rule is proposed for shortening controlling time and utilizing the experiential knowledge sufficiently. Finally, experiment results verify that the control strategy and controller proposed have the practicability and reliability in actual complicated system.

1. Introduction

With the tightening supplies of energy and deterioration of environment pollution in the globe, the reliability, cleanliness, and security help the nuclear technology gain increasing traction. The majority of marine nuclear power plants adopt integrated design. Since vessels request strong mobility and operational condition changes dramatically, the nuclear reactor and system device need to accommodate themselves to the radical load changing. The nuclear reactor power control dominates the whole system of nuclear power plant and decides the security and reliability of the global device. Hence, the operation control strategy and controller design should accord with its particularities, which is convenient for power controlling and adjusting and ensures the steady operation. The reactor is a time delay system that is similar to other industrial systems. The time delay effect influences the system performance, which possibly brings oscillation and even destroys the instability [1, 2]. Thus, it is practically significant to research the time delay control of nuclear reactor.

In numerous problems encountered in modern control, the time delay issue is of central importance and also a difficult point to elaborate. There is an increasing interest in time delay problem of industrial system from scientists and engineers. Classical control theory utilizes plenty of Smith control, Dahlin control, and so on [3–5]. During recent years, some novel control algorithms also have obtained achievements of time delay systems [6, 7]; for instance, Yang and Zhang [8] studied a robust H_∞ network control method for T-S fuzzy systems with uncertainty and time delay. Besides, Shi and Yu proposed the H_2/H_∞ control for the two-mode-dependent robust control synthesis of networked control systems with random delay [9]. Moreover, a class of uncertain singular time delay systems is controlled by sliding mode [10, 11], and an optimal sliding surface is designed for sliding model control in time delay uncertain systems [12]. The industrial system, especially the large-scaled complex industrial systems, possesses the feature of process delay. The control of these systems depends on both current state and past state, which makes the control issue difficult.

As a mature controller with strong availability, PID and the controllers have been applied in industrial processes for sophisticated dynamic systems which have large scale. The primary reason is its relatively simple structure that could be easily comprehended and implemented [13–15]. In recent years, considerable interests in time delay have been aroused in fractional-order PID controller and many researchers have presented beneficial methods to design stabilizing fractional-order PID (FOPID) controllers for varying time delay system [16–19]. This paper adopts neural network (NN) to tune the FOPID online to control the nuclear reactor power, which keeps the coolant average temperature constant on the condition of load changing. NN is an effective method for PID parameters tuning, and the wavelet neural network (WNN) improves the efficiency of NN by transforming the nonlinear excitation function to the wavelet function. However, WNN still follows the gradient descent method to adjust weights, which inevitably causes the problems of deficient generalization performance, ill-posedness, and overfitting. Motivated by the drawback above, it should take advantage of the algorithm core, LMS, to improve WNN. Both LMS and KLMS are derived from mean squared error cost function [20–22]. LMS algorithm is famous for its easy comprehending and implementing but has no ability to be utilized in nonlinear domain. With the wide spread of kernel theory, the novel LMS algorithm with kernel trick has drawn considerable attention to learning machines including neural network [23–26]. Recently, kernel functions (such as Gaussian kernel) are applied in support vector machine (SVM) mostly, which projects the linear nonseparable data to another space for being separable in the new space. RBF network is a sort of NN which use Gaussian kernel frequently. But this kernel could not approach the target appropriately when it encounters more complex signal in the calculation process. There are some literatures indicating that Gaussian kernel in RBF network and SVM could not generate a set of complete bases in the subspace ideally by translation [27]. Some potential wavelet functions could be transformed as kernel functions, which are competitive to produce a set of complete bases in $L_2(R)$ space (square and integrable space) by stretching and translating. Meanwhile, RBF network needs to identify the model of object and obtain the Jacobian information when it tunes the network weights, and it is hard to gain the Jacobian information of complex industrial system. But the conventional WNN could tune the PID controller without such information and is more appropriate to be the parent body for improvement. Thus, this paper proposes a novel neural network based on wavelet kernel function and combines with fuzzy logic rules to achieve the online tuning of FOPID. Experiment results validate that the control strategy and controller design improve the accuracy and speed, which could control the nuclear reactor power operation and enhances the overall efficiency.

The next content is composed of six sections. Section 2 provides a short introduction to LMS algorithm and kernel LMS algorithm. Section 3 presents the related theory of wavelet kernel function and proposes the method of wavelet kernel neural network including the discussion of adaptive parameters. The model of nuclear reactor is structured by RELAP 5 program in Section 4. Then the control strategy and the fractional-order PID controller are designed with fuzzy logic and wavelet kernel neural network in Section 5. In Section 6, experiments illustrate the effectiveness of the wavelet kernel neural network and the fractional-order PID controller. As some controllers are faced with the problem of losing efficacy in practice, Section 6 also emulates and analyzes the control process in the basis of reactor model. Finally, conclusion of this paper has been described in Section 7.

2. The Description of Kernel LMS Algorithm

In this section, the algorithm of kernel LMS is introduced as a show review for wavelet kernel neural network. Kernel LMS is an approach to perform the LMS algorithm in the kernel feature space. In the LMS algorithm, the hypothesis space H_1 is composed by all the linear operators on U (mapping $w : U \rightarrow R$) [28], which is embedded in a Hilbert space, and the linear operator represents standard inner product. The LMS algorithm minimizes the total instantaneous error of the system output:

$$E(n) = \frac{1}{2} \sum \mathbf{e}^2(n), \tag{1}$$

where $\mathbf{e}(n) = \mathbf{norm}(n) - \mathbf{x}(n)\mathbf{w}(n)$, **norm** is the desired value, $\mathbf{w}(n)$ is the weight vector, $\mathbf{x}(n)$ is the input vector, and n is the instant. Using the straightforward stochastic gradient descent update rule, the weight vector $\mathbf{w}$ is approximated based on input vector $\mathbf{x}$:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \eta \mathbf{e}(n)\mathbf{x}(n), \tag{2}$$

where η is the step-size. By training step by step, the weight $\mathbf{w}$ is updated as a linear combination of the previous and present input vector.

This LMS algorithm learns linear patterns satisfactorily while it could not approach the same sound effect in nonlinear pattern. Motivated by the drawback, Pokharel with other scholars extended the LMS algorithm to RKHS by the kernel trick [28, 29]. Kernel methods map the input into a high dimensional space (HDS) [30]. And any kernel function based on Mercer's theorem has the follow mapping [31]:

$$k\left(x, x'\right) = \left\langle \phi(x), \phi\left(x'\right) \right\rangle = \phi^T(x)\phi\left(x'\right). \tag{3}$$

When the input in (2) consists of $\{\phi(\mathbf{x}(1)), \phi(\mathbf{x}(2)), \ldots, \phi(\mathbf{x}(n))\}$ but not $\{\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(n)\}$, $\phi(\mathbf{x}(n))$ is the infinite feature vector transformed from input vector $\mathbf{x}(n)$. The LMS algorithm is no longer applicable, and the total instantaneous error of the system output is as follows:

$$E = \frac{1}{2} \sum \left(\mathbf{norm} - \phi(\mathbf{x}(n))\mathbf{w}(n)\right)^2, \tag{4}$$

where $\phi(\mathbf{x}(n))$ is matrixes of input vector, **norm** is desired value vector, and $\mathbf{w}(n)$ is weight vector in RKHS.

Then build the hypothesis that $\mathbf{w}(0) = 0$ for convenience, and (2) could convert to

$$\mathbf{w}(n) = \eta \sum_{i=0}^{n-1} \mathbf{e}(i)\phi(\mathbf{x}(i)), \tag{5}$$

and the final output in n step updating of the learning algorithm is

$$\mathbf{z}(n) = \eta \sum_{i=0}^{n-1} \mathbf{e}(i) \left\langle \phi(\mathbf{x}(i)), \phi(\mathbf{x}(n)) \right\rangle = \eta \mathbf{e}^T (n-1)\, \mathbf{k}, \quad (6)$$

and $\mathbf{k} = [k(\mathbf{x}(1), \mathbf{x}(n)), k(\mathbf{x}(2), \mathbf{x}(n)), \ldots, k(\mathbf{x}(n-1), \mathbf{x}(n))]$. From (4)–(6), the KLMS algorithm remedies the defect of the LMS algorithm of its relatively small dimensional input space and improves the well-posedness study in the infinite dimensional space with finite training data [28]. As the application of adaptive filter theory, the neural network could take advantage of the KLMS algorithm, which endows the learning of neural network with more efficiency and reliability.

3. Wavelet Kernel Neural Network

Following the principle above, this paper acquires the wavelet kernel function by Mercer theorem and proposes a more competent method to update parameters on basis of conventional WNN. With the kernel LMS above, the wavelet kernel neural network is proposed in this section on the basis of wavelet kernel function and conventional wavelet neural network. First, the wavelet kernel function is constructed, which is admissible neural network kernel. Second, the wavelet kernel neural network is proposed including neuron explanation and weight updating. Then the modulations of adaptive variable step-size and wavelet kernel width are discussed in Sections 3.3 and 3.4.

3.1. The Wavelet Kernel Function. The wavelet analysis is executed to express or approximate functions or signals generated family functions given by dilations and translations from the mother wavelet $\psi_{a,b}(x) \in L_2(R)$. Hypothesize that $L^1(R)$ is linear integrable space and $L^2(R)$ is square integrable space. When the wavelet function $\psi(x) \in L^1(R) \cap L^2(R)$ ($\widehat{\psi}(0) = 0$), the wavelet function group could be defined as

$$\psi_{a,b}(x) = a^{-1/2} \psi\left(\frac{x-b}{a}\right), \quad (7)$$

where $x, a, b \in R$, a is a dilation factor, and b is a translation factor. And $\psi(x)$ satisfy the condition [32, 33]

$$C_\psi = \int_R \frac{\left|\widehat{\psi}(w)\right|^2}{|w|} dw < \infty, \quad (8)$$

where $\widehat{\psi}(w)$ is the Fourier transform of $\psi(x)$. For a common multidimensional wavelet function, if the wavelet function of one dimension is $\psi(x)$ which satisfies the condition (8), the product of one-dimensional (1D) wavelet functions could be described by tensor theory [29, 31]:

$$\psi_d(x) = \prod_{i=1}^{d} \psi(x_i), \quad (9)$$

where $\{\mathbf{x} = (x_1, \ldots, x_d) \in R^d\}$.

The conception above is the foundation of wavelet analysis and theory. Note that the wavelet kernel function for neural network is a special kernel function and comes from wavelet theory; it would be the horizontal function $k(x, x') = k(x - x')$ and satisfied the condition of Mercer theorem [34], which could be summarized as follows.

Lemma 1. *A continuous symmetric kernel is $k(x, x')$, and x is bounded, likewise for x'. Then $k(x, x')$ is the valid kernel function for wavelet neural network if and only if it holds for any nonzero function $h(x)$ which makes $\int_{R^d} h^2(\xi) d\xi < \infty$, and the following condition will be satisfied:*

$$\iint_{R^d \otimes R^d} k(x, x') h(x) h(x') \, dx dx' \geq 0. \quad (10)$$

With regard to the horizontal floating function, there is an analogous theory for support vector machine (SVM) [35]. Allowing for the affiliation between SVM and NN [34], an approved horizontal floating kernel function of SVM is effective in NN. When $\psi(x)$ is a mother wavelet, a is a dilation factor and b is a translation factor, $x, a \in R$; the horizontal floating translation-invariant wavelet kernels could be expressed as [36]

$$k(x, x') = \prod_{i=1}^{d} \psi\left(\frac{x_i - x_i'}{a_i}\right). \quad (11)$$

Since the number of wavelet kernel functions which satisfy the horizontal floating function condition [37] is not much up to now, the neural network in this paper adopts an existent wavelet kernel function, Morlet wavelet function:

$$\psi(x) = \cos(w_0 x) \exp\left(\frac{-x^2}{2}\right). \quad (12)$$

Then the Morlet wavelet kernel function is defined as

$$k(x, x') = \prod_{i=1}^{d} \cos\left(w_0 \left(\frac{x_i - x'_i}{a_i}\right)\right) \exp\left(-\frac{(x_i - x_i')^2}{2a_i^2}\right). \quad (13)$$

Figure 1 displays the Morlet wavelet kernel function in the four-dimensional space. As a continuously differentiable function, the derived function of Morlet kernel is illustrated in Figure 2.

3.2. The Wavelet Kernel-LMS Neural Network. The neural network on basis of wavelet kernel function focuses on the matter relating to the performance of a multilayer perception trained with the backpropagation algorithm by adding KLMS theory to wavelet neural network. This paper utilizes the multilayer perception (MLP) fully connected structure and chooses three layers for understanding. The network consists of input layer I, hidden layer J, and output layer K. The activation functions in hidden layer and output layer are

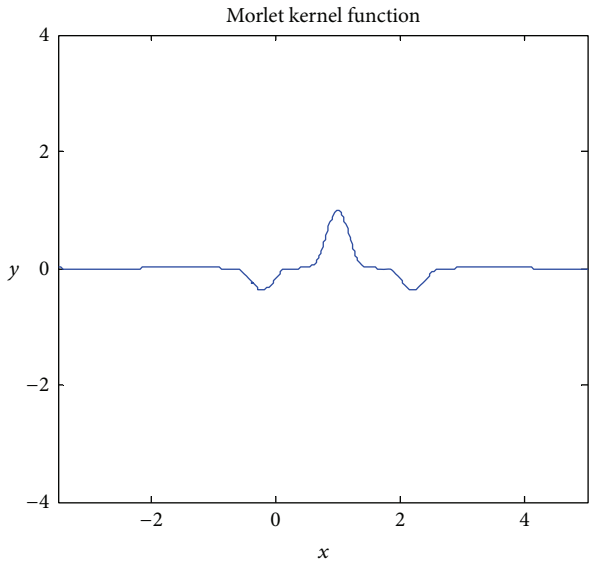

Figure 1: Morlet kernel function.

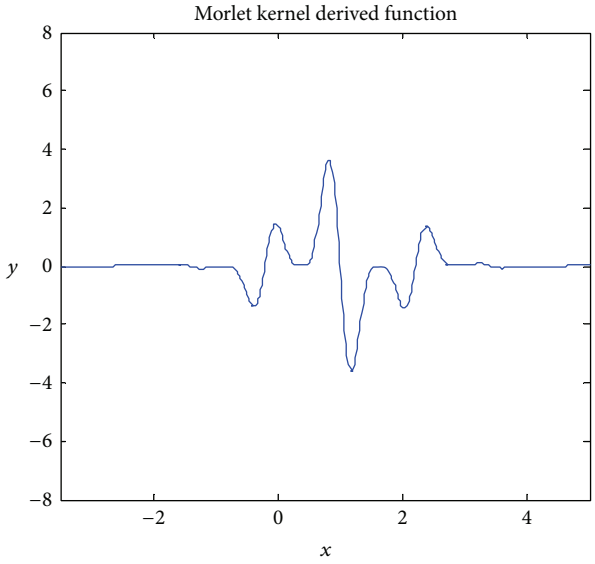

Figure 2: Morlet kernel derived function.

wavelet kernel function $\varphi(x)$ and sigmoid function $f(x)$, respectively. Weights between input layer and hidden layer are regarded as $\mathbf{w}_{JI}$, and the one between hidden layer and output layer is $\mathbf{w}_{KJ}$.

In the WKNN, hypothesize that the input set is $\{\mathbf{x}_I(1), \mathbf{x}_I(2), \ldots, \mathbf{x}_I(n)\}$, practical output set and **norm** output set are $\{\mathbf{z}_K(1), \mathbf{z}_K(2), \ldots, \mathbf{z}_K(n)\}$ and $\{\mathbf{norm}_K(1), \mathbf{norm}_K(2), \ldots, \mathbf{norm}_K(n)\}$, respectively, and then the cost function is defined as $\varepsilon(n)$:

$$\varepsilon(n) = \frac{1}{2}\mathbf{e}^2(n), \tag{14}$$

where $\mathbf{e}(n) = \mathbf{norm}(n) - \mathbf{z}_K(n)$.

As the hidden layer J, the signal of single neuron j is provided by input layer, and the induced local field $\mathbf{v}_J$ and output of hidden layer $\mathbf{x}_J$ could be displayed as

$$\begin{aligned} \mathbf{v}_J(n) &= \sum_{i=1}^{I} w_{ji}(n)\, x_i(n) = \mathbf{x}_I^T(n)\, \mathbf{w}_{JI}(n), \\ \mathbf{x}_J(n) &= \varphi\left(\mathbf{v}_J(n)\right), \end{aligned} \tag{15}$$

where $\mathbf{v}_J(n)$ is weights, $w_{ij}(n)$ is the input of input layer neurons, $\mathbf{w}_{JI}(n)$ and $\mathbf{x}_I(n)$ are the matrix or vector constituted by $w_{ij}(n)$ and $x_i(n)$, I represents the sum of the neurons in the input layer, and $\varphi(\cdot)$ is the wavelet kernel function in J layer.

As the output layer K, the signal of single neuron k is provided by hidden layer, and the induced local field $\mathbf{v}_K$ and output of output layer $\mathbf{z}_K$ could be displayed as

$$\mathbf{v}_K(n) = \sum_{j=1}^{J} w_{kj}(n)\, x_j(n) = \mathbf{x}_J^T(n)\, \mathbf{w}_{KJ}(n), \tag{16}$$

$$\mathbf{z}_K = f\left(\mathbf{v}_K(n)\right) = f\left(\varphi^T\left(\mathbf{v}_J(n)\right)\mathbf{w}_{KJ}(n)\right), \tag{17}$$

where $w_{kj}(n)$ is weights, $x_j(n)$ is the input of hidden layer neurons, $\mathbf{w}_{KJ}(n)$ and $\mathbf{x}_J(n)$ are the matrix or vector constituted by $w_{kj}(n)$ and $x_j(n)$, I represents the sum of the neurons in the input layer, and $f(\cdot)$ is the sigmoid function in K layer.

Similar to the WNN algorithm, this paper achieves a gradient search to solve the optimum weight. For convenience, it chooses $\mathbf{w}_{KJ}(0)$ to be zero. Then, based on (17), the weight matrix $\mathbf{w}_{KJ}(n)$ between output layer and hidden layer is given as follows:

$$\mathbf{w}_{KJ}(n) = \eta_K(n)\, \varphi\left(\mathbf{v}_J(n)\right) \sum_{p=0}^{n-1} \mathbf{e}(p)\, f'\left(\varphi^T\left(\mathbf{v}_J(p)\right)\mathbf{w}_{KJ}(p)\right), \tag{18}$$

where $\eta_K(n)$ is adaptive variable step-size, and the method to find appropriate step-size will be illustrated in the following subsection. By the kernel trick and (17), $\mathbf{z}_K(n)$ could be determined as

$$\begin{aligned} \mathbf{z}_K(n) &= f\left(\eta_K(n) \sum_{p=0}^{n-1} \boldsymbol{\alpha}_p k\left(\mathbf{v}_J(p), \mathbf{v}_J(n)\right)\right) \\ &= f\left(\eta_K(n)\, \boldsymbol{\alpha}^T(n-1)\, \mathbf{k}_K\right), \end{aligned} \tag{19}$$

where $\boldsymbol{\alpha}(n) = \mathbf{e}(n) f'(\eta_K(n-1)\boldsymbol{\alpha}^T(n-1)\mathbf{k}_K)$.

Then the weight matrix $\mathbf{w}_{JI}(n)$ between hidden layer and input layer is obtained on the basis of (18) as follows:

$$\begin{aligned} \mathbf{w}_{JI}(n) = \eta_J(n)\, \mathbf{x}_I(n) &\sum_{p=0}^{n-1} \varphi'\left(\mathbf{x}_I^T(p)\, \mathbf{w}_{JI}(p)\right) \\ &\times \sum_{K} \mathbf{e}(p)\, f'\left(\varphi^T\left(\mathbf{v}_J(p)\right)\mathbf{w}_{KJ}(p)\right)\mathbf{w}_{KJ}(p), \end{aligned} \tag{20}$$

and the output of hidden neuron could be determined as

$$\begin{aligned}\mathbf{x}_J &= \varphi\left(\left\langle \mathbf{w}_{JI}(n), \mathbf{x}_I(n)\right\rangle\right)\\ &= \varphi\left(\eta_J(n)\sum_{p=0}^{n-1}\boldsymbol{\beta}(p)\, k_J\left(\mathbf{x}_I(p), \mathbf{x}_I(n)\right)\right)\\ &= \varphi\left(\eta_J(n)\,\boldsymbol{\beta}^T(n-1)\,\mathbf{k}_J\right),\end{aligned} \tag{21}$$

where $\boldsymbol{\beta}(n) = \varphi'(\eta_j(n)\boldsymbol{\beta}^T(n-1)\,\mathbf{k}_J)\boldsymbol{\alpha}(n-1)\mathbf{w}_{KJ}(n-1)$.

3.3. The Updating of Adaptive Variable Step. According to the algorithm presented above (see Section 2), it stresses that the adaptive variable step could be regarded as a significant coefficient with the ability to influence the adaptivity and effectivity of the algorithm. The stability and convergence of the algorithm are influenced by step-size parameter and the choice of this parameter reflects a compromise between the dual requirements of rapid convergence and small misadjustment. A large prediction error would cause the step-size to increase to provide faster tracking while a small prediction error will result in a decrease in the step-size to yield smaller misadjustment [37]. Therefore, this paper improves the NLMS algorithm derived from LMS algorithm, which is propitious to multidimensional nonlinear structure and avoids the limitation of linear structure.

NLMS focuses on the main problem of LMS algorithm's constant step-size, which designs an adaptive step-size to avoid instability and picks the convergence speedup:

$$w(n+1) = w(n) + \frac{1}{\langle u(n), u(n)\rangle} e(n) \times u(n), \tag{22}$$

where $1/\langle u(n), u(n)\rangle$ is regarded as the variable step-size compared with the LMS algorithm. But the NLMS has little ability to perform in the nonlinear space. By the kernel trick, the kernel function could not only improve the wavelet neural network, but also update the step-size of neural network self-adaptively. As what (17) reveals, $\mathbf{z}_K$ is a consequence of nonlinear function $f(\cdot)$. Since $f(\cdot)$ is continuously differentiable, this function could be matched by innumerable linear function following the principle of the definition of differential calculus.

Theorem 2. *The weights updating satisfies the NLMS condition in the linear integrable space $L^1(R)$ as follows:*

$$w(n+1) = w(n) + \eta(n) \times e(n) \times u(n), \tag{23}$$

where $\eta(n) = 1/\langle u(n), u(n)\rangle$ is the adaptive variable step-size. Then weights are expanded to K dimension matrix $\mathbf{w}_K(n)$ and adjusted as

$$\begin{aligned}\mathbf{w}_K(n+1) = \mathbf{w}_K(n) + \boldsymbol{\eta}_K(n) \times f'\left(\mathbf{w}_K(n) \times \mathbf{u}_K(n)\right)\\ \times \mathbf{e}_K(n) \times \mathbf{u}_K(n),\end{aligned} \tag{24}$$

where the nonlinear function $f(\cdot)$ is continuously differentiable everywhere, the $\boldsymbol{\eta}_K(n)$ will have K elements, and each step-size $\eta_k(n)$ $(k = 1, 2 \ldots K)$ is given as

$$\eta_k(n) = \frac{1}{f'\left(\mathbf{w}_K(n) \times \mathbf{u}_K(n)\right)\left\langle \mathbf{u}_K(n), \mathbf{u}_K(n)\right\rangle}. \tag{25}$$

Proof. Hypothesize that ε is small enough and linear function $F(n) = \kappa(n)\mathbf{u}_K(n) + \gamma(n)$ exists, which has the relationship that $F(n) - f(n) < \varepsilon$ in the closed interval $[a, b]$; therefore, regard that $F(n) \approx f(n)$ and $f'(n) \approx \kappa$ are satisfied.

Consider the equations

$$\begin{aligned}h\left(\boldsymbol{\eta}_K(n)\right) &= \mathbf{norm} - f\left(\mathbf{w}_K(n) \times \mathbf{u}_K(n)\right)\\ &\approx \mathbf{norm} - \kappa(n)\left(\mathbf{w}_K(n) \times \mathbf{u}_K(n)\right) - b(n)\\ &= \mathbf{E}_K^T(n)\left[I - \boldsymbol{\eta}_K(n)\,\kappa^2(n)\,\mathbf{u}_K^T(n)\,\mathbf{u}_K(n)\right]\mathbf{E}_K,\\ g\left(\boldsymbol{\eta}_K(n)\right) &= \frac{1}{2}h^2\left(\boldsymbol{\eta}_K(n)\right),\end{aligned} \tag{26}$$

where $\boldsymbol{\eta}_K(n) = \begin{bmatrix} \eta_1(n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \eta_K(n) \end{bmatrix}$, and minimize the functions $g(\boldsymbol{\eta}_K(n))$ and $g'(\boldsymbol{\eta}_K(n)) = 0$, and then $\boldsymbol{\eta}_K(n)$ $(k = 1, 2, \ldots, K)$ could be inferred as follows:

$$\boldsymbol{\eta}_K(n) = \begin{bmatrix} \eta_1(n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \eta_K(n) \end{bmatrix}, \tag{27}$$

$$\eta_k(n) = \frac{1}{\kappa_k^2(n)\left\langle \mathbf{u}_K(n), \mathbf{u}_K(n)\right\rangle}. \tag{28}$$

Equations (27) and (28) reveal that the $\boldsymbol{\eta}_K(n)$ is independent of the linear function intercept $\gamma(n)$ and only relates to the input and slope $\kappa_k(n)$. Motivated by the drawback of slight error ε from $F(n) - f(n)$, the $\kappa_k^2(n)$ in (28) is replaced by $f'(\mathbf{w}_K(n) \times \mathbf{u}_K(n))$, which improves the accuracy and completes the proof. □

On the basis of the discussion above, (22) could be represented as a special situation by Theorem 2 when the slope of function $f(\cdot)$ is 1. Theorem 2 expands the NLMS to the multidimensional nonlinear space from the linear space and broadens the application of NLMS. Compared with the variable step-size in (22), (27) and (28) could be defined as the adaptive variable step-size of NLMS in the extended space.

When $\langle \mathbf{u}_K(n), \mathbf{u}_K(n)\rangle$ is too small, the problem caused by numerical calculation has to be considered. To overcome this difficulty, (28) is altered to

$$\eta_k(n) = \frac{1}{\kappa_k^2(n)\left\langle \mathbf{u}_K(n), \mathbf{u}_K(n)\right\rangle + \sigma_v^2}, \tag{29}$$

where $\sigma_v^2 > 0$ which is set as a pretty small value. The weights updating in (20) is more complicated than that between output layer and hidden layer. Because the updating in (20) is based on the output layer, meanwhile the existence of $\sum_K \mathbf{e}(p) f'(\varphi^T(\mathbf{v}_J(p))\mathbf{w}_{KJ}(p))\mathbf{w}_{KJ}(p)$ helps every neuron of output layer have the relationship with the updating process between ith neuron of hidden layer and the previous layer; in other words, every neuron of output layer contributes to $\boldsymbol{\eta}_J(n)$. Therefore, this paper adopts the same step-size in $\boldsymbol{\eta}_J(n)$

Input: Input signal $\mathbf{x}_I$ Ideal output **norm**
Output: Observe output $\mathbf{z}_k(n)$ Weights $\mathbf{w}(n)$
Initialize: The number of layers in WNN structure N_{layer}
The number of neurons in every layer $M_1, M_2, \ldots$
The activation function of every layer $f_1(\cdot), f_2(\cdot), \ldots$
Loop count Num
Begin:
Initialize weights $\mathbf{w}(0)$
Repeat
Wavelet kernel neural network output $\mathbf{z}_K(n)$
Output layer step-size $\eta_K(n)$
Hidden layer step-size $\eta_J(n)$
Synaptic weight updating $\mathbf{w}_{KJ}$
Synaptic weight updating $\mathbf{w}_{JI}$
Wavelet kernel Width modulating $a(n)$
Until $n > Num$
End

Algorithm 1: Algorithm process.

based on $\boldsymbol{\eta}_K(n)$, and the elements $\eta_j(n)$ of $\boldsymbol{\eta}_J(n)$ could be given as follows:

$$\eta_j(n) = \frac{1}{K}\sum_{k=1}^{K}\eta_k(n). \tag{30}$$

3.4. The Modulation of Wavelet Kernel Width. The goal of minimizing the mean square error could be achieved by gradient search. If kernel space structure cost function is derivable, the wavelet kernel of neural network could also be deduced by gradient algorithm. In consideration of conventional wavelet neural network, the gradient search method could be utilized for updating wavelet kernel weight due to the derivability of the wavelet kernel function. Besides the synaptic weights among various layers of network, there is another coefficient—dilation factor a—in (11) which also influences the mean square error. With respect to the cost function (14), the dilation factor a is corrected by

$$a(n+1) = a(n) - \mu' \nabla_{a(n)}\left(\mathbf{e}^2(n)\right), \tag{31}$$

where μ' is the constant step-size and $\nabla_{a(n)}$ is the partial derivative function, that is, gradient function. According to the chain rule of calculus, this gradient is expressed as

$$\nabla_{a(n)} = \mathbf{e}(n)\left(\frac{\partial \mathbf{z}_K}{\partial a(n)}\right) = \mathbf{e}(n)\left(\frac{\partial \mathbf{z}_K}{\partial u(n)}\frac{\partial u(n)}{\partial a(n)}\right),$$

$$\frac{\partial u(n)}{\partial a(n)} = \frac{\partial u(n)}{\partial k(n)}\frac{\partial k(n)}{\partial a(n)} = \eta_K(n)\sum_{p=0}^{n-1}\boldsymbol{\alpha}(p)\,Hk\left(x(p), x(n)\right), \tag{32}$$

where $H = \partial k/\partial a$ and $u = \eta_K(n)\sum_{p=0}^{n-1}\boldsymbol{\alpha}(p)k(x(p), x(n))$. Then $\nabla_{a(n)}$ could be represented as

$$\nabla_{a(n)} = \eta_K(n)\,\mathbf{e}(n)\,f'(u)\sum_{p=0}^{n-1}\boldsymbol{\alpha}(p)\,Hk\left(x(p), x(n)\right). \tag{33}$$

Allowing for (19) and (33), the updating formulation of dilation factor $a(n)$ is

$$a(n+1) = a(n) - \mu(n)\,\boldsymbol{\alpha}(n)\sum_{p=0}^{n-1}\boldsymbol{\alpha}(p)\,Hk\left(x(p), x(n)\right), \tag{34}$$

where $\mu(n) = \mu'\eta_K(n)$.

The WKNN process is presented as Algorithm 1 as follows.

4. The Model Design of IPWR

In order to reduce the volume of the pressure vessel, the integration reactor generally uses compact OTSG. Since there is no separator in the steam generator, the superheated steam produced by OTSG could improve the thermal efficiency of the secondary loop systems. Compared with natural circulation steam generator U-tube, the OTSG is designed with simpler structure but less water volume in heat transfer tube. Since it is difficult to measure the water level in OTSG especially during variable load operation, the OTSG needs a high-efficiency control system to ensure safe operation. Primary coolant average temperature is an important parameter to ensure steam generator heat exchange. The control scheme with a constant average temperature of the coolant can achieve accurate control of the reactor power and avoid generation of large volume coolant loop changes.

In the integrated pressurized water reactor (IPWR) system, the essential equation of thermal-hydraulic model is listed as follows.

(1) Mass continuity equations:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\alpha_g\rho_g\right) + \frac{1}{A}\frac{\partial}{\partial x}\left(\alpha_g\rho_g v_g A\right) &= \Gamma_g,\\ \frac{\partial}{\partial t}\left(\alpha_f\rho_f\right) + \frac{1}{A}\frac{\partial}{\partial x}\left(\alpha_f\rho_f v_f A\right) &= \Gamma_f.\end{aligned} \tag{35}$$

(2) Momentum conservation equations:

$$\begin{aligned}
&\alpha_g\rho_g A\frac{\partial v_g}{\partial t}+\frac{1}{2}\alpha_g\rho_g A\frac{\partial v_g^2}{\partial x}\\
&\quad=-\alpha_g A\frac{\partial P}{\partial x}+\alpha_g\rho_g B_x A-\left(\alpha_g\rho_g A\right)\mathrm{FWG}\left(v_g\right)\\
&\qquad+\Gamma_g A\left(v_{gI}-v_g\right)-\left(\alpha_g\rho_g A\right)\mathrm{FIG}\left(v_g-v_f\right)\\
&\qquad-C\alpha_g\alpha_f\rho_m A\left[\frac{\partial\left(v_g-v_f\right)}{\partial t}+v_f\frac{\partial v_g}{\partial x}-v_g\frac{\partial v_f}{\partial x}\right],\\
&\alpha_f\rho_f A\frac{\partial v_f}{\partial t}+\frac{1}{2}\alpha_f\rho_f A\frac{\partial v_f^2}{\partial x}\\
&\quad=-\alpha_f A\frac{\partial P}{\partial x}+\alpha_f\rho_f B_x A-\left(\alpha_f\rho_f A\right)\mathrm{FWF}\left(v_f\right)\\
&\qquad-\Gamma_g A\left(v_{fI}-v_f\right)-\left(\alpha_f\rho_f A\right)\mathrm{FIF}\left(v_f-v_g\right)\\
&\qquad-C\alpha_f\alpha_g\rho_m A\left[\frac{\partial\left(v_f-v_g\right)}{\partial t}+v_g\frac{\partial v_f}{\partial x}-v_f\frac{\partial v_g}{\partial x}\right].
\end{aligned}\tag{36}$$

(3) Energy conservation equations:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_g\rho_g U_g\right)+\frac{1}{A}\frac{\partial}{\partial x}\left(\alpha_g\rho_g U_g v_g A\right)\\
&\quad=-P\frac{\partial\alpha_g}{\partial t}-\frac{P}{A}\frac{\partial}{\partial x}\left(\alpha_g v_g A\right)+Q_{wg}\\
&\qquad+Q_{ig}+\Gamma_{ig}h_g^*+\Gamma_w h_g'+\mathrm{DISS}_g,\\
&\frac{\partial}{\partial t}\left(\alpha_f\rho_f U_f\right)+\frac{1}{A}\frac{\partial}{\partial x}\left(\alpha_f\rho_f U_f v_f A\right)\\
&\quad=-P\frac{\partial\alpha_f}{\partial t}-\frac{P}{A}\frac{\partial}{\partial x}\left(\alpha_f v_f A\right)+Q_{wf}\\
&\qquad+Q_{if}-\Gamma_{ig}h_f^*-\Gamma_w h_f'+\mathrm{DISS}_f.
\end{aligned}\tag{37}$$

(4) Noncondensables in the gas phase:

$$\frac{\partial}{\partial t}\left(\alpha_g\rho_g X_n\right)+\frac{1}{A}\frac{\partial}{\partial x}\left(\alpha_g\rho_g v_g X_n A\right)=0.\tag{38}$$

(5) Boron concentration in the liquid field:

$$\frac{\partial\rho_b}{\partial t}+\frac{1}{A}\frac{\partial\left(\rho_b v_f A\right)}{\partial x}=0.\tag{39}$$

(6) The point-reactor kinetics equations are

$$\begin{aligned}
\frac{dn}{dt}&=\frac{\rho-\beta}{l}n+\sum_{i=1}^{6}\lambda_i C_i+q,\\
\frac{dC_i}{dt}&=\frac{\beta_i}{l}n-\lambda_i C_i.
\end{aligned}\tag{40}$$

The RELAP5 thermal-hydraulic model solves eight field equations for eight primary dependent variables. The primary dependent variables are pressure (P), phasic specific internal energies (U_g, U_f), vapor volume fraction (void fraction) (α_g), phasic velocities (v_g, v_f), noncondensable quality (X_n), and boron density (ρ_b). The independent variables are time (t) and distance (x). The secondary dependent variables used in the equations are phasic densities (ρ_g, ρ_f), phasic temperatures (T_g, T_f), saturation temperature (T_s), and noncondensable mass fraction in noncondensable gas phase (X_{ni}) [38]. The point-reactor kinetics model in RELAP5 computes both the immediate fission power and the power from decay of fission fragments.

5. The Control Strategy and FOPID Controller Design

New control law is demanded in nuclear device [39]. When the nuclear power plant operates stably on different conditions, the variation of main parameters in primary and secondary side is indeterminate. The operation strategy employs the coolant average temperature T_{av} to be constant with load changing. The WKNN fuzzy fractional-order PID system achieves coolant average temperature invariability by controlling reactor power system (Figure 4).

The five parameters, including three coefficients and two exponents, are acquired in two forms—fuzzy logic and wavelet kernel neural network. The idea of this method combination is born from the fact that the power level of nuclear reactor tends to alter dramatically in a short while and then maintain slight variation within a certain period. Due to the boundedness of the WKNN excitation function, the output range of FOPID coefficients could not be wide enough. The initial parameters of FOPID are set by WKNN and fuzzy logic. Moreover, the exponents of FOPID have no need to vary with the small changing. As a consequence, the fuzzy logic rule is executed when the reactor power transforms remarkably to tune the five parameters of FOPID, and the WKNN retains updating and tuning of the coefficients based on the fuzzy logic all the while. The process is illustrated in Figure 5.

Based on theoretical analysis and design experience, fuzzy enhanced fractional-order PID controller is adopted to achieve the reactor power changing with steam flow and maintain the coolant average temperature T_{av} constant. Power control system regards the average temperature deviation and steam flow as the main control signal and assistant control signal, respectively. Since the thermal output of reactor is proportional to the steam output of steam generator, the consideration of steam flow variation could calculate the power need immediately, which impels the reactor power to follow secondary loop load changing and improves the mobility of equipment. When the load alters, reactor power could be determined by the following equation:

$$\begin{aligned}
n_0&=K_1G_s+K_{Pp}\Delta T_m(n)+K_{Pi}s^{\lambda}\sum_{j=0}^{n}q_j\Delta T_m(n-j)\\
&\quad+K_{Pd}s^{-\mu}\sum_{j=0}^{n}d_j\Delta T_m(n-j),
\end{aligned}\tag{41}$$

where $q_j = (1 - ((1 + \lambda)/j))q_{j-1}$, $d_j = (1 - ((1 - \mu)/j))d_{j-1}$, s is time step, K_1 is conversion coefficient, K_{Pp}, K_{Pi}, and K_{Pd} are design parameters, G_s is new steam flow of steam generator, and ΔT_m is the coolant average temperature deviation. Consider

$$\Delta T_m = T_m^{(0)} - T_m, \tag{42}$$

where $T_m^{(0)}$ is **norm** value and T_m is measuring temperature:

$$T_m = \frac{T_o + T_i}{2}, \tag{43}$$

where T_o and T_i are the coolant temperature of core inlet and outlet, respectively. The signal from signal processor is generated by

$$\Delta n = n_0 - n. \tag{44}$$

Following the fuzzy logic principle, the fuzzy FOPID controller considers the error and its rate of change as incoming signals. It adjusts K_{Pp-F}, K_{Pi-F}, K_{Pd-F}, λ, and μ online and produces them with the fractional factors satisfying conditions $0 < \lambda, \mu < 2$. The coordination principle is establishing the relationship among e, ec, and parameters. The fuzzy subset of fuzzy quantity chooses {NB, NM, NS, ZO, PS, PM, PB} and the membership functions adopt normal distribution functions. Larger K_{Pp-F}, smaller K_{Pd-F}, and μ could obtain rapider system response and avoid the differential overflow by initial e saltus step when e is large. Then adopt smaller K_{Pi-F} and λ to prevent exaggerated overshooting. When e is medium, K_{Pp-F} should reduce properly; meanwhile, K_{Pi-F}, K_{Pd-F}, and λ should be moderate to decrease the system overshooting and ensure the response fast. If e is small, the larger K_{Pp-F}, K_{Pi-F}, and λ have the ability to fade the steady-state error. At the same time, moderate K_{Pd-F} and μ keep output away from vibration around the **norm** value and guarantee anti-interference performance of the system. For more information, one could refer to the literature [40].

The WKNN proposed above in Section 3 has been adopted to improve the performance of FOPID parameter tuning, and three-layer structure has been applied. The WKNN input and output could be displayed as follows:

$$\begin{aligned}
\mathbf{x}_1(n) &= \mathrm{rin}(n), \\
\mathbf{x}_2(n) &= \mathrm{yout}(n), \\
\mathbf{x}_3(n) &= \mathrm{rin}(n) - \mathrm{yout}(n), \\
\mathbf{x}_4(n) &= 1, \\
\mathbf{z}_1(n) &= K_{Pp-W}(n), \\
\mathbf{z}_2(n) &= K_{Pi-W}(n), \\
\mathbf{z}_3(n) &= K_{Pd-W}(n).
\end{aligned} \tag{45}$$

As the boundedness of sigmoid function, WKNN output parameters for enhanced PID are maintained in a range of $[-1, 1]$. To solve this problem, the expertise improvement unit is added between WKNN and power require counter in Figure 4, which corrects parameters order of magnitudes. This control strategy overcomes the limitation problem of NN activation function and takes advantage of a priori knowledge from the researchers in reactor power domain, which endows the algorithm with reliability and simplifies the fuzzy logic rules setting to reduce the control complexity.

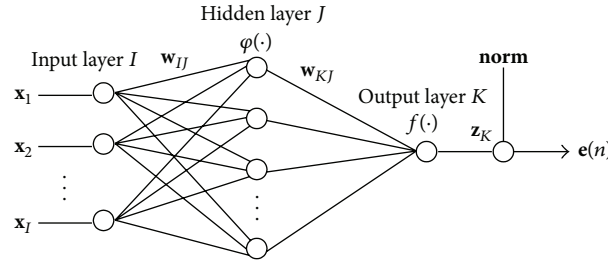

Figure 3: The structure of WKNN.

6. Simulation Results and Discussion

In this section, simulations of WKNN and WKNN fractional-order PID controller have been carried out aiming to examine performance and assess accuracy of the algorithm proposed in the previous sections, and it compares experimentally the performance of the WKNN algorithm and WNN algorithm. To display the results clearly, simulations are divided into two parts. One is achieved as numerical examples in Matlab program which includes WKNN algorithm and its fractional-order PID application in time delay system. Another is written in FORTRAN language and connected with RELAP 5 which implements the reactor power controlling.

6.1. Numerical Examples. To demonstrate the performance of WKNN algorithm, two simulations are conducted comparing with the WNN algorithm. The structure of neural network could be consulted in Figure 3, and the initial weights are generated randomly. The first experiment shows the behavior of WKNN and WNN in a stationary environment of which **norm** output is 0.6 (Figures 5 and 6). And the second one operates in the variable time process as $0.3\sin(0.02t) + 0.5$ (Figures 7 and 8).

Figures 6–9 adopt WKNN and WNN to calculate the regression process in the constant and time varying environment, respectively. And both methods bring relatively large system error in initial period. In Figures 6 and 7, the constant **norm** output and straight regression process emphasize the faster convergence of algorithms. In Figures 8 and 9, the time varying impels the regression process more difficult than the previous experiment. Firstly, there is overshoot from 0 s to 50 s in the WNN, which impels the error of WNN to decrease slowly. When the error of WNN starts to reduce, the one of WKNN has approached zero. Secondly, comparing the two error curves, the phenomenon could not be ignored in which the WKNN curve has sharp points, while the WNN curve has the relative smoothness. Thirdly, although the initial parameters have randomness, the regression process is periodical. The figures imply that the convergence rate and

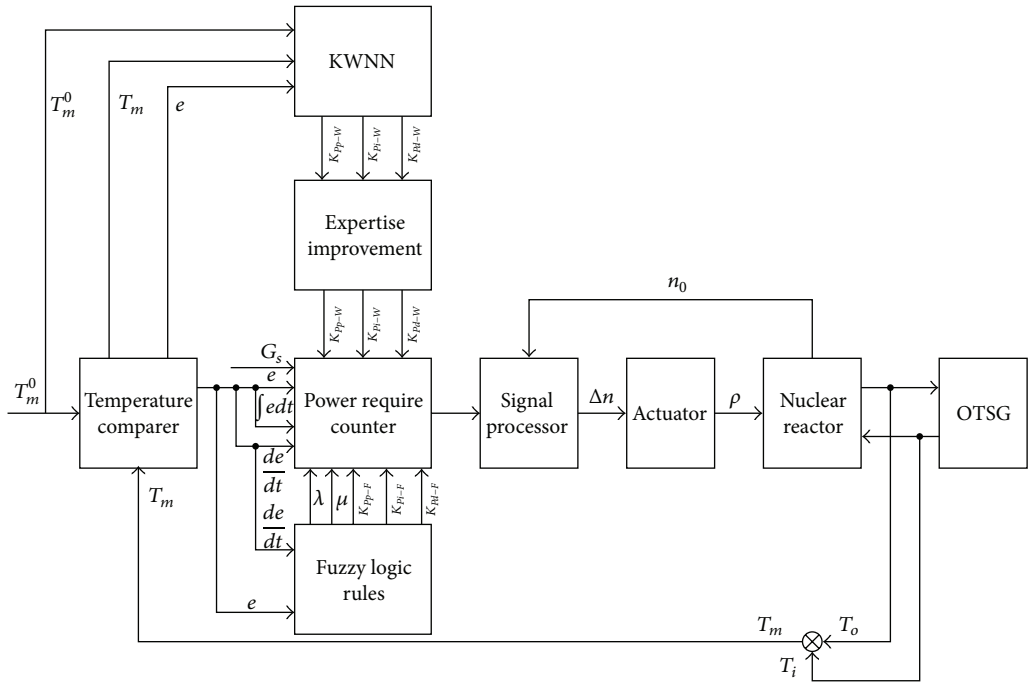

Figure 4: Control system of nuclear reactor power.

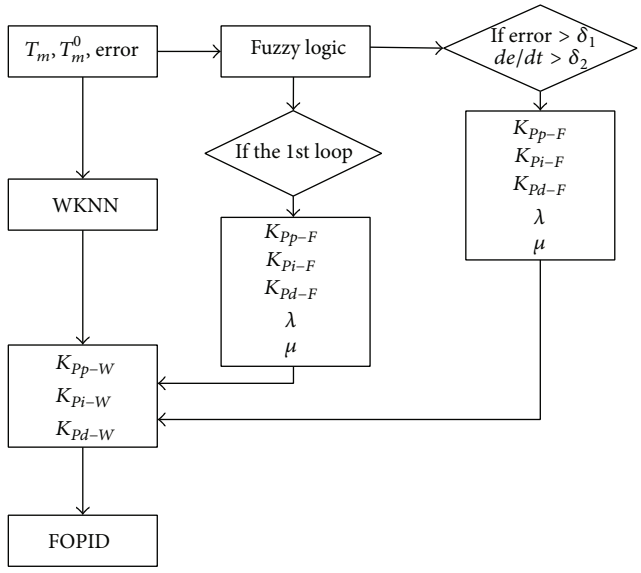

Figure 5: The control strategy of FOPID.

regression quality of WKNN are better than that of WNN and initial condition brings no benefit to WKNN. From the argument above, it indicates preliminary that WKNN algorithm reduces the amplitude of error variation and has better performance than WNN. However, it remains to be verified whether or not the WKNN is more appropriate for FOPID controller tuning in time delay system. Then, in this paper, the algorithm proposed is designed to examine the effectiveness of tuning the FOPID controller in time delay control. To illustrate clearly, a known dynamic time delay control system is proposed as follows:

$$G(s) = \frac{4}{5s+1} e^{-3s}, \tag{46}$$

and then analyze the performance of unit step response. The control process is displayed in Figures 10 and 11.

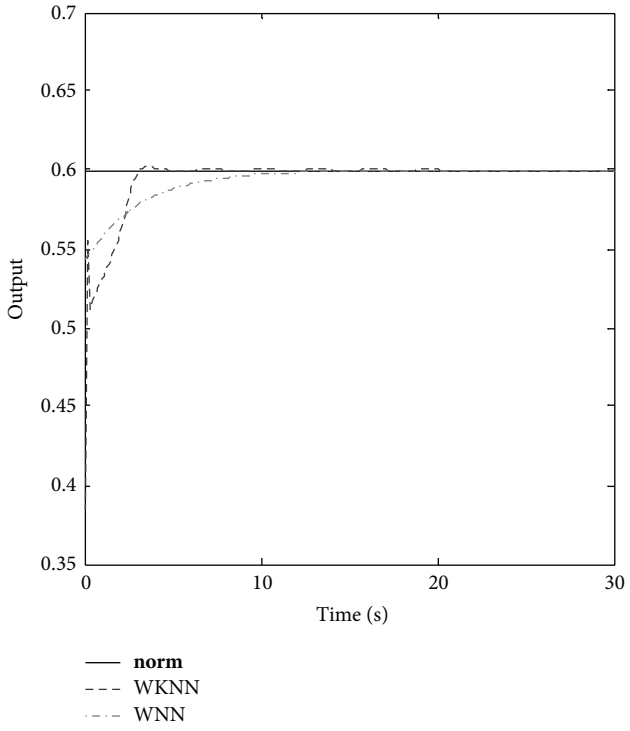

Figure 6: The comparison of output.

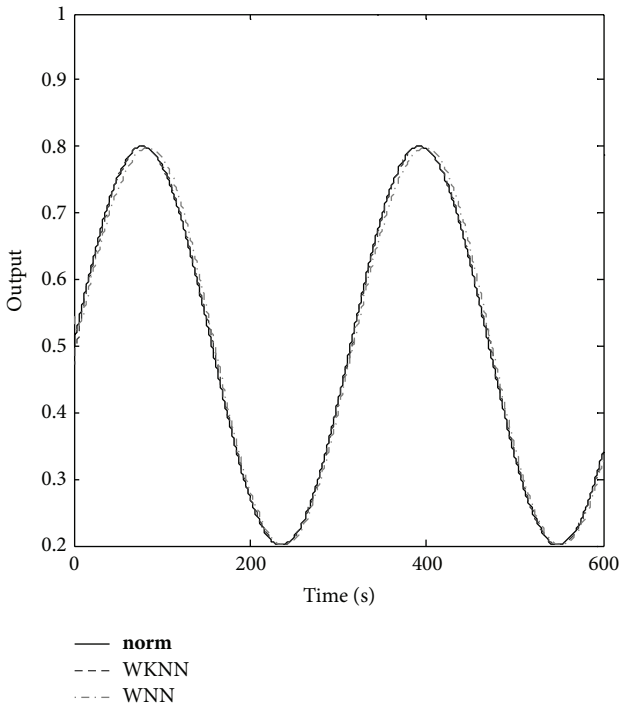

Figure 8: The comparison of output.

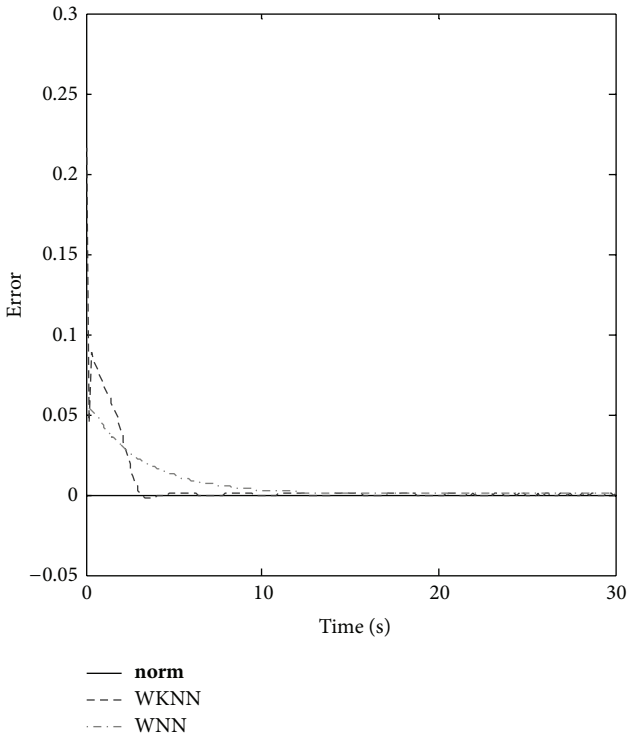

Figure 7: The comparison of error.

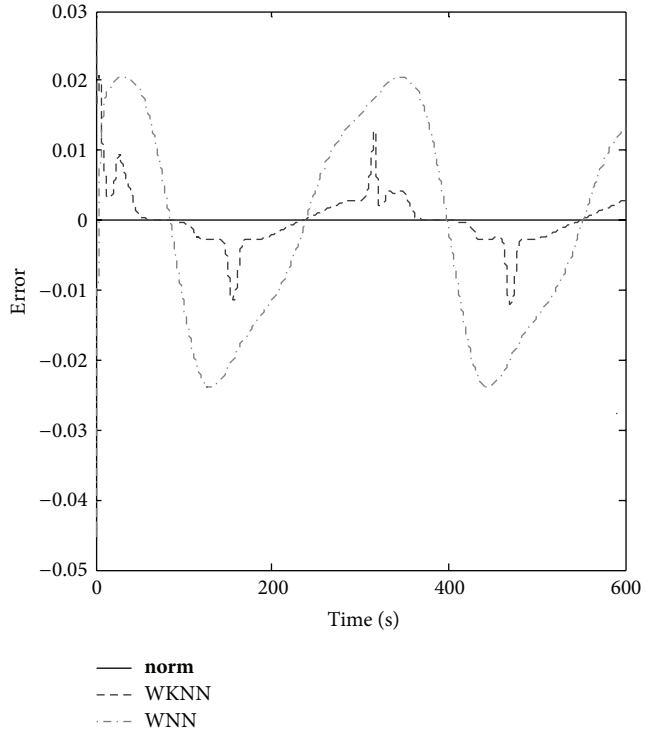

Figure 9: The comparison of error.

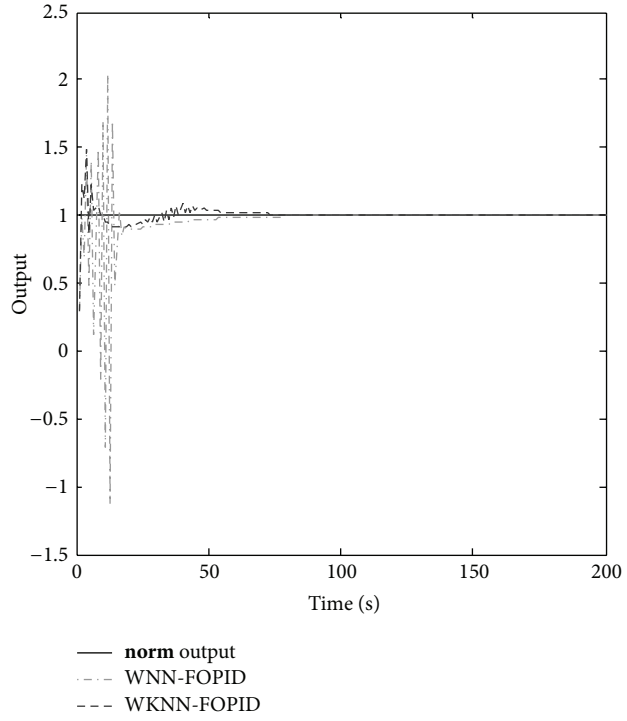

Figure 10: The comparison of controller output.

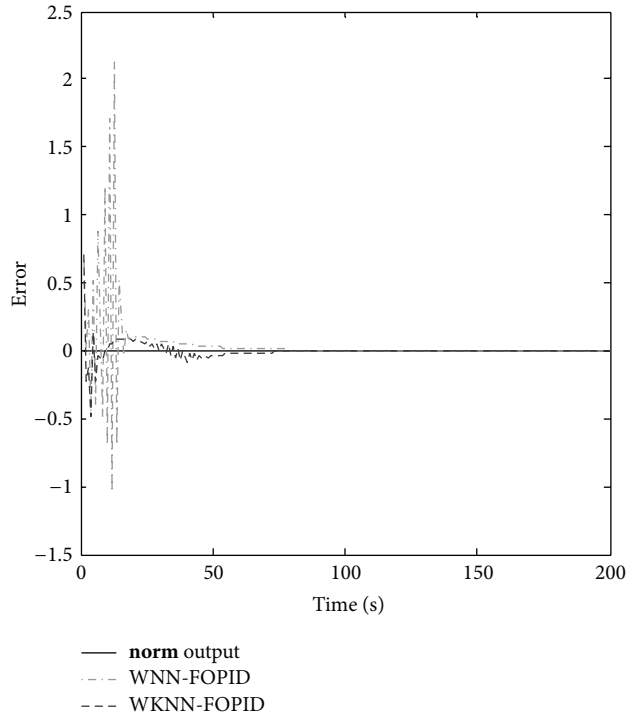

Figure 11: The comparison of controller error.

From the information of Figures 10 and 11, when the time delay system utilizes FOPID controller with the same order, the overshooting of WNN-FOPID controller and the output oscillation frequency and amplitude are all higher than WKNN-FOPID obviously. The control curves and error curves of different algorithm confirm that the whole tuning performance of WKNN is better than that of WNN. To validate the algorithm practicability and independence from model, the next subsection will apply and control a more complex nuclear reactor model with the strategy of Section 5.

6.2. RELAP5 Operation Example. This paper establishes the integrated pressurized water reactor (IPWR) model by RELAP5 transient analysis program, and the control strategy in Figure 5 is applied to control the reactor power. As the boundedness of sigmoid function, WNN output parameters for enhanced PID remained in a range of $[-1, 1]$. To solve this problem, the expertise improvement unit is added between KWNN and power require counter in Figure 1, which corrects parameters order of magnitudes. Through the analysis of load shedding limiting condition of nuclear power unit, the reliability and availability of the control method proposed have been demonstrated. In the process of load shedding, secondary feedwater reduces from 100% FP to 60% FP in 10 s and then reduces to 20% FP. The operating characteristic could be displayed in Figure 7.

Figure 12(a) displays the variation of once-through steam generator (OTSG) secondary feedwater flow and steam flow. Due to the small water capacity in OTSG, the steam flow decreased rapidly when water flow decreases. Then the decrease in steam flow leads to the reduction of the heat by feedwater, and primary coolant average temperature increases. The reactor power control system accords with the deviation of average temperature and introduces negative reactivity to bring the power down and maintain the temperature constant. The variation of outlet and inlet temperature is shown in Figure 12(c). The primary coolant flow stays the same and reduces the core temperature difference to guarantee the heat derivation. In the process of rapid load variation, the change of coolant temperature causes the motion of coolant loading and then leads to the variation of voltage regulator pressure (Figure 12(d)). The pressure maximum approaches 15.62 MPa; however, it drops rapidly with the power decrease and volume compression. Then pressure comes to the stable state when coolant temperature stabilizes.

7. Conclusions

The goal of this paper is to present a novel wavelet kernel neural network. Before the proposed WKNN, the wavelet kernel function has been confirmed to be valid in neural network. The WKNN takes advantage of KLMS and NLMS, which endow the algorithm with reliability, stability, and well convergence. On basis of the WKNN, a novel control strategy for FOPID controller design has been proposed. It overcomes the boundary problem of activation functions in network and utilizes the experiential knowledge of researchers sufficiently. Finally, by establishing the model of IPWR, the method above is applied in a certain simulation of load shedding, and simulation results validate the fact that the method proposed has the practicability and reliability in actual complicated system.

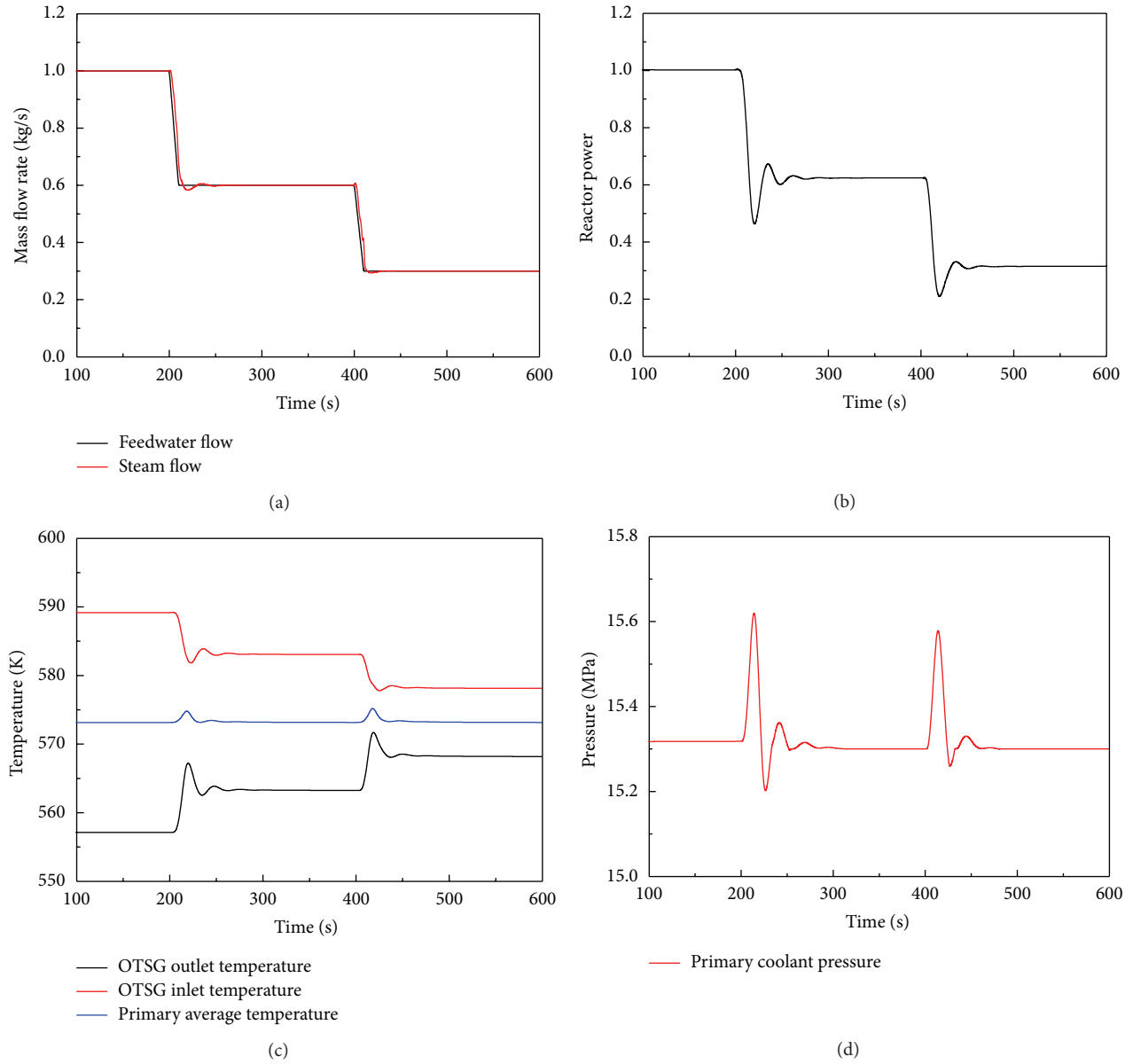

Figure 12: Operating characteristic.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

New Results on Stability and Stabilization of Markovian Jump Systems with Time Delay

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This technical paper deals with the problem of stochastic stability and stabilization for a class of linear Markovian jumping systems with discrete time-varying delay. A novel delay-dependent stochastic stability criterion for Markovian delay systems is established based on new augmented Lyapunov-Krasovskii functional and delay fractioning techniques. Then a state feedback controller is designed to guarantee the stochastic stability of the resulting closed-loop system. Numerical examples are provided to illustrate the effectiveness of the proposed design approach in this paper.

1. Introduction

Markovian jump systems are a special class of hybrid systems with both modes and state variables. Since many dynamical systems such as the chemical processes, transportation systems, and manufacturing systems can be modeled as Markovian jump systems, considerable attention has been devoted to the problem of analysis and synthesis for Markovian jump systems over the past decades; see, for example, [1–4] and the references therein.

On the other hand, time delays are often encountered in various practical systems such as biological, electronics, and economical systems, which are frequently one of the main causes of poor performance, oscillation, or even instability [5–10]. Thus the analysis and synthesis of Markovian jump systems with time delay are of theoretical and practical importance and have received considerable research interest in the past few decades (e.g., [1–4, 10–13]).

In the literature, different design approaches have been proposed to deal with the delays. Generally, to get stability criteria, a Lyapunov function (or functional) is involved. For more details, one can see [10–14] and references therein. In those results, the information of delay is included in some criteria, which is so-called delay dependent, while others are irrelevant to the delay [15–17], which is called delay independent. It is well known that the former ones are generally much less conservative than the latter ones due to using the information of delay such as its length or rules of changes [18–20]. Therefore, how to choose a Lyapunov function (or functional) and derive a less conservative condition from the time derivative of such a function (or functional) is one of the most important techniques in the analysis and synthesis process.

Since cross products between two vectors are commonly encountered, different bound techniques are introduced. For example, Park's inequality and Moon's inequality were frequently used for bounding some cross terms. Moreover, many researchers tried to construct special forms of Lyapunov functional to obtain a feasible result. Some delay-dependent conditions were obtained through model transformation approaches; more details can be found in [18–22] and references therein. But these choices of specific functional and overbounding techniques are the origin of conservatism [20–23].

Slack matrices technique has been proven to be an effective approach to establish an improved result. To mention a few, by introducing two auxiliary matrices a new LMI representation of H∞ performance is presented for

the linear systems with a single time-varying state delay in [19]. Wu et al. [20] and He et al. [21] tried to seek the relationship between the terms in the Newton-Leibniz formula and to introduce some free weighting matrices to express this relationship and some delay-dependent stability conditions with less conservativeness have been established. These methods are quite different from the conventional one of using model transformation which may introduce some superfluous dynamics into the original system. Additional studies can be found in [22–26] and references cited therein.

To improve the delay-dependent conditions, some efforts on constructing new Lyapunov functional were made by employing the delay partitioning method [21, 23–26]. To mention a few, a delay partitioning method has been developed for the delay-dependent stability analysis of fuzzy time-varying state delay systems based on a novel fuzzy Lyapunov-Krasovskii functional in [23]. In [24], by choosing an augmented Lyapunov-Krasovskii functional and utilizing the delay-partitioning method, novel delay-dependent mean square asymptotic stability conditions are derived in terms of LMIs. With the idea of delay partitioning, in [25] a new delay-dependent stochastic stability criterion on the stochastic stability of Markovian jump systems with time-delay has been proposed, which is less conservative than most existing results. However, when revisiting those problems, we find that there is still much room left for improvement, which motivates the present study.

In this context, we are interested in investigating the delay-dependent stability analysis and stabilization problem for a class of Markovian jump systems with time-varying delay. By employing the delay partitioning method and introducing some slack matrix variables based on the Newton-Leibniz formula, a less conservative delay-dependent stability condition is proposed. Then a state feedback controller, which guarantees the stochastic stability of the corresponding closed-loop system, is designed.

The rest of the paper is organized as follows. Section 2 gives problem formulation and some necessary lemmas. In Section 3, we present our main results of stability analysis and stabilization problem. Numerical examples are given in Section 4, followed by the conclusions, which are presented in Section 5.

Notations. The notations used throughout the paper are standard except where otherwise stated. The superscript "T" stands for matrix transposition. "I" and "0" denote the identity matrix and a zero matrix with compatible dimensions, respectively. The notation "$P > 0$ ($P \geq 0$)" means that P is real symmetric and positive definite (positive semidefinite); $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$ stands for a block-diagonal matrix with matrices $M_1, M_2, \ldots, M_n$ on the diagonal. "$*$" represents the elements below the main diagonal of a symmetric matrix. $\|\cdot\|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulations and Preliminaries

Consider the following stochastic linear state-delay systems with Markovian jumping parameters:

$$\begin{aligned}\dot{x}(t) &= A(r(t))x(t) + A_d(r(t))x(t-\tau(t)) \\ &\quad + B(r(t))u(t), \quad t > 0, \\ x(t) &= \varphi(t), \quad t \in [-\overline{\tau}, 0], \; r(0) = 0,\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the system state vector, $u(t) \in R^m$ is the controlled input vector, and $A(r(t))$, $A_d(r(t))$, and $B(r(t))$ are matrix functions of the random jumping process $\{r(t)\}$. $r(t)$ is a continuous-time Markovian jump process with right continuous trajectories taking discrete values in a given finite set $S = \{1, 2, \ldots, N\}$ with transition probability matrix $\Gamma \triangleq \{\pi_{ij}\}$ given by

$$\Pr\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases}\pi_{ij}\Delta + o(\Delta), & i \neq j \\ 1 + \pi_{ij}\Delta + o(\Delta), & i = j,\end{cases} \tag{2}$$

where $\Delta > 0$ and $\pi_{ij} \geq 0$, for $j \neq i$, is the transition rate from mode i at time t to mode j at time $t + \Delta$ and

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}, \tag{3}$$

for each mode $i \in S$, $\lim_{h\to 0}(o(\Delta)/\Delta) = 0$. $\varphi(t)$ ($t \in [-\overline{\tau}, 0]$) is the system's initial function which is continuous differentiable on $[-\overline{\tau}, 0]$ and $r_0 \in S$ are the initial conditions of the continuous state and mode. $\tau(t)$ is a continuously differentiable function, which denotes the time-varying delay when the mode is in $r(t) = i$, satisfying the conditions as follows:

$$0 \leq \tau(t) \leq \overline{\tau}, \qquad \dot{\tau}(t) \leq \mu \qquad \forall i \in S, \tag{4}$$

where $\overline{\tau}$ and μ are known positive real constants.

Remark 1. It should be pointed out that, when a time-varying delay $\tau(t)$ is considered, it is usually assumed that $\dot{\tau}(t) \leq \mu < 1$, while we only require $\dot{\tau}(t) \leq \mu$ in this paper, which is more general, as discussed by Xu et al. [22].

For notational simplicity, in the sequel, for each possible $r(t) = i$, $i \in S$, a matrix $\aleph(r(t))$ will be denoted as $\aleph_i$; for example, $A(r(t))$ and $A_d(r(t))$ are denoted as A_i and A_{di}, respectively, and so on.

To facilitate further developments, we introduce the following lemma and definition that will be frequently used in deriving the main results.

Lemma 2 (Schur complement). *Given constant matrices S_1, S_2, and S_3 with appropriate dimensions, where $S_1 = S_1^T$ and $S_2 = S_2^T > 0$, then $S_1 + S_3^T S_2^{-1} S_3 < 0$ if and only if*

$$\begin{bmatrix} S_1 & S_3^T \\ * & -S_2 \end{bmatrix} < 0 \qquad or \qquad \begin{bmatrix} -S_2 & S_3^T \\ * & S_1 \end{bmatrix} < 0. \tag{5}$$

Definition 3. The Markovian jump system

$$\dot{x}(t) = A(r(t))x(t) + A_d(r(t))x(t-\tau(t)), \quad t > 0, \tag{6}$$
$$x(t) = \varphi(t), \quad \forall t \in [-\overline{\tau}, 0], \; r(0) = r_0$$

is said to be stochastically stable, if for any finite $\varphi(t) \in R^n$ defined on $[-\overline{\tau}, 0]$ and $r_0 \in S$ the following condition is satisfied:

$$\lim_{t\to\infty} E\left\{\int_0^t x^T(t)x(t)\,dt \mid \varphi, r_0\right\} < \infty. \tag{7}$$

3. Main Results

In this section, we will consider the stability and stabilization problem of the system (1). First of all, we introduce the following results which will be used in deriving the main results.

By the Leibniz-Newton formula, we have

$$x(t-\tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)\,ds; \tag{8}$$

and then for any appropriate dimensional matrices N_{1i}, N_{2i}, N_{3i}, N_{4i}, N_{5i}, $(i \in S)$, we have that the following equation holds:

$$2\left[x^T(t)N_{1i} + x^T\left(t - \frac{\tau(t)}{2}\right)N_{2i} + x^T(t-\tau(t))N_{3i} + x^T\left(t - \frac{\tau(t)+\overline{\tau}}{2}\right)N_{4i} + x^T(t-\overline{\tau})N_{5i}\right] \times \left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right] = 0. \tag{9}$$

Moreover, since $0 \le \tau(t) \le \overline{\tau}$, $\dot{\tau}(t) \le \mu \; \forall i \in S$, for any appropriate dimensional matrix $X \ge 0$,

$$\overline{\tau}\eta^T(t)X\eta(t) - \int_{t-\tau(t)}^{t}\eta^T(t)X\eta(t)\,ds \ge 0, \tag{10}$$

where

$$\eta(t) = \left[x^{\mathrm{T}}(t) \;\; x^T\left(t - \frac{\tau(t)}{2}\right) \;\; x^T(t-\tau(t)) \;\; x^T\left(t - \frac{\tau(t)+\overline{\tau}}{2}\right) \;\; x^T(t-\overline{\tau})\right]^T. \tag{11}$$

3.1. Stability Analysis. Combining delay partitioning approach and free-weighting matrices technique, a new delay-dependent stochastic stability condition is established in the theorem as follows.

Theorem 4. *For given positive scalars $\overline{\tau} > 0$ and $\mu > 0$, the autonomous Markovian jump system* (1) *with $u(t) \equiv 0$ is stochastically stable if there exist matrices $P_i = P_i^T > 0$, $Q_{1i} = Q_{1i}^T \ge 0$, $Q_{2i} = Q_{2i}^T \ge 0$, $Q_{3i} = Q_{3i}^T \ge 0$, $Q_{4i} = Q_{4i}^T \ge 0$, $(i \in S)$, $Z = Z^T > 0$,*

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ * & X_{22} & X_{23} & X_{24} & X_{25} \\ * & * & X_{33} & X_{34} & X_{35} \\ * & * & * & X_{44} & X_{45} \\ * & * & * & * & X_{55} \end{bmatrix} \ge 0, \tag{12}$$

and free-weighting matrices $N_{1i}, N_{2i}, N_{3i}, N_{4i}, N_{5i}, (i \in S)$ with appropriate dimensions, such that the following LMIs hold:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \overline{\tau}A_i^T Z \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & 0 \\ * & * & \Phi_{33} & \Phi_{34} & \Phi_{35} & \overline{\tau}A_{di}^T Z \\ * & * & * & \Phi_{44} & \Phi_{45} & 0 \\ * & * & * & * & \Phi_{55} & 0 \\ * & * & * & * & * & -\overline{\tau}Z \end{bmatrix} < 0, \tag{13}$$

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & N_{1i} \\ * & X_{22} & X_{23} & X_{24} & X_{25} & N_{2i} \\ * & * & X_{33} & X_{34} & X_{35} & N_{3i} \\ * & * & * & X_{44} & X_{45} & N_{4i} \\ * & * & * & * & X_{55} & N_{5i} \\ * & * & * & * & * & Z \end{bmatrix} \ge 0, \tag{14}$$

$$\sum_{j=1}^{N}\pi_{ij}Q_{1j} \le 0, \quad \sum_{j=1}^{N}\pi_{ij}Q_{2j} \le 0, \quad \sum_{j=1}^{N}\pi_{ij}Q_{3j} \le 0, \quad \sum_{j=1}^{N}\pi_{ij}Q_{4j} \le 0, \tag{15}$$

where

$$\Phi_{11} = P_iA_i + A_i^TP_i + \sum_{j=1}^{N}\pi_{ij}P_j + Q_{1i} + N_{1i} + N_{1i}^T + \overline{\tau}X_{11},$$
$$\Phi_{12} = N_{2i}^T + \overline{\tau}X_{12},$$

$$\begin{aligned}
\Phi_{13} &= P_i A_{di} - N_{1i} + N_{3i}^T + \overline{\tau} X_{13},\\
\Phi_{14} &= N_{4i}^T + \overline{\tau} X_{14},\\
\Phi_{15} &= N_{5i}^T + \overline{\tau} X_{15},\\
\Phi_{22} &= -\left(1 - \frac{\mu}{2}\right) Q_{1i} + Q_{2i} + \overline{\tau} X_{22},\\
\Phi_{23} &= -N_{2i} + \overline{\tau} X_{23},\\
\Phi_{24} &= \overline{\tau} X_{24},\\
\Phi_{25} &= \overline{\tau} X_{25},\\
\Phi_{33} &= -(1-\mu) Q_{2i} + Q_{3i} - N_{3i} - N_{3i}^T + \overline{\tau} X_{33},\\
\Phi_{34} &= -N_{4i}^T + \overline{\tau} X_{34},\\
\Phi_{35} &= -N_{5i}^T + \overline{\tau} X_{35},\\
\Phi_{44} &= -\left(1 - \frac{\mu}{2}\right) Q_{3i} + Q_{4i} + \overline{\tau} X_{44},\\
\Phi_{45} &= \overline{\tau} X_{45},\\
\Phi_{55} &= -Q_{4i} + \overline{\tau} X_{55}.
\end{aligned} \tag{16}$$

Proof. We define the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned}
V(x(t), r(t)) = {} & V_1(x(t), r(t)) + V_2(x(t), r(t))\\
& + V_3(x(t), r(t)) + V_4(x(t), r(t))\\
& + V_5(x(t), r(t)) + V_6(x(t), r(t)),
\end{aligned} \tag{17}$$

with

$$\begin{aligned}
V_1(x(t), r(t)) &= x^T(t) P_i x(t),\\
V_2(x(t), r(t)) &= \int_{t-(\tau(t)/2)}^{t} x^T(s) Q_{1i} x(s)\, ds,\\
V_3(x(t), r(t)) &= \int_{t-\tau(t)}^{t-(\tau(t)/2)} x^T(s) Q_{2i} x(s)\, ds,\\
V_4(x(t), r(t)) &= \int_{t-((\tau(t)+\overline{\tau})/2)}^{t-\tau(t)} x^T(s) Q_{3i} x(s)\, ds,\\
V_5(x(t), r(t)) &= \int_{t-\overline{\tau}}^{t-((\tau(t)+\overline{\tau})/2)} x^T(s) Q_{4i} x(s)\, ds,\\
V_6(x(t), r(t)) &= \int_{-\overline{\tau}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z \dot{x}(s)\, ds d\theta,
\end{aligned} \tag{18}$$

where $P_i = P_i^T > 0$, $Q_{1i} = Q_{1i}^T \geq 0$, $Q_{2i} = Q_{2i}^T \geq 0$, $Q_{3i} = Q_{3i}^T \geq 0$, $Q_{4i} = Q_{4i}^T \geq 0$, $(i \in S)$, and $Z = Z^T > 0$ are matrices to be determined.

Let $\wp$ be the weak infinitesimal generator of the random process $\{x(t), r(t)\}$. Then, for each $r(t) = i$, $i \in S$, we have

$$\begin{aligned}
\wp V_1(x(t), i) &= 2x^T(t) P_i \dot{x}(t) + x^T(t) \sum_{j=1}^{N} \pi_{ij} P_j x(t)\\
&= 2x^T(t) P_i A_i x(t) + 2x^T(t) P_i A_{di}\\
&\quad \times x(t - \tau(t)) + x^T(t) \left(\sum_{j=1}^{N} \pi_{ij} P_j\right) x(t),
\end{aligned}$$

$$\begin{aligned}
&\wp V_2(x(t), i)\\
&\quad = x^T(t) Q_{1i} x(t)\\
&\qquad - \left(1 - \frac{\dot{\tau}(t)}{2}\right) x^T\left(t - \frac{\tau(t)}{2}\right) Q_{1i} x\left(t - \frac{\tau(t)}{2}\right)\\
&\qquad + \int_{t-(\tau(t)/2)}^{t} x^T(s) \left(\sum_{j=1}^{N} \pi_{ij} Q_{1j}\right) x(s)\, ds,
\end{aligned}$$

$$\begin{aligned}
&\wp V_3(x(t), i)\\
&\quad = \int_{t-\tau(t)}^{t-(\tau(t)/2)} x^T(s) \left(\sum_{j=1}^{N} \pi_{ij} Q_{2j}\right) x(s)\, ds\\
&\qquad + \left(1 - \frac{\dot{\tau}(t)}{2}\right) x^T\left(t - \frac{\tau(t)}{2}\right) Q_{2i} x\left(t - \frac{\tau(t)}{2}\right)\\
&\qquad - (1 - \dot{\tau}(t)) x^T(t - \tau(t)) Q_{2i} x(t - \tau(t)),
\end{aligned}$$

$$\begin{aligned}
&\wp V_4(x(t), i)\\
&\quad = \int_{t-((\tau(t)+\overline{\tau})/2)}^{t-\tau(t)} x^T(s) \left(\sum_{j=1}^{N} \pi_{ij} Q_{3j}\right) x(s)\, ds\\
&\qquad + (1 - \dot{\tau}(t)) x^T(t - \tau(t)) Q_{3i} x(t - \tau(t))\\
&\qquad - \left(1 - \frac{\dot{\tau}(t)}{2}\right) x^T\left(t - \frac{\tau(t) + \overline{\tau}}{2}\right) Q_{3i} x\left(t - \frac{\tau(t) + \overline{\tau}}{2}\right),
\end{aligned}$$

$$\begin{aligned}
&\wp V_5(x(t), i)\\
&\quad = \left(1 - \frac{\dot{\tau}(t)}{2}\right) x^T\left(t - \frac{\tau(t) + \overline{\tau}}{2}\right) Q_{4i} x\left(t - \frac{\tau(t) + \overline{\tau}}{2}\right)\\
&\qquad - x^T(t - \overline{\tau}) Q_{4i} x(t - \overline{\tau})\\
&\qquad + \int_{t-\overline{\tau}}^{t-((\tau(t)+\overline{\tau})/2)} x^T(s) \left(\sum_{j=1}^{N} \pi_{ij} Q_{4j}\right) x(s)\, ds,
\end{aligned}$$

$$\wp V_6(x(t), i) = \overline{\tau} \dot{x}^T(t) Z \dot{x}(t) - \int_{t-\overline{\tau}}^{t} \dot{x}^T(s) Z \dot{x}(s)\, ds. \tag{19}$$

According to (8)–(11) and considering $0 \leq \tau(t) \leq \overline{\tau}$, $\dot{\tau}(t) \leq \mu$ $\forall i \in S$ in (4), we have

$$\begin{aligned}
&\wp V(x(t), i)\\
&\quad \leq 2x^T(t) P_i A_i x(t) + 2x^T(t) P_i A_{di} x(t - \tau(t))
\end{aligned}$$

$$
\begin{aligned}
&+ x^T(t)\left(\sum_{j=1}^{N}\pi_{ij}P_j\right)x(t) + x^T(t)Q_{1i}x(t)\\
&-\left(1-\frac{\overline{\tau}(t)}{2}\right)x^T\left(t-\frac{\tau(t)}{2}\right)Q_{1i}x\left(t-\frac{\tau(t)}{2}\right)\\
&+\int_{t-(\tau(t)/2)}^{t}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{1j}\right)x(s)\,ds\\
&+\int_{t-\tau(t)}^{t-(\tau(t)/2)}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)x(s)\,ds\\
&+\left(1-\frac{\dot{\tau}(t)}{2}\right)x^T\left(t-\frac{\tau(t)}{2}\right)Q_{2i}x\left(t-\frac{\tau(t)}{2}\right)\\
&-(1-\overline{\tau}(t))\,x^T(t-\tau(t))\,Q_{2i}x(t-\tau(t))\\
&+\int_{t-((\tau(t)+\overline{\tau})/2)}^{t-\tau(t)}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{3j}\right)x(s)\,ds\\
&+(1-\dot{\tau}(t))\,x^T(t-\tau(t))\,Q_{3i}x(t-\tau(t))\\
&-\left(1-\frac{\overline{\tau}(t)}{2}\right)x^T\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right)Q_{3i}x\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right)\\
&+\left(1-\frac{\dot{\tau}(t)}{2}\right)x^T\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right)Q_{4i}x\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right)\\
&-x^T(t-\overline{\tau})\,Q_{4i}x(t-\overline{\tau})\\
&+\int_{t-\overline{\tau}}^{t-((\tau(t)+\overline{\tau})/2)}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{4j}\right)x(s)\,ds\\
&+\overline{\tau}\dot{x}^T(t)\,Z\dot{x}(t)-\int_{t-\overline{\tau}}^{t}\dot{x}^T(s)\,Z\dot{x}(s)\,ds\\
&+2\left[x^T(t)\,N_{1i}+x^T\left(t-\frac{\tau(t)}{2}\right)N_{2i}+x^T(t-\tau(t))\,N_{3i}\right.\\
&\qquad\left.+x^T\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right)N_{4i}+x^T(t-\overline{\tau})\,N_{5i}\right]\\
&\times\left[x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\right]\\
&+\overline{\tau}\eta^T(t)\,X\eta(t)-\int_{t-\tau(t)}^{t}\eta^T(t)\,X\eta(t)\,ds\\
&=\eta^T(t)\,\Lambda_1\eta(t)-\int_{t-\tau(t)}^{t}\xi^T(t,s)\,\Psi\xi(t,s)\,ds+\Lambda_2+\Lambda_3,
\end{aligned}
\tag{20}
$$

where

$$
\xi(t)=\left[\eta^T(t)\quad\dot{x}^T(s)\right]^T, \tag{21}
$$

$$
\Lambda_1=\begin{bmatrix}
\overline{\Phi}_{11} & \Phi_{12} & \overline{\Phi}_{13} & \Phi_{14} & \Phi_{15}\\
* & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25}\\
* & * & \overline{\Phi}_{33} & \Phi_{34} & \Phi_{35}\\
* & * & * & \Phi_{44} & \Phi_{45}\\
* & * & * & * & \Phi_{55}
\end{bmatrix}, \tag{22}
$$

$$
\begin{aligned}
\overline{\Phi}_{11}&=\Phi_{11}+A_i^THA_i,\\
\overline{\Phi}_{13}&=\Phi_{13}+A_i^THA_{di},\\
\overline{\Phi}_{33}&=\Phi_{33}+A_{di}^THA_{di}.
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2=&\int_{t-(\tau(t)/2)}^{t}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{1j}\right)x(s)\,ds\\
&+\int_{t-\tau(t)}^{t-(\tau(t)/2)}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{2j}\right)x(s)\,ds\\
&+\int_{t-((\tau(t)+\overline{\tau})/2)}^{t-\tau(t)}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{3j}\right)x(s)\,ds\\
&+\int_{t-\overline{\tau}}^{t-((\tau(t)+\overline{\tau})/2)}x^T(s)\left(\sum_{j=1}^{N}\pi_{ij}Q_{4j}\right)x(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_3=&-\frac{\dot{\tau}(t)}{2}x^T\left(t-\frac{\tau(t)}{2}\right)Q_{2i}x\left(t-\frac{\tau(t)}{2}\right)\\
&-\dot{\tau}(t)\,x^T(t-\tau(t))\,Q_{3i}x(t-\tau(t))\\
&-\frac{\dot{\tau}(t)}{2}x^T\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right)Q_{4i}x\left(t-\frac{\tau(t)+\overline{\tau}}{2}\right),
\end{aligned}
\tag{23}
$$

and Ψ is denoted in (14).

By Schur complement (Lemma 2) and some matrices primary manipulations, it is easy to see that the abovementioned matrix $\Lambda_1 < 0$ in (22) is equivalent to (13).

Since $Q_{2i} \geq 0$, $Q_{3i} \geq 0$, $Q_{4i} \geq 0$, we have $\Lambda_3 \leq 0$.

It is easy to see that $\wp V(x(t), i) < 0$ if LMIs (13)–(15) hold.

Set $\lambda_1 = \min\{\lambda_{\min}(-\Lambda_1), i \in S\}$; then $\lambda_1 > 0$. For any $t \geq \overline{\tau}$, we have

$$
\wp V(x(t), i) \leq -\lambda_1\eta^T(t)\,\eta(t) \leq -\lambda_1 x^T(t)\,x(t). \tag{24}
$$

By Dynkin's formula, we have

$$
\begin{aligned}
&E\{V(x(t), i)\} - E\{V(\varphi(t), r_0)\}\\
&\quad= E\left\{\int_0^t \wp V(x(s), r(s))\,ds\right\} \leq -\lambda_1\int_0^t E\left\{x^T(s)\,x(s)\right\}ds.
\end{aligned}
\tag{25}
$$

Hence

$$
\int_0^t E\left\{x^T(s)\,x(s)\right\}ds \leq \lambda_1^{-1}V(\varphi(t), r_0). \tag{26}
$$

Set $\lambda_2 = \min\{\lambda_{\min}(P_i), i \in S\}$; then $\lambda_2 > 0$. From (17), it is easy to see that for any $t \geq 0$

$$
\begin{aligned}
&E\{V(x(t), i)\}\\
&\quad= E\{V_1(x(t), i)\} + E\{V_2(x(t), i)\} + E\{V_3(x(t), i)\}\\
&\qquad+ E\{V_4(x(t), i)\} + E\{V_5(x(t), i)\} + E\{V_6(x(t), i)\}\\
&\quad\geq \lambda_2 E\left\{x^T(t)\,x(t)\right\}.
\end{aligned}
\tag{27}
$$

From inequalities (26) and (27), we know that

$$E\left\{x^T(t)\,x(t)\right\} \le -\chi_1 E\left\{\int_0^t x^T(s)\,x(s)\,ds\right\} + \chi_2 V(x_0, r_0), \tag{28}$$

where $\chi_1 = \lambda_1\lambda_2^{-1}$, $\chi_2 = \lambda_2^{-1}$.

By Gronwell-Bellman lemma, we have

$$E\left\{x^T(t)\,x(t)\right\} \le \exp(-\chi_1 t)\,\chi_2 V(x_0, r_0). \tag{29}$$

Thus, according to (28), we have

$$\begin{aligned} &E\left\{\int_0^t x^T(s)\,x(s)\,ds \mid \varphi, r_0\right\} \\ &\quad\le \chi_1^{-1}\chi_2\left(1 - \exp(-\chi_1 t)\right)V(x_0, r_0). \end{aligned} \tag{30}$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} E\left\{\int_0^t x^T(s)\,x(s)\,ds \mid \varphi, r_0\right\} \le \chi_1^{-1}\chi_2 V(x_0, r_0). \tag{31}$$

For there always exists a scalar $\varepsilon > 0$, such that

$$\begin{aligned} &\lim_{t\to\infty} E\left\{\int_0^t x^T(s)\,x(s)\,ds \mid \varphi, r_0\right\} \\ &\quad\le \chi_1^{-1}\chi_2 V(x_0, r_0) \le \varepsilon \sup_{-\overline{\tau}\le s\le 0}\left|\varphi(s)\right|^2. \end{aligned} \tag{32}$$

From Definition 3, we know that the Markovian jump system in (1) with $u(t) \equiv 0$ is stochastically stable.

The proof is thus completed. □

Remark 5. Benefiting from a novel LKF construction, Theorem 4 presents a delay-dependent stochastic stability criterion which is much less conservative than most existing results in the literature. The idea of delay portioning used here is different from those approaches in the literature. For example, assuming the time delay can be denoted as a constant part and time-varying part, the idea of delay partitioning is employed to the constant part and some LKF are constructed in [21, 23–26]. As a further improved result, an appropriate LKF is established with the idea of partitioning the lower bound of time delay in [26]. However, in this brief, we divided the whole time varying delay interval $[0, \overline{\tau}]$ into multi-parts, with which a novel LKF is constructed.

If the system mode set $S = \{1\}$, the jump linear system (1) is simplified into a general linear system. Then we have the following simplified result.

Corollary 6. *For given positive scalars $\overline{\tau} > 0$ and $\mu > 0$, the autonomous Markovian jump system* (1) *with $i \in S = \{1\}$ is stochastically stable if there exist matrices $P = P^T > 0$, $Q_k = Q_k^T \ge 0$, $(k = 1, 2, 3, 4)$, $Z = Z^T > 0$,*

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ * & X_{22} & X_{23} & X_{24} & X_{25} \\ * & * & X_{33} & X_{34} & X_{35} \\ * & * & * & X_{44} & X_{45} \\ * & * & * & * & X_{55} \end{bmatrix} \ge 0, \tag{33}$$

and free-weighting matrices N_j, $(j = 1, 2, \ldots, 5)$ with appropriate dimensions, such that the following LMIs hold:

$$\begin{aligned} &\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \overline{\tau}A^T Z \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & 0 \\ * & * & \Phi_{33} & \Phi_{34} & \Phi_{35} & \overline{\tau}A_d^T Z \\ * & * & * & \Phi_{44} & \Phi_{45} & 0 \\ * & * & * & * & \Phi_{55} & 0 \\ * & * & * & * & * & -\overline{\tau}Z \end{bmatrix} < 0, \\ &\Psi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & N_1 \\ * & X_{22} & X_{23} & X_{24} & X_{25} & N_2 \\ * & * & X_{33} & X_{34} & X_{35} & N_3 \\ * & * & * & X_{44} & X_{45} & N_4 \\ * & * & * & * & X_{55} & N_5 \\ * & * & * & * & * & Z \end{bmatrix} \ge 0, \\ &\sum_{j=1}^N \pi_{ij} Q_1 \le 0, \\ &\sum_{j=1}^N \pi_{ij} Q_2 \le 0, \\ &\sum_{j=1}^N \pi_{ij} Q_3 \le 0, \\ &\sum_{j=1}^N \pi_{ij} Q_4 \le 0, \end{aligned} \tag{34}$$

where

$$\begin{aligned} \Phi_{11} &= PA + A^T P + Q_1 + N_1 + N_1^T + \overline{\tau}X_{11}, \\ \Phi_{12} &= N_2^T + \overline{\tau}X_{12}, \\ \Phi_{13} &= PA_d - N_1 + N_3^T + \overline{\tau}X_{13}, \\ \Phi_{14} &= N_4^T + \overline{\tau}X_{14}, \\ \Phi_{15} &= N_5^T + \overline{\tau}X_{15}, \\ \Phi_{22} &= -\left(1 - \frac{\mu}{2}\right)Q_1 + Q_2 + \overline{\tau}X_{22}, \\ \Phi_{23} &= -N_2 + \overline{\tau}X_{23}, \\ \Phi_{24} &= \overline{\tau}X_{24}, \\ \Phi_{25} &= \overline{\tau}X_{25}, \\ \Phi_{33} &= -(1-\mu)Q_2 + Q_3 - N_3 - N_3^T + \overline{\tau}X_{33}, \\ \Phi_{34} &= -N_4^T + \overline{\tau}X_{34}, \\ \Phi_{35} &= -N_5^T + \overline{\tau}X_{35}, \end{aligned}$$

$$\begin{aligned}
\Phi_{44} &= -\left(1-\frac{\mu}{2}\right)Q_3 + Q_4 + \overline{\tau}X_{44},\\
\Phi_{45} &= \overline{\tau}X_{45},\\
\Phi_{55} &= -Q_4 + \overline{\tau}X_{55}.
\end{aligned} \tag{35}$$

3.2. Stabilization Controller Design. Now we are in a position to present a solution to the stabilization problem for system (1). We assume that the state feedback controller is in the general formula as follows:

$$u_i = K_i x(t) + K_{di} x(t-\tau(t)), \tag{36}$$

where K_i and K_{di} are the gain matrices of the state feedback controller to be designed.

Then, for each $r(t) = i \in S$, the resulting close-loop system of (1) with (36) becomes

$$\dot{x}(t) = \overline{A}_i x(t) + \overline{A}_{di} x(t-\tau(t)), \tag{37}$$

where $\overline{A}_i = A_i + B_i K_i$, $\overline{A}_{di} = A_{di} + B_i K_{di}$. Thus, representing A_i by $\overline{A}_i$, A_{di} by $\overline{A}_{di}$ in Theorem 4, and by some matrices primary manipulations, we can obtain the following delay-dependent theorem.

Theorem 7. *For given positive scalars $\overline{\tau} > 0$ and $\mu > 0$, the autonomous Markovian jump system* (1) *is stochastically stable if there exist matrices $\widetilde{P}_i = \widetilde{P}_i^T > 0$, $\widetilde{Q}_{1i} = \widetilde{Q}_{1i}^T \geq 0$, $\widetilde{Q}_{2i} = \widetilde{Q}_{2i}^T \geq 0$, $\widetilde{Q}_{3i} = \widetilde{Q}_{3i}^T \geq 0$, $\widetilde{Q}_{4i} = \widetilde{Q}_{4i}^T \geq 0$, $(i \in S)$, $\widetilde{Z} = \widetilde{Z}^T > 0$,*

$$\widetilde{X} = \begin{bmatrix}
\widetilde{X}_{11} & \widetilde{X}_{12} & \widetilde{X}_{13} & \widetilde{X}_{14} & \widetilde{X}_{15}\\
* & \widetilde{X}_{22} & \widetilde{X}_{23} & \widetilde{X}_{24} & \widetilde{X}_{25}\\
* & * & \widetilde{X}_{33} & \widetilde{X}_{34} & \widetilde{X}_{35}\\
* & * & * & \widetilde{X}_{44} & \widetilde{X}_{45}\\
* & * & * & * & \widetilde{X}_{55}
\end{bmatrix} \geq 0, \tag{38}$$

$\widetilde{K}_i, \widetilde{K}_{di}$ and free-weighting matrices $\widetilde{N}_{1i}, \widetilde{N}_{2i}, \widetilde{N}_{3i}, \widetilde{N}_{4i}, \widetilde{N}_{5i}$, $(i \in S)$ with appropriate dimensions, such that the following LMIs hold:

$$\widetilde{\Phi} = \begin{bmatrix}
\widetilde{\Phi}_{11} & \widetilde{\Phi}_{12} & \widetilde{\Phi}_{13} & \widetilde{\Phi}_{14} & \widetilde{\Phi}_{15} & \widetilde{\Phi}_{16}\\
* & \widetilde{\Phi}_{22} & \widetilde{\Phi}_{23} & \widetilde{\Phi}_{24} & \widetilde{\Phi}_{25} & 0\\
* & * & \widetilde{\Phi}_{33} & \widetilde{\Phi}_{34} & \widetilde{\Phi}_{35} & \widetilde{\Phi}_{36}\\
* & * & * & \widetilde{\Phi}_{44} & \widetilde{\Phi}_{45} & 0\\
* & * & * & * & \widetilde{\Phi}_{55} & 0\\
* & * & * & * & * & \widetilde{\Phi}_{66}
\end{bmatrix} < 0, \tag{39}$$

$$\widetilde{\Psi} = \begin{bmatrix}
\widetilde{X}_{11} & \widetilde{X}_{12} & \widetilde{X}_{13} & \widetilde{X}_{14} & \widetilde{X}_{15} & \widetilde{N}_{1i}\\
* & \widetilde{X}_{22} & \widetilde{X}_{23} & \widetilde{X}_{24} & \widetilde{X}_{25} & \widetilde{N}_{2i}\\
* & * & \widetilde{X}_{33} & \widetilde{X}_{34} & \widetilde{X}_{35} & \widetilde{N}_{3i}\\
* & * & * & \widetilde{X}_{44} & \widetilde{X}_{45} & \widetilde{N}_{4i}\\
* & * & * & * & \widetilde{X}_{55} & \widetilde{N}_{5i}\\
* & * & * & * & * & \widetilde{Z}
\end{bmatrix} \geq 0, \tag{40}$$

$$\begin{aligned}
&\sum_{j=1}^{N} \pi_{ij} \widetilde{Q}_{1j} \leq 0,\\
&\sum_{j=1}^{N} \pi_{ij} \widetilde{Q}_{2j} \leq 0,\\
&\sum_{j=1}^{N} \pi_{ij} \widetilde{Q}_{3j} \leq 0,\\
&\sum_{j=1}^{N} \pi_{ij} \widetilde{Q}_{4j} \leq 0,
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
\widetilde{\Phi}_{11} &= A_i\widetilde{P}_i + \widetilde{P}_i A_i^T + B_i\widetilde{K}_i + \widetilde{K}_i^T B_i^T + \widetilde{Q}_{1i}\\
&\quad + \widetilde{N}_{1i} + \widetilde{N}_{1i}^T + \overline{\tau}\widetilde{X}_{11},\\
\widetilde{\Phi}_{12} &= \widetilde{N}_{2i}^T + \overline{\tau}\widetilde{X}_{12},\\
\widetilde{\Phi}_{13} &= A_{di}\widetilde{P}_i + B_i\widetilde{K}_{di} - \widetilde{N}_{1i} + \widetilde{N}_{3i}^T + \overline{\tau}\widetilde{X}_{13},\\
\widetilde{\Phi}_{14} &= \widetilde{N}_{4i}^T + \overline{\tau}\widetilde{X}_{14},\\
\widetilde{\Phi}_{15} &= \widetilde{N}_{5i}^T + \overline{\tau}\widetilde{X}_{15},\\
\widetilde{\Phi}_{16} &= \widetilde{P}A_i^T + \widetilde{K}_i^T B_i^T,\\
\widetilde{\Phi}_{22} &= -\left(1-\frac{\mu}{2}\right)\widetilde{Q}_{1i} + \widetilde{Q}_{2i} + \overline{\tau}\widetilde{X}_{22},\\
\widetilde{\Phi}_{23} &= -\widetilde{N}_{2i} + \overline{\tau}\widetilde{X}_{23},\\
\widetilde{\Phi}_{24} &= \overline{\tau}\widetilde{X}_{24},\\
\widetilde{\Phi}_{25} &= \overline{\tau}\widetilde{X}_{25},\\
\widetilde{\Phi}_{33} &= -(1-\mu)\widetilde{Q}_{2i} + \widetilde{Q}_{3i} - \widetilde{N}_{3i} - \widetilde{N}_{3i}^T + \overline{\tau}\widetilde{X}_{33},\\
\widetilde{\Phi}_{34} &= -\widetilde{N}_{4i}^T + \overline{\tau}\widetilde{X}_{34},\\
\widetilde{\Phi}_{35} &= -\widetilde{N}_{5i}^T + \overline{\tau}\widetilde{X}_{35},\\
\widetilde{\Phi}_{36} &= \widetilde{P}A_{di}^T + \widetilde{K}_{di}^T B_i^T,\\
\widetilde{\Phi}_{44} &= -\left(1-\frac{\mu}{2}\right)\widetilde{Q}_{3i} + \widetilde{Q}_{4i} + \overline{\tau}\widetilde{X}_{44},\\
\widetilde{\Phi}_{45} &= \overline{\tau}\widetilde{X}_{45},
\end{aligned}$$

$$\widetilde{\Phi}_{55} = -\widetilde{Q}_{4i} + \overline{\tau}\widetilde{X}_{55},$$
$$\widetilde{\Phi}_{66} = \overline{\tau}^{-1}\left(\widetilde{Z} - 2\widetilde{P}_i\right). \tag{42}$$

In this case, if the conditions abovementioned are feasible, the matrices K_i, K_{di} in (36) are given by $K_i = \widetilde{K}_i\widetilde{P}_i^{-1}$ and $K_{di} = \widetilde{K}_{di}\widetilde{P}_i^{-1}$. That is, the state feedback control law (36) can be designed as

$$u_i = \widetilde{K}_i\widetilde{P}_i^{-1}x(t) + \widetilde{K}_{di}\widetilde{P}_i^{-1}x(t - \tau(t)). \tag{43}$$

Proof. By performing a congruence transformation $\mathrm{diag}\{P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}\}$ to both sides of (12), and defining matrices $\widetilde{X}_{mn} \triangleq P_i^{-1}X_{mn}P_i^{-1}$, $(m, n = 1, 2, 3, 4, 5, i \in S)$, we can obtain (38).

Similarly, by performing a congruence transformation to (14) by $\mathrm{diag}\{P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}\}$ and defining matrices $\widetilde{Z} \triangleq P_i^{-1}ZP_i^{-1}$, $\widetilde{N}_{ki} \triangleq P_i^{-1}N_{ki}P_i^{-1}$, $(k = 1, 2, 3, 4, 5, i \in S)$, we have (40).

By performing a congruence transformation to (13) with $\mathrm{diag}\{P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}, P_i^{-1}, Z^{-1}\}$ and defining matrices $\widetilde{P}_i \triangleq P_i^{-1}$, $\widetilde{Q}_{ki} \triangleq P_i^{-1}Q_{ki}P_i^{-1}$, $(k = 1, 2, 3, 4)$, $\widetilde{K}_i = K_iP_i^{-1}$, $\widetilde{K}_{di} = K_{di}P_i^{-1}$, we have

$$\widetilde{\Phi} = \begin{bmatrix} \widetilde{\Phi}_{11} & \widetilde{\Phi}_{12} & \widetilde{\Phi}_{13} & \widetilde{\Phi}_{14} & \widetilde{\Phi}_{15} & \left(A_i\widetilde{P}_i + B_i\widetilde{K}_i\right)^T \\ * & \widetilde{\Phi}_{22} & \widetilde{\Phi}_{23} & \widetilde{\Phi}_{24} & \widetilde{\Phi}_{25} & 0 \\ * & * & \widetilde{\Phi}_{33} & \widetilde{\Phi}_{34} & \widetilde{\Phi}_{35} & \left(A_{di}\widetilde{P}_i + B_i\widetilde{K}_{di}\right)^T \\ * & * & * & \widetilde{\Phi}_{44} & \widetilde{\Phi}_{45} & 0 \\ * & * & * & * & \widetilde{\Phi}_{55} & 0 \\ * & * & * & * & * & -\overline{\tau}^{-1}\widetilde{P}_i\widetilde{Z}^{-1}\widetilde{P}_i \end{bmatrix} < 0. \tag{44}$$

According to the definition of $\widetilde{P}_i$ and $\widetilde{Z}$, the following equation always holds:

$$\left(\widetilde{P}_i - \widetilde{Z}\right)^T\widetilde{Z}^{-1}\left(\widetilde{P}_i - \widetilde{Z}\right) \geq 0, \tag{45}$$

which implies that

$$-\widetilde{P}_i\widetilde{Z}^{-1}\widetilde{P}_i \leq \widetilde{Z} - 2\widetilde{P}_i, \tag{46}$$

This completes the proof. □

Remark 8. The stabilization controller design problem is solved in Theorem 7, which guarantees the addressed Markovian jump system to be stochastically stable. Sometimes, we would like to design a memory-less state feedback control law $u_i = K_ix(t)$; in this case, just set $K_{di} = 0$ in Theorem 7.

Remark 9. It is worth pointing out that our main results are based on the LMI conditions, which can be easily obtained using LMI Control Toolbox.

4. Numerical Examples

In this section, some examples are presented to demonstrate the effectiveness of our proposed theoretical results.

Table 1: Allowable upper bound of $\overline{\tau}$ for various μ.

μ	0	0.1	0.2	0.5	0.8	1
$\overline{\tau}$	1.3384	1.3224	1.3165	1.3775	1.5342	1.1651

Example 1. Consider a Markovian jump system in (1) with two modes and the following parameters [27]:

$$A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & -3 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -5 & 1 \\ 1 & 0.2 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.3 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}. \tag{47}$$

The generator matrix of the stochastic process Γ is

$$\Gamma = \begin{bmatrix} \pi_{11} & -\pi_{11} \\ -\pi_{22} & \pi_{22} \end{bmatrix}. \tag{48}$$

To compare our stability result in Theorem 4 with those in [27–30], we assume $\mu = 0$. When $\pi_{11} = -7$ and $\pi_{22} \geq -6$, the result of [28] cannot be applied to stability. When $\pi_{11} = -7$ and $\pi_{22} = -6$, based on the result of [29], the system is found to be delay-independent stable. If we increase π_{22}, for example, $\pi_{11} = -7$ and $\pi_{22} = -3$, the result of [29] cannot guarantee system stability. By the results in [27, 30], we can obtain a feasible solution with $\tau \leq \overline{\tau} = 0.404$ and $\tau \leq \overline{\tau} = 0.7316$, respectively. By Theorem 4 of this paper, we can obtain a feasible solution with $\tau \leq \overline{\tau} = 1.3384$, which is much larger than that of [27, 30]. We can see that our criterion gives a less conservative result than those in [27–30].

Example 2. Consider the Markovian jump system (1) as Example 1. For $\pi_{11} = -7$ and $\pi_{22} = -3$ and various μ, the results obtained by the proposed method in Theorem 4 for the admissible upper bound value of $\overline{\tau}$ for which the system (1) remains stochastic stable are shown in Table 1.

Example 3. Consider the Markovian jump system (1) as Example 1. Suppose $\pi_{11} = -7$, $\pi_{22} = -3$, $\overline{\tau} = 1.32$, and $\mu = 0.1$; according to Theorem 7, one can obtain a state feedback controller designed as

$$u_i = \begin{cases} \begin{bmatrix} -3.7084 \\ 0.9821 \end{bmatrix}x(t) + \begin{bmatrix} -0.3845 \\ -0.0491 \end{bmatrix}x(t - d(t)), & i = 1 \\ \begin{bmatrix} -1.1560 \\ -0.6338 \end{bmatrix}x(t) + \begin{bmatrix} -0.8394 \\ 0.5502 \end{bmatrix}x(t - d(t)), & i = 2. \end{cases} \tag{49}$$

The simulation results are illustrated in Figures 1–3 with the initial condition $\varphi(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T (t \in \begin{bmatrix} -\overline{\tau} & 0 \end{bmatrix})$. Among them, Figure 1 shows the Markovian jumping signal. The control input signal $u(t)$ is given in Figure 2, while Figure 3 plots the state responses of system (1) under the presented control law $u(t)$. From the simulation, it is shown that the designed control scheme is feasible and effective since the resulting closed-loop system is stable.

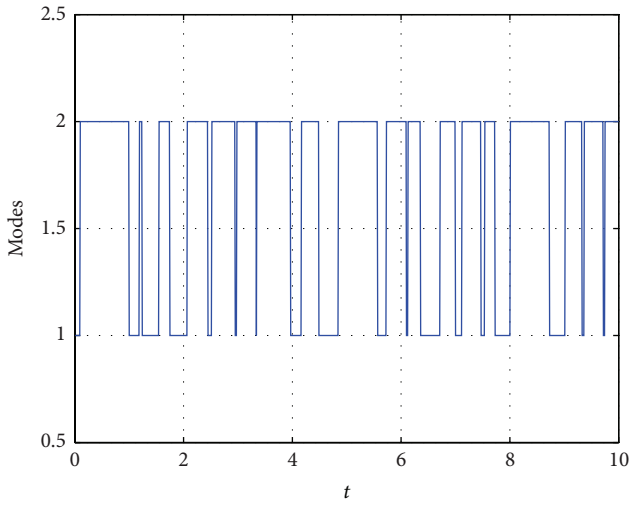

Figure 1: Markovian jumping signal.

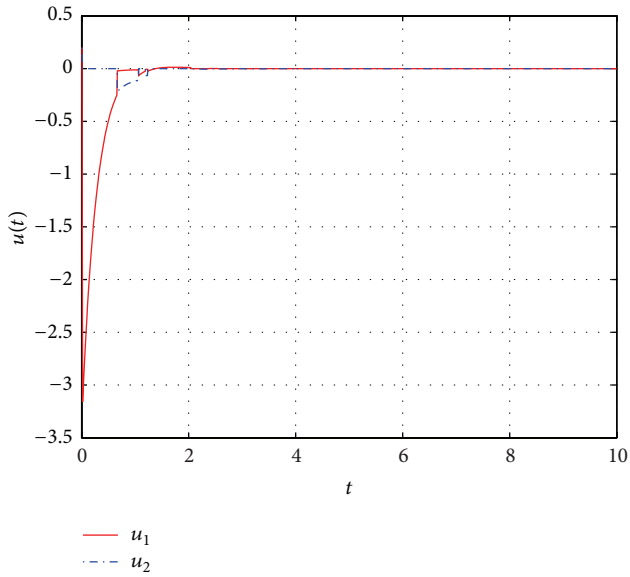

Figure 2: Control input.

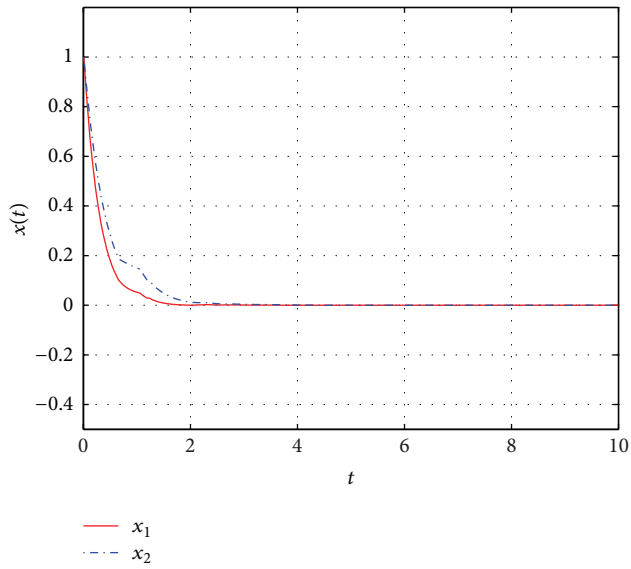

Figure 3: States of the closed-loop system.

5. Conclusion

In this paper, new delay-dependent conditions for stochastic stability and stabilization of a class of Markovian jump systems with time delay are presented. By the idea of delay fractioning approach and introducing some zero equations, a sufficient condition is given to guarantee the stochastic stability of the Markovian jump systems, which is less conservative than most existing results. Furthermore, based on the new criterion, an approach has been shown to design a state feedback controller, which guarantees the resulting closed-loop system to be stochastically stable. Finally, numerical examples have been provided to demonstrate the effectiveness of the proposed methods in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Experimental Evaluation of FSM-Based Testing Cost for Time-Delay Systems

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With time-delay systems arising, time-delay system testing has attracted much attention. Additionally, evaluating the cost and effectiveness is required to make a good test strategy in practice. In this paper, we take time-delay and other five factors (state number, input number, output number, completeness degree, and accessibility degree) into account and present a timer embedded FSM (TEFSM) model to design a comparative strategy for assessing the coverage criteria and test suites generation methods for time-delay systems. We explore the impact on the average length of test suites, in which the test suites generation methods, coverage criteria, and TEFSM model parameters are involved.

1. Introduction

In real world, time-delay exists in various engineering systems such as chemical processes and long transmission lines in pneumatic systems; the study of time-delay systems has gained considerable attention over the past years [1, 2].

The attendant increase in time-delay system complexity has made the timely development of reliable systems extremely challenging. To ensure the quality and reliability of time-delay systems, model based testing technique has been used widely. El-Fakih et al. present a method for deriving test suites with the guaranteed fault coverage for deterministic possibly partial timed finite state machines (TFSMs) [3]. Abramovici and Stroud present a delay-fault testing approach for Field Programmable Gate Arrays based on built-in self-test (BIST) [4]. And many conformance test derivation methods are based on a specification given in the form of a finite state machine (FSM), such as W [3, 5], partial W (W_p) [6, 7], HIS [8], and H [9] test derivation methods.

However, the project schedule is always tight and it is impossible to test a system completely. Thus, the efficiency of testing method becomes a key factor for ensuring system reliability. There are some researchers studying the evaluation of the effectiveness of system testing. Briand [10] presents a critical analysis of empirical research in software testing and provides a structured overview and discussion of the validity issues most commonly encountered while performing empirical studies of testing techniques. It gives us some guides to do experiments and analyze the validity of our research at last. Besides, there are some researchers exploring methods in the field of empirical research. For example, the literature [11] based on state charts proposes a precise simulation and analysis procedure and investigates the cost effectiveness of four adequacy criteria: all transitions (AT), all transition pairs (ATP), and all the paths in the transition tree (TT).

However, they do not have a further analysis on the effect of the parameters of FSMs to the derivation methods. Testers cannot determine derivation methods to use. Moreover, there are no papers to research these in combination (generation methods to coverage criteria, parameters to generation methods/coverage criteria).

Based on the previous discussion, aiming at decreasing the cost of time-delay systems, we present an experimental evaluation method for obtaining minimum test suite for time-delay systems. This method incorporates recent advances in THE literature [10–12]. We take time-delay and other five factors (state number, input number, output number, completeness degree, AND accessibility degree) into account and present a timer embedded FSM (TEFSM) model firstly. Moreover, by taking time-delay as an element of input, the TEFSM can be converted as normal FSM model, and

this makes the evaluation task much easier. Furthermore, the goal of our investigation is focused on minimizing the length of test suite since the minor the test sequence is, the less the system cost is. We present a novel method to evaluate the efficiency of FSM-based testing method and explore the situation that a test sequence generation method may have relatively good effect for a particular or some coverage criteria and how the parameters of a FSM model affect the FSM test suite. Namely, we investigate the relationship between generation methods, coverage criteria, FSM's parameters, and testing cost.

The rest of the paper is organized as follows. Section 2 describes some fundamental concepts such as FSM, TEFSM, and some classic coverage criteria of FSM-based systems. Section 3 presents the research method to covert TEFSM as FSM and decrease the size of test sequence for saving test cost. Section 4 presents experiments and analyzes the results. Finally, this paper makes a conclusion and presents directions for the further research.

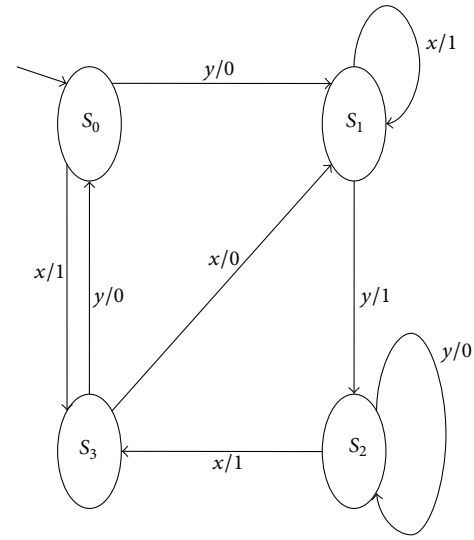


Figure 1: FSM.

2. Related Concepts Definition

In this section, some basic concepts are described which will be used in the proposed approach, and TEFSM model is especially described.

2.1. FSM Model. Formally, an FSM is a system $M = (X, Y, S, s_0, \delta, \lambda)$ where X is a finite input character set, Y is a finite output character set, S is a finite state set, $s_0 \in S$ is an initial state, is a transition function, and is a output controlled function [3].

If each state of one FSM respondes to all the inputs, tis is a complete FSM (CFSM); otherwise, it is a partial FSM. That is, if for all $s_i \in S$, $x \in X$, $\exists s_j \in S$, $\delta(s_i, x) = s_j$, it is CFSM.

If one state of a FSM reaches different states for the same input, this is a nondeterministic FSM; otherwise, it is a determined FSM. That is, if $\exists s_i, s_j \in S$, $x \in X$, $i \neq j$, $\delta(s_i, x) = \delta(s_j, x)$, it is nondeterministic FSM.

If the states of FSM are pair-wise distinguishable, this is a reduced FSM. Otherwise, it is nonreduced. That is, if $\exists s_i, s_j \in S$, for all $\gamma \in T$, $i \neq j$, $\lambda(s_i, \gamma) = \lambda(s_j, \gamma)$, where T indicates a set of inputs sequences, it is nonreduced FSM.

2.2. TEFSM. TEFSM model is an extension of FSM; it is used for dealing with time-delay system. In this model, a timer is embedded as it is called timer embedded FSM model. Timer means delay time of a state, time-delay is considered as a factor during state transition, timers and entry actions are associated with states, conditions and actions are associated with transitions, and time is implicitly associated with inputs and outputs [12]. The state transition occurs automatically after timeout unless it receives further inputs before timeout. Given a TEFSM as $M_t = (X_t, Y, S, S_0, \delta, \lambda)$, in which X_t indicates input that contains a time-delay, a pair of (x, t), $x \in X$ and t is a nonnegative rational and is a time-delay of state transaction, and input x is applied after time t.

2.3. Coverage Criteria. Coverage criteria rule the sufficient conditions for deciding when to end the testing process; that is, one test suite must satisfy one kind of testing requirements [13]. If K indicates one coverage criterion and T indicates a test suite, $\mathrm{TR}_K(M)$ is the testing requirements set of a given M for criterion K, and $\mathrm{TS}_K(M, T)$ is the requirement sets satisfied by a test suite T. Obviously, it is the subset of $\mathrm{TR}_K(M)$. $C_K(M, T)$ indicates the coverage rate of T; that is $C_K(M, T) = |\mathrm{TS}_K(M, T)|/|\mathrm{TR}_K(M)|$. If $\mathrm{TS}_K(M, T) = \mathrm{TR}_K(M)$ or $C_K(M, T) = 1$, T is a test suite adequate for the criterion K of M.

Figure 1 is an example of FSM model which has four states S_0, S_1, S_2, S_3, respectively, and eight transactions; it will be used for the following coverage criteria explained in this section.

(1) State Coverage (SC) Criterion. Cover all states of M; hence, $\mathrm{TR}_{\mathrm{SC}}(M) = S$, where S is the state set of M and $\mathrm{TS}_{\mathrm{SC}}(M, T) = \{s \mid s \in S\}$. For the FSM in Figure 1, there are four states S_0, S_1, S_2, and S_3, and the test suite $\{x, yy\}$ is SC-adequate for all the four states that could be covered with this test suite. Note that the initial state is reachable with the empty transfer sequence, while the prefix y of the test yy is a transfer sequence to state S_1.

(2) Transition Coverage (TC) Criterion. A transition is a state change triggered by an input event. Such as in Figure 1, there are eight transitions between the four states [11]. To cover all transitions of M, conditions $\mathrm{TR}_{\mathrm{TC}}(M) = D$ and $\mathrm{TS}_{\mathrm{TC}}(M, T) = \{(s, x) \in D \mid \exists \pi \in T, ax \leq \pi, \delta(S_0, \alpha) = s\}$ need to be satisfied.

The test suite $\{yx, yyy, yyx, xy, xx\}$ is TC-adequate for Figure 1 because this test suite could cover all the 8 transitions.

(3) Initialization Fault (IF) Coverage Criterion. Avoid confusing other states with the initial state, this is for checking whether the behaviors of other states are same as the initial

state, denotes as $\mathrm{TR}_{\mathrm{IF}}(M) = \{s \in S \mid s \neq S_0\}$. There might be no states that can be distinguished from the initial state [11]. However, for a minimum M, there are states which are different from the initial state. We can express $\mathrm{TS}_{\mathrm{IF}}(M, T)$ formally as follows:

$$\mathrm{TS}_{\mathrm{IF}}(M,T) = \{s \in S \mid s \neq S_0,\ \exists \chi \in T,\ \beta\gamma \leq \chi, \delta(S_0,\beta) = s,\ \lambda(S_0,\gamma) \neq \lambda(s,\gamma)\}. \quad (1)$$

It indicates that there are some sequences in T to distinguish the initial state from other states.

For the TEFSM in Figure 1, the test suite $\{yyy, yyyy, xx\}$ is IF-adequate. We know that the input sequence y is a transfer sequence to state and is followed by y, which distinguishes state s_1 from the initial state. Similarly, the sequences $yyyy$ and xx also satisfy the requirements related to state S_3 and S_4.

(4) Transition Fault (TF) Coverage Criterion. Detect whether the output or the reachable state of one transition is correct to certain input and whether the ending state could be distinguished from other states [14]. We can express $\mathrm{TR}_{\mathrm{IF}}(M)$ formally in mathematics as follows:

$$\mathrm{TR}_{\mathrm{IF}}(M) = \left\{\left(s, x, s'\right) \in D \times S \delta(s,x) \neq s'\right\}. \quad (2)$$

$\mathrm{TS}_{\mathrm{IF}}(M, T)$ defines whether there are some sequences in T that could distinguish the ending states of all transitions of M from other states. We can express it formally as follows:

$$\mathrm{TS}_{\mathrm{IF}}(M,T) = \left\{\left(s,x,s'\right) \in D \times S \mid \delta(s,x) \neq s', \exists \pi, \chi \in T, \alpha x \gamma \leq \pi, \beta\gamma \leq \chi, \delta(S_0,\alpha) = s', \lambda(\delta(S_0,\alpha x),\gamma) \neq \lambda\left(s',\gamma\right)\right\}. \quad (3)$$

For the example TEFSM in Figure 1, the test suite $\{xxy, yyy, xxx\}$ is TF-adequate. Consider, for instance, the transition (S_3, x) whose tail state is S_1. Test xx covers this transition. States S_1 and S_2 are distinguished by y, which follows xx and yy; states S_1 and S_3 are distinguished by x, which follows xx and x; and states S_1 and S_0 are distinguished by y, which follows xx and empty sequence. Thus, all requirements related to transition (S_3, x) are satisfied. One can check the other requirements that are also satisfied.

2.4. Test Sequences Generation Methods. There are many methods reviewed in Section 5 of the literature [15]. This part describes some test sequences generation methods.

(1) W-Method. It uses a characterization set W to distinguish each state of the specification FSM [5]. For the FSM M of Figure 1, $W = \{x, y, yy\}$ and M has the state cover set $Q = \{q_0, q_1, q_2, q_3\} = \{\varepsilon, y, x, yy\}$ (ε means empty sequence), after removing those sequences that are proper prefixes of other sequences; the test suite is

$$\mathrm{TS} = Q\{x,y\}W = \{xxx, xxyy, xyx, xyyy, yxx, yxyy, yyxx, yyxyyy, yyyy\}. \quad (4)$$

(2) W_p-Method. It serves as an improvement of W-method [7]. In the testing phase of transition, while it does not use the characterization set W, it uses subset instead. According to the ending state of one transition, choose the corresponding state identifiers set W_i, which uses fewer test sequences than W-method. For the FSM in Figure 1, $W_0 = \{yy\}$, $W_1 = \{y\}$, $W_2 = \{x, yy\}$, and $W_3 = \{x\}$. In the first phase, test sequences that check the state are $Q \cdot W$.

(3) HIS-Method. It utilizes a separating family from the identifiers of each state to harmonize these identifiers [8]. For the example of Figure 1, we can obtain the separating family set $F = \{H_0, H_1, H_2, H_3\}$, where $H_0 = \{x, yy\}$, $H_1 = \{x, y\}$, $H_2 = \{x, yy\}$, and $H_3 = \{x\}$. So, in the first phase, we get $q_j H_j$; in the second phase, we compute $q_0 x H_3 + q_0 y H_1 + q_1 x H_1 + q_1 y H_2 + q_2 x H_3 + q_2 y H_2 + q_3 x H_2 + q_3 y H_0$. The test suite is

$$\{xxx, xxy, xyx, xyyy, yxx, yxy, yyxx, yyyx, yyyyy\}. \quad (5)$$

Besides, there are methods such as UIO presented in literature [8], DS, and H proposed in the literature [16]. And also some researchers think about the derivation methods of nondeterministic FSM.

3. Research Method

3.1. Conversion between TEFSM and FSM. According to the definition of TEFSM presented in Section 2.2, TEFSM diagram could be described as in Table 1. A number of symbols are used in order to increase readability: "?" symbol for input; "!" for output; "%" for probability; "~" for condition and probability; and "Δ" for time-delay [12].

Figure 2 is an example of TEFSM model, which has three states named S_0 (initial state), S_1, and S_2.

There are transactions between each two states with different transaction conditions. Δt_1 and Δt_2 indicate the minimum delay time from state S_0 to state S_1 with transition conditions a and b, respectively. Set $a' = a \cup (t > \Delta t_1)$ and $b' = b \cup (t > \Delta t_2)$; then, the TEFSM in Figure 2 can be converted as Figure 3, which is now a normal FSM model.

In this way, theory of normal FSM could be applied for TEFSM model reasonably.

3.2. Testing Cost. Testing cost is a main aspect to evaluate testing technique performance. Briand presents that the testing cost involves two dimensions at least [10]: human effort and machine CPU time. In addition, Namin and Andrews describe the relationship among three properties of test suites: size, structural coverage, and fault location effectiveness [17].

We consider a technique more effective if it could generate minimum test suite and be highly active; hence, we make

Table 1: Conventions for constructing a TEFSM diagram.

Primitives	Notations
External transition edge	Condition; %(probability) ?(input); T-action
Internal transition edge	Condition; %(probability); T-action
State node	Initial state; State Δt E-action; Final state

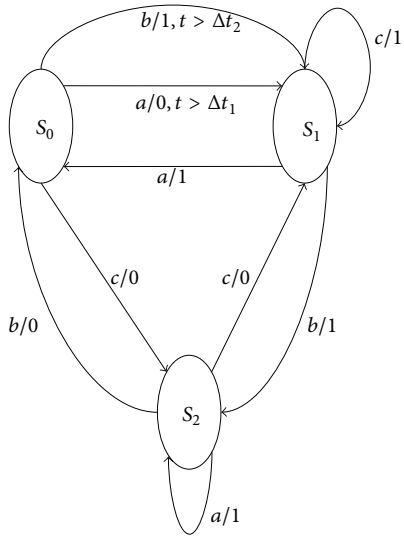


Figure 2: One example TEFSM.

use of the length of test suites as a measure of the cost. The length of test suites means the total length of test cases within the test suite. In the literature [4], it does not consider the accessibility degree and only changes one factor to observe the experiments each time. Here, we set five parameters of FSM model as the effect factors: the number of states, the number of inputs, the number of outputs, the completeness degree, and the accessibility.

3.3. Evaluation Method. This paper mainly focus on three test derivation methods (W, W_P, HIS). W-method, as the most classic method, should be taken into consideration. W_P-method improves the W-method for reducing the number of test suites and thus should be considered. HIS-method is the method used in the literature [4, 14], so it is necessary to consider this method here for comparison.

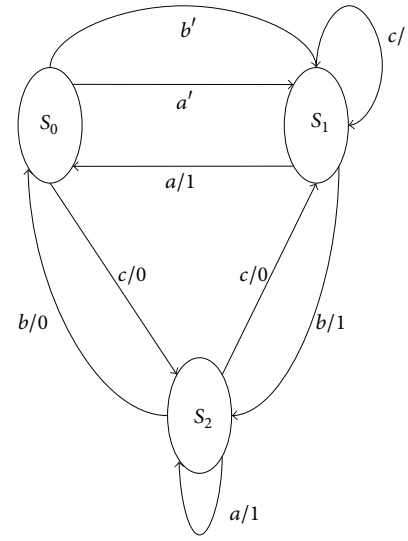


Figure 3: FSM converted from TEFSM of Figure 2.

Moreover, we use the coverage criteria as those considered in the literature [13]: state coverage (SC), transition coverage (TC), initialization fault (IF), and transition fault (TF) coverage criteria. When an FSM is the specification for testing, tests covering an FSM specification target one or several elements such as inputs, outputs, states, and fragments of its transition graph. Paths are typical fragments of the transition graph considered for coverage; therefore, path coverage has to be taken into account. We choose the traditional TC criterion. The latter two criteria (IF, TF) are proposed to direct at the fault models. IF criterion finds that the only possible faults in FSM implementation are related to a wrong initial state of a specification FSM, while TF criterion states that implementation faults occur in transitions. The four criteria could be representative of criteria for the FSM model.

For evaluation method, there are three ways.

(1) Similar to the method proposed in the literature [1], generate a test set which is adequate for all coverage criteria as a test pool first; then, minimize it to get the minimum test set which is adequate for one specific coverage criteria.

(2) Generate the test set which satisfies each kind of coverage criteria by generation method directly.

(3) According to the relation among the coverage criteria shown in Table 3 of the literature [4], the test suite covering TC/IF/TF criteria almost covers the test requirements of SC (the coverage rate and the standard deviation in the first row are 1.000/0.000, 0.970/0.064 and 0.994/0.034). Since there may be some states that are not distinguishable from any other states, the latter two (IF/TF) do not cover all the states. However, we can generate the approximate minimum test set, which is adequate for the SC criterion, and then add some test sequences to the minimum test suite to satisfy other criteria. Contrast to (1), it resembles

Inputs:
FSM
T: A set of initial test set
K: A coverage criterion
Output:
T': The minimum test suite satisfy the criterion K
Process:
(1) Set $T' = \emptyset$;
(2) Initial $\mathrm{TR}_K(M)$; //test requirement set satisfy
//criterion K.
(3) Initial test coverage info TS by calculating covering
weight $(w(t))/|\mathrm{TR}_K(M)|$ of T.
(4) While $\mathrm{TS}_K(M, T') \neq \mathrm{TS}_K(M, T)$ do
(5) If K in(SC||TC||IF) and NOT the 1st time
(6) Re-adjust covering weights of the remaining
sequences;
(7) Select the minimum weight from the weights of all sequences in T;
(8) For each t in T
(9) If $(\mathrm{TS}_K(M, T' + \{t\}))/(\mathrm{TS}_K(M, T')) \neq \emptyset$
$T' = T' + \{t\}$
(10) End of While
(11) return T'

Algorithm 1: MinTSByWeight.

an incremental method. Because we can obtain the test suite adequate for some coverage criteria when utilizing the generation methods, so it is better to use this incremental method if the coverage criterion is very strict and one generation method is not able to generate the test set adequate for this coverage criterion. We use the first evaluation method in this paper, which we are coming to illustrate in Section 4 in detail.

After generating the final test set, we analyze the experimental data using the orthogonal testing method. The orthogonal test is useful when many factors influence one thing. Because at that situation the inputs would be many values, it would be effective through choosing a suitable orthogonal table that has advantage of equilibrium and dispersion. In this study, we choose the orthogonal table containing four levels and five factors which actually can be written as $L_{16}(4^5)$, which is shown in Section 4.2 in detail. The whole evaluation process is a general process for assessment as well. We can choose a way to solve the problem according to the situation. Figure 4 describes the process in detail.

How can we select a minimum test suite T' from T to satisfy the test coverage criteria? The literature [13] applies greedy algorithm to deal with this problem. Test suite selection for SC/TC criteria is still a weighted set-cover problem, while the selection for IF/TF criteria is defined as set-cover with pairs. However, for the definition of IF criterion, we can consider the selection of the minimum test suite for IF criterion as the weight set-cover problem as well.

We also define the cost $w(\alpha)$ of a sequence α in test suite T as $|\alpha| + 1$, that is, the length of the sequence plus the reset symbol used to bring the FSM back to the initial state.

In the literature [13], every time pick up the min ratio between the cost and coverage increments induced by t in T', that is, $(\omega(T' \cup \{t\}) - \omega(T'))/|\mathrm{TS}_{\mathrm{SC}}(M, T' \cup \{t\})/\mathrm{TS}_{\mathrm{SC}}(M, T')|$. However, it may generate redundant test sequences. For example, in the case of SC criterion, the FSM model M shown in Figure 3 has three states S_0, S_1 and S_2. The initial test set is: ba, bb. The first sequence "ba" could cover state S_0 and state S_1 with min ratio = $3/2 = 1.5$. So, we can add it to T', and then check the second sequence "bb", because the second input can make the system reach state S_2, which is not covered before, the ratio = $3/1 = 3 >$ min ratio, so it should be ignored in this round. For the second loop, add "bb" to it; as a result, there are "ba" and "bb." In fact, "bb" is enough to satisfy the SC criterion.

To deal with this problem, we present a novel algorithm MinTSByWeight (minimum test suite by weight), which determines test sequence by weight. The algorithm is mainly based on the theory that the minor test suite it has, the less it costs. We do not need to consider time-delay in this algorithm since we already presented a method to covert TEFSM as FSM by taking time-delay as an element of input.

As shown in Algorithm 1, there are 3 inputs: FSM model, initial test set T, and test coverage criteria K; and one output T', which is the minimum test sequence of test object M and satisfies the coverage criteria K. At step (3), the weight of each test sequence or each pair test sequence is calculated based on the formula $w(t)/|\mathrm{TR}_K(M)|$. In steps (4) to (7), select the minimum weight from the weights of all sequences or pairs and then add such sequence or pair "t" or "$t1, t2$" into the result set T' with steps (8)~(10). From the second round, the weights of the remaining sequences or pairs need to readjust in the initial test set by removing the cover points in t or $t1/t2$ and then go to the select min weight as before. Cycle

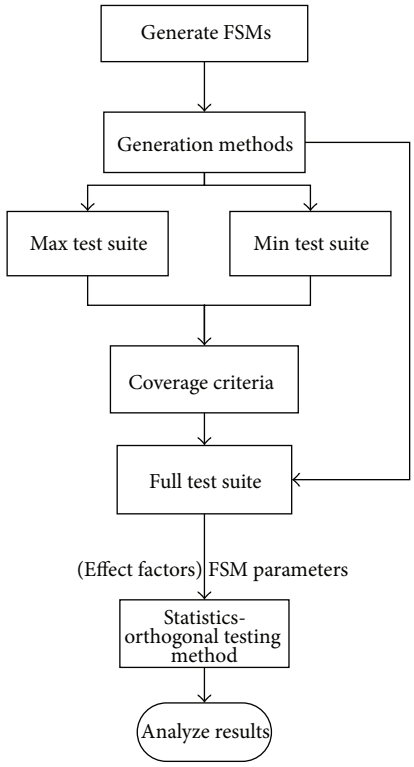


Figure 4: Assessment process.

this process until the selected sequences are adequate for the criterion or having checked all the sequences in the initial set T.

4. Experiments and Results Analysis

4.1. Coverage Criteria for Different Method

Premise. FSM is deterministic and completed finite state machine. When the generated FSM is nonreduced, the program does not proceed. In other words, although the automatic process of generating FSMs may produce nonreduced FSMs, this kind of model is not considered in the experiment. Therefore, we generate deterministic and completed models directly.

Coverage Criteria and Generation Method. This method includes average length of the test suite satisfying four coverage criteria (SC/TC/IF/TF) generated by W-method/W_P-method/HIS-method

Experiment Application Scenario. Considering that in practical applications, it is impossible to model the whole system, but some important parts, and the experiment contains eight experimental groups and sets the number of states: 3, 6, 8, 10, 12, 14, 16, and 20. Besides, we set the number of inputs to be 2, 5, and 7, respectively, and the number of outputs to be also 2, 5, and 7, respectively. Each group automatically generated 50 FSMs.

Experiment Result. The results of the three types under the three generation methods are shown in Tables 2, 3, and 4.

Figures 5(a)~5(e) show impact of kinds of methods to 4 criteria.

Result Analysis from Test Suite Length. Figures 5(a) and 5(c) show that the test suite length is, in low degree, affected by the generation methods. Almost all groups can obtain the sequence-suites with nearly the same length in average. Figure 5(b) shows that the gap between W_p-method and HIS-method under TC-adequacy is small, and W-method can generate a test set with shorter length. Figure 5(d) shows that the impact trend of these three methods on the TF-criterion is similar, but the average length of test suite with TF-adequacy generated by W-method and HIS-method is relatively longer. Figure 5(e) shows the difference of the initial test suite average length among the three generation methods. The W-method generates the longest test suite length, and the length gap of suite derived by HIS-method and W_p-method is insignificant, but W_p-method could generate less test suite because it does not need to harmonize the state identifiers. Moreover, W-method generates shorter TC-adequacy test suite and TF-adequacy test suite as well as shorter initial test suite.

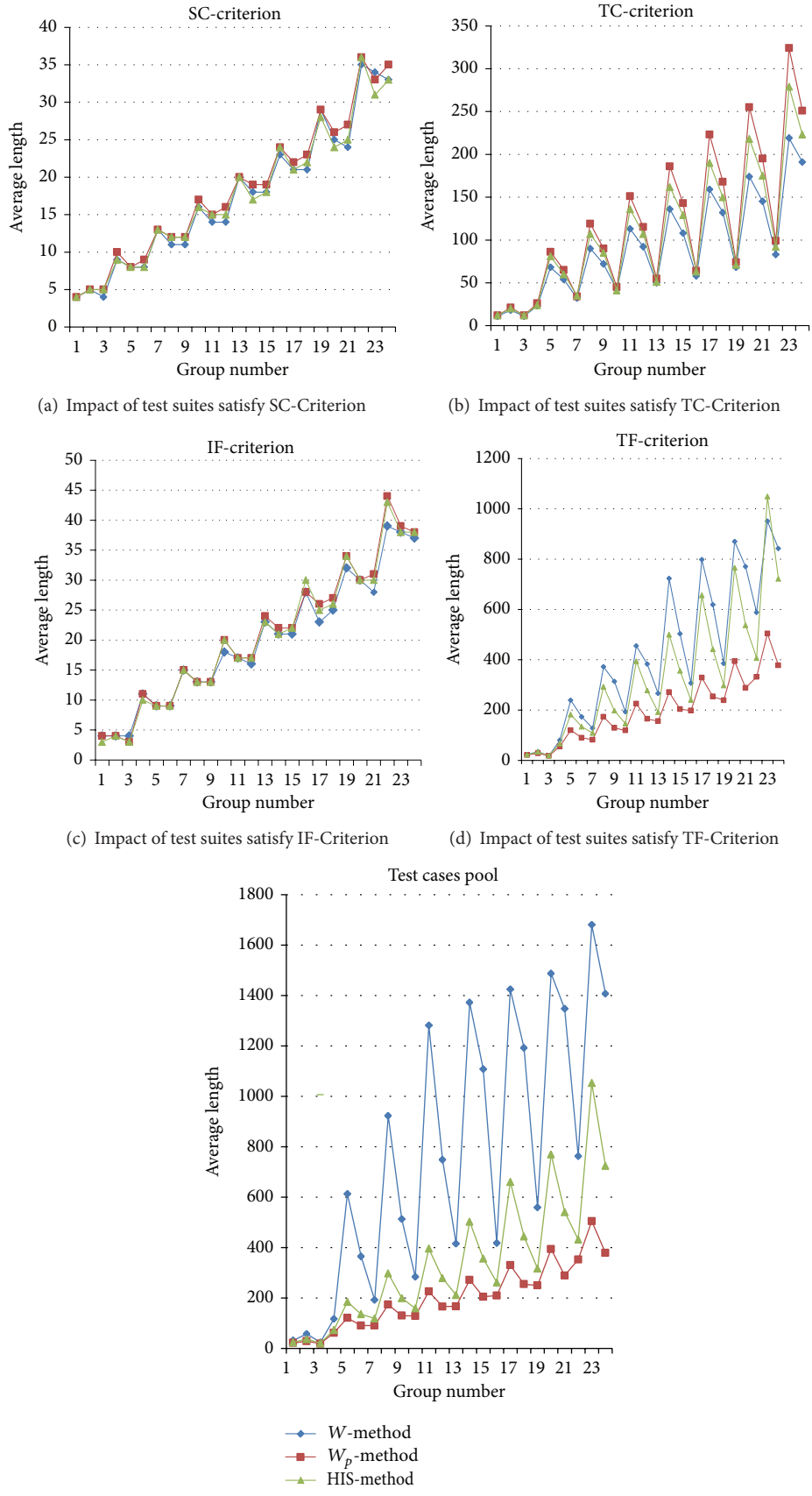
Result Analysis from Effort. When choosing the minimum set from a large pool suiting to each coverage criteria, the reduction rate is different. If the reduction rate is low, we could directly use the initial test suite instead of choosing the minimization set to improve the test efficiency, because the SC/IF criteria can be covered by fewer test sequences, while the TC/TF criteria need more test sequences to be covered. We only consider the latter two criteria under different derivation methods. The impact of each derivation method on the TC-criterion is shown in Figure 6(a), and the impact on TF-criterion is shown in Figure 6(b).

From Figure 6, we know that choosing the minimization test suite needs the largest effort. For the TC-criterion, Figures 5(b) and 5(e) show the reduction intensity to remove the useless sequences and Figure 6(a) reflects the high reduction rate, while there is little difference of the reduction rate between the other two generation methods (W_p-method and HIS-method) when choosing the minimum set to satisfy the TF-criterion. However, as for TC-criterion, when the number of inputs is small, the gap is small. But, when it is large, there is a clear gap. For example, if the groups of 7, 8 and 9 (the number of states, the number of inputs, and the number of 22 outputs) are 8/2/2, 8/4/7 and 8/5/5, respectively, in group NO. 7, the results selection rate is 0.62637 for W_p-method and 0.7059 for HIS-method, while in the NO. 8 the results selection rate is 0.31609 for W_p-method and 0.6409 for HIS-method. So, if the input number is large, W_p-method can be used to improve the quality of the test sequences.

4.2. The Impact of FSM Parameters on Coverage Criteria.

Consider five factors: the number of states, the number of inputs, the number of outputs, the completeness degree, and the accessibility degree. From the results above, for CFSMs, the minimization cost of W_p-method is the minimum. Hence, here, we choose W_p-method as generation method.

Using the orthogonal experimental method, we choose some representative samples from all level combinations for



(a) Impact of test suites satisfy SC-Criterion

(b) Impact of test suites satisfy TC-Criterion

(c) Impact of test suites satisfy IF-Criterion

(d) Impact of test suites satisfy TF-Criterion

(e) Impact of original test suites

Figure 5: Impact of kinds of methods to criteria.

Table 2: The experimental data of different coverage criteria for W-method.

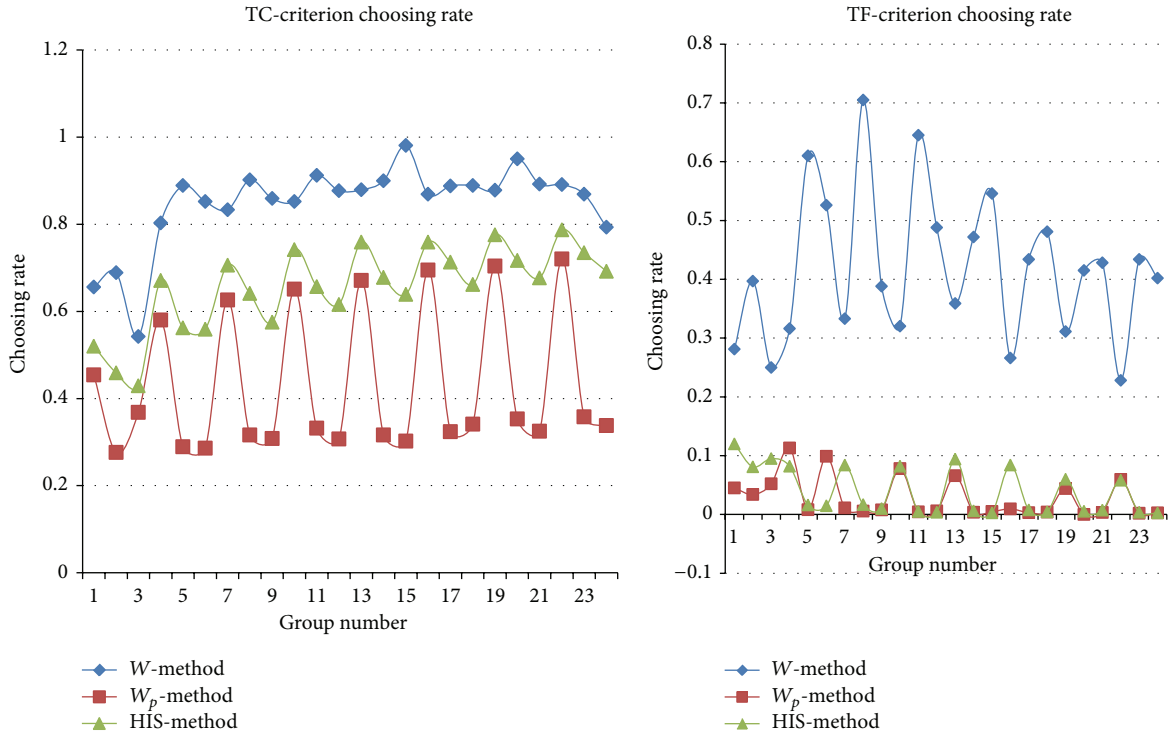
Group number	Number of states	Number of inputs	Number of outputs	Average length of initial test suites	W-method/average length of test suites satisfying each coverage criteria			
					SC	TC	IF	TF
1	3	2	2	32	4	11	4	23
2	3	3	5	58	5	18	4	35
3	3	2	7	24	4	11	4	18
4	6	2	2	117	9	23	11	80
5	6	5	5	613	8	68	9	239
6	6	4	7	365	8	54	9	173
7	8	2	2	192	13	32	15	128
8	8	5	5	923	11	90	13	372
9	8	4	7	513	11	72	13	314
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	20	4	7	1407	33	191	37	842

Table 3: The experimental data of different coverage criteria for W_p-method.

Group number	Number of states	Number of inputs	Number of outputs	Average length of initial test suites	W_p-method/average length of test suites satisfying each coverage criteria			
					SC	TC	IF	TF
1	3	2	2	22	4	12	4	21
2	3	3	5	29	5	21	4	28
3	3	2	7	19	5	12	3	18
4	6	2	2	62	10	26	11	55
5	6	5	5	121	8	86	9	120
6	6	4	7	91	9	65	9	90
7	8	2	2	91	13	34	15	82
8	8	5	5	174	12	119	13	173
9	8	4	7	130	12	90	13	129
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	20	4	7	379	35	251	38	378

Table 4: The experimental data of different coverage criteria for HIS-method.

Group number	Number of states	Number of inputs	Number of outputs	Average length of initial test suites	HIS-method/average length of test suites satisfying each coverage criteria			
					SC	TC	IF	TF
1	3	2	2	25	4	12	3	22
2	3	3	5	37	5	20	4	34
3	3	2	7	21	5	12	3	19
4	6	2	2	73	9	24	10	67
5	6	5	5	185	8	81	9	182
6	6	4	7	136	8	60	9	134
7	8	2	2	119	13	35	15	109
8	8	5	5	298	12	107	13	293
9	8	4	7	200	12	85	13	198
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	20	4	7	724	33	223	38	722



(a) Efforts of minimizing test suite satisfied TC Criterion

(b) Efforts of minimizing test suite satisfied TF Criterion

Figure 6: Screening ratio.

experiments and then find out the optimal level combination by analyzing the results of these samples to comprehend experiments completely.

In this experiment, there are 4 levels and 5 factors. Each group generates 500 FSMs automatically. We control the level of the number of inputs and accessibility degree related to the number of states to make the FSMs more similar to the reality. As one level among these factors is on the same situation, though the level values among each factor may be different, the control would not lead to the inequality among the factors.

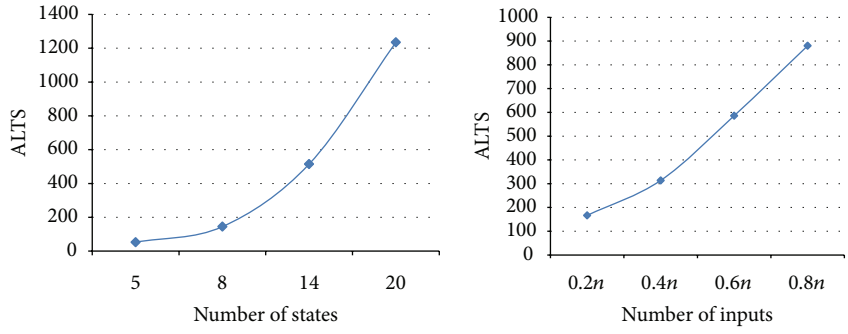
Table 5 shows the levels for each factor and Table 6 shows the design scheme (number in brackets indicates level number).

Table 5: Levels for each factor.

Factor	Levels	Level range
Number of states (n)	5, 8, 14, 20	3~20
Number of inputs (k)	$0.2n$, $0.4n$, $0.6n$, $0.8n$	$0.2n$~$0.8n$
Number of outputs (m)	2, 4, 7, 10	2~10
Completeness degree (d)	1.0	—
Accessibility degree (r)	$0.6 * (n-1)$, $0.8 * (n-1)$, $1.0 * (n-1)$	~$(n-1)$

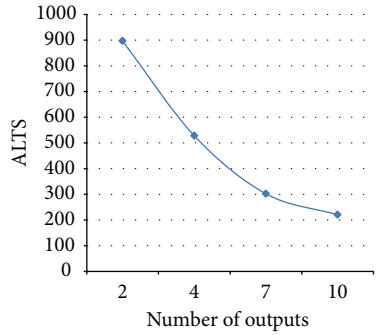
Experimental Results Analysis. Applying the range-analysis method, we can obtain some values such as $R1 = 1181.750$, $R2 = 713.500$, $R3 = 676$, $R4 = 374.000$, and $R5 = 402.5$; from these range values, we can easily know $R1 > R2 > R3 > R5 > R4$. Also, the factor of the number of states has the greatest impact on the test sequences length, followed by the number of inputs and the number of outputs (outputs will affect the verification of transitions). By the orthogonal design tool, we can get the effect curve shown in Figure 7(a). The effect curve describes the impact for SC-criterion as shown in Figure 7(b).

Figure 7 shows that the number of states for the average length has a great impact, while the other three parameters have little effect on the average length of the test suites which satisfy the SC-criterion. The number of inputs has a positive influence on the average length as Figure 7(a) shows. And the number of outputs increased results in the growth of the length of the test sequences, because the more the inputs, the shorter the length of characterization sequences and the fewer the number of such sequences. As each derivation method is closely related to the sequences reaching each state, the impact of accessibility degree emerges as a wave form but is not positively correlated. Figure 7(a) shows that when the accessibility degree (r) is 0.8 times as long as the maximum length (the number of states − 1), the test suite length nearly reaches the maximum. When it equals the 60 percent of the maximum length, the test suite length nearly reaches the minimum. After all, r represents the length of each sequence reaching one state.

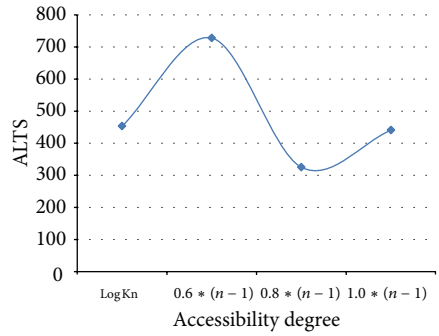


(a1) x-axis for number of states

(a2) x-axis for number of inputs

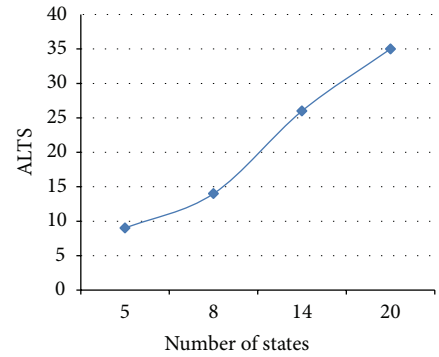


(a3) x-axis for number of outputs

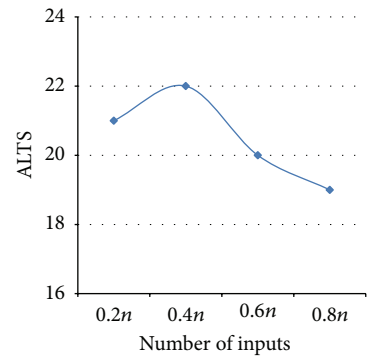


(a4) x-axis for accessibility degree

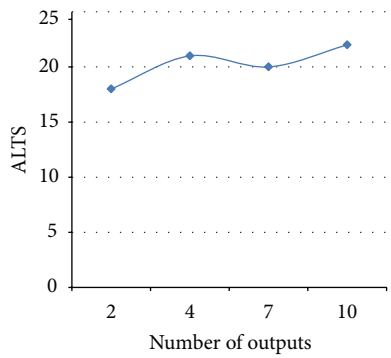
(a) Effect curve 1



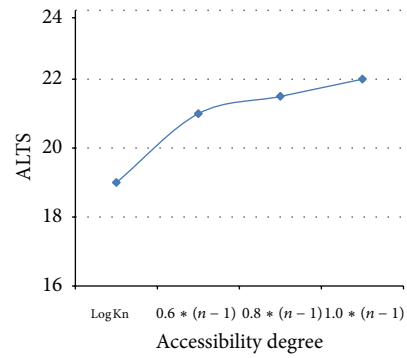
(b1) x-axis for number of states



(b2) x-axis for number of inputs



(b3) x-axis for number of outputs



(b4) x-axis for accessibility degree

(b) Effect curve 2

Figure 7: Effect curve.

Table 6: (a) Experimental design and data recording. (b) Parameter in different level.

(a)

Group number	Factors					Average length of test suites	
	Number of states (n)	Number of inputs (k)	Number of outputs (m)	Completeness degree (d)	Accessibility degree (r)	W_p test suites	SC
1	5 (1)	2 (1)	2 (1)	1.0	2 (1)	48	9
2	5 (1)	2 (2)	4 (2)	1.0	2 (2)	37	10
3	5 (1)	3 (3)	7 (3)	1.0	3 (3)	55	8
4	5 (1)	4 (4)	10 (4)	1.0	4 (4)	71	8
5	8 (2)	2 (1)	4 (2)	1.0	7 (4)	69	14
6	8 (2)	3 (2)	2 (1)	1.0	5 (3)	157	13
7	8 (2)	5 (3)	10 (4)	1.0	4 (2)	158	13
8	8 (2)	6 (4)	7 (3)	1.0	2 (1)	195	15
9	14 (3)	3 (1)	7 (3)	1.0	7 (2)	188	26
10	14 (3)	5 (2)	10 (4)	1.0	2 (1)	291	32
11	14 (3)	8 (3)	2 (1)	1.0	13 (4)	853	22
12	14 (3)	10 (4)	4 (2)	1.0	10 (3)	727	22
13	20 (4)	4 (1)	10 (4)	1.0	15 (3)	362	35
14	20 (4)	8 (2)	7 (3)	1.0	19 (4)	770	32
15	20 (4)	12 (3)	4 (2)	1.0	3 (1)	1278	38
16	20 (4)	16 (4)	2 (1)	1.0	11 (2)	2528	30

(b)

Level number	Number of states (n)	Number of inputs (k)	Number of outputs (m)	Completeness degree (d)	Accessibility degree (r)
1	52.750	166.750	896.500	425.750	453.00
2	144.750	313.750	527.750	361.750	727.750
3	514.750	586.00	302.000	735.750	325.250
4	1234.500	880.250	220.500	423.500	440.750
Worst	1181.750	713.500	676.000	374.000	402.500

Table 7: The results of variance analysis.

Source of variance	Sum of squared deviations	Free degree	F ratio
Number of states (n)	S_n = 3460936.188	3	9.898
Number of inputs (k)	S_k = 1188087.188	3	3.398
Number of outputs (m)	S_m = 1098390.688	3	3.141
Accessibility degree (r)	S_r = 349673.188	3	0.976
error	S_e = 349673.188	3	1.000
Fa	$F0.05(3, 3) = 9.28$		

Because the range analysis cannot distinguish the data changing caused by test conditions during the testing from that caused by data errors, without respect to whether the factor effect is significant, we cannot figure out the accuracy of analysis. Next, we perform the variance analysis of the impact of factors and errors, and let the column of accessibility degree be the column here that the experimental errors exist. Result is shown in Table 7.

We can see that the number of states (n) has a most significant impact on the length of the test suites.

To sum up the above results, we draw the following conclusions. (a) For SC-criterion, IF-criterion, the length of the test suites has little difference among the derivation methods. W_p-method can produce a relatively shorter length of the test suites for TC coverage criterion adequacy, but it needs to pick out a relatively small part of the whole initial test suite. To satisfy the TF-criterion, W_p-method has some advantages. Relatively speaking, it generates the smallest test set length, and when choosing the minimum test suite, the number of removed sequences is few, so when testing, if the

coverage criterion is TF-criterion, we can use W_p-method directly and do not need to select the minimal criterion adequacy test suite. Moreover, if we only need to satisfy TC-criteria, we can make a trade-off and consider the HIS-method. Because W_p-method is influenced by the number of inputs of FSM in the process of selecting the minimum test set for TC-criterion, and in case the number of test sequences derived by HIS-method is more than that derived by W_p-method, the rate of the sequences needing to be removed is relatively stable. (b) One of the FSM model parameters, the number of states (n), has a most significant impact on the length of the test suites. With the number of states increasing, the length increases much faster. In addition, the increase of the number of inputs will cause an increase in length, but the change rate is lower than that in the situation of increasing the number of states. And as the number of outputs increases, the length decreases. Because the increase of the outputs makes the length of the sequences distinct, the states are short and the number of distinguishable sequences is limited; and when the accessibility degree is up, the length changes as a wave form. So, when designing the complete FSM models, in the conditions of the same number of states, inputs, and outputs, if we try to make the accessibility degree be 0.6 times as long as the maximum length (the number of states − 1), the length of the generated test set will be fluctuating around the minimum.

4.3. Threats to Validity. The FSM considered in the experiments is generated randomly, and we only consider the complete and deterministic FSMs together with time-delay factor, so we could not make sure these would be close to the FSM specifications in the actual project. Therefore, the conclusions drawn from the random FSMs may not be well applied in the actual situation. In the future work, it needs further validation in real situation and to extend our study on the nondeterministic FSM and the partial FSM.

Measuring the cost of testing in such an experimental context is also a challenge. Briand presents that the testing cost involves at least two dimensions: human effort and machine CPU time, details of which are in Figure 3 of the literature [10]. It is rather clear that test suite size, regardless of how it is measured, is a very rough cost measure. Moreover, the literature [11] use the cumulative length of a test set to measure the cost for it's easy to count. Though many follow the simple way to measure the cost, it would be better to consider other factors.

We can assess the ability of the test criteria not only by the test suites average length but also through the malfunction versions generated by the use of mutation operators from a specification FSM together with time-delay. In this case, we compare their variation scores that measure the number of variants killed and use it to assess the fault detection capabilities of various test suites generation methods. However, it is only used for finding the faults. From here, it can be reflected that the former (the test suites average length) mainly evaluate the cost and the later (variation scores) mainly evaluate the effectiveness. Whether there are some valuable suggestions on the problem of finding the balance point of cost-effectiveness is not probed here.

5. Conclusion

The main contribution of this paper is that we propose a comparative strategy for assessing the coverage criteria and test cost for time-delay systems test. A TEFSM model is adopted for time-delay system's test suite generation. In the future, if we want to assess the coverage criteria or the test suites generation methods that are not applied in this paper, the coverage criteria or the test suites generation methods can be added directly to the comparative architecture. Besides, we take many factors of the FSM model (FSM state number, input number, output number, completeness degree, accessibility degree, and time-delay which are not considered in the literature [13]) into account. Through empirical study, we explore the test suites generation methods, coverage criteria, and FSM model parameters in terms of the influence of the average length of test suites. Based on these relationships, test engineers can make a good test strategy on the part of the conformance testing based on the TEFSM model. For example, when testing, if the coverage criterion is TF-criterion, we can use that directly for its relatively minimum cost. When designing the complete FSM models, in the conditions of the same number of states, inputs, and outputs, and if we try to make the accessibility degree be 0.6 times as long as the maximum length (the number of states − 1), the length of the generated test set will be fluctuating around the minimum. In industrial practice, one may design FSM models for critical components and automatically obtain test sequences by suitable methods. If, based on the requirements, we could judge in which situation the test cost is lower than others' cost, one could know which method to choose or how to change the models to be better. However, due to the random generation of the FSMs for experiments, we only consider the complete and deterministic FSM; it may vary with the actual situation. Next, we will extend our work on the nondeterministic FSM and the partial FSM. It requires further validation in more practical situation. The literature [13] explores the correlation between the structure testing and the model testing. We will benefit from the efforts, if we solve these problems, such as, how to extract the commons of the testing phases, understand the kinds of errors detected by each of the various stages, and use the various test adequacy criteria and test suites generation methods reasonably to test efficiently.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Application of Time-Delay-Dependent H_∞ Control in the Transmission Effect of Monetary Policy on Real Estate Market

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This paper uses a time-delay-dependent H_∞ control model to analyze the transmission effect of monetary policy on the real estate market. We establish a theoretical framework between monetary policy and real estate market based on the three channels, that is, interest rate channel, monetary supply channel, and credit channel. Then we construct a basic model using H_∞ control method. By analyzing the effect of the time-delay characteristics of monetary policy on the real estate market, we introduce time variables and propose a time-delay-dependent H_∞ control model. We test the robustness of the model using the Chinese data of the monetary policy and the real estate market and prove that this model can be well employed in the reality.

1. Introduction

The real estate market is increasingly becoming a mature monetary policy transmission channel. Many empirical studies find that the potential influence of monetary policies on assets prices, especially on real estate market. Ben and Bernanke find that real estate investments react greatly to the impact of short-term interest rate, while they react mildly to the impact of long-term interest rate [1]. Aoki et al. introduce BGG model [2] to study how the real estate market reacts in the transmission of monetary policy. The result is that, in some circumstances, monetary policy is influenced by financial accelerator effect that its impact on real estate price, real estate investment, and consumption is magnified.

The empirical studies in this field can be sorted into two types. The first type of studies employs econometrics models and the other type of studies uses DSGE model. For the first type of studies, many scholars would like to use VAR models. For example, MatteoIa and Raoul construct multiple VAR models and analyze the different reactions of real estate markets to the impact of the monetary policies of Finland, UK, Norway, and Germany [3]. Liu and He, Gao and Wang use Chinese data and make empirical studies using VAR model or SVAR model in the related field [4, 5]. For the other type of studies using DSGE model, the first study can be traced to Kyland and Prescott, who use DSGE to analyze economic cycles [6]. After the Lucas critique, DSGE model has been developed from RBC theory framework into the theory framework of new Keynes. For example, Liang introduces the DSGE model into the real estate market and finds that the monetary policy has a strong effect on the real estate market [7].

However, the research methods in the transmission effect of are supposed to develop. Take DSGE model as an example, Cogley and Nason, Su propose that DSGE model is not well implemented in China because the assumptions are not designed and related to Chinese economic situations [8, 9]. Meanwhile, the above models are most based on discrete system, which is lack of the continuous data.

Economic system can be viewed as a complex system which is full of uncertain factors. H_∞ control method can minimize the effect of disturbance on the state, which indicates that H_∞ control can be used to solve economic questions. H-infinite control is widely used in the field of engineering. Saifia et al., Krokavec and Filasová, Wu et al., and Wu and Zheng have carried some detailed research [10–13]. Meanwhile there are several articles using H-infinite control in economy, such as Li et al. [14]. And others like that of

Huang and Zhong employ H_∞ control in Nash equilibrium and find that this method has a good explanation ability in economic problems [15]. H_∞ control can manage risks and thus minimize the disturbing effect in an uncertain environment. Also, through the state feedback controller, the monetary policy can keep the house price stable as optimal as possible.

However, there are few articles introducing robust control into monetary policy. Always, the articles on monetary policy and real estate market focus on positive transmission mechanism, but few focus on reverse. Therefore, this paper introduces H_∞ control method and constructs a time-delay-dependent model to adjust the monetary policy on the real estate market, which is a new attempt. The advantage of this model is that it can deal with the continuous system in economic data; that is, the previous studies usually simplify economic problem as a discrete system; it is a new attempt to study the feedback of real state marker to design optimal monetary policy. Also being different from the usual measuring method and empirical method, this paper mainly adopts the method of simulation. Section 2 proposes the theoretical framework. Section 3 delineates the basic H_∞ control model of monetary policy on real estate market and Section 4 explains the time-delay-dependent feedback model. Section 5 puts forward a design example. Section 6 is conclusion.

2. Theoretical Framework

Monetary policy is supposed to manage real estate market, so the transmission effect of monetary policy is a crucial problem in economy. There have been some theories in this field. Many key issues of the theories agree that the following channels are well accepted ways for transmission of monetary policy, such as interest rate channel, credit channel, and monetary supply channel.

As for interest rate channel, the most famous theory is Keynes's interest rate channel theory. When the monetary policy maker sets a new interest rate, the demand, capital supply, and capital cost will change, and thus the supply and demand of the real estate will change. Take the loose monetary policy as an example, interest rates decrease and thus the cost of capital will decrease. So the cost of real estate will decrease and thus the real estate market will boom.

In the aspect of credit channel, under loose monetary policy, the credit amount grows with the increase of money supply, which leads to high currency liquidity in society. During boom years, land agents will buy lands at a high price that the house price in future will grow due to the increasing cost, which causes the rise of house price anticipation. Besides, when credit amount grows with the increase of money supply, it is easier for residents to get housing loans. This will expand house demands and raise house prices and, at the same time, encourage people's desire to earn more money. In a word, under loose monetary policy, real estate investment and house prices promote each other and grow together.

About the monetary supply channel, Friedman and Schwartz propose that monetary supply will change the economic cycles and thus change the real estate market [16]. With the increase of the monetary supply, the gross revenue will increase; that is, this study proves the existence of monetary supply channel. Zhao et al. and Sun use VAR model to test the monetary supply channel and find that this channel has a very important role in the monetary policy transmission effect [17, 18].

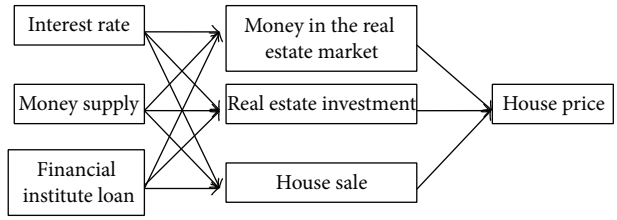

Figure 1: The framework model for the transmission of monetary policy on real estate market.

Meanwhile, the real estate market can be explained using demand and supply theory, which means that the house price is the trade balance result of house demand and supply and can be affected by factors related to supply and demand.

For house supply, it means the quantity of house which can be provided in a certain time and price. This indicates two aspects: the willingness and ability that real estate companies can supply. The willingness can be seen from the investment of real estate developers. Besides the general real estate companies' supply, the government can assist and affect the supply based on its purpose of manage people's living. In order to reflect this supply, this paper uses the monetary in the real estate market to reflect how many houses can be supplied. For the demand, it means the quantity of houses which can be bought in a certain price and time. In China, the house demand is quite rigid because Chinese people prefer buying not renting houses. Especially in current situation, the high profit attracts more and more demand in real estate market.

Based on the above theories, we construct a framework model for the transmission of monetary policy on real estate market. In Figure 1, the monetary policy affects the real estate market via interest rate channel, monetary supply channel, and credit channel. And then monetary policy can manage the real estate investment, the money in the real estate market, and the sales in the real estate market. Finally, the house price can be set.

What is more, the relevant indicators are as follows: we select interest rate (R), money supply (M), and financial institute loan (L) to represent the monetary policy; the monetary in the real estate market (HM), the real estate investment (HI), and house sale (HS) to reflect the supply and demand of real estate market; the house price (HP) is the target item.

Then, assume that we have known the details of this framework model above. While monetary policy changes someday, we can infer how the house price goes. Also, if there is disturbance out of system affecting the house price and deviating from what is excepted, we can design new monetary policies to keep the price stable by the feedback of real estate market. The latter is the main idea of this paper. The following will be introduced in detail.

3. Basic Model

In this section, the theoretical framework above will be converted into mathematical model, and the method of state feedback controller will be introduced.

The key idea is to build a continuous H_∞ control, as

$$\begin{aligned} \dot{X}(t) &= AX(t) + Bu(t) + B_1\omega(t), \\ Y(t) &= CX(t) + D\omega(t), \end{aligned} \tag{1}$$

where $u(t)$ is the control input and $X(t)$ is the measured output or the so called state variable. $Y(t)$ is the control output. So we can get

$$\begin{aligned} \begin{bmatrix} \dot{\mathrm{HM}}(t) \\ \dot{\mathrm{HI}}(t) \\ \dot{\mathrm{HS}}(t) \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} R(t) \\ M(t) \\ L(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \omega(t), \\ \mathrm{HP}(t) &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix} + D\omega(t), \end{aligned} \tag{2}$$

where HP means the house price, HM means the money supply in the real estate market, HI means the investment in the real estate market, HS means the house sell, R means the interest rate, M means the money supply, L means the financial institute loans, ω means the disturbance term, and the others are the undetermined coefficients.

Then, C. Romer and D. Romer find that, in the long term, the monetary policy is "neutral" [19, 20], which means that the monetary policy does not have a real impact on output or other actual economic indicators; though in the short term, it can affect the economy. Obviously, in the long term, the output depends on the input of resources, the level of technology, and the work efficiency. The monetary policy mainly causes short-term impact, which would finally decay to the equilibrium state. What is more, Wu and Hu do an impulse response research on the transmission of monetary policy in China's real estate market [21]. They find that, in the short term, the monetary policy's conduction in the real estate market has a time lag. Therefore, in this section, we will add a lag period (h) in the monetary policy. So, we can get a delay-dependent H_∞ control as

$$\begin{aligned} \begin{bmatrix} \dot{\mathrm{HM}}(t) \\ \dot{\mathrm{HI}}(t) \\ \dot{\mathrm{HS}}(t) \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} R(t-h) \\ M(t-h) \\ L(t-h) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \omega(t), \\ \mathrm{HP}(t) &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix} + D\omega(t). \end{aligned} \tag{3}$$

According to the economic theory, we can build such assumptions.

Assumption 1. Supply and demand are independent of each other. It means that a_{13}, a_{23}, a_{31}, and a_{32} are zero, while HM, HI are not affected by HS, and HS is not affected by HM, HI.

Assumption 2. HM means money supply in real estate market. So the key influencing factor is money supply. Then we can set b_{11} and b_{13} to be zero.

Assumption 3. b_{22} is zero, while HI is not affected by M. It is because money supply is determined by central bank but the investment in real estate is determined by estate agent. The estate agent invests in the real estate market with the money of his own or borrowed from the bank.

With the assumptions above, we can get

$$\begin{aligned} \begin{bmatrix} \dot{\mathrm{HM}}(t) \\ \dot{\mathrm{HI}}(t) \\ \dot{\mathrm{HS}}(t) \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & b_{12} & 0 \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} R(t-h) \\ M(t-h) \\ L(t-h) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \omega(t), \\ \mathrm{HP}(t) &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix} + D\omega(t). \end{aligned} \tag{4}$$

In the uncertain environment, the economic system will be affected by error, asymmetric information, credit risk, psychological factors, and so on [22]. To avoid these disturbances, we use the continuous delay-dependent H_∞ control. So that we can make an optimal control of monetary policy's conduction in the real estate market at the "worst case," which means $\|Y(t)\|_2 < \gamma^2\|\omega\|_2$. Then the differential equation can be further simplified into

$$\begin{aligned} \dot{X}(t) &= AX(t) + Bu(t-h) + B_1\omega(t), \\ Y(t) &= CX(t) + D\omega(t), \end{aligned} \tag{5}$$

where

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} \dot{\mathrm{HM}}(t) \\ \dot{\mathrm{HI}}(t) \\ \dot{\mathrm{HS}}(t) \end{bmatrix}, \qquad X(t) = \begin{bmatrix} \mathrm{HM}(t) \\ \mathrm{HI}(t) \\ \mathrm{HS}(t) \end{bmatrix}, \\ u(t-h) &= \begin{bmatrix} R(t-h) \\ M(t-h) \\ L(t-h) \end{bmatrix}, \qquad Y(t) = \mathrm{HP}(t), \\ A &= \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & b_{12} & 0 \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \\ C &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \end{aligned} \tag{6}$$

For the design of an H_∞ control, we are interested in designing a memoryless state feedback controller

$$u(t) = KX(t), \tag{7}$$

where K is the state feedback gain matrix to be designed, so that

(1) the closed-loop system is asymptotically stable;

(2) under zero initial condition, the closed-loop system guarantees that $\|Y(t)\|_2 < \gamma^2\|\omega\|_2$ for all nonzero $\omega \in L_2[0,\infty)$, where $\gamma > 0$ is a prescribed constant. Thus, the closed-loop system is given by

$$\begin{aligned} \dot{X}(t) &= AX(t) + BKX(t-h) + B_1\omega(t), \\ Y(t) &= CX(t) + D\omega(t). \end{aligned} \tag{8}$$

4. Delay-Dependent H_∞ Control Model

The following theorem presents the sufficient conditions for the existence of a delay-dependent H_∞ state feedback control.

Theorem 1. *Given $\gamma > 0$ and $h > 0$, if there exit matrices $P = P^T > 0$, $Q = Q^T > 0$, and S_i, satisfying matrix inequalities* (5)*. Moreover, a desired delay-dependent H_∞ state feedback control law is given by $u(t) = KX(t)$:*

$$\begin{bmatrix} A^TP + PA + S_1 + S_1^T & PBK - S_1 + S_2^T & PB_1 + S_3^T & \sqrt{h}S_1 & \sqrt{h}A^T & C^T \\ * & -S_2 - S_2^T & -S_3^T & \sqrt{h}S_2 & \sqrt{h}K^TB^T & 0 \\ * & * & -\gamma^2 I & \sqrt{h}S_3 & \sqrt{h}B_1^T & D^T \\ * & * & * & -Q & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0. \tag{9}$$

Proof. Under the condition of the theorem, firstly, we choose a Lyapunov functional candidate for system (5) as

$$V(t) = X^T(t)PX(t) + \int_{-h}^{0}\int_{t+\beta}^{t} \dot{X}^T(\alpha)Q\dot{X}(\alpha)\,d\alpha\,d\beta, \tag{10}$$

where $P > 0$ and $Q > 0$ are matrices to be determined.

The derivative of $V(t)$ satisfies

$$\begin{aligned} \dot{V}(t) = {} & \dot{X}^T(t)PX(t) + X^T(t)P\dot{X}(t) + h\dot{X}^T(t)Q\dot{X}(t) \\ & - \int_{t-h}^{t} \dot{X}^T(\alpha)Q\dot{X}(\alpha)\,d\alpha. \end{aligned} \tag{11}$$

So we design that

$$Z(t) = \dot{V}(t) + Y^T(t)Y(t) - \gamma^2\omega^T(t)\omega(t); \tag{12}$$

then,

$$\begin{aligned} Z(t) &= \dot{V}(t) + Y^T(t)Y(t) - \gamma^2\omega^t(t)\omega(t) \\ &= \left(AX + BKX_1 + B_1\omega\right)^T PX \\ &\quad + X^TP\left(AX + BKX_1 + B_1\omega\right) \\ &\quad + (CX + D\omega)^T(CX + D\omega) \\ &\quad + h\left(AX + BKX_1 + B_1\omega\right)^T Q\left(AX + BKX_1 + B_1\omega\right) \\ &\quad - \int_{t-h}^{t} \dot{X}^T(\alpha)Q\dot{X}(\alpha)\,d\alpha - \gamma^2\omega^t\omega \\ &= X^T\left(A^TP + PA + hA^TQA + C^TC\right)X \\ &\quad + X^T\left(PBK + hA^TQBK\right)X_1 \\ &\quad + X^T\left(PB_1 + hA^TQB_1 + C^TD\right)\omega \\ &\quad + X_1^T\left(K^TB^TP + hK^TB^TQA\right)X \\ &\quad + X_1^T\left(hK^TB^TQBK\right)X_1 + X_1^T\left(hK^TB^TQB_1\right)\omega \\ &\quad - \int_{t-h}^{t} \dot{X}^T(\alpha)Q\dot{X}(\alpha)\,d\alpha \\ &\quad + \omega^T\left(B_1^TP + hB_1^TQA + D^TC\right)X \\ &\quad + \omega^T\left(hB_1^TQBK\right)X_1 \\ &\quad + \omega^T\left(hB_1QB_1 + D^TD - \gamma^2I\right)\omega, \end{aligned} \tag{13}$$

where $X = X(t)$, $X_1 = X(t-h)$, and $\omega = \omega(t)$.

For simplification, we make

$$\psi_1 = \begin{bmatrix} A^TP + PA + C^TC & PBK & PB_1 + C^TD \\ K^TB^TP & 0 & 0 \\ B_1^TP + D^TC & 0 & D^TD - \gamma^2I \end{bmatrix}, \tag{14}$$

$$\psi_2 = h\begin{bmatrix} A & BK & B_1 \end{bmatrix}^T Q \begin{bmatrix} A & BK & B_1 \end{bmatrix}.$$

Thus, we have

$$Z = \zeta^T(t)\left(\psi_1 + \psi_2\right)\zeta(t) - \int_{t-h}^{t} \dot{X}^T(\alpha)Q\dot{X}(\alpha)\,d\alpha. \tag{15}$$

In addition, by the Leibniz-Newton formula, for any appropriate matrices $S = \sum_{i=1}^{t} k_i S_i = \sum_{i=1}^{t} \left[S_{1i}^T \;\; S_{2i}^T \;\; S_{3i}^T\right]^T$, we have

$$\zeta^T(t) S \left[X(t) - X(t-h) - \int_{t-h}^{t} \dot{X}(\alpha)\, d\alpha \right] = 0, \tag{16}$$

where $\zeta(t) = \left[X^T(t) \;\; X^T(t-h) \;\; \omega^T(t)\right]^T$. So we can get

$$\begin{aligned} Z &= \zeta^T(t)\left(\psi_1 + \psi_2\right)\zeta(t) - \int_{t-h}^{t} \dot{X}^T(\alpha) Q \dot{X}(\alpha)\, d\alpha \\ &\quad + \zeta^T(t) S \left[X(t) - X(t-h) - \int_{t-h}^{t} \dot{X}(\alpha)\, d\alpha \right] \\ &\quad + \left[X(t) - X(t-h) - \int_{t-h}^{t} \dot{X}(\alpha) d\alpha \right]^T S^T \zeta(t) \\ &= \zeta^T(t)\left(\psi_1 + \psi_2\right)\zeta(t) + \zeta^T(t) S \left(X - X_1\right) \\ &\quad + \left(X^T - X_1^T\right) S^T \zeta(t) \\ &\quad - \int_{t-h}^{t} \left[\dot{X}^T(\alpha) Q \dot{X}(\alpha) + \zeta^T(t) S \dot{X}(\alpha) \right. \\ &\qquad \left. + \dot{X}^T(\alpha) S^T \zeta(t) \right] d\alpha. \end{aligned} \tag{17}$$

With the addition of $h\zeta^T(t) S Q^{-1} S^T \zeta(t) - \int_{t-h}^{t} \zeta^T(t) S Q^{-1} S^T \zeta(t)\, d\alpha = 0$, then we have

$$\begin{aligned} Z &= \zeta^T(t)\left(\psi_1 + \psi_2 + \psi_3 + \psi_3^T\right)\zeta(t) + h\zeta^T(t) S Q^{-1} S^T \zeta(t) \\ &\quad - \int_{t-h}^{t} \left[\zeta^T(t) S + \dot{X}(\alpha) Q\right] Q^{-1} \left[Q\dot{X}^T(\alpha) + S^T \zeta(t)\right] d\alpha, \end{aligned} \tag{18}$$

where $\psi_3 = [S \;\; -S \;\; 0]$, $S = \left[S_1^T \;\; S_2^T \;\; S_3^T\right]^T$.

Note that $Q > 0$; thus $-\int_{t-h}^{t} [\zeta^T(t) S + \dot{X}(\alpha) Q] Q^{-1} [Q\dot{X}^T(\alpha) + S^T \zeta(t)] d\alpha$ is positive. By Schur complement, inequalities (9) guarantee

$$\psi_1 + \psi_2 + \psi_3 + \psi_3^T + h S Q^{-1} S^T < 0. \tag{19}$$

Therefore, we have

$$Z(t) = \dot{V}(t) + Y^T(t) Y(t) - \gamma^2 \omega^T(t)\, \omega(t) < 0, \tag{20}$$

for all nonzero $\omega \in L_2[0, \infty)$, and the asymptotic stability of system (8) is established. Under zero initial condition, we have $V(0) = 0$ and $V(\infty) \geq 0$. Integrating both sides of inequality (20), we obtain

$$0 < V(\infty) - V(0) < \gamma^2 \|\omega\|_2 - \|Y(t)\|_2. \tag{21}$$

So, the inequality above means $\|Y(t)\|_2 < \gamma^2 \|\omega\|_2$ for all nonzero $\omega \in L_2[0, \infty)$, and the H_∞ performance is established. This completes the proof. □

5. A Design Example

The time-delay-dependent H_∞ control has been proved to be useful in many aspects, [23–26]. In this section, we will provide an example to illustrate the effectiveness of the proposed delay-dependent state feedback H_∞ control based on the theories above. Through the demonstration of the quarterly data from 2005 to 2013, we have the parameters as the following values:

$$\begin{aligned} A &= \begin{bmatrix} 0.401404 & 2.504153 & 0 \\ -0.368434 & 2.013139 & 0 \\ 0 & 0 & 0.914773 \end{bmatrix}; \\ B &= \begin{bmatrix} 0 & -3.821928 & 0 \\ -0.283832 & 0 & -2.102299 \\ 0.331962 & -11.82723 & 10.80898 \end{bmatrix}; \\ C &= \begin{bmatrix} -0.230125 & 0.310886 & 0.256653 \end{bmatrix}; \\ B_1 \omega(t) &= \begin{bmatrix} 28.51301 \\ 14.20208 \\ 10.01136 \end{bmatrix}; \qquad D\omega(t) = 5.219788. \end{aligned} \tag{22}$$

Now, we design a continuous delay-dependent H_∞ control, based on the theory of system (5). To further research, we set the lag phase (h) to be 0.2 and the D to be 1. So we can choose $B_1 = \left[5.46248 \;\; 2.72082 \;\; 1.91796\right]^T$. By solving the matrices inequalities (9) with LMI tools of MATLAB, while there exit matrices $P > 0$, $Q > 0$, and S with a given scalar $\gamma > 0$, the state feedback controller is established. Then we can find that, when the best $\gamma = 1.0923$, the linear matrix inequality system is feasible. And we can get the following controller gain matrix based on system (7):

$$K = \begin{bmatrix} 13.4161 & -17.0838 & -21.1605 \\ -4.7350 & 7.6431 & 5.9833 \\ -4.2677 & 7.1089 & 5.4628 \end{bmatrix}. \tag{23}$$

Assume that the original system is stable and the fluctuations of house price have been within the expected range. But suddenly, some disturbance out of system happens, such as system (5), with the parameters of (22). It causes the price gravely deviating from the forecast. Then the robust control militates. We have to design some new monetary policy to keep its stability, where the design is mentioned in Section 3 and the strict proof in Section 4. So we can build a state feedback controller as system (7) with the coefficient of (23). Then we can find that $\|Y(t)\|_2 < \gamma^2 \|\omega\|_2$ with the smallest possible $\gamma = 1.0923$, and the house price is asymptotically stable.

It means that this example further proves the possibility of delay-dependent H-infinity control's application in monetary policy. And the economy system has robustness. In other words, through the state feedback controller, we can adjust the monetary policy in time to keep the house price stable as optimal as possible, in uncertain case. Also, if the lag phase changes or the external disturbance is different, we can find a new state feedback controller to make the best decision.

6. Conclusion

This paper uses a time-delay-dependent H_∞ control model to analyze the transmission effect of monetary policy on the real estate market and finds the result which meets the optimal feedback requirement. We test the robustness of the model using the Chinese data of the monetary policy and the real estate market and prove that this model has a good ability to be employed in the reality. Based on the simulation test, we find that under an uncertain circumstance, the model can give an optimal result to fulfill the goal of controlling the risk and improve the policy's effectiveness. In the real estate market, the monetary policy can have a good effect on the market when the policies are well combined based on the simulation test.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Monitoring of Nonlinear Time-Delay Processes Based on Adaptive Method and Moving Window

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A new adaptive kernel principal component analysis (KPCA) algorithm for monitoring nonlinear time-delay process is proposed. The main contribution of the proposed algorithm is to combine adaptive KPCA with moving window principal component analysis (MWPCA) algorithm, and exponentially weighted principal component analysis (EWPCA) algorithm respectively. The new algorithm prejudges the new available sample with MKPCA method to decide whether the model is updated. Then update the KPCA model using EWKPCA method. And also extend MPCA and EWPCA from linear data space to nonlinear data space effectively. Monitoring experiment is performed using the proposed algorithm. The simulation results show that the proposed method is effective.

1. Introduction

Fault detection and diagnosis are very important aspects in modern industrial processes because they concern the execution of planned operations and process productivity. In order to meet the need of production, data-based methods have been deeply developed, such as principal component analysis (PCA), partial least squares (PLS), and independent component analysis (ICA) [1–5].

PCA as a multivariable statistical method is widely used for damage detection and diagnosis of structures in industrial systems [6–8]. Because PCA is an orthogonal transformation of the coordinate system, it is a linear monitoring method. However, most industrial processes have strong-nonlinearity characteristics [9, 10]. Document [11] shows that application of linear monitoring approaches to nonlinear processes may lead to unreliable process monitoring, because a linear method is inappropriate to extract the nonlinearities within the process variables. To solve the nonlinear problem, several nonlinear methods have been proposed in the past decades [12–17].

Kernel principal component analysis (KPCA), as a nonlinear version of PCA method which is developed by many researchers, is more proper in nonlinear process monitoring [18–20]. KPCA can efficiently compute principal components (PCs) by nonlinear kernel functions in a high-dimensional feature spaces. The main advantages of KPCA are that it only solves an eigenvalue problem and does not involve nonlinear optimization [21]. However, a KPCA monitoring model requires kernel matrix, whose dimension is given by the number of reference samples. In addition, the KPCA with fixed model may lead to large error because of the gradual change of parameters with the operation of process [22–26]. Because the old samples are not representative of the current process status, the adaptation of KPCA model is necessary.

To date, MWPCA algorithm and EWPCA algorithm are two representative adaptive methods [27–31]. Performing a moving window approach, which was proposed by Hoegaerts et al., overcomes this problem and produces a constant scale of the kernel matrix and a fixed speed of adaptation [32, 33]. Choosing a proper weighting factor is an important issue, which determines the influence that the older data has on the model [31]. Paper [34] has proposed a moving window KPCA formulation, which has some advantages such as using the Gram matrix instead of the kernel matrix, incorporation of an adaptation method for the eigendecomposition of the Gram matrix.

In this paper, a new algorithm combining these two representative methods is proposed. The proposed algorithm mapped the sample set in feature space and prejudges the new available sample with MKPCA method to decide whether the model is updated. Then update the KPCA model using EWKPCA method. The proposed algorithm can reduce negative impact of outliers on model; the updating after prejudgment can reduce computational complexity and be more efficient. The remaining sections of this paper are organized as follows. The KPCA method based on loss function and the online monitoring strategy are introduced in Section 2. The iterative KPCA algorithm with penalty factor is presented in Section 3. The proposed adaptive KPCA method combining two representative methods is described in Section 4. The simulation results are presented in Section 5. Finally, the conclusion is given in Section 6.

2. Kernel Principal Component Analysis Based on Loss Function

2.1. Kernel Principal Component Analysis. KPCA is an extension of PCA, and it can be solved as an eigenvalue problem of its kernel matrix. KPCA algorithm makes use of the following idea.

Via a nonlinear mapping

$$\mathbf{\Phi} : \mathbf{R}^N \longrightarrow F, \tag{1}$$

data set $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \in \mathbf{R}^N$ is mapped into a potentially much higher dimensional feature space F; the mapped data set $\mathbf{\Phi}(\mathbf{X}) = \{\mathbf{\Phi}(\mathbf{x}_1), \mathbf{\Phi}(\mathbf{x}_2), \ldots, \mathbf{\Phi}(\mathbf{x}_N)\}$ is obtained. $\overline{\mathbf{\Phi}}(\mathbf{X})$ is gotten by centering $\mathbf{\Phi}(\mathbf{X})$ with the corresponding mean s.t. Consider

$$\frac{1}{N}\sum_{i=1}^{N}\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right) = 0. \tag{2}$$

A principal component $\mathbf{V}$ can be computed by solving the eigenvalue problem

$$\lambda\mathbf{V} = \frac{1}{N}\overline{\mathbf{\Phi}}\left(\mathbf{X}\right)\overline{\mathbf{\Phi}}(\mathbf{X})^T\mathbf{V} = \mathbf{C}^F\mathbf{V}, \tag{3}$$

where $\mathbf{C}^F = (1/N)\sum_{k=1}^{N}\overline{\mathbf{\Phi}}(\mathbf{x}_k)\overline{\mathbf{\Phi}}(\mathbf{x}_k)^T = (1/N)\overline{\mathbf{\Phi}}(\mathbf{X})\overline{\mathbf{\Phi}}(\mathbf{X})^T$ is the sample covariance matrix of $\overline{\mathbf{\Phi}}(\mathbf{X})$ and $\lambda > 0$ is the eigenvalue corresponding to $\mathbf{V}$.

By defining the kernel matrix $\mathbf{K}$ of dimension $N \times N$,

$$[\mathbf{K}]_{ij} = \mathbf{K}_{ij} = \left\langle\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right), \overline{\mathbf{\Phi}}\left(\mathbf{x}_j\right)\right\rangle = k\left(\mathbf{x}_i, \mathbf{x}_j\right), \tag{4}$$

the eigenvalue problem can be put in the form

$$N\lambda\boldsymbol{\alpha} = \mathbf{K}\boldsymbol{\alpha}. \tag{5}$$

Regarding the kernel function, radial basis function (RBF) is chosen in this paper as

$$k\left(\mathbf{x}, \mathbf{y}\right) = \exp\left(-\frac{\left\|\mathbf{x} - \mathbf{y}\right\|^2}{c}\right), \tag{6}$$

where c is the width of the Gaussian kernel; it can be very small (<1) or quite large.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ be the eigenvalues and the corresponding eigenvectors of the kernel matrix $\mathbf{K}$, respectively. As $\mathbf{V}^k = \sum_{i=1}^{N}\alpha_i^k\overline{\mathbf{\Phi}}(\mathbf{x}_i)$, the score of the kth eigenvector $\mathbf{V}^k$ can be calculated as follows:

$$\begin{aligned}\left\langle\mathbf{V}^k, \overline{\mathbf{\Phi}}\left(\mathbf{X}\right)\right\rangle &= \sum_{i=1}^{N}\alpha_i^k\left\langle\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right), \overline{\mathbf{\Phi}}\left(\mathbf{X}\right)\right\rangle \\ &= \sum_{i=1}^{N}\alpha_i^k \cdot k\left(\mathbf{x}_i, \mathbf{X}\right).\end{aligned} \tag{7}$$

2.2. Loss Function in Feature Space. Reconstruct sample data set $\overline{\mathbf{\Phi}}(\mathbf{X})$ in feature space; the reconstructed data set is given by

$$\left\langle\mathbf{W}, \overline{\mathbf{\Phi}}\left(\mathbf{X}\right)\right\rangle\mathbf{W} = \mathbf{W}\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{X}\right), \tag{8}$$

where $\mathbf{W}$ is the transformation matrix, with $\|\mathbf{W}\| = 1$.

Between the original sample data set and the reconstructed data set exists the reconstructed error

$$\begin{aligned}\left\|\mathbf{e}\left(\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right)\right\|^2 &= \left\|\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right) - \mathbf{W}\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right\|^2 \\ &= \overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\cdot\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right) - 2\mathbf{W}\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right) \\ &\quad + \left(\mathbf{W}\mathbf{W}^T\right)\left(\mathbf{W}\mathbf{W}^T\right)\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right) \\ &= k\left(\mathbf{x}_i, \mathbf{x}_i\right) - 2\mathbf{W}\mathbf{W}^T k\left(\mathbf{x}_i, \mathbf{x}_i\right) \\ &\quad + \left(\mathbf{W}\mathbf{W}^T\right)^2 k\left(\mathbf{x}_i, \mathbf{x}_i\right).\end{aligned} \tag{9}$$

Define the loss function in feature space as follows:

$$\begin{aligned}J_1\left(\mathbf{W}\right) &= \sum_{i=1}^{N}\left\|\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right) - \mathbf{W}\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right\|^2 \\ &= \sum_{i=1}^{N}\left(\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right)^2\|\mathbf{W}\|^2 - 2\sum_{i=1}^{N}\left(\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right)^2 \\ &\quad + \sum_{i=1}^{N}\left(\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right)^2 \\ &= -\sum_{i=1}^{N}\left(\mathbf{W}^T\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right)^2 + \sum_{i=1}^{N}\left\|\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right)\right\|^2,\end{aligned} \tag{10}$$

where $\sum_{i=1}^{N}\left\|\overline{\mathbf{\Phi}}(\mathbf{x}_i)\right\|^2$ is a constant value. Thus, when the transformation matrix $\mathbf{W}$ maximizes $\mathbf{W}^T\overline{\mathbf{\Phi}}(\mathbf{x}_i)$, the loss function $J_1(\mathbf{W})$ reaches its minimum value. Similarly, the score of the kth PC $\mathbf{W}^k$ is calculated as follows:

$$\begin{aligned}\mathbf{t}_k &= \left\langle\mathbf{W}^k, \overline{\mathbf{\Phi}}\left(\mathbf{x}\right)\right\rangle \\ &= \sum_{i=1}^{N}\alpha_i^k\left\langle\overline{\mathbf{\Phi}}\left(\mathbf{x}_i\right), \overline{\mathbf{\Phi}}\left(\mathbf{X}\right)\right\rangle = \sum_{i=1}^{N}\alpha_i^k k\left(\mathbf{x}_i, \mathbf{X}\right).\end{aligned} \tag{11}$$

2.3. Online Monitoring Strategy of KPCA. A measure of the variation within the KPCA model is given by Hotelling's T^2 statistic. The measure of goodness of fit of a sample to the model is the squared prediction error (SPE), also known as the Q statistic. T^2 is given by

$$T^2 = \left[\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_A\right] \boldsymbol{\Delta}^{-1} \left[\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_A\right]^T, \tag{12}$$

where A is the PC number, N is the number of samples, $\mathbf{t}_A$ is the Ath nonlinear principal component which can be calculated with (11), $\boldsymbol{\Delta}^{-1}$ is the inverse matrix of the diagonal matrix composed of eigenvalues, and $\boldsymbol{\Delta} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_A\} = (1/(N-1))\mathbf{T}^T\mathbf{T} = (1/(N-1))\mathbf{W}^T\mathbf{K}^T\mathbf{K}\mathbf{W}$.

Control limit of Hotelling's T^2 is as follows:

$$T^2_{A,N,\alpha} \sim \frac{A(N-1)}{N-A} F_{N,N-A,\alpha}. \tag{13}$$

Once a new sample $\boldsymbol{\Phi}(\mathbf{x}_{\text{new}})$ is available, T^2 new can be calculated with the following equation:

$$T^2_{\text{new}} = \mathbf{t}_{\text{new}} \boldsymbol{\Delta}^{-1} \mathbf{t}^T_{\text{new}}. \tag{14}$$

The SPE and the corresponding control limit are obtained using the following equations:

$$\begin{aligned} \text{SPE} &= \left\| \overline{\boldsymbol{\Phi}}(\mathbf{x}) - \widehat{\overline{\boldsymbol{\Phi}}}(\mathbf{x}) \right\|^2 = \sum_{i=1}^{N} \mathbf{t}_j^2 - \sum_{i=1}^{A} \mathbf{t}_j^2, \\ \text{SPE}_\alpha &\sim g\chi^2_h, \end{aligned} \tag{15}$$

where g and h are two relevant parameters. For new sample $\boldsymbol{\Phi}(\mathbf{x}_{\text{new}})$, SPE is given by

$$\begin{aligned} \text{SPE} &= \left\| \boldsymbol{\Phi}(\mathbf{x}_{\text{new}}) - \widehat{\boldsymbol{\Phi}}(\mathbf{x}_{\text{new}}) \right\|^2 \\ &= \boldsymbol{\Phi}(\mathbf{x}_{\text{new}}) \boldsymbol{\Phi}(\mathbf{x}_{\text{new}})^T - 2\boldsymbol{\Phi}(\mathbf{x}_{\text{new}}) \widehat{\boldsymbol{\Phi}}(\mathbf{x}_{\text{new}})^T \\ &\quad + \widehat{\boldsymbol{\Phi}}(\mathbf{x}_{\text{new}}) \widehat{\boldsymbol{\Phi}}(\mathbf{x}_{\text{new}})^T \\ &= k(\mathbf{x}_{\text{new}}, \mathbf{x}_{\text{new}}) - 2\boldsymbol{\Phi}(\mathbf{x}_{\text{new}}) \mathbf{W}\mathbf{t}^T_{\text{new}} + \mathbf{t}_{\text{new}} \mathbf{W}\mathbf{W}^T \mathbf{t}_{\text{new}} \\ &= k(\mathbf{x}_{\text{new}}, \mathbf{x}_{\text{new}}) - 2\mathbf{k}_{\text{new}} \mathbf{T}\mathbf{t}^T_{\text{new}} + \mathbf{t}_{\text{new}} \mathbf{T}^T \mathbf{K}\mathbf{T}\mathbf{t}^T_{\text{new}}, \end{aligned} \tag{16}$$

where $k(\mathbf{x}_{\text{new}}, \mathbf{x}_{\text{new}}) = 1 - (2/N)\sum_{i=1}^{N} k_{\text{new},i} + (1/N)\sum_{i=1}^{N} \sum_{j=1}^{N} k_{ij}$ and $k_{\text{new},i}$ is the ith element of $\mathbf{k}_{\text{new}}$.

3. Iterative Kernel Principal Component Analysis

KPCA algorithm based on eigenvalue decomposition is a batch learning method. This algorithm is not available for online monitoring as all sample data has to be known before modeling. In addition, no outlier is allowed in modeling process; this requires all sample data to be normal. But the requirement is hard to reach in actual production process. Outliers always exist in sample set even after mapping in feature space. In the sense of least squared error, an iterative KPCA algorithm is proposed in this paper. Furthermore, penalty factor is added to solve the outlier problem.

Use stochastic gradient descent algorithm to solve the problem proposed in (10):

$$\frac{dJ_1(\mathbf{W})}{d\mathbf{W}} = 2\left(\mathbf{W}\mathbf{W}^T\mathbf{K} - \mathbf{K}\right)\mathbf{W}. \tag{17}$$

Iterative formula is given by

$$\mathbf{W}_{n+1} = \mathbf{W}_n + \mu_n \left(\mathbf{K} - \mathbf{W}\mathbf{W}^T\mathbf{K}\right)\mathbf{W}_n, \tag{18}$$

where μ_n is the iteration step, s.t. $0 < \mu_n < 1$, and $\mathbf{W}_n$ converges to the first nonlinear principal component.

Because nonlinear principal components are mutually orthogonal, the Schmidt orthogonal method is used to calculate the other nonlinear principal components:

$$\begin{aligned} \mathbf{W}^j_{n+1} &= \mathbf{W}^j_n + \mu_n \left(\mathbf{K}^j - \mathbf{W}^j_n \mathbf{W}^{jT}_n \mathbf{K}^j\right)\mathbf{W}^j_n, \\ \mathbf{K}^j &= \mathbf{K} - \sum_{i=1}^{j-1} \mathbf{W}^i \mathbf{W}^{iT} \mathbf{K}, \end{aligned} \tag{19}$$

where $\mathbf{W}^j$ is the jth nonlinear principal component ($j = 2, 3, \ldots, A$) and A is the PC number.

To solve the outlier problem, penalty factor is added in (10):

$$J_2(\mathbf{W}) = \sum_{i=1}^{N} \left(C_i \left\| \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) - \mathbf{W}\mathbf{W}^T \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) \right\|^2 + \eta(1 - C_i)\right), \tag{20}$$

where η is the predefined threshold, $\eta > 0$, and C_i satisfies the following formula:

$$C_i = \begin{cases} 1 & \left\| \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) - \mathbf{W}\mathbf{W}^T \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) \right\|^2 \le \eta, \\ 0 & \left\| \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) - \mathbf{W}\mathbf{W}^T \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) \right\|^2 > \eta. \end{cases} \tag{21}$$

As is shown above, when $\left\| \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) - \mathbf{W}\mathbf{W}^T \overline{\boldsymbol{\Phi}}(\mathbf{x}_i) \right\|^2 > \eta$, $\overline{\boldsymbol{\Phi}}(\mathbf{x}_i)$ is regarded as outlier. The value of $J_2(\mathbf{W})$ is set to η to reduce impact on KPCA model. Note that C_i in the penalty factor is discrete; thus the continuous Sigmoid function is used to approach C_i. Iteration formula with penalty factor is shown as follows:

$$\mathbf{W}_{n+1} = \mathbf{W}_n + \mu_n \frac{1}{1 + e^{\beta\left(\left\|\mathbf{e}_n\left(\overline{\boldsymbol{\Phi}}(\mathbf{X})\right)\right\|^2 - \eta\right)}} \left(\mathbf{K} - \mathbf{W}\mathbf{W}^T\mathbf{K}\right)\mathbf{W}_n, \tag{22}$$

where μ_n is the iteration step, $0 < \mu_n < 1$, and $1/(1 + e^{\beta\left(\left\|\mathbf{e}_n\left(\overline{\boldsymbol{\Phi}}(\mathbf{X})\right)\right\|^2 - \eta\right)})$ is the continuous Sigmoid function. As (16) shows, the number of outliers is determined by the value of η in the penalty factor. The smaller the threshold value is, the more the sample points which will be treated as outlier are. Therefore, the value of η can be determined by the proportion of outliers. Sort the sample reconstruction error in descending order, and set the percentage of outliers to r.

Select the largest r percent reconstruction error data to be outliers. Thus, the value of η can be set to the smallest reconstruction error value in all the outliers.

Similarly, the other nonlinear principal components are given by

$$\mathbf{W}_{n+1}^{j} = \mathbf{W}_{n}^{j} + \mu_n \frac{1}{1 + e^{\beta(\|\mathbf{e}_n(\overline{\Phi}(\mathbf{x}))\|^2 - \eta)}} \left(\mathbf{K}^{j} - \mathbf{W}\mathbf{W}^{T}\mathbf{K}^{j}\right)\mathbf{W}_{n}^{j},$$
$$\mathbf{K}^{j} = \mathbf{K}_0 - \sum_{i=1}^{j-1} \mathbf{W}^{i}\mathbf{W}^{iT}\mathbf{K}_0. \tag{23}$$

The iterative kernel principal component analysis algorithm with penalty factor is summarized as follows.

(1) Given an initial standardized block of sample data set $\overline{\mathbf{\Phi}}(\mathbf{X})$ and maximum iterations N and ε, set initial iteration steps $n = 1$ and principal component number $j = 1$.

(2) Construct the kernel matrix and scale it to obtain $\mathbf{K}_0$.

(3) Calculate the jth nonlinear principal component ($j = 1, 2, 3, \ldots, A$). If $j = 1$, calculate the first PC $\mathbf{W}^1$ with (22); if $j = 2, 3, \ldots, A$, calculate the jth PC $\mathbf{W}^j$ with (23).

(4) If $|\mathbf{W}_{n+1}^{j} - \mathbf{W}_{n}^{j}| > \varepsilon$ as well as $n < N$, make $n = n + 1$ and go back to Step (3) to do the next iteration; otherwise, $\mathbf{W}^j$ is obtained.

(5) Make $j = j + 1$; go back to Step (3) to calculate the next PC.

4. Adaptive Kernel Principal Component Analysis

In practice, the size of sample data set is gradually increased during dynamic process. Using the obtained KPCA model for online monitoring may cause large error. Therefore, in order to improve the ability to adapt new samples, dynamic performance is added in KPCA algorithm.

In this paper, an adaptive kernel PCA algorithm combining two representative methods is proposed. Kernel matrix and corresponding control limits of two statistics can be updated in real time.

In MKPCA, once a new sample data is available, a data window of fixed length is moving in real time to update the KPCA model.

In EWKPCA algorithm proposed by Li et al., the updating of the covariance matrix is as follows:

$$\left(\mathbf{X}^{T}\mathbf{X}\right)_t = \lambda_t\left(\mathbf{X}^{T}\mathbf{X}\right)_{t-1} + (1 - \lambda_t)\left(\mathbf{x}^{T}\mathbf{x}\right)_t, \tag{24}$$

where $\mathbf{x}_t$ is a new sample and λ_t is the corresponding weighting factor, $0 < \lambda_t < 1$. As is shown above, the weight on the latest observation is increasing with the decrease of λ_t. So the latest observation contributes most to the model. Introduce the idea to KPCA algorithm. Once a new sample is available, a new kernel vector $\mathbf{k}_t$ can be calculated. In this paper, $\mathbf{k}_t$ is used to update the kernel matrix with the following formula:

$$\mathbf{K}_{\text{rec}}^{t} = \gamma_t \mathbf{K}_{\text{rec}}^{t-1} + (1 - \gamma_t)\,\mathbf{k}_t^{T}\mathbf{k}_t, \tag{25}$$

where $\mathbf{K}_{\text{rec}}^{t-1}$ is the kernel matrix calculated at time $t - 1$, γ_t is the weighting factor, $0 < \gamma_t < 1$, and $\mathbf{k}_t$ is the standardized kernel vector at time t. As t increases, old samples affect the model less and less until the affection can almost be negligible. Therefore, old samples can be abandoned automatically instead of manually.

One important step of the EWKPCA algorithm is determination of the weighting factor γ_t. Fixed forgetting factor is not applicable because industrial process does not change in a constant rate. In the algorithm proposed by Choi et al., two forgetting factors α and β are used to update the sample mean and covariance matrix, respectively. The forgetting factor used to update the covariance matrix is given by

$$\beta_t = \beta_{\max} - (\beta_{\max} - \beta_{\min})\left[1 - \exp\left(-k\left(\frac{\|\Delta\mathbf{R}_{t-1}\|}{\|\Delta\mathbf{R}_{\text{nor}}\|}\right)^{n}\right)\right], \tag{26}$$

where $\beta_{\max}$ and $\beta_{\min}$ are the maximum and minimum values of forgetting factor, respectively, $\|\Delta\mathbf{R}\|$ is the Euclidean matrix norm of the difference between two consecutive correlation matrices, and $\|\Delta\mathbf{R}_{\text{nor}}\|$ is the average $\|\Delta\mathbf{R}\|$ obtained using historical data. k and n are two parameters which control the sensitiveness needed to be determined.

Let $\mathbf{X}_{t-1} = \{\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_{N+t-1}\} \in \mathbf{R}^N$ be the sample data set, let $\mathbf{K}_{\text{rec}}^{t-1}$ be the kernel matrix and $\mathbf{K}_{\text{rec}}^{0} = \mathbf{K}_0$, let $\mathbf{W}_{t-1}$ be the transformation matrix, let $\mathbf{\Delta}_{t-1}$ be the principal eigenvalue diagonal matrix, and let $T_{\text{lim},t-1}^2$ and $\text{SPE}_{\text{lim},t-1}$ be the control limits at time $t - 1$, respectively.

The adaptive kernel principal component analysis algorithm can be concluded as follows.

(1) At time t, a new sample is available. $\mathbf{\Phi}(\mathbf{x}_{N+t})$ is obtained after scaling it with the mean and covariance gotten at time $t - 1$.

(2) Calculate kernel vector $\mathbf{k}_t = \mathbf{\Phi}(\mathbf{x}_{N+t})^{T}\mathbf{\Phi}(t - 1)$, and scale it, that is, $\overline{\mathbf{k}}_t$.

(3) Hotelling's T^2 and SPE statistics of $\mathbf{\Phi}(\mathbf{x}_{N+t})$ are obtained with (14) and (16). Compare them with the control limits $T_{\text{lim},t-1}^2$ and $\text{SPE}_{\text{lim},t-1}$ at time $t - 1$. $\mathbf{\Phi}(\mathbf{x}_{N+t})$ is accepted if control limits are not exceeded; go to Step (4) for model updating. Otherwise, go to Step (6) for further judgment.

(4) Kernel matrix $\mathbf{K}_{\text{rec}}^{t}$ is calculated with (25); the forgetting factor is

$$\gamma_t = \gamma_{\max} - (\gamma_{\max} - \gamma_{\min})\left[1 - \exp\left(-k\|\Delta\mathbf{R}_{t-1}\|^{n}\right)\right], \tag{27}$$

where $\gamma_{\max} = 0.99$, $\gamma_{\min} = 0.80$, and k and n are adjusted to control the model sensitiveness.

(5) Update KPCA model, and transformation matrix $\mathbf{W}_t$ is gotten. Control limits of Hotelling's T^2 and SPE statistics are updated at the same time. Then go back to Step (1) for the next adaptation.

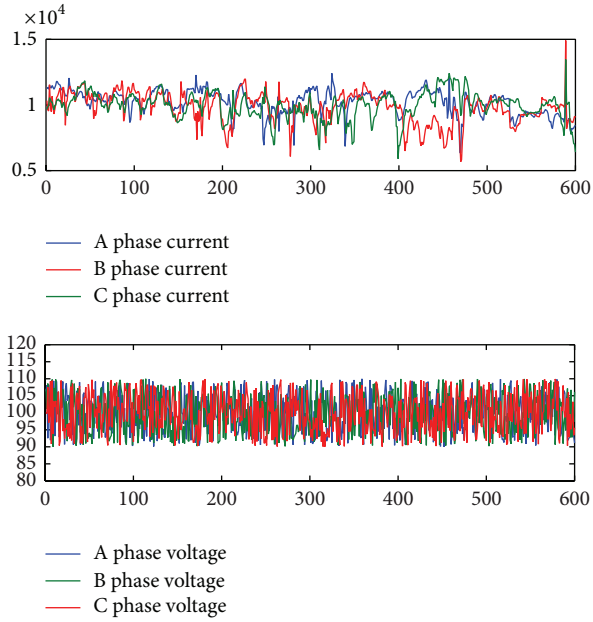

Figure 1: Sample points for training the proposed algorithm.

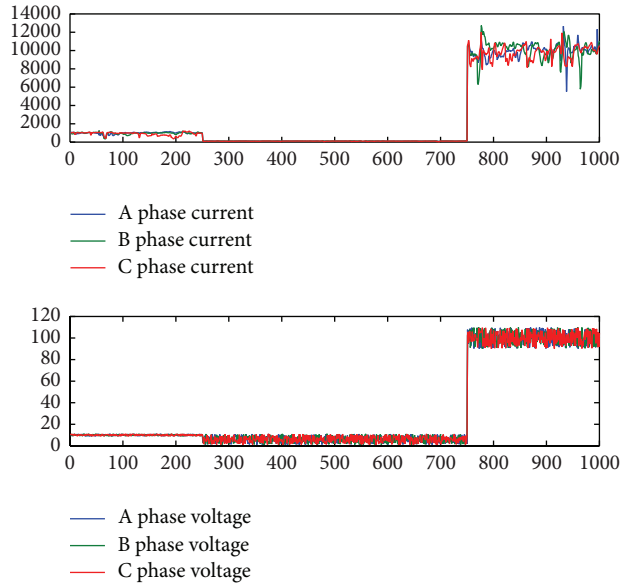

Figure 2: Sample points for testing the proposed algorithm.

(6) Model updating will not conduct if the control limits are exceeded. According to the number of continuous transfinite samples, they are judged to be outlier or fault. Let i be the number of continuous transfinite samples, initially $i = 0$. The value of i keeps adding by 1 until the next normal sample is obtained. Transfinite samples are judged to be outlier if $i < 3$; otherwise, they are judged to be fault. After the judgment, go back to Step (1) for the next sample.

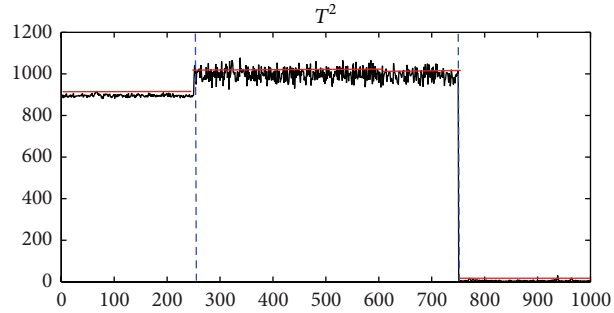

Figure 3: T^2 statistics of testing data.

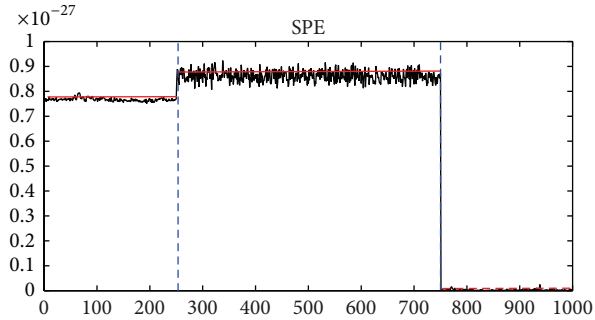

Figure 4: SPE statistics of testing data.

5. Experimental Results

5.1. Electrofused Magnesium Furnace Process. The purpose of the section is to test the performance of the proposed algorithm. We choose the electrofused magnesium furnace (EFMF) process to be monitoring object, because the working conditions of the EFMF are complex and changed frequently such as strong nonlinearity [35]. The shell of EFMF is round and slightly tapered, which can facilitate melting process. There are rings on the furnace wall and trolley under the furnace. When melting process has completed, move the trolley to cool. The control object is to ensure the temperature of EFMF can meet the set value. The average time of the whole EFMF multimode processes is 10 h. The current value and voltage value of three phases and the temperature of furnace all can be online measured, which provide abundant process information. The "healthy" process data is used for modeling. The "faulty" process data is used for monitoring.

In this experiment, the data set for training has 600×6 sample points and the data set for testing has 1000×6 sample points, which contains the current value and voltage value, respectively. The two data sets are printed in Figures 1 and 2. The size of the moving window is 50; the length of every step is 1. According to the steps which are given in Section 4, Hotelling's T^2 and SPE statistics are shown in Figures 3 and 4. The results indicate that the algorithm which this paper proposed can follow the variation of the object well.

5.2. Continuous Annealing Process. Continuous annealing process is a highly efficient heat treatment process after cold

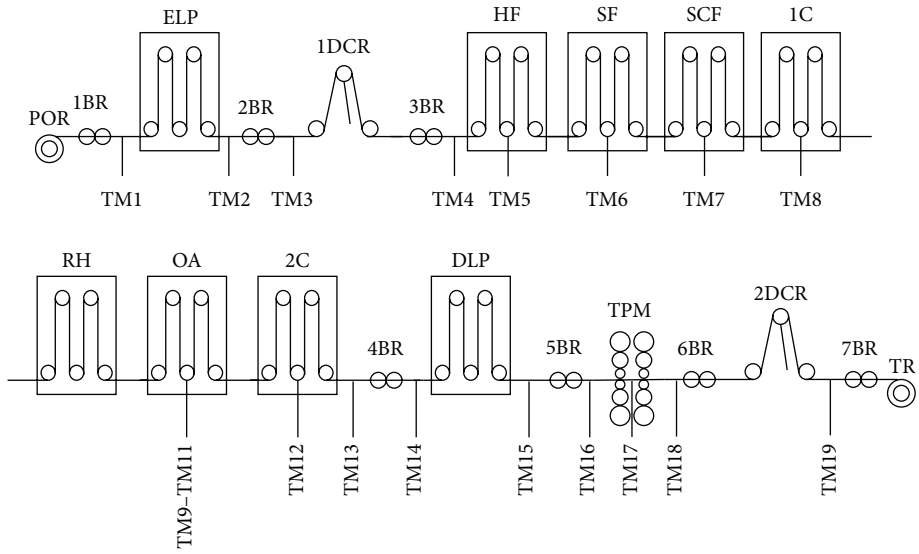

Figure 5: Schematic diagram of continuous annealing process.

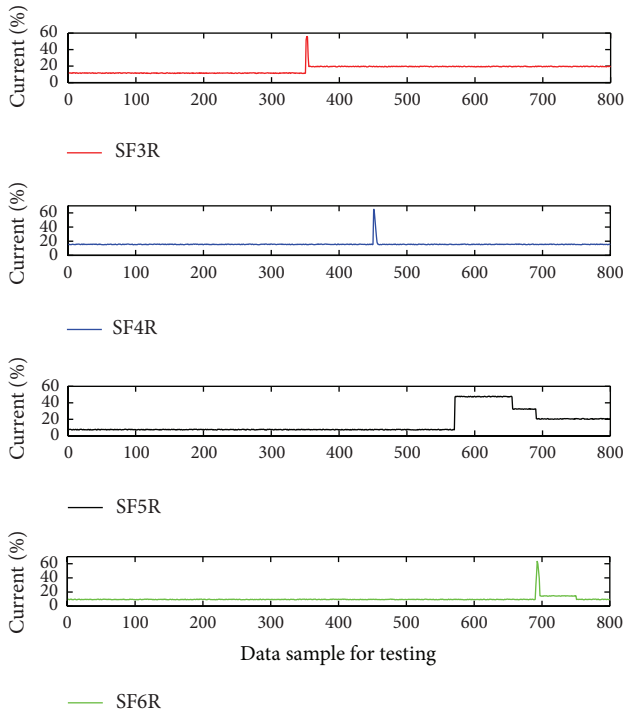

Figure 6: Data sample points for monitoring.

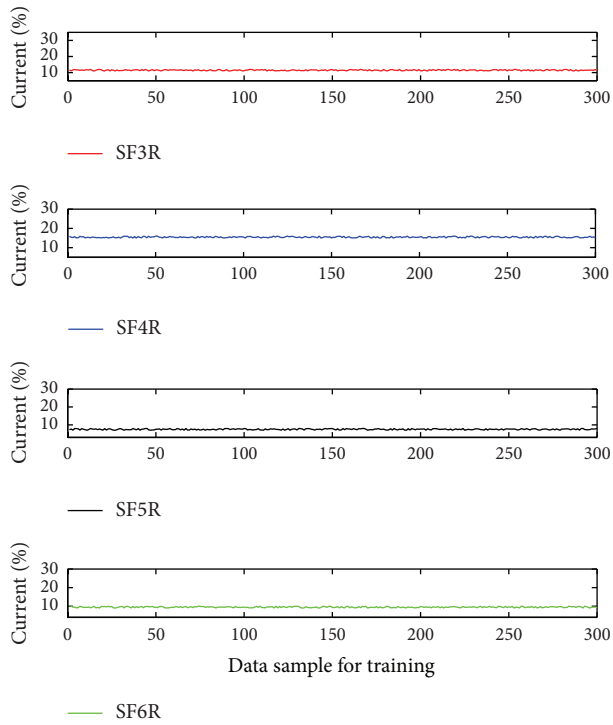

Figure 7: Data sample for training the algorithm.

rolling in steel works. As there are furnace zones, stable operation is necessary to product quality and continuous processing of the upstream and downstream. Process monitoring and fault diagnosis have always been the primary concern [36].

The strip in the continuous annealing line is heated in order to arrange the internal crystal of the strip. The material for annealing is a cold-rolled strip coil, which is put on a payoff reel on the entry side of the line. The head end of the coil is then pulled out and welded with the tail end of the preceding coil. Then the strip runs through the process with a certain line speed. On the delivery side, the strip is cut into a product length by a shear machine and is coiled again by a tension reel. The schematic diagram of continuous annealing process is shown in Figure 5, in which some abbreviations are listed in Table 1.

The strip-break is a frequent fault in continuous annealing process [37]. So a strip-break fault is considered here. Four

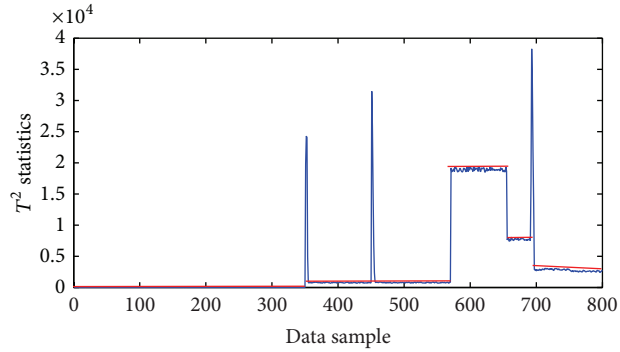

Figure 8: Hotelling's T^2 statistics for the algorithm.

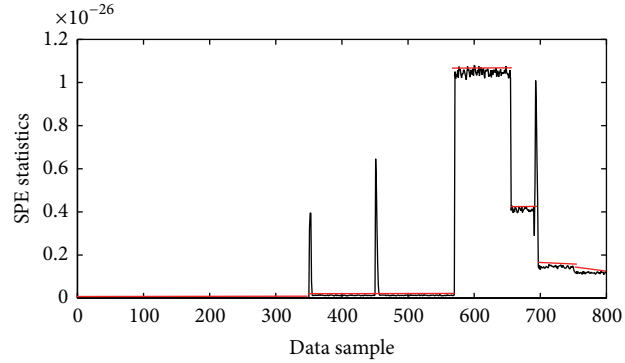

Figure 9: SPE statistics for the algorithm.

Table 1: Meanings of some abbreviations in Figure 5.

Abbreviation	Meaning
POR	Payoff reel
BR	Bridle roll
ELP	Entrance loop
DCR	Dancer roll
HF	Heating furnace
SF	Soaking furnace
SCF	Slow cooling furnace
C	Cooling furnace
RF	Reheating furnace
DLP	Delivery loop
TPM	Temper rolling machine
TR	Roll type reel
OA	Overageing furnace
TM	Tension model

current data sets with faults are measured from SF3R-SF6R; every data set contains 800 sample points, which is shown in Figure 6. And we choose 300 normal data (Figure 7) to train the algorithm. For the sake of easiness to calculate, normalization has been performed before. The size of the moving window is 60; the length of every step is 1. Hotelling's T^2 and SPE statistics are shown in Figures 8 and 9.

6. Conclusion

This paper has studied an algorithm for monitoring nonlinear time-delay process. In the paper, we propose an adaptive KPCA algorithm, which is combined with MWPCA and EWPCA. The proposed algorithm mapped the sample set in feature space and prejudges the new available sample with MKPCA method to decide whether the model is updated. Then update the KPCA model using EWKPCA method. This algorithm can reduce negative impact of outliers on model; the updating after prejudgment can reduce computational complexity and be more efficient. The experimental results show that the proposed algorithm is effective.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Delay Decomposition Approach to the Stability Analysis of Singular Systems with Interval Time-Varying Delay

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This paper investigates delay-dependent stability problem for singular systems with interval time-varying delay. An appropriate Lyapunov-Krasovskii functional is constructed by decomposing the delay interval into multiple equidistant subintervals, where both the information of every subinterval and time-varying delay have been taken into account. Employing the Lyapunov-Krasovskii functional, improved delay-dependent stability criteria for the considered systems to be regular, impulse-free, and stable are established. Finally, two numerical examples are presented to show the effectiveness and less conservativeness of the proposed method.

1. Introduction

During the last few decades, there has been a growing research interest in the analysis and synthesis of time-delay systems, which widely exist in various practical systems such as biological systems, chemical systems, electronic systems, and network control systems [1]. Usually, the range of delays considered in most of existing references is from zero to an upper bound [2–6]. In practice, however, the delay range may have a nonzero lower bound, and such systems are referred to as interval time-varying delay systems. For this reason, the stability of the systems with such interval time-varying delays has attracted considerable attention. As we know, in order to reduce conservatism of the stability criteria, many approaches were developed; for example, by using free-weighting matrices, [7, 8] present some stability conditions for systems with interval time-varying delays. In [9], via a Lyapunov-Krasovskii functional with fewer matrix variables whose derivative is estimated using the convex analysis method, a simple stability criterion was obtained; this result is improved in [10] by using the reciprocally convex approach. Recently, some new methods were proposed to the stability analysis of interval time-varying delay systems. New classes of Lyapunov-Krasovskii functionals and augmented Lyapunov-Krasovskii functionals were introduced in [11–15], where some multi-integral terms were introduced in Lyapunov-Krasovskii functionals. In [16–18], combining with a decomposition approach, the upper bound of the derivative of Lyapunov-Krasovskii functional was estimated tightly and new stability results were obtained. It has been shown that these new methods can be applied efficiently to derive less conservative stability results for systems with interval time-varying delay.

On the other hand, singular systems have been extensively studied in the past years due to the fact that singular systems can better describe the behavior of some physical systems than regular ones. Singular systems are also referred to as descriptor systems, differential algebraic systems, or semistate systems. A great number of fundamental results based on the theory of regular systems have been extended to the area of singular systems [19]. It is well known that the stability analysis for singular systems is much more complicated than that for regular systems because it requires considering not only stability but also regularity and absence of impulse (for continuous singular systems) [20–30] or causality (for discrete singular systems) [31–34]. Recently, more and more attention has been paid to singular systems with delay. To obtain delay-dependent conditions, many efforts have been made in the literature. Via different Lyapunov-Krasovskii functionals, some stability criteria were obtained in [20, 21], and the results in [20, 21] were improved in [22, 23]

using the free-weighting matrices and discretized Lyapunov-Krasovskii functional, respectively. However, the involved time delays in [20–23] are all time invariant, which limits the scope of applications of the given results. In the case where time-varying delays appear in singular systems, some stability results were proposed in [24–27]. The range of time-varying delay considered in [24–27] is from zero to an upper bound. For singular systems with interval time-varying delay, there are fewer results [28–30] and there still exists some room for further investigation.

In this paper, our purpose is to present some new delay-dependent stability criteria for singular systems with interval time-varying delay. An appropriate Lyapunov-Krasovskii functional is constructed by using the idea of "delay decomposition." Employing this Lyapunov-Krasovskii functional, some new delay-dependent sufficient conditions ensuring the stability for the considered systems are obtained in terms of LIMs. The main contribution of this paper is in two aspects. First, the new results will be less conservative than the existing ones, which will be demonstrated by two numerical examples. Second, the new results will involve fewer decision variables than some existing ones; hence, they are mathematically less complex and more computationally efficient.

Notations. Throughout this paper, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, while $\mathbf{R}^{m\times n}$ refers to the set of all real matrices with m rows and n columns. A^T represents the transpose of the matrix A. For real symmetric matrices X and Y, the notation $X \geqslant Y$ (resp., $X > Y$) means matrix $X - Y$ is positive-semidefinite (resp., positive-definite). I is the identity matrix with appropriate dimensions. $\|x\|$ refers to the Euclidean norm of the vector x; that is, $\|x\| = \sqrt{x^T x}$.

2. Problem Formulation and Preliminaries

Consider the following singular system with interval time-varying delay described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t - d(t)), \\ x(t) &= \varphi(t), \quad t \in [-d_2, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector and the initial condition $\varphi(t) \in \mathbf{R}^n$ is a continuously differentiable vector-valued function. The matrix $E \in \mathbf{R}^{n\times n}$ may be singular and ranks $E = r \leqslant n$, $A, A_d \in \mathbf{R}^{n\times n}$ are known real constant matrices. $d(t)$ is the time-varying delay satisfying

$$d_1 \leqslant d(t) \leqslant d_2, \quad \dot{d}(t) \leqslant \mu, \tag{2}$$

where $0 < d_1 < d_2$ and $0 \leqslant \mu < 1$ are known constant scalars.

In this paper, our objective is to establish new delay-dependent stability conditions for singular time-delay system (1). The following definitions and lemma will be used in the proof of our main results.

Definition 1 (see [19]). (i) The pair (E, A) is said to be regular if $\det(sE - A)$ is not identically zero. (ii) The pair (E, A) is said to be impulse-free if $\deg(\det(sE - A)) = \operatorname{rank} E$.

Definition 2 (see [28, 30]). (i) The singular time-delay system (1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free. (ii) The singular time-delay system (1) is said to be stable if, for any $\varepsilon > 0$, there exists a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\varphi(t)$ satisfying $\sup_{-d_2 \leqslant t \leqslant 0} \|\varphi(t)\| \leqslant \delta(\varepsilon)$, the solution $x(t)$ of system (1) satisfies $\|x(t)\| \leqslant \varepsilon$ for any $t \geqslant 0$; moreover, $\lim_{t\to\infty} x(t) = 0$.

Lemma 3 (see [1]). *For any symmetric positive defined matrix $R > 0$, scalars $\gamma_2 > \gamma_1 \geqslant 0$, and vector function $x : [\gamma_1, \gamma_2] \mapsto \mathbf{R}^n$ such that the following integrations are well defined, the following inequality holds*

$$-(\gamma_2 - \gamma_1)\int_{\gamma_1}^{\gamma_2} x^T(s) R x(s)\, ds \leqslant -\int_{\gamma_1}^{\gamma_2} x^T(s)\, ds R \int_{\gamma_1}^{\gamma_2} x(s)\, ds. \tag{3}$$

3. Main Results

In this section, we consider the stability problem of singular time-delay system (1). We first present a delay-dependent stability criterion for singular time-delay system (1) as follows.

Theorem 4. *Given scalars $0 < d_1 < d_2$ and $0 \leqslant \mu < 1$ for any delay $d(t)$ satisfying* (2)*, singular time-delay system* (1) *is regular, impulse-free, and stable if there exist the following matrices:*

$$\begin{aligned} &P > 0, \quad W > 0, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} > 0, \\ &R = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} > 0, \quad S_i > 0, \\ &Z_i > 0, \quad M_i \ (i = 1, 2), \end{aligned} \tag{4}$$

and U, such that the LMIs

$$\Xi + \Xi_k < 0, \tag{5}$$

$$\begin{bmatrix} Z_k & M_k \\ M_k^T & Z_k \end{bmatrix} > 0 \tag{6}$$

hold for any $k \in \{1, 2\}$, where

$$\begin{aligned} \Xi &= e_1\left(E^T P + U\Psi^T\right)e_0^T + e_0\left(PE + \Psi U^T\right)e_1^T \\ &\quad + e_0\left(\left(\frac{d_1}{2}\right)^2 (S_1 + S_2) + \left(\frac{d_2 - d_1}{2}\right)^2 (Z_1 + Z_2)\right)e_0^T \\ &\quad + e_3 W e_3^T - (1 - \mu)\, e_6 W e_6^T + [e_1 \ \ e_2]\, Q\, [e_1 \ \ e_2]^T \end{aligned}$$

$$
\begin{aligned}
&- [e_2 \;\; e_3] Q [e_2 \;\; e_3]^T + [e_3 \;\; e_4] R [e_3 \;\; e_4]^T \\
&- [e_4 \;\; e_5] R [e_4 \;\; e_5]^T - (e_1 - e_2) E^T S_1 E (e_1 - e_2)^T \\
&- (e_2 - e_3) E^T S_2 E (e_2 - e_3)^T, \\
\Xi_1 = \; &- (e_4 - e_5) E^T Z_2 E (e_4 - e_5)^T \\
&- \begin{bmatrix} e_6^T - e_4^T \\ e_3^T - e_6^T \end{bmatrix}^T \begin{bmatrix} E^T Z_1 E & E^T M_1 E \\ E^T M_1^T E & E^T Z_1 E \end{bmatrix} \begin{bmatrix} e_6^T - e_4^T \\ e_3^T - e_6^T \end{bmatrix}, \\
\Xi_2 = \; &- (e_3 - e_4) E^T Z_1 E (e_3 - e_4)^T \\
&- \begin{bmatrix} e_6^T - e_5^T \\ e_4^T - e_6^T \end{bmatrix}^T \begin{bmatrix} E^T Z_2 E & E^T M_2 E \\ E^T M_2^T E & E^T Z_2 E \end{bmatrix} \begin{bmatrix} e_6^T - e_5^T \\ e_4^T - e_6^T \end{bmatrix},
\end{aligned} \tag{7}
$$

and $\Psi \in \mathbf{R}^{n\times(n-r)}$ *is any full-column rank matrix satisfying* $E^T\Psi = 0$, $e_i = [0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(6-i)n}]^T$ $(i = 1, 2, \ldots, 6)$ *are block entry matrices, and* $e_0 = (Ae_1^T + A_d e_6^T)^T$.

Proof. The proof is divided into two parts. The first one deals with the regularity and impulse-free property, and the second one treats the stability property of the studied class of systems. Let us first of all show that the singular time-delay system (1) is regular and impulse-free. Using (5), it is easy to see that the following inequality holds:

$$
\begin{aligned}
&E^T PA + A^T PE + U\Psi^T A + A^T \Psi U^T + \left(\frac{d_1}{2}\right)^2 A^T (S_1 + S_2) A \\
&+ \left(\frac{d_2 - d_1}{2}\right)^2 A^T (Z_1 + Z_2) A + Q_1 - E^T S_1 E < 0.
\end{aligned} \tag{8}
$$

Using the fact that $S_i > 0$, $Z_i > 0$ $(i = 1, 2)$, and $Q_1 > 0$, from (8), we have

$$
E^T PA + A^T PE + U\Psi^T A + A^T \Psi U^T - E^T S_1 E < 0. \tag{9}
$$

It follows from (9) and Lemma 3.5 in [32] that the pair (E, A) is regular and impulse-free. Thus, according to Definition 2, singular time-delay system (1) is regular and impulse-free.

Next, we will show the stability of the singular time-delay system (1). To the end, the following Lyapunov-Krasovskii functional for system (1) is considered:

$$
V(x_t) = \sum_{i=1}^{4} V_i(x_t), \tag{10}
$$

where

$$
\begin{aligned}
V_1(x_t) &= x^T(t) E^T P E x(t) + \int_{t-d(t)}^{t-d_1} x^T(s) W x(s)\, \mathrm{d}s, \\
V_2(x_t) &= \int_{t-(d_1/2)}^{t} \xi_1^T(s) Q \xi_1(s)\, \mathrm{d}s \\
&\quad + \int_{t-((d_1+d_2)/2)}^{t-d_1} \xi_2^T(s) R \xi_2(s)\, \mathrm{d}s, \\
V_3(x_t) &= \frac{d_1}{2} \left(\int_{-d_1/2}^{0} \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T S_1 E \dot{x}(\beta)\, \mathrm{d}\beta \mathrm{d}\alpha \right. \\
&\quad \left. + \int_{-d_1}^{-d_1/2} \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T S_2 E \dot{x}(\beta)\, \mathrm{d}\beta \mathrm{d}\alpha \right), \\
V_4(x_t) &= \frac{d_2 - d_1}{2} \\
&\quad \times \left(\int_{-(d_1+d_2)/2}^{-d_1} \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T Z_1 E \dot{x}(\beta)\, \mathrm{d}\beta \mathrm{d}\alpha \right. \\
&\quad \left. + \int_{-d_2}^{-(d_1+d_2)/2} \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T Z_2 E \dot{x}(\beta)\, \mathrm{d}\beta \mathrm{d}\alpha \right),
\end{aligned} \tag{11}
$$

with

$$
\begin{aligned}
\xi_1(s) &= \left[x^T(s) \;\; x^T\left(s - \frac{d_1}{2}\right) \right]^T, \\
\xi_2(s) &= \left[x^T(s) \;\; x^T\left(s - \frac{d_2 - d_1}{2}\right) \right]^T.
\end{aligned} \tag{12}
$$

Define $\xi(t) = [x^T(t) \;\; x^T(t-(d_1/2)) \;\; x^T(t-d_1) \;\; x^T(t-((d_1+d_2)/2)) \;\; x^T(t-d_2) \;\; x^T(t-d(t))]^T$. Now calculating the derivative of $V(x_t)$ along the trajectory of the system (1), we derive

$$
\begin{aligned}
\dot{V}_1(x_t) &= 2x^T(t) E^T P E \dot{x}(t) + x^T(t-d_1) W x(t-d_1) \\
&\quad - \left(1 - \dot{d}(t)\right) x^T(t-d(t)) W x(t-d(t)) \\
&\leqslant \xi^T(t) \left(e_1 E^T P e_0^T + e_0 P E e_1^T + e_3 W e_3^T \right. \\
&\quad \left. - (1-\mu) e_6 W e_6^T \right) \xi(t), \\
\dot{V}_2(x_t) &= \xi_1^T(t) Q \xi_1(t) - \xi_1^T\left(t - \frac{d_1}{2}\right) Q \xi_1\left(t - \frac{d_1}{2}\right) \\
&\quad + \xi_2^T(t-d_1) R \xi_2(t-d_1) - \xi_2^T\left(t - \frac{d_1+d_2}{2}\right) \\
&\quad \times R \xi_2\left(t - \frac{d_1+d_2}{2}\right)
\end{aligned}
$$

$$= \xi^T(t)\Big([e_1\ \ e_2]Q[e_1\ \ e_2]^T - [e_2\ \ e_3] \times Q[e_2\ \ e_3]^T + [e_3\ \ e_4]R[e_3\ \ e_4]^T - [e_4\ \ e_5]R[e_4\ \ e_5]^T\Big)\xi(t),$$

$$\dot V_3(x_t) = \left(\frac{d_1}{2}\right)^2 \dot x^T(t)E^T(S_1+S_2)E\dot x(t) - \frac{d_1}{2}\int_{t-(d_1/2)}^{t}\dot x^T(s)E^TS_1E\dot x(s)\,ds - \frac{d_1}{2}\int_{t-d_1}^{t-(d_1/2)}\dot x^T(s)E^TS_2E\dot x(s)\,ds. \tag{13}$$

By Lemma 3, it can be shown that

$$\begin{aligned}\dot V_3(x_t) &\leqslant \left(\frac{d_1}{2}\right)^2 \dot x^T(t)E^T(S_1+S_2)E\dot x(t) - \left(x(t)-x\left(t-\frac{d_1}{2}\right)\right)^T \times E^TS_1E\left(x(t)-x\left(t-\frac{d_1}{2}\right)\right) - \left(x\left(t-\frac{d_1}{2}\right)-x(t-d_1)\right)^T E^TS_2E \times \left(x\left(t-\frac{d_1}{2}\right)-x(t-d_1)\right)\\ &= \xi^T(t)\left(\left(\frac{d_1}{2}\right)^2 e_0(S_1+S_2)e_0^T - (e_1-e_2)E^TS_1E(e_1-e_2)^T - (e_2-e_3)E^TS_2E(e_2-e_3)^T\right)\xi(t),\end{aligned} \tag{14}$$

$$\dot V_4(x_t) = \left(\frac{d_2-d_1}{2}\right)^2 \dot x^T(t)E^T(Z_1+Z_2)E\dot x(t) - \frac{d_2-d_1}{2}\int_{t-((d_1+d_2)/2)}^{t-d_1}\dot x^T(s)E^TZ_1E\dot x(s)\,ds - \frac{d_2-d_1}{2}\int_{t-d_2}^{t-((d_1+d_2)/2)}\dot x^T(s)E^TZ_2E\dot x(s)\,ds. \tag{15}$$

When $d_1 \leqslant d(t) \leqslant (d_1+d_2)/2$, we have

$$\dot V_4(x_t) = \left(\frac{d_2-d_1}{2}\right)^2 \dot x^T(t)E^T(Z_1+Z_2)E\dot x(t) - \frac{d_2-d_1}{2}\int_{t-d_2}^{t-((d_1+d_2)/2)}\dot x^T(s)E^TZ_2E\dot x(s)\,ds - \frac{d_2-d_1}{2}\int_{t-((d_1+d_2)/2)}^{t-d(t)}\dot x^T(s)E^TZ_1E\dot x(s)\,ds - \frac{d_2-d_1}{2}\int_{t-d(t)}^{t-d_1}\dot x^T(s)E^TZ_1E\dot x(s)\,ds. \tag{16}$$

Using Lemma 3 again, we get

$$\begin{aligned}\dot V_4(x_t) \leqslant{}& \left(\frac{d_2-d_1}{2}\right)^2 \dot x^T(t)E^T(Z_1+Z_2)E\dot x(t) - \left(x\left(t-\frac{d_1+d_2}{2}\right)-x(t-d_2)\right)^T E^TZ_2E \times \left(x\left(t-\frac{d_1+d_2}{2}\right)-x(t-d_2)\right) - \left(1+\frac{2(d(t)-d_1)}{d_1+d_2-2d(t)}\right) \times \left(x(t-d(t))-x\left(t-\frac{d_1+d_2}{2}\right)\right)^T E^TZ_1E \times \left(x(t-d(t))-x\left(t-\frac{d_1+d_2}{2}\right)\right) - \left(1+\frac{d_1+d_2-2d(t)}{2(d(t)-d_1)}\right) \times (x(t-d_1)-x(t-d(t)))^T E^TZ_1E \times (x(t-d_1)-x(t-d(t))).\end{aligned} \tag{17}$$

Define

$$J = \left[\sqrt{\frac{2(d(t)-d_1)}{d_1+d_2-2d(t)}} \times \left(x(t-d(t))-x\left(t-\frac{d_1+d_2}{2}\right)\right)^T E^T - \sqrt{\frac{d_1+d_2-2d(t)}{2(d(t)-d_1)}}(x(t-d_1)-x(t-d(t)))^T E^T\right], \tag{18}$$

with the condition (6); then, premultiplying and postmultiplying $\left[\begin{smallmatrix} Z_1 & M_1 \\ M_1^T & Z_1 \end{smallmatrix}\right] > 0$ by J and J^T, respectively, we can obtain

$$-\frac{2(d(t)-d_1)}{d_1+d_2-2d(t)}\left(x(t-d(t))-x\left(t-\frac{d_1+d_2}{2}\right)\right)^T \times E^TZ_1E\left(x(t-d(t))-x\left(t-\frac{d_1+d_2}{2}\right)\right) - \frac{d_1+d_2-2d(t)}{2(d(t)-d_1)}(x(t-d_1)-x(t-d(t)))^T$$

$$\times E^T Z_1 E\left(x\left(t-d_1\right)-x\left(t-d\left(t\right)\right)\right)$$
$$\leqslant -\left(x\left(t-d\left(t\right)\right)-x\left(t-\frac{d_1+d_2}{2}\right)\right)^T$$
$$\times E^T M_1 E\left(x\left(t-d_1\right)-x\left(t-d\left(t\right)\right)\right)$$
$$-\left(x\left(t-d_1\right)-x\left(t-d\left(t\right)\right)\right)^T E^T M_1^T E$$
$$\times\left(x\left(t-d\left(t\right)\right)-x\left(t-\frac{d_1+d_2}{2}\right)\right); \quad (19)$$

hence,

$$\dot{V}_4\left(x_t\right)\leqslant\left(\frac{d_2-d_1}{2}\right)^2\dot{x}^T\left(t\right)E^T\left(Z_1+Z_2\right)E\dot{x}\left(t\right)$$
$$-\left(x\left(t-\frac{d_1+d_2}{2}\right)-x\left(t-d_2\right)\right)^T E^T Z_2 E$$
$$\times\left(x\left(t-\frac{d_1+d_2}{2}\right)-x\left(t-d_2\right)\right)$$
$$-\begin{bmatrix} x(t-d(t))-x\left(t-\frac{d_1+d_2}{2}\right)\\ x(t-d_1)-x(t-d(t))\end{bmatrix}^T$$
$$\times\begin{bmatrix} E^T Z_1 E & E^T M_1 E\\ E^T M_1^T E & E^T Z_1 E\end{bmatrix}$$
$$\times\begin{bmatrix} x\left(t-d\left(t\right)\right)-x\left(t-\frac{d_1+d_2}{2}\right)\\ x\left(t-d_1\right)-x\left(t-d\left(t\right)\right)\end{bmatrix}$$
$$=\xi^T\left(t\right)\left(\left(\frac{d_2-d_1}{2}\right)^2 e_0\left(Z_1+Z_2\right)e_0^T\right.$$
$$-\left(e_4-e_5\right)E^T Z_2 E$$
$$\times\left(e_4-e_5\right)^T-\begin{bmatrix} e_6^T-e_4^T\\ e_3^T-e_6^T\end{bmatrix}^T$$
$$\times\begin{bmatrix} E^T Z_1 E & E^T M_1 E\\ E^T M_1^T E & E^T Z_1 E\end{bmatrix}$$
$$\left.\times\begin{bmatrix} e_6^T-e_4^T\\ e_3^T-e_6^T\end{bmatrix}\right)\xi\left(t\right). \quad (20)$$

Note that when $d(t)=d_1$ or $d(t)=(d_1+d_2)/2$, we have $\xi^T(t)(e_3-e_6)=0$ or $\xi^T(t)(e_6-e_4)=0$, respectively. So relation (20) still holds.

When $(d_1+d_2)/2\leqslant d(t)\leqslant d_2$, (15) can be rewritten as

$$\dot{V}_4\left(x_t\right)=\left(\frac{d_2-d_1}{2}\right)^2\dot{x}^T\left(t\right)E^T\left(Z_1+Z_2\right)E\dot{x}\left(t\right)$$
$$-\frac{d_2-d_1}{2}\int_{t-((d_1+d_2)/2)}^{t-d_1}\dot{x}^T\left(s\right)E^T Z_1 E\dot{x}\left(s\right)\mathrm{d}s$$
$$-\frac{d_2-d_1}{2}\int_{t-d_2}^{t-d(t)}\dot{x}^T\left(s\right)E^T Z_2 E\dot{x}\left(s\right)\mathrm{d}s$$
$$-\frac{d_2-d_1}{2}\int_{t-d(t)}^{t-((d_1+d_2)/2)}\dot{x}^T\left(s\right)E^T Z_2 E\dot{x}\left(s\right)\mathrm{d}s. \quad (21)$$

From condition (6) and Lemma 3, proceeding in a similar manner as above, we can get

$$\dot{V}_4\left(x_t\right)\leqslant\xi^T\left(t\right)\left(\left(\frac{d_2-d_1}{2}\right)^2 e_0\left(Z_1+Z_2\right)e_0^T\right.$$
$$-\left(e_3-e_4\right)E^T Z_1 E\left(e_3-e_4\right)^T$$
$$-\begin{bmatrix} e_6^T-e_5^T\\ e_4^T-e_6^T\end{bmatrix}^T\begin{bmatrix} E^T Z_2 E & E^T M_2 E\\ E^T M_2^T E & E^T Z_2 E\end{bmatrix} \quad (22)$$
$$\left.\times\begin{bmatrix} e_6^T-e_5^T\\ e_4^T-e_6^T\end{bmatrix}\right)\xi\left(t\right).$$

Noting $E^T\Psi=0$, we can deduce

$$0=2x^T\left(t\right)U\Psi^T E\dot{x}\left(t\right)=2\xi^T\left(t\right)e_1 U\Psi^T e_0^T\xi\left(t\right), \quad (23)$$

where U is any matrix with appropriate dimensions.

Combining (10)–(14), (20)–(23) yield

$$\dot{V}\left(x_t\right)\leqslant\begin{cases}\xi^T\left(t\right)\left(\Xi+\Xi_1\right)\xi\left(t\right), & d_1\leqslant d\left(t\right)\leqslant\frac{d_1+d_2}{2},\\ \xi^T\left(t\right)\left(\Xi+\Xi_2\right)\xi\left(t\right), & \frac{d_1+d_2}{2}\leqslant d\left(t\right)\leqslant d_2.\end{cases} \quad (24)$$

Therefore, we can see if LMIs (5) and (6) hold for any $k\in\{1,2\}$; then $\dot{V}(x_t)<0$ holds for any $d(t)\in[d_1,d_2]$. Hence, there exist a sufficiently small scalar $\varepsilon>0$, such that

$$\dot{V}\left(t\right)\leqslant-\varepsilon\|x(t)\|^2. \quad (25)$$

By (25), the following steps are similar to the proofs of Theorem 3.1 in [25] and Proposition 1 in [28]; we deduce that singular time-delay system (1) is stable. This completes the proof. □

Remark 5. Based on the Lyapunov-Krasovskii functional (10), Theorem 4 proposed a new delay-dependent criterion guaranteeing the singular time-delay system (1) to be regular, impulse-free, and stable. Lyapunov-Krasovskii functional

(10) is constructed by using the idea of "delay decomposition" [28–30]. We decompose the delay intervals $[0, d_1]$ and $[d_1, d_2]$ into two equidistant subintervals, respectively, such that the information of delay states $x(t - (d_1/2))$ and $x(t - ((d_1 + d_2)/2))$ is all taken into account. Compared with [29], in which the Lyapunov-Krasovskii functional was constructed by decomposing delay intervals $[0, d_1]$ and $[0, d_2]$ into two equidistant subintervals, respectively, our Lyapunov-Krasovskii functional contains the delay state $x(t - ((d_1 + d_2)/2))$. Thus, by checking the variation of $\dot{V}(x_t)$ for the case when $d(t) \in [d_1, (d_1 + d_2)/2]$ or $d(t) \in [(d_1 + d_2)/2, d_2]$, respectively, we introduce a new estimation on the upper bound of $\dot{V}(x_t)$. That is to say, the delay state $x(t - ((d_1 + d_2)/2))$ plays a key role in the further reduction of conservatism. Furthermore, our Lyapunov-Krasovskii functional can be easily extended to the case as the number of delay partitions grows, which can be seen from the following discussion.

In the following, by decomposing the delay intervals $[0, d_1]$ and $[d_1, d_2]$ into $N(N \geqslant 2)$ equidistant subintervals, respectively, we will derive a more general stability result than Theorem 4. For this purpose, the following Lyapunov-Krasovskii functional for singular time-delay system (1) is considered:

$$\begin{aligned}
\widetilde{V}(x_t) &= x^T(t) E^T P E x(t) + \int_{t-d(t)}^{t-d_1} x^T(s) W x(s)\, \mathrm{d}s \\
&\quad + \sum_{i=1}^{N-1} \left(\int_{t-(i/N)d_1}^{t-((i-1)/N)d_1} \eta_1^T(s) Q_i \eta_1(s)\, \mathrm{d}s \right. \\
&\qquad \left. + \int_{t-d_1-(i/N)(d_2-d_1)}^{t-d_1-((i-1)/N)(d_2-d_1)} \eta_2^T(s) R_i \eta_2(s)\, \mathrm{d}s \right) \\
&\quad + \sum_{i=1}^{N} \left(\frac{d_1}{N} \int_{-(i/N)d_1}^{-((i-1)/N)d_1} \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T \right. \\
&\qquad \times S_i E \dot{x}(\beta)\, \mathrm{d}\beta \mathrm{d}\alpha \\
&\qquad + \frac{d_2 - d_1}{N} \int_{-d_1-(i/N)(d_2-d_1)}^{-d_1-((i-1)/N)(d_2-d_1)} \\
&\qquad \times \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T Z_i E \dot{x} \\
&\qquad \left. \times (\beta)\, \mathrm{d}\beta \mathrm{d}\alpha \right),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
\eta_1(s) &= \left[x^T(s) \quad x^T\left(s - \frac{1}{N} d_1\right) \right]^T, \\
\eta_2(s) &= \left[x^T(s) \quad x^T\left(s - \frac{1}{N}(d_2 - d_1)\right) \right]^T,
\end{aligned}$$

$$\begin{aligned}
&P > 0, \quad W > 0, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} > 0, \\
&R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \\
&S_j > 0, \quad Z_j > 0 \quad (j = 1, 2, \ldots, N).
\end{aligned} \tag{27}$$

Using Lyapunov-Krasovskii functional (26), the following theorem can be obtained.

Theorem 6. *Given an integer $N \geqslant 2$ and scalars $0 < d_1 < d_2$, $0 \leqslant \mu < 1$, for any delay $d(t)$ satisfying (2), singular time-delay system (1) is regular, impulse-free, and stable if there exist matrices*

$$\begin{aligned}
&P > 0, \quad W > 0, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} > 0, \\
&R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \\
&S_j > 0, \quad Z_j > 0, \quad M_j \ (j = 1, 2, \ldots, N)
\end{aligned} \tag{28}$$

and U, such that LMIs

$$\Gamma + \Gamma_k < 0, \tag{29}$$

$$\begin{bmatrix} Z_k & M_k \\ M_k^T & Z_k \end{bmatrix} > 0 \tag{30}$$

hold for any $k \in \{1, 2, \ldots, N\}$, where

$$\begin{aligned}
\Gamma &= e_1 \left(E^T P + U \Psi^T \right) e_0^T + e_0 \left(PE + \Psi U^T \right) e_1^T \\
&\quad + e_{N+1} W e_{N+1}^T - (1 - \mu) e_{2N+2} W e_{2N+2}^T \\
&\quad + \sum_{i=1}^{N-1} \left(\left[e_i \quad e_{i+1} \right] Q_i \left[e_i \quad e_{i+1} \right]^T \right. \\
&\qquad - \left[e_{i+1} \quad e_{i+2} \right] Q_i \left[e_{i+1} \quad e_{i+2} \right]^T \\
&\qquad + \left[e_{N+i} \quad e_{N+i+1} \right] R_i \left[e_{N+i} \quad e_{N+i+1} \right]^T \\
&\qquad \left. - \left[e_{N+i+1} \quad e_{N+i+2} \right] R_i \left[e_{N+i+1} \quad e_{N+i+2} \right]^T \right) \\
&\quad + \sum_{i=1}^{N} \left(e_0 \left(\left(\frac{d_1}{N} \right)^2 S_i + \left(\frac{d_2 - d_1}{N} \right)^2 Z_i \right) e_0^T \right. \\
&\qquad - (e_i - e_{i+1}) E^T S_i E (e_i - e_{i+1})^T \\
&\qquad \left. - (e_{N+i} - e_{N+i+1}) E^T Z_i E (e_{N+i} - e_{N+i+1})^T \right),
\end{aligned}$$

$$\Gamma_k = \left(e_{N+k} - e_{N+k+1}\right) E^T Z_k E \left(e_{N+k} - e_{N+k+1}\right)^T - \begin{bmatrix} e_{2N+2}^T - e_{N+k+1}^T \\ e_{N+k}^T - e_{2N+2}^T \end{bmatrix}^T \begin{bmatrix} E^T Z_k E & E^T M_k E \\ E^T M_k^T E & E^T Z_k E \end{bmatrix} \times \begin{bmatrix} e_{2N+2}^T - e_{N+k+1}^T \\ e_{N+k}^T - e_{2N+2}^T \end{bmatrix}, \tag{31}$$

and $\Psi \in \mathbf{R}^{n\times(n-r)}$ *is any full-column rank matrix satisfying* $E^T\Psi = 0$, $e_i = \left[0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(2N-i+2)n}\right]^T$ $(i = 1, 2, \ldots, 2N+2)$ *are block entry matrices, and* $e_0 = (Ae_1^T + A_d e_{2N+2}^T)^T$.

Proof. The proof of Theorem 6 can be carried out using methods in the proof of Theorem 4. Hence, it is omitted. □

Remark 7. Based on the Lyapunov-Krasovskii functional (26), Theorem 6 proposed an improved delay-dependent criterion guaranteeing the singular time-delay system (1) to be regular, impulse-free, and stable. Lyapunov-Krasovskii functional (26) is constructed by decomposing the delay intervals $[0, d_1]$ and $[d_1, d_2]$ into N equidistant subintervals, respectively, such that the information of delay states $x(t - (i/N)d_1)$ and $x(t - d_1 - (i/N)(d_2 - d_1))$ $(i = 1, 2, \ldots, N)$ is all taken into account. Compared with [28, 30], the augmented vectors $\eta_1(s)$ and $\eta_2(s)$ were introduced in Lyapunov-Krasovskii functional (26), so (26) is more general than the ones in [28, 30].

Remark 8. If the integer N is set to 2, then Theorem 6 reduces to Theorem 4. That is, Theorem 6 is an extension of Theorem 4. It should be pointed out that the conservatism of Theorem 6 lies in the parameter N, which refers to the number of delay partitioning; that is, the conservatism is reduced as the partitions grow. On the other hand, the computational complexity also depends on the partition number N; that is, the computational complexity is increased as the partitions become thinner. Generally taking the small value of N such as $N = 2$ and $N = 3$, we can obtain less conservative and simple results. This can be seen from the simulation results in the sequel.

For system (1) with the routine delay case described by

$$0 \leqslant d \leqslant (t)\, d, \quad \dot{d}(t) \leqslant \mu, \tag{32}$$

where $d > 0$ and $0 \leqslant \mu < 1$ are known constant scalars, the corresponding Lyapunov-Krasovskii reduces to

$$\overline{V}(x_t) = x^T(t) E^T P E x(t) + \int_{t-d(t)}^{t} x^T(s) W x(s)\, \mathrm{d}s + \sum_{i=1}^{N-1} \int_{t-(i/N)d}^{t-((i-1)/N)d} \zeta^T(s) R_i \zeta(s)\, \mathrm{d}s + \frac{d}{N} \sum_{i=1}^{N} \int_{-(i/N)d}^{-((i-1)/N)d} \int_{t+\alpha}^{t} \dot{x}^T(\beta) E^T Z_i E \dot{x}(\beta)\, \mathrm{d}\beta \mathrm{d}\alpha, \tag{33}$$

where

$$\zeta(s) = \left[x^T(s) \;\; x^T\left(s - \frac{1}{N}d\right)\right]^T, \quad P > 0, \quad W > 0, \quad R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \quad Z_j > 0 \quad (j = 1, 2, \ldots, N). \tag{34}$$

Similar to the proof of Theorem 4, we can obtain the following delay-dependent stability criterion for singular time-delay system (1) with $d(t)$ satisfying (32).

Corollary 9. *Given an integer* $N \geqslant 2$ *and scalars* $d > 0$, $0 \leqslant \mu < 1$, *for any delay* $d(t)$ *satisfying* (32), *singular time-delay system* (1) *is regular, impulse-free, and stable if there exist matrices*

$$P > 0, \quad W > 0, \quad R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \quad Z_j > 0, \quad M_j \;\; (j = 1, 2, \ldots, N), \tag{35}$$

and U, *such that the LMI* (30) *and LMI*

$$\Upsilon + \Upsilon_k < 0 \tag{36}$$

hold for any $k \in \{1, 2, \ldots, N\}$, *where*

$$\Upsilon = e_1\left(E^T P + U\Psi^T\right) e_0^T + e_0\left(PE + \Psi U^T\right) e_1^T + e_1 W e_1^T - (1-\mu)\, e_{N+2} W e_{N+2}^T + \sum_{i=1}^{N-1} \left(\left[e_i \;\; e_{i+1}\right] R_i \left[e_i \;\; e_{i+1}\right]^T - \left[e_{i+1} \;\; e_{i+2}\right] R_i \left[e_{i+1} \;\; e_{i+2}\right]^T \right) + \sum_{i=1}^{N} \left(\left(\frac{d}{N}\right)^2 e_0 Z_i e_0^T - \left(e_i - e_{i+1}\right) E^T Z_i E \left(e_i - e_{i+1}\right)^T \right),$$

$$\Upsilon_k = \left(e_k - e_{k+1}\right) E^T Z_k E \left(e_k - e_{k+1}\right)^T - \begin{bmatrix} e_{N+2}^T - e_{k+1}^T \\ e_k^T - e_{N+2}^T \end{bmatrix}^T \times \begin{bmatrix} E^T Z_k E & E^T M_k E \\ E^T M_k^T E & E^T Z_k E \end{bmatrix} \begin{bmatrix} e_{N+2}^T - e_{k+1}^T \\ e_k^T - e_{N+2}^T \end{bmatrix}, \tag{37}$$

and $\Psi \in \mathbf{R}^{n\times(n-r)}$ *is any full-column rank matrix satisfying* $E^T\Psi = 0$, $e_i = \left[0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(N-i+2)n}\right]^T$ $(i = 1, 2, \ldots, N+2)$ *are block entry matrices, and* $e_0 = (Ae_1^T + A_d e_{N+2}^T)^T$.

Remark 10. Theorem 6 and Corollary 9 give stability criteria of system (1) with $d(t)$ satisfying (2) and (32), respectively. It is noted that the conditions in Theorem 6 and

Corollary 9 are both delay-dependent and rate-dependent. However, the information of delay rate may not be known in many cases, or $d(t)$ even may not be differentiable; then Theorem 6 and Corollary 9 fail to work. Regarding these circumstances, delay-dependent and rate-independent criteria can be derived by choosing $W = 0$ in Theorem 6 and Corollary 9.

For the case when $d(t)$ satisfies $0 < d_1 \leqslant d(t) \leqslant d_2$ and μ is unknown, the following result can be obtained from Theorem 6 by setting $W = 0$.

Corollary 11. *Given an integer $N \geqslant 2$ and scalars $0 < d_1 < d_2$, for any delay $d(t)$ satisfying $d_1 \leqslant d(t) \leqslant d_2$, singular time-delay system* (1) *is regular, impulse-free, and stable if there exist matrices*

$$P > 0, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} > 0,$$
$$R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \tag{38}$$
$$S_j > 0, \quad Z_j > 0, \quad M_j \ (j = 1, 2, \ldots, N),$$

and U, such that the LMI (30) *and LMI*

$$\Gamma_0 + \Gamma_k < 0 \tag{39}$$

hold for any $k \in \{1, 2, \ldots, N\}$, where

$$\begin{aligned}
\Gamma_0 = {} & e_1 \left(E^T P + U\Psi^T\right) e_0^T + e_0 \left(PE + \Psi U^T\right) e_1^T \\
& + \sum_{i=1}^{N-1} \Big(\left[e_i \ \ e_{i+1}\right] Q_i \left[e_i \ \ e_{i+1}\right]^T \\
& \quad - \left[e_{i+1} \ \ e_{i+2}\right] Q_i \left[e_{i+1} \ \ e_{i+2}\right]^T \\
& \quad + \left[e_{N+i} \ \ e_{N+i+1}\right] R_i \left[e_{N+i} \ \ e_{N+i+1}\right]^T \\
& \quad - \left[e_{N+i+1} \ \ e_{N+i+2}\right] R_i \left[e_{N+i+1} \ \ e_{N+i+2}\right]^T \Big) \\
& + \sum_{i=1}^{N} \left(e_0 \left(\left(\frac{d_1}{N}\right)^2 S_i + \left(\frac{d_2 - d_1}{N}\right)^2 Z_i \right) e_0^T \right. \\
& \quad - \left(e_i - e_{i+1}\right) E^T S_i E \left(e_i - e_{i+1}\right)^T \\
& \quad \left. - \left(e_{N+i} - e_{N+i+1}\right) E^T Z_i E \left(e_{N+i} - e_{N+i+1}\right)^T \right),
\end{aligned} \tag{40}$$

and Γ_k, Ψ, e_i $(i = 0, 1, 2, \ldots, 2N + 2)$ are the same as those defined in Theorem 6.

For the case when $d(t)$ satisfies $0 \leqslant d(t) \leqslant d$ and μ is unknown, the following result can be obtained from Corollary 9 by setting $W = 0$.

Corollary 12. *Given an integer $N \geqslant 2$ and scalar $d > 0$, for any delay $d(t)$ satisfying $0 \leqslant d(t) \leqslant d$, singular time-delay system* (1) *is regular, impulse-free, and stable if there exist matrices*

$$P > 0, \quad R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \tag{41}$$
$$Z_j > 0, \quad M_j \ (j = 1, 2, \ldots, N),$$

and U, such that the LMI (30) *and LMI*

$$\Upsilon_0 + \Upsilon_k < 0 \tag{42}$$

hold for any $k \in \{1, 2, \ldots, N\}$, where

$$\begin{aligned}
\Upsilon_0 = {} & e_1 \left(E^T P + U\Psi^T\right) e_0^T + e_0 \left(PE + \Psi U^T\right) e_1^T \\
& + \sum_{i=1}^{N-1} \Big(\left[e_i \ \ e_{i+1}\right] R_i \left[e_i \ \ e_{i+1}\right]^T - \left[e_{i+1} \ \ e_{i+2}\right] \\
& \quad \times R_i \left[e_{i+1} \ \ e_{i+2}\right]^T \Big) \\
& + \sum_{i=1}^{N} \left(\left(\frac{d}{N}\right)^2 e_0 Z_i e_0^T - \left(e_i - e_{i+1}\right) E^T Z_i E \left(e_i - e_{i+1}\right)^T \right),
\end{aligned} \tag{43}$$

and Υ_k, Ψ, e_i $(i = 0, 1, 2, \ldots, N + 2)$ are the same as those defined in Corollary 9.

Remark 13. Compared with the results in [20–25], the LMIs in Theorems 4 and 6 and Corollaries 9 and 11, 12 are all strict LMIs; thus, they can be directly solved by using any LMI toolbox like the one of Matlab or the one of Scilab.

When the matrix E is nonsingular, the stability problem of singular time-delay system (1) is reduced to analyzing the stability of the following regular system:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + A_d x(t - d(t)), \\
x(t) &= \varphi(t), \quad t \in [-d_2, 0].
\end{aligned} \tag{44}$$

This problem has been widely studied in the recent literature (see, e.g., [7–15]). Choose the following Lyapunov-Krasovskii functional:

$$\begin{aligned}
\widetilde{\widetilde{V}}\left(x_t\right) = {} & x^T(t) P x(t) + \int_{t-d(t)}^{t-d_1} x^T(s) W x(s)\, \mathrm{d}s \\
& + \sum_{i=1}^{N-1} \left(\int_{t-(i/N)d_1}^{t-((i-1)/N)d_1} \eta_1^T(s) Q_i \eta_1(s)\, \mathrm{d}s \right. \\
& \quad \left. + \int_{t-d_1-(i/N)(d_2-d_1)}^{t-d_1-((i-1)/N)(d_2-d_1)} \eta_2^T(s) R_i \eta_2(s)\, \mathrm{d}s \right)
\end{aligned}$$

Table 1: Allowable upper bound d_2 with $d_1 = 0$ for different μ.

Methods	$\mu = 0$	$\mu = 0.3$	$\mu = 0.7$
Zhong and Yang [20]	0.5567	—	—
Wu and Zhou [22]	1.0660	—	—
Li and Jia [24]	1.0660	1.0130	0.9510
Lin and Fei [28]	1.0660	1.0260	1.0220
Corollary 9 ($N = 2$)	1.0937	1.0597	1.0597
Corollary 9 ($N = 3$)	1.1000	1.0689	1.0689

Table 2: Allowable upper bound d_2 with given d_1 for different μ.

d_1	Methods	$\mu = 0$	$\mu = 0.3$	$\mu = 0.7$
0.3	Lin and Fei [28]	1.0660	1.0310	1.0290
	Jiao [29]	1.1075	1.0607	1.0607
	Theorem 4	1.1095	1.0704	1.0704
	Theorem 6 ($N = 3$)	1.1122	1.0729	1.0729
0.6	Lin and Fei [28]	1.0660	1.0420	1.0420
	Jiao [29]	1.1075	1.0679	1.0679
	Theorem 4	1.1166	1.0727	1.0727
	Theorem 6 ($N = 3$)	1.1184	1.0748	1.0748
1.1	Lin and Fei [28]	—	—	—
	Jiao [29]	1.1075	1.1066	1.1066
	Theorem 4	1.1078	1.1066	1.1066
	Theorem 6 ($N = 3$)	1.1155	1.1133	1.1133

$$
\begin{aligned}
&+\sum_{i=1}^{N}\left(\frac{d_1}{N}\int_{-(i/N)d_1}^{-((i-1)/N)d_1}\int_{t+\alpha}^{t}\dot{x}^T(\beta)\,S_i\dot{x}(\beta)\,\mathrm{d}\beta\mathrm{d}\alpha\right.\\
&\quad+\frac{d_2-d_1}{N}\int_{-d_1-(i/N)(d_2-d_1)}^{-d_1-((i-1)/N)(d_2-d_1)}\\
&\quad\left.\times\int_{t+\alpha}^{t}\dot{x}^T(\beta)\,Z_i\dot{x}(\beta)\,\mathrm{d}\beta\mathrm{d}\alpha\right),
\end{aligned}
\tag{45}
$$

where vectors $\eta_1(s), \eta_2(s)$ and matrices P, W, Q_i, R_i ($i = 1, 2, \ldots, N-1$), S_j, Z_j ($j = 1, 2, \ldots, N$) are defined the same as those defined in (26).

By employing the Lyapunov-Krasovskii functional (45) and using the similar proof of Theorem 4, we can obtain the following delay-dependent stability criterion for time-delay system (44).

Corollary 14. *Given an integer $N \geqslant 2$ and scalars $0 < d_1 < d_2$, $0 \leqslant \mu < 1$, for any delay $d(t)$ satisfying (2), time-delay system (44) is stable if there exist matrices*

$$
\begin{aligned}
&P > 0, \quad W > 0, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} > 0,\\
&R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1),\\
&S_j > 0, \quad Z_j > 0, \quad M_j \ \ (j = 1, 2, \ldots, N),
\end{aligned}
\tag{46}
$$

such that the LMI (30) and LMI (47)

$$
\widetilde{\Gamma} + \widetilde{\Gamma}_k < 0 \tag{47}
$$

hold for any $k \in \{1, 2, \ldots, N\}$, where

$$
\begin{aligned}
\widetilde{\Gamma} &= e_1 P e_0^T + e_0 P e_1^T + e_{N+1} W e_{N+1}^T - (1-\mu)\, e_{2N+2} W e_{2N+2}^T\\
&\quad+\sum_{i=1}^{N-1}\Big(\left[e_i \ \ e_{i+1}\right] Q_i \left[e_i \ \ e_{i+1}\right]^T - \left[e_{i+1} \ \ e_{i+2}\right]\\
&\qquad\times Q_i \left[e_{i+1} \ \ e_{i+2}\right]^T + \left[e_{N+i} \ \ e_{N+i+1}\right] R_i\\
&\qquad\times \left[e_{N+i} \ \ e_{N+i+1}\right]^T - \left[e_{N+i+1} \ \ e_{N+i+2}\right] R_i\\
&\qquad\times \left[e_{N+i+1} \ \ e_{N+i+2}\right]^T\Big)\\
&\quad+\sum_{i=1}^{N}\left(e_0\left(\left(\frac{d_1}{N}\right)^2 S_i + \left(\frac{d_2-d_1}{N}\right)^2 Z_i\right) e_0^T\right.\\
&\qquad-(e_i - e_{i+1})\, S_i (e_i - e_{i+1})^T\\
&\qquad\left.-(e_{N+i} - e_{N+i+1})\, Z_i (e_{N+i} - e_{N+i+1})^T\right),\\
\widetilde{\Gamma}_k &= (e_{N+k} - e_{N+k+1})\, Z_k (e_{N+k} - e_{N+k+1})^T\\
&\quad-\begin{bmatrix} e_{2N+2}^T - e_{N+k+1}^T \\ e_{N+k}^T - e_{2N+2}^T \end{bmatrix}^T\\
&\quad\times\begin{bmatrix} Z_k & M_k \\ M_k^T & Z_k \end{bmatrix}\begin{bmatrix} e_{2N+2}^T - e_{N+k+1}^T \\ e_{N+k}^T - e_{2N+2}^T \end{bmatrix},
\end{aligned}
\tag{48}
$$

and e_i ($i = 0, 1, 2, \ldots, 2N+2$) are the same as those defined in Theorem 6.

Based on Corollary 9, we can get the following stability criterion for system (44) with delay $d(t)$ satisfying (32).

Corollary 15. *Given an integer $N \geqslant 2$ and scalars $d > 0$, $0 \leqslant \mu < 1$, for any delay $d(t)$ satisfying (32), time-delay system (44) is stable if there exist matrices*

$$
\begin{aligned}
&P > 0, \quad W > 0,\\
&R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1),\\
&Z_j > 0, \quad M_j \ \ (j = 1, 2, \ldots, N),
\end{aligned}
\tag{49}
$$

such that the LMI (30) and LMI

$$
\widetilde{\Upsilon} + \widetilde{\Upsilon}_k < 0 \tag{50}
$$

hold for any $k \in \{1, 2, \ldots, N\}$, *where*

$$\begin{aligned}
\widetilde{\Upsilon} &= e_1 P e_0^T + e_0 P e_1^T + e_1 W e_1^T - (1-\mu)\, e_{N+2} W e_{N+2}^T \\
&\quad + \sum_{i=1}^{N-1} \Big([e_i \;\; e_{i+1}]\, R_i [e_i \;\; e_{i+1}]^T \\
&\qquad - [e_{i+1} \;\; e_{i+2}]\, R_i [e_{i+1} \;\; e_{i+2}]^T \Big) \\
&\quad + \sum_{i=1}^{N} \left(\left(\frac{d}{N} \right)^2 e_0 Z_i e_0^T - (e_i - e_{i+1})\, Z_i (e_i - e_{i+1})^T \right), \\
\widetilde{\Upsilon}_k &= (e_k - e_{k+1})\, Z_k (e_k - e_{k+1})^T - \begin{bmatrix} e_{N+2}^T - e_{k+1}^T \\ e_k^T - e_{N+2}^T \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} Z_k & M_k \\ M_k^T & Z_k \end{bmatrix} \begin{bmatrix} e_{N+2}^T - e_{k+1}^T \\ e_k^T - e_{N+2}^T \end{bmatrix},
\end{aligned} \tag{51}$$

and e_i $(i = 0, 1, 2, \ldots, N+2)$ *are the same as those defined in Corollary 9.*

When the information of μ is unknown, the following results can be obtained directly from Corollaries 11 and 12 for system (44) for two cases: $0 < d_1 \leqslant d(t) \leqslant d_2$ and $0 \leqslant d(t) \leqslant d$.

Corollary 16. *Given an integer* $N \geqslant 2$ *and scalars* $0 < d_1 < d_2$, *for any delay* $d(t)$ *satisfying* $d_1 \leqslant d(t) \leqslant d_2$, *time-delay system* (44) *is stable if there exist matrices*

$$\begin{aligned}
&P > 0, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} > 0, \\
&R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \\
&S_j > 0, \quad Z_j > 0, \quad M_j \;\; (j = 1, 2, \ldots, N),
\end{aligned} \tag{52}$$

such that the LMI (30) *and LMI*

$$\widetilde{\Gamma}_0 + \widetilde{\Gamma}_k < 0 \tag{53}$$

hold for any $k \in \{1, 2, \ldots, N\}$, *where*

$$\begin{aligned}
\widetilde{\Gamma}_0 &= e_1 P e_0^T + e_0 P e_1^T \\
&\quad + \sum_{i=1}^{N-1} \Big([e_i \;\; e_{i+1}]\, Q_i [e_i \;\; e_{i+1}]^T \\
&\qquad - [e_{i+1} \;\; e_{i+2}]\, Q_i [e_{i+1} \;\; e_{i+2}]^T \\
&\qquad + [e_{N+i} \;\; e_{N+i+1}]\, R_i [e_{N+i} \;\; e_{N+i+1}]^T \\
&\qquad - [e_{N+i+1} \;\; e_{N+i+2}]\, R_i [e_{N+i+1} \;\; e_{N+i+2}]^T \Big) \\
&\quad + \sum_{i=1}^{N} \left(e_0 \left(\left(\frac{d_1}{N} \right)^2 S_i + \left(\frac{d_2 - d_1}{N} \right)^2 Z_i \right) e_0^T \right. \\
&\qquad - (e_i - e_{i+1})\, S_i (e_i - e_{i+1})^T \\
&\qquad \left. - (e_{N+i} - e_{N+i+1})\, Z_i (e_{N+i} - e_{N+i+1})^T \right), \\
\widetilde{\Gamma}_k &= (e_{N+k} - e_{N+k+1})\, Z_k (e_{N+k} - e_{N+k+1})^T \\
&\quad - \begin{bmatrix} e_{2N+2}^T - e_{N+k+1}^T \\ e_{N+k}^T - e_{2N+2}^T \end{bmatrix}^T \begin{bmatrix} Z_k & M_k \\ M_k^T & Z_k \end{bmatrix} \begin{bmatrix} e_{2N+2}^T - e_{N+k+1}^T \\ e_{N+k}^T - e_{2N+2}^T \end{bmatrix},
\end{aligned} \tag{54}$$

and e_i $(i = 0, 1, 2, \ldots, 2N+2)$ *are the same as those defined in Theorem 6.*

Corollary 17. *Given an integer* $N \geqslant 2$ *and scalar* $d > 0$, *for any delay* $d(t)$ *satisfying* $0 \leqslant d(t) \leqslant d$, *time-delay system* (44) *is stable if there exist matrices*

$$\begin{aligned}
&P > 0, \quad R_i = \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix} > 0 \quad (i = 1, 2, \ldots, N-1), \\
&Z_j > 0, \quad M_j \;\; (j = 1, 2, \ldots, N),
\end{aligned} \tag{55}$$

such that the LMI (30) *and LMI*

$$\widetilde{\Upsilon}_0 + \widetilde{\Upsilon}_k < 0 \tag{56}$$

hold for any $k \in \{1, 2, \ldots, N\}$, *where*

$$\begin{aligned}
\widetilde{\Upsilon}_0 &= e_1 P e_0^T + e_0 P e_1^T \\
&\quad + \sum_{i=1}^{N-1} \Big([e_i \;\; e_{i+1}]\, R_i [e_i \;\; e_{i+1}]^T \\
&\qquad - [e_{i+1} \;\; e_{i+2}]\, R_i [e_{i+1} \;\; e_{i+2}]^T \Big)
\end{aligned}$$

Table 3: Allowable upper bound d_2 with given d_1 for different μ.

d_1	Methods	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.9$
2	Zhang et al. [8]	2.81	2.61	2.61
	Sun et al. [12]	3.01	2.56	2.56
	Qian and Liu [15]	3.02	2.69	2.69
	Corollary 14 ($N = 2$)	3.18	2.75	2.75
	Corollary 14 ($N = 3$)	3.21	2.77	2.77
3	Zhang et al. [8]	3.31	3.31	3.31
	Sun et al. [12]	3.34	3.34	3.34
	Qian and Liu [15]	3.41	3.41	3.41
	Corollary 14 ($N = 2$)	3.46	3.46	3.46
	Corollary 14 ($N = 3$)	3.48	3.48	3.48
4	Zhang et al. [8]	4.09	4.09	4.09
	Sun et al. [12]	4.16	4.16	4.16
	Qian and Liu [15]	4.20	4.20	4.20
	Corollary 14 ($N = 2$)	4.25	4.25	4.25
	Corollary 14 ($N = 3$)	4.28	4.28	4.28
5	Zhang et al. [8]	—	—	—
	Sun et al. [12]	5.02	5.02	5.02
	Qian and Liu [15]	5.03	5.03	5.03
	Corollary 14 ($N = 2$)	5.09	5.09	5.09
	Corollary 14 ($N = 3$)	5.12	5.12	5.12

$$+\sum_{i=1}^{N}\left(\left(\frac{d}{N}\right)^2 e_0 Z_i e_0^T - (e_i - e_{i+1}) Z_i (e_i - e_{i+1})^T\right),$$

$$\tilde{\Upsilon}_k = (e_k - e_{k+1}) Z_k (e_k - e_{k+1})^T - \begin{bmatrix} e_{N+2}^T - e_{k+1}^T \\ e_k^T - e_{N+2}^T \end{bmatrix}^T \begin{bmatrix} Z_k & M_k \\ M_k^T & Z_k \end{bmatrix} \begin{bmatrix} e_{N+2}^T - e_{k+1}^T \\ e_k^T - e_{N+2}^T \end{bmatrix}, \tag{57}$$

and e_i $(i = 0, 1, 2, \ldots, N + 2)$ *are the same as those defined in Corollary 9.*

4. Numerical Examples

In the section, numerical examples are given to illustrate the effectiveness and the less conservatism of obtained results in this paper.

Example 18. Consider the singular time-delay system (1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix}. \tag{58}$$

For $d_1 = 0$, the allowable upper bound d_2, which guarantees regular, impulse-free, and stable systems (1) for different μ, is listed in Table 1. From Table 1, it can be seen that the stability criteria in Corollary 9 are less conservative than those in [20, 22, 24, 28].

For systems with interval time-varying delay, the allowable upper bound d_2, which guarantees regular, impulse-free, and stable systems (1) with given lower bound d_1 for different μ, is listed in Table 2. From Table 2, it can be seen that the stability criteria in Theorems 4 and 6 are less conservative than those in [28, 29]. Especially, when $d_1 = 1.1$, the result in [28] is not feasible while the allowable upper bound d_2 can also be obtained from Theorems 4 and 6 in this paper.

Table 4: Number of decision variables.

Methods	Number of decision variables
Zhang et al. [8]	$13n^2 + 5n$
Sun et al. [12]	$18n^2 + 8n$
Qian and Liu [15]	$22n^2 + 8n$
Corollary 14 ($N = 2$)	$9n^2 + 5n$
Corollary 14 ($N = 3$)	$15n^2 + 8n$

Example 19. Consider the time-delay system (44) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{59}$$

This example is used in many recent papers such as [7–15, 29]. The purpose is to compare the allowable upper bound d_2 which guarantees the stability of the considered system for given lower bound d_1.

For various μ, the allowable upper bound d_2 is listed in Table 3. Moreover, the number of decision variables involved in the stability criteria is given in Table 4. From Tables 3 and 4, it can be seen that Corollary 14 ($N = 2$) in this paper has fewer decision variables and less conservatism than those results in [8, 12, 15].

For unknown μ, the allowable upper bound d_2 is listed in Table 5. From Table 5, it can be seen that Corollary 16 in this paper is less conservative than those results in [7–15, 29].

5. Conclusion

This paper deals with the problem of stability for singular systems with interval time-varying delay. By using a delay decomposition approach, new stability criteria for singular time-delay systems to be regular, impulse-free, and stable are proposed in terms of LMIs. Based on the obtained criterion, some improved stability results for the regular systems with interval time-varying delay are also given. The obtained results in this paper have been shown to be less conservative than recently reported results. Moreover, the proposed method decreases the computational complexity comparable to some existing methods. Two numerical examples are given to illustrate the applicability of the results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Table 5: Allowable upper bound d_2 with given d_1 for unknown μ.

Methods	$d_1 = 1$	$d_1 = 2$	$d_1 = 3$	$d_1 = 4$
He et al. [7]	1.74	2.43	3.22	4.06
Shao [9]	1.76	2.50	3.25	4.07
Zhang et al. [8] and Park et al. [10]	2.06	2.61	3.31	4.09
Sun et al. [12]	1.90	2.56	3.34	4.16
Qian et al. [14]	1.90	2.56	3.34	4.17
Sun et al. [11]	1.90	2.56	3.41	4.17
Qian and Liu [15]	2.09	2.69	3.41	4.20
Liu et al. [13]	2.09	2.69	3.41	4.20
Jiao [29]	2.12	2.72	3.45	4.25
Corollary 16 ($N = 2$)	2.21	2.75	3.46	4.26
Corollary 16 ($N = 3$)	2.25	2.77	3.48	4.28

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Research Article

Robust Predictive Control of Water Level in an Experimental Pilot Plant with Uncertain Input Delay

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This paper is concerned with the predictive control of water level in an experimental pilot plant, in the presence of input constraints and uncertain time delays. Robustness is achieved by casting the delay uncertainty into a polytopic form and using a predictive control formulation based on linear matrix inequalities. Integral action is introduced into the controller to ensure offset-free tracking of step changes in the setpoint. Moreover, an integrator resetting procedure based on the concept of regions of guaranteed cost is proposed to improve the resulting transient response. The results show that the introduction of robustness into the predictive control formulation is indeed of value to avoid closed-loop instability in the presence of time delays. In addition, the integrator resetting procedure was found to provide substantial performance improvements in terms of overshoot and settling time.

1. Introduction

Time delays, often associated with fluid transport processes, measurement procedures, communication lags, and computational processing, may degrade the performance or even cause instability of a control loop [1]. Under such conditions, the design of an appropriate controller can be a challenging task [2, 3], especially if the delay is uncertain or subject to unpredictable alterations. In this context, much research effort has been devoted to the development of robust control strategies that guarantee appropriate closed-loop behavior if the delay remains within known bounds [4–6]. Within this scope, robust model-based predictive control (RMPC) may be of particular value in cases where actuator and/or process variable constraints are to be enforced.

The term RMPC refers to a body of receding-horizon optimal control techniques aimed at ensuring stability and constraint satisfaction in the presence of model uncertainties [7, 8]. Early RMPC formulations involved the online solution of a min-max optimization problem, where the worst case value of the cost function was evaluated over a set of uncertain plants [9–11]. However, the computational cost could be prohibitive for real-time implementation. Within this scope, a significant step forward was made in [12] by casting the optimization problem into a semidefinite programming (SDP) form [13], with model uncertainties and operational constraints incorporated into a set of linear matrix inequalities (LMIs). The control actions can thus be obtained by using efficient numerical methods for convex optimization. Subsequent developments of this RMPC approach have included LMI formulations with less conservatism [14–17], use of output feedback [18–20], extensions to nonlinear systems [21], and handling of asymmetric output constraints [22], among others.

Time-delay systems were briefly mentioned in [12] as a remark concerning possible applications of the proposed RMPC technique. This possibility was formalized in a more recent paper [23], which concerned systems with polytopic uncertainty description and both input and state delays. The proposed formulation was later extended to handle multiple input delays [24], as well as uncertainties in both the model parameters and delay values [25]. A simpler approach can also be adopted if the uncertainty is restricted to the input delay. In this case, the original RMPC formulation of [12] can be employed by using an augmented state vector including current values of the state and past values of

the control input [26–28]. As can be seen, much research effort has been devoted to the control of time delay systems with RMPC techniques derived from [12]. However, to the best of the authors' knowledge, experimental studies of this RMPC approach for systems with uncertain delays have not been reported, either in laboratory scale or actual industrial practice. Such a gap motivated the present work, which is concerned with the control of water level in an experimental pilot plant in the presence of uncertain input delay.

Tank level control plays an important role in several industrial settings, including, for instance, chemical, pharmaceutical, and paper manufacturing processes [29]. This application has been widely employed in the literature and is often considered a benchmark control problem [30–32]. It is worth noting that the work reported in [32] involved the control of a laboratory tank system by using the RMPC formulation of [12]. However, that work employed a didactic plant of reduced scale and was not concerned with possible time delays. In contrast, the present investigation involves a pilot plant with actual industrial components, with emphasis on potential problems associated with the presence of time delay.

The RMPC approach adopted in this paper follows the formulation proposed in [12]. Uncertainty in the input delay is handled by using an augmented state vector, as in [28]. Moreover, offset-free tracking of step changes in the setpoint is ensured by introducing an integrator associated with the process variable of interest (water level). In order to reduce the overshoot and settling time of the transient response, an integrator resetting procedure is employed [33]. An improved resetting criterion is formulated herein, by using the concept of regions of guaranteed cost [34].

The remainder of this paper is organized as follows. Section 2 describes the LMI-based RMPC formulation adopted in the present work. Section 2.1 presents an overview of the RMPC formulation. Section 2.2 discusses the concept of regions of guaranteed cost, which will be used in the integrator resetting procedure. The handling of uncertain input delays is addressed in Section 2.3. The proposed integrator resetting scheme is presented in Section 3. The case study involving the experimental pilot plant is described in Section 4. Simulation and experimental results are discussed in Section 5. Lastly, concluding remarks are presented in Section 6.

2. LMI-Based Robust Model Predictive Control

2.1. Overview of the RMPC Formulation. The RMPC approach under consideration is concerned with discrete-time, uncertain state-space models of the form as follows:

$$x(k+1) = Ax(k) + Bu(k), \quad (A,B) \in \Omega, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the state and input variables, respectively. It is assumed that matrices A and B belong to an uncertainty polytope Ω with known vertices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, L$; that is,

$$(A,B) = \sum_{i=1}^{L} \lambda_i (A_i, B_i), \tag{2}$$

for some set of scalars $\lambda_1, \lambda_2, \ldots, \lambda_L$ such that

$$\sum_{i=1}^{L} \lambda_i = 1, \quad \lambda_i \geq 0 \tag{3}$$

Let $J_\infty(k)$ denote the following infinite-horizon cost function:

$$J_\infty(k) = \sum_{j=0}^{\infty} \left[x(k+j \mid k)^T S x(k+j \mid k) + u(k+j \mid k)^T R u(k+j \mid k) \right], \tag{4}$$

where $S \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are positive-definite weight matrices and $(\bullet \mid k)$ denotes a predicted value, which is computed on the basis of the information available at time k. It is assumed that the system state is directly measured, so that $x(k \mid k) = x(k)$.

The optimization problem to be solved at time k can be formulated as

$$\min_{u(k+j|k), j\geq 0} \max_{(A,B)\in\Omega} J_\infty(k), \tag{5}$$

subject to

$$\left| u_r(k+j \mid k) \right| \leq u_{r,\max}, \quad r = 1, 2, \ldots, m, \ j \geq 0, \tag{6}$$

where u_r denotes the rth input variable and $u_{r,\max}$ is the corresponding bound on its magnitude.

In the RMPC approach proposed in [12], an upper bound γ on the cost $J_\infty(k)$ is minimized by solving the following semidefinite programming (SDP) problem:

$$\min_{\gamma, Q>0, Y, X} \gamma, \tag{7}$$

subject to

$$\begin{bmatrix} Q & x(k) \\ * & 1 \end{bmatrix} \geq 0 \tag{8}$$

$$\begin{bmatrix} Q & 0 & 0 & A_iQ + B_iY \\ * & \gamma I & 0 & S^{1/2}Q \\ * & * & \gamma I & R^{1/2}Y \\ * & * & * & Q \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, L \tag{9}$$

$$\begin{bmatrix} X & Y \\ * & Q \end{bmatrix} \geq 0 \tag{10}$$

$$X_{rr} \leq u_{r,\max}^2, \quad r = 1, 2, \ldots, m, \tag{11}$$

where X_{rr} are the diagonal elements of X and symbol $*$ represents the elements below the main diagonal of a symmetric matrix. In what follows the SDP problem given by (7)–(11) will be termed $\mathbb{P}(x(k))$.

The resulting control action is given by $u(k) = Fx(k)$, with $F = YQ^{-1}$. By applying this control law in a receding horizon manner, that is, by solving $\mathbb{P}(x(k))$ in order to obtain a new gain matrix F at each sampling time k, the closed-loop system can be shown to be robustly asymptotically stable, provided that $\mathbb{P}(x(k))$ is feasible at the initial time $k = 0$. A detailed proof of this property can be found in [12].

Henceforth, a subscript k will be included in the solution $(\gamma_k, Q_k, Y_k, X_k)$ of $\mathbb{P}(x(k))$, as well as in the resulting gain matrix $F_k = Y_k Q_k^{-1}$, to indicate the corresponding time index.

Remark 1 (*tracking a constant reference*). In cases where the target state is an equilibrium point $x_{p,\text{ref}} \neq 0$, associated with a steady-state control $u_{p,\text{ref}}$, the RMPC control law can be implemented by expressing $x(k)$ as a deviation from the equilibrium value; that is,

$$x(k) = x_p(k) - x_{p,\text{ref}}, \tag{12}$$

where $x_p(k)$ denotes the actual state of the plant. The actual control $u_p(k)$ is obtained by solving problem $\mathbb{P}(x(k))$ and calculating

$$\begin{aligned} u(k) &= F_k x(k) \\ u_p(k) &= u(k) + u_{p,\text{ref}}. \end{aligned} \tag{13}$$

It is worth noting that the constraints on $u_p(k)$ must be translated in terms of symmetrical bounds on each component of $u(k)$ for use in the optimization problem $\mathbb{P}(x(k))$.

2.2. Regions of Guaranteed Cost. A region of guaranteed cost $\mathscr{D}^{\overline{\gamma}} \subset \mathbb{R}^n$ can be defined as the set of state vectors $x \in \mathbb{R}^n$ for which $\mathbb{P}(x)$ is feasible, with optimal cost γ^* being smaller or equal to a given bound $\overline{\gamma} > 0$. This region is convex and symmetric about the origin, as demonstrated in [34].

An extreme point of $\mathscr{D}^{\overline{\gamma}}$ along the direction of a given vector $\xi \in \mathbb{R}^n$ can be obtained by solving the following SDP problem [34]:

$$\min_{\gamma, Q>0, \beta<0, Y, X} \beta, \tag{14}$$

subject to

$$\begin{aligned} &\gamma \leq \overline{\gamma} \\ &\begin{bmatrix} Q & \beta\xi \\ * & 1 \end{bmatrix} \geq 0 \end{aligned} \tag{15}$$

and the remaining LMIs (9)–(11) of the original RMPC optimization problem. The resulting extreme point is given by $\beta^*\xi$, where β^* is the minimal value found for β. Due to the property of symmetry, $-\beta^*\xi$ will also be an extreme point. An inner approximation of $\mathscr{D}^{\overline{\gamma}}$ can be generated as the convex hull of extreme points obtained with different ξ vectors.

2.3. Handling of Uncertain Input Delays. In the present work, the time delay is considered to be the main source of uncertainty for the design of the controller. It is assumed that the dynamics of the plant can be described by a model of the form

$$z(k+1) = \Phi z(k) + \Gamma u(k-\tau), \tag{16}$$

where $z(k) \in \mathbb{R}^{n_z}$ denotes the plant state at time k, $\Phi \in \mathbb{R}^{n_z \times n_z}$, $\Gamma \in \mathbb{R}^{n_z \times m}$ are known matrices, and $\tau \in \mathbb{N}$ is an uncertain delay, which may take values in the range $[0, \overline{\tau}]$. For simplicity, the formulation presented herein is derived for a single-input system (i.e., $m = 1$), as in [28]. However, the extension to the multiple-input case does not pose substantial difficulties, involving mostly changes in notation [35].

The RMPC formulation can be applied by adopting the following state vector $x(k) \in \mathbb{R}^{n_z+\overline{\tau}}$, which comprises the plant state $z(k)$ and $\overline{\tau}$ past values of control:

$$x(k) \triangleq \begin{bmatrix} z(k) \\ u(k-1) \\ u(k-2) \\ \vdots \\ u(k-\overline{\tau}+1) \\ u(k-\overline{\tau}) \end{bmatrix}. \tag{17}$$

The dynamics of the plant with uncertain input delay can then be described by a model in the form (1) with

$$\begin{aligned} A &= \begin{bmatrix} \Phi & \lambda_1\Gamma & \lambda_2\Gamma & \cdots & \lambda_{\overline{\tau}-1}\Gamma & \lambda_{\overline{\tau}}\Gamma \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} \lambda_0\Gamma^T & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \end{aligned} \tag{18}$$

where the scalars $\lambda_1, \lambda_2, \ldots, \lambda_{\overline{\tau}}$ are such that

$$\lambda_d = \begin{cases} 1, & \text{if } d = \tau \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

From the definition of λ_d in (19), it follows that

$$\begin{aligned} &\lambda_d \in \{0, 1\}, \quad d = 0, 1, \ldots, \overline{\tau} \\ &\sum_{d=0}^{\overline{\tau}} \lambda_d = 1. \end{aligned} \tag{20}$$

Now, let $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{\overline{\tau}})$. In view of (20), Λ can only assume $\overline{\tau} + 1$ values; namely,

$$\begin{aligned} \Lambda_0 &= (1, 0, \ldots, 0) \\ \Lambda_1 &= (0, 1, \ldots, 0) \\ &\vdots \\ \Lambda_{\overline{\tau}} &= (0, 0, \ldots, 1). \end{aligned} \tag{21}$$

Therefore, the matrices (A, B) in (18) belong to a polytope Ω with $L = (\overline{\tau} + 1)$ vertices associated with $\Lambda_0, \Lambda_1, \ldots, \Lambda_{\overline{\tau}}$.

The control algorithm can be summarized in the following form. Given the matrices Φ and Γ of the model (16), as well as the maximum value $\overline{\tau}$ of the time delay, the vertices (A_i, B_i) of the uncertainty polytope Ω are written as

$$A_1 = \begin{bmatrix} \Phi & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \Gamma \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \Phi & \Gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{22}$$

$$\vdots$$

$$A_L = \begin{bmatrix} \Phi & 0 & 0 & \cdots & 0 & \Gamma \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_L = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

At the kth sampling time, the state vector $z(k)$ is measured, the augmented state vector $x(k)$ is built as in (17), and the optimization problem (7)–(11) is solved in order to obtain the matrix F_k. The control $u(k) = F_k x(k)$ is then calculated and applied to the plant.

Remark 2. This polytopic description of the input delay uncertainty was adopted in [26–28] as a direct extension of the original RMPC formulation of [12]. This approach can also be employed if the delay is lower-bounded by a known value $\underline{\tau} > 0$, in which case the vertices of the uncertainty polytope Ω corresponding to $\Lambda_0, \Lambda_1, \ldots, \Lambda_{\underline{\tau}-1}$ should be removed. If both the time delay and the model parameters were subject to uncertainties, an alternative would consist of using the more elaborate formulation proposed in [25]. However, as will be seen in Section 5, the input delay is indeed the main source of uncertainty in the present work.

3. Integrator Resetting

3.1. Integral Control Action. As discussed in Remark 1, the state of the plant can be steered to a desired equilibrium point by expressing the RMPC control law in terms of deviations from the equilibrium values. However, the presence of constant disturbances acting on the plant may lead to steady-state errors. Therefore, if offset-free tracking is required for a particular process variable y, some form of integral action should be introduced in the controller [33, 36].

For this purpose, assume that y is related to the state x by an output equation of the form

$$y(k) = Cx(k), \tag{23}$$

where $C \in \mathbb{R}^{1 \times n}$ is a known matrix. If both y and x denote deviations from desired target values, as in Remark 1, offset-free tracking will be achieved if $y(k) \to 0$ as $k \to \infty$. Integral control action can be introduced by augmenting the plant state with an additional variable w associated with a discrete-time integrator:

$$w(k+1) = w(k) - y(k). \tag{24}$$

If the control loop is stable, $w(k)$ will converge to a constant value as $k \to \infty$ and thus $y(k)$ will converge to zero [36]. It is worth noting that the use of a minus sign in (24) is an arbitrary convention, as the same reasoning holds if the equation is rewritten as $w(k+1) = w(k) + y(k)$. The minus sign is employed herein for consistency with the notation adopted in [33].

From (1) and (24), the augmented model can be written as

$$\begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k), \tag{25}$$

where 0 denotes either a scalar zero or a matrix of zeros with appropriate dimensions. The RMPC formulation presented in Section 2 can then be employed by replacing the original state $x(k)$ with the augmented state $[x^T(k)\ w(k)]^T$. The optimal control at each time k will be given by

$$u(k) = F_k \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{26}$$

with F_k obtained from the solution of $\mathbb{P}([x^T(k)\ w(k)]^T)$.

3.2. Integrator Resetting Procedure. The inclusion of an integrator in a control loop may cause undesirable effects in the transient response, such as increase in overshoot and/or settling time. Indeed, a control law such as (26) is aimed at forcing $x(k) \xrightarrow{k \to \infty} 0$ and $w(k) \xrightarrow{k \to \infty} 0$. The variable w corresponds to the accumulated value of $-y$ (in terms of deviations from the desired target), as defined in (24). Therefore, after the target value is reached (i.e., when y reaches zero), the controller will need to steer the process variable to the opposite side of the target, in order to switch the sign of y and thus move w progressively to zero (possibly in a slow manner). A speed up in this process can be achieved by means of integrator resetting.

A resetting procedure that retains the robust stability properties of the RMPC formulation was introduced in [33]. Such a procedure consisted of replacing the integrator state $w(k)$ with zero when the process variable crossed the setpoint for the first time, provided that the RMPC optimization problem remained feasible.

Herein, an improved scheme of integrator resetting is proposed by using a criterion based on the overall state vector,

instead of the process variable alone. More specifically, the integrator state $w(k)$ will be replaced with 0 if such a procedure reduces the RMPC cost below a prespecified threshold $\overline{\gamma}$. This condition is tested by checking whether the augmented state $[x^T(k)\ 0]^T$ belongs to the region of guaranteed cost $\mathscr{D}^{\overline{\gamma}}$. In this manner, the subsequent transient performance of the closed-loop system, as evaluated by the cost function, is taken into account. In contrast, it may be argued that the procedure described in [33], which was based on the difference between a single variable and its corresponding setpoint, does not consider the transient response resulting from the resetting intervention.

The proposed procedure can be described as follows.

Step 0 (initialization). Let $\mathscr{D}^{\overline{\gamma}} \subset \mathbb{R}^{n+1}$ be a region of guaranteed cost associated with an upper bound $\overline{\gamma}$ on the cost value.

Let $k = 0$, $w(0) = 0$, and $FLAG = 0$.

Step 1. Read state $x(k)$.

Step 2 (integrator resetting). If the following conditions are simultaneously satisfied:

$$\begin{bmatrix} x(k) \\ 0 \end{bmatrix} \in \mathscr{D}^{\overline{\gamma}} \tag{27}$$

$$FLAG = 0 \tag{28}$$

then let $w(k) = 0$ and $FLAG = 1$.

Step 3. Obtain the solution $(\gamma_k, Q_k, Y_k, X_k)$ for the problem $\mathbb{P}([x^T(k)\ w(k)]^T)$.

Step 4. Calculate the gain matrix $F_k = Y_k(Q_k)^{-1}$ and the control input $u(k) = F_k[x^T(k)\ w(k)]^T$.

Step 5. Apply $u(k)$ to the plant and let $w(k+1) = w(k) - Cx(k)$.

Step 6. Let $k = k + 1$; wait until the next sampling time and return to Step 1.

Remark 3 (resetting conditions). Condition (27), together with the definition of regions of guaranteed cost in Section 2.2, ensures that problem $\mathbb{P}([x^T(k)\ w(k)]^T)$ will be feasible with $w(k) = 0$. The *FLAG* variable indicates whether the integrator has already been reset. Thus, condition (28) ensures that the resetting operation will be carried out only once. This safeguard is of value to retain the stability properties of the RMPC formulation. Indeed, after the integrator is reset, all subsequent control actions will be obtained by following the standard RMPC control law, with no more resetting interventions, and thus the associated robust stability guarantees will apply.

Remark 4 (region of guaranteed cost $\mathscr{D}^{\overline{\gamma}}$: simplified test). Condition (27) can be replaced by a simpler sufficient test, as described below. Let $x_1, x_2, \ldots, x_n$ denote the components of the state vector x. Assume that the following n vectors in $\mathbb{R}^{n+1}$ are extreme points of $\mathscr{D}^{\overline{\gamma}}$ along the directions of these components:

$$\begin{bmatrix} p_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ p_2 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p_n \\ 0 \end{bmatrix}. \tag{29}$$

These extreme points can be obtained by using the search method described in Section 2.2 with vectors ξ of the form $[1\ 0\ \cdots\ 0\ 0]^T, [0\ 1\ \cdots\ 0\ 0]^T, \ldots, [0\ 0\ \cdots\ 1\ 0]^T$. Since $\mathscr{D}^{\overline{\gamma}}$ is symmetrical about the origin, the following vectors are also extreme points:

$$\begin{bmatrix} -p_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -p_2 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -p_n \\ 0 \end{bmatrix}. \tag{30}$$

The convexity property ensures that $\mathscr{D}^{\overline{\gamma}}$ contains the convex hull of the extreme points in (29) and (30). Therefore, if

$$\frac{|x_1(k)|}{|p_1|} + \frac{|x_2(k)|}{|p_2|} + \cdots + \frac{|x_n(k)|}{|p_n|} \leq 1 \tag{31}$$

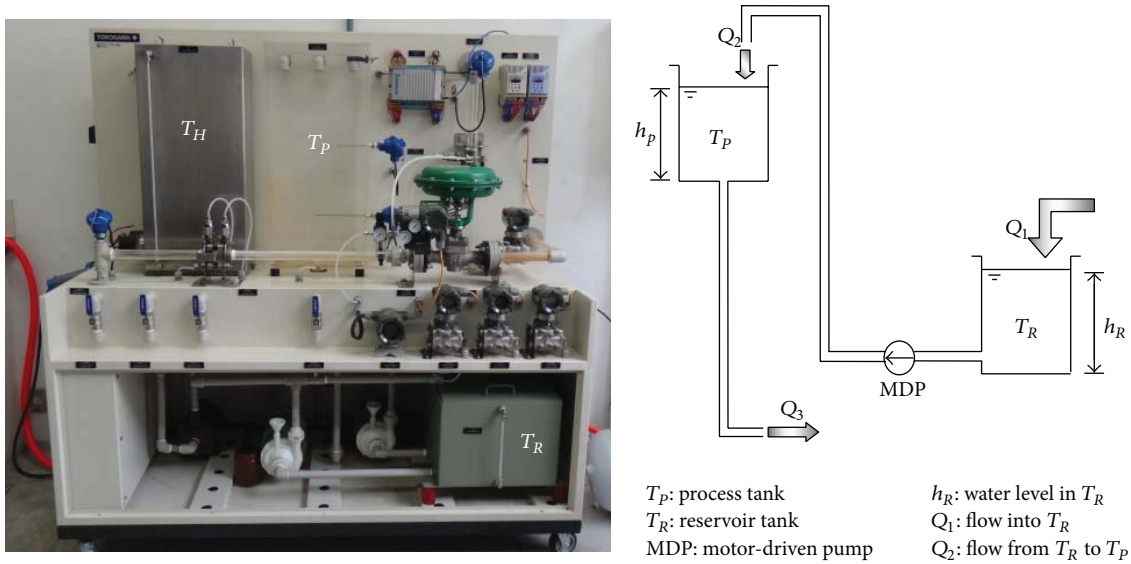
then $[x^T(k)\ 0]^T \in \mathscr{D}^{\overline{\gamma}}$.

It is worth noting that the integrator resetting procedure described herein can be directly applied to the case of uncertain input delay. It is sufficient to adopt the state vector $x(k)$ as in (17), with the associated polytopic model described in Section 2.3.

4. Case Study

The pilot plant employed herein was developed by Yokogawa Electric Corporation for process control experiments involving actual industrial components. As depicted in Figure 1(a), the plant comprises a reservoir tank (T_R), a process tank (T_P), and a heating tank (T_H). Since the present work was only concerned with the control of water level in the process tank T_P, the connection valves to the heating tank T_H were closed. A simplified scheme of the remaining components of the plant can be seen in Figure 1(b).

The water level in the reservoir tank T_R is regulated by means of a simple on/off switch, which was designed to keep this tank at its maximum capacity. A pump connected to a three-phase induction motor (MDP in Figure 1(b)) transfers water from T_R to the process tank T_P. The water leaves T_P through a valve at the bottom of the tank. A frequency inverter is used to drive the induction motor and a differential pressure transmitter is employed to measure the water level in T_P. A stardom programmable logic controller (PLC) is responsible for sending commands to the frequency inverter and reading the level measurements through a fieldbus



(a) (b)

Figure 1: (a) Experimental pilot plant and (b) schematic representation of the plant section involved in the level control experiments.

network. In addition, the PLC communicates with a personal computer over an Ethernet channel. In the present work, the RMPC control algorithm was implemented in the personal computer by using an OPC (object linking and embedding for process control) protocol to acquire sensor readings (water level in T_P) and send actuator commands (rotation speed of the motor-driven pump) at each sampling time. The LMI toolbox of the Matlab software environment was employed to solve the semidefinite programming problem involved in the RMPC implementation.

4.1. Plant Model. The values of the control input u_p (rotation speed) and the process variable of interest h_p (water level) were normalized between 0% and 100%, corresponding to a range of [0–3450] rpm for u_p and [0–70] cm for h_p. A model of the plant was obtained from open-loop step responses around the equilibrium values $u_{p,\text{ref}} = 65\%$ and $h_{p,\text{ref}} = 30\%$. By using a sampling time $T_s = 5$ s, the resulting model can be written as

$$h(k+1) = h(k) + 0.0865u(k-1), \tag{32}$$

where h and u denote deviations from h_p and u_p from the equilibrium values; that is,

$$h(k) = h_p(k) - h_{p,\text{ref}}, \tag{33}$$

$$u(k) = u_p(k) - u_{p,\text{ref}}, \tag{34}$$

It is worth noting that the model (32) comprises a delay of one sampling period at the plant input. As discussed in [37], this delay is associated with the relation between the actuator command and the water flow to the process tank. Uncertainty was introduced by including an additional delay in the control code running within the Matlab software environment. In practice, such a delay could arise, for example, from failures in communication links of networked control systems [38]. The plant model (32) then becomes

$$h(k+1) = h(k) + 0.0865u(k-\tau), \tag{35}$$

where τ is an uncertain delay with possible values in the range $[1, \overline{\tau}]$. In the present case study, the maximum delay value was taken as $\overline{\tau} = 6$.

Equation (35) is in the form (16) with $z(k) = h(k)$, $\Phi = 1$, and $\Gamma = 0.0865$. Therefore, the delay uncertainty can be handled by using the formulation in Section 2.3 with the state vector $x(k)$ defined as

$$x(k) = [h(k) \quad u(k-1) \quad u(k-2) \quad \cdots \quad u(k-6)]^T. \tag{36}$$

In order to ensure offset-free tracking of a constant reference for the water level, an additional state $w(k)$ was included as in (25), with $C = [1 \quad 0 \quad \cdots \quad 0]$.

4.2. Controller Design. The control task in this case study consists of increasing the water level from an initial value $h_p(0) = 12.5\%$ up to a setpoint $h_{p,\text{ref}} = 30\%$. As mentioned above, the corresponding steady-state rotation of the motor-driven pump is approximately $u_{p,\text{ref}} = 65\%$. The precise value of u_p required to keep the water level at the desired setpoint depends on the settings of the output valve at the bottom of the process tank. However, possible differences with respect to the adopted value ($u_{p,\text{ref}} = 65\%$) are compensated by the inclusion of integral action in the RMPC controller.

As will be seen in the Results section, this seemingly simple control task may become very challenging because of the uncertain input delay.

Constraints on the control input u_p were imposed in view of the following physical limitations. Experimental tests

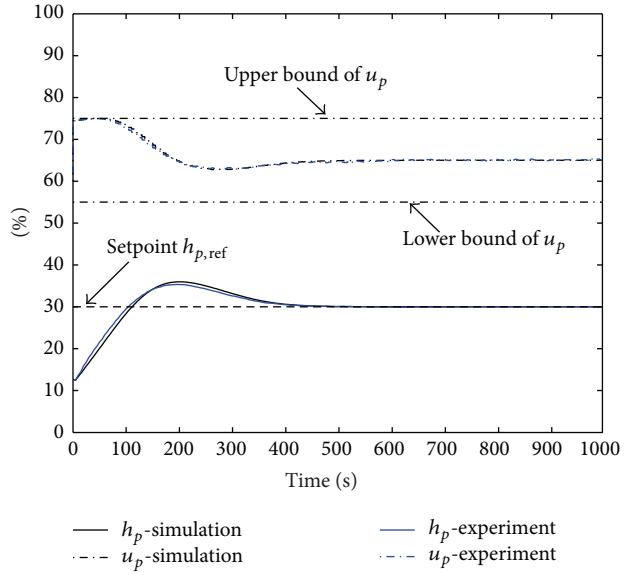


FIGURE 2: Results for the plant with nominal delay ($\tau_p = 1$), using the nominal predictive controller ($\tau = 1$).

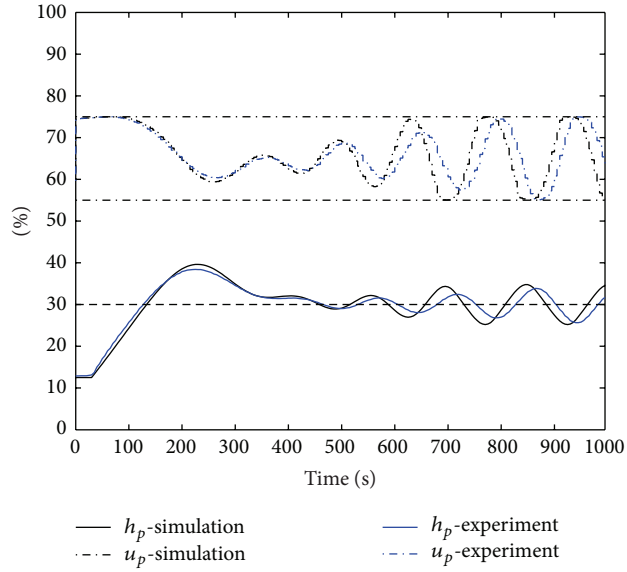


FIGURE 3: Results for the plant with additional delay ($\tau_p = 6$), using the nominal predictive controller ($\tau = 1$).

revealed that the pump was unable to elevate water from the reservoir tank T_R to the process tank T_P if the motor rotation was smaller than 40%. On the other hand, if the rotation was increased beyond 75%, the flow from T_R to T_P became larger than the input flow to the reservoir tank, which would eventually cause the reservoir to become empty. For these reasons, it is convenient to constrain the control input to the range $40\% \leq u_p \leq 75\%$. Therefore, in view of (34) with $u_{p,\mathrm{ref}} = 65\%$, the upper and lower bounds on the variable u should be -25% and 10%. Since the input constraints must be symmetric in the adopted RMPC approach, the bounds on u were set to $\pm 10\%$ (corresponding to $55\% \leq u_p \leq 75\%$).

The robust predictive controller (RMPC) was designed by using the state vector (36) augmented with the integrator state $w(k)$. The weight matrices S and R were set to

$$S = \mathrm{diag}\left[\rho_h, \frac{\rho_u}{(\overline{\tau}+1)}, \ldots, \frac{\rho_u}{(\overline{\tau}+1)}, \rho_w\right], \qquad R = \frac{\rho_u}{(\overline{\tau}+1)}, \tag{37}$$

where diag[·] indicates the elements of a diagonal matrix and $\rho_h = 1$, $\rho_u = 0.1$, and $\rho_w = 0.01$ are the weights corresponding to the water level $h(k)$, control $u(k)$, and integrator state $w(k)$. A normalization factor $1/(\overline{\tau}+1)$ is included in (37) to account for the fact that the control appears $(\overline{\tau}+1)$ times in the cost function, once in the control cost itself and $\overline{\tau}$ times in the state cost, since $u(k-1), u(k-2), \ldots, u(k-\overline{\tau})$ are used as state variables.

For comparison, a nominal predictive controller was designed by using the formulation presented in Section 2.3 with $\overline{\tau} = \underline{\tau} = 1$. In this case, the polytopic model (1) has a single vertex, which corresponds to the plant without the additional delay. The state vector employed in this formulation is given by

$$x(k) = \begin{bmatrix} h(k) & u(k-1) \end{bmatrix}^T \tag{38}$$

which is also augmented with the integrator state $w(k)$. The weight matrices S and R were set to

$$S = \mathrm{diag}\left[\rho_h, \frac{\rho_u}{2}, \rho_w\right], \qquad R = \frac{\rho_u}{2} \tag{39}$$

with $\rho_h = 1$, $\rho_u = 0.1$, and $\rho_w = 0.01$, as in the design of the robust controller.

The following section begins with a comparison between the nominal and robust predictive controllers, in order to illustrate the problems that may arise if the delay uncertainty is not taken into account. Afterwards, the effects of the proposed integrator resetting procedure are discussed. For this purpose, the simplified resetting test (31) was employed, with a cost threshold $\overline{\gamma} = 7.5 \times 10^6$.

5. Results and Discussion

In what follows, the delay considered in the design of the controller will be denoted by τ, whereas the actual delay of the plant will be denoted by τ_p. The nominal delay case corresponds to $\tau_p = 1$ and the case with additional delay introduced in the software environment corresponds to $\tau_p = 6$.

In addition to the experimental outcomes, simulation results are depicted in all graphs to illustrate the validity of the model presented in Section 4.1.

Figure 2 shows the results obtained by using the nominal predictive controller, which considers a delay $\tau = 1$ in the plant model. In this case, no additional delay was introduced in the plant and thus τ_p was also equal to 1. As can be seen, the water level h_p converges to the setpoint $h_{p,\mathrm{ref}}$ and the input constraints ($55\% \leq u_p \leq 75\%$) are properly enforced. The simulation and experimental results are in good agreement, which indicates that the adopted model provides an appropriate description of the plant dynamics.

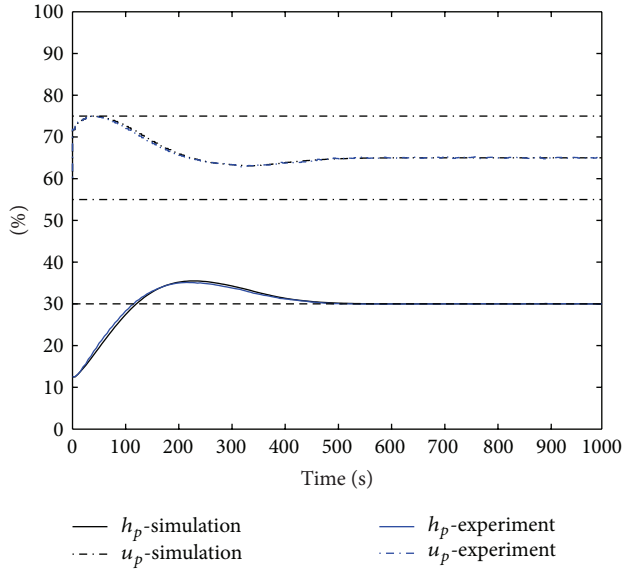


Figure 4: Results for the plant with nominal delay ($\tau_p = 1$), using the robust predictive controller ($1 \leq \tau \leq 6$).

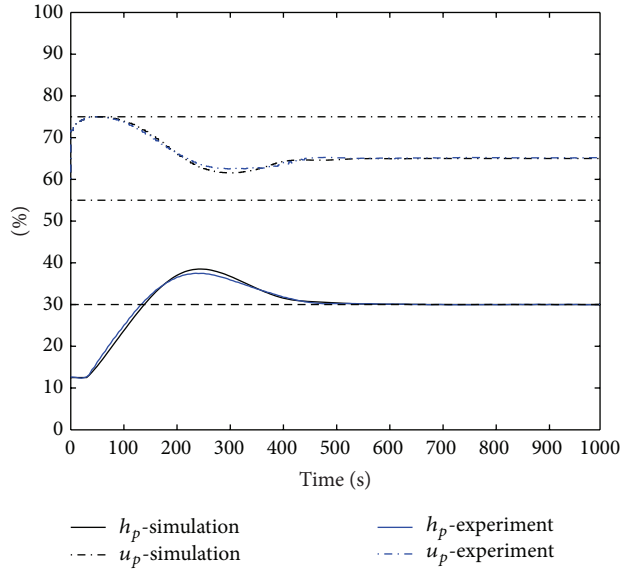


Figure 5: Results for the plant with additional delay ($\tau_p = 6$), using the robust predictive controller ($1 \leq \tau \leq 6$).

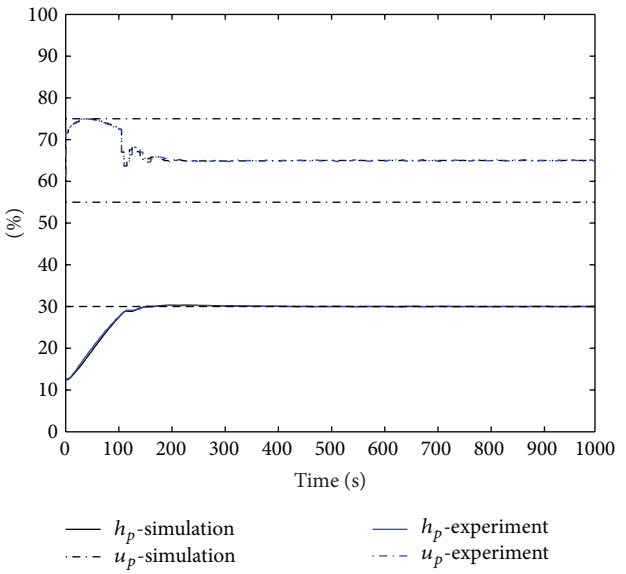


Figure 6: Results for the plant with nominal delay ($\tau_p = 1$), using the robust predictive controller ($1 \leq \tau \leq 6$) with integrator resetting.

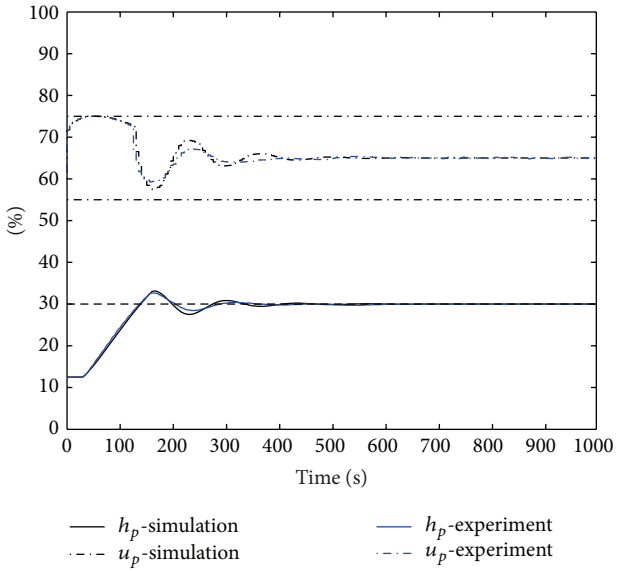


Figure 7: Results for the plant with additional delay ($\tau_p = 6$), using the robust predictive controller ($1 \leq \tau \leq 6$) with integrator resetting.

Figure 3 shows the case where an additional delay of five sampling periods was introduced in the software environment. Since the plant already has a nominal delay of one sampling period, the overall delay becomes $\tau_p = 6$. The mismatch between this overall delay and the delay assumed in the design of the nominal predictive controller ($\tau = 1$) results in the loss of closed-loop stability. Indeed, the system response enters a limit cycle rather than converging to the setpoint. The difference between simulation and experimental results can be explained because the model presented in Section 4.1, which was experimentally obtained from small step disturbances around the equilibrium, does not fully capture the dynamics involved in the transition to a limit cycle.

As can be seen, even for this simple water level control task, an increase in the time delay can have serious effects on the closed-loop operation. This finding underlines the convenience of using a robust predictive controller (RMPC) to handle possible delays in the range $1 \leq \tau \leq 6$, as described in Section 4.2.

Figures 4 and 5 show the RMPC results obtained in the extreme cases, which correspond to the plant with minimum delay ($\tau_p = 1$) and maximum delay ($\tau_p = 6$), respectively. As can be seen, in both figures the tank level h_p converges

Table 1: Experimental values of overshoot (M_p) and settling time (t_s) obtained by using the robust predictive controller with and without integrator resetting.

Delay τ_p	Resetting	M_p (%)	t_s (s)
1	No	5.1	425
6	No	7.5	440
1	Yes	0.3	130
6	Yes	2.7	270

to the setpoint, with satisfaction of the input constraints. Moreover, the transient responses in the two cases are very similar, which corroborates the robustness of the predictive controller. A slightly larger overshoot is observed for the case of maximum delay (Figure 5), which is to be expected since an increase in the time delay tends to deteriorate the closed-loop performance.

The transient performance obtained with the RMPC controller can be improved by using the integrator resetting procedure described in Section 3.2. The results are shown in Figures 6 and 7 for the cases of minimum delay ($\tau_p = 1$) and maximum delay ($\tau_p = 6$), respectively. A comparison with Figures 4 and 5 reveals that a substantial reduction was achieved in terms of overshoot and settling time, as summarized in Table 1.

It is worth noting that the overshoot (M_p) indicated in Table 1 corresponds to the difference between the maximum value of h_p and the setpoint $h_{p,\text{ref}} = 30\%$. For example, the first entry ($M_p = 5.1\%$) indicates that h_p reached a maximum value of 35.1%. The settling time t_s was determined by using a 0.05 criterion. Since the step size from the initial value $h_p(0) = 12.5\%$ to the setpoint $h_{p,\text{ref}} = 30\%$ was 17.5%, the settling time t_s is the time required for h_p to reach and stay within the range $30\% \pm (0.05 \times 17.5\%)$, that is, $29.125\% \leq h_p \leq 30.875\%$.

In short, the results obtained in this case study show that the RMPC formulation is indeed of value to ensure closed-loop stability in the presence of input constraints and uncertain time delays. Moreover, the integrator resetting procedure is convenient to improve the transient response, by reducing both overshoot and settling time in a substantial manner.

6. Conclusions

This paper reported experimental results involving the use of an RMPC technique for the control of water level in a pilot plant with actual industrial components. Stability and offset-free tracking of the setpoint were achieved in the presence of uncertain time delays and control input constraints. Moreover, an integrator resetting procedure based on the concept of regions of guaranteed cost was proposed to improve the closed-loop transient response. The resulting control formulation provided fast and well damped responses, either in the case of nominal delay, or in the case where additional delay was introduced in the plant.

Future research could be concerned with the use of less conservative LMI formulations [14–17] for the deployment of RMPC controllers in experimental settings. Moreover, it would be interesting to carry out studies with other laboratory or field systems that exhibit actual time-delay variations in inputs or state variables. Finally, computational issues may need to be taken into account to allow the application of RMPC to plants with fast dynamics, such as electromechanical systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Modeling and Bifurcation Research of a Worm Propagation Dynamical System with Time Delay

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Both vaccination and quarantine strategy are adopted to control the Internet worm propagation. By considering the interaction infection between computers and external removable devices, a worm propagation dynamical system with time delay under quarantine strategy is constructed based on anomaly intrusion detection system (IDS). By regarding the time delay caused by time window of anomaly IDS as the bifurcation parameter, local asymptotic stability at the positive equilibrium and local Hopf bifurcation are discussed. Through theoretical analysis, a threshold τ_0 is derived. When time delay is less than τ_0, the worm propagation is stable and easy to predict; otherwise, Hopf bifurcation occurs so that the system is out of control and the containment strategy does not work effectively. Numerical analysis and discrete-time simulation experiments are given to illustrate the correctness of theoretical analysis.

1. Introduction

Internet worms, a great threat to the network security, can spread quickly among hosts via wired or wireless networks. In real network environment, many intelligent worms, such as Conficker, Stuxnet, and Flamer, can also spread themselves via external removable devices (USB drives, CD/DVD drives, external hard drives, etc.), which have become one of the main means of infection transmission as well as networks. Conficker can copy itself as the autorun.inf to removable media drives in the system, thereby forcing the executable to be launched every time a removable drive is inserted into a system [1, 2]. Discovered in the summer of 2010, Stuxnet is a threat targeting a specific industrial control system (ICS) likely in Iran, such as a gas pipeline or power plant. Removable device is one of the main pathways for Stuxnet to migrate from the outside world to supposedly isolated and secure ICS [3–5]. Discovered in May 2012, Flamer can spread via removable drives using a special folder that hides the files and can result in automatic execution on viewing the removable drive when combined with the Microsoft Windows Shortcut "LNK/PIF" File Automatic File Execution Vulnerability (CVE-2010-2568) [6, 7]. Therefore, it is time to analyze the dynamic behavior and containment strategy of such worms.

Worm propagation dynamical system plays an important role in predicting the spread of worms. It aids in identifying the weakness in the worm spreading chain and provides accurate prediction for the purpose of damage assessment for a new worm threat. Over decades of years, many researches on worms' dynamical behavior have been done. Kermack and Mckendrick [8] proposed the classical SIR model to explain the rapid rise and fall in the number of infected patients observed in epidemics, which also suits the worm spread. Based on the classical SIR model, Zou et al. derived an Internet worm model called the two-factor model [9]. Quarantine strategy, which borrows from the method of epidemic disease control, has been widely used in worm containment and produced a tremendous effect on controlling worm propagation [10–14]. Zou et al. proposed a worm propagation model under dynamic quarantine defense based on the principle "assume [*sic*] guilty before proven

innocent" [10]. Wang et al. proposed a novel epidemic model named SEIQV model which combines both vaccinations and dynamic quarantine methods [11]. However, there is time delay in actual network environment, which may lead to bifurcation phenomenon. Much research has been done on time delay and bifurcation [15–25]. Han and Tan studied the dynamic spread behavior of worms by incorporating the delay factor [19]. Dong et al. proposed a computer virus model with time delay based on SEIR model and regarded time delay as bifurcating parameter to study the dynamical behaviors including local asymptotical stability and local Hopf bifurcation [20]. Yao et al. constructed a model with time delay under quarantine strategy [21]. Wu et al. investigated the problem of sliding mode control of Markovian jump singular time-delay systems [23]. Li and Zhang established a delay-dependent bounded real lemma for singular linear parameter-varying systems with time-variant delay [24]. The problems of D-stability and nonfragile control for a class of discrete-time descriptor Takagi-Sugeno fuzzy systems with multiple state delays are discussed in [25].

However, the above works consider less of the effect of removable devices on worm propagation. As mentioned above, removable devices have become a main pathway for some worms to intrude those hosts not connected to the Internet. Song et al. presented a worm model incorporating specific features to worms spreading via both web-based scanning and removable devices [26]. Zhu et al. studied the dynamics of interaction infection between computers and removable devices in [27]. However, time delay and bifurcation research are not considered in their work. In this paper, by considering the interaction infection between hosts and removable devices, we model a delayed worm propagation dynamical system which combines both vaccination and quarantine strategy. Local asymptotic stability of the positive equilibrium and local Hopf bifurcation are discussed to analyze the influence of time delay on worm propagation dynamical system.

The main contributions of this paper can be summarized as follows.

(1) Considering the influence of removable devices on Internet worm propagation and the time delay caused by anomaly IDS, we propose a novel worm propagation dynamical system with time delay.

(2) We analyze the system stability at positive equilibrium and derive the time delay threshold at which Hopf bifurcation occurs.

(3) By numerical analysis, we illustrate the correctness of theoretical analysis.

(4) The discrete-time simulation is adopted to simulate the worm propagation in real network environment. The results demonstrate the reasonableness of the worm propagation model.

The rest of the paper is organized as follows. In Section 2, considering the influence of removable devices, a worm propagation dynamical system with time delay under quarantine strategy is constructed. In Section 3, local stability of the positive equilibrium and local Hopf bifurcation are investigated. In Section 4, several numerical analyses supporting the theoretical analysis are given. Section 5 makes a comparison between simulation experiments and numerical ones. Finally, we give our conclusions in Section 6.

2. Model Formulation

The system contains both hosts and removable devices. In this model, all hosts are in one of following five states: susceptible (S), infectious (I), delayed (D), quarantined (Q), and removed (R). All removable devices are divided into two groups: susceptible (R_S) and infectious (R_I). N and R_N denote the total number of hosts and removable devices, respectively. That is, $S + I + D + Q + R = N$; $R_S + R_I = R_N$. Susceptible (S) hosts, which are vulnerable to the attack from worms, will be infected by infectious hosts or removable devices; then they will infect other hosts connected to them or removable devices plugged into them. Infectious (I) hosts will be immunized by antivirus software at the rate of γ_1. Removed (R) hosts, which have been immunized by antivirus software, will become susceptible at reassembly rate ω. Hosts whose behavior looks anomaly will be quarantined by IDS and then they will become in a quarantined (Q) state. A susceptible removable device (R_S) will be infected when inserted into an infectious host. Worm in an infectious removable device (R_I) will be eliminated when connected to removed hosts; then it will become in a susceptible state.

The quarantine strategy is an effective measure to defend against worms' attack and make up the deficiency of vaccination strategy. In this paper, anomaly intrusion detection system is chosen for applying quarantine strategy. Comparing with misuse IDS, anomaly IDS has great advantage in detecting unknown intrusion or the variants of known intrusion. However, anomaly IDS judges whether a detected behavior is an attack or not via comparing detected behavior with the normal or expected behavior of system and user. If a deviation occurs, the detected behavior is treated as an intrusion immediately. Because of the difficulty in collecting and building the normal behavior database, high false-alarm rate is considered the main drawback of anomaly IDS. In order to reduce the false alarm of anomaly IDS, the mechanism of time window is adopted. A suspicious behavior will not trigger an alarm immediately. On the contrary, anomaly IDS has a period of time to analyze the accumulated behavior. Therefore, an intermediate state, delayed (D) state, is added into the propagation model. The larger the value of time window, the less the false alarm aroused by anomaly IDS, because there is enough time for anomaly IDS to recognize whether a behavior is an intrusion or not. However, the overlarge time window may lead to worm propagation dynamical system being unstable and out of control. The main notations and definitions are listed in Table 1. The state transition diagram is given by Figure 1.

On the basis of current research, we present a delayed worm propagation model which combines both vaccination and quarantine strategy. Several appropriate assumptions are given as follows.

Table 1: Notations and definitions of the model.

Notations	Definitions
N	Total number of hosts in the network
R_N	Total number of removable devices in the network
$S(t)$	Number of susceptible hosts at time t
$I(t)$	Number of infectious hosts at time t
$D(t)$	Number of delayed hosts at time t
$Q(t)$	Number of quarantined hosts at time $t-\tau$
$R(t)$	Number of removed hosts at time t
$R_S(t)$	Number of susceptible removable devices at time t
$R_I(t)$	Number of infectious removable devices at time t
β_1	Infection ratio of infectious hosts
β_2	Contact infection rate between computers and removable devices
γ_1	Recovery rate of infectious hosts
γ_2	Recovery rate of infectious removable devices
ω	Reassembly rate of immunized hosts
θ_1	Quarantine rate of susceptible hosts
θ_2	Quarantine rate of infectious hosts
δ	Immunization rate of quarantined hosts
τ	Time delay of detection by anomaly intrusion detection system

(1) β_1 denotes the infection ratio of infectious hosts. Therefore, at time t, the infection force of infectious computers to susceptible computers is given by $\beta_1 S(t) I(t)$.

(2) Infectious removable devices have the same infectious ability as the infectious hosts. β_2 is the contact infection rate between computers and removable devices, that is, the interactive infection rate when a removable device links to a host. The probability of connecting removable devices for every host is R_N/N, and the probability of removable device exactly being in the infectious state is $R_I(t)/R_N$. Therefore, the infection force of infectious removable devices to susceptible hosts is $\beta_2(R_N/N)(R_I(t)/R_N)S(t)$.

(3) Susceptible removable devices will be infected when connecting to an infectious host, and then they will infect any other hosts to which they are connected. Meanwhile, worms of infectious removable devices will be eliminated when connecting to one immunized host. That is, the infection force of infectious hosts to susceptible removable devices is $\beta_2(I(t)/N)R_S(t)$, and the recovery force of removed hosts to infectious removable devices is $\gamma_2(R(t)/N)R_I(t)$.

(4) Owing to the influence of time delay τ, the increment of the number of quarantined hosts is the ones quarantined at time $t-\tau$. Therefore, the increment is $\theta_1 S(t-\tau) + \theta_2 I(t-\tau)$.

(5) The time window mechanism leads to an intermediate state, delayed state (D). The increment of the number of delayed hosts at time t is given by $\theta_1 S(t) + \theta_2 I(t)$; the decrement of delayed hosts is the number of those being quarantined, that is, $\theta_1 S(t-\tau) + \theta_2 I(t-\tau)$.

Based on the analyses and assumptions above, the delayed differential equations of the model are formulated as (1). The differential on the left of equations means the change rate of related states at time t. Consider

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta_1 S(t) I(t) - \beta_2 \frac{R_I(t)}{N} S(t) + \omega R(t) - \theta_1 S(t),\\
\frac{dI(t)}{dt} &= \beta_1 S(t) I(t) + \beta_2 \frac{R_I(t)}{N} S(t) - \gamma_1 I(t) - \theta_2 I(t),\\
\frac{dR(t)}{dt} &= \gamma_1 I(t) - \omega R(t) + \delta Q(t),\\
\frac{dD(t)}{dt} &= \theta_1 S(t) - \theta_1 S(t-\tau) + \theta_2 I(t) - \theta_2 I(t-\tau),\\
\frac{dQ(t)}{dt} &= \theta_1 S(t-\tau) + \theta_2 I(t-\tau) - \delta Q(t),\\
\frac{dR_S(t)}{dt} &= -\beta_2 \frac{I(t)}{N} R_S(t) + \gamma_2 \frac{R(t)}{N} R_I(t),\\
\frac{dR_I(t)}{dt} &= \beta_2 \frac{I(t)}{N} R_S(t) - \gamma_2 \frac{R(t)}{N} R_I(t).
\end{aligned} \tag{1}$$

3. Stability at the Positive Equilibrium and Bifurcation Analysis

Theorem 1. *The system* (1) *has a unique positive equilibrium* $E^* = (S^*, I^*, D^*, Q^*, R^*, R_S^*, R_I^*)$*, where*

$$\begin{aligned}
I^* &= \frac{b_2 S^{*2} + b_3 S^*}{b_4 - b_1 S^*}, \qquad D^* = \theta_1 S^* \tau + \theta_2 I^* \tau,\\
Q^* &= \frac{\theta_1 S^* + \theta_2 I^*}{\delta}, \qquad R^* = \frac{\gamma_1 I^* + \theta_1 S^* + \theta_2 I^*}{\omega},\\
R_I^* &= \frac{\beta_2 R_N I^*}{\beta_2 I^* + \gamma_2 R^*}.
\end{aligned} \tag{2}$$

Proof. For system (1), according to [28], if all the derivatives on the left of equal sign of the system are set to 0, which implies that the system becomes stable, we can derive

$$\begin{aligned}
I &= \frac{b_2 S^{*2} + b_3 S^*}{b_4 - b_1 S^*},\\
Q &= \frac{\theta_1 S^* + \theta_2 I^*}{\delta},\\
R &= \frac{\gamma_1 I^* + \theta_1 S^* + \theta_2 I^*}{\omega},\\
R_I &= \frac{\beta_2 R_N I^*}{\beta_2 I^* + \gamma_2 R^*},
\end{aligned} \tag{3}$$

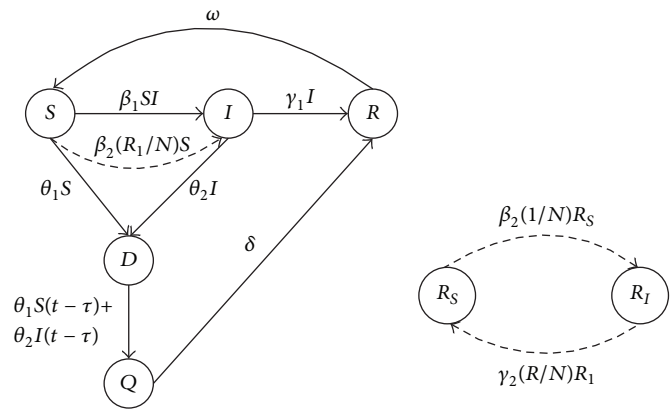


Figure 1: The state transition diagram.

where

$$b_1 = \omega\beta_1\beta_2 + \gamma_1\gamma_2\beta_1 + \beta_1\gamma_2\theta_2,$$

$$b_2 = \beta_1\gamma_2\theta_1, \qquad b_3 = \frac{\beta_2^2\omega R_N}{N - \theta_1\gamma_2(\gamma_1 + \theta_2)}, \tag{4}$$

$$b_4 = (\gamma_1 + \theta_2)(\omega\beta_2 + \gamma_1\gamma_2 + \gamma_2\theta_2).$$

Assume that system (1) becomes stable at time T. By integrating the fourth equation of system (1) with time t from 0 to $T + \tau$, we can get

$$D = \theta_1 S^*\tau + \theta_2 I^*\tau. \tag{5}$$

Since $S + I + D + Q + R = N$,

$$S^* + \frac{b_2 S^{*2} + b_3 S^*}{b_4 - b_1 S^*} + \theta_1 S^*\tau + \theta_2 I^*\tau + \frac{\theta_1 S^* + \theta_2 I^*}{\delta} + \frac{\gamma_1 I^* + \theta_1 S^* + \theta_2 I^*}{\omega} = N. \tag{6}$$

Obviously, (6) has one unique positive root I^*. So there is one unique positive equilibrium $E^* = (S^*, I^*, D^*, Q^*, R^*, R_S^*, R_I^*)$ of system (1). The proof is completed. □

Since $S + I + D + Q + R = N$, $R_S + R_I = R_N$, $Q = N - S - I - D - R$, $R_S = R_N - R_I$. System (1) can be simplified to

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta_1 S(t) I(t) - \beta_2 \frac{R_I(t)}{N} S(t) + \omega R(t) - \theta_1 S(t),\\
\frac{dI(t)}{dt} &= \beta_1 S(t) I(t) + \beta_2 \frac{R_I(t)}{N} S(t) - \gamma_1 I(t) - \theta_2 I(t),\\
\frac{dR(t)}{dt} &= \gamma_1 I(t) - \omega R(t) + \delta(N - S(t) - I(t) - D(t) - R(t)),\\
\frac{dD(t)}{dt} &= \theta_1 S(t) - \theta_1 S(t - \tau) + \theta_2 I(t) - \theta_2 I(t - \tau),\\
\frac{dR_I(t)}{dt} &= \beta_2 \frac{I(t)}{N}(R_N - R_I(t)) - \gamma_2 \frac{R(t)}{N} R_I(t).
\end{aligned} \tag{7}$$

The Jacobian matrix of (7) about $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ is given by

$$J(E^*) = \begin{pmatrix}
-\beta_1 I^* - \beta_2 \frac{R_I^*}{N} - \theta_1 & -\beta_1 S^* & \omega & 0 & -\frac{\beta_2 S^*}{N}\\
\beta_1 I^* + \beta_2 \frac{R_I^*}{N} & \beta_1 S^* - \gamma_1 - \theta_2 & 0 & 0 & \frac{\beta_2 S^*}{N}\\
-\delta & \gamma_1 - \delta & -\omega - \delta & -\delta & 0\\
\theta_1 - \theta_1 e^{-\lambda\tau} & \theta_2 - \theta_2 e^{-\lambda\tau} & 0 & 0 & 0\\
0 & \frac{\beta_2(R_N - R_I^*)}{N} & -\frac{\gamma_2 R_I^*}{N} & 0 & -\frac{\beta_2 I^* + \gamma_2 R^*}{N}
\end{pmatrix}. \tag{8}$$

Let

$$c_1 = \beta_1 I^* + \beta_2 \frac{R_I^*}{N}, \qquad c_2 = \beta_1 S^*, \qquad c_3 = \frac{\beta_2 S^*}{N}, \qquad c_4 = \frac{\beta_2(R_N - R_I^*)}{N},$$

$$c_5 = \frac{\gamma_2 R_I^*}{N}, \quad c_6 = \frac{\beta_2 I^* + \gamma_2 R^*}{N},$$

$$p_4 = c_6 - c_2 + \gamma_1 + \theta_2 + \omega + \delta + c_1 + \theta_1,$$

$$p_3 = c_6(-c_2 + \gamma_1 + \theta_2) + (\omega + \delta + c_1 + \theta_1)(c_6 - c_2 + r_1 + \theta_2) + (c_1 + \theta_1)(\omega + \delta) - c_3c_4 + c_1c_2 + \delta\omega,$$

$$p_2 = c_6(\omega + \delta + c_1 + \theta_1)(-c_2 + \gamma_1 + \theta_2) + ((c_1 + \theta_1)(\omega + \delta) + \delta\omega)(c_6 - c_2 + \gamma_1 + \theta_2) - c_3c_4(\theta_1 + \omega + \delta) + c_1c_2(c_6 + \omega + \delta) + (c_1\omega - c_3c_5)(\delta - \gamma_1) + \delta(c_3 + \omega\theta_1),$$

$$p_1 = (c_6(c_1 + \theta_1)(\omega + \delta) + c_6\delta\omega + \delta c_3)(-c_2 + \gamma_1 + \theta_2) + (c_1c_2c_6 - \theta_1c_3c_4)(\omega + \delta) + (c_1c_6\omega - \theta_1c_3c_5)(\delta - \gamma_1) - \delta\omega(c_3c_4 + \theta_2 + \theta_1(c_6 - c_2 + \gamma_1 + \theta_2)) + \delta c_5(c_2c_3 + c_3\theta_1 - \theta_2\theta_3),$$

$$p_0 = \delta\omega(\theta_2c_6 - \theta_1c_3c_4 + \theta_1c_6(-c_2 + \gamma_1 + \theta_2)) + \delta c_5(\theta_1c_2c_3 + \theta_2c_1c_3 + c_3\theta_1(-c_2 + \gamma_1 + \theta_2) - c_3\theta_2(c_1 + \theta_1)),$$

$$q_2 = -\delta\theta_1\omega,$$

$$q_1 = -\delta\omega(\theta_2 + \theta_1(c_6 - c_2 + \gamma_1 + \theta_2)) - \delta c_5(c_3\theta_1 - c_3\theta_2),$$

$$q_0 = \delta\omega(\theta_2c_6 - \theta_1c_3c_4 + c_6\theta_1(-c_2 + \gamma_1 + \theta_2)) + \delta c_5(\theta_1c_2c_3 + \theta_2c_1c_3 + \theta_1c_3(-c_2 + \gamma_1 + \theta_2) - c_3\theta_2(c_1 + \theta_1)). \tag{9}$$

The characteristic equation of system (8) can be obtained by

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{10}$$

where

$$P(\lambda) = \lambda^5 + p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0,$$
$$Q(\lambda) = q_2\lambda^2 + q_1\lambda + q_0. \tag{11}$$

Theorem 2. *The positive equilibrium E^* is locally asymptotically stable without time delay, if condition (H_1) is satisfied:*

$$H_1\text{: } p_4 > 0, \quad d_1 > 0, \quad d_2 > 0, \quad (p_2 + q_2)d_1 - p_4^2d_2 > 0, \tag{12}$$

where

$$d_1 = p_3p_4 - (p_2 + q_2), \quad d_2 = p_1 + q_1. \tag{13}$$

Proof. When $\tau = 0$, (10) reduces to

$$\lambda^5 + p_4\lambda^4 + p_3\lambda^3 + (p_2 + q_2)\lambda^2 + (p_1 + q_1)\lambda + (p_0 + q_0) = 0. \tag{14}$$

According to Routh-Hurwitz criterion, all roots of (14) have negative real parts. Therefore, it can be concluded that the positive equilibrium $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ is locally asymptotically stable without time delay. The proof is completed. □

If $\lambda = i\omega$ ($\omega > 0$) is the root of (10), separating the real and imaginary parts, the following two equations can be obtained:

$$p_4\omega^4 - p_2\omega^2 + p_0 + q_1\omega\sin(\omega\tau) - q_2\omega^2\cos(\omega\tau) + q_0\cos(\omega\tau) = 0,$$
$$\omega^5 - p_3\omega^3 + p_1\omega + q_1\omega\cos(\omega\tau) + q_2\omega^2\sin(\omega\tau) - q_0\sin(\omega\tau) = 0. \tag{15}$$

From (15), the following equation can be obtained:

$$q_1^2\omega^2 + \left(q_0 - q_2\omega^2\right)^2 = \left(p_4\omega^4 - p_2\omega^2 + p_0\right)^2 + \left(\omega^5 - p_3\omega^3 + p_1\omega\right)^2. \tag{16}$$

That is,

$$\omega^8 + D_3\omega^6 + D_2\omega^4 + D_1\omega^2 + D_0 = 0, \tag{17}$$

where

$$D_3 = p_4^2 - 2p_3, \quad D_2 = p_3^2 + 2p_1 - 2p_2p_4,$$
$$D_1 = p_2^2 - q_2^2 + 2p_0p_4 - 2p_1p_3,$$
$$D_0 = p_1^2 - q_1^2 + 2q_0q_2 - 2p_0p_2. \tag{18}$$

Letting $z = \omega^2$, (17) can be written as

$$h(z) = z^4 + D_3z^3 + D_2z^2 + D_1z + D_0. \tag{19}$$

Zhang et al. [18] obtained the following results on the distribution of roots of (19). Denote

$$m = \frac{1}{2}D_2 - \frac{3}{16}D_3^2, \quad n = \frac{1}{32}D_3^3 - \frac{1}{8}D_3D_2 + D_1,$$
$$\Delta = \left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3, \quad \sigma = \frac{-1 + \sqrt{3}i}{2},$$
$$y_1 = \sqrt[3]{-\frac{n}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\Delta}},$$
$$y_2 = \sqrt[3]{-\frac{n}{2} + \sqrt{\Delta}}\sigma + \sqrt[3]{-\frac{n}{2} - \sqrt{\Delta}}\sigma^2,$$

$$y_3 = \sqrt[3]{-\frac{n}{2} + \sqrt{\Delta}\sigma^2} + \sqrt[3]{-\frac{n}{2} - \sqrt{\Delta}\sigma},$$

$$z_i = y_i - \frac{3D_3}{4}, \quad (i = 1, 2, 3). \tag{20}$$

Lemma 3. *For the polynomial equation* (19),

(1) *if* $D_0 < 0$, *then* (19) *has at least one positive root;*

(2) *if* $D_0 \geq 0$ *and* $\Delta \geq 0$, *then* (19) *has positive root if and only if* $z_1 > 0$ *and* $h(z_1) < 0$;

(3) *if* $D_0 \geq 0$ *and* $\Delta < 0$, *then* (19) *has positive root if and only if there exists at least one* $z^* \in (z_1, z_2, z_3)$, *such that* $z^* > 0$ *and* $h(z^*) \leq 0$.

Lemma 4. *Suppose that condition* H_1: $p_4 > 0$, $d_1 > 0$, $d_2 > 0$, $(p_2 + q_2)d_1 - p_4^2 d_2 > 0$ *is satisfied.*

(1) *If one of the followings holds, (a)* $D_0 < 0$; *(b)* $D_0 \geq 0$, $\Delta \geq 0$, $z_1 > 0$, *and* $h(z_1) < 0$; *(c)* $D_0 \geq 0$, *and* $\Delta < 0$, *and there exits at least a* $z^* \in (z_1, z_2, z_3)$ *such that* $z^* > 0$ *and* $h(z^*) \leq 0$, *then all roots of* (10) *have negative real parts when* $\tau \in [0, \tau_0)$; *here,* τ_0 *is a certain positive constant.*

(2) *If conditions (a)–(c) of (1) are not satisfied, then all roots of* (10) *have negative real parts for all* $\tau \geq 0$.

Proof. When $\tau = 0$, (10) can be reduced to

$$\lambda^4 + p_4\lambda^3 + p_3\lambda^2 + (p_2 + q_2)\lambda + (p_1 + q_1) = 0. \tag{21}$$

According to the Routh-Hurwitz criterion, all roots of (21) have negative real parts if and only if $p_4 > 0$, $d_1 > 0$, $d_2 > 0$, and $(p_2 + q_2)d_1 - p_4^2 d_2 > 0$.

From Lemma 3, it can be known that if (a)–(c) are not satisfied, then (10) has no roots with zero real part for all $\tau \geq 0$; if one of (a)–(c) holds, when $\tau \neq \tau_k^{(j)}$, $k = 1, 2, 3, 4$, $j > 1$, (10) has no roots with zero real part and τ_0 is the minimum value of τ, so (10) has purely imaginary roots. According to [18], one obtains the conclusion of the lemma.

Let $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ be the root of (10), $\nu(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$.

From Lemmas 3 and 4, the following are obtained.

When conditions (a)–(c) of Lemma 4(1) are not satisfied, $h(z)$ always has no positive root. Therefore, under these conditions, (10) has no purely imaginary roots for any $\tau > 0$, which implies that the positive equilibrium $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ of system (7) is absolutely stable. Therefore, the following theorem on the stability of positive equilibrium $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ can be easily obtained. □

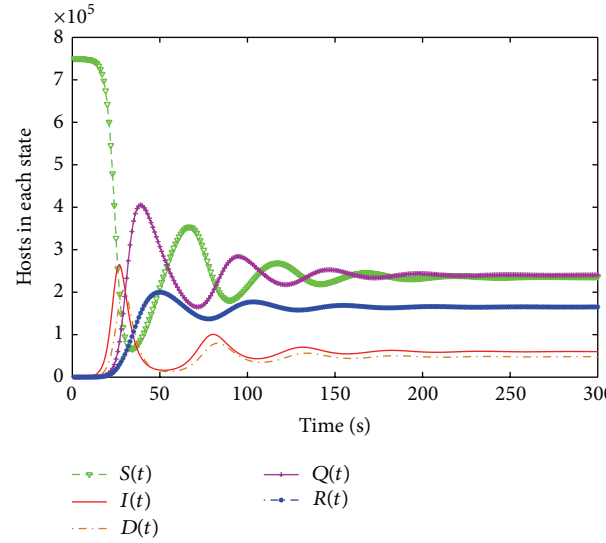


Figure 2: Propagation trend of the five kinds of hosts when $\tau < \tau_0$.

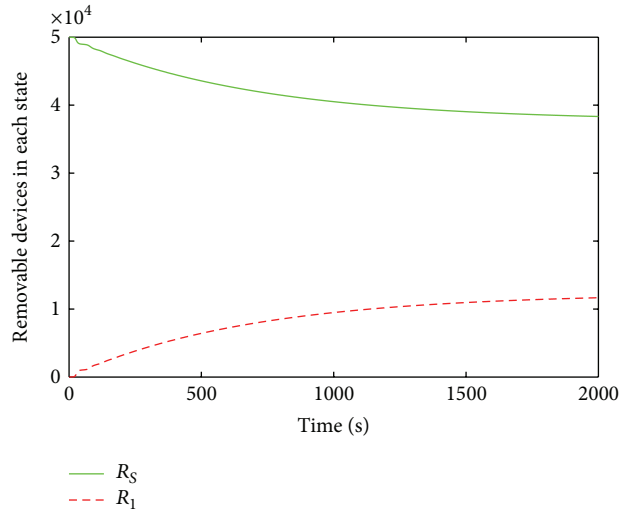


Figure 3: Propagation trend of the two kinds of removable devices when $\tau < \tau_0$.

Theorem 5. *Supposing that condition* (H_1) *is satisfied, (a)* $D_0 \geq 0$, $\Delta \geq 0$, $z_1 < 0$, *and* $h(z_1) > 0$; *(b)* $D_0 \geq 0$ *and* $\Delta < 0$, *and there is no* $z^* \in (z_1, z_2, z_3)$ *such that* $z^* > 0$ *and* $h(z^*) \leq 0$, *then the positive equilibrium* $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ *of system* (7) *is absolutely stable.*

In what follows, it is assumed that the coefficients in $h(z)$ *satisfy the condition*

(H_2) *(a)* $D_0 \geq 0$, $\Delta \geq 0$, $z_1 < 0$, *and* $h(z_1) > 0$; *(b)* $D_0 \geq 0$, $\Delta < 0$, *and there is no* $z^* \in (z_1, z_2, z_3)$ *such that* $z^* > 0$ *and* $h(z^*) \leq 0$.

According to [29], it is known that (19) *has at least a positive root* ω_0, *which implies that characteristic equation* (10) *has a pair of purely imaginary roots* $\pm i\omega_0$.

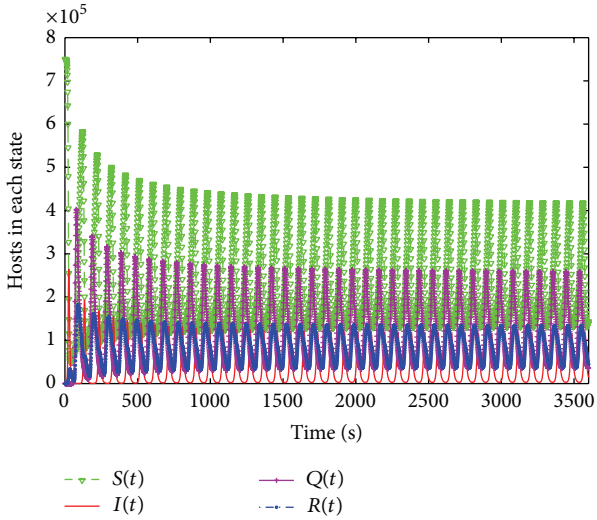


Figure 4: Propagation trend of the four kinds of hosts when $\tau > \tau_0$.

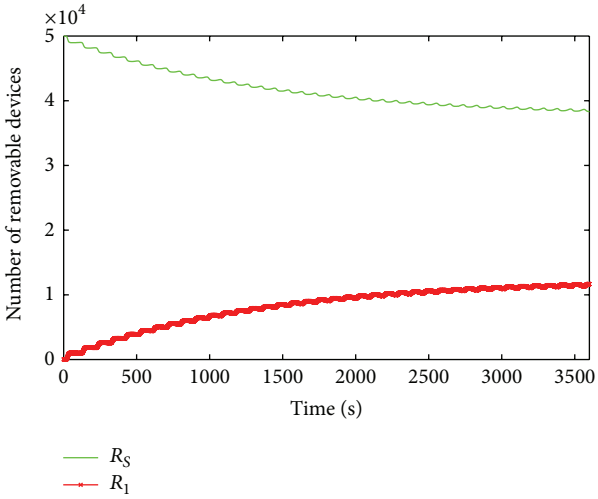


Figure 5: Propagation trend of the two kinds of removable devices when $\tau > \tau_0$.

Since (10) *has a pair of purely imaginary roots* $\pm i\omega_0$*, the corresponding* $\tau_k > 0$ *is given by* (15)*. Consider*

$$\tau_k = \frac{1}{\omega_0}\arccos\left[\left(\left(q_0 - q_2\omega^2\right)\left(p_2\omega_0^2 - p_4\omega_0^4 - p_0\right) + q_1\omega_0\left(p_3\omega_0^3 - \omega_0^5 - p_1\omega_0\right)\right) \times\left(\left(q_0 - q_2\omega^2\right)^2 + q_1^2\omega_0^2\right)^{-1}\right] + \frac{2k\pi}{\omega_0}, \quad (k = 0, 1, 2, 3, \ldots). \tag{22}$$

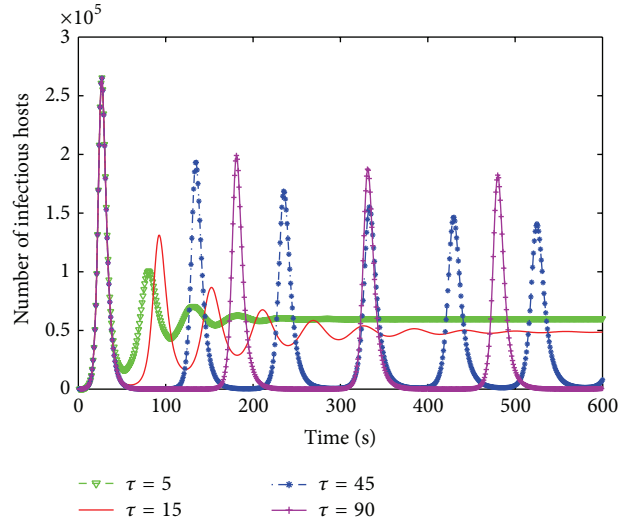


Figure 6: The number of infectious hosts when τ is changed in one coordinate.

Let $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ *be the root of* (10)*.* $\nu(\tau_k) = 0$ *and* $\omega(\tau_k) = \omega_0$ *are satisfied when* $\tau = \tau_k$*.*

Lemma 6. *Suppose that* $h'(z_0) \neq 0$*. If* $\tau = \tau_0$*, then* $\pm i\omega_0$ *is a pair of purely imaginary roots of* (10)*. In addition, if the conditions of Lemma 4(1) are satisfied, then* $d\,\mathrm{Re}\,\lambda(\tau_0)/d\tau > 0$*.*

It is claimed that

$$\mathrm{sgn}\left[\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right]_{\tau=\tau_k} = \mathrm{sgn}\left\{h'\left(\omega_0^2\right)\right\}. \tag{23}$$

This signifies that there is at least one eigenvalue with positive real part for $\tau > \tau_k$*.*

Differentiating two sides of (10) *with respect to* τ*, it can be written as*

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left(\left(5\lambda^4 + 4p_4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1\right) + \left(2q_2\lambda + q_1\right)e^{-\lambda\tau} - \left(q_2\lambda^2 + q_1\lambda + q_0\right)\tau e^{-\lambda\tau}\right) \times\left(\left(q_2\lambda^2 + q_1\lambda + q_0\right)\lambda e^{-\lambda\tau}\right)^{-1} = \frac{\left(5\lambda^4 + 4p_4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1\right)e^{\lambda\tau}}{\left(q_2\lambda^2 + q_1\lambda + q_0\right)\lambda} + \frac{2q_2\lambda + q_1}{\left(q_2\lambda^2 + q_1\lambda + q_0\right)\lambda} - \frac{\tau}{\lambda}. \tag{24}$$

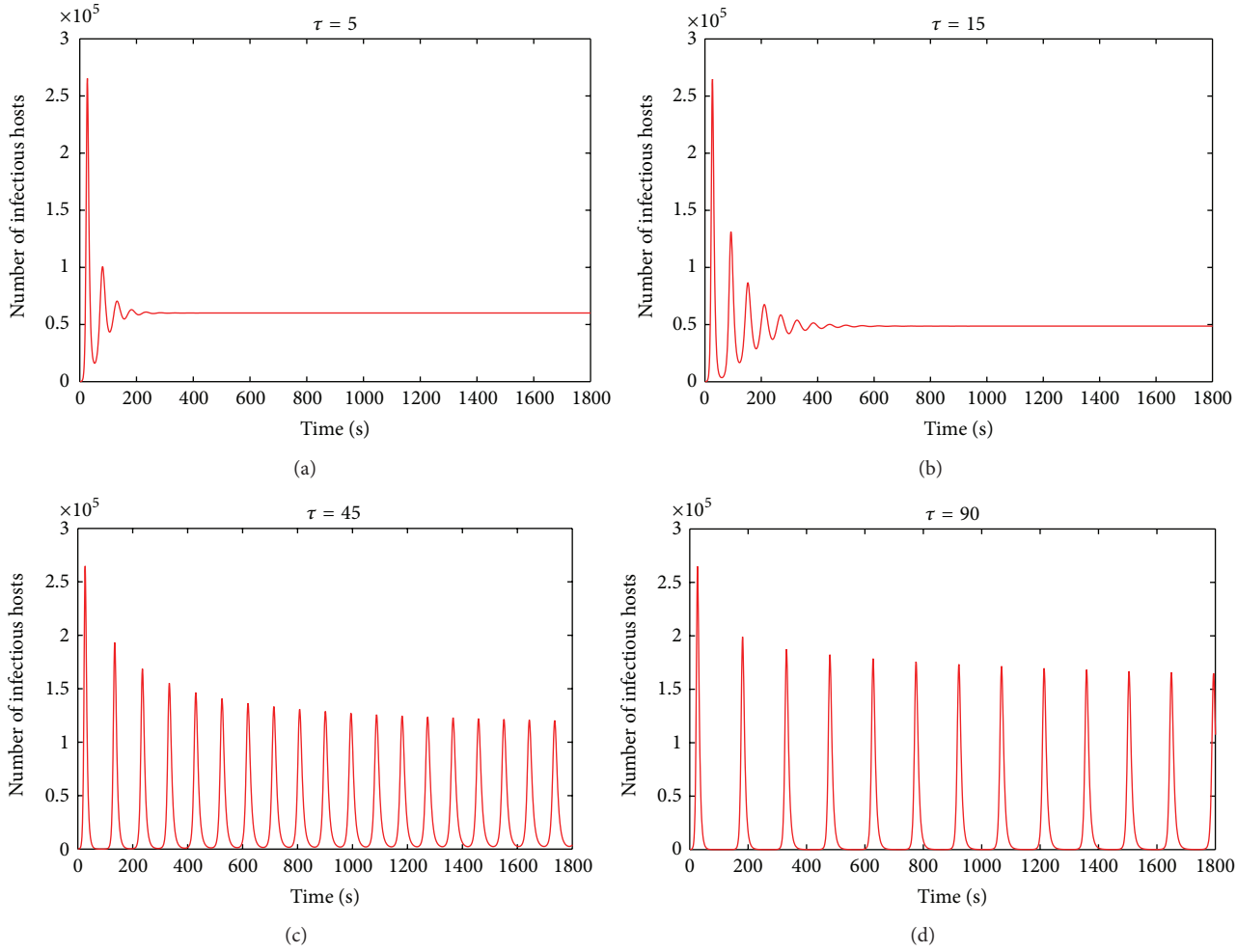


Figure 7: The number of infectious hosts when τ is changed in four coordinates.

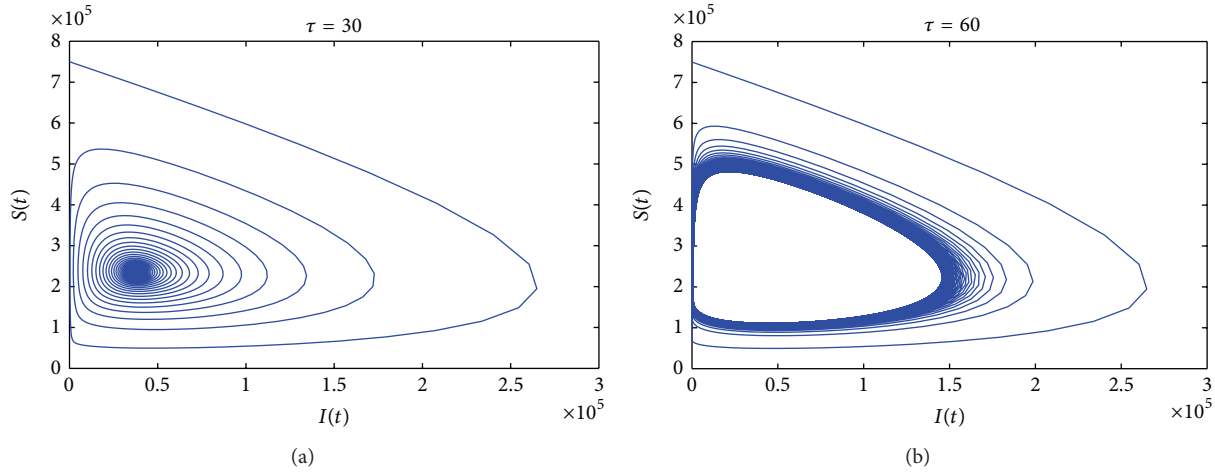
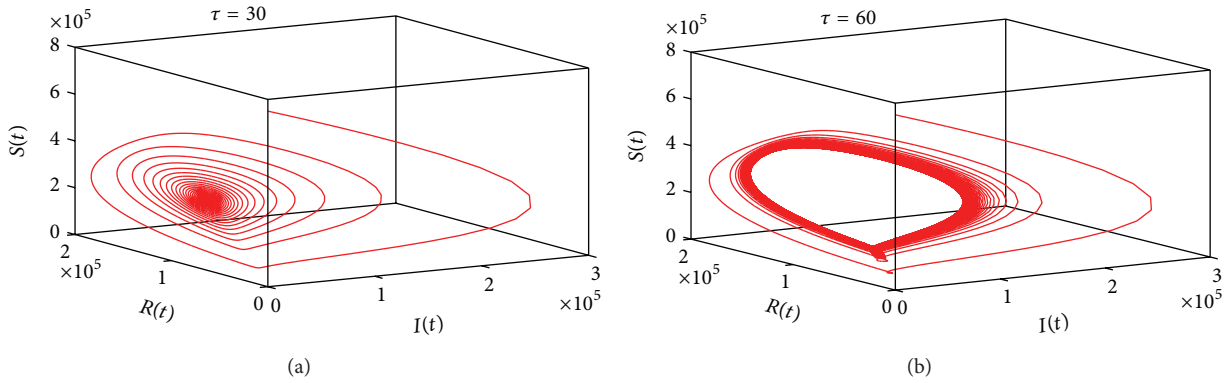


Figure 8: The phase portrait of susceptible hosts $S(t)$ and infectious hosts $I(t)$.



(a) (b)

FIGURE 9: The projection of the phase portrait of system (1) in (S, I, R)-space.

Therefore

$$\begin{aligned}
&\operatorname{sgn}\left[\frac{d\operatorname{Re}\lambda}{d\tau}\right]_{\tau=\tau_k} \\
&= \operatorname{sgn}\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} \\
&= \operatorname{sgn}\left[\operatorname{Re}\left(\frac{\left(5\lambda^4+4p_4\lambda^3+3p_3\lambda^2+2p_2\lambda+p_1\right)e^{\lambda\tau}}{\left(q_2\lambda^2+q_1\lambda+q_0\right)\lambda}\right.\right. \\
&\qquad \left.\left.+\frac{2q_2\lambda+q_1}{\left(q_2\lambda^2+q_1\lambda+q_0\right)\lambda}-\frac{\tau}{\lambda}\right)\right]_{\lambda=i\omega_0} \\
&= \operatorname{sgnRe}\left\{\left(\left(5\omega_0^4-4p_4\omega_0^3 i-3p_3\omega_0^2+2p_2\omega_0 i+p_1\right)\right.\right. \\
&\qquad \times\left[\cos\left(\omega_0\tau_k\right)+i\sin\left(\omega_0\tau_k\right)\right]\Big) \\
&\qquad \times\left(\left(q_1\omega_0 i+q_0-q_2\omega^2\right)\omega_0 i\right)^{-1} \\
&\qquad \left.\left.+\frac{2q_2\omega_0 i+q_1}{\left(q_1\omega_0 i+q_0-q_2\omega^2\right)\omega_0 i}\right\}\right. \\
&= \operatorname{sgn}\frac{\omega_0^2}{K}\left[4\omega_0^6+\left(3p_4^2-6p_3\right)\omega_0^4\right. \\
&\qquad +\left(2p_3^2+4p_1-4p_2p_4\right)\omega_0^2 \\
&\qquad \left.+\left(p_2^2+2p_0p_4-2p_1p_3\right)\right] \\
&= \operatorname{sgn}\frac{\omega_0^2}{\Gamma} = \operatorname{sgn}\frac{\omega_0^2}{\Gamma}\left\{h'\left(\omega_0^2\right)\right\} = \operatorname{sgn}\left\{h'\left(\omega_0^2\right)\right\},
\end{aligned} \tag{25}$$

where $K = q_1^2\omega_0^4+(q_0\omega_0-q_2\omega_0^3)^2$. *It follows from the hypothesis* (H_2) *that* $h'(\omega_0^2)\neq 0$ *and therefore the transversality condition holds. It can be obtained that*

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_k} > 0. \tag{26}$$

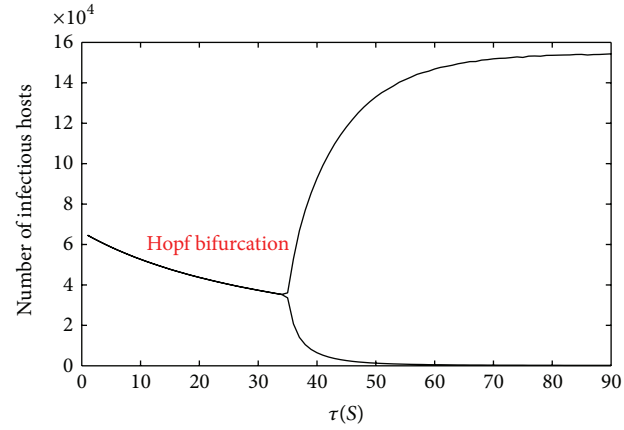


FIGURE 10: Bifurcation diagram of system (1) with τ ranging from 1 to 90.

The root of characteristic equation (10) *crosses from the left to the right on the imaginary axis as* τ *continuously varies from a value less than* τ_k *to one greater than* τ_k *according to Rouche's theorem [15]. Therefore, according to the Hopf bifurcation theorem [30] for functional differential equations, the transversality condition holds and the conditions for Hopf bifurcation are satisfied at* $\tau = \tau_k$. *Then the following result can be obtained.*

Theorem 7. *Supposing that condition* (H_1) *is satisfied,*

(1) *if* $\tau \in [0, \tau_0)$, *then the positive equilibrium* $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ *of system* (7) *is asymptotically stable and unstable when* $\tau > \tau_0$;

(2) *if condition* (H_2) *is satisfied, system* (7) *will undergo a Hopf bifurcation at the positive equilibrium* $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ *when* $\tau = \tau_k$ $(k = 0, 1, 2, \ldots)$, *where* τ_k *is defined by* (22).

This implies that when the time delay $\tau < \tau_0$, the system will stabilize at its infection equilibrium point, which is beneficial for us to implement a containment strategy; when time delay $\tau > \tau_0$, the system will be unstable and worms cannot be effectively controlled.

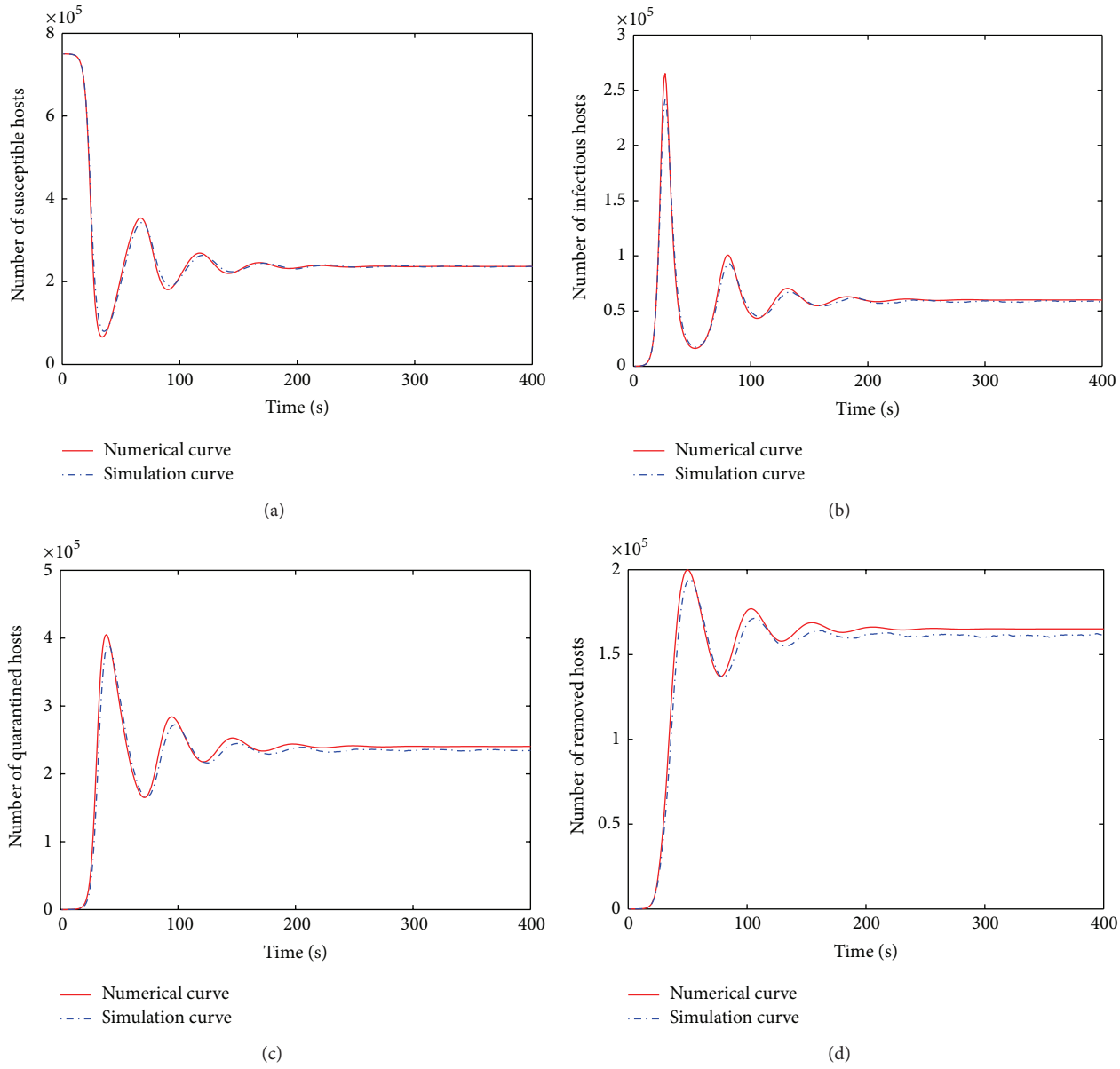


FIGURE 11: Comparisons between numerical curves and simulation curves when $\tau < \tau_0$.

4. Numerical Analysis

In this section, several numerical results are presented to prove the correctness of theoretical analysis above. 750,000 hosts and 50,000 removable devices are selected as the population size; the worm's average scan rate is $\eta = 4000$ per second. The worm infection rate can be calculated as $\alpha = \eta N/2^{32} = 0.698$, which means that average 0.698 hosts of all the hosts can be scanned by one infectious host. The infection ratio is $\beta_1 = \eta/2^{32} = 0.00000093$. The contact infection rate between hosts and removable devices is $\beta_2 = 0.0045$. The recovery rates of infectious hosts and removable devices are $\gamma_1 = 0.02$ and $\gamma_2 = 0.005$, respectively. The immunization rate of quarantined hosts is $\delta = 0.05$ and the reassembly rate of immunization hosts is $\omega = 0.08$. At the beginning, there are 50 infectious hosts and 20 infectious removable devices, while the rest of hosts and removable devices are susceptible.

In anomaly intrusion detection system, the rate at which infected hosts are detected and quarantined is $\theta_2 = 0.2$ per second. It means that an infected host can be detected and quarantined in about 5 s. The rate at which susceptible hosts are detected and quarantined is $\theta_1 = 0.00002315$ per second; that is, about two false alarms are generated by the anomaly intrusion detection system per day.

When $\tau = 5 < \tau_0$, Figure 2 presents the changes of the number of five kinds of hosts and Figure 3 shows the curves of two kinds of removable devices. According to Theorem 5, the positive equilibrium $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ is asymptotically stable when $\tau \in [0, \tau_0)$, which is illustrated by the numerical simulations in Figures 2 and 3. Finally, the

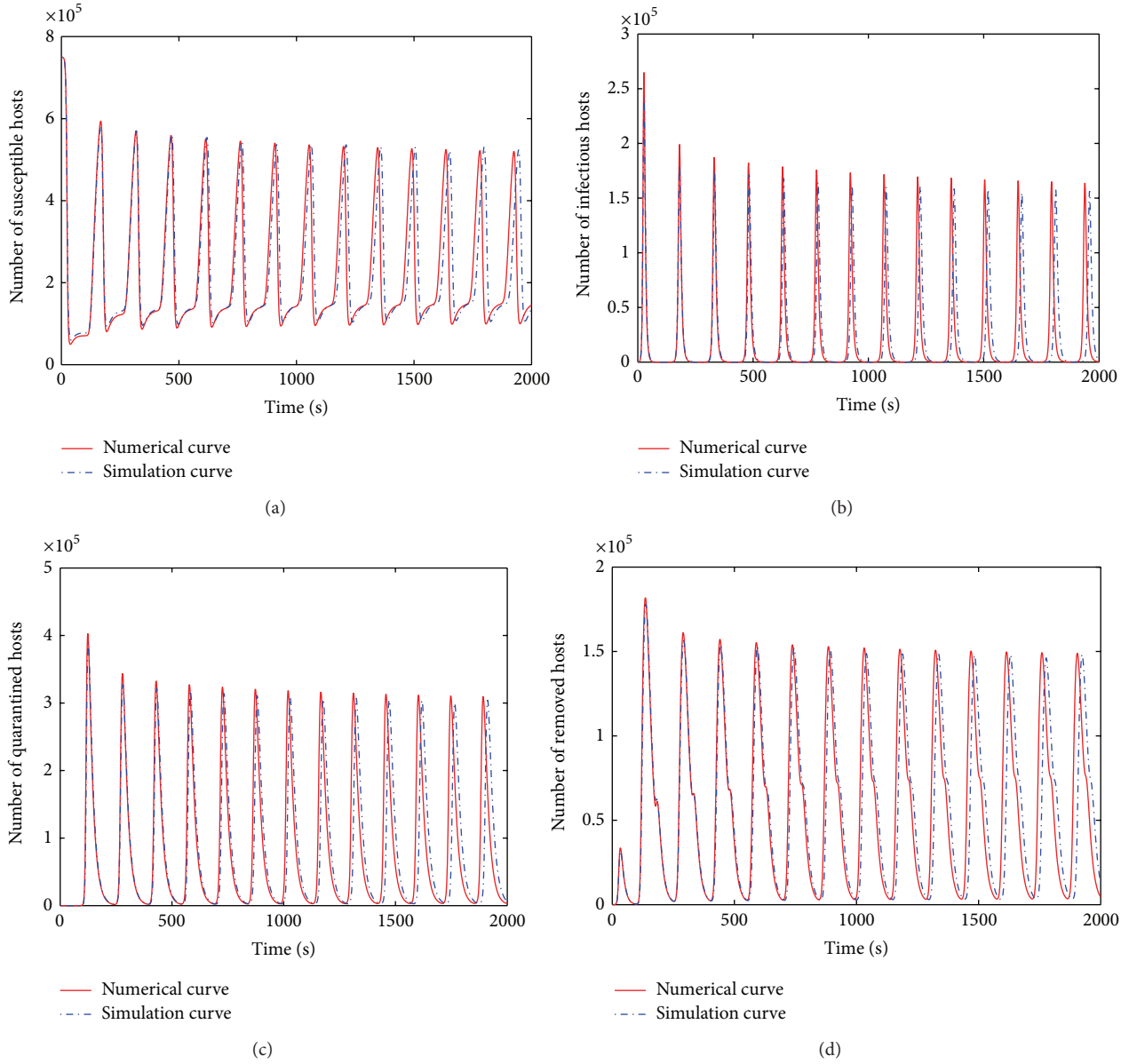


Figure 12: Comparisons between numerical curves and simulation curves when $\tau > \tau_0$.

number of every kind of host and removable device keeps stable.

When τ gets increased and passes through the threshold τ_0, the positive equilibrium $E^* = (S^*, I^*, D^*, R^*, R_I^*)$ will lose its stability and a Hopf bifurcation will occur. A family of periodic solution bifurcates from the positive equilibrium E^*. When $\tau = 45 > \tau_0$, Figure 4 shows the curves of susceptible, infectious, quarantined, and removed hosts and the numerical simulation results of two kinds of removable devices are depicted by Figure 5. From Figures 4 and 5, we can clearly see that every state of hosts and removable devices is unstable. Figure 4 shows that the number of infectious hosts will outburst after a short period of peace and repeat again and again.

In order to state the influence of time delay, the delay τ is set to a different value each time with other parameters remaining unchanged. Figure 6 shows four curves of the number of infectious hosts in the same coordinate with four delays: $\tau = 5$, $\tau = 15$, $\tau = 45$, and $\tau = 90$, respectively. Figures 7(a)–7(c) show four curves of the number of infectious hosts in four coordinates. Initially, the four curves are overlapped which means that the time delay has little effect on the initial state of worm spread. With the increase of the time T, the time delay affects the number of infectious hosts. With the increase of time delay, the curve begins to oscillate. The system becomes unstable as time delay passes through the critical value τ_0. At the same time, it can be discovered that the amplitude and period of the number of infectious hosts gradually increase.

Figures 8(a) and 8(b) show the phase portraits of susceptible hosts $S(t)$ and infectious hosts $I(t)$ with $\tau = 30 < \tau_0$ and $\tau = 60 > \tau_0$, respectively. Figures 9(a) and 9(b) show

the projection of the phase portrait of system (1) in (S, I, R)-space when $\tau = 30 < \tau_0$ and $\tau = 60 > \tau_0$, respectively. It is clear that the curve converges to a fixed point when $\tau < \tau_0$, which means that the system is stable. When $\tau > \tau_0$, the curve converges to a limit circle, which implies that the system is unstable and the worm propagation is out of control.

Figure 10 shows the bifurcation diagram with τ from 1 to 90. It is clear that Hopf bifurcation will occur when $\tau = \tau_0 = 35$.

5. Simulation Experiments

In our simulation experiments, the discrete-time simulation is adopted because of its accuracy and is less time-consuming. The discrete-time simulation is an expanded version of Zou's program simulating Code Red worm propagation. All of the parameters are consistent with the numerical experiments.

Figures 11(a)–11(d) show the comparisons between numerical and simulation curves of susceptible, infectious, quarantined, and removed hosts when $\tau = 5 < \tau_0$, respectively. It is clearly seen that the simulation curves match the numerical ones very well. Figures 12(a)–12(d) show the comparisons between numerical and simulation results of four kinds of hosts when $\tau = 90 > \tau_0$. In this figure, two curves are still matched well. It fully illustrates the correctness of our theoretical analysis.

6. Conclusions

In this paper, considering the influence of removable devices, a delayed worm propagation dynamical system based on anomaly IDS has been constructed. By regarding the time delay caused by time window of anomaly IDS as the bifurcation parameter, the local asymptotic stability at the positive equilibrium and local Hopf bifurcation were discussed. Through theoretical analysis and related experiments, the main conclusions can be summarized as follows.

(a) The critical time delay τ_0 where Hopf bifurcation appears is derived:

$$\tau_0 = \frac{1}{\omega_0}\arccos\Bigg[\left(\left(q_0 - q_2\omega^2\right)\left(p_2\omega_0^2 - p_4\omega_0^4 - p_0\right) + q_1\omega_0\left(p_3\omega_0^3 - \omega_0^5 - p_1\omega_0\right)\right) \times\left(\left(q_0 - q_2\omega^2\right)^2 + q_1^2\omega_0^2\right)^{-1}\Bigg]. \quad (27)$$

(b) When the time delay $\tau < \tau_0$, worm propagation system is stable and worms' behavior is easy to predict, which is beneficial for us to implement containment strategy to control and eliminate the worm.

(c) When time delay $\tau \geq \tau_0$, Hopf bifurcation occurs, which implies that the system will be unstable and containment strategy does not work effectively.

Thus, in order to control and even eliminate the worm, the size of time window of anomaly IDS must be less than τ_0. In real network environment, various factors can affect worm propagation. This paper concentrates on analyzing the influence of time delay caused by anomaly IDS; other factors having an impact on worm propagation will be the center of our future study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Optimal Control of a Nonlinear Time-Delay System in Batch Fermentation Process

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The main control goal in batch process is to get a high yield of products. In this paper, to maximize the yield of 1,3-propanediol (1,3-PD) in bioconversion of glycerol to 1,3-PD, we consider an optimal control problem involving a nonlinear time-delay system. The control variables in this problem include the initial concentrations of biomass and glycerol and the terminal time of the batch process. By a time-scaling transformation, we transcribe the optimal control problem into a new one with fixed terminal time, which yields a new nonlinear system with variable time-delay. The gradients of the cost and constraint functionals with respect to the control variables are derived using the costate method. Then, a gradient-based optimization method is developed to solve the optimal control problem. Numerical results show that the yield of 1,3-PD at the terminal time is increased considerably compared with the experimental data.

1. Introduction

Batch processing represents the natural way to scale-up processes from the laboratory to the production environment [1]. During the batch operation, no substrate is added to the initial charge and no product is removed until the end of the process [2]. Optimal control of batch processes has received attention recently because it is a choice for reducing production costs, improving product quality, and meeting safety requirements [3–5].

1,3-Propanediol (1,3-PD) is one of the important products used in the chemical industry. Using glycerol for producing 1,3-PD is effective from both the economical and ecological point of view. The operations of glycerol bioconversion to 1,3-PD consist of batch, continuous, and fed-batch cultures. Compared with continuous and fed-batch cultures, glycerol fermentation in batch process can obtain the highest production concentration and molar yield 1,3-PD to glycerol [6]. For this process, many studies have been carried out including the quantitative description of the cell growth kinetics [7, 8], enzyme-catalytic kinetics [9], the parameter identification problem [10–13], and the pathway optimization problem [14]. In particular, considering the existence of time-delays in the fermentation process [15, 16], a nonlinear time-delay system was recently proposed in [13]. Numerical simulations showed that the nonlinear time-delay system can describe the batch process better than previous mathematical models. However, in batch process of glycerol bioconversion to 1,3-PD, the aim is to obtain as much 1,3-PD as possible at the terminal time. Mathematically, this is an optimal control problem. Although the achieved results are interesting, such optimal control problem is ignored in the above researches.

In this paper, taking the initial concentrations of biomass and glycerol and the terminal time of the fermentation process as control variables and the yield of 1,3-PD as the performance index, we propose an optimal control model involving nonlinear time-delay system in [13] and subject to continuous state constraints. In fact, this problem is a free time delayed optimal control problem. By the way, there has been a mounting interest in optimal control of time-delay systems. A maximum principle for optimal control problem with a constant delay was provided in [17]. It should, however, be noted that many time-delay systems including the nonlinear time-delay system in this paper are highly nonlinear. Therefore, it is often impossible to obtain analytical solutions of the delayed optimal control problem and one has

to resort to numerical solution methods. As a result, some successful algorithms, such as iterative dynamic programming method [18], control parameterization method [19], and measure theoretical approach [20], have been developed. However, terminal time in above optimal control problems is fixed. The presence of free terminal time makes the delayed optimal control problem much more complicated. Thus, to solve the optimal control problem in this work, these existing computational methods cannot be used directly and new computational techniques should be explored.

In this paper, by a time-scaling transformation, we equivalently transcribe the free time delayed optimal control problem into one with fixed terminal time. It is significant to mention that this transformation yields a more complex dynamic system in which the time-delays are variable. By the costate method, we derived the gradients of the cost functional and constraints with respect to the control variables. Then, a gradient-based optimization technique is developed. Finally, numerical results show that the yield of 1,3-PD at the terminal time is increased considerably compared with the experiment data.

2. Nonlinear Time-Delay Systems

In batch process, a proper quantity of biomass and glycerol is added to the reactor only once and stirred uniformly under given conditions. Then, 1,3-PD is removed at the end of the process. Based on the work [13], the following nonlinear time-delay system can be used to describe the batch process:

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-h)), \quad t \in (0, T], \\ x(0) &= \zeta, \\ x(t) &= \phi(t, \zeta), \quad t \le 0, \end{aligned} \tag{1}$$

where $x(t) := (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))^\top \in R^5$ is the state vector whose components are, respectively, the concentrations of biomass, glycerol, 1,3-PD, acetate, and ethanol in the reactor at time t; h is a delay argument; $T > 0$ is the terminal time; $\zeta := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^\top$ is the initial state; $\phi : R \times R^5 \to R^5$ is a given initial function; and the dynamics of the batch process is given by

$$f(x(t), x(t-h)) := \begin{pmatrix} \mu x_1(t-h) \\ -q_2 x_1(t-h) \\ q_3 x_1(t-h) \\ q_4 x_1(t-h) \\ q_5 x_1(t-h) \end{pmatrix}. \tag{2}$$

In (2), the specific growth rate of cells μ, the specific consumption rate of substrate q_2, and the specific formation rates of products q_ℓ, $\ell = 3, 4, 5$, can be expressed as

$$\begin{aligned} \mu &= \frac{\mu_m x_2(t)}{k_1 + x_2(t)} \prod_{\ell=2}^{5} \left(1 - \frac{x_\ell(t)}{x_\ell^*}\right), \\ q_\ell &= m_\ell + \mu Y_\ell, \quad \ell = 2, 3, \ldots, 5, \end{aligned} \tag{3}$$

where μ_m, k_1, m_ℓ, and Y_ℓ are kinetic parameters and x_* and x^* are the critical concentrations for cells growth.

The terminal time in system (1) is a control variable. Define

$$\mathcal{T} := \{\gamma \in R : T_{\min} \le \gamma \le T_{\max}\}, \tag{4}$$

where $T_{\min}$ and $T_{\max}$ are the lower and upper bounds for the terminal time, respectively. Any $T \in \mathcal{T}$ is called an *admissible terminal time*. Moreover, since no product comes into being at the initial point, $\zeta_\ell = 0$, $\ell = 3, 4, 5$. Hence, let $u = (\zeta_1, \zeta_2)^\top$ and define

$$\mathcal{U} := \{v \in R^2 : a_i \le v_i \le b_i\}, \tag{5}$$

where a_i and b_i are real numbers such that $a_i < b_i$. Any $u \in \mathcal{U}$ is called an *admissible initial vector of biomass and glycerol*.

For the nonlinear time-delay system (1), there exists a unique continuous solution $x(\cdot \mid u)$ corresponding to each $u \in \mathcal{U}$ on $[0, \infty)$ [21]. Furthermore, there exist critical concentrations of biomass, glycerol, and products, outside which cells cease to grow. Hence, it is biologically meaningful to restrict the concentrations of biomass, glycerol, and products within a set W defined as

$$x(t \mid u) \in W := \prod_{\ell=1}^{5} [x_{*\ell}, x_\ell^*], \quad t \in [0, T]. \tag{6}$$

3. Optimal Control Problems

In the batch process, it is desired that the yield of 1,3-PD should be maximized at the end of the process. This is achieved by manipulating some control variables u and T. Thus, the optimal control problem in batch process can be formulated as

$$\begin{aligned} \text{(OCP)} \qquad \min \quad & J(u, T) = -\frac{x_3(T \mid u)}{T} \\ \text{s.t.} \quad & x(t \mid u) \in W, \quad t \in [0, T], \\ & (u, T) \in \mathcal{U} \times \mathcal{T}. \end{aligned} \tag{7}$$

Note that (OCP) is of nonstandard type because the terminal time and the initial values of biomass and glycerol are variables to be determined. Thus, the (OCP) is actually a free time optimal control problem. It is difficult to solve the (OCP) using existing numerical techniques [18–20]. The main difficulty is the implicit dependence of the system state on the terminal time. We now employ a time-scaling transformation from $[0, T]$ to $[0, 1]$ as follows:

$$t = t(s) = Ts, \tag{8}$$

where $s \in [0, 1]$ is a new time variable. Clearly, $s = 0$ corresponds to $t = 0$, and $s = 1$ corresponds to $t = T$. Let $\tilde{x}(s) = x(t(s))$. Then

$$\begin{aligned} \dot{\tilde{x}}(s) &= \frac{d}{ds}\{x(t(s))\} \\ &= \frac{dx(t(s))}{dt}\frac{dt(s)}{ds} \\ &= Tf(x(Ts), x(Ts - h)) \\ &= Tf\left(\tilde{x}(s), \tilde{x}\left(s - hT^{-1}\right)\right). \end{aligned} \tag{9}$$

The initial conditions become

$$\begin{aligned} &\widetilde{x}(0) = \zeta, \\ &\widetilde{x}(s) = \phi(Ts, \zeta), \quad s \le 0. \end{aligned} \tag{10}$$

Thus, system (1) is transformed into the following form:

$$\begin{aligned} &\dot{\widetilde{x}}(s) = Tf\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right), \quad s \in [0,1], \\ &\widetilde{x}(0) = \zeta, \\ &\widetilde{x}(s) = \phi(Ts, \zeta), \quad s \le 0. \end{aligned} \tag{11}$$

Let $\widetilde{x}(\cdot \mid u, T)$ denote the solution of system (11) corresponding to each $(u, T) \in \mathcal{U} \times \mathcal{T}$ on $[0, 1]$. Then, (OCP) can be transcribed into the following equivalently optimal control problem with fixed terminal time:

$$\text{(EOCP)} \quad \min \quad J(u, T) = -\frac{\widetilde{x}_3(1 \mid u, T)}{T} \tag{12}$$

$$\begin{aligned} &\text{s.t.} \quad \widetilde{x}(s \mid u, T) \in W, \quad s \in [0,1], \\ &\qquad (u, T) \in \mathcal{U} \times \mathcal{T}. \end{aligned} \tag{13}$$

Note that the time-delay in system (11) depends on the control variable T. Thus, when applied to time-delay systems, the time-scaling transformation (8) yields a more complex dynamic system in which the time-delay is variable.

4. Computational Approaches

(EOCP) is in essence an optimization problem with continuous state inequality constraint (13), which has an infinite number of constraints and can be viewed as a semi-infinite programming problem. An efficient algorithm transforming this type of problems to standard mathematical programming problems was discussed in [22]. We will now briefly discuss the application of this algorithm to (EOCP).

Let

$$\begin{aligned} &g_\ell(\widetilde{x}(s \mid u, T)) := x_\ell^* - \widetilde{x}_\ell(s \mid u, T), \\ &g_{5+\ell}(\widetilde{x}(s \mid u, T)) := \widetilde{x}_\ell(s \mid u, T) - x_{*\ell}, \quad \ell = 1, 2, \ldots, 5. \end{aligned} \tag{14}$$

The continuous state constraint (13) becomes

$$G(u, T) = 0, \tag{15}$$

where $G(u,T) := \sum_{l=1}^{10} \int_0^1 \min\{0, g_l(\widetilde{x}(s \mid u, T))\} ds$. However, the equality constraint (15) is nondifferentiable at the points when $g_l = 0$. We replace (15) with the following inequality constraint:

$$\widetilde{G}_{\epsilon,\tau}(u, T) := \tau + \sum_{l=1}^{10} \int_0^1 \pi_\epsilon\left(g_l(\widetilde{x}(s \mid u, T))\right) ds \ge 0, \tag{16}$$

where $\epsilon > 0$, $\tau > 0$, and

$$\pi_\epsilon(\eta) = \begin{cases} \eta, & \text{if } \eta < -\epsilon, \\ -\dfrac{(\eta - \epsilon)^2}{4\epsilon}, & \text{if } -\epsilon \le \eta \le \epsilon, \\ 0, & \text{if } \eta > \epsilon. \end{cases} \tag{17}$$

Thus, (EOCP) is approximated by a sequence of standard mathematical programming problems $\{(\text{EOCP}_{\epsilon,\tau})\}$ defined by replacing constraint (15) with (16). Moreover, each of $\{(\text{EOCP}_{\epsilon,\tau})\}$ can be solved by a gradient-based optimization method (e.g., sequential quadratic programming (SQP) [23]). However, this optimization method requires the gradients of the cost functional and constraints. Now, we will derive these required gradients using the so-called *costate method*, which is a commonly used technique in the optimal control domain [22, 24, 25].

Define

$$\psi(s) := \begin{cases} \dfrac{\partial \phi(Ts, \zeta)}{\partial s}, & s \le 0, \\ Tf\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right), & s \in [0,1], \end{cases} \tag{18}$$

$$\chi_I(s) := \begin{cases} 1, & s \in I, \\ 0, & \text{otherwise.} \end{cases}$$

The gradients of the cost functional $J(\cdot, \cdot)$ defined in (12) with respect to the control variables are given in the following theorem.

Theorem 1. *Let $(u, T) \in \mathcal{U} \times \mathcal{T}$. Then*

$$\begin{aligned} \frac{\partial J(u, T)}{\partial u} &= \lambda^\top(0) \frac{\partial \widetilde{x}(0)}{\partial u} \\ &\quad + \int_{-hT^{-1}}^0 T\lambda^\top\left(s + hT^{-1}\right) \\ &\qquad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \frac{\partial \phi(Ts, \zeta)}{\partial u} ds, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial J(u, T)}{\partial T} &= \widetilde{x}_3(1) T^{-2} + \int_0^1 \lambda^\top(s) f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right) ds \\ &\quad + \int_0^1 hT^{-1}\lambda^\top(s) \\ &\qquad \times \frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right)}{\partial \widetilde{x}\left(s - hT^{-1}\right)} \psi\left(s - hT^{-1}\right) ds \\ &\quad + \int_{-hT^{-1}}^0 T\lambda^\top\left(s + hT^{-1}\right) \\ &\qquad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \frac{\partial \phi(Ts, \zeta)}{\partial T} ds, \end{aligned} \tag{20}$$

where $\lambda(\cdot)$ is the solution of the following costate system:

$$\begin{aligned} &\dot{\lambda}(s) \\ &= -T\left(\frac{\partial f(\widetilde{x}(s), \widetilde{x}(s - hT^{-1}))}{\partial \widetilde{x}(s)}\right)^\top \lambda(s) \\ &\quad - T\chi_{[0,1-hT^{-1}]}(s) \left(\frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\right)^\top \lambda\left(s + hT^{-1}\right), \\ &\qquad\qquad s \in [0,1], \end{aligned} \tag{21}$$

with the initial conditions

$$\lambda(1) = \left(0, 0, -T^{-1}, 0, 0\right)^{\top},$$
$$\lambda(s) = (0, 0, 0, 0, 0)^{\top}, \quad s > 1. \tag{22}$$

Proof. The derivations of the gradients of the cost functional $J(u,T)$ with respect to u and T are similar. Thus, only the derivation of the gradient of the cost functional $J(u,T)$ with respect to u is given below.

Let $w : [0,\infty] \to R^5$ be an arbitrary function that is continuous and differentiable almost everywhere. Then, we may express the cost functional J as follows:

$$\begin{aligned} &J(u,T) \\ &\quad = -T^{-1}\widetilde{x}_3(1) \\ &\qquad + \int_0^1 w^{\top}(s)\left(Tf\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right) - \dot{\widetilde{x}}(s)\right) ds \\ &\quad = -T^{-1}\widetilde{x}_3(1) + \int_0^1 Tw^{\top}(s) f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right) ds \\ &\qquad - \int_0^1 w^{\top}(s)\dot{\widetilde{x}}(s)\, ds. \end{aligned} \tag{23}$$

Applying integration by parts to the last integral term gives

$$\begin{aligned} &J(u,T) \\ &\quad = -T^{-1}\widetilde{x}_3(1) \\ &\qquad + \int_0^1 \left(Tw^{\top}(s) f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right) + \dot{w}^{\top}(s)\widetilde{x}(s)\right) ds \\ &\qquad - w^{\top}(1)\widetilde{x}(1) + w^{\top}(0)\widetilde{x}(0). \end{aligned} \tag{24}$$

Differentiating (24) with respect to u yields

$$\begin{aligned} &\frac{\partial J(u,T)}{\partial u} \\ &= -T^{-1}\frac{\partial \widetilde{x}_3(1)}{\partial u} - w^{\top}(1)\frac{\partial \widetilde{x}(1)}{\partial u} + w^{\top}(0)\frac{\partial \widetilde{x}(0)}{\partial u} \\ &\quad + \int_0^1 \left(Tw^{\top}(s)\frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right)}{\partial \widetilde{x}(s)} + \dot{w}^{\top}(s)\right)\frac{\partial \widetilde{x}(s)}{\partial u} ds \\ &\quad + \int_0^1 Tw^{\top}(s)\frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right)}{\partial \widetilde{x}\left(s - hT^{-1}\right)}\frac{\partial \widetilde{x}\left(s - hT^{-1}\right)}{\partial u} ds. \end{aligned} \tag{25}$$

Performing a change of variable in the last term on the right-hand side of (25) gives

$$\begin{aligned} &\int_0^1 Tw^{\top}(s)\frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right)}{\partial \widetilde{x}\left(s - hT^{-1}\right)}\frac{\partial \widetilde{x}\left(s - hT^{-1}\right)}{\partial u} ds \\ &\quad = \int_0^1 T\chi_{[0,1-hT^{-1}]}(s)\, w^{\top}\left(s + hT^{-1}\right) \\ &\qquad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\frac{\partial \widetilde{x}(s)}{\partial u} ds \\ &\qquad + \int_{-hT^{-1}}^0 Tw^{\top}\left(s + hT^{-1}\right) \\ &\qquad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\frac{\partial \phi(Ts, \zeta)}{\partial u} ds. \end{aligned} \tag{26}$$

Substituting (26) into (25) yields

$$\begin{aligned} &\frac{\partial J(u,T)}{\partial u} \\ &\quad = -T^{-1}\frac{\partial \widetilde{x}_3(1)}{\partial u} - w^{\top}(1)\frac{\partial \widetilde{x}(1)}{\partial u} + w^{\top}(0)\frac{\partial \widetilde{x}(0)}{\partial u} \\ &\qquad + \int_0^1 \left(\dot{w}^{\top}(s) + Tw^{\top}(s)\frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right)}{\partial \widetilde{x}(s)}\right. \\ &\qquad\quad + T\chi_{[0,1-hT^{-1}]}(s)\, w^{\top}\left(s + hT^{-1}\right) \\ &\qquad\quad \left.\times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\right)\frac{\partial \widetilde{x}(s)}{\partial u} ds \\ &\qquad + \int_{-hT^{-1}}^0 Tw^{\top}\left(s + hT^{-1}\right) \\ &\qquad\quad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\frac{\partial \phi(Ts, \zeta)}{\partial u} ds. \end{aligned} \tag{27}$$

Choosing $w(\cdot) = \lambda(\cdot)$ and substituting (21)-(22) into the above equation, we obtain the conclusion (19). The gradient formula (20) can be derived similarly. The proof is complete. □

The gradients of the constraint $\widetilde{G}_{\epsilon,\tau}(\cdot,\cdot)$ defined in (16) with respect to the control variables are given in the next theorem.

Theorem 2. *Let $(u,T) \in \mathcal{U} \times \mathcal{T}$. Then*

$$\begin{aligned} \frac{\partial \widetilde{G}_{\epsilon,\tau}(u,T)}{\partial u} &= \widetilde{\lambda}^{\top}(0)\frac{\partial \widetilde{x}(0)}{\partial u} \\ &\quad + \int_{-hT^{-1}}^0 T\widetilde{\lambda}^{\top}\left(s + hT^{-1}\right) \\ &\quad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\frac{\partial \phi(Ts, \zeta)}{\partial u} ds, \end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial \widetilde{G}_{\epsilon,\tau}(u,T)}{\partial T} = & \int_0^1 \widetilde{\lambda}^\top(s)\, f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right) ds \\ & + \int_0^1 hT^{-1}\widetilde{\lambda}^\top(s)\, \frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right)}{\partial \widetilde{x}\left(s-hT^{-1}\right)} \\ & \quad \times \psi\left(s-hT^{-1}\right) ds \\ & + \int_{-hT^{-1}}^0 T\widetilde{\lambda}^\top\left(s+hT^{-1}\right) \\ & \quad \times \frac{\partial f\left(\widetilde{x}\left(s+hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \frac{\partial \phi(Ts,\zeta)}{\partial T} ds,\end{aligned} \tag{29}$$

where $\widetilde{\lambda}(\cdot)$ is the solution of the following costate system:

$$\begin{aligned}\dot{\widetilde{\lambda}}(s) = & -\sum_{l=1}^{10} \frac{\partial \pi_\epsilon\left(g_l(\widetilde{x}(s))\right)}{\partial \widetilde{x}(s)} \\ & - T\left(\frac{\partial f(\widetilde{x}(s), \widetilde{x}(s-hT^{-1}))}{\partial \widetilde{x}(s)}\right)^\top \widetilde{\lambda}(s) \\ & - T\chi_{[0,1-hT^{-1}]}(s)\left(\frac{\partial f\left(\widetilde{x}\left(s+hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)}\right)^\top \\ & \times \widetilde{\lambda}\left(s+hT^{-1}\right), \\ & s \in [0,1],\end{aligned} \tag{30}$$

with the initial condition

$$\widetilde{\lambda}(s) = (0,0,0,0,0)^\top, \quad s \geq 1. \tag{31}$$

Proof. The derivations of the gradients of the constraint $\widetilde{G}_{\epsilon,\tau}(u,T)$ with respect to u and T are similar. Thus, only the derivation of the gradient of the constraint $\widetilde{G}_{\epsilon,\tau}(u,T)$ with respect to u is given below.

Let $w : [0,\infty] \to R^5$ be an arbitrary function that is continuous and differentiable almost everywhere. Then, we may express the constraint $\widetilde{G}_{\epsilon,\tau}$ as follows:

$$\begin{aligned}\widetilde{G}_{\epsilon,\tau}(u,T) = & \tau + \sum_{l=1}^{10} \int_0^1 \pi_\epsilon\left(g_l(\widetilde{x}(s))\right) ds \\ & + \int_0^1 Tw^\top(s)\, f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right) ds \\ & - \int_0^1 w^\top(s)\, \dot{\widetilde{x}}(s)\, ds.\end{aligned} \tag{32}$$

Applying integration by parts to the last integral term gives

$$\begin{aligned}\widetilde{G}_{\epsilon,\tau}(u,T) = & \tau - w^\top(1)\,\widetilde{x}(1) + w^\top(0)\,\widetilde{x}(0) \\ & + \sum_{l=1}^{10} \int_0^1 \pi_\epsilon\left(g_l(\widetilde{x}(s))\right) ds \\ & + \int_0^1 \left(Tw^\top(s)\, f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right)\right. \\ & \quad \left. + \dot{w}^\top(s)\,\widetilde{x}(s)\right) ds.\end{aligned} \tag{33}$$

Differentiating (33) with respect to u yields

$$\begin{aligned}& \frac{\partial \widetilde{G}_{\epsilon,\tau}(u,T)}{\partial u} \\ & = -w^\top(1)\frac{\partial \widetilde{x}(1)}{\partial u} + w^\top(0)\frac{\partial \widetilde{x}(0)}{\partial u} \\ & \quad + \int_0^1 \left(\sum_{l=1}^{10} \frac{\partial \pi_\epsilon\left(g_l(\widetilde{x}(s))\right)}{\partial \widetilde{x}(s)}\right. \\ & \qquad + Tw^\top(s)\, \frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right)}{\partial \widetilde{x}(s)} \\ & \qquad \left. + \dot{w}^\top(s)\right) \frac{\partial \widetilde{x}(s)}{\partial u} ds \\ & \quad + \int_0^1 Tw^\top(s)\, \frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right)}{\partial \widetilde{x}\left(s-hT^{-1}\right)} \frac{\partial \widetilde{x}\left(s-hT^{-1}\right)}{\partial u} ds.\end{aligned} \tag{34}$$

Performing a change of variable in the last term on the right-hand side of (34) gives

$$\begin{aligned}& \int_0^1 Tw^\top(s)\, \frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s-hT^{-1}\right)\right)}{\partial \widetilde{x}\left(s-hT^{-1}\right)} \frac{\partial \widetilde{x}\left(s-hT^{-1}\right)}{\partial u} ds \\ & = \int_0^1 T\chi_{[0,1-hT^{-1}]}(s)\, w^\top\left(s+hT^{-1}\right) \\ & \qquad \times \frac{\partial f\left(\widetilde{x}\left(s+hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \frac{\partial \widetilde{x}(s)}{\partial u} ds \\ & \quad + \int_{-hT^{-1}}^0 Tw^\top\left(s+hT^{-1}\right) \\ & \qquad \times \frac{\partial f\left(\widetilde{x}\left(s+hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \frac{\partial \phi(Ts,\zeta)}{\partial u} ds.\end{aligned} \tag{35}$$

Substituting (35) into (34) yields

$$\begin{aligned}
&\frac{\partial \widetilde{G}_{\epsilon,\tau}(u,T)}{\partial u} \\
&= -w^{\top}(1)\frac{\partial \widetilde{x}(1)}{\partial u} + w^{\top}(0)\frac{\partial \widetilde{x}(0)}{\partial u} \\
&\quad + \int_0^1 \left(\sum_{l=1}^{10} \frac{\partial \pi_{\epsilon}\left(g_l\left(\widetilde{x}(s)\right)\right)}{\partial \widetilde{x}(s)} \right. \\
&\qquad + \dot{w}^{\top}(s) + Tw^{\top}(s)\frac{\partial f\left(\widetilde{x}(s), \widetilde{x}\left(s - hT^{-1}\right)\right)}{\partial \widetilde{x}(s)} \\
&\qquad + T\chi_{[0,1-hT^{-1}]}(s)\, w^{\top}\left(s + hT^{-1}\right) \\
&\qquad \left. \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \right) \frac{\partial \widetilde{x}(s)}{\partial u} ds \\
&\quad + \int_{-hT^{-1}}^{0} Tw^{\top}\left(s + hT^{-1}\right) \\
&\qquad \times \frac{\partial f\left(\widetilde{x}\left(s + hT^{-1}\right), \widetilde{x}(s)\right)}{\partial \widetilde{x}(s)} \frac{\partial \phi(Ts, \zeta)}{\partial u} ds.
\end{aligned} \tag{36}$$

Choosing $w(\cdot) = \widetilde{\lambda}(\cdot)$ and substituting (30) and (31) into the above equation, we obtain the conclusion (28). The gradient formula (29) can be derived similarly. Thus, the proof is complete. □

On the basis of Theorems 1 and 2, we can develop the following algorithm to solve the (OCP).

Algorithm 1.

Step 1. Choose initial values of u, T, ϵ, and τ.

Step 2. Solve (EOCP$_{\epsilon,\tau}$) using SQP [23] to give $(u^*_{\epsilon,\tau}, T^*_{\epsilon,\tau})$.

Step 3. Check feasibility of $g_l(\widetilde{x}(s \mid u^*_{\epsilon}, T^*_{\epsilon})) \geq 0$, for $l = 1, 2, \ldots, 10$.

Step 4. If $(u^*_{\epsilon,\tau}, T^*_{\epsilon,\tau})$ is feasible, then go to Step 5. Otherwise, set $\tau := \alpha\tau$, where α is a given positive constant. If $\tau < \overline{\tau}$, where $\overline{\tau}$ is a prespecified positive constant, then go to Step 6. Otherwise, go to Step 2.

Step 5. Set $\epsilon := \beta\epsilon$, where β is a given positive constant. If $\epsilon > \overline{\epsilon}$, then go to Step 2. Otherwise, go to Step 6.

Step 6. Output $(u^*_{\epsilon,\tau}, T^*_{\epsilon,\tau})$ and stop.

At the conclusion of Steps 1–6, $(u^*_{\epsilon,\tau}, T^*_{\epsilon,\tau})$ is an approximate optimal solution of the (OCP).

5. Numerical Results

Algorithm 1 was applied to seek the optimal control variables in (OCP) and all computations were implemented in Fortran 6.5. We obtained that $u^* = (0.201867, 418.376)^{\top}$ and $T^* = 7.2578$ h. Here, nonlinear time-delay system (1) was solved using the six-order Runge-Kutta method with Lagrange interpolation [26]. The time-delay $h = 0.26$ h, the initial state $\zeta = (0.102, 418.26087, 0, 0, 0)^{\top}$, the initial function $\phi(t) = \zeta$, the terminal time $T = 6.92$ h, and the kinetic parameters and critical concentrations are listed in Table 1. In the optimization process, we assume that $u_1 \in [0.02, 0.202]$, $u_2 \in [400, 500]$, and $T \in [5.0, 8.0]$. In addition, the smoothing and feasible parameters were initially selected as $\epsilon = 0.1$ and $\tau = 0.01$. The parameters α and β were chosen as 0.1 and 0.01 until the solution obtained was feasible for the (OCP). The process terminated when $\overline{\epsilon} = 1.0 \times 10^{-8}$ and $\overline{\tau} = 1.0 \times 10^{-7}$.

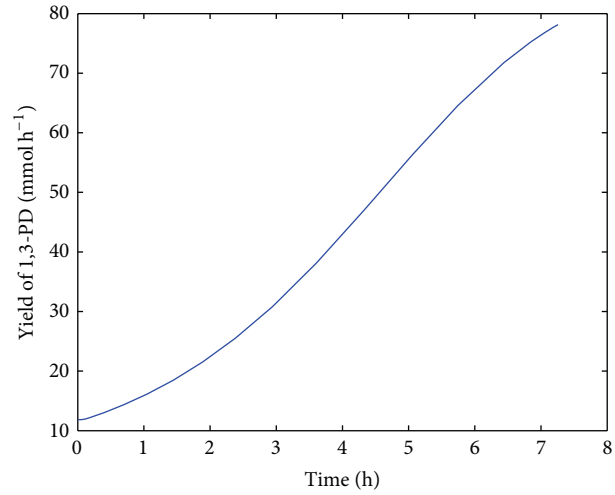

Figure 1: The yield change of 1,3-PD with respect to fermentation time.

Table 1: The kinetic parameters and critical concentrations in the system (1) [13].

μ_m	k_1	m_2	m_3	m_4	m_5	Y_2	Y_3	Y_4	Y_5
0.994	0.368	3.24	−3.679	−0.491	7.309	117.647	76	35.54	14.78
x_{*1}	x_{*2}	x_{*3}	x_{*4}	x_{*5}	x^*_1	x^*_2	x^*_3	x^*_4	x^*_5
0.01	0	0	0	0	6	2039	939.5	1026	360.9

Under the obtained optimal control strategy, we obtained the optimal yield of 1,3-PD at the optimal terminal time is 78.1586 mmol h^{-1}, which is increased by 90.04% compared with experiment data [13]. Furthermore, we plotted the optimal yield change of 1,3-PD with respect to the fermentation time in Figure 1. From Figure 1, we can see that the yield of 1,3-PD at the terminal time is actually increased.

6. Conclusion

In this paper, the optimal control problem in batch process was investigated. We presented the optimal control problem involving a nonlinear time-delay system and with free terminal time. We then transcribe the free time optimal control problem into a new one with fixed terminal time and variable time-delay. We developed a computational method based

on the gradients of the cost and constraint functionals with respect to the control variables. Numerical results showed the effectiveness of the developed computational method.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Modelling and Simulation of Volume Controlled Mechanical Ventilation System

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Volume controlled mechanical ventilation system is a typical time-delay system, which is applied to ventilate patients who cannot breathe adequately on their own. To illustrate the influences of key parameters of the ventilator on the dynamics of the ventilated respiratory system, this paper firstly derived a new mathematical model of the ventilation system; secondly, simulation and experimental results are compared to verify the mathematical model; lastly, the influences of key parameters of ventilator on the dynamics of the ventilated respiratory system are carried out. This study can be helpful in the VCV ventilation treatment and respiratory diagnostics.

1. Introduction

Mechanical ventilation is an important treatment which is usually utilized to ventilate patients who cannot breathe adequately on their own [1]. As an alternative mode of ventilation, volume controlled ventilation (VCV) can be used in respiratory failure [2].

As well known, the VCV ventilation system is a typical time-delay system. Due to the respiratory resistance and compliance, the dynamics of the ventilated lung always lags behind the output dynamics of the VCV ventilator. Time delays are usually the main causes of instability and poor performance of system [3–6]; therefore, in clinical application, the dynamics of the outlet of the ventilator is monitored to estimate the dynamics of the patient's respiratory system. So the precision of the estimation is limited [7].

Because the dynamics of the ventilated lung is influenced by the parameters of the VCV ventilator, in order to lay a foundation for the VCV ventilation treatment, the influences of the VCV ventilator's parameters on the dynamics of the ventilated lung should be illustrated. But the dynamics of the ventilated lung cannot be measured directly and precisely. Therefore, a simulation study of the ventilation system (including a VCV ventilator and a patient's respiratory system) is needed.

In the present modelling and simulation studies of the mechanical ventilation system, the system is commonly considered as an electrical system [8–11]. However, these models have many serious shortcomings, which limit their versatility and applicability [10, 12–15].

In this paper, in order to improve the versatility and applicability of the mathematical models of the VCV ventilation systems, the VCV ventilation system is considered as an equivalent pneumatic system. Then a new mathematical model of the VCV ventilation system can be derived.

Furthermore, through the simulation study on the VCV ventilation system, its dynamic characteristics can be obtained. Simulation and experimental results [7] are compared to verify the mathematical model.

Lastly, in order to provide guidance for the VCV ventilation treatment, influences of key parameters of the VCV ventilator on the dynamics of the ventilated respiratory system are studied.

2. Methods of Study on the VCV Ventilation System

2.1. Introduction of VCV Ventilation System. A VCV ventilation system is composed of a human lung, a respiratory

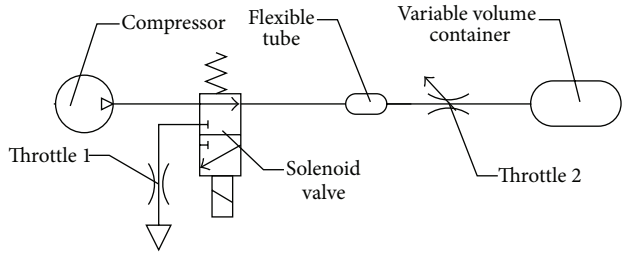


Figure 1: Structure of the equivalent pneumatic system.

tract, a flexible tube, and a VCV ventilator. In the inspiration process, positive pressure ventilation (generated by the ventilator) is utilized to force airflow into the lung. In the expiration process, due to the elasticity of the lung, air is expelled to the atmosphere through an exhalation valve, which is embedded in the ventilator. Therefore, according to their functions, the ventilator can be regarded as an air compressor, the exhalation valve can be considered as a throttle, and the human lung can be regarded as a variable volume container. The minimum inner diameter of the respiratory tracts is influenced by the secretion deposition, so the respiratory tract can be considered as a throttle.

Therefore, the VCV ventilation system can be regarded as a pneumatic system, as shown in Figure 1. The compressor, the container, and throttles 1 and 2 represent the ventilator, the human lung, the exhalation valve, and the respiratory tract, respectively. The effective area of the exhalation valve and respiratory tract are represented by A_{ev} and A_{rt}, respectively.

When the variable volume container is ventilated, the flexible tube and the compressor are connected with the solenoid valve. Then the compressed air, output from the compressor, flows into the variable container through the solenoid valve, flexible tube, and throttle 2. During the expiration, the flexible tube and throttle 1 are connected with the solenoid valve. Then the compressed air, output from the variable volume container, flows into the atmosphere through the solenoid valve, flexible tube, and throttles 2 and 1.

Therefore, the versatility and applicability of the mathematical models of the VCV ventilation systems are better than the mathematical models when the VCV ventilation system is commonly considered as an electrical system.

2.2. Modeling of Mechanical Ventilation System

2.2.1. Flow Equations. To facilitate research, the following assumptions are made:

(1) air of the system follows all ideal gas laws;

(2) the dynamic process is a quasi-balanced process;

(3) there is no air leakage during the working process.

(1) Output Flow of the VCV Ventilator. According to the working principle of the VCV ventilation, the output flow of the VCV ventilator can be given as

$$Q_{\text{vo}} = \begin{cases} Q_T & V_{\text{vo}} < V_T, \\ 0 & V_{\text{vo}} \geq V_T, \end{cases} \qquad Q_t = \frac{V_T}{T_{\text{vo}}} = \frac{V_T}{T_i - T_p}. \tag{1}$$

(2) Air Flow through the Exhalation Valve and the Endotracheal Tube. When air flows through the exhalation valve and the endotracheal tube, their mass flow can be calculated by the equation when air flows through the LAVAL nozzle. As the pressure of the studied ventilation system is in the range of 0~40 cm H_2O, the ratio of the downstream pressure to the upstream pressure is always bigger than 0.528; the air flow through the exhalation valve and the respiratory tract can be given by

$$Q_{\text{ev}} = \frac{A_{\text{ev}} p_t}{\rho_a \sqrt{\theta}} \sqrt{\frac{2\kappa}{\kappa - 1} \frac{1}{R} \left[\left(\frac{p_a}{p_t} \right)^{2/\kappa} - \left(\frac{p_a}{p_t} \right)^{(\kappa+1)/k} \right]},$$

$$Q_{\text{rt}} = \begin{cases} \dfrac{A_{\text{rt}} p_t}{\rho_a \sqrt{\theta}} \sqrt{\dfrac{2\kappa}{\kappa - 1} \dfrac{1}{R} \left[\left(\dfrac{p_l}{p_t} \right)^{2/\kappa} - \left(\dfrac{p_l}{p_t} \right)^{(\kappa+1)/k} \right]} & p_t > p_l \\ \dfrac{A_{\text{rt}} p_l}{\rho_a \sqrt{\theta}} \sqrt{\dfrac{2\kappa}{\kappa - 1} \dfrac{1}{R} \left[\left(\dfrac{p_t}{p_l} \right)^{2/\kappa} - \left(\dfrac{p_t}{p_l} \right)^{(\kappa+1)/k} \right]} & p_l \leq p_t. \end{cases} \tag{2}$$

2.2.2. Pressure Equation. The VCV ventilation system can be considered as an open thermodynamic system, and its work can be regarded as an isothermal process. The differential expression of the Clapeyron equation ($PV = mR\theta$) is given by

$$\frac{dp}{dt} = \frac{1}{V} R\theta q - \frac{mR\theta}{V^2} \frac{dV}{dt}. \tag{3}$$

After transformation of (3), the pressure in the flexible tube and the ventilated lung can be given by

$$\frac{dp_t}{dt} = \frac{R\theta q V_t}{V_t^2 + CmR\theta}, \qquad \frac{dp_l}{dt} = \frac{R\theta q V_l}{V_l^2 + CmR\theta}. \tag{4}$$

2.2.3. Volume Equation. The compliance C of the lung can be described as [16]

$$C_l = \frac{dV_l}{dp_l}. \tag{5}$$

Then, the volume of the lung can be calculated by the following formula:

$$dV_l = C_l dp_l. \tag{6}$$

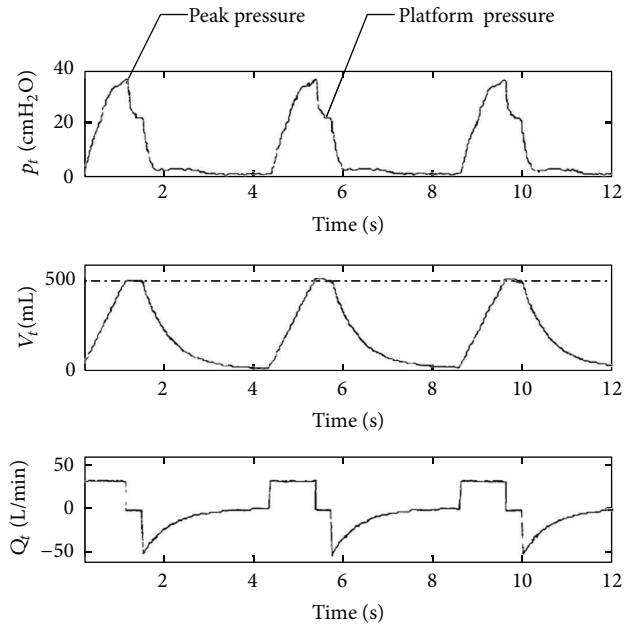


FIGURE 2: Experimental results in the reference.

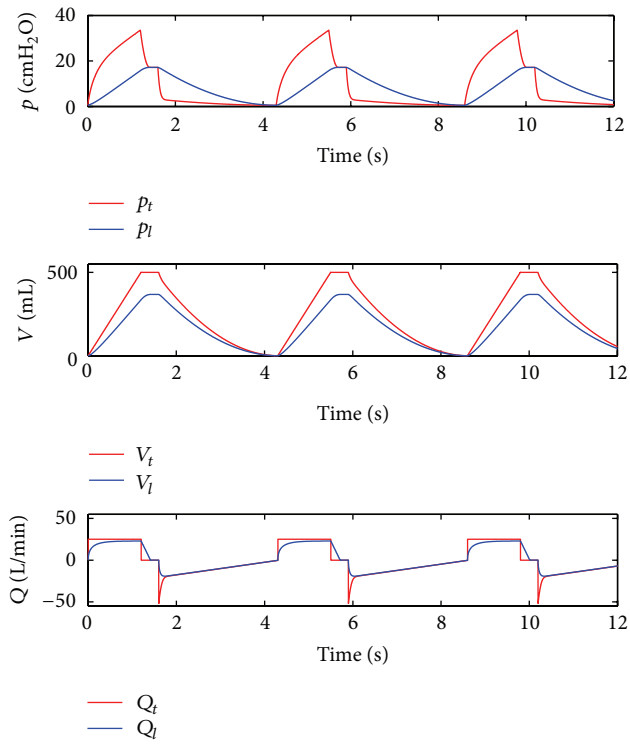


FIGURE 3: Simulation results.

3. Study on the Dynamics of the VCV Ventilation System

3.1. Experimental Verification of the Mathematical Model. In [7], an experimental study on a VCV ventilation system was carried out; T_i, T_p, T_e, and V_T are set to 1.6 s, 0.4 s, 2.7 s, and 500 mL. The compliance of the flexible tube and that of the ventilated lung are 4 mL/cmH$_2$O and 25.7 mL/cmH$_2$O. The effective areas of the exhalation valve and respiratory tract are 16 mm^2 and 9 mm^2. The experimental VCV ventilation system [7] was simulated with the mathematical model, which is coded in an S-function of Matlab/simulink. The experimental and the simulation results are shown in Figures 2 and 3.

The curves of the pressure (p_t) in the flexible tube, the pressure (p_l) in the lung, the volume (V_t) of the air which flows through the inlet of the flexible tube, the volume (V_l) of the air which flows into and out of the lung, the air flow (Q_t) through the inlet of the flexible tube, and the air flow (Q_l) into and out of the lung are shown in Figures 2 and 3.

As can be seen in Figures 2 and 3, it is obvious that

(1) the simulation results are consistent with the experimental results, and this verifies the mathematical model; therefore, the mathematical model can be used in the study on the VCV system;

(2) as can be seen, the air pressure in the lung always lags behind the pressure in the flexible tube and that is why the pressure in the lung cannot be maintained precisely; the main reason of the difference between the simulation and the experimental results is that the respiratory resistance and compliance block the fluctuation of the pressure in the lung simulator.

3.2. Influence of the Key Parameters of the VCV Ventilator on the Dynamics. The key parameters of the VCV ventilator consist of its structure parameters (A_{ev}) and its settings (such as V_T and T_{vo}).

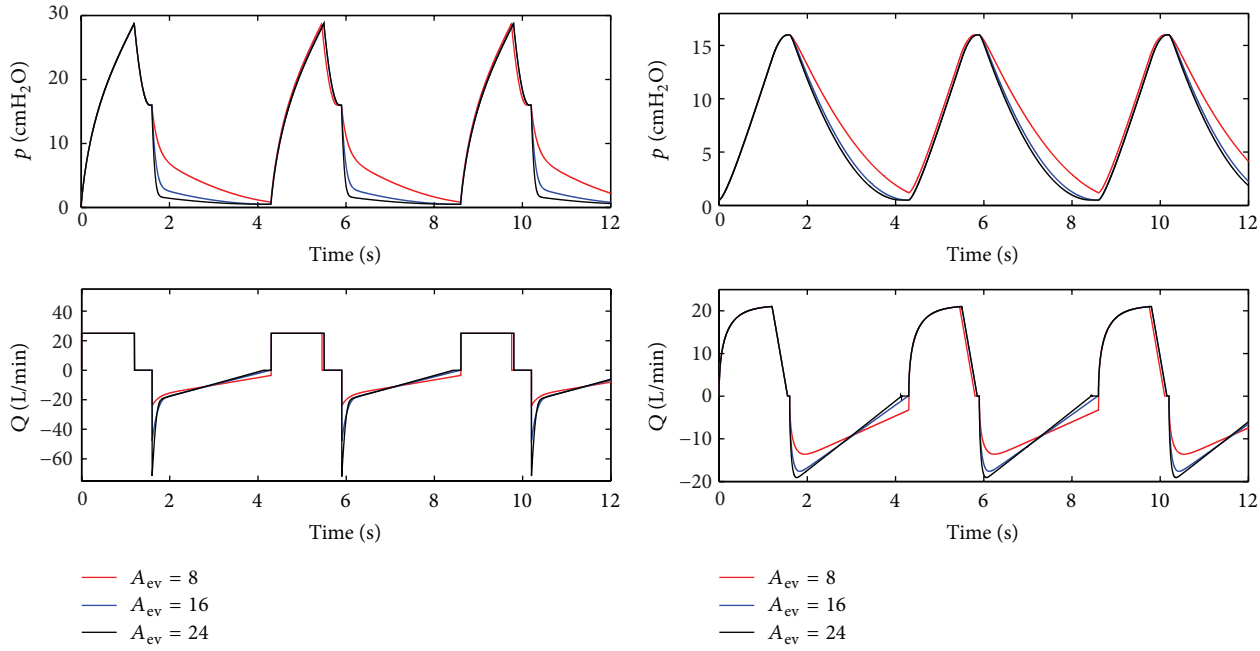
According to the simulation above, each parameter can be changed for comparison while all other parameters are kept constant, and the simulation results for varying each parameter are illustrated in Figures 4, 5, and 6.

3.2.1. Influence of the Effective Area (A_{ev}) of the Exhalation Valve. The effective area (A_{ev}) of the exhalation valve is set to 6 mm^2, 8 mm^2, 12 mm^2, 16 mm^2, 20 mm^2, and 24 mm^2, and the simulation results are illustrated in Figure 4.

As shown in Figure 4, it can be seen that only the exhaust flows of the flexible tube and the ventilated lung are influenced by the effective area (A_{ev}) of the exhalation valve. With an increase in the effective area (A_{ev}) of the exhalation valve, the maximum exhaust air flow of the flexible tube increases proportionally, but the maximum exhaust air flow of the lung increases more and more slowly.

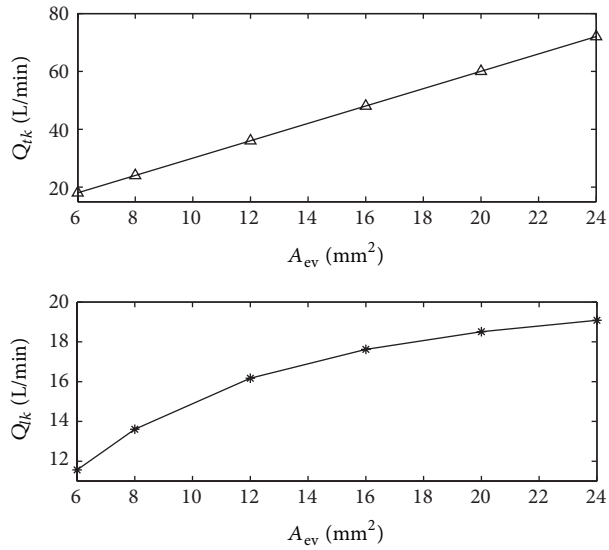
3.2.2. Influence of the Tidal Volume (V_T) of the Ventilator. The tidal volume (V_T) of the ventilator is set to 200 mL, 300 mL, 400 mL, 500 mL, 600 mL, and 700 mL, and the simulation results are illustrated in Figure 5.

As shown in Figure 5, it can be seen that with an increase in the tidal volume (V_T) of the ventilator, the peak pressures in the flexible tube and the lung increase proportionally, but when the tidal volume (V_T) is larger than 600 mL, the peak pressure in the lung increases slowly.



(a) Influences of the A_{ev} on the dynamics of the tube

(b) Influences of the A_{ev} on the dynamics of the lung



(c) Relationship between the Q_{tk} and Q_{lk} and the A_{ev}

FIGURE 4: Influences of the effective area of the exhalation valve.

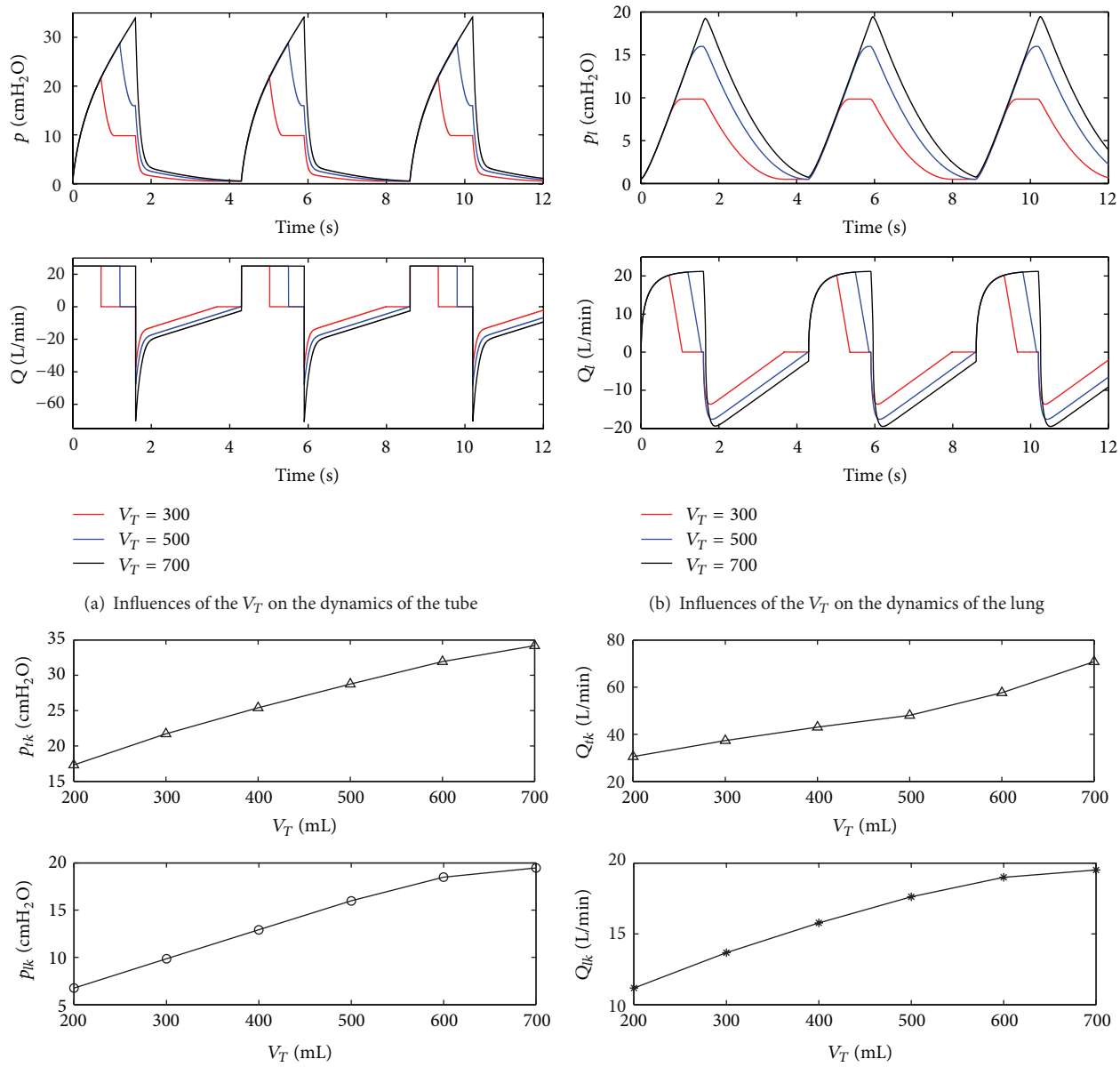
When the tidal volume (V_T) is smaller than 600 mL, the maximum exhaust air flows of the flexible tube and the lung proportionally increase with an increase in the tidal volume (V_T). When the tidal volume (V_T) is larger than 600 mL, the maximum exhaust flow of the flexible tube increases faster and faster; however, the maximum exhaust flow of the lung increases more and more slowly.

3.2.3. Influence of the Exhaust Time (T_{vo}) of the Ventilator. The exhaust time (T_{vo}) of the ventilator is set to 1 s, 1.2 s, 1.4 s, 1.6 s, 1.8 s, and 2 s, and the simulation results are illustrated in Figure 6.

As shown in Figure 6, it can be seen that when the exhaust time (T_{vo}) of the ventilator is smaller than 1.6 s, the platform of the pressure in the tube may not appear.

Furthermore, with an increase in the exhaust time (T_{vo}) of the ventilator, the peak pressure in the lung increases. And when the exhaust time (T_{vo}) of the ventilator is longer than 1.6 s, the peak pressure in the lung is constant.

Moreover, with an increase in the exhaust time (T_{vo}) of the ventilator, the maximum flow of the flexible tube increases and it peaks when the exhaust time (T_{vo}) of the ventilator is 1.2 s. After the peak, the maximum flow of the flexible tube falls. When the exhaust time (T_{vo}) of the ventilator is 1.6 s,



(a) Influences of the V_T on the dynamics of the tube

(b) Influences of the V_T on the dynamics of the lung

(c) Relationship between the p_{tk} and p_{lk} and the V_T

(d) Relationship between the Q_{tk} and Q_{lk} and the V_T

FIGURE 5: Influences of the tidal volume (V_T) of the ventilator.

the maximum flow of the flexible tube reaches its bottom; after that it keeps constant.

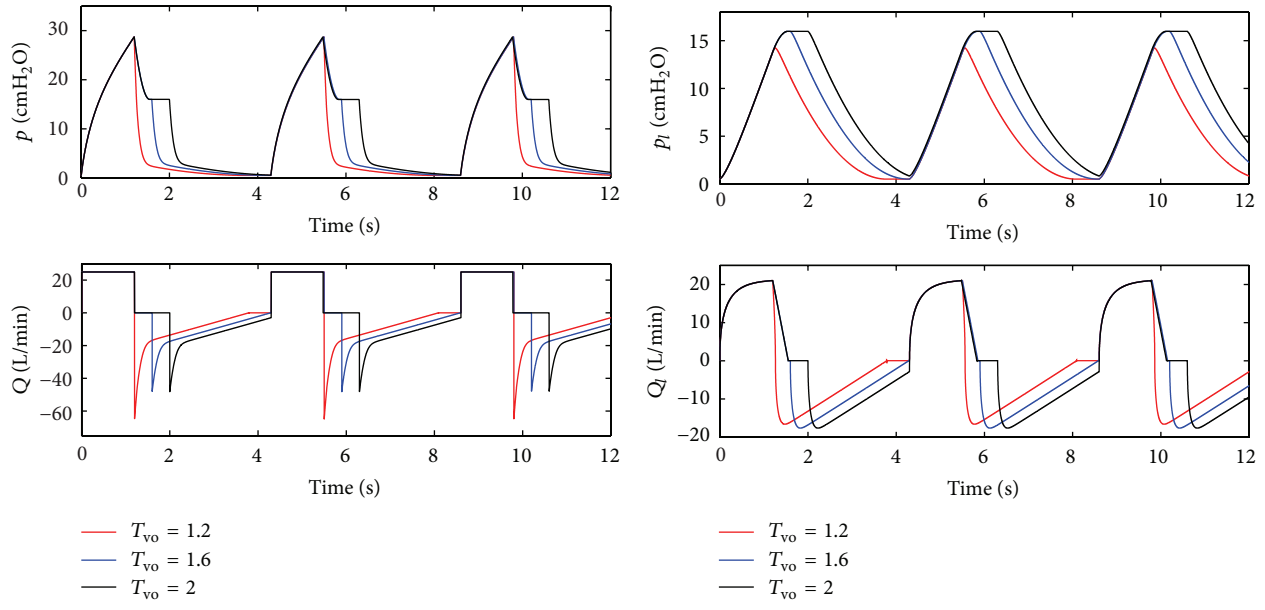
Finally, when the exhaust time (T_{vo}) of the ventilator is shorter than 1.6 s, the maximum flow of the lung increases with an increase in the exhaust time (T_{vo}). When the exhaust time (T_{vo}) of the ventilator is longer than 1.6 s, the maximum flow is constant.

4. Conclusions

In this paper, the VCV ventilation system was compared to a pneumatic system, and then a new mathematical model of the VCV ventilation system was derived. In order to verify the mathematical model, an experimental prototype VCV ventilation system was simulated mathematically. Simulation studies on the influences of the key parameters of the VCV ventilator on the dynamics of the ventilated respiratory system were done, and the conclusions of this study are summarized as follows.

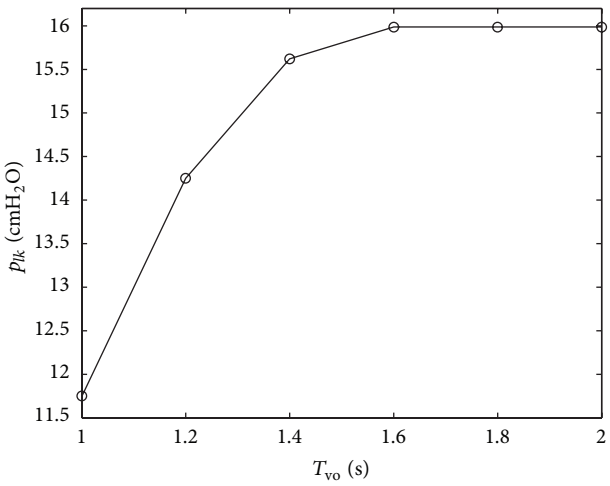
(1) The simulation results are consistent with the experimental results, which verify the mathematical model. Therefore, the mathematical model can be used in the study on the VCV system, and it has better versatility and applicability.

(2) With an increase in the effective area of the exhalation valve, the maximum exhaust air flow of the lung increases more and more slowly.

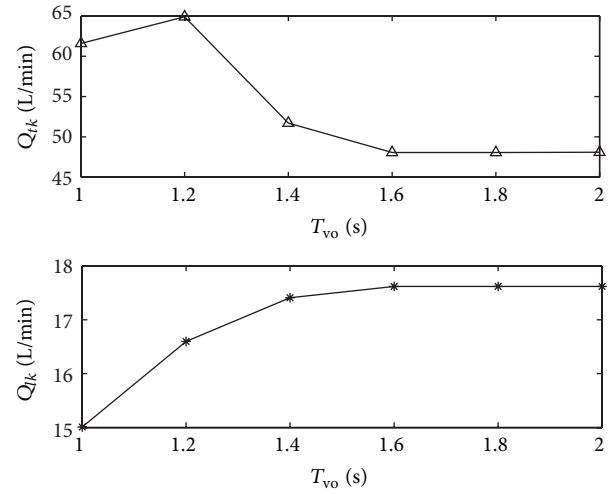


(a) Influences of the T_{vo} on the dynamics of the tube

(b) Influences of the T_{vo} on the dynamics of the lung



(c) Relationship between the p_{lk} and the T_{vo}



(d) Relationship between the Q_{tk} and Q_{lk} and the T_{vo}

FIGURE 6: Influences of the exhaust time (T_{vo}) of the ventilator.

(3) With an increase in the tidal volume of the ventilator, the peak pressure in the lung increases proportionally, but when the tidal volume is larger than 600 mL, the peak pressure in the lung increases slowly.

(4) When the tidal volume is smaller than 600 mL, the maximum exhaust air flow of the lung proportionally increases with an increase in the tidal volume. When the tidal volume is larger than 600 mL, the maximum exhaust flow of the lung increases more and more slowly.

(5) When the exhaust time of the ventilator is shorter than 1.6 s, the maximum flow of the lung increases with an increase in the exhaust time. When the exhaust time of the ventilator is longer than 1.6 s, the maximum flow is constant.

This study can be of use in the VCV ventilation treatment and respiratory diagnostics. In the future, the clinical study will be done to verify the conclusions.

Nomenclature

A: Equivalent effective area [m^2]
C: Respiratory compliance [L/cmH_2O]
m: Mass [kg]
p: Pressure [Pa]
q: Air mass flow [kg/s]
Q: Air volume flow [m^3/s]
R: Gas constant = 287 [J/(kg·K)]
t: Time [s]
V: Volume [m^3]

ρ: Density [kg/m^3]
κ: Specific heat ratio = 1.4
θ: Temperature [K].

Subscripts

a: The standard reference atmosphere state
ev: Exhalation valve
frc: Functional residual capacity
i: Inspiration/input
k: Peak/maximum
l: Lung
o: Output
p: Platform
rt: Respiratory tract
T: Tidal
t: Tube
v: Ventilator.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Robust Stability Analysis of Nonlinear Fractional-Order Time-Variant Systems

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This paper presents a stability theorem for a class of nonlinear fractional-order time-variant systems with fractional order α ($0 \leq \alpha < 1$) by using the Gronwall-Bellman lemma. Based on this theorem, a sufficient condition for designing a state feedback controller to stabilize such fractional-order systems is also obtained. Finally, a numerical example demonstrates the validity of this approach.

1. Introduction

Recently, fraction-order system or system containing fractional derivatives and integrals has been studied widely [1–5]. It was found that many systems in interdisciplinary fields could be elegantly described with the help of fractional derivatives and integrals, such as viscoelastic system, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves [2–4]. In fact, real word processes generally or most likely are fractional-order systems. Moreover, fractional-order controllers [6, 7] have so far been implemented to enhance the robustness and the performance of the closed loop control system.

The problem of stability is a very essential and crucial issue for control systems including fractional-order system. Very recently, the stability problem of fractional-order system has been investigated both from an algebraic and from an analytic point of view [8–11]. By analyzing the characteristic equation of the Jacobian matrix, an asymptotically stable critical was proposed in [12]. As a way of efficiently solving the robust stability and stabilization problem, the linear-matrix-inequality (LMI) approach was presented [13–15] that provided the sufficient condition and the designing method of stabilizing controllers for fractional-order system. Note that the existing LMI-based control methods for fractional-order system only focus on the linear system but not on the case of nonlinear system. To account this problem, based on the generalization of Gronwall-Bellman lemma, the analytical stability conditions and state feedback stabilization problem of nonlinear affine fractional-order system have been investigated in [16–18]. In [19], by using of Mittag-Leffler function, Laplace transform, and the generalized Gronwall inequality, a new sufficient condition ensuring local asymptotic stability and stabilization of a class of fractional-order nonlinear systems with fractional-order α ($1 < \alpha < 2$) was proposed. Based on Lyapunov's second method, a novel stability criterion for a class of nonlinear fractional differential system was presented in [20].

Motivated by the above mentioned works, the main purpose of this this paper is to consider the stability problem of a class of nonlinear fractional-order time-variant systems. The main contribution of this paper is as follows. First, the time-variant uncertainty was discussed for fractional-order nonlinear system. Second, by using Gronwall-Bellman lemma, a stability condition for such fractional-order time-variant systems is presented. The rest of this paper is organized as follows. In Section 2, the problem formulation and some preliminaries are presented. The main results are derived in Section 3. The efficiency of the approach is shown through an illustrative example in Section 4. Finally, some conclusions are drawn in Section 5.

Throughout this paper, $\mathbb{R}^n$ denotes an n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$

real matrices. The matrix norms are defined as $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$ $(i, j = 1, 2, \ldots, n)$. $a_{i,j}$ is the element of matrix A, and $\overline{\lim}_{t\to\infty} A(t) = (\overline{\lim}_{t\to\infty} a_{i,j}(t))_{n\times n}$, $\underline{\lim}_{t\to\infty} A(t) = (\underline{\lim}_{t\to\infty} a_{i,j}(t))_{n\times n}$, $(i, j = 1, 2, \ldots, n)$.

2. Fractional Derivative and Preliminaries

In this paper, the following Caputo definition [1] is adopted for fractional derivative of order α for function $f(t)$:

$$D^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad (1)$$

where m is an integer satisfying $m - 1 < a \le m$ and $\Gamma(\cdot)$ is the well-known Gamma function. The Laplace transform of the Caputo fractional derivative (1) of order α is

$$\int_0^{\infty} D^{\alpha} f(t) e^{-st} dt = s^{\alpha} L\{f(t)\} - \sum_{k=0}^{m-1} s^{-\alpha-k+1} f^{(k)}(t)\Big|_{t=0}, \quad (2)$$

where $s \in \mathbb{C}$ denotes the Laplace operator. Note that upon considering all the initial conditions to be zero, (2) can be reduced to

$$L\{D^{\alpha} f(t)\} = s^{\alpha} L\{f(t)\}. \quad (3)$$

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, which plays a very important role in the fractional calculus, is introduced as follows.

Definition 1 (see [1]). The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0. \quad (4)$$

The Laplace transform of the Mittag-Leffler function is

$$\int^{\infty} e^{-st} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\alpha t^{\alpha}) dt = \frac{k! s^{\alpha-\beta}}{(s^{\alpha} - \alpha)^{k+1}}, \quad \mathrm{Re}(s) > |\alpha|^{1/\alpha}, \quad (5)$$

where $E_{\alpha,\beta}^{(k)} = (d^k/dt^k) E_{\alpha,\beta}$.

To prove the main results in the next section, the following lemmas are needed.

Lemma 2 (see [1]). *If $\alpha < 2$, β is an arbitrary real number, η is a constant satisfying $\alpha\pi/2 < \eta < \min\{\pi, \pi\alpha\}$, and $M > 0$ is a real constant, then*

$$\left|E_{\alpha,\beta}(z)\right| < \frac{M}{1 + |z|}, \quad \eta \le |\arg(z)| \le \pi, \quad |z| > 0. \quad (6)$$

For the n-dimension matrix, one has the following lemma.

Lemma 3 (see [17]). *If $A \in \mathbb{C}^{n\times n}$ and $\alpha < 2$, β is an arbitrary real number, η is such that $\alpha\pi/2 < \eta < \min\{\pi, \pi\alpha\}$ and $C > 0$ is a real constant, then*

$$\left\|E_{\alpha,\beta}(A)\right\| < \frac{M}{1 + \|A\|}, \quad \eta \le \left|\arg \lambda_i(A)\right| \le \pi, \quad i = 1, 2, \ldots, n, \quad (7)$$

where $\lambda_i(A)$ denotes the ith eigenvalue of matrix A and $\|A\|$ denotes the l_2-norm.

Lemma 4 (Gronwall-Bellman lemma [21]). *If*

$$x(t) \le h(t) + \int_{t_0}^{t} k(s) x(s) ds, \quad t \in [t_0, T), \quad (8)$$

where all the functions involved are continuous on $[t_0, T)$, $T \le \infty$, and $k(t) \ge 0$, then $x(t)$ satisfies

$$x(t) \le h(t) + \int_{t_0}^{t} h(s) k(s) \exp\left[\int_s^t k(u) du\right] ds, \quad t \in [t_0, T). \quad (9)$$

3. Stability Analysis of Nonlinear Fractional-Order Time-Variant Systems

Consider the n-dimensional nonlinear fractional-order time-variant system described by the following form:

$$D^{\alpha} x(t) = A(t) x(t) + f(t, x(t)) + Bu(t), \quad (10)$$

where $0 \le \alpha < 1$, $A(t) \in \mathbb{R}^{n\times n}$ is a time-variant regular matrix, and $f(t, x(t))$ is a nonlinear function of state $x(t)$, which is Lebesgue measurable with respect to t on $(0, \infty)$. Let $f(t, 0) \equiv 0$, then $x(t) = 0$ is a zero solution of nonlinear fractional-order time-variant system (10) with $u(t) = 0$.

Based on the aforementioned definition and lemmas, we first give out the stability theorem for nonlinear fractional-order time-variant system (10).

Theorem 5. *For the nonlinear fractional-order system* (10) *with $u(t) = 0$, assume that*

(i) $\overline{\lim}_{t\to\infty} A(t) = \overline{A}$, $\underline{\lim}_{t\to\infty} A(t) = \underline{A}$, *and* $\|\overline{A} - \underline{A}\| < b$;

(ii) $f(t, 0) = 0$, *and* $\lim_{\|x\|\to 0} (\|f(t, x(t))\|/\|x(t)\|) = 0$;

(iii) $|\arg(\mathrm{spec}(\overline{A}))| > a\pi/2$, $\alpha\rho(\overline{A}) > 1$,

then the zero solution of system (10) *is asymptotically stable.*

Proof. By the assumption (i), it yields

$$\overline{\lim_{t\to\infty}} a_{ij}(t) = \overline{a}_{ij}, \quad \underline{\lim_{t\to\infty}} a_{ij}(t) = \underline{a}_{ij}, \quad (i, j = 1, 2, \ldots, n), \quad (11)$$

that is, for any given $0 < \varepsilon < (b - \|\overline{A} - \underline{A}\|)/2n^2$, there exists $t_{ij} \ge t_0 > 0$, when $t \ge t_{ij}$, it has

$$a_{ij}(t) < \overline{a}_{ij} + \varepsilon, \quad a_{ij}(t) > \underline{a}_{ij} - \varepsilon. \quad (12)$$

Setting $T_1 = \max\{t_{ij} \mid i, j = 1, 2, \ldots, n\}$, when $t \geq T_1$, then

$$\underline{a}_{ij} - \varepsilon < a_{ij}(t) < \overline{a}_{ij} + \varepsilon. \tag{13}$$

As a result,

$$\underline{a}_{ij} - \varepsilon < \overline{a}_{ij} < \overline{a}_{ij} + \varepsilon. \tag{14}$$

Without loss of generality, assume that $|\arg(\mathrm{spec}(\overline{A}))| > a\pi/2$, then from (13) and (14), it obtains

$$\left|a_{ij}(t) - \overline{a}_{ij}\right| < \left|\overline{a}_{ij} - \underline{a}_{ij} + 2\varepsilon\right| < \left|\overline{a}_{ij} - \underline{a}_{ij}\right| + 2\varepsilon, \quad (t \geq T_1). \tag{15}$$

It follows that,

$$\sum_{i,j=1}^{n} \left|a_{ij}(t) - \overline{a}_{ij}\right| < \sum_{i,j=1}^{n} \left|\overline{a}_{ij} - \underline{a}_{ij}\right| + \sum_{i,j=1}^{n} 2\varepsilon, \quad (t \geq T_1), \tag{16}$$

that is

$$\left\|A(t) - \overline{A}\right\| < \left\|\overline{A} - \underline{A}\right\| + 2n^2\varepsilon < b, \quad (t \geq T_1). \tag{17}$$

On the other hand, by the assumption (ii) of Theorem 5, for any given $\varepsilon_0 > 0$, there exists $T_2 \geq t_0 > 0$, when $t \geq T_2$, it has

$$\left\|f(t, x(t))\right\| < \varepsilon_0 \|x(t)\|. \tag{18}$$

Rewriting system (10) with $u(t) = 0$ as

$$D^{\alpha}x(t) = \overline{A}x(t) + \left(A(t) - \overline{A}\right)x(t) + f(t, x(t)), \tag{19}$$

taking Laplace transform on the system (19), it obtains

$$X(s) = \left(Is^{\alpha} - \overline{A}\right)^{-1}\left(s^{\alpha-1}x(0) + L\left\{\left(A(t) - \overline{A}\right)x(t) + f(t, x(t))\right\}\right). \tag{20}$$

Then, taking Laplace inverse transform for (20) by using the inverse Laplace transform formula of the Mittag-Leffler function, it yields

$$x(t) = E_{\alpha,1}\left(\overline{A}t^{\alpha}\right)x(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\overline{A}(t-\tau)^{\alpha}\right) \times \left(\left(A(\tau) - \overline{A}\right)x(\tau) + f(\tau, x(\tau))\right)d\tau. \tag{21}$$

Taking $T^* = \max\{T_1, T_2\}$, when $t \geq T^*$, we have

$$\|x(t)\| \leq \left\|E_{\alpha,1}\left(\overline{A}t^{\alpha}\right)\right\|\|x(0)\| + \int_0^t \|(t-\tau)\|^{\alpha-1}\left\|E_{\alpha,\alpha}\left(\overline{A}(t-\tau)^{\alpha}\right)\right\| \times \left(\left\|\left(A(\tau) - \overline{A}\right)\right\|\|x(\tau)\| + \left\|f(\tau, x(\tau))\right\|\right)d\tau. \tag{22}$$

Substituting (17) and (18) into (22), it results

$$\|x(t)\| \leq \left\|E_{\alpha,1}\left(\overline{A}t^{\alpha}\right)\right\|\|x(0)\| + \int_0^t C_1\|(t-\tau)\|^{\alpha-1}\left\|E_{\alpha,\alpha}\left(\overline{A}(t-\tau)^{\alpha}\right)\right\|\|x(\tau)\|d\tau, \tag{23}$$

where $C_1 = b + \varepsilon_0$.

Since the eigenvalues of $\overline{A}$ satisfy the assumption (iii) of Theorem 5, then from Lemma 3 there exists a real constant $C_2 > 0$ such that

$$\left\|E_{\alpha,1}\left(\overline{A}t^{\alpha}\right)\right\| \leq \frac{C_2}{1 + \left\|\overline{A}t^{\alpha}\right\|}, \qquad \left\|E_{\alpha,1}\left(\overline{A}(t-\tau)^{\alpha}\right)\right\| \leq \frac{C_2}{1 + \left\|\overline{A}(t-\tau)^{\alpha}\right\|}. \tag{24}$$

Substituting (24) into (23), it gives

$$\begin{aligned}\|x(t)\| &\leq \frac{C_2}{1 + \left\|\overline{A}t^{\alpha}\right\|}\|x(0)\| + C_1C_2\int_0^t \frac{\|(t-\tau)\|^{\alpha-1}}{1 + \left\|\overline{A}(t-\tau)^{\alpha}\right\|}\|x(\tau)\|d\tau \\ &= \frac{C_2}{1 + \left\|\overline{A}\right\|t^{\alpha}}\|x(0)\| + C_1C_2\int_0^t \frac{\|(t-\tau)\|^{\alpha-1}}{1 + \left\|\overline{A}\right\|(t-\tau)^{\alpha}}\|x(\tau)\|d\tau.\end{aligned} \tag{25}$$

Setting

$$h(t) = \frac{C_2}{1 + \left\|\overline{A}t^{\alpha}\right\|}\|x(0)\|, \qquad k(t) = \frac{(t-\tau)^{\alpha-1}}{1 + \left\|\overline{A}\right\|(t-\tau)^{\alpha}}, \tag{26}$$

and using Gronwall-Bellman lemma, it follows from (25) that

$$\|x(t)\| \leq \frac{C_2}{1 + \left\|\overline{A}\right\|t^{\alpha}}\|x(0)\| + C_1C_2 \times \int_0^t \frac{\|x(0)\|}{1 + \left\|\overline{A}\right\|\tau^{\alpha}} \cdot \frac{(t-\tau)^{\alpha-1}}{\left(1 + \left\|\overline{A}\right\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}}d\tau. \tag{27}$$

The integral in (27) equals the sum of the two parts

$$\int_0^{t/2} \frac{\|x(0)\|}{1 + \left\|\overline{A}\right\|\tau^{\alpha}} \cdot \frac{(t-\tau)^{\alpha-1}}{\left(1 + \left\|\overline{A}\right\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}}d\tau + \int_{t/2}^t \frac{\|x(0)\|}{1 + \left\|\overline{A}\right\|\tau^{\alpha}} \cdot \frac{(t-\tau)^{\alpha-1}}{\left(1 + \left\|\overline{A}\right\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}}d\tau. \tag{28}$$

Since $\alpha < 1$ and $(t - \tau) \geq \tau$ when $\tau \in [0, t/2]$, we obtain

$$\int_0^{t/2} \frac{\|x(0)\|}{1 + \left\|\overline{A}\right\|\tau^{\alpha}} \cdot \frac{(t-\tau)^{\alpha-1}}{\left(1 + \left\|\overline{A}\right\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}}d\tau \leq \int_{t/2}^t \frac{\|x(0)\|}{1 + \left\|\overline{A}\right\|\tau^{\alpha}} \cdot \frac{\tau^{\alpha-1}}{\left(1 + \left\|\overline{A}\right\|\tau^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}}d\tau. \tag{29}$$

Similarly,

$$\int_{t/2}^{t} \frac{\|x(0)\|}{1+\left\|\overline{A}\right\|\tau^{\alpha}} \cdot \frac{(t-\tau)^{\alpha-1}}{\left(1+\left\|\overline{A}\right\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}} d\tau$$

$$\leq \int_{t/2}^{t} \frac{\|x(0)\|}{1+\left\|\overline{A}\right\|(t-\tau)^{\alpha-1}} \cdot \frac{(t-\tau)^{\alpha-1}}{\left(1+\left\|\overline{A}\right\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}} d\tau$$

$$\underline{\underline{t-\tau=s}} \int_{0}^{t/2} \frac{\|x(0)\| s^{\alpha-1}}{\left(1+\left\|\overline{A}\right\| s^{\alpha}\right)\left(1+\left\|\overline{A}\right\| s^{\alpha}\right)^{1-1/(\alpha\|\overline{A}\|)}} ds. \tag{30}$$

From (28) to (30), and with $a\rho(\overline{A}) > 1$, (27) gives

$$\begin{aligned}\|x(t)\| &\leq \frac{C_2\|x(0)\|}{1+\left\|\overline{A}\right\| t^{\alpha}} \\ &\quad + 2C_1C_2 \int_{0}^{t/2} \frac{\|x(0)\| s^{\alpha-1}}{\left(1+\left\|\overline{A}\right\| s^{\alpha}\right)^{2-1/(\alpha\|\overline{A}\|)}} ds \\ &= \frac{2C_1C_2\|x(0)\|}{\alpha\left\|\overline{A}\right\|-1} + \frac{C_2\|x(0)\|}{1+\left\|\overline{A}\right\| t^{\alpha}} \\ &\quad + \frac{2C_1C_2\|x(0)\|}{\left(1-\alpha\left\|\overline{A}\right\|\right)\left(1+\left\|\overline{A}\right\|(t^{\alpha}/2)\right)^{1-1/(\alpha\|\overline{A}\|)}}.\end{aligned} \tag{31}$$

Now for arbitrarily small $\delta > 0$, it can be proved that

$$\|x(0)\| < \delta$$

$$\Longrightarrow \|x(t)\| < \frac{2C_1C_2\delta}{\alpha\left\|\overline{A}\right\|-1} + \frac{C_2\delta}{1+\left\|\overline{A}\right\| t^{\alpha}} + \frac{2C_1C_2\delta}{\left(1-\alpha\left\|\overline{A}\right\|\right)\left(1+\left\|\overline{A}\right\|(t^{\alpha}/2)\right)^{1-1/(\alpha\|\overline{A}\|)}}, \tag{32}$$

which implies stability of the zero solution. This completes the proof. □

In particular, if $\lim_{t\to\infty} A(t) = A$, the following corollary is easily derived.

Corollary 6. *For the nonlinear fractional-order system* (10) *with* $u(t) = 0$, *assume that*

(i) $\lim_{t\to\infty} A(t) = A$ *and* $|\arg(\mathrm{spec}(\mathrm{A}))| > \mathrm{a}\pi/2$, $\alpha\rho(a) > 1$;

(ii) $f(t,0) = 0$, *and* $\lim_{\|x\|\to 0}(\|f(t,x(t))\|/\|x(t)\|) = 0$,

then the zero solution of system (10) *is asymptotically stable.*

Based on Corollary 6 and according to the stability theory of linear fractional-order system and the pole placement technique of the linear control theory (see, e.g., [16, 17, 22]), it is easy to get the stabilization theorem of nonlinear fractional-order system (10).

Corollary 7. *Assume that*

(i) $\lim_{t\to\infty} A(t) = A$;

(ii) $f(t,0) = 0$, *and* $\lim_{\|x\|\to 0}(\|f(t,x(t))\|/\|x(t)\|) = 0$,

and if K *is selected to, such that,* $|\arg(\mathrm{spec}(\mathrm{A} + \mathrm{BK}))| > \mathrm{a}\pi/2$, *then the nonlinear fractional-order system* (10) *can be stabilized by the state feedback controller* $u(t) = Kx(t)$.

4. Numerical Example

Consider the following fractional-order time-variant Lorenz system:

$$\begin{aligned} D^{\alpha}x_1 &= a(t)(x_2 - x_1) + u(t) \\ D^{\alpha}x_2 &= bx_1 - x_1x_3 - c(t)x_2 + u(t) \\ D^{\alpha}x_3 &= x_1x_2 - dx_3, \end{aligned} \tag{33}$$

where $0 \leq \alpha < 1$. Lorenz system (33) can be rewritten as

$$D^{\alpha}x(t) = A(t)x(t) + f(t,x(t)) + B(t)u(t), \tag{34}$$

where $x(t) = [x_1(t), x_2(t), x_3(t)]^T$ and

$$A(t) = \begin{pmatrix} -a(t) & a(t) & 0 \\ b & -c(t) & 0 \\ 0 & 0 & -d \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$
$$f(t,x(t)) = \begin{pmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{pmatrix}. \tag{35}$$

Setting $a(t) = 9 + \sin t$, $b = 28$, $c(t) = -8$, $d = 8/3$, $\alpha = 0.90$. To simulate the fractional order systems, we use Oustaloup's recursive poles/zeros filter [3] (an integer order system) of order three to approximate the fractional operator $1/s^{0.9}$, which has an error about 2 dB in the frequency range $\omega = 10^{-2}$ rad/s to $\omega = 10^{2}$ rad/s. We use this recursive poles/zeros filter of order three to approximate the fractional operator. The initial value of the system $[(x_1(0), x_2(0), x_3(0)] = (0.22, 0.24, 0.12)$. The commensurate fractional-order Lorenz system (34) is chaotic without the controller, which is shown in Figure 1.

Note that $f(t,0) = 0$ and

$$\begin{aligned}\lim_{\|x\|\to 0} \frac{\|f(t,x(t))\|}{\|x(t)\|} &= \lim_{\|x\|\to 0} \frac{\sqrt{(x_1(t)x_2(t))^2 + (-x_1(t)x_3(t))^2}}{\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}} \\ &\leq \lim_{\|x\|\to 0} \frac{\sqrt{(x_1(t)x_2(t))^2 + (-x_1(t)x_3(t))^2}}{\sqrt{x_1^2(t)}} \\ &= \lim_{\|x\|\to 0} \sqrt{x_2^2(t) + x_3^2(t)} = 0.\end{aligned} \tag{36}$$

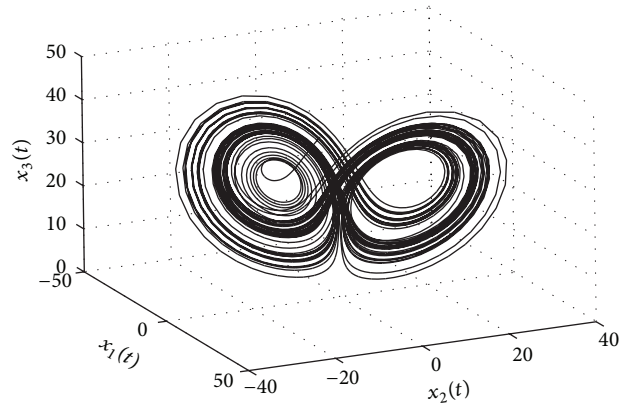

Figure 1: Time response of the selected systems with $u(t) = 0$.

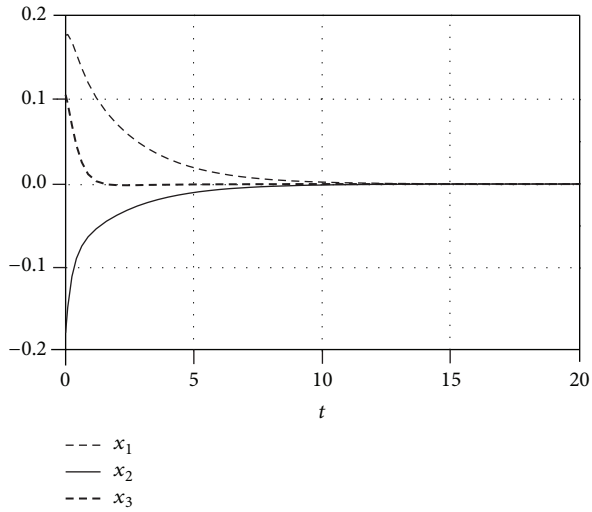

Figure 2: Time response of the selected systems with the control input $u(t) = Kx(t)$.

To stabilize the fractional-order system (33), we select $K = [1\ -5\ -1]$, which makes $\|\overline{A} - \underline{A}\| = 4$ and $|\arg(\overline{A} + BK)| > \alpha\pi/2$, by the eigenvalues of $(\overline{A} + BK)$ being $(-20.3516, -8.1380, -0.1771)$. So the conditions of Corollary 7 are satisfied. Thus, the fractional-order system (33) can be stabilized by the state feedback controller $u(t) = Kx(t)$. The simulation result (Figure 2) shows that it is asymptotically stable and its states converge to zero, which shows that the obtained theoretic results are feasible and efficient for the nonlinear fractional-order system.

5. Conclusion

In this paper, based on the Gronwall-Bellman lemma and the property of fractional calculus, a stabilization theorem of a class of nonlinear fractional-order time-variant systems with fractional order α $(0 \le \alpha < 1)$ has been proven theoretically. Furthermore, the sufficient condition for designing a state feedback controller to stabilize such fractional-order systems was also obtained. Finally, a numerical example demonstrates the validity of this approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Improved Results on Robust Stability for Systems with Interval Time-Varying Delays and Nonlinear Perturbations

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This paper investigated delay-dependent robust stability criteria for systems with interval time-varying delays and nonlinear perturbations. A delay-partitioning approach is used in this paper, the delay-interval is partitioned into multiple equidistant subintervals, a new Lyapunov-Krasovskii (L-K) functional contains some triple-integral terms, and augment terms are introduced on these intervals. Then, by using integral inequalities method together with free-weighting matrix approach, a new less conservative delay-dependent stability criterion is formulated in terms of linear matrix inequalities (LMIs), which can be easily solved by optimization algorithms. Numerical examples are given to show the effectiveness and the benefits of the proposed method.

1. Introduction

Time-delay phenomena are ubiquitous in many practical systems such as communication systems, nuclear reactors, aircraft stabilization, and process control systems, which are often major sources of instability and poor performance. Hence, stability analysis and stabilization of systems with time-delays have received considerable attention in the past few decades [1–14]. Recently, many researchers pay attention to the interval time-varying delay, wherein the delay varies in a range for which the lower bound is not restricted to be zero. A typical example with interval time delay is the network control systems [2].

The basic framework for stability analysis and synthesis of stabilizing controllers is L-K functional and linear matrix inequality (LMI). Under this framework, an important issue was to enlarge the feasible region of stability criteria. To derive the delay-dependent stability conditions, many methods have been reported in the literature. For example, the free-weighting matrix method was used in [3–6]; Jensen's integral inequality method was adopted in [7–13]. Recently, inspired by the discretized Lyapunov method, delay-partitioning approach was proposed in [14, 15], wherein the delay-interval was uniformly divided into multiple segments, choosing proper functional with different weighted matrices corresponding to different segments. A new technique called delay-central point method was proposed in [16]. Based on the delay-central point method and decomposition technique, [17] proposed a less conservative stability criterion for computing the maximum allowable bound of the delay range. As an extension of delay-central point method, a new delay-partitioning approach was proposed in [18] for the uncertain stochastic systems with interval time-varying delay. Referring to the nonlinearities, as time delays, also can cause instability and poor performance of practical systems. Therefore, the stability problem of time-delay systems with nonlinear perturbations has received increasing attention [19–24]. A descriptor model transformation was employed in [19]. The free-weighting matrices approach was adopt in [20, 21]. Recently, a less conservative delay-dependent stability criterion was provided in [22] by partitioning the delay-interval into two segments of equal length and evaluating the time-derivative of a candidate L-K functional in each segment of the delay-interval. The main advantage of the method [22] is that more information on the variation interval of the delay is employed, but we can employ information of time-delay much more if we partition the delay-interval into more segments.

Inspired by the idea of [18, 22], in this paper, we divide the variation interval of the delay into N parts with equal

length and construct a new L-K functional with triple-integral terms and augment terms for this delay-interval. Based on integral inequalities method together with free-weighting matrix approach, a new delay-dependent stability criterion for the system is formulated in terms of linear matrix inequalities, which can be easily calculated by using MATLAB LMI control toolbox. Numerical examples are given to illustrate the effectiveness and less conservatism of the proposed method.

2. Problem Description and Preliminaries

Consider the following neutral system with mixed time-varying delays and nonlinear perturbations:

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}x(t-h(t)) \\ &\quad + f_1(x(t),t) + f_2(x(t-h(t)),t), \\ x(\theta) &= \phi(\theta), \quad \dot{x}(\theta) = \varphi(\theta), \\ &\forall\theta \in [-\max(h_M,\tau),0],\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{A}$ and $\mathbf{B}$ are constant matrices with appropriate dimensions, and $h(t)$ is a time-varying delay satisfying

$$0 \le h_m \le h(t) \le h_M, \quad \dot{h}(t) \le \mu, \quad \forall t \ge 0, \tag{2}$$

where h_m and h_M represent the lower and upper bounds of the time-varying delay $h(t)$, respectively, μ is the bound on the delay-derivative, and initial condition $\varphi(t)$ is a continuous vector-valued function. The functions $f_1(x(t),t)$ and $f_2(x(t-h(t)),t)$ are unknown nonlinear perturbations with respect to the current state $x(t)$ and in the delay state $x(t-h(t))$, respectively. They satisfy $f_1(0,t) = 0$, $f_2(0,t) = 0$, and

$$\begin{aligned}&f_1^T(x(t),t)\, f(x(t),t) \le \alpha^2 x^T(t)\, x(t), \\ &f_2^T(x(t-h(t)),t)\, f_2(x(t-h(t)),t) \\ &\le \beta^2 x^T(t-h(t))\, x(t-h(t)),\end{aligned} \tag{3}$$

where $\alpha \ge 0$, $\beta \ge 0$ are given constants; for simplicity we denote $f_1 = f_1(x(t),t)$, $f_2 = f_2(x(t-h(t)),t)$.

Before moving on, we need the following instrumental lemma.

Lemma 1 (see [10]). *For any scalar $h(t) \ge 0$ and constant matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, $\mathbf{Q} = \mathbf{Q}^T > 0$, the following inequality holds:*

$$\begin{aligned}-\int_{t-h(t)}^{t} \dot{x}^T(s)\,\mathbf{Q}\dot{x}(s)\,ds &\le h(t)\,\zeta^T(t)\,\mathbf{V}\mathbf{Q}^{-1}\mathbf{V}^T\zeta(t) \\ &\quad + 2\zeta^T(t)\,\mathbf{V}\left[x(t) - x(t-h(t))\right],\end{aligned} \tag{4}$$

where

$$\begin{aligned}\zeta^T(t) = \Bigg[&x^T(t) \quad x^T(t-h_i) \quad x^T(t-h(t)) \quad x^T(t-h_{i+1}) \\ &\dot{x}^T(t) \quad \int_{t-h_i}^{t} x^T(s)\,ds \quad \int_{t-h_{i+1}}^{t-h_i} x^T(s)\,ds \quad f_1^T \quad f_2^T\Bigg].\end{aligned} \tag{5}$$

$\mathbf{V}$ *is a free-weighting matrix with appropriate dimensions.*

Lemma 2 (see [9]). *For any constant matrix $\mathbf{W} \in \mathbb{R}^{n\times n}$, $\mathbf{W} = \mathbf{W}^T > 0$, a scalar function $h := h(t) > 0$ and a vector-valued function $\dot{x} : [-h,0] \to R^n$, such that the following integrations are well defined; then*

$$\begin{aligned}&-h\int_{t-h}^{t} \dot{x}^T(s)\,\mathbf{W}\dot{x}(s)\,ds \le \zeta_1^T(t)\begin{bmatrix}-\mathbf{W} & \mathbf{W} \\ \mathbf{W} & -\mathbf{W}\end{bmatrix}\zeta_1(t), \\ &-\frac{h^2}{2}\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(t)\,\mathbf{W}\dot{x}(t)\,ds \le \zeta_2^T(t)\begin{bmatrix}-\mathbf{W} & \mathbf{W} \\ \mathbf{W} & -\mathbf{W}\end{bmatrix}\zeta_2(t)\end{aligned} \tag{6}$$

with $\zeta_1^T(t) = \left[x^T(t) \quad x^T(t-h)\right]$ *and* $\zeta_2^T(t) = \left[hx^T(t) \quad \int_{t-h}^{t} x^T(s)ds\right]$.

Lemma 3 (see [25]). *Suppose $\gamma_1 \le \gamma(t) \le \gamma_2$, where $\gamma(\cdot) : R_+ \to R_+$. Then, for any constant matrices Ξ_1, Ξ_2, and $\mathbf{\Omega}$ with proper dimensions, the following matrix inequality*

$$\mathbf{\Omega} + (\gamma(t) - \gamma_1)\,\Xi_1 + (\gamma_2 - \gamma(t))\,\Xi_2 < 0 \tag{7}$$

holds, if and only if

$$\begin{aligned}&\mathbf{\Omega} + (\gamma_2 - \gamma_1)\,\Xi_1 < 0, \\ &\mathbf{\Omega} + (\gamma_2 - \gamma_1)\,\Xi_2 < 0.\end{aligned} \tag{8}$$

3. Main Result

In this section, we study the delay-dependent robust stability of system (1) based on the delay-partitioning approach.

Theorem 4. *For given positive scalars h_m, h_M, μ, α, and β system* (1) *with uncertainty* (3) *is asymptotically stable if there exist symmetric matrices*

$$\mathbf{P} = \begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ * & \mathbf{P}_{22} & \mathbf{P}_{23} \\ * & * & \mathbf{P}_{33}\end{bmatrix} \tag{9}$$

and $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{R}_2$, $\mathbf{R}_3$, any matrices $\mathbf{K}_a$, $\mathbf{V}_a$, $\mathbf{W}_a$ ($a = 1,2$), and scalars $\varepsilon_i \ge 0$ ($i = 1,2$) such that the following LMIs hold:

$$\begin{bmatrix}\mathbf{\Phi} & \sqrt{h_\delta}\mathbf{W} \\ * & -\mathbf{Z}_3\end{bmatrix} < 0, \quad i = 1,2,\ldots,N, \tag{10}$$

$$\begin{bmatrix}\mathbf{\Phi} & \sqrt{h_\delta}\mathbf{V} \\ * & -\mathbf{Z}_3\end{bmatrix} < 0, \quad i = 1,2,\ldots,N, \tag{11}$$

where $\mathbf{\Phi} = (\mathbf{\Phi}_{i,j})_{9\times 9}$ *with*

$$\mathbf{\Phi}_{11} = \mathbf{P}_{12} + \mathbf{P}_{12}^T + \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 - \mathbf{Z}_2 - h_i^2\mathbf{R}_2 - h_\delta^2\mathbf{R}_3 + \mathbf{K}_1\mathbf{A} + \mathbf{A}^T\mathbf{K}_1^T + \varepsilon_1 a^2,$$

$$\mathbf{\Phi}_{12} = \mathbf{Z}_2 - \mathbf{P}_{12} + \mathbf{P}_{13}, \qquad \mathbf{\Phi}_{13} = \mathbf{K}_1\mathbf{B},$$

$$\mathbf{\Phi}_{14} = -\mathbf{P}_{13}, \qquad \mathbf{\Phi}_{15} = \mathbf{P}_{11} - \mathbf{K}_1 + \mathbf{A}^T\mathbf{K}_2^T,$$

$$\mathbf{\Phi}_{16} = \mathbf{P}_{22}^T + h_i\mathbf{R}_2, \qquad \mathbf{\Phi}_{17} = \mathbf{P}_{32}^T + h_\delta\mathbf{R}_3,$$

$$\mathbf{\Phi}_{18} = \mathbf{\Phi}_{19} = \mathbf{K}_1, \qquad \mathbf{\Phi}_{22} = -\mathbf{Q}_2 - \mathbf{Z}_2 + \mathbf{V}_1 + \mathbf{V}_1^T,$$

$$\mathbf{\Phi}_{23} = -\mathbf{V}_1 + \mathbf{V}_2^T, \qquad \mathbf{\Phi}_{24} = \mathbf{\Phi}_{25} = 0,$$

$$\mathbf{\Phi}_{26} = -\mathbf{P}_{22}^T + \mathbf{P}_{23}^T, \qquad \mathbf{\Phi}_{27} = -\mathbf{P}_{32}^T + \mathbf{P}_{33}^T,$$

$$\mathbf{\Phi}_{28} = \mathbf{\Phi}_{29} = 0,$$

$$\mathbf{\Phi}_{33} = -(1-\mu)\mathbf{Q}_1 - \mathbf{V}_2 - \mathbf{V}_2^T + \mathbf{W}_1 + \mathbf{W}_1^T + \varepsilon_2\beta^2,$$

$$\mathbf{\Phi}_{34} = -\mathbf{W}_1 + \mathbf{W}_2^T, \qquad \mathbf{\Phi}_{35} = \mathbf{B}^T\mathbf{K}_2^T,$$

$$\mathbf{\Phi}_{36} = \cdots = \mathbf{\Phi}_{39} = 0, \qquad \mathbf{\Phi}_{44} = -\mathbf{Q}_3 - \mathbf{W}_2 - \mathbf{W}_2^T,$$

$$\mathbf{\Phi}_{45} = 0, \qquad \mathbf{\Phi}_{46} = -\mathbf{P}_{23}^T,$$

$$\mathbf{\Phi}_{47} = -\mathbf{P}_{33}^T, \qquad \mathbf{\Phi}_{48} = \mathbf{\Phi}_{49} = 0,$$

$$\mathbf{\Phi}_{55} = h_i^2\mathbf{Z}_2 + h_\delta\mathbf{Z}_3 + \frac{1}{4}h_i^4\mathbf{R}_2 + \frac{1}{4}\left(h_{i+1}^2 - h_i^2\right)^2\mathbf{R}_3 - \mathbf{K}_2 - \mathbf{K}_2^T,$$

$$\mathbf{\Phi}_{56} = \mathbf{P}_{21}^T, \qquad \mathbf{\Phi}_{57} = \mathbf{P}_{31}^T,$$

$$\mathbf{\Phi}_{58} = \mathbf{\Phi}_{5,9} = \mathbf{K}_2, \qquad \mathbf{\Phi}_{66} = -\mathbf{R}_2,$$

$$\mathbf{\Phi}_{67} = \cdots = \mathbf{\Phi}_{69} = 0,$$

$$\mathbf{\Phi}_{77} = -\mathbf{R}_3, \qquad \mathbf{\Phi}_{78} = \mathbf{\Phi}_{79} = 0,$$

$$\mathbf{\Phi}_{88} = -\varepsilon_1\mathbf{I}, \qquad \mathbf{\Phi}_{89} = 0, \qquad \mathbf{\Phi}_{99} = -\varepsilon_2\mathbf{I},$$

$$h_\delta = h_{i+1} - h_i = \frac{(h_M - h_m)}{N},$$

$$h_i = h_1 + \frac{(i-1)(h_M - h_m)}{N},$$

$$\mathbf{K} = \left[\mathbf{K}_1^T \;\; 0 \;\; 0 \;\; 0 \;\; \mathbf{K}_2^T \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T,$$

$$\mathbf{V} = \left[0 \;\; \mathbf{V}_1^T \;\; \mathbf{V}_2^T \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T,$$

$$\mathbf{W} = \left[0 \;\; 0 \;\; \mathbf{W}_1^T \;\; \mathbf{W}_2^T \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T. \tag{12}$$

Proof. First, we decompose the delay-interval $[h_m, h_M]$ into N equidistant subinterval, where N is a given integer; that is,

$$[h_m, h_M] = [h_1, h_2] \cup [h_2, h_3] \cup \cdots \cup [h_i, h_{i+1}] \cup \cdots \cup [h_N, h_{N+1}]. \tag{13}$$

Then, the L-K functional corresponding to the time-variation $h(t) \in [h_2, h_3]$ is chosen as

$$\begin{aligned} V_2(t) = {} & \xi^T(t)\mathbf{P}\xi(t) + \int_{t-h(t)}^{t} x^T(s)\mathbf{Q}_1 x(s)\,ds \\ & + \int_{t-h_2}^{t} x^T(s)\mathbf{Q}_2 x(s)\,ds + \int_{t-h_3}^{t} x^T(s)\mathbf{Q}_3 x(s)\,ds \\ & + h_2\int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\mathbf{Z}_2\dot{x}(s)\,ds\,d\theta \\ & + \int_{-h_3}^{-h_2}\int_{t+\theta}^{t} \dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds\,d\theta \\ & + \frac{h_2^2}{2}\int_{-h_2}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^T(s)\mathbf{R}_2\dot{x}(s)\,ds\,d\lambda\,d\theta \\ & + \frac{\left(h_3^2 - h_2^2\right)}{2}\int_{-h_3}^{-h_2}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^T(s)\mathbf{R}_3\dot{x}(s)\,ds\,d\lambda\,d\theta \end{aligned} \tag{14}$$

with $\xi^T(t) = \left[x^T(t) \quad \int_{t-h_2}^{t} x^T(t)ds \quad \int_{t-h_3}^{t-h_2} x^T(t)ds\right]$.

The time-derivative of the L-K functional along the trajectory of (1) is given by

$$\begin{aligned} \dot{V}_2(t) = {} & 2\xi^T(t)\mathbf{P}\dot{\xi}(t) + x^T(t)\left(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3\right)x(t) \\ & - (1-\mu)x^T(t-h(t))\mathbf{Q}_1 x(t-h(t)) \\ & - x^T(t-h_2)\mathbf{Q}_2 x(t-h_2) - x^T(t-h_3)\mathbf{Q}_3 x(t-h_3) \\ & + \dot{x}^T(t)\left[h_2^2\mathbf{Z}_2 + (h_3 - h_2)\mathbf{Z}_3\right]\dot{x}(t) \\ & - h_2\int_{t-h_2}^{t} \dot{x}^T(s)\mathbf{Z}_2\dot{x}(s)\,ds - \int_{t-h_3}^{t-h_2} \dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds \\ & + \frac{1}{4}h_2^4\dot{x}^T(t)\mathbf{R}_2\dot{x}(t) + \frac{1}{4}\left(h_3^2 - h_2^2\right)^2\dot{x}^T(t)\mathbf{R}_3\dot{x}(t) \\ & - \frac{h_2^2}{2}\int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\mathbf{R}_2\dot{x}(t)\,ds\,d\theta \\ & - \frac{\left(h_3^2 - h_2^2\right)}{2}\int_{-h_3}^{-h_2}\int_{t+\theta}^{t} \dot{x}^T(s)\mathbf{R}_3\dot{x}(t)\,ds\,d\theta. \end{aligned} \tag{15}$$

Note that

$$-\int_{t-h_2}^{t-h_2}\dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds = -\int_{t-h_3}^{t-h(t)}\dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds - \int_{t-h(t)}^{t-h_2}\dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds. \tag{16}$$

From Lemmas 1 and 2, we have

$$\begin{aligned}
&-\int_{t-h(t)}^{t-h_2}\dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds \\
&\quad\le (h(t)-h_2)\,\zeta^T(t)\,\mathbf{V}\mathbf{Z}_3^{-1}\mathbf{V}^T\zeta(t) \\
&\qquad + 2\zeta^T(t)\,\mathbf{V}\left[x(t-h_2)-x(t-h(t))\right], \\
&-\int_{t-h_3}^{t-h(t)}\dot{x}^T(s)\mathbf{Z}_3\dot{x}(s)\,ds \\
&\quad\le (h_3-h(t))\,\zeta^T(t)\,\mathbf{W}\mathbf{Z}_3^{-1}\mathbf{W}^T\zeta(t) \\
&\qquad + 2\zeta^T(t)\,\mathbf{W}\left[x(t-h(t))-x(t-h_3)\right], \\
&-h_2\int_{t-h_2}^{t}\dot{x}^T(s)\mathbf{Z}_2\dot{x}(s)\,ds \\
&\quad\le \begin{bmatrix} x(t) \\ x(t-h_2)\end{bmatrix}^T \begin{bmatrix} -\mathbf{Z}_2 & \mathbf{Z}_2 \\ * & -\mathbf{Z}_2\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-h_2)\end{bmatrix}, \\
&-\frac{h_2^2}{2}\int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)\mathbf{R}_2\dot{x}(t)\,ds\,d\theta \\
&\quad\le \begin{bmatrix} h_2x(t) \\ \int_{t-h_2}^{t}x(s)\,ds\end{bmatrix}^T \begin{bmatrix} -\mathbf{R}_2 & \mathbf{R}_2 \\ \mathbf{R}_2 & -\mathbf{R}_2\end{bmatrix}\begin{bmatrix} h_2x(t) \\ \int_{t-h_2}^{t}x(s)\,ds\end{bmatrix}, \\
&-\frac{\left(h_3^2-h_2^2\right)}{2}\int_{-h_3}^{-h_2}\int_{t+\theta}^{t}\dot{x}^T(s)\mathbf{R}_3\dot{x}(t)\,ds\,d\theta \\
&\quad\le \begin{bmatrix} (h_3-h_2)x(t) \\ \int_{t-h_3}^{t-h_2}x(s)\,ds\end{bmatrix}\begin{bmatrix} -\mathbf{R}_3 & \mathbf{R}_3 \\ \mathbf{R}_3 & -\mathbf{R}_3\end{bmatrix}\begin{bmatrix} (h_3-h_2)x(t) \\ \int_{t-h_3}^{t-h_2}x(s)\,ds\end{bmatrix}.
\end{aligned} \tag{17}$$

$$\tag{18}$$

From (3), we can obtain, for any scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$,

$$\begin{aligned}
&\varepsilon_1\left[\alpha^2x^T(t)x(t)-f_1^T(x(t),t)\,f_1(x(t),t)\right]\ge 0, \\
&\varepsilon_2\left[\beta^2x^T(t-h(t))\,x(t-h(t))\right. \\
&\qquad\left.-f_2^T(x(t-h(t)),t)\,f_2(x(t-h(t)),t)\right]\ge 0.
\end{aligned} \tag{19}$$

From the system (1), we have the following equation:

$$\begin{aligned}
&2\left[x^T(t)\mathbf{K}_1+\dot{x}(t)\mathbf{K}_2\right] \\
&\times\left[\mathbf{A}x(t)+\mathbf{B}x(t-h(t))\right. \\
&\qquad\left.+f_1(x(t),t)+f_2(x(t-h(t)),t)-\dot{x}(t)\right]=0.
\end{aligned} \tag{20}$$

By substituting (17)~(20) in (15) and defining an augmented state vector,

$$\begin{aligned}
\zeta^T(t) = \Big[&x^T(t)\quad x^T(t-h_i)\quad x^T(t-h(t))\quad x^T(t-h_{i+1}) \\
&\dot{x}^T(t)\quad \int_{t-h_{i+1}}^{t}x^T(s)\,ds\quad \int_{t-h_{i+1}}^{t-h_i}x^T(s)\,ds\quad f_1^T\quad f_2^T\Big].
\end{aligned} \tag{21}$$

Then the time-derivative $\dot{V}_2(x(t),t)$ can be expressed as follows:

$$\begin{aligned}
\dot{V}_2(x(t),t) \le \zeta^T(t)\Big(&\mathbf{\Phi}+(h_3-h(t))\,\mathbf{W}\mathbf{Z}_3^{-1}\mathbf{W}^T \\
&+(h(t)-h_2)\,\mathbf{V}\mathbf{Z}_3^{-1}\mathbf{V}^T\Big)\zeta(t).
\end{aligned} \tag{22}$$

One can see that if $h(t) \in [h_2, h_3]$,

$$\mathbf{\Phi}+(h_3-h(t))\,\mathbf{W}\mathbf{Z}_3^{-1}\mathbf{W}^T+(h(t)-h_2)\,\mathbf{V}\mathbf{Z}_3^{-1}\mathbf{V}^T<0. \tag{23}$$

Then $\dot{V}(t) < -\varepsilon\|x(t)\|^2$ for some scalar $\varepsilon > 0$, from which we conclude that system (1) is asymptotically stable according to L-K stability theory [1].

Applying Lemma 3 to (23) yields the following inequality:

$$\begin{aligned}
&\mathbf{\Phi}+(h_3-h(t))\,\mathbf{W}\mathbf{Z}_3^{-1}\mathbf{W}^T<0, \\
&\mathbf{\Phi}+(h(t)-h_2)\,\mathbf{V}\mathbf{Z}_3^{-1}\mathbf{V}^T<0.
\end{aligned} \tag{24}$$

By Schur complement on (24), we can obtain (10) and (11) with $i = 2$.

Without loss of generality, when $h(t) \in [h_i, h_{i+1}]$ ($i = 1, 3, \ldots, N$), construct the following L-K functional:

$$\begin{aligned}
V_i(t) = {}&\xi^T(t)\,\mathbf{P}\xi(t)+\int_{t-h(t)}^{t}x^T(s)\,\mathbf{Q}_1x(s)\,ds \\
&+\int_{t-h_i}^{t}x^T(s)\,\mathbf{Q}_2x(s)\,ds+\int_{t-h_{i+1}}^{t}x^T(s)\,\mathbf{Q}_3x(s)\,ds \\
&+h_i\int_{-h_i}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)\,\mathbf{Z}_2\dot{x}(s)\,ds\,d\theta \\
&+\int_{-h_{i+1}}^{-h_i}\int_{t+\theta}^{t}\dot{x}^T(s)\,\mathbf{Z}_3\dot{x}(s)\,ds\,d\theta \\
&+\frac{h_i^2}{2}\int_{-h_i}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)\,\mathbf{R}_2\dot{x}(s)\,ds\,d\lambda\,d\theta \\
&+\frac{\left(h_{i+1}^2-h_i^2\right)}{2}\int_{-h_{i+1}}^{-h_i}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)\,\mathbf{R}_3\dot{x}(s)\,ds\,d\lambda\,d\theta
\end{aligned} \tag{25}$$

with $\xi^T(t) = [x^T(t) \quad \int_{t-h_i}^{t} x^T(t)ds \quad \int_{t-h_{i+1}}^{t-h_i} x^T(t)ds]$, and $\mathbf{P}, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{R}_2, \mathbf{R}_3$ are the same matrix variables used in $V_2(t)$. Now, define an augmented state vector:

$$\zeta^T(t) = \left[x^T(t) \quad x^T(t-h_i) \quad x^T(t-h(t)) \quad x^T(t-h_{i+1}) \right.$$
$$\left. \dot{x}^T(t) \quad \int_{t-h_{i+1}}^{t} x^T(s)\,ds \quad \int_{t-h_{i+1}}^{t-h_i} x^T(s)\,ds \quad f_1^T \quad f_2^T \right]. \tag{26}$$

Using the same method, we have the conditions (10) and (11) as that for $h(t) \in [h_i, h_{i+1}]$ $(i = 1, 3, \ldots, N)$. The proof is completed. □

Remark 5. As an extension of the method used in [18, 22], we divide the delay-interval into N subintervals, constructing a new L-K functional that contains some triple-integral terms and augment term for each delay-interval. This treatment makes us employ more information on the time delay and yields less conservative delay-range bounds.

Remark 6. If there is no perturbation, that is, $f = 0$, $g = 0$, then the stability problem of system (1) is reduced to analyze the stability of the system:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}x(t-h(t)),$$
$$x(\theta) = \phi(\theta), \quad \forall\theta \in [-h_M, 0]. \tag{27}$$

According to Theorem 4, we can obtain the following corollary for the delay-dependent stability of system (27).

Corollary 7. *For given positive scalars, h_m, h_M, μ, system* (27) *with uncertainty* (10) *is asymptotically stable if there exist real symmetric positive definitive matrices*

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ * & \mathbf{P}_{22} & \mathbf{P}_{23} \\ * & * & \mathbf{P}_{33} \end{bmatrix} \tag{28}$$

and $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{R}_2, \mathbf{R}_3$, any matrices $\mathbf{K}_a, \mathbf{V}_a, \mathbf{W}_a$ $(a = 1, 2)$, and scalars $\varepsilon_i \geq 0$ $(i = 1, 2)$ such that the following LMIs hold:

$$\begin{bmatrix} \overline{\mathbf{\Phi}} & \sqrt{h_\delta}\overline{\mathbf{W}} \\ * & -\mathbf{Z}_3 \end{bmatrix} < 0, \quad i = 1, 2, \ldots, N,$$
$$\begin{bmatrix} \overline{\mathbf{\Phi}} & \sqrt{h_\delta}\overline{\mathbf{V}} \\ * & -\mathbf{Z}_3 \end{bmatrix} < 0, \quad i = 1, 2, \ldots, N, \tag{29}$$

where $\overline{\mathbf{\Phi}} = (\mathbf{\Phi}_{i,j})_{7\times 7}$ with

$$\overline{\mathbf{\Phi}}_{11} = \mathbf{P}_{12} + \mathbf{P}_{12}^T + \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 - \mathbf{Z}_2 - h_i^2\mathbf{R}_2 - h_\delta^2\mathbf{R}_3 + \mathbf{K}_1\mathbf{A} + \mathbf{A}^T\mathbf{K}_1^T,$$
$$\overline{\mathbf{\Phi}}_{12} = \mathbf{\Phi}_{12}, \quad \overline{\mathbf{\Phi}}_{13} = \mathbf{\Phi}_{13}, \quad \overline{\mathbf{\Phi}}_{14} = \mathbf{\Phi}_{14},$$
$$\overline{\mathbf{\Phi}}_{15} = \mathbf{\Phi}_{15}, \quad \overline{\mathbf{\Phi}}_{16} = \mathbf{\Phi}_{16}, \quad \overline{\mathbf{\Phi}}_{17} = \mathbf{\Phi}_{17},$$
$$\overline{\mathbf{\Phi}}_{22} = \mathbf{\Phi}_{22}, \quad \overline{\mathbf{\Phi}}_{23} = \mathbf{\Phi}_{23}, \quad \overline{\mathbf{\Phi}}_{24} = \overline{\mathbf{\Phi}}_{25} = 0,$$
$$\overline{\mathbf{\Phi}}_{26} = \mathbf{\Phi}_{26}, \quad \overline{\mathbf{\Phi}}_{27} = \mathbf{\Phi}_{27},$$
$$\overline{\mathbf{\Phi}}_{33} = -(1-\mu)\mathbf{Q}_1 - \mathbf{V}_2 - \mathbf{V}_2^T + \mathbf{W}_1 + \mathbf{W}_1^T,$$
$$\overline{\mathbf{\Phi}}_{34} = \mathbf{\Phi}_{34}, \quad \overline{\mathbf{\Phi}}_{35} = \mathbf{\Phi}_{35}, \quad \overline{\mathbf{\Phi}}_{36} = \overline{\mathbf{\Phi}}_{37} = 0,$$
$$\overline{\mathbf{\Phi}}_{44} = \mathbf{\Phi}_{44}, \quad \overline{\mathbf{\Phi}}_{45} = 0, \quad \overline{\mathbf{\Phi}}_{46} = \mathbf{\Phi}_{46},$$
$$\overline{\mathbf{\Phi}}_{47} = \mathbf{\Phi}_{47}, \quad \overline{\mathbf{\Phi}}_{55} = \mathbf{\Phi}_{55}, \quad \overline{\mathbf{\Phi}}_{56} = \mathbf{\Phi}_{56},$$
$$\overline{\mathbf{\Phi}}_{57} = \mathbf{\Phi}_{57}, \quad \overline{\mathbf{\Phi}}_{66} = \mathbf{\Phi}_{66}, \quad \mathbf{\Phi}_{67} = 0,$$
$$\mathbf{\Phi}_{77} = -\mathbf{R}_3,$$
$$h_\delta = h_{i+1} - h_i = \frac{(h_M - h_m)}{N},$$
$$h_i = h_1 + \frac{(i-1)(h_M - h_m)}{N},$$
$$\overline{\mathbf{K}} = \left[\mathbf{K}_1^T \quad 0 \quad 0 \quad 0 \quad \mathbf{K}_2^T \quad 0 \quad 0\right]^T,$$
$$\overline{\mathbf{V}} = \left[0 \quad \mathbf{V}_1^T \quad \mathbf{V}_2^T \quad 0 \quad 0 \quad 0 \quad 0\right]^T,$$
$$\overline{\mathbf{W}} = \left[0 \quad 0 \quad \mathbf{W}_1^T \quad \mathbf{W}_2^T \quad 0 \quad 0 \quad 0\right]^T. \tag{30}$$

4. Numerical Examples

Example 1. Consider the following neutral time-delay system with

$$\mathbf{A} = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}. \tag{31}$$

For given values of α, β, and μ, we apply Theorem 4 to calculate the maximal allowable value h_M that guarantees that the asymptotical stability of the system is listed in Table 1. From Table 1, it is easy to see that our proposed stability criterion gives much less conservative results than those in [21, 22].

Example 2. Consider the system (27) with the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}. \tag{32}$$

For given h_m and μ, we calculate the allowable upper bound of h_M that guarantees the asymptotical stability of system (27). Using different methods, computational results are obtained and these are listed in Table 2. From Table 2, it can be seen that our results are less conservative than the existing criteria.

5. Conclusion

This paper studies the problem of robust delay-dependent stability for a class of linear systems with interval time-varying delay and nonlinear perturbations. Based on the delay-partitioning approach, the less conservative delay-dependent stability conditions are derived. The reduction in

Table 1: Admissible upper bounds h_2 for various μ and $h_m = 0, 0.5, 1$.

h_m	α, β	$\alpha = 0, \beta = 0.1$			$\alpha = 0.1, \beta = 0.1$		
	μ	0.5	0.9	1.1	0.5	0.9	1.1
0	[21]	1.442	1.280	1.280	1.284	1.209	1.209
	[22]	1.443	1.408	1.408	1.287	1.279	1.279
	Theorem 4 ($N = 2$)	1.6299	1.4938	1.4938	1.4743	1.3943	1.3943
	Theorem 4 ($N = 3$)	1.6866	1.6866	1.6866	1.5304	1.5304	1.5304
0.5	[21]	1.442	1.338	1.338	1.284	1.245	1.245
	[22]	1.558	1.558	1.558	1.384	1.384	1.384
	Theorem 4 ($N = 2$)	1.6618	1.6618	1.6618	1.5272	1.5272	1.5272
	Theorem 4 ($N = 3$)	1.8599	1.8599	1.8599	1.6622	1.6622	1.6622
1	[21]	1.543	1.543	1.543	1.408	1.408	1.408
	[22]	1.760	1.760	1.760	1.532	1.532	1.532
	Theorem 4 ($N = 2$)	1.8824	1.8824	1.8824	1.7059	1.7059	1.7059
	Theorem 4 ($N = 3$)	2.0650	2.0650	2.0650	1.8188	1.8188	1.8188

Table 2: Maximum bounds h_M for a given h_m.

μ	Method	$h_m = 1$	$h_m = 2$	$h_m = 3$	$h_m = 4$	$h_m = 5$
0.3	[4]	2.2125	2.4091	3.3342	4.2799	5.2393
	[8]	2.2474	2.4798	3.3896	4.3250	5.2773
	[18] ($N = 2$)	2.3564	3.0484	3.8779	4.7481	5.6475
	Corollary 7 ($N = 2$)	2.5378	3.0804	3.9171	4.7933	5.6967
	[18] ($N = 3$)	2.7077	3.4408	4.2307	5.0463	5.9319
	Corollary 7 ($N = 3$)	2.7469	3.4900	4.2899	5.1313	6.0030

the conservatism of the proposed stability criteria is mainly attributed to the new L-K functional which contains some triple-integral terms and augment terms for each divided segment. Numerical examples have illustrated the effectiveness of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stability Analysis and H_∞ Output Tracking Control for Linear Systems with Time-Varying Delays

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The problem of stability analysis and H_∞ output tracking control for linear systems with time-varying delays is studied. First, by construction of a newly augmented Lyapunov-Krasovskii functional, a delay-dependent stability criterion for nominal systems with time-varying delays is established in terms of linear matrix inequalities (LMIs). Second, based on the H_∞ sense, the proposed method is extended to solve the problem of designing an H_∞ output tracking controller to track the output of a given reference model. Finally, three examples are included to show the validity and effectiveness of the presented delay-dependent stability and the H_∞ output tracking controller design.

1. Introduction

The problem of output tracking control has received a great deal of attention since this issue is an important requirement in many systems [1–5]. In detail, output tracking controller design methods are utilized in robot systems [1, 2], flight systems [3, 4], and so on. Moreover, in [5], the problem of output tracking control is derived in H_∞ sense for time-delayed systems with nonlinear perturbations. The main purpose of output tracking control is to design a closed-loop feedback controller such that the output of plant tracks the output of a given reference model as close as possible. Moreover, disturbances should be considered in designing output tracking control since disturbances can lead to adverse effects on the performance of systems. To minimize the effects of the disturbances on systems, one possible approach is to design a tracking controller in H_∞ sense. Since H_∞ control which has been used to minimize the effects of the disturbances was firstly introduced by Zames [6], the study on H_∞ controller design has been attracted by many researchers [7–10]. Similarly, H_∞ output tracking controller has an objective of designing a controller such that the closed-loop system is asymptotically stable and the tracking error by the effects of disturbances does not exceed a prescribed level.

On the other hand, time-delay is one of the sources of instability and poor performance in various systems such as physical and chemical systems and industrial and engineering systems. The reason is that time-delays frequently occur in various systems due to the finite capabilities of information processing and transmission. For this reason, during the past several decades, stability analysis for time-delayed systems has been widely studied by many researchers [11–23]. One of the aims of analysis for time-delayed systems is to establish a less conservative stability criterion which can find maximum upper bounds of time-delays for guaranteeing the asymptotic stability of systems. Stability analysis for time-delayed systems can be classified into delay-independent criteria and delay-dependent ones. Since delay-dependent stability criteria use information of time-delays such as lower bounds and upper bounds of delays while delay-independent

ones do not have them, delay-dependent stability criteria are generally less conservative than delay-independent ones especially when the sizes of time-delays are small.

In line with this thinking, according to Zhang and Yu [5], it has been well acknowledged that output tracking control design for time-delayed system is more general and more difficult than stabilization. However, most existing results on output tracking control have focused on systems without time-delays. From the practical point of view, it is worth designing output tracking controllers for time-delayed systems since it is well known that there exist time-delays in many practical systems. Therefore, to establish less conservative stability criteria and design output tracking controllers for time-delayed systems are still challenging.

Motivated by the matters mentioned above, this paper investigates the problem of stability analysis and H_∞ tracking controller designing for linear systems with time-varying delays and disturbances. First, in Theorem 5, by constructing a newly augmented Lyapunov-Krasovskii functional, an improved delay-dependent stability criterion is derived by utilizing reciprocally convex approach [15] and some new zero equalities with LMI framework [24] which can be formulated as convex optimization algorithms. Second, based on the results of Theorem 5, an H_∞ tracking controller design method for linear systems with time-varying delays and disturbances will be proposed in Theorem 7. Finally, through three examples, validity and effectiveness of the proposed theorem will be verified.

Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. For symmetric matrices X and Y, $X > Y$ (resp., $X \geq Y$) means that the matrix $X-Y$ is positive definite (resp., nonnegative). X^T denotes the transposition of X. I_n denotes the $n \times n$ identity matrix. 0_n and $0_{n\times m}$ denote the $n \times n$ zero matrix and the $n \times m$ zero matrix, respectively. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. diag$\{\cdots\}$ denotes the block diagonal matrix, respectively. $*$ represents the elements below the main diagonal of a symmetric matrix. $\mathscr{L}_2[0,\infty)$ is the space of square integrable vector. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of the matrix $X_{[f(t)]}$ include the value of $f(t)$; for example, $X_{[f_0]} \equiv X_{[f_{(t)}=f_0]}$.

2. Problem Statements

Consider the following system with time-varying delays:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d(t)\,x(t-h(t)) + Bu(t) + B_w w(t), \\ z(t) &= C_1 x(t) + C_2 x(t-h(t)) + B_1 u(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^r$ is the disturbance input which belongs to $\mathscr{L}_2[0,\infty)$, $z(t) \in \mathbb{R}^s$ is the vector of controlled output, and $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_w \in \mathbb{R}^{n\times r}$, $C_1 \in \mathbb{R}^{s\times n}$, $C_2 \in \mathbb{R}^{s\times n}$, and $B_1 \in \mathbb{R}^{s\times m}$ are known real constant matrices. $h(t)$ means a time-delay satisfying time-varying continuous function as follows:

$$0 \leq h(t) \leq h_M, \qquad \left|\dot{h}(t)\right| \leq h_d, \tag{2}$$

where $h_M > 0$ and $0 \leq h_d < 1$ are known scalars.

Designing the output tracking control has an objective to design a controller such that the output of the system tracks a reference signal. Therefore, the reference signal $z_r(t)$ is assumed to be generated by reference model as follows:

$$\begin{aligned} \dot{x}_r(t) &= Fx_r(t) + r(t), \\ z_r(t) &= Hx_r(t), \end{aligned} \tag{3}$$

where $z_r(t) \in \mathbb{R}^s$ is the reference signal which has the same dimension as $z(t)$, $x_r(t) \in \mathbb{R}^p$ is the reference state, and $r(t) \in \mathbb{R}^p$ is the energy bounded reference input. $G \in \mathbb{R}^{p\times p}$ and $H \in \mathbb{R}^{s\times p}$ are known real constant matrices with the assumption that the matrix G is Hurwitz. The state-feedback controller is considered as the following form:

$$u(t) = K_1 x(t) + K_2 x_r(t), \tag{4}$$

where $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times p}$ are gain matrices of the state-feedback controller. Let us define $\xi(t) = [x^T(t), x_r^T(t)]^T \in \mathbb{R}^q$, $e(t) = z(t) - z_r(t)$, and $v(t) = [w^T(t), r^T(t)]^T$. Then the following augmented systems can be obtained as

$$\begin{aligned} \dot{\xi}(t) &= \left(\overline{A} + \overline{B}\,\overline{K}\right)\xi(t) + \overline{A}_d(t)\,\xi(t-h(t)) + \overline{B}_w v(t), \\ e(t) &= \overline{C}_1 \xi(t) + \overline{C}_2 \xi(t-h(t)), \end{aligned} \tag{5}$$

where

$$\begin{aligned} \overline{A} &= \begin{bmatrix} A & 0_{n\times p} \\ 0_{p\times n} & F \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix}, \qquad \overline{K} = [K_1, K_2], \\ \overline{A}_d &= \begin{bmatrix} A_d & 0_{n\times p} \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}, \qquad \overline{B}_w = \begin{bmatrix} B_w & 0_{n\times p} \\ 0_{p\times r} & I_p \end{bmatrix}, \\ \overline{C}_1 &= [C_1 + B_1 K_1, -H + B_1 K_2], \qquad \overline{C}_2 = \left[C_2, 0_{s\times p}\right]. \end{aligned} \tag{6}$$

From the H_∞ sense, let us consider the two requirements of the output tracking controller as follows.

(i) With $v(t) = 0$, the closed-loop system (5) with control input $u(t)$ is asymptotically stable.

(ii) The effects of $w(t)$ and $r(t)$ on the tracking error $e(t)$ are attenuated below a desired level in the H_∞ sense; that is,

$$\begin{aligned} J &= \left\{\int_0^\infty e^T(s)\,e(s) - \gamma^2 v^T(s)\,v(s)\,ds\right\} \leqslant 0, \\ &\left(\text{i.e.} \sup_{v(t)\neq 0, v(t)\in\mathscr{L}_2[0,\infty)} \frac{\|e(t)\|_2}{\|v(t)\|_2} \leq \gamma\right), \end{aligned} \tag{7}$$

where $\gamma > 0$ is a prescribed scalar. Then, if the obtained controller $u(t)$ is satisfied with two requirements, then it is said to be an H_∞ output tracking controller.

The objective of this paper is to design a state-feedback controller (4) such that system (5) achieves the H_∞ output tracking below a prescribed level and the H_∞ output tracking performance γ is minimized.

Before proceeding further, the following lemmas will be utilized in deriving main results.

Lemma 1 (see [17]). *For a positive matrix Z, scalars $h_2 > h_1 > 0$ such that the integrations are well defined; then*

$$\begin{aligned}
&-(h_2-h_1)\int_{t-h_2}^{t-h_1} x^T(s)Zx(s)\,ds \\
&\quad\le -\left(\int_{t-h_2}^{t-h_1} x(s)\,ds\right)^T Z\left(\int_{t-h_2}^{t-h_1} x(s)\,ds\right), \\
&-\frac{\left(h_2^2-h_1^2\right)}{2}\int_{t-h_2}^{t-h_1}\int_{\theta}^{t} x^T(s)Zx(s)\,ds\,d\theta \\
&\quad\le -\left(\int_{t-h_2}^{t-h_1}\int_{\theta}^{t} x(s)\,ds\,d\theta\right)^T Z\left(\int_{t-h_2}^{t-h_1}\int_{\theta}^{t} x(s)\,ds\,d\theta\right), \\
&-\frac{\left(h_2^3-h_1^3\right)}{6}\int_{t-h_2}^{t-h_1}\int_{\theta}^{t}\int_{\lambda}^{t} x^T(s)Zx(s)\,ds\,d\lambda\,d\theta \\
&\quad\le -\left(\int_{t-h_2}^{t-h_1}\int_{\theta}^{t}\int_{\lambda}^{t} x(s)\,ds\,d\lambda\,d\theta\right)^T \\
&\qquad\times Z\left(\int_{t-h_2}^{t-h_1}\int_{\theta}^{t}\int_{\lambda}^{t} x(s)\,ds\,d\lambda\,d\theta\right).
\end{aligned} \tag{8}$$

Lemma 2 (see [15]). *For a scalar α in the interval $(0,1)$, a given matrix $R \in \mathbb{R}^{n\times n} > 0$, two matrices $W_1 \in \mathbb{R}^{n\times m}$ and $W_2 \in \mathbb{R}^{n\times m}$, for all vector $\xi \in \mathbb{R}^m$, let one define a function $\Theta(\alpha, R)$ given by*

$$\Theta(\alpha,R) = \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi. \tag{9}$$

Then, if there exists a matrix X in $\mathbb{R}^{n\times n}$ such that $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] > 0$, then the following inequality holds

$$\min_{\alpha\in(0,1)} \Theta(\alpha,R) \ge \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}. \tag{10}$$

Lemma 3 (Finsler's lemma [25]). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that* $\operatorname{rank}(B) < n$. *The following statements are equivalent:*

(1) $\zeta^T\Phi\zeta < 0, \forall B\zeta = 0, \zeta \ne 0$.

(2) $(B^\perp)^T\Phi B^\perp < 0$, *where $B^\perp$ is a right orthogonal complement of B.*

Lemma 4 (see [21]). *For a positive matrix Λ, a symmetric matrix Ξ, and a matrix Γ, the following two statements are equivalent:*

(1) $\Xi - \Gamma^T\Lambda\Gamma < 0$;

(2) *There exists a matrix of appropriate dimension U such that*

$$\begin{bmatrix} \Xi + U^T\Gamma + U^T\Gamma & U^T \\ U & -\Lambda \end{bmatrix} \le 0. \tag{11}$$

3. Main Results

This section consists of two subsections. The first section introduces an improved stability criterion for system (1) with $u(t) = 0$ and $w(t) = 0$ as Theorem 5. The second subsection will investigate the problem of an H_∞ output tracking controller design method for the augmented system (5) based on the results of Theorem 5.

3.1. Stability Analysis. Let us consider the system (1) with $w(t) = 0$ and $u(t) = 0$ given by

$$\dot{x}(t) = Ax(t) + A_d x(t - h(t)). \tag{12}$$

In this subsection, a delay-dependent stability criterion for system (12) is derived. For simplicity of matrix and vector representation, e_i $(i = 1,\dots,10) \in \mathbb{R}^{10n\times n}$ are defined as block entry matrices which will be used in Theorem 5. For example, $e_1 = [I_n, 0_{n\times 9n}]^T$ and $e_5 = [0_{n\times 4n}, I_n, 0_{n\times 5n}]^T$. The other notations are defined as

$$\begin{aligned}
\zeta^T(t) = \Bigg[& x^T(t), x^T(t-h(t)), x^T\left(t-h_M\right), \dot{x}^T(t), \\
& \dot{x}^T\left(t-h_M\right), \int_{t-h(t)}^{t} x^T(s)\,ds, \\
& \int_{t-h_M}^{t-h(t)} x^T(s)\,ds, \int_{t-h(t)}^{t}\int_{s}^{t} x^T(u)\,du\,ds, \\
& \int_{t-h_M}^{t-h(t)}\int_{s}^{t} x^T(u)\,du\,ds, \dot{x}^T(t-h(t)) \Bigg],
\end{aligned}$$

$$\Theta = \left(Ae_1^T + A_d e_2^T - e_4^T\right),$$

$$\Psi_{11} = [e_1, e_2, e_3, e_6, e_7, e_8, e_9],$$

$$\Psi_{12} = [e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_{10}],$$

$$\Psi_{21} = [e_1, e_4, 0_{10n\times n}, 0_{10n\times n}],$$

$$\Psi_{22} = [e_3, e_5, e_1 - e_3, e_6 + e_7],$$

$$\Psi_{23} = [e_6 + e_7, e_1 - e_3, h_M e_1 - e_6 - e_7, e_8 + e_9],$$

$$\Psi_{24} = [0_{10n\times n}, 0_{10n\times n}, e_4, e_1],$$

$$\Psi_{31} = [e_1, e_4, 0_{10n\times n}, 0_{10n\times n}],$$

$$\Psi_{32} = [e_2, e_{10}, e_1 - e_2, e_6],$$

$$\Psi_{33[h(t)]} = [e_6, e_1 - e_3, h(t)e_1 - e_6, e_8],$$

$$\Psi_{34} = [0_{10n\times n}, 0_{10n\times n}, e_4, e_1],$$

$$\Psi_{35} = [e_2, e_{10}, e_1 - e_2, e_6],$$

$$\Psi_{36} = [e_3, e_5, e_1 - e_3, e_6 + e_7],$$

$$\Psi_{37[h(t)]} = [e_7, e_2 - e_3, (h_M - h(t)) e_1 - e_7, e_9],$$

$$\Psi_{38} = [0_{10n\times n}, 0_{10n\times n}, e_4, e_1],$$

$$\Psi_{41} = [e_1, e_4],$$

$$\Psi_{42} = [e_6, e_1 - e_2, e_7, e_2 - e_3],$$

$$\Psi_5 = [e_1, e_4],$$

$$\Psi_6 = \left(\left(\frac{h_M^2}{2}\right) e_1 - e_8 - e_9\right),$$

$$\Upsilon_1 = \begin{bmatrix} Q_1 + \begin{bmatrix} P_3 & P_1 \\ P_1 & 0_n \end{bmatrix} & S_1 \\ S_1^T & Q_1 + \begin{bmatrix} P_4 & P_2 \\ P_2 & 0_n \end{bmatrix} \end{bmatrix},$$

$$\Upsilon_2 = \begin{bmatrix} Q_2 + \left(\frac{4}{h_M}\right) \begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix} & S_2 \\ S_2^T & Q_2 + \left(\frac{4}{h_M}\right) \begin{bmatrix} 0_n & P_4 \\ P_4 & 0_n \end{bmatrix} \end{bmatrix},$$

$$\Gamma_{1[h(t),\dot{h}(t)]} = \begin{bmatrix} 0_{n\times 3n}, I_n, 0_{n\times 4n} \\ \hline 0_{n\times 7n}, \left(1 - \dot{h}(t)\right) I_n \\ \hline 0_{n\times 4n}, I_n, 0_{n\times 3n} \\ \hline I_n, -\left(1 - \dot{h}(t)\right) I_n, 0_{n\times 6n} \\ \hline 0_n, \left(1 - \dot{h}(t)\right) I_n, -I_n, 0_{n\times 5n} \\ \hline h(t) I_n, 0_{n\times 4n}, -\left(1 - \dot{h}(t)\right) I_n, 0_{n\times 2n} \\ \hline (h_M - h(t)) I_n, 0_{n\times 4n}, -\dot{h}(t) I_n, 0_{n\times 2n} \end{bmatrix},$$

$$\Gamma_{2[h(t)]} = \begin{bmatrix} 0_{n\times 3n}, I_n, 0_n \\ \hline h(t) I_n, -I_n, 0_{n\times 3n} \\ \hline 0_{n\times 4n}, I_n \\ \hline (h_M - h(t)) I_n, 0_n, -I_n, 0_{n\times 2n} \end{bmatrix} \begin{bmatrix} e_1^T \\ e_6^T \\ e_7^T \\ e_8^T \\ e_9^T \end{bmatrix},$$

$$\Xi_{1[h(t),\dot{h}(t)]} = \Psi_{11}\Gamma_{1[h(t),\dot{h}(t)]} R \Psi_{12}^T + \Psi_{12} R \Gamma_{1[h(t),\dot{h}(t)]}^T \Psi_{11}^T,$$

$$\Xi_2 = \Psi_{21} N \Psi_{21}^T - \Psi_{22} N \Psi_{22}^T + \Psi_{23} N \Psi_{24}^T + \Psi_{24} N \Psi_{23}^T,$$

$$\begin{aligned} \Xi_{3[h(t),\dot{h}(t)]} = {} & \Psi_{31} G \Psi_{31}^T - \left(1 - \dot{h}(t)\right) \Psi_{32} G \Psi_{32}^T \\ & + \Psi_{33[h(t),\dot{h}(t)]} G \Psi_{34}^T + \Psi_{34} G \Psi_{33[h(t),\dot{h}(t)]}^T \\ & + \left(1 - \dot{h}(t)\right) \Psi_{35} M \Psi_{35}^T - \Psi_{36} M \Psi_{36}^T \\ & + \Psi_{37[h(t),\dot{h}(t)]} M \Psi_{38}^T + \Psi_{38} M \Psi_{37[h(t),\dot{h}(t)]}^T, \end{aligned}$$

$$\begin{aligned} \Xi_{\text{zero}[h(t)]} = {} & h_M \left(e_1 P_1 e_1^T - e_2 P_1 e_2^T + e_2 P_2 e_2^T - e_3 P_2 e_3^T\right) \\ & + h_M \left(h(t) e_1 P_3 e_1^T - (h_M - h(t)) e_1 P_4 e_1^T\right), \end{aligned}$$

$$\Xi_{4[h(t)]} = h_M^2 \Psi_{41} Q_1 \Psi_{41}^T - \Psi_{42} \Upsilon_1 \Psi_{42}^T + \Xi_{\text{zero}[h(t)]},$$

$$\Xi_5 = \left(\frac{h_M^2}{2}\right)^2 \Psi_5 Q_2 \Psi_5^T,$$

$$\Xi_6 = \left(\frac{h_M^3}{6}\right)^2 e_4 Q_3 e_4^T - \Psi_6 Q_3 \Psi_6^T,$$

$$\begin{aligned} \Xi_{[h(t),\dot{h}(t)]} = {} & \Xi_{1[h(t),\dot{h}(t)]} + \Xi_2 + \Xi_{3[h(t),\dot{h}(t)]} \\ & + \Xi_{4[h(t)]} + \Xi_5 + \Xi_6, \end{aligned}$$

$$\begin{aligned} L_{[h(t),\dot{h}(t)]} = {} & \Theta^{\perp T} \Xi_{[h(t),\dot{h}(t)]} \Theta^{\perp} + \Theta^{\perp T} \Gamma_{2[h(t)]}^T \Lambda \\ & + \Lambda^T \Gamma_{2[h(t)]} \Theta^{\perp}, \end{aligned}$$

$$\Phi_{[h(t),\dot{h}(t)]} = \begin{bmatrix} L_{[h(t),\dot{h}(t)]} & \Lambda^T \\ \Lambda & -\Upsilon_2 \end{bmatrix}, \tag{13}$$

where $\Theta^{\perp}$ is the right orthogonal complement of Θ.

Now, a delay-dependent stability of the system (12) is given as Theorem 5.

Theorem 5. *For given scalars $0 \le h_M$ and $0 \le h_d < 1$, the system (12) is asymptotically stable for $0 \le h(t) \le h_M$ and $|\dot{h}(t)| \le h_d$, if there exist positive definite matrices $R \in \mathbb{R}^{7n\times 7n}$, $N \in \mathbb{R}^{4n\times 4n}$, $G \in \mathbb{R}^{4n\times 4n}$, $M \in \mathbb{R}^{4n\times 4n}$, $Q_1 \in \mathbb{R}^{2n\times 2n}$, $Q_2 \in \mathbb{R}^{2n\times 2n}$, and $Q_3 \in \mathbb{R}^{n\times n}$, symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, \ldots, 4)$, and any matrices $S_1 \in \mathbb{R}^{2n\times 2n}$, $S_2 \in \mathbb{R}^{2n\times 2n}$, and $\Lambda \in \mathbb{R}^{4n\times 9n}$, satisfying the following LMIs:*

$$\begin{aligned} \Phi_{j,k} < 0, \quad & \left(j, k = 1, 2\right), \\ \Upsilon_i \ge 0, \quad & (i = 1, 2), \end{aligned} \tag{14}$$

where $\Phi_{j,k}$ are the four vertices of $\Phi_{[h(t),\dot{h}(t)]}$ with the bounds of $h(t)$ and $\dot{h}(t)$, that is, h_M and h_D when $j = k = 1$, h_M and $-h_D$ when $j < k$, 0 and h_D when $j > k$, and 0 and $-h_D$ when $j = k = 2$.

Proof. For positive definite matrices $R \in \mathbb{R}^{7n\times 7n}$, $N \in \mathbb{R}^{4n\times 4n}$, $G \in \mathbb{R}^{4n\times 4n}$, $Q_1 \in \mathbb{R}^{2n\times 2n}$, $Q_2 \in \mathbb{R}^{2n\times 2n}$, and $Q_3 \in \mathbb{R}^{n\times n}$, let us consider the following Lyapunov-Krasovskii functional candidate as

$$V(t) = \sum_{i=1}^{6} V_i(t), \tag{15}$$

where

$$V_1(t) = \Omega_1^T(t) R \Omega_1(t),$$

$$V_2(t) = \int_{t-h_M}^{t} \Omega_2^T(t,s) N \Omega_2(t,s)\, ds,$$

$$\begin{aligned} V_3(t) = {} & \int_{t-h(t)}^{t} \Omega_2^T(t,s) G \Omega_2(t,s)\, ds \\ & + \int_{t-h_M}^{t-h(t)} \Omega_2^T(t,s) M \Omega_2(t,s)\, ds, \end{aligned}$$

$$V_4(t) = h_M \int_{t-h_M}^{t} \int_{s}^{t} \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}^T Q_1 \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds,$$

$$V_5(t) = \left(\frac{h_M^2}{2}\right) \int_{t-h_M}^{t} \int_{s}^{t} \int_{u}^{t} \begin{bmatrix} x(v) \\ \dot{x}(v) \end{bmatrix}^T Q_2 \begin{bmatrix} x(v) \\ \dot{x}(v) \end{bmatrix} dv\, du\, ds,$$

$$V_6(t) = \left(\frac{h_M^3}{6}\right) \int_{t-h_M}^{t} \int_{s}^{t} \int_{u}^{t} \int_{v}^{t} \dot{x}^T(\lambda) Q_3 \dot{x}(\lambda)\, d\lambda\, dv\, du\, ds, \tag{16}$$

with $\Omega_1(t) = [x^T(t), x^T(t-h(t)), x^T(t-h_M), \int_{t-h(t)}^{t} x^T(s)ds, \int_{t-h_M}^{t} x^T(s)ds, \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u)du\, ds, \int_{t-h_M}^{t} \int_{s}^{t} x^T(u)du\, ds]^T$ and $\Omega_2(t,s) = [x^T(s), \dot{x}^T(s), \int_{s}^{t} \dot{x}^T(u)du, \int_{s}^{t} x^T(u)du]^T$.

Now, calculating the time-derivative of $V_1(t)$ yields

$$\begin{aligned} &\dot{V}_1(t) \\ &= 2\Omega_1(t) R \\ &\times \begin{bmatrix} \dot{x}(t) \\ \left(1-\dot{h}(t)\right)\dot{x}(t-h(t)) \\ \dot{x}(t-h_M) \\ x(t) - \left(1-\dot{h}(t)\right)x(t-h_M) \\ \left(1-\dot{h}(t)\right)x(t-h(t)) - x(t-h_M) \\ h(t)\,x(t) - \left(1-\dot{h}(t)\right)\int_{t-h(t)}^{t} x(s)\,ds \\ \begin{pmatrix} \left(h_M - h(t)\right)x(t) + \left(1-\dot{h}(t)\right)\int_{t-h(t)}^{t} x(s)\,ds \\ -\int_{t-h(t)}^{t} x(s)\,ds - \int_{t-h_M}^{t} x(s)\,ds \end{pmatrix} \end{bmatrix} \\ &= \zeta^T(t)\, \Xi_{1[h(t),\dot{h}(t)]}\, \zeta(t). \end{aligned} \tag{17}$$

Also, $\dot{V}_2(t)$ can be represented as

$$\begin{aligned} \dot{V}_2(t) = &\begin{bmatrix} x(t) \\ \dot{x}(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix}^T N \begin{bmatrix} x(t) \\ \dot{x}(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix} - \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^{t} x(u)\,du \end{bmatrix}^T \\ &\times N \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^{t} x(u)\,du \end{bmatrix} \\ &+ 2\int_{t-h_M}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ x(t) - x(s) \\ \int_{s}^{t} x(u)du \end{bmatrix}^T ds N \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \dot{x}(t) \\ x(t) \end{bmatrix} \\ = &\zeta^T(t)\, \Xi_2 \zeta(t). \end{aligned} \tag{18}$$

The time-derivative of $V_3(t)$ can be given as follows:

$$\begin{aligned} \dot{V}_3(t) = &\begin{bmatrix} x(t) \\ \dot{x}(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix}^T G \begin{bmatrix} x(t) \\ \dot{x}(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix} \\ &- \left(1-\dot{h}(t)\right) \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \\ x(t) - x(t-h(t)) \\ \int_{t-h(t)}^{t} x(s)ds \end{bmatrix}^T \\ &\times G \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \\ x(t) - x(t-h(t)) \\ \int_{t-h(t)}^{t} x(s)\,ds \end{bmatrix} \\ &+ 2\int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ \dot{x}(t) \\ \int_{s}^{t} \dot{x}(u)du \\ \int_{s}^{t} x(u)du \end{bmatrix}^T ds G \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \dot{x}(t) \\ x(t) \end{bmatrix} \\ &+ \left(1-\dot{h}(t)\right) \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \\ x(t) - x(t-h(t)) \\ \int_{t-h(t)}^{t} x(s)ds \end{bmatrix}^T \\ &\times M \begin{bmatrix} x(t-h(t)) \\ \dot{x}(t-h(t)) \\ x(t) - x(t-h(t)) \\ \int_{t-h(t)}^{t} x(s)\,ds \end{bmatrix} \\ &- \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^{t} x(s)ds \end{bmatrix}^T M \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^{t} x(s)\,ds \end{bmatrix} \\ &+ 2\begin{bmatrix} \int_{t-h_M}^{t-h(t)} x(s) \\ x(t-h(t)) - x(t-h_M) \\ \left(h_M - h(t)\right)x(t) - \int_{t-h_M}^{t-h(t)} x(s)\,ds \\ \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x(u)\,du\,ds \end{bmatrix}^T \\ &\times M \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \dot{x}(t) \\ x(t) \end{bmatrix} \\ = &\zeta^T(t)\, \Xi_{3[h(t),\dot{h}(t)]}\, \zeta(t). \end{aligned} \tag{19}$$

Calculating $\dot{V}_4(t)$ gives

$$\dot{V}_4(t) = h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} - h_M \int_{t-h_M}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T Q_1 \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds. \tag{20}$$

Inspired by the work in [22], the following four zero equalities with symmetric matrices P_1, P_2, P_3, and P_4 are considered:

$$\begin{aligned}
0 &= x^T(t)\left(h_M P_1\right) x(t) - x^T(t-h(t))\left(h_M P_1\right) x(t-h(t)) \\
&\quad - 2h_M \int_{t-h(t)}^{t} x^T(s) P_1 \dot{x}(s)\, ds, \\
0 &= x^T(t-h(t))\left(h_M P_2\right) x(t-h(t)) \\
&\quad - x^T(t-h_M)\left(h_M P_2\right) x(t-h_M) \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} x^T(s) P_2 \dot{x}(s)\, ds, \\
0 &= x^T(t)\left(h_M h(t) P_3\right) x(t) - h_M \int_{t-h(t)}^{t} x^T(s) P_3 x(s)\, ds \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds, \\
0 &= x^T(t)\left(h_M\left(h_M - h(t)\right) P_4\right) x(t) \\
&\quad - h_M \int_{t-h(t)}^{t} x^T(s) P_4 x(s)\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{u}(s)\, du\, ds.
\end{aligned} \tag{21}$$

By adding (21) into (20), $\dot{V}_4(t)$ can be represented as follows:

$$\begin{aligned}
\dot{V}_4(t) &= h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - h_M \int_{t-h_M}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T Q_1 \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\
&\quad + x^T(t)\left(h_M P_1\right) x(t) - x^T(t-h(t)) \\
&\quad \times \left(h_M P_1\right) x(t-h(t)) \\
&\quad - 2h_M \int_{t-h(t)}^{t} x^T(s) P_1 \dot{x}(s)\, ds \\
&\quad + x^T(t-h(t))\left(h_M P_2\right) x(t-h(t)) \\
&\quad - x^T(t-h_M)\left(h_M P_2\right) x(t-h_M) \\
&\quad - 2h_M \int_{t-h(t)}^{t-h(t)} x^T(s) P_2 \dot{x}(s)\, ds \\
&\quad + x^T(t)\left(h_M h(t) P_3\right) x(t) \\
&\quad - h_M \int_{t-h(t)}^{t} x^T(s) P_3 x(s)\, ds \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad + x^T(t)\left(h_M\left(h_M - h(t)\right) P_4\right) x(t) \\
&\quad - h_M \int_{t-h(t)}^{t} x^T(s) P_4 x(s)\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{u}(s)\, du\, ds \\
&= h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - h_M \int_{t-h_M}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \left(Q_1 + \begin{bmatrix} P_3 & P_1 \\ P_1 & 0_n \end{bmatrix} \right) \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\
&\quad - h_M \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \left(Q_1 + \begin{bmatrix} P_4 & P_2 \\ P_2 & 0_n \end{bmatrix} \right) \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\
&\quad + \zeta^T(t)\, \Xi_{\text{zero}[h(t)]} \zeta(t) \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds.
\end{aligned} \tag{22}$$

From Lemma 1, an upper-bound of $\dot{V}_4(t)$ can be calculated as

$$\begin{aligned}
&\dot{V}_4(t) \\
&\le h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - \frac{h_M}{h(t)} \int_{t-h_M}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \left(Q_1 + \begin{bmatrix} P_3 & P_1 \\ P_1 & 0_n \end{bmatrix} \right) \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\
&\quad - \frac{h_M}{h_M - h(t)} \\
&\quad \times \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \left(Q_1 + \begin{bmatrix} P_4 & P_2 \\ P_2 & 0_n \end{bmatrix} \right) \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\
&\quad + \zeta^T(t)\, \Xi_{\text{zero}[h(t)]} \zeta(t) \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds
\end{aligned} \tag{23}$$

$$
\begin{aligned}
&= h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_{t-h(t)}^{t} x(s)ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} \frac{1}{\phi_1(t)} \left(Q_1 + \begin{bmatrix} P_3 & P_1 \\ P_1 & 0_n \end{bmatrix} \right) & 0_{2n} \\ 0_{2n} & \frac{1}{1-\phi_1(t)} \left(Q_1 + \begin{bmatrix} P_4 & P_2 \\ P_2 & 0_n \end{bmatrix} \right) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \int_{t-h(t)}^{t} x(s)\,ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)\,ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix} \\
&\quad + \zeta^T(t)\, \Xi_{\text{zero}[h(t)]} \zeta(t) \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds,
\end{aligned}
\tag{24}
$$

where $\phi_1(t) = h(t)/h_M$ which satisfies $0 \le \phi_1(t) \le 1$. It should be noted that if $h(t)$ is zero, then the term of $\int_{t-h(t)}^{t} x(s)ds$ and $x(t) - x(t-h(t))$ are zero. Also, if $h(t)$ is h_M, then the term of $\int_{t-h_M}^{t-h(t)} x(s)ds$ and $x(t-h(t)) - x(t-h_M)$ are zero. Thus, inequality (24) still holds.

By using Lemma 2 with $\Upsilon_1 \ge 0$ which is defined in (13), a new upper-bound of $\dot{V}_4(t)$ can be obtained as

$$
\begin{aligned}
&\dot{V}_4(t) \\
&\le h_M^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - \begin{bmatrix} \int_{t-h(t)}^{t} x(s)ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix}^T \\
&\quad \times \Upsilon_1 \begin{bmatrix} \int_{t-h(t)}^{t} x(s)\,ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)\,ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \int_{t-h(t)}^{t} x(s)\,ds \\ x(t) - x(t-h(t)) \\ \int_{t-h_M}^{t-h(t)} x(s)\,ds \\ x(t-h(t)) - x(t-h_M) \end{bmatrix} \\
&\quad + \zeta^T(t)\, \Xi_{\text{zero}[h(t)]} \zeta(t) \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds \\
&= \zeta^T(t) \underbrace{\left(h_M^2 \Psi_{41} Q_1 \Psi_{41}^T - \Psi_{42} \Upsilon_1 \Psi_{42}^T + \Xi_{\text{zero}[h(t)]} \right)}_{\Xi_{4[h(t)]}} \zeta(t) \\
&\quad - 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad - 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds.
\end{aligned}
\tag{25}
$$

The $\dot{V}_5(t)$ is calculated as

$$
\begin{aligned}
\dot{V}_5(t) &= \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} - \frac{h_M^2}{2} \int_{t-h_M}^{t} \int_{s}^{t} \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} Q_2 \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds \\
&\quad \underbrace{-2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds + 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds}_{0} \\
&\quad \underbrace{-2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds + 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds}_{0} \\
&= \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} - \frac{h_M^2}{2} \int_{t-h(t)}^{t} \int_{s}^{t} \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} \left(Q_2 + \frac{4}{h_M} \begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix} \right) \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{h_M^2}{2} \int_{t-h_M}^{t-h(t)} \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} \left(Q_2 + \frac{4}{h_M} \begin{bmatrix} 0_n & P_4 \\ P_4 & 0_n \end{bmatrix} \right) \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds \\
&+ 2h_M \int_{t-h(t)}^{t} \int_s^t x^T(u) P_3 \dot{x}(u)\, du\, ds + 2h_M \int_{t-h_M}^{t-h(t)} \int_s^t x^T(u) P_4 \dot{x}(s)\, du\, ds.
\end{aligned} \tag{26}
$$

Applying Lemma 1 to (26), the upper-bound of $\dot{V}_5(t)$ can be derived as

$$
\begin{aligned}
&\dot{V}_5(t) \\
&\le \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - \left(\frac{h_M}{h(t)} \right)^2 \left(\int_{t-h(t)}^{t} \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds \right)^T \\
&\quad \times \left(Q_2 + \frac{4}{h_M} \begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix} \right) \left(\int_{t-h(t)}^{t} \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds \right) \\
&\quad - \left(\frac{h_M^2}{h_M^2 - h(t)^2} \right) \left(\int_{t-h_M}^{t-h(t)} \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds \right)^T \\
&\quad \times \left(Q_2 + \frac{4}{h_M} \begin{bmatrix} 0_n & P_4 \\ P_4 & 0_n \end{bmatrix} \right) \left(\int_{t-h_M}^{t-h(t)} \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\, ds \right) \\
&\quad + 2h_M \int_{t-h(t)}^{t} \int_s^t x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad + 2h_M \int_{t-h_M}^{t-h(t)} \int_s^t x^T(u) P_4 \dot{x}(s)\, du\, ds \\
&= \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - \begin{bmatrix} \int_{t-h(t)}^{t} \int_s^t x(u)\, du\, ds \\ \int_{t-h(t)}^{t} \int_s^t \dot{x}\, du\, ds \\ \int_{t-h_M}^{t-h(t)} \int_s^t x(u)\, du\, ds \\ \int_{t-h_M}^{t-h(t)} \int_s^t \dot{x}(u)\, du\, ds \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} \frac{1}{\phi_2(t)} \left(Q_2 + \frac{4}{h_M} \begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix} \right) & 0_{2n} \\ 0_{2n} & \frac{1}{1-\phi_2(t)} \left(Q_2 + \frac{4}{h_M} \begin{bmatrix} 0_n & P_4 \\ P_4 & 0_n \end{bmatrix} \right) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \int_{t-h(t)}^{t} \int_s^t x(u)\, du\, ds \\ \int_{t-h(t)}^{t} \int_s^t \dot{x}\, du\, ds \\ \int_{t-h_M}^{t-h(t)} \int_s^t x(u)\, du\, ds \\ \int_{t-h_M}^{t-h(t)} \int_s^t \dot{x}(u)\, du\, ds \end{bmatrix} \\
&\quad + 2h_M \int_{t-h(t)}^{t} \int_s^t x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad + 2h_M \int_{t-h_M}^{t-h(t)} \int_s^t x^T(u) P_4 \dot{x}(s)\, du\, ds,
\end{aligned} \tag{27}
$$

where $\phi_2(t) = (h(t)/h_M)^2$ satisfies $0 \le \phi_2(t) \le 1$. Note that when $h(t)$ is zero, the terms of $\int_{t-h(t)}^{t} \int_s^t x(u) du\, ds$ and $\int_{t-h(t)}^{t} \int_s^t \dot{x}\, du\, ds$ are zero, and when $h(t)$ is h_M, the terms of $\int_{t-h_M}^{t-h(t)} \int_s^t x(u) du\, ds$ and $\int_{t-h_M}^{t-h(t)} \int_s^t \dot{x}(u) du\, ds$ are zero. Thus, inequality (27) still holds.

By the use of Lemma 2 to (27) with $\Upsilon_2 \ge 0$ defined in (13), the following inequality can be obtained:

$$
\begin{aligned}
\dot{V}_5(t) &\le \left(\frac{h_M^2}{2} \right)^2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&\quad - \begin{bmatrix} \int_{t-h(t)}^{t} \int_s^t x(u)\, du\, ds \\ h(t) x(t) - \int_{t-h(t)}^{t} x(s) ds \\ \int_{t-h_M}^{t-h(t)} \int_s^t x(u)\, du\, ds \\ (h(t) - h_M) x(t) - \int_{t-h_M}^{t-h(t)} x(s) ds \end{bmatrix}^T \Upsilon_2 \\
&\quad \times \begin{bmatrix} \int_{t-h(t)}^{t} \int_s^t x(u)\, du\, ds \\ h(t) x(t) - \int_{t-h(t)}^{t} x(s)\, ds \\ \int_{t-h_M}^{t-h(t)} \int_s^t x(u)\, du\, ds \\ (h(t) - h_M) x(t) - \int_{t-h_M}^{t-h(t)} x(s)\, ds \end{bmatrix} \\
&\quad + 2h_M \int_{t-h(t)}^{t} \int_s^t x^T(u) P_3 \dot{x}(u)\, du\, ds \\
&\quad + 2h_M \int_{t-h_M}^{t-h(t)} \int_s^t x^T(u) P_4 \dot{x}(s)\, du\, ds \\
&= \zeta^T(t) \left(\Xi_5 - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{[2h(t)]} \right) \zeta(t)
\end{aligned}
$$

$$+ 2h_M \int_{t-h(t)}^{t} \int_{s}^{t} x^T(u) P_3 \dot{x}(u)\, du\, ds + 2h_M \int_{t-h_M}^{t-h(t)} \int_{s}^{t} x^T(u) P_4 \dot{x}(s)\, du\, ds. \tag{28}$$

The time-derivative of $V_6(t)$ is obtained as follows:

$$\dot{V}_6(t) = \left(\frac{h_M^3}{6}\right)^2 \dot{x}^T(t) Q_3 \dot{x}(t) - \left(\frac{h_M^3}{6}\right) \times \int_{t-h_M}^{t} \int_{s}^{t} \int_{u}^{t} \dot{x}^T(v) Q_3 \dot{x}(v)\, dv\, du\, ds. \tag{29}$$

Using Lemma 1 to (29) yields a new upper-bound of $\dot{V}_6(t)$ as

$$\begin{aligned} \dot{V}_6(t) &\le \left(\frac{h_M^3}{6}\right)^2 \dot{x}^T(t) Q_3 \dot{x}(t) - \left(\frac{h_M^2}{2} x(t) - \int_{t-h_M}^{t} \int_{s}^{t} x(u)\, du\, ds\right)^T \\ &\quad \times Q_3 \left(\frac{h_M^2}{2} x(t) - \int_{t-h_M}^{t} \int_{s}^{t} x(u)\, du\, ds\right) \\ &= \zeta^T(t) \Xi_6 \zeta(t). \end{aligned} \tag{30}$$

By summation of (17)–(30), an upper-bound of $\dot{V}(t)$ is obtained as follows:

$$\dot{V}(t) \le \zeta^T(t) \left(\Xi_{[h(t),\dot{h}(t)]} - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{2[h(t)]}\right) \zeta(t). \tag{31}$$

From the dynamic equation of the system (12), zero equality $0 = \Theta\zeta(t)$ can be obtained. Therefore, from Lemma 3, a stability criterion for the system (12) can be obtained as follows:

$$\Theta^{\perp T} \left(\Xi_{[h(t),\dot{h}(t)]} - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{2[h(t)]}\right) \Theta^{\perp} < 0. \tag{32}$$

By using Lemma 4 with any matrix $U \in \mathbb{R}^{4n\times 9n}$, inequality (32) is equivalent to

$$\begin{aligned} &\Phi_{[h(t),\dot{h}(t)]} \\ &= \begin{bmatrix} \Theta^{\perp T} \Xi_{[h(t),\dot{h}(t)]} \Theta^{\perp} + \Theta^{\perp T} \Gamma_{2[h(t)]}^T \Lambda + \Lambda^T \Gamma_{2[h(t)]} \Theta^{\perp} & \Lambda^T \\ \Lambda & -\Upsilon_2 \end{bmatrix} \\ &= \begin{bmatrix} L_{[h(t),\dot{h}(t)]} & \Lambda^T \\ \Lambda & -\Upsilon_2 \end{bmatrix} < 0, \end{aligned} \tag{33}$$

where $L_{[h(t),\dot{h}(t)]}$ is defined in (13). Note that inequality (33) is affinely dependent on $h(t)$ and $\dot{h}(t)$ where $0 \le h(t) \le h_M$ and $|\dot{h}(t)| \le h_d < 1$. So, $\Phi_{[h(t),\dot{h}(t)]} < 0$ is satisfied if

$$\begin{aligned} &\Phi_{[h(t)=0,\dot{h}(t)=-h_d]} < 0, \\ &\Phi_{[h(t)=0,\dot{h}(t)=h_d]} < 0, \\ &\Phi_{[h(t)=h_M,\dot{h}(t)=-h_d]} < 0, \\ &\Phi_{[h(t)=h_M,\dot{h}(t)=h_d]} < 0. \end{aligned} \tag{34}$$

This completes our proof. □

Remark 6. Compared with the existing works, the main differences of Lyapunov-Krasovskii functionals are $V_2(t)$ and $V_3(t)$. In detail, $V_2(t)$ and $V_3(t)$ have augmented vectors $\int_{t-h(t)}^{t} \dot{x}(s)ds$, respectively. With this consideration, some new cross terms are calculated and utilized in stability condition. Furthermore, the time-varying delayed state $x(t - h(t))$ and double integral terms such as $\int_{t-h(t)}^{t} \int_{s}^{t} x(u)du\, ds$ and $\int_{t-h_M}^{t-h(t)} \int_{s}^{t} x(u)du\, ds$ are utilized as an element of the augmented vectors of $V_1(t)$. In next section, the results for Theorem 5 will show larger delay bounds than the works in other literature to confirm the less conservatism of proposed criterion.

3.2. H_∞ Output Tracking Controller Design. In this subsection, an H_∞ output tracking controller design method for the system (5) will be introduced based on the result of Theorem 5. For simplicity of matrix and vector representation, the scalar q which will be used in representing dimensions of matrices and vectors is defined as $q = n + r$. For example, block entry matrices e_i $(i = 1, \ldots, 10) \in \mathbb{R}^{10q\times q}$ will be used in Theorem 7. The other notations are defined as

$$\begin{aligned} \overline{\zeta}^T(t) = \Big[&\xi^T(t), \xi^T(t - h(t)), \xi^T(t - h_M), \dot{\xi}^T(t), \\ &\dot{\xi}^T(t - h_M), \int_{t-h(t)}^{t} \xi^T(s)\, ds, \\ &\int_{t-h_M}^{t-h(t)} \xi^T(s)\, ds, \int_{t-h(t)}^{t} \int_{s}^{t} \xi^T(u)\, du\, ds, \\ &\int_{t-h_M}^{t-h(t)} \int_{s}^{t} \xi^T(u)\, du\, ds, \dot{\xi}^T(t - h(t)) \Big], \end{aligned}$$

$$\overline{\Theta} = \left(\overline{A} + \overline{B}\,\overline{K}\right) e_1^T + \overline{A}_d e_2^T - e_4^T,$$

$$\overline{\Upsilon}_1 = \begin{bmatrix} \overline{Q}_1 + \begin{bmatrix} \overline{P}_3 & \overline{P}_1 \\ \overline{P}_1 & 0_q \end{bmatrix} & \overline{S}_1 \\ \overline{S}_1^T & \overline{Q}_1 + \begin{bmatrix} \overline{P}_4 & \overline{P}_2 \\ \overline{P}_2 & 0_q \end{bmatrix} \end{bmatrix},$$

$$\overline{\Upsilon}_2 = \begin{bmatrix} \overline{Q}_2 + \left(\frac{4}{h_M}\right) \begin{bmatrix} 0_q & \overline{P}_3 \\ \overline{P}_3 & 0_q \end{bmatrix} & \overline{S}_2 \\ \overline{S}_2^T & \overline{Q}_2 + \left(\frac{4}{h_M}\right) \begin{bmatrix} 0_q & \overline{P}_4 \\ \overline{P}_4 & 0_q \end{bmatrix} \end{bmatrix},$$

$$\overline{\Xi}_{1[h(t),\dot{h}(t)]} = \Psi_{11} \Gamma_{1[h(t),\dot{h}(t)]} \overline{R} \Psi_{12}^T + \Psi_{12} \overline{R} \Gamma_{1[h(t),\dot{h}(t)]}^T \Psi_{11}^T,$$

$$\overline{\Xi}_2 = \Psi_{21} \overline{N} \Psi_{21}^T - \Psi_{22} \overline{N} \Psi_{22}^T + \Psi_{23} \overline{N} \Psi_{24}^T + \Psi_{24} \overline{N} \Psi_{23}^T,$$

$$\begin{aligned} \overline{\Xi}_{3[h(t),\dot{h}(t)]} = {} & \Psi_{31} \overline{G} \Psi_{31}^T - \left(1 - \dot{h}(t)\right) \Psi_{32} \overline{G} \Psi_{32}^T \\ & + \Psi_{33[h(t),\dot{h}(t)]} \overline{G} \Psi_{34}^T + \Psi_{34} \overline{G} \Psi_{33[h(t),\dot{h}(t)]}^T \\ & + \left(1 - \dot{h}(t)\right) \overline{\Psi}_{35} \overline{M} \Psi_{35}^T - \Psi_{36} \overline{M} \Psi_{36}^T \\ & + \Psi_{37[h(t),\dot{h}(t)]} \overline{G} \Psi_{38}^T + \Psi_{38} \overline{G} \Psi_{37[h(t),\dot{h}(t)]}^T, \end{aligned}$$

$$\begin{aligned}
\overline{\Xi}_{4[h(t)]} &= h_M^2 \Psi_{41} \overline{Q}_1 \Psi_{41}^T \Psi_{42} \Upsilon_1 \Psi_{42}^T \\
&\quad + h_M \left(e_1 \overline{P}_1 e_1^T - e_2 \overline{P}_1 e_2^T + e_2 \overline{P} 2 e_2^T - e_3 \overline{P}_2 e_3^T \right) \\
&\quad + h_M \left(h(t)\, e_1 \overline{P}_3 e_1^T - \left(h_M - h(t) \right) e_1 \overline{P}_4 e_1^T \right), \\
\overline{\Xi}_5 &= \left(\frac{h_M^2}{2} \right)^2 \Psi_5 \overline{Q}_2 \Psi_5^T, \\
\overline{\Xi}_6 &= \left(\frac{h_M^3}{6} \right)^2 e_4 \overline{Q}_3 e_4^T - \Psi_6 \overline{Q}_3 \Psi_6^T, \\
\overline{\Xi}_7 &= \alpha \Big(\left(e_1 + \overline{e}_4 \right) \left(\left(\overline{A}_1 X + \overline{B} Y \right) e_1^T + \overline{A}_d e_2^T - e_4^T \right) \\
&\quad + \left(\left(e_1 + \overline{e}_4 \right) \left(\left(\overline{A}_1 X + \overline{B} Y \right) e_1^T + \overline{A}_d e_2^T - e_4^T \right) \right)^T \Big), \\
\overline{\Xi} &= \overline{\Xi}_{1[h(t),\dot{h}(t)]} + \overline{\Xi}_2 + \overline{\Xi}_{3[h(t),\dot{h}(t)]} + \overline{\Xi}_{4[h(t)]} \\
&\quad + \overline{\Xi}_5 + \overline{\Xi}_6 + \overline{\Xi}_7, \\
\overline{L}_{1[h(t),\dot{h}(t)]} &= \overline{\Xi}_{[h(t),\dot{h}(t)]} + \Gamma_{2[h(t)]}^T \overline{\Lambda} + \overline{\Lambda}^T \Gamma_{2[h(t)]}, \\
\overline{L}_2 &= \left[C_1, -H \right] X e_1^T + B_1 Y e_1^T + \left[C_2, 0_{s\times p} \right] X e_2^T, \\
\overline{\Phi}_{[h(t),\dot{h}(t)]} &= \begin{bmatrix} \overline{L}_{1[h(t),\dot{h}(t)]} & \overline{B}_w & \overline{\Lambda}^T & \overline{L}_2^T \\ \star & -\gamma^2 I_{p+r} & 0 & 0 \\ \star & \star & -\overline{\Upsilon}_2 & 0 \\ \star & \star & \star & -I_s \end{bmatrix},
\end{aligned} \tag{35}$$

and other notations are the same ones as (13). Now, the following theorem is given by the second result.

Theorem 7. *For given scalars $0 \le h_M$, $0 \le h_d < 1$, and $\alpha > 0$, the augmented system (5) achieves the H_∞ output tracking performance γ for $0 \le h(t) \le h_M$ and $|\dot{h}(t)| \le h_d$, if there exist positive definite matrices $\overline{R} = [\overline{R}_{ij}]_{7\times 7} \in \mathbb{R}^{7q\times 7q}$, $X = \overline{R}_{11} \in \mathbb{R}^{q\times q}$ $\overline{N} \in \mathbb{R}^{4q\times 4q}$, $\overline{G} \in \mathbb{R}^{4q\times 4q}$, $\overline{Q}_1 \in \mathbb{R}^{2q\times 2q}$, $\overline{Q}_2 \in \mathbb{R}^{2q\times 2q}$, and $\overline{Q}_3 \in \mathbb{R}^{q\times q}$, symmetric matrices $\overline{P}_i \in \mathbb{R}^{q\times q}$ $(i = 1, \ldots, 4)$, and any matrices $\overline{S}_1 \in \mathbb{R}^{2q\times 2q}$, $\overline{S}_2 \in \mathbb{R}^{2q\times 2q}$, $\overline{\Lambda} \in \mathbb{R}^{4q\times 9q}$, and $Y \in \mathbb{R}^{m\times q}$ satisfying the following LMIs:*

$$\begin{aligned}
&\overline{\Phi}_{j,k} < 0, \quad \left(j, k = 1, 2 \right), \\
&\overline{\Upsilon}_i \ge 0, \quad \left(i = 1, 2 \right),
\end{aligned} \tag{36}$$

where $\overline{\Phi}_{j,k}$ are the four vertices of $\overline{\Phi}_{[h(t),\dot{h}(t)]}$ with the bounds of $h(t)$ and $\dot{h}(t)$, that is h_M and h_D when $j = k = 1$, h_M and $-h_D$ when $j < k$, 0 and h_D when $j > k$, and 0 and $-h_D$ when $j = k = 2$. Moreover, if the above conditions are feasible, a desired H_∞ tracking controller gain matrix is obtained by $\overline{K} = \left[K_1, K_2 \right] = YX^{-1}$.

Proof. For positive definite matrices $R \in \mathbb{R}^{7q\times 7q}$, $N \in \mathbb{R}^{4q\times 4q}$, $G \in \mathbb{R}^{4q\times 4q}$, $Q_1 \in \mathbb{R}^{2q\times 2q}$, $Q_2 \in \mathbb{R}^{2q\times 2q}$, and $Q_3 \in \mathbb{R}^{q\times q}$, let us consider the following Lyapunov-Krasovskii functional candidate as

$$\widehat{V}(t) = \sum_{i=1}^{6} \widehat{V}_i(t), \tag{37}$$

where

$$\begin{aligned}
\widehat{V}_1(t) &= \widehat{\Omega}_1^T(t) R \widehat{\Omega}_1(t), \\
\widehat{V}_2(t) &= \int_{t-h_M}^{t} \widehat{\Omega}_2^T(t,s) N \widehat{\Omega}_2(t,s)\, ds, \\
\widehat{V}_3(t) &= \int_{t-h(t)}^{t} \widehat{\Omega}_2^T(t,s) G \widehat{\Omega}_2(t,s)\, ds \\
&\quad + \int_{t-h_M}^{t-h(t)} \widehat{\Omega}_2^T(t,s) M \widehat{\Omega}_2(t,s)\, ds, \\
\widehat{V}_4(t) &= h_M \int_{t-h_M}^{t} \int_{s}^{t} \begin{bmatrix} \xi(u) \\ \dot{\xi}(u) \end{bmatrix}^T Q_1 \begin{bmatrix} \xi(u) \\ \dot{\xi}(u) \end{bmatrix} du\, ds, \\
\widehat{V}_5(t) &= \left(\frac{h_M^2}{2} \right) \int_{t-h_M}^{t} \int_{s}^{t} \int_{u}^{t} \begin{bmatrix} \xi(v) \\ \dot{\xi}(v) \end{bmatrix}^T Q_2 \begin{bmatrix} \xi(v) \\ \dot{\xi}(v) \end{bmatrix} dv\, du\, ds, \\
\widehat{V}_6(t) &= \left(\frac{h_M^3}{6} \right) \int_{t-h_M}^{t} \int_{s}^{t} \int_{u}^{t} \int_{v}^{t} \dot{\xi}^T(\lambda) Q_3 \dot{\xi}(\lambda)\, d\lambda\, dv\, du\, ds,
\end{aligned} \tag{38}$$

with $\widehat{\Omega}_1(t) = [\xi^T(t), \xi^T(t-h(t)), \xi^T(t-h_M), \int_{t-h(t)}^{t} \xi^T(s) ds, \int_{t-h_M}^{t} \xi^T(s) ds, \int_{t-h(t)}^{t} \int_{s}^{t} \xi^T(u) du\, ds, \int_{t-h_M}^{t} \int_{s}^{t} \xi^T(u) du\, ds]^T$, and $\widehat{\Omega}_2(t,s) = [\xi^T(s), \dot{\xi}^T(s), \int_{s}^{t} \dot{\xi}^T(u) du, \int_{s}^{t} \xi^T(u) du]^T$. By using similar method in the proof of Theorem 5, an upper-bound of $\dot{\widehat{V}}(t)$ can be obtained as follows:

$$\dot{\widehat{V}}(t) \le \overline{\zeta}^T(t) \left(\Xi_{[h(t),\dot{h}(t)]} - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{2[h(t)]} \right) \overline{\zeta}(t), \tag{39}$$

with

$$\begin{aligned}
&\Upsilon_1 \ge 0, \\
&\Upsilon_2 \ge 0.
\end{aligned} \tag{40}$$

Since zero equality $\overline{\Theta}\overline{\zeta}(t) + \overline{B}_w v(t) = 0$ can be obtained from the dynamic equation of the augmented system (5), the following zero equality for any matrix $Z \in \mathbb{R}^{q\times q}$ is considered:

$$0 = 2\overline{\zeta}^T(t) \left(e_1^T + e_4^T \right)^T Z \left(\overline{\Theta}\overline{\zeta}(t) + \overline{B}_w v(t) \right). \tag{41}$$

For $R = [R_{ij}]_{7\times 7} \in \mathbb{R}^{7q\times 7q}$ and a given scalar $\alpha > 0$, let us define $Z = \alpha R_{11}$. By adding (41) into (39), a new upper-bound of $\dot{\widehat{V}}(t)$ can be

$$\begin{aligned}
\dot{\widehat{V}}(t) &\le \overline{\zeta}^T(t) \left(\Xi_{[h(t),\dot{h}(t)]} - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{2[h(t)]} \right) \overline{\zeta}(t) \\
&\quad + 2\overline{\zeta}^T(t) \left(e_1^T + e_4^T \right)^T \alpha R_{11} \left(\overline{\Theta}\overline{\zeta}(t) + \overline{B}_w v(t) \right).
\end{aligned} \tag{42}$$

Since $\widehat{V}(t)|_{t=0} = 0$ and $\widehat{V}(t)|_{t=\infty} \geq 0$ are satisfied under the zero initial condition, H_∞ performance index in (7) leads to the following inequality:

$$\begin{aligned} J &= \int_0^\infty \left[e^T(s)\,e(s) - \gamma^2 v^T(s)\,v(s)\right] ds \\ &= \int_0^\infty \left[e^T(s)\,e(s) - \gamma^2 v^T(s)\,v(s) + \dot{\widehat{V}}(s)\right] ds - \int_0^\infty \dot{\widehat{V}}(s)\,ds \\ &= \int_0^\infty \left[e^T(s)\,e(s) - \gamma^2 v^T(s)\,v(s) + \dot{\widehat{V}}(s)\right] ds \\ &\quad - \widehat{V}(t)|_{t=\infty} + \widehat{V}(t)|_{t=0} \\ &\leq \int_0^\infty \left[e^T(s)\,e(s) - \gamma^2 v^T(s)\,v(s) + \dot{\widehat{V}}(s)\right] ds < 0. \end{aligned} \tag{43}$$

If inequality (43) is satisfied, the augmented system (5) achieves the H_∞ output tracking performance γ under the obtained H_∞ tracking controller (4). With $e^T(t)e(t) = \begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \end{bmatrix}^T \begin{bmatrix} \overline{C}_1^T \\ \overline{C}_2^T \end{bmatrix} \begin{bmatrix} \overline{C}_1^T \\ \overline{C}_2^T \end{bmatrix}^T \begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \end{bmatrix}$, (43) is equivalent to the following inequality:

$$\begin{aligned} &\overline{\zeta}^T(t)\left(\Xi_{[h(t),\dot{h}(t)]} - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{2[h(t)]}\right)\overline{\zeta}(t) \\ &\quad + 2\overline{\zeta}^T(t)\left(e_1^T + e_4^T\right)^T \alpha R_{11}\left(\overline{\Theta}\overline{\zeta}(t) + \overline{B}_w v(t)\right) \\ &\quad + \begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \end{bmatrix}^T \begin{bmatrix} \overline{C}_1^T \\ \overline{C}_2^T \end{bmatrix} \begin{bmatrix} \overline{C}_1^T \\ \overline{C}_2^T \end{bmatrix}^T \begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \end{bmatrix} \\ &\quad - \gamma^2 v^T(t)\,v(t) \\ &= \begin{bmatrix} \overline{\zeta}(t) \\ v(t) \end{bmatrix}^T \begin{bmatrix} \Pi_{1[h(t),\dot{h}(t)]} & \Pi_2 \\ \Pi_2^T & \Pi_3 \end{bmatrix} \begin{bmatrix} \overline{\zeta}(t) \\ v(t) \end{bmatrix} < 0, \end{aligned} \tag{44}$$

where

$$\begin{aligned} \Pi_{1[h(t),\dot{h}(t)]} &= \left(\Xi_{[h(t),\dot{h}(t)]} - \Gamma_{2[h(t)]}^T \Upsilon_2 \Gamma_{2[h(t)]}\right) \\ &\quad + \alpha\left(e_1^T + e_4^T\right)^T R_{11}\overline{\Theta} + \alpha\overline{\Theta}^T R_{11}\left(e_1^T + e_4^T\right) \\ &\quad + \left(\overline{C}_1 e_1^T + \overline{C}_2 e_2^T\right)^T \left(\overline{C}_1 e_1^T + \overline{C}_2 e_2^T\right), \\ \Pi_2 &= \alpha\left(e_1^T + e_4^T\right)^T R_{11} B_w, \\ \Pi_3 &= -\gamma^2 I_{p+r}. \end{aligned} \tag{45}$$

Therefore, inequality (44) is equivalent to the following condition:

$$\begin{bmatrix} \Pi_{1[h(t),\dot{h}(t)]} & \Pi_2 \\ \Pi_2^T & \Pi_3 \end{bmatrix} < 0. \tag{46}$$

Let us define $X = R_{11}^{-1} > 0$ and $\Sigma_i = \text{diag}\{\underbrace{X, \ldots, X}_{i}\} \in \mathbb{R}^{iq \times iq}$. For example, $\Sigma_1 = X$ and $\Sigma_2 = \text{diag}\{X, X\}$. And other notations are defined as $\overline{R} = \Sigma_7^T R \Sigma_7$, $\overline{N} = \Sigma_4^T N \Sigma_4$, $\overline{G} = \Sigma_4^T G \Sigma_4$, $\overline{M} = \Sigma_4^T M \Sigma_4$, $\overline{Q}_1 = \Sigma_2^T Q_1 \Sigma_2$, $\overline{Q}_2 = \Sigma_2^T Q_2 \Sigma_2$, $\overline{Q}_3 = \Sigma_1^T Q_3 \Sigma_1$, $\overline{P}_1 = \Sigma_1^T P_1 \Sigma_1$, $\overline{P}_2 = \Sigma_1^T P_1 \Sigma_1$, $\overline{P}_3 = \Sigma_1^T P_3 \Sigma_1$, $\overline{P}_4 = \Sigma_1^T P_4 \Sigma_1$, $\overline{S}_1 = \Sigma_2^T S_1 \Sigma_2$, $\overline{S}_2 = \Sigma_2^T S_1 \Sigma_2$, and $Y = \overline{K}X$. By pre- and post-multiplying (40) and (46), respectively, by Σ_4 and $\text{diag}\{\Sigma_{10}, I_{p+r}\}$, the following inequalities can be obtained:

$$\begin{aligned} &0 \leq \Sigma_4 \Upsilon_1 \Sigma_4 = \overline{\Upsilon}_1, \\ &0 \leq \Sigma_4 \Upsilon_2 \Sigma_4 = \overline{\Upsilon}_2, \end{aligned} \tag{47}$$

$$\begin{aligned} &\begin{bmatrix} \Sigma_{10} & 0_{10q\times(p+r)} \\ 0_{(p+r)\times 10q} & I_{p+r} \end{bmatrix} \begin{bmatrix} \Pi_{1[h(t),\dot{h}(t)]} & \Pi_2 \\ \Pi_2^T & \Pi_3 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \Sigma_{10} & 0_{10q\times(p+r)} \\ 0_{(p+r)\times 10q} & I_{p+r} \end{bmatrix} \\ &= \begin{bmatrix} \overline{\Pi}_{1[h(t),\dot{h}(t)]} & \overline{\Pi}_2 \\ \overline{\Pi}_2^T & \overline{\Pi}_3 \end{bmatrix} < 0, \end{aligned} \tag{48}$$

where

$$\begin{aligned} \overline{\Pi}_{1[h(t),\dot{h}(t)]} &= \left(\overline{\Xi}_{[h(t),\dot{h}(t)]} - \Gamma_{[h(t)]}^T \overline{\Upsilon}_2 \Gamma_{[h(t)]}\right) \\ &\quad + \left(\left[C_1, -H\right] X e_1^T + D_1 Y e_1^T + \left[C_2, 0_{s\times p}\right] X e_2^T\right)^T \\ &\quad \times \left(\left[C_1, -H\right] X e_1^T + D_1 Y e_1^T + \left[C_2, 0_{s\times p}\right] X e_2^T\right) \\ \overline{\Pi}_2 &= \alpha\left(e_1^T + e_4^T\right)^T B_w, \\ \overline{\Pi}_3 &= -\gamma^2 I_{p+r}. \end{aligned} \tag{49}$$

Also, by using Schur complements and Lemma 4 with any matrix $\overline{\Lambda} \in \mathbb{R}^{4q\times(10q+p+r)}$, inequality (47) is equivalent to

$$\overline{\Phi}_{[h(t),\dot{h}(t)]} = \begin{bmatrix} \overline{L}_{1[h(t),\dot{h}(t)]} & \overline{B}_w & \overline{\Lambda}^T & \overline{L}_2^T \\ \star & -\gamma^2 I_{p+r} & 0 & 0 \\ \star & \star & -\overline{\Upsilon}_2 & 0 \\ \star & \star & \star & -I_s \end{bmatrix} < 0, \tag{50}$$

where $\overline{L}_{1[h(t),\dot{h}(t)]}$ and $\overline{L}_2$ are defined in (35). Note that inequality (50) is affinely dependent on $h(t)$ and $\dot{h}(t)$. Therefore, $\overline{\Phi}_{[h(t),\dot{h}(t)]} < 0$ holds if

$$\begin{aligned} &\overline{\Phi}_{[h(t)=0,\dot{h}(t)=-h_d]} < 0, \\ &\overline{\Phi}_{[h(t)=0,\dot{h}(t)=h_d]} < 0, \\ &\overline{\Phi}_{[h(t)=h_M,\dot{h}(t)=-h_d]} < 0, \\ &\overline{\Phi}_{[h(t)=h_M,\dot{h}(t)=h_d]} < 0. \end{aligned} \tag{51}$$

This completes our proof. □

Remark 8. This subsection concludes the criterion of H_∞ output tracking controller design for linear systems with time-varying delays and disturbances. To obtain an H_∞ output tracking controller gain which provides enhanced feasible region, the authors introduce the improved delay-dependent stability criterion as shown in Theorem 5. Then, based on the results of Theorem 5, the designing method of H_∞ output tracking controller gain was introduced in Theorem 7.

Remark 9. Theorems 5 and 7 have some restriction such as $h_d < 1$. In order to remove the restrictive condition $h_d < 1$ in Theorem 5, $V_1(t)$ and $V_3(t)$ of Lyapunov-Krasovskii functional (15) should be changed as follows:

$$V_1(t) = \Omega_1^T(t) R \Omega_1(t),$$

$$V_3(t) = \int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \\ \int_s^t x(u)du \end{bmatrix}^T G \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)\,du \\ \int_s^t x(u)\,du \end{bmatrix} ds + \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \\ \int_s^t x(u)du \end{bmatrix}^T M \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)\,du \\ \int_s^t x(u)\,du \end{bmatrix} ds, \tag{52}$$

where $R \in \mathbb{R}^{6n\times 6n} > 0$, $G \in \mathbb{R}^{3n\times 3n} > 0$, $M \in \mathbb{R}^{3n\times 3n} > 0$, and $\Omega_1(t) = [x^T(t), x^T(t-h_M), \int_{t-h(t)}^t x^T(s)ds, \int_{t-h_M}^t x^T(s)ds, \int_{t-h(t)}^t \int_s^t x^T(u)du\,ds, \int_{t-h_M}^t \int_s^t x^T(u)du\,ds]^T$. Then, by utilizing the following vector,

$$\zeta(t) = \Big[x^T(t), x^T(t-h(t)), x^T(t-h_M), \dot{x}^T(t), \dot{x}^T(t-h_M), \int_{t-h(t)}^t x^T(s)\,ds, \int_{t-h_M}^{t-h(t)} x^T(s)\,ds, \int_{t-h(t)}^t \int_s^t x^T(u)\,du\,ds, \int_{t-h_M}^{t-h(t)} \int_s^t x^T(u)\,du\,ds \Big]^T, \tag{53}$$

as an augmented vector in deriving a stability condition, condition $h_d < 1$ can be removed. Similarly, in Theorem 7, the restrictive condition $h_d < 1$ can be also removed by using the following $\widehat{V}_1(t)$ and $\widehat{V}_3(t)$ of Lyapunov-Krasovskii functional (37) with positive matrices $R \in \mathbb{R}^{6q\times 6q}$, $G \in \mathbb{R}^{3q\times 3q}$, and $M \in \mathbb{R}^{3q\times 3q}$ as

$$\widehat{V}_1(t) = \widehat{\Omega}_1^T(t) R \widehat{\Omega}_1(t),$$

$$\widehat{V}_3(t) = \int_{t-h(t)}^{t} \begin{bmatrix} \zeta(s) \\ \int_s^t \dot{\zeta}(u)du \\ \int_s^t \zeta(u)du \end{bmatrix}^T G \begin{bmatrix} \zeta(s) \\ \int_s^t \dot{\zeta}(u)\,du \\ \int_s^t \zeta(u)\,du \end{bmatrix} ds + \int_{t-h_M}^{t-h(t)} \begin{bmatrix} \zeta(s) \\ \int_s^t \dot{\zeta}(u)du \\ \int_s^t \zeta(u)du \end{bmatrix}^T M \begin{bmatrix} \zeta(s) \\ \int_s^t \dot{\zeta}(u)\,du \\ \int_s^t \zeta(u)\,du \end{bmatrix} ds, \tag{54}$$

where $\widehat{\Omega}_1(t) = [\xi^T(t), \xi^T(t-h_M), \int_{t-h(t)}^t \xi^T(s)ds, \int_{t-h_M}^t \xi^T(s)ds, \int_{t-h(t)}^t \int_s^t \xi^T(u)du\,ds, \int_{t-h_M}^t \int_s^t \xi^T(u)du\,ds]$.

Table 1: The maximum h_M in Example 1.

h_d	0	0.05	0.1	0.5
Wu et al. [11]	4.47	3.98	3.60	2.0
He et al. [12]	4.47	3.98	3.60	2.4
Park and Ko [13]	4.47	4.01	3.66	2.33
Qian and Karimi [14]	4.60	4.12	3.70	2.33
Park et al. [15]	4.66	4.17	3.76	2.12
Kim [16]	4.97	4.35	3.86	2.33
Theorem 5	5.2468	4.6423	4.2320	2.8768

4. Numerical Examples

In this section, three numerical examples demonstrate the effectiveness of the proposed criteria.

Example 1. Consider the system (12) with following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{55}$$

The results for maximum upper-bound of $h(t)$ with different h_d are listed in Table 1 which conducts the comparison of the obtained results by Theorem 5 with the previous results. Table 1 shows that when h_d is bigger, the maximum upper-bound of $h(t)$ is smaller. It means that h_d affects the stability region of the system. Moreover, from the results in Table 1, it can be confirmed that Theorem 5 gives less conservative results than the existing results in [11–16].

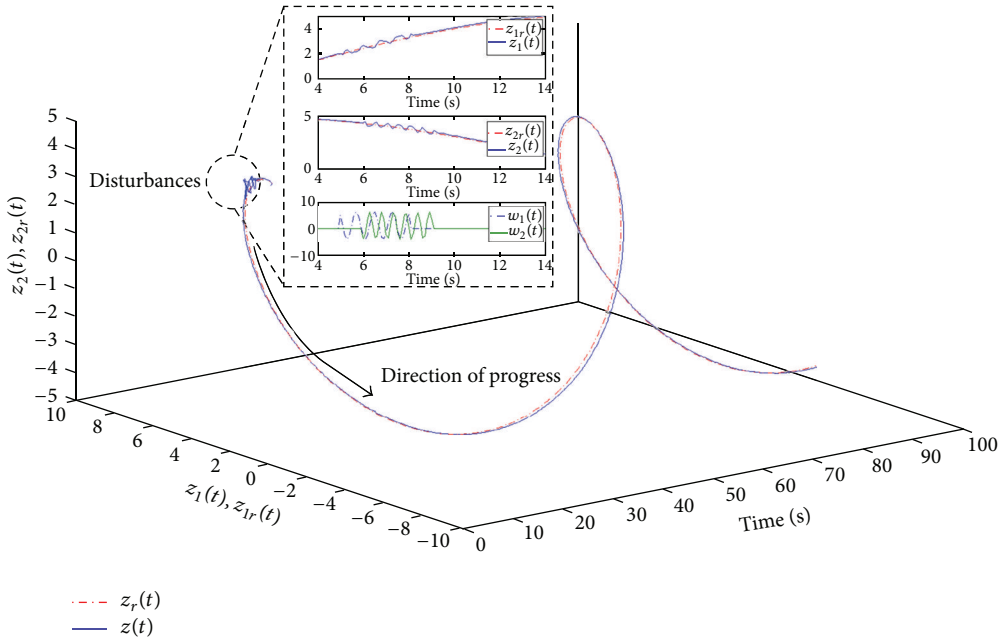

FIGURE 1: Trajectories of outputs $z(t)$ and $z_r(t)$ for Example 2.

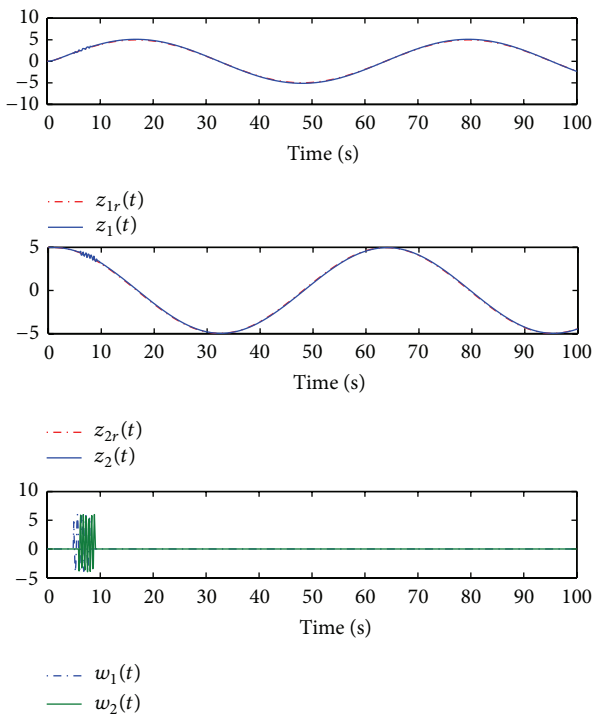

FIGURE 2: Trajectories of $z(t)$, $z_r(t)$, and $w(t)$ in 2D plot for Example 2.

Example 2. Consider the system (5) with the following parameters:

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & -0.3 \\ 0.2 & -1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_w = 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{56}$$

Also, the reference model is given by

$$\dot{x}_r(t) = -x_r(t) + r(t),$$
$$z_r(t) = 0.5 x_r(t). \tag{57}$$

Disturbances are assumed to be

$$w_1(t) = \begin{cases} \sin(8t), & 5 \le t \le 8, \\ 0, & \text{otherwise}, \end{cases}$$
$$w_2(t) = \begin{cases} \cos(12t), & 6 \le t \le 9, \\ 0, & \text{otherwise}. \end{cases} \tag{58}$$

Reference signal $r^T(t) = [r_1^T(t), r_2^T(t)]$ is defined as

$$r_1(t) = \sin(0.1t), \qquad r_2(t) = \cos(0.1t). \tag{59}$$

In this example, Theorem 7 is used to obtain feedback controller gain K for the system (5) with (56). By applying Theorem 7, the feedback controller gain,

$$K = \begin{bmatrix} -1.8931 & -0.0057 & 0.4983 & 0 \\ 0.0455 & -1.9317 & -0.0003 & 0.5000 \end{bmatrix}, \tag{60}$$

can be obtained with minimum $\gamma = 0.1068$ when $h_M = 0.1$, $h_d = 0.5$, and the tuning parameter $\alpha = 1$. Figure 1 illustrates the trajectories of outputs of the system (5) with disturbances (58) and time-delay $h(t) = 0.5h_M(\cos(2h_d t/h_M) + 1)$ in

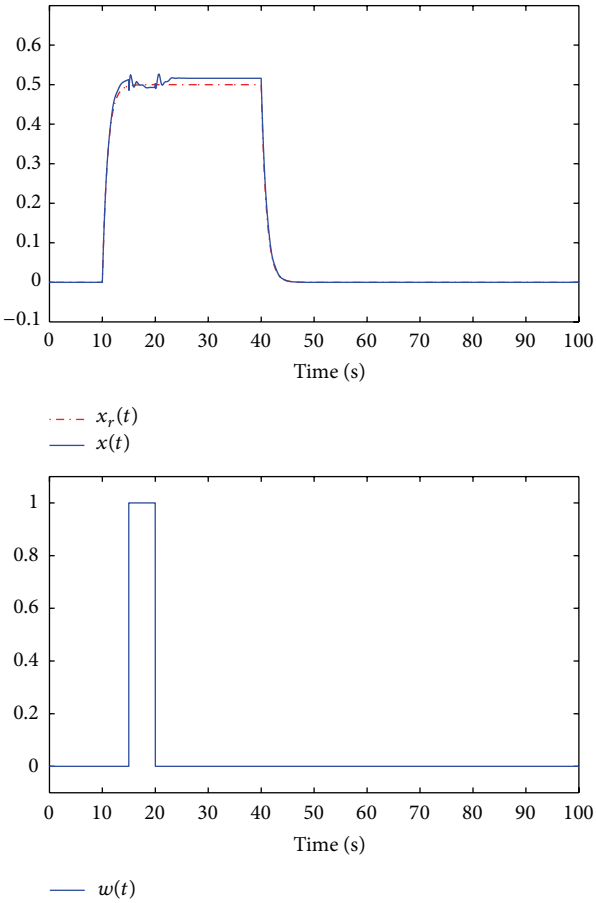

FIGURE 3: Outputs $z(t)$ and $z_r(t)$ with disturbances $w(t)$ for Example 3.

the 3D plot. The simulation result in Figure 1 shows that the $z(t)$ follows the $z_r(t)$ under the obtained H_∞ output tracking controller. Also, to confirm the results in detail, Figure 2 is added to show the trajectories of $z(t)$ and $z_r(t)$ with disturbances $w(t)$ in 2D plot.

Example 3. Let us consider the system (5) as follows:

$$A = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & -0.1 \\ -0.3 & -1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{61}$$
$$C_1 = C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = 1.$$

Moreover, the reference model is given by

$$\dot{x}_r(t) = -x_r(t) + r(t),$$
$$z_r(t) = 0.5x_r(t). \tag{62}$$

Disturbances are assumed to be

$$w(t) = \begin{cases} 1, & 15 \le t \le 20, \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

The reference signal $r(t)$ is as follows:

$$r(t) = \begin{cases} 1, & 10 \le t \le 40, \\ 0, & \text{otherwise.} \end{cases} \tag{64}$$

And assume that $x(0) = [0,0]^T$ and $x_r(0) = 0$. By applying Theorem 7, the feedback controller gain,

$$K = \begin{bmatrix} -1.9454 & -2.0123 & 0.4929 \end{bmatrix}, \tag{65}$$

can be obtained with minimum $\gamma = 0.1020$ when $\alpha = 1$, $h_M = 0.1$, and $h_d = 0.2$. Figure 3 shows the outputs of the systems when the time-delay is $h(t) = 0.05(\sin(4t) + 1)$ which satisfied with $0 \le h(t) \le 0.1$ and $|\dot{h}(t)| < 0.2$. By calculation, $\|e(t)\|_2 = 0.0794$ and $\|v(t)\|_2 = 5.9160$, which yields $\|e(t)\|_2/\|v(t)\|_2 = 0.0134 < \gamma$. This results show the effectiveness of the proposed H_∞ output tacking controller design.

5. Conclusions

This paper has investigated the problem of stability analysis and H_∞ output tracking control for linear systems with time-varying delays and disturbances. Main results were made up of two subsections which were stability analysis criterion and H_∞ output tracking controller design criterion. First, in Theorem 5, the stability criterion for the nominal systems with time-varying delays was derived by utilizing the newly augmented Lyapunov-Krasovskii functional and by using reciprocally convex approach. Second, Theorem 5 was extended to the problem of H_∞ output tracking controller design for the augmented system by using the H_∞ sense in Theorem 7. Finally, three examples were given to illustrate the effectiveness of the presented criteria. Based on these results, future research will focus on H_∞ output tracking control for various systems such as Markovian jump systems [26, 27], fault diagnosis problem [28], and discrete-time descriptor systems [29].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Less Conservative Stability Criteria for Neural Networks with Time-Varying Delays Utilizing Wirtinger-Based Integral Inequality

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This paper investigates the problem of stability analysis for neural networks with time-varying delays. By utilizing the Wirtinger-based integral inequality and constructing a suitable augmented Lyapunov-Krasovskii functional, two less conservative delay-dependent criteria to guarantee the asymptotic stability of the concerned networks are derived in terms of linear matrix inequalities (LMIs). Three numerical examples are included to explain the superiority of the proposed methods by comparing maximum delay bounds with the recent results published in other literature.

1. Introduction

Since neural networks are generally recognized as one of the simplified models of neural processing in the human brain and can provide their good performance and strong capability of information processing, they have been successfully applied in many fields such as image and signal processing, pattern recognition, fixed-point computations, optimization, feedback control, medical diagnosis, financial applications, and other scientific areas [1–3]. Due to the finite switching speed of amplifiers and the inherent communication time between the neurons, it is well known that time delays exist and may cause oscillation or deteriorate system performance. Therefore, during the last few decades, many researchers [4–9] put their times and efforts into stability analysis of neural networks with time delays because it is a prerequisite and an important job to check whether the equilibrium point of the concerned networks is stable or not due to the fact that the application of these networks is heavily dependent on the dynamic behavior of the equilibrium points. Methodologically, the delay-derivative dependent [10], the weighting-delay based [11], delay-slope dependent [12], and delay-partitioning [13] analyses were taken by the use of the information for time-delay. Also, Faydasicok and Arik [14, 15] addressed the stability of neural networks with multiple time delays. The asymptotic stability analysis was dealt with in the aforesaid works, while, in [16–19], the exponential stability analysis was studied. In addition to this, the stochastic perturbation condition was considered in [20]. Furthermore, more attentions have been received in delay-dependent stability analysis since time delays encountered in neural networks are usually not very big [21].

One of the hot issues in delay-dependent stability analysis for dynamic systems such as linear systems, neutral systems, neural networks, and so on is to reduce the conservatism of stability criteria or to enhance the feasible region of the concerned criteria. To check the enhancement of the feasible region of stability criteria, maximum delay bounds for guaranteeing the asymptotic stability of the concerned systems are compared with the other methods. The Jensen inequality [22],

Park's inequality [23], model transformation [24, 25], free-weighting matrices techniques [26, 27], convex combination technique [28], and reciprocally convex optimization [29] are well recognized and have been utilized in many fields as tools of reducing the conservatism of stability and stabilization criteria. Very recently, Seuret and Gouaisbaut [30] proposed the Wirtinger-based integral inequality and the advantages of the proposed integral inequality were shown via the comparison of maximum delay bounds for various systems such as systems with constant and known delay, systems with a time-varying delay, and sampled-data systems.

Another remarkable approach to reduce the conservatism of stability criteria is the delay-partitioning idea which was firstly proposed by Gu [22]. The advantage of this method provides larger delay bounds when the delay-partitioning number increases. The idea of the work [22] has been utilized in stability analysis of neural networks with time delays by many researchers [4, 6, 7, 9–11, 13, 21]. In [9], with the recent techniques such as free-weighting matrices techniques and reciprocally convex optimization, a delay-partitioning approach was presented by considering the time-varying delay at each subinterval. Zhang et al. [11] proposed a new delay-partitioning stability analysis by introducing a tuning parameter adjusting delay interval. Zhang et al. [21] proposed new Lyapunov-Krasovskii functionals and delay-partitioning method to investigate the problem of delay-dependent stability analysis for neural networks with two additive time-varying delays and some discussions about the recent works were described. However, when the delay-partitioning number increases, the computational burden and time-consumption increase while the increasing rate of maximum delay bounds is decreased.

In addition to the techniques mentioned above, the choice of Lyapunov-Krasovskii functional and augmented state vectors also play important roles to enhance the feasible region of stability criteria. Since the introduction of the triple integral Lyapunov-Krasovskii functional [32, 33], some new results in stability analysis of neural networks with time-varying delays have been presented in [4, 19]. Furthermore, in the authors' previous works [5, 8] with the addition of zero equalities [34], it was shown that larger delay bounds can be obtained by constructing newly augmented Lyapunov-Krasovskii functional and introducing some new methods in the activation function condition. However, the application of the Wirtinger-based integral inequality to the terms obtained by calculating the augmented Lyapunov-Krasovskii functional has not been fully investigated yet, and thus there is room for further improvements on the reduction of conservatism.

With motivation mentioned above, in this paper, two improved delay-dependent stability criteria for neural networks with time-varying delays will be proposed. First, in Theorem 6, by constructing a suitable augmented Lyapunov-Krasovskii functional, an improved stability condition such that the considered neural networks are asymptotically stable is derived in terms of linear matrix inequalities (LMIs) by applying Wirtinger-based integral inequality to the augmented quadratic integral term and utilizing the zero equalities [34]. Second, based on the results of Theorem 6 and motivated by the works [35–37], a further improved stability criterion will be proposed in Theorem 9 by ensuring the positiveness of the Lyapunov-Krasovskii functional and utilizing the Wirtinger-based integral inequality. Through three numerical examples utilized in many previous works to check the conservatism of stability criteria, it will be shown that the proposed stability criteria can provide larger delay bounds than the recent existing results. By extension, the developed methods can be applied into the networked control [38, 39], the filtering problem [40, 41], the uncertain systems of neutral type [42], and so on.

Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ means the set of all $m \times n$ real matrices. For symmetric matrices X and Y, $X > Y$ (resp., $X \geq Y$) means that the matrix $X - Y$ is positive definite (resp., nonnegative). $X^\perp$ denotes a basis for the null-space of X. I_n, 0_n, and $0_{m\cdot n}$ denote $n \times n$ identity matrix and $n \times n$ and $m \times n$ zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. diag$\{\cdots\}$ denotes the block diagonal matrix. For square matrix S, Sym$\{S\}$ means the sum of S and its symmetric matrix S^T; that is, Sym$\{S\} = S+S^T$. $\star$ represents the elements below the main diagonal of a symmetric matrix. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$.

2. Problem Statement and Preliminaries

Consider the following neural networks with discrete time-varying delays:

$$\dot{y}(t) = -Ay(t) + W_0 g(y(t)) + W_1 g(y(t - h(t))) + b, \quad (1)$$

where $y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $g(y(t)) = [g_1(y_1(t)), \ldots, g_n(y_n(t))]^T \in \mathbb{R}^n$ denotes the neuron activation function, $g(y(t - h(t))) = [g_1(y_1(t - h(t))), \ldots, g_n(y_n(t - h(t)))]^T \in \mathbb{R}^n$, $A = \text{diag}\{a_i\} \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $W_0 = (w_{ij}^0)_{n\times n} \in \mathbb{R}^{n\times n}$ and $W_1 = (w_{ij}^1)_{n\times n} \in \mathbb{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, and $b = [b_1, b_2, \ldots, b_n]^T \in \mathbb{R}^n$ represents a constant input vector.

The delay, $h(t)$, is a time-varying continuous function satisfying

$$0 \leq h(t) \leq h_U, \quad \dot{h}(t) \leq h_D, \quad (2)$$

where h_U is a known positive scalar and h_D is any constant one.

The neuron activation functions satisfy the following assumption.

Assumption 1. The neuron activation functions $g_i(\cdot)$, $i = 1, \ldots, n$, with $g_i(0) = 0$ are continuous, bounded, and they satisfy

$$k_i^- \leq \frac{g_i(u) - g_i(v)}{u - v} \leq k_i^+, \quad u, v \in \mathbb{R}, \quad u \neq v, \quad i = 1, \ldots, n, \quad (3)$$

where k_i^+ and k_i^- are constants.

Remark 2. In Assumption 1, k_i^+ and k_i^- can be allowed to be positive, negative, or zero. As mentioned in [12], Assumption 1 describes the class of globally Lipschitz continuous and monotone nondecreasing activation when $k_i^- = 0$ and $k_i^+ > 0$. And the class of globally Lipschitz continuous and monotone increasing activation functions can be described when $k_i^+ > k_i^- > 0$.

For simplicity, in stability analysis of the neural networks (1), the equilibrium point $y^* = [y_1^*, \ldots, y_n^*]^T$ whose uniqueness has been reported in [17] is shifted to the origin by utilizing the transformation $x(\cdot) = y(\cdot) - y^*$, which leads the system (1) to the following form:

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 f(x(t - h(t))), \quad (4)$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed system, $f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T$, and $f_j(x_j(t)) = g_j(x_j(t) + y_j^*) - g_j(y_j^*)$ with $f_j(0) = 0$ $(j = 1, \ldots, n)$.

It should be noted that the activation functions $f_i(\cdot)$ $(i = 1, \ldots, n)$ satisfy the following condition [16]:

$$k_i^- \le \frac{f_i(u) - f_i(v)}{u - v} \le k_i^+, \quad u, v \in \mathbb{R}, \quad u \ne v, \quad i = 1, \ldots, n; \quad (5)$$

$$k_i^- \le \frac{f_i(u)}{u} \le k_i^+, \quad \forall u \ne 0, \ i = 1, \ldots, n. \quad (6)$$

The objective of this paper is to investigate the delay-dependent stability analysis of system (4) which will be introduced in Section 3.

Before deriving our main results, the following lemmas will be utilized in deriving the main results.

Lemma 3 (see [30]). *Consider a given matrix $R > 0$. Then, for all continuous function ω in $[a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\begin{aligned} &\int_a^b \omega^T(s) R \omega(s)\, ds \\ &\ge \frac{1}{b-a}\left(\int_a^b \omega(s)\, ds\right)^T \left(\int_a^b \omega(s)\, ds\right) \\ &\quad + \frac{3}{b-a}\left(\int_a^b \omega(s)\, ds - \frac{2}{b-a}\int_a^b \int_a^s \omega(u)\, du\, ds\right)^T \\ &\quad \times R\left(\int_a^b \omega(s)\, ds - \frac{2}{b-a}\int_a^b \int_a^s \omega(u)\, du\, ds\right). \end{aligned} \quad (7)$$

Lemma 4 (see [43]). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that rank$(B) < n$. Then, the following statements are equivalent:*

(1) $\zeta^T \Phi \zeta < 0$, $B\zeta = 0$, $\zeta \ne 0$,

(2) $(B^\perp)^T \Phi B^\perp < 0$, *where $B^\perp$ is a right orthogonal complement of B.*

Lemma 5 (see [44]). *For the symmetric appropriately dimensional matrices $\Omega > 0$, Ξ, matrix Λ, the following two statements are equivalent:*

(1) $\Xi - \Lambda^T \Omega \Lambda < 0$,

(2) *there exists a matrix of appropriate dimension Ψ such that*

$$\begin{bmatrix} \Xi + \Lambda^T \Psi + \Psi^T \Lambda & \Psi^T \\ \Psi & -\Omega \end{bmatrix} < 0. \quad (8)$$

3. Main Results

In this section, two new stability criteria for system (4) will be proposed. For the sake of simplicity of matrix and vector representation, e_i $(i = 1, 2, \ldots, 14) \in \mathbb{R}^{14n\times n}$ which will be used in Theorems 6 and 9 are defined as block entry matrices (e.g., $e_3 = [0_{n\cdot 2n}, I_n, 0_{n\cdot 11n}]^T$). The other notations for some vectors and matrices are defined as follows:

$$\begin{aligned} &\zeta(t) \\ &= \Bigg[x^T(t), x^T(t - h(t)), x^T(t - h_U), \dot{x}^T(t), \dot{x}^T(t - h_U), \\ &\quad \frac{1}{h(t)}\int_{t-h(t)}^{t} x^T(s)\, ds, \frac{1}{h_U - h(t)}\int_{t-h_U}^{t-h(t)} x^T(s)\, ds, \\ &\quad f^T(x(t)), f^T(x(t - h(t))), f^T(x(t - h_U)), \\ &\quad \int_{t-h(t)}^{t} f^T(x(s))\, ds, \int_{t-h_U}^{t-h(t)} f^T(x(s))\, ds, \\ &\quad \frac{1}{h(t)}\int_{t-h(t)}^{t}\int_{t-h(t)}^{s} x(u)\, du\, ds, \\ &\quad \frac{1}{h_U - h(t)}\int_{t-h_U}^{t-h(t)}\int_{t-h_U}^{s} x^T(u)\, du\, ds\Bigg]^T, \end{aligned}$$

$$\Gamma = [-A, 0_{n\cdot 2n}, -I_n, 0_{n\cdot 3n}, W_0, W_1, 0_{n\cdot 5n}],$$

$$\begin{aligned} \Xi_{1[h(t)]} = \mathrm{Sym}\{&[e_1, e_3, h(t)e_6 + (h - h(t))e_7, e_{11} + e_{12}] \\ &\times \mathscr{R}[e_4, e_5, e_1 - e_3, e_8 - e_{10}]^T\}, \end{aligned}$$

$$\begin{aligned} \Xi_2 = {} & [e_1, e_4, e_8]\mathscr{N}[e_1, e_4, e_8]^T - [e_3, e_5, e_{10}]\mathscr{N}[e_3, e_5, e_{10}]^T \\ & + \mathrm{Sym}\{[e_8 - K_m e_1]\Lambda_1 e_4^T\} + \mathrm{Sym}\{[K_p e_1 - e_8]\Delta_1 e_4^T\} \\ & + \mathrm{Sym}\{[e_{10} - K_m e_3]\Lambda_2 e_5^T\} + \mathrm{Sym}\{[K_p e_3 - e_{10}]\Delta_2 e_5^T\}, \end{aligned}$$

$$\begin{aligned} \Xi_3 = {} & [e_1, e_8, 0_{14n\cdot n}]\mathscr{G}[e_1, e_8, 0_{14n\cdot n}] \\ & - (1 - h_D)[e_2, e_9, e_1 - e_2]\mathscr{G}[e_2, e_9, e_1 - e_2]^T \\ & + \mathrm{Sym}\{e_{11} G_{23} e_4^T\}, \end{aligned}$$

$$\Xi_4 = h_U^2 \left[e_4, e_1\right] \mathcal{Q}_1 \left[e_4, e_1\right]^T,$$

$$\Xi_5 = h_U^2 e_8 Q_2 e_8^T - \left[e_{11}, e_{12}\right] \begin{bmatrix} Q_2 & S_2 \\ S_2^T & Q_2 \end{bmatrix} \left[e_{11}, e_{12}\right]^T,$$

$$\Xi_6 = h_U \left(e_1 P_1 e_1^T - e_2 P_1 e_2^T + e_2 P_2 e_2^T - e_3 P_2 e_3^T\right),$$

$$\Upsilon_{[h(t)]} = \mathrm{Sym}\left\{e_6 \left(h\left(t\right) G_{13}\right) e_4^T + \left[e_1 - e_6\right] \left(h\left(t\right) G_{33}\right) e_4^T\right\},$$

$$\Pi_{[h(t)]} = \begin{bmatrix} e_1^T - e_2^T \\ h\left(t\right) e_6^T \\ e_1^T + e_2^T - 2e_6^T \\ h\left(t\right) e_6^T - 2e_{13}^T \\ e_2^T - e_3^T \\ \left(h_U - h\left(t\right)\right) e_7^T \\ e_2^T + e_3^T - 2e_7^T \\ \left(h_U - h\left(t\right)\right) e_7^T - 2e_{14}^T \end{bmatrix},$$

$$\begin{aligned} \Theta_1 = &- \mathrm{Sym}\left\{\left[e_8 - e_1 K_m\right] H_1 \left[e_8 - e_1 K_p\right]^T\right\} \\ &- \mathrm{Sym}\left\{\left[e_9 - e_2 K_m\right] H_2 \left[e_9 - e_2 K_p\right]^T\right\} \\ &- \mathrm{Sym}\left\{\left[e_{10} - e_3 K_m\right] H_3 \left[e_{10} - e_3 K_p\right]^T\right\}, \end{aligned}$$

$$\begin{aligned} \Theta_2 = &- \mathrm{Sym}\left\{\left[e_8 - e_9 - \left(e_1 - e_2\right) K_m\right]\right. \\ &\left. \times H_4 \left[e_8 - e_9 - \left(e_1 - e_2\right) K_p\right]^T\right\} \\ &- \mathrm{Sym}\left\{\left[e_9 - e_{10} - \left(e_2 - e_3\right) K_m\right]\right. \\ &\left. \times H_5 \left[e_9 - e_{10} - \left(e_2 - e_3\right) K_p\right]^T\right\}, \end{aligned}$$

$$\mathcal{Q}_{\mathrm{Aug1}} = \mathcal{Q}_1 + \begin{bmatrix} 0_n & P_1 \\ P_1 & 0_n \end{bmatrix}, \qquad \mathcal{Q}_{\mathrm{Aug2}} = \mathcal{Q}_2 + \begin{bmatrix} 0_n & P_2 \\ P_2 & 0_n \end{bmatrix},$$

$$\Omega = \left[\begin{array}{c|c} \begin{bmatrix} \mathcal{Q}_{\mathrm{Aug1}} & 0_{2n} \\ 0_{2n} & 3\mathcal{Q}_{\mathrm{Aug1}} \end{bmatrix} & \mathcal{S}_1 \\ \hline \mathcal{S}_1^T & \begin{bmatrix} \mathcal{Q}_{\mathrm{Aug2}} & 0_{2n} \\ 0_{2n} & 3\mathcal{Q}_{\mathrm{Aug2}} \end{bmatrix} \end{array}\right],$$

$$\begin{aligned} \Sigma_{[h(t)]} = &\ \Xi_{1[h(t)]} + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 \\ &+ \Xi_6 + \Upsilon_{[h(t)]} + \Theta_1 + \Theta_2, \end{aligned}$$

$$\Phi_{[h(t)]} = \left(\Gamma^{\perp}\right)^T \Sigma_{[h(t)]} \left(\Gamma^{\perp}\right) + \mathrm{Sym}\left\{\left(\Gamma^{\perp}\right)^T \Pi_{[h(t)]}^T \Psi\right\}. \tag{9}$$

Now, the following theorem is given by the main result.

Theorem 6. *For given scalars $h_U > 0$, h_D, and diagonal matrices $K_p = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$ and $K_m = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$, system (4) is asymptotically stable for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$, if there exist positive diagonal matrices $\Lambda_i = \mathrm{diag}\{\lambda_{1i}, \ldots, \lambda_{ni}\}$ $(i = 1, 2)$, $\Delta_i = \mathrm{diag}\{\delta_{1i}, \ldots, \delta_{ni}\}$ $(i = 1, 2)$, and $H_i = \mathrm{diag}\{h_{1i}, \ldots, h_{ni}\}$ $(i = 1, \ldots, 5)$, positive definite matrices $\mathcal{R} \in \mathbb{R}^{4n\times 4n}$, $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} = [G_{ij}] \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_1 \in \mathbb{R}^{2n\times 2n}$, and $Q_2 \in \mathbb{R}^{n\times n}$, any matrices $\mathcal{S}_1 \in \mathbb{R}^{4n\times 4n}$, $S_2 \in \mathbb{R}^{n\times n}$, and $\Psi \in \mathbb{R}^{8n\times 13n}$, and any symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$ satisfying the following LMIs:*

$$\begin{bmatrix} \Phi_{[h(t)=0]} & \Psi^T \\ \Psi & -\Omega \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \Phi_{[h(t)=h_U]} & \Psi^T \\ \Psi & -\Omega \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} Q_2 & S_2 \\ S_2^T & Q_2 \end{bmatrix} > 0, \tag{12}$$

where $\Phi_{[h(t)]}$, Ψ, Ω, and other notations are defined in (9).

Proof. Let us consider the following candidate for the appropriate Lyapunov-Krasovskii functional:

$$V\left(t\right) = \sum_{i=1}^{5} V_i\left(t\right), \tag{13}$$

where

$$V_1\left(t\right) = \begin{bmatrix} x\left(t\right) \\ x\left(t - h_U\right) \\ \int_{t-h_U}^{t} x\left(s\right) ds \\ \int_{t-h_U}^{t} f\left(x\left(s\right)\right) ds \end{bmatrix}^T \mathcal{R} \begin{bmatrix} x\left(t\right) \\ x\left(t - h_U\right) \\ \int_{t-h_U}^{t} x\left(s\right) ds \\ \int_{t-h_U}^{t} f\left(x\left(s\right)\right) ds \end{bmatrix},$$

$$V_2\left(t\right) = \int_{t-h_U}^{t} \begin{bmatrix} x\left(s\right) \\ \dot{x}\left(s\right) \\ f\left(x\left(s\right)\right) \end{bmatrix}^T \mathcal{N} \begin{bmatrix} x\left(s\right) \\ \dot{x}\left(s\right) \\ f\left(x\left(s\right)\right) \end{bmatrix} ds$$

$$\begin{aligned} &+ 2\sum_{i=1}^{n} \left(\lambda_{i1} \int_0^{x_i(t)} \left(f_i\left(s\right) - k_i^- s\right) ds \right. \\ &\qquad \left. + \delta_{i1} \int_0^{x_i(t)} \left(k_i^+ s - f_i\left(s\right)\right) ds\right) \\ &+ 2\sum_{i=1}^{n} \left(\lambda_{i2} \int_0^{x_i(t-h_U)} \left(f_i\left(s\right) - k_i^- s\right) ds \right. \\ &\qquad \left. + \delta_{i2} \int_0^{x_i(t-h_U)} \left(k_i^+ s - f_i\left(s\right)\right) ds\right) \end{aligned}$$

$$V_3\left(t\right) = \int_{t-h(t)}^{t} \begin{bmatrix} x\left(s\right) \\ f\left(x\left(s\right)\right) \\ \int_s^t \dot{x}\left(u\right) du \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x\left(s\right) \\ f\left(x\left(s\right)\right) \\ \int_s^t \dot{x}\left(u\right) du \end{bmatrix},$$

$$V_4\left(t\right) = h_U \int_{t-h_U}^{t} \int_s^t \begin{bmatrix} \dot{x}\left(u\right) \\ x\left(u\right) \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \dot{x}\left(u\right) \\ x\left(u\right) \end{bmatrix} du\, ds,$$

$$V_5\left(t\right) = h_U \int_{t-h_U}^{t} \int_s^t f^T\left(x\left(u\right)\right) Q_2 f\left(x\left(u\right)\right) du\, ds. \tag{14}$$

It should be noted that

$$\begin{aligned}
&\begin{bmatrix} x(t) \\ x(t-h_U) \\ \int_{t-h_U}^{t} x(s)\,ds \\ \int_{t-h_U}^{t} f(x(s))\,ds \end{bmatrix} \\
&= \begin{bmatrix} x(t) \\ x(t-h_U) \\ h(t)\left(\frac{1}{h(t)}\int_{t-h(t)}^{t} x(s)\,ds\right) + (h_U - h(t))\left(\frac{1}{h_U - h(t)}\int_{t-h_U}^{t-h(t)} x(s)\,ds\right) \\ \int_{t-h(t)}^{t} f(x(s))\,ds + \int_{t-h_U}^{t-h(t)} f(x(s))\,ds \end{bmatrix} \\
&= \left[e_1, e_3, h(t)\,e_6 + (h_U - h(t))\,e_7, e_{11} + e_{12}\right]^T \zeta(t), \\
&\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_U) \\ x(t) - x(t-h_U) \\ f(x(t)) - f(x(t-h_U)) \end{bmatrix} = \left[e_4, e_5, e_1 - e_3, e_8 - e_{10}\right]^T \zeta(t).
\end{aligned} \tag{15}$$

From (15), $\dot{V}_1(t)$ can be represented as

$$\dot{V}_1(t) = \zeta^T(t)\,\Xi_{1[h(t)]}\,\zeta(t). \tag{16}$$

The result of the time-derivative of $V_2(t)$ is as follows:

$$\begin{aligned}
\dot{V}_2(t) &= \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(x(t)) \end{bmatrix}^T \mathcal{N} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(x(t)) \end{bmatrix} \\
&\quad - \begin{bmatrix} x(t-h_U) \\ \dot{x}(t-h_U) \\ f(x(t-h_U)) \end{bmatrix}^T \mathcal{N} \begin{bmatrix} x(t-h_U) \\ \dot{x}(t-h_U) \\ f(x(t-h_U)) \end{bmatrix} \\
&\quad + 2\left[f(x(t)) - K_m x(t)\right]^T \Sigma_1 \dot{x}(t) \\
&\quad + 2\left[K_p x(t) - f(x(t))\right]^T \Delta_1 \dot{x}(t) \\
&\quad + 2\left[f(x(t-h_U)) - K_m x(t-h_U)\right]^T \Sigma_2 \dot{x}(t-h_U) \\
&\quad + 2\left[K_p x(t-h_U) - f(x(t-h_U))\right]^T \Delta_2 \dot{x}(t-h_U) \\
&= \zeta^T(t)\,\Xi_2\,\zeta(t).
\end{aligned} \tag{17}$$

By calculating $\dot{V}_3(t)$, it follows that

$$\begin{aligned}
&\dot{V}_3(t) \\
&= \frac{d}{dt}\left(\int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix} ds\right) \\
&= \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}\Bigg|_{s=t} \times \frac{d}{dt}(t) \\
&\quad - \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}\Bigg|_{s=t-h(t)} \times \frac{d}{dt}(t - h(t)) \\
&\quad + \int_{t-h(t)}^{t} \frac{d}{dt}\left(\begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}\right) ds \\
&\le \begin{bmatrix} x(t) \\ f(x(t)) \\ 0_n \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x(t) \\ f(x(t)) \\ 0_n \end{bmatrix} \\
&\quad - (1 - h_D)\begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \\ x(t) - x(t-h(t)) \end{bmatrix}^T \mathcal{G} \begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \\ x(t) - x(t-h(t)) \end{bmatrix} \\
&\quad + \int_{t-h(t)}^{t} 2\begin{bmatrix} x(s) \\ f(x(s)) \\ \int_s^t \dot{x}(u)\,du \end{bmatrix}^T \mathcal{G} \begin{bmatrix} 0_n \\ 0_n \\ \dot{x}(t) \end{bmatrix} ds \\
&= \zeta^T(t)\left\{\left[e_1, e_8, 0_{14n\cdot n}\right]\mathcal{G}\left[e_1, e_8, 0_{14n\cdot n}\right]^T \right. \\
&\qquad \left. - (1 - h_D)\left[e_2, e_9, e_1 - e_2\right]\mathcal{G}\left[e_2, e_9, e_1 - e_2\right]^T\right\}\zeta(t)
\end{aligned}$$

$$
\begin{aligned}
&+2\left(\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right)\left(h(t)\,G_{13}\dot{x}(t)\right)\\
&+2\left(\int_{t-h(t)}^{t}f(x(s))\,ds\right)\left(G_{23}\dot{x}(t)\right)\\
&+2\left(x(t)-\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right)^{T}\left(h(t)\,G_{33}\dot{x}(t)\right)\\
&=\zeta^{T}(t)\left(\Xi_{3}+\Upsilon_{[h(t)]}\right)\zeta(t).
\end{aligned}
\tag{18}
$$

Inspired by the work of [34], the following two zero equalities with any symmetric matrices P_i $(i=1,2)$ are considered:

$$
\begin{aligned}
0&=h_U\left\{x^{T}(t)\,P_1x(t)-x^{T}(t-h(t))\,P_1x(t-h(t))\right.\\
&\qquad\left.-2\int_{t-h(t)}^{t}x^{T}(s)\,P_1\dot{x}(s)\,ds\right\},\\
0&=h_U\left\{x^{T}(t-h(t))P_2x(t-h(t))-x^{T}(t-h_U)P_2x(t-h_U)\right.\\
&\qquad\left.-2\int_{t-h_U}^{t-h(t)}x^{T}(s)\,P_2\dot{x}(s)\,ds\right\}.
\end{aligned}
\tag{19}
$$

Summing the two zero equalities presented at (19) leads to

$$
\begin{aligned}
0&=\zeta^{T}(t)\,\Xi_6\zeta(t)-2h_U\int_{t-h(t)}^{t}x^{T}(s)\,P_1\dot{x}(s)\,ds\\
&\quad-2h_U\int_{t-h_U}^{t-h(t)}x^{T}(s)\,P_2\dot{x}(s)\,ds.
\end{aligned}
\tag{20}
$$

The result of the $\dot{V}_4(t)$ is given by

$$
\begin{aligned}
\dot{V}_4(t)&=h_U^2\begin{bmatrix}\dot{x}(t)\\x(t)\end{bmatrix}^{T}\mathcal{Q}_1\begin{bmatrix}\dot{x}(t)\\x(t)\end{bmatrix}\\
&\quad-h_U\int_{t-h_U}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}^{T}\mathcal{Q}_1\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds.
\end{aligned}
\tag{21}
$$

The integral term $-h_U\int_{t-h_U}^{t}\left[\begin{smallmatrix}\dot{x}(s)\\x(s)\end{smallmatrix}\right]^{T}\mathcal{Q}_1\left[\begin{smallmatrix}\dot{x}(s)\\x(s)\end{smallmatrix}\right]ds$ with the addition of the two integral terms at (20) can be described as

$$
\begin{aligned}
&-h_U\int_{t-h_U}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}^{T}\mathcal{Q}_1\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\\
&\quad-2h_U\int_{t-h(t)}^{t}x^{T}(s)\,P_1\dot{x}(s)\,ds\\
&\quad-2h_U\int_{t-h_U}^{t-h(t)}x^{T}(s)\,P_2\dot{x}(s)\,ds\\
&=-h_U\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}^{T}\underbrace{\left(\mathcal{Q}_1+\begin{bmatrix}0_n & P_1\\P_1 & 0_n\end{bmatrix}\right)}_{\mathcal{Q}_{\mathrm{Aug1}}}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\\
&\quad-h_U\int_{t-h_U}^{t-h(t)}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}^{T}\underbrace{\left(\mathcal{Q}_1+\begin{bmatrix}0_n & P_2\\P_2 & 0_n\end{bmatrix}\right)}_{\mathcal{Q}_{\mathrm{Aug2}}}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds.
\end{aligned}
\tag{22}
$$

Here, by applying Lemma 3, we have

$$
\begin{aligned}
&-h_U\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}^{T}\mathcal{Q}_{\mathrm{Aug1}}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\\
&\le-\frac{h_U}{h(t)}\left(\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\right)^{T}\mathcal{Q}_{\mathrm{Aug1}}\left(\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\right)\\
&\quad-\frac{3h_U}{h(t)}\left(\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\right.\\
&\qquad\left.-\frac{2}{h(t)}\int_{t-h(t)}^{t}\int_{t-h(t)}^{s}\begin{bmatrix}\dot{x}(u)\\x(u)\end{bmatrix}du\,ds\right)^{T}\\
&\quad\times\mathcal{Q}_{\mathrm{Aug1}}\left(\int_{t-h(t)}^{t}\begin{bmatrix}\dot{x}(s)\\x(s)\end{bmatrix}ds\right.\\
&\qquad\left.-\frac{2}{h(t)}\int_{t-h(t)}^{t}\int_{t-h(t)}^{s}\begin{bmatrix}\dot{x}(u)\\x(u)\end{bmatrix}du\,ds\right)\\
&=-\frac{h_U}{h(t)}\left(\begin{bmatrix}x(t)-x(t-h(t))\\h(t)\left(\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right)\end{bmatrix}\right)^{T}\\
&\quad\times\mathcal{Q}_{\mathrm{Aug1}}\left(\begin{bmatrix}x(t)-x(t-h(t))\\h(t)\left(\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right)\end{bmatrix}\right)\\
&\quad-\frac{3h_U}{h(t)}\left(\begin{bmatrix}x(t)+x(t-h(t))\\h(t)\left(\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right)\end{bmatrix}\right.\\
&\qquad\left.-2\begin{bmatrix}\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\\\frac{1}{h(t)}\int_{t-h(t)}^{t}\int_{t-h(t)}^{s}x(u)\,du\,ds\end{bmatrix}\right)^{T}\\
&\quad\times\mathcal{Q}_{\mathrm{Aug1}}\left(\begin{bmatrix}x(t)+x(t-h(t))\\h(t)\left(\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right)\end{bmatrix}\right.\\
&\qquad\left.-2\begin{bmatrix}\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\\\frac{1}{h(t)}\int_{t-h(t)}^{t}\int_{t-h(t)}^{s}x(u)\,du\,ds\end{bmatrix}\right)\\
&=-\frac{h_U}{h(t)}\zeta^{T}(t)\,\$_{1[h(t)]}\zeta(t),
\end{aligned}
\tag{23}
$$

where

$$\begin{aligned}\$_{1[h(t)]} &= \left[e_1 - e_2, h(t)\, e_6, e_1 + e_2 - 2e_6, h(t)\, e_6 - e_{13}\right] \\ &\times \begin{bmatrix} \mathcal{Q}_{\text{Aug1}} & 0_{2n} \\ 0_{2n} & 3\mathcal{Q}_{\text{Aug1}} \end{bmatrix} \\ &\times \left[e_1 - e_2, h(t)\, e_6, e_1 + e_2 - 2e_6, h(t)\, e_6 - e_{13}\right]^T.\end{aligned} \tag{24}$$

With the similar process presented in (23), it can be obtained that

$$\begin{aligned}&- h_U \int_{t-h_U}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T \mathcal{Q}_{\text{Aug2}} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds \\ &\le -\frac{h_U}{h_U - h(t)} \zeta^T(t)\, \$_{2[h(t)]} \zeta(t),\end{aligned} \tag{25}$$

where

$$\begin{aligned}\$_{2[h(t)]} &= \left[e_2 - e_3, (h - h(t))\, e_7, e_2 + e_3 - 2e_7,\right. \\ &\left.(h_U - h(t))\, e_7 - 2e_{14}\right] \begin{bmatrix} \mathcal{Q}_{\text{Aug2}} & 0_{2n} \\ 0_{2n} & 3\mathcal{Q}_{\text{Aug2}} \end{bmatrix} \\ &\times \left[e_2 - e_3, (h - h(t))\, e_7, e_2 + e_3 - 2e_7,\right. \\ &\left.(h_U - h(t))\, e_7 - 2e_{14}\right]^T.\end{aligned} \tag{26}$$

With (23)–(26) and utilizing reciprocally convex optimization [29], if $\Omega > 0$ which is defined in (9), an upper bound of the term (22) can be obtained as

$$\begin{aligned}&- h_U \int_{t-h(t)}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T \mathcal{Q}_{\text{Aug1}} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds \\ &\quad - h_U \int_{t-h_U}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T \mathcal{Q}_{\text{Aug2}} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds \\ &\le -\frac{h_U}{h(t)} \zeta^T(t)\, \$_{1[h(t)]} \zeta(t) \\ &\quad - \frac{h_U}{h_U - h(t)} \zeta^T(t)\, \$_{2[h(t)]} \zeta(t) \\ &\le -\zeta^T(t)\, \Pi_{[h(t)]}^T \Omega \Pi_{[h(t)]} \zeta(t),\end{aligned} \tag{27}$$

where $\Pi_{[h(t)]}$ and Ω are defined in (9).

Thus, from (21) to (27), the following inequality holds:

$$\begin{aligned}&\dot{V}_4(t) - 2h_U \int_{t-h(t)}^{t} x^T(s)\, P_1 \dot{x}(s)\, ds \\ &\quad - 2h_U \int_{t-h_U}^{t-h(t)} x^T(s)\, P_2 \dot{x}(s)\, ds \\ &\le \zeta^T(t) \left(\Xi_4 - \Pi_{[h(t)]}^T \Omega \Pi_{[h(t)]}\right) \zeta(t).\end{aligned} \tag{28}$$

By utilizing Jensen's inequality [22] and reciprocally convex optimization [29], if the inequality (12) holds, then the estimation of $\dot{V}_5(t)$ can be

$$\begin{aligned}&\dot{V}_5(t) \\ &= h_U^2 f^T(x(t))\, Q_2 f(x(t)) \\ &\quad - h_U \int_{t-h(t)}^{t} f^T(x(s))\, Q_2 f(x(s))\, ds \\ &\quad - h_U \int_{t-h_U}^{t-h(t)} f^T(x(s))\, Q_2 f(x(s))\, ds \\ &\le h_U^2 f^T(x(t))\, Q_2 f(x(t)) \\ &\quad - \frac{h_U}{h(t)} \left(\int_{t-h(t)}^{t} f(x(s))\, ds\right)^T Q_2 \left(\int_{t-h(t)}^{t} f(x(s))\, ds\right) \\ &\quad - \frac{h_U}{h_U - h(t)} \left(\int_{t-h_U}^{t-h(t)} f(x(s)) ds\right)^T Q_2 \left(\int_{t-h(t)}^{t} f(x(s))\, ds\right) \\ &\le h_U^2 f^T(x(t))\, Q_2 f(x(t)) \\ &\quad - \begin{bmatrix} \int_{t-h(t)}^{t} f(x(s))\, ds \\ \int_{t-h_U}^{t-h(t)} f(x(s))\, ds \end{bmatrix}^T \begin{bmatrix} Q_2 & S_2 \\ S_2^T & Q_2 \end{bmatrix} \begin{bmatrix} \int_{t-h(t)}^{t} f(x(s))\, ds \\ \int_{t-h_U}^{t-h(t)} f(x(s))\, ds \end{bmatrix} \\ &= \zeta^T(t)\, \Xi_5 \zeta(t).\end{aligned} \tag{29}$$

From (6), for any positive diagonal matrices $H_i = \text{diag}\{h_{1i}, \ldots, h_{ni}\}$ $(i = 1, \ldots, 3)$, the following inequality holds:

$$\begin{aligned}0 \le &-2\sum_{i=1}^{n} h_{i1} \left[f_i(x_i(t)) - k_i^- x_i(t)\right] \left[f_i(x_i(t)) - k_i^+ x_i(t)\right] \\ &- 2\sum_{i=1}^{n} h_{i2} \left[f_i(x_i(t - h(t))) - k_i^- x_i(t - h(t))\right] \\ &\times \left[f_i(x_i(t - h(t))) - k_i^+ x_i(t - h(t))\right] \\ &- 2\sum_{i=1}^{n} h_{i3} \left[f_i(x_i(t - h_U)) - k_i^- x_i(t - h_U)\right] \\ &\times \left[f_i(x_i(t - h_U)) - k_i^+ x_i(t - h_U)\right] \\ &= \zeta^T(t)\, \Theta_1 \zeta(t),\end{aligned} \tag{30}$$

where Θ is defined in (9).

Inspired by the authors' work of [5], from (5), the following conditions hold:

$$\begin{aligned}k_i^- \le \frac{f_i(x_i(t)) - f_i(x_i(t - h(t)))}{x_i(t) - x_i(t - h(t))} \le k_i^+, \\ k_i^- \le \frac{f_i(x_i(t - h(t))) - f_j(x_i(t - h_U))}{x_i(t - h(t)) - x_i(t - h_U)} \le k_i^+, \\ i = 1, \ldots, n.\end{aligned} \tag{31}$$

Therefore, for any positive diagonal matrices $H_i = \mathrm{diag}\{h_{1i}, \ldots, h_{ni}\}$ $(i = 4, 5)$, the following inequality holds:

$$\begin{aligned}
0 \le & -2\sum_{i=1}^{n} \{h_{i4}\left[f_i\left(x_i\left(t\right)\right) - f_i\left(x_i\left(t-h\left(t\right)\right)\right)\right. \\
& \left. - k_i^{-}\left(x_i\left(t\right) - x_i\left(t-h\left(t\right)\right)\right)\right] \\
& \times \left[f_i\left(x_i\left(t\right)\right) - f_i\left(x_i\left(t-h\left(t\right)\right)\right)\right. \\
& \left. - k_i^{+}\left(x_i\left(t\right) - x_i\left(t-h\left(t\right)\right)\right)\right]\} \\
& - 2\sum_{i=1}^{n} \{h_{i5}\left[f_i\left(x_i\left(t-h\left(t\right)\right)\right) - f_i\left(x_i\left(t-h_U\right)\right)\right. \\
& \left. - k_i^{-}\left(x_i\left(t-h\left(t\right)\right) - x_i\left(t-h_U\right)\right)\right] \\
& \times \left[f_i\left(x_i\left(t-h\left(t\right)\right)\right) - f_i\left(x_i\left(t-h_U\right)\right)\right. \\
& \left. - k_i^{+}\left(x_i\left(t-h\left(t\right)\right) - x_i\left(t-h_U\right)\right)\right]\} \\
= & \, \zeta^T\left(t\right)\Theta_2\zeta\left(t\right).
\end{aligned} \tag{32}$$

From (13) to (32) and by application of the S-procedure [45], an upper bound of $\dot{V}(t) = \sum_{i=1}^{5} V_i(t)$ with the addition of (20) can be written as

$$\begin{aligned}
& \dot{V}\left(t\right) + \zeta^T\left(t\right)\Xi_6\zeta\left(t\right) - 2h_U\int_{t-h(t)}^{t} x^T\left(s\right)P_1\dot{x}\left(s\right)ds \\
& \quad - 2h_U\int_{t-h_U}^{t-h(t)} x^T\left(s\right)P_2\dot{x}\left(s\right)ds \\
& \le \zeta^T\left(t\right)\left(\underbrace{\Xi_{1[h(t)]} + \sum_{i=6}^{2}\Xi_i + \Upsilon_{[h(t)]} + \Theta_1 + \Theta_2}_{\Sigma_{[h(t)]}}\right. \\
& \quad \left. - \Pi_{[h(t)]}^T\Omega\Pi_{[h(t)]}\right)\zeta\left(t\right).
\end{aligned} \tag{33}$$

By Lemma 4, $\zeta^T(t)(\Sigma_{[h(t)]} - \Pi_{[h(t)]}^T\Omega\Pi_{[h(t)]})\zeta(t) < 0$ with $0 = \Gamma\zeta(t)$ is equivalent to

$$\left(\Gamma^{\perp}\right)^T\left(\Sigma_{[h(t)]} - \Pi_{[h(t)]}^T\Omega\Pi_{[h(t)]}\right)\left(\Gamma^{\perp}\right) < 0. \tag{34}$$

Then, by Lemma 5, condition (34) is equivalent to the following inequality with any matrix $\Psi \in \mathbb{R}^{8n\times 13n}$:

$$\left[\begin{array}{c|c} \underbrace{\left(\Gamma^{\perp}\right)^T\left(\Sigma_{[h(t)]}\right)\Gamma^{\perp} + \mathrm{Sym}\left\{\left(\Gamma^{\perp}\right)^T\Pi_{[h(t)]}^T\Psi\right\}}_{\Phi_{[h(t)]}} & \Psi^T \\ \hline \Psi & -\Omega \end{array}\right] < 0. \tag{35}$$

The above condition is affinely dependent on $h(t)$. Therefore, if inequalities (10) and (11) hold, then inequality (35) is satisfied, which means that system (4) is asymptotically stable for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$. It should be noted that $\Omega > 0$ holds if inequalities (11) and (12) are feasible. This completes our proof. □

Remark 7. As mentioned in the Introduction section, Lemma 3 was first introduced in [30] to reduce the conservatism of delay-dependent stability criteria. In [30], an upper bound of the integral form of $-\int_{t-h_U}^{t}\dot{x}^T(s)Qx(s)ds$ $(Q > 0)$ was obtained by Lemma 3 with augmented vectors $(1/h(t))\int_{t-h(t)}^{t}x(s)ds$ and $(1/(h_U - h(t)))\int_{t-h_U}^{t-h(t)}x(s)ds$. However, unlike the presented method in [30], the augmented integral term of $-\int_{t-h_U}^{t}\left[\begin{smallmatrix}\dot{x}(s)\\ x(s)\end{smallmatrix}\right]^T\mathcal{Q}_1\left[\begin{smallmatrix}\dot{x}(s)\\ x(s)\end{smallmatrix}\right]ds$ $(\mathcal{Q}_1 > 0)$ was estimated in (23)–(26) with the consideration of the two integral terms obtained by zero equality (20). Thus, by utilizing the newly introduced state vectors such as $(1/h(t))\int_{t-h(t)}^{t}\int_{t-h(t)}^{s}x(u)du\,ds$ and $(1/(h_U - h(t)))\int_{t-h_U}^{t-h(t)}\int_{t-h_U}^{s}x(u)du\,ds$, more relaxed conditions can be expected since more information about the past history of states and some new cross-terms which may play roles to reduce the conservatism of stability condition were considered in Theorem 6. In the authors' future works, this method will be extended to various problems such as state estimation, H_∞ performance analysis, filtering, synchronization between two chaotic systems, and stability and stabilization of other dynamic systems which are receiving much attention in the control society.

Remark 8. Another novelty in Theorem 6 is $V_3(t)$ introduced in (13). In many works dealing with stability of neural networks with time-varying delays, the proposed Lyapunov-Krasovskii functionals having information of time-varying delays have been of the form of $\int_{t-h(t)}^{t}x^T(s)Q_1x(s)ds + \int_{t-h(t)}^{t}f^T(x(s))Q_2x(s)ds$ $(Q_i > 0)$ or $\int_{t-h(t)}^{t}\left[\begin{smallmatrix}x(s)\\ f(x(s))\end{smallmatrix}\right]^T Q\left[\begin{smallmatrix}x(s)\\ f(x(s))\end{smallmatrix}\right]ds$ $(Q > 0)$. However, the proposed Lyapunov-Krasovskii functional is $\int_{t-h(t)}^{t}\left[\begin{smallmatrix}x(s)\\ f(x(s))\\ \int_s^t\dot{x}(u)du\end{smallmatrix}\right]^T\mathcal{G}\left[\begin{smallmatrix}x(s)\\ f(x(s))\\ \int_s^t\dot{x}(u)du\end{smallmatrix}\right]ds$ which is different from the previous works. Thus, the results of time-derivative of the proposed $V_3(t)$ contain some new cross-terms such as

$$\begin{aligned}
& 2\left(\frac{1}{h\left(t\right)}\int_{t-h(t)}^{t}x\left(s\right)ds\right)\left(h\left(t\right)G_{13}\dot{x}\left(t\right)\right), \\
& 2\left(\int_{t-h(t)}^{t}f\left(x\left(s\right)\right)ds\right)\left(G_{23}\dot{x}\left(t\right)\right), \\
& 2\left(x\left(t\right) - \frac{1}{h\left(t\right)}\int_{t-h(t)}^{t}x\left(s\right)ds\right)^T\left(h\left(t\right)G_{33}\dot{x}\left(t\right)\right),
\end{aligned} \tag{36}$$

which were presented in (18) and does not be used in existing works.

In the proposed Theorem 6, the positiveness of $V(t)$ is included such as $\mathscr{R} > 0$, $\mathscr{N} > 0$, $\Sigma_i > 0$ $(i = 1, 2)$, $\Delta_i > 0$ $(i = 1, 2)$, $\mathscr{G} > 0$, $\mathscr{Q}_1 > 0$, and $Q_2 > 0$. These conditions guarantee the positiveness of each $V_i(t)$ $(i = 1, \ldots, 5)$. However, as mentioned in [35–37], by incorporating some functional of $V(t)$, the positiveness of $V(t)$ can be relaxed which will be introduced in Theorem 9. For the sake of simplicity of matrix and vector representation in Theorem 9, $\widetilde{e}_i$ $(i = 1, \ldots, 6) \in \mathbb{R}^{6n \times n}$ which will be used are defined as block entry matrices (e.g., $\widetilde{e}_3 = [0_{n\cdot 2n}, I_n, 0_{n\cdot 3n}]^T$). Assume that $\mathscr{N} > 0$, $\Sigma_i > 0$ $(i = 1, 2)$, $\Delta_i > 0$ $(i = 1, 2)$, $\mathscr{G} > 0$, $\mathscr{Q}_1 > 0$, and $Q_2 > 0$. Then, $V(t)$ has a lower bound as follows:

$$V(t) > \begin{bmatrix} x(t) \\ x(t - h_U) \\ \int_{t-h_U}^{t} x(s)\, ds \\ \int_{t-h_U}^{t} f(x(s))\, ds \end{bmatrix}^T \mathscr{R} \begin{bmatrix} x(t) \\ x(t - h_U) \\ \int_{t-h_U}^{t} x(s)\, ds \\ \int_{t-h_U}^{t} f(x(s))\, ds \end{bmatrix} + \int_{t-h_U}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix}^T \mathscr{N} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix} ds. \tag{37}$$

By Lemma 3, the lower bound of the second integral term at the right side of the inequality (35) can be obtained as

$$\begin{aligned}
&\int_{t-h_U}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix}^T \mathscr{N} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix} ds \\
&\geq \left(\frac{1}{h_U}\right) \left(\int_{t-h_U}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix} ds \right)^T \mathscr{N} \left(\int_{t-h_U}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix} ds \right) \\
&\quad + \left(\frac{3}{h_U}\right) \left(\int_{t-h_U}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix} ds - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} \int_{t-h_U}^{s} \begin{bmatrix} x(u) \\ \dot{x}(u) \\ f(x(us)) \end{bmatrix} du\, ds \right)^T \\
&\quad \times \mathscr{N} \left(\int_{t-h_U}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(x(s)) \end{bmatrix} ds - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} \int_{t-h_U}^{s} \begin{bmatrix} x(u) \\ \dot{x}(u) \\ f(x(us)) \end{bmatrix} du\, ds \right) \\
&= \left(\frac{1}{h_U}\right) \left(\begin{bmatrix} \int_{t-h_U}^{t} x(s)\, ds \\ x(t) - x(t - h_U) \\ \int_{t-h_U}^{t} f(x(s))\, ds \end{bmatrix} \right)^T \mathscr{N} \left(\begin{bmatrix} \int_{t-h_U}^{t} x(s)\, ds \\ x(t) - x(t - h_U) \\ \int_{t-h_U}^{t} f(x(s))\, ds \end{bmatrix} \right) \\
&\quad + \left(\frac{3}{h_U}\right) \left(\begin{bmatrix} \int_{t-h_U}^{t} x(s)\, ds - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} \int_{t-h_U}^{s} x(u)\, du\, ds \\ x(t) + x(t - h_U) - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} x(s)\, ds \\ \int_{t-h_U}^{t} f(x(s))\, ds - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} \int_{t-h_U}^{s} f(x(u))\, du\, ds \end{bmatrix} \right)^T \\
&\quad \times \mathscr{N} \left(\begin{bmatrix} \int_{t-h_U}^{t} x(s)\, ds - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} \int_{t-h_U}^{s} x(u)\, du\, ds \\ x(t) + x(t - h_U) - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} x(s)\, ds \\ \int_{t-h_U}^{t} f(x(s))\, ds - \left(\frac{2}{h_U}\right) \int_{t-h_U}^{t} \int_{t-h_U}^{s} f(x(u))\, du\, ds \end{bmatrix} \right) \\
&= \alpha^T(t)\, \widetilde{\Pi} \alpha(t),
\end{aligned} \tag{38}$$

where

$$\alpha(t) = \left[x^T(t), x^T(t-h_U), \int_{t-h_U}^{t} x^T(s)\,ds, \int_{t-h_U}^{t} f^T(x(s))\,ds, \int_{t-h_U}^{t}\int_{t-h_U}^{s} x^T(u)\,du\,ds, \int_{t-h_U}^{t}\int_{t-h_U}^{s} f^T(x(u))\,du\,ds\right]^T, \tag{39}$$

$$\widetilde{\Pi} = \left(\frac{1}{h_U}\right)[\widetilde{e}_3, \widetilde{e}_1 - \widetilde{e}_2, \widetilde{e}_5]\,\mathcal{N}[\widetilde{e}_3, \widetilde{e}_1 - \widetilde{e}_2, \widetilde{e}_5]^T + \left(\frac{3}{h}\right)\left[\widetilde{e}_3 - \left(\frac{2}{h_U}\right)\widetilde{e}_5, \widetilde{e}_1 + \widetilde{e}_2 - \left(\frac{2}{h}\right)\widetilde{e}_3, \widetilde{e}_5 - \left(\frac{2}{h_U}\right)\widetilde{e}_6\right] \times \mathcal{N}\left[\widetilde{e}_3 - \left(\frac{2}{h_U}\right)\widetilde{e}_5, \widetilde{e}_1 + \widetilde{e}_2 - \left(\frac{2}{h}\right)\widetilde{e}_3, \widetilde{e}_5 - \left(\frac{2}{h_U}\right)\widetilde{e}_6\right]^T. \tag{40}$$

Therefore, if the following inequality holds

$$[\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4]\,\mathcal{R}[\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4]^T + \widetilde{\Pi} > 0, \tag{41}$$

then the lower bound of $V(t)$ can be guaranteed to be of the positiveness. Thus, by deleting the positiveness of the matrix $\mathcal{R}$ and adding the inequality (40) into the stability condition of Theorem 6, we have the following theorem.

Theorem 9. *For given scalars $h_U > 0$, h_D, and diagonal matrices $K_p = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$ and $K_m = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$, system (4) is asymptotically stable for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$, if there exist positive diagonal matrices $\Lambda_i = \mathrm{diag}\{\lambda_{1i}, \ldots, \lambda_{ni}\}$ $(i = 1, 2)$, $\Delta_i = \mathrm{diag}\{\delta_{1i}, \ldots, \delta_{ni}\}$ $(i = 1, 2)$, and $H_i = \mathrm{diag}\{h_{1i}, \ldots, h_{ni}\}$ $(i = 1, \ldots, 5)$, positive definite matrices $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} = [G_{ij}] \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_1 \in \mathbb{R}^{2n\times 2n}$, and $Q_2 \in \mathbb{R}^{n\times n}$, any matrices $\mathcal{S}_1 \in \mathbb{R}^{4n\times 4n}$, $S_2 \in \mathbb{R}^{n\times n}$, and $\Psi \in \mathbb{R}^{8n\times 13n}$, and any symmetric matrices $\mathcal{R} \in \mathbb{R}^{4n\times 4n}$, $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$ satisfying the LMIs (10)–(12) and (41).*

Remark 10. Unlike Theorem 6, Theorem 9 does not require the positiveness condition of $\mathcal{R} > 0$. Instead, the condition (41) was added. It should be noted that the positive definiteness of a chosen Lyapunov-Krasovskii functional does not necessarily require all the involved symmetric matrices in the Lyapunov-Krasovskii functional to be positive definite [36]. As presented in (39), by taking the states $\int_{t-h_U}^{t}\int_{t-h_U}^{s} x(u)du\,ds$ and $\int_{t-h_U}^{t}\int_{t-h_U}^{s} f(x(u))du\,ds$ as elements of the augmented vector $\alpha(t)$, Lemma 3 was applied to obtain a lower bound of $V(t)$ in (38) for the first time. Through three numerical examples, it will be shown that larger delay bounds comparing with the results of Theorem 6 can be obtained.

Remark 11. When information about the upper bound of $\dot{h}(t)$ is unknown, then Theorems 6 and 9 can provide delay-dependent stability criteria for $0 \le h(t) \le h_U$ by choosing $\mathcal{G} = 0_{3n}$.

Table 1: Delay bounds h_U with different h_D (Example 1).

Method	h_D		
	0.8	0.9	Unknown or ≥1
[4] $(m = 2)^*$	3.0604	1.9956	1.7860
[19] $(m = 2)^*$	3.0640	2.0797	1.9207
[10] $(m = 2)^*$	3.1150	2.1153	1.3189
[13] $(m = 2)^*$	2.8991	2.0087	—**
[9] $(m = 2)^*$	2.9541	1.9654	1.7839
[5]	3.7174	2.8339	2.8222
Theorem 6	4.8104	3.4716	3.4420
Theorem 9	5.0945	3.4978	3.4506

*m is the delay-partitioning number.
**means that the corresponding result is not presented.

4. Numerical Examples

In this section, three numerical examples are introduced to show the improvements of the proposed methods. In the examples, MATLAB, YALMIP, and SeDuMi 1.3 are used to solve LMI problems.

Example 1. Consider the neural networks (4) with the parameters as follows:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad K_p = \mathrm{diag}\{0.4, 0.8\}, \quad K_m = \mathrm{diag}\{0, 0\}. \tag{42}$$

When h_D is 0.8, 0.9, and unknown, maximum delay bounds for guaranteeing the asymptotic stability are listed in Table 1 which conducts the comparison of the previous results [4, 5, 9, 10, 13, 19]. In spite of not employing delay-partitioning approach, the obtained delay bounds by applying Theorem 6 are larger than those of [4, 9, 10, 13, 19] which divided the delay interval into two subintervals. Furthermore, comparing with the results of Theorem 6, it can be confirmed that Theorem 9 enhances the feasible region of stability criteria which shows the effectiveness of the lower bound of Lyapunov-Krasovskii functional as mentioned in Remark 10.

Example 2. Consider the neural networks (4) where

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

Table 2: Delay bounds h_U with different h_D (Example 2).

Method	h_D			
	0.1	0.5	0.9	Unknown or ≥1
[11] ($\rho = 0.6$)	3.3574	2.5915	2.1306	2.0779
[6] ($m = 2$)*	3.5546	2.6438	2.1349	—**
[7] ($m = 2$)*	3.91	2.79	2.33	2.2047
[5]	3.7857	3.0546	2.6703	2.6575
[13] ($m = 2$)*	3.9151	2.8049	2.5583	—**
[9] ($m = 2$)*	4.1840	2.8387	2.3423	—**
Theorem 6	4.3711	3.2891	3.0482	3.0437
Theorem 9	4.3726	3.2899	3.0557	3.0526

*m is the delay-partitioning number.
**means that the corresponding result is not presented.

Table 3: Delay bounds h_U with different h_D (Example 3).

Method	h_D			
	0.4	0.45	0.5	0.55
[7] ($m = 2$)*	4.39	3.67	3.46	3.41
[4] ($m = 2$)*	5.2420	4.4301	4.1055	3.9231
[8] (Corollary 1)	4.8748	4.2702	4.0551	3.9369
[31]	7.6312	6.4033	5.9944	5.7963
Corollary 1	8.5174	7.5914	7.2411	7.0260
Theorem 6	9.3397	8.2439	7.7893	7.5068
Theorem 9	9.6800	8.5192	8.0535	7.7707

*m is the delay-partitioning number.

$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$$K_p = \operatorname{diag}\{0.1137, 0.1279, 0.7994, 0.2368\},$$

$$K_m = \operatorname{diag}\{0, 0, 0, 0\}. \tag{43}$$

In Table 2, maximum delay bounds for guaranteeing the asymptotic stability of system (43) are listed for various conditions of $\dot{h}(t)$. As presented in Table 2, one can see that Theorem 6 effectively reduces the conservatism of stability criteria compared with the previous results [5–7, 9, 11, 13]. Also, it can be confirmed that Theorem 9 slightly enhances the feasible region of Theorem 6.

Example 3. Consider the neural networks (4) with the following parameters:

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}, \tag{44}$$

$$K_p = \operatorname{diag}\{0.3, 0.8\}, \quad K_m = \operatorname{diag}\{0, 0\}.$$

For this example, one can see that maximum delay bounds are much larger than those of [4, 7, 8] in Table 3 which shows maximum delay bounds when $\dot{h}(t)$ is 0.4, 0.45, 0.5, and 0.55. It can be also confirmed that the feasible region of Theorem 9 is effectively enhanced compared with those of Theorem 6, which also supports the effectiveness of the statements in Remark 10. To explain the contribution of $V_3(t)$ mentioned in Remark 8, let us refer Corollary 1 when $V_3(t)$ in Theorem 6 is replaced by the traditional Lyapunov-Krasovskii functional $\int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T Q \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds$ ($Q > 0$). From the results of Corollary 1 listed in Table 3, one can confirm that the results of Corollary 1 is smaller than those of Theorem 6. This means that the proposed $V_3(t)$ in Theorem 6 is effective in reducing the conservatism of stability condition.

5. Conclusion

In this paper, two improved delay-dependent stability criteria for neural networks with time-varying delays have been proposed by the use of the Lyapunov stability theorem and LMI framework. In Theorem 6, by constructing the suitable augmented Lyapunov-Krasovskii functional and utilizing some novel Lyapunov-Krasovskii functionals and techniques mentioned in Remarks 7 and 8, a delay-dependent sufficient condition for asymptotic stability of the concerned network was derived. Based on the results of Theorem 6 and by taking lower bound of Lyapunov-Krasovskii functional and utilizing the property of its positiveness with the newly constructed augmented vectors, the further improved stability condition was derived in Theorem 9. Via three numerical examples dealt with in many previous works to check the conservatism of stability criteria, the improvements of the feasible region of the three proposed stability criteria have been successfully verified. Moreover, in [46], the triple integral forms of Lyapunov-Krasovskii functional were shown effectiveness in reducing the conservatism of stability sufficient conditions. Thus, by grafting such an approach onto the proposed idea of this paper, further improved results will be investigated in the near future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Fixed Points and Stability of a Class of Integrodifferential Equations

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We study a class of integrodifferential functional differential equations $\ddot{x} + f(t, x, \dot{x})\dot{x} + \sum_{j=1}^{N} \int_{t-r_j(t)}^{t} a_j(t,s) g_j(s, x(s)) ds = 0$ with variable delay. By using the fixed point theory, we establish necessary and sufficient conditions ensuring that the zero solution of this equation is asymptotically stable.

1. Introduction

It is well known that differential equations have many applications in engineering, control theory, neural networks, biology, and so on. The stability of the solution of functional differential equations has drawn a lot of attention for many years. Liapunov's direct method has been widely applied to study the stability problems for a long time; see, for example, [1–3]. Nowadays, many authors have applied the fixed points theory to study the stability of solution of functional differential equations; some authors even use fixed point theory to study stability conditions for stochastic equations with delay; see, for example, [4–7] and the references therein.

Burton studied stability of a nonconvolution equation

$$\dot{x} = -\int_{t-r}^{t} a(t,s)\, g(x(s))\, ds, \tag{1}$$

where r was a positive constant. He obtained sufficient conditions ensuring that the zero solution was asymptotically stable via fixed point theorem; see [8].

Becker and Burton studied the following differential equation:

$$\dot{x} = -\int_{t-r(t)}^{t} a(t,s)\, g(x(s))\, ds \tag{2}$$

with variable delay and its special case

$$\dot{x} = -\int_{t-r}^{t} a(t-s)\, g(x(s))\, ds, \tag{3}$$

for r, is a positive constant, $t \geq 0$, where function $r(t) : [0,\infty) \to [0,\infty)$. They found sufficient conditions ensuring that the zero solution was asymptotically stable by changing the supremum metric to an exponentially weighted metric. Moreover, they hoped to reduce condition that the function $t - r(t)$ was strictly increasing; see [9]. Jin and Luo reduced this condition in their paper. Moreover, they established necessary and sufficient conditions that could ensure that the zero solution of this equation is asymptotically stable; see, for example, [10].

Ardjouni et al. studied the stability of the following neutral integrodifferential equation:

$$\dot{x}(t) = -\sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t,s)\, x(s)\, ds + \sum_{j=1}^{N} c_j(t)\, x'\left(t - \tau_j(t)\right) \tag{4}$$

via fixed point theorem. Also $m_j(0) = \inf\{t - \tau_j(t) : t \geq 0\}$. Their main result was as follows.

Theorem A. *Suppose that τ_j is twice differentiable and $\tau_j'(t) \neq 1$ for all $t \in R^+$ and there exist continuous functions*

$h_j : [m_j(0), \infty) \to R$ *for* $j = 1, 2, \ldots, N$ *and a constant* $\alpha \in (0, 1)$ *such that, for* $t \geq 0$,

(i) $\liminf_{t \to \infty} \int_0^t H(s)ds > -\infty$,

(ii)

$$\sum_{j=1}^{N} \left| \frac{c_j(t)}{1 - \tau_j'(t)} \right| + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left| h_j(s) + B_j(t,s) \right| ds + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left| h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s) - r_j(s)\right) \right| ds + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} |H(s)| \times \int_{s-\tau_j(s)}^{s} \left| h_j(u) + B_j(s,u) \right| du\, ds \leq \alpha, \tag{5}$$

where

$$H(t) = \sum_{j=1}^{N} h_j(t),$$

$$r_j(t) = \frac{\left[c_j(t) H(t) + c_j'(t)\right]\left(1 - \tau_j'(t)\right) + c_j(t)\tau_j''(t)}{\left(1 - \tau_j'(t)\right)^2},$$

$$B_j(t,s) = \int_t^s a_j(u,s)\, du, \tag{6}$$

$$B_j\left(t, t - \tau_j(t)\right) = \int_t^{t-\tau_j(t)} a_j\left(u, t - \tau_j(t)\right) du.$$

Then the zero solution is asymptotically stable if and only if

$$\int_0^t H(s)\, ds \longrightarrow \infty \quad as\ t \longrightarrow \infty. \tag{7}$$

They improved the results of [9]; see [11]. Other results on fixed points and stability properties in equations with variable delays can be found in [5].

Recently, Zhao and Yuan investigate the stability of a generalized Volterra-Levin equation:

$$\dot{x} = -\int_{t-L}^{t} p(s-t)\, g(s, x(s))\, ds, \quad t > t_0,$$
$$x(t) = \psi(t), \quad t_0 - L \leq t \leq t_0. \tag{8}$$

They give new conditions on the stability of zero solution of this equation; their conditions are different from [5]; see, for example, [12].

The stability of second order equation has always been hot issue of research. Levin and Nohel [13] studied the global asymptotic stability of a class of nonlinear systems:

$$\ddot{x}(t) + h(t, x, \dot{x})\dot{x} + f(x) = e(t). \tag{9}$$

They obtained asymptotic stability by constructing proper Lyapunov function.

Burton [14] considered the following equation with delay:

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)\, g(x(t - L)) = 0, \tag{10}$$

where L is a positive constant, and obtained some sufficient conditions under which each solution $x(t)$ satisfies $(x(t), \dot{x}(t)) \to 0$ via the fixed point theorem.

We generalized the above equation to an equation with a variable delay [15] as follows:

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)\, g(x(t - \tau(t))) = 0, \tag{11}$$

and obtained some results on asymptotic stability of the zero solution. Before we introduce our new results we recall the main results in [15]. There are *basic assumptions* on the delay function $\tau(t)$: $(\mathscr{A})$ $t - \tau(t)$ which is strictly increasing and $\lim_{t \to \infty} t - \tau(t) = \infty$. $p(t)$ is the inverse of $t - \tau(t)$. Moreover $0 \leq b(t) \leq M$ for some constant $M > 0$.

The main results in [15] can be stated as follows.

Theorem B. *Suppose* $(\mathscr{A})$ *and the following conditions hold.*

(i) *There exists a constant* $l > 0$ *such that* $g(x)$ *satisfies the Lipschitz condition on* $[-l, l]$. *The function* $g(x)$ *is odd and is strictly increasing on* $[-l, l]$, *and* $x - g(x)$ *is nondecreasing on* $[0, l]$.

(ii) *There exist an* $\alpha \in (0, 1)$ *and a continuous function* $a(t) : [0, \infty) \to [0, \infty)$ *such that* $f(t, x, y) \geq a(t)$ *for* $t \geq 0$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $\int_0^\infty a(t)dt = \infty$, *and*

$$2\sup_{t \geq 0} \int_t^{p(t)} \int_0^\infty e^{-\int_s^{w+s} a(v)dv} b(s)\, dw\, ds + 2\sup_{t \geq 0} \int_0^t \int_{t-s}^\infty e^{-\int_s^{w+s} a(v)dv} b(s)\, dw\, ds \leq \alpha. \tag{12}$$

(iii) *There exist constants* $a_0 > 0$ *and* $Q > 0$ *such that, for each* $t \geq 0$, *if* $J \geq Q$, *then*

$$\int_t^{t+J} a(v)\, dv \geq a_0 J. \tag{13}$$

(iv) *There exist continuous functions* $F : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *and* $c(t) : [0, \infty) \to [0, \infty)$ *such that, for all* $t \geq 0$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $f(t, x, y) \leq F(x, y)c(t)$. *The function* $g'(x)$ *is continuous on* $[-l, l]$, $g'(0) \neq 0$.

If, for each $\gamma > 0$, $\iint_0^\infty e^{-\int_s^{w+s} \gamma c(v)} b(s)dw\, ds = \infty$, then the zero solution of (11) is asymptotically stable.

In [16], we have studied the following equation:

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + \int_{t-r(t)}^{t} a(t, s)\, g(x(s))\, ds = 0 \tag{14}$$

with variable delay. By using the fixed point theory, we obtain conditions which ensure that the zero solution of this equation is stable under an exponentially weighted metric.

Moreover, we establish necessary and sufficient conditions ensuring that the zero solution is asymptotically stable.

In this paper we consider

$$\ddot{x} + f(t,x,\dot{x})\,\dot{x} + \sum_{j=1}^{N} \int_{t-r_j(t)}^{t} a_j(t,s)\, g_j(s,x(s))\, ds = 0 \quad (15)$$

for $t \geq 0$, where functions $r_j(t) : [0,\infty) \to [0,\infty)$, $a_j(t,s) : [0,\infty) \times [-r(0),\infty) \to R$, $g_j : [-r(0),\infty) \times R \to R$, and $f : R \times R \times R \to R^+$ are all continuous, where $R^+ = [0,+\infty)$. We assume that $t - r_j(t) \to \infty$ as $t \to \infty$. $j = 1,2,\ldots,N$.

For each $t_0 \geq 0$, define $m_j(t_0) = \inf\{s - r_j(s) : s \geq t_0\}$, $j = 1,2,\ldots,N$. Let $m(t_0) = \min\{m_1(t_0),\ldots,m_N(t_0)\}$. Also $C(t_0) = C([m(t_0),t_0],R)$ is a function space endowed with function norm $\|\cdot\|$, where $\|\psi\| = \sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}$. We will also use $\|\phi\|$ to express the supremum on $[m(t_0),\infty)$ later. It is well known that, in [2], for a given continuous function ϕ, there exists a solution of (15) on an interval $[t_0,T)$; if the solution remains bounded, then $T = \infty$. We denote by $(x(t),y(t))$ the solution $(x(t,t_0,\phi),y(t,t_0,\phi))$. Denote by $A(t) := f(t,x(t),y(t))$.

We can write (15) in an equivalent form as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -A(t)\,y - \sum_{j=1}^{N} \int_{t-r_j(t)}^{t} a_j(t,s)\, g_j(s,x(s))\, ds. \end{aligned} \quad (16)$$

We will give a necessary and sufficient condition ensuring that the zero solution of this equation is asymptotically stable. To our knowledge, there are few results about its stability. This paper is organized as follows. In the next section we will state our main results. Their proofs will be given in Sections 3 and 4.

2. Statement of Main Results

First of all, we consider a special case of (15). We consider the following equation:

$$\ddot{x} + f(t,x,\dot{x})\,\dot{x} + \int_{t-L}^{t} p(s-t)\, g(s,x(s))\, ds = 0. \quad (17)$$

Rewrite this equation as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -A(t)\,y - \int_{t-L}^{t} p(s-t)\, g(s,x(s))\, ds. \end{aligned} \quad (18)$$

Our main results are as follows.

Theorem 1. *Assume that $h(t) \in C(R^+,R^+)$ and the following conditions hold.*

(i) *Consider that $\int_{-L}^{0} p(s)ds = 1$. There exist a constant $\lambda > 0$ and a function $Q(x) \in C(R,R^+)$ such that, for $\forall x, y_1, y_2 \in R$, $|g(x,y_1) - g(x,y_2)| \leq Q(x)|y_1 - y_2|$ and the inequality $\sup_{t\geq 0} \int_{-L}^{0} |p(s)| \int_{t-L}^{t+s} Q(u)du\, ds \leq \lambda$ hold. There exists a constant $K > 0$ such that $Q(t) \leq K$.*

(ii) *There exist $\alpha \in (0,1)$ and a continuous function $A_1(t) : [0,\infty) \to [0,\infty)$ such that $f(t,x,y) \geq A_1(t)$ for $t \geq 0$. For $t \geq 0$, $x \in R$, $y \in R$*

$$\begin{aligned} &\int_0^t e^{-\int_u^t h(s)ds} \int_0^u e^{-\int_s^u A_1(v)dv} Q(s-L)\, ds\, du \\ &\quad + 2\lambda \int_0^t e^{-\int_u^t h(s)ds} du \leq \alpha. \end{aligned} \quad (19)$$

(iii) *There exist constants $a_0 > 0$ and $Q_0 > 0$ such that, for each $t \geq 0$, if $J \geq Q_0$, then*

$$\int_t^{t+J} A_1(s)\, ds \geq a_0 J. \quad (20)$$

Then the zero solution of (17) is stable.

In addition, we have the following.

Theorem 2. *Assume that $\lim_{t\to\infty} t - r_j(t) = \infty$, $r_j(t)$ is differentiable, $j = 1,2,\ldots,N$, and there exist some functions $h_j(t) \in C(R^+,R^+)$ such that for $t \geq 0$ the following conditions hold.*

(i) *There exist a constant $\Gamma > 0$ and some functions $Q_j(x) \in C(R,R^+)$ such that, for $\forall x, y_1, y_2 \in R$,*

$$\begin{aligned} &\left|g_j(x,y_1) - g_j(x,y_2)\right| \leq Q_j(x)\left|y_1 - y_2\right|, \quad j = 1,2,\ldots,N, \\ &\sup_{t\geq 0} \sum_{j=1}^{N} \int_{t-r_j(t)}^{t} \left(\int_t^u \left|a_j(v,t)\right| dv\right) Q_j(u)\, du \leq \Gamma. \end{aligned} \quad (21)$$

There exists a constant $K^* > 0$ such that $Q_j(t) \leq K^*$, $j = 1,2,\ldots,N$.

(ii) *Let $H(t) = \sum_{j=1}^{N} h_j(t)$. There exist a constant $\alpha \in (0,1)$ and a continuous function $A_1(t) : R^+ \to R^+$ such that $f(t,x,y) \geq A_1(t)$. For $\forall t \geq 0$, $x \in R$, $y \in R$,*

$$\begin{aligned} &2\sum_{j=1}^{N} \int_{t-r_j(t)}^{t} h_j(v)\, dv + 2\Gamma \int_0^t e^{-\int_u^t H(v)dv} du \\ &\quad + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} h_j\left(s - r_j(s)\right)\left|1 - r_j'(s)\right| ds \\ &\quad + \sum_{j=1}^{N} \int_0^t e^{-\int_u^t H(v)dv} \left(\int_0^u e^{-\int_s^u A_1(v)dv} K^* \left|W_j(s)\right| ds\right) du \leq \alpha, \end{aligned} \quad (22)$$

where $W_j(s) = (1 - r_j'(s)) \int_s^{s-r_j(s)} a_j(v, s - r_j(s))dv$.

(iii) *There exist constants $a_0 > 0$ and $Q_0 > 0$ such that, for each $t \geq 0$, if $J \geq Q_0$, then*

$$\int_t^{t+J} A_1(s)\, ds \geq a_0 J. \quad (23)$$

Then the zero solution of (15) is asymptotically stable if and only if

$$\int_0^t H(s)\, ds \longrightarrow \infty \quad as\ t \longrightarrow \infty. \quad (24)$$

3. Proof of Theorem 1

In this section, we will prove Theorem 1 by applying the fixed point theory. We will give the expression of the solution of the related equation. The following result can be found in [12].

Lemma 3. *Let the function* $p : [-L, 0] \to R$, $g \in C(R \times R, R)$, $\int_{-L}^{0} p(s)ds = 1$. *Then*

$$\dot{x}(t) = -\int_{t-L}^{t} p(s-t)\, g(s, x(s))\, ds \tag{25}$$

is equivalent to

$$\begin{aligned}\dot{x}(t) &= -g(t-L, x(t-L)) \\ &\quad + \frac{d}{dt}\int_{-L}^{0} p(s) \int_{t+s}^{t-L} g(u, x(u))\, du\, ds.\end{aligned} \tag{26}$$

Lemma 4. *Let* $\psi : [-L, 0] \to R$ *be a given continuous function; if* $(x(t), y(t))$ *is the solution of* (17) *satisfying* $x(t) = \psi(t)$, $t \in [-L, 0]$, *and* $y(0) = \dot{x}(0)$, *then* $x(t)$ *is the solution of the following integral equation:*

$$\begin{aligned}&x(t) \\ &= \psi(0)\, e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u)\, du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_0^u e^{-\int_s^u A(v)dv} g(s-L, x(s-L))\, ds \right) du \\ &\quad + \int_0^t e^{-\int_u^t h(s)ds} \left(e^{-\int_0^u A(v)dv} \right. \\ &\qquad \left. \times \int_{-L}^{0} p(v) \int_{v}^{-L} g(w, \psi(w))\, dw\, dv \right) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^{0} p(v) \int_{u+v}^{u-L} g(w, x(w))\, dw\, dv \right) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left[\int_0^u e^{-\int_s^u A(v)dv} A(s) \right. \\ &\qquad \times \left(\int_{-L}^{0} p(v) \right. \\ &\qquad \left. \left. \times \int_{s+v}^{s-L} g(w, x(w))\, dw\, dv \right) ds \right] du,\end{aligned} \tag{27}$$

where $B(t) = \dot{x}(0) e^{-\int_0^t A(s)ds}$.

Proof. We apply the variation of parameters formula to the second equation of (18); we obtain

$$\begin{aligned}\dot{x}(t) &= \dot{x}(0)\, e^{-\int_0^t A(s)ds} \\ &\quad - \int_0^t e^{-\int_s^t A(v)dv} \left(\int_{s-L}^{s} p(v-s)\, g(v, x(v))\, dv \right) ds.\end{aligned} \tag{28}$$

By Lemma 3, (28) can be written as

$$\begin{aligned}\dot{x}(t) &= B(t) - \int_0^t e^{-\int_s^t A(v)dv} g(s-L, x(s-L))\, ds \\ &\quad + \int_0^t e^{-\int_s^t A(v)dv} \\ &\qquad \times \left(\frac{d}{ds} \int_{-L}^{0} p(s) \int_{s+v}^{s-L} g(u, x(u))\, du\, dv \right) ds;\end{aligned} \tag{29}$$

multiplying both sides of the above equation by $e^{\int_0^t h(s)ds}$, we see that

$$\begin{aligned}&x(t) \\ &= \psi(0)\, e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u)\, du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_0^u e^{-\int_s^u A(v)dv} g(s-L, x(s-L))\, ds \right) du \\ &\quad + \int_0^t e^{-\int_u^t h(s)ds} \\ &\qquad \times \left[\int_0^u e^{-\int_s^u A(v)dv} \right. \\ &\qquad\quad \times \left(\frac{d}{ds} \int_{-L}^{0} p(s) \right. \\ &\qquad\quad \left. \left. \times \int_{s+v}^{s-L} g(w, x(w))\, dw\, dv \right) ds \right] du.\end{aligned} \tag{30}$$

If we integrate the last term by parts, we have

$$\begin{aligned}&x(t) \\ &= \psi(0)\, e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u)\, du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_0^u e^{-\int_s^u A(v)dv} g(s-L, x(s-L))\, ds \right) du \\ &\quad + \int_0^t e^{-\int_u^t h(s)ds} \left(e^{-\int_0^u A(v)dv} \right. \\ &\qquad \left. \times \int_{-L}^{0} p(v) \int_{v}^{-L} g(w, \psi(w))\, dw\, dv \right) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^{0} p(v) \int_{u+v}^{u-L} g(w, x(w))\, dw\, dv \right) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \\ &\qquad \times \left[\int_0^u e^{-\int_s^u A(v)dv} A(s) \right. \\ &\qquad\quad \times \left(\int_{-L}^{0} p(v) \right. \\ &\qquad\quad \left. \left. \times \int_{s+v}^{s-L} g(w, x(w))\, dw\, dv \right) ds \right] du.\end{aligned} \tag{31}$$

□

Let $(C, \|\cdot\|)$ be the Banach space of bounded continuous functions on $[-L, \infty)$ with the supremum norm. For a given continuous initial function $\psi : [-L, 0] \to R$, define the set $C_\psi \subset C$ by

$$C_\psi = \{\phi : [-L, \infty) \longrightarrow R \mid \phi \in C, \phi(t) = \psi(t), t \in [-L, 0]\} \tag{32}$$

and its subset

$$C_\psi^l = \{\phi : [-L, \infty) \longrightarrow R \mid \phi \in C, \phi(t) = \psi(t), t \in [-L, 0], |\phi(t)| \le l, t > 0\}, \tag{33}$$

where $\psi : [-L, 0] \to [-l, l]$ is a given initial function and l is a positive constant. Let P_1 be a mapping defined on C_ψ^l as follows: for $\phi \in C_\psi^l$, if $t \in [-L, 0]$, $(P_1\phi)(t) = \psi(t)$. If $t > 0$,

$$\begin{aligned}
&(P_1\phi)(t) \\
&= \psi(0) e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u)\, du \\
&\quad - \int_0^t e^{-\int_u^t h(s)ds} \int_0^u e^{-\int_s^u A(v)dv} g(s - L, \phi(s - L))\, ds\, du \\
&\quad + \int_0^t e^{-\int_u^t h(s)ds} \left(e^{-\int_0^u A(v)dv} \times \int_{-L}^0 p(v) \int_v^{-L} g(w, \psi(w))\, dw\, dv \right) du \\
&\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^0 p(v) \int_{u+v}^{u-L} g(w, \phi(w))\, dw\, dv \right) du \\
&\quad - \int_0^t e^{-\int_u^t h(s)ds} \left[\int_0^u e^{-\int_s^u A(v)dv} A(s) \times \left(\int_{-L}^0 p(v) \times \int_{s+v}^{s-L} g(w, \phi(w))\, dw\, dv \right) ds \right] du.
\end{aligned} \tag{34}$$

Easy calculation shows that

$$\begin{aligned}
&|(P_1\phi)(t) - (P_1\eta)(t)| \\
&\le \int_0^t e^{-\int_u^t h(s)ds} \int_0^u e^{-\int_s^u A(v)dv} |g(s - L, \phi(s - L)) - g(s - L, \eta(s - L))|\, ds\, du \\
&\quad + \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^0 p(v) \int_{u-L}^{u+v} |g(w, \phi(w)) - g(w, \eta(w))|\, dw\, dv \right) du \\
&\quad + \int_0^t e^{-\int_u^t h(s)ds} \times \left(\int_0^u e^{-\int_s^u A(v)dv} A(s) \times \left(\int_{-L}^0 p(v) \times \int_{s-L}^{s+v} |g(w, \phi(w)) - g(w, \eta(w))|\, dw\, dv \right) ds \right) du.
\end{aligned} \tag{35}$$

We can easily obtain that $(P_1\phi)(t)$ is a contraction mapping by using condition (ii). We continue to prove Theorem 1. Choose $\psi : [-L, 0] \to R$ and $\dot{x}(0)$ such that

$$\left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) |\dot{x}(0)| + \delta + \lambda\delta \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \le (1 - \alpha) l. \tag{36}$$

Condition (i) implies that $g(t, 0) = 0$, since $g(t, x)$ satisfies the Lipschitz condition with respect to x, such that δ exists and $\delta < l$. By the expression of $(P_1\phi)(t)$, condition (ii), and condition (iii) of Theorem 1, we have

$$\begin{aligned}
\int_0^t e^{-\int_0^u A(s)ds} du &= \int_0^{Q_0} e^{-\int_0^u A(s)ds} du + \int_{Q_0}^t e^{-\int_0^u A(s)ds} du \\
&\le \int_0^{Q_0} du + \int_{Q_0}^t e^{-a_0 u} du \\
&\le Q_0 + \frac{e^{-a_0 Q_0}}{a_0}.
\end{aligned} \tag{37}$$

According to condition (ii) and (36), we have

$$|(P_1\phi)(t)| \le (1 - \alpha) l + \alpha l = l. \tag{38}$$

Note that if $t \in [-L, 0]$, then $(P_1\phi)(t) = \psi(t)$. We obtain that $|(P_1\phi)(t)| \le l$, $t \in [-L, \infty)$. Thus $P_1 : C_\psi^l \to C_\psi^l$. Since we have proved that P_1 is a contraction mapping, hence P_1 has a unique fixed point $x(t)$ and $|x(t)| \le l$.

Recalling (28), we have

$$\begin{aligned}
|y(t)| &\le |\dot{x}(0)| e^{-\int_0^t A(s)ds} \\
&\quad + \int_0^t e^{-\int_s^t A(v)dv} \int_{s-L}^s |p(s - v)|\, |g(v, x(v))|\, dv\, ds;
\end{aligned} \tag{39}$$

then

$$\begin{aligned}
|y(t)| &\le |\dot{x}(0)| + Kl \cdot \int_0^t e^{-\int_u^t A(s)ds} du \\
&< l \left(1 + K \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \right).
\end{aligned} \tag{40}$$

It follows that

$$|x(t)| + |y(t)| < l\left(2 + K\left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0}\right)\right). \quad (41)$$

For any given $\epsilon > 0$; we only need to replace ϵ by l. Then we obtain the stability of zero solution. This completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section, we will prove Theorem 2. First of all, we will obtain a useful expression for the solution of (15) in the following proof.

We apply the variation of parameters formula to the second equation of (15); then

$$\dot{x}(t) = B_1(t) - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \times \left(\int_{s-r_j(s)}^{s} a_j(s,v)\, g_j(v, x(v))\, dv\right) ds, \quad (42)$$

where $B_1(t) = \dot{x}(t_0) e^{-\int_{t_0}^{t} A(s)ds}$.

Lemma 5. *The equation*

$$\dot{x}(t) = -\sum_{j=1}^{N} \int_{t-r_j(t)}^{t} a_j(t,s)\, g_j(s, x(s))\, ds \quad (43)$$

is equivalent to

$$\begin{aligned}\dot{x}(t) &= \sum_{j=1}^{N} C_j\left(t, t-r_j(t)\right)\left(1 - r_j'(t)\right) g_j\left(t - r_j(t), x\left(t - r_j(t)\right)\right) \\ &\quad + \sum_{j=1}^{N} \frac{d}{dt} \int_{t-r_j(t)}^{t} C_j(t,s)\, g_j(s, x(s))\, ds,\end{aligned} \quad (44)$$

where

$$\begin{aligned}C_j(t,s) &= \int_t^s a_j(u,s)\, du, \\ C_j\left(t, t-r_j(t)\right) &= \int_t^{t-r_j(t)} a_j\left(u, t-r_j(t)\right) du.\end{aligned} \quad (45)$$

Proof. Differentiating the integral term in (44), we have

$$\begin{aligned}&\frac{d}{dt} \int_{t-r_j(t)}^{t} C_j(t,s)\, g_j(s, x(s))\, ds \\ &= \int_{t-r_j(t)}^{t} \frac{\partial}{\partial t} C_j(t,s)\, g_j(s, x(s))\, ds + C_j(t,t)\, g_j(t, x(t)) \\ &\quad - C_j\left(t, t-r_j(t)\right) g_j\left(t - r_j(t), x\left(t - r_j(t)\right)\right) \\ &\quad \times \left(1 - r_j'(t)\right).\end{aligned} \quad (46)$$

It follows that if $C_j(t,t) = 0$, $(\partial/\partial t) C_j(t,s) = -a_j(t,s)$, then (43) is equivalent to (44). Easy calculation shows that

$$\begin{aligned}C_j(t,s) &= \int_t^s a_j(u,s)\, du, \\ C_j\left(t, t-r_j(t)\right) &= \int_t^{t-r_j(t)} a_j\left(u, t-r_j(t)\right) du.\end{aligned} \quad (47)$$

□

By Lemma 5, (42) can be written as

$$\begin{aligned}\dot{x}(t) &= B_1(t) + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \\ &\quad \times \left(\frac{d}{ds} \int_{s-r_j(s)}^{s} C_j(s,v) \times g_j(v, x(v))\, dv\right) ds \\ &\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \left(C_j\left(s, s-r_j(s)\right)\left(1 - r_j'(s)\right)\right. \\ &\quad \left. \times g_j\left(s - r_j(s), x\left(s - r_j(s)\right)\right)\right) ds.\end{aligned} \quad (48)$$

We define

$$\begin{aligned}C_\psi^0 = \{&\phi : [m(t_0), \infty) \longrightarrow R \mid \phi \in C, \phi(t) = \psi(t), \\ &t \in [m(t_0), t_0], \phi(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty\}.\end{aligned} \quad (49)$$

Multiplying both sides of the above equation by $e^{\int_{t_0}^{t} H(u)du}$, easy calculation shows that

$$\begin{aligned}x(t) &= \psi(t_0)\, e^{-\int_{t_0}^{t} H(u)du} + \int_{t_0}^{t} e^{-\int_s^t H(u)du} B_1(s)\, ds \\ &\quad + \int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s)\, x(s)\, ds \\ &\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t H(v)dv} \\ &\quad \times \left(\int_{t_0}^{u} e^{-\int_s^u A(v)dv} \frac{d}{ds} \right. \\ &\quad \left. \times \int_{s-r_j(s)}^{s} C_j(s,v)\, g_j(v, x(v))\, dv\right) ds\, du \\ &\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t H(v)dv} \left(\int_{t_0}^{u} e^{-\int_s^u A(v)dv} W_j(s) \right. \\ &\quad \left. \times g_j\left(s - r_j(s), x\left(s - r_j(s)\right)\right) ds\right) du,\end{aligned} \quad (50)$$

where $W_j(s) = C_j(s, s - r_j(s))(1 - r_j'(s))$. Performing an integration by parts, then

$$
\begin{aligned}
&x(t)\\
&= \psi(t_0)\, e^{-\int_{t_0}^{t} H(s)ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)du} B_1(s)\, ds\\
&\quad + \sum_{j=1}^{N} \int_{t-r_j(t)}^{t} h_j(v)\, x(v)\, dv\\
&\quad - \sum_{j=1}^{N} \int_{t_0-r_j(t_0)}^{t_0} h_j(u)\, \psi(u)\, du \cdot e^{-\int_{t_0}^{t} H(u)du}\\
&\quad - \sum_{j=1}^{N} \int_{t_0}^{t} \left(\int_{s-r_j(s)}^{s} h_j(v)\, x(v)\, dv \right) e^{-\int_{s}^{t} H(u)du} H(s)\, ds\\
&\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)du} h_j\left(s - r_j(s)\right) x\left(s - r_j(s)\right)\\
&\qquad \times \left(1 - r_j'(s)\right) ds\\
&\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{u}^{t} H(v)dv} \left(\int_{u-r_j(u)}^{u} C_j(u, v)\, g_j(v, x(v))\, dv \right) du\\
&\quad - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{u}^{t} H(v)dv} \left[\int_{t_0}^{u} e^{-\int_{t_0}^{u} A(v)dv} \right.\\
&\qquad \times \left(\int_{t_0-r_j(t_0)}^{t_0} C_j(t_0, v)\right.\\
&\qquad \left.\left. \times g_j(v, \psi(v))\, dv \right) ds \right] du\\
&\quad - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)du} \left(\int_{t_0}^{s} e^{-\int_{v}^{s} A(r)dr} A(v)\right.\\
&\qquad \times \int_{v-r_j(v)}^{v} C_j(v, u)\\
&\qquad \left. \times g_j(u, x(u))\, du\, dv \right) ds\\
&\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{u}^{t} H(v)dv} \left(\int_{t_0}^{u} e^{-\int_{s}^{u} A(v)dv} W_j(s)\right.\\
&\qquad \left. \times g_j\left(s - r_j(s), x\left(s - r_j(s)\right)\right) ds \right) du.
\end{aligned}
\tag{51}
$$

Let P_2 be a mapping defined on C_ψ^0 as follows: for $\phi \in C_\psi$, if $t \in [m(t_0), t_0]$, $(P_2\phi)(t) = \psi(t)$. If $t > t_0$, we define

$$
\begin{aligned}
&(P_2\phi)(t)\\
&= \psi(t_0)\, e^{-\int_{t_0}^{t} H(s)ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)du} B_1(s)\, ds\\
&\quad + \sum_{j=1}^{N} \int_{t-r_j(t)}^{t} h_j(v)\, \phi(v)\, dv\\
&\quad - \sum_{j=1}^{N} \int_{t_0-r_j(t_0)}^{t_0} h_j(u)\, \psi(u)\, du \cdot e^{-\int_{t_0}^{t} H(u)du}\\
&\quad - \sum_{j=1}^{N} \int_{t_0}^{t} \left(\int_{s-r_j(s)}^{s} h_j(v)\, \phi(v)\, dv \right) e^{-\int_{s}^{t} H(u)du} H(s)\, ds\\
&\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)du} h_j(s - r(s))\, \phi\left(s - r_j(s)\right)\\
&\qquad \times \left(1 - r_j'(s)\right) ds\\
&\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{u}^{t} H(v)dv} \left(\int_{u-r_j(u)}^{u} C_j(u, v)\, g_j(v, \phi(v))\, dv \right) du\\
&\quad - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{u}^{t} H(v)dv}\\
&\qquad \times \left[\int_{t_0}^{u} e^{-\int_{t_0}^{u} A(v)dv} \right.\\
&\qquad \times \left(\int_{t_0-r_j(t_0)}^{t_0} C_j(t_0, v)\right.\\
&\qquad \left.\left. \times g_j(v, \psi(v))\, dv \right) ds \right] du\\
&\quad - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{s}^{t} H(u)du}\\
&\qquad \times \left(\int_{t_0}^{s} e^{-\int_{v}^{s} A(r)dr} A(v)\right.\\
&\qquad \left. \times \int_{v-r_j(v)}^{v} C_j(v, u)\, g_j(u, \phi(u))\, du\, dv \right) ds\\
&\quad + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_{u}^{t} H(v)dv}\\
&\qquad \times \left(\int_{t_0}^{u} e^{-\int_{s}^{u} A(v)dv} \right.\\
&\qquad \left. \times W_j(s)\, g_j\left(s - r_j(s), \phi\left(s - r_j(s)\right)\right) ds \right) du.
\end{aligned}
\tag{52}
$$

If $\phi \in C_\psi^0$, since $\int_0^t H(s)ds \to \infty$ as $t \to \infty$, $t_0 \geq 0$, hence $\int_{t_0}^{t} H(s)ds \to \infty$ as $t \to \infty$. The first term and fourth term of $(P_2\phi)(t) \to 0$, as $t \to \infty$.

Note that

$$
\int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds} B_1(u)\, du = \dot{x}(t_0) \int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds} e^{-\int_{t_0}^{u} A(s)ds} du.
\tag{53}
$$

Since $A(t) = f(t, x(t), y(t)) \geq A_1(t) \geq 0$, then

$$\int_{t_0}^{t} e^{-\int_u^t H(s)ds} e^{-\int_{t_0}^u A(s)ds} du \leq \int_{t_0}^{t} e^{-\int_{t_0}^u A(s)ds} du. \tag{54}$$

For a given $\epsilon > 0$, there exists $T_1 > Q_0 + t_0$ such that $e^{a_0 t_0 - a_0 T} < a_0 \epsilon$. For $T_1 < t$ and $t_0 < Q_0 < T_1 \leq u$, we have

$$\begin{aligned}\int_{T_1}^{t} e^{-\int_{t_0}^u A(s)ds} du &\leq \int_{T_1}^{t} e^{-a_0(u-t_0)} du = \frac{e^{a_0 t_0}}{a_0}\left(e^{-a_0 T_1} - e^{-a_0 t}\right) \\ &< \frac{e^{a_0 t_0 - a_0 T_1}}{a_0} < \epsilon.\end{aligned} \tag{55}$$

For $t > T_1$, we have

$$\begin{aligned}&\int_{t_0}^{T_1} e^{-\int_u^t H(s)ds} e^{-\int_{t_0}^u A(s)ds} du \\ &\quad = e^{-\int_{T_1}^t H(s)ds} \cdot \int_{t_0}^{T_1} e^{-\int_u^{T_1} H(s)ds} e^{-\int_{t_0}^u A(s)ds} du.\end{aligned} \tag{56}$$

The second term of $(P_2\phi)(t) \to 0$, as $t \to \infty$.

Note that $\int_{t_0 - r_j(t_0)}^{t_0} C_j(t_0, v) g_j(v, \psi(v)) dv$ is bounded; we can check the eighth term of $(P_2\phi)(t) \to 0$, as $t \to \infty$ similarly.

Then we can easily check that P_2 is a contraction mapping on C_ψ^0 by using condition (ii). By the contraction mapping principle, P_2 has a unique fixed point $x(t)$ in C_ψ^0. Also $x(t) \to 0$ as $t \to +\infty$.

In order to obtain the asymptotic stability, we still need to show that the zero solution is stable. Let $\epsilon > 0$ be given, we choose $\psi : [m(t_0), t_0] \to R$ and $\dot{x}(t_0)$ such that

$$\begin{aligned}&\delta + \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0}\right)\left|\dot{x}(t_0)\right| \\ &\quad + \sum_{j=1}^{N} \int_{t_0 - r_j(t_0)}^{t_0} h_j(s)\, ds \cdot \delta + \alpha\Gamma\left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\delta \\ &\quad \leq (1 - \alpha)\epsilon.\end{aligned} \tag{57}$$

Then we have

$$\begin{aligned}|x(t)| &\leq \delta + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} \left|\dot{x}(t_0)\right| e^{-\int_{t_0}^u A(s)ds} du \\ &\quad + \sum_{j=1}^{N} \int_{t_0 - r_j(t_0)}^{t_0} h_j(s)\, ds \cdot \delta + \alpha\Gamma\left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\delta + \alpha\epsilon.\end{aligned} \tag{58}$$

Easy calculation shows that

$$\begin{aligned}\int_{t_0}^{t} e^{-\int_{t_0}^u A(s)ds} du &\leq \int_{t_0}^{t_0 + Q_0} du + \int_{t_0 + Q_0}^{t} e^{-a_0(u - t_0)} du \\ &\leq Q_0 + \frac{e^{-a_0 Q_0}}{a_0}.\end{aligned} \tag{59}$$

Hence

$$|x(t)| \leq \epsilon. \tag{60}$$

Therefore

$$\begin{aligned}&|\dot{x}(t)| \\ &\leq \left|\dot{x}(t_0)\right| + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \\ &\qquad \times \left(\int_{s - r_j(s)}^{s} \left|a_j(s, v)\, g_j(v, x(v))\right| dv\right) ds.\end{aligned} \tag{61}$$

It follows that

$$|x(t)| + |y(t)| < \epsilon\left(2 + NK^*\left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0}\right)\right). \tag{62}$$

Therefore the zero solution is stable; since we have obtained that $x(t) \to 0$ as $t \to \infty$, it follows that the zero solution is asymptotically stable.

A necessary condition is as follows. For each $t_0 \geq 0$, we denote that $M = \sup_{t \geq 0}\{e^{-\int_0^t H(s)ds}\}$. We will prove that $\int_0^\infty H(s)ds = \infty$ by way of contradiction. If $\int_0^\infty H(s)ds < \infty$, since $H(t) \geq 0$, $\exists$ being a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n \to \infty} \int_0^{t_n} H(s)ds = \sigma$, where σ is a finite number. Choose η such that $-\eta \leq \int_0^{t_n} H(s)ds \leq \eta$ holds, for all $n \geq 1$.

Denote that

$$\begin{aligned}\omega(u) &\triangleq H(u) \sum_{j=1}^{N} \int_{u - r_j(u)}^{u} \left|h_j(v)\right| dv \\ &\quad + \sum_{j=1}^{N} \left|h_j\left(s - r_j(s)\right)\left(1 - r_j'(s)\right)\right| \\ &\quad + \sum_{j=1}^{N} \int_{t_0}^{u} e^{-\int_v^u A(r)dr} A(v) \\ &\qquad \times \int_{v - r_j(v)}^{v} \left|C_j(v, w)\right| Q_j(w)\, dw\, dv \\ &\quad + \sum_{j=1}^{N} \int_0^t e^{-\int_u^t H(v)dv} \left(\int_0^u e^{-\int_s^u A_1(v)dv} K^* \right. \\ &\qquad \left. \times \left|W_j(s)\right| ds\right) du \\ &\quad + \sum_{j=1}^{N} \int_{u - r_j(u)}^{u} \left|C_j(u, v)\right| Q_j(v)\, dv\, du,\end{aligned} \tag{63}$$

where $W_j(s) = (1 - r_j'(s)) \int_s^{s - r_j(s)} a_j(v, s - r_j(s)) dv$.

By conditions of Theorem 2, we have

$$0 \leq \int_0^{t_n} \omega(u)\, e^{-\int_u^{t_n} H(s)ds} du \leq \alpha. \tag{64}$$

Then

$$\int_0^{t_n} \omega(u)\, e^{\int_0^u H(s)ds} du \le \alpha e^{\int_0^{t_n} H(s)ds} \le \alpha e^{\eta}. \tag{65}$$

Thus the sequence $\{\int_0^{t_n} \omega(u) e^{\int_0^u H(s)ds} du\}$ is bounded; there exists a convergent subsequence; we assume that $\lim_{n\to\infty} \int_0^{t_n} \omega(u) e^{\int_0^u H(s)ds} du = I$, $I \ge 0$. We can choose a positive integer $\overline{k}$ large enough such that

$$\int_{t_{\overline{k}}}^{t_n} \omega(u)\, e^{\int_0^u H(s)ds} du < \frac{\delta_0}{8M} \tag{66}$$

for $\forall n \ge \overline{k}$, where $0 < \delta_0\ (\delta_0 < \epsilon)$ is satisfying

$$\begin{aligned} &\delta_0 + \max\left\{Q_0 + \frac{e^{-a_0 Q_0}}{a_0}, 1\right\} \delta_0 + \delta_0 \sum_{j=1}^{N} \int_{t_{\overline{k}} - r_j(t_{\overline{k}})}^{t_{\overline{k}}} h_j(s)\, ds \\ &\quad + \alpha\Gamma\left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0}\right)\delta_0 \le (1-\alpha). \end{aligned} \tag{67}$$

Now we consider the solution $x(t) = x(t, \psi, \dot{x}(t_{\overline{k}}))$ of (11) which satisfies

$$\begin{aligned} &\psi(t_{\overline{k}}) = \frac{3\delta_0}{4}, \qquad \dot{x}(t_{\overline{k}}) = \frac{\delta_0}{4}, \\ &|\psi(s)| + |\dot{x}(s)| \le \delta_0, \quad s \le t_{\overline{k}}. \end{aligned} \tag{68}$$

We can obtain that $|x(t)| \le 1$ by a similar argument with (60) if we replace ϵ by 1. Consider

$$\begin{aligned} &\psi(t_{\overline{k}}) - \sum_{j=1}^{N} \int_{t_{\overline{k}} - r_j(t_{\overline{k}})}^{t_{\overline{k}}} h_j(s)\, \psi(s)\, ds \\ &\ge \psi(t_{\overline{k}}) - \sum_{j=1}^{N} \int_{t_{\overline{k}} - r_j(t_{\overline{k}})}^{t_{\overline{k}}} h_j(s) |\psi(s)|\, ds \\ &\ge \frac{3\delta_0}{4} - \frac{\delta_0}{2} = \frac{\delta_0}{4}. \end{aligned} \tag{69}$$

By (27) and $|x(t)| \le 1$, if $t_n \ge t_{\overline{k}}$, we have

$$\begin{aligned} &\left| x(t_n) - \sum_{j=1}^{N} \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, x(s)\, ds \right| \\ &\ge \left| \psi(t_{\overline{k}})\, e^{-\int_{t_{\overline{k}}}^{t_n} H(s)ds} + \int_{t_{\overline{k}}}^{t_n} e^{-\int_u^{t_n} H(s)ds} B(u)\, du \right. \\ &\quad \left. - \sum_{j=1}^{N} \int_{t_{\overline{k}} - r_j(t_{\overline{k}})}^{t_{\overline{k}}} h_j(s)\, \psi(s)\, ds \cdot e^{-\int_{t_{\overline{k}}}^{t_n} H(u)du} \right| \\ &\quad - \left| \int_{t_{\overline{k}}}^{t_n} e^{-\int_u^{t_n} H(s)ds} \omega(u)\, du \right|. \end{aligned} \tag{70}$$

This implies that

$$\begin{aligned} &\left| x(t_n) - \sum_{j=1}^{N} \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, x(s)\, ds \right| \\ &\ge e^{-\int_{t_{\overline{k}}}^{t_n} H(s)ds} \left[\left| \frac{\delta_0}{4} + \int_{t_{\overline{k}}}^{t_n} e^{\int_{t_{\overline{k}}}^{u} H(s)ds} \dot{x}(t_{\overline{k}})\, e^{\int_{t_{\overline{k}}}^{u} A(s)ds} du \right| \right. \\ &\quad \left. - \int_{t_{\overline{k}}}^{t_n} \omega(u)\, e^{\int_{t_{\overline{k}}}^{u} H(s)ds} du \right]. \end{aligned} \tag{71}$$

Note that $\dot{x}(t_{\overline{k}}) = \delta_0/4 > 0$; it follows that

$$\begin{aligned} &\left| x(t_n) - \sum_{j=1}^{N} \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, x(s)\, ds \right| \\ &\ge e^{-\int_{t_{\overline{k}}}^{t_n} h(s)ds} \left[\frac{\delta_0}{4} - e^{-\int_0^{t_{\overline{k}}} H(s)ds} \right. \\ &\quad \left. \cdot \int_{t_{\overline{k}}}^{t_n} \omega(u)\, e^{\int_0^u H(s)ds} du \right] \\ &\ge \frac{\delta_0}{8} e^{-\int_{t_{\overline{k}}}^{t_n} H(s)ds} \ge \frac{\delta_0}{8} e^{-2\eta} > 0. \end{aligned} \tag{72}$$

If the zero solution $x(t)$ is asymptotically stable, then $x(t) \to 0$ as $t \to \infty$. By the mean value theorem and condition (ii), we have

$$\begin{aligned} \left| \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, x(s)\, ds \right| &= \left| (x(\varpi)) \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, ds \right| \\ &\le |x(\varpi)|. \end{aligned} \tag{73}$$

Thus

$$\sum_{j=1}^{N} \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, x(s)\, ds \longrightarrow 0 \quad \text{as } t_n \longrightarrow \infty. \tag{74}$$

Consequently,

$$\lim_{t_n \to \infty} \left(x(t_n) - \sum_{j=1}^{N} \int_{t_n - r_j(t_n)}^{t_n} h_j(s)\, x(s)\, ds \right) = 0. \tag{75}$$

This is a contradiction to (72).

Hence

$$\int_0^{\infty} H(s)\, ds = \infty. \tag{76}$$

This completes the proof of Theorem 2.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Study of Robust H_∞ Filtering Application in Loosely Coupled INS/GPS System

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Since a celebrate linear minimum mean square (MMS) Kalman filter in integration GPS/INS system cannot guarantee the robustness performance, a H_∞ filtering with respect to polytopic uncertainty is designed. The purpose of this paper is to give an illustration of this application and a contrast with traditional Kalman filter. A game theory H_∞ filter is first reviewed; next we utilize linear matrix inequalities (LMI) approach to design the robust H_∞ filter. For the special INS/GPS model, unstable model case is considered. We give an explanation for Kalman filter divergence under uncertain dynamic system and simultaneously investigate the relationship between H_∞ filter and Kalman filter. A loosely coupled INS/GPS simulation system is given here to verify this application. Result shows that the robust H_∞ filter has a better performance when system suffers uncertainty; also it is more robust compared to the conventional Kalman filter.

1. Introduction

Inertial navigation system (INS) is a self-contained system that had been widely used in military, mining, and tunnel construction in the recent decades. However, the INS parameter update functions are nonlinear, the process biases are coupled, and twice integration procedure definitely enlarges the noise influence, the dynamic error of INS keeps increasing with the time going on. Due to hardware and sensor limitation, low cost INS device can only provide an accurate navigation solution for a short period. To conquer this hardship, INS/GPS integrate system is proposed. The GPS is a time bias-free and precise system under less interference circumstance. Measurement of GPS can contribute a calibration to INS, which maintains the accuracy constant.

Estimation theory is widely used in integrate navigation field that blends and fuses information to get an optimal solution. For a typical INS/GPS system, the filter dynamic equation is normally chosen by the INS error state for a linear propagation property. It calls up various investigations based on the modification of a Kalman filter to improve the accuracy or to reduce the computation load of the system [1–3].

However, Kalman filter is the optimal estimation of minimum mean square error (MSE) for a linear time invariant (LTI) system; it has a strict requirement for the process noise covariance **Q** and measurement noise covariance **R**. If the noise is Gaussian, Kalman filter is the linear minimum variance estimator and if the noise is not Gaussian, it is linear minimum variance estimator. In practical engineering problem, we cannot absolutely assure either the statistical character of the noise or of the dynamic model. And if the objective is to achieve a minimum worst-case estimator rather than minimum variance estimator, Kalman filter will not meet the requirement. In other words, Kalman filter cannot guarantee the robustness if the system contains structure or model uncertainty and lack of noise statistical character.

Motivated by this problem, precursors tried to improve the robustness performance although at the sacrifice of somewhat accuracy such as an EKF approach for parameter identification system [4]. H_∞ filter is undoubtedly the best choice since it calls for no prior knowledge of the noise

distribution. Also we see application in INS/GPS area [5–7], including bounded filter theory [8], and we will review the game theory which is also called the minimax theory used in this area. It is well known that no models can all the way precisely describe a system; huge efforts have been put into the research of Robust H_∞ filer. This theory assumes that there exists some uncertainty in the system and tries to solve this problem by a quadratic stabilized constrain. Riccati equation and linear matrix inequality (LMI) are two main approaches to deal with robust problems. Compared to Riccati process, LMI does not make any assumptions about the parameters and can be solved efficiently with various MATLAB toolboxes [9, 10]. H_∞ theory was first applied in the control area [11]; state space realization is a bond connecting it with a estimation theory. In fact, most of research of robust H_∞ filter is based on the "conservatism," an important index in robust control. To the authors' knowledge, there is no investigation of the robust H_∞ filter considered unstable factor in INS/GPS model; here we merely discuss this application and how it can be implemented.

In this paper, we firstly analyse the relationship between H_∞ filter and Kalman filter and then modify the LMI design approach, making it possible to apply the robust H_∞ filter for INS/GPS model; a classical 15-element (incorporating errors of attitude, velocity, position, and gyro and accelerator biases) state vector is used to check filter performance. The purpose of this paper is to make a summary and an easy access to H_∞ filtering implementation in INS/GPS system.

The rest of the paper is organized as follows. Derivation of recursive H_∞ filter formulation based on "game theory" [12] is introduced in Section 2; in Section 3 gives designing procedure for robust H_∞ filter based on LMI; Section 4 discusses the Kalman divergence issue with uncertain system and method in dealing with unstable model when using LMI; in Section 5, we compare the robust H_∞ filter and Kalman filter performance in different situation for loosely coupled INS/GPS model.

The notation used here is fairly standard: $\|\cdot\|_2$ is a L_2 norm; $\overline{\sigma}(D)$ denotes the largest singular value of D; T_{ab} denotes a transform operator from a to b; and $\mathbf{R^n}$ is nth dimensional Euclidian state space. Tr(M) stands for the trace of matrix M, symbol $*$ represents the symmetry term in a symmetric matrix, sup means the supremum, and A^T represents the transposed matrix A.

2. Game Theory Recursive H_∞ Filter

Since many previous authors had utilized H_∞ filter in INS/GPS model, we discuss the game theory H_∞ approach in this section as a fundamental illustration of the robust H_∞ filter. In order to give a definition of H_∞ norm. We begin with the L_2 norm. For a continuous infinite systems during the period $(0, T)$, L_2 norm is defined as

$$\|x(t)\|_2 = \sqrt{\int_0^T x(t)\, x^T(t)\, dt}, \tag{1}$$

where to a discrete system, suppose the sampling number is N in the period T; then

$$\|x(k)\|_2 = \sqrt{\sum_{k=1}^{N} x(k)\, x^T(k)}. \tag{2}$$

L_2 reflects the meaning of "energy" of the variable. Then the H_∞ norm of a linear, time-invariant operator $g = z(w)$ is defined as

$$\left\|T_{gw}\right\|_\infty = \sup \frac{\|z\|_2}{\|w\|_2} = \sup \overline{\sigma}\left(T_{gw}(j\omega)\right), \tag{3}$$

where $\overline{\sigma}(\cdot)$ denotes the max singular value of the square matrix $T * T^T$ and $j\omega$ is the frequency parameter in the range of $[-\pi, \pi]$. In frequency domain, H_∞ norm means a peak value of all the frequency ranges, while in time domain, it means the largest energy "transfer" form variable w to variable z; in other words, w has the largest influence on z. So what is the H_∞ filter problem? Consider the following state-space system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \qquad y(k) = Cx(k) + v(k), \\ z(k) &= Lx(k), \qquad x(0) = x_0, \end{aligned} \tag{4}$$

where $x \in \mathbf{R^n}$ is the dynamic state, $y \in \mathbf{R^v}$ is the measurement, $z \in \mathbf{R^r}$ is the output to be estimated, $Lx(t)$ is linear combination of x, and $w \in \mathbf{R^q}$ and $v \in \mathbf{R^p}$ are the process and measurement noise, respectively. (A, B) is controllable and (A, C) is detectable. Note that it is detectable but not "observable"; in an INS/GPS model, the measurement is often "detectable" not "observable." Now, consider such a cost function:

$$J = \frac{\sum_{k=1}^{N} \|\widehat{z}(k) - z(k)\|_{2S_k}^2}{\left\|\widehat{x}_0 - x_0\right\|_{2P_k^{-1}}^2 + \sum_{k=1}^{N}\left(\|w\|_{2Q_k^{-1}}^2 + \|v\|_{2R_k^{-1}}^2\right)}, \tag{5}$$

where P_k, Q_k, R_k, and S_k are symmetric, positive definite weighting matrices, and formulation $\|x\|_{2P_k}^2 = xP_kx'$. Note that these weighting matrix are not noise covariance matrix in a kalman filter, they can be set arbitrary by the filter designer, of cause, can set to be equal with Kalman filter's Q, R, P.

The H_∞ filter problem is to find an estimation strategy $\widehat{z}(t)$ which minimizes J. The famous game theory depicted the estimation of cost function as a dynamic, two-person game: one is the engineering and the other say the nature. The goal of engineering is to find a proper estimation that minimizes J, while his opponent, the nature tries to maximize J from the disturbances and initial state adversely. As a contrast to Kalman filter (KF), what KF pursuits is to minimize $\sum_{k=1}^{N} \|\widehat{z}(k) - z(k)\|_{2S_k}^2$, but it cannot guarantee the J under some fixed extent, meaning that there may be a huge noise influence on the estimation result when achieves the minimum MSE.

Substituting, $\|v\|_{2R_k^{-1}}^2 = \|y(k) - Cx(k)\|_{2R_k^{-1}}^2$; this problem can turn to seeking a saddle point $J(x_0^*, w^*, y^*, \widehat{x}^*)$, satisfying

$$J(x_0, w, y, \widehat{x}) < J(x_0^*, w^*, y, \widehat{x}) < J(x_0^*, w^*, y^*, \widehat{x}^*). \tag{6}$$

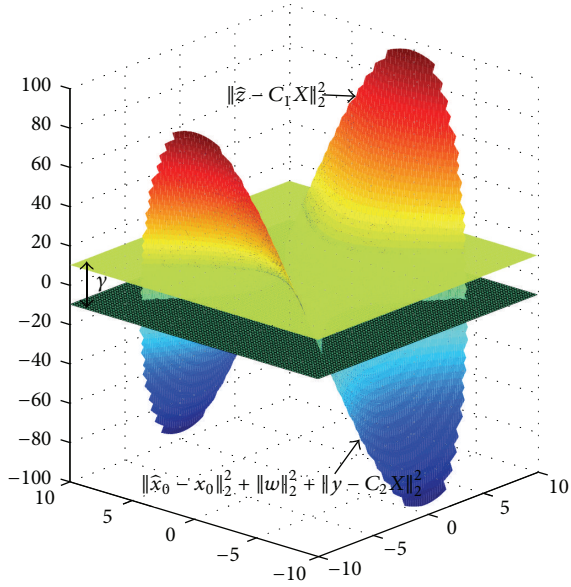

FIGURE 1: Game theory H_∞ filter model.

In fact, this saddle point, that is, the smallest J, is not tractable in normal case. In general, people look for a closed form solution to the suboptimal estimation such that in Figure 1: the sole saddle point, which lies in the crossing of the upside and bottom surfaces, is difficult to achieve, while suboptimal question guarantees the solution in the range of the two planes that contains the saddle point.

A summary for the game theory H_∞ filter approach is: fix a prescribed parameter γ satisfying $J = \gamma^2$, $\gamma > 0$; try to achieve minimum $\sum_{k=1}^{N} \|\hat{z}(k) - z(k)\|_{2S_k}^2$ subject to (5). This derivation process can be implemented by using the Lagrange multiplier [13, 14]. The steady-state H_∞ filter formulations are given by

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + A_k K_k \left(y_k - C_k x_k\right), \\ K_k &= P_k \left(I - \gamma^2 P_k + C_k^T R_k^{-1} C_k P_k\right)^{-1} C_k^T R_k^{-1}, \\ P_{k+1} &= A_k P_k \left(I - \gamma^2 P_k + C_k^T R_k^{-1} C_k P_k\right)^{-1} A_k^T + Q_k. \end{aligned} \tag{7}$$

For an intuitive contrast with recursive Kalman filter, some transformations are carried out here. A typical Kalman gain K_k can be written as

$$K_k = \left(I + P_k^- C_k^T R_k^{-1} C_k\right)^{-1} P_k^- C_k^T R_k^{-1}. \tag{8}$$

Premultiplying outside by P_k^- and postmultiplying each term inside by the inverse of P_k^-, we obtain

$$K_k = P_k^- \left(P_k^- + P_k^- C_k^T R_k^{-1} C_k P_k^-\right)^{-1} P_k^- C_k^T R_k^{-1}. \tag{9}$$

Postmultiplying outside the parentheses by the inverse of P_k^- and premultiplying each term inside parentheses by the inverse of P_k^-, we have

$$K_k = P_k^- \left(I + C_k^T R_k^{-1} C_k P_k^{-1}\right)^{-1} C_k^T R_k^{-1}. \tag{10}$$

So the reformed Kalman filter is given as

$$\begin{aligned} \hat{x}_{k+1}^- &= A_k \hat{x}_k^- + A_k K_k \left(y_k - C_k x_k^-\right), \\ K_k &= P_k^- \left(I + C_k^T R_k^{-1} C_k P_k^-\right)^{-1} C_k^T R_k^{-1}, \\ P_{k+1}^- &= A_k P_k^- \left(I + C_k^T R_k^{-1} C_k P_k^-\right)^{-1} A_k^T + Q_k. \end{aligned} \tag{11}$$

Note that there is P_k^- in (10) while P_k in (11). For a LTI system, it is easy to prove that the P_k^- and P_k^+ converse to the same stable constant when $T \to \infty$, proof procedure is omitted here for brief. Now suppose that $\gamma^2 = 0$; the H_∞ filter reduces to Kalman filter fantastically. Again, analogous to Figure 1, if the up half-surface represents the performance of the Kalman filter, this filter will pursuit the minimum MMS point, that is, ellipsoid endpoint without concerning the noise transform level. We treat the H_∞ as a filter criterion, and minimax game theory provides a guarantee about this, and the noise attenuation parameter γ is also the key index in robust H_∞ filter.

3. Robust H_∞ Filter

Consider a system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \qquad y(t) = Cx(t) + Dw(t), \\ z(t) &= Lx(t), \qquad x(0) = x_0. \end{aligned} \tag{12}$$

Note that the difference with system (1) lies in the description of noise. Setting $B = [B \quad 0]$, $D = [0 \quad D]$, and $w = [w \quad v]'$ we can get the above formula. In this term, the disturbance process and measurement noise are seen as a united element.

For a system, it is impossible to obtain an exact system model due to the presence of inherent parameter uncertainty instinct. The matrices A, B, C, D are not exactly known or depend on some time-varying parameter that vary in a fixed bound set. This is called "polytopic model" with the character

$$\begin{aligned} S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Bigg\{ & \begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} \\ & = \sum_{i=1}^{N} \alpha(i) \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} \Bigg\} = \Lambda. \end{aligned} \tag{13}$$

The vector time-varying parameter $\alpha(t)$ belongs to the unit simplex as

$$\Omega = \left\{ \alpha(i) \in R^n, \sum_{i=1}^{N} \alpha(i) = 1; \alpha(i) \geq 0 \right\}. \tag{14}$$

Any uncertain matrices set $S(\alpha)$ belongs to a polytopic model with convex bound uncertain domain Λ; hence any $S(\alpha) \in \Lambda$, $S(\alpha)$ can be written as a combination of vertices $S(i)$ of the convex polytope. And the model is supposed to be entirely known if the vertices number $N = 1$.

Another important descriptor is the norm-bounded uncertainty. The mode S subject to norm-bounded parameter uncertainty is written in the form

$$S = S_0 + \mathrm{HFE} \tag{15}$$

for the uncertain matrix parameter F such that $FF' = I$, H, and E are known matrices.

The purpose of the filter problem is to design an estimation z which is given by $z = \mathcal{F} \cdot y$, where $\mathcal{F}$ is an operator filter set with the form

$$\dot{\widehat{x}}(t) = A_f \widehat{x}(t) + B_f y(t), \qquad \widehat{z}(t) = C_f \widehat{x}(t), \qquad \widehat{x}(0) = 0 \tag{16}$$

for all the admissible uncertainty Λ such that $\{A_f, B_f, C_f\} \in \mathcal{F}$. The estimation error e can easily be yielded as $e = z - \widehat{z}$. Construct state variable $\xi = [x \quad \widehat{x}]^T$, the closed loop state-space form is

$$\dot{\widehat{\xi}}(t) = A_a \xi(t) + B_a w(t); \qquad e(t) = C_a xi(t), \tag{17}$$

where

$$A_a = \begin{bmatrix} A & 0 \\ K_f C & A_f \end{bmatrix}, \qquad B_a = \begin{bmatrix} B \\ K_f D \end{bmatrix}, \qquad C_a = \begin{bmatrix} L & -C_f \end{bmatrix}. \tag{18}$$

Defining the transfer function from the noise input w to estimation error e is $T_{we}(s)$; hence $T_{we}(s) = C_a(sI - A_a)^{-1}B_a$. *Robust H_∞ filtering problem*: determining A_f, B_f, C_f, for all the admissible uncertainty Λ, satisfy the following:

(i) model (17) is asymptotic stabilization;

(ii) transfer function infinity norm holds that $\|T_{we}(s)\|_\infty \le \gamma$.

According to the *bounded real lemma*, the closed loop system (17) with the upper bounded γ is asymptotically stable if and only if there exists a matrix $P = P^T$ to such the algebraic Riccati inequality:

$$A_a^T P_a + P_a A_a + \gamma^{-2} P_a B_a B_a^T P_a + C_a^T C_a < 0. \tag{19}$$

Implement Schur complement lemma, this inequality is equivalent with the LMI:

$$\begin{bmatrix} A_a P + P A^T & P_a C_a^T & B_a \\ C_a P_a & -\gamma I & 0 \\ B_a^T & 0 & -\gamma I \end{bmatrix} < 0. \tag{20}$$

3.1. LMI Approach to Robust H_∞ Filtering. The LMI can be solved efficiently by the Matlab LMI Toolbox; however, two problems have to be considered.

(i) Solutions to LMI (20) are A_a, B_a, C_a, so the relationship between inequality and variables A_f, B_f, C_f must be founded.

(ii) The inequality should be transformed into linear convex optimisation problem, so it can be addressed by LMI approach.

Considering the formulation (20), partition P together with its inverse as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} \widehat{P}_{11} & \widehat{P}_{12} \\ \widehat{P}_{12}^T & \widehat{P}_{22} \end{bmatrix}, \tag{21}$$

substituting the partitioned matrix P in (17) with P_{11}, P_{22}, $\widehat{P}_{11}$, $\widehat{P}_{22}$; the inequality changes into

$$\begin{bmatrix} AP_{11} + P_{11}A^T & * & * & * \\ B_f C P_{11} + A_f P_{12}^T + P_{12}^T A^T & B_f C P_{12} + A_f P_{22} + P_{12}^T C^T B_f^T + P_{22} A_f^T & * & * \\ L P_{11} - C_f P_{12}^T & L P_{12} - C_f P_{22} & -\gamma I & * \\ B^T & D^T B_f^T & 0 & -\gamma I \end{bmatrix}, \tag{22}$$

where $*$ denotes the symmetric block. The next step is converting this joint nonlinear matrix into LMI form [15].

Making an equivalent transform by permuting third and fourth columns and rows, we obtain

$$\begin{bmatrix} AP_{11} + P_{11}A^T & * & * & * \\ B_f C P_{11} + A_f P_{12}^T + P_{12}^T A^T & B_f C P_{12} + A_f P_{22} + P_{12}^T C^T B_f^T + P_{22} A_f^T & * & * \\ B^T & D^T B_f^T & -\gamma I & * \\ L P_{11} - C_f P_{12}^T & L P_{12} - C_f P_{22} & 0 & -\gamma I \end{bmatrix}. \tag{23}$$

Define the nonsingular matrix

$$\Psi = \begin{bmatrix} \widehat{P}_{11}^{-1} & 0 & 0 & 0 \\ \widehat{P}_{11} & \widehat{P}_{12} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{24}$$

Applying the congruence transformation, pre- and postmultiplying Ψ^T and Ψ, the inequality can be obtained as

$$\begin{bmatrix} AX + XA^T & * & * & * \\ A^TX + P_{11}A + YC + K & \widehat{P}_{11}A + A^T\widehat{P}_{11} + YC + C^TY^T & * & * \\ B^TX & B^T\widehat{P}_{11} + D^TY^T & -\gamma I & * \\ L - Z & L & 0 & -\gamma I \end{bmatrix} > 0, \tag{25}$$

where X, Y, Z, K are matrix variables defined as

$$\begin{gathered} X = P_{11}^{-1}, \qquad Y = \widehat{P}_{12}, \qquad Z = C_f P_{12}^T X, \\ K = \widehat{P}_{12} A_f P_{12}^T X, \\ \begin{bmatrix} X & X \\ X & \widehat{P}_{11} \end{bmatrix} > 0. \end{gathered} \tag{26}$$

The objective is to determine the feasible solutions to the LMI; hence the filter matrices can be obtained:

$$Af = \widehat{P}_{12}^{-1} K \left(P_{12}^T X\right)^{-1}, \qquad B_f = \widehat{P}_{12}^{-1} Y, \qquad Cf = Z\left(P_{12}^T X\right)^{-1} \tag{27}$$

and matrices $\widehat{P}_{12}$, P_{12} can be set arbitrarily to be nonsingular as

$$P_{12}\widehat{P}_{12}^T = I - P_{11}\widehat{P}_{11}. \tag{28}$$

Here the filtering problem can be solved by a convex LMI optimisation approach. Give a summary to this filter problem. *LMI H_∞ filtering*: minimise γ subject to inequality (26) hold. Based on the inherent properties of polytopic system, this approach can be extended to a robust filter problem. Substitute A, B, C, D by the vertex matrices A_i, B_i, C_i, D_i of Λ, and guaranteeing that all the vertex LMIs hold, the robust H_∞ problem can be solved.

It should be highlighted that this formulation can reduced to standard Kalman filter when restricting some special case. Suppose that $BD^T = 0$ and $DD^T = I$; a standard Kalman state description (4) can recovered from (12) when replace this strict LMI (21) by a non strict one through choosing that $X \to 0$, $K = -YC - \widehat{P}_{11}A$, and $L = Z$; then the remaining variable should hold the LMI as

$$\begin{bmatrix} \widehat{P}_{11}A + A^T\widehat{P}_{11} + YC + C^TY^T & * & * \\ B^T\widehat{P}_{11} + D^TY^T & -\gamma I & * \\ L & 0 & -\gamma I \end{bmatrix}. \tag{29}$$

Define the new variable $P = \widehat{P}_{11}^{-1}$ and $Y = -C^T$. Applying the Schur complement to (29), we obtain

$$AP + PA^T - P\left(C'C - \gamma^{-2}L^TL\right)P + BB^T < 0. \tag{30}$$

If the parameter $\gamma \to \infty$ (30) is reduced to

$$AP + PA^T - PC'CP + BB^T < 0, \tag{31}$$

the optimal solution P is arbitrarily close to the solution of this algebraic Riccati equation when replacing the inequality by a same equality. And this Riccati equation solution is our familiar estimation error covariance $P(k)$, when $k \to \infty$, also called stable state of the Kalman filter for a LTI system. Defining $P_{11} = P_{12}^T$ and $\widehat{P}_{12}^{-1} = -\widehat{P}_{11}^{-1}$ and holding (28) we have

$$\begin{gathered} C_f = Z\left(P_{12}^T X\right)^{-1} = Z\left(P_{12}^T P_{11}^{-1}\right)^{-1} = Z = L, \\ B_f = \widehat{P}_{12}^{-1} Y = -\widehat{P}_{11}^{-1} Y = PC, \\ A_f = \widehat{P}_{12}^{-1} K\left(P_{12}^T X\right)^{-1} = \widehat{P}_{12}^{-1} K = \widehat{P}_{12}^{-1}\left(-YC - \widehat{P}_{11}A\right) \\ = \left(-\widehat{P}_{12}^{-1}\widehat{P}_{11}\right)A - \left(\widehat{P}_{12}^{-1}Y\right)C = A - B_f C. \end{gathered} \tag{32}$$

4. Robust H_∞ to INS/GPS Loosely Coupled Model

4.1. INS Error Model. The estimation objective in GPS/INS system is the INS error. Filter for this INS dynamic error model consists of 15 elements such as

$$x = \left[\delta\phi_x \;\; \delta\phi_y \;\; \delta\phi_z \;\; \delta V_x \;\; \delta V_y \;\; \delta V_z \;\; \delta L \;\; \delta\lambda \;\; \delta h \;\; \varepsilon_{xb} \;\; \varepsilon_{yb} \;\; \varepsilon_{zb} \;\; \nabla_{xb} \;\; \nabla_{yb} \;\; \nabla_{zb}\right], \tag{33}$$

where $\delta\phi_x$, $\delta\phi_y$, $\delta\phi_z$ are the pitch, roll, and yaw errors, V_x, δV_y, δV_z are the east, north, and up velocity errors, δL, $\delta\lambda$, δh are the latitude, longitude, and height errors, ε_{xb}, ε_{yb}, ε_{zb}, and ∇_{xb}, ∇_{yb}, ∇_{zb} are the gyroscope and accelerator bias errors in subscript orientation. Assume that the gyroscope and accelerator have a Gaussian white noise. The transition matrix can be written as

$$F(t) = \begin{bmatrix} F_N(t)_{9\times 9} & F_S(t)_{9\times 6} \\ 0_{6\times 9} & I_{6\times 6} \end{bmatrix}. \tag{34}$$

$F_s(t)$ is the 9 fundamental parameter transfer matrix of which element detailed definition can be found in [16], I is identity matrix, and $F_S(t)$ is the transfer matrix between fundamental parameter and INS sensors bias, written as

$$F_S(t) = \begin{bmatrix} 0_{3\times 3} & C_b^n \\ C_b^n & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}. \tag{35}$$

C_b^n is the rotation matrix to transform the states from body frame to navigation frame. It is obviously that $F(t)$ is a time-varying matrix determined by the real attitude ϕ, velocity V, and position P, although can be seen as a piecewise approximate constant matrix when implementing filtering for each epoch. H_∞ estimation calls for no prestatistical character of noise, but accurate system matrices like A, B, C, D. If the INS output is corrupt, then F_S cannot be guaranteed, neither is the estimation state accuracy X. And that is why a robust H_∞ filter is designed for such a case.

4.2. Alternative LMI Approach to Unstable Model. It can be easy found that in a LMI approach to solve the H_∞ filter context, a precondition that transfer matrix $A(F_S)$ be stable must be satisfied. The reason for this is that since the state-space model cannot be completely cancelled in the estimation error, they must be stable to guarantee the boundness of estimation error. And to apply LMI (21), it must guarantee diagonal element strictly less than 0. If A is unstable, $P > 0$, $A^TP + PA < 0$ the first element of (21) is infeasible. To deal with this tough situation, an alternative LMI approach is proposed for coping with unstable model [17, 18]. Consider system (12), its state vector can be partitioned as

$$x(t) = \left[x_s(t) \;\; x_u(t)\right]^T \tag{36}$$

and the system matrices partitions are

$$\begin{aligned} A(\alpha) &= \begin{bmatrix} A_s(\alpha) & 0 \\ A_{us}(\alpha) & A_u \end{bmatrix}, \qquad B = \begin{bmatrix} B_s \\ B_u \end{bmatrix}, \\ C(\alpha) &= \left[C_s(\alpha) \;\; C_u\right], \qquad L = \left[L_s(\alpha) \;\; L_u\right], \end{aligned} \tag{37}$$

where the subscript "s" denotes the stable component and "u" denotes the unstable component.

Assumption. Consider the following.

(i) The subsystem $\dot{x}_s(t) = A_s x_s(t)$ is quadratically stable.

(ii) The matrix A_u is unstable.

(iii) (C_s, A_s) is detectable and (A, B) is controllable.

Next, partition the filter matrices, and we obtain

$$\begin{aligned} \widehat{x}(t) &= \begin{bmatrix} \widehat{x}_s(t) \\ \widehat{x}_u(t) \end{bmatrix}, \qquad A_f = \begin{bmatrix} A_{fs} & A_{f1} \\ A_{f2} & A_{fu} \end{bmatrix}, \\ B_f &= \left[B_{fs} \;\; B_{fu}\right], \qquad C_f = \left[C_{fs} \;\; C_{fu}\right]. \end{aligned} \tag{38}$$

Change the variable ξ with a augmented state, ξ will become

$$\xi = \left[x_s \;\; \widehat{x}_x \;\; (x_u - \widehat{x}_u)\right]. \tag{39}$$

Equation (17) is reformed as

$$\begin{aligned} \dot{\xi}(t) &= A_a(\alpha)\xi(t) + A_{au}x_u(t) + B_a w(t), \\ e(t) &= C_a(\alpha)\xi(t) + C_{au}x_u(t), \end{aligned} \tag{40}$$

where

$$\begin{aligned} A_a(\alpha) &= \begin{bmatrix} A_s(\alpha) & 0 & 0 \\ B_{fs}C_s(\alpha) & A_{fs} & -A_{f1} \\ A_{us} - B_{fu}C_s(\alpha) & -A_{f2} & A_{fu} \end{bmatrix}, \\ A_{au} &= \begin{bmatrix} 0 \\ A_{f1} + B_{fs}C_u \\ A_u - B_{fu}C_u - A_{fu} \end{bmatrix}, \qquad B_a = \begin{bmatrix} B_s \\ B_{fs}D \\ B_u - B_{fu}D \end{bmatrix}, \\ C_a(\alpha) &= \left[L_s \;\; -C_{fs} \;\; C_{fu}\right], \qquad C_{au} = \left[L_u - C_{fu}\right]. \end{aligned} \tag{41}$$

So the partial state x_s is bounded while x_u is unbounded. If $\xi(t)$ and $e(t)$ are bounded, there must be assured that $A_{au} = 0$ and $C_{au} = 0$, also meaning that

$$A_{f1} = -B_{fs}C_u, \qquad A_{fu} = A_u - B_{fu}C_u, \qquad C_{fu} = L_u. \tag{42}$$

Now, we have solved the unstable problems and the remaining processor can use a typical LMI approach to the robust H_∞ filtering. Since we cannot remove the unstable part from an entire system, the only choice is to try a decomposition of the stable and unstable parts; the key factor lies in eliminating the unbounded estimation influence rendering that the final result is not corrupted.

4.3. Uncertainty to Kalman Filter Divergency Issue. Another drawback when applying Kalman filter to an uncertain and unstable system is divergency issue. The steady state of (4) is

$$\widehat{x}(k+1) = (A - KC)\widehat{x}(t) + Ky(t). \tag{43}$$

The estimation error dynamic function is

$$\widetilde{x}(k+1) = (A - KC)\widetilde{x}(k+1) + w(t) + Kv(t). \tag{44}$$

If $(A-KC)$ is stable, then error dynamic function is stable, meaning KF is stable. Now, suppose that A is uncertain, and the real matrix is A', and A is the nominal matrix in filtering. We obtain

$$\begin{aligned} \widetilde{x}(k+1) &= x(k+1) - \widehat{x}(k+1) \\ &= A'x(k) - A\widehat{x}(k) + KC\widehat{x}(k) - Ky(k) + w(k) \\ &= \left(A' - A\right)x(k) + A\left(x(k) - \widehat{x}(k)\right) \\ &\quad + KC\left(\widehat{x}(k) - x(k)\right) - Kv(k) + w(k) \\ &= \left(A' - A\right)x(k) + (A - KC)\widetilde{x}(k) \\ &\quad + w(k) - Kv(k). \end{aligned} \tag{45}$$

The above formulation can be written as

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ \widetilde{x}(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ A' - A & A - KC \end{bmatrix} \begin{bmatrix} x(k) \\ \widetilde{x}(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} w(k) \\ w(k) - Kv(k) \end{bmatrix}. \end{aligned} \tag{46}$$

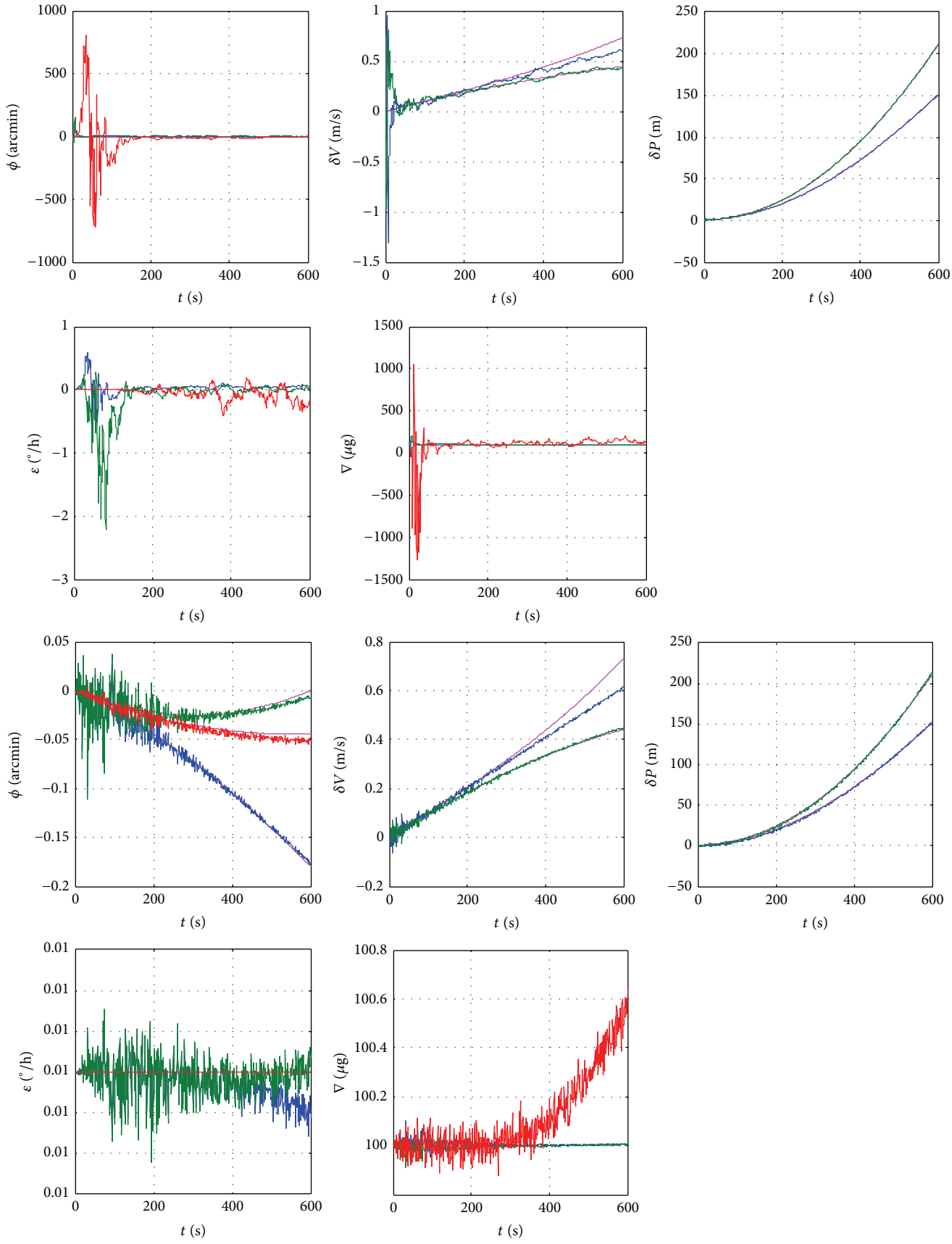

Figure 2: Normal situation Kalman filter and robust H_∞ filter results.

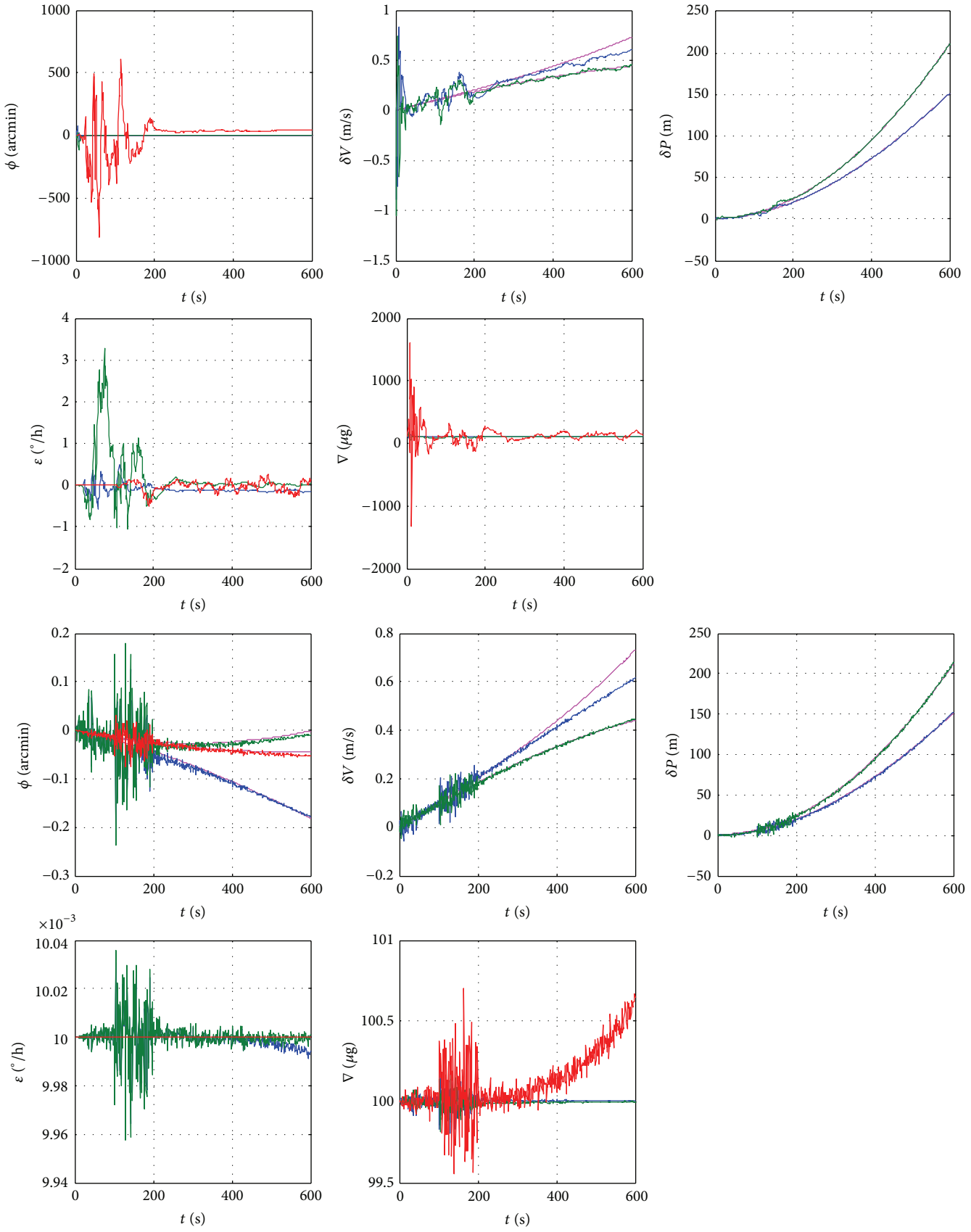

Figure 3: Interfered situation Kalman filter and robust H_∞ filter results.

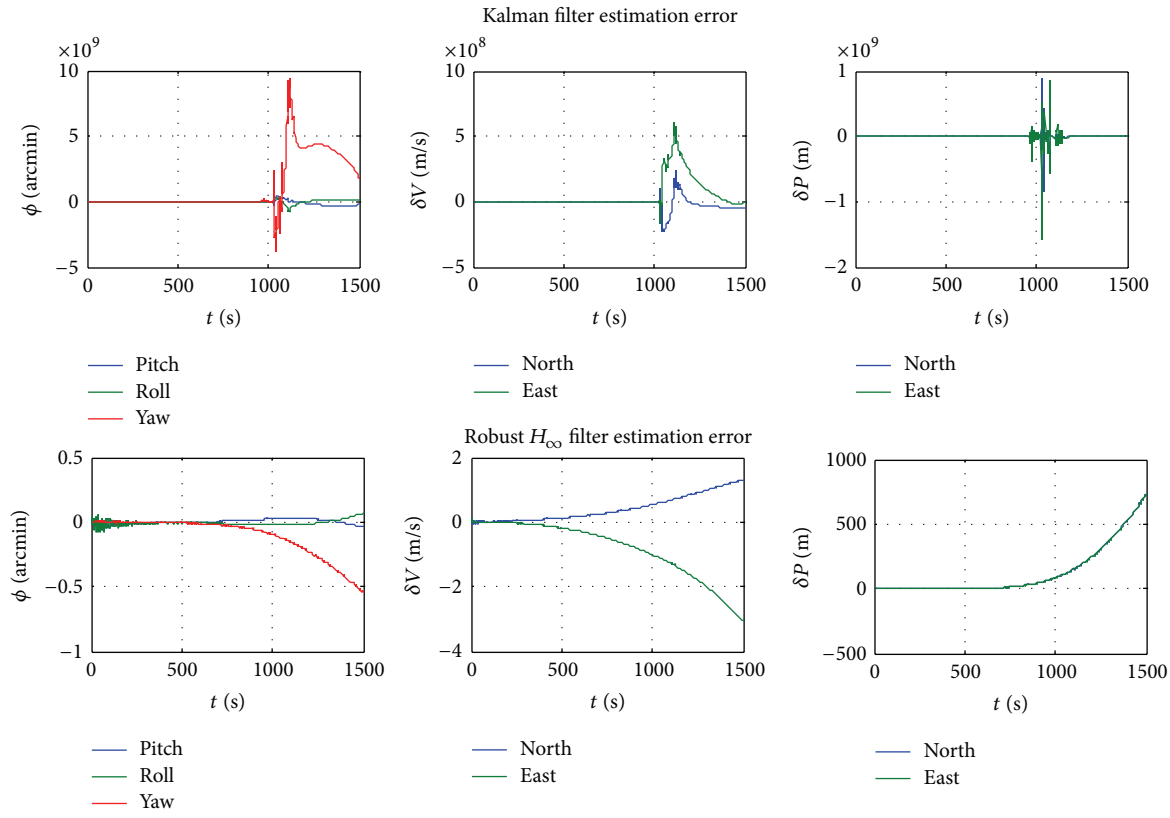

Figure 4: Normal situation Kalman filter and robust H_∞ filter results for 1500 s.

If A' and A are both unstable, when $k \to \infty$, $x(t)$ can be arbitrarily large, so even $A' - A$ is small; hence $\tilde{x}(k)$ can be arbitrarily large, and the filer is divergent.

5. Simulation Result

The state model is as (34), observations are the difference between INS and GPS position measurement. This simulation chooses a static condition, and its period lasts for 500 s. Gyros constant drift and random white noise 1 sigma are 0.02°/h and 0.01°/h; accelerator constant drift and random white noise 1 sigma are 100 μg and 10 μg. GPS position measurement white noise 1 sigma is 1 m/s. Corresponding correlation noise matrices are set as

$$\begin{aligned} Q = {} & \operatorname{diag}\left(0_{6\times1}; [0.01\,\text{m}, 0.01\,\text{m}, 0.01\,\text{m}]; \right. \\ & [0.01^\circ/\text{h}, 0.01^\circ/\text{h}, 0.01^\circ/\text{h}]; \\ & \left. [10\,\mu\text{g}, 10\,\mu\text{g}, 10\,\mu\text{g}]\right)^2, \\ R = {} & \operatorname{diag}\left([1\,\text{m}, 1\,\text{m}, 1\,\text{m}]\right)^2. \end{aligned} \tag{47}$$

Initial estimation error correlation matrix can be given as

$$\begin{aligned} P = {} & \operatorname{diag}\left([1^\circ, 1^\circ, 1^\circ], [1\,\text{m/s}, 1\,\text{m/s}, 1\,\text{m/s}], \right. \\ & [10\,\text{m}, 10\,\text{m}, 10\,\text{m}], [0.1^\circ/\text{h}, 0.1^\circ/\text{h}, 0.1^\circ/\text{h}], \\ & \left. [100\,\mu\text{g}, 100\,\mu\text{g}, 100\,\mu\text{g}]\right). \end{aligned} \tag{48}$$

This simulation software details can be found in [19], since compared to INS, GPS measurement is more easily interfered; two scenarios are considered.

(i) Normal situation means that there is an accurate prior information for the noise character.

(ii) Interfered situation means that the GPS measurement noise is 5 times compared to nominal noise character during 100–200 s.

Results are shown in Figures 2 and 3; pink curve represents the real value. Subpictures show the error attitude, error velocity, error position, gyro bias, and accelerator bias in turn. Because the up direction is divergent, only north and east results are shown here.

Triple INS compensate parameters $\delta\phi$, δV, δP statistical index STD is shown in Tables 1 and 2 for these two situations.

In normal situation, KF shows a more "stable" and "accurate" performance than H_∞ filter; the H_∞ filter exhibits a jitter character in the tracking process of real value, whereas in interfered situation, H_∞ filter is more robust, takes account of the north δV, H_∞ filter STD varies from 0.0366 to 0.0372 while KF from 0.0296 to 0.0473. Note that H_∞ filter contributes a better performance to $\delta\phi$ in both situations because H_∞ filter has a "weak" effect on estimation error matrix P which means that the initial value P has a weak influence. In the follow scenario, we extend the simulation time to 1500 s without calibration to INS using the filter output, result is shown in Figure 4. The Kalman filter suffers a

Table 1: Normal situation STD index.

STD	Kalman			Robust H_∞		
	Pitch	Roll	Yaw	Pitch	Roll	Yaw
ϕ^o	0.1130	0.1386	2.4796	0.0129	0.0149	0.035
	North	East	Up	North	East	Up
δV m/s	0.0296	0.0266	Nan	0.0366	0.0519	Nan
δP m	0.2806	0.3487	Nan	0.6351	0.8284	Nan

Table 2: Interfered situation STD index.

STD	Kalman			Robust H_∞		
	Pitch	Roll	Yaw	Pitch	Roll	Yaw
ϕ^o	0.1430	0.1786	2.9196	0.0159	0.0199	0.055
	North	East	Up	North	East	Up
δV m/s	0.0473	0.0565	Nan	0.0372	0.0568	Nan
δP m	0.4206	0.5787	Nan	0.8321	0.9674	Nan

divergence. Although there is a large trend of the estimation error, H_∞ is stable during this period.

6. Conclusion

In this paper, we design a robust H_∞ filter for a loosely coupled INS/GPS model. Firstly, a game theory gives a H_∞ approach in a Kalman form; it can be easily transformed by adjusting the noise attenuation parameter γ. Next, a robust H_∞ filter based LMI is discussed; the unstable case is considered. We also illustrate that the robust H_∞ filter based on LMI approach can convert into standard KF in special conditions. For a direct performance contrast, simulation result shows the INS error estimate state that is not compensated for INS. This proposed approach may be applied in a system under uncertainty such as low-cost IMU and interfered GPS environment.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Global Existence and Uniqueness of Solutions for a Free Boundary Problem Modeling the Growth of Tumors with a Necrotic Core and a Time Delay in Process of Proliferation

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We study a mathematical model for the growth of necrotic tumors with time delays in proliferation. By transforming this problem into an initial-boundary value problem in fixed domain of a coupled system of a parabolic equation and one integrodifferential equation with time delays, in which all equations involve discontinuous terms, and using the approximation method combined with Schauder fixed point theorem, we prove that this problem has a unique global solution in any time interval $[0, T]$.

1. Introduction

Tumor is a major threat in public health; it can cause serious problems for people at all ages. However, tumor progression is a complex process. The understanding of its dynamics is one of the great challenges of modern medical science. To describe the growth of solid tumors, an increasing number of mathematical models in forms of free boundary problems of partial differential equations have been proposed and studied during the past several decades (see, e.g., [1–12] and the references cited therein). Most of those models are based on the reaction-diffusion equations and mass conservation law. Analysis of such mathematical models has drawn great interest, and many results have been established; compare [13–22] and references therein. Many numerical results have been performed; compare [1, 2, 13, 23, 24] and references therein. Analysis of such models can help us to examine and distinguish different functions of different mechanisms involved in tumor-growth process and may also assist us in assessing effects of various drug treatments and chemotherapy.

The first equation of the model which we will study in this paper is a reaction-diffusion equation for the nutrient concentration $\sigma(r,t)$, which we present in dimensionless form:

$$c\frac{\partial\sigma}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma}{\partial r}\right) - F(\sigma)H(\sigma-\sigma_N), \quad 0<r<R(t), \quad t>0, \tag{1}$$

where H is the Heaviside function and c represents the time scale of diffusion of nutrient compared to the time scale of the tumor doubling within the tumor. $F(\sigma)$ in (1) is the consumption rate of nutrient in a unit volume in the region $\rho(t) < r < R(t)$, $t > 0$; (1) means dead cells do not consume the nutrient. σ_N is the minimal nutrient concentration needed for proliferation; that is, σ_N is a positive constant representing a threshold value for necrosis. In the region where $\sigma > \sigma_N$, nutrient is enough to sustain (at least a portion of) tumor cells alive, whereas in the region where $\sigma \le \sigma_N$, nutrient is insufficient to sustain any tumor cells alive and all cells there are dead cells.

Let $d = (1/c)$ and $F(\sigma) = cf(\sigma)$; we have

$$\frac{\partial\sigma}{\partial t} = d\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma}{\partial r}\right) - f(\sigma)H(\sigma-\sigma_N), \quad 0<r<R(t), \quad t>0. \tag{2}$$

The initial and boundary conditions for the necrotic phase of the tumor are

$$\begin{aligned}
&\frac{\partial\sigma}{\partial r}(0,t)=0, \quad \sigma(R(t),t)=\sigma_\infty, \quad t>0,\\
&\frac{\partial\sigma}{\partial r}(\rho(t)-0,t)=\frac{\partial\sigma}{\partial r}(\rho(t)+0,t), \quad t>0,\\
&\sigma(\rho(t)-0,t)=\sigma(\rho(t)+0,t)=\sigma_N, \quad t>0,\\
&\sigma(r,t)=\psi(r,t), \quad 0\le r\le R(t), \ -\tau\le t\le 0,\\
&\rho(t)=\phi(t), \quad -\tau\le t\le 0,\\
&R(t)=\varphi(t), \quad -\tau\le t\le 0.
\end{aligned} \tag{3}$$

The second equation of the model

$$\begin{aligned}
\frac{dR}{dt}=\frac{s}{R^2}\Bigg(&\int_{\rho(t-\tau)}^{R(t-\tau)}\sigma(r,t-\tau)\,r^2dr\\
&-\int_{\rho(t)}^{R(t)}\widetilde{\sigma}r^2dr-\int_0^{\rho(t)}\lambda r^2dr\Bigg),\\
&t>0,
\end{aligned} \tag{4}$$

describes the evolution of the tumor radius $R(t)$ and is obtained by applying mass balance equations with adequate constitutive laws, where $R(t)$ and $\rho(t)$ denote the external and necrotic radius of tumor at time t, respectively, and σ_∞ denotes the external concentration of nutrients, which is assumed to be a constant. The three terms on the right-hand side of (4) have the following explanation. The first term is the total tumor volume increase in a unit time interval induced by cell proliferation; the rate of cell proliferation in a unit volume is assumed to be proportional to σ. The second term is the total volume shrinkage in a unit time interval caused by cell apoptosis; the rate of cell apoptosis in a unit volume is assumed to be $s\widetilde{\sigma}$. The third term is the total volume shrinkage induced by dead cell dissolution; the rate of cell dissolution in a unit volume is assumed to be $s\lambda$. s is a scaling constant.

Equations (1) and (3) are from Bueno et al. [15]. And (4) is from [1]. The process of tumor growth can be divided into several different stages, starting from the very early stage of solid tumor without necrotic core inside (cf. [19, 20, 22, 25, 26]). In the present paper we focus on the next stage, that is, the process of necrotic core formation. In this stage there are three main cellular processes: proliferation, apoptosis, and necrosis (cf. [1, 2, 8, 13, 14, 16, 27, 28]). For process of necrotic core formation, the study by using the method of mathematical models was initiated by Byrne in [1]. Recently this study has drawn the attention of some other researchers; compare Bueno et al. [15], Cui [16], Foryś and Mokwa-Borkowska [13], and references cited therein. But all of them are related to quasistationary version; that is, $c=0$, and Cui [16] and Foryś and Mokwa-Borkowska [13] studied the particular cases in which $F(\sigma)=\Gamma\sigma$ and $F(\sigma)=a$, respectively, where Γ and a are two constants.

The model we studied in this paper is established by modifying the model studied in [15] by introducing a time delay in proliferation as that in Byrne [1]. In their model, the cell proliferation rate in unit volume is $S(\sigma)$, where S is a continuous function which satisfies some conditions; that is, (4) is as follows ($\tau=0$):

$$\begin{aligned}
\frac{dR}{dt}=\frac{s}{R^2}\Bigg(&\int_{\rho(t)}^{R(t)}S(\sigma(r,t))\,r^2dr\\
&-\int_{\rho(t)}^{R(t)}\widetilde{\sigma}r^2dr-\int_0^{\rho(t)}\lambda r^2dr\Bigg),\\
&t>0.
\end{aligned} \tag{5}$$

But for simplicity we assume that the cell proliferation rate in unit volume is proportional to its concentration; that is, $S(\sigma)=\mu\sigma$. The methods presented in this paper can be extended to the case that the cell proliferation rate in unit volume is $S(\sigma)$, where S is a continuous function that satisfies some conditions. It should be pointed out that, in [15], the authors only study the existence and uniqueness of stationary solutions to the model (2)–(4), but in our study we mainly discuss the existence and uniqueness of the solution to the model (2)–(4).

2. Main Results

In this paper, we always use the following notation

$$\begin{aligned}
&\overline{B}_1(0)=\left\{y\in R^3,|y|\le 1\right\}, \qquad \overline{Q}_T=\overline{B}_1(0)\times[0,T],\\
&W_p^{2,1}\left(\overline{Q}_T\right)=\left\{u\in L^p\left(\overline{Q}_T\right):\nabla u,\nabla^2u,u_t\in L^p\left(\overline{Q}_T\right)\right\}.
\end{aligned} \tag{6}$$

We will prove a global existence and uniqueness theorem for the problem (2)–(4) under the following assumptions:

(A_1) $f(\sigma)=\sigma f_1(\sigma)$, where f_1 is defined and Lipschitz is continuous on $[0,\infty)\times[0,\infty)$, and $f_1(\sigma)\ge 0$, for $\sigma\ge 0$;

(A_2) $\varphi\in C[-\tau,0]$, $\varphi(0)>\rho(0)>0$, for $-\tau\le t\le 0$, $\psi\in C([0,\infty]\times[-\tau,0])$, and $\psi(r,t)=\sigma_\infty$, for $r\ge R(t)$; $0\le\psi(r,t)\le\sigma_\infty$, for $r\le R(t)$;

(A_3) $\psi(r,0)=\psi_0(r)$ is twice weakly differentiable on $[0,R(0)]$, $\psi_0''\in L^\infty[0,R(0)]$, $\psi_0'(0)=0$, $\psi_0(R(0))=\sigma_\infty$.

Our main result is as follows.

Theorem 1. *Assume that conditions* (A_1), (A_2), *and* (A_3) *are satisfied. Then for any* $T>0$, *the problem* (2)–(4) *has a solution* $(\sigma(r,t),R(t))$, *for all* $0\le t\le T$, *satisfying* $\sigma(|x|,t)\in W_p^{2,1}(Q_T)$ $(5<p<\infty)$ *and* $0\le\sigma(|x|,t)\le\sigma_\infty$.

3. Transformation to a Fixed Domain

We introduce a transformation of $(r,t,\sigma,R)\to(z,t,u,R)$ as follows:

$$\begin{aligned}
&z=\frac{r}{R(t)}, \quad t=t,\\
&u(z,t)=\sigma(zR(t),t), \quad R(t)=R(t).
\end{aligned} \tag{7}$$

Then the problem (2)–(4) is transformed into the following problem:

$$\frac{\partial u}{\partial t} = d\frac{1}{R^2(t)}\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial u}{\partial z}\right) + \frac{R'(t)}{R(t)}zu_z - f(u)H(u-\sigma_N), \quad 0<z<1,\ t>0, \tag{8}$$

$$\frac{\partial u}{\partial z}(0,t) = 0, \quad u(1,t) = \sigma_\infty, \quad t>0, \tag{9}$$

$$\frac{dR}{dt} = sR(t)\left(\int_0^1 \frac{R_\tau^3}{R^3}u_\tau H(u_\tau-\sigma_N)z^2 - \widetilde{\sigma}H(u-\sigma_N)z^2 - \lambda\left[1-H(u-\sigma_N)\right]z^2dz\right), \quad t>0, \tag{10}$$

$$u(z,t) = \psi(zR(t),t), \quad 0\le z\le 1,\ -\tau\le t\le 0, \tag{11}$$

$$\frac{\partial u}{\partial z}(z\rho(t)-0,t) = \frac{\partial u}{\partial z}(z\rho(t)+0,t), \quad 0\le z\le 1, \quad t>0, \tag{12}$$

$$u(z\rho(t)-0,t) = u(z\rho(t)+0,t) = \sigma_N, \quad 0\le z\le 1, \quad t>0, \tag{13}$$

$$\rho(t) = \phi(t), \quad -\tau\le t\le 0, \tag{14}$$

$$R(t) = \varphi(t), \quad -\tau\le t\le 0, \tag{15}$$

where $R_\tau = R(t-\tau)$, $u_\tau = u(z,t-\tau)$. Let $X_T = C([0,1]\times[0,T])\times C([0,1]\times[-\tau,T])$. Denote the function G as follows

$$G(u,u_\tau,t) = s\left(\int_0^1 \frac{R_\tau^3}{R^3}u_\tau H(u_\tau-\sigma_N)z^2 - \widetilde{\sigma}H(u-\sigma_N)z^2 - \lambda\left[1-H(u-\sigma_N)\right]z^2dz\right) \tag{16}$$

for $t>0$. Using this notation, (10) can be simply written as follows:

$$R'(t) = R(t)G(u,u_\tau,t), \quad t>0. \tag{17}$$

It is clear that $R(t)$ can be expressed by

$$R(t) = R_0\exp\left\{\int_0^t G(u,u_\tau,\xi)\,d\xi\right\}, \quad t>0. \tag{18}$$

Hence the problem (8)–(15) is reduced to the problem below:

$$\frac{\partial u}{\partial t} = dR_0^{-2}\exp\left\{-2\int_0^t G(u,u_\tau,\xi)\,d\xi\right\} \times \frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial u}{\partial z}\right) + G(u,u_\tau,t)zu_z - f(u)H(u-\sigma_N), \quad 0<z<1, \quad t>0, \tag{19}$$

$$\frac{\partial u}{\partial z}(0,t) = 0, \quad u(1,t) = \sigma_\infty, \quad t>0, \tag{20}$$

$$u(z,t) = \psi(zR(t),t), \quad 0\le z\le 1,\ -\tau\le t\le 0, \tag{21}$$

$$\frac{\partial u}{\partial z}(z\rho(t)-0,t) = \frac{\partial u}{\partial z}(z\rho(t)+0,t), \quad 0\le z\le 1,\ t>0, \tag{22}$$

$$u(z\rho(t)-0,t) = u(z\rho(t)+0,t) = \sigma_N, \quad 0\le z\le 1,\ t>0, \tag{23}$$

$$\rho(t) = \phi(t), \quad -\tau\le t\le 0, \tag{24}$$

$$R(t) = \varphi(t), \quad -\tau\le t\le 0. \tag{25}$$

Then we conclude the following.

Lemma 2. *Under the variable transformation* (7)*, the free boundary problem* (2)–(4) *is equivalent to the initial-boundary value problem* (19)–(25).

4. The Proof of Main Results

First we consider the following approximation problem:

$$\frac{\partial u}{\partial t} = dR_0^{-2}\exp\left\{-2\int_0^t G_\varepsilon(u,u_\tau,\xi)\,d\xi\right\} \times \frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial u}{\partial z}\right) + G_\varepsilon(u,u_\tau,t)zu_z - f(u)H(u-\sigma_N), \quad 0<z<1,\ t>0, \tag{26}$$

with initial and boundary condition as (20)–(25), where ε is an arbitrary positive number,

$$G_\varepsilon(u,u_\tau,t) = s\left(\int_0^1 \frac{R_\tau^3}{R^3}u_\tau H_\varepsilon(u_\tau-\sigma_N)z^2 - \widetilde{\sigma}H_\varepsilon(u-\sigma_N)z^2 - \lambda\left[1-H_\varepsilon(u-\sigma_N)\right]z^2dz\right), \tag{27}$$

$$H_\varepsilon(x) = \begin{cases} 1, & x\ge\varepsilon, \\ \dfrac{\varepsilon}{x}, & 0<x<\varepsilon, \\ 0, & x\le 0. \end{cases}$$

It is clear that H_ε is a Lipschitz continuous function.

Recall that $u(z,t) = \psi(zR(t),t)$, $0 \le z \le 1$, $-\tau \le t \le 0$; by assumption (A_3) we know that $u(r,0) = \psi_0(r) = \psi(rR(0),0)$ is twice weakly differentiable on $[0,1]$, $\psi_0'' \in L^\infty[0,1]$, $\psi_0'(0) = 0$, and $\psi_0(1) = \sigma_\infty$.

Lemma 3. *If the assumptions (A_1)–(A_3) hold, then for any $T > 0$ the problem (26), (20)–(25) has a unique solution u_ε on $[0,1] \times [-\tau, T]$ satisfying the following assertions:*

(i) *$u_\varepsilon(y,t) \in W_p^{2,1}(\overline{Q}_T)$, $p \in (1,\infty)$ and for any $\varepsilon > 0$ there holds*

$$\left\| u_\varepsilon\left(|y|,t\right)\right\|_{W_p^{2,1}(\overline{Q}_T)} \le C\left(T,p\right)\left(\left\|u_0\left(|y|,0\right)\right\|_{W^{2,\infty}(0,1)} + \sigma_\infty\right), \tag{28}$$

where $C(T,p)$ is a constant independent of ε, $u_0(y,0) = \psi(zR(0),0)$;

(ii) *For any $\varepsilon > 0$, there holds $0 \le |u_\varepsilon(|y|,t)| \le \sigma_\infty$, $0 \le y \le 1$, $-\tau \le t \le T$.*

Remark 4. From (i) and assumption (A_2) it is easy to get

$$\begin{aligned}&\left\| u_\varepsilon\left(|y|,t-\tau\right)\right\|_{W_p^{2,1}(\overline{Q}_T)}\\ &\quad\le C\left(T,p\right)\left(\left\|u_0\left(|y|,0\right)\right\|_{W^{2,\infty}(0,1)} + \sigma_\infty\right).\end{aligned} \tag{29}$$

Proof. Endow X_T with the following norm:

$$\left\|(u,u_\tau)\right\| = \max_{[0,1]\times[-\tau,T]} |u|. \tag{30}$$

Then X_T is clearly a Banach space. Define a mapping $F : X_T \to X_T$ as follows: for any $(u,u_\tau) \in X_T$, define $F(u,u_\tau) = (\widetilde{u},\widetilde{u}_\tau)$, where $\widetilde{u}(z,t) = \widetilde{\widetilde{u}}(zy^o,t)$, $y^o \in R^3$, $|y^o| = 1$, $u_\tau = \widetilde{u}(z,t-\tau)$, and $\widetilde{\widetilde{u}}$ is the solution of the following problem:

$$\begin{aligned}&\frac{\partial\widetilde{\widetilde{u}}}{\partial t} = dR_0^{-2}\exp\left\{-2\int_0^t G_\varepsilon\left(u,u_\tau,\xi\right)d\xi\right\}\Delta\widetilde{\widetilde{u}}\\ &\qquad + G_\varepsilon\left(u,u_\tau,t\right)z\cdot\nabla\widetilde{\widetilde{u}}\\ &\qquad -\widetilde{\widetilde{u}}f_1\left(u\right)H\left(u-\sigma_N\right),\\ &\qquad \widetilde{\widetilde{u}}\left(y,t\right) = \sigma_\infty, \quad |y| = 1,\ t > 0,\\ &\frac{\partial\widetilde{\widetilde{u}}}{\partial z}\left(z\rho\left(t\right)-0,t\right) = \frac{\partial\widetilde{\widetilde{u}}}{\partial z}\left(z\rho\left(t\right)+0,t\right),\\ &\qquad\qquad 0 \le z \le 1, \quad t > 0,\\ &\widetilde{\widetilde{u}}\left(z\rho\left(t\right)-0,t\right) = \widetilde{\widetilde{u}}\left(z\rho\left(t\right)+0,t\right) = \sigma_N,\\ &\qquad\qquad 0 \le z \le 1, \quad t > 0,\\ &\widetilde{\widetilde{u}}\left(y,t\right) = u_0\left(y,t\right), \quad |y| \le 1,\ -\tau \le t \le 0,\end{aligned} \tag{31}$$

where $u_0(y,t) = \psi(zR(t),t)$. First we prove that F is well defined. Since all coefficients in the first equation of (31) are bounded continuous functions and

$$R_0^{-2}\exp\left\{-2\int_0^t G_\varepsilon\left(u,u_\tau,\xi\right)d\xi\right\} \ge \lambda_\varepsilon\left(T,\left\|(u,u_\tau)\right\|,\tau\right) > 0, \quad 0 \le t \le T. \tag{32}$$

Using the L^p theory linear parabolic equations [29] we infer that the problem (31) has a unique solution $\widetilde{\widetilde{u}}$. Since the initial and boundary values are symmetric in space variable y, it is easy to see that, for any orthogonal transformation $O : R^3 \to R^3$, $\widetilde{\widetilde{u}}_1(y,t) = \widetilde{\widetilde{u}}(Oy,t)$ are also solutions of this problem which implies, by uniqueness of the solution, that there exists functions $\widetilde{u} \in C([0,1]\times[-\tau,T])$ such that

$$\widetilde{\widetilde{u}}\left(y,t\right) = \widetilde{u}\left(|y|,t\right), \quad |y| \le 1,\ -\tau \le t \le T. \tag{33}$$

By L^p estimates we have, for $p \in (1,\infty)$,

$$\begin{aligned}\left\|\widetilde{\widetilde{u}}\left(y,t\right)\right\| &\le C\left(T,p,\|u\|\right)\left\|u_0\left(|y|,0\right)\right\|_{W^{2,\infty}(B_1(0))}\\ &\quad + \sigma_\infty + \left\|\widetilde{\widetilde{u}}\left(|y|,t\right)\right\|_{L^\infty(\overline{Q}_T)}.\end{aligned} \tag{34}$$

By the maximum principle one can get

$$0 \le \widetilde{\widetilde{u}}\left(y,t\right) \le \sigma_\infty, \quad |y| \le 1,\ 0 \le t \le T. \tag{35}$$

Hence

$$\begin{aligned}&\left\|\widetilde{u}\left(|y|,t\right)\right\|_{W_p^{2,1}(\overline{Q}_T)}\\ &\quad\le C\left(T,p,\left\|(u,u_\tau)\right\|\right)\\ &\qquad\times\left(\left\|u_0\left(|y|,0\right)\right\|_{W^{2,\infty}(0,1)} + \sigma_\infty\right),\end{aligned} \tag{36}$$

$$0 \le \widetilde{u}\left(y,t\right) \le \sigma_\infty, \quad |y| \le 1,\ -\tau \le t \le T. \tag{37}$$

Taking particularly $p > (5/2)$, by the embedding $W_p^{2,1}(\overline{Q}_T) \hookrightarrow\hookrightarrow C(\overline{Q}_T)$, $(p > (5/2))$, we infer that $(\widetilde{u},\widetilde{u}_\tau) \in X_T$. Hence the mapping F is well defined.

Let $E = \{(u,u_\tau) : (u,u_\tau) \in C([0,1]\times[0,T]) \times C([0,1]\times[-\tau,T]), 0 \le u, u_\tau \le \sigma_\infty\}$. E is clearly a bounded closed convex subset of X_T. We first prove that $F(E)$ is precontract in X_T. Actually, since, for any $(u,u_\tau) \in E$, $\|(u,u_\tau)\| \le \sigma_\infty$ by (36) we infer that for any $1 < p < \infty$ there hold

$$\left\|\widetilde{u}\left(|y|,t\right)\right\| \le C\left(T,p\right). \tag{38}$$

Taking $p > (5/2)$ and using the compact embedding $W_p^{2,1}(\overline{Q}_T) \hookrightarrow\hookrightarrow C(\overline{Q}_T)$, $(p > (5/2))$, we conclude that $F(E)$ is precontract in X_T.

Next we prove that F is continuous. Let $(u_1, u_{1\tau})$, $(u_2, u_{2\tau}) \in X_T$ and denote $(\widetilde{u}_1, \widetilde{u}_{1\tau}) = F(u_1, u_{1\tau})$, $(\widetilde{u}_2, \widetilde{u}_{2\tau}) = F(u_2, u_{2\tau})$, $\widetilde{\widetilde{u}}^* = \widetilde{u}_1 - \widetilde{u}_2$, $\widetilde{\widetilde{u}}^*_{1\tau} = \widetilde{u}_{1\tau} - \widetilde{u}_{2\tau}$. It is obvious that $\widetilde{\widetilde{u}}^*$ is a solution of the following problem:

$$\begin{aligned}
\frac{\partial \widetilde{\widetilde{u}}^*}{\partial t} &= dR_0^{-2} \exp\left\{-2\int_0^t G_\varepsilon\left(u_1, u_{1\tau}, \xi\right) d\xi\right\} \Delta \widetilde{\widetilde{u}}^* \\
&\quad + G_\varepsilon\left(u_1, u_{1\tau}, t\right)\left(y \cdot \nabla \widetilde{\widetilde{u}}^*\right) \\
&\quad - \widetilde{\widetilde{u}}^* f\left(u_1\right) H_\varepsilon\left(u_1 - \sigma_N\right) + h\left(y, t\right), \\
&\quad \widetilde{\widetilde{u}}\left(y, t\right) = 0, \quad |y| = 1, \ t > 0, \\
\frac{\partial \widetilde{\widetilde{u}}}{\partial z}\left(z\rho\left(t\right) - 0, t\right) &= \frac{\partial \widetilde{\widetilde{u}}}{\partial z}\left(z\rho\left(t\right) + 0, t\right), \\
&\quad 0 \le z \le 1, \quad t > 0, \\
\widetilde{\widetilde{u}}\left(z\rho\left(t\right) - 0, t\right) &= \widetilde{\widetilde{u}}\left(z\rho\left(t\right) + 0, t\right) = 0, \\
&\quad 0 \le z \le 1, \quad t > 0, \\
\widetilde{\widetilde{u}}\left(y, t\right) &= 0, \quad |y| \le 1, \ -\tau \le t \le 0,
\end{aligned} \tag{39}$$

where $h(y,t) = dR_0^{-2}[\exp\{-2\int_0^t G_\varepsilon(u_1, u_{1\tau}, \xi)d\xi\} - \exp\{-2\times \int_0^t G_\varepsilon(u_2, u_{2\tau}, \xi)d\xi\}]\widetilde{\widetilde{u}}_2 + [G_\varepsilon(u_1, u_{1\tau}, t) - G_\varepsilon(u_2, u_{2\tau}, t)](y \cdot \nabla\widetilde{\widetilde{u}}_2) - \widetilde{\widetilde{u}}_2[f_1(u_1)H_\varepsilon(u_1 - \sigma_N) - f_1(u_2)H_\varepsilon(u_2 - \sigma_N)]$.

Clearly,

$$G_\varepsilon\left(u, u_\tau, t\right) \le C\left(\left\|\left(u, u_\tau\right)\right\|, T\right) \tag{40}$$

for $(u, u_\tau) \in X_T$, $t \in [0, T]$, where $C(\|(u, u_\tau)\|)$ is independent of ε. Then one can get

$$\begin{aligned}
&e^{-2TC(\|(u,u_\tau)\|,T)} \\
&\quad \le \exp\left\{-2\int_0^t G_\varepsilon\left(u, u_\tau, \xi\right)\right\} d\xi \le e^{2TC(\|(u,u_\tau)\|,T)}, \\
&\left|\exp\left\{-2\int_0^t G_\varepsilon\left(u, u_\tau, \xi\right)\right\} d\xi - \exp\left\{-2\int_0^{t'} G_\varepsilon\left(u, u_\tau, \xi\right)\right\} d\xi\right| \\
&\quad \le C\left(\left\|\left(u, u_\tau\right)\right\|\right)\left|t - t'\right|
\end{aligned} \tag{41}$$

for $u, v \in X_T$, $t \in [0, T]$. Besides, it is also clear that

$$0 \le H_\varepsilon\left(u - \sigma_N\right) \le 1, \quad |y| \le 1, \ 0 \le t \le T. \tag{42}$$

Hence, by the L^p estimate one can get

$$\left\|\widetilde{\widetilde{u}}^*\right\|_{W_p^{2,1}(\overline{Q}_T)} \le C\left(T, p, \left\|\left(u_1, u_{1\tau}\right)\right\|\right) \|h\|_{L^p(\overline{Q}_T)}. \tag{43}$$

By the fact that H_ε is Lipschitz continuous and (36), one can verify

$$\begin{aligned}
\|h\|_{L^p(\overline{Q}_T)} &\le C\left(T, p, \left\|\left(u_1, u_{1\tau}\right)\right\|, \left\|\left(u_2, u_{2\tau}\right)\right\|, \varepsilon\right) \\
&\quad \times \left\|\left(u_1 - u_2, u_{1\tau} - u_{2\tau}\right)\right\|.
\end{aligned} \tag{44}$$

Substituting these estimates into (43) we have

$$\begin{aligned}
\left\|\widetilde{\widetilde{u}}^*\right\|_{W_p^{2,1}(\overline{Q}_T)} &\le C\left(T, p, \left\|\left(u_1, u_{1\tau}\right)\right\|, \left\|\left(u_2, u_{2\tau}\right)\right\|, \varepsilon\right) \\
&\quad \times \left\|\left(u_1 - u_2, u_{1\tau} - u_{2\tau}\right)\right\|.
\end{aligned} \tag{45}$$

Taking $p > (5/2)$ and using the embedding $W_p^{2,1}(\overline{Q}_T) \hookrightarrow\hookrightarrow C(\overline{Q}_T)$ we get

$$\begin{aligned}
&\left\|\left(\widetilde{u}_1 - \widetilde{u}_2, \widetilde{u}_{1\tau} - \widetilde{u}_{2\tau}\right)\right\| \\
&\quad \le \left\|\widetilde{\widetilde{u}}^*\right\|_{W_p^{2,1}(\overline{Q}_T)} \\
&\quad \le C\left(T, p, \left\|\left(u, u_\tau\right)\right\|, \varepsilon\right)\left\|\left(u_1 - u_2, u_{1\tau} - u_{2\tau}\right)\right\|.
\end{aligned} \tag{46}$$

Hence, F is continuous.

Since we have proven that F maps E into itself, using the Schauder fixed point theorem we conclude that F has a fixed point in E. Therefore, the problem (19)–(25) has a solution $u = u_\varepsilon$.

Next, we prove uniqueness. Let $u^* = u_1 - u_2$. Then u^* satisfy

$$\begin{aligned}
\frac{\partial u^*}{\partial t} &= dR_0^{-2} \exp\left\{-2\int_0^t G_\varepsilon\left(u_1, u_{1\tau}, \xi\right) d\xi\right\} \Delta u^* \\
&\quad + G_\varepsilon\left(u_1, u_{1\tau}, t\right)\left(y \cdot u^*\right) + p\left(y, t\right)
\end{aligned} \tag{47}$$

for $|y| < 1$, $t > 0$, where $p(y,t) = dR_0^{-2}[\exp\{-2\int_0^t G_\varepsilon(u_1, u_{1\tau}, \xi)d\xi\} - \exp\{-2\int_0^t G_\varepsilon(u_2, u_{2\tau}, \xi)d\xi\}]\Delta u_2 + [G_\varepsilon(u_1, u_{1\tau}, t) - G_\varepsilon(u_2, u_{2\tau}, t)](y\cdot\nabla u_2) - [f(u_1) - f(u_2)]H_\varepsilon(u_1(|y|, t) - \sigma_N) - f(u_2)[H_\varepsilon(u_1(|y|, t) - \sigma_N) - H_\varepsilon(u_2(|y|, t) - \sigma_N)]$.

Multiplying (47) with u^* and integrating in y, one can get

$$\begin{aligned}
&\frac{1}{2}\int_{\|y\|\le 1} \left(u^*\left(y, t\right)\right)^2 dy \\
&\quad = -dR_0^{-2} \exp\left\{-2\int_0^t G_\varepsilon\left(u_1, u_{1\tau}, \xi\right) d\xi\right\} \\
&\qquad \times \int_{\|y\|\le 1} \left|\nabla u^*\left(y, t\right)\right|^2 dy \\
&\qquad - \frac{3}{2} G_\varepsilon\left(u_1, u_{1\tau}, t\right) \int_{\|y\|\le 1} \left(u^*\left(y, t\right)\right)^2 dy \\
&\qquad + \int_{\|y\|\le 1} p\left(y, t\right) u^*\left(y, t\right) dy \\
&\quad \le -dR_0^{-2} \exp\left(-2TC\left(\left\|\left(u_1, u_{1\tau}\right)\right\|\right)\right) \left\|\nabla u^*\left(\cdot, t\right)\right\|^2_{L^2(B_1(0))} \\
&\qquad + \frac{3}{2} C\left(\left\|\left(u_1, u_{1\tau}\right)\right\|\right) \left\|u^*\left(\cdot, t\right)\right\|^2_{L^2(B_1(0))} + I\left(t\right),
\end{aligned} \tag{48}$$

where $I(t) = I_1 + I_2 + I_3 + I_4$ and

$$\begin{aligned} I_1 &= dR_0^{-2}\left[\exp\left\{-2\int_0^t G_\varepsilon(u_1, u_{1\tau}, \xi)\,d\xi\right\}\right. \\ &\qquad \left. - \exp\left\{-2\int_0^t G_\varepsilon(u_2, u_{2\tau}, \xi)\,d\xi\right\}\right] \\ &\quad \times \int_{\|y\|\le 1} \Delta u_2 \cdot \nabla u^*(y,t)\,dy, \\ I_2 &= [G_\varepsilon(u_1, u_{1\tau}, t) - G_\varepsilon(u_2, u_{2\tau}, t)] \\ &\quad \times \int_{\|y\|\le 1} (y\cdot\nabla u_2)\nabla u^*(y,t)\,dy, \\ I_3 &= -\int_{\|y\|\le 1} [f(u_1) - f(u_2)]\, H_\varepsilon(u_1(|y|,t) - \sigma_N) \\ &\qquad \times u^*(y,t)\,dy, \\ I_4 &= -\int_{\|y\|\le 1} f(u_2)\,[H_\varepsilon(u_1(|y|,t) - \sigma_N) \\ &\qquad - H_\varepsilon(u_2(|y|,t) - \sigma_N)]\,u^*(y,t)\,dy. \end{aligned} \tag{49}$$

By the mean value theorem and the fact that $f(u) \ge 0$, we have

$$I_4 = -\int_{\|y\|\le 1} f(u_2)\int_0^1 H_\varepsilon' d\theta (u^*)^2 dy \le 0. \tag{50}$$

Using (28) and embedding $W_p^{2,1}(\overline{Q}_T) \hookrightarrow\hookrightarrow C(\overline{Q}_T)$, ($p > (5/2)$), we get

$$\max_{\overline{Q}_T} |\nabla u_2(|y|,t)| \le C(T), \qquad \max_{\overline{Q}_T} |\nabla u_{2\tau}(|y|,t)| \le C(T). \tag{51}$$

From definition of G_ε and Lipschitz continuous of H_ε, we have

$$|G_\varepsilon(u_1, u_{1\tau}, t) - G_\varepsilon(u_2, u_{2\tau}, t)| \le C(\varepsilon)\left(\|u^*\|_2 + \|u_\tau^*\|_2\right). \tag{52}$$

Then by using (51) one can get

$$I_2 \le C(\varepsilon)\left(\|u^*\|_2 + \|u_\tau^*\|_2\right)\int_{\|y\|\le 1} |u^*(y,t)|\,dy. \tag{53}$$

By a direct computation and (52) one can get

$$I_1 \le C(\varepsilon, T)\int_0^t \left(\|u^*\|_2 + \|u_\tau^*\|_2\right)d\xi \int_{\|y\|\le 1} |\nabla u^*(y,t)|\,dy. \tag{54}$$

Lipschitz continuity of $f(u)$ combined with (51) yields

$$I_3 \le C(\varepsilon, T)\int_{\|y\|\le 1} |u^*(y,t)|^2 dy. \tag{55}$$

Summing up (51), (53)–(55) into $I(t)$, one can get

$$\begin{aligned} I(t) &\le C(\varepsilon)\left(\|u^*\|_2 + \|u_\tau^*\|_2\right)\int_{\|y\|\le 1} |u^*(y,t)|\,dy \\ &\quad + C(\varepsilon, T)\int_0^t \left(\|u^*\|_2 + \|u_\tau^*\|_2\right)d\xi \int_{\|y\|\le 1} |\nabla u^*(y,t)|\,dy \\ &\quad + C(\varepsilon, T)\int_{\|y\|\le 1} |u^*(y,t)|^2 dy. \end{aligned} \tag{56}$$

Using Cauchy inequality and summing up (51), (53)–(55) into $I(t)$, one can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^*(\cdot,t)\|_2^2 \\ &\quad \le -C_1(T)\|\nabla u^*(\cdot,t)\|_2^2 \\ &\qquad + C_2\|u^*(\cdot,t)\|_2^2 + C(\varepsilon)\left(\|u^*\|_2 + \|u_\tau^*\|_2\right) \\ &\qquad \times \int_{\|y\|\le 1} |u^*(y,t)|\,dy \\ &\qquad + C(\varepsilon, T)\int_0^t \left(\|u^*\|_2 + \|u_\tau^*\|_2\right)d\xi \int_{\|y\|\le 1} |\nabla u^*(y,t)|\,dy \\ &\qquad + C(\varepsilon, T)\int_{\|y\|\le 1} |u^*(y,t)|^2 dy. \end{aligned} \tag{57}$$

Denoting $w(t) = \|u^*(\cdot,t)\|_2^2$ and $w_1(t) = \|\nabla u^*(\cdot,t)\|_2^2$, one can get

$$\begin{aligned} \frac{1}{2}\frac{dw}{dt} &\le -C_1(T)w_1(t) + C_2(T)w(t) \\ &\quad + J(t) + H(t) + C(\varepsilon, T)w(t), \end{aligned} \tag{58}$$

where $J(t) = C(\varepsilon)(\|u^*\|_2 + \|u_\tau^*\|_2)\int_{\|y\|\le 1} |u^*(y,t)|dy = C(\varepsilon)[\sqrt{w(t)} + \sqrt{w(t-\tau)}]\sqrt{w(t)}$, $H(t) = C(\varepsilon, T)\int_0^t (\|u^*\|_2 + \|u_\tau^*\|_2)d\xi \int_{\|y\|\le 1} |\nabla u^*(y,t)|dy = C(\varepsilon, T)\int_0^t \sqrt{w(t)} + \sqrt{w(t-\tau)}d\xi\sqrt{w_1(t)}$. By the fact that $u^* = 0$, for $-\tau \le t \le 0$, then

$$\begin{aligned} w(t-\tau) &= \|u^*(\cdot, t-\tau)\|_2^2 = \int_{\|y\|\le 1, t>0} |u^*(\cdot, t-\tau)|^2 dy \\ &= \int_{\|y\|\le 1, t>0} |u^*(\cdot,t)|^2 dy + \int_{\|y\|\le 1, -\tau\le t\le 0} |u^*(\cdot,t)|^2 dy \\ &= \int_{\|y\|\le 1, t>0} |u^*(\cdot,t)|^2 dy = w(t) = \|u^*(\cdot,t)\|_2^2. \end{aligned} \tag{59}$$

It follows that

$$\begin{aligned} J(t) &\le 2C(\varepsilon)w(t), \\ H(t) &\le C(\varepsilon, T)C(\delta)\int_0^t w(\xi)\,d\xi + C(\varepsilon, T)\delta w_1(t), \end{aligned} \tag{60}$$

where we have used the δ-Cauchy inequality, δ is an arbitrary positive number, and $C(\delta)$ is a positive constant depending on δ. Then we have

$$\frac{1}{2}\frac{dw}{dt} \le -C_1(T)\,w_1(t) + C_3(T)\,w(t) + 2C(\varepsilon)\,C(\delta)\int_0^t w(\xi)\,d\xi + C(\varepsilon,T)\,\delta w_1(t), \tag{61}$$

where $C_3(T) = C_2(T)+2C(\varepsilon)+C(\varepsilon,T)$. Therefore, by choosing δ sufficiently small such that $C(\varepsilon,T)\delta < C_1(T)$ one can get

$$\frac{1}{2}\frac{dw}{dt} \le C_3(T)\,w(t) + 2C(\varepsilon)\,C(\delta)\int_0^t w(\xi)\,d\xi. \tag{62}$$

Since $w(0) = 0$, by Gronwall lemma we conclude that $w(t) \equiv 0$, for $0 \le t \le T$. Hence $u_1 = u_2$. □

Lemma 5. *If conditions (A_1)–(A_3) are satisfied, then for any $T > 0$ the problem (19)–(25) has a unique solution (u, u_τ) on X_T satisfying*

$$u(|y|,t), u(|y|,t-\tau) \in W_p^{2,1}\left(\overline{Q}_T\right), \quad p \in (1,\infty), \quad 0 \le u(z,t) \le \sigma_\infty, \quad 0 \le z \le 1,\ -\tau \le t \le T. \tag{63}$$

Proof. By Lemma 3 for any $\varepsilon > 0$ the problem (26), (20)–(25) has a unique solution u_ε on $[0,1] \times [-\tau, T]$. Take a number $p > 5$, by compact embedding $W_p^{2,1}(\overline{Q}_T) \hookrightarrow\hookrightarrow C^{\alpha,(\alpha/2)}(\overline{Q}_T)$ $(0 < \alpha < 2-(5/p))$. It follows that one can find a sequence of positive numbers $\varepsilon_k \to 0, k \to \infty$, and function $u \in W_p^{2,1}(\overline{Q}_T)$ such that if we denote $u_k = u_{\varepsilon_k}$, then

$u_k(|y|,t) \to u(|y|,t)$, $\nabla u_k(|y|,t) \to \nabla u(|y|,t)$ uniformly for $(y,t) \in \overline{Q}_T$,

$u_k(|y|,t-\tau) \to u(|y|,t-\tau)$, $\nabla u_k(|y|,t-\tau) \to \nabla u(|y|,t-\tau)$ uniformly for $(y,t) \in \overline{Q}_T$,

$u_{kt}(|y,t|)\overrightarrow{w}u_t(|y|,t)$ $(L^p(\overline{Q}_T))$, $(\partial^2 u_k/\partial y_i \partial y_j)\overrightarrow{w}(\partial^2 u/\partial y_i \partial y_j)(L^p(\overline{Q}_T))$.

Clearly $\{H_{\varepsilon k}(u-\sigma_N)\}$ is a bounded sequence in $L^\infty(\overline{Q}_T)$, so we can find a subsequence of $\{\varepsilon_k\}$ which is still denoted as $\{\varepsilon_k\}$ and a function $h \in L^\infty(\overline{Q}_T)$, such that

$$H_{\varepsilon k}(u-\sigma_N)\,\overrightarrow{w}\,h\left(L^\infty\left(\overline{Q}_T\right)\right), \quad G_{\varepsilon k}\left(u_{\varepsilon_k}, u_{\tau\varepsilon_k}, t\right)\overrightarrow{*w}\,m(t)\left(L^\infty\left(\overline{Q}_T\right)\right), \tag{64}$$

where

$$m(t) = s\left(\int_0^1 \frac{R_\tau^3}{R^3}u_\tau h(z,t-\tau)\,z^2 - \widetilde{\sigma}h(z,t)\,z^2 - \lambda[1-h(z,t)]\,z^2 dz\right). \tag{65}$$

Taking $\varepsilon = \varepsilon_k$ in (26), replacing u, u_τ with $u_k, u_{\tau k}$, respectively, and letting $k \to \infty$, one can get

$$\frac{\partial u}{\partial t} = dR_0^{-2}\exp\left\{-2\int_0^t m(\xi)\,d\xi\right\}\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial u}{\partial z}\right) + m(t)\,zu_z - f(u)\,h, \quad 0 < z < 1,\ t > 0, \tag{66}$$

$$\frac{\partial u}{\partial z}(0,t) = 0, \quad u(1,t) = \sigma_\infty, \quad t > 0, \tag{67}$$

$$u(z,t) = \psi(zR(t),t), \quad 0 \le z \le 1,\ -\tau \le t \le 0 \tag{68}$$

$$\frac{\partial u}{\partial z}(z\rho(t)-0,t) = \frac{\partial u}{\partial z}(z\rho(t)+0,t), \quad 0 \le z \le 1,\ t > 0, \tag{69}$$

$$u(z\rho(t)-0,t) = u(z\rho(t)+0,t) = \sigma_N, \quad 0 \le z \le 1,\ t > 0, \tag{70}$$

$$\rho(t) = \phi(t), \quad -\tau \le t \le 0, \tag{71}$$

$$R(t) = \varphi(t), \quad -\tau \le t \le 0. \tag{72}$$

We assert that

$$h(|y|,t) = \begin{cases} 1, & u(|y|,t) > \sigma_N, \\ 0, & u(|y|,t) < \sigma_N, \end{cases} \quad \text{a.e.} \tag{73}$$

Actually, it is easy to verify that, for any $\delta > 0, h(|y|,t) = 1$ for a.e. $(y,t) \in \{(y,t) \in \overline{Q}_T) : u(|y|,t) \ge \sigma_N + \delta\}$. By the arbitrariness of δ, we infer that $h(|y|,t) = 1$ for a.e. $(y,t) \in \{(y,t) \in \overline{Q}_T) : u(|y|,t) > \sigma_N\}$. Similarly we can prove that $h(|y|,t) = 0$ for a.e. $(y,t) \in \{(y,t) \in \overline{Q}_T : u(|y|,t) < \sigma_N\}$. This proves the assertion □

Since $u_t = 0$, $\nabla u = 0$, and $\Delta u = 0$ a.e. on the set $\{(y,t) \in \overline{Q}_T : u(|y|,t) = \sigma_N\}$, by (66) it follows that $h = 0$ a.e. on the set $\{(y,t) \in \overline{Q}_T : u(|y|,t) = \sigma_N\}$. Hence

$$h(|y|,t) = H(u(|y|,t) - \sigma_N), \quad (y,t) \in \overline{Q}_T. \tag{74}$$

Substituting (74) into (65), one can get

$$m(t) = G(u, u_\tau, t). \tag{75}$$

By (66), (74), and (75) we conclude that $u = u(z,t)$ is a solution of (19)–(25). This proves the existence of a solution. The rest of the assertions of Lemma 5 follow immediately from taking the weak limit in (26).

By Lemmas 2 and 5, we see that Theorem 1 immediately follows.

5. Conclusions

In this paper a mathematical model for the growth of necrotic tumors with time delays in proliferation is studied. By transforming this problem into an initial-boundary value problem in fixed domain of a coupled system of a parabolic equation

and one integrodifferential equation with time delays, in which all equations involve discontinuous terms, and using the approximation method combined with Schauder fixed point theorem, we prove that this problem has a unique global solution in any time interval $[0, T]$.

The model we study in this paper is established by modifying the model studied in [15] by introducing a time delay in proliferation as that in Byrne [1]. It should be pointed out that, in [15], the authors only studied the existence and uniqueness of stationary solutions to the model (2)–(4), but in our study we mainly discuss the existence and uniqueness of the solution to the model (2)–(4) under some assumptions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Simplified Stability Criteria for Delayed Neutral Systems

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For a class of linear time-invariant neutral systems with neutral and discrete constant delays, several existing asymptotic stability criteria in the form of linear matrix inequalities (LMIs) are simplified by using matrix analysis techniques. Compared with the original stability criteria, the simplified ones include fewer LMI variables, which can obviously reduce computational complexity. Simultaneously, it is theoretically shown that the simplified stability criteria and original ones are equivalent; that is, they have the same conservativeness. Finally, a numerical example is employed to verify the theoretic results investigated in this paper.

1. Introduction

Some practical systems, such as population ecology, neural networks, heat exchangers, and robots in contract with rigid environments, have been modeled by functional differential equations of neutral type (e.g., [1–6]). Since stability analysis is the primary task of analyzing and synthesizing a system, many researchers have paid more and more attention to establish stability criteria for delayed neutral systems (see [7, 8] and the references therein).

For a class of linear time-invariant neutral systems with neutral and discrete constant delays, both neutral- and discrete-delay-dependent stability criteria have been investigated in [9–27]. Many approaches have been provided for obtaining these stability criteria. Here we mention only some critical and representative approaches. Model transformation approach first transforms the neutral system with discrete delay into the one with distributed delays, and then delay-dependent stability criteria can be obtained by constructing a Lyapunov-Krasovskii functional [9]. However, model transformation approach introduces some additional dynamics, which leads still to conservative results [28]. To overcome this shortage, the so-called descriptor model transformation approach is introduced in [15, 24], which transforms the original system to an equivalent descriptor form. The descriptor model transformation approach will not introduce additional dynamics in the sense defined in [28]. Although the results obtained from descriptor model transformation approach may be less conservative than some existing ones, they can be improved by employing some other approaches (e.g., free weighting matrix approach [21, 22] and augmented Lyapunov-Krasovskii functional approach [14, 23, 27]) to get a larger bound for discrete delay.

By employing free weighting matrices and the Leibniz-Newton formula, He et al. [21] presented a neutral- and discrete-delay-dependent stability criterion for neutral systems with neutral and discrete constant delays, which reduces the conservativeness of methods involving a fixed model transformation. For the same system models, Qian et al. [27] established neutral- and discrete-delay-dependent stability criteria by constructing a Lyapunov-Krasovskii functional with additional functional parameters and employing free weighting matrices. To the best of our knowledge, the large-scale employment of functional parameters and/or free weighting matrices can cause complex stability criteria, and thereby the computational complexity increases obviously. Therefore, in order to reduce computational complexity, it is necessary to reduce the number of functional parameters and free weighting matrices. This motivates the present study.

The aim of this paper is to simplify several stability criteria proposed by He et al. [21] and Qian et al. [27]. The main contributions of the paper are as follows. (i) In the premise of not increasing conservativeness, a pair of stability criteria proposed in [27, Theorem 1 and Corollary 1] are

simplified by using matrix analysis techniques; (ii) one of the pair of simplified stability criteria is theoretically presented to be less conservative than [26, Corollary 1], and another is theoretically proven to be equivalent to [21, Theorem 1]; and (iii) by one numerical example, our theoretic results are verified to be more effective than many existing results.

2. Problem Statement and Preliminary Results

Consider a class of delayed neutral systems described as

$$\dot{x}(t) - C\dot{x}(t-\eta) = Ax(t) + A_1 x(t-\tau), \quad t \geq 0, \tag{1a}$$

$$x(t) = \phi(t), \quad t \in [-\omega, 0], \tag{1b}$$

where $x(t)$ is the n-dimensional state vector, A, A_1, and C are known real constant matrices with appropriate dimensions, $\omega = \max\{\eta, \tau\}$, $\phi(s)$ is an $\mathbb{R}^n$-valued continuous initial function defined on $[-\omega, 0]$, and positive scalars η and τ represent the neutral and discrete delays, respectively.

Define $C([-\eta, 0], \mathbb{R}^n)$ as the set of continuous $\mathbb{R}^n$ valued function on the interval $[-\eta, 0]$, and let $x_t \in C([-\eta, 0], \mathbb{R}^n)$ be a segment of system trajectory defined as

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\eta, 0]. \tag{2}$$

Define an operator $\mathscr{D} : C([-\eta, 0], \mathbb{R}^n) \to \mathbb{R}^n$ by $\mathscr{D}x_t = x(t) - Cx(t-\eta)$. The definitions on stability of operator $\mathscr{D}$ and systems (1a) and (1b) can be seen in [29].

As mentioned in Introduction, neutral- and discrete-delay-dependent stability criteria for systems (1a) and (1b) have been investigated by He et al. [21] and Qian et al. [27]. In this paper, we will simplify the stability criteria [21, Theorem 1] and [27, Theorem 1 and Corollary 1] in the premise of not increasing conservativeness of stability criteria.

In order to present conveniently our main results, the following lemmas are required.

Lemma 1 (Schur complement lemma [30]). *Given constant matrices Ω_1, Ω_2, and Ω_3 of appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2$, then*

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0 \tag{3}$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0. \tag{4}$$

Lemma 2 (see [31]). *Given a real symmetric matrix Ψ and a pair of real matrices P and Q, the following LMI problem*

$$\Psi + P^T X^T Q + Q^T X P < 0 \tag{5}$$

is solvable with respect to decision variable X if and only if

$$\mathcal{N}_P^T \Psi \mathcal{N}_P < 0, \qquad \mathcal{N}_Q^T \Psi \mathcal{N}_Q < 0, \tag{6}$$

where $\mathcal{N}_P$ and $\mathcal{N}_Q$ are matrices whose columns form a basis of the right null spaces of P and Q, respectively.

3. Several Stability Criteria in the Literature

In this section we introduce several existing stability criteria for systems (1a) and (1b), which will be useful to present conveniently the main results of this paper in the next section.

Proposition 3 (see [21, Theorem 1]). *For given scalars $\tau > 0$ and $\eta > 0$, systems* (1a) *and* (1b) *are asymptotically stable, if the operator $\mathscr{D}$ is stable and there exist real $n \times n$ matrices $P^T = P > 0$, $Q_i^T = Q_i > 0$ $(i = 1, 2)$, $R^T = R > 0$, $\check{X}_{ii}^T = \check{X}_{ii}$, $\check{Y}_{ii} = \check{Y}_{ii}$ $(i = 1, \ldots, 5)$, $\check{X}_{ij}$, and $\check{Y}_{ij}$ $(1 \leq i < j \leq 5)$ such that the following LMIs are feasible:*

$$\begin{bmatrix} \check{X} & \check{X}_5^T \\ \check{X}_5 & \check{X}_{55} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \check{Y} & \check{Y}_5^T \\ \check{Y}_5 & \check{Y}_{55} \end{bmatrix} \geq 0, \tag{7}$$

$$\begin{bmatrix} \check{\Phi} + \tau\check{X} + \eta\check{Y} + \check{\Phi}_X + \check{\Phi}_Y & \check{A}^T S \\ S\check{A} & -S \end{bmatrix} < 0, \tag{8}$$

where

$$\begin{aligned}
\check{X} &= \begin{bmatrix} \check{X}_{11} & \check{X}_{12} & \check{X}_{13} & \check{X}_{14} \\ \check{X}_{12}^T & \check{X}_{22} & \check{X}_{23} & \check{X}_{24} \\ \check{X}_{13}^T & \check{X}_{23}^T & \check{X}_{33} & \check{X}_{34} \\ \check{X}_{14}^T & \check{X}_{24}^T & \check{X}_{34}^T & \check{X}_{44} \end{bmatrix}, \\
\check{Y} &= \begin{bmatrix} \check{Y}_{11} & \check{Y}_{12} & \check{Y}_{13} & \check{Y}_{14} \\ \check{Y}_{12}^T & \check{Y}_{22} & \check{Y}_{23} & \check{Y}_{24} \\ \check{Y}_{13}^T & \check{Y}_{23}^T & \check{Y}_{33} & \check{Y}_{34} \\ \check{Y}_{14}^T & \check{Y}_{24}^T & \check{Y}_{34}^T & \check{Y}_{44} \end{bmatrix}, \\
\check{X}_5 &= \begin{bmatrix} \check{X}_{15}^T & \check{X}_{25}^T & \check{X}_{35}^T & \check{X}_{45}^T \end{bmatrix}, \\
\check{Y}_5 &= \begin{bmatrix} \check{Y}_{15}^T & \check{Y}_{25}^T & \check{Y}_{35}^T & \check{Y}_{45}^T \end{bmatrix}, \\
\check{\Phi}_X &= \check{X}_5^T \Pi_1 + \Pi_1^T \check{X}_5, \qquad \check{\Phi}_Y = \check{Y}_5^T \Pi_2 + \Pi_2^T \check{Y}_5, \\
\Pi_1 &= \begin{bmatrix} I_n & -I_n & 0 & 0 \end{bmatrix}, \\
\Pi_2 &= \begin{bmatrix} I_n & 0 & -I_n & 0 \end{bmatrix}, \qquad S = R + \tau\check{X}_{55} + \eta\check{Y}_{55}, \\
\check{A} &= \begin{bmatrix} A & A_1 & 0 & C \end{bmatrix}, \\
\check{\Phi} &= \begin{bmatrix} \check{\Phi}_{11} & PA_1 & -A^T PC & 0 \\ A_1^T P & -Q_1 & -A_1^T PC & 0 \\ -C^T PA & -C^T PA_1 & -Q_2 & 0 \\ 0 & 0 & 0 & -R \end{bmatrix}, \\
\check{\Phi}_{11} &= PA + A^T P + Q_1 + Q_2.
\end{aligned} \tag{9}$$

Proposition 4 (see [27, Theorem 1]). *For given scalars $\tau > 0$ and $\eta > 0$, systems* (1a) *and* (1b) *are asymptotically stable, if*

the operator $\mathscr{D}$ is stable and there exist real $n \times n$ matrices U, $P^T = P$, $W^T = W$, $Q_k^T = Q_k$, $N_k^T = N_k$, $R_k^T = R_k$, $T_k^T = T_k$, M_k, $Z_k (k = 1, 2)$, E_i, F_i, H_i, Y_{ii} $(i = 1, \ldots, 5)$, and Y_{ij} $(1 \le i < j \le 5)$ such that the following LMIs are feasible:

$$\begin{bmatrix} R_2 & Z_2 \\ Z_2^T & T_2 \end{bmatrix} \ge 0, \tag{10}$$

$$Q_k \ge 0, \qquad k = 1, 2, \tag{11}$$

$$W > 0, \qquad \begin{bmatrix} R_1 & Z_1 \\ Z_1^T & T_1 \end{bmatrix} > 0, \tag{12}$$

$$\begin{bmatrix} P & M_1 & M_2 \\ M_1^T & \tau^{-1}Q_1 + N_1 & U^T \\ M_2^T & U & \eta^{-1}Q_2 + N_2 \end{bmatrix} > 0, \tag{13}$$

$$\Xi := \begin{bmatrix} \Phi + \eta Y & \tau G_1 & -\tau F \\ \tau G_1^T & -\tau R_1 & -\tau Z_1 \\ -\tau F^T & -\tau Z_1^T & -\tau T_1 \end{bmatrix} < 0, \tag{14}$$

$$\Omega := \begin{bmatrix} Y & -G_2 & H \\ -G_2^T & R_2 & Z_2 \\ H^T & Z_2^T & T_2 \end{bmatrix} \ge 0, \tag{15}$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{15} \\ \cdots & \cdots & \cdots \\ \Phi_{15}^T & \cdots & \Phi_{55} \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_{11} & \cdots & Y_{15} \\ \cdots & \cdots & \cdots \\ Y_{15}^T & \cdots & Y_{55} \end{bmatrix},$$

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_5 \end{bmatrix}, \qquad H = \begin{bmatrix} H_1 \\ \vdots \\ H_5 \end{bmatrix},$$

$$\begin{aligned}
\Phi_{11} &= PA + A^T P + Q_1 + Q_2 + \tau R_1 + \eta R_2 \\
&\quad + M_1 + M_1^T + M_2 + M_2^T + F_1 + F_1^T \\
&\quad + H_1 + H_1^T - E_1 A - A^T E_1^T, \\
\Phi_{12} &= PA_1 - M_1 + H_2^T - A^T E_2^T + F_2^T - F_1 - E_1 A_1, \\
\Phi_{13} &= -A^T PC - M_1^T C - M_2^T C - M_2 \\
&\quad - H_1 + H_3^T + F_3^T - A^T E_3^T, \\
\Phi_{14} &= -A^T E_4^T + E_1 + F_4^T + H_4^T + \tau Z_1 + \eta Z_2, \\
\Phi_{15} &= -A^T E_5^T + F_5^T + H_5^T - E_1 C, \\
\Phi_{22} &= -Q_1 - E_2 A_1 - A_1^T E_2^T - F_2 - F_2^T, \\
\Phi_{23} &= -A_1^T PC + M_1^T C - H_2 - F_3^T - A_1^T E_3^T, \\
\Phi_{24} &= E_2 - A_1^T E_4^T - F_4^T, \qquad \Phi_{25} = -E_2 C - A_1^T E_5^T - F_5^T, \\
\Phi_{33} &= -Q_2 + C^T M_2 + M_2^T C - H_3 - H_3^T, \\
\Phi_{34} &= E_3 - H_4^T, \\
\Phi_{35} &= -E_3 C - H_5^T, \qquad \Phi_{44} = E_4 + E_4^T + W + \tau T_1 + \eta T_2, \\
\Phi_{45} &= -E_4 C + E_5^T, \qquad \Phi_{55} = -W - E_5 C - C^T E_5^T, \\
G_1 &= \begin{bmatrix} N_1 + U^T & -N_1 & -U^T & M_1^T & -M_1^T C \end{bmatrix}^T, \\
G_2 &= \begin{bmatrix} N_2 + U & -U & -N_2 & M_2^T & -M_2^T C \end{bmatrix}^T.
\end{aligned} \tag{16}$$

Proposition 5 (see [27, Corollary 1]). *When $\eta = \tau$, for a given scalar $\tau > 0$, systems* (1a) *and* (1b) *are asymptotically stable, if the operator $\mathscr{D}$ is stable and there exist real $n \times n$ matrices $W^T = W > 0$, $Q_1^T = Q_1 > 0$, $P^T = P$, $N_1^T = N_1$, $R_1^T = R_1$, $T_1^T = T_1$, M_1, Z_1, E_i, and F_i $(i = 1, 2, 4, 5)$ such that the following LMIs are feasible:*

$$\begin{bmatrix} R_1 & Z_1 \\ Z_1^T & T_1 \end{bmatrix} > 0, \qquad \begin{bmatrix} P & M_1 \\ M_1^T & \tau^{-1}Q_1 + N_1 \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \tau N_1 & -\tau F_1 \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} & \Pi_{24} & -\tau N_1 & -\tau F_2 \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} & \Pi_{34} & \tau M_1 & -\tau F_4 \\ \Pi_{14}^T & \Pi_{24}^T & \Pi_{34}^T & \Pi_{44} & -\tau C^T M_1 & -\tau F_5 \\ \tau N_1 & -\tau N_1 & \tau M_1^T & -M_1^T C & -\tau R_1 & -\tau Z_1 \\ -\tau F_1^T & -\tau F_2^T & -\tau F_4^T & -\tau F_5^T & -\tau Z_1^T & -\tau T_1 \end{bmatrix} < 0, \tag{18}$$

where

$$\begin{aligned}
\Pi_{11} &= PA + A^T P + M_1 + M_1^T + Q_1 + \tau R_1 \\
&\quad + F_1 + F_1^T - E_1 A - A^T E_1^T, \\
\Pi_{12} &= PA_1 - A^T PC - M_1 - M_1^T C \\
&\quad - A^T E_2^T - E_1 A_1 + F_2^T - F_1, \\
\Pi_{13} &= -A^T E_4^T + E_1 + F_4^T + \tau Z_1, \\
\Pi_{14} &= -A^T E_5^T + F_5^T - E_1 C, \\
\Pi_{22} &= -Q_1 - A_1^T PC - C^T PA_1 + C^T M_1 + M_1^T C \\
&\quad - E_2 A_1 - A_1^T E_2^T - F_2 - F_2^T, \\
\Pi_{23} &= E_2 - A_1^T E_4^T - F_4^T, \qquad \Pi_{24} = -E_2 C - A_1^T E_5^T - F_5^T, \\
\Pi_{33} &= E_4 + E_4^T + \tau T_1 + W, \qquad \Pi_{34} = -E_4 C + E_5^T, \\
\Pi_{44} &= -E_5 C - C^T E_5^T - W.
\end{aligned} \tag{19}$$

Remark 6. Proposition 3 simplifies the notations in [21, Theorem 1].

Remark 7. Propositions 4 and 5 correct some slips of the pen in [27, Theorem 1 and Corollary 1]. After [27, Theorem 1] is amended as in Remark 7, [27, Theorem 2] will be correct.

4. Simplified Stability Criteria

In this section we will simplify the stability criteria introduced in the previous section. Firstly, the stability criterion presented in Proposition 4 can be simplified by the following theorem.

Theorem 8. *For given scalars $\tau > 0$ and $\eta > 0$, systems* (1a) *and* (1b) *are asymptotically stable, if the operator $\mathscr{D}$ is stable and one of the following cases, (i)–(iv), holds.*

(i) *There exist real $n \times n$ matrices U, $P^T = P$, $W^T = W$, $Q_k^T = Q_k$, $N_k^T = N_k$, $R_k^T = R_k$, $T_k^T = T_k$, M_k, $Z_k (k = 1, 2)$, E_i, F_i, H_i, $Y_{ii} (i = 1, \ldots, 5)$, and $Y_{ij} (1 \le i < j \le 5)$ such that the LMIs* (10)–(15) *hold.*

(ii) *There exist real $n \times n$ matrices U, $P^T = P$, $W^T = W$, $Q_k^T = Q_k$, $N_k^T = N_k$, $R_k^T = R_k$, $T_k^T = T_k$, M_k, Z_k $(k = 1, 2)$, E_i, F_i, H_i, $Y_{ii} (i = 1, \ldots, 5)$, and $Y_{ij} (1 \le i < j \le 5)$ such that the LMIs* (11)–(15) *and the following LMI* (20) *hold:*

$$\begin{bmatrix} R_2 & Z_2 \\ Z_2^T & T_2 \end{bmatrix} > 0. \tag{20}$$

(iii) *There exist real $n \times n$ matrices U, $P^T = P$, $W^T = W$, $Q_k^T = Q_k$, $N_k^T = N_k$, $R_k^T = R_k$, $T_k^T = T_k$, M_k, and Z_k $(k = 1, 2)$ such that the LMIs* (11), (12), *and* (13) *and the following LMI* (21) *hold:*

$$\begin{bmatrix} \Phi & \tau G_1 & -\tau F & -\eta G_2 & \eta H \\ \tau G_1^T & -\tau R_1 & -\tau Z_1 & 0 & 0 \\ -\tau F^T & -\tau Z_1^T & -\tau T_1 & 0 & 0 \\ -\eta G_2^T & 0 & 0 & -\eta R_2 & -\eta Z_2 \\ \eta H^T & 0 & 0 & -\eta Z_2^T & -\eta T_2 \end{bmatrix} < 0. \tag{21}$$

(iv) *There exist real $n \times n$ matrices U, $P^T = P$, $W^T = W$, $Q_k^T = Q_k$, $N_k^T = N_k$, $R_k^T = R_k$, $T_k^T = T_k$, M_k, and Z_k $(k = 1, 2)$ such that the LMIs* (11), (12), (13), *and* (20) *and the following LMI* (22) *hold:*

$$\widetilde{\Sigma} := \begin{bmatrix} \widetilde{\Phi} & \widetilde{G}_1 & \widetilde{G}_2 \\ \widetilde{G}_1^T & -\tau R_1 & 0 \\ \widetilde{G}_2^T & 0 & -\eta R_2 \end{bmatrix} < 0, \tag{22}$$

where

$$\widetilde{\Phi} = \begin{bmatrix} \widetilde{\Phi}_{11} & \widetilde{\Phi}_{12} & \widetilde{\Phi}_{13} & \widetilde{\Phi}_{14} \\ \widetilde{\Phi}_{12}^T & \widetilde{\Phi}_{22} & \widetilde{\Phi}_{23} & A_1^T SC \\ \widetilde{\Phi}_{13}^T & \widetilde{\Phi}_{23}^T & \widetilde{\Phi}_{33} & 0 \\ \widetilde{\Phi}_{14}^T & C^T SA_1 & 0 & C^T SC - W \end{bmatrix},$$

$$\widetilde{\Phi}_{11} = PA + A^T P + Q_1 + Q_2 + \tau R_1 + \eta R_2 + M_1 + M_1^T + M_2 + M_2^T + A^T SA + \tau A^T Z_1^T + \tau Z_1 A + \eta A^T Z_2^T + \eta Z_2 A - \tau^{-1} T_1 - \eta^{-1} T_2,$$

$$\widetilde{\Phi}_{12} = PA_1 - M_1 + \tau Z_1 A_1 + \eta Z_2 A_1 + A^T SA_1 + \tau^{-1} T_1,$$

$$\widetilde{\Phi}_{13} = -A^T PC - M_1^T C - M_2^T C - M_2 + \eta^{-1} T_2,$$

$$\widetilde{\Phi}_{14} = \tau Z_1 C + \eta Z_2 C + A^T SC,$$

$$\widetilde{\Phi}_{22} = -Q_1 - \tau^{-1} T_1 + A_1^T SA_1, \qquad \widetilde{\Phi}_{23} = -A_1^T PC + M_1^T C,$$

$$\widetilde{\Phi}_{33} = -Q_2 + C^T M_2 + M_2^T C - \eta^{-1} T_2,$$

$$S = W + \tau T_1 + \eta T_2,$$

$$\widetilde{G}_1 = \begin{bmatrix} \tau N_1^T + \tau U - Z_1^T + \tau A^T M_1 \\ -\tau N_1^T + Z_1^T + \tau A_1^T M_1 \\ -\tau U \\ 0 \end{bmatrix},$$

$$\widetilde{G}_2 = \begin{bmatrix} -\eta N_2^T - \eta U^T + Z_2^T - \eta A^T M_2 \\ \eta U^T - \eta A_1^T M_2 \\ \eta N_2^T - Z_2^T \\ 0 \end{bmatrix}. \tag{23}$$

Proof. Due to Proposition 4, it suffices to show that the conditions (i)–(iv) are equivalent.

(i)⇒(ii) Since the eigenvalues of a matrix are continuous functions of its elements, it follows from (i) that there exists a sufficiently small positive scalar ϵ such that

$$\begin{gathered} \Xi + \operatorname{diag}(\eta\epsilon I, 0, 0, \eta\epsilon I, 0, 0, 0) < 0, \\ \Omega + \operatorname{diag}(0, \epsilon I, \epsilon I) \ge 0, \\ \begin{bmatrix} R_2 + \epsilon I & Z_2 \\ Z_2^T & T_2 + \epsilon I \end{bmatrix} > 0. \end{gathered} \tag{24}$$

Therefore, the LMIs (11)–(15) and (20) are feasible.

(ii)⇒(i) The proof is very easy, and hence it is omitted.

(ii)⇒(iii) By Lemma 1, it follows from (20) and (15) that

$$Y \geq \begin{bmatrix} -G_2 & H \end{bmatrix} \begin{bmatrix} R_2 & Z_2 \\ Z_2^T & T_2 \end{bmatrix}^{-1} \begin{bmatrix} -G_2 & H \end{bmatrix}^T. \tag{25}$$

This, together with (14) and Lemma 1, implies that (21) holds.

(iii)⇒(ii) It follows from (21) and Lemma 1 that (20) holds and

$$\begin{bmatrix} \Phi + \eta Y & \tau G_1 & -\tau F \\ \tau G_1^T & -\tau R_1 & -\tau Z_1 \\ -\tau F^T & -\tau Z_1^T & -\tau T_1 \end{bmatrix} < 0, \tag{26}$$

where

$$Y = \begin{bmatrix} -G_2 & H \end{bmatrix} \begin{bmatrix} R_2 & Z_2 \\ Z_2^T & T_2 \end{bmatrix}^{-1} \begin{bmatrix} -G_2 & H \end{bmatrix}^T, \tag{27}$$

and hence (ii) holds.

(iii)⇔(iv) Let

$$\begin{aligned} \widetilde{F} &= \begin{bmatrix} F & H & E \end{bmatrix}, \qquad E = \begin{bmatrix} E_1^T & E_2^T & E_3^T & E_4^T & E_1^5 \end{bmatrix}^T, \\ K &= \begin{bmatrix} e_1^T & e_2^T & e_3^T & e_4^T & e_5^T \end{bmatrix}^T, \\ L &= \begin{bmatrix} (e_1 - e_2 - \tau e_7)^T & (e_1 - e_3 + \eta e_9)^T & \widetilde{A}^T \end{bmatrix}^T, \\ \widetilde{A} &= \begin{bmatrix} -A & -A_1 & 0 & I & -C & 0_{n\times 4n} \end{bmatrix}, \\ e_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I & 0_{n\times(9-i)n} \end{bmatrix}, \quad i = 1, 2, \ldots, 9. \end{aligned} \tag{28}$$

Then LMI (21) can be written as

$$\Phi_1 + K^T \widetilde{F} L + L^T \widetilde{F}^T K < 0, \tag{29}$$

where the matrix Φ_1 is obtained from the matrix on the left of (21) by deleting all parts containing one of F, H, and E (e.g., the (5, 5)th subblock of Φ_1 equals $-W$, which is obtained from Φ_{55} by deleting $-E_5C - C^T E_5^T$). Choose

$$\begin{aligned} \mathcal{N}_K &= \begin{bmatrix} e_6^T & e_7^T & e_8^T & e_9^T \end{bmatrix}, \\ \mathcal{N}_L &= \begin{bmatrix} e_1 + A^T e_4 + \tau^{-1} e_7 - \eta^{-1} e_9 \\ e_2 + A_1^T e_4 - \tau^{-1} e_7 \\ e_3 + \eta^{-1} e_9 \\ C^T e_4 + e_5 \\ e_6 \\ e_8 \end{bmatrix}^T. \end{aligned} \tag{30}$$

Noting that $\mathcal{N}_K^T \Phi_1 \mathcal{N}_K < 0$, one can derive from (12) and Lemma 2 that the LMI (29) (i.e., (21)) is feasible if and only if LMIs (22) and (20) are feasible.

The proof is completed. □

Clearly, the stability condition (iv) in Theorem 8 is more simpler than (i) in Theorem 8 (i.e., Proposition 4). By a process similar to investigating Theorem 8, one can easily obtain the following theorem which simplifies the stability criterion presented in Proposition 5.

Theorem 9. *When $\eta = \tau$, for a given scalar $\tau > 0$, systems* (1a) *and* (1b) *are asymptotically stable, if the operator $\mathcal{D}$ is stable, and one of the following cases, (i)-(ii), holds.*

(i) *There exist real $n \times n$ matrices $W^T = W > 0$, $Q_1^T = Q_1 > 0$, $P^T = P$, $N_1^T = N_1$, $R_1^T = R_1$, $T_1^T = T_1$, M_1, Z_1, E_i, and F_i ($i = 1, 2, 4, 5$) such that the LMIs* (17) *and* (18) *hold.*

(ii) *There exist real $n \times n$ matrices $W^T = W > 0$, $Q_1^T = Q_1 > 0$, $P^T = P$, $N_1^T = N_1$, $R_1^T = R_1$, $T_1^T = T_1$, M_1, and Z_1 such that* (17) *and the following LMI* (31) *hold:*

$$\begin{bmatrix} \widetilde{\Pi}_{11} & \widetilde{\Pi}_{12} & \widetilde{\Pi}_{13} & \widetilde{\Pi}_{14} \\ \widetilde{\Pi}_{12}^T & \widetilde{\Pi}_{22} & \widetilde{\Pi}_{23} & \widetilde{\Pi}_{24} \\ \widetilde{\Pi}_{13}^T & \widetilde{\Pi}_{23}^T & \widetilde{\Pi}_{33} & 0 \\ \widetilde{\Pi}_{14}^T & \widetilde{\Pi}_{24}^T & 0 & -\tau R_1 \end{bmatrix} < 0, \tag{31}$$

where

$$\begin{aligned} \widetilde{\Pi}_{11} &= PA + A^T P + M_1 + M_1^T + Q_1 + \tau R_1 \\ &\quad + \tau Z_1 A + \tau A^T Z_1^T + A^T S_0 A - \tau^{-1} T_1, \\ \widetilde{\Pi}_{12} &= PA_1 - A^T PC - M_1 - M_1^T C \\ &\quad + \tau Z_1 A_1 + A^T S_0 A_1 + \tau^{-1} T_1, \\ \widetilde{\Pi}_{13} &= \tau Z_1 C + A^T S_0 C, \qquad \widetilde{\Pi}_{14} = \tau N_1 + \tau A^T M_1 - Z_1^T, \\ \widetilde{\Pi}_{22} &= -Q_1 - A_1^T PC - C^T PA_1 + C^T M_1 \\ &\quad + M_1^T C + A_1^T S_0 A_1 - \tau^{-1} T_1, \\ \widetilde{\Pi}_{23} &= A_1^T S_0 C, \qquad \widetilde{\Pi}_{24} = -\tau N_1 + \tau A_1^T M_1 + Z_1^T, \\ \widetilde{\Pi}_{33} &= C^T S_0 C - W, \qquad S_0 = W + \tau T_1. \end{aligned} \tag{32}$$

Remark 10. Theorems 8 and 9 indicate that, without increasing conservativeness of stability criteria, the free weighting matrices required in Propositions 4 and 5 (i.e., [27, Theorem 1 and Corollary 1]) can be eliminated. This will obviously reduce the computational complexity, and hence the stability conditions (iv) of Theorem 8 and (ii) of Theorem 9 are more effective.

Remark 11. In the special case that $M_1 = 0$, $N_1 = 0$, $Z_1 = 0$, and $R_1 = \epsilon I_n$ (ϵ is a sufficiently small positive number), Theorem 9 reduces to [26, Corollary 1]. So, by choosing suitable M_1, N_1, Z_1, and R_1, Theorem 9 could reduce the conservativeness of [26, Corollary 1], by which the improvement over [26, Corollary 1] is theoretically demonstrated.

If we choose $U = 0$, $M_i = 0$, $N_i = 0$, $Z_i = 0$, and $R_i = \epsilon I_n$ (ϵ is a sufficiently small positive number) ($i = 1, 2$) in Theorem 8, the following corollary can be easily derived.

Corollary 12. *For given scalars $\tau > 0$ and $\eta > 0$, systems* (1a) *and* (1b) *are asymptotically stable, if the operator $\mathscr{D}$ is stable and there exist real $n \times n$ matrices $P^T = P > 0$, $W^T = W > 0$, $Q_k^T = Q_k > 0$, and $T_k^T = T_k > 0$ ($k = 1, 2$) such that LMI* (33) *holds:*

$$\widehat{\Phi} := \begin{bmatrix} \widehat{\Phi}_{11} & \widehat{\Phi}_{12} & \widehat{\Phi}_{13} & A^T SC \\ \widehat{\Phi}_{12}^T & \widehat{\Phi}_{22} & \widehat{\Phi}_{23} & A_1^T SC \\ \widehat{\Phi}_{13}^T & \widehat{\Phi}_{23}^T & \widehat{\Phi}_{33} & 0 \\ C^T SA & C^T SA_1 & 0 & C^T SC - W \end{bmatrix} < 0, \tag{33}$$

where

$$\begin{aligned} \widehat{\Phi}_{11} &= PA + A^T P + Q_1 + Q_2 + A^T SA - \tau^{-1} T_1 - \eta^{-1} T_2, \\ \widehat{\Phi}_{12} &= PA_1 + A^T SA_1 + \tau^{-1} T_1, \qquad \widehat{\Phi}_{13} = -A^T PC + \eta^{-1} T_2, \\ \widehat{\Phi}_{22} &= -Q_1 - \tau^{-1} T_1 + A_1^T SA_1, \qquad \widehat{\Phi}_{23} = -A_1^T PC, \\ \widehat{\Phi}_{33} &= -Q_2 - \eta^{-1} T_2, \qquad S = W + \tau T_1 + \eta T_2. \end{aligned} \tag{34}$$

Next we will show that the pair of stability criteria presented in Proposition 3 (i.e., [21, Theorem 1]) and Corollary 12 have the same conservativeness. However, the stability criterion presented in Corollary 12 is more simpler than the one in Proposition 3 (i.e., [21, Theorem 1]), which reduces greatly the computational complexity.

Theorem 13. *For given scalars $\tau > 0$ and $\eta > 0$, LMIs* (7) *and* (8) *are feasible if and only if the LMI* (33) *is feasible.*

Proof. The "Only If" Part. It follows from (7) and (8) that

$$\begin{aligned} &\check{X} \geq \check{X}_5^T \check{X}_{55}^{-1} \check{X}_5, \qquad \check{Y} \geq \check{Y}_5^T \check{Y}_{55}^{-1} \check{Y}_5, \\ &\check{\Phi} + \tau \check{X} + \eta \check{Y} + \check{\Phi}_X + \check{\Phi}_Y + \check{A}^T S \check{A} < 0, \end{aligned} \tag{35}$$

and hence

$$\check{\Phi} + \tau \check{X}_5^T \check{X}_{55}^{-1} \check{X}_5 + \eta \check{Y}_5^T \check{Y}_{55}^{-1} \check{Y}_5 + \check{\Phi}_X + \check{\Phi}_Y + \check{A}^T S \check{A} < 0. \tag{36}$$

Noting that

$$\begin{aligned} \tau \check{X}_5^T \check{X}_{55}^{-1} \check{X}_5 + \check{\Phi}_X + \tau^{-1} \Pi_1^T \check{X}_{55} \Pi_1 \geq 0, \\ \eta \check{Y}_5^T \check{Y}_{55}^{-1} \check{Y}_5 + \check{\Phi}_Y + \eta^{-1} \Pi_2^T \check{Y}_{55} \Pi_2 \geq 0, \end{aligned} \tag{37}$$

one can conclude from (36) that

$$\check{\Phi} - \tau^{-1} \Pi_1^T \check{X}_{55} \Pi_1 - \eta^{-1} \Pi_2^T \check{Y}_{55} \Pi_2 + \check{A}^T S \check{A} < 0, \tag{38}$$

Table 1: The maximum allowable bounds of $\tau = \eta$.

Method	max τ
[9]	0.3
[10]	0.5658
[11]	0.5937
[12]	0.6054
[13]	0.6612
[14]	0.7039
[15]	0.74
[16]	0.8844
[17]	1.0402
[18]	1.3718
[19]	1.6014
[20]	1.61
[21]	1.6527
[22]	1.7191
[23]	1.7220
[24]	1.78
[25]	1.7844
[26, Corollary 1]	1.7856
[27] and Theorem 9	1.7890

which becomes LMI (33) by letting $W = R$, $T_1 = \check{X}_{55}$, and $T_2 = \check{T}_{55}$. That is, LMI (33) is feasible.

The "If" Part. Since LMI (33) is feasible, there exist real $n \times n$ matrices $P^T = P > 0$, $W^T = W > 0$, $Q_k^T = Q_k > 0$, and $T_k^T = T_k > 0$ ($k = 1, 2$) such that LMI (33) is satisfied. Set $R = W$, $\check{X}_{55} = T_1$, and $\check{Y}_{55} = T_2$. Then it follows from (33) and Lemma 1 that

$$\begin{bmatrix} \check{\Phi} - \tau^{-1} \Pi_1^T \check{X}_{55} \Pi_1 - \eta^{-1} \Pi_2^T \check{Y}_{55} \Pi_2 & \check{A}^T S \\ S\check{A} & -S \end{bmatrix} < 0. \tag{39}$$

Let

$$\begin{aligned} \check{X}_5 &= -\tau^{-1} \check{X}_{55} \Pi_1, \qquad \check{X} = \check{X}_5^T \check{X}_{55}^{-1} \check{X}_5, \\ \check{Y}_5 &= -\eta^{-1} \check{Y}_{55} \Pi_2, \qquad \check{Y} = \check{Y}_5^T \check{Y}_{55}^{-1} \check{Y}_5. \end{aligned} \tag{40}$$

Then LMI (7) holds, and

$$\begin{aligned} \tau \check{X}_5^T \check{X}_{55}^{-1} \check{X}_5 + \check{\Phi}_X + \tau^{-1} \Pi_1^T \check{X}_{55} \Pi_1 = 0, \\ \eta \check{Y}_5^T \check{Y}_{55}^{-1} \check{Y}_5 + \check{\Phi}_Y + \eta^{-1} \Pi_2^T \check{Y}_{55} \Pi_2 = 0. \end{aligned} \tag{41}$$

Furthermore, LMI (8) is derived from (39).

The proof is completed. □

5. Numerical Comparisons of Stability Criteria

In this section we will present one example to demonstrate effectiveness of the theoretic results described above.

Table 2: The maximum allowable bounds of τ for different η.

Method	η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	10000	100000
[21] and Corollary 12	τ	1.7100	1.6987	1.6883	1.6792	1.6718	1.6664	1.6624	1.6591	1.6563	1.6543	1.6531	1.6527	1.6527	1.6527	1.6527	1.6527	1.6527
[25]	τ	1.7844	1.7757	1.7669	1.7581	1.7495	1.7413	1.7338	1.7273	1.7226	1.7201	1.7193	1.7191	1.7191	1.7191	1.7191	1.7191	1.7191
[27] and Theorem 8	τ	1.8305	1.8174	1.8038	1.7897	1.7755	1.7616	1.7484	1.7366	1.7272	1.7213	1.7202	1.7202	1.7202	1.7202	1.7202	1.7193	1.7193

Example 1. Consider the stability of systems (1a) and (1b) with

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad C = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}. \tag{42}$$

Case 1 ($\tau = \eta$). The upper bounds on the delay τ under which robust stability of this system can be guaranteed using the methods in [9–27] and Theorem 9 of this paper are listed in Table 1. It is clear that our results are significantly better than those in [9–26] because our allowable maximum time delay τ is larger.

Case 2 ($\tau \neq \eta$). For different values of η, Table 2 lists the maximum allowable bounds on τ that guarantee the stability of the system. It can be seen that the maximum allowable bound on τ decreases as η increases when η is small but that τ remains almost unchanged when η increases to certain extent.

6. Conclusions

By using matrix analysis techniques, several stability criteria (i.e., [21, Theorem 1] and [27, Theorem 1 and Corollary 1]) for delayed neutral systems have been simplified. The numbers of LMI variables in [21, Theorem 1] and [27, Theorem 1 and Corollary 1] are reduced into the simplified ones, which can obviously reduce computational complexity. Furthermore, it is theoretically proven that the simplified stability criteria have the same conservativeness as the original ones. A numerical example is given to illustrate the theoretic results investigated in this paper.

Extending the idea of this paper to other system models, including singular delayed systems [32–35], stochastic systems [36], Markovian jump systems [37, 38], and genetic regulatory networks [39–41], is under consideration.

Conflict of Interests

The authors declare that there is no commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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Research Article

Adaptive Fuzzy Output-Feedback Method Applied to Fin Control for Time-Delay Ship Roll Stabilization

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The ship roll stabilization by fin control system is considered in this paper. Assuming that angular velocity in roll cannot be measured, an adaptive fuzzy output-feedback control is investigated. The fuzzy logic system is used to approximate the uncertain term of the controlled system, and a fuzzy state observer is designed to estimate the unmeasured states. By utilizing the fuzzy state observer and combining the adaptive backstepping technique with adaptive fuzzy control design, an observer-based adaptive fuzzy output-feedback control approach is developed. It is proved that the proposed control approach can guarantee that all the signals in the closed-loop system are semiglobally uniformly ultimately bounded (SGUUB), and the control strategy is effective to decrease the roll motion. Simulation results are included to illustrate the effectiveness of the proposed approach.

1. Introduction

Stabilization of ship roll motion induced by wave disturbances has received considerable attention. This is because excessive roll motion would make the crew feel uncomfortable and may also cause damage to the cargoes and equipment on board. Various types of antirolling devices are introduced to reduce undesirable wave-induced roll motion [1]. Stabilizing fins are the most effective and popular antirolling devices in use. Stabilizing fins have been used extensively for high speed vessels, particularly on war ships and cruise ships. Lift forces are generated by the fins and a couple is produced to counteract the wave-induced roll moment. Since the lift force depends on the relative inflow speed, the stabilizing fins are effective only when the ships are sailing at relatively high speed. Some methods have been introduced. A multivariable control approach to the design of antirolling fins for a war ship has been proposed, where the rudder and the fins are considered simultaneously to reduce the wave-induced roll motion without sacrificing the yaw control performance [2]. Fuzzy control method basing on empirical if-then rules has also been introduced to the design of the fin stabilization system [3]. Application of the adaptive LQ method to the stabilization for a monohull ship is reported in [4], and an H-∞ control design method has been employed in the design of a robust stabilizing fin controller [5]. A novel ship stabilizing fin controller based on the internal model control (IMC) method was proposed in [6].

Adaptive backstepping technique is an important control method in the last twenty years. The backstepping design provides a systematic framework for the design of tracking and regulation strategies, suitable for a large class of nonlinear systems [7–11]. Some fin control methods based on backstepping technique have been proposed [12, 13]. A nonlinear robust controller was designed by using backstepping and closed-loop gain shaping algorithms to improve the robustness of fin stabilizer controller. A novel sliding backstepping controller was designed to decrease the roll motion of fin stabilizer control system for the system with nonlinear disturbance observer. However, the control methods in [12, 13] are all based on the assumption that angular velocity in roll is measured. As what authors stated in [14–16], in practice, the state variables are often unmeasured for many nonlinear systems. Therefore, the existing approaches in [12, 13] cannot be implemented for the fin control system, in which the angular velocity is not measured.

Motivated by the above observation, this paper investigates the adaptive fuzzy output-feedback control problem of the time-delay ship roll stabilization. The fuzzy logic system can be used to approximate the unmodeled plant [17–19]. In this paper, assuming that angular velocity in roll cannot be measured, an adaptive fuzzy output-feedback control design problem is investigated, and a fuzzy state observer is designed to estimate the unmeasured states. By utilizing the fuzzy state observer and combining the adaptive backstepping technique with adaptive fuzzy control design, an observer-based adaptive fuzzy output-feedback control approach is developed. It is proved that the proposed control approach can guarantee that all the signals in the closed-loop system are SGUUB.

The main contribution of this paper is the innovation and application value of the proposed controller for the ship roll motion. Compared with other control strategies, the unmeasured angular velocity is considered during the design process of the controller, and the control strategy is effective to decrease the roll motion.

2. Nonlinear Model Descriptions of Fin Control System

In this paper, we consider the following nonlinear systems [12, 13]:

$$\begin{aligned}
&\left(I_{xx}+J_{xx}\right)\ddot{\varphi}+\delta_N\dot{\varphi}+\delta_W\dot{\varphi}\left|\dot{\varphi}\right|+Dh\varphi\left[1-\left(\frac{\varphi}{\varphi_v}\right)^2\right]\\
&\quad=M_c+M_W,\\
&I_{xx}+J_{xx}=\frac{DB^2}{g}\left(0.3085+\frac{0.0227B}{d}-\frac{0.00043L}{100}\right)^2,\\
&\delta_N=\frac{2n_1\sqrt{Dh\left(I_{xx}+J_{xx}\right)}}{\pi},\\
&\delta_W=\frac{3n_2\left(I_{xx}+J_{xx}\right)}{4},\\
&M_C=-\rho v^2A_f l_f C_L^a\left(a_f+\hbar+\frac{\dot{\varphi}l_f}{v}\right),\\
&M_W=-Dha_e,
\end{aligned}\tag{1}$$

where φ is rolling angle of ship; I_{xx} and J_{xx} are inertias moments and the added inertia moments of the own ship; δ_N and δ_W are damping factors; D is the tonnage of ship; h is initial metacentric height; φ_v is flooding angle; M_C is control moment of the fin stabilizer; M_W is the moment of sea wave act on ship; g is gravitational acceleration; B is width of ship; L is length between tow-column of ship; d is draught; n_1 and n_2 are test coefficients; ρ is fluid density; v is ship's speed; A_f is the area of fin; l_f is the acting force arm of the stabilizer fin; C_L^a is the slope of lift coefficient; a_f is the rotation angle of the stabilizer fin; a_e is significant wave angle. $\hbar=\varphi(t-\tau(t))$ is a time-delay term, and $\tau(t)$ is an unknown bounded time delay satisfying $|\tau(t)|\le\overline{\tau}$ and $\dot{\tau}(t)\le\tau^*\le 1$, where $\overline{\tau}$ and τ^* are known constants.

From (1), we have

$$\ddot{\varphi}=a_1\varphi+a_2\varphi^3+a_3\dot{\varphi}+a_4\dot{\varphi}\left|\dot{\varphi}\right|+ba_f+b\hbar+ca_e,\tag{2}$$

where $a_1=-Dh/(I_{xx}+J_{xx})$, $a_2=Dh/\varphi_v^2(I_{xx}+J_{xx})$, $a_3=-\delta_N/(I_{xx}+J_{xx})-\rho v^2A_f l_f^2C_L^a/(I_{xx}+J_{xx})v$, $a_4=-\delta_W/(I_{xx}+J_{xx})$, $b=-\rho v^2A_f l_f C_L^a/(I_{xx}+J_{xx})$ and $c=-Dh/(I_{xx}+J_{xx})$.

Define $x=[x_1,x_2]^T=[\varphi,\dot{\varphi}]^T$, $y=\varphi=x_1$, and $u=a_f$; (2) can be rewritten as

$$\begin{aligned}
\dot{x}_1&=x_2,\\
\dot{x}_2&=bu+f\left(x_1,x_2\right)+b\hbar+\Delta,\\
y&=x_1,
\end{aligned}\tag{3}$$

where $f(x_1,x_2)=a_1x_1+a_2x_1^3+a_3x_2+a_4x_2|x_2|$ and $\Delta=ca_e$.

3. Fuzzy Logic Systems

We introduce the fuzzy logic systems. A fuzzy logic system (FLS) consists of four parts: the knowledge base, the fuzzifier, the fuzzy inference engine working on fuzzy rules, and the defuzzifier. The knowledge base for FLS comprises a collection of fuzzy if-then rules of the following form:

$$\begin{aligned}
R^l:\quad&\text{If } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l \text{ and} \ldots \text{and } x_n \text{ is } F_n^l,\\
&\text{Then } y \text{ is } G^l,\quad l=1,2,\ldots,N,
\end{aligned}\tag{4}$$

where $x=[x_1,\ldots,x_n]^T$ and y are the fuzzy logic system input and output, respectively. Fuzzy sets F_i^l and $G^l$are associated with the fuzzy functions $\mu_{F_i^l}(x_i)$ and $\mu_{G^l}(y)$, respectively. N is the rules number.

Through singleton function, center average defuzzification, and product inference [20], the fuzzy logic system can be expressed as

$$y\left(x\right)=\frac{\sum_{l=1}^N\overline{y}_l\prod_{i=1}^n\mu_{F_i^l}\left(x_i\right)}{\sum_{l=1}^N\left[\prod_{i=1}^n\mu_{F_i^l}\left(x_i\right)\right]},\tag{5}$$

where $\overline{y}_l=\max_{y\in R}\mu_{G^l}(y)$.

Define the fuzzy basis functions as

$$\overline{\phi}_l=\frac{\prod_{i=1}^n\mu_{F_i^l}\left(x_i\right)}{\sum_{l=1}^N\left(\prod_{i=1}^n\mu_{F_i^l}\left(x_i\right)\right)}.\tag{6}$$

Denote $\theta=[\overline{y}_1,\overline{y}_2,\ldots,\overline{y}_N]^T=[\theta_1,\theta_2,\ldots,\theta_N]^T$ and $\phi^T(x)=[\overline{\phi}_1(x),\ldots,\overline{\phi}_N(x)]$; then, fuzzy logic system (5) can be rewritten as

$$y\left(x\right)=\theta^T\phi\left(x\right).\tag{7}$$

Lemma 1 (see [20]). *Let $f(x)$ be a continuous function defined on a compact set Ω. Then, for any constant $\varepsilon>0$, there exists an FLS* (5) *such as*

$$\sup_{x\in\Omega}\left|f\left(x\right)-\theta^T\phi\left(x\right)\right|\le\varepsilon.\tag{8}$$

The optimal parameter vector θ_i^* is defined as

$$\theta^* := \arg\min_{\theta\in R^l}\left\{\sup_{x\in D}\left|f(x)-\theta^T\phi(x)\right|\right\}. \tag{9}$$

4. Fuzzy State Observer Design

The state $x_2 = \dot{\varphi}$ in system (3) is not available for feedback. Therefore, a state observer should be established to estimate the states

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + k_1\left(y-\hat{x}_1\right),\\ \dot{\hat{x}}_2 &= k_2\left(y-\hat{x}_1\right)+\hat{f}\left(x_1,\hat{x}_2\right)+bu,\\ \hat{y} &= \hat{x}_1,\end{aligned} \tag{10}$$

where $\hat{f}(x_1,\hat{x}_2) = \theta^T\phi(x_1,\hat{x}_2)$ is the fuzzy logic system and is used to approximate the nonlinear term $f(x_1,\hat{x}_2)$ in (11). System (10) can be rewritten as

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x}+Ky+E_2\hat{f}+E_2bu+E_2b\hbar,\\ \hat{y} &= E_1^T\hat{x},\end{aligned} \tag{11}$$

where $\hat{x} = [\hat{x}_1,\hat{x}_2]^T$, $A = \begin{bmatrix}-k_1 & 1\\ -k_2 & 0\end{bmatrix}$, $K = [k_1,k_2]^T$, $E_1^T = [1\ \ 0]$, and $E_2 = [0\ \ 1]^T$.

The coefficients k_1 and k_2 are chosen such that the polynomial $p(s) = s^2 + k_1 s + k_2$ is a Hurwitz. Thus, given a $Q^T = Q > 0$, there exists a positive definite matrix $P^T = P > 0$ such that

$$A^TP + PA = -Q. \tag{12}$$

Let $e = x - \hat{x}$ be observer error. Then, from (3) and (11), we have the observer errors equation

$$\dot{e} = Ae + E_2\Delta + E_2\tilde{f} + E_2b\hbar, \tag{13}$$

where $\tilde{f} = f(x_1,x_2) - \hat{f}(x_1,\hat{x}_2)$. Consider the following Lyapunov candidate V_0 for (13) as

$$V_0 = e^TPe. \tag{14}$$

The time derivative of V_0 along the solutions of (13) is

$$\dot{V}_0 = -e^TQe + 2e^TP\left(E_2\Delta + E_2\tilde{f} + E_2b\hbar\right). \tag{15}$$

Since $f(x_1,x_2) = a_1x_1 + a_2x_1^3 + a_3x_2 + a_4x_2|x_2|$ satisfies lipschitz condition, we have

$$\left|f(x_1,x_2)-f(x_1,\hat{x}_2)\right| \le L_f\left|x_2-\hat{x}_2\right|, \tag{16}$$

where L_f is a constant. By using Young's inequality and $\phi^T(x_1,\hat{x}_2)\phi(x_1,\hat{x}_2) \le 1$, we have

$$\begin{aligned}2e^TPE_2\left(\Delta+\tilde{f}+b\hbar\right) \le{}& \overline{L}\|e\|^2 + \overline{D} + \|P\|^2\tilde{\theta}^T\tilde{\theta}\\ &+ x_1^2\left(t-\tau(t)\right),\end{aligned} \tag{17}$$

where $\overline{L} = 3 + 2\|P\|L_f + \|P\|^2$, $\tilde{\theta} = \theta^* - \theta$, $\overline{D} = \|P\|^2\Delta^2 + \|P\|^2\varepsilon^{*2}$, and ε is the fuzzy minimum approximation error, which satisfies $|\varepsilon| \le \varepsilon^*$, and ε^* is an unknown constant. Substituting (17) into (15) yields

$$\dot{V}_0 \le -e^TQe + \overline{L}\|e\|^2 + \overline{D} + \|P\|^2\tilde{\theta}^T\tilde{\theta}. \tag{18}$$

5. Adaptive Fuzzy Backstepping Control Design and Stability Analysis

Define the following change of coordinates:

$$\begin{aligned}\chi_1 &= y,\\ \chi_2 &= \hat{x}_2 - \alpha_1,\end{aligned} \tag{19}$$

where α_1 is an intermediate control function which will be designed in the following.

Step 1. The time derivative of χ_1 is

$$\dot{\chi}_1 = x_2 = \hat{x}_2 + e_2 = \chi_2 + \alpha_1 + e_2. \tag{20}$$

Consider the following Lyapunov function candidate V_1:

$$V_1 = V_0 + \chi_1^2. \tag{21}$$

The time derivative of V_1 is

$$\begin{aligned}\dot{V}_1 \le{}& -e^TQe + \left(\overline{L}+1\right)\|e\|^2 + \overline{D} + \|P\|^2\tilde{\theta}^T\tilde{\theta} + x_1^2\left(t-\tau(t)\right)\\ &+ 2\chi_1\left(\chi_2+\alpha_1+\frac{1}{2}\chi_1\right).\end{aligned} \tag{22}$$

Choose the intermediate control function α_1:

$$\alpha_1 = -\frac{1}{2}\chi_1 - c_1\chi_1 - \frac{e^{r\overline{\tau}}}{1-\tau^*}\chi_1. \tag{23}$$

By substituting (23) into (22), we have

$$\begin{aligned}\dot{V}_1 \le{}& -e^TQe + \left(\overline{L}+1\right)\|e\|^2 + \overline{D} + \|P\|^2\tilde{\theta}^T\tilde{\theta}\\ &+ x_1^2\left(t-\tau(t)\right) + 2\chi_1\chi_2 - 2c_1\chi_1^2 - \frac{e^{r\overline{\tau}}}{1-\tau^*}\chi_1^2.\end{aligned} \tag{24}$$

Step 2. The time derivative of χ_2 is

$$\begin{aligned}\dot{\chi}_2 &= \dot{\hat{x}}_2 - \dot{\alpha}_1\\ &= bu + \theta^T\varphi\left(x_1,\hat{x}_2\right) + \tilde{\theta}^T\varphi\left(x_1,\hat{x}_2\right) - \tilde{\theta}^T\varphi\left(x_1,\hat{x}_2\right)\\ &\quad + k_2\left(y-\hat{x}_1\right) - \frac{\partial\alpha_1}{\partial x_1}e_2 - \frac{\partial\alpha_1}{\partial x_1}\hat{x}_2.\end{aligned} \tag{25}$$

Consider the following Lyapunov-Krasovskii functional

$$V = V_1 + \chi_2^2 + \overline{V} + \frac{1}{2\gamma}\tilde{\theta}^T\tilde{\theta}, \tag{26}$$

where γ is a positive design constant and $\overline{V} = (e^{r(\overline{\tau}-t)}/(1-\tau^*))\int_{t-\tau(t)}^{t} e^{rs}x_1^2(s)ds$ with r being a positive constant.

The time derivative of V is

$$\begin{aligned}\dot{V} = \dot{V}_1 + 2\chi_2 \Bigg(& bu + \theta^T \phi(x_1, \hat{x}_2) + k_2 (y - \hat{x}_1) \\ & + \widetilde{\theta}^T \phi(x_1, \hat{x}_2) - \widetilde{\theta}^T \phi(x_1, \hat{x}_2) \\ & - \frac{\partial \alpha_1}{\partial x_1} e_2 - \frac{\partial \alpha_1}{\partial x_1} \hat{x}_2 \Bigg) + \frac{1}{\gamma} \widetilde{\theta}^T \dot{\widetilde{\theta}} - r\overline{V} \\ & + \frac{e^{r\overline{\tau}}}{1 - \tau^*} x_1^2 - x_1^2 (t - \tau(t)).\end{aligned} \tag{27}$$

By using Young's inequality, we have

$$\begin{aligned} -2\chi_2 \frac{\partial \alpha_1}{\partial x_1} e_2 &\le \left(\chi_2 \frac{\partial \alpha_1}{\partial x_1} \right)^2 + \|e\|^2, \\ -2\chi_2 \widetilde{\theta}^T \phi(x_1, \hat{x}_2) &\le \chi_2^2 + \widetilde{\theta}^T \widetilde{\theta}. \end{aligned} \tag{28}$$

Substituting (28) into (27) yields

$$\begin{aligned}\dot{V} \le & - e^T Q e + \left(\overline{L} + 2 \right) \|e\|^2 - r\overline{V} + \overline{D} + \left(\|P\|^2 + 1 \right) \widetilde{\theta}^T \widetilde{\theta} \\ & - 2c_1 \chi_1^2 + 2\chi_2 \Bigg(\chi_1 + \frac{1}{2}\chi_2 + bu + \theta^T \phi(x_1, \hat{x}_2) \\ & + k_2 (y - \hat{x}_1) + \frac{1}{2}\chi_2 \left(\frac{\partial \alpha_1}{\partial x_1} \right)^2 - \frac{\partial \alpha_1}{\partial x_1} \hat{x}_2 \Bigg) \\ & + \widetilde{\theta}^T \left(2\chi_2 \phi(x_1, \hat{x}_2) \right) - \frac{1}{\gamma} \dot{\theta}.\end{aligned} \tag{29}$$

Choose control input u and the adaptive function θ as

$$\begin{aligned} u = \frac{1}{b} \Bigg[& - c_2 \chi_2 - \chi_1 - \frac{1}{2}\chi_2 - \theta^T \phi(x_1, \hat{x}_2) - k_2 (y - \hat{x}_1) \\ & - \frac{1}{2}\chi_2 \left(\frac{\partial \alpha_1}{\partial x_1} \right)^2 + \frac{\partial \alpha_1}{\partial x_1} \hat{x}_2 \Bigg], \\ \dot{\theta} = & \gamma \left[2\chi_2 \phi(x_1, \hat{x}_2) - \sigma\theta \right], \end{aligned} \tag{30}$$

where c_2 and σ are positive design constants. Substituting (30) into (29) yields

$$\begin{aligned}\dot{V} \le & - e^T Q e + \left(\overline{L} + 2 \right) \|e\|^2 + \overline{D} - r\overline{V} - 2c_1 \chi_1^2 - 2c_2 \chi_2^2 \\ & + \sigma \widetilde{\theta}^T \theta + \left(\|P\|^2 + 1 \right) \widetilde{\theta}^T \widetilde{\theta}.\end{aligned} \tag{31}$$

By using Young's inequality, the following inequality can be obtained:

$$\sigma \widetilde{\theta}^T \theta = \sigma \widetilde{\theta}^T \left(\theta^* - \widetilde{\theta} \right) \le -\frac{\sigma}{2} \widetilde{\theta}^T \widetilde{\theta} + \frac{\sigma}{2} \theta^{*T} \theta^*. \tag{32}$$

Therefore, we have

$$\begin{aligned}\dot{V} \le & - \left[\lambda_{\min}(Q) - \overline{L} - 2 \right] \|e\|^2 + \overline{D} - 2c_1 \chi_1^2 - 2c_2 \chi_2^2 \\ & - \left(\frac{\sigma}{2} - \left(\|P\|^2 + 1 \right) \right) \widetilde{\theta}^T \widetilde{\theta} + \frac{\sigma}{2} \theta^{*T} \theta^*,\end{aligned} \tag{33}$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q. Choose $\rho = \lambda_{\min}(Q) - \overline{L} - 2 > 0$ and $q = (\sigma/2) - (\|P\|^2 + 1) > 0$. Let $\beta = \min\{\rho/\lambda_{\max}(P), c_1/2, c_2/2, \gamma q, r\}$, $\overline{\pi} = \overline{D} + (\sigma/2)\theta^{*T}\theta^*$, and (33) can be expressed as

$$\dot{V} \le -\beta V + \overline{\pi}. \tag{34}$$

Equation (34) can be rewritten as

$$V(t) \le V(0) e^{-\beta t} + \frac{\overline{\pi}}{\beta}. \tag{35}$$

According to (35), it can be shown for the signals that x, e, χ_2, θ_2, $\widehat{w}$, and u are SGUUB. $|y(t)| \le \sqrt{V(0)} e^{-\beta t} + \sqrt{\pi/\beta}$ and $\|e(t)\| \le (\sqrt{V(0)} e^{-\beta t} + \sqrt{\pi/\beta})/\sqrt{\lambda_{\max}(P)}$. It is worth noting that $t \to \infty$, $e^{-\beta t} \to 0$, $\lim_{t \to \infty} |y(t) - y_r(t)| = \sqrt{\pi/\beta}$, and $\lim_{t \to \infty} \|e(t)\| = \sqrt{\pi/\beta}/\sqrt{\lambda_{\max}(P)}$.

6. Simulation Study

The parameters of a certain war ship are as follows: the length between perpendiculars is 98 m, the width of ship is 10.2 m, draught is 3.1 m, tonnage is 1458 t, the area of fin is 5.22 m^2, the acting force arm of fin stabilizer is 3.46 m, the lift coefficient of fin stabilizer is 3.39, flooding angle is 43°, initial metacentric height is 1.15 m, designed speed is 18 kt, test coefficients are $n_1 = 0.031$ and $n_2 = 0.051$, and $\tau = 0.5(1 + \sin(t))$.

In the simulation studies, the if-then rules are chosen as

$$\begin{aligned} R^1&: \quad \text{If } x_1 \text{ is } F_1^1 \text{ and } \hat{x}_2 \text{ is } F_2^1, \text{ then } y \text{ is } G^1; \\ R^2&: \quad \text{If } x_1 \text{ is } F_1^2 \text{ and } \hat{x}_2 \text{ is } F_2^2, \text{ then } y \text{ is } G^2; \\ R^3&: \quad \text{If } x_1 \text{ is } F_1^3 \text{ and } \hat{x}_2 \text{ is } F_2^3, \text{ then } y \text{ is } G^3; \\ R^4&: \quad \text{If } x_1 \text{ is } F_1^4 \text{ and } \hat{x}_2 \text{ is } F_2^4, \text{ then } y \text{ is } G^4; \\ R^5&: \quad \text{If } x_1 \text{ is } F_1^5 \text{ and } \hat{x}_2 \text{ is } F_2^5, \text{ then } y \text{ is } G^5, \end{aligned} \tag{36}$$

where fuzzy sets are chosen as F_1^1 = (NL), F_2^1 = (NL), F_1^2 = (NS), F_2^2 = (NS), F_1^3 = (ZE), F_2^3 = (ZE), F_1^4 = (PS), F_2^4 = (PS), F_1^5 = (PL), F_2^5 = (PL), which are defined over the interval $[-2, 2]$ for variables x_1 and $\hat{x}_2$, respectively. NL, NS, ZO, PS, and PL denote negative large, negative small, zero, positive small, and positive large, respectively.

In this section, the fuzzy membership functions are determined by using the expertise and experimental result. Center points of these membership functions are selected as −2, −1, 0, 1, and 2, respectively. The corresponding fuzzy membership functions are given by

$$\begin{aligned} \mu_{F_1^l}(x_1) &= \exp\left[-\frac{(x_1 - 3 + l)^2}{4} \right], \\ \mu_{F_2^l}(\hat{x}_2) &= \exp\left[-\frac{(\hat{x}_2 - 3 + l)^2}{4} \right], \end{aligned} \quad l = 1, \ldots, 5. \tag{37}$$

Define fuzzy basis functions as

$$\phi(x_1, \hat{x}_2) = \frac{\mu_{F_1^l} \mu_{F_2^l}}{\sum_{k=1}^{5} \left[\mu_{F_1^k} \mu_{F_2^k} \right]}, \quad l = 1, \ldots, 5. \tag{38}$$

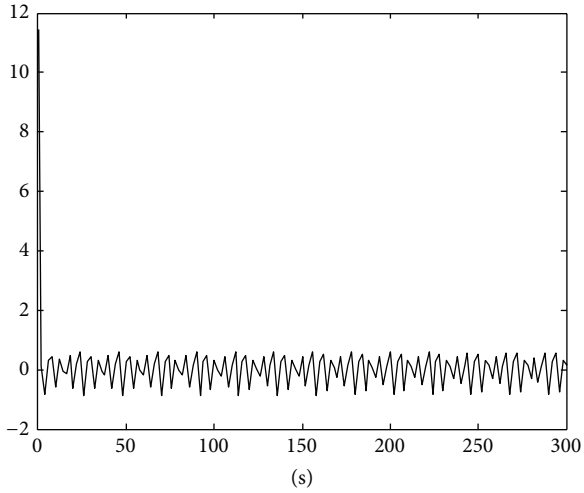

Figure 1: Ship roll angle φ.

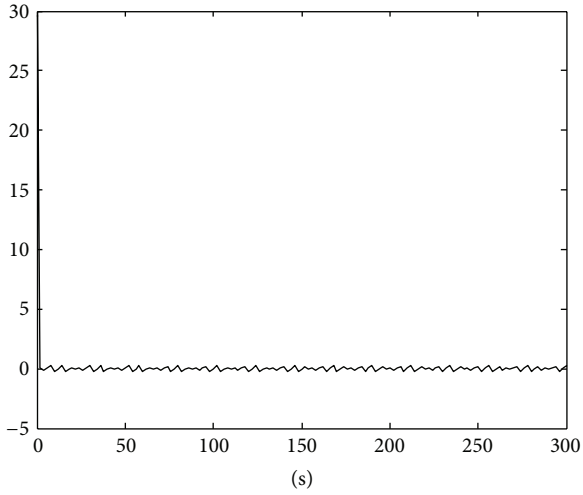

Figure 2: Fin control angle a_f.

The design parameters are chosen as $Q = \text{diag}\{30, 30\}$, $k_1 = 2$, $k_2 = 2$, $\sigma = 32$, $\gamma = 20$, $r = 1$, $\tau^* = 0.6$, $\overline{\tau} = 1.1$, $c_1 = c_2 = 4$. The initial conditions are given as $\varphi(0) = 11.26^\circ$, $\dot{\varphi}(0) = 5.63^\circ$, and the other initial values are chosen as zeros. The simulation results are shown in Figures 1 and 2.

7. Conclusions

In this paper, the time-delay ship roll stabilization by fin control system has been considered. Assuming that angular velocity in roll cannot be measured, an adaptive fuzzy output-feedback control has been investigated. The fuzzy logic system was used to approximate the uncertain term of the controlled system, and a fuzzy state observer was designed to estimate the unmeasured states. By utilizing the fuzzy state observer, an adaptive fuzzy output-feedback control approach was developed for fin control system. It is proven that the proposed control approach can guarantee that all the signals in the closed-loop system are SGUUB.

In the future research, the input constraints of the fin control system should be considered. In fact, the input variable of the fin control system is limited in a certain range. The adaptive fuzzy output-feedback controller with the input saturation will be researched in the future.

Conflict of Interests

The author of the paper has declared no conflict of interests.

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Research Article

Stability and Hopf Bifurcation of a Predator-Prey Model with Distributed Delays and Competition Term

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A class of predator-prey system with distributed delays and competition term is considered. By considering the time delay as bifurcation parameter, we analyze the stability and the Hopf bifurcation of the predator-prey system. According to the theorem of Hopf bifurcation, some sufficient conditions are obtained for the local stability of the positive equilibrium point.

1. Introduction

In general, delay differential equations display more complicated dynamics than ordinary differential equations as a time delay could bring a switch in the stability of equilibria and induce various oscillations and periodic solutions. Neural Networks with time delay or distributed delays have been investigated by some researchers [1–4]. Different from [1–4], distributed delays are introduced into a Lotka-Volterra predator-prey system. Lotka-Volterra predator-prey system is always the hot question in mathematical ecology, and many scholars have made deep and broad study in this area [5–10]. May [5] first proposed a model describing a predator-prey system with delay:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r_1 - a_{11}x(t-\tau) - a_{12}y(t)\right], \\ \dot{y}(t) &= y(t)\left[-r_2 - a_{21}x(t) - a_{22}y(t)\right], \end{aligned} \tag{1}$$

where $x(t)$, $y(t)$ denote the densities of prey and predator at time t, respectively, $\tau > 0$ denotes growth time of prey, $r_1 > 0$, $r_2 > 0$ denote the growth rates of prey and predator at time t, respectively, $a_{ii} > 0$ $(i = 1, 2)$ denote the inner impact factors in two populations, respectively, and $a_{ij} > 0$ $(i \neq j)$ denote the mutual impact factors in two populations, respectively.

Recently, Song and Wei [9] developed a Logistic predator-prey model with discrete and distributed delays:

$$\dot{x}(t) = rx(t)\left[1 - a_1 x(t-\tau) - a_2 \int_{-\infty}^{t} f(t-s)\, x(s)\, ds\right], \tag{2}$$

where r, τ, a_1, and a_2 are positive constants and $f(t)$ is the kernel function satisfying $f(t) \geq 0$ $(t \geq 0)$ and $\int_0^\infty f(t)dt = 1$.

In nature, living space and resources are limited. Therefore it is advisable to study a predator-prey model with competition term. The competition term describes the competitive interactions between the two species. Associated with these works in [5, 9, 10], in this paper, a predator-prey model with distributed delays and competition term is considered:

$$\begin{aligned} \dot{\bar{\bar{x}}}(t) &= \bar{\bar{x}}(t)\left[r - r_1\bar{\bar{x}}(t-\tau) - r_2\int_{-\infty}^{t} f(t-s)\,\bar{\bar{x}}(s)\,ds \right. \\ &\qquad \left. - a_1\bar{\bar{y}}(t-\tau)\right], \\ \dot{\bar{\bar{y}}}(t) &= \bar{\bar{y}}(t)\left[-r_3 + a_2\bar{\bar{x}}(t-\tau) - r_4\bar{\bar{y}}(t-\tau)\right] \\ &\quad - c\bar{\bar{y}}(t)\,\bar{\bar{y}}^2(t-\tau). \end{aligned} \tag{3}$$

Obviously, system (3) improves and contains most of existing models. For (3), we define the kernel function

$$f(t) = \sigma e^{-\sigma t}, \quad \sigma > 0. \tag{4}$$

Denote

$$\bar{\bar{u}}(t) = \int_{-\infty}^{t} \sigma e^{-\sigma(t-s)}\bar{\bar{x}}(s)\,ds. \tag{5}$$

Then we have

$$\begin{aligned}
\dot{\overline{\overline{x}}}(t) &= \overline{\overline{x}}(t)\left[r - r_1\overline{\overline{x}}(t-\tau) - r_2\overline{\overline{u}}(t) - a_1\overline{\overline{y}}(t-\tau)\right]\\
\dot{\overline{\overline{u}}}(t) &= \sigma\overline{\overline{x}}(t) - \sigma\overline{\overline{u}}(t)\\
\dot{\overline{\overline{y}}}(t) &= \overline{\overline{y}}(t)\left[-r_3 + a_2\overline{\overline{x}}(t-\tau) - r_4\overline{\overline{y}}(t-\tau)\right]\\
&\quad - c\overline{\overline{y}}(t)\overline{\overline{y}}^2(t-\tau).
\end{aligned} \tag{6}$$

Let $\overline{\overline{x}}(t) = (r/r_1)\overline{x}(t)$, $\overline{\overline{u}}(t) = (r/r_1)\overline{u}(t)$, and $\overline{\overline{y}}(t) = (r_3/r_4)\overline{y}(t)$; by (6),

$$\begin{aligned}
\dot{\overline{x}}(t) &= r\overline{x}(t)\left[1 - \overline{x}(t-\tau) - \frac{r_2}{r_1}\overline{u}(t) - \frac{a_1 r_3}{r r_4}\overline{y}(t-\tau)\right]\\
\dot{\overline{u}}(t) &= \sigma\overline{x}(t) - \sigma\overline{u}(t)\\
\dot{\overline{y}}(t) &= r_3\overline{y}(t)\left[-1 + \frac{a_2 r}{r_1 r_3}\overline{x}(t-\tau)\right.\\
&\quad \left. - \overline{y}(t-\tau) - \frac{c r_3}{r_4^2}\overline{y}^2(t-\tau)\right].
\end{aligned} \tag{7}$$

Denote $a = r_2/r_1$, $b = a_1 r_3/r r_4$, $c = a_2 r/r_1 r_3$, and $d = c r_3/r_4^2$; from (7), it follows that

$$\begin{aligned}
\dot{\overline{x}}(t) &= r\overline{x}(t)\left[1 - \overline{x}(t-\tau) - a\overline{u}(t) - b\overline{y}(t-\tau)\right]\\
\dot{\overline{u}}(t) &= \sigma\overline{x}(t) - \sigma\overline{u}(t)\\
\dot{\overline{y}}(t) &= r_3\overline{y}(t)\\
&\quad \times\left[-1 + c\overline{x}(t-\tau) - \overline{y}(t-\tau) - d\overline{y}^2(t-\tau)\right].
\end{aligned} \tag{8}$$

2. Main Results

To ensure system (8) with positive equilibrium points, let us suppose system (8) with conditions

$$G_1:\ \sqrt{\Delta} - (1+a)d > 0,\ (2+b)(1+a)d - \sqrt{\Delta} > 0,$$

where $\Delta = (1 + a + bc)^2 - 4(1+a)d(1+a-c)$.

Clearly, $z^* = (x^*, u^*, y^*)$ is a positive equilibrium point of system (8) if and only if $z^* = (x^*, u^*, y^*)$ satisfies

$$\begin{aligned}
0 &= 1 - x^* - au^* - by^*\\
0 &= bx^* - bu^*\\
0 &= -1 + cx^* - y^* - dy^{*2}.
\end{aligned} \tag{9}$$

Obviously, under the assumption G_1, system (8) has unique positive equilibrium points $z^* = (x^*, u^*, y^*)$, where

$$\begin{aligned}
x^* &= \frac{(2+b)(1+a)d - b\sqrt{\Delta}}{2(1+a)^2 d}\\
u^* &= x^*\\
y^* &= \frac{\sqrt{\Delta} - (1+a+bc)}{2(1+a)d}.
\end{aligned} \tag{10}$$

Let $\overline{x}(t) = x(t) - x^*$, $\overline{u}(t) = u(t) - u^*$, and $\overline{y}(t) = y(t) - y^*$; then

$$\begin{aligned}
\dot{x}(t) &= r\left[x(t) + x^*\right]\left[-x(t-\tau) - au(t) - by(t-\tau)\right]\\
\dot{u}(t) &= bx(t) - bu(t)\\
\dot{y}(t) &= r_3\left[y(t) + y^*\right]\\
&\quad \times\left[cx(t-\tau) - \left(1 + 2dy^*\right)y(t) - dy^2(t)\right].
\end{aligned} \tag{11}$$

Linearizing system (11) at $(0, 0, 0)$,

$$\begin{aligned}
\dot{x}(t) &= rx^*\left[-x(t-\tau) - au(t) - by(t-\tau)\right]\\
\dot{u}(t) &= bx(t) - bu(t)\\
\dot{y}(t) &= r_3 y^*\left[cx(t-\tau) - \left(1 + 2dy^*\right)y(t)\right],
\end{aligned} \tag{12}$$

then the characteristic equation of (12) is

$$\begin{aligned}
&\lambda^3 + d_1\lambda^2 + d_2\lambda - d_3 + \left(d_4\lambda^2 + d_5\lambda - d_6\right)e^{-\lambda\tau}\\
&\quad + \left(d_7\lambda + d_8\right)e^{-2\lambda\tau} = 0,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
d_1 &= b - r_3 y^*\left(1 + 2dy^*\right),\\
d_2 &= rabx^* - rr_3 x^* y^*\left(1 + 2dy^*\right),\\
d_3 &= rr_3 abx^* y^*\left(1 + 2dy^*\right), \qquad d_4 = rx^*,\\
d_5 &= rbx^* - rr_3 x^* y^*\left(1 + 2dy^*\right),\\
d_6 &= rr_3 bx^* y^*\left(1 + 2dy^*\right),\\
d_7 &= rr_3 bcx^* y^*, \qquad d_8 = rr_3 b^2 cx^* y^*.
\end{aligned} \tag{14}$$

By (13), we can get

$$\begin{aligned}
&\left(\lambda^3 + d_1\lambda^2 + d_2\lambda - d_3\right)e^{\lambda\tau} + \left(d_4\lambda^2 + d_5\lambda - d_6\right)\\
&\quad + \left(d_7\lambda + d_8\right)e^{-\lambda\tau} = 0.
\end{aligned} \tag{15}$$

Clearly, $i\omega$ ($\omega > 0$) is a solution of (15) if and only if ω satisfies

$$\begin{aligned}
&\left(-i\omega^3 - d_1\omega^2 + d_2 i\omega - d_3\right)e^{i\omega} + \left(-d_4\omega^2 + d_5 i\omega - d_6\right)\\
&\quad + \left(d_7 i\omega + d_8\right)e^{-i\omega} = 0.
\end{aligned} \tag{16}$$

Substituting $e^{i\omega} = \cos\omega\tau + i\sin\omega\tau$, $e^{-i\omega} = \cos\omega\tau - i\sin\omega\tau$ into (16), then

$$\begin{aligned}
&\left(-i\omega^3 - d_1\omega^2 + d_2 i\omega - d_3\right)(\cos\omega\tau + i\sin\omega\tau)\\
&\quad + \left(-d_4\omega^2 + d_5 i\omega - d_6\right)\\
&\quad + \left(d_7 i\omega + d_8\right)(\cos\omega\tau - i\sin\omega\tau) = 0;
\end{aligned} \tag{17}$$

that is,

$$\begin{aligned}&\left[\left(\omega^3 - d_2\omega\right) - d_7\omega\right]\cos\omega\tau \\ &\quad + \left[d_8 + \left(d_1\omega + d_3\right)\right]\sin\omega\tau = d_5\omega, \\ &\left[\left(\omega^3 - d_2\omega\right) + d_7\omega\right]\sin\omega\tau \\ &\quad + \left[d_8 - \left(d_1\omega + d_3\right)\right]\cos\omega\tau = d_4\omega^2 + d_6;\end{aligned} \tag{18}$$

thus

$$\begin{aligned}\sin\omega\tau &= \left(d_4\omega^5 + \left[d_6 - d_4\left(d_2 + d_7\right)\right]\omega^3 + d_1d_5\omega^2\right. \\ &\quad \left. + \left[d_5\left(d_3 - d_8\right) - d_6\left(d_2 + d_7\right)\right]\omega\right) \\ &\quad \times \left(\omega^6 - 2d_2\omega^4 + \left(d_1^2 + d_2^2 - d_7^2\right)\omega^2\right. \\ &\quad \left. + 2d_1d_3\omega + d_3^2 - d_8^2\right)^{-1}, \\ \cos\omega\tau &= \left(d_5\omega^4 - d_1d_4\omega^3 + \left(d_5d_7 - d_2d_5 - d_4d_8 - d_3d_4\right)\omega^2\right. \\ &\quad \left. - d_1d_6\omega - \left(d_3 + d_8\right)d_6\right) \\ &\quad \times \left(\omega^6 - 2d_2\omega^4 + \left(d_1^2 + d_2^2 - d_7^2\right)\omega^2\right. \\ &\quad \left. + 2d_1d_3\omega + d_3^2 - d_8^2\right)^{-1}.\end{aligned} \tag{19}$$

Let

$$\begin{aligned}&e_1 = -2d_2, \quad e_2 = d_1^2 + d_2^2 - d_7^2, \\ &e_3 = 2d_1d_3, \quad e_4 = d_3^2 - d_8^2, \quad e_5 = d_4, \\ &e_6 = d_6 - d_4(d_2 + d_7), \quad e_7 = d_1d_5, \\ &e_8 = \left[d_5\left(d_3 - d_8\right) - d_6\left(d_2 + d_7\right)\right], \\ &e_9 = d_5, \quad e_{10} = -d_1d_4, \\ &e_{11} = d_5d_7 - d_2d_5 - d_4d_8 - d_3d_4, \\ &e_{12} = -d_1d_6, \quad e_{13} = -\left(d_3 + d_8\right),\end{aligned} \tag{20}$$

so

$$\begin{aligned}\sin\omega\tau &= \frac{e_5\omega^5 + e_6\omega^3 + e_7\omega^2 + e_8\omega}{\omega^6 + e_1\omega^4 + e_2\omega^2 + e_3\omega + e_4}, \\ \cos\omega\tau &= \frac{e_9\omega^4 + e_{10}\omega^3 + e_{11}\omega^2 + e_{12}\omega + e_{13}}{\omega^6 + e_1\omega^4 + e_2\omega^2 + e_3\omega + e_4}.\end{aligned} \tag{21}$$

Combined with $\sin^2\omega\tau + \cos^2\omega\tau = 1$, we get

$$\begin{aligned}&\omega^{12} + f_1\omega^{10} + f_2\omega^8 + f_3\omega^7 + f_4\omega^6 + f_5\omega^5 + f_6\omega^4 + f_7\omega^3 \\ &\quad + f_8\omega^2 + f_9\omega + f_{10} = 0,\end{aligned} \tag{22}$$

where

$$\begin{aligned}&f_1 = 2e_1 - e_5^2, \quad f_2 = 2e_2 + e_1^2 - 2e_5e_6 - e_9^2, \\ &f_3 = 2\left(e_3 - e_5e_7 - e_9e_{10}\right), \\ &f_4 = 2e_4 + 2e_1e_2 - e_6^2 - e_{10}^2 - 2e_5e_8 - 2e_9e_{11}, \\ &f_5 = 2\left(e_1e_3 - e_6e_7 - e_9e_{12} - e_{10}e_{11}\right), \\ &f_6 = e_2^2 + 2e_1e_4 - e_7^2 - e_{11}^2 - 2e_6e_8 - 2e_9e_{13} - 2e_{10}e_{12}, \\ &f_7 = 2\left(e_2e_3 - e_7e_8 - e_{10}e_{13} - e_{11}e_{12}\right), \\ &f_8 = e_3^2 + 2e_2e_4 - e_8^2 - e_{12}^2 - 2e_{11}e_{13}, \\ &f_9 = 2\left(e_3e_4 - e_{12}e_{13}\right), \quad f_{10} = e_4^2 - e_{13}^2.\end{aligned} \tag{23}$$

Let

$$\begin{aligned}G(\omega) &= \omega^{12} + f_1\omega^{10} + f_2\omega^8 + f_3\omega^7 + f_4\omega^6 + f_5\omega^5 \\ &\quad + f_6\omega^4 + f_7\omega^3 + f_8\omega^2 + f_9\omega + f_{10}.\end{aligned} \tag{24}$$

Assume

G_2: system (22) has at least one positive real solution.

On the other hand,

$$\cos\omega_k\tau = \frac{e_9\omega_k^4 + e_{10}\omega_k^3 + e_{11}\omega_k^2 + e_{12}\omega_k + e_{13}}{\omega_k^6 + e_1\omega_k^4 + e_2\omega_k^2 + e_3\omega_k + e_4}; \tag{25}$$

thus

$$\begin{aligned}\tau_k^{(j)} &= \frac{1}{\omega_k} \\ &\quad \times \left\{\arccos\frac{e_9\omega_k^4 + e_{10}\omega_k^3 + e_{11}\omega_k^2 + e_{12}\omega_k + e_{13}}{\omega_k^6 + e_1\omega_k^4 + e_2\omega_k^2 + e_3\omega_k + e_4} + 2j\pi\right\},\end{aligned} \tag{26}$$

where $k = 1, \ldots, 12$, $j = 0, 1, \ldots, \pm i\omega_k$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,\ldots,12\}}\left\{\tau_k^{(0)}\right\}, \quad \omega_0 = \omega_{k_0}. \tag{27}$$

When $\tau = 0$, we can obtain

$$\lambda^3 + \left(d_1 + d_4\right)\lambda^2 + \left(d_2 + d_5 + d_7\right)\lambda + d_8 - d_3 - d_6 = 0. \tag{28}$$

By Hurwitz criterion, it follows that the solutions of (28) are all of negative real parts when $\tau = 0$, so the positive equilibrium point Z^* is locally asymptotically stable.

In the following, we assume

G_3: $\mathrm{Re}(d\lambda/d\tau)|_{\tau=\tau_0} \neq 0$.

From (13),

$$\begin{aligned}&\left(3\lambda^2 + 2d_1\lambda + d_2\right)e^{\lambda\tau} + \left(\lambda^3 + d_1\lambda^2 + d_2\lambda - d_3\right)e^{\lambda\tau} \\ &\quad \times \left(\lambda + \tau\frac{d\lambda}{d\tau}\right) + \left(2d_4\lambda + d_5\right)\frac{d\lambda}{d\tau} + d_7e^{-\lambda\tau}\frac{d\lambda}{d\tau} \\ &\quad - \left(d_7\lambda + d_8\right)e^{-\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) = 0;\end{aligned} \tag{29}$$

therefore

$$\begin{aligned}
&\frac{d\lambda}{d\tau} \\
&= \left(-\lambda\left(\lambda^3 + d_1\lambda^2 + d_2\lambda - d_3\right)e^{\lambda\tau} + \lambda\left(d_7\lambda + d_8\right)e^{-\lambda\tau}\right) \\
&\quad \times \left(\left(3\lambda^2 + 2d_1\lambda + d_2\right)e^{\lambda\tau} + \left(\lambda^3 + d_1\lambda^2 + d_2\lambda - d_3\right)\tau e^{\lambda\tau}\right. \\
&\qquad \left. + 2d_4\lambda + d_5 + d_7e^{-\lambda\tau} - \left(d_7\lambda + d_8\right)e^{-\lambda\tau}\right)^{-1},
\end{aligned} \tag{30}$$

and then

$$\begin{aligned}
&\left(\frac{d\lambda}{d\tau}\right)^{-1} \\
&= \frac{\left(3\lambda^2 + 2d_1\lambda + d_2\right)e^{\lambda\tau} + 2d_4\lambda + d_5 + d_7e^{-\lambda\tau}}{-\lambda\left(\lambda^3 + d_1\lambda^2 + d_2\lambda - d_3\right)e^{\lambda\tau} + \lambda\left(d_7\lambda + d_8\right)e^{-\lambda\tau}} \\
&\quad - \frac{\tau}{\lambda} \\
&= \frac{\left(3\lambda^2 + 2d_1\lambda + d_2\right)e^{\lambda\tau} + 2d_4\lambda + d_5 + d_7e^{-\lambda\tau}}{d_4\lambda^3 + d_5\lambda^2 - d_6\lambda + 2\lambda\left(d_7\lambda + d_8\right)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\
&= \frac{\left(-3\omega^2 + 2d_1i\omega + d_2\right)e^{\lambda\tau} + 2d_4i\omega + d_5 + d_7e^{-\lambda\tau}}{-d_4i\omega^3 - d_5\omega^2 - d_6i\omega + 2i\omega\left(d_7i\omega + d_8\right)e^{-\lambda\tau}} \\
&\quad - \frac{\tau}{i\omega}.
\end{aligned} \tag{31}$$

Substituting $e^{i\omega} = \cos\omega\tau + i\sin\omega\tau$, $e^{-i\omega} = \cos\omega\tau - i\sin\omega\tau$ into (31), we get

$$\begin{aligned}
&\left(\frac{d\lambda}{d\tau}\right)^{-1} \\
&= \left(\left(-3\omega^2 + 2d_1i\omega + d_2\right)(\cos\omega\tau + i\sin\omega\tau)\right. \\
&\qquad \left. + 2d_4i\omega + d_5 + d_7\left(\cos\omega\tau - i\sin\omega\tau\right)\right) \\
&\quad \times \left(-d_4i\omega^3 - d_5\omega^2 - d_6i\omega\right. \\
&\qquad \left. + 2i\omega\left(d_7i\omega + d_8\right)\left(\cos\omega\tau - i\sin\omega\tau\right)\right)^{-1} - \frac{\tau}{i\omega} \\
&= \left(\left(-3\omega^2 + d_2 + d_7\right)\cos\omega\tau - 2d_1\omega\sin\omega\tau + d_5\right) \\
&\quad \times \left(\left(-d_5\omega^2 - 2d_7\omega^2\cos\omega\tau + 2d_8\omega\sin\omega\tau\right)\right. \\
&\qquad \left. + i\left(-d_4\omega^3 - d_6\omega + 2d_8\omega\cos\omega\tau + 2d_7\omega^2\sin\omega\tau\right)\right)^{-1} \\
&\quad + \left(i\left[2d_1\omega\cos\omega\tau + \left(-3\omega^2 + d_2 - d_7\right)\sin\omega\tau + 2d_4\omega\right]\right) \\
&\quad \times \left(\left(-d_5\omega^2 - 2d_7\omega^2\cos\omega\tau + 2d_8\omega\sin\omega\tau\right)\right. \\
&\qquad \left. + i\left(-d_4\omega^3 - d_6\omega + 2d_8\omega\cos\omega\tau + 2d_7\omega^2\sin\omega\tau\right)\right)^{-1} \\
&\quad - \frac{\tau}{i\omega}.
\end{aligned} \tag{32}$$

Let

$$\begin{aligned}
Q &= \left(-d_5\omega^2 - 2d_7\omega^2\cos\omega\tau + 2d_8\omega\sin\omega\tau\right)^2 \\
&\quad + \left(-d_4\omega^3 - d_6\omega + 2d_8\omega\cos\omega\tau + 2d_7\omega^2\sin\omega\tau\right)^2 \\
&> 0
\end{aligned}$$

$$\begin{aligned}
&Q\,\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \\
&= \left[\left(-3\omega^2 + d_2 + d_7\right)\cos\omega\tau - 2d_1\omega\sin\omega\tau + d_5\right] \\
&\quad \times \left[\left(-d_5\omega^2 - 2d_7\omega^2\cos\omega\tau + 2d_8\omega\sin\omega\tau\right)\right] \\
&\quad + \left[2d_1\omega\cos\omega\tau + \left(-3\omega^2 + d_2 - d_7\right)\sin\omega\tau + 2d_4\omega\right] \\
&\quad \times \left[-d_4\omega^3 - d_6\omega + 2d_8\omega\cos\omega\tau + 2d_7\omega^2\sin\omega\tau\right].
\end{aligned} \tag{33}$$

Combined with

$$\mathrm{sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|^{-1}_{\tau=\tau_0}\right\}, \tag{34}$$

then we can derive the following theorem.

Theorem 1. *If assumptions G_2, G_3 hold, then*

(i) *when $\tau \in [0, \tau_0)$, the positive equilibrium point $z^*(x^*, u^*, y^*)$ of system* (11) *is asymptotically stable;*

(ii) *when $\tau = \tau_0$, the region near the positive equilibrium point $z^*(x^*, u^*, y^*)$ of system* (11) *can appear Hopf bifurcation.*

3. Conclusion

As we know, time delay has a strong impact on the dynamic evolution of a population, which could bring a switch in the stability of equilibria and induce various oscillations and periodic solutions. However, these effects would induce ecosystem unbalance and even biological disaster. In order to maintain the sustainable development of the biological resources, we investigate a predator-prey model with delays. In this paper, on the basis of Lotka-Volterra predator-prey system, a general class of predator-prey models with distributed delays and competition term is formulated and studied. Some sufficient conditions in terms of algebraic criterion are obtained, in order to achieve stability and Hopf bifurcation. These theoretical analyses can characterize the fundamental properties of ecological system and provide convenience for applications. By using the same method, similar results may be obtained for the predator-prey model with functional response which describes prey eaten per predator per unit of time.

In the future, we will investigate the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions based on the normal form approach theory and center manifold theory introduced by Hassard et al. [11].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Output-Feedback Control for a Class of Stochastic High-Order Feedforward Nonlinear Systems with Delay

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The problem of global output-feedback stabilization for a class of stochastic high-order time-delay feedforward nonlinear systems with different power orders is investigated. By combining the adding one power integrator technique with the homogeneous domination approach, an output-feedback controller design is proposed, which ensures the global asymptotical stability in probability of the closed-loop system.

1. Introduction

In the last decades, stochastic systems have received much more attention since stochastic modeling has come to play an important role in many branches of science and engineering applications. For this type of systems, the authors in [1–10] presented the basic stability theory of the stochastic control systems.

Global stabilization of triangular structural stochastic nonlinear systems has been a major issue in control theory over the last decades. For lower-triangular systems (namely, feedback systems), the stochastic asymptotic stabilization has been studied by using backstepping approach; see [11–13] and the references therein. Upper-triangular systems, which are also called feedforward systems, have been fully used to model many physical devices, such as the ball-beam with a friction term [14] and the cart-pendulum system [15]. By using a homogeneous domination approach, the problem of global output-feedback stabilization was addressed in [16] for a class of upper-triangular systems with higher-order nonlinearities. Then, the case of lower-order nonlinearities was considered in [17]. With the help of a generalized definition of homogeneity, the problem of using small controls to globally stabilize a class of upper-triangular systems was investigated in [18]. Inspired by the works [19, 20], the problem of using a sampled data controller to globally stabilize a class of uncertain upper-triangular systems was considered in [21]. When parameter uncertainties appeared in a system model, an adaptive stabilizer for feedforward nonlinear systems with general dynamic order in [22] was extended to [23].

On the other hand, time delays may arise naturally, which are usually the key factors that influence the stability of nonlinear systems. Many researchers have paid more attention to studying the stochastic nonlinear time-delay systems over the last decades and various results concerning stochastic lower-triangular nonlinear systems with time delays have been reported in [24–27]. For the case where the delays are of unknown length, an adaptive control design was presented in [28] for a system in feedforward form. An approach for compensating input delay of arbitrary length was presented [29] for forward complete and strict-feedforward nonlinear systems. When the nonlinear functions were higher order in states, the problem of global stabilization by state feedback was addressed in [30, 31]. It should be pointed out that these results were obtained for the case of state measurement. Naturally, one may ask a challenging and interesting question: if not all the states are measurable, how do we design an output-feedback controller for stochastic high-order feedforward systems with time delay? To the best of our knowledge, no output-feedback stabilization control scheme has so far been proposed for stochastic high-order feedforward systems with time delay.

Inspired by the aforementioned discussion, we deal with the problem of global output-feedback stabilization for a class of stochastic high-order feedforward nonlinear systems with time-varying delay and different power orders. Firstly, we design a state-feedback controller for the nominal system using the adding one power integrator technique. Then, by designing a homogeneous observer for the nominal system, we explicitly construct a homogeneous output-feedback stabilizer under the certainty equivalence principle. At last, we propose a scaled controller which guarantees global asymptotic stability in probability of the closed-loop system.

The outline of this paper is as follows. Sections 2 and 3 offer some preliminary results and problem formulation, respectively. The output-feedback controller is designed and analyzed in Section 4. This paper is concluded in Section 5.

Notations. The following standard notations are used throughout this paper. R^n denotes the real n-dimensional space. $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ for any $x \in R^n$. For any matrix $Q \in R^n \times R^m$, $|Q|$ denotes the Frobenius norm $|Q| = (\sum_{i,j} q_{ij}^2)^{1/2}$ and $|Q|_\infty = \max_{1\le i\le n}(\sum_{j=1}^{m} q_{ij}^2)^{1/2}$. For the sake of simplicity, sometimes function $x(t)$ is denoted by x; x_i represents the ith element of vector x and $\overline{x}_i = (x_1, \ldots, x_i)^T$. C^i denotes the set of all functions with continuous ith partial derivative. $\mathcal{K}$ denotes the set of all functions: $R^+ \to R^+$ which are continuous, strictly increasing, and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded.

2. Preliminary Results

Consider the following stochastic time-delay system

$$dx(t) = f(x(t), x(t-\sigma(t)))\,dt + g(x(t), x(t-\sigma(t)))^T d\omega \tag{1}$$

with an initial condition $\{x(s) : -\sigma_0 \le s \le 0\} = \xi \in C^b_{F_0} \times ([-\sigma_0, 0], R^n)$, where $\sigma(t) : R^+ \to [0, \sigma_0]$ is a Borel measurable function; $x(t) \in R^n$ denotes the state vector and $x(t-\sigma(t))$ is the state vectors with time-delay; ω is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, F, \{F_t\}_{t\ge 0}, P)$ with Ω being a sample space, F being a σ-field, $\{F_t\}_{t\ge 0}$ being a filtration, and P being a probability measure; $f(\cdot) \in R^n$, $g(\cdot) \in R^{n\times m}$ are locally Lipschitz functions and satisfy $f(0,0) = 0$, $g(0,0) = 0$.

In the following, we borrow some definitions and lemmas which play an important role in stabilizing and analyzing the stochastic time-delay systems for later development in this paper.

Definition 1 (see [26]). Define the infinitesimal generator $\mathcal{L}$ of function $V(x) \in C^2$ along system (1) as follows:

$$\mathcal{L}V(x) = \frac{\partial V}{\partial z} f(x, x(t-\sigma(t))) + \frac{1}{2}\mathrm{Tr}\left\{ g(x, x(t-\sigma(t))) \frac{\partial^2 V}{\partial x^2} g \times (x, x(t-\sigma(t)))^T \right\}, \tag{2}$$

in which $(1/2)\,\mathrm{Tr}\{g(x, x(t-\sigma(t)))(\partial^2 V/\partial x^2) g(x, x(t-\sigma(t)))^T\}$ is called the Hessian term of $\mathcal{L}$.

Definition 2 (see [26]). The equilibrium $x = 0$ of system (1) is globally asymptotically stable (GAS) in probability if for any $\epsilon > 0$, there exists a function $\gamma(\cdot,\cdot) \in \mathcal{K}\mathcal{L}$ such that $P\{|x| \le \gamma(\|\zeta\|, t)\} \ge 1 - \epsilon$ for any $t \ge 0$, $\zeta \in C^b_{F_0} \times ([-\tau, 0], R^n) \setminus \{0\}$, where $\|\zeta\| = \sup_{\theta\in[-\tau,0]}|\zeta(\theta)|$.

Definition 3 (see [16]). For fixed coordinates $(x_1, \ldots, x_n)$ and real numbers $r_i > 0$, $i = 1, \ldots, n$, one has the following.

(i) The dilation $\Delta_\varepsilon(x)$ is defined by $\Delta_\varepsilon(x) = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n)$, for all $\varepsilon > 0$, with r_i being called as the weights of the coordinates. For simplicity of notation, we define dilation weight $\Delta = (r_1, \ldots, r_n)$.

(ii) A function $V \in C(R^n, R)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $V(\Delta_\varepsilon(x)) = \varepsilon^\tau V(x_1, \ldots, x_n)$, for $\forall x \in R^n \setminus \{0\}$, $\varepsilon > 0$.

(iii) A vector field $f \in C(R^n, R^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $f_i(\Delta_\varepsilon(x)) = \varepsilon^{\tau+r_i} f_i(x_1, \ldots, x_n)$, for $\forall x \in R^n \setminus \{0\}$, $\varepsilon > 0$, $i = 1, \ldots, n$.

(iv) A homogeneous p-norm is defined as $\|x\|_{\Delta,p} = (\sum_{i=1}^{n} |x_i|^{p/r_i})^{1/p}$, for all $x \in R^n$, for a constant $p \ge 1$. For the simplicity, we choose $p = 2$ and write $\|x\|_\Delta$ for $\|x\|_{\Delta,2}$ in this paper.

Lemma 4 (see [26]). *Consider the stochastic time-delay system* (1)*; if there exist a function $V(x) \in C^2([-\sigma_0, \infty] \times R^n)$ and class $\mathcal{K}_\infty$ functions α_1, α_2 satisfying the following inequalities:*

$$\alpha_1(|x(t)|) \le V(x) \le \alpha_2\left(\sup_{(-\sigma_0\le s\le 0)} |x(t+s)|\right), \qquad \mathcal{L}V(x) \le -W(x(t)), \tag{3}$$

where $W(x(t))$ is continuous and positive definite, then

(1) *there exists a unique solution on $[-\sigma_0, \infty)$,*

(2) *the equilibrium $x = 0$ of the system* (1) *is GAS in probability and $P\{\lim_{t\to\infty}|x(t)| = 0\} = 1$.*

Lemma 5 (see [16]). *Given a dilation weight $\Delta = (r_1, \ldots, r_n)$, suppose $V_1(x)$ and $V_2(x)$ are homogeneous functions of degree τ_1 and τ_2, respectively. Then, $V_1(x) \cdot V_2(x)$ is also homogeneous with respect to the same dilation weight Δ. Moreover, the homogeneous degree of $V_1(x) \cdot V_2(x)$ is $\tau_1 + \tau_2$.*

Lemma 6 (see [16]). *Suppose $V : R^n \to R$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then, the following holds.*

(i) $\partial V/\partial x_i$ *is homogeneous of degree* $\tau - r_i$ *with* r_i *being the homogeneous weight of* x_i.

(ii) *There is a constant* c *such that* $V(x) \le c\|x\|_\Delta^\tau$. *Moreover, if* $V(x)$ *is positive definite, then*

$$\underline{c}\|x\|_\Delta^\tau \le V(x), \quad \text{for a constant } \underline{c} > 0. \tag{4}$$

Lemma 7 (see [16]). *For any* $x \in R$, $y \in R$, *and* $p \ge 1$, *the following inequalities hold:*

$$\begin{aligned} |x+y|^p &\le 2^{p-1}\left|x^p + y^p\right|, \\ \left(|x| + |y|\right)^{1/p} &\le |x|^{1/p} + |y|^{1/p}. \end{aligned} \tag{5}$$

Moreover, when $p \in R_{odd}^{\ge 1}$,

$$\begin{aligned} |x-y|^p &\le 2^{p-1}\left|x^p - y^p\right|, \\ \left|x^p - y^p\right| &\le p\,|x-y|\left|x^{p-1} + y^{p-1}\right| \\ &\le c\,|x-y|\left|(x-y)^{p-1} + y^{p-1}\right|, \end{aligned} \tag{6}$$

where $c > 0$ *is a constant.*

Lemma 8 (see [27]). *Let* x *and* y *be any real numbers and* $p \in R_{odd}^{\ge 1}$; *then*

$$-(x-y)\left(x^p - y^p\right) \le -\frac{1}{2^{p-1}}(x-y)^{p+1}. \tag{7}$$

Lemma 9 (see [27]). *Let* c *and* d *be two positive real numbers. For any positive number* $\gamma > 0$, *the following inequality holds*

$$|x|^c|y|^d \le \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-c/d}|y|^{c+d}. \tag{8}$$

3. Problem Formulation

In this paper, we will consider the following stochastic high-order feedforward nonlinear system with time-varying delay in the form

$$\begin{aligned} dz_1 &= z_2^{p_1}dt + f_1(z, z(t-\sigma(t))) \\ &\quad + g_1^T(z, z(t-\sigma(t)))\,d\omega, \\ dz_2 &= z_3^{p_2}dt + f_2(z, z(t-\sigma(t))) \\ &\quad + g_2^T(z, z(t-\sigma(t)))\,d\omega, \\ &\vdots \\ dz_n &= v^{p_n} + f_n(z, z(t-\sigma(t))) \\ &\quad + g_n^T(z, z(t-\sigma(t)))\,d\omega, \\ y &= z_1, \end{aligned} \tag{9}$$

where $z = [z_1, \ldots, z_n]^T \in R^n$, $y \in R$, $v \in R$ denote the system state, output, and input, respectively; $z(t-\sigma(t)) = [z_1(t-\sigma(t)), \ldots, z_n(t-\sigma(t))]$ is the state vector with time-delay; $\sigma(t) : R^+ \to [0, \sigma_0]$ is a time-varying delay. ω is an m-dimensional standard Wiener process. For $i = 1, \ldots, n$, $p_i \in R_{\text{odd}}^{\ge 3} \triangleq \{p \in R : p \ge 3$ and p is a ratio of odd integers$\}$. The drift functions $f_i : R^n \times R^n \to R$ and the diffusion functions $g_i : R^n \times R^n \to R^m$ are locally Lipschitz in $(z, z(t-\sigma(t)))$ and vanish at the origin.

In order to obtain the main results of this paper, the following assumptions are needed.

Assumption 10. For $i = 1, \ldots, n-1$, there are constants $a_i > 0$, $b_i > 0$, such that

$$\begin{aligned} &\left|f_i(z, z(t-\sigma(t)))\right| \\ &\quad \le a_i \sum_{j=i+2}^{n+1}\left(\left|z_j(t)\right|^{(r_i+\tau)/r_j} + \left|z_j(t-\sigma(t))\right|^{(r_i+\tau)/r_j}\right), \\ &\left|g_i(z, z(t-\sigma(t)))\right| \\ &\quad \le b_i \sum_{j=i+2}^{n+1}\left(\left|z_j(t)\right|^{(2r_i+\tau)/2r_j} + \left|z_j(t-\sigma(t))\right|^{(2r_i+\tau)/2r_j}\right), \end{aligned} \tag{10}$$

where $p_i \in R_{\text{odd}}^{\ge 3}$, $\tau \in [d_0, \infty)$, $r_1 = 1$, $r_{i+1} = (r_i + \tau)/p_i$, $d_1 = (2 - 1/(p_1 \cdots p_{n-1}))/(1 + \sum_{s=1}^{n-1}(1/(p_s \cdots p_{n-1})))$, $d_{i+1} = (2/(p_1 \cdots p_i) - 1/(p_1 \cdots p_{n-1}))/(1 + \sum_{s=1}^{n-1}(1/(p_s \cdots p_{n-1})) - \sum_{s=1}^{i}(2/(p_s \cdots p_i)))$, and $d_0 = \max_{j=1,\ldots,n}\{d_j\}$.

Assumption 11. The time-varying delay $\sigma(t)$ satisfies $\dot\sigma(t) \le \sigma < 1$ for a constant σ.

In this paper, we aim to constructively design a homogeneous output-feedback controller

$$\begin{aligned} \dot\chi(t) &= \nu(\chi(t), y(t)), \\ u(t) &= \omega(\chi(t), y(t)), \end{aligned} \tag{11}$$

for system (9) under Assumptions 10-11, such that the closed-loop system is GAS in probability.

4. Controller Design and Stability Analysis

In this section, an output-feedback controller will be explicitly constructed for the nonlinear system (9). We will combine the adding one power integrator technique with the homogeneous domination approach for output-feedback stabilization. The design procedure can be divided into three steps: (i) we first design a state-feedback controller for the nominal system without the drift and diffusion functions using the adding one power integrator technique; (ii) then, by designing a homogeneous observer for the nominal system, we explicitly construct a homogeneous output-feedback stabilizer under the certainty equivalence principle; (iii) in the end, we propose a scaled controller which guarantees global asymptotic stability in probability of the closed-loop system.

For simplicity, we assume $\tau = m/n$ in this paper with m being an even integer and n being an odd integer. Based on this assumption, it is obvious that r_i is a ratio of odd integers.

4.1. Homogeneous State-Feedback Control of Nominal Nonlinear System. In this subsection, we firstly introduce a key lemma, which avoids the zero-division problem and serves as a basis in the following design procedure. The proof is similar to that in [32].

Lemma 12. *If $p_i \in R_{odd}^{\geq 3}$, $\tau \in [d_0, +\infty)$ are guaranteed, then we can find $l_0, \mu_0 \in R_{odd}^{+}$, such that*

$$\max_{1\leq i\leq n}\left\{2r_i, \frac{r_i+\tau}{l_0}\right\} \leq \mu_0 \leq r_n + \tau \tag{12}$$

holds.

Now, we design a homogeneous state-feedback controller for the nominal system

$$\begin{aligned} dx_1 &= x_2^{p_1}dt,\\ dx_2 &= x_3^{p_2}dt,\\ &\vdots\\ dx_n &= u^{p_n}dt. \end{aligned} \tag{13}$$

Lemma 13. *For system* (13)*, there are positive definite, proper, and C^2 Lyapunov function V_n, a state-feedback controller u^*, and two positive constants, such that*

$$\mathscr{L}V_n \leq -\sum_{j=1}^{n}\pi_{n,j}\eta_j^{4l_0} + \eta_n^{q_n/\mu_0}\left(u^{p_n} - u^{*p_n}\right), \tag{14}$$

where V_n, u^* are defined in the following form:

$$\begin{aligned} V_n(\overline{x}_n) &= \sum_{i=1}^{n}\int_{x_i^*}^{x_i}\left(s^{\mu_0/r_i} - x_i^{*\mu_0/r_i}\right)^{q_i/\mu_0}ds,\\ x_1^* &= 0, \qquad \eta_1 = x_1^{\mu_0/r_1},\\ x_2^* &= -\beta_1^{r_2/\mu_0}\eta_1^{r_2/\mu_0}, \qquad \eta_2 = x_2^{\mu_0/r_2} - x_2^{*\mu_0/r_2},\\ &\vdots \qquad\qquad\qquad \vdots\\ x_n^* &= -\beta_{n-1}^{r_n/\mu_0}\eta_{n-1}^{r_n/\mu_0}, \qquad \eta_n = x_n^{\mu_0/r_n} - x_n^{*\mu_0/r_n},\\ u^* &= -\left(\pi_{nn} + b_{n1} + b_{n2}\right)^{1/p_n}\eta_n^{r_{n+1}/\mu_0}\\ &\triangleq -\beta_n^{r_{n+1}/\mu_0}\eta_n^{r_{n+1}/\mu_0}\\ &= -\left(\overline{\beta}_n x_n^{\mu_0/r_n} + \overline{\beta}_{n-1}x_{n-1}^{\mu_0/r_{n-1}} + \cdots + \overline{\beta}_1 x_1^{\mu_0/r_1}\right)^{r_{n+1}/\mu_0},\\ \overline{\beta}_j &= \beta_n\cdots\beta_j, \quad j = 1,\ldots,n. \end{aligned} \tag{15}$$

Remark 14. For the deterministic work, it is well known that there only needs a positive definite, proper, and C^1 Lyapunov function for the controller design and analysis. However, for the stochastic system, the Lyapunov function must be positive definite, proper, and at least C^2 because of the appearance of the Hessian term.

4.2. Construction of a Homogeneous Observer. Since $x_2, \ldots, x_n$ are unmeasurable, we design a reduced-order homogeneous observer

$$\begin{aligned} \dot{\widehat{\eta}}_2 &= -k_2\widehat{x}_2^{p_1}, & \widehat{x}_2 &= \left[\widehat{\eta}_2 + k_2\widehat{x}_1\right]^{r_2/r_1},\\ \dot{\widehat{\eta}}_3 &= -k_3\widehat{x}_3^{p_2}, & \widehat{x}_3 &= \left[\widehat{\eta}_3 + k_3\widehat{x}_2\right]^{r_3/r_2},\\ &\vdots & &\vdots\\ \dot{\widehat{\eta}}_n^* &= -k_n\widehat{x}_n^{p_{n-1}}, & \widehat{x}_n &= \left[\widehat{\eta}_n + k_n\widehat{x}_{n-1}\right]^{r_n/r_{n-1}}, \end{aligned} \tag{16}$$

where $k_2 > 0, \ldots, k_n > 0$ are constant gains to be determined later and $\widehat{x}_1 = x_1$.

According to the certainty equivalence principle, we use the estimated states $\widehat{x}_2, \ldots, \widehat{x}_n$ to construct the implementable controller

$$u(\widehat{x}) = -\left(\overline{\beta}_n\widehat{x}_n^{\mu_0/r_n} + \overline{\beta}_{n-1}\widehat{x}_{n-1}^{\mu_0/r_{n-1}} + \cdots + \overline{\beta}_1\widehat{x}_1^{\mu_0/r_1}\right)^{r_{n+1}/\mu_0}, \tag{17}$$

where $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)$.

In order to prove the global stability for the closed-loop system (13)–(16)-(17), we define

$$U_j = \int_{\lambda_j^{q_{j-1}/r_{j-1}}}^{x_j^{q_{j-1}/r_j}}\left(s^{r_{j-1}/q_{j-1}} - \lambda_j\right)ds, \quad \lambda_j = \widehat{\eta}_j + k_j\widehat{x}_{j-1}, \tag{18}$$

and let

$$e_j = \left(x_j^{p_{j-1}} - \widehat{x}_j^{p_{j-1}}\right)^{\mu_0/r_j p_{j-1}}, \tag{19}$$

for $j = 2, \ldots, n$.

Therefore, the differential operator $\mathscr{L}$ of U_j is

$$\begin{aligned} \mathscr{L}\left(U_j\right) &= \frac{q_{j-1}}{r_j}x_j^{q_{j-1}/r_j-1}\left(x_j^{r_{j-1}/r_j} - \lambda_j\right)x_{j+1}^{p_j}\\ &\quad - k_j\left(x_j^{q_{j-1}/r_j} - \lambda_j^{q_{j-1}/r_{j-1}}\right)x_j^{p_{j-1}}\\ &\quad + k_j\left(x_j^{q_{j-1}/r_j} - \lambda_j^{q_{j-1}/r_{j-1}}\right)\widehat{x}_j^{p_{j-1}}\\ &= \frac{q_{j-1}}{r_j}x_j^{q_{j-1}/r_j-1}\left(x_j^{r_{j-1}/r_j} - \lambda_j\right)x_{j+1}^{p_j}\\ &\quad - k_j\left[\left(x_j^{q_{j-1}/r_j} - \widehat{x}_j^{q_{j-1}/r_j}\right)\right.\\ &\qquad \left. + \left(\widehat{x}_j^{q_{j-1}/r_j} - \lambda_j^{q_{j-1}/r_{j-1}}\right)\right]e_j^{r_j p_{j-1}/\mu_0}, \end{aligned} \tag{20}$$

where $x_{n+1} = u(\widehat{x})$.

Just as in [16], each term of the right-hand side of (20) can be estimated by the following propositions, whose proofs are omitted here.

Proposition 15. *For $j = 2, \ldots, n-1$,*

$$\frac{q_{j-1}}{r_j} x_j^{q_{j-1}/r_j - 1} \left(x_j^{r_{j-1}/r_j} - \lambda_j \right) x_{j+1}^{p_j} \leq \sum_{k=j-1}^{j+1} \delta_{j,k,3} \eta_k^{4l_0} + g_{j1}\left(k_j\right) e_{j-1}^{4l_0} + b_{j3} e_j^{4l_0}, \tag{21}$$

where $\delta_{j,k,3} > 0$, $b_{j3} > 0$, $g_{j1}(k_j)$ is a continuous function with $g_{21}(k_2) = 0$.

Proposition 16. *For the implementable controller $u(\widehat{x})$ in (17), one obtains*

$$\frac{q_{j-1}}{r_j} x_j^{q_{j-1}/r_j - 1} \left(x_j^{r_{j-1}/r_j} - \lambda_j \right) u(\widehat{x})^{p_n} \leq \sum_{j=1}^{n} \delta_{n,j,4} \eta_j^{4l_0} + g_{n1}\left(k_n\right) e_{n-1}^{4l_0} + \sum_{j=2}^{n} b_{j4} e_j^{4l_0} \tag{22}$$

with positive constants $\delta_{n,j,4}$, b_{j4} and with $g_{n1}(k_n)$ being a continuous function.

Proposition 17. *For $j = 2, \ldots, n$, there exists a constant $\varpi_j = 2^{1-q_{j-1}/r_j p_{j-1}} > 0$ such that*

$$-k_j \left(x_j^{q_{j-1}/r_j} - \widehat{x}_j^{q_{j-1}/r_j} \right) e_j^{r_j p_{j-1}/\mu_0} \leq -k_j \varpi_j e_j^{4l_0}. \tag{23}$$

Proposition 18. *For $j = 3, \ldots, n$,*

$$-k_j \left(\widehat{x}_j^{q_{j-1}/r_j} - \lambda_j^{q_{j-1}/r_{j-1}} \right) e_j^{r_j p_{j-1}/\mu_0} \leq \sum_{k=j-1}^{j} \delta_{j,k,5} \eta_k^{4l_0} + g_{j2}\left(k_j\right) e_{j-1}^{4l_0} + b_{j5} e_j^{4l_0} \tag{24}$$

with positive constants $\delta_{j,k,5}$, b_{j5} and with $g_{j2}(k_j)$ being a continuous function.

Considering the Lyapunov function $U = \sum_{j=2}^{n} U_j$ and substituting (21)–(24) into (20), one concludes

$$\begin{aligned} \mathscr{L}U \leq & \sum_{j=1}^{n} \epsilon_{j1} \eta_j^{4l_0} + e_2^{4l_0} \\ & \times \left[-k_2 \varpi_2 + b_{23} + b_{24} + g_{31}\left(k_3\right) + g_{32}\left(k_3\right) \right] \\ & + \sum_{j=3}^{n-1} e_j^{4l_0} \left[-k_j \varpi_j + b_{j3} + b_{j4} + b_{j5} \right. \\ & \qquad \left. + g_{j+1,1}\left(k_{j+1}\right) + g_{j+1,2}\left(k_{j+1}\right) \right] \\ & + e_n^{4l_0} \left[-k_n \varpi_n + b_{n4} + b_{n5} \right]. \end{aligned} \tag{25}$$

To determinate the observer gain k_j in (16), we have the following proposition to deal with the redundant term $\eta_n^{q_n/\mu_0}(u^{p_n}(\widehat{x}) - u^{*p_n})$ in (14).

Proposition 19. *There are constants $\epsilon_{j2} > 0, b_{j6} > 0$ such that*

$$\eta_n^{q_n/\mu_0} \left(u^{p_n}(\widehat{x}) - u^{*p_n} \right) \leq \sum_{j=1}^{n} \epsilon_{j2} \eta_j^{4l_0} + \sum_{j=2}^{n} b_{j6} e_j^{4l_0}. \tag{26}$$

Defining the Lyapunov function $V = V_n + U$ and combining (14), (25), and (26) together yield

$$\begin{aligned} \mathscr{L}V \leq & -\sum_{j=1}^{n} \pi_{n,j} \eta_j^{4l_0} + \sum_{j=1}^{n} \epsilon_{j2} \eta_j^{4l_0} + \sum_{j=1}^{n} \epsilon_{j1} \eta_j^{4l_0} \\ & + \sum_{j=2}^{n} b_{j6} e_j^{4l_0} + e_2^{4l_0} \\ & \times \left[-k_2 \varpi_2 + b_{23} + b_{24} + g_{31}\left(k_3\right) + g_{32}\left(k_3\right) \right] \\ & + \sum_{j=3}^{n-1} e_j^{4l_0} \left[-k_j \varpi_j + b_{j3} + b_{j4} + b_{j5} \right. \\ & \qquad \left. + g_{j+1,1}\left(k_{j+1}\right) + g_{j+1,2}\left(k_{j+1}\right) \right] \\ & + e_n^{4l_0} \left[-k_n \varpi_n + b_{n4} + b_{n5} \right]. \end{aligned} \tag{27}$$

It is clear that by choosing

$$\begin{aligned} l_j & \triangleq \pi_{n,j} - \epsilon_{j1} - \epsilon_{j2} > 0, \quad (j = 1, \ldots, n) \\ k_n & \geq \frac{1}{\varpi_n} \left(\varphi_n + b_{n4} + b_{n5} + b_{n6} \right), \\ k_{n-1} & \geq \frac{1}{\varpi_{n-1}} \left(\varphi_{n-1} + b_{n-1,3} + b_{n-1,4} + b_{n-1,5} \right. \\ & \qquad \left. + b_{n-1,6} + g_{n,1}\left(k_n\right) + g_{n,2}\left(k_n\right) \right), \\ & \vdots \\ k_2 & \geq \frac{1}{\varpi_2} \left(\varphi_2 + b_{2,3} + b_{2,4} + b_{2,6} + g_{3,1}\left(k_3\right) + g_{3,2}\left(k_3\right) \right), \end{aligned} \tag{28}$$

one leads to

$$\mathscr{L}V \leq -\sum_{j=1}^{n} l_j \eta_j^{4l_0} - \sum_{j=2}^{n} \varphi_j e_j^{4l_0} \tag{29}$$

with positive constants $\varphi_2, \ldots, \varphi_n$.

By denoting $\Phi = (x_1, \ldots, x_n, \widehat{\eta}_2, \ldots, \widehat{\eta}_n)^T$, the closed-loop system (13), (16), and (17) can be rewritten as

$$d\Phi = E(\Phi)\, dt = \left[x_2^{p_1}, \ldots, x_{n-1}^{p_{n-2}}, u^{p_n}(\widehat{x}), \dot{\widehat{\eta}}_2, \ldots, \dot{\widehat{\eta}}_n \right] dt, \tag{30}$$

which is homogeneous of degree τ with respect to $\Delta = (\underbrace{r_1, \ldots, r_n}_{\text{for } x_1,\ldots,x_n}, \underbrace{r_1, \ldots, r_{n-1}}_{\text{for } \bar{\eta}_2,\ldots,\bar{\eta}_n})$. Following a similar way in [16], it can be shown that

$$V(\Phi) = \sum_{j=1}^{n} \int_{x_j^*}^{x_j} \left(s^{\mu_0/r_j} - x_i^{*\mu_0/r_j} \right)^{q_j/\mu_0} ds + \sum_{j=2}^{n} \int_{\lambda_j^{q_{j-1}/r_{j-1}}}^{x_j^{q_{j-1}/r_j}} \left(s^{r_{j-1}/q_{j-1}} - \lambda_j \right) ds \tag{31}$$

is homogeneous of degree $4l_0\mu_0 - \tau$ about Δ.

4.3. Stability Analysis. To state the main result in this paper, we first introduce the following coordinate transformation:

$$x_1 = z_1, \qquad x_i = \frac{z_i}{L^{s_i}}, \qquad u^{p_n} = \frac{v^{p_n}}{L^{s_n+1}}, \tag{32}$$

where $s_1 = 0$, $s_i = (s_{i-1} + 1)/p_{i-1}$, $i = 2, \ldots, n$, $0 < L < 1$ is a constant to be determined later.

It follows from (32) that the system (9) can be rewritten as

$$\begin{aligned} dx_1 &= Lx_2^{p_1} dt + f_1(x, x(t-\sigma(t))) + g_1^T(x, x(t-\sigma(t)))\, d\omega, \\ dx_2 &= Lx_3^{p_2} dt + \frac{f_2(x, x(t-\sigma(t)))}{L^{s_2}} + \frac{g_2^T(x, x(t-\sigma(t)))}{L^{s_2}} d\omega, \\ &\vdots \\ dx_n &= Lu^{p_n} + \frac{f_n(x, x(t-\sigma(t)))}{L^{s_n}} + \frac{g_n^T(x, x(t-\sigma(t)))}{L^{s_n}} d\omega, \\ y &= z_1. \end{aligned} \tag{33}$$

Next, we construct a scaled homogeneous observer

$$\begin{aligned} \dot{\hat{\eta}}_2 &= -Lk_2 \hat{x}_2^{p_1}, & \hat{x}_2 &= \left[\hat{\eta}_2 + k_2 \hat{x}_1\right]^{r_2/r_1}, \\ \dot{\hat{\eta}}_3 &= -Lk_3 \hat{x}_3^{p_2}, & \hat{x}_3 &= \left[\hat{\eta}_3 + k_3 \hat{x}_2\right]^{r_3/r_2}, \\ &\vdots & &\vdots \\ \dot{\hat{\eta}}_n^* &= -Lk_n \hat{x}_n^{p_{n-1}}, & \hat{x}_n &= \left[\hat{\eta}_n + k_n \hat{x}_{n-1}\right]^{r_n/r_{n-1}}, \end{aligned} \tag{34}$$

where $k_2 > 0, \ldots, k_n > 0$ are the gains selected in (28) and the controller u is designed with the same construction of (17). The closed-loop system (33), (34), and (17) can be expressed as

$$d\Phi = LE(\Phi)\, dt + Fdt + G^T d\omega, \tag{35}$$

where $F = (f_1, f_2/L^{s_2}, \ldots, f_n/L^{s_n}, 0, \ldots, 0)$, $G = (g_1, g_2/L^{s_2}, \ldots, g_n/L^{s_n}, 0, \ldots, 0)$. Based on the above discussion, we are ready to prove the main result of this paper.

Theorem 20. *Under Assumptions 10 and 11, there is an output-feedback controller $v^{p_n} = u^{p_n} L^{s_n+1}$ such that the closed-loop system consisting of* (9), (16), *and* (17) *has a global unique solution and the equilibrium $x = 0$ is GAS in probability.*

Proof. We prove Theorem 20 as follows.

Step 1. From (17), for $j = 1, \ldots, n$, one obtains

$$\begin{aligned} \frac{\partial u^{p_n}(\hat{x})}{\partial \hat{x}_j} &= -\frac{r_n + \tau}{r_j} \overline{\beta}_j \hat{x}_j^{\mu_0/r_j - 1} \\ &\quad \times \left(\overline{\beta}_n \hat{x}_n^{\mu_0/r_n} + \overline{\beta}_{n-1} \hat{x}_{n-1}^{\mu_0/r_{n-1}} + \cdots + \overline{\beta}_1 \hat{x}_1^{\mu_0/r_1} \right)^{(r_n+\tau)/\mu_0 - 1}. \end{aligned} \tag{36}$$

By Lemma 12, we know $\mu_0/r_j - 1 \geq 0$, $(r_n + \tau)/\mu_0 - 1 \geq 0$, which implies that $u^{p_n}(\hat{x})$ is C^1. Therefore, the closed-loop system satisfies the locally Lipschitz condition.

Step 2. Construct a Lyapunov-Krasovskii functional

$$W(\Phi) = V(\Phi) + \frac{L^{1+\varepsilon_0}(a_0 + b_0)}{1 - \sigma} \int_{t-\sigma(t)}^{t} \|\Phi(s)\|_{\Delta}^{4l_0\mu_0} ds, \tag{37}$$

which is positive definite, proper, and C^2 on Φ, where ε_0, a_0, b_0 are positive parameters to be determined later. According to Lemma 4.3 in [33], there are two $\mathcal{K}_\infty$ functions α_1 and α_{21} such that

$$\alpha_1(|\Phi(t)|) \leq V(\Phi) \leq \alpha_{21}(|\Phi(t)|). \tag{38}$$

Then, with the notation in [26], it is easy to verify that

$$\frac{L^{1+\varepsilon_0}(a_0 + b_0)}{1 - \sigma} \int_{t-\sigma(t)}^{t} \|\Phi(s)\|_{\Delta}^{4l_0\mu_0} ds \leq \alpha_{22}\left(\sup_{-\sigma_0 \leq s \leq 0} |\Phi(t+s)| \right), \tag{39}$$

where α_{22} is a $\mathcal{K}_\infty$ function. Because $\alpha_{21}(|\Phi(t)|) \leq \alpha_{21}(\sup_{-\sigma_0 \leq s \leq 0} |\Phi(t+s)|)$, then

$$\alpha_1(|\Phi(t)|) \leq W(\Phi) \leq \alpha_2\left(\sup_{-\sigma_0 \leq s \leq 0} |\Phi(t+s)| \right) \tag{40}$$

with $\alpha_2 = \alpha_{21} + \alpha_{22}$.

Step 3. Since $V(\Phi)$ and $E(\Phi)$ are homogenous of degree $4l_0\mu_0 - \tau$ and τ, respectively, by Lemmas 5 and 6, one has

$$\frac{\partial W(\Phi)}{\partial X} LE(\Phi) = \frac{\partial V(\Phi)}{\partial X} LE(X) \leq -L\gamma_1 \|\Phi(s)\|_{\Delta}^{4l_0\mu_0}, \tag{41}$$

where $\gamma_1 > 0$ is a constant.

Using Assumption 10 and the new coordinates in (32), we have

$$\left|\frac{f_i}{L^{s_i}}\right| \le a_i \sum_{j=i+2}^{n+1}\left(\frac{\left|L^{s_j}x_j(t)\right|^{(r_i+\tau)/r_j}+\left|L^{s_j}x_j(t-\sigma(t))\right|^{(r_i+\tau)/r_j}}{L^{s_i}}\right) \tag{42}$$

and the power of L in (42) is

$$\begin{aligned} s_j\frac{r_i+\tau}{r_j}-s_i &= 1+\frac{s_j(r_i+\tau)-(s_i+1)r_j}{r_j} \\ &= 1+\frac{p_i}{r_j}\left(s_jr_{i+1}-s_{i+1}r_j\right). \end{aligned} \tag{43}$$

Owing to $s_1=0$, $s_i=(s_{i-1}+1)/p_{i-1}$, one has

$$s_i=\sum_{s=1}^{i-1}\frac{1}{p_s\cdots p_{i-1}},\quad i=2,\ldots,n. \tag{44}$$

For $j\ge i+2$, (44) gives

$$\begin{aligned} &s_jr_{i+1}-s_{i+1}r_j \\ &= s_j\left(\frac{1}{p_1\cdots p_i}+s_{i+1}\tau\right)-s_{i+1}\left(\frac{1}{p_1\cdots p_{j-1}}+s_j\tau\right) \\ &= \frac{s_j}{p_1\cdots p_i}-\frac{s_{i+1}}{p_1\cdots p_{j-1}}>0. \end{aligned} \tag{45}$$

In summary, it can be concluded that there exists a positive constant

$$\varepsilon_1=\min\left\{\frac{p_i}{r_j}\left(s_jr_{i+1}-s_{i+1}r_j\right):1\le i\le n-1, i+2\le j\le n\right\} \tag{46}$$

such that

$$\begin{aligned} \left|\frac{f_i}{L^{s_i}}\right| &\le a_iL^{1+\varepsilon_1}\sum_{j=i+2}^{n+1}\left(\left|x_j(t)\right|^{(r_i+\tau)/r_j}+\left|x_j(t-\sigma(t))\right|^{(r_i+\tau)/r_j}\right) \\ &\le \widehat{a}_iL^{1+\varepsilon_1}\left(\|\Phi(t)\|_\Delta^{r_i+\tau}+\|\Phi(t-\sigma(t))\|_\Delta^{r_i+\tau}\right) \end{aligned} \tag{47}$$

with $\widehat{a}_i>0$. According to Lemma 6, we deduce that

$$\begin{aligned} \left|\frac{\partial V(\Phi)}{\partial\Phi}F\right| &\le \sum_{i=1}^{n-1}\left|\frac{\partial V(\Phi)}{\partial\Phi_i}\right|\left|\frac{f_i}{L^{s_i}}\right| \\ &\le \sum_{i=1}^{n-1}\widehat{a}_iL^{1+\varepsilon_1}\|\Phi(t)\|_\Delta^{4l_0\mu_0-r_i-\tau} \\ &\quad\times\left(\|\Phi(t)\|_\Delta^{r_i+\tau}+\|\Phi(t-\sigma(t))\|_\Delta^{r_i+\tau}\right) \\ &\le a_0L^{1+\varepsilon_1}\left(\|\Phi(t)\|_\Delta^{4l_0\mu_0}+\|\Phi(t-\sigma(t))\|_\Delta^{4l_0\mu_0}\right), \end{aligned} \tag{48}$$

where a_0 is a positive constant.

Similarly, there is a positive constant

$$\varepsilon_2=\min\left\{\frac{s_j(2r_i+\tau)-(2s_i+1)r_j}{2r_j}:1\le i\le n-1, i+2\le j\le n\right\}, \tag{49}$$

such that

$$\begin{aligned} \left|\frac{g_i}{L^{s_i}}\right| &\le b_iL^{1/2+\varepsilon_2} \\ &\quad\times\sum_{j=i+2}^{n+1}\left(\left|\Phi_j(t)\right|^{(2r_i+\tau)/2r_j}+\left|\Phi_j(t-\sigma(t))\right|^{(2r_i+\tau)/2r_j}\right) \\ &\le \widehat{b}_iL^{1/2+\varepsilon_2}\left(\|\Phi(t)\|_\Delta^{r_i+\tau/2}+\|\Phi(t-\sigma(t))\|_\Delta^{r_i+\tau/2}\right) \end{aligned} \tag{50}$$

with $\widehat{b}_i>0$. Then,

$$\begin{aligned} &\frac{1}{2}\operatorname{Tr}\left\{G\frac{\partial^2V(\Phi)}{\partial\Phi^2}G^T\right\} \\ &\le \frac{1}{2}m\sqrt{m}\sum_{k,l=1}^{n-1}\left|\frac{\partial^2V(\Phi)}{\partial\Phi_k\partial\Phi_l}\right|\left|\frac{g_k}{L^{s_k}}\right|\left|\frac{g_l}{L^{s_l}}\right| \\ &\le \widetilde{b}L^{1+2\varepsilon_2} \\ &\quad\times\sum_{k,l=1}^{n-1}\|\Phi(t)\|^{4l_0\mu_0-\tau-r_k-r_l} \\ &\qquad\times\left(\|\Phi(t)\|_\Delta^{r_k+\tau/2}+\|\Phi(t-\sigma(t))\|_\Delta^{r_k+\tau/2}\right) \\ &\qquad\times\left(\|\Phi(t)\|_\Delta^{r_i+\tau/2}+\|\Phi(t-\sigma(t))\|_\Delta^{r_i+\tau/2}\right) \\ &\le b_0L^{1+2\varepsilon_2}\left(\|\Phi(t)\|_\Delta^{4l_0\mu_0}+\|\Phi(t-\sigma(t))\|_\Delta^{4l_0\mu_0}\right), \end{aligned} \tag{51}$$

where $\widetilde{b}$, b_0 are positive constants.

By choosing $\varepsilon_0=\min\{\varepsilon_1,2\varepsilon_2\}$ and Definition 1, Assumption 11, (41), (48), and (51), one can obtain

$$\begin{aligned} \mathscr{L}W(\Phi) &\le \frac{\partial W(\Phi)}{\partial X}LE(\Phi)+\frac{\partial V(\Phi)}{\partial X}F \\ &\quad+\frac{1}{2}\operatorname{Tr}\left\{G\frac{\partial^2V(\Phi)}{\partial\Phi^2}G^T\right\}+L^{1+\varepsilon_0}(a_0+b_0) \\ &\quad\times\left(\frac{1}{1-\sigma}\|\Phi(t)\|_\Delta^{4l_0\mu_0}-\|\Phi(t-\sigma(t))\|_\Delta^{4l_0\mu_0}\right) \\ &\le -L\left(\gamma_1-L^{\varepsilon_0}\left(a_0+b_0+\frac{a_0+b_0}{1-\sigma}\right)\right)\|\Phi(t)\|_\Delta^{4l_0\mu_0}, \end{aligned} \tag{52}$$

where $0<L<1$. Apparently, by choosing $0<L<\min\{(\gamma_1/(a_0+b_0+(a_0+b_0)/(1-\sigma)))^{1/\varepsilon_0},1\}$, there is a constant δ, such that $\mathscr{L}W(\Phi)\le-\delta\|\Phi(t)\|_\Delta^{4l_0\mu_0}$.

From Steps 1 to 3 and Lemma 4, it is obtained that the closed-loop system (33), (34), and (17) has a global unique solution and the equilibrium $\Phi = 0$ is GAS in probability.

Step 4. Because of the equivalence in (32), then there is a global unique solution for the closed-loop system consisting of (9), (16), and (17) and $v^{p_n} = u^{p_n} L^{s_n+1}$ and the equilibrium $z = 0$ is GAS in probability. □

Remark 21. Using the adding one power integrator method and the homogeneous domination technique, [34] has designed a state-feedback stabilizer for a class of stochastic high-order feedforward nonlinear systems. However, to deal with the unmeasurable states and time-delay terms in this paper, we choose a reduced-order homogeneous observer and an appropriate Lyapunov-Krasovskii functional, which is not an easy work.

5. Conclusions

For a class of stochastic high-order feedforward nonlinear systems with different power orders and time-varying delay, an output-feedback stabilizer has been designed by virtue of the adding one power integrator technique and homogeneous domination approach. It globally stabilizes the origin of the closed-loop system. The proof of stability we have adopted depends on the construction of a Lyapunov-Krasovskii functional.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

H_∞ Filtering for Networked Systems with Bounded Measurement Missing

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This paper is concerned with the H_∞ filtering problem for networked systems with bounded measurement missing. A switched linear system model is proposed to describe the considered filtering error system. A sufficient condition is derived for the filtering error system to be exponentially stable and achieve a prescribed H_∞ filtering performance level. The obtained condition establishes quantitative relations among the H_∞ performance level and two parameters characterizing the measurement missing, namely, the measurement missing rate bound and the maximal number of consecutive measurement missing. A convex optimization problem is presented to design the linear H_∞ filters. Finally, an illustrative example is given to show the effectiveness of the proposed results.

1. Introduction

A sensor network consists of spatially distributed autonomous sensors to cooperatively monitor physical or environmental conditions. The purpose of a sensor network is to provide users with the information of interest from data gathered by spatially distributed sensors, and it has found applications in a variety of areas, such as moving target localization and tracking, environment monitoring, intelligent transportation systems, and sensor-actuator network-based control (see [1, 2] and the literature therein). Thus, it is not surprising that signal estimation has been one of the most fundamental collaborative information processing problems in sensor networks [3–10]. In the environment of sensor networks, the measurements may be unavailable to the estimators intermittently. The corresponding estimation problem is usually termed estimation with missing, incomplete, or intermittent measurements and has received increasing research attention; see, for example, [11–17] and the references therein.

Measurement missing degrades the performance of the filtering system, and one may be concerned with the problem about how much the filtering performance declines for a certain amount of measurement missing, what is the maximal number of consecutive measurement missing that the filtering system can tolerant to guarantee the desired filtering performance, and so forth. This gives rise to the idea of revealing the relations, especially the quantitative relations, among the filtering performance level and some parameters characterizing the measurement missing, such as the missing rate/probability bound and the maximal number of consecutive measurement missing. Such relations may provide useful guidelines for designing the filtering systems with measurement missing. For example, in the sensor-network-based filtering system, one may purposely suspend some sensor nodes intermittently to save node power without destroying the stability and desired performance level of the filtering systems; in some network-based fast sampling filtering systems, one may consider dropping certain amount of data packets to ensure that the set of the network-based filtering systems is schedulable and meanwhile to guarantee that the overall network-based filtering systems are stable and achieve prescribed filtering performance level and so forth. In these applications, establishing the relations among the filtering performance level and the parameters characterizing the measurement missing is of prior importance. Existing results on this topic can be generally classified into two frameworks, namely, the stochastic framework [18–22] and the deterministic framework [23].

A general structure of the filtering system with measurement missing is shown in Figure 1, where y is the measurement and $\bar{y}$ is the filter input. In the stochastic

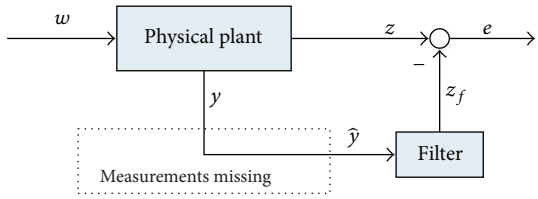

Figure 1: Diagram of the filtering problem with measurement missing.

framework, the measurement missing is usually described by a Bernoulli sequence or a Markov chain. For those results using the Bernoulli sequence, such as [14, 15], the filter input is usually described as $\widehat{y}(k) = \gamma(k)y(k) + (1 - \gamma(k))\widehat{y}(k-1)$ or as $\widehat{y}(k) = \gamma(k)y(k)$ by introducing the binary random variable $\gamma(k)$, where $\text{Prob}\{\gamma(k) = 1\} = \gamma$, and γ is usually called the measurement arrival probability while $1 - \gamma$ is called the measurement missing probability. For those results using Markov chain technique, such as [12, 21], the measurement missing and the measurement arrival are defined as two modes of the Markov chain, and the mode transition probabilities are assumed to be known. In these ways, the stochastic parameters (either the Bernoulli sequence or the Markov chain) are incorporated into the filtering system models, and the relation between the filtering performance level and the measurement missing probability is implicitly established. Note that the maximal number of consecutive measurement missing is an important parameter that characterizes the measurement missing, and it also plays an important role in affecting the filtering performance. However, the relation between the filtering performance level and the maximal number of consecutive measurement missing is not established in all the aforementioned results using stochastic framework. Being different from the stochastic framework, the effects of the measurement missing on the filtering system are treated as time delays in the filter input in the deterministic framework, and the filtering error system with bounded measurement missing is usually described as a deterministic system with bounded time-varying delays. In this way, the relation between the filtering performance level and the maximal number of consecutive measurement missing is established. Some results on this topic can be found, for example, in [7, 23], where the H_∞ filtering was investigated for linear continuous-time systems with bounded measurement missing and delays. However, the relation between the filtering performance level and the measurement missing rate/probability is not established in the existing results using deterministic framework.

By a closer inspection, it is found that the relations among the filtering performance level and the parameters characterizing the measurement missing revealed in the existing results using either the stochastic framework or the deterministic framework are quite implicit. Actually, besides of determining the maximal allowable measurement missing rate/probability bound, one may be interested more in the quantitative results about how much the filtering performance declines for a certain amount of measurement missing and for a certain extent of increase on the maximal number of consecutive measurement missing, or in other words, the quantitative relations among the performance level and the parameters characterizing the measurement missing. Specifically, this gives rise to the problem of how to express the filtering performance level as a function of the parameters characterizing the measurement missing. To the best of the authors' knowledge, such quantitative relations for the filtering systems with bounded measurement missing have not yet been established in the existing results, which motivates the presented research.

In this paper, the H_∞ filtering problem is investigated for networked systems with bounded measurement missing. The filtering error system is described as a switched linear system by using the augmentation technique and by including the numbers of consecutive measurement missing as switching parameters. Based on the obtained switched system model, a sufficient condition is derived for the filtering system to be exponentially stable and achieve a prescribed H_∞ performance level. A convex optimization problem is also presented to design the linear H_∞ filters which guarantee that the considered filtering system achieves a suboptimal H_∞ performance level. The main contributions of the paper are summarized as follows. (1) A switched system model with finite number of subsystems is established to describe the networked filtering system with bounded missing measurements. (2) A quantitative relation is established between the filtering performance and parameters characterizing the measurement missing. Specifically, the exponential decay rate of the filtering error system is given as a monotonic increasing function of the measurement missing rate bound, while the H_∞ performance level is given as a monotonic increasing function of both the measurement missing rate bound and the maximal number of consecutive measurement missing. (3) The established relation gives quantitative results about how much the H_∞ filtering performance level declines for a certain amount of measurement missing and for a certain extent of increase on the maximal number of consecutive measurement missing. They theoretically reveal that measurement missing degrades the H_∞ filtering performance.

2. Modelling of the Filtering System

The considered filtering problem is shown in Figure 1, where the physical plant is described by

$$\begin{aligned} x(k+1) &= A_p x(k) + B_p w(k), \\ y(k) &= C_p x(k) + D_p w(k), \\ z(k) &= L_p x(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathfrak{R}^n$ is the system state, $y(k) \in \mathfrak{R}^p$ is the measured output, $z(k) \in \mathfrak{R}^q$ is the signal to be estimated, and $w(k) \in \mathfrak{R}^m$ is the noise signal which belongs to $l_2[0, \infty)$. A_p, B_p, C_p, D_p, and L_p are constant matrices with appropriate

dimensions, and A_p is assumed to be stable. The full-order linear filter used to estimate the signal $z(k)$ is given by

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f \hat{y}(k), \\ z_f(k) &= C_f x_f(k), \end{aligned} \tag{2}$$

where $x_f(k) \in \Re^n$ is the filter state, $\hat{y}(k) \in \Re^p$ is the filter input, and $z_f(k) \in \Re^q$ is the estimated signal. A_f, B_f, and C_f are filter matrices to be determined.

Since the measurements may be missed during the transmission from the sensor to the filter, the filter input $\hat{y}$ may not be equal to the measured output y. Suppose that the number of consecutive measurement missing i is upper bounded by $i \le d$ and the filter input holds at its last available value if a measurement is missed, where d is a known constant. Then, we have

$$\hat{y}(k) = y(k-i) \tag{3}$$

if there are i consecutive measurement losses before the time step $k+1$, where $i \in Z_0 = \{0, 1, \ldots, d\}$. It can be seen from (3) that the system model of the filter varies over different sampling intervals since the number of consecutive measurement missing is time varying over different sampling periods. By defining the filtering error signal as $e(k) = z(k) - z_f(k)$ and denoting

$$\begin{aligned} \xi(k) &= \left[x^T(k) \quad x^T(k-1) \quad \cdots \quad x^T(k-d) \quad x_f^T(k) \right]^T, \\ \upsilon(k) &= \left[w^T(k) \quad w^T(k-1) \quad \cdots \quad w^T(k-d) \right]^T, \end{aligned} \tag{4}$$

we obtain the following filtering error system model:

$$S_{\sigma(k)} : \begin{cases} \xi(k+1) = A_{\sigma(k)} \xi(k) + B_{\sigma(k)} \upsilon(k) \\ e(k) = C\xi(k), \end{cases} \tag{5}$$

where $\sigma(k) \in Z_0$, and

$$\begin{aligned} A_{\sigma(k)} &= \begin{bmatrix} \begin{bmatrix} A_p \Lambda_a \\ \Pi \end{bmatrix} & \begin{bmatrix} 0 \\ O_a \end{bmatrix} \\ B_f C_p I_{a\sigma(k)} & A_f \end{bmatrix}, \\ B_{\sigma(k)} &= \begin{bmatrix} \begin{bmatrix} B_p \Lambda_b \\ O_b \end{bmatrix} \\ B_f D_p I_{b\sigma(k)} \end{bmatrix}, \\ C &= \begin{bmatrix} [L_p \quad O_c] & -C_f \end{bmatrix}, \\ \Lambda_a &= \begin{bmatrix} I_1 & O_a^T \end{bmatrix}, \qquad \Lambda_a = \begin{bmatrix} I_3 & O_d^T \end{bmatrix}, \\ \Pi &= \begin{bmatrix} I_2 & O_a \end{bmatrix}, \qquad O_a = 0 \in R^{dn \times n}, \\ O_b &= 0 \in R^{dn \times (d+1)n}, \\ O_c &= 0 \in R^{p \times dn}, \qquad O_d = 0 \in R^{dm \times m}, \\ I_1 &= I \in R^{n \times n}, \qquad I_2 = I \in R^{dn \times dn}, \\ I_3 &= I \in R^{m \times m}, \\ I_{a\sigma(k)} &= \begin{bmatrix} \Delta_{a0} & \Delta_{a1} & \cdots & \Delta_{ad} \end{bmatrix}, \\ I_{b\sigma(k)} &= \begin{bmatrix} \Delta_{b0} & \Delta_{b1} & \cdots & \Delta_{bd} \end{bmatrix}, \\ \Delta_{ai} &= \begin{cases} I_1, & \sigma(k) = i \\ 0, & \sigma(k) \ne i, \end{cases} \qquad \Delta_{bi} = \begin{cases} I_3, & \sigma(k) = i \\ 0, & \sigma(k) \ne i. \end{cases} \end{aligned} \tag{6}$$

By applying the augmentation technique, the filtering error system is finally described as a nondelayed switched system by including the numbers of consecutive measurement missing as switching parameters. $\sigma(k)$ serves as the switching signal of system (5) which contains $d+1$ subsystems. The switching of the subsystems is determined by the measurement missing status. Specifically, for $i = 0, 1, \ldots, d$, we have $\sigma(k) = i$ and system (5) resides in the subsystem S_i during the interval $[k, k+1)$ if there is i consecutive measurement missing before time step $k+1$. This is illustrated in Figure 2, where $\bullet$ represents the measurement that is successfully received by the filter, while $\circ$ stands for the one that is missed. At time step $k-1$, the measurement $y(k-1)$ is successfully received by the filter. Then, $\hat{y}(k-1) = y(k-1)$, and $S_{\sigma(k)}$ resides in S_0. At time step k, the measurement $y(k)$ is missed and the filter input holds at its previous value; that is, $\hat{y}(k)$ takes $y(k-1)$, and $S_{\sigma(k)}$ resides in S_1. Then, at time step $k+1$, the measurement $y(k+1)$ is missed again, and the filter input remains at $y(k-1)$; that is, $\hat{y}(k+1) = y(k-1)$, and $S_{\sigma(k)}$ resides in S_2. The rest can be deduced by analogy. It is seen from the above analysis that the filtering error system $S_{\sigma(k)}$ takes on the following characteristics.

(a) For $i = 1, \ldots, d-1$, subsystem S_{i+1} always appears after the subsystem S_i, and subsystem S_0 always appears before S_1 or after S_d.

(b) Switching between the subsystems of $S_{\sigma(k)}$ occurs only when the subsystems S_j ($j = 1, \ldots, d$) appear, and the number of switches of $S_{\sigma(k)}$ is determined by the total activation times of S_j. Moreover, at most two switches may be involved when S_i appears one time.

For $i = 0, 1, \ldots, d$ and $0 \le k_1 \le k \le k_2$, let $N_\sigma[k_1, k_2]$ denote the number of switches of $\sigma(k)$ and $n_i[k_1, k_2]$ the times that the subsystem S_i is activated on the interval $[k_1, k_2]$. Then, it follows from the characteristics of $S_{\sigma(k)}$ that the following relation holds:

$$N_\sigma[k_1, k_2] \le 2 \sum_{i=1}^{d} n_i[k_1, k_2]. \tag{7}$$

Definition 1. For any switching signal $\sigma(k)$ and $0 \le k_1 \le k \le k_2$, let $N_\sigma[k_1, k_2]$ denote the number of switches of $\sigma(k)$ on

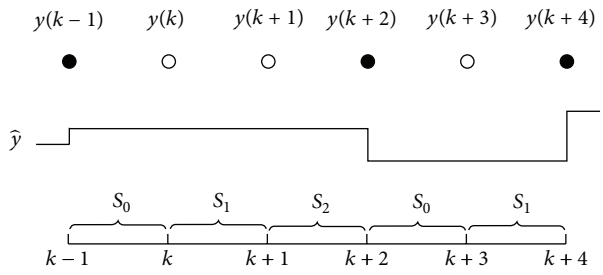

Figure 2: An example of measurement missing.

the interval $[k_1, k_2]$. Then, $\theta[k_1, k_2] = N_\sigma[k_1, k_2]/(k_2 - k_1)$ is called the switching frequency of $\sigma(k)$ on the interval $[k_1, k_2]$.

Definition 2. For $i \in Z_0$ and $0 \le k_1 < k_2$, let $\alpha_i[k_1, k_2] = n_i[k_1, k_2]/(k_2 - k_1)$. Then $\alpha_i[k_1, k_2]$ is called the occurrence rate of the subsystem S_i on the interval $[k_1, k_2]$.

The subsystem S_j, $j \in \{1, \ldots, d\}$ is activated when a measurement missing occurs, so the measurement missing rate on any interval $[k_1, k_2]$ can be defined as $\alpha[k_1, k_2] = \sum_{i=1}^{d} \alpha_i[k_1, k_2]$. Then, we have by relation (7) and Definitions 1 and 2 that

$$\theta[k_1, k_2] \le 2\alpha[k_1, k_2]. \tag{8}$$

Moreover, we have by the characteristics of $S_{\sigma(k)}$ that there exists a constant $\alpha < 1$ such that the following relations are true for any $t \ge 0$ and $l \ge 0$:

$$\begin{aligned} \alpha[t, t+l] &\le 1, \quad l \le d \\ \alpha[t, t+l] &\le \alpha < 1, \quad l > d. \end{aligned} \tag{9}$$

Remark 3. The considered filtering system with bounded measurement missing is finally described as a switched linear system $S_{\sigma(k)}$ with $d + 1$ subsystems, where S_0 describes the filtering system without measurement missing, while S_j ($j = 1, \ldots, d$) describes those with measurement missing. It is reasonable to see that the filtering system will achieve a better H_∞ performance level if S_0 appears with higher frequency (or, in other words, the measurement missing rate is lower). This is the motivation of the proposed switched system model of the filtering error system, which may enable us to establish the quantitative relation between the H_∞ filtering performance and the measurement missing rate bound. Note that the augmentation technique is used to obtain the switched system model, which increases computational cost. From the energy-efficient perspective, the proposed method is applicable to the case where the number of consecutive packet losses is bounded and not too large.

The following definition and assumptions are needed in the derivation of the main results.

Definition 4. System (5) is said to be exponentially stable with decay rate $\lambda < 1$ if, for any finite initial state $\xi(0) \in \mathfrak{R}^{(d+2)n}$, there exist a constant $c > 0$ such that $\|\xi(k)\| \le c\lambda^k \|\xi(0)\|$ holds.

Assumption 5. For any $t \ge 0$ and $l > d$, $\alpha[t, t + l]$ is upper bounded by a constant measurement missing rate bound α; that is, $\alpha[t, t + l] \le \alpha$, where $0 \le \alpha < 1$.

Assumption 6. $\xi(k) = 0$ and $w(k) = 0$, for all $k \le 0$.

The H_∞ filtering problem to be addressed in this paper is expressed as follows.

Problem HFBMM (H_∞ Filtering with Bounded Measurement Missing). For the filtering problem in Figure 1 and a given system (1), determine the matrices A_f, B_f, and C_f in filter (2), such that the system (5) with $v(k) = 0$ is exponentially stable, and the estimation error e satisfies a prescribed H_∞ performance level γ (i.e., $\|e\|_2 \le \gamma\|v\|_2$) under zero initial conditions (i.e., $\xi(k) = 0$, for all $k \le 0$) for all admissible bounded measurement missing.

3. H_∞ Filtering Analysis

An H_∞ performance condition for the filtering error system (5) is presented in the following theorem.

Theorem 7. *For given scalars $\mu > 1$ and $0 < \lambda < 1$ satisfying $\mu\lambda < 1$, if there exist matrices $P_i > 0$, $i = 0, 1, \ldots, d$, and a scalar $\gamma_0 > 0$, such that the following inequalities hold:*

$$\Omega_i = \begin{bmatrix} -\lambda^2 P_i + C^T C & 0 \\ 0 & -\gamma_0^2 I \end{bmatrix} + \begin{bmatrix} A_i^T \\ B_i^T \end{bmatrix} P_i \begin{bmatrix} A_i & B_i \end{bmatrix} < 0, \tag{10}$$

$$P_\alpha \le \mu P_\beta, \quad \alpha, \beta \in Z_0, \tag{11}$$

then system (5) *is exponentially stable with decay rate $\delta(\alpha) = \lambda\mu^\alpha$ and achieves a prescribed H_∞ performance level $\gamma(\alpha, d) = \gamma_0((1-\lambda^2)(1-(\mu\lambda)^{2(d+1)})/(1-(\mu\lambda)^2) + (1-\lambda^2)(\mu^\alpha\lambda)^{2(d+1)}/(1-(\mu^\alpha\lambda)^2))^{1/2}$, where α is the measurement missing rate bound.*

Proof. Choose the Lyapunov function $V_{\sigma(k)}(k) = \xi^T(k)P_{\sigma(k)}\xi(k)$ and define $J(k) = \|e(k)\|^2 - \gamma_0^2\|v(k)\|^2$. For $i \in Z_0$, it follows from (5) and (10) that

$$V_i(k+1) - \lambda V_i(k) + J(k) = \eta^T(k)\,\Omega_i\,\eta(k) < 0, \tag{12}$$

where $\eta(k) = \left[\xi^T(k) \;\; v^T(k)\right]^T$. For any given integer $k \ge 1$, let $k_1 < \cdots < k_i$, $i \ge 1$ denotes the switching instants of the switching signal $\sigma(k)$ on the interval $[0, k]$. Then, we have by (11) that

$$\begin{aligned} V_{\sigma(k_i)}(k_i) &= \xi^T(k_i) P_{\sigma(k_i)} \xi(k_i) \le \mu \xi^T(k_i) P_{\sigma(k_i - 1)} \xi(k_i) \\ &= \mu V_{\sigma(k_i - 1)}(k_i), \quad \forall i \ge 1. \end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned} V_{\sigma(k)}(k) &\le \lambda^{2(k-k_i)} V_{\sigma(k_i)}(k_i) - \sum_{t=k_i}^{k-1} \lambda^{2(k-1-t)} J(t) \\ &\le \lambda^{2(k-k_i)} \cdot \mu \cdot V_{\sigma(k_i-1)}(k_i) - \sum_{t=k_i}^{k-1} \lambda^{2(k-1-t)} J(t) \\ &\le \lambda^{2(k-k_i)} \cdot \mu \cdot \left[\lambda^{2(k_i-k_{i-1})} V_{\sigma(k_{i-1})}(k_{i-1}) - \sum_{t=k_{i-1}}^{k_i-1} \lambda^{2(k_i-1-t)} J(t) \right] \\ &\quad - \sum_{t=k_i}^{k-1} \lambda^{2(k-1-t)} J(t) \\ &\quad \vdots \\ &\le \mu^{N_\sigma[0,k-1]} \lambda^{2(k-k_i)} \lambda^{2(k_i-k_{i-1})} \cdots \lambda^{2k_1} V_{\sigma(0)}(0) \\ &\quad - \Gamma(J(t)), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Gamma(J(t)) &= \mu^{N_\sigma[0,k-1]} \lambda^{2(k-k_i)} \lambda^{2(k_i-k_{i-1})} \cdots \lambda^{2(k_2-k_1)} \\ &\quad \times \sum_{t=0}^{k_1-1} \lambda^{2(k_1-1-t)} J(t) \\ &\quad + \mu^{N_\sigma[0,k-1]-1} \lambda^{2(k-k_i)} \lambda^{2(k_i-k_{i-1})} \cdots \lambda^{2(k_3-k_2)} \\ &\quad \times \sum_{t=k_1}^{k_2-1} \lambda^{2(k_2-1-t)} J(t) \\ &\quad \vdots \\ &\quad + \mu^{0} \sum_{t=k_i}^{k-1} \lambda^{2(k-1-t)} J(t) \\ &= \Gamma\left(\|e(t)\|^2\right) - \gamma_0^2 \Gamma\left(\|v(t)\|^2\right). \end{aligned} \tag{15}$$

First, we consider the exponential stability of system (5) for $v(k) = 0$. It follows from (8), (14), $v(k) = 0$, and $N_\sigma[0, k-1] = (k-1) \cdot \theta[0, k-1] \le k \cdot \theta[0, k-1]$ that

$$\begin{aligned} V_{\sigma(k)}(k) &\le \mu^{N_\sigma[0,k-1]} \lambda^{2(k-k_i)} \lambda^{2(k_i-k_{i-1})} \cdots \lambda^{2k_1} V_{\sigma(0)}(0) \\ &\le \mu^{2k\cdot(\theta[0,k-1]/2)} \lambda^{2k} V_{\sigma(0)}(0) \\ &\le \left(\mu^{\alpha[0,k-1]} \lambda\right)^{2k} V_{\sigma(0)}(0). \end{aligned} \tag{16}$$

It is reasonable in practice that the time horizon k is larger than d. So, it can be further obtained from (16) by assumption 1 that

$$V_{\sigma(k)}(k) \le \left(\mu^{\alpha} \lambda\right)^{2k} V_{\sigma(0)}(0) = \delta^{2k} V_{\sigma(0)}(0) \tag{17}$$

which yields $\|\xi(k)\| \le \sqrt{\pi_M/\pi_m}\,\delta^k \|\xi(0)\|$, where $\pi_M = \max_{i\in Z_0}\{\lambda_{\max}(P_i)\}$ and $\pi_m = \min_{i\in Z_0}\{\lambda_{\min}(P_i)\}$. $\lambda_{\max}(\star)$ and $\lambda_{\min}(\star)$ represent the maximum and the minimum eigenvalues of the matrix $\star$, respectively. On the other hand, $\alpha < 1$ and $\mu\lambda < 1$ guarantee that $\delta < 1$. Thus, it is concluded by Definition 4 that system (5) is exponentially stable with decay rate δ.

Next, to prove the H_∞ performance, we consider $v(k) \ne 0$. Then, under zero initial condition, we have, by (14) and the fact that $V_{\sigma(k)}(k) \ge 0$, that $\Gamma(J(t)) = \Gamma(\|e(t)\|^2) - \gamma_0^2 \Gamma(\|v(t)\|^2) \le 0$, and thus

$$\begin{aligned} \sum_{t=0}^{k-1} \mu^{N_\sigma[t,k-1]} \lambda^{2(k-1-t)} \|e(t)\|^2 &= \Gamma\left(\|e(t)\|^2\right) \\ &\le \gamma_0^2 \Gamma\left(\|v(t)\|^2\right) \\ &= \gamma_0^2 \sum_{t=0}^{k-1} \mu^{N_\sigma[t,k-1]} \lambda^{2(k-1-t)} \|v(t)\|^2. \end{aligned} \tag{18}$$

It follows from Definition 1 that $N_\sigma[t, k-1] = (k-1-t) \cdot \theta[t, k-1] \le 2(k-1-t)\alpha[t, k-1]$, which leads to

$$1 \le \mu^{N_\sigma[t,k-1]} \le \mu^{2(k-1-t)\alpha[t,k-1]}. \tag{19}$$

We have by (18) and (19) that

$$\sum_{t=0}^{k-1} \lambda^{2(k-1-t)} \|e(t)\|^2 \le \gamma_0^2 \sum_{t=0}^{k-1} \left(\mu^{\alpha[t,k-1]} \lambda\right)^{2(k-1-t)} \|v(t)\|^2. \tag{20}$$

Summing both sides of (20) from $k = 1$ to $k = +\infty$ and changing the order of the summation, taking (9) into account, we obtain

$$\begin{aligned} \left(1-\lambda^2\right)^{-1} \sum_{t=0}^{+\infty} \|e(t)\|^2 &= \sum_{t=0}^{+\infty} \|e(t)\|^2 \cdot \left(\sum_{k=t+1}^{+\infty} \lambda^{2(k-1-t)} \right) \\ &= \sum_{k=1}^{+\infty} \sum_{t=0}^{k-1} \lambda^{2(k-1-t)} \|e(t)\|^2 \\ &\le \gamma_0^2 \sum_{k=1}^{+\infty} \sum_{t=0}^{k-1} \left(\mu^{\alpha[t,k-1]} \lambda\right)^{2(k-1-t)} \|v(t)\|^2 \\ &= \gamma_0^2 \sum_{t=0}^{+\infty} \|v(t)\|^2 \cdot \left(\sum_{k=t+1}^{+\infty} \left(\mu^{\alpha[t,k-1]} \lambda\right)^{2(k-1-t)} \right) \\ &= \gamma_0^2 \sum_{t=0}^{+\infty} \|v(t)\|^2 \cdot \left(\sum_{i=0}^{d} \left(\mu^{\alpha[t,t+i]} \lambda\right)^{2i} + \sum_{i=d+1}^{+\infty} \left(\mu^{\alpha[t,t+i]} \lambda\right)^{2i} \right) \end{aligned}$$

$$\begin{aligned}
&\le \gamma_0^2 \sum_{t=0}^{+\infty} \|v(t)\|^2 \\
&\quad \cdot \left(\sum_{i=0}^{d} (\mu\lambda)^{2i} + \sum_{i=d+1}^{+\infty} (\mu^\alpha \lambda)^{2i} \right) \\
&= \gamma_0^2 \left(\frac{1-(\mu\lambda)^{2(d+1)}}{1-(\mu\lambda)^2} + \frac{(\mu^\alpha\lambda)^{2(d+1)}}{1-(\mu^\alpha\lambda)^2} \right) \\
&\quad \times \sum_{t=0}^{+\infty} \|v(t)\|^2.
\end{aligned} \tag{21}$$

It follows from (21) that $\|e\|_2 \le \gamma(\alpha, d)\|v\|_2$, which implies that the filtering error system (5) achieves the H_∞ performance level $\gamma(\alpha, d)$. The proof is completed. □

Remark 8. Theorem 7 establishes the quantitative relations among the H_∞ filtering performance level, the measurement missing rate bound, and the maximal number of consecutive measurement missing. Specifically, the H_∞ filtering performance level $\gamma(\alpha, d)$ is given as a monotonic increasing function of both the measurement missing rate bound α and the maximal number of consecutive measurement missing d, which implies that some smaller measurement missing rate and maximal number of consecutive measurement missing lead to a better H_∞ filtering performance.

Remark 9. It is obtained from (12) that

$$V_i(k) < \lambda^k V(0) - \sum_{t=0}^{k-1} \lambda^{(k-1-t)} J(t). \tag{22}$$

It follows from (22) that $V_i(k) < \lambda^k V(0)$ for $v(k) = 0$, which implies that the subsystem S_i ($i \in Z_0$) is exponentially stable with decay rate λ. On the other hand, we have by (22) that

$$\sum_{t=0}^{k-1} \lambda^{(k-1-t)} J(t) < 0 \tag{23}$$

under zero initial condition. Summing both sides of (23) from $k = 1$ to $k = +\infty$, we obtain $(1-\lambda^2)^{-1} \sum_{t=0}^{+\infty} J(t) < 0$, which yields $\|e\|_2 < \gamma_0 \|v\|_2$. It implies that the subsystem S_i achieves the H_∞ performance level γ_0. In summary, if (10) is true for $0 < \lambda < 1$, then the subsystem S_0 is exponentially stable with decay rate λ and achieves the H_∞ performance level γ_0. Note that if there is no measurement missing, then $S_{\sigma(k)}$ resides in S_0, and we have $\alpha = 0$, $d = 0$ and $\mu = 1$. In this case, $\delta(\alpha)$ and $\gamma(\alpha, d)$ in Theorem 7 are, respectively, reduced to λ and γ_0, which are just the decay rate and the H_∞ performance level of S_0, respectively. So, Theorem 7 theoretically reveals that measurement missing degrades the H_∞ filtering performance and presents the quantitative result about the effect of the measurement missing on the H_∞ filtering performance.

4. H_∞ Filter Design

Theorem 10. *Consider the filtering problem in Figure 1. For given scalars $\mu > 1$ and $0 < \lambda < 1$ satisfying $\mu\lambda < 1$, if there exist matrices $Q_{i11} > 0$, $Q_{i22} > 0$, Q_{i12}, $i = 0, 1, \ldots, d$, U_1, U_2, U_3, $\widehat{A}_f$, $\widehat{B}_f$, and $\widehat{C}_f$ of appropriate dimensions and a scalar $\gamma_0 > 0$, such that the following linear matrix inequalities hold:*

$$\begin{bmatrix}
\lambda^2 Q_{i11} & \lambda^2 Q_{i12} & 0 & \Delta_a^T U_1 + I_{ai}^T C_p^T \widehat{B}_f^T E^T & \Delta_a^T U_3 + I_{ai}^T C_p^T \widehat{B}_f^T & \begin{bmatrix} L_p O_c \end{bmatrix}^T \\
* & \lambda^2 Q_{i22} & 0 & \widehat{A}_f^T E^T & \widehat{A}_f^T & -\widehat{C}_f^T \\
* & * & \gamma_0^2 I & \Delta_b^T U_1 + I_{bi}^T D_p^T \widehat{B}_f^T E^T & \Delta_b^T U_3 + I_{bi}^T D_p^T \widehat{B}_f^T & 0 \\
* & * & * & U_1 + U_1^T - Q_{i11} & U_3 + E U_2 - Q_{i12} & 0 \\
* & * & * & * & U_2 + U_2^T - Q_{i22} & 0 \\
* & * & * & * & * & I
\end{bmatrix} > 0, \tag{24}$$

$$Q_\alpha \le \mu Q_\beta, \quad \alpha, \beta \in Z_0, \tag{25}$$

then the filters of form (2) guarantee that the filtering error system (5) is exponentially stable with decay rate $\delta(\alpha)$ and achieves a prescribed H_∞ performance level $\gamma(\alpha, d)$, and the filter matrices are given by

$$\begin{aligned}
A_f = U_2^{-1} \widehat{A}_f, \qquad B_f = U_2^{-1} \widehat{B}_f, \\
C_f = \widehat{C}_f,
\end{aligned} \tag{26}$$

where $\delta(\alpha)$ and $\gamma(\alpha, d)$ are given in Theorem 7, and

$$\begin{aligned}
\Delta_a &= \begin{bmatrix} A_p \Lambda_a \\ \Pi \end{bmatrix}, \qquad \Delta_b = \begin{bmatrix} B_p \Lambda_b \\ O_b \end{bmatrix}, \\
Q_i &= \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i12}^T & Q_{i22} \end{bmatrix}, \qquad E = \begin{bmatrix} I_{n\times n} \\ 0_{dn\times dn} \end{bmatrix}.
\end{aligned} \tag{27}$$

Proof. By Lemma 1 in [24], (10) is true if and only if there exists a matrix G such that the following inequality holds:

$$\begin{bmatrix} \begin{bmatrix} \lambda_i^2 P_i - C^T C & 0 \\ 0 & \gamma^2 I \end{bmatrix} & \begin{bmatrix} A_i^T \\ B_i^T \end{bmatrix} G \\ * & G + G^T - P_i \end{bmatrix} > 0, \tag{28}$$

On the other hand, (24) indicates that U_2 is nonsingular, and thus we can always find square and nonsingular matrices $M \in R^{n\times n}$ and $G_{22} \in R^{n\times n}$ satisfying $U_2 = M^T G_{22}^{-1} M$. Now, introduce the variables $U_1 \in R^{(d+1)n\times(d+1)n}$ and $G_{12} \in R^{(d+1)n\times n}$, and let

$$\begin{aligned} G &= \begin{bmatrix} U_1 & G_{12} \\ ME^T & G_{22} \end{bmatrix}, \qquad U_3 = G_{12} G_{22}^{-1} M, \\ J_1 &= \begin{bmatrix} I & 0 \\ 0 & G_{22}^{-1} M \end{bmatrix}, \\ \begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} &= \begin{bmatrix} M^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{A}_f & \widehat{B}_f \\ \widehat{C}_f & 0 \end{bmatrix} \begin{bmatrix} M^{-1} G_{22} & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{29}$$

Then, pre- and postmultiplying (28) and (11) by $\mathrm{diag}\{J_1, I, J_1\}$ and J_1, respectively, following some routine matrix manipulations, yield (24) and (25). Furthermore, denote the filter transfer function from $\widehat{y}(k)$ to $z_f(k)$ by $T_f(z) = C_f(zI - A_f)^{-1} B_f$. Substituting the filter matrices with (29) and taking $U_2 = M^T G_{22}^{-1} M$ into account, we obtain $T_f(z) = \widehat{C}_f(zI - U_2^{-1}\widehat{A}_f)^{-1} U_2^{-1}\widehat{B}_f$. So, the filter matrices are given by (26). Then, it follows from Theorem 7 that Theorem 10 is true. The proof is completed. □

The condition in Theorem 10 is convex in γ_0^2. So, the convex optimization problem

$$\begin{aligned} &\min \quad \rho \\ &\text{s.t.} \quad (24) \text{ and } (25) \text{ with } \rho = \gamma_0^2 \end{aligned} \tag{30}$$

can be formulated to design the H_∞ filters. If ρ^* is the solution of the optimization problem (30), then the designed filters guarantee that system (5) achieves the following H_∞ performance level:

$$\gamma^*(\alpha, d) = \left(\frac{\rho^* \left(1 - \lambda^2\right)\left(1 - (\mu\lambda)^{2(d+1)}\right)}{1 - (\mu\lambda)^2} + \frac{\rho^* \left(1 - \lambda^2\right)(\mu^\alpha \lambda)^{2(d+1)}}{1 - (\mu^\alpha \lambda)^2} \right)^{1/2}. \tag{31}$$

Remark 11. Note that $\gamma^*(\alpha, d)$ is monotonic increasing on μ and λ. Therefore, μ and λ should be chosen as small as possible to obtain a better H_∞ filtering performance and meanwhile to make the condition $\mu\lambda < 1$ easier to be satisfied. Nevertheless, it can be seen from (24) and (25) that some smaller μ and λ will result in a larger value of γ_0 and ultimately yield a larger value of $\gamma^*(\alpha, d)$. The following algorithm provides a method on determining the parameters μ and λ that will result in a suboptimal H_∞ performance level $\gamma^*(\alpha, d)$.

Algorithm 12.

Step 1. Choose some sufficiently large initial μ and λ such that $\mu\lambda < 1$ holds. Set $\mu^0 = \mu$ and $\lambda^0 = \lambda$. Solve the optimization problem (30) and calculate $\gamma^*(\alpha, d, \mu^0, \lambda^0)$ for given α and d.

Step 2. Decrease λ^0 by a certain step length; say, $\Delta\lambda$; that is, set $\lambda^0 = \lambda^0 - \Delta\lambda$. Then, solve the optimization problem (30) and calculate $\gamma^*(\alpha, d, \mu^0, \lambda^0)$ for given α and d.

Step 3. If $\gamma^*(\alpha, d, \mu^0, \lambda^0) < \gamma^*(\alpha, d, \mu^0, \lambda^0 + \Delta\lambda)$, then return to Step 2. Otherwise, exit and set $\lambda^* = \lambda^0 + \Delta\lambda$.

Step 4. Decrease μ^0 by a certain step length; say, $\Delta\mu$; that is, set $\mu^0 = \mu^0 - \Delta\mu$. Then, solve the optimization problem (30) and calculate $\gamma^*(\alpha, d, \mu^0, \lambda^*)$ for the obtained λ^* and given α and d.

Step 5. If $\gamma^*(\alpha, d, \mu^0, \lambda^*) < \gamma^*(\alpha, d, \mu^0 + \Delta\mu, \lambda^*)$, then return to Step 4. Otherwise, exit and set $\mu^* = \mu^0 + \Delta\mu$. Then, μ^* and λ^* are the desired values of the parameters μ and λ, respectively, and $\gamma^*(\alpha, d, \mu^*, \lambda^*)$ is the desired suboptimal H_∞ performance level for given α and d.

5. Illustrative Examples

Example 1. Consider a mechanical system with two masses and two springs as that studied in [7]. Its state-space model is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{c}{m_1} & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & -\dfrac{c}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix} w(t), \tag{32}$$

where x_1 and x_2 are the positions of masses m_1 and m_2, respectively. Choose $m_1 = 1$, $m_2 = 0.5$, $k_1 = k_2 = 1$, and $c = 0.5$, and set $T = 200$ ms; then we obtain the discrete-time system model (1) with

$$\begin{aligned} A_p &= \begin{bmatrix} 0.9617 & 0.0191 & 0.1878 & 0.001228 \\ 0.03697 & 0.9629 & 0.002456 & 0.1789 \\ -0.3732 & 0.1853 & 0.8678 & 0.01787 \\ 0.3528 & -0.3553 & 0.03574 & 0.784 \end{bmatrix}, \\ B_p &= \begin{bmatrix} 0.01922 \\ 0.000125 \\ 0.1878 \\ 0.002456 \end{bmatrix}. \end{aligned} \tag{33}$$

Suppose that x_1 is measured by a device with noise $w(k)$ and x_2 is to be estimated by using the H_∞ filters; then we have $C_p = [1\ 0\ 0\ 0]$ and $L_p = [0\ 1\ 0\ 0]$. Furthermore, we choose $D_p = 0.1$.

Choose $\mu = 1.01$ and $\lambda = 0.96$. Suppose that $d = 2$ and $\alpha = 50\%$. Then, by solving the optimization problem (23),

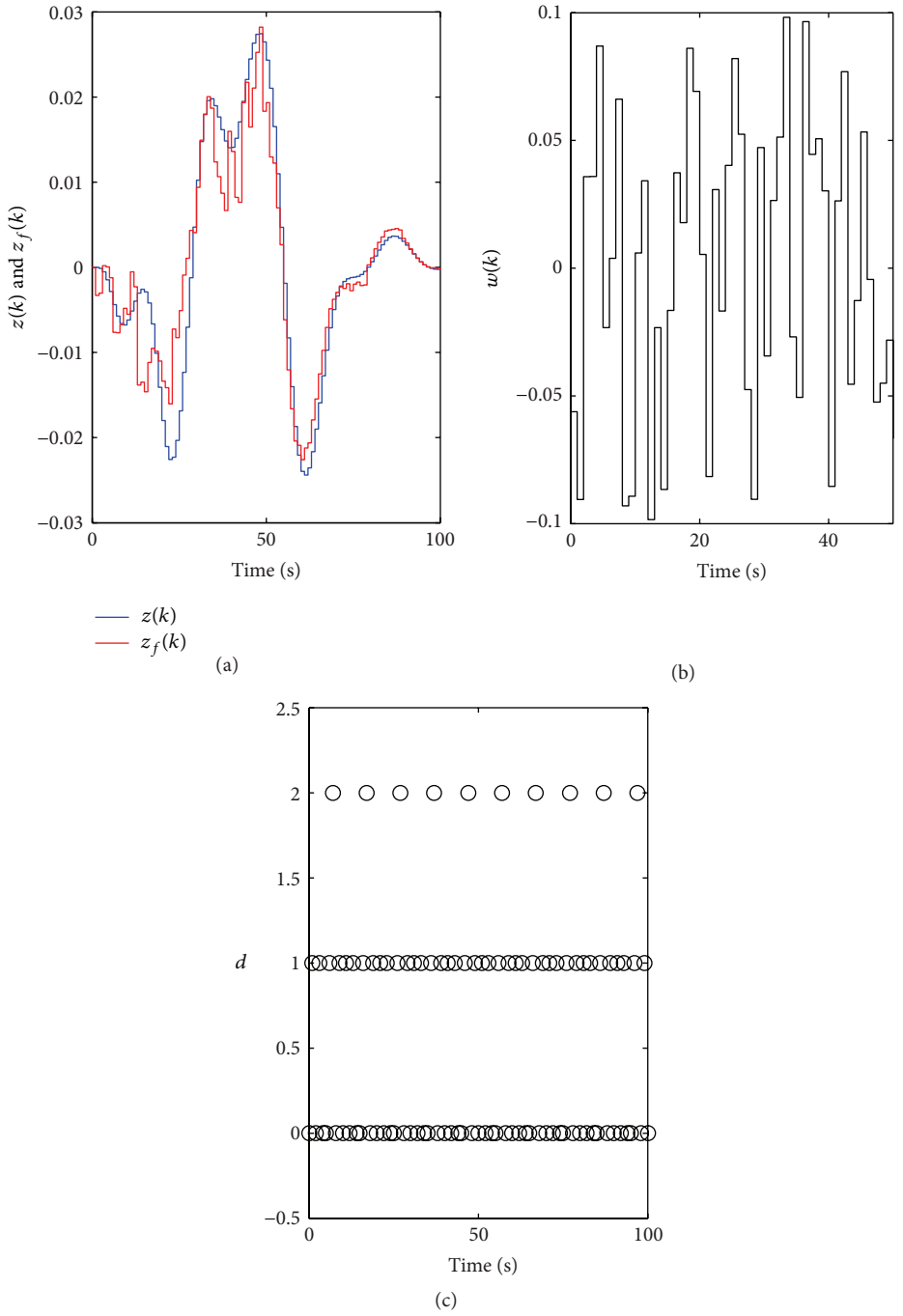

FIGURE 3: Trajectories of $z(k)$ and $z_f(k)$, the noise signal $w(k)$, and the measurement missing status.

we obtain the H_∞ performance level $\gamma^* = 0.8001$ with the following filter gain matrices:

$$A_f = \begin{bmatrix} -0.0691 & -0.0666 & 0.3304 & -0.0409 \\ -0.5288 & 0.9180 & 0.0793 & 0.1596 \\ -1.1896 & 0.1687 & 1.0783 & -0.0035 \\ 0.8709 & -0.3034 & -0.0219 & 0.8090 \end{bmatrix},$$
$$B_f = \begin{bmatrix} -1.0655 \\ -0.5858 \\ -0.7988 \\ 0.5298 \end{bmatrix}, \quad C_f = \begin{bmatrix} 0.0003 \\ -1.0002 \\ -0.0002 \\ -0.0001 \end{bmatrix}. \tag{34}$$

In the simulation, the noise signal $w(k)$ is assumed to be uniformly distributed with $[-0.1, 0.1]$ for the interval $[0, 50]$. The measurement missing is generated randomly and is shown in Figure 3(b), where the measurement missing rate is 50%. The trajectories of $z(k)$ and $z_f(k)$ are depicted in Figure 3(a). By calculation, we obtain from the results in Figure 3(a) that $\gamma_a = \|e\|_2/\|v\|_2 = 0.0460 < \gamma^* = 0.8001$, showing the effectiveness of the H_∞ filter design. In what follows, we will show the relations among the H_∞ performance level, the measurement missing rate bound, and the maximal number of consecutive measurement missing.

Table 1: The relation between the H_∞ performance level and the measurement missing rate bound for $d = 2$.

α	0%	20%	50%	80%
γ^*	0.7525	0.7704	0.8001	0.8339
γ_a	0.0470	0.0475	0.0565	0.0926

Table 2: The relation between the H_∞ performance level and the maximal number of consecutive measurement missing for $\alpha = 50\%$.

d	1	2	3
γ^*	0.5529	0.8001	1.0027
γ_a	0.0218	0.0424	0.0506

For $\mu = 1.01$, $\lambda = 0.96$, and $d = 2$, the relation between the H_∞ performance level and the measurement missing rate bound is given in Table 1 by solving the optimization problem (30). For $\mu = 1.01$, $\lambda = 0.96$, and $\alpha = 50\%$, the relation between the H_∞ performance level and the maximal number of consecutive measurement missing is presented in Table 2 by solving the optimization problem (30). In calculating the actual noise attenuation level γ_a, the noise signal $w(k)$ in Figure 3 is adopted, and the measurement missing is generated randomly under the constraint of the corresponding missing rate bound. It can be seen from Table 1 that the larger the missing rate bound, the worse the H_∞ filtering performance. Similarly, Table 2 shows that the larger the maximal number of consecutive measurement missing, the worse the H_∞ filtering performance. These verify the statements in Theorem 7. Moreover, γ_a is always smaller than γ^* for different values of α in Table 1, and it is always smaller than γ^* for different values of d in Table 2, showing the effectiveness of the H_∞ filter design. On the other hand, it can also be seen from Tables 1 and 2 that γ^* is much larger than γ_a for different values of α and d, which indicates that there exists certain extent of conservatism in the H_∞ filter design and the estimation of the H_∞ performance level γ^*.

6. Conclusions

In this paper, the H_∞ filtering problem was investigated for networked systems with bounded measurement missing. A switched system model was proposed to describe the considered system, which helped establish the quantitative relation between the H_∞ performance level and two parameters, namely, the measurement missing rate bound and the maximal number of consecutive measurement missing. It has been shown by the example that there exists certain extent of conservatism in the proposed H_∞ filter design, and the conservatism is partly introduced by the bounding on the switching frequency of the switched filtering error system given in (8). A tighter bounding on the switching frequency may help reduce the conservatism, which requires more detailed online information about the measurement missing status.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stability Analysis of Fractional-Order Nonlinear Systems with Delay

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Stability analysis of fractional-order nonlinear systems with delay is studied. We propose the definition of Mittag-Leffler stability of time-delay system and introduce the fractional Lyapunov direct method by using properties of Mittag-Leffler function and Laplace transform. Then some new sufficient conditions ensuring asymptotical stability of fractional-order nonlinear system with delay are proposed firstly. And the application of Riemann-Liouville fractional-order systems is extended by the fractional comparison principle and the Caputo fractional-order systems. Numerical simulations of an example demonstrate the universality and the effectiveness of the proposed method.

1. Introduction

Fractional calculus is a 300-year-old mathematical topic. Although it has a long history, it has not been used in physics and engineering for many years. However, during the last twenty years or so, fractional calculus starts to attract increasing attentions of physicists and engineers from an application point of view [1, 2]. It was found that many systems in interdisciplinary fields could be elegantly described with the help of fractional derivatives. There are two essential differences between fractional-order derivation and integer-order derivation. Firstly, the fractional-order derivative is concerned with the whole time domain for a mechanical or physical process, while the integer-order derivative indicates a variation or certain attribute at particular time. Secondly, the fractional-order derivative is related to the whole space for a physical process, while the integer-order derivative describes the local properties of a certain position. It is the reason that many real world physical systems are well characterized by the fractional-order state equations [3–6], such as fractional-order Lotka-Volterra equation [3] in biological systems, fractional-order Schrödinger equation [4] in quantum mechanics, fractional-order Langevin equation [5] in anomalous diffusion, and fractional-order oscillator equation [6] in damping vibration.

Time delays are inherent phenomena in interconnected real systems or processes, including feedback control systems, due to transportation of material, energy, or information. The presence of time delays in a feedback control system leads to a closed-loop characteristic equation which involves the exponential type transcendental terms. The exponential transcendentality brings infinitely many isolated roots, and hence it makes the stability analysis of time-delay systems a challenging task. Integer-order delay systems have been thoroughly investigated during the past decades [7–12], but there is no general stability theory for fractional-order dynamic systems with delay [13]. The necessary and sufficient stability conditions for linear fractional-order differential equations and linear time-delayed fractional differential equations have already been obtained in [13–15]. Reference [16] investigated the stability of n-dimensional linear fractional-order differential systems with order $1 < \alpha < 2$. However, only under some special circumstances or in certain cases, the practical problems may be regarded as linear systems. Therefore, stability of nonlinear dynamic systems not only is of great significance but also has important value in application.

In [17], the authors proposed the finite-time stabilization of a class of multistate time delay of fractional nonlinear systems. In [18, 19], the authors studied the stability of fractional nonlinear dynamic systems using Lyapunov direct method with the introductions of Mittag-Leffler stability and generalized Mittag-Leffler stability notions.

Although there are some works about stability of fractional-order time-delay systems and Lyapunov's second method, few attempts were done in order to combine these two powerful concepts and to observe what the benefits of this combination are. The aim of this paper is to present several analytical and numerical approaches for the stability analysis of fractional nonlinear time-delay systems. For extending the application of fractional calculus in nonlinear systems, we introduce the fractional comparison principle and some properties of Mittag-Leffler function. By the properties of Mittag-Leffler function, Laplace transform, and some fractional inequalities, some sufficient conditions for the asymptotical stability of fractional nonlinear systems with delay are proposed firstly. And the application of Riemann-Liouville fractional-order time-delay systems is extended by using fractional comparison principle and Caputo fractional-order systems. The numerical example illustrates principal results of the paper.

This paper is organized as follows. In Section 2 some basic definitions of fractional calculus and properties of Mittag-Leffler functions are presented. Mittag-Leffler stability and fractional-order extension of Lyapunov direct method are proposed in Section 3. An illustrative example is given in Section 4. Finally, conclusion is in Section 5.

2. Fractional-Order Derivatives and Mittag-Leffler Functions

2.1. Definition of Fractional Derivatives and Mittag-Leffler Functions. Fractional calculus plays an important role in modern science [1, 20, 21]. There are some definitions for fractional derivatives. In this paper, we give three commonly used definitions [1]: Riemann-Liouville (RL), Grünwald-Letnikov (GL), and Caputo definition.

Definition 1 (see [1]). The fractional integral ${}_aD_t^{-\alpha}$ of function $f(t)$ is defined as follows:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{1} $$

where fractional-order $\alpha > 0$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function.

Definition 2 (see [1]). The Riemann-Liouville derivative with order α of function $f(t)$ is defined as

$$ \begin{aligned} {}^{\mathrm{RL}}_{a}D_t^{\alpha} f(t) &= \frac{d^n}{dt^n}\, {}_aD_t^{-(n-\alpha)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \end{aligned} \tag{2} $$

where $n-1 < \alpha < n, n \in Z^+$.

Also, there are other definitions of fractional derivative introduced by Caputo and Grünwald-Letnikov.

Definition 3 (see [1]). The Caputo derivative with order α of function $f(t)$ is given as

$$ \begin{aligned} {}^{C}_{a}D_t^{\alpha} f(t) &= {}_aD_t^{-(n-\alpha)} \frac{d^n}{dt^n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \end{aligned} \tag{3} $$

where $n-1 < \alpha < n, n \in Z^+$.

The formulas for Laplace transform of Riemann-Liouville derivative ${}^{\mathrm{RL}}_{a}D_t^{\alpha} f(t)$ and Caputo fractional derivative ${}^{C}_{a}D_t^{\alpha} f(t)$ have the following forms [1]:

$$ \begin{aligned} \mathfrak{L}t\left\{ {}^{\mathrm{RL}}_{a}D^{\alpha} f(t); s \right\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \left[{}^{\mathrm{RL}}_{a}D_t^{\alpha-k-1} f(t) \right]_{t=0}, \\ \mathfrak{L}\left\{ {}^{C}_{a}D_t^{\alpha} f(t); s \right\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \end{aligned} \tag{4} $$

where $n-1 \le \alpha < n$ and $F(s) = \mathfrak{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt$.

Definition 4. The Grünwald-Letnikov derivative with order α of function $f(t)$ is defined as

$$ \begin{aligned} {}^{\mathrm{GL}}_{a}D_t^{\alpha} f(t) &= \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh) \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)\,(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} \\ &\quad + \frac{1}{\Gamma(m-\alpha+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(m+1)}(\tau)\, d\tau, \end{aligned} \tag{5} $$

where $m < \alpha < m+1$.

Throughout the papers about these definitions, we obtain the following conclusions. Grünwald-Letnikov definition is suitable for numerical calculations, Riemann-Liouville definition plays an important role in theory analysis, and Caputo definition is well used since its Laplace transform allows for initial conditions taking the same forms as those for integer-order derivatives, which have clear physical interpretations and have a wide range of applications in the process of factual modeling. As a generalization of the exponential function which is frequently used in the solutions of integer-order systems, the Mittag-Leffler function is frequently used in the solutions of the fractional differential equations. The definition and properties are given in the following.

Definition 5 (see [1]). The Mittag-Leffler function is given as

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{6} $$

where $\alpha > 0$ and $z \in \mathbb{C}$.

The generalization of Mittag-Leffler function with two parameters is widely used and defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{7}$$

where $\alpha > 0$, $\beta > 0$ and $z \in \mathbb{C}$.

Remark 6. If $\beta = 1$, we have $E_{\alpha,1}(z) = E_\alpha(z)$, especially $E_{1,1}(z) = E_1(z) = e^z$.

2.2. Properties of Fractional Derivatives and Mittag-Leffler Functions. In this section, some important properties of the fractional derivatives and the Mittag-Leffler functions are given, which are used in this paper.

Lemma 7. *Let $\alpha \in (0,1)$ and let $f(0) \geq 0$; then*

$$ {}_0^C D_t^\alpha f(t) \leq {}_0^{RL} D_t^\alpha f(t), \tag{8}$$

where ${}_0^C D_t^\alpha$ and ${}_0^{RL} D_t^\alpha$ are the Caputo and the Riemann-Liouville fractional derivatives.

Proof. By using the definitions of fractional derivatives, we have ${}_0^{\mathrm{RL}} D_t^\alpha f(t) = {}_0^C D_t^\alpha f(t) + ((f(0)t^{-\alpha})/(\Gamma(1-\alpha)))$. Since $\alpha \in (0,1)$ and $f(0) \geq 0$, we have the conclusion ${}_0^C D_t^\alpha f(t) \leq {}_0^{\mathrm{RL}} D_t^\alpha f(t)$. □

Lemma 8. *If ${}_0^C D_t^\alpha x(t) \geq 0$ and $x(0) \geq 0$, $0 < \alpha < 1$, then $x(t) \geq 0$.*

Proof. Suppose that ${}_0^C D_t^\alpha x(t) = f(t,x) \geq 0$. Using the equivalent Volterra integral equation [14]

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau, \tag{9}$$

since $t - \tau \geq 0$, $\Gamma(\alpha) > 0$, and $f(t,x) > 0$, we can get $x(t) \geq x(0) \geq 0$; that is, $x(t) \geq 0$.

According to the properties of the fractional derivatives and Lemma 8, we obtain the comparison theorem of the fractional derivatives. □

Theorem 9 (comparison theorem). *Let $0 < \alpha < 1$ and let $x(0) = y(0)$; then $x(t) \geq y(t)$, if ${}_0^C D_t^\alpha x(t) \geq {}_0^C D_t^\alpha y(t)$.*

Proof. The fractional differentiation and fractional integration are linear operations; then ${}_0^C D_t^\alpha (x(t) - y(t)) \geq 0$. By the Lemma 8, we can easily get $x(t) - y(t) \geq 0$; that is, $x(t) \geq y(t)$. □

Lemma 10 (see [1]). *Consider the Laplace transform of Mittag-Leffler function with two parameters; one has*

$$\mathfrak{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \left(\Re(s) > |\lambda|^{1/\alpha}\right), \tag{10}$$

where t and s are, respectively, the variables in the time domain and the Laplace domain, $\Re(s)$ stands for the real part of s, $\lambda \in \mathbb{R}$, and $\mathfrak{L}\{\cdot\}$ denotes the Laplace transform.

Proof. By the definitions of Laplace transform and Mittag-Leffler function, we have

$$\begin{aligned}
\mathfrak{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\} &= \int_0^\infty e^{-st} t^{\beta-1} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k t^{\alpha k}}{\Gamma(k\alpha+\beta)} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{\Gamma(k\alpha+\beta)} \int_0^\infty e^{-st} t^{\alpha k+\beta-1} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{\Gamma(k\alpha+\beta)} \frac{\Gamma(\alpha k+\beta)}{s^{\alpha k+\beta}} \\
&= \sum_{k=0}^{\infty} (-1)^k s^{-\beta} \left(\frac{\lambda}{s^\alpha}\right)^k \\
&= \frac{s^{\alpha-\beta}}{s^\alpha + \lambda},
\end{aligned} \tag{11}$$

where $\Re(s) > |\lambda|^{1/\alpha}$. □

3. Fractional-Order Extension of Lyapunov Direct Method

3.1. Fractional-Order Nonlinear Systems with Delay and Mittag-Leffler Stability. We also use functional differential equations to describe time-delay systems [22–24]. To formally introduce the concept of functional differential equations, let $\mathscr{C} = \mathscr{C}([a,b], \mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a maximum time-delay d of a system. In this case, we are often interested in the set of continuous functions mapping $[-d,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $\mathscr{C} = \mathscr{C}([-d,0], \mathbb{R}^n)$. For any $T > 0$ and any continuous function of time $x \in \mathscr{C}([t_0 - d, t_0 + T], \mathbb{R}^n)$, and $t_0 \leq t \leq t_0 + T$, let $x_t \in \mathscr{C}$ be a segment of the function x defined as $x_t(\theta) = x(t+\theta)$ and $-d \leq \theta \leq 0$.

Firstly, the Caputo fractional nonlinear time-delay system is given by

$$ {}_{t_0}^C D_t^\alpha x(t) = f(t, x_t), \tag{12}$$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$, and $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$. In equation (12), the Caputo derivative of the state variable x is on $[t_0, t]$ and $x(\tau)$ is for $t - d \leq \tau \leq t$. As such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length d, from $t_0 - d$ to t_0; that is,

$$x_{t_0}(\theta) = x(t_0 + \theta) = \varphi(\theta), \quad \forall \theta \in [-d, 0], \tag{13}$$

where $\varphi(\cdot)$ is an element of the Banach space $\mathscr{C}$ with the standard uniform norm: $\|\varphi\|_\infty = \max_{\theta \in [-d,0]} \|\varphi(\theta)\|$. And the vector norm $\|\cdot\|$ represents the 2-norm $\|\cdot\|_2$.

For given initial conditions of the form (13), let $y(t) = y(t, t_0, \varphi(\cdot))$ be a solution of the fractional-order time-delay system (12). The stability of the solution concerns the system's behavior when the system trajectory $x(t)$ deviates from $y(t, t_0, \varphi)$. In the following, without loss of generality, we will

assume that the functional differential equation (12) admits the solution $x(t) = 0$, which will be referred to as the trivial solution. Indeed, if it is desirable to study the stability of a nontrivial solution $y(t)$, then we may resort to the variable transformation $z(t) = x(t) - y(t)$, so that the new system

$$ {}_{t_0}^{C}D_t^{\alpha} z(t) = f(t, z_t + y_t) - f(t, y_t) \tag{14} $$

has the trivial solution $z(t) = 0$.

Definition 11 (see [24]). The system (12) is said to be stable if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\varphi(\cdot)\|_{\infty} < \delta$, then $\|x(t, t_0, \varphi)\| < \epsilon$ for all $t \geq t_0$.

Definition 12 (see [24]). If the value δ from the above definition may be chosen such that in addition$x(t, t_0, \varphi(\cdot)) \to 0$, when $t \to \infty$, then the system (12) is said to be asymptotically stable.

Definition 13 (exponential stability [23, 25]). The system (12) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu \geq 0$ such that for every solution $x(t, t_0, \varphi(\cdot))$ of system satisfy the following inequality:

$$ \left\| x(t, t_0, \varphi(\cdot)) \right\| \leq \mu \|\varphi\|_{\infty} e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0. \tag{15} $$

Definition 14 (Mittag-Leffler stability). The solution (12) is said to be Mittag-Leffler stable if there exist $\lambda \geq 0$ and $b > 0$ such that

$$ \left\| x(t, t_0, \varphi(\cdot)) \right\| \leq \left\{ m(\varphi) E_{\alpha}\left(-\lambda (t - t_0)^{\alpha}\right) \right\}^{b}, \tag{16} $$

where $\alpha \in (0, 1)$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally the Lipschitz on $x \in \mathscr{C}$ with the Lipschitz constant m_0.

Lemma 15. *Consider the real-valued continuous $f(t, x)$ in system* (12)*, one obtains*

$$ \left\| {}_{a}D_t^{-\alpha} f(t, x) \right\| \leq {}_{a}D_t^{-\alpha} \left\| f(t, x) \right\|, \tag{17} $$

where $\alpha > 0$ and $\|\cdot\|$ denotes an arbitrary norm.

Proof. It follows from the definition of fractional integral (1) that

$$ \begin{aligned} \left\| {}_{a}D_t^{-\alpha} f(t, x(t)) \right\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left\| \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} \right\| d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\left\| f(\tau, x(\tau)) \right\|}{(t-\tau)^{1-\alpha}} d\tau \\ &= {}_{a}D_t^{-\alpha} \left\| f(t, x(t)) \right\|. \end{aligned} \tag{18} $$

□

3.2. Fractional-Order Extension of Lyapunov Direct Method. It is well known that Lyapunov stability provides an important tool for stability analysis in nonlinear systems. We primarily study the Lyapunov direct method which involves finding a Lyapunov function candidate for a given nonlinear system. If there exists such function, the system is stable. Applying Lyapunov direct method is to search for an appropriate function. However, Lyapunov direct method is a sufficient condition. It means that if one cannot find a Lyapunov function, the system may still be stable and one cannot claim the system is not stable. In the following, we extend the Lyapunov direct method to the fractional-order nonlinear time-delay system and propose some sufficient conditions of stability for the fractional-order time-delay system.

Theorem 16. *Suppose $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ in system* (12) *takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$, and $V(t, x(t, 0, \varphi)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and locally Lipschitz with respect to x such that*

$$ \alpha_1 \|x(t)\|^a \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^{ab} \tag{19} $$

and the fractional derivative of V along the solution $x(t) = x(t, 0, \varphi)$ of system (12) *satisfies*

$$ {}_{0}^{C}D_t^{\beta} V(t, x(t)) \leq -\alpha_3 \|x(t)\|^{ab}, \tag{20} $$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $0 < \beta < 1$, and α_i $(i = 1, 2, 3)$, a and b are arbitrary positive constants. Then system is asymptotically stable.

Proof. By (19) and (20), we can easily get ${}_{0}^{C}D_t^{\beta} V(t, x(t)) \leq -\alpha_3 \alpha_2^{-1} V(t, x(t))$. So there is a nonnegative function $W(t)$ satisfying

$$ {}_{0}^{C}D_t^{\beta} V(t, x(t)) + W(t) = -\alpha_3 \alpha_2^{-1} V(t, x(t)). \tag{21} $$

Taking the Laplace transform of (21), we have

$$ s^{\beta} V(s) - V(\varphi) s^{\beta-1} + W(s) = -\alpha_3 \alpha_2^{-1} V(s), \tag{22} $$

where nonnegative constant $V(\varphi) = V(0, x(0)) = V(0, \varphi(\cdot))$ and $V(s) = \mathfrak{L}\{V(t, x(t)); s\}$. Then

$$ V(s) = \frac{V(\varphi) s^{\beta-1} - W(s)}{s^{\beta} + (\alpha_3/\alpha_2)}. \tag{23} $$

If $x(0) = \varphi = 0$, namely, $V(0) = 0$, the solution of (12) is $x = 0$. If $x(0) \neq 0$, then, $V(\varphi) > 0$. Since $V(t, x(t)) > 0$ is Lipschitz with respect to x, it follows from the fractional uniqueness and the existence theorem [13] and the inverse Laplace transform that the unique solution of (23) is

$$ \begin{aligned} V(t, x(t)) = {} & V(\varphi) E_{\beta}\left(-\frac{\alpha_3}{\alpha_2} t^{\beta}\right) \\ & - W(t) * \left[t^{\beta-1} E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2} t^{\beta}\right) \right]. \end{aligned} \tag{24} $$

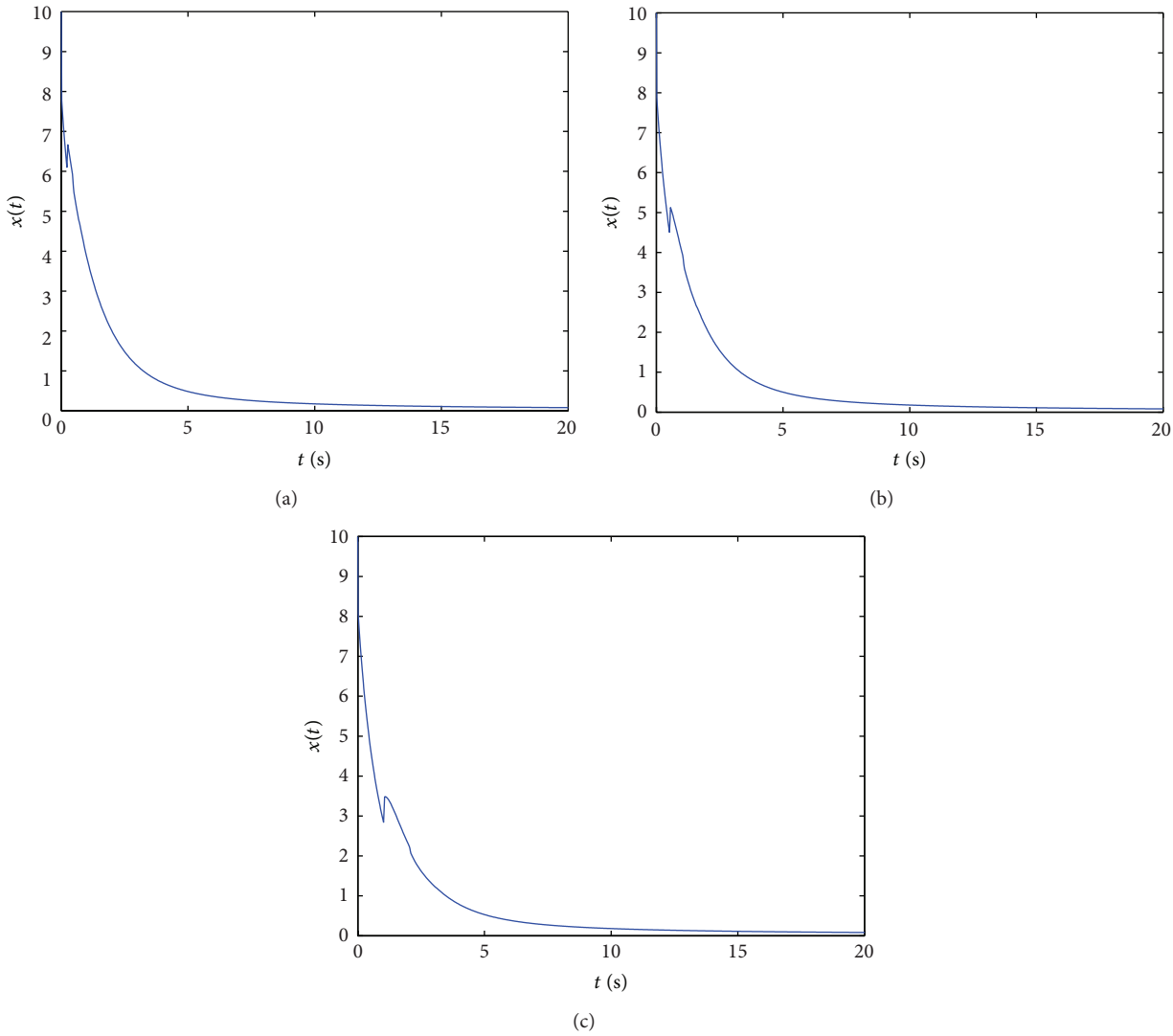

Figure 1: Time waveforms of numerical solutions $x(t)$ with $\alpha = 0.9$ and $\varphi(t) = t + 10$. (a), (b), and (c) show the solutions with $d = 0.2$ (a), $d = 0.5$ (b), and $d = 1$ (c), respectively.

Because both $W(t)$ and $t^{\beta-1}E_{\beta,\beta}(-(\alpha_3/\alpha_2)t^\beta)$ are nonnegative functions, we have

$$V(t,x) \le V(\varphi) E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right). \tag{25}$$

Then we substitute (25) into (19),

$$\|x(t)\| \le \left(\frac{V(\varphi)}{\alpha_1}E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right)\right)^{1/a}. \tag{26}$$

Let $m(\varphi) = (V(\varphi)/\alpha_1)^{1/a} \ge 0$; then

$$\|x(t)\| \le \left(m(\varphi) E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right)\right)^{1/a}, \tag{27}$$

where $m(\varphi) = 0$ holds if and only if $\varphi = 0$. Because $V(t, x(t))$ is Lipschitz with respect to x and $V(0, x(0)) = 0$ if and only if $x(0) = 0, m = (V(0, x(0)))/\alpha_1$ is also Lipschitz with respect to $x(0)$ and $m(0) = 0$, which imply the Mittag-Leffler stability of system (12). □

Theorem 17. *Suppose $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ in system (12) takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$, and $V(t, x(t, 0, \varphi)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1 \|x(t)\|^a \le V(t, x(t)) \le \alpha_2 \|x(t)\| \tag{28}$$

and the derivative of V along the solution $x(t, t_0, \varphi)$ of system (12) satisfies

$$\frac{dV(t, x(t))}{dt} \le -\alpha_3 \|x(t)\|, \tag{29}$$

where $t \ge 0$, α_i $(i = 1, 2, 3)$, and a are arbitrary positive constants. Then $\|x(t)\| \le (m(\varphi)E_\beta(-(\alpha_3/\alpha_2)t^{1-\alpha}))^{1/a}$; that is, system (12) is asymptotically stable.

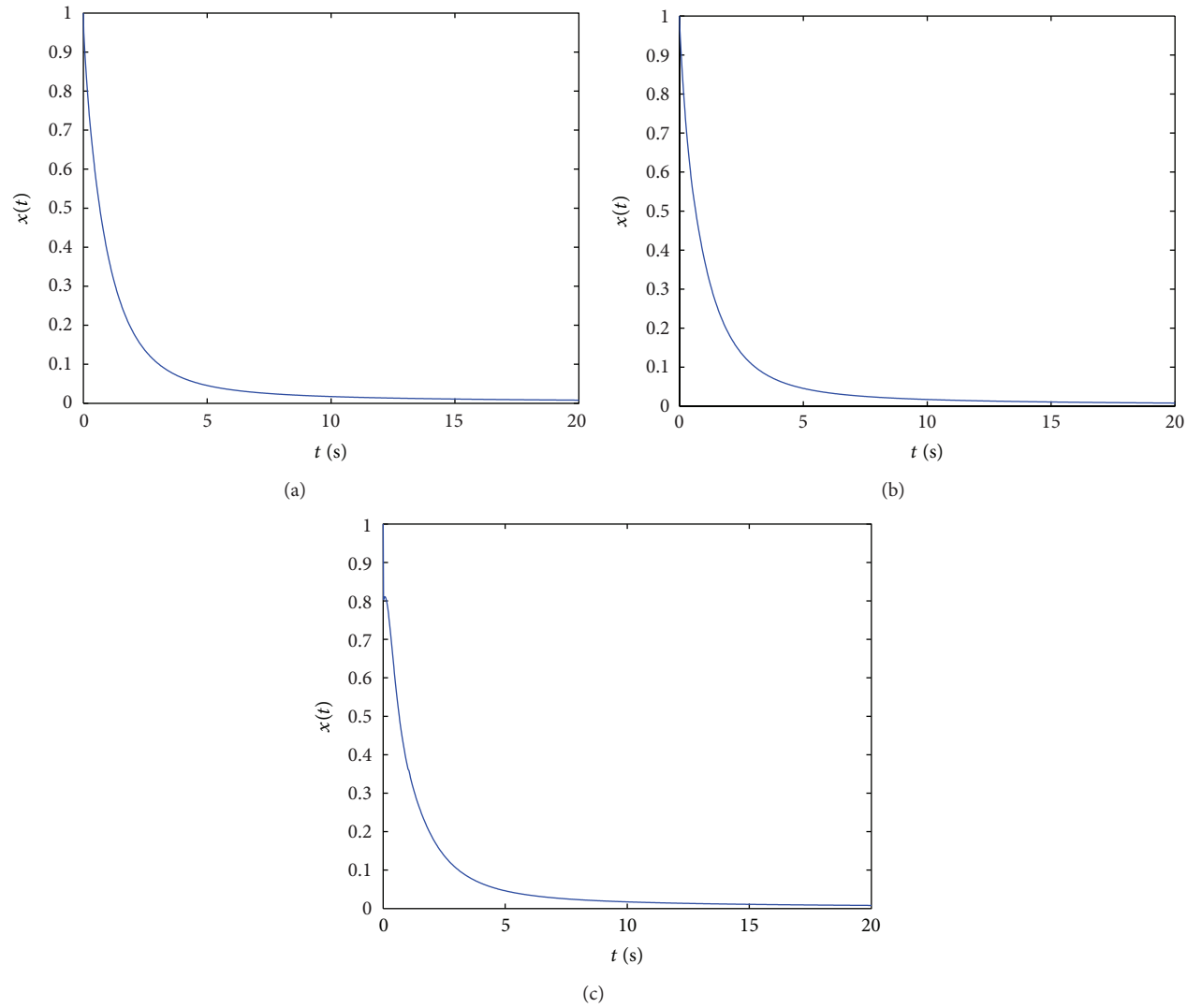

FIGURE 2: Time waveforms of numerical solutions $x(t)$with $\alpha = 0.8$ and $\varphi(t) = 2t^3 + 1$. (a), (b), and (c) show the solutions with $d = 0.2$ (a), $d = 0.5$ (b), and $d = 1$ (c), respectively.

Proof. It follows from the properties of Caputo derivative and Lemma 15 that

$$\begin{aligned} {}_0^C D_t^{1-\alpha} V(t, x(t)) &= {}_0 D_t^{-\alpha} \frac{dV(t, x(t))}{dt} \\ &\le -\alpha_3 {}_0 D_t^{-\alpha} \|x(t)\| \\ &\le -\frac{\alpha_3}{L} {}_0 D_t^{-\alpha} \|f(t, x(t))\| \\ &\le -\frac{\alpha_3}{L} \| {}_0 D_t^{-\alpha} f(t, x(t)) \| \\ &\le -\frac{\alpha_3}{L} \|x(t)\|, \end{aligned} \tag{30}$$

where $[{}_0 D_t^{-(1-\alpha)} x(t)]_{t=0} = 0$. Let $\beta = 1 - \alpha$, let $\alpha_3' = \alpha_3 / L$, and let $b = a^{-1}$; the conclusion can be easily obtained by using Theorem 16.

In the following, we study the Riemann-Liouville fractional dynamic system

$$ {}_{t_0}^{\mathrm{RL}} D_t^{\alpha} x(t) = f(t, x_t) \tag{31}$$

with the initial condition $x_{t_0} = \varphi(\theta)$, where $\alpha \in (0, 1)$ and f is piecewise continuous in t and locally Lipschitz in x. □

Theorem 18. *Let the assumptions in Theorem 16 be satisfied except for replacing ${}_0^C D_t^{\alpha}$ by ${}_0^{RL} D_t^{\alpha}$; then one has the same conclusion $\|x(t)\| \le (m(\varphi) E_{\beta}(-(\alpha_3/\alpha_2) t^{\beta}))^{1/a}$; that is, system (31) is asymptotically stable.*

Proof. It follows from $V(t, x(t)) \ge 0$ and Lemma 7 that ${}_0^C D_t^{\alpha} V(t, x(t)) \le {}_0^{\mathrm{RL}} D_t^{\alpha} V(t, x(t))$. Then ${}_0^C D_t^{\alpha} V(t, x(t)) \le {}_0^{\mathrm{RL}} D_t^{\alpha} V(t, x(t)) \le -\alpha_3 \|x(t)\|$. Therefore the conclusion can be obtained by Theorem 16. □

4. An Illustrative Example

In this section an illustrative example is used as proofs of concept. And the numerical simulations of a fractional-order delay-time system demonstrate the universality and the effectiveness of the proposed method.

Example 19. For a fractional-order delay-time nonlinear system,

$$ {}_{0}^{\mathrm{RL}}D_{t}^{\alpha}\left(x\left(t\right)\operatorname{sgn}\left(x\left(t\right)\right)\right) = -x\left(t\right)\operatorname{sgn}\left(x\left(t\right)\right) - x^{2}\left(t-d\right), \tag{32} $$

with the initial condition $x_{t_0} = \varphi(t)$ $(-d \le t \le 0)$, where $0 < \alpha < 1$ and $\operatorname{sgn}(\cdot)$ is the sign function. Choose the Lyapunov function $V(t, x(t)) = x\operatorname{sgn}(x)$; then ${}_{0}^{\mathrm{RL}}D_{t}^{\alpha}V(t, x(t)) = -x(t)\operatorname{sgn}(x(t)) - x^2(t-d) \le 0$. When selecting $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $a = b = 1$, it follows from Theorem 18 that $\|x(t)\| \le m(\varphi)E_{\alpha}(-t^{\alpha})$; that is, system (32) is asymptotically stable. In the following, we give the numerical simulations of system (32).

(a) When we select $\alpha = 0.9$ and $\varphi(t) = t + 10$, the numerical solutions of the fractional differential equations (32) are shown in Figure 1, which demonstrates the applicability of the proposed approach. In Figures 1(a)–1(c), the time waveforms of numerical solutions are given with $d = 0.2$ s, $d = 0.5$ s, and $d = 1$ s, respectively.

(b) When we select $\alpha = 0.8$ and $\varphi(t) = 4t^3 + 1$, the numerical solutions of the fractional differential equations (32) are shown in Figure 2, which demonstrates the applicability of the proposed approach. In Figures 2(a)–2(c), the time waveforms of numerical solutions are given with $d = 0.2$ s, $d = 0.5$ s, and $d = 1$ s, respectively.

Furthermore, when selecting other α and $\varphi(t)$, the fractional nonlinear time-delay system (32) is also asymptotically stable, which demonstrates the universality and the effectiveness of the proposed method.

5. Conclusion

Stability analysis of time-delay systems is one of the most fundamental and important issues for control systems. And stability of the nonlinear systems is important for scientists and engineers. Fractional dynamic systems were used intensively during the last decade in order to describe the behavior of complex systems in physics and engineering. In this paper the stabilization of fractional-order nonlinear time-delay systems is studied. We discussed the properties of the Caputo and Riemann-Liouville derivatives and the comparison theorem. And by the properties of Mittag-Leffler function and Laplace transform, we proposed the extending Lyapunov direct method which is the sufficient condition of stability for fractional-order time-delay systems. This enriches the knowledge of both the system theory and the fractional calculus. We partly extended the application of Riemann-Liouville fractional-order systems by fractional comparison theorem and Caputo fractional-order systems. And the numerical simulations of a fractional-order delay-time system demonstrate the universality and the effectiveness of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Less Conservative Stability Criterion for Delayed Stochastic Genetic Regulatory Networks

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This paper concerns the problem of stability analysis for delayed stochastic genetic regulatory networks. By introducing an appropriate Lyapunov-Krasovskii functional and employing delay-range partition approach, a new stability criterion is given to ensure the mean square stability of genetic regulatory networks with time-varying delays and stochastic disturbances. The stability criterion is given in the form of linear matrix inequalities, which can be easily tested by the LMI Toolbox of MATLAB. Moreover, it is theoretically shown that the obtained stability criterion is less conservative than the one in W. Zhang et al., 2012. Finally, a numerical example is presented to illustrate our theory.

1. Introduction

With the further progress of gene expression, researchers find that a gene expression is affected by other genes; conversely, it also influences others. Based on this reciprocal impact relation, gene expression forms a complex network—genetic regulatory network (GRN). GRNs are dynamical systems, which consist of an interaction of genes, proteins, and small molecules. In the past two decades, scholars have established mathematical models to represent GRNs. Basically, there are four types of GRN models, that is, Petri net model [1], Bayesian network model [2], Boolean model [3, 4], and (functional) differential equation model [5, 6]. The concentrations of mRNA and protein are described as the state variables in the functional differential equation model.

As dynamical systems, stability analysis is the first priority to explore GRNs. On the one hand, time delay inevitably occurs in GRNs due to the slow process of transcription, translation, and translocation [7]. On the other hand, internal noises of cells caused by random birth and death of the individual molecules and external noises from environmental fluctuations make the gene expression be best viewed as a stochastic process [8, 9]. So, it is very necessary to analyze the stability of GRNs with time-varying delays and stochastic disturbances [6, 10–18].

Recently, for a class of GRNs with interval time-varying delays and stochastic disturbances (see (17a), (17b) below), Wu et al. [18] established several delay-range-dependent and/or rate-dependent global stochastic asymptotical stability criteria in terms of linear matrix inequalities (LMIs) by using the stochastic analysis approach, employing some free-weighting matrices and introducing a type of Lyapunov-Krasovskii functional which includes the items like $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s)Ph_1(s)\mathrm{d}s\,\mathrm{d}\theta$ and $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_3^T(s)Qh_3(s)\mathrm{d}s\,\mathrm{d}\theta$, where P and Q are real symmetric positive definite matrices. Furthermore, in [18], the item $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ in the stochastic differential of $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s)Ph_1(s)\mathrm{d}s\,\mathrm{d}\theta$ was first estimated by employing Leibniz-Newton formula and the inequality

$$\begin{aligned} &-2\xi^T(t)M\int_{t-\tau(t)}^{t-\tau_1} h_1(s)\,\mathrm{d}s \\ &\le (\tau(t)-\tau_1)\,\xi^T(t)MP^{-1}M^T\xi(t) \\ &\quad+\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\,\mathrm{d}s, \end{aligned} \tag{1}$$

where M is a free-weighting matrix, and then $\tau(t) - \tau_1$ was enlarged to $\tau_2 - \tau_1$. Clearly, the conservatism would be produced because $\tau(t) - \tau_1$ is enlarged to $\tau_2 - \tau_1$. In order to reduce the conservatism, Zhang et al. [17] first estimated the item $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ by using Leibniz-Newton formula and the inequality

$$\begin{aligned} &-2\xi^T(t)S\int_{t-\tau(t)}^{t-\tau_1} h_1(s)\,\mathrm{d}s \\ &\quad\le (1-\alpha)\left(\tau(t)-\tau_1\right)\xi^T(t)SP^{-1}S^T\xi(t) \\ &\qquad+\alpha\left(\tau(t)-\tau_1\right)\xi^T(t)NP^{-1}N^T\xi(t) \\ &\qquad+\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\,\mathrm{d}s, \end{aligned} \tag{2}$$

where S and N are free-weighting matrices and α is an adjusting parameter with $0 < \alpha < 1$, and then a so-called convex combination technique was employed to obtain less conservative delay-range-dependent and/or rate-dependent global stochastic asymptotical stability criteria. It should be emphasized that the Lyapunov-Krasovskii functional used in [17] includes not only the items like $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s)Ph_1(s)\mathrm{d}s\,\mathrm{d}\theta$ and $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_3^T(s)Qh_3(s)\mathrm{d}s\,\mathrm{d}\theta$, but also the items like $\int_{t-\alpha\tau_1}^{t} x^T(s)Q_4x(s)\mathrm{d}s$ and $\int_{t-\tau_1}^{t-\alpha\tau_1} x^T(s)Q_4x(s)\mathrm{d}s$ involved in the adjustable parameter α, which can be viewed as a generalized delay-partition approach due to the adjustable parameter α. Generally, an appropriate delay-partition approach can bring less conservative stability criteria (see [19] and the references therein).

Note that free-weighting matrices (e.g., matrices M, S, and N above) have been employed in both [17, 18] to obtain less conservative stability criteria. However, it is emphasized in [18, Remark 2] that free-weighting matrices may produce a super-high amount of computation for the feasible solutions of LMIs. In order to overcome the disadvantage, in this paper we will propose a delay-range partition (DRP) approach to estimate accurately the item $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ (see (27) and (30)), where no free-weighting matrix is involved. By employing an appropriate Lyapunov-Krasovskii functional and introducing a DRP approach, a mean square stability criterion for GRNs with time-varying delays and stochastic disturbances is first established. Then it is theoretically shown that the proposed stability criterion is less conservative than [17, Theorem 1]. Finally, a numerical example is given to illustrate the theoretical results proposed here. The main contribution of this paper can be listed as follows: (i) the Lyapunov-Krasovskii functional employed in this paper does not include the items like $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_3^T(s)Qh_3(s)\mathrm{d}s\,\mathrm{d}\theta$ which are required in [17, 18]; (ii) the items like $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ in the stochastic differential of Lyapunov-Krasovskii functional are estimated accurately by proposing a DRP approach; (iii) theoretical comparison of the stability criterion [17, Theorem 1] and the one proposed in this paper is given; and (iv) there is no free-weighting matrix involved, which reduces the computational complexity.

The rest of the paper is organized as follows. In Section 2, the model of GRNs to be studied is described. A DRP-based mean square stability criterion (Theorem 3 below) for GRNs with time-varying delays and stochastic disturbances is established in Section 3. The theoretical comparison of Theorem 3 and [17, Theorem 1] is presented in Section 4. In Section 5, an example is given to show the validity of the obtained results. Finally, in Section 6, the conclusions are drawn.

Notation. For a positive integer n, set $\langle n\rangle = \{1, 2, \ldots, n\}$. R^n denotes the n-dimensional Euclidean space. We denote by $R^{m\times n}$ the set of $m \times n$ matrices over R. A^T and A^{-1} represent the transpose and inverse of a matrix A, respectively. For real symmetric matrices X and Y, the notation $X \ge Y$ ($X > Y$) means that the matrix $X - Y$ is positive semidefinite (positive definite). I_n is the $n\times n$ identity matrix. In a symmetric matrix, $\star$ denotes the entries implied by symmetry.

2. Model Description

The following differential equations have been used recently to describe GRNs [7]:

$$\begin{aligned} \dot{m}_i(t) = &-a_i m_i(t) + b_i\left(p_1(t-\sigma(t)), p_2(t-\sigma(t)), \ldots,\right. \\ &\left. p_n(t-\sigma(t))\right), \end{aligned} \tag{3a}$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t-\tau(t)), \qquad i \in \langle n\rangle, \tag{3b}$$

where $m_i(t)$ and $p_i(t)$ are the concentrations of the ith mRNA and protein at time t, respectively; $a_i > 0, c_i > 0$, and $d_i > 0$ are constants, representing the degradation rate of the ith mRNA, the degradation rate of the ith protein, and the translation rate of the ith mRNA to ith protein, respectively; both $\sigma(t)$ and $\tau(t)$ are transcriptional and translational delays, respectively; b_i is the regulatory function of the ith gene, which is generally a nonlinear function of the variables $p_1(t), p_2(t), \ldots, p_n(t)$, but it is monotonic with each variable.

For convenience, we give the following assumptions throughout the paper.

Assumption 1. The delays $\sigma(t)$ and $\tau(t)$ are differentiable functions satisfying

$$0 \le \sigma_1 \le \sigma(t) \le \sigma_2, \qquad 0 \le \tau_1 \le \tau(t) \le \tau_2, \tag{4a}$$

$$\dot{\sigma}(t) \le \sigma_d < +\infty, \qquad \dot{\tau}(t) \le \tau_d < +\infty, \tag{4b}$$

where σ_1, σ_2, σ_d, τ_1, τ_2, and τ_d are constants.

Assumption 2. The function b_i is taken as

$$b_i\left(p_1(t), p_2(t), \ldots, p_n(t)\right) = \sum_{j=1}^{n} b_{ij}\left(p_j(t)\right), \tag{5}$$

which is called SUM logic. Here, b_{ij} is a monotonic function of the Hill form; that is,

$$b_{ij}(x) = \begin{cases} \alpha_{ij} \dfrac{\left(x/\beta_j\right)^{H_j}}{1+\left(x/\beta_j\right)^{H_j}} & \text{if transcription factor } j \text{ is} \\ & \text{an activator of gene } i, \\ \alpha_{ij} \dfrac{1}{1+\left(x/\beta_j\right)^{H_j}} & \text{if transcription factor } j \text{ is} \\ & \text{a represssor of gene } i, \end{cases} \tag{6}$$

where H_j is the Hill coefficient, β_j is a scalar, and α_{ij} is a bounded constant, which denotes the dimensionless transcriptional rate of transcription factor j to gene i.

Clearly, GRN ((3a), (3b)) can be rewritten as

$$\dot{m}_i(t) = -a_i m_i(t) + \sum_{j=1}^{n} w_{ij} h_j\left(p_j(t-\sigma(t))\right) + v_i, \tag{7a}$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t-\tau(t)), \quad i \in \langle n \rangle, \tag{7b}$$

where

$$w_{ij} = \begin{cases} \alpha_{ij} & \text{if transcription factor } j \text{ is} \\ & \text{an activator of gene } i, \\ 0 & \text{if there is no connection} \\ & \text{between } j \text{ and } i, \\ -\alpha_{ij} & \text{if transcription factor } j \text{ is} \\ & \text{a represssor of gene } i, \end{cases} \tag{8}$$

$$h_j(x) = \frac{\left(x/\beta_j\right)^{H_j}}{1+\left(x/\beta_j\right)^{H_j}}, \qquad v_i = \sum_{j \in \mathscr{V}_i} \alpha_{ij},$$

and $\mathscr{V}_i$ is the set of all the transcription factors j which is a repressor of gene i.

Rewriting GRN ((7a), (7b)) into compact matrix form, we obtain

$$\dot{m}(t) = -Am(t) + Wh\left(p(t-\sigma(t))\right) + v, \tag{9a}$$

$$\dot{p}(t) = -Cp(t) + Dm(t-\tau(t)), \tag{9b}$$

where

$$\begin{aligned} A &= \operatorname{diag}(a_1, a_2, \ldots, a_n), \\ W &= \left[w_{ij}\right]_{n\times n}, \\ C &= \operatorname{diag}(c_1, c_2, \ldots, c_n), \\ D &= \operatorname{diag}(d_1, d_2, \ldots, d_n), \\ v &= \operatorname{col}(v_1, v_2, \ldots, v_n), \\ m(t) &= \operatorname{col}(m_1(t), m_2(t), \ldots, m_n(t)), \\ p(t) &= \operatorname{col}(p_1(t), p_2(t), \ldots, p_n(t)), \\ h(p(t)) &= \operatorname{col}(h_1(p_1(t)), h_2(p_2(t)), \ldots, h_n(p_n(t))). \end{aligned} \tag{10}$$

Let (m^*, p^*) be an equilibrium point of ((9a), (9b)); that is, it is a solution of the following equation:

$$-Am^* + Wh(p^*) + v = 0, \qquad -Cp^* + Dm^* = 0. \tag{11}$$

For convenience, we shift the equilibrium point (m^*, p^*) to the origin by using the transformations $x(t) = m(t) - m^*$ and $y(t) = p(t) - p^*$; then we have

$$\dot{x}(t) = -Ax(t) + Wf\left(y(t-\sigma(t))\right), \tag{12a}$$

$$\dot{y}(t) = -Cy(t) + Dx(t-\tau(t)), \tag{12b}$$

where $f(y(t)) = h(y(t) + p^*) - h(p^*)$.

From the relationship between h and f, one can easily find that f satisfies the following sector condition:

$$f_i(0) = 0, \qquad l_i^- \le \frac{f_i(s)}{s} \le l_i^+, \quad i \in \langle n \rangle, \ \forall 0 \ne s \in R, \tag{13}$$

where l_i^- and l_i^+ are a pair of nonnegative scalars and $f_i(s)$ is the ith entry of $f(s)$. Since h_i is a monotonically increasing and differentiable function with saturation, we have to choose l_i^- as zero or a small positive number. Let $L^- = \operatorname{diag}(l_1^-, l_2^-, \ldots, l_n^-)$ and $L^+ = \operatorname{diag}(l_1^+, l_2^+, \ldots, l_n^+)$.

As shown in [15–17] the gene regulation is an intrinsically noisy process. For this reason, in this paper, we consider a class of GRNs with both time delays and noise disturbances by the following model:

$$\begin{aligned} \mathrm{d}x(t) = {} & \left[-Ax(t) + Wf\left(y(t-\sigma(t))\right)\right]\mathrm{d}t \\ & + H\left(x(t), x(t-\tau(t)), y(t), y(t-\sigma(t))\right)\mathrm{d}\omega(t), \end{aligned} \tag{14a}$$

$$\mathrm{d}y(t) = \left[-Cy(t) + Dx(t-\tau(t))\right]\mathrm{d}t, \tag{14b}$$

where $\omega(t)$ is an m-dimensional Brown motion, $m \ge 1$, and $H(x(t), x(t-\tau(t)), y(t), y(t-\sigma(t)))$ is the noise intensity matrix at time t such that

$$\begin{aligned} \operatorname{trace}\left(HH^T\right) \le {} & x^T(t) H_1 x(t) + y^T(t) H_2 y(t) \\ & + x^T(t-\tau(t)) H_3 x(t-\tau(t)) \\ & + y^T(t-\sigma(t)) H_4 y(t-\sigma(t)), \end{aligned} \tag{15}$$

where H_i $(i = 1, 2, 3, 4)$ are real symmetric positive semidefinite matrices.

For simplicity, set

$$\begin{aligned} h_1(t) &= -Ax(t) + Wf\left(y(t-\sigma(t))\right), \\ h_2(t) &= -Cy(t) + Dx(t-\tau(t)), \\ h_3(t) &= H\left(x(t), x(t-\tau(t)), y(t), y(t-\sigma(t))\right). \end{aligned} \tag{16}$$

Then, GRN ((14a), (14b)) can be represented as

$$\mathrm{d}x(t) = h_1(t)\,\mathrm{d}t + h_3(t)\,\mathrm{d}\omega(t), \tag{17a}$$

$$\mathrm{d}y(t) = h_2(t)\,\mathrm{d}t. \tag{17b}$$

3. Stability Criterion

In the following theorem, we will propose a DRP approach to present an asymptotical stability criterion in the mean square sense for GRNs with time-varying delays and stochastic disturbance.

Theorem 3. *For given scalars $\alpha \in (0,1)$, $\tau_2 > \tau_1 > 0$, $\sigma_2 > \sigma_1 > 0$, τ_d and σ_d, and positive integers s_1 and s_2, under the conditions* (15) *and (*(4a), (4b)*), we can conclude that GRN (*(14a), (14b)*) is asymptotically stable in the sense of mean square, if there exist a scalar $\rho > 0$ and matrices $U_i^T = U_i > 0$ $(i = 1,2)$, $V_j^T = V_j > 0$, $W_j^T = W_j > 0$ $(j = 1,2,\ldots,7)$, and $\Lambda := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0$ such that the following LMIs hold:*

$$U_1 \le \rho I, \tag{18}$$

$$\begin{aligned} \Psi_{k,l} :&= \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6 \\ &+ \Psi_k + \Psi_l < 0, \quad \forall k \in \langle s_1 \rangle,\ l \in \langle s_2 \rangle, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Psi_1 &= \widetilde{\Psi}_1 + \widetilde{\Psi}_1^T + \rho\left(e_1^T H_1 e_1 + e_3^T H_2 e_3 + e_7^T H_3 e_7 + e_9^T H_4 e_9\right), \\ \widetilde{\Psi}_1 &= e_1^T U_1 \left(-A e_1 + W e_{13}\right) + e_7^T U_2 \left(-C e_7 + D e_3\right), \\ \Psi_2 &= e_1^T \left(V_1 + V_5\right) e_1 + e_4^T \left(V_2 - V_1\right) e_4 + e_5^T \left(V_3 - V_2\right) e_5 \\ &\quad + e_2^T \left[\left(1 + \alpha\tau_d\right) V_4 - \left(1 - \alpha\tau_d\right) V_3\right] e_2 \\ &\quad - \left(1 - \tau_d\right) e_3^T V_4 e_3 - e_6^T V_5 e_6, \\ \Psi_3 &= e_7^T \left(W_1 + W_5\right) e_7 + e_{10}^T \left(W_2 - W_1\right) e_{10} \\ &\quad + e_{11}^T \left(W_3 - W_2\right) e_{11} \\ &\quad + e_8^T \left[\left(1 + \alpha\sigma_d\right) W_4 - \left(1 - \alpha\sigma_d\right) W_3\right] e_8 \\ &\quad - \left(1 - \sigma_d\right) e_9^T W_4 e_9 - e_{12}^T W_5 e_{12}, \\ \Psi_4 &= \left(-A e_1 + W e_{13}\right)^T \left(\tau_1 V_6 + \tau_{12} V_7\right) \left(-A e_1 + W e_{13}\right) \\ &\quad + \left(-C e_7 + D e_3\right)^T \left(\sigma_1 W_6 + \sigma_{12} W_7\right) \left(-C e_7 + D e_3\right), \\ \Psi_5 &= \widetilde{\Psi}_5 + \widetilde{\Psi}_5^T, \\ \widetilde{\Psi}_5 &= e_{13}^T (L^+)^- \Lambda L^- e_9 - e_{13}^T (L^+)^- \Lambda e_{13} - e_9^T \Lambda L^- e_9 + e_9^T \Lambda e_{13}, \\ \Psi_6 &= -\frac{1}{\alpha\tau_1} \left(e_4 - e_1\right)^T V_6 \left(e_4 - e_1\right) \\ &\quad - \frac{1}{\left(1 - \alpha\right)\tau_1} \left(e_5 - e_4\right)^T V_6 \left(e_5 - e_4\right) \\ &\quad - \frac{1}{\alpha\sigma_1} \left(e_{10} - e_7\right) W_6 \left(e_{10} - e_7\right) \\ &\quad - \frac{1}{\left(1 - \alpha\right)\sigma_1} \left(e_{11} - e_{10}\right)^T W_6 \left(e_{11} - e_{10}\right), \\ \Psi_k &= -\frac{s_1}{k\alpha\tau_{12}} \left(e_2 - e_5\right)^T V_7 \left(e_2 - e_5\right) \\ &\quad - \frac{s_1}{\left(1 - \alpha\right) k\tau_{12}} \left(e_3 - e_2\right)^T V_7 \left(e_3 - e_2\right) \\ &\quad - \frac{s_1}{\left(s_1 - k + 1\right)\tau_{12}} \left(e_6 - e_3\right)^T V_7 \left(e_6 - e_3\right), \\ \Psi_l &= -\frac{s_2}{l\alpha\sigma_{12}} \left(e_8 - e_{11}\right)^T W_7 \left(e_8 - e_{11}\right) \\ &\quad - \frac{s_2}{\left(1 - \alpha\right) l\sigma_{12}} \left(e_9 - e_8\right)^T W_7 \left(e_9 - e_8\right) \\ &\quad - \frac{s_2}{\left(s_2 - l + 1\right)\sigma_{12}} \left(e_{12} - e_9\right)^T W_7 \left(e_{12} - e_9\right), \\ e_i &= \left[\underbrace{0 \ \cdots \ 0}_{\text{number } i-1} \ I_n \ \underbrace{0 \ \cdots \ 0}_{\text{number } n-i}\right], \quad i = 1,2,\ldots,n, \\ \sigma_{12} &= \sigma_2 - \sigma_1, \qquad \tau_{12} = \tau_2 - \tau_1. \end{aligned} \tag{20}$$

Proof. Let $\beta(t) = \tau_1 + \alpha(\tau(t) - \tau_1)$ and $\gamma(t) = \sigma_1 + \alpha(\sigma(t) - \sigma_1)$. Choose a Lyapunov-Krasovskii functional candidate as

$$V(t) = \sum_{i=1}^{4} V_i(t), \tag{21}$$

where

$$\begin{aligned} V_1(t) &= x^T(t) U_1 x(t) + y^T(t) U_2 y(t), \\ V_2(t) &= \int_{t-\alpha\tau_1}^{t} x^T(s) V_1 x(s)\,ds \\ &\quad + \int_{t-\tau_1}^{t-\alpha\tau_1} x^T(s) V_2 x(s)\,ds \\ &\quad + \int_{t-\beta(t)}^{t-\tau_1} x^T(s) V_3 x(s)\,ds \\ &\quad + \int_{t-\tau(t)}^{t-\beta(t)} x^T(s) V_4 x(s)\,ds \\ &\quad + \int_{t-\tau_2}^{t} x^T(s) V_5 x(s)\,ds, \\ V_3(t) &= \int_{t-\alpha\sigma_1}^{t} y^T(s) W_1 y(s)\,ds \\ &\quad + \int_{t-\sigma_1}^{t-\alpha\sigma_1} y^T(s) W_2 y(s)\,ds \\ &\quad + \int_{t-\gamma(t)}^{t-\sigma_1} y^T(s) W_3 y(s)\,ds \\ &\quad + \int_{t-\sigma(t)}^{t-\gamma(t)} y^T(s) W_4 y(s)\,ds \\ &\quad + \int_{t-\sigma_2}^{t} y^T(s) W_5 y(s)\,ds, \end{aligned}$$

$$
\begin{aligned}
V_4(t) = & \int_{-\tau_1}^{0}\int_{t+\theta}^{t} h_1^T(s) V_6 h_1(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s) V_7 h_1(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-\sigma_1}^{0}\int_{t+\theta}^{t} h_2^T(s) W_6 h_2(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-\sigma_2}^{-\sigma_1}\int_{t+\theta}^{t} h_2^T(s) W_7 h_2(s)\,\mathrm{d}s\,\mathrm{d}\theta,
\end{aligned}
\tag{22}
$$

and the matrices $U_i^T = U_i > 0$ $(i = 1, 2)$, $V_j^T = V_j > 0$, and $W_j^T = W_j > 0$ $(j = 1, 2, \ldots, 7)$ are taken from a feasible solution to (18) and (19). By Itô's formula, we can obtain the following stochastic differential:

$$
\mathrm{d}V(t) = \sum_{i=1}^{4} \mathscr{L}V_i(t)\,\mathrm{d}t + 2x^T(t) U_1 h_3(t)\,\mathrm{d}\omega(t), \tag{23}
$$

where $\mathscr{L}$ is the weak infinitesimal operator and

$$
\begin{aligned}
\mathscr{L}V_1(t) &= 2x^T(t) U_1 h_1(t) + 2y^T(t) U_2 h_2(t) \\
&\quad + h_3^T(t) U_1 h_3(t) \\
&\le \eta^T(t) \Psi_1 \eta(t),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\mathscr{L}V_2(t) &= x^T(t)\left(V_1 + V_5\right) x(t) \\
&\quad + x^T\left(t - \alpha\tau_1\right)\left(V_2 - V_1\right) x\left(t - \alpha\tau_1\right) \\
&\quad + x^T\left(t - \tau_1\right)\left(V_3 - V_2\right) x\left(t - \tau_1\right) \\
&\quad + \left(1 - \dot{\beta}(t)\right) x^T\left(t - \beta(t)\right)\left(V_4 - V_3\right) x\left(t - \beta(t)\right) \\
&\quad - \left(1 - \dot{\tau}(t)\right) x^T\left(t - \tau(t)\right) V_4 x\left(t - \tau(t)\right) \\
&\quad - x^T\left(t - \tau_2\right) V_5 x\left(t - \tau_2\right) \\
&\le \eta^T(t) \Psi_2 \eta(t),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\mathscr{L}V_3(t) &= y^T(t)\left(W_1 + W_5\right) y(t) \\
&\quad + y^T\left(t - \alpha\sigma_1\right)\left(W_2 - W_1\right) y\left(t - \alpha\sigma_1\right) \\
&\quad + y^T\left(t - \sigma_1\right)\left(W_3 - W_2\right) y\left(t - \sigma_1\right) \\
&\quad + \left(1 - \dot{\gamma}(t)\right) y^T\left(t - \gamma(t)\right)\left(W_4 - W_3\right) y\left(t - \gamma(t)\right) \\
&\quad - \left(1 - \dot{\sigma}(t)\right) y^T\left(t - \sigma(t)\right) W_4 y\left(t - \sigma(t)\right) \\
&\quad - y^T\left(t - \sigma_2\right) W_5 y\left(t - \sigma_2\right) \\
&\le \eta^T(t) \Psi_3 \eta(t),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\mathscr{L}V_4(t) &= h_1^T(t)\left(\tau_1 V_6 + \tau_{12} V_7\right) h_1(t) \\
&\quad - \int_{t-\tau_1}^{t} h_1^T(s) V_6 h_1(s)\,\mathrm{d}s - \int_{t-\tau_2}^{t-\tau_1} h_1^T(s) V_7 h_1(s)\,\mathrm{d}s \\
&\quad + h_2^T(t)\left(\sigma_1 W_6 + \sigma_{12} W_7\right) h_2(t) \\
&\quad - \int_{t-\sigma_1}^{t} h_2^T(s) W_6 h_2(s)\,\mathrm{d}s \\
&\quad - \int_{t-\sigma_2}^{t-\sigma_1} h_2^T(s) W_7 h_2(s)\,\mathrm{d}s \\
&\le \eta^T(t) \Psi_4 \eta(t) \\
&\quad - \frac{1}{\alpha\tau_1}\int_{t-\alpha\tau_1}^{t} h_1^T(s)\,\mathrm{d}s\, V_6 \int_{t-\alpha\tau_1}^{t} h_1(s)\,\mathrm{d}s \\
&\quad - \frac{1}{(1-\alpha)\tau_1}\int_{t-\tau_1}^{t-\alpha\tau_1} h_1^T(s)\,\mathrm{d}s\, V_6 \int_{t-\tau_1}^{t-\alpha\tau_1} h_1(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\beta(t)-\tau_1}\int_{t-\beta(t)}^{t-\tau_1} h_1^T(s)\,\mathrm{d}s\, V_7 \int_{t-\beta(t)}^{t-\tau_1} h_1(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\tau(t)-\beta(t)}\int_{t-\tau(t)}^{t-\beta(t)} h_1^T(s)\,\mathrm{d}s\, V_7 \int_{t-\tau(t)}^{t-\beta(t)} h_1(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} h_1^T(s)\,\mathrm{d}s\, V_7 \int_{t-\tau_2}^{t-\tau(t)} h_1(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\alpha\sigma_1}\int_{t-\alpha\sigma_1}^{t} h_2^T(s)\,\mathrm{d}s\, W_6 \int_{t-\alpha\sigma_1}^{t} h_2(s)\,\mathrm{d}s \\
&\quad - \frac{1}{(1-\alpha)\sigma_1}\int_{t-\sigma_1}^{t-\alpha\sigma_1} h_2^T(s)\,\mathrm{d}s\, W_6 \int_{t-\sigma_1}^{t-\alpha\sigma_1} h_2(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\gamma(t)-\sigma_1}\int_{t-\gamma(t)}^{t-\sigma_1} h_2^T(s)\,\mathrm{d}s\, W_7 \int_{t-\gamma(t)}^{t-\sigma_1} h_2(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\sigma(t)-\gamma(t)}\int_{t-\sigma(t)}^{t-\gamma(t)} h_2^T(s)\,\mathrm{d}s\, W_7 \int_{t-\sigma(t)}^{t-\gamma(t)} h_2(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\sigma_2-\sigma(t)}\int_{t-\sigma_2}^{t-\sigma(t)} h_2^T(s)\,\mathrm{d}s\, W_7 \int_{t-\sigma_2}^{t-\sigma(t)} h_2(s)\,\mathrm{d}s.
\end{aligned}
\tag{27}
$$

For any scalars a, b with $a < b$, it follows from (16) that $\int_a^b h_1(t)\mathrm{d}t = x(b) - x(a) - \int_a^b h_3(t)\mathrm{d}\omega(t)$ and $\int_a^b h_2(t)\mathrm{d}t = y(b) - y(a)$, and hence

$$
\begin{aligned}
&\mathscr{E}\left(\int_a^b h_1^T(t)\,\mathrm{d}t V_j \int_a^b h_1(t)\,\mathrm{d}t\right) \\
&\quad \ge [x(b) - x(a)]^T V_j [x(b) - x(a)], \quad j = 6, 7,
\end{aligned}
\tag{28a}
$$

$$
\begin{aligned}
&\int_a^b h_2^T(t)\,\mathrm{d}t W_j \int_a^b h_2(t)\,\mathrm{d}t \\
&\quad = [y(b) - y(a)]^T W_j [y(b) - y(a)], \quad j = 6, 7,
\end{aligned}
\tag{28b}
$$

where $\mathscr{E}$ represents the mathematical expectation operator. Next, from the sector condition (13), we can obtain that

$$\begin{aligned} 0 \leq & -2\left[\left(L^{+}\right)^{-} f\left(y\left(t-\sigma\left(t\right)\right)\right)-y\left(t-\sigma\left(t\right)\right)\right]^{T} \\ & \times \Lambda\left[f\left(y\left(t-\sigma\left(t\right)\right)\right)-L^{-} y\left(t-\sigma\left(t\right)\right)\right] \\ & =\eta^{T}\left(t\right) \Psi_{5} \eta\left(t\right). \end{aligned} \tag{29}$$

When $\tau(t) \in [\tau_1 + ((k-1)/s_1)\tau_{12},\ \tau_1 + (k/s_1)\tau_{12}]$ for some positive integer $k \in \langle s_1 \rangle$ and $\sigma(t) \in [\sigma_1 + ((l-1)/s_2)\sigma_{12},\ \sigma_1 + (l/s_2)\sigma_{12}]$ for some positive integer $l \in \langle s_2 \rangle$, it is easy to see that

$$\begin{aligned} \frac{1}{\beta\left(t\right)-\tau_{1}} & \geq \frac{s_{1}}{k\alpha\tau_{12}}, \frac{1}{\tau\left(t\right)-\beta\left(t\right)} \geq \frac{s_{1}}{\left(1-\alpha\right)k\tau_{12}}, \frac{1}{\tau_{2}-\tau\left(t\right)} \\ & \geq \frac{s_{1}}{\left(s_{1}-k+1\right)\tau_{12}}, \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{1}{\gamma\left(t\right)-\sigma_{1}} & \geq \frac{s_{2}}{l\alpha\sigma_{12}}, \frac{1}{\sigma\left(t\right)-\gamma\left(t\right)} \geq \frac{s_{2}}{\left(1-\alpha\right)l\sigma_{12}}, \frac{1}{\sigma_{2}-\sigma\left(t\right)} \\ & \geq \frac{s_{2}}{\left(s_{2}-l+1\right)\sigma_{12}}. \end{aligned} \tag{31}$$

Then, the combination of (21)–(31) gives

$$\mathscr{E}\mathscr{L}V\left(t\right) \leq \eta^{T}\left(t\right) \Psi_{k,l} \eta\left(t\right), \tag{32}$$

where

$$\begin{aligned} \eta\left(t\right) = \operatorname{col}\{ & x\left(t\right), x\left(t-\beta\left(t\right)\right), x\left(t-\tau\left(t\right)\right), \\ & x\left(t-\alpha\tau_{1}\right), x\left(t-\tau_{1}\right), x\left(t-\tau_{2}\right), \\ & y\left(t\right), y\left(t-\gamma\left(t\right)\right), y\left(t-\sigma\left(t\right)\right), y\left(t-\alpha\sigma_{1}\right), \\ & y\left(t-\sigma_{1}\right), y\left(t-\sigma_{2}\right), f\left(y\left(t-\sigma\left(t\right)\right)\right)\}. \end{aligned} \tag{33}$$

Due to (19), we have $\mathscr{E}\mathscr{L}V(t) < 0$, and hence GRN ((14a), (14b)) is asymptotically stable in the mean square sense. □

Remark 4. In the above theorem a DRP approach has been proposed to establish an asymptotic mean square stability criterion for GRN ((14a), (14b)). Both DRP approach and the so-called piecewise analysis method (see, e.g., [20]) divide the delay-varying intervals into some parts with equal length. Then DRP approach enlarges the expectation of weak infinitesimal operator of the same Lyapunov-Krasovskii functional in every subinterval, while the piecewise analysis method constructs different Lyapunov-Krasovskii functional in every subinterval.

Remark 5. Comparing with the Lyapunov-Krasovskii functionals employed in Theorem 3 and [17, Theorem 1], we remove the items $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} \operatorname{trace}[f_1^T(s)Z_2 f_1^T(s)]\,ds\,d\theta$ and $\int_{-\sigma_2}^{-\sigma_1}\int_{t+\theta}^{t} \operatorname{trace}[f_2^T(s)Z_4 f_2^T(s)]\,ds\,d\theta$, which is required in [17]. This will reduce the number of LMI variables to be solved, and hence Theorem 3 requires less computer time than [17, Theorem 1]. Furthermore, it will be shown in the next section that Theorem 3 is certainly less conservative than [17, Theorem 1].

4. Theoretical Comparisons

In this part we will offer a theoretical comparison on conservativeness of Theorem 3 and [17, Theorem 1]. For this reason, we introduce [17, Theorem 1] as follows.

Lemma 6 (see [17, Theorem 1]). *When $L^{-} = 0$ and $L^{+} = K$, GRN ((14a), (14b)) subject to ((4a), (4b)) is asymptotically stable in the mean square sense, if there exist positive definite matrices P_i, R_i $(i = 1, 2)$, Q_j $(j = 1, 2, \ldots, 8)$, and Z_k $(k = 1, 2, \ldots, 6)$, a diagonal positive matrix Λ, matrices S_i, J_i, N_i, M_i, U_i, V_i, L_i, and T_i $(i = 1, 2)$ of appropriate sizes, and positive scalars ρ_1, ρ_2, and ρ_3 such that the following LMIs hold:*

$$\begin{aligned} & P_{1} \leq \rho_{1} I, \qquad Z_{3} \leq \rho_{2} I, \qquad Z_{4} \leq \rho_{3} I, \\ & \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Psi_{i14} \\ \star & -\Omega_{22} & 0 & 0 \\ \star & \star & -\Omega_{33} & 0 \\ \star & \star & \star & -\Psi_{i44} \end{bmatrix} < 0, \qquad i = 1, 2, 3, 4, \end{aligned} \tag{34}$$

where

$$\Psi_{114} = \begin{bmatrix} \tau_{12}\left(1-\alpha\right)S & \tau_{12}\alpha N & \sigma_{12}\left(1-\alpha\right)L & \sigma_{12}\alpha U \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix},$$

$$\Psi_{214} = \begin{bmatrix} \tau_{12}\left(1-\alpha\right)S & \tau_{12}\alpha N & \sigma_{12}T \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix},$$

$$\Psi_{314} = \begin{bmatrix} \tau_{12}J & \sigma_{12}\left(1-\alpha\right)L & \sigma_{12}\alpha U \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix},$$

$$\Psi_{414} = \begin{bmatrix} \tau_{12}J & \sigma_{12}T \\ 0_{2n\times n} & 0_{2n\times n} \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & S_1^T & S_2^T & 0_{n\times 10n} \end{bmatrix}^T,$$

$$N = \begin{bmatrix} N_1^T & N_2^T & 0_{n\times 11n} \end{bmatrix}^T,$$

$$L = \begin{bmatrix} 0_{n\times 7n} & L_1^T & L_2^T & 0_{n\times 4n} \end{bmatrix}^T,$$

$$U = \begin{bmatrix} 0_{n\times 6n} & U_1^T & U_2^T & 0_{n\times 5n} \end{bmatrix}^T,$$

$$T = \begin{bmatrix} 0_{n\times 6n} & T_1^T & 0 & T_2^T & 0_{n\times 4n} \end{bmatrix}^T,$$

$$J = \begin{bmatrix} J_1^T & 0 & J_2^T & 0_{n\times 10n} \end{bmatrix}^T,$$

$$\Psi_{144} = \operatorname{diag}\left(\tau_{12}\left(1-\alpha\right)Z_2, \tau_{12}\alpha Z_2, \sigma_{12}\left(1-\alpha\right)Z_6, \sigma_{12}\alpha Z_6\right),$$

$$\Psi_{244} = \operatorname{diag}\left(\tau_{12}\left(1-\alpha\right)Z_2, \tau_{12}\alpha Z_2, \sigma_{12}Z_6\right),$$

$$\Psi_{344} = \operatorname{diag}\left(\tau_{12}Z_2, \sigma_{12}\left(1-\alpha\right)Z_6, \sigma_{12}\alpha Z_6\right),$$

$$\Psi_{444} = \operatorname{diag}\left(\tau_{12}Z_2, \sigma_{12}Z_6\right),$$

$$\Omega_{11} = \begin{bmatrix} \Sigma_{11} & \Gamma_1^T\phi_1 & \Gamma_2^T\phi_2 \\ \star & -\phi_1 & 0 \\ \star & \star & -\phi_2 \end{bmatrix},$$

$$\Sigma_{11} = \begin{bmatrix} \Theta_1 & \Theta_3 \\ \star & \Theta_2 \end{bmatrix},$$

$$\Omega_{22} = \operatorname{diag}\left(\tau_1 Z_1, \sigma_1 Z_5\right),$$

$$\Omega_{12} = \begin{bmatrix} \tau_1 M & \sigma_1 V \\ 0_{2n\times n} & 0_{2n\times n} \end{bmatrix},$$

$$M = \begin{bmatrix} M_1^T & 0 & M_2^T & 0_{n\times 10n} \end{bmatrix}^T,$$

$$V = \begin{bmatrix} 0_{n\times 6n} & V_1^T & 0 & V_2^T & 0_{n\times 4n} \end{bmatrix}^T,$$

$$\Omega_{13} = \begin{bmatrix} S & N & M & J \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix},$$

$$\Omega_{33} = \operatorname{diag}\left(Z_4, Z_4, Z_3, Z_4\right),$$

$$\Theta_1 = \begin{bmatrix} \Pi_{11} & -N_1 & M_2^T + J_1 & 0 & N_1 - M_1 & -J_1 \\ \star & \Pi_{22} & -S_1 + S_2^T & 0 & N_2 & 0 \\ \star & \star & \Pi_{33} & 0 & -M_2 & -J_2 \\ \star & \star & \star & \Pi_{44} & 0 & 0 \\ \star & \star & \star & \star & \Pi_{55} & 0 \\ \star & \star & \star & \star & \star & -R_1 \end{bmatrix},$$

$$\Theta_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & P_1 W \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D^T P_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \Pi_{11} &= -P_1 A - A^T P_1 + Q_4 + \left(\rho_1 + \tau_1\rho_2 + \tau_{12}\rho_3\right) H_1 \\ &\quad + M_1 + M_1^T + R_1, \\ \Pi_{22} &= \left(1 + \alpha\tau_d\right) Q_1 - \left(1 - \alpha\tau_d\right) Q_2 + S_1 + S_1^T - N_2 - N_2^T, \\ \Pi_{33} &= -\left(1 - \tau_d\right) Q_1 + \left(\rho_1 + \tau_1\rho_2 + \tau_{12}\rho_3\right) H_2 \\ &\quad + J_2 + J_2^T - S_2 - S_2^T, \\ \Pi_{44} &= Q_3 - Q_4, \qquad \Pi_{55} = Q_2 - Q_3, \\ \Gamma_1 &= \begin{bmatrix} 0 & 0 & D & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \Gamma_2 &= \begin{bmatrix} -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \end{bmatrix}, \\ \phi_1 &= \sigma_1 Z_5 + \sigma_{12} Z_6, \qquad \phi_2 = \tau_1 Z_1 + \tau_{12} Z_2, \end{aligned}$$

Θ_2

$$= \begin{bmatrix} \Xi_{11} & -U_1 & V_2^T + T_1 & 0 & U_1 - V_1 & -T_1 & 0 \\ \star & \Xi_{22} & -L_1 + L_2^T & 0 & U_2 & 0 & 0 \\ \star & \star & \Xi_{33} & 0 & -V_2 & -T_2 & \Lambda \\ \star & \star & \star & \Xi_{44} & 0 & 0 & 0 \\ \star & \star & \star & \star & \Xi_{55} & 0 & 0 \\ \star & \star & \star & \star & \star & -R_2 & 0 \\ \star & \star & \star & \star & \star & \star & -2\Lambda K^{-1} \end{bmatrix},$$

$$\begin{aligned} \Xi_{11} &= -P_2 C - C^T P_2 + Q_8 + \left(\rho_1 + \tau_1\rho_2 + \tau_{12}\rho_3\right) H_3 \\ &\quad + V_1 + V_1^T + R_2, \\ \Xi_{22} &= \left(1 + \alpha\sigma_d\right) Q_5 - \left(1 - \alpha\sigma_d\right) Q_6 + L_1 + L_1^T - U_2 - U_2^T, \\ \Xi_{33} &= -\left(1 - \sigma_d\right) Q_5 + \left(\rho_1 + \tau_1\rho_2 + \tau_{12}\rho_3\right) H_4 - L_2 \\ &\quad - L_2^T + T_2 + T_2^T, \\ \Xi_{44} &= Q_7 - Q_8, \\ \Xi_{55} &= Q_6 - Q_7. \end{aligned} \tag{35}$$

In order to show that Theorem 3 is less conservative than [17, Theorem 1], the following propositions are required.

Proposition 7. *Let $\Sigma^T = \Sigma$, J_j, X_j $(j = 1, 2, 3)$, and $W^T = W > 0$ be given real matrices of appropriate sizes. For given scalars $\alpha \in (0,\ 1)$ and $c > 0$, set*

$$\begin{aligned} S_1 &= \begin{bmatrix} (1-\alpha) J_1 & \alpha J_2 \end{bmatrix}, \qquad T_1 = \operatorname{diag}\left((1-\alpha) W, \alpha W\right), \\ S_2 &= J_3, \qquad T_2 = W, \\ \Sigma_0 &= J_1 X_1 + J_2 X_2 + J_3 X_3. \end{aligned} \tag{36}$$

If

$$\begin{bmatrix} \Sigma + \Sigma_0 + \Sigma_0^T & \sqrt{c} S_i \\ \star & -T_i \end{bmatrix} < 0, \quad i = 1, 2, \tag{37}$$

then there exists a (sufficiently large) positive integer s such that

$$\begin{aligned} &\Sigma - \frac{s}{k(1-\alpha)} c^{-1} X_1^T W X_1 - \frac{s}{k\alpha} c^{-1} X_2^T W X_2 \\ &\quad - \frac{s}{s-k+1} c^{-1} X_3^T W X_3 < 0, \quad \forall k \in \langle s \rangle. \end{aligned} \tag{38}$$

Proof. It follows from (37) and the Schur complementary lemma [21] that

$$\Sigma + \Sigma_0 + \Sigma_0^T + c S_i T_i^{-1} S_i^T < 0, \quad i = 1, 2, \tag{39}$$

and hence there exists a (sufficiently large) positive integer s such that

$$\Sigma + \Sigma_0 + \Sigma_0^T + \frac{s+1}{s} c S_i T_i^{-1} S_i^T < 0, \quad i = 1, 2. \tag{40}$$

For an arbitrary but fixed $k \in \langle s \rangle$, one can derive from (36) and (40) that

$$\begin{aligned} &\Sigma + J_1 X_1 + X_1^T J_1^T + \frac{k}{s} c (1-\alpha) J_1 W^{-1} J_1^T \\ &\quad + J_2 X_2 + X_2^T J_2^T + \frac{k}{s} c\alpha J_2 W^{-1} J_2^T \\ &\quad + J_3 X_3 + X_3^T J_3^T + \frac{s-k+1}{s} c J_3 W^{-1} J_3^T < 0. \end{aligned} \tag{41}$$

Since

$$\begin{aligned} &J_1X_1 + X_1^TJ_1^T + \frac{k(1-\alpha)}{s}cJ_1W^{-1}J_1^T + \frac{s}{k(1-\alpha)}c^{-1}X_1^TWX_1 \\ &\quad \geq 0, \\ &J_2X_2 + X_2^TJ_2^T + \frac{k\alpha}{s}cJ_2W^{-1}J_2^T + \frac{s}{k\alpha}c^{-1}X_2^TWX_2 \geq 0, \\ &J_3X_3 + X_3^TJ_3^T + \frac{s-k+1}{s}cJ_3W^{-1}J_3^T + \frac{s}{s-k+1}c^{-1}X_3^TWX_3 \\ &\quad \geq 0, \end{aligned} \tag{42}$$

we obtain from (41) that (38) holds. The proof is completed. □

Proposition 8. *Let $\Omega^T = \Omega$, J_i, L_i, X_i, Y_i $(i = 1, 2, 3)$, $W^T = W > 0$, and $Z^T = Z > 0$ be given real matrices of appropriate sizes and $c_i > 0$ $(i = 1, 2)$ and $\alpha \in (0, 1)$ given scalars. Set*

$$\begin{aligned} S_{11} &= \left[c_1(1-\alpha)J_1 \;\; c_1\alpha J_2 \;\; c_2(1-\alpha)L_1 \;\; c_2\alpha L_2\right], \\ S_{12} &= \left[c_1(1-\alpha)J_1 \;\; c_1\alpha J_2 \;\; c_2L_3\right], \\ S_{21} &= \left[c_1J_3 \;\; c_2(1-\alpha)L_1 \;\; c_2\alpha L_2\right], \\ S_{22} &= \left[c_1J_3 \;\; c_2L_3\right], \\ T_{11} &= \operatorname{diag}\left(c_1(1-\alpha)W, c_1\alpha W, c_2(1-\alpha)Z, c_2\alpha Z\right), \\ T_{12} &= \operatorname{diag}\left(c_1(1-\alpha)W, c_1\alpha W, c_2Z\right), \\ T_{21} &= \operatorname{diag}\left(c_1W, c_2(1-\alpha)Z, c_2\alpha Z\right), \\ T_{22} &= \operatorname{diag}\left(c_1W, c_2Z\right), \\ \Omega_1 &= J_1X_1 + J_2X_2 + J_3X_3, \\ \Omega_2 &= L_1Y_1 + L_2Y_2 + L_3Y_3. \end{aligned} \tag{43}$$

If

$$\begin{bmatrix} \Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T & S_{ij} \\ \star & -T_{ij} \end{bmatrix} < 0, \quad i, j = 1, 2, \tag{44}$$

then there exists a pair of (sufficiently large) positive integers s_1 and s_2 such that

$$\begin{aligned} &\Omega - \frac{s_1}{k(1-\alpha)}c_1^{-1}X_1^TWX_1 - \frac{s_1}{k\alpha}c_1^{-1}X_2^TWX_2 \\ &\quad - \frac{s_1}{s_1-k+1}c_1^{-1}X_3^TWX_3 \\ &\quad - \frac{s_2}{l(1-\alpha)}c_2^{-1}Y_1^TZY_1 - \frac{s_2}{l\alpha}c_2^{-1}Y_2^TZY_2 \\ &\quad - \frac{s_2}{s_2-l+1}c_2^{-1}Y_3^TZY_3 < 0, \quad \forall k \in \langle s_1\rangle,\ l \in \langle s_2\rangle. \end{aligned} \tag{45}$$

Proof. It follows from (44) that

$$\Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + S_{ij}T_{ij}^{-1}S_{ij}^T < 0, \quad i, j = 1, 2, \tag{46}$$

and hence

$$\begin{aligned} &\Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + \delta S_{i1}T_{i1}^{-1}S_{i1}^T \\ &\quad + (1-\delta)S_{i2}T_{i2}^{-1}S_{i2}^T < 0, \quad i = 1, 2,\ \forall\delta \in [0, 1]; \end{aligned} \tag{47}$$

that is,

$$\begin{bmatrix} \Omega_\delta + \Omega_1 + \Omega_1^T & \sqrt{c_1}\widehat{S}_i \\ \star & \widehat{T}_i \end{bmatrix} < 0, \quad i = 1, 2,\ \forall\delta \in [0, 1], \tag{48}$$

where

$$\begin{aligned} \widehat{S}_1 &= \left[(1-\alpha)J_1 \;\; \alpha J_2\right], \qquad \widehat{T}_1 = \operatorname{diag}\left((1-\alpha)W, \alpha W\right), \\ \widehat{S}_2 &= J_3, \qquad \widehat{T}_2 = W, \\ \Omega_\delta &= \Omega + \Omega_2 + \Omega_2^T + \delta c_2\widetilde{S}_1\widetilde{T}_1^{-1}\widetilde{S}_1^T + (1-\delta)c_2\widetilde{S}_2\widetilde{T}_2^{-1}\widetilde{S}_2^T, \\ \widetilde{S}_1 &= \left[(1-\alpha)L_1 \;\; \alpha L_2\right], \qquad \widetilde{T}_1 = \operatorname{diag}\left((1-\alpha)Z, \alpha Z\right), \\ \widetilde{S}_2 &= L_3, \qquad \widetilde{T}_2 = Z. \end{aligned} \tag{49}$$

Applying Proposition 7 to $\Sigma = \Omega_\delta$, $\Sigma_0 = \Omega_1$, $S_i = \widehat{S}_i$, $T_i = \widehat{T}_i$, and $c = c_1$, we obtain that

$$\begin{aligned} &\Omega_\delta - \frac{s_1}{k(1-\alpha)}c_1^{-1}X_1^TWX_1 - \frac{s_1}{k\alpha}c_1^{-1}X_2^TWX_2 - \frac{s_1}{s_1-k+1} \\ &\quad \times c_1^{-1}X_3^TWX_3 < 0, \quad \forall\delta \in [0, 1],\ \forall k \in \langle s_1\rangle,\ \delta \in [0, 1] \end{aligned} \tag{50}$$

for some (sufficiently large) positive integer s_1.

By the Schur complementary lemma, one can easily derive from (50) that

$$\begin{bmatrix} \Omega_k + \Omega_2 + \Omega_2^T & \sqrt{c_2}\widetilde{S}_i \\ \star & -\widetilde{T}_i \end{bmatrix} < 0, \quad i = 1, 2,\ \forall k \in \langle s_1\rangle, \tag{51}$$

where

$$\begin{aligned} \Omega_k &= \Omega - \frac{s_1}{k(1-\alpha)}c_1^{-1}X_1^TWX_1 - \frac{s_1}{k\alpha}c_1^{-1}X_2^TWX_2 \\ &\quad - \frac{s_1}{s_1-k+1}c_1^{-1}X_3^TWX_3. \end{aligned} \tag{52}$$

Again applying Proposition 7 to $\Sigma = \Omega_k$, $\Sigma_0 = \Omega_2$, $S_i = \widetilde{S}_i$, $T_i = \widetilde{T}_i$, and $c = c_2$, one can complete the proof. □

Proposition 9. *Let $p, q \in R^n$ and $M \in R^{n\times n}$ satisfying $M^T = M > 0$. Then*

$$(p+q)^TM(p+q) \leq \frac{1}{\alpha}p^TMp + \frac{1}{1-\alpha}q^TMq, \quad \forall\alpha \in (0, 1). \tag{53}$$

Proof. One has

$$\begin{aligned} &\frac{1}{\alpha}p^TMp + \frac{1}{1-\alpha}q^TMq - (p+q)^TM(p+q) \\ &\quad = \frac{1-\alpha}{\alpha}p^TMp + \frac{\alpha}{1-\alpha}q^TMq - p^TMq - q^TMp \end{aligned}$$

$$= \left(\sqrt{\frac{1-\alpha}{\alpha}}p - \sqrt{\frac{\alpha}{1-\alpha}}q\right)^T M \left(\sqrt{\frac{1-\alpha}{\alpha}}p - \sqrt{\frac{\alpha}{1-\alpha}}q\right) \geq 0. \tag{54}$$

□

Now it is time to show that Theorem 3 is less conservative than [17, Theorem 1] in theory.

Theorem 10. *Set $L^- = 0$ and $L^+ = K$. If the LMIs in* (34) *are feasible, then the LMIs in* (18) *and* (19) *are feasible.*

Proof. Set $\rho = \rho_1$, $U_i = P_i$ $(i = 1, 2)$, $V_j = Q_{5-j}$, $W_j = Q_{9-j}$ $(j = 1, 2, 3, 4)$, $V_5 = R_1$, $W_5 = R_2$, $V_k = Z_{k-5}$, and $W_k = Z_{k-1}$ $(k = 6, 7)$. Then it follows from (34) and the Schur complementary lemma that (18) holds and

$$\Sigma_{11} + \Psi_4 + \tau_1 M V_6^{-1} M^T + \sigma_1 V W_6^{-1} V^T + \widetilde{S}_{ij} \widetilde{T}_{ij} \widetilde{S}_{ij}^T < 0, \quad i, j = 1, 2, \tag{55}$$

where

$$\begin{aligned}
\widetilde{S}_{11} &= \left[\tau_{12}(1-\alpha)S \;\; \tau_{12}\alpha N \;\; \sigma_{12}(1-\alpha)L \;\; \sigma_{12}\alpha U\right], \\
\widetilde{S}_{12} &= \left[\tau_{12}(1-\alpha)S \;\; \tau_{12}\alpha N \;\; \sigma_{12}T\right], \\
\widetilde{S}_{21} &= \left[\tau_{12}J \;\; \sigma_{12}(1-\alpha)L \;\; \sigma_{12}\alpha U\right], \\
\widetilde{S}_{22} &= \left[\tau_{12}J \;\; \sigma_{12}T\right], \\
\widetilde{T}_{11} &= \operatorname{diag}\left(\tau_{12}(1-\alpha)V_7, \tau_{12}\alpha V_7, \sigma_{12}(1-\alpha)W_7, \sigma_{12}\alpha W_7\right), \\
\widetilde{T}_{12} &= \operatorname{diag}\left(\tau_{12}(1-\alpha)V_7, \tau_{12}\alpha V_7, \sigma_{12}W_7\right), \\
\widetilde{T}_{21} &= \operatorname{diag}\left(\tau_{12}V_7, \sigma_{12}(1-\alpha)W_7, \sigma_{12}\alpha W_7\right), \\
\widetilde{T}_{22} &= \operatorname{diag}\left(\tau_{12}V_7, \sigma_{12}W_7\right),
\end{aligned} \tag{56}$$

and $S, N, L, U, T, J, M, V, \Psi_4$, and Σ_{11} are defined as noted previously.

By simple computation one can derive from (55) that

$$\begin{bmatrix} \widetilde{\Omega} + \widetilde{\Omega}_1 + \widetilde{\Omega}_1^T + \widetilde{\Omega}_2 + \widetilde{\Omega}_2^T & \widetilde{S}_{ij} \\ \star & -\widetilde{T}_{ij} \end{bmatrix} < 0, \quad i, j = 1, 2, \tag{57}$$

where

$$\begin{aligned}
\widetilde{\Omega}_1 &= S(e_2 - e_3) + N(e_5 - e_2) + J(e_3 - e_6), \\
\widetilde{\Omega}_2 &= L(e_8 - e_9) + U(e_{11} - e_8) + T(e_9 - e_{12}), \\
\widetilde{\Omega} &= \widetilde{\Psi} + V(e_7 - e_{11}) + (e_7 - e_{11})^T V^T + M(e_1 - e_5) \\
&\quad + (e_1 - e_5)^T M^T + \tau_1 M V_6^{-1} M^T + \sigma_1 V W_6^{-1} V^T, \\
\widetilde{\Psi} &= \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5,
\end{aligned} \tag{58}$$

and Ψ_i $(i = 1, 2, \ldots, 5)$ are defined as in (19).

By Proposition 8, there exists a pair of (sufficiently large) positive integers s_1 and s_2 such that

$$\widetilde{\Omega} + \Psi_k + \Psi_l < 0, \quad \forall k \in \langle s_1 \rangle,\ l \in \langle s_2 \rangle, \tag{59}$$

where Ψ_k and Ψ_l are defined as in (19). This, together with Proposition 9, implies that

$$\begin{aligned}
\widetilde{\Omega} &\geq \widetilde{\Psi} - \tau_1^{-1}(e_1 - e_5)^T V_6 (e_1 - e_5) \\
&\quad - \sigma_1^{-1}(e_7 - e_{11})^T W_6 (e_7 - e_{11}) \geq \widetilde{\Psi} + \Psi_6,
\end{aligned} \tag{60}$$

and hence the LMIs in (19) are feasible. The proof is completed. □

Remark 11. In Theorem 10, it has been theoretically investigated that Theorem 3 is certainly less conservative than [17, Theorem 1]. On the other hand, the numbers of LMI variables to be solved in Theorem 3 and [17, Theorem 1] are $8n^2 + 9n + 1$ and $25n^2 + 10n + 3$, respectively, which implies that Theorem 3 will require less computer time than [17, Theorem 1].

5. An Illustrative Example

In this section, a numerical example is given to illustrate the effectiveness and less conservativeness of our theoretical results.

We consider a delayed GRN with stochastic disturbances, with the parameters described as

$$\begin{aligned}
A &= \operatorname{diag}(3, 3, 3), \qquad C = \operatorname{diag}(2.5, 2.5, 2.5), \\
D &= \operatorname{diag}(0.8, 0.8, 0.8), \\
W &= \begin{bmatrix} 0 & 0 & -2.5 \\ -2.5 & 0 & 0 \\ 0 & -2.5 & 0 \end{bmatrix}.
\end{aligned} \tag{61}$$

Let $G_1 = G_2 = G_3 = G_4 = 0.4I$ and $f(x) = x^2/(1 + x^2)$, which means that $L^- = 0$ and $L^+ = K = 0.65I$. When $s_1 = s_2 = 10$, for $\tau_1 = \sigma_1 = 1$, $\tau_2 = \sigma_2 = 10$, $\tau_d = \sigma_d = 0.5$, and $\alpha = 0.1$, by using the MATLAB Toolbox, inequalities (18) and (19) have feasible solutions. By the theorem, the system is asymptotically stable in the mean square sense. Here, some of the solution matrices are given as follows:

$$\begin{aligned}
U_1 &= \begin{bmatrix} 3.9297 & 0.0000 & 0.0000 \\ 0.0000 & 3.9297 & 0.0000 \\ 0.0000 & 0.0000 & 3.9297 \end{bmatrix}, \\
U_2 &= \begin{bmatrix} 7.8311 & -0.0000 & -0.0000 \\ -0.0000 & 7.8311 & -0.0000 \\ -0.0000 & -0.0000 & 7.8311 \end{bmatrix},
\end{aligned} \tag{62}$$

and show the trajectories of the $x(t)$ and $y(t)$ in Figure 1.

When $\tau_1 = \sigma_1 = 1$ and $s_1 = s_2 = 10$, the maximal allowable upper bounds of $\tau_2 = \sigma_2$ for different values of $\tau_d = \sigma_d$ obtained by Theorem 3 and [17, Theorem 1] are shown in Table 1. It can be seen from Table 1 that Theorem 3 is the less conservative than [17, Theorem 1].

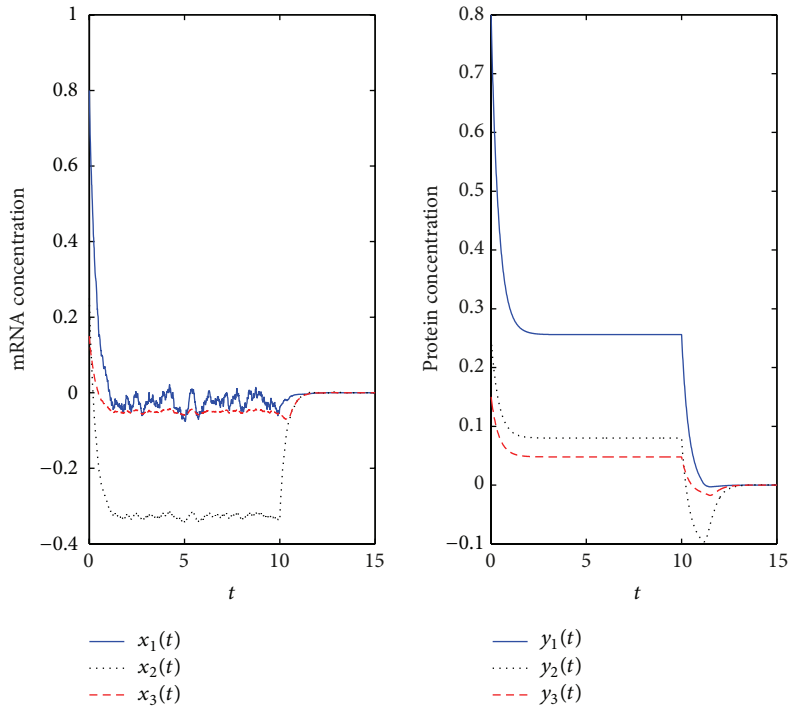

Figure 1: State trajectories of the GRN in the considered example.

Table 1: The maximal allowable bounds of $\tau_2 = \sigma_2$ for different $\tau_d = \sigma_d$.

$\tau_d = \sigma_d$	0.8	1	1.5
[17, Theorem 1]	1.5602	1.5500	1.5450
Theorem 3	2.095	2.053	2.047

6. Conclusions

In this paper, the stability problem for a class of GRNs with time-varying delays and stochastic disturbances has been investigated. By constructing an appropriate Lyapunov-Krasovskii functional and proposing a DRP approach, a mean square stability criterion is given in terms of LMIs, which can be easily tested by the LMI Toolbox of MATLAB. It is theoretically shown that the proposed result is less conservative than [17, Theorem 1]. Moreover, the number of LMI variables in this paper is more less than the one in [17, Theorem 1]. A numerical example has been provided to illustrate the theoretical results given in this paper.

Extending the idea of this paper to other system models, including singular delayed systems [19, 22–24], stochastic systems [25], Markovian jump systems [20, 26], and genetic regulatory networks [5, 10, 27], is under consideration.

Conflict of Interests

The authors declare that there is no commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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Research Article

The Identification of Convex Function on Riemannian Manifold

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The necessary and sufficient condition of convex function is significant in nonlinear convex programming. This paper presents the identification of convex function on Riemannian manifold by use of Penot generalized directional derivative and the Clarke generalized gradient. This paper also presents a method for judging whether a point is the global minimum point in the inequality constraints. Our objective here is to extend the content and proof the necessary and sufficient condition of convex function to Riemannian manifolds.

1. Introduction

Manifold is the space with the local property of Euclidean space. We often judge the warp of the space by measuring its space. The standard of the measurement is called metric. Metric is an intrinsic property and the space with metric is called Riemannian space. Riemann manifold is a differential manifold which has Riemannian metric. Manifold learning attempts to obtain the intrinsic structure of non-linearly distributed data, which can be used in non-linear dimensionality reduction (NLDR). We can find the meaningful low dimensional structure hidden in the high-dimensional observation data by the non-linear dimension reduction of high data space [1]. Expanding the optimization based on linear space to the nonlinear space (such as Riemannian manifold) is a hot topic in this research field [2]. Convex function is a kind of special function that has a broad application, especially has a wide range of applications in the areas of optimization and mathematical programming, cybernetics, and other fields. Since the useful data is usually not a linear combination of the features in the real world that the nonlinear convex programming problem has become a new research subject [3–5]. In order to have a more in-depth study of nonsmooth and its related problems, several important tools for analyzing nonsmooth problem have been extended from European space to Riemannian Manifold.

In 2004, Ledyacv and Zhu established the concept and algorithm of subdifferential of nonsmooth function in Riemannian manifold and proved that the solution of Hamilton-Jacobi equation defined on Riemannian manifold is unique. They also discussed the concept and variational principle of approximate subdifferential in Riemannian manifold [6, 7]. In 2005, Azagra and others discussed the J. Ferrera subdifferential which was defined on finite dimensional Riemannian manifold and relevant limit subdifferential and analyzed the issues of differential inclusion which were defined on differential manifold by using the concept of generalized derivative [8, 9]. Liu and others expanded (MP) problem from linear space to differential manifold and Fritz John necessary optimality conditions on Riemann manifold was given [10]. In 2008, Gang and Sanyang solved the nondifferentiable multiobjective optimization problems with equality and inequality constraints from Euclidean space to Riemannian manifolds and derived the Fritz-John necessary conditions with generalized gradient formula for weak Pareto optimal solutions from Ekeland variational principle [11]. Wang et al. established the semilocal convergence of sequences which was generated by the Gauss-Newton method (with quasiregular initial points) for the convex composite function $h \circ F$ on Riemannian manifold [12] by using the majorizing function technique. Bento et al. presented a steepest descent method with Armijo's rule for multicriteria optimization in

the Riemannian context. With assuming quasiconvexity of the multicriteria function and nonnegative curvature of the Riemannian manifold, they proved full convergence of the sequence to a critical Pareto point [13].

Convex programming is of great significance in the study of nonlinear programming theory. Based on the above research results, we will parallel extend the identification of convex function and the optimality conditions of constraint problems from linear space to Riemannian manifold in order to solve the convex programming problem in Riemannian manifold. This paper is set out as follows. In Section 2, we briefly review some preliminary concepts. In Section 3, we introduce the identification of convex function on Riemannian manifold. We make some concluding remarks and suggest future research in this area in Section 4.

2. Preliminaries

Let R^n be an n-dimensional Euclidean space; the ith coordinate of point $p \in R^n$ is denoted by $(p)^i$; that is $(\,)^i$ is the ith coordinate function of R^n.

Definition 1 (see [14]). Let M be a Hausdorff topological space. If every point p of M has an open neighborhood $U \subset M$, such that an open subset of U and n-dimensional Euclidean space is homeomorphous, then M is called an n-dimensional topological manifold.

Definition 2 (see [10]). Let $f(x)$ be a real-valued function defined on Riemannian manifold, $x_0 \in U \subset M$, and let (U, φ) be a coordinate plot which contains x_0. If for all $x, y \in U$, there exists a constant L such that

$$\left|f(y) - f(x)\right| \le L\left|\varphi(y) - \varphi(x)\right|. \tag{1}$$

Then, $f(x)$ is a function which meets the local Lipschitz condition in the neighborhood of x_0.

Definition 3 (see [10]). Let function $f : M \to R$ be a meet local Lipschitz condition in the neighborhood of the point $x \in M$ and (U, φ) be a coordinate plot which contains x. The *Clarke* generalized directional derivative of f at x along the direction of $v \in T_xM$ is denoted by

$$\begin{aligned} &f^{\circ}(x; v) \\ &= \lim_{y \to \varphi(x)} \sup_{t \downarrow 0} \frac{f \circ \varphi^{-1}\left(y + t\varphi_{*_x}(v)\right) - f \circ \varphi^{-1}(y)}{t} \\ &= \left(f \circ \varphi^{-1}\right)^{\circ}\left(\varphi(x); \varphi_{*_x}(v)\right), \end{aligned} \tag{2}$$

where $t \to 0$.

Definition 4 (see [10]). Let $f(x)$ be a function denoted in Riemannian manifold (M, g) and meet local Lipschitz condition in the neighborhood of x. Then, the element of the subset of cotangent space ${T_x}^*M$ of x

$$\partial f(x) = \left\{\xi \in {T_x}^*M \mid f^{\circ}(x; v) \ge (\xi, v),\ \forall v \in T_xM\right\} \tag{3}$$

is said to be Clarke generalized gradient in which f is at x.

The norm of generalized gradient is denoted by

$$\left\|\xi\right\|_{\varphi} := \sup\left\{(\xi, v), v \in T_xM, \|v\| \le 1\right\}. \tag{4}$$

Definition 5 (see [15]). Let $f : M \to R$ meet local Lipschitz condition in the neighborhood of $x \in M$ and let (U, φ) be a coordinate plot which contains x. We define Penot generalized directional derivative of f at x along the direction of $v \in T_xM$ as follows:

$$\begin{aligned} &\underline{d} f(x; v) \\ &= \liminf_{u \to v} \liminf_{t \downarrow 0} \frac{f \circ \varphi^{-1}\left(y + t\varphi_{*_x}(u)\right) - f \circ \varphi^{-1}(y)}{t} \\ &= \underline{d}\left(f \circ \varphi^{-1}\right)\left(\varphi(x); \varphi_{*_x}(v)\right), \\ &\overline{d} f(x; v) \\ &= \limsup_{u \to v} \limsup_{t \downarrow 0} \frac{f \circ \varphi^{-1}\left(y + t\varphi_{*_x}(u)\right) - f \circ \varphi^{-1}(y)}{t} \\ &= \overline{d}\left(f \circ \varphi^{-1}\right)\left(\varphi(x); \varphi_{*_x}(v)\right), \\ &\underline{d}^2 f(x; v; w) \\ &= \liminf_{u \to w} \liminf_{t \downarrow 0} \left(f \circ \varphi^{-1}\left(y + t\varphi_{*_x}(v) + t^2\varphi_{*_x}(u)\right)\right. \\ &\qquad \left. - f \circ \varphi^{-1}(y) - t\underline{d} f(x; v)\right)\left(t^{-2}\right), \\ &\overline{d}^2 f(x; v; w) \\ &= \limsup_{u \to w} \limsup_{t \downarrow 0} \left(f \circ \varphi^{-1}\left(y + t\varphi_{*_x}(v) + t^2\varphi_{*_x}(u)\right)\right. \\ &\qquad \left. - f \circ \varphi^{-1}(y) - t\overline{d} f(x; v)\right)\left(t^{-2}\right). \end{aligned} \tag{5}$$

If $\underline{d} f(x; v) = \overline{d} f(x; v)$, we define $df(x; v)$ as Penot generalized directional derivative of f at x along the direction of $v \in T_xM$.

If $\underline{d}^2 f(x; v; \omega) = \overline{d}^2 f(x; v; \omega)$, we define $d^2 f(x; v; \omega)$ to represent second-order Penot generalized directional derivative of f at x along the direction of $v \in T_xM$.

If both $df(x; v)$ and $d^2 f(x; v; \omega)$ exist, then

$$\begin{aligned} &d^2 f(x; v; \omega) \\ &= \lim_{y \to \varphi(x), t \downarrow 0} \left(f \circ \varphi^{-1}\left(y + t\varphi_{*_x}(v) + t^2\varphi_{*_x}(\omega)\right)\right. \\ &\qquad \left. - f \circ \varphi^{-1}(y) - t df(x; v)\right)\left(t^{-2}\right). \end{aligned} \tag{6}$$

3. The Identification of Convex Function on Riemannian Manifold

Theorem 6 (first-order necessary and sufficient condition of convex function). *Let M be an m-dimensional Riemannian manifold and let U be a geodesic convex set, where $U \subset M$. If $f : U \to R$ meets local Lipschitz condition in U, then the necessary*

and sufficient condition that f is geodesic convex function in U is as follows. For $\forall x, y \in U$, there exists

$$f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) \geq f \circ \varphi^{-1}(y) + \nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x). \tag{7}$$

Proof (necessary condition). For $\forall t \in [0,1]$, there exists

$$\varphi^{-1}(y) + t\left[\varphi^{-1}\left(y + \varphi_{*_x}(v)\right) - \varphi^{-1}(y)\right] = t\varphi^{-1}\left(y + \varphi_{*_x}(v)\right) + (1-t)\varphi^{-1}(y) \in U. \tag{8}$$

Then, according to the first-order Taylor expansion,

$$f \circ \left(\varphi^{-1}(y) + t\left[\varphi^{-1}\left(y + \varphi_{*_x}(v)\right) - \varphi^{-1}(y)\right]\right) = f \circ \varphi^{-1}(y) + t\nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x) + o(t). \tag{9}$$

Since f is geodesic convex function in U, then

$$f \circ \left(\varphi^{-1}(y) + t\left[\varphi^{-1}\left(y + \varphi_{*_x}(v)\right) - \varphi^{-1}(y)\right]\right) = f \circ \left(t\varphi^{-1}\left(y + \varphi_{*_x}(v)\right) + (1-t)\varphi^{-1}(y)\right) \leq tf \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) + (1-t) f \circ \varphi^{-1}(y). \tag{10}$$

Then, from two simultaneous equations of (9) and (10), we get

$$f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) + (1-t) f \circ \varphi^{-1}(y) \geq f \circ \varphi^{-1}(y) + t\nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x) + o(t); \tag{11}$$

that is

$$f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) \geq f \circ \varphi^{-1}(y) + \nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x) + \frac{o(t)}{t}. \tag{12}$$

It follows from $t \to 0^+$ that $(o(t)/t) \to 0$; hence,

$$f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) \geq f \circ \varphi^{-1}(y) + \nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x). \tag{13}$$

Sufficient Condition. For $\forall t \in [0,1]$, let $z = tx + (1-t)y$.

From the known conditions it follows that

$$\begin{aligned} f \circ \varphi^{-1}(y) &\geq f \circ \varphi^{-1}\left(y + (1-t)\varphi_{*_x}(v)\right) + \nabla\left(f \circ \varphi^{-1}\right)\left(y + (1-t)\varphi_{*_x}(v)\right)^T (x - z), \\ f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) &\geq f \circ \varphi^{-1}\left(y + (1-t)\varphi_{*_x}(v)\right) + \nabla\left(f \circ \varphi^{-1}\right)\left(y + (1-t)\varphi_{*_x}(v)\right)^T (y - z). \end{aligned} \tag{14}$$

Hence,

$$\begin{aligned} tf \circ \varphi^{-1}(y) &\geq tf \circ \varphi^{-1}\left(y + (1-t)\varphi_{*_x}(v)\right) + t\nabla\left(f \circ \varphi^{-1}\right)\left(y + (1-t)\varphi_{*_x}(v)\right)^T (x - z), \\ (1-t) f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) &\geq (1-t) f \circ \varphi^{-1}\left(y + (1-t)\varphi_{*_x}(v)\right) + (1-t)\nabla\left(f \circ \varphi^{-1}\right)\left(y + (1-t)\varphi_{*_x}(v)\right)^T (y - z). \end{aligned} \tag{15}$$

Then, from (15) it follows that

$$tf \circ \varphi^{-1}(y) + (1-t) f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) \geq f \circ \varphi^{-1}\left(y + (1-t)\varphi_{*_x}(v)\right) + \nabla\left(f \circ \varphi^{-1}\right)\left(y + (1-t)\varphi_{*_x}(v)\right)^T \left(tx + (1-t)y - z\right). \tag{16}$$

Since $z = tx + (1-t)y$, then

$$\begin{aligned} tf \circ \varphi^{-1}(y) + (1-t) f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) &\geq f \circ \varphi^{-1}\left(y + (1-t)\varphi_{*_x}(v)\right) \\ &= f \circ \left(t\varphi^{-1}(y) + (1-t)\varphi^{-1}\left(y + \varphi_{*_x}(v)\right)\right). \end{aligned} \tag{17}$$

Thus, f is geodesic convex function in U. □

Theorem 7 (first-order necessary and sufficient condition of strictly convex function). *Let M be an m-dimensional Riemannian manifold and U be a geodesic convex set. Let $U \subset M$, $f : U \to R$ meet local Lipschitz condition in U, and then the necessary and sufficient condition that f is geodesic strictly convex function in U is as follows. For $\forall x, y \in U$, there exists*

$$f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) > f \circ \varphi^{-1}(y) + \nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x). \tag{18}$$

Proof (necessary condition). In a similar way with proof of Theorem 6, we get

$$f \circ \varphi^{-1}\left(y + \varphi_{*_x}(v)\right) > f \circ \varphi^{-1}(y) + \nabla\left(f \circ \varphi^{-1}\right)(y)^T (y - x). \tag{19}$$

Sufficient Condition. Since f is a strictly convex function, then f is a convex function. For any two different points x, y in which $x, y \in U$, there exists $z = (1/2)x + (1/2)y$, and then $z \in U$.

From Theorem 6 it follows that

$$\begin{aligned} & f\circ\varphi^{-1}\left(y+\frac{1}{2}\varphi_{*_x}(v)\right) \\ & \geq f\circ\varphi^{-1}(y)+\nabla\left(f\circ\varphi^{-1}\right)(y)^T(z-x) \\ & = f\circ\varphi^{-1}(y)+\nabla\left(f\circ\varphi^{-1}\right)(y)^T\left(\frac{1}{2}x+\frac{1}{2}y-x\right) \\ & = f\circ\varphi^{-1}(y)+\frac{1}{2}\nabla\left(f\circ\varphi^{-1}\right)(y)^T(y-x). \end{aligned} \tag{20}$$

Since f is a strictly convex function, then

$$\begin{aligned} & f\circ\varphi^{-1}\left(y+\frac{1}{2}\varphi_{*_x}(v)\right) \\ & = f\circ\left(\frac{1}{2}\varphi^{-1}(y)+\frac{1}{2}\varphi^{-1}\left(y+\varphi_{*_x}(v)\right)\right) \\ & < \frac{1}{2}f\circ\varphi^{-1}(y)+\frac{1}{2}f\circ\varphi^{-1}\left(y+\varphi_{*_x}(v)\right). \end{aligned} \tag{21}$$

Then, it follows from (20) and (21) that

$$\begin{aligned} & \frac{1}{2}f\circ\varphi^{-1}(y)+\frac{1}{2}f\circ\varphi^{-1}\left(y+\varphi_{*_x}(v)\right) \\ & > f\circ\varphi^{-1}(y)+\frac{1}{2}\nabla\left(f\circ\varphi^{-1}\right)(y)^T(y-x); \end{aligned} \tag{22}$$

that is,

$$\begin{aligned} & f\circ\varphi^{-1}\left(y+\varphi_{*_x}(v)\right) \\ & > f\circ\varphi^{-1}(y)+\nabla\left(f\circ\varphi^{-1}\right)(y)^T(y-x). \end{aligned} \tag{23}$$

□

Theorem 8 (second-order necessary and sufficient condition of convex function). *Let M be an m-dimensional Riemannian manifold and U be a geodesic convex set. Let $U \subset M$, $f : U \to R$ meet local Lipschitz condition in U and let (U,φ) be a coordinate plot which contains x, and second-order Penot generalized directional derivative of f at x along the direction of $v \in T_xM$ exists. Then, the necessary and sufficient condition that f is geodesic convex function in U is as follows. For any $x \in U$, there exists $d^2 f(x;v,w) \geq 0$.*

Proof (necessary condition). Assume that for any $x \in U$, there exists $d^2 f(x;v,w) \geq 0$. For any two different points x, y, from Taylor expansion, it follows that

$$\begin{aligned} & f\circ\varphi^{-1}\left(y+\varphi_{*_x}(v)\right) \\ & = f\circ\varphi^{-1}(y)+\nabla\left(f\circ\varphi^{-1}\right)(y)^T(y-x) \\ & \quad +\frac{1}{2}(y-x)^T\nabla^2\left(f\circ\varphi^{-1}\right)(z)(y-x). \end{aligned} \tag{24}$$

Note that $z = tx + (1-t)y$, $0 \leq t \leq 1$.

Since U is a geodesic convex set, then $z \in U$. From the known conditions it follows that

$$d^2 f(x;v,w) \geq 0. \tag{25}$$

Hence,

$$(y-x)^T\nabla^2\left(f\circ\varphi^{-1}\right)(z)(y-x) \geq 0, \tag{26}$$

so that

$$\begin{aligned} & f\circ\varphi^{-1}\left(y+\varphi_{*_x}(v)\right) \\ & \geq f\circ\varphi^{-1}(y)+\nabla\left(f\circ\varphi^{-1}\right)(y)^T(y-x). \end{aligned} \tag{27}$$

Therefore, f is geodesic convex function in U according to Theorem 7.

Sufficient Condition. Since U is a geodesic convex set, thus for any $x \in U$ and any given nonzero vector l, there exists a sufficiently small positive number t and thus $x + tl \in U$; it follows from Taylor expansion that

$$\begin{aligned} f\circ\varphi^{-1}(x+tl) = {} & f\circ\varphi^{-1}(x)+t\nabla\left(f\circ\varphi^{-1}\right)(x)^T l \\ & +\frac{t^2}{2}l^T\nabla^2\left(f\circ\varphi^{-1}\right)(\overline{x})l+o\left(t^2\right). \end{aligned} \tag{28}$$

Since f is geodesic convex function in U, using Theorem 7, we have

$$f\circ\varphi^{-1}(x+tl) \geq f\circ\varphi^{-1}(x)+t\nabla\left(f\circ\varphi^{-1}\right)(x)^T l. \tag{29}$$

Hence,

$$\frac{t^2}{2}l^T\nabla^2\left(f\circ\varphi^{-1}\right)(\overline{x})l+o\left(t^2\right) \geq 0. \tag{30}$$

Let $t \to 0^+$; then,

$$\frac{o\left(t^2\right)}{t^2} \longrightarrow 0. \tag{31}$$

Thus,

$$l^T\nabla^2\left(f\circ\varphi^{-1}\right)(\overline{x})l \geq 0. \tag{32}$$

That is, for any $x \in U$, there exists $d^2 f(x;v,w) \geq 0$. □

Theorem 9 (second-order necessary and sufficient condition of strictly convex function). *Let M be an m-dimensional Riemannian manifold and U be a geodesic convex set. Let $U \subset M$, $f : U \to R$ meet local Lipschitz condition in U and let (U,φ) be a coordinate plot which contains x and second-order Penot generalized directional derivative of f at x along the direction of $v \in T_xM$ exists. Then, the necessary and sufficient condition that f is geodesic strictly convex function in U is as follows. For any $x \in U$, there exists $d^2 f(x;v,w) > 0$.*

Proof. It is similar to the proof of Theorem 8. □

Definition 10. For the mathematical programming problem

$$\begin{aligned} \min \quad & f(x) \\ \text{st.} \quad & g_i(x) \leq 0, \quad i \in I \\ & x \in M, \end{aligned} \tag{33}$$

where $f : M \to R$, $g : M \to R^l$ are geodesic convex functions in U, U is an open subset of M, (U, φ) is a coordinate plot which contains x, and the feasible region of the problem is denoted by

$$D := \{x \in M \mid g_i(x) \le 0,\ i \in I\}. \tag{34}$$

Theorem 11. *Let M be an m-dimensional Riemannian manifold and let x^* be the optimal solution of the problem. If $f : M \to R$, $g_i : M \to R$, $i \in I$ meet local Lipschitz condition in the neighborhood of $x^* \in M$ and g_i is continuous in x^*, where $i \notin I$, or the vector group $\nabla(g_i \circ \varphi^{-1})(\varphi(x^*))$ is linearly independent, where $i \in I$, then there exists constant u_i $(i \in I)$, such that*

$$\nabla\left(f \circ \varphi^{-1}\right)(\varphi(x^*)) + \sum_{i \in I} u_i \nabla\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*)) = 0, \qquad u_i \ge 0, \quad i \in I. \tag{35}$$

In particular, if g_i meets local Lipschitz condition in the neighborhood of $x^* \in M$, where $i \notin I$, then

$$\nabla\left(f \circ \varphi^{-1}\right)(\varphi(x^*)) + \sum_{i=1}^{m} u_i \nabla\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*)) = 0, \qquad u_i\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*)) = 0, \quad i = 1, 2, \ldots, m, \qquad u_i \ge 0, \quad i = 1, 2, \ldots, m. \tag{36}$$

Theorem 11 is called Kuhn-Tucker theorem or K-T theorem for short. The necessary condition of the minimum point is called K-T condition and the point which meets the K-T condition is called K-T point.

Theorem 12 (sufficient condition of the minimum point). *Let the minimization problem* (33) *be a convex programming problem and let x^* be a K-T point. Then, x^* is the global minimum point.*

Proof. For $\forall x$, there exists

$$g_i \circ \varphi^{-1}(\varphi(x)) \le 0. \tag{37}$$

Since $g_i \circ \varphi^{-1}(\varphi(x^*))$ is geodesic convex function, then

$$g_i \circ \varphi^{-1}(\varphi(x)) \ge g_i \circ \varphi^{-1}(\varphi(x^*)) + \nabla\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*))^T (x - x^*), \qquad i = 1, 2, \ldots, m. \tag{38}$$

For $i \in I$, there exists

$$g_i \circ \varphi^{-1}(\varphi(x^*)) = 0. \tag{39}$$

Then,

$$\nabla\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*))^T (x - x^*) \le 0. \tag{40}$$

Since x^* is a K-T point, then

$$\nabla\left(f \circ \varphi^{-1}\right)(\varphi(x^*)) = -\sum_{i \in I} u_i \nabla\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*)), \qquad u_i \ge 0, \quad i \in I,$$
$$\nabla\left(f \circ \varphi^{-1}\right)\left(\varphi(x^*)^T\right)(x - x^*) = -\sum_{i \in I} u_i \nabla\left(g_i \circ \varphi^{-1}\right)(\varphi(x^*))^T (x - x^*) \ge 0. \tag{41}$$

Since $f \circ \varphi^{-1}(\varphi(x))$ is geodesic convex function, then

$$f \circ \varphi^{-1}(\varphi(x)) \ge f \circ \varphi^{-1}(\varphi(x^*)) + \nabla\left(f \circ \varphi^{-1}\right)\left(\varphi(x^*)^T\right)(x - x^*), \qquad f \circ \varphi^{-1}(\varphi(x)) \ge f \circ \varphi^{-1}(\varphi(x^*)). \tag{42}$$

Thus, x^* is the global minimum point. □

4. Conclusions

In this paper, we introduce the classical Clarke generalized directional derivative and generalized gradient. Then, we extend the identification of convex function and prove the theorem of the first-order necessary and sufficient condition of convex function and strictly convex function to Riemannian manifolds by use of Penot generalized directional derivative. We have also given the second-order necessary and sufficient condition of convex function and strictly convex function and the Kuhn-Tucker theorem and sufficient condition of the minimum point of the inequality constrained optimization problems. In this way, we can do more research about optimization method on differential manifold. In the future, the theorem and the application of the equality constrained optimization problems and the inequality constrained optimization problems will be explored.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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