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Completely Monotonic and Related Functions: Their Applications

Guest Editors: Senlin Guo, Andrea Laforgia, Necdet Batir, and Qiu-Ming Luo

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Editorial Board

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Editorial

Completely Monotonic and Related Functions: Their Applications

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Completely monotonic and related functions are important function classes in mathematical analysis. It was Bernstein [1] who in 1914 first introduced the notion of *completely monotonic function*. This year we celebrate its 100th anniversary. In 1921, Hausdorff [2] gave the notion of *completely monotonic sequence*, which is related to the notion of completely monotonic function. Widder [3] in 1931 introduced the notion of *minimal completely monotonic sequence*. The terminology: *logarithmically completely monotonic function*, was first used in [4] in 1988. In 1989, the authors [5] introduced the notion of *strongly completely monotonic function*. In 2009, the authors [6] introduced the notion of *strongly logarithmically completely monotonic function*. Also, the terminology: *almost strongly completely monotonic function*, was introduced in [6] in order to simplify the statement of the main results of the article [6]. In 2012, the terminology of *almost completely monotonic function* was defined and used in [7] in order to simplify the statement of the main results of the article [7]. Researchers did a lot of investigations, in both theory and applications, on these functions. More recently, for example, the authors [6] showed that a strongly logarithmically completely monotonic function must be almost strongly completely monotonic. Also, in [6], the following fact that a strongly logarithmically completely monotonic function cannot be strongly completely monotonic, or, equivalently, a strongly completely monotonic function cannot be strongly logarithmically completely monotonic was established. Figure 1 illustrates the inclusion relationships among these classes of functions on $(0, \infty)$. In these inclusion relationships, the fact that a logarithmically completely monotonic function must be completely monotonic was, in fact, first proved in [8] although the author did not use the terminology of *logarithmically completely monotonic function*; all others are from [6] or directly from their definitions. Also note that in [6] counter examples were presented to show that some function classes are not included in some other function classes or to show that the intersection sets of two function classes are not empty.

We invited investigators to contribute original research articles as well as comprehensive review articles to this special issue which will stimulate the continuing efforts to understand these functions. This special issue mainly focuses on the applications, in all fields, of completely monotonic and related functions. The topics included in this special issue are as follows:

(i) completely monotonic functions and their applications,

(ii) almost completely monotonic functions and their applications,

(iii) logarithmically completely monotonic functions and their applications,

(iv) strongly logarithmically completely monotonic functions and their applications,

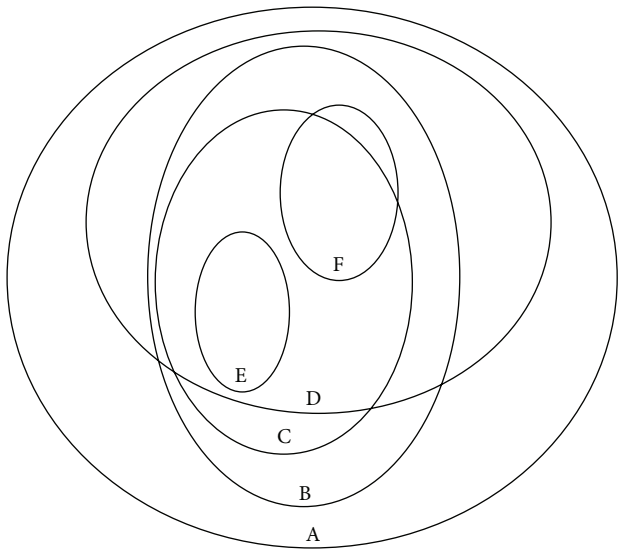

Figure 1: Inclusion relationships among classes of completely monotonic and related functions on $(0, \infty)$. (A) Almost completely monotonic function class. (B) Completely monotonic function class. (C) Logarithmically completely monotonic function class. (D) Almost strongly completely monotonic function class. (E) Strongly logarithmically completely monotonic function class. (F) Strongly completely monotonic function class.

(v) strongly completely monotonic functions and their applications,

(vi) almost strongly completely monotonic functions and their applications,

(vii) absolutely monotonic functions and their applications,

(viii) Bernstein functions and their applications,

(ix) completely convex functions and their applications,

(x) moment sequences and their applications,

(xi) Laplace (Stieltjes) transforms associated with completely monotonic or related functions,

(xii) interpolation or fitting of data by completely monotonic or related functions,

(xiii) approximation of (by) completely monotonic or related functions,

(xiv) analytic inequalities associated with completely monotonic or related functions,

(xv) numerical methods for completely monotonic or related functions, explicit and implicit,

(xvi) ordinary, partial, and fractional differential equations and difference equations related to completely monotonic or related functions,

(xvii) mathematical modeling using completely monotonic or related functions.

Dozens of manuscripts were received for this special issue, but only a few peer-reviewed, high-quality articles were published. We hope that the interested readers will continue to investigate the applications of completely monotonic and related functions in probability and statistics, numerical and asymptotic analysis, statistical physics, physical chemistry, and so forth.

Acknowledgments

The guest editors would like to express their gratitude to all authors who submitted their papers for possible publication in this special issue and to the reviewers whose comments and suggestions on the manuscripts were important to them. They also would like to thank the editorial board members for their support to this special issue.

Senlin Guo
Andrea Laforgia
Necdet Batir
Qiu-Ming Luo

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Research Article

Existence of Solutions for a Baby-Skyrme Model

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The existence of the energy-minimizing solutions for a baby-Skyrme model on the sphere is proved using variational method. Some properties of the solutions are also established.

1. Introduction

Half a century ago, Skyrme [1] firstly suggested that the soliton in the nonlinear σ-model [2] may be explained by the baryon number, which is corresponding to the winding number of soliton.

The Skyrmions were originally introduced to describe baryons in three spatial dimensions [1]. In a nonlinear scalar field theory, a Skyrmion is a classical static field configuration of minimal energy. The scalar field is the pion field, and the Skyrmion represents a baryon. The Skyrmion has a topological charge which prevents continuously deforming to the vacuum field configuration. This charge is identified with the conserved baryon number which prevents a baryon from decaying into pions [1, 3].

Skyrmions have been shown to exist for a very wide class of geometries [4], which are now playing an increasing role in other areas of physics as well. For example, in certain condense matter systems, Skyrmions are used to model the bubbles that appear in the presence of an external magnetic field in two dimensions; they could provide a mechanism associated with the disappearance of antiferromagnetism, the onset of HTc superconductivity, and so on. In condensed matter physics [5], the model [6] has direct applications which may give an effective description in quantum Hall systems. In the context of condensed matter physics [7, 8], direct experimental observations can be made. In three spatial dimensions [6], baby Skyrmions have been studied in the context of strong interactions as a toy-model in order to understand the more complicated dynamics of usual Skyrmions which live.

In the present paper we consider a baby-Skyrme model, that is, Skyrmional model in two spatial dimensions, which was introduced in [9]. Our purpose of this paper is to establish the existence of the energy-minimizing solutions for this baby-Skyrme model rigorously by the variational method. In Section 2, we will present the mathematical structure of the model and the main existence theorem. In Section 3, we will show the existence of the energy-minimizing solutions by the variational method and establish some properties of the solutions.

2. The Mathematical Structure and Existence Theorem

Baby Skyrmions are obtained as the nontrivial solutions of the well-known nonlinear $O(3)$ model. The model consists of three real scalars $\phi_a (a = 1, 2, 3)$ subject to the constraint

$$\vec{\phi} \cdot \vec{\phi} = 1. \tag{1}$$

The equation of motion admits solutions with finite energy which represents a mapping of $\mathbf{R}^2_{\text{spat}}$ into $\mathbf{S}^2_{\text{int}}$. They are characterized by the density ρ,

$$\rho \equiv \epsilon_{ij} \vec{\phi} \cdot \left(\partial_i \vec{\phi} \times \partial^i \vec{\phi} \right), \tag{2}$$

and the winding number W,

$$W = \frac{1}{8\pi}\int d^2 r\rho. \tag{3}$$

The energy functional of this model is as follows:

$$E = E^{(2)} + E^{(4)} + E^{(p)}, \tag{4}$$

with

$$\begin{aligned} E^{(2)} &= \frac{1}{2}\int \partial_i\vec{\phi}\partial^i\vec{\phi}d^2r, \\ E^{(4)} &= \frac{1}{8}\int \rho\rho d^2r, \\ E^{(p)} &= \frac{\alpha}{2}\int \left(\hat{n}_3 - \vec{\phi}\right)^2 d^2r, \end{aligned} \tag{5}$$

where $\hat{n}_3$ is a unit vector in the third derivation in internal space and α is a parameter that is assumed positive.

By using the inequality

$$\begin{aligned} 0 \le \int \Bigg\{ &\frac{1}{4}\left(\partial_i\vec{\phi} \pm \epsilon_{ij}\vec{\phi}\times\partial^j\vec{\phi}\right)^2 \\ &+\frac{1}{2}\left(\frac{1}{2}\epsilon_{ij}\partial^i\vec{\phi}\times\partial^j\vec{\phi} \pm \sqrt{\alpha}\left(\hat{n}_3 - \vec{\phi}\right)\right)^2 \Bigg\} d^2r, \end{aligned} \tag{6}$$

we may find the Bogomol'nyi bound

$$E \ge 4\pi k\left(1+\sqrt{\alpha}\right). \tag{7}$$

We are to extend the model above by going from $\mathbf{R}^2_{\mathrm{sp}}$ to $\mathbf{S}^2_{\mathrm{sp}}(L)$ where L is the radius of the two-sphere. By the polar coordinates $\theta, \varphi (0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi)$,

$$x = L\sin\theta\cos\varphi; \qquad y = L\sin\theta\sin\varphi. \tag{8}$$

And the Jacobian of the transformation and the metric associated with the polar coordinates are

$$\begin{aligned} J &= -L^2\sin\theta, \\ ds^2 &= L^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right). \end{aligned} \tag{9}$$

In order to obtain explicit static solutions in the winding number $W = k$ sector, we introduce the hedgehog parameterization

$$\begin{aligned} \phi_1 &= \sin f\cos k\varphi; \\ \phi_2 &= \sin f\sin k\varphi; \\ \phi_3 &= \cos f, \end{aligned} \tag{10}$$

where

$$f = f(\theta), \tag{11}$$

is subject to the boundary conditions

$$f(0) = \pi, \qquad f(\pi) = 0. \tag{12}$$

The energy functional is as follows:

$$\begin{aligned} &E_k(f) \\ &= \frac{1}{4k}\int_0^\pi d\theta\sin\theta\left\{f'^2 + k^2\left(\frac{\sin f}{\sin\theta}\right)^2 + \frac{k^2}{L^2}f'^2\left(\frac{\sin f}{\sin\theta}\right)^2 \right. \\ &\qquad \left. + 2\alpha L^2\left(1-\cos f\right)\right\}, \end{aligned} \tag{13}$$

while the winding number density results in

$$\rho_k = -\frac{2k}{L^2}f'\frac{\sin f}{\sin\theta}. \tag{14}$$

It is not difficult to show that the Euler-Lagrange equation of (13) is

$$\begin{aligned} &\left[1 + \frac{k^2}{L^2}\left(\frac{\sin f}{\sin\theta}\right)^2\right]f'' \\ &\quad + \left[f' - k^2\frac{\sin 2f}{\sin 2\theta} + \frac{k^2}{L^2}f'\frac{\sin f}{\sin\theta}\left(f'\frac{\cos f}{\cos\theta} - \frac{\sin f}{\sin\theta}\right)\right] \\ &\quad \times\cot\theta - \alpha L^2\sin f = 0. \end{aligned} \tag{15}$$

Next we are to find a solution of the boundary problem (15) and (12). We will establish the existence of solutions by the indirect variational method.

Here is our main existence theorem, which solves the above problem.

Theorem 1. *The boundary value problem* (15) *and* (12) *has a solution* $f(\theta)$ *such that*

$$0 < f(\theta) < \pi, \quad \forall\theta \in (0,\pi), \tag{16}$$

and there hold the sharp asymptotic estimates

$$\begin{aligned} \pi - f(\theta) &= O\left(\theta^{3/2}\right) \quad (\text{as } \theta \longrightarrow 0), \\ f(\theta) &= O\left((\pi-\theta)^{3/2}\right) \quad (\text{as } \theta \longrightarrow \pi). \end{aligned} \tag{17}$$

3. The Proof of Theorem 1

In this section, we will divide the proof of Theorem 1 into two lemmas.

Lemma 2. *The boundary value problem* (15) *and* (12) *has a solution* $f(\theta)$ *such that*

$$\begin{aligned} \pi - f(\theta) &= O\left(\theta^{3/2}\right) \quad (\text{as } \theta \longrightarrow 0), \\ f(\theta) &= O\left((\pi-\theta)^{3/2}\right) \quad (\text{as } \theta \longrightarrow \pi). \end{aligned} \tag{18}$$

Proof. In order to get a solution of (15) with the boundary condition (12), we may look for the minimizers of the functional (13).

We first introduce the admissible space

$$\begin{aligned}\mathscr{A} = \{ & f \mid f(\theta) \text{ is continuous on } [0,\pi] \\ & \text{and absolutely continuous on every compact} \\ & \text{subinterval of } [0,\pi] \text{ such that it satisfies} \\ & \text{the boundary condition (12) and } E_k(f) < \infty\}.\end{aligned} \tag{19}$$

Obviously the set $\mathscr{A}$ is not empty.

We intend to find a solution of (15) and (12) by solving the minimization problem:

$$\eta \equiv \min\{E_k(f) \mid f \in \mathscr{A}\}. \tag{20}$$

Let $\{f_n(\theta)\}$ be a minimizing sequence of (20). Without loss of generality, we may assume that

$$0 \le f_n(\theta) \le \pi, \qquad 0 < \theta < \pi. \tag{21}$$

Otherwise, we may modify the sequence to fulfill (21) meanwhile without enlarging the energy. From the inequality

$$\begin{aligned}&|1+\cos f_n(\theta)| \\ &\le \int_0^\theta \left|(-\sin f_n(s))\, f_n'(s)\right| ds \\ &\le \int_0^\theta |\sin s| \cdot \frac{\left|\sin f_n(s)\cdot f_n'(s)\right|}{|\sin s|} ds \\ &\le \left(\int_0^\theta |\sin s|^2 ds\right)^{1/2} \left(\int_0^\theta \left(f_n'(s)\right)^2 \left(\frac{\sin f_n(s)}{\sin s}\right)^2 ds\right)^{1/2} \\ &\le \frac{L}{\sqrt{3}k}\theta^{3/2}\left(E_k(f_n)\right)^{1/2} \\ &= C_1\theta^{3/2}, \quad C_1 > 0,\end{aligned} \tag{22}$$

we may see that $f_n(\theta) \to \pi$ uniformly as $\theta \to 0$.

Similarly, we have

$$\begin{aligned}&|1-\cos f_n(\theta)| \\ &\le \int_\theta^\pi \left|\sin f_n(s)\, f_n'(s)\right| ds \\ &\le \int_\theta^\pi |\sin s| \cdot \frac{\left|\sin f_n(s)\cdot f_n'(s)\right|}{|\sin s|} ds \\ &\le \left(\int_0^\theta |\sin s|^2 ds\right)^{1/2} \left(\int_0^\theta \left(f_n'(s)\right)^2 \left(\frac{\sin f_n(s)}{\sin s}\right)^2 ds\right)^{1/2} \\ &\le \frac{L}{\sqrt{3}k}(\pi-\theta)^{3/2}\left(E_k(f_n)\right)^{1/2} \\ &= C_2(\pi-\theta)^{3/2}, \quad C_2 > 0.\end{aligned} \tag{23}$$

Then, we may find that $f_n(\theta) \to 0$ uniformly as $\theta \to \pi$.

In view of (22) and (23), letting $n \to \infty$, we have

$$\begin{aligned} f_n(\theta) &\longrightarrow \pi \quad (\text{as } \theta \longrightarrow 0), \\ f_n(\theta) &\longrightarrow 0 \quad (\text{as } \theta \longrightarrow \pi), \\ \pi - f(\theta) &= O\left(\theta^{3/2}\right) \quad (\text{as } \theta \longrightarrow 0), \\ f(\theta) &= O\left((\pi-\theta)^{3/2}\right) \quad (\text{as } \theta \longrightarrow \pi). \end{aligned} \tag{24}$$

We may get that the sequence $\{f_n(\theta)\}$ is bounded in $W^{1,2}(a,b)$ for any

$$0 < a < b < \pi. \tag{25}$$

Using weak compactness, we may assume that $\{f_n(\theta)\}$ (in fact, a subsequence in it) is weakly convergent in $W^{1,2}(a,b)$. Applying a diagonal subsequence argument, we may assume there is an

$$f(\theta) \in W^{1,2}_{\text{loc}}(o,\pi), \tag{26}$$

such that

$$f_n(\theta) \longrightarrow f(\theta) \quad \text{as } n \longrightarrow \infty, \tag{27}$$

weakly in $W^{1,2}(a,b)$. In view of the compact embedding theorem, we may get

$$W^{1,2}(a,b) \hookrightarrow C[a,b]. \tag{28}$$

That is, $W^{1,2}(a,b)$ can be compactly embedded into $C[a,b]$. So we see that the convergence (27) is strong in $C[a,b]$. Consequently, we know that $f_n(\theta)$ is absolutely continuous in any compact subinterval of (a,b) and continuous on $(0,\pi)$.

Let

$$\begin{aligned}F = \frac{1}{4k}\sin\theta \Bigg[& f'^2 + k^2\left(\frac{\sin f}{\sin\theta}\right)^2 + \frac{k^2}{L^2} f'^2 \left(\frac{\sin f}{\sin\theta}\right)^2 \\ & + 2\alpha L^2 (1-\cos f)\Bigg].\end{aligned} \tag{29}$$

Using the weak lower semicontinuity property of the functional, we obtain the inequality

$$\int_a^b F(f(\theta)) \le \liminf_{n\to\infty} \int_a^b F(f_n(\theta))\, d\theta \le \liminf_{n\to\infty} E_k(f_n) = \eta, \tag{30}$$

for any

$$0 < a < b < \pi. \tag{31}$$

Letting

$$a \longrightarrow 0^+, \qquad b \longrightarrow \pi^-, \tag{32}$$

we have

$$E(f) = \int_0^\pi F(f)\, d\theta \le \eta \le E(f). \tag{33}$$

Thus we see that $f(\theta)$ fulfills the complete boundary conditions (12). Therefore

$$f(\theta) \in \mathscr{A}, \tag{34}$$

and (30) allows us to obtain

$$E(f) = \eta. \tag{35}$$

That is, f is found to be a solution of (20). As a consequence, f is a finite-energy solution of (12) and (15). □

Next we will establish some properties of the energy-minimizing solutions.

Lemma 3. *Let f be the energy-minimizing solution obtained in Lemma 2. Then*

$$0 < f(\theta) < \pi, \quad \forall \theta \in (0, \pi). \tag{36}$$

Proof. Evidently, $f(\theta) \equiv 0$ is an equilibrium point of (15). We assume that there is θ_0 such that

$$f(\theta_0) = 0. \tag{37}$$

Hence, $f(\theta)$ attains its global minimum, so

$$f'(\theta_0) = 0. \tag{38}$$

Using the uniqueness theorem for the initial value problem of ordinary differential equations, we can get

$$f(\theta) \equiv 0, \tag{39}$$

which contradicts

$$f(0) = \pi, \tag{40}$$

so

$$f(\theta) > 0, \quad \forall \theta \in (0, \pi). \tag{41}$$

Similarly, we may find that

$$f(\theta) < \pi, \quad \forall \theta \in (0, \pi). \tag{42}$$

Combining Lemmas 2 and 3, we complete the proof of Theorem 1. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Review Article

A Class of Logarithmically Completely Monotonic Functions and Their Applications

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We study the recent investigations on a class of functions which are logarithmically completely monotonic. Two open problems are also presented.

1. Introduction

Recall [1] that a positive function f is said to be logarithmically completely monotonic (LCM) on an open interval I if f has derivatives of all orders on I and for all $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$,

$$(-1)^n [\ln f(x)]^{(n)} \geq 0. \tag{1}$$

LCM functions are related to completely monotonic (CM) functions [2], strongly logarithmically completely monotonic (SLCM) functions [3], almost strongly completely monotonic (ASCM) functions [3], almost completely monotonic (ACM) functions [4], Laplace transforms, and Stieltjes transforms and have wide applications. It is evident that the set of SLCM functions is a nontrivial subset of the set of LCM functions, which is a nontrivial subset of the set of CM functions, and that the set of CM functions is a nontrivial subset of the set of ACM functions. It was established [3] that the set of SLCM functions is a nontrivial subset of the set of ASCM functions and that the set of SLCM functions on the interval $(0, \infty)$ is disjoint with the set of strongly completely monotonic (SCM) functions (see [5] for its definition) on the interval $(0, \infty)$.

It is well known that the classical Euler gamma function is defined for $x > 0$ by

$$\Gamma(z) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t. \tag{2}$$

The logarithmic derivative of $\Gamma(z)$, denoted by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{3}$$

is called psi function, and $\psi^{(k)}$ for $k \in \mathbb{N}$ are called polygamma functions.

For $\alpha, \gamma \in \mathbb{R}$ and $\beta \geq 0$, define

$$f_{\alpha,\beta,\gamma}(x) := \left[\frac{e^x \Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right]^\gamma, \quad x \in (0, \infty), \tag{4}$$

which is encountered in probability and statistics.

Since $f_{\alpha,\beta,\gamma}(x)$ $(\gamma > 0)$ is logarithmically completely monotonic if and only if $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic and $f_{\alpha,\beta,\gamma}(x)$ $(\gamma < 0)$ is logarithmically completely monotonic if and only if $f_{\alpha,\beta,-1}(x)$ is logarithmically completely monotonic, we only need to study the logarithmically complete monotonicity of the function

$$f_{\alpha,\beta,\pm1}(x) = \left[\frac{e^x \Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right]^{\pm1}, \quad x \in (0, \infty). \tag{5}$$

In [6, Theorem 3.2], it was proved that the function $f_{1/2,0,1}(x)$ is decreasing and logarithmically convex from $(0, \infty)$ onto $(\sqrt{2\pi}, \infty)$ and that the function $f_{1,0,1}(x)$ is increasing and logarithmically concave from $(0, \infty)$ onto $(1, \infty)$.

In [7, Theorem 1], for showing

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b} \tag{6}$$

for

$$b > a > 1, \tag{7}$$

monotonic properties of the functions $\ln f_{\alpha,0,1}(x)$ and $\ln f_{\alpha,0,1}(x)$ on the interval $(1,\infty)$ were obtained.

In [8, Theorem 2], it was presented that the function $f_{\alpha,0,1}(x)$ is decreasing on the interval (c,∞) for $c \geq 0$ if and only if

$$\alpha \leq \frac{1}{2} \tag{8}$$

and increasing on the interval (c,∞) if and only if

$$\alpha \geq \begin{cases} c\left[\ln c - \psi(c)\right] & \text{if } c > 0, \\ 1 & \text{if } c = 0. \end{cases} \tag{9}$$

In [9], after proving the logarithmically completely monotonic property of the functions $f_{1/2,0,1}(x)$ and $f_{1,0,-1}(x)$, in virtue of Jensen's inequality for convex functions, the upper and lower bounds for the Gurland's ratio were established: for positive numbers x and y, the inequality

$$\frac{x^{x-1/2}y^{y-1/2}}{\left[(x+y)/2\right]^{x+y-1}} \leq \frac{\Gamma(x)\Gamma(y)}{\left[\Gamma((x+y)/2)\right]^2} \leq \frac{x^{x-1}y^{y-1}}{\left[(x+y)/2\right]^{x+y-2}} \tag{10}$$

holds true, where the middle term in (10) is called Gurland's ratio [10].

In [11] the authors proved the following result.

Theorem 1 (see [11]). *If*

$$2\alpha \leq 1 \leq \beta, \tag{11}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

The necessary and sufficient conditions for the functions $f_{\alpha,0,1}(x)$ and $f_{\alpha,0,-1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ were also given in [11].

Using monotonic properties of the functions $f_{1/2,0,1}(x)$ and $f_{1,0,-1}(x)$, the inequality (6) was extended (see [11, Remark 1]) from

$$b > a > 1 \tag{12}$$

to

$$b > a > 0. \tag{13}$$

In [12] the authors proved the following results.

Theorem 2 (see [12]). *If $\beta > 0$ and $\alpha \leq 0$, then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.*

Theorem 3 (see [12]). *For $\beta > 0$, a necessary condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that*

$$\alpha \leq \min\left\{\beta, \frac{1}{2}\right\}. \tag{14}$$

Theorem 4 (see [12]). *For $\beta \geq 1$, a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that*

$$\alpha \leq \frac{1}{2}. \tag{15}$$

As direct consequences of the above results, the following Kečkić-Vasić-type inequality is deduced.

Theorem 5 (see [12]). *Let x and y be positive numbers with $x \neq y$.*

(1) *For $\beta \geq 1$, the following inequality*

$$I(x,y) > \left[\left(\frac{x}{y}\right)^{\alpha-\beta}\frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \tag{16}$$

holds true if and only if $\alpha \leq 1/2$, where

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \quad (a > 0,\ b > 0,\ a \neq b) \tag{17}$$

is the identric or exponential mean.

(2) *For $\beta > 0$, the inequality (16) holds true also if $\alpha \leq 0$.*

In [13], the following result was established.

Theorem 6 (see [13]). (1) *For $\beta \in [0, 1/2)$, if*

$$\alpha \leq \beta - e^{-4}(1-\beta)^2 \exp\left(\frac{2}{1-\beta}\right), \tag{18}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

(2) *For $\beta \in [1/2, 1]$, if*

$$\alpha \leq \min\left\{3\beta^2 - 3\beta + 1, \frac{1}{2}\right\}, \tag{19}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

From Theorem 6 we can directly obtain the following new result.

Corollary 7. (1) *For $\beta \in [1/4, 1/2]$, if*

$$\alpha \leq \beta - \frac{1}{4}, \tag{20}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

(2) *For* $\beta \in (1/2, 3/4]$, *if*

$$\alpha \le \beta - \frac{1}{3}, \tag{21}$$

then the function $f_{\alpha,\beta,1}(x)$ *is logarithmically completely monotonic on the interval* $(0,\infty)$.

(3) *For* $\beta \in (3/4, 1]$, *if*

$$\alpha \le \beta - \frac{1}{2}, \tag{22}$$

then the function $f_{\alpha,\beta,1}(x)$ *is logarithmically completely monotonic on the interval* $(0,\infty)$.

A necessary and sufficient condition is obtained in [13] as follows.

Theorem 8 (see [13]). *For*

$$\beta \in \{0\} \cup \left[\frac{1}{2} + \frac{\sqrt{3}}{6}, \infty\right), \tag{23}$$

a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ *to be logarithmically completely monotonic on the interval* $(0,\infty)$ *is that*

$$\alpha \le \frac{1}{2}. \tag{24}$$

Regarding the logarithmically complete monotonicity for the function $f_{\alpha,\beta,-1}(x)$ and their applications. In [14], the authors proved the following results.

Theorem 9 (see [14]). *If the function* $f_{\alpha,\beta,-1}(x)$ *is logarithmically completely monotonic on the interval* $(0,\infty)$, *then either*

$$\beta > 0, \qquad \alpha \ge \max\left\{\beta, \frac{1}{2}\right\} \tag{25}$$

or

$$\beta = 0, \qquad \alpha \ge 1. \tag{26}$$

Theorem 10 (see [14]). *For*

$$\beta \ge \frac{1}{2}, \tag{27}$$

the necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ *to be logarithmically completely monotonic on the interval* $(0,\infty)$ *is that*

$$\alpha \ge \beta. \tag{28}$$

As first application, the following inequalities are derived by using logarithmically completely monotonic properties of the function $f_{\alpha,\beta,\pm1}(x)$ on the interval $(0,\infty)$.

Theorem 11 (see [14]). (1) *For* $k \in \mathbb{N}$, *double inequalities*

$$\begin{gathered} \ln x - \frac{1}{x} \le \psi(x) \le \ln x - \frac{1}{2x}, \\ \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \le (-1)^{k+1}\psi^{(k)}(x) \le \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \end{gathered} \tag{29}$$

hold true on the interval $(0,\infty)$.

(2) *When* $\beta > 0$, *inequalities*

$$\begin{gathered} \psi(x+\beta) \le \ln x + \frac{\beta}{x}, \\ (-1)^k\psi^{(k-1)}(x+\beta) \ge \frac{(k-2)!}{x^{k-1}} - \frac{\beta(k-1)!}{x^k} \end{gathered} \tag{30}$$

hold true on the interval $(0,\infty)$ *for* $k \ge 2$.

(3) *When* $\beta \ge 1/2$, *inequalities*

$$\begin{gathered} \psi(x+\beta) \ge \ln x, \\ (-1)^k\psi^{(k-1)}(x+\beta) \le \frac{(k-2)!}{x^{k-1}} \end{gathered} \tag{31}$$

hold true on the interval $(0,\infty)$ *for* $k \ge 2$.

(4) *When* $\beta \ge 1$, *inequalities*

$$\begin{gathered} \psi(x+\beta) \le \ln x + \frac{\beta - 1/2}{x}, \\ (-1)^k\psi^{(k-1)}(x+\beta) \ge \frac{(k-2)!}{x^{k-1}} - \frac{(\beta-1/2)(k-1)!}{x^k} \end{gathered} \tag{32}$$

hold true on the interval $(0,\infty)$ *for* $k \ge 2$.

As second application, the following inequalities are derived by using logarithmically convex properties of the function $f_{\alpha,\beta,\pm1}(x)$ on $(0,\infty)$.

Theorem 12 (see [14]). *Let* $n \in \mathbb{N}$ *and*

$$x_k > 0 \quad (1 \le k \le n). \tag{33}$$

Suppose also that

$$\sum_{k=1}^{n} p_k = 1 \quad (p_k \ge 0). \tag{34}$$

If either

$$\beta > 0, \qquad \alpha \le 0 \tag{35}$$

or

$$\beta \ge 1, \qquad \alpha \le \frac{1}{2}, \tag{36}$$

then

$$\frac{\prod_{k=1}^{n}[\Gamma(x_k+\beta)]^{p_k}}{\Gamma\left(\sum_{k=1}^{n} p_k x_k + \beta\right)} \ge \frac{\prod_{k=1}^{n} x_k^{p_k(x_k+\beta-\alpha)}}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k + \beta - \alpha}}. \tag{37}$$

If

$$\alpha \ge \beta \ge \frac{1}{2}, \tag{38}$$

then the inequality (37) *reverses.*

As final application, the following inequality can be derived by using the decreasingly monotonic property of the function $f_{\alpha,\beta,-1}(x)$ on $(0,\infty)$.

Theorem 13 (see [14]). *If*

$$\alpha \geq \beta \geq \frac{1}{2}, \tag{39}$$

then

$$I(x,y) < \left[\left(\frac{x}{y}\right)^{\alpha-\beta} \frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \tag{40}$$

holds true for $x, y \in (0,\infty)$ with $x \neq y$, where $I(x,y)$, defined by (17)*, is the identric or exponential mean.*

The following results were shown in [15].

Theorem 14 (see [15]). *For*

$$\beta \geq 0, \tag{41}$$

a sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that

$$\alpha \geq \max\left\{\frac{1}{2}, \beta, 3\beta^2 - 3\beta + 1\right\}. \tag{42}$$

Remark 15. From Theorems 9 and 14 we see that the necessary and sufficient condition for the function $f_{\alpha,0,-1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that

$$\alpha \geq 1. \tag{43}$$

This result is Theorem 2 in [11]. Here we recovered it.

Theorem 16 (see [15]). *Let*

$$\beta \in \left[\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right]. \tag{44}$$

Then the necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that

$$\alpha \geq \frac{1}{2}. \tag{45}$$

The following results are applications of the above theorems.

Theorem 17 (see [15]). *When*

$$\frac{1}{2} - \frac{\sqrt{3}}{6} \leq \beta \leq \frac{1}{2}, \tag{46}$$

the following inequalities

$$\begin{gathered} \psi(x+\beta) \geq \ln x - \frac{1/2-\beta}{x}, \\ (-1)^k \psi^{(k-1)}(x+\beta) \leq \frac{(k-2)!}{x^{k-1}} + \frac{(1/2-\beta)(k-1)!}{x^k} \\ (k \geq 2) \end{gathered} \tag{47}$$

hold true on the interval $(0,\infty)$.

Theorem 18 (see [15]). *Let $n \in \mathbb{N}$ and*

$$x_k > 0 \quad (1 \leq k \leq n). \tag{48}$$

Suppose also that

$$\sum_{k=1}^{n} p_k = 1 \quad (p_k \geq 0). \tag{49}$$

If

$$\begin{gathered} 0 \leq \beta \leq \frac{1}{2}, \\ \alpha \geq \max\left\{\frac{1}{2}, 3\beta^2 - 3\beta + 1\right\}, \end{gathered} \tag{50}$$

then

$$\frac{\prod_{k=1}^{n}[\Gamma(x_k+\beta)]^{p_k}}{\Gamma\left(\sum_{k=1}^{n} p_k x_k + \beta\right)} \leq \frac{\prod_{k=1}^{n} x_k^{p_k(x_k+\beta-\alpha)}}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k + \beta - \alpha}}. \tag{51}$$

Theorem 19 (see [15]). *If*

$$\begin{gathered} 0 \leq \beta \leq \frac{1}{2}, \\ \alpha \geq \max\left\{\frac{1}{2}, 3\beta^2 - 3\beta + 1\right\}, \end{gathered} \tag{52}$$

then

$$\begin{gathered} I(x,y) < \left[\left(\frac{x}{y}\right)^{\alpha-\beta} \frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \\ (x > 0;\ y > 0;\ x \neq y), \end{gathered} \tag{53}$$

where in (53) *$I(x,y)$, defined by* (17)*, is the identric or exponential mean.*

2. Open Problems

2.1. Open Problem 1. From Theorem 8 we have already known, for

$$\beta \in \{0\} \cup \left[\frac{1}{2} + \frac{\sqrt{3}}{6}, \infty\right), \tag{54}$$

a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$.

For

$$\beta \in \left(0, \frac{1}{2} + \frac{\sqrt{3}}{6}\right), \tag{55}$$

what is a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$?

Already Known. Theorem 3 gave a necessary condition; Theorem 6 provided a sufficient condition.

2.2. Open Problem 2. From Remark 15, Theorems 10 and 16 we have already known, for

$$\beta \in \{0\} \cup \left[\frac{1}{2} - \frac{\sqrt{3}}{6}, \infty\right), \tag{56}$$

a necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$.

For

$$\beta \in \left(0, \frac{1}{2} - \frac{\sqrt{3}}{6}\right), \tag{57}$$

what is a necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$?

Already Known. Theorem 9 gave a necessary condition; Theorem 14 provided a sufficient condition.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

T-Stability of the Heun Method and Balanced Method for Solving Stochastic Differential Delay Equations

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This paper studies the T-stability of the Heun method and balanced method for solving stochastic differential delay equations (SDDEs). Two T-stable conditions of the Heun method are obtained for two kinds of linear SDDEs. Moreover, two conditions under which the balanced method is T-stable are obtained for two kinds of linear SDDEs. Some numerical examples verify the theoretical results proposed.

1. Introduction

Stochastic differential delay equations (SDDEs) are the promotion of stochastic differential equations (SDEs) and differential delay equations (DDEs). These kinds of equations consider not only the stochastic factors in the process of the development of a system, but also the impact of the delay. As an important mathematical model, SDDEs have been applied widely in many areas, such as stochastic control, economics, and biology. Since it is difficult to find the analytic solutions to SDDEs, to get the numerical solutions to SDDEs generated by some numerical methods is commonly used. For a numerical method, it is important to analyze its stability.

The general form of SDDEs with Gaussian white noise is

$$\begin{aligned} \mathrm{d}X(t) &= f(t, X(t), X(t-\tau))\,\mathrm{d}t \\ &\quad + g(t, X(t), X(t-\tau))\,\mathrm{d}W(t), \quad t \in [0, T], \\ X(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $\tau > 0$, $\varphi(t) \in \mathbf{C}([-\tau, 0], \mathbf{R}^m)$, $f : \mathbf{R}^+ \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m$, $g : \mathbf{R}^+ \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^{m\times d}$, and $W(t)$ is a standard d-dimensional Wiener process. Equation (1) has a unique solution if f and g are sufficiently smooth and satisfy the following conditions:

$$\begin{aligned} &|f(t, x_1, y_1) - f(t, x_2, y_2)| \vee |g(t, x_1, y_1) - g(t, x_2, y_2)| \\ &\qquad \le L(|x_1 - x_2| + |y_1 - y_2|), \end{aligned} \tag{2}$$

$$|f(t, x, y)|^2 \vee |g(t, x, y)|^2 \le K\left(1 + |x|^2 + |y|^2\right), \tag{3}$$

where $t \ge 0$, x, y, x_1, y_1, x_2, $y_2 \in \mathbf{R}^m$, and L and K are constants. The condition in (2) is called *Lipschitz condition*, and the condition in (3) is called the *linear growth condition*.

The main numerical methods for SDDEs are Euler-Maruyama method [1, 2] and Milstein method [3] at present. The mean square stability of these methods for SDDEs has been well studied. Cao et al. [1] studied the mean square stability of Euler-Maruyama method for linear SDDEs. Liu et al. [2] studied the mean square stability of the semi-implicit Euler method for linear SDDEs. Wang and Zhang [3] discussed the mean square stability of Milstein method for linear SDDEs. Wang and Chen [4, 5] studied the mean square

stability of semi-implicit Euler method for nonlinear neutral SDDEs and that of Heun methods for nonlinear SDDEs. Tan et al. [6] discussed the mean square stability of balanced methods for SDDEs. T-stability is introduced by Saito et al. in [7–9], and it is another kind of stability with respect to the approximate sequence of sample path. Cao [10] studied the T-stability of the semi-implicit Euler method for delay differential equations with multiplicative noise. Rathinasamy and Balachandran [11] studied the T-stability of the split-step θ-methods for linear SDDEs. Yang and Liu [12] discussed the T-stability of the θ-method for a stochastic pantograph differential equation.

Applying the Heun method [5] to (1) gives

$$\begin{aligned} X_{n+1} &= X_n + \frac{1}{2}\left[f\left(t_n, X_n, X_{n-m}\right)\right. \\ &\quad \left. + f\left(t_{n+1}, X_n + h\, f\left(t_n, X_n, X_{n-m}\right), X_{n-m+1}\right)\right] h \\ &\quad + g\left(t_n, X_n, X_{n-m}\right) \Delta W_n, \end{aligned} \tag{4}$$

where $h > 0$ is a step size with $\tau = mh$ for a positive integer m and $t_n = nh$. X_n is an approximation of $X(t_n)$, and $X_n = \varphi(t_n)$ if $t_n \le 0$. $\Delta W_n = W(t_{n+1}) - W(t_n) \sim N(0, h)$.

The mean square stability of the Heun method in (4) was studied in [5], but there is no result about T-stability of the method at present. This paper gives two T-stable conditions of the Heun method (4) for two kinds of linear SDDEs.

The balanced method for solving (1) is

$$\begin{aligned} X_{n+1} &= X_n + f\left(t_n, X_n, X_{n-m}\right) h + g\left(t_n, X_n, X_{n-m}\right) \Delta W_n \\ &\quad + C\left(X_n, X_{n-m}\right)\left(X_n - X_{n+1}\right), \end{aligned} \tag{5}$$

where $C(X_n, X_{n-m}) = C_0(X_n, X_{n-m})h + C_1(X_n, X_{n-m})|\Delta W_n|$ and C_0, C_1 are $d \times d$ real matrix functions.

Let $M(x, y) = I + \beta_0 C_0(x, y) + \beta_1 C_1(x, y)$, where I is a unit matrix, $\beta_0 \in [0, \alpha]$, $\alpha \ge h$, $\beta_1 \ge 0$, and $(x, y) \in R^d \times R^d$. Assume that $M(x, y)$ is invertible with $|(M(x, y))^{-1}| \le K < \infty$.

The mean square stability of the balanced method for SDEs and SDEs with jumps was studied in [13, 14], respectively. In 2011, Tan et al. [6] applied the balanced method to SDDEs and discussed the mean square convergence and stability of this method. However, there is no research result about T-stability of the balanced method (5) at present. In this paper, the conditions under which the balanced method (5) is T-stable are obtained for two kinds of linear SDDEs.

Section 2 introduces the stochastically asymptotically stable conditions in the large for two kinds of linear SDDEs. In Section 3, T-stability of the Heun method equipped with a specified driving process is discussed and the corresponding step size range is given. Section 4 studies T-stability of the balanced method, and Section 5 uses some numerical examples to verify the results given in this paper.

2. Asymptotic Stability of Analytical Solution

Consider the following two scalar linear test equations:

$$\begin{aligned} \mathrm{d}X(t) &= [aX(t) + bX(t-\tau)]\,\mathrm{d}t \\ &\quad + cX(t)\,\mathrm{d}W(t), \quad t \in [0, T] \\ X(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{6}$$

$$\begin{aligned} \mathrm{d}X(t) &= [aX(t) + bX(t-\tau)]\,\mathrm{d}t \\ &\quad + [cX(t) + dX(t-\tau)]\,\mathrm{d}W(t), \quad t \in [0, T] \\ X(t) &= \varphi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7}$$

Let $(\Omega, \mathbf{F}, \{F_t\}_{t\ge0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\ge0}$, which is right continuous, and each F_t contains all P-null sets in $\mathbf{F}$. In (6) and (7), $a, b, c, d \in \mathbf{R}$; $\tau > 0$; $W(t)$ is a one-dimensional standard Wiener process; initial function $\varphi(t) \in \mathbf{C}([-\tau, 0], \mathbf{R}^m)$ is F_0-measurable and $E\|\varphi\|^2 < \infty$. Equations (6) and (7) have unique strong solution if (6) and (7) meet Lipschitz condition in (2) and the linear growth condition in (3).

Definition 1 (see [10]). The solution of (1) is *stochastically asymptotically stable in the large* if

$$P\left(\lim_{t\to\infty} |X(t, \varphi)| = 0\right) = 1 \tag{8}$$

for all initial functions φ.

From Corollary 3.2 in [15], we get Lemmas 2 and 3 as follows.

Lemma 2. *The solution of* (6) *is stochastically asymptotically stable in the large if parameters a, b, and c in* (6) *satisfy*

$$a < -|b| - \frac{1}{2}c^2. \tag{9}$$

Lemma 3. *The solution of* (7) *is stochastically asymptotically stable in the large if parameters a, b, c, and d in* (7) *satisfy*

$$a < -|b| - \frac{1}{2}(|c| + |d|)^2. \tag{10}$$

3. T-Stability of the Heun Method

Definition 4 (see [10]). Suppose that the condition in (9) or in (10) is fulfilled. A numerical scheme equipped with a specified driving process is said to be T-stable if

$$\lim_{n\to\infty} |X_n| \longrightarrow 0 \quad \text{a.s.} \tag{11}$$

for the driving process, where X_n is the numerical solution generated by the numerical scheme applied to the test equation (6) or (7).

For analyzing T-stability, we focus our attention on the trajectory of numerical solution. A specified driving

process proposed in Definition 4 is used to approximate the Wiener increment ΔW_n of the numerical methods. This paper analyzes the Heun method equipped with two-point random variables for the driving process. So $\Delta W_n := \eta_n \sqrt{h}$ and $P(\eta_n = \pm 1) = 1/2$, where P denotes the probability.

The Heun method applied to (6) and (7), respectively, gives

$$\begin{aligned} X_{n+1} &= \left(1 + ah + \frac{1}{2}a^2h^2 + c\Delta W_n\right) X_n \\ &\quad + \frac{1}{2}bh(1+ah) X_{n-m} + \frac{1}{2}bhX_{n-m+1}, \end{aligned} \tag{12}$$

$$\begin{aligned} X_{n+1} &= \left(1 + ah + \frac{1}{2}a^2h^2 + c\Delta W_n\right) X_n \\ &\quad + \left[\frac{1}{2}bh(1+ah) + d\Delta W_n\right] X_{n-m} + \frac{1}{2}bhX_{n-m+1}, \end{aligned} \tag{13}$$

where $h > 0$ is a step size with $\tau = mh$ for a positive integer m and $t_n = nh$, $X_n \approx X(t_n)$ if $t_n \le 0$, $X_n = \varphi(t_n)$.

For the Heun method in (12), we have

$$\begin{aligned} |X_{n+1}| &\le \left(\left|1 + ah + \frac{1}{2}a^2h^2 + c\Delta W_n\right| \right. \\ &\quad \left. + \left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right|\right) \\ &\quad \times \max\{|X_n|, |X_{n-m}|, |X_{n-m+1}|\} \\ &= \left(\left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} + c\eta_n\sqrt{h}\right| \right. \\ &\quad \left. + \left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right|\right) \\ &\quad \times \max\{|X_n|, |X_{n-m}|, |X_{n-m+1}|\}. \end{aligned} \tag{14}$$

Let

$$\begin{aligned} R(h,a,b,c) &= \left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right| \\ &\quad + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} + c\eta_n\sqrt{h}\right|. \end{aligned} \tag{15}$$

It is clear that $|X_n| \to 0$, a.s. $(n \to \infty)$ if $P(R(h,a,b,c) < 1) = 1$, and therefore the Heun method in (12) is T-stable. Denote

$$\begin{aligned} &R_T(h,a,b,c) \\ &\quad = \left(\left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right| \right. \\ &\qquad \left. + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} + c\sqrt{h}\right|\right) \\ &\qquad \times \left(\left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right| \right. \\ &\qquad \left. + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} - c\sqrt{h}\right|\right) \end{aligned} \tag{16}$$

Since η_n's follow two-point distribution, we get $|X_n| \to 0$, a.s. $(n \to \infty)$ if $R_T(h,a,b,c) < 1$, which means that the Heun method in (12) is T-stable.

Similarly, for the Heun method in (13), we can get $\widetilde{R}_T(h,a,b,c,d)$ as follows:

$$\begin{aligned} &\widetilde{R}_T(h,a,b,c,d) \\ &\quad = \left(\left|\frac{1}{2}bh(1+ah) + d\sqrt{h}\right| + \left|\frac{1}{2}bh\right| \right. \\ &\qquad \left. + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} + c\sqrt{h}\right|\right) \\ &\qquad \times \left(\left|\frac{1}{2}bh(1+ah) - d\sqrt{h}\right| + \left|\frac{1}{2}bh\right| \right. \\ &\qquad \left. + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} - c\sqrt{h}\right|\right), \end{aligned} \tag{17}$$

and $|X_n| \to 0$, a.s. $(n \to \infty)$ if $\widetilde{R}_T(h,a,b,c,d) < 1$, which means that the Heun method in (13) is T-stable.

Theorem 5. *Suppose* (6) *meets the condition in* (9). *The Heun method in* (12) *is T-stable if $h < H$, where*

$$H = \min\left\{\frac{2}{|b| - a}, \frac{1}{4c^2}\right\}. \tag{18}$$

Proof. The condition in (9) gives $a < 0$. Denote

$$\begin{aligned} R_1(h,a,b,c) &= \left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right| \\ &\quad + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} + c\sqrt{h}\right|, \\ R_2(h,a,b,c) &= \left|\frac{1}{2}bh(1+ah)\right| + \left|\frac{1}{2}bh\right| \\ &\quad + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} - c\sqrt{h}\right|, \end{aligned} \tag{$*$}$$

and $R_T(h,a,b,c) = R_1(h,a,b,c)R_2(h,a,b,c)$.

For $b \ne 0$, we have $a + b < 0$, $a - b < 0$ from the condition in (9). Let

$$h_1 = -\frac{1}{a}, \quad h_2 = \min\left\{-\frac{1}{a}, \frac{1}{4c^2}\right\}. \tag{19}$$

Only the conclusion of the theorem when $b < 0, c > 0$ is proven, and that of the theorem when $b < 0, c < 0$ or $b > 0, c > 0$ or $b > 0, c < 0$ can be proven similarly.

If $h < h_2$, then $h < -1/a$ and $h < 1/4c^2$, and we have $1 + ah > 0$ and $(1/2) - c\sqrt{h} > 0$. Hence

$$\begin{aligned} &R_1(h,a,b,c) + R_2(h,a,b,c) \\ &\quad = -bh(1+ah) - bh + (1+ah)^2 + 1 \\ &\quad = (a-b)(ah+2)h + 2. \end{aligned} \tag{20}$$

Since $ah + 1 > 0$ and $a - b < 0$, we have $ah + 2 > 0$ and

$$0 < R_1(h,a,b,c) + R_2(h,a,b,c) < 2. \tag{21}$$

Consequently,

$$R_T(h,a,b,c) = R_1(h,a,b,c)\,R_2(h,a,b,c) \le \left[\frac{R_1(h,a,b,c)+R_2(h,a,b,c)}{2}\right]^2 < 1, \tag{22}$$

which means that the Heun method (12) is T-stable if $h < h_2$.

(a) When $h_1 = -1/a < H$, if $H = 1/4c^2$, then $h_2 = \min\{-1/a, 1/4c^2\} = -1/a = h_1$.

If $H = 2/(|b| - a)$, then $H \le 1/4c^2$, which yields $h_1 = -1/a < H \le 1/4c^2$, and $h_2 = -1/a = h_1$. Hence $h_2 = h_1$ if $h_1 < H$.

For obtaining the result of this theorem in this case, the left work is to prove the method is T-stable for $h_1 \le h < H$.

If $h_1 \le h < H$, then $-1/a \le h < 1/4c^2$, $h < 2/(|b| - a)$; that is,

$$1 + ah \le 0, \quad \frac{1}{2} - c\sqrt{h} > 0, \quad (a - |b|)\,h + 2 > 0. \tag{23}$$

Since $a - |b| \le a + b$, we get $(a+b)h + 2 \ge (a - |b|)h + 2 > 0$. Since

$$\begin{aligned} &R_1(h,a,b,c) + R_2(h,a,b,c) \\ &\quad = bh(1+ah) - bh + (1+ah)^2 + 1 \\ &\quad = ah[(a+b)h + 2] + 2, \end{aligned} \tag{24}$$

we have $0 < R_1(h,a,b,c) + R_2(h,a,b,c) < 2$ from $a < 0$ and $(*)$, and therefore

$$R_T(h,a,b,c) = R_1(h,a,b,c)\,R_2(h,a,b,c) \le \left[\frac{R_1(h,a,b,c)+R_2(h,a,b,c)}{2}\right]^2 < 1, \tag{25}$$

which means that the Heun method (12) is T-stable if $h < H$.

(b) When $h_1 = H$, it is easy to know $h_2 = h_1$, and the Heun method (12) is T-stable if $h < h_2 = H$.

(c) When $h_1 = -1/a > H$, if $H = 1/4c^2$, then $h_2 = \min\{-1/a, 1/4c^2\} = 1/4c^2 = H$. If $H = 2/(|b| - a)$, then $-1/a > 2/(|b| - a)$. Solving this gives $a > -|b|$, which contradicts the condition in (9). So $h_2 = H$ if $h_1 > H$. Hence the Heun method in (12) is T-stable if $h < h_2 = H$.

The discussion above shows that the Heun method in (12) is T-stable provided that $h < H$, which completes the proof. □

Theorem 6. *Suppose* (7) *meets the condition in* (10). *The Heun method in* (13) *is T-stable if* $H_1 < h < H_2$, *where*

$$H_1 = \frac{4d^2}{(a+|b|)^2}, \qquad H_2 = \min\left\{-\frac{1}{a}, \frac{1}{4c^2}, \frac{4d^2}{b^2}\right\}. \tag{26}$$

Proof. The condition in (10) gives $a < 0$. Denote

$$\begin{aligned} \widetilde{R}_1(h,a,b,c,d) &= \left|\frac{1}{2}bh(1+ah) + d\sqrt{h}\right| \\ &\quad + \left|\frac{1}{2}bh\right| + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} + c\sqrt{h}\right|, \\ \widetilde{R}_2(h,a,b,c,d) &= \left|\frac{1}{2}bh(1+ah) - d\sqrt{h}\right| \\ &\quad + \left|\frac{1}{2}bh\right| + \left|\frac{1}{2}(1+ah)^2 + \frac{1}{2} - c\sqrt{h}\right|, \end{aligned} \tag{$**$}$$

and then $\widetilde{R}_T(h,a,b,c,d) = \widetilde{R}_1(h,a,b,c,d)\widetilde{R}_2(h,a,b,c,d)$ from (17).

(a) Suppose $b < 0$, $c > 0$, $d > 0$, and $H_1 < H_2$. With $h < H_2$, we have

$$1 + ah > 0, \qquad \frac{1}{2}bh + d\sqrt{h} > 0, \qquad \frac{1}{2} - c\sqrt{h} > 0, \tag{27}$$

and $\widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) = (2a-b)h + 2d\sqrt{h} + a^2h^2 + 2$ from $(**)$. With $h > H_1$, we have $h > 4d^2/(a-b)^2$; that is, $(2a-b)h + 2d\sqrt{h} < ah$. The inequality $1 + ah > 0$ with $a < 0$ gives $ah < -a^2h^2$, and therefore $(2a-b)h + 2d\sqrt{h} < -a^2h^2$. With this and $(**)$, we get $0 < \widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) < 2$. Consequently,

$$\widetilde{R}_T(h,a,b,c,d) = \widetilde{R}_1(h,a,b,c,d)\,\widetilde{R}_2(h,a,b,c,d) < \left[\frac{\widetilde{R}_1(h,a,b,c,d)+\widetilde{R}_2(h,a,b,c,d)}{2}\right]^2 < 1, \tag{28}$$

which means the Heun method in (13) is T-stable.

In the same way, we can prove that the Heun method in (13) is T-stable when $b < 0, c > 0, d < 0$ or $b < 0, c < 0, d > 0$ or $b < 0, c < 0, d < 0$.

(b) Suppose $b > 0$, $c > 0$, $d > 0$, and $H_1 < H_2$. With $h < H_2$, we get

$$1 + ah > 0, \qquad \frac{1}{2}bh - d\sqrt{h} < 0, \qquad \frac{1}{2} - c\sqrt{h} > 0, \tag{29}$$

and $\widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) = (2a+b)h + 2d\sqrt{h} + a^2h^2 + 2$ from $(**)$. With $h > H_1$, we have $h > 4d^2/(a+b)^2$; that is, $(2a+b)h + 2d\sqrt{h} < ah$. The inequality $1 + ah > 0$ with $a < 0$ gives $ah < -a^2h^2$, and therefore $(2a+b)h + 2d\sqrt{h} < -a^2h^2$. With this and $(**)$, we get $0 < \widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) < 2$. Consequently,

$$\widetilde{R}_T(h,a,b,c,d) = \widetilde{R}_1(h,a,b,c,d)\,\widetilde{R}_2(h,a,b,c,d) < \left[\frac{\widetilde{R}_1(h,a,b,c,d)+\widetilde{R}_2(h,a,b,c,d)}{2}\right]^2 < 1, \tag{30}$$

which means that the Heun method in (13) is T-stable.

In the same way, we can prove that the Heun method in (13) is T-stable when $b > 0, c > 0, d < 0$ or $b > 0, c < 0, d > 0$ or $b > 0, c < 0, d < 0$.

The proof of the theorem is complete. □

4. T-Stability of the Balanced Method

The balanced method applied to (6) and (7), respectively, gives

$$X_{n+1} = \left(1 + \frac{ah + c\Delta W_n}{1 + C}\right) X_n + \left(\frac{bh}{1 + C}\right) X_{n-m}, \tag{31}$$

$$X_{n+1} = \left(1 + \frac{ah + c\Delta W_n}{1 + C}\right) X_n + \left(\frac{bh + d\Delta W_n}{1 + C}\right) X_{n-m}, \tag{32}$$

where $C = C_0 h + C_1 |\Delta W_n|$ and C_0 and C_1 are real numbers. In the analysis of T-stability of balanced method, we also use two-point random variables for the driving process.

For the balanced method in (31), we have

$$|X_{n+1}| \le \left(\left|1 + \frac{ah + c\Delta W_n}{1 + C_0 h + C_1 |\Delta W_n|}\right| + \left|\frac{bh}{1 + C_0 h + C_1 |\Delta W_n|}\right|\right) \max\{|X_n|, |X_{n-m}|\}. \tag{33}$$

Denote

$$\begin{aligned} R(h,a,b,c) &= \left|1 + \frac{ah + c\eta_n \sqrt{h}}{1 + C_0 h + C_1 |\eta_n| \sqrt{h}}\right| + \left|\frac{bh}{1 + C_0 h + C_1 |\eta_n| \sqrt{h}}\right|, \\ R_T(h,a,b,c) &= \left(\left|1 + \frac{ah + c\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right| + \left|\frac{bh}{1 + C_0 h + C_1 \sqrt{h}}\right|\right) \\ &\quad \times \left(\left|1 + \frac{ah - c\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right| + \left|\frac{bh}{1 + C_0 h + C_1 \sqrt{h}}\right|\right). \end{aligned} \tag{34}$$

The discussion about the Heun method in Section 3 implies $|X_n| \to 0$, a.s. $(n \to \infty)$ if $R_T(h,a,b,c) < 1$, which means that the balanced method in (31) is T-stable if $R_T(h,a,b,c) < 1$.

Similarly, for the balanced method in (32), we get

$$\begin{aligned} &\widetilde{R}_T(h,a,b,c,d) \\ &= \left(\left|1 + \frac{ah + c\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right| + \left|\frac{bh + d\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right|\right) \\ &\quad \times \left(\left|1 + \frac{ah - c\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right| + \left|\frac{bh - d\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right|\right), \end{aligned} \tag{35}$$

and $|X_n| \to 0$, a.s. $(n \to \infty)$ if $\widetilde{R}_T(h,a,b,c,d) < 1$, which means that the Heun method in (32) is T-stable if $\widetilde{R}_T(h,a,b,c,d) < 1$.

Theorem 7. *Suppose* (6) *meets the condition in* (9). *The balanced method in* (31) *is T-stable if* $F(h) \ge 0$, *where*

$$F(h) = (C_0 + a) h + (C_1 - |c|) \sqrt{h}. \tag{36}$$

Proof. The condition in (9) gives $a < 0$. Denote

$$\begin{aligned} R_1(h,a,b,c) &= \left|1 + \frac{ah + c\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right| + \left|\frac{bh}{1 + C_0 h + C_1 \sqrt{h}}\right|, \\ R_2(h,a,b,c) &= \left|1 + \frac{ah - c\sqrt{h}}{1 + C_0 h + C_1 \sqrt{h}}\right| + \left|\frac{bh}{1 + C_0 h + C_1 \sqrt{h}}\right|, \end{aligned} \tag{$***$}$$

and $R_T(h,a,b,c) = R_1(h,a,b,c) R_2(h,a,b,c)$ from (34). It is easy to know $a + b < 0, a - b < 0$ for $b \ne 0$.

If $c > 0, b > 0$, we have

$$\begin{aligned} C_0 h + C_1 \sqrt{h} + ah - c\sqrt{h} &\ge 0, \\ C_0 h + C_1 \sqrt{h} + ah + c\sqrt{h} &\ge 0, \end{aligned} \tag{37}$$

from the assumption $F(h) \ge 0$, which applies $1 + C_0 h + C_1 \sqrt{h} > 0$. Then $R_1(h,a,b,c)$ and $R_2(h,a,b,c)$ in $(***)$ become

$$\begin{aligned} R_1(h,a,b,c) &= \frac{1 + C_0 h + C_1 \sqrt{h} + ah + c\sqrt{h} + bh}{1 + C_0 h + C_1 \sqrt{h}}, \\ R_2(h,a,b,c) &= \frac{1 + C_0 h + C_1 \sqrt{h} + ah - c\sqrt{h} + bh}{1 + C_0 h + C_1 \sqrt{h}}, \\ R_1(h,a,b,c) + R_2(h,a,b,c) &= 2 + \frac{2(a+b)h}{1 + C_0 h + C_1 \sqrt{h}}. \end{aligned} \tag{38}$$

Since $a + b < 0$, $1 + C_0 h + C_1 \sqrt{h} > 0$, we have $0 < R_1(h,a,b,c) + R_2(h,a,b,c) < 2$ and

$$\begin{aligned} R_T(h,a,b,c) &= R_1(h,a,b,c) R_2(h,a,b,c) \\ &< \left[\frac{R_1(h,a,b,c) + R_2(h,a,b,c)}{2}\right]^2 < 1, \end{aligned} \tag{39}$$

which means that balanced method in (31) is T-stable.

In the same way, we can prove that the balanced method in (31) is T-stable when

$$\begin{aligned} &c > 0, \quad b < 0 \quad \text{or} \quad c < 0, \quad b > 0, \\ &\text{or} \quad c < 0, \quad b < 0. \end{aligned} \tag{40}$$

The proof of the theorem is complete. □

Theorem 8. *Suppose* (7) *meets the condition in* (10). *The balanced method in* (32) *is T-stable if* $h > H$ *and* $F(h) \ge 0$, *where*

$$H = \frac{d^2}{b^2}, \qquad F(h) = (C_0 + a) h + (C_1 - |c|) \sqrt{h}. \tag{41}$$

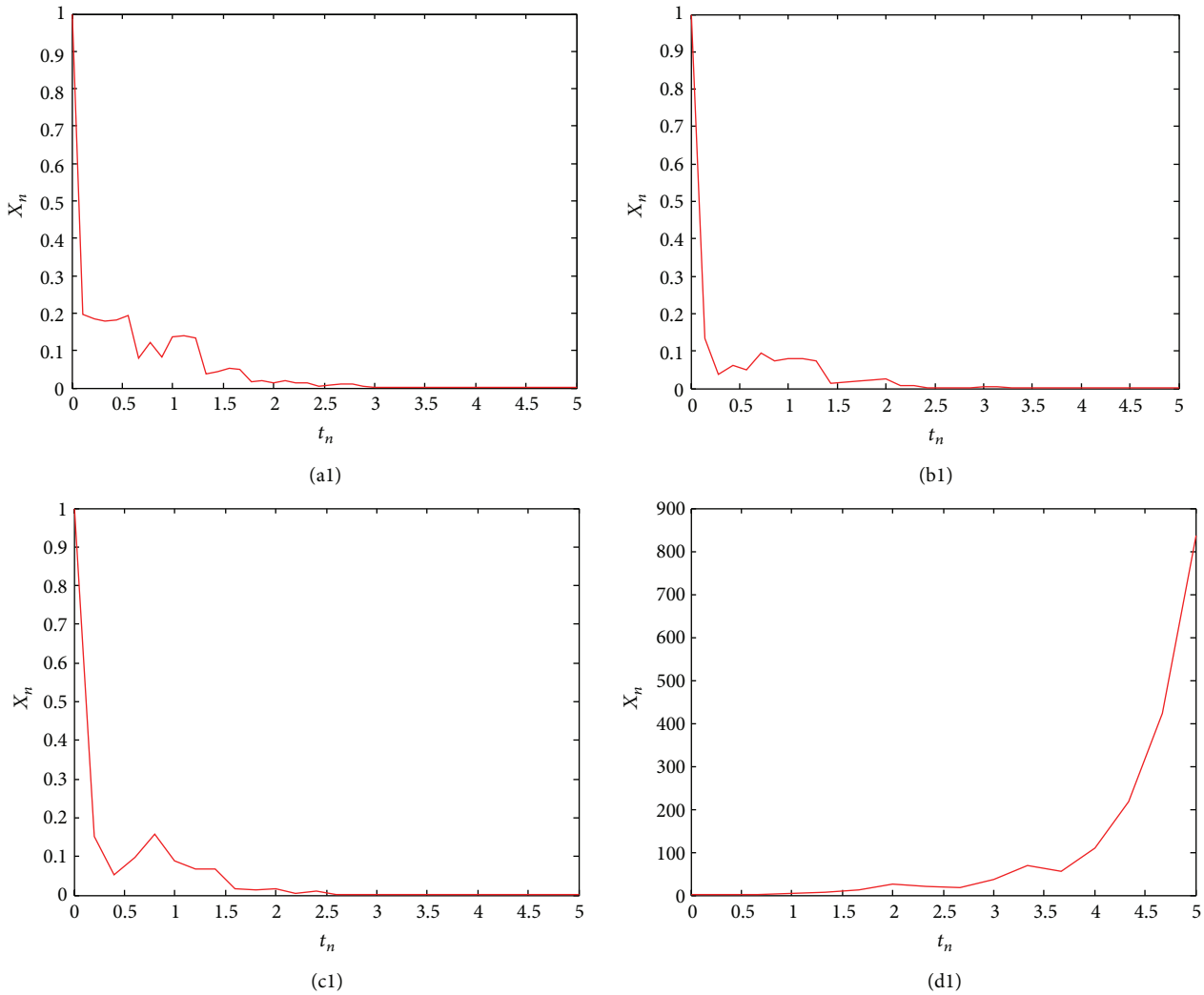

FIGURE 1: Simulations with the Heun method for (6) with $a = -7$, $b = 1$, and $c = 1$. (a1) $h = \frac{1}{9}$, (b1) $h = \frac{1}{7}$, (c1) $h = \frac{1}{5}$, and (d1) $h = 1/3$.

Proof. The condition in (10) gives $a < 0$. Denote

$$\begin{aligned}\widetilde{R}_1(h,a,b,c,d) &= \left|1 + \frac{ah + c\sqrt{h}}{1 + C_0 h + C_1\sqrt{h}}\right| \\ &\quad + \left|\frac{bh + d\sqrt{h}}{1 + C_0 h + C_1\sqrt{h}}\right|, \\ \widetilde{R}_2(h,a,b,c,d) &= \left|1 + \frac{ah - c\sqrt{h}}{1 + C_0 h + C_1\sqrt{h}}\right| \\ &\quad + \left|\frac{bh - d\sqrt{h}}{1 + C_0 h + C_1\sqrt{h}}\right|,\end{aligned} \tag{$***$}$$

and then $\widetilde{R}_T(h,a,b,c,d) = \widetilde{R}_1(h,a,b,c,d)\widetilde{R}_2(h,a,b,c,d)$ from (35). It is easy to know $a + b < 0$, $a - b < 0$ for $b \neq 0$.

If $c > 0, b > 0, d > 0$, we have $bh - d\sqrt{h} > 0$ from $h > H$ and

$$\begin{aligned} C_0 h + C_1\sqrt{h} + ah - c\sqrt{h} &\geq 0, \\ C_0 h + C_1\sqrt{h} + ah + c\sqrt{h} &\geq 0, \end{aligned} \tag{42}$$

from the assumption $F(h) \geq 0$, which yields $1 + C_0 h + C_1\sqrt{h} > 0$. Then $\widetilde{R}_1(h,a,b,c,d)$ and $\widetilde{R}_2(h,a,b,c,d)$ in $(***)$ become

$$\begin{aligned}\widetilde{R}_1(h,a,b,c,d) &= \frac{1 + C_0 h + C_1\sqrt{h} + ah + c\sqrt{h} + bh + d\sqrt{h}}{1 + C_0 h + C_1\sqrt{h}}, \\ \widetilde{R}_2(h,a,b,c,d) &= \frac{1 + C_0 h + C_1\sqrt{h} + ah - c\sqrt{h} + bh - d\sqrt{h}}{1 + C_0 h + C_1\sqrt{h}}, \\ \widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) &= 2 + \frac{2(a+b)h}{1 + C_0 h + C_1\sqrt{h}}.\end{aligned} \tag{43}$$

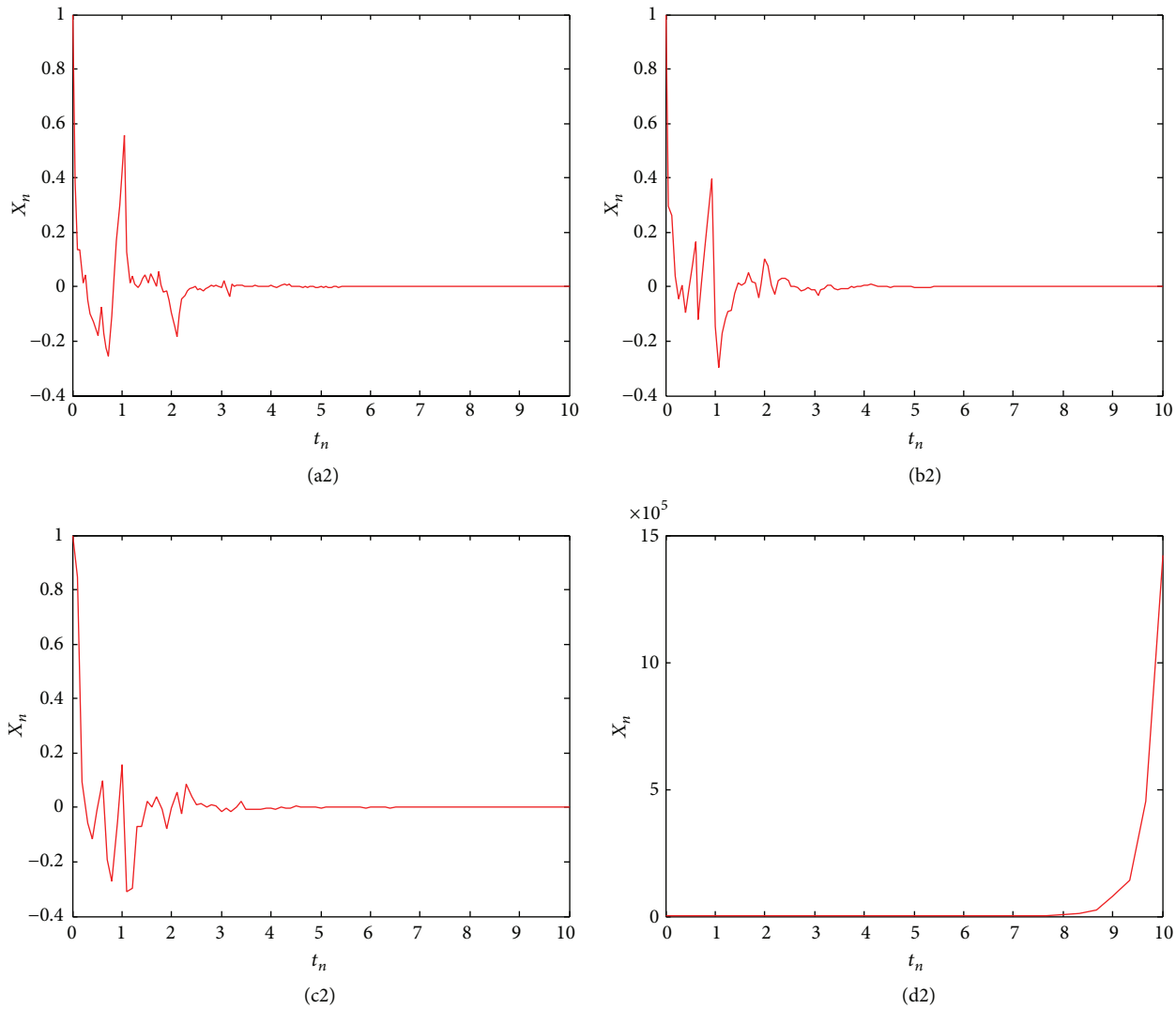

FIGURE 2: Simulations with the Heun method for (7) with $a = -9$, $b = -1$, $c = 1.1$, and $d = 0.9$. (a2) $h = 1/19$, (b2) $h = 1/15$, (c2) $h = 1/11$, and (d2) $h = 1/3$.

Since $a + b < 0, 1 + C_0 h + C_1 \sqrt{h} > 0$, we have $0 < \widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) < 2$, and $\widetilde{R}_T(h,a,b,c,d) = \widetilde{R}_1(h,a,b,c,d)\widetilde{R}_2(h,a,b,c,d) < 1$, which means that the balanced method in (32) is T-stable.

In the same way, we can prove that the balanced method in (32) is T-stable when

$$\begin{gathered} c > 0, \quad b > 0, \quad d < 0 \quad \text{or} \quad c > 0, \quad b < 0, \quad d > 0 \\ \text{or} \quad c > 0, \quad b < 0, \quad d < 0. \end{gathered} \tag{44}$$

If $c < 0, b > 0, d > 0$, we get $bh - d\sqrt{h} > 0$ from $h > H$ and

$$\begin{gathered} C_0 h + C_1 \sqrt{h} + ah + c\sqrt{h} \geq 0, \\ C_0 h + C_1 \sqrt{h} + ah - c\sqrt{h} \geq 0 \end{gathered} \tag{45}$$

from the assumption $F(h) \geq 0$, which applies $1+C_0 h+C_1\sqrt{h} > 0$. Then

$$\widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) = 2 + \frac{2(a+b)h}{1 + C_0 h + C_1\sqrt{h}} \tag{46}$$

from $(****)$. Since $a + b < 0, 1 + C_0 h + C_1\sqrt{h} > 0$, we have

$$0 < \widetilde{R}_1(h,a,b,c,d) + \widetilde{R}_2(h,a,b,c,d) < 2; \tag{47}$$

consequently, $\widetilde{R}_T(h,a,b,c,d) < 1$, which means that the balanced method in (32) is T-stable.

In the same way, we can prove that the balanced method in (32) is T-stable when

$$\begin{gathered} c < 0, \quad b > 0, \quad d < 0 \quad \text{or} \quad c < 0, \quad b < 0, \quad d > 0 \\ \text{or} \quad c < 0, \quad b < 0, \quad d < 0. \end{gathered} \tag{48}$$

The proof of Theorem 8 is complete. □

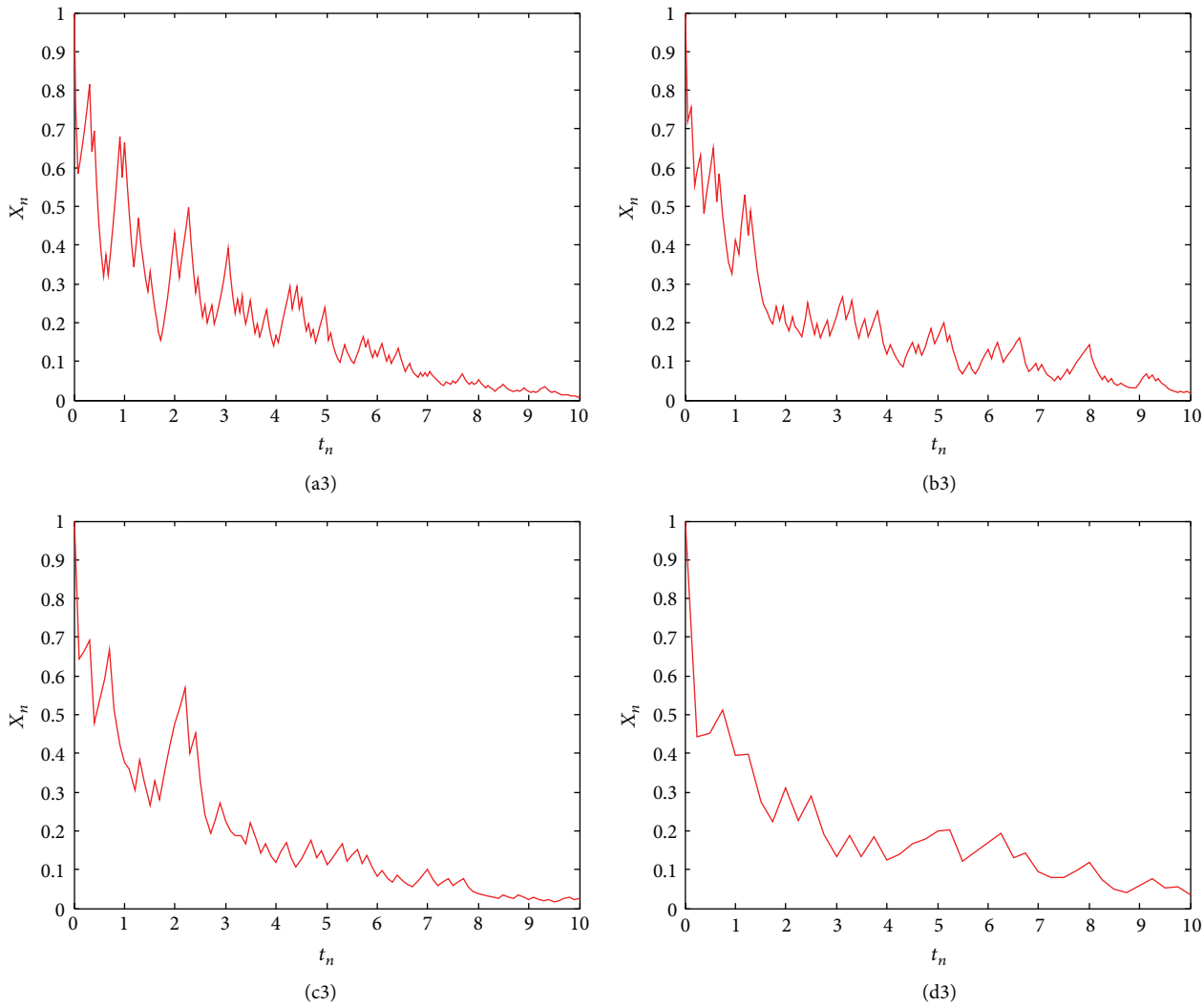

Figure 3: Simulations with the balanced method for (6) with $a = -3$, $b = 2$, and $c = 1$. (a3) $h = 1/22$, (b3) $h = 1/16$, (c3) $h = 1/10$, and (d3) $h = 1/4$.

5. Numerical Examples

Consider test equations (6) and (7) with $\tau = 1$, $\varphi(t) = t + 1$, $t \in [-1, 0]$. In the following figures t_n's are nodes, and X_n denotes the numerical solution at $t = t_n$.

Take $a = -7, b = 1$, and $c = 1$ in (6). From (18), we get $H = 1/4$, which means that the Heun method in (12) is T-stable if $h < 1/4$. Figure 1 shows that the Heun method in (12) is T-stable when $h = 1/9$, $h = 1/7$, and $h = 1/5$ but is unstable when $h = 1/3$, since $h = 1/3$ exceeds the range of h in Theorem 5, which verifies Theorem 5.

Take $a = -9, b = -1, c = 1.1$, and $d = 0.9$ in (7). From (26), we get $H_1 \approx 1/20$, $H_2 = 1/9$, which means that the Heun method in (13) is T-stable if $1/20 < h < 1/9$. Figure 2 shows that the Heun method in (13) is T-stable when $h = 1/19$, $h = 1/15$, and $h = 1/11$, but it is unstable when $h = 1/3$, since $h = 1/3$ exceeds the range of h in Theorem 6, which verifies Theorem 6.

Take $a = -3, b = 2$, and $c = 1$ in (6) and $C_0 = 1, C_1 = 2$ in the balanced method in (31). Then the balanced method in (31) is T-stable if $h \leq 1/4$ from Theorem 7. Figure 3 shows that the balanced method in (31) is T-stable when $h = 1/22$, $h = 1/16$, $h = 1/10$, and $h = 1/4$, since these h's are in the range of h in Theorem 7, which verifies Theorem 7.

Take $a = -5$, $b = 1$, $c = 1$, and $d = 0.2$ in (7) and $C_0 = 2$, $C_1 = 2$ in the balanced method (32). Then the balanced method in (32) is T-stable if $1/25 < h \leq 1/9$ from Theorem 8. Figure 4 shows that the balanced method in (32) is T-stable when $h = 1/15, h = 1/13$, $h = 1/11$, and $h = 1/9$, since these h's are in the range of h in Theorem 8, which verifies Theorem 8.

6. Conclusion

In this paper, T-stability of the Heun methods and the balanced methods for two kinds of linear SDDEs is studied.

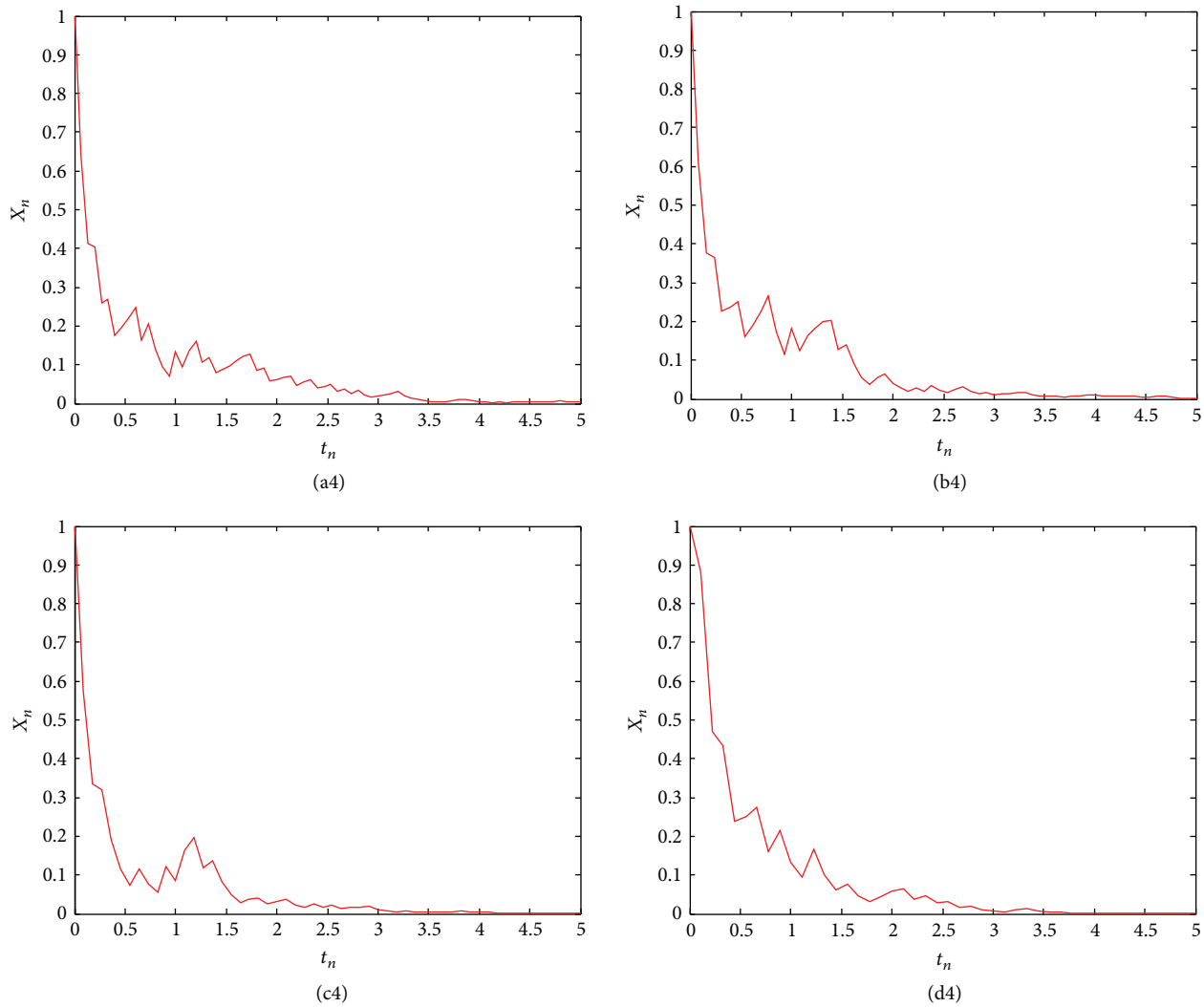

FIGURE 4: Simulations with the balanced method for (7) with $a = -5$, $b = 1$, $c = 1$, and $d = 0.2$. (a4) $h = 1/15$, (b4) $h = 1/13$, (c4) $h = 1/11$, and (d4) $h = 1/9$.

The Wiener increment of the numerical methods in this paper is approximated by a discrete random variable with two-point distribution in the process of the study of T-stability. The T-stable conditions for the Heun methods and balanced methods are given, respectively, and are verified by some numerical examples.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Some New Generating Functions for q-Hahn Polynomials

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We obtain some new generating functions for q-Hahn polynomials and give their proofs based on the homogeneous q-difference operator.

1. Introduction

Throughout this paper we suppose that $q \in \mathbb{C}$, $|q| < 1$, and the q-shifted factorials are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad n \geq 1. \tag{1}$$

Clearly,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2}$$

We also adopt the following compact notation for the multiple q-shifted factorials:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \qquad (a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{3}$$

The basic hypergeometric series or q-series ${}_r\phi_s$ are defined by

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_s; \end{matrix} \; q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \tag{4}$$

Euler identity is as follows:

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}. \tag{5}$$

The q-binomial theorem is as follows:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{6}$$

The usual q-differential operator or q-derivative operator D_q is defined by (see [1, Page 177, (2.1)])

$$D_q\{f(a)\} = \frac{f(a)-f(aq)}{a}, \qquad D_q^n\{f(a)\} = D_q\left\{D_q^{n-1}\{f(a)\}\right\}. \tag{7}$$

In [1], Chen and Liu introduced the q-exponential $T(bD_q)$ operator as follows (see [1, Page 17, (2.5)]):

$$T\left(bD_q\right) = \sum_{n=0}^{\infty}\frac{\left(bD_q\right)^n}{(q;q)_n}, \tag{8}$$

and they get the q-operator identity of $T(bD_q)$ (see [1, Page 178, Theorems 2.2 and 2.3]) as follows:

$$T\left(bD_q\right)\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{1}{(at,bt;q)_\infty}\quad |bt|<1,$$
$$T\left(bD_q\right)\left\{\frac{1}{(as,at;q)_\infty}\right\}=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}\quad |bt|<1. \tag{9}$$

Recently Chen et al. [2] introduced the following homogeneous q-difference D_{xy}

$$D_{xy}\{f(x,y)\}=\frac{f\left(x,q^{-1}y\right)-f(qx,y)}{x-q^{-1}y} \tag{10}$$

and the homogeneous q-difference operator $E(D_{xy})$:

$$E\left(D_{xy}\right)=\sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}. \tag{11}$$

They obtained some properties of D_{xy} as follows:

$$\begin{gathered}D_{xy}\{P_n(x,y)\}=\left(1-q^n\right)P_{n-1}(x,y),\\ D_{xy}\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=t\frac{(yt;q)_\infty}{(xt;q)_\infty},\\ D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=t^k\frac{(yt;q)_\infty}{(xt;q)_\infty},\\ E\left(D_{xy}\right)\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}.\end{gathered} \tag{12}$$

The classical Rogers-Szegö polynomial is defined by means of the generating function:

$$\sum_{n=0}^{\infty}h_n(x\mid q)\frac{t^n}{(q;q)_n}=\frac{1}{(t,xt;q)_\infty},\quad |t|<1; \tag{13}$$

obviously, we have

$$T\left(D_q\right)\{x^n\}=h_n(x\mid q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k. \tag{14}$$

The homogeneous Rogers-Szegö polynomial is defined by

$$h_n(x,y\mid q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,y), \tag{15}$$

where $P_n(x,y)=(x-y)(x-yq)\cdots(x-yq^{n-1})$. Clearly, $h_n(x,y\mid q)=\Phi_n^{(y/x)}(x)$ are the Cauchy polynomials with the following generating function:

$$\sum_{k=0}^{\infty}P_k(x,y)\frac{z^k}{(q;q)_k}=\frac{(yz;q)_\infty}{(xz;q)_\infty},\quad |xz|<1. \tag{16}$$

From the above properties, we have

$$E\left(D_{xy}\right)\{P_n(x,y)\}=h_n(x,y\mid q), \tag{17}$$

$$\sum_{n=0}^{\infty}h_n(x,y\mid q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}. \tag{18}$$

Lemma 1 (see [3, Lemma 2.3]). *For $|t|,|xt|<1$,*

$$\begin{aligned}&E\left(D_{xy}\right)\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_n(x,y)}{(yt;q)_n}\right\}\\ &\quad=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(y,xt;q)_k}{(yt;q)_k}x^{n-k}.\end{aligned} \tag{19}$$

q-Hahn polynomial is defined by [4]

$$\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\frac{t^n}{(q;q)_n}=\frac{(axt;q)_\infty}{(t,xt;q)_\infty}. \tag{20}$$

We have

$$\Phi_n^{(a)}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_kx^k. \tag{21}$$

Clearly, $\Phi_n^{(0)}(x)=h_n(x\mid q)$.

Recently, Chen et al. [3] gave some new proofs of the following results based on the method of homogeneous q-difference operator $E(D_{xy})$.

Theorem 2. *Consider the following:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\Phi_n^{(b)}(y)\frac{t^n}{(q;q)_n}\\ &\quad=\frac{(xat,ybt;q)_\infty}{(t,xt,yt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}t,a,b\\xat,ybt;\end{matrix}\;q,xyt\right).\end{aligned} \tag{22}$$

Theorem 3. *Consider the following:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Phi_{m+n}^{(a)}(x)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\ &\quad=\frac{(xas;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}xa,xs\\xas;\end{matrix}\;q,t\right).\end{aligned} \tag{23}$$

For more references on the q-difference operators, see [1, 5–16].

In the present paper, we obtain some new generating functions for q-Hahn polynomials and give their proofs based on the homogeneous q-difference operator.

2. Some New Generating Functions for q-Hahn Polynomial

In the present section we obtain the following new generating functions of q-Hahn polynomial.

Theorem 4. *For* $|z| < 1$,

$$\sum_{k=0}^{\infty}\Phi_{n+k}^{(a)}(x)\frac{z^k}{(q;q)_k}=\frac{(axz;q)_\infty}{(z,xz;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a,z;q)_k}{(axz;q)_k}x^k. \tag{24}$$

Proof. Let $x\mapsto y$ and $a\mapsto b$ in (21), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\Phi_n^{(b)}(y)\frac{z^n}{(q;q)_n}&=\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(b;q)_k y^k\frac{z^n}{(q;q)_n}\\&=\sum_{k,n=0}^{\infty}\Phi_{n+k}^{(a)}(x)\frac{(b;q)_k z^n}{(q;q)_n(q;q)_k}(yz)^k.\end{aligned} \tag{25}$$

By the q-binomial theorem (6) and noting that $(b;q)_{n+k}=(bq^k;q)_n(b;q)_k$, we have

$$\begin{aligned}&\frac{(xaz,ybz;q)_\infty}{(z,xz,yz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,z;q)_k}{(axz,byz,q;q)_k}(xyz)^k\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,z;q)_k}{(axz,q;q)_k}(xyz)^k\frac{(byzq^k;q)_\infty}{(yz;q)_\infty}\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,z;q)_k}{(axz,q;q)_k}(xyz)^k\sum_{n=0}^{\infty}\frac{(bq^k;q)_n}{(q;q)_n}(yz)^n\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{n,k=0}^{\infty}\frac{(a,z;q)_k}{(axz,q;q)_k}\frac{(b;q)_{n+k}}{(q;q)_n}x^k(yz)^{n+k}.\end{aligned} \tag{26}$$

By (17), (25), and (26), we obtain

$$\begin{aligned}&\sum_{k,n=0}^{\infty}\Phi_{n+k}^{(a)}(x)\frac{(b;q)_k z^n}{(q;q)_n(q;q)_k}(yz)^k\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{n,k=0}^{\infty}\frac{(a,z;q)_k}{(axz,q;q)_k}\frac{(b;q)_{n+k}}{(q;q)_n}x^k(yz)^{n+k}.\end{aligned} \tag{27}$$

Comparing the coefficients of $y^k/(q;q)_k$ on both sides of (27), we obtain the formula (24) immediately. This proof is complete. □

Theorem 5. *For* $|t| < 1$,

$$\begin{aligned}&\sum_{n=0}^{\infty}\Phi_{m+n}^{(a)}(x)\Phi_n^{(b)}(y)\frac{t^n}{(q;q)_n}\\&=\frac{(xat;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(b;q)_k(xyt)^k}{(q;q)_k}\sum_{j=0}^{m+k}\begin{bmatrix}m+k\\j\end{bmatrix}\frac{(xa,xt;q)_j}{(xat;q)_j}x^{m-j}.\end{aligned} \tag{28}$$

Proof. By (17) and (19), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}h_{m+n}(x,y\mid q)h_n(u,v\mid q)\frac{t^n}{(q;q)_n}\\&=\sum_{n=0}^{\infty}E(D_{xy})\{P_{m+n}(x,y)\}h_n(u,v\mid q)\frac{t^n}{(q;q)_n}\\&=E(D_{xy})\left\{\sum_{n=0}^{\infty}P_{m+n}(x,y)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(u,v)\frac{t^n}{(q;q)_n}\right\}\\&=E(D_{xy})\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{P_{m+n+k}(x,y)P_k(u,v)t^{n+k}}{(q;q)_k(q;q)_n}\right\}\\&=E(D_{xy})\left\{\sum_{k=0}^{\infty}\frac{P_{m+k}(x,y)P_k(u,v)t^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{P_n(x,yq^{m+k})t^n}{(q;q)_n}\right\}\\&=E(D_{xy})\left\{\sum_{k=0}^{\infty}\frac{P_{m+k}(x,y)P_k(u,v)t^k}{(q;q)_k}\frac{(yt;q)_\infty}{(xt;q)_\infty(yt;q)_{m+k}}\right\}\\&=\sum_{k=0}^{\infty}\frac{P_k(u,v)t^k}{(q;q)_k}E(D_{xy})\left\{\frac{(yt;q)_\infty P_{m+k}(x,y)}{(xt;q)_\infty(yt;q)_{m+k}}\right\}\\&=\sum_{k=0}^{\infty}\frac{P_k(u,v)t^k}{(q;q)_k}\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{m+k}\begin{bmatrix}m+k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{m+k-j}\\&=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(v/u;q)_k(utx)^k}{(q;q)_k}\\&\quad\times\sum_{j=0}^{m+k}\begin{bmatrix}m+k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{m-j}.\end{aligned} \tag{29}$$

Setting $y/x=a$, $v/u=b$, $u=y$ in the last sum, we obtain the formula (28) of Theorem 5. This proof is complete. □

Theorem 6. *For* $|l| < 1$, $|s| < 1$, $|t| < 1$,

$$\begin{aligned}&\sum_{m,n,k=0}^{\infty}\Phi_{m+k}^{(a)}(x)\Phi_{n+k}^{(b)}(y)\frac{l^m s^n t^k}{(q;q)_m(q;q)_n(q;q)_k}\\&=\frac{(xal,ybs;q)_\infty}{(l,xl,s,ys;q)_\infty}\sum_{k=0}^{\infty}\frac{t^k}{(q;q)_k}\\&\quad\times\sum_{i,j=0}^{\infty}\begin{bmatrix}k\\i\end{bmatrix}\begin{bmatrix}k\\j\end{bmatrix}\frac{(xa,xl;q)_i}{(xal;q)_i}\frac{(yb,ys;q)_j}{(ybs;q)_j}x^{k-i}y^{k-j}.\end{aligned} \tag{30}$$

Proof. By (17) and (19), we have

$$\begin{aligned}&\sum_{m,n,k=0}^{\infty}h_{m+k}(x,y\mid q)h_{n+k}(u,v\mid q)\\&\quad\times\frac{l^m s^n t^k}{(q;q)_m(q;q)_n(q;q)_k}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m,n,k=0}^{\infty} E\left(D_{xy}\right)\left\{P_{m+k}(x,y)\right\} E\left(D_{uv}\right)\left\{P_{n+k}(u,v)\right\} \\
&\quad \times \frac{l^m s^n t^k}{(q;q)_m (q;q)_n (q;q)_k} \\
&= E\left(D_{xy}\right) E\left(D_{uv}\right)\left\{\sum_{m,n,k=0}^{\infty} \frac{P_{m+k}(x,y) P_{n+k}(u,v) l^m s^n t^k}{(q;q)_m (q;q)_n (q;q)_k}\right\} \\
&= E\left(D_{xy}\right) E\left(D_{uv}\right)\left\{\sum_{k=0}^{\infty} \frac{P_k(x,y) P_k(u,v) t^k}{(q;q)_k}\right. \\
&\qquad \left. \times \sum_{m=0}^{\infty} \frac{P_m\left(x, yq^k\right) l^m}{(q;q)_m} \sum_{n=0}^{\infty} \frac{P_n\left(u, vq^k\right) s^n}{(q;q)_n}\right\} \\
&= E\left(D_{xy}\right) E\left(D_{uv}\right)\left\{\sum_{k=0}^{\infty} \frac{P_k(x,y) P_k(u,v) t^k}{(q;q)_k}\right. \\
&\qquad \left. \times \frac{\left(ylq^k;q\right)_\infty}{(xl;q)_\infty} \frac{\left(vsq^k;q\right)_\infty}{(us;q)_\infty}\right\} \\
&= \sum_{k=0}^{\infty} E\left(D_{xy}\right)\left\{\frac{(yl;q)_\infty P_k(x,y)}{(xl;q)_\infty (yl;q)_k}\right\} \\
&\quad \times E\left(D_{uv}\right)\left\{\frac{(vs;q)_\infty P_k(u,v)}{(us;q)_\infty (vs;q)_k}\right\} \frac{t^k}{(q;q)_k} \\
&= \sum_{k=0}^{\infty}\left\{\frac{(yl;q)_\infty}{(l,xl;q)_\infty} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \frac{(y,xl;q)_i}{(yl;q)_i} x^{k-i}\right\} \\
&\quad \times \left\{\frac{(vs;q)_\infty}{(s,us;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(v,us;q)_j}{(vs;q)_j} u^{k-j}\right\} \frac{t^k}{(q;q)_k} \\
&= \frac{(yl,vs;q)_\infty}{(l,s,xl,us;q)_\infty} \\
&\quad \times \sum_{k=0}^{\infty} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(y,xl;q)_i (v,us;q)_j x^{k-i} u^{k-j} t^k}{(yl;q)_i (vs;q)_j (q;q)_k}.
\end{aligned} \tag{31}
$$

Setting $y/x = a$, $v/u = b$, $u = y$ in the last sum, we obtain the formula (30) of Theorem 6. This proof is complete. □

Theorem 7. *For $|t| < 1$,*

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \Phi_{m+k}^{(a)}(x) \Phi_{n+k}^{(b)}(y) \frac{t^k}{(q;q)_k} \\
&\quad = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{\infty} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (by;q)_j \\
&\qquad \times x^{k-i} y^{k-j} \Phi_m^{(a)}\left(xq^i\right) \Phi_n^{(b)}\left(yq^j\right).
\end{aligned} \tag{32}
$$

Proof. Applying (2) and the Euler identity (5) and noting (21), then the right-hand side is equal to (30) as follows:

$$
\begin{aligned}
&\frac{(xal, ybs;q)_\infty}{(l,xl,s,ys;q)_\infty} \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \\
&\quad \times \sum_{i,j=0}^{\infty} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(xa,xl;q)_i}{(xal;q)_i} \frac{(yb,ys;q)_j}{(ybs;q)_j} x^{k-i} y^{k-j} \\
&\quad = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \\
&\qquad \times \frac{\left(xalq^i, ybsq^j;q\right)_\infty}{(l,s,xlq^i,ysq^j;q)_\infty} \\
&\quad = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \\
&\qquad \times \sum_{u,v,m,n=0}^{\infty} \frac{(a;q)_m (b;q)_n \left(xlq^i\right)^m \left(ysq^j\right)^n l^u s^v}{(q;q)_m (q;q)_n (q;q)_u (q;q)_v} \\
&\quad = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \\
&\qquad \times \sum_{u,v,m,n=0}^{\infty} \frac{(a;q)_m (b;q)_n \left(xlq^i\right)^m \left(ysq^j\right)^n}{(q;q)_m (q;q)_n (q;q)_u (q;q)_v} l^{m+u} s^{n+v}.
\end{aligned} \tag{33}
$$

By (30) and (33), we have

$$
\begin{aligned}
&\sum_{m,n,k=0}^{\infty} \Phi_{m+k}^{(a)}(x) \Phi_{n+k}^{(b)}(y) \frac{l^m s^n t^k}{(q;q)_m (q;q)_n (q;q)_k} \\
&\quad = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \\
&\qquad \times \sum_{u,v,m,n=0}^{\infty} \frac{(a;q)_m (b;q)_n \left(xlq^i\right)^m \left(ysq^j\right)^n}{(q;q)_m (q;q)_n (q;q)_u (q;q)_v} l^{m+u} s^{n+v}.
\end{aligned} \tag{34}
$$

Comparing the coefficients of $l^m s^n/(q;q)_m (q;q)_n$ on both sides of (34), we obtain the formula (32) immediately. □

Theorem 8. *For $|t| < 1$,*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \Phi_{m+n}^{(a)}(x) \Phi_n^{(b)}(y) \frac{t^n}{(q;q)_n} \\
&\quad = \frac{(xyat, xybt;q)_\infty}{(xyt, xt, yt;q)_\infty} \\
&\qquad \times \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \frac{(yt;q)_s}{(xyat;q)_s} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} xyt, xa, yb \\ xyatq^s, xybt; \end{matrix} \; q, tq^s\right).
\end{aligned} \tag{35}
$$

Proof. Set $n = 0$ and then let $k \mapsto n$ in (32) and note that $\Phi_0^{(b)}(x) = 1$; by (21) and (22), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Phi_{m+n}^{(a)}(x)\, \Phi_n^{(b)}(y)\, \frac{t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{i,j=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} (xa;q)_i \\
&\qquad \times (yb;q)_j x^{n-i} y^{n-j} \Phi_m^{(a)}\left(xq^i\right) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{i,j=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{n-i} y^{n-j} \\
&\quad \times \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s \left(xq^i\right)^s \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s y^n \\
&\quad \times \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (ax;q)_i x^{s+n} \left(\frac{q^s}{x}\right)^i \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (by;q)_j \left(\frac{1}{y}\right)^j \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^{s+n} y^n \Phi_n^{(xa)}\left(\frac{q^s}{x}\right) \Phi_n^{(yb)}\left(\frac{1}{y}\right) \\
&= \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \sum_{n=0}^{\infty} \Phi_n^{(xa)}\left(\frac{q^s}{x}\right) \Phi_n^{(yb)}\left(\frac{1}{y}\right) \frac{(xyt)^n}{(q;q)_n} \\
&= \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \frac{(xytaq^s, xybt;q)_\infty}{(xyt, ytq^s, xt;q)^\infty} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} xyt, xa, yb \\ xyatq^s, xybt; \end{matrix} \quad q, tq^s\right) \\
&= \frac{(xyat, xybt;q)_\infty}{(xyt, xt, yt;q)_\infty} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \frac{(yt;q)_s}{(xyat;q)_s} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} xyt, xa, yb \\ xyatq^s, xybt; \end{matrix} \quad q, tq^s\right).
\end{aligned} \tag{36}$$

This proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Bargmann Type Systems for the Generalization of Toda Lattices

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Under a constraint between the potentials and eigenfunctions, the nonlinearization of the Lax pairs associated with the discrete hierarchy of a generalization of the Toda lattice equation is proposed, which leads to a new symplectic map and a class of finite-dimensional Hamiltonian systems. The generating function of the integrals of motion is presented, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Finally, the representation of solutions for a lattice equation in the discrete hierarchy is obtained.

1. Introduction

Differential difference equations have very remarkable applications in modern mathematics and physics; they can model a number of physically interesting phenomena, such as the vibration of particle in lattice [1], the quantum spin chains [2, 3], the Toda lattice [4], the vibration of pulse [5, 6], the nonlinear self-dual network [7], and others. After Toda [8] showed that the Toda lattice was associated with a discretization of the Schrödinger spectral problem, various discrete soliton equations are found, for instance, the discrete nonlinear Schrödinger equation [9], the discrete sine-Gordon equation [10], the discrete KdV equation [11], the discrete mKdV equation [12], and so forth. Recently, the authors have obtained a new discrete hierarchy associated with fourth-order discrete spectral problem, in which a typical member is a generalization of the Toda lattice equation [13].

It has been known that the key to complete integrability of a finite-dimensional Hamiltonian system is the existence of an involutive system of conserved integrals according to the Liouville-Arnold theorem. Many researchers have tried to construct complete integrable Hamiltonian systems. Recently, there are active researches on soliton hierarchies associated with so $(3, \mathbb{R})$ [14]. However, it is a difficult work to search for an involutive system of conserved integrals for a given finite-dimensional Hamiltonian system. An effective method, the nonlinearization of Lax pairs [15, 16], has been developed and applied to various soliton hierarchies associated with 2×2 matrix spectral problems to get finite-dimensional completely integrable systems many years ago, such as the nonlinearization of the AKNS hierarchy [15], the coupled KdV hierarchy [17], the discrete Ablowitz-Ladik hierarchy [18], the Heisenberg hierarchy [19], and the Kac-van Moerbeke hierarchy [20]. Subsequently, this method has been generalized to discuss the nonlinearization of Lax pairs and adjoint Lax pairs of soliton hierarchies [21–23]. Moreover, there are attempts to apply the nonlinearization method to the Lax pairs and adjoint Lax pairs of $(2+1)$-dimensional soliton systems, such as the Kadomtsev-Petviashvili equation and the Davey-Stewartson equation, in order to get $(1+1)$-dimensional integrable systems [24]. And it is proved that the binary nonlinearization will be more natural to carry out in the case of higher-order matrix spectral problems [25].

Discrete versions of classical integrable systems have become the focus of common concern in recent years because of their importance. However, the known discrete integrable systems are few compared with the continuous case. In the present paper, the nonlinearization approach is developed and applied to the discrete hierarchy associated with a 4×4 discrete eigenvalue problem. Such transformations are adjoint symmetry constraints [26] and a general scheme for doing nonlinearization for lattice soliton hierarchies was presented in [27]. We propose a constraint between the

potentials and eigenfunctions. The nonlinearization of the Lax pairs for the discrete hierarchy leads to a new integrable symplectic map and a class of finite-dimensional integrable Hamiltonian systems.

The outline of this paper is as follows. In Section 2, depending on the spectral problems given in [13], the Bargmann constraint between the potentials and eigenfunctions is introduced, from which a new symplectic map and a class of finite-dimensional Hamiltonian systems are obtained. In Section 3, the generating function approach is used to calculate the involutivity of integrals, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Finally, in Section 4, the representation of solutions for a lattice equation in the discrete hierarchy is obtained.

2. A New Symplectic Map

Consider the discrete 4×4 spectral problem given in [13]

$$E\psi_n = U_n\psi_n, \qquad U_n = \begin{pmatrix} -\frac{a_n}{c_n} & -\frac{1}{c_n} & 0 & \frac{\lambda - b_n}{c_n} \\ 0 & 0 & 1 & a_n \\ 0 & 0 & 0 & c_n \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \\ \psi_n^4 \end{pmatrix}, \tag{1}$$

where a_n, b_n, c_n are three potentials and λ is a constant spectral parameter; E is a translation operator defined by $Ef_n = f_{n+1}$. For the sake of convenience, we usually denote $f_{n+k} = E^k f_n$, $f_{n-k} = E^{-k} f_n$. In order to derive the hierarchy of Lattice equations associated with (1), authors of [13] first solve the stationary discrete zero-curvature equation:

$$V_{n+1}U_n - U_nV_n = 0, \qquad V_n = \left(V_{n,ij}\right)_{4\times 4}, \tag{2}$$

where the entries V_{ij} of the matrix V_n are Laurent expansions of λ. Let ψ_n satisfy the spectral problem (1) and its auxiliary problem:

$$\psi_{n,t} = V_n^{(m)}\psi_n, \qquad V_n^{(m)} = \left(\lambda^m V_n\right)_+; \tag{3}$$

then the zero-curvature equation $U_{n,t} = V_{n+1}^{(m)}U_n - U_nV_n^{(m)}$ yields the discrete hierarchy of a generalization of Toda lattices. The first system of evolution equations in this hierarchy is

$$\begin{aligned} a_{n,t} &= \frac{1}{2}a_n\left(b_n - b_{n+1}\right) + a_{n-1}c_{n-1} - a_{n+1}c_n, \\ b_{n,t} &= a_{n-1}^2 - a_n^2 + c_{n-2}^2 - c_n^2, \\ c_{n,t} &= \frac{1}{2}c_n\left(b_n - b_{n+2}\right), \end{aligned} \tag{4}$$

which is a generalization of Toda lattice equation.

Let $\lambda_1, \ldots, \lambda_N$ be N distinct nonzero eigenvalues of (1), and the associated eigenfunctions are denoted by

$$\begin{aligned} q_j^1 &= \psi_n^3\left(\lambda_j\right), \qquad q_j^2 = \psi_n^2\left(\lambda_j\right), \\ p_j^1 &= \psi_n^1\left(\lambda_j\right), \qquad p_j^2 = \psi_n^4\left(\lambda_j\right), \end{aligned} \tag{5}$$

where we denote $q_j^k = q_j^k(n)$ and $p_j^k = p_j^k(n)$ $(k = 1, 2)$ for convenience. Then the system associated with (1) can be written in the form

$$\begin{aligned} Eq_j^1 &= c_n p_j^2, \qquad Eq_j^2 = q_j^1 + a_n p_j^2, \\ Ep_j^1 &= -\frac{1}{c_n}\left(a_n p_j^1 + q_j^2 - \lambda_j p_j^2 + b_n p_j^2\right), \qquad Ep_j^2 = p_j^1. \end{aligned} \tag{6}$$

Now we consider the Bargmann constraint

$$\sum_{j=1}^{N} \nabla\lambda_j = G_n^{(0)}, \tag{7}$$

where $G_n^{(0)} = (2a_n, b_n, 2c_n)^{\mathrm{T}}$ and $\nabla\lambda_j$ is the functional gradient of the eigenvalue λ_j with regard to the potentials a_n, b_n, and c_n; that is,

$$\nabla\lambda_j = \begin{pmatrix} \frac{\delta\lambda_j}{\delta a_n} \\ \frac{\delta\lambda_j}{\delta b_n} \\ \frac{\delta\lambda_j}{\delta c_n} \end{pmatrix} = \begin{pmatrix} 2p_j^1 p_j^2 \\ p_j^2 p_j^2 \\ -\frac{2}{c_n}\left[a_n p_j^1 p_j^2 + q_j^2 p_j^2 + \left(\lambda_j - b_n\right) p_j^2 p_j^2\right] \end{pmatrix}. \tag{8}$$

Combining (7) and (8), it is easy to see that

$$\begin{aligned} a_n &= \langle p^1, p^2\rangle, \qquad b_n = \langle p^2, p^2\rangle, \\ c_n &= \left(-\langle p^1, p^2\rangle^2 - \langle q^2, p^2\rangle + \langle \Lambda p^2, p^2\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{1/2}, \end{aligned} \tag{9}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $\langle\cdot,\cdot\rangle$ is the standard inner-product in $\mathbb{R}^N$, $q^i = (q_1^i, \ldots, q_N^i)^{\mathrm{T}}$ and $p^i = (p_1^i, \ldots, p_N^i)^{\mathrm{T}}$. Substituting (9) into (6), we can get the following system:

$$\begin{aligned} Eq_j^1 &= \left(-\left\langle p^1, p^2\right\rangle^2 - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{1/2} p_j^2, \\ Eq_j^2 &= q_j^1 + \left\langle p^1, p^2\right\rangle p_j^2, \\ Ep_j^1 &= -\left(-\left\langle p^1, p^2\right\rangle^2 - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{-1/2} \\ &\quad \times \left(\left\langle p^1, p^2\right\rangle p_j^1 + q_j^2 - \lambda_j p_j^2 + \left\langle p^2, p^2\right\rangle p_j^2\right), \\ Ep_j^2 &= p_j^1. \end{aligned} \tag{10}$$

Through tedious calculations one infers

$$\sum_{j=1}^{N}\sum_{i=1}^{2} d\left(Eq_j^i\right)\wedge d\left(Eq_j^i\right)=\sum_{j=1}^{N}\sum_{i=1}^{2} dq_j^i\wedge dq_j^i. \tag{11}$$

Therefore, (10) determines a symplectic map H of the Bargmann type:

$$\left(Eq^1, Eq^2, Ep^1, Ep^2\right)=H\left(q^1, q^2, p^1, p^2\right). \tag{12}$$

3. Liouville Integrability

Introducing a matrix $\mathscr{V}_\lambda$,

$$\mathscr{V}_\lambda=\left(\mathscr{V}_\lambda^{ij}\right)_{4\times4}=\begin{pmatrix} -Q_\lambda\left(q^1,p^1\right)-\dfrac{\lambda}{2} & Q_\lambda\left(p^1,p^2\right) & Q_\lambda\left(p^1,p^1\right)+1 & -Q_\lambda\left(q^2,p^1\right) \\ -Q_\lambda\left(q^1,q^2\right)-\left\langle q^1,p^2\right\rangle & Q_\lambda\left(q^2,p^2\right)+\dfrac{\lambda}{2} & Q_\lambda\left(q^2,p^1\right) & -Q_\lambda\left(q^2,q^2\right)-\left\langle q^2,p^2\right\rangle \\ -Q_\lambda\left(q^1,q^1\right)-\left\langle q^1,p^1\right\rangle & Q_\lambda\left(q^1,p^2\right) & Q_\lambda\left(q^1,p^1\right)+\dfrac{\lambda}{2} & -Q_\lambda\left(q^1,q^2\right)-\left\langle q^1,p^2\right\rangle \\ -Q_\lambda\left(q^1,p^2\right) & Q_\lambda\left(p^2,p^2\right)+1 & Q_\lambda\left(p^1,p^2\right) & -Q_\lambda\left(q^2,p^2\right)-\dfrac{\lambda}{2} \end{pmatrix}, \tag{13}$$

where

$$Q_\lambda\left(q^i,p^j\right)=\sum_{k=1}^{N}\frac{q_k^i p_k^j}{\lambda-\lambda_k},\qquad Q_\lambda\left(p^i,p^j\right)=\sum_{k=1}^{N}\frac{p_k^i p_k^j}{\lambda-\lambda_k},$$
$$Q_\lambda\left(q^i,q^j\right)=\sum_{k=1}^{N}\frac{q_k^i q_k^j}{\lambda-\lambda_k}. \tag{14}$$

We can find that $\mathscr{V}_\lambda$ and $\mu I-\mathscr{V}_\lambda$ are two solutions of the stationary discrete zero-curvature equation (2) under the Bargmann constraint (7), where μ is a parameter and I is a 4×4 unit matrix. Then we assert that $\det\mathscr{V}_\lambda$ and $\det(\mu I-\mathscr{V}_\lambda)$ are independent constants of the discrete variable n. On the other hand,

$$\det\left(\mu I-\mathscr{V}_\lambda\right)=\mu^4+\mathscr{F}_\lambda^{(1)}\mu^2+\mathscr{F}_\lambda^{(2)}, \tag{15}$$

where

$$\mathscr{F}_\lambda^{(1)}=\sum_{1\le i<j\le4}\begin{vmatrix}\mathscr{V}_\lambda^{ii} & \mathscr{V}_\lambda^{ij}\\ \mathscr{V}_\lambda^{ji} & \mathscr{V}_\lambda^{jj}\end{vmatrix},\qquad \mathscr{F}_\lambda^{(2)}=\det\mathscr{V}_\lambda. \tag{16}$$

Substituting the Laurent expansion of $Q_\lambda(q^i,p^j)$, $Q_\lambda(p^i,p^j)$, $Q_\lambda(q^i,q^j)$ into (16) we have

$$\mathscr{F}_\lambda^{(1)}=-\frac{1}{2}\lambda^2+\sum_{m\ge1}F_m^{(1)}\lambda^{-m},\qquad \widehat{\mathscr{F}}_\lambda^{(2)}=\sum_{m\ge1}F_m^{(2)}\lambda^{-m}, \tag{17}$$

where

$$\widehat{\mathscr{F}}_\lambda^{(2)}=\mathscr{F}_\lambda^{(2)}+\frac{1}{4}\mathscr{F}_\lambda^{(1)}\lambda^2+\frac{1}{16}\lambda^4,$$

$$\begin{aligned}
F_m^{(1)} &= \sum_{\substack{l+k=m-2\\ l,k\ge0}}\Big(-\left\langle\Lambda^j q^1,p^1\right\rangle\left\langle\Lambda^k q^1,p^1\right\rangle-\left\langle\Lambda^j q^2,p^2\right\rangle\left\langle\Lambda^k q^2,p^2\right\rangle \\
&\quad+\left\langle\Lambda^j p^1,p^1\right\rangle\left\langle\Lambda^k p^1,p^1\right\rangle \\
&\quad+\left\langle\Lambda^j p^2,p^1\right\rangle\left\langle\Lambda^k q^2,q^2\right\rangle+2\left\langle\Lambda^j p^1,p^2\right\rangle \\
&\quad\times\left\langle\Lambda^k q^1,q^2\right\rangle-2\left\langle\Lambda^j q^1,p^2\right\rangle\left\langle\Lambda^k q^2,p^1\right\rangle\Big) \\
&\quad+\left\langle q^1,p^1\right\rangle\left\langle\Lambda^{m-1}p^1,p^1\right\rangle+\left\langle q^2,p^2\right\rangle\left\langle\Lambda^{m-1}p^2,p^2\right\rangle \\
&\quad+2\left\langle q^1,p^2\right\rangle\left\langle\Lambda^{m-1}p^1,p^2\right\rangle+\left\langle\Lambda^{m-1}q^1,q^1\right\rangle \\
&\quad+\left\langle\Lambda^{m-1}q^2,q^2\right\rangle-\left\langle\Lambda^m q^1,p^2\right\rangle-\left\langle\Lambda^m q^2,p^2\right\rangle,
\end{aligned}$$

$$\begin{aligned}
F_m^{(2)} &= \sum_{\substack{k+j+s+i=m-4\\ j,k,s,i\ge0}}\begin{vmatrix}\left\langle\Lambda^k q^1,p^1\right\rangle & \left\langle\Lambda^k p^1,p^2\right\rangle & \left\langle\Lambda^k p^1,p^1\right\rangle & \left\langle\Lambda^k q^2,p^1\right\rangle\\ \left\langle\Lambda^j q^1,q^2\right\rangle & \left\langle\Lambda^j q^2,p^2\right\rangle & \left\langle\Lambda^j q^2,p^1\right\rangle & \left\langle\Lambda^j q^2,q^2\right\rangle\\ \left\langle\Lambda^i q^1,q^1\right\rangle & \left\langle\Lambda^i q^1,p^2\right\rangle & \left\langle\Lambda^i q^1,p^1\right\rangle & \left\langle\Lambda^i q^1,q^2\right\rangle\\ \left\langle\Lambda^s q^1,p^2\right\rangle & \left\langle\Lambda^s p^2,p^2\right\rangle & \left\langle\Lambda^s p^1,p^2\right\rangle & \left\langle\Lambda^s q^2,p^2\right\rangle\end{vmatrix} \\
&\quad+\sum_{\substack{k+j+s=m-2\\ j,k,s\ge0}}\left(\begin{vmatrix}\left\langle\Lambda^j q^2,p^2\right\rangle & \left\langle\Lambda^j q^2,p^1\right\rangle & \left\langle\Lambda^j q^2,q^2\right\rangle\\ \left\langle\Lambda^k q^1,p^2\right\rangle & \left\langle\Lambda^k q^1,p^1\right\rangle & \left\langle\Lambda^k q^1,q^2\right\rangle\\ \left\langle\Lambda^s p^2,p^2\right\rangle & \left\langle\Lambda^s p^1,p^2\right\rangle & \left\langle\Lambda^s q^2,p^2\right\rangle\end{vmatrix}\right. \\
&\quad\left.+\begin{vmatrix}\left\langle\Lambda^k q^1,p^1\right\rangle & \left\langle\Lambda^k p^1,p^1\right\rangle & \left\langle\Lambda^k q^2,p^1\right\rangle\\ \left\langle\Lambda^j q^1,q^1\right\rangle & \left\langle\Lambda^j q^1,p^1\right\rangle & \left\langle\Lambda^j q^1,q^2\right\rangle\\ \left\langle\Lambda^s q^1,p^2\right\rangle & \left\langle\Lambda^s p^1,p^2\right\rangle & \left\langle\Lambda^s q^2,p^2\right\rangle\end{vmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{k+j+s=m-3\\ j,k,s\geq 0}}\left(\begin{vmatrix}\langle\Lambda^j q^1,q^2\rangle & \langle\Lambda^j q^2,p^2\rangle & \langle\Lambda^j q^2,q^2\rangle\\ \langle\Lambda^k q^1,q^1\rangle & \langle\Lambda^k q^1,p^2\rangle & \langle\Lambda^k q^1,q^2\rangle\\ \langle\Lambda^s q^1,p^2\rangle & \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s q^2,p^2\rangle\end{vmatrix}\right.\\
&\quad+\begin{vmatrix}\langle\Lambda^k q^1,p^1\rangle & \langle\Lambda^k p^1,p^1\rangle & \langle\Lambda^k q^2,p^1\rangle\\ \langle\Lambda^j q^1,q^2\rangle & \langle\Lambda^j q^2,p^1\rangle & \langle\Lambda^j q^2,q^2\rangle\\ \langle\Lambda^s q^1,q^1\rangle & \langle\Lambda^s q^1,p^1\rangle & \langle\Lambda^s q^1,q^2\rangle\end{vmatrix}\\
&\quad+\langle q^2,p^2\rangle\\
&\quad\times\begin{vmatrix}\langle\Lambda^k q^1,p^1\rangle & \langle\Lambda^k p^1,p^2\rangle & \langle\Lambda^k p^1,p^1\rangle\\ \langle\Lambda^j q^1,q^1\rangle & \langle\Lambda^j q^1,p^2\rangle & \langle\Lambda^j q^1,p^1\rangle\\ \langle\Lambda^s q^1,p^2\rangle & \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s p^1,p^2\rangle\end{vmatrix}\\
&\quad+\langle q^1,p^1\rangle\\
&\quad\times\begin{vmatrix}\langle\Lambda^k p^1,p^2\rangle & \langle\Lambda^k p^1,p^1\rangle & \langle\Lambda^k q^2,p^1\rangle\\ \langle\Lambda^j q^2,p^2\rangle & \langle\Lambda^j q^2,p^1\rangle & \langle\Lambda^j q^2,q^2\rangle\\ \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s p^1,p^2\rangle & \langle\Lambda^s q^2,p^2\rangle\end{vmatrix}\\
&\quad-2\langle q^1,p^2\rangle\\
&\quad\left.\times\begin{vmatrix}\langle\Lambda^k p^1,p^2\rangle & \langle\Lambda^k p^1,p^1\rangle & \langle\Lambda^k q^2,p^1\rangle\\ \langle\Lambda^j q^1,p^2\rangle & \langle\Lambda^j q^1,p^1\rangle & \langle\Lambda^j q^1,q^2\rangle\\ \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s p^1,p^2\rangle & \langle\Lambda^s q^2,p^2\rangle\end{vmatrix}\right)\\
&+\sum_{\substack{k+s=m-2\\ k,s\geq 0}}\left[\langle\Lambda^k q^1,q^1\rangle\langle\Lambda^s q^1,q^1\rangle\right.\\
&\quad-\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s q^1,q^2\rangle+2\langle q^1,p^2\rangle\\
&\quad\times\left(\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^2,p^1\rangle-\langle\Lambda^k p^1,p^1\rangle\right.\\
&\qquad\times\langle\Lambda^s q^1,q^2\rangle+\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\qquad\left.-\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s p^2,p^2\rangle\right)-\langle q^1,p^1\rangle\\
&\quad\times\left(\langle\Lambda^k q^2,p^1\rangle\langle\Lambda^s q^2,p^1\rangle-\langle\Lambda^k p^1,p^1\rangle\right.\\
&\qquad\times\langle\Lambda^s q^2,q^2\rangle+\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s q^2,p^2\rangle\\
&\qquad\left.-\langle\Lambda^k q^2,q^2\rangle\langle\Lambda^s p^2,p^2\rangle\right)-\langle q^2,p^2\rangle\\
&\quad\times\left(\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^1,p^1\rangle-\langle\Lambda^k p^1,p^1\rangle\right.\\
&\qquad\times\langle\Lambda^s q^1,q^1\rangle+\langle\Lambda^k q^1,p^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\qquad\left.-\langle\Lambda^k q^1,q^1\rangle\langle\Lambda^s p^2,p^2\rangle\right)\\
&\quad+\left(\langle q^1,p^2\rangle^2-\langle q^1,p^1\rangle\langle q^2,q^2\rangle\right)\\
&\quad\times\left(\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s p^1,p^2\rangle\right.\\
&\qquad\left.\left.-\langle\Lambda^k p^1,p^1\rangle\langle\Lambda^s p^2,p^2\rangle\right)\right]\\
&+\sum_{\substack{k+s=m-1\\ k,s\geq 0}}\left[\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s q^2,p^1\rangle\right.\\
&\quad-\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^2,q^2\rangle\\
&\quad+\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\quad-\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s q^1,q^1\rangle+\langle q^1,p^2\rangle\\
&\quad\times\left(\langle\Lambda^k q^1,p^2\rangle\langle\Lambda^s p^1,p^1\rangle\right.\\
&\qquad-\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s p^1,p^2\rangle\\
&\qquad+\langle\Lambda^k q^2,p^1\rangle\langle\Lambda^s p^2,p^2\rangle\\
&\qquad\left.-\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s p^1,p^2\rangle\right)\\
&\quad-\langle q^1,p^1\rangle\left(\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s p^1,p^1\rangle\right.\\
&\qquad\left.-\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s q^2,p^1\rangle\right)\\
&\quad+\langle q^2,p^2\rangle\left(\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s q^1,p^2\rangle\right.\\
&\qquad\left.\left.-\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s p^2,p^2\rangle\right)\right]\\
&+\sum_{\substack{k+s=m\\ k,s\geq 0}}\left(\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^2,p^2\rangle\right.\\
&\qquad\left.-\langle\Lambda^k q^2,p^1\rangle\langle\Lambda^s q^1,p^2\rangle\right)\\
&+\langle q^1,p^2\rangle\left(\langle\Lambda^m q^1,p^2\rangle-\langle\Lambda^m q^2,p^1\rangle\right)\\
&-\langle q^1,p^1\rangle\langle\Lambda^m q^2,p^2\rangle-\langle q^2,p^2\rangle\langle\Lambda^m q^1,p^1\rangle\\
&-2\langle q^1,p^2\rangle\langle\Lambda^{m-1}q^1,p^2\rangle-\langle\Lambda^m q^1,q^2\rangle\\
&+\langle q^1,p^1\rangle\langle\Lambda^{m-1}q^2,q^2\rangle+\langle q^2,p^2\rangle\langle\Lambda^{m-1}q^1,q^1\rangle\\
&-\left(\langle q^1,p^2\rangle^2-\langle q^1,p^1\rangle\langle q^2,q^2\rangle\right)\\
&\times\left(\langle\Lambda^{m-1}p^1,p^1\rangle+\langle\Lambda^{m-1}p^2,p^2\rangle\right),\quad m\geq 1.
\end{aligned}
\tag{18}
$$

In the above equations, the Poisson bracket of two functions is defined as

$$
\begin{aligned}
\{f,g\}&=\sum_{j=1}^{N}\sum_{i=1}^{2}\left(\frac{\partial f}{\partial q_j^i}\frac{\partial g}{\partial p_j^i}-\frac{\partial g}{\partial q_j^i}\frac{\partial f}{\partial p_j^i}\right)\\
&=\sum_{i=1}^{2}\left(\left\langle\frac{\partial f}{\partial q^i},\frac{\partial g}{\partial p^i}\right\rangle-\left\langle\frac{\partial g}{\partial q^i},\frac{\partial f}{\partial p^i}\right\rangle\right).
\end{aligned}
\tag{19}
$$

Then we can prove the following assertions.

Proposition 1. *The functions* $\{F_m^{(i)} \mid i = 1,2, m \geq 1\}$ *are in involution in pairs; that is,*

$$\left\{F_m^{(i)}, F_l^{(j)}\right\} = 0, \quad \forall m, l \geq 1, \ 1 \leq i, \ j \leq 2. \tag{20}$$

Proof. Through tedious calculation we can obtain

$$\left\{\mathscr{F}_\lambda^{(1)}, \mathscr{F}_\mu^{(1)}\right\} = \left\{\mathscr{F}_\lambda^{(1)}, \mathscr{F}_\mu^{(2)}\right\} = \left\{\mathscr{F}_\lambda^{(2)}, \mathscr{F}_\mu^{(2)}\right\} = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \tag{21}$$

Then we have

$$\left\{\mathscr{F}_\lambda^{(1)}, \mathscr{F}_\mu^{(1)}\right\} = \left\{\mathscr{F}_\lambda^{(1)}, \widehat{\mathscr{F}}_\mu^{(2)}\right\} = \left\{\widehat{\mathscr{F}}_\lambda^{(2)}, \widehat{\mathscr{F}}_\mu^{(2)}\right\} = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \tag{22}$$

Then relation (20) follows by comparison of power of λ^N in (22) with (17) taken into account. □

Proposition 2. *The* $2N$ *1-forms* $dF_j^{(i)}$ $(1 \leq j \leq N, i = 1,2)$ *are linearly independent.*

Proof. Assuming that there exist $2N$ constants $b_l^{(i)}$, so that

$$\sum_{j=1}^{N} \left(b_j^{(1)} \frac{\partial F_j^{(1)}}{\partial q^i} + b_j^{(2)} \frac{\partial F_j^{(2)}}{\partial q^i} \right) = 0, \quad i = 1,2. \tag{23}$$

It is easy to obtain

$$\left.\frac{\partial F_j^{(1)}}{\partial q^1}\right|_{(p^1,p^2,q^2)=0} = 2\Lambda^{j-1} q^1, \qquad \left.\frac{\partial F_j^{(2)}}{\partial q^1}\right|_{(p^1,p^2,q^2)=0} = 0, \quad 1 \leq j \leq N. \tag{24}$$

Then we have

$$\sum_{j=1}^{N} b_j^{(1)} \lambda_k^{j-1} = 0, \quad 1 \leq k \leq N, \tag{25}$$

which gives rise to $b_j^{(1)} = 0$, $1 \leq j \leq N$, by utilizing the fact that the Vandermonde determinant is not zero. Therefore, (23) is reduced to

$$\sum_{j=1}^{N} b_j^{(2)} \frac{\partial F_j^{(2)}}{\partial q^i} = 0, \quad i = 1,2. \tag{26}$$

Take $P_0 \in \mathbb{R}^{4N}$ with the coordinates $q^2 = p^1 = 0$, $q^1 = O(\varepsilon)$, and $p^2 = O(\varepsilon)$, where ε is a small real number. Then, at P_0,

$$\begin{aligned} \left.\frac{\partial F_j^{(1)}}{\partial q^1}\right|_{P_0} &= \langle \Lambda^j q^1, p^2 \rangle p^2 + \langle q^1, p^2 \rangle \Lambda^j p^2 \\ &\quad - 2 \left\langle q^1, p^2 \right\rangle \left\langle \Lambda^{j-1} p^2, p^2 \right\rangle p^2 \\ &= \left\langle \Lambda^j q^1, p^2 \right\rangle p^2 + \left\langle q^1, p^2 \right\rangle \Lambda^j p^2 + O\left(\varepsilon^5\right), \end{aligned} \tag{27}$$

and the determinant of the coefficients of the linear system of equations

$$\sum_{j=1}^{N} b_j^{(2)} \frac{\partial F_j^{(2)}}{\partial q_k^1} = 0, \quad 1 \leq k \leq N, \tag{28}$$

is

$$A = \begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle p_1^2 + \left\langle q^1, p^2 \right\rangle \lambda_1 p_1^2 & \left\langle \Lambda^2 q^1, p^2 \right\rangle p_1^2 + \left\langle q^1, p^2 \right\rangle \lambda_1^2 p_1^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle p_1^2 + \left\langle q^1, p^2 \right\rangle \lambda_1^N p_1^2 \\ \left\langle \Lambda q^1, p^2 \right\rangle p_2^2 + \left\langle q^1, p^2 \right\rangle \lambda_2 p_2^2 & \left\langle \Lambda^2 q^1, p^2 \right\rangle p_2^2 + \left\langle q^1, p^2 \right\rangle \lambda_2^2 p_2^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle p_2^2 + \left\langle q^1, p^2 \right\rangle \lambda_2^N p_2^2 \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \Lambda q^1, p^2 \right\rangle p_N^2 + \left\langle q^1, p^2 \right\rangle \lambda_N p_N^2 & \left\langle \Lambda^2 q^1, p^2 \right\rangle p_N^2 + \left\langle q^1, p^2 \right\rangle \lambda_N^2 p_N^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle p_N^2 + \left\langle q^1, p^2 \right\rangle \lambda_N^N p_N^2 \end{vmatrix} + O\left(\varepsilon^{5N}\right)$$

$$= \prod_{j=1}^{N} p_j^2 \begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle + \left\langle q^1, p^2 \right\rangle \lambda_1 & \left\langle \Lambda^2 q^1, p^2 \right\rangle + \left\langle q^1, p^2 \right\rangle \lambda_1^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle + \left\langle q^1, p^2 \right\rangle \lambda_1^N \\ \left\langle q^1, p^2 \right\rangle (\lambda_2 - \lambda_1) & \left\langle q^1, p^2 \right\rangle \left(\lambda_2^2 - \lambda_1^2\right) & \cdots & \left\langle q^1, p^2 \right\rangle \left(\lambda_2^N - \lambda_1^N\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle q^1, p^2 \right\rangle (\lambda_N - \lambda_1) & \left\langle q^1, p^2 \right\rangle \left(\lambda_N^2 - \lambda_1^2\right) & \cdots & \left\langle q^1, p^2 \right\rangle \left(\lambda_N^N - \lambda_1^N\right) \end{vmatrix} + O\left(\varepsilon^{5N}\right)$$

$$= \prod_{j=1}^{N} p_j^2 \left(\left\langle q^1, p^2 \right\rangle^{N-1} \begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle & \left\langle \Lambda^2 q^1, p^2 \right\rangle & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle \\ \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^N - \lambda_1^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N - \lambda_1 & \lambda_N^2 - \lambda_1^2 & \cdots & \lambda_N^N - \lambda_1^N \end{vmatrix} + \left\langle q^1, p^2 \right\rangle^N \begin{vmatrix} \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^N \\ \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N & \lambda_N^2 & \cdots & \lambda_N^N \end{vmatrix} \right) + O\left(\varepsilon^{5N}\right), \tag{29}$$

where

$$\begin{aligned} &\begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle & \left\langle \Lambda^2 q^1, p^2 \right\rangle & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle \\ \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^N - \lambda_1^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N - \lambda_1 & \lambda_N^2 - \lambda_1^2 & \cdots & \lambda_N^N - \lambda_1^N \end{vmatrix} \\ &= \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^N \\ 0 & \left\langle \Lambda q^1, p^2 \right\rangle & \left\langle \Lambda^2 q^1, p^2 \right\rangle & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^N \end{vmatrix} \\ &= \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^N \\ -\left\langle q^1, p^2 \right\rangle & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^N \end{vmatrix} \\ &= \left\langle q^1, p^2 \right\rangle \prod_{\substack{i,j=1 \\ i>j}}^{N} \left(\lambda_i - \lambda_j\right). \end{aligned} \tag{30}$$

Therefore,

$$A = 2\left\langle q^1, p^2 \right\rangle^N \left(\prod_{j=1}^{N} p_j^2 \right) \left(\prod_{\substack{i,j=1 \\ i>j}}^{N} \left(\lambda_i - \lambda_j\right) \right) + O\left(\varepsilon^{5N}\right) \neq 0. \tag{31}$$

Then we obtain $b_j^{(2)} = 0$, $1 \leq j \leq N$. The proof is complete. □

Combining Propositions 1 and 2, we have immediately the following conclusions.

Proposition 3. *The symplectic map of the Bargmann type defined by* (10) *is completely integrable in the Liouville sense.*

Proposition 4. *The systems defined as follows are completely integrable in the Liouville sense:*

$$\frac{\partial q^i}{\partial t} = \frac{\partial F_m^{(1)}}{\partial p^i}, \quad \frac{\partial p^i}{\partial t} = -\frac{\partial F_m^{(1)}}{\partial q^i}, \quad m \geq 1, \ i = 1, 2. \tag{32}$$

4. The Representation of Solutions

Consider the following initial value problem:

$$\begin{gathered} q_t^i = \frac{\partial H_1}{\partial p^i}, \qquad p_t^i = -\frac{\partial H_1}{\partial q^i}, \\ \left(q^i, p^i\right)\big|_{t=0} = \left(q^i(0), p^i(0)\right), \\ i = 1, 2, \end{gathered} \tag{33}$$

where $H_1 = -(1/2)F_1^{(1)}$. In fact, the first two equations in (33) are

$$\begin{aligned} q_t^1 &= \frac{1}{2}\left(\Lambda - \left\langle p^1, p^1 \right\rangle\right) q^1 - \left\langle q^1, p^1 \right\rangle p^1 - \left\langle q^1, p^2 \right\rangle p^2, \\ q_t^2 &= -\left\langle p^1, p^2 \right\rangle q^1 + \frac{1}{2}\left(\Lambda - \left\langle p^2, p^2 \right\rangle\right) q^2 \\ &\quad - \left\langle q^1, p^2 \right\rangle p^1 - \left\langle q^2, p^2 \right\rangle p^2, \\ p_t^1 &= q^1 + \frac{1}{2}\left(\left\langle p^1, p^1 \right\rangle - \Lambda\right) p^1 + \left\langle p^1, p^2 \right\rangle p^2, \\ p_t^2 &= q^2 + \frac{1}{2}\left(\left\langle p^2, p^2 \right\rangle - \Lambda\right) p^2. \end{aligned} \tag{34}$$

Then we can obtain the presentation of solutions for the lattice equation (4).

Proposition 5. *Let $q^i(t)$ and $p^i(t)$ $(1 \leq i \leq 3)$ be a solution of* (33)*; define*

$$\begin{aligned} &\left(q^1(n,t), q^2(n,t), p^1(n,t), p^2(n,t)\right) \\ &\quad = H^n\left(q^1(t), q^2(t), p^1(t), p^2(t)\right). \end{aligned} \tag{35}$$

Then

$$\begin{gathered} a_n = \left\langle p^1, p^2 \right\rangle, \qquad b_n = \left\langle p^2, p^2 \right\rangle, \\ c_n = \left(-\left\langle p^1, p^2 \right\rangle^2 - \left\langle q^2, p^2 \right\rangle + \left\langle \Lambda p^2, p^2 \right\rangle - \left\langle p^2, p^2 \right\rangle^2\right)^{1/2}, \end{gathered} \tag{36}$$

and solve the lattice equation (4).

Proof. It is easy to see that (35) is equivalent to (12), that is, (10) with $(q^i(0,t), p^i(0,t)) = (q^i(t), p^i(t))$. Using (33), (36), and (10), a direct calculation shows that

$$\begin{aligned}
a_{n,t} &= \frac{1}{2}\left\langle p^1, p^2\right\rangle\left(\left\langle p^1, p^1\right\rangle + \left\langle p^2, p^2\right\rangle\right) + \left\langle p^2, p^2\right\rangle\left\langle p^1, p^2\right\rangle \\
&\quad + \left\langle q^1, p^2\right\rangle + \left\langle q^2, p^1\right\rangle - \left\langle \Lambda p^1, p^2\right\rangle \\
&= \frac{1}{2}\left(E^{-1}c_n - c_n E\right)\left(2\langle p^1, p^2\rangle\right) + \frac{1}{2}a_n(1-E)\left\langle p^2, p^2\right\rangle, \\
b_{n,t} &= \left\langle p^2, p^2\right\rangle\left\langle p^2, p^2\right\rangle + 2\left\langle \Lambda q^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle \\
&= \frac{1}{2}\left(E^{-1}-1\right)a_n\left(2\left\langle p^1, p^2\right\rangle\right) + \left(E^{-2}-1\right) \\
&\quad \times\left(-a_n\left\langle p^1, p^2\right\rangle - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - b_n\left\langle p^2, p^2\right\rangle\right), \\
c_{n,t} &= \frac{1}{2}\left(-\left\langle p^1, p^2\right\rangle^2 - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{-1/2} \\
&\quad \times\Big[-\left\langle p^2, p^2\right\rangle\Big(\left\langle p^1, p^2\right\rangle^2 + \left\langle q^2, p^2\right\rangle - \left\langle \Lambda p^2, p^2\right\rangle \\
&\qquad + 2\left\langle p^2, p^2\right\rangle^2\Big) - \left\langle p^1, p^2\right\rangle \\
&\quad \times\Big(\left\langle p^1, p^2\right\rangle\left\langle p^1, p^1\right\rangle + 2\left\langle p^2, p^2\right\rangle\left\langle p^1, p^2\right\rangle \\
&\qquad - 2\left\langle \Lambda p^1, p^2\right\rangle + 2\left\langle q^2, p^1\right\rangle\Big) - 2\left\langle p^2, p^2\right\rangle \\
&\quad \times\left(\left\langle \Lambda q^2, p^2\right\rangle - \left\langle \Lambda p^2, p^2\right\rangle\right) \\
&\quad - \left\langle q^2, q^2\right\rangle + 2\left\langle \Lambda q^2, p^2\right\rangle - \left\langle \Lambda^2 p^2, p^2\right\rangle\Big] \\
&= \frac{1}{2}c_n\left(1-E^2\right)\left\langle p^2, p^2\right\rangle.
\end{aligned} \tag{37}$$

Therefore, we have

$$\frac{\partial}{\partial t}(a_n, b_n, c_n)^{\mathrm{T}} = J_n\sum_{j=1}^{N}\nabla\lambda_j = J_n G_n^{(0)}, \tag{38}$$

where

$$J_n = \begin{pmatrix} \frac{1}{2}\left(E^{-1}c_n - c_n E\right) & \frac{1}{2}a_n(1-E) & 0 \\ \frac{1}{2}\left(E^{-1}-1\right)a_n & 0 & \frac{1}{2}\left(E^{-2}-1\right)c_n \\ 0 & \frac{1}{2}c_n\left(1-E^2\right) & 0 \end{pmatrix}. \tag{39}$$

Then (38) is equivalent to the generalization of Toda lattice equation (4). This proves Proposition 5. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

q-Extensions for the Apostol Type Polynomials

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The aim of this work is to introduce an extension for q-standard notations. The q-Apostol type polynomials and study some of their properties. Besides, some relations between the mentioned polynomials and some other known polynomials are obtained.

1. Introduction, Preliminaries, and Definitions

Throughout this research we always apply the following notations. $\mathbb{N}$ indicates the set of natural numbers, $\mathbb{N}_0$ indicates the set of nonnegative integers, $\mathbb{R}$ indicates the set of all real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We refer the readers to [1] for all the following q-standard notations. The q-shifted factorial is defined as

$$\begin{gathered}(a;q)_0 = 1,\\ (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n\in\mathbb{N},\\ (a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right),\\ |q|<1, \quad a\in\mathbb{C}.\end{gathered} \tag{1}$$

The q-numbers and q-factorials are defined by

$$\begin{gathered}[a]_q = \frac{1-q^a}{1-q} \quad (q\neq 1);\\ [0]! = 1;\\ [n]_q! = [1]_q[2]_q\cdots[n]_q,\\ n\in\mathbb{N}, \quad a\in\mathbb{C},\end{gathered} \tag{2}$$

respectively. The q-polynomial coefficient is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{3}$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{(1/2)k(k-1)}x^{n-k}y^k, \quad n\in\mathbb{N}_0. \tag{4}$$

The q-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{(1/2)k(k-1)}(-1)^k a^k. \tag{5}$$

In the standard approach to the q-calculus, two exponential functions are used:

$$\begin{gathered}e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \prod_{j=0}^{\infty}\frac{1}{\left(1-(1-q)q^j z\right)},\\ 0<|q|<1, \quad |z|<\frac{1}{|1-q|},\end{gathered} \tag{6}$$

$$\begin{gathered}E_q(z) = \sum_{k=0}^{\infty}\frac{q^{(1/2)k(k-1)}z^k}{[k]_q!} = \prod_{j=0}^{\infty}\left(1+(1-q)q^j z\right),\\ 0<|q|<1, \quad z\in\mathbb{C}.\end{gathered} \tag{7}$$

As an immediate result of these two definitions, we have $e_q(z)E_q(-z) = 1$.

Recently, Luo and Srivastava [2] introduced and studied the generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x;\lambda)$ and the generalized Apostol-Euler polynomials $E_n^{\alpha}(x;\lambda)$. Kurt [3] gave the generalization of the Bernoulli polynomials $B_n^{[m-1,\alpha]}(x)$ of order α and studied their properties. They also studied these polynomials systematically; see [2, 4–9]. There are numerous recent investigations on this subject by many other authors; see [3, 10–20]. More recently, Tremblay et al. [10] further gave the definition of $B_n^{[m-1,\alpha]}(x;\lambda)$ and studied their properties. On the other hand, Mahmudov and Keleshteri [21, 22] studied various two dimensional q-polynomials. Motivated by these papers, we define generalized Apostol type q-polynomials as follows.

Definition 1. Let $q,\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, and $0 < |q| < 1$. The generalized q-Apostol-Bernoulli numbers $B_{n,q}^{[m-1,\alpha]}$ and polynomials $B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ in x, y of order α are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions:

$$\left(\frac{t^m}{\lambda e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(\lambda)\frac{t^n}{[n]_q!},$$
$$\left(\frac{t^m}{\lambda e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!}, \tag{8}$$

where $T_{m-1,q}(t) = \sum_{k=0}^{m-1}(t^k/[k]q!)$.

Definition 2. Let $q,\alpha \in \mathbb{C}$, $0 < |q| < 1$, and $m \in \mathbb{N}$. The generalized q-Apostol-Euler numbers $E_{n,q}^{[m-1,\alpha]}$ and polynomials $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ in x, y of order α are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions:

$$\left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} = \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(\lambda)\frac{t^n}{[n]_q!},$$
$$\left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!}. \tag{9}$$

Definition 3. Let $q,\alpha \in \mathbb{C}, 0 < |q| < 1$, and $m \in \mathbb{N}$. The generalized q-Apostol-Genocchi numbers $G_{n,q}^{[m-1,\alpha]}$ and polynomials $G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ in x, y of order α are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions:

$$\left(\frac{2^m t^m}{\lambda e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} = \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(\lambda)\frac{t^n}{[n]_q!},$$
$$\left(\frac{2^m t^m}{\lambda e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!}. \tag{10}$$

Clearly, for $m = 1$, one has

$$\begin{aligned} B_{n,q}^{[0,\alpha]}(x,y;\lambda) &= B_{n,q}^{(\alpha)}(x,y;\lambda), \\ E_{n,q}^{[0,\alpha]}(x,y;\lambda) &= E_{n,q}^{(\alpha)}(x,y;\lambda), \\ G_{n,q}^{[0,\alpha]}(x,y;\lambda) &= G_{n,q}^{(\alpha)}(x,y;\lambda). \end{aligned} \tag{11}$$

For $m = 1$ and $\lambda = 1$, one has

$$\begin{aligned} B_{n,q}^{[0,\alpha]}(x,y;1) &= B_{n,q}^{(\alpha)}(x,y), \\ E_{n,q}^{[0,\alpha]}(x,y;1) &= E_{n,q}^{(\alpha)}(x,y), \\ G_{n,q}^{[0,\alpha]}(x,y;1) &= G_{n,q}^{(\alpha)}(x,y). \end{aligned} \tag{12}$$

For $x = y = 0$, one has

$$\begin{aligned} B_{n,q}^{[m-1,\alpha]}(0,0;\lambda) &= B_{n,q}^{[m-1,\alpha]}(\lambda), \\ E_{n,q}^{[m-1,\alpha]}(0,0;\lambda) &= E_{n,q}^{[m-1,\alpha]}(\lambda), \\ G_{n,q}^{[m-1,\alpha]}(0,0;\lambda) &= G_{n,q}^{[m-1,\alpha]}(\lambda). \end{aligned} \tag{13}$$

2. Properties of the Apostol Type q-Polynomials

In this section, we show some basic properties of the generalized q-polynomials. We only prove the facts for one of them. Obviously, by applying the similar technique, other ones can be proved.

Proposition 4. *The generalized q-polynomials* $B_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, $E_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, *and* $G_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ *satisfy the following relations:*

$$\begin{aligned} B_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}^{[m-1,\alpha]}(x, 0; \lambda) \times B_{n-k,q}^{[m-1,\beta]}(0, y; \lambda), \\ E_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(x, 0; \lambda) \times E_{n-k,q}^{[m-1,\beta]}(0, y; \lambda), \\ G_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{[m-1,\alpha]}(x, 0; \lambda) \times G_{n-k,q}^{[m-1,\beta]}(0, y; \lambda). \end{aligned} \tag{14}$$

Proof. We only prove the second identity. By using Definition 2, we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) \frac{t^n}{[n]_q!} &= \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha+\beta} e_q(tx) E_q(ty) \\ &= \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} e_q(tx) \times \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\beta} E_q(ty) \\ &= \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x, 0; \lambda) \frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\beta]}(0, y; \lambda) \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(x, 0; \lambda) E_{n-k,q}^{[m-1,\beta]}(0, y; \lambda) \frac{t^n}{[n]_q!}. \end{aligned} \tag{15}$$

Comparing the coefficients of the term $t^n/[n]_q!$ in both sides gives the result. □

Corollary 5. *The generalized q-polynomials* $B_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, $E_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, *and* $G_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ *satisfy the following relations:*

$$\begin{aligned} B_{n,q}^{[m-1,\alpha]}(x, y; \lambda) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}^{[m-1,\alpha]}(0, y; \lambda) x^{n-k}, \\ E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0, y; \lambda) x^{n-k}, \\ G_{n,q}^{[m-1,\alpha]}(x, y; \lambda) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{[m-1,\alpha]}(0, y; \lambda) x^{n-k}. \end{aligned} \tag{16}$$

Proposition 6. *The generalized q-polynomials* $B_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, $E_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, *and* $G_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ *satisfy the following relations:*

$$\lambda B_{n,q}^{[m-1,\alpha]}(1, y; \lambda) - B_{n,q}^{[m-1,\alpha]}(0, y; \lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q B_{k,q}^{[m-1,\alpha]}(0, y; \lambda) B_{n-k,q}^{[0,-1]}(\lambda), \quad \text{for } n \geq 1, \tag{17}$$

$$\lambda E_{n,q}^{[m-1,\alpha]}(1, y; \lambda) + E_{n,q}^{[m-1,\alpha]}(0, y; \lambda) = 2\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0, y; \lambda) E_{n-k,q}^{[0,-1]}(\lambda), \tag{18}$$

$$\lambda G_{n,q}^{[m-1,\alpha]}(1, y; \lambda) + G_{n,q}^{[m-1,\alpha]}(0, y; \lambda) = 2\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q G_{k,q}^{[m-1,\alpha]}(0, y; \lambda) G_{n-k,q}^{[0,-1]}(\lambda), \quad \text{for } n \geq 1. \tag{19}$$

Proof. We only prove (18). By using Definition 2 and starting from the left hand side of the relation (18), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\lambda E_{n,q}^{[m-1,\alpha]}(1, y; \lambda) + E_{n,q}^{[m-1,\alpha]}(0, y; \lambda) \right) \frac{t^n}{[n]_q!} &= \lambda \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} e_q(t) E_q(ty) \\ &\quad + \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \\ &= \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \left(\lambda e_q(t) + 1 \right) \\ &= 2 \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \left(\frac{2}{\lambda e_q(t) + 1} \right)^{-1} \\ &= 2 \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(0, y; \lambda) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} E_{n,q}^{[0,-1]}(\lambda) \frac{t^n}{[n]_q!} \end{aligned}$$

$$= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\, E_{n-k,q}^{[0,-1]}(\lambda)\,\frac{t^n}{[n]_q!}. \tag{20}$$

Comparing the coefficients of the term $t^n/[n]_q!$ in both sides gives the result. □

3. q-Analogue of the Luo-Srivastava Addition Theorem

In this section, we state and prove a q-generalization of the Luo-Srivastava addition theorem.

Theorem 7. *The following relation holds between generalized q-Apostol-Euler and q-Apostol-Bernoulli polynomials:*

$$\begin{aligned} &E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) \\ &\quad= \sum_{j=0}^{n}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \\ &\qquad\times\left(\lambda\sum_{k=0}^{n-j+1}\frac{1}{[n+1]_q}\begin{bmatrix} n-j+1 \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha-1]}(0,y;\lambda)\right. \\ &\qquad\qquad\left.-\,E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right) \\ &\qquad\times B_{j,q}(x,0;\lambda)+\frac{\lambda-1}{[n+1]_q} \\ &\qquad\times\left(\frac{2^m}{\lambda+1}\right)^{\alpha} B_{n+1,q}(x,0;\lambda). \end{aligned} \tag{21}$$

Proof. We take aid of the following identity to prove (21):

$$\begin{aligned} &\lambda\frac{t}{\lambda e_q(t)-1}e_q(tx)\,e_q(t)-\frac{t}{\lambda e_q(t)-1}e_q(tx) \\ &\quad= \frac{te_q(tx)}{\lambda e_q(t)-1}\left(\lambda e_q(t)-1\right)=te_q(tx). \end{aligned} \tag{22}$$

Therefore, we can write

$$\begin{aligned} &\lambda\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}(x,0;\lambda)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}B_{n,q}(x,0;\lambda)\frac{t^n}{[n]_q!} \\ &\quad= \sum_{n=0}^{\infty}x^n\frac{t^{n+1}}{[n+1]_q!}[n+1]_q \\ &\quad= \sum_{n=0}^{\infty}[n]_q x^{n-1}\frac{t^n}{[n]_q!}. \end{aligned} \tag{23}$$

From that we can conclude the following:

$$\lambda\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}(x,0;\lambda)-B_{n,q}(x,0;\lambda)=[n]_q x^{n-1}. \tag{24}$$

That is,

$$x^n=\frac{1}{[n+1]_q}\left(\lambda\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q B_{k,q}(x,0;\lambda)-B_{n+1,q}(x,0;\lambda)\right). \tag{25}$$

Substituting (25) into the right hand side of (16), we obtain

$$\begin{aligned} &E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) \\ &\quad= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{1}{[n-k+1]_q} \\ &\qquad\times\left(\lambda\sum_{j=0}^{n-k+1}\begin{bmatrix} n-k+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda)\right. \\ &\qquad\qquad\left.-B_{n-k+1,q}(x,0;\lambda)\right) \\ &\quad= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{1}{[n-k+1]_q} \\ &\qquad\times\left(\lambda\sum_{j=0}^{n-k}\begin{bmatrix} n-k+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda)\right. \\ &\qquad\qquad\left.+(\lambda-1)B_{n-k+1,q}(x,0;\lambda)\right) \\ &\quad= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{\lambda}{[n-k+1]_q} \\ &\qquad\times\sum_{j=0}^{n-k}\begin{bmatrix} n-k+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda) \\ &\qquad+\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{\lambda-1}{[n-k+1]_q} \\ &\qquad\times B_{n-k+1,q}(x,0;\lambda):=I_1+I_2. \end{aligned} \tag{26}$$

Thus, from one hand, we can write

$$\begin{aligned} I_1 &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{\lambda}{[n-k+1]_q} \\ &\quad\times\sum_{j=0}^{n-k}\begin{bmatrix} n-k+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda) \\ &= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{\lambda}{[n+1]_q}\begin{bmatrix} n+1 \\ n-k+1 \end{bmatrix}_q\begin{bmatrix} n-k+1 \\ j \end{bmatrix}_q \\ &\quad\times E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\,B_{j,q}(x,0;\lambda). \end{aligned} \tag{27}$$

As we know that

$$\begin{bmatrix} m \\ l \end{bmatrix}_q\begin{bmatrix} l \\ n \end{bmatrix}_q=\begin{bmatrix} m \\ n \end{bmatrix}_q\begin{bmatrix} m-n \\ m-l \end{bmatrix}_q, \quad \text{for } m\ge l\ge n, \tag{28}$$

we can continue as

$$\begin{aligned} I_1 &= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{\lambda}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \begin{bmatrix} n-j+1 \\ k \end{bmatrix}_q \\ &\quad \times E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\, B_{j,q}(x,0;\lambda) \\ &= \sum_{j=0}^{n}\frac{\lambda}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda) \\ &\quad \times \sum_{k=0}^{n-j}\begin{bmatrix} n-j+1 \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &= \sum_{j=0}^{n}\frac{\lambda}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda) \\ &\quad \times \left(E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda) - E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right). \end{aligned} \tag{29}$$

On the other hand, for I_2, we can write

$$\begin{aligned} I_2 &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\,\frac{\lambda-1}{[n-k+1]_q}B_{n-k+1,q}(x,0;\lambda) \\ &= \sum_{k=0}^{n}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{\lambda-1}{[n+1]_q}B_{n-k+1,q}(x,0;\lambda)\, E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &= \sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{\lambda-1}{[n+1]_q}B_{n-k+1,q}(x,0;\lambda)\, E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &\quad - \frac{\lambda-1}{[n+1]_q}B_{0,q}(x,0;\lambda)\, E_{k,q}^{[m-1,\alpha]}(0,y;\lambda), \end{aligned} \tag{30}$$

and, as $B_{0,q}(x,0;\lambda)=0$, we have

$$\begin{aligned} I_2 &= \sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{\lambda-1}{[n+1]_q}B_{n-k+1,q}(x,0;\lambda)\, E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &= \sum_{j=0}^{n+1}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \frac{\lambda-1}{[n+1]_q}B_{j,q}(x,0;\lambda)\, E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &= \sum_{j=0}^{n}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \frac{\lambda-1}{[n+1]_q}B_{j,q}(x,0;\lambda)\, E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &\quad + \frac{\lambda-1}{[n+1]_q}B_{n+1,q}(x,0;\lambda)\, E_{0,q}^{[m-1,\alpha]}(0,y;\lambda). \end{aligned} \tag{31}$$

Adding I_2 to I_1 we obtain

$$\begin{aligned} &E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) \\ &\quad = I_1 + I_2 \\ &\quad = \sum_{j=0}^{n}\frac{\lambda}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q B_{j,q}(x,0;\lambda) \\ &\quad \times \left(E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda) - E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right) \\ &\quad + \sum_{j=0}^{n}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \frac{\lambda-1}{[n+1]_q}B_{j,q}(x,0;\lambda)\, E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &\quad + \frac{\lambda-1}{[n+1]_q}B_{n+1,q}(x,0;\lambda)\, E_{0,q}^{[m-1,\alpha]}(0,y;\lambda). \end{aligned} \tag{32}$$

Consequently,

$$\begin{aligned} &E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) \\ &\quad = \sum_{j=0}^{n}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \\ &\quad \times \left(\lambda E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda) - \lambda E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right. \\ &\qquad \left. + (\lambda-1)\, E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right) \\ &\quad \times B_{j,q}(x,0;\lambda) + \frac{\lambda-1}{[n+1]_q} \\ &\quad \times B_{n+1,q}(x,0;\lambda)\, E_{0,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &\quad = \sum_{j=0}^{n}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \\ &\quad \times \left(\lambda E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda) - E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right) \\ &\quad \times B_{j,q}(x,0;\lambda) + \frac{(\lambda-1)}{[n+1]_q}B_{n+1,q}(x,0;\lambda) \\ &\quad \times E_{0,q}^{[m-1,\alpha]}(0,y;\lambda) \\ &\quad = \sum_{j=0}^{n}\frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ j \end{bmatrix}_q \\ &\quad \times \left(\lambda \sum_{k=0}^{n-j+1}\begin{bmatrix} n-j+1 \\ k \end{bmatrix}_q E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right. \\ &\qquad \left. - E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right) \\ &\quad \times B_{j,q}(x,0;\lambda) + \frac{(\lambda-1)}{[n+1]_q}\left(\frac{2^m}{\lambda+1}\right)^{\alpha} B_{n+1,q}(x,0;\lambda). \end{aligned} \tag{33}$$

□

Taking $m = 1$ in Theorem 7, we get a q-generalization of the Luo-Srivastava addition theorem [2].

Corollary 8. *The following relation holds between generalized q-Apostol-Euler and q-Apostol-Bernoulli polynomials:*

$$\begin{aligned} E_{n,q}^{(\alpha)}(x,y;\lambda) = {} & \sum_{j=0}^{n} \frac{2}{[j+1]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q \\ & \times \left(E_{j+1,q}^{(\alpha)}(0,y;\lambda) - E_{j+1,q}^{(\alpha)}(0,y;\lambda) \right) \\ & \times B_{n-j,q}(x,0;\lambda) + \frac{\lambda-1}{[n+1]_q} \left(\frac{2}{\lambda+1} \right)^{\alpha} \\ & \times B_{n+1,q}(x,0;\lambda). \end{aligned} \tag{34}$$

Letting $q \uparrow 1$, we get the Luo-Srivastava addition theorem (see [12]):

$$\begin{aligned} E_{n}^{(\alpha)}(x+y;\lambda) = {} & \sum_{j=0}^{n} \frac{2}{j+1} \binom{n}{j} \\ & \times \left(E_{j+1}^{(\alpha)}(y;\lambda) - E_{j+1}^{(\alpha)}(y;\lambda) \right) \\ & \times B_{n-j,q}(x;\lambda) + \frac{\lambda-1}{n+1} \left(\frac{2}{\lambda+1} \right)^{\alpha} \\ & \times B_{n+1}(x;\lambda). \end{aligned} \tag{35}$$

Next theorem gives relationship between $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ and $G_{n,q}(x,0)$.

Theorem 9. *The following relation holds between generalized q-Apostol-Euler and q-Apostol-Genocchi polynomials:*

$$\begin{aligned} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = {} & \frac{1}{2} \sum_{k=0}^{n} \frac{1}{[k+1]_q} \\ & \times \left(\lambda \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q E_{n-j,q}^{[m-1,\alpha]}(0,y;\lambda) \right. \\ & \left. + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{n-k,q}^{[m-1,\alpha]}(0,y;\lambda) \right) \\ & \times G_{k+1,q}(x,0). \end{aligned} \tag{36}$$

Proof. The proof follows from the following identity:

$$\begin{aligned} & \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} e_q(tx) E_q(ty) \\ & = \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \frac{2t}{e_q(t)+1} \\ & \quad \times e_q(tx) \frac{e_q(t)+1}{2t}. \end{aligned} \tag{37}$$

□

Theorem 10. *The following relation holds between generalized q-Apostol-Euler and q-Stirling polynomials $S_q(i,j)$ of the second kind:*

$$\begin{aligned} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = {} & \sum_{k=0}^{n} \sum_{j=k}^{n} \begin{bmatrix} n \\ n-j \end{bmatrix}_q \\ & \times E_{n-j,q}^{[m-1,\alpha]}(0,y;\lambda) S_q(j,k) x_k(x). \end{aligned} \tag{38}$$

Proof. The q-Stirling polynomials $S_q(n,k)$ of the second kind are defined by means of the following generating function:

$$x^n = \sum_{k=0}^{n} S_q(n,k) x_k(x), \tag{39}$$

where $x_k(x) = x(x-[1]_q)(x-[2]_q)\cdots(x-[k-1]_q)$; see [23]. Replacing identity (39) in the right hand side of (16), we have

$$\begin{aligned} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = {} & \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\ & \times \sum_{k=0}^{n-k} S_q(n-k,k) x_k(x) \\ = {} & \sum_{k=0}^{n} \sum_{j=k}^{n} \begin{bmatrix} n \\ n-j \end{bmatrix}_q \\ & \times E_{n-j,q}^{[m-1,\alpha]}(0,y;\lambda) S_q(j,k) x_k(x). \end{aligned} \tag{40}$$

□

Theorem 11. *The relationship*

$$\begin{aligned} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = {} & \sum_{k=0}^{[n/2]} \sum_{j=0}^{n-2k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \frac{[k]_q!}{[2]_q^n [k]_{q^2}!} \\ & \times E_{j,q}^{[m-1,\alpha]}(0,y;\lambda) H_{n-2k-j,q}(x) \end{aligned} \tag{41}$$

holds between the polynomials $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ and the q-Hermite polynomials defined by (see [24])

$$e_q(tx) E_{q^2}\left(-\frac{t^2}{[2]_q} \right) = \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{42}$$

Proof. Indeed,

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!} \\
&\quad= \left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,E_q(ty) \\
&\quad= \left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} E_q(ty)\,e_q(tx) \\
&\qquad\times E_{q^2}\left(-\frac{t^2}{[2]_q}\right) e_{q^2}\left(\frac{t^2}{[2]_q}\right) \\
&\quad= \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{t^n}{[n]_q!} \\
&\qquad\times \sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{2n}}{[2]_q^n[n]_{q^2}!} \\
&\quad= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q E_{j,q}^{[m-1,\alpha]}(0,y;\lambda)\,H_{n-j,q}(x)\frac{t^n}{[n]_q!} \\
&\qquad\times \sum_{n=0}^{\infty}\frac{t^{2n}}{[2]_q^n[n]_{q^2}!} \\
&\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{[n]_q!}{[2]_q^n[k]_{q^2}![n-2k]_q!} \\
&\qquad\times \sum_{j=0}^{n-2k}\begin{bmatrix} n-2k \\ j \end{bmatrix}_q E_{j,q}^{[m-1,\alpha]}(0,y;\lambda)\,H_{n-2k-j,q}(x)\frac{t^n}{[n]_q!} \\
&\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\sum_{j=0}^{n-2k}\begin{bmatrix} n \\ k \end{bmatrix}_q\begin{bmatrix} n-2k \\ j \end{bmatrix}_q\frac{[k]_q!}{[2]_q^n[k]_{q^2}!} \\
&\qquad\times E_{j,q}^{[m-1,\alpha]}(0,y;\lambda)\,H_{n-2k-j,q}(x)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{43}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Nonexistence of Global Solutions to the Initial Boundary Value Problem for the Singularly Perturbed Sixth-Order Boussinesq-Type Equation

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We are concerned with the singularly perturbed Boussinesq-type equation including the singularly perturbed sixth-order Boussinesq equation, which describes the bidirectional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to 1/3. The nonexistence of global solution to the initial boundary value problem for the singularly perturbed Boussinesq-type equation is discussed and two examples are given.

1. Introduction

In the numerical study of the ill-posed Boussinesq equation,

$$u_{tt} = u_{xx} + \left(u^2\right)_{xx} + u_{xxxx}. \tag{1}$$

Darapi and Hua [1] proposed the singularly perturbed Boussinesq equation

$$u_{tt} = u_{xx} + \left(u^2\right)_{xx} + u_{xxxx} + \delta u_{xxxxxx} \tag{2}$$

as a dispersive regularization of the ill-posed classical Boussinesq equation (1), where $\delta > 0$ is a small parameter. The authors use both filtering and regularization techniques to control growth of the errors and to provide better approximate solutions of this equation. Dash and Daripa [2] presented a formal derivation of (2) from two-dimensional potential flow equations for water waves through an asymptotic series expansion for small amplitude and long wave length. The physical relevance of (2) in the context of water waves was also addressed in [2]; it was shown that (2) actually describes the bidirectional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to 1/3. On the basis of far-field analysis and heuristic arguments, Daripa and Dash [3] proved that the traveling wave solutions of (2) are weakly nonlocal solitary waves characterized by small amplitude fast oscillations in the far-field and obtained weakly nonlocal solitary wave solutions of (2). Feng [4] investigated the generalized Boussinesq equation including the singularly perturbed Boussinesq equation

$$u_{tt} = [Q(u)]_{xx} + \sum_{i=1}^{n} b_i u_{(2i+2)x}, \tag{3}$$

where $Q(u) = u + b_0 u^r$, $u_{(2i+2)x} = (\partial^{2i+2}u)/(\partial x^{2i+2})$, r and b_i $(i = 1, 2, \ldots, n)$ are all real constants. It is easily seen that the choices $b_0 = 1$, $r = 2$, $n = 2$, $b_1 = 1$, and $b_2 = \delta$ lead (3) to the singularly perturbed Boussinesq equation (2). By the means of two proper ansatzs, the author obtained explicit traveling solitary wave solutions of the generalized Boussinesq equation (3). To the best of our knowledge, however, there have not been any discussions on global solutions of the initial boundary value problem for (2) in the literature; recently, Song et al. [5] discussed the initial boundary value problem for the singularly perturbed Boussinesq-type equation

$$u_{tt} = u_{xx} + \sigma(u)_{xx} + \alpha u_{x^4} + \beta u_{x^6}, \quad x \in \Omega,\ t > 0, \tag{4}$$

with the initial boundary value conditions

$$\begin{aligned} u_x(0,t) &= u_x(1,t) = u_{x^3}(0,t) = u_{x^3}(1,t) = u_{x^5}(0,t) \\ &= u_{x^5}(1,t) = 0, \quad t > 0, \\ u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{5}$$

or with

$$\begin{aligned} u(0,t) &= u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = u_{x^4}(0,t) \\ &= u_{x^4}(1,t) = 0, \quad t > 0, \\ u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{6}$$

where, and in the sequel $u_{x^i} = \partial^i u/\partial x^i$, $\sigma(s)$ is a given nonlinear function, $\alpha > 0$ and $\beta > 0$ are real numbers, $u_0(x)$ and $u_1(x)$ are given initial value functions, and $\Omega = (0,1)$. By virtue of the Galerkin method and prior estimates, under the assumption "$\sigma'(s)$ is bounded below and $\sigma(s)$ satisfies some smooth condition," the existence and uniqueness of the global generalized solution and the global classical solution of the initial boundary value problem (4), (5) and (4), (6) are proved, respectively. But if $\sigma'(s)$ is not bounded below, does the above-mentioned problem have any global solution? In this paper, we employ the energy method and the Jensen inequality to prove that the global solutions of the initial boundary value problem (4), (5) and (4), (6) cease to exist in a finite time, respectively. At last, we show that the global solution of the initial boundary value problem (2), (6) blows up in a finite time.

The paper is organized as follows. In Section 2, the main results are stated. The nonexistence of global solution of problem (4), (5) and (4), (6) is discussed in Section 3. In Section 4, we study the initial boundary problem (2), (6) and give two examples satisfying the theorems (Theorems 1–6).

2. Main Theorems

Throughout this paper, we use the abbreviations $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. In the following we state the main results of this paper, where the existence of Theorems 1–4 has been proved in [5].

Theorem 1 (see [5]). *Assume that $u_0 \in H^6(\Omega)$, $u_1 \in H^3(\Omega)$, $\int_0^1 u_0(x)dx = \int_0^1 u_1(x)dx = 0$, $u_{0x^{2k+1}}(0,t) = u_{0x^{2k+1}}(1,t) = u_{1x^{2k+1}}(0,t) = u_{1x^{2k+1}}(1,t) = 0$ $(k = 0,1,2)$, $\sigma \in C^5(\mathbf{R})$, and $\sigma'(s)$ is bounded below; namely, there exists a constant C_0 such that $\sigma'(s) \geq C_0$, for any $s \in \mathbf{R}$. Then, for any $T > 0$, the initial boundary value problem* (4), (5) *admits a unique global generalized solution $u(x,t)$ with*

$$\begin{aligned} u \in{}& C\left([0,T]; H^6(\Omega)\right) \cap C^1\left([0,T]; H^3(\Omega)\right) \\ &\cap C^2\left([0,T]; L^2(\Omega)\right). \end{aligned} \tag{7}$$

Theorem 2 (see [5]). *Assume that the assumptions of Theorem 1 hold, $u_0 \in H^{10}(\Omega)$, $u_1 \in H^7(\Omega)$, and $\sigma \in C^9(\mathbf{R})$. Then, the initial boundary value problem* (4), (5) *admits a unique global classical solution $u(x,t)$.*

Theorem 3 (see [5]). *Assume that $u_0 \in H^6(\Omega)$, $u_1 \in H^3(\Omega)$, $u_{0x^{2k}}(0,t) = u_{0x^{2k}}(1,t) = u_{1x^{2k}}(0,t) = u_{1x^{2k}}(1,t) = 0$ $(k = 0,1,2)$, $\sigma \in C^5(\mathbf{R})$, $\sigma^{(2i)}(0) = 0$ $(i = 1,2)$, and $\sigma'(s)$ is bounded below. Then, for any $T > 0$, the initial boundary value problem* (4), (6) *admits a unique global generalized solution $u(x,t)$ with*

$$\begin{aligned} u \in{}& C\left([0,T]; H^6(\Omega)\right) \cap C^1\left([0,T]; H^3(\Omega)\right) \\ &\cap C^2\left([0,T]; L^2(\Omega)\right). \end{aligned} \tag{8}$$

Theorem 4 (see [5]). *Assume that the assumptions of Theorem 3 hold, $u_0 \in H^{10}(\Omega)$, $u_1 \in H^7(\Omega)$, $\sigma \in C^9(\mathbf{R})$, and $\sigma^{(2i)}(0) = 0$ $(i = 3,4)$. Then, the initial boundary value problem* (4), (6) *admits a unique global classical solution $u(x,t)$.*

Theorem 5. *Assume that (1) $\sigma(s)s \leq \mu\Gamma(s)$, $\Gamma(s) \leq -\gamma|s|^{m+1}$, where $\Gamma(s) = \int_0^s \sigma(\tau)d\tau$, $\mu > 2$, $\gamma > 0$, and $m > 1$ are constants, and (2) $u_0 \in H^2$, $u_1 \in L^2$, $\int_0^1 u_0(x)dx = \int_0^1 u_1(x)dx = 0$, and*

$$\begin{aligned} E_0 ={}& \int_0^1 \left(\int_0^x u_1(\xi)\,d\xi\right)^2 dx + \|u_0\|^2 - \alpha\|u_0'\|^2 + \beta\|u_0''\|^2 \\ &+ 2\int_0^1 \int_0^{u_0} \sigma(s)\,ds\,dx \leq -\left[\frac{2}{D\left(1 - e^{(1-m)/4}\right)^2}\right]^{2/(m-1)}, \end{aligned} \tag{9}$$

where $D = \gamma(\mu-2)/[2^{(m-7)/2}(m+3)]$. Then the solution $u(x,t)$ of initial boundary value problem (4), (5) *blows up in a finite time T_0; namely,*

$$\|u(t)\|_{L^1(\Omega)}^2 + \int_0^t \|u(\tau)\|^2 d\tau \longrightarrow +\infty \quad \text{as } t \longrightarrow T_0^-, \tag{10}$$

where T_0 is defined in the proof.

Theorem 6. *Assume that (1) $\sigma \in C^2(\mathbf{R})$, $\sigma(0) = 0$, and one of the following conditions holds: (i) $\sigma(s)$ is a convex and even function, $\sigma(s) \geq as^m$, where $a > 0$ and $m > 1$ are real numbers, (ii) $\sigma(s)$ is a convex function, $\sigma(s) \geq as^m$, where $a > 0$ is a real number and $m \geq 2$ is an even number, and (2) $-(\pi/2)\int_0^1 u_0(x)\sin\pi x\,dx > \max\{0, ((\beta\pi^4 - \alpha\pi^2 + 1)/a)^{1/(m-1)}\}$, $-(\pi/2)\int_0^1 u_1(x)\sin\pi x\,dx > 0$. Then the solution $u(x,t)$ of the initial boundary value problem* (4), (6) *blows up in a finite time T_1; namely,*

$$\|u(t)\| \longrightarrow +\infty, \quad \text{as } t \longrightarrow T_1^-, \tag{11}$$

where T_1 is defined in the proof.

3. Nonexistence of Global Solutions of Problem (4), (5) and (4), (6)

We first quote the following lemmas.

Lemma 7 (see Li [6]). *Assume that $\dot{u} = G(t,u)$, $\dot{v} \geq G(t,v)$, $G \in C([0,+\infty) \times \mathbf{R})$, $t_0 \leq t < +\infty$, and $u(t_0) = v(t_0)$. Then $u(t) \geq v(t)$ as $t \geq t_0$.*

Lemma 8 (Jensen inequality [7]). *Assume that $\varphi(u) : u \in [\alpha,\beta] \mapsto \mathbf{R}$ is a convex function, $f : x \in \Omega \mapsto [\alpha,\beta]$, and $P(x)$ is a continuous function, $P(x) \geq 0$, $P(x) \not\equiv 0$. Then*

$$\varphi\left(\frac{\int_\Omega f(x)P(x)\,dx}{\int_\Omega P(x)\,dx}\right) \leq \frac{\int_\Omega \varphi(f(x))P(x)\,dx}{\int_\Omega P(x)\,dx}. \tag{12}$$

Integrating both sides of (4) over $(0,1)$ and using (5) and the assumption of Theorem 1, we obtain $\int_0^1 u(x,t)dx = 0$, $t \geq 0$. Let $v(x,t) = \int_0^x u(\xi,t)d\xi$; then $u = v_x$ and v satisfies

$$v_{tt} = v_{xx} + \sigma(v_x)_x + \alpha v_{x^4} + \beta v_{x^6}, \quad x \in \Omega,\ t > 0, \tag{13}$$

$$\begin{aligned} v(0,t) &= v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = v_{x^4}(0,t) \\ &= v_{x^4}(1,t) = 0, \quad t > 0, \end{aligned} \tag{14}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \overline{\Omega}, \tag{15}$$

where $v_0(x) = \int_0^x u_0(\xi)d\xi$ and $v_1(x) = \int_0^x u_1(\xi)d\xi$.

Proof of Theorem 5. Multiplying both sides of (13) by $2v_t$, integrating by parts, and using condition (2) of Theorem 5, we have

$$\dot{E}(t) = 0, \quad E(t) = E(0) = E_0 < 0, \quad t > 0, \tag{16}$$

where

$$\begin{aligned} E(t) &= \|v_t(t)\|^2 + \|v_x(t)\|^2 - \alpha\|v_{xx}(t)\|^2 + \beta\|v_{x^3}(t)\|^2 \\ &\quad + 2\int_0^1 \Gamma(v_x(x,t))\,dx. \end{aligned} \tag{17}$$

Let

$$F(t) = \|v(t)\|^2 + \int_0^t \int_0^\tau \|v_x(s)\|^2 ds\,d\tau. \tag{18}$$

By virtue of condition (1) of Theorem 5 and noting that

$$\begin{aligned} \mu\int_0^1 \Gamma(v_x(x,t))\,dx &= E_0 - \|v_t(t)\|^2 - \|v_x(t)\|^2 + \alpha\|v_{xx}(t)\|^2 \\ &\quad - \beta\|v_{x^3}(t)\|^2 + (\mu-2)\int_0^1 \Gamma(v_x)\,dx, \end{aligned} \tag{19}$$

we obtain

$$\begin{aligned} \ddot{F}(t) &= 2\int_0^1 v v_{tt}dx + 2\|v_t(t)\|^2 + \|v_x(t)\|^2 \\ &= -2\int_0^1 \left(v_x^2 - \alpha v_{xx}^2 + \beta v_{x^3}^2 + \sigma(v_x)v_x\right)dx \\ &\quad + 2\|v_t(t)\|^2 + \|v_x(t)\|^2 \\ &\geq -2E_0 + \|v_x(t)\|^2 + 4\|v_t(t)\|^2 - 2(\mu-2)\int_0^1 \Gamma(v_x)\,dx \\ &\geq -2E_0 + 2\gamma(\mu-2)\int_0^1 |v_x|^{m+1}dx. \end{aligned} \tag{20}$$

It follows from (20) that

$$\dot{F}(t) \geq -2E_0 t + 2\gamma(\mu-2)\int_0^t\int_0^1 |v_x(x,\tau)|^{m+1}dx\,d\tau + \dot{F}(0), \tag{21}$$

$$\begin{aligned} F(t) &\geq -E_0 t^2 + 2\gamma(\mu-2)\int_0^t\int_0^\tau\int_0^1 |v_x(x,s)|^{m+1}dx\,ds\,d\tau \\ &\quad + \dot{F}(0)t + F(0), \end{aligned} \tag{22}$$

where $\dot{F}(0) = 2\int_0^1(\int_0^x u_0(\xi)d\xi \int_0^x u_1(\xi)d\xi)dx$ and $F(0) = \|\int_0^x u_0(\xi)d\xi\|^2$. Combining (20) with (22) leads to

$$\begin{aligned} &\ddot{F}(t) + F(t) \\ &\quad \geq -E_0 t^2 + \dot{F}(0)t + F(0) - 2E_0 \\ &\qquad + 2\gamma(\mu-2)\left(\int_0^1 |v_x|^{m+1}dx \right. \\ &\qquad\qquad \left. + \int_0^t\int_0^\tau\int_0^1 |v_x(x,s)|^{m+1}dx\,ds\,d\tau\right). \end{aligned} \tag{23}$$

Making use of the Hölder inequality, we get

$$\begin{aligned} &\int_0^1 |v_x|^{m+1}dx \geq \|v_x(t)\|^{m+1}, \\ &\int_0^t\int_0^\tau\int_0^1 |v_x(x,s)|^{m+1}dx\,ds\,d\tau \\ &\quad \geq \left(\frac{t^2}{2}\right)^{(1-m)/2}\left(\int_0^t\int_0^\tau\int_0^1 |v_x(x,s)|^2 dx\,ds\,d\tau\right)^{(1+m)/2}. \end{aligned} \tag{24}$$

Substituting (24) into (23) and using the Poincaré inequality ($\|v_x(t)\| \geq \|v(t)\|$) and the inequality $a^n + b^n \geq 2^{1-n}(a+b)^n$ ($a, b \geq 0, n \geq 1$), we conclude that when $t \geq 1$,

$$\begin{aligned}
&\ddot{F}(t) + F(t) \\
&\quad \geq 2\gamma(\mu-2)t^{1-m} \\
&\qquad \times \left[\left(\|v_x(t)\|^2\right)^{(m+1)/2}\right. \\
&\qquad\quad \left. + \left(\int_0^t \int_0^\tau \int_0^1 |v_x(x,s)|^2 dx\, ds\, d\tau\right)^{(m+1)/2}\right] \\
&\qquad - E_0 t^2 + \dot{F}(0)t + F(0) - 2E_0 \\
&\quad \geq 2^{(3-m)/2}\gamma(\mu-2)t^{1-m}F^{(1+m)/2}(t) \\
&\qquad - E_0 t^2 + \dot{F}(0)t + F(0) - 2E_0.
\end{aligned} \tag{25}$$

Choose $t_0 \geq 1$ such that

$$-2E_0 t_0 + \dot{F}(0) \geq 0, \qquad -E_0 t_0^2 + \dot{F}(0)t_0 + F(0) - 2E_0 \geq 0, \tag{26}$$

and thus (21) and (22) imply that $\dot{F}(t) \geq 0$ and $F(t) \geq 0$, as $t \geq t_0$. Multiplying both sides of (25) by $2\dot{F}(t)$, we have

$$\frac{d}{dt}\left[\dot{F}^2(t) + F^2(t)\right] \geq Dt^{1-m}\frac{d}{dt}F^{(m+3)/2}(t) + H(t), \quad t \geq t_0, \tag{27}$$

where

$$\begin{aligned}
D &= \frac{\gamma(\mu-2)}{2^{(m-7)/2}(m+3)}, \\
H(t) &= 2\dot{F}(t)\left(-E_0 t^2 + \dot{F}(0)t + F(0) - 2E_0\right).
\end{aligned} \tag{28}$$

Equation (27) implies that

$$\frac{d}{dt}\left[t^{m-1}\left(\dot{F}^2(t) + F^2(t)\right) - DF^{(m+3)/2}(t)\right] \geq t^{m-1}H(t), \quad t \geq t_0. \tag{29}$$

Since

$$\begin{aligned}
&\int_{t_0}^t \tau^{m-1}H(\tau)\,d\tau \\
&\quad \geq \int_{t_0}^t 2\left(-2E_0\tau + \dot{F}(0)\right)\left(-E_0\tau^2 + \dot{F}(0)\tau + F(0)\right. \\
&\qquad \left. -2E_0\right)d\tau \longrightarrow +\infty, \quad t \longrightarrow +\infty,
\end{aligned} \tag{30}$$

there exists a $t_1 > t_0$ such that

$$\begin{aligned}
&\int_{t_0}^t \tau^{m-1}H(\tau)\,d\tau + t_0^{m-1}\left(\dot{F}^2(t_0) + F^2(t_0)\right) \\
&\quad - DF^{(m+3)/2}(t_0) \geq 0, \quad t > t_1.
\end{aligned} \tag{31}$$

By (31), integrating both sides of (29) over (t_0, t), we obtain

$$t^{m-1}\left[\dot{F}^2(t) + F^2(t)\right] \geq DF^{(m+3)/2}(t), \quad t \geq t_1. \tag{32}$$

Namely,

$$\dot{F}(t) + F(t) \geq \sqrt{2D}t^{(1-m)/2}F^{(m+3)/4}(t), \quad t \geq t_1. \tag{33}$$

In order to use Lemma 7, we consider the following initial value problem of the Bernoulli equation:

$$\begin{aligned}
\dot{X} + X &= \sqrt{2D}t^{(1-m)/2}X^{(m+3)/4}, \quad t > t_1, \\
X(t_1) &= F(t_1).
\end{aligned} \tag{34}$$

We can obtain the solution of the initial value problem (34) as follows:

$$\begin{aligned}
X(t) &= e^{-(t-t_1)}\left[F^{(1-m)/4}(t_1) - \frac{m-1}{4}\sqrt{2D}\right. \\
&\qquad \left. \times \int_{t_1}^t \tau^{(1-m)/2}e^{((1-m)/4)(\tau-t_1)}d\tau\right]^{4/(1-m)} \\
&= e^{-(t-t_1)}F(t_1)I^{4/(1-m)}(t), \quad t \geq t_1,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
I(t) &= 1 - \frac{m-1}{4}\sqrt{2D}F^{(m-1)/4}(t_1) \\
&\qquad \times \int_{t_1}^t \tau^{(1-m)/2}e^{((1-m)/4)(\tau-t_1)}d\tau.
\end{aligned} \tag{36}$$

By (36), we know that $I(t_1) = 1 > 0$ and

$$\begin{aligned}
J(t) &= \frac{m-1}{4}\sqrt{2D}F^{(m-1)/4}(t_1)\int_{t_1}^t \tau^{(1-m)/2}e^{((1-m)/4)(\tau-t_1)}d\tau \\
&\geq \frac{m-1}{4}\sqrt{2D}F^{(m-1)/4}(t_1)(1+t_1)^{(1-m)/2} \\
&\qquad \times \int_{t_1}^{t_1+1} e^{((1-m)/4)(\tau-t_1)}d\tau \\
&= \sqrt{2D}F^{(m-1)/4}(t_1)(1+t_1)^{(1-m)/2} \\
&\qquad \times \left(1 - e^{-(m-1)/4}\right), \quad t \geq t_1 + 1.
\end{aligned} \tag{37}$$

It follows from (22) that

$$\begin{aligned}
&F^{(m-1)/4}(t)(1+t)^{(1-m)/2} \\
&\quad \geq \left[\frac{-E_0 t^2 + \dot{F}(0)t + F(0)}{(t+1)^2}\right]^{(m-1)/4} \longrightarrow (-E_0)^{(m-1)/4}, \\
&\qquad t \longrightarrow +\infty.
\end{aligned} \tag{38}$$

Choose t_1 sufficiently large such that $F^{(m-1)/4}(t_1)(1+t_1)^{(1-m)/2} \geq (-E_0)^{(m-1)/4}/2$. Combining (37) with (9), we obtain

$$J(t) \geq \frac{1}{2}\sqrt{2D}(-E_0)^{(m-1)/4}\left(1-e^{(1-m)/4}\right) \geq 1, \quad t \geq t_1+1. \tag{39}$$

Hence

$$I(t) = 1 - J(t) \leq 0, \quad t \geq t_1+1. \tag{40}$$

By using the continuity of $I(t)$, there exists a finite time T_0, $t_1 < T_0 \leq t_1+1$ such that $I(T_0)=0$. Therefore, $X(t) \to +\infty$ as $t \to T_0^-$. By virtue of Lemma 7, we deduce that $F(t) \geq X(t)$, $t \geq t_1$. Hence

$$F(t) = \int_0^1 \left(\int_0^x u(\xi,t)\,d\xi\right)^2 dx + \int_0^t \int_0^\tau \int_0^1 u^2(\xi,s)\,d\xi\,ds\,d\tau \longrightarrow +\infty \tag{41}$$

as $t \to T_0^-$. It follows from (41) that

$$F(t) \leq \left(\int_0^1 |u(\xi,t)|\,d\xi\right)^2 + t\int_0^t \|u(\tau)\|^2 d\tau. \tag{42}$$

Therefore,

$$\|u(t)\|^2_{L^1(\Omega)} + \int_0^t \|u(\tau)\|^2 d\tau \longrightarrow +\infty \quad \text{as } t \longrightarrow T_0^-. \tag{43}$$

Theorem 5 is proved. □

Proof of Theorem 6. Let

$$y(t) = -\frac{\pi}{2}\int_0^1 u(x,t)\sin\pi x\,dx. \tag{44}$$

Multiplying both sides of (4) by $(\pi/2)\sin\pi x$, integrating by parts over $[0,1]$, and making use of the Jensen inequality and condition (1) of Theorem 6, we have

$$\begin{aligned}\ddot{y} + \left(\pi^2 - \alpha\pi^4 + \beta\pi^6\right)y &= \frac{\pi^3}{2}\int_0^1 \sigma(u)\sin\pi x\,dx \\ &\geq \pi^2\sigma\left(\frac{\pi}{2}\int_0^1 u(x,t)\sin\pi x\,dx\right) \\ &\geq a\pi^2 y^m, \quad t>0,\end{aligned} \tag{45}$$

and, from (6) and condition (2) of Theorem 6, we get

$$\begin{aligned} y(0) &= -\frac{\pi}{2}\int_0^1 u_0(x)\sin\pi x\,dx = y_0 > 0, \\ \dot{y}(0) &= -\frac{\pi}{2}\int_0^1 u_1(x)\sin\pi x\,dx = y_1 > 0.\end{aligned} \tag{46}$$

Thus, we claim that

$$y(t) > 0, \quad \dot{y}(t) > 0, \quad t > 0. \tag{47}$$

In fact, if it is not true, then there exists a t^* such that $\dot{y}(t) > 0$, $t \in [0,t^*)$ and $\dot{y}(t^*) = 0$. Then $y(t)$ is monotonically increasing on $[0,t^*]$; that is, $y(t) \geq y_0$, $t \in [0,t^*]$. By using (45) and condition (2) of Theorem 6, we obtain

$$\begin{aligned}\ddot{y}(t) &\geq \pi^2 y\left(ay^{m-1} - \beta\pi^4 + \alpha\pi^2 - 1\right) \\ &> \pi^2 y_0\left(ay_0^{m-1} - \beta\pi^4 + \alpha\pi^2 - 1\right) > 0, \quad (0,t^*],\end{aligned} \tag{48}$$

and hence $\dot{y}(t)$ is monotonically increasing on $[0,t^*]$, which contradicts the assumption $y(t^*) = 0$. So claim (47) is valid.

Multiplying both sides of (45) by $2\dot{y}$ and integrating the product over $[0,t]$ lead to

$$\begin{aligned}\dot{y}^2 &\geq \frac{2a\pi^2}{m+1}\left(y^{m+1} - y_0^{m+1}\right) \\ &\quad - \left(\pi^2 - \alpha\pi^4 + \beta\pi^6\right)\left(y^2 - y_0^2\right) + y_1^2 = G(y).\end{aligned} \tag{49}$$

Since $G(y_0) = y_1^2 > 0$ and

$$\begin{aligned}G'(y) &= 2\pi^2 y\left[ay^{m-1} - \left(1 - \alpha\pi^2 + \beta\pi^4\right)\right] \\ &\geq 2\pi^2 y_0\left[ay_0^{m-1} - \left(1 - \alpha\pi^2 + \beta\pi^4\right)\right] \geq 0,\end{aligned} \tag{50}$$

$G(y) > G(y_0) > 0$, $t > 0$. It follows from (49) that

$$\dot{y} \geq \left[\frac{2a\pi^2}{m+1}\left(y^{m+1} - y_0^{m+1}\right) - \left(\pi^2 - \alpha\pi^4 + \beta\pi^6\right)\left(y^2 - y_0^2\right) + y_1^2\right]^{1/2}, \quad t > 0, \tag{51}$$

and (51) implies that the interval $[0,T_1)$ of the existence of $y(t)$ is finite; namely,

$$T_1 \leq \int_{y_0}^{+\infty}\left[\frac{2a\pi^2}{m+1}\left(y^{m+1} - y_0^{m+1}\right) - \left(\pi^2 - \alpha\pi^4 + \beta\pi^6\right)\left(y^2 - y_0^2\right) + y_1^2\right]^{-1/2} dy < +\infty, \tag{52}$$

and $y(t) \to +\infty$ as $t \to T_1^-$; that is, $\int_0^1 u(x,t)\sin\pi x\,dx \to -\infty$ as $t \to T_1^-$. By the Hölder inequality, we have $\|u(t)\| \to +\infty$, as $t \to T_1^-$. Theorem 6 is proved. □

4. Initial Boundary Value Problem (2), (6) and Some Examples

By virtue of the Galerkin method [8] we can prove that initial boundary value problem (2), (6) admits a unique local

generalized solution and a unique local classical solution. Moreover, by using Theorem 6, we obtain the following theorem.

Theorem 9. *Assume that $u(x,t)$ is the generalized solution of initial boundary value problem* (2), (6) *and the following condition holds:*

$$y_0 = -\frac{\pi}{2}\int_0^1 u_0(x)\sin\pi x\,dx > \max\left\{0, \left(\delta\pi^4 - \pi^2 + 1\right)\right\},$$
$$y_1 = -\frac{\pi}{2}\int_0^1 u_1(x)\sin\pi x\,dx > 0. \tag{53}$$

Then

$$\|u(t)\| \longrightarrow +\infty, \quad \text{as } t \longrightarrow T_2^-, \tag{54}$$

where

$$T_2 \le \int_{y_0}^{+\infty}\left[\frac{2\pi^2}{3}\left(y^3 - y_0^3\right) - (\pi^2 - \pi^4 + \delta\pi^6)(y^2 - y_0^2) + y_1^2\right]^{-1/2} dy < +\infty. \tag{55}$$

Proof. A simple verification shows that all conditions of Theorem 6 are satisfied and thus Theorem 9 is proved immediately. □

Example 1. We consider the following equation:

$$u_{tt} = u_{xx} + a\left(|u|^{m-1}u\right)_{xx} + \alpha u_{x^4} + \beta u_{x^6}, \quad x \in \Omega,\ t > 0, \tag{56}$$

with the initial boundary value conditions

$$\begin{aligned} &u_x(0,t) = u_x(1,t) = u_{x^3}(0,t) = u_{x^3}(1,t) = u_{x^5}(0,t) \\ &\quad = u_{x^5}(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{57}$$

or with

$$\begin{aligned} &u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = u_{x^4}(0,t) \\ &\quad = u_{x^4}(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{58}$$

where $a \ne 0$ and $m > 1$ are all real numbers, $u_0(x) = u_1(x) = K_0\cos\pi x$, and $K_0 > 0$ is a constant.

(1) If $a > 0$ and $m \ge 9$, a simple calculation shows that $\sigma(= a|s|^{m-1}s) \in C^9(\mathbf{R})$, $\sigma^{(2i)}(0) = 0$ $(i = 1,2,3,4)$, and $\sigma'(s) = am|s|^{m-1} \ge 0$; that is, $\sigma'(s)$ is bounded below. And $u_0(x)$ and $u_1(x)$ satisfy the conditions of Theorems 2 and 4, respectively; then by Theorems 2 and 4 we know that the initial boundary value problem (56), (57) and (56), (58) admits a unique global classical solution, respectively.

(2) If $a < 0$ and $m > 1$, we have $\sigma(s)s = a|s|^{m-1}$, $\Gamma(s) = a|s|^{m-1}/(m+1)$; taking $\mu = m+1 > 2$ and $\gamma = -a/(m+1)$, then $\sigma(s)s = \mu\Gamma(s)$, $\Gamma(s) = -\gamma|s|^{m-1}$; obviously, $u_0 \in H^2$, $u_1 \in L^2$, $\int_0^1 u_0(x)dx = \int_0^1 u_1(x)dx = 0$, and

$$\begin{aligned} E_0 &= \int_0^1\left(\int_0^x u_1(\xi)\,d\xi\right)^2 dx + \|u_0\|^2 - \alpha\|u_0'\|^2 \\ &\quad + \beta\|u_0''\|^2 + \frac{2a}{m+1}\|u_0\|_{L^{m+1}(\Omega)}^{m+1} \\ &= \left(\frac{1}{\pi^2} + 1 - \alpha\pi^2 + \beta\pi^4\right)\frac{K_0^2}{2} \\ &\quad + \frac{2aK_0^{m+1}}{m+1}\int_0^1 |\cos\pi x|^{m+1}dx. \end{aligned} \tag{59}$$

We can take K_0 suitable large such that

$$E_0 \le -\left[\frac{2}{D\left(1 - e^{(1-m)/4}\right)^2}\right]^{2/(m-1)}, \tag{60}$$

where $D = -a(m-1)/[2^{(m-7)/2}(m+1)(m+3)]$. Thus, all assumptions of Theorem 5 are satisfied; then by Theorem 5 we conclude that the solution of initial boundary value problem (56), (57) must blow up in a finite time T_0; namely,

$$\|u(t)\|_{L^1(\Omega)} + \int_0^t \|u(\tau)\|^2 d\tau \longrightarrow +\infty \quad \text{as } t \longrightarrow T_0^-. \tag{61}$$

Example 2. We consider the following equation:

$$u_{tt} = u_{xx} + a\left(u^m\right)_{xx} + \alpha u_{x^4} + \beta u_{x^6}, \quad x \in \Omega,\ t > 0, \tag{62}$$

with the initial boundary value conditions

$$\begin{aligned} &u_x(0,t) = u_x(1,t) = u_{x^3}(0,t) = u_{x^3}(1,t) = u_{x^5}(0,t) \\ &\quad = u_{x^5}(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{63}$$

or with

$$\begin{aligned} &u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = u_{x^4}(0,t) \\ &\quad = u_{x^4}(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{64}$$

where $a > 0$ is a real number and $m > 1$ is a positive integer, $u_0(x) = u_1(x) = -K_1$, and $K_1 > 0$ is a constant.

(1) If $a > 0$ and m is an odd number, a simple verification shows that all conditions of Theorems 2 and 4 are satisfied; then by Theorems 2 and 4 we know that the initial boundary value problem (62), (63) and (62), (64) admits a unique global classical solution, respectively.

(2) If $a > 0$ and m is an even number, then $\sigma(s)$ $(= as^m)$ is a convex and even function, and we can take K_1 suitable large such that

$$-\frac{\pi}{2}\int_0^1 u_0(x)\sin\pi x\,dx = K_1 > \max\left\{0, \left(\frac{\beta\pi^4 - \alpha\pi^2 + 1}{a}\right)^{1/(m-1)}\right\}, \quad (65)$$

$$-\frac{\pi}{2}\int_0^1 u_1(x)\sin\pi x\,dx = K_1 > 0.$$

Thus, by Theorem 6, we deduce that the solution of initial boundary value problem (62), (64) must blow up in a finite time T_1; namely,

$$\|u(t)\| \longrightarrow +\infty, \quad \text{as } t \longrightarrow T_1^-. \quad (66)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stability and Hopf Bifurcation Analysis of a Vector-Borne Disease with Time Delay

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A delay-differential modelling of vector-borne is investigated. Its dynamics are studied in terms of local analysis and Hopf bifurcation theory, and its linear stability and Hopf bifurcation are demonstrated by studying the characteristic equation. The stability and direction of Hopf bifurcation are determined by applying the normal form theory and the center manifold argument.

1. Introduction

Vector-borne diseases are an important public health problem. Vector-borne diseases are infectious diseases caused by virus, bacteria, and so on which are primarily transmitted by disease biological agents, called vector carrying the disease.

Malaria is the most prevalent vector-borne disease, which is transmitted to the human host through a bite by an infected mosquito. It can lead to serous affecting the brain, lungs, kidneys, and other organs, and it caused the greatest number of deaths. Approximately, 40 percent of the world's population is at risk, and 2 million deaths per year can be attributed to malaria, half of those in children under 5 years old. Especially in Africa, more than one million children mostly under 5 years die each year. No effective vaccines are available for the disease. In many years, the effective way to prevent the malaria and other mosquito-borne disease is to control mosquito.

Several theoretical studies have proposed vector-borne models. Reference [1] used a mathematical model to show that bringing a mosquito population below a certain threshold was sufficient to eliminate malaria. Reference [2] studied both a baseline ODE version of the model and a model with a discrete time delay and gave the conditions under which equilibrium is globally stable and the disease dies out. Reference [3] showed that reducing the number of mosquitoes is an inefficient control strategy that would have little effect on the epidemiology of malaria in areas of intense transmission. Reference [4] used a mathematical model to evaluate the impact from the programs of selective mass drug administration and vector control through mosquito nets. References [5, 6] models took into account the acquired immunity to malaria depends on exposure (i.e., that immunity is boosted by additional infections).

For a long time, it has been recognized that delay may have very complicated impact on the dynamics of a system. Delay can cause the loss of stability and can bifurcate various periodic solutions. Recently, there has been extensive work dealing with time delay systems (see, e.g., [7–11]). As far as we know, there are few works on the delayed vector-borne system, let alone the existence of Hopf bifurcation, and the stability and direction of bifurcating periodic solutions. In this paper, we focus on investigating these problems.

This paper is organized as follows. In Section 2, we provide a vector-borne model and analyze the property of the nonnegative equilibria. In Section 3, we get the existence of the Hopf bifurcation. In Section 4, the stability and direction of periodic solutions bifurcating from the Hopf bifurcation are determined by using the normal form theory and center manifold argument introduced by Hassard et al. [12].

2. Property of the Nonnegative Equilibria

We can describe the dynamics of the disease in the host population as follows:

$$\begin{aligned}\dot{S}(t) &= b_1 - \lambda_1 S(t) I(t) - \lambda_2 S(t) V(t) - \mu_1 S(t),\\ \dot{I}(t) &= \lambda_1 S(t) I(t) + \lambda_2 S(t) V(t) - \gamma I(t) - \mu_1 I(t),\\ \dot{R}(t) &= \gamma I(t) - \mu_1 R(t).\end{aligned} \tag{1}$$

Here $S(t)$, $I(t)$, and $R(t)$ represent the population density of susceptible, infectious, and recovered at time t, respectively. The total host population size at time t is given by $N_1(t)$. The host population dies at a natural rate μ_1, and the host grows with intrinsic growth rate b_1. λ_1 is the rate of direct transmission, while λ_2 is the biting rate of a pathogen-carrier vector. The host recovers at the rate γ. The recovered individuals are assumed to acquire permanent immunity.

The system that describes the dynamics of the vector is given by

$$\begin{aligned}\dot{M}(t) &= b_2 - \lambda_3 M(t) I(t) - \mu_2 M(t),\\ \dot{V}(t) &= \lambda_3 M(t) I(t) - \mu_2 V(t).\end{aligned} \tag{2}$$

Here, $V(t)$ is the number of vectors at time t carrying the pathogen at time t, and $M(t)$ represents the population density of pathogen-free vector at time t. The total vectors population size at time t is given by $N_2(t)$. b_2 and μ_2 are the birth rate and death rate of vector population, respectively. Suspectable vectors start carrying the pathogen after getting into contact with an infective host at a rate λ_3. We assume that the vectors carry the microparasite for life once they became carrier of it.

The time delay $\tau > 0$ is introduced in the system (2) to describe the dynamics of the vector. At time t, the susceptible vectors bite the host τ time ago, and the vector became infectious. The delay model of the system takes the following form:

$$\begin{aligned}\dot{M}(t) &= b_2 - \lambda_3 M(t) I(t-\tau) - \mu_2 M(t),\\ \dot{V}(t) &= \lambda_3 M(t) I(t-\tau) - \mu_2 V(t).\end{aligned} \tag{3}$$

The systems (1) and (3) satisfy the initial conditions: $S(\theta) = S_0$, $I(\theta) = I_0$, $R(\theta) = R_0$, $M(\theta) = M_0$, $V(\theta) = V_0$, and $\theta \in [-\tau, 0)$. The total host population size $N_1(t)$ can be determined by $N_1(t) = S(t) + I(t) + R(t)$ or

$$\dot{N}_1(t) = b_1 - \mu_1 N_1(t). \tag{4}$$

The total number of vectors $N_2(t)$ can be determined by $N_2(t) = M(t) + V(t)$ or

$$N_2(t) = b_2 - \mu_2 N_2(t). \tag{5}$$

The total population size of both host and vector populations are asymptotically constant; that is, $\lim_{t\to\infty} N_1(t) = b_1/\mu_1$ and $\lim_{t\to\infty} N_2(t) = b_2/\mu_2$. Without loss of generality, we assume that $N_1(t) = b_1/\mu_1$ and $N_2(t) = b_2/\mu_2$ for all $t \geq 0$ provided that $S_0 + I_0 + R_0 = b_1/\mu_1$ and $M_0 + V_0 = b_2/\mu_2$.

The systems (1) and (3) are equivalent to the dynamics of the following system:

$$\begin{aligned}\dot{S}(t) &= b_1 - \lambda_1 S(t) I(t) - \lambda_2 S(t) V(t) - \mu_1 S(t),\\ \dot{I}(t) &= \lambda_1 S(t) I(t) + \lambda_2 S(t) V(t) - \gamma I(t) - \mu_1 I(t),\\ \dot{V}(t) &= \lambda_3 \left(\frac{b_2}{\mu_2} - V(t)\right) I(t-\tau) - \mu_2 V(t).\end{aligned} \tag{6}$$

The initial condition of system (6) is

$$\left(S(\theta), I(\theta), V(\theta) \in C_+ = C\left((-\tau, 0], R_+^3\right)\right), \quad S_0, I_0, V_0 > 0. \tag{7}$$

System (6) has two equilibria $E_{10} = (b_1/\mu_1, 0, 0)$ and $E^* = (S^*, I^*, V^*)$, where

$$S^* = \frac{b_1 - (\gamma + \mu_1) I^*}{\mu_1}, \qquad V^* = \frac{\lambda_3 b_2 I^*}{\mu_2 (\mu_2 + \lambda_3 I^*)}. \tag{8}$$

I^* is determined by the following equation:

$$\frac{b_1 - (\gamma + \mu_1) I}{\mu_1}\left(\lambda_1 + \frac{\lambda_2 \lambda_3 b_2}{\mu_2 (\lambda_3 I + \mu_2)}\right) = \gamma + \mu_1. \tag{9}$$

Equation (9) has a unique positive root, when $(\gamma + \mu_1)\mu_1\mu_2^2 < b_1(\lambda_1\mu_2^2 + \lambda_2\lambda_3 b_2)$.

For the equilibrium E_{10}, the characteristic equation is

$$(\mu_1 + \lambda)\left(\gamma + \mu_1 - \frac{\lambda_1 b_1}{\mu_1} + \lambda\right)(\mu_2 + \lambda) = 0. \tag{10}$$

We can easily get the following theorem by some calculation.

Theorem 1. *E_{10} is asymptotically stable if $\lambda_1 b_1/\mu_1 - \gamma - \mu_1 < 0$; it is unstable if $\lambda_1 b_1/\mu_1 - \gamma - \mu_1 > 0$.*

3. Existence of Hopf Bifurcation

We study E^* under the condition $(\gamma + \mu_1)\mu_1\mu_2^2 < b_1(\lambda_1\mu_2^2 + \lambda_2\lambda_3 b_2)$. The characteristic equation of E^* is

$$\begin{aligned}&\lambda^3 + (2\mu_1 + \mu_2 + \gamma + \lambda_1 I^* + \lambda_3 I^* + \lambda_2 V^* - \lambda_1 S^*)\lambda^2\\ &\quad + [(\mu_2 + \lambda_3 I^*)(2\mu_1 + \gamma + \lambda_1 I^* + \lambda_2 V^* - \lambda_1 S^*)\\ &\qquad + \mu_1(\mu_1 + \gamma - \lambda_1 S^*) + (\lambda_1 I^* + \lambda_2 V^*)(\mu_1 + \gamma)]\lambda\\ &\quad + (\mu_2 + \lambda_3 I^*)[\mu_1(\mu_1 + \gamma - \lambda_1 S^*)\\ &\qquad + (\lambda_1 I^* + \lambda_2 V^*)(\mu_1 + \gamma)]\\ &\quad + \lambda_2\lambda_3 S^* V^*(\lambda + \mu_1) e^{-\lambda\tau} = 0.\end{aligned} \tag{11}$$

Let

$$A = 2\mu_1 + \mu_2 + \gamma + \lambda_1 I^* + \lambda_3 I^* + \lambda_2 V^* - \lambda_1 S^*, \tag{12}$$

$$\begin{aligned} B &= (\mu_2 + \lambda_3 I^*)(2\mu_1 + \gamma + \lambda_1 I^* + \lambda_2 V^* - \lambda_1 S^*) \\ &\quad + \mu_1(\mu_1 + \gamma - \lambda_1 S^*) + (\lambda_1 I^* + \lambda_2 V^*)(\mu_1 + \gamma), \end{aligned} \tag{13}$$

$$\begin{aligned} C &= \mu_1(\mu_2 - \lambda_3 I^*)(\mu_1 + \gamma - \lambda_1 S^*) \\ &\quad + (\lambda_1 I^* + \lambda_2 V^*)(\mu_2 - \lambda_3 I^*), \end{aligned} \tag{14}$$

$$D = \lambda_2 \lambda_3 S^* V^*, \qquad E = \mu_1 \lambda_2 \lambda_3 S^* V^*. \tag{15}$$

Then (11) can be rewritten as

$$\lambda^3 + A\lambda^2 + B\lambda + C + [D\lambda + E]\, e^{-\lambda\tau} = 0. \tag{16}$$

Lemma 2. *Equation* (11) *has a unique pair of purely imaginary roots if* $A^2 - 2B > 0$ *and* $C^2 - E^2 < 0$.

Proof. If $\lambda = iw$, $w > 0$ is a root of (16), separating real and imaginary parts, we have the following:

$$\begin{aligned} Aw^2 - C &= E\cos w\tau + Dw\sin w\tau, \\ w^3 - Bw &= Dw\cos w\tau - E\sin w\tau. \end{aligned} \tag{17}$$

Squaring and adding both equations, we have

$$w^6 + Q_1 w^4 + Q_2 w^2 + Q_3 = 0, \tag{18}$$

where

$$Q_1 = A^2 - 2B, \qquad Q_2 = B^2 - 2AC - D^2, \qquad Q_3 = C^2 - E^2. \tag{19}$$

We know that

$$Q_1 = A^2 - 2B > 0, \tag{20}$$

$$Q_3 = C^2 - E^2 < 0. \tag{21}$$

Then this lemma implies that there is a unique positive root w_0 satisfying (16). That is, (11) has a unique pair of purely imaginary roots $\pm iw_0$.

From (17), τ_n can be obtained:

$$\tau_n = \frac{1}{w_0}\cos^{-1}\frac{Dw_0^4 + (AE - BD)\,w_0^2 - EC}{E^2 + D^2 w_0^2} + \frac{2n\pi}{w_0}, \qquad n = 0, 1, 2, \ldots. \tag{22}$$

□

Theorem 3. *If the following conditions*

$$A^2 - 2B > 0, \qquad C^2 - E^2 < 0, \qquad A(B + D) > C + E, \tag{23}$$

$$\left(B^2 - 2AC\right)E^2 > C^2 D^2 \tag{24}$$

are satisfied, system (3) *undergoes Hopf bifurcation at* E^* *when* $\tau = \tau_n$, $n = 0, 1, 2, \ldots$; *furthermore,* E^* *is locally asymptotically stable if* $\tau \in [0, \tau_0)$ *and unstable if* $\tau > \tau_0$.

Proof. Differentiating (16) with respect to τ, we get

$$\begin{aligned} &\left[3\lambda^2 + 2A\lambda + B + De^{-\lambda\tau} - \tau(D\lambda + E)\,e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} \\ &\quad = \lambda(D\lambda + E)\,e^{-\lambda\tau}. \end{aligned} \tag{25}$$

That is

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{3\lambda^2 + 2A\lambda + B}{\lambda(\lambda^3 + A\lambda^2 + B\lambda + C)} + \frac{D}{\lambda(D\lambda + E)} - \frac{\tau}{\lambda}. \tag{26}$$

Thus,

$$\begin{aligned} &\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda = iw_0} \\ &= \operatorname{Re}\left(-\frac{B - 3w_0^2 + i2Aw_0}{iw_0\left[(C - Aw_0^2) + iw_0(B - w_0^2)\right]} + \frac{D}{iw_0(iDw_0 + E)}\right) \\ &= \left(2D^2 w_0^6 + \left(3E^2 + A^2D^2 - 2BD^2\right)w_0^4 + \left(2A^2 - 4B\right)E^2 w_0^2 \right. \\ &\quad \left. + \left(B^2E^2 - C^2D^2 - 2ACE^2\right)\right) \\ &\quad \times \left(\left[w_0^2\left(B - w_0^2\right)^2 + \left(C - Aw_0^2\right)^2\right]\left[(Dw_0)^2 + E^2\right]\right)^{-1}. \end{aligned} \tag{27}$$

We can rewrite the numerator as follows. Let

$$\begin{aligned} V &= w^2, \\ f(w) &= 2D^2 w^6 + \left(3E^2 + A^2D^2 - 2BD^2\right)w^4 \\ &\quad + \left(2A^2 - 4B\right)E^2 w^2 + \left(B^2E^2 - C^2D^2 - 2ACE^2\right), \end{aligned} \tag{28}$$

and then

$$\begin{aligned} G(V) &= f(w) \\ &= 2D^2V^3 + \left(3E^2 + A^2D^2 - 2BD^2\right)V^2 \\ &\quad + \left(2A^2 - 4B\right)E^2 V + \left(B^2E^2 - C^2D^2 - 2ACE^2\right), \\ G' &= 2\left[3D^2V^2 + \left(3E^2 + \left(A^2 - 2B\right)D^2\right)V + \left(A^2 - 2B\right)E^2\right]. \end{aligned} \tag{29}$$

For G',

$$\begin{aligned} \Delta &= \left(3E^2 + A^2D^2 - 2BD^2\right)^2 - 12\left(A^2 - 2B\right)D^2E^2 \\ &= \left[3E^2 - \left(A^2 - 2B\right)D^2\right]^2 \geq 0. \end{aligned} \tag{30}$$

G' has two real roots, which take the form

$$\begin{aligned} V_1 &= \frac{-\left(3E^2 + \left(A^2 - 2B\right)D^2\right) + \sqrt{\Delta}}{6D^2} < 0, \\ V_2 &= \frac{-\left(3E^2 + \left(A^2 - 2B\right)D^2\right) - \sqrt{\Delta}}{6D^2} < 0. \end{aligned} \tag{31}$$

Then we know that $G(V)$ monotonously increases in $(V_1, +\infty)$, that is to say, that $f(w)$ monotonously increases in $(0, +\infty)$. And as we know $f(0) = B^2E^2 - C^2D^2 - 2ACE^2 > 0$, we have $f(w) > 0$ for $w > 0$. Then we obtain

$$\operatorname{sign} \left.\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right|_{\tau=\tau_n} = \operatorname{sign}\operatorname{Re}\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=iw_0} > 0. \quad (32)$$

Therefore, the transversality condition holds and hence Hopf bifurcation occurs. For (16), when $\tau = 0$, the characteristic equation is

$$\lambda^3 + A\lambda^2 + (B+D)\lambda + C + E = 0. \quad (33)$$

Under the conditions of the theorem, $A(B+D) > C+E$ and Routh-Hurwitz criterion, we know that all roots of (16) have negative real part; that is to say, the equilibrium E^* is locally stable for $\tau = 0$, while τ_0 is the minimum τ_n at which the real parts of these roots are zero. So, E^* is locally asymptotically stable if $\tau \in [0, \tau_0)$ and unstable if $\tau > \tau_0$. □

4. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtain the conditions that a family periodic solutions bifurcate from the steady state E^* at the critical value τ_n. Throughout this section, we assume that these conditions hold. As pointed by Hassard et al. [12], it is interesting to determine the direction, stability, and period of these periodic solutions bifurcating from the steady state. In this section, we will follow the idea of Ross [1] to derive the explicit formulas determining these factors. Let $u_1 = S - S^*$, $u_2 = I - I^*$, and $u_3 = V - V^*$. Equation (3) becomes

$$\begin{aligned}
\dot{S}(t) &= b_1 - \lambda_1(u_1 + S^*)(u_2 + I^*) \\
&\quad - \lambda_2(u_1 + S^*)(u_3 + V^*) - \mu_1(u_1 + S^*), \\
\dot{I}(t) &= \lambda_1(u_1 + S^*)(u_2 + I^*) + \lambda_2(u_1 + S^*)(u_3 + V^*) \\
&\quad - \gamma(u_2 + I^*) - \mu_1(u_2 + I^*), \\
\dot{V}(t) &= \lambda_3\left(\frac{b_2}{\mu_2} - u_3 - V^*\right)(u_2(t-\tau) + I^*) \\
&\quad - \mu_2(u_3 + V^*).
\end{aligned} \quad (34)$$

The linearization of (34) at $u = (u_1, u_2, u_3) = (0, 0, 0)$ is

$$\begin{aligned}
\dot{u}_1 &= -l_1u_1 - l_2u_2 - l_3u_3, \\
\dot{u}_2 &= l_1u_1 + m_2u_2 + m_3u_3, \\
\dot{u}_3 &= -n_2u_2(t-\tau) - n_3u_3,
\end{aligned} \quad (35)$$

where $l_1 = \lambda_1 I^* + \lambda_2 V^* + \mu_1$, $l_2 = \lambda_1 S^*$, $m_2 = \lambda_1 S^* - \gamma - \mu_1$, $m_3 = \lambda_2 S^*$, $n_2 = \lambda_3(b_2/\mu_2 - V^*)$, and $n_3 = \lambda_3 I^* - \mu_2$. Let $x_i(t) = u_i(t\tau)$ and $\tau = \tau_n + \mu$ $(n = 0, 1, 2, \ldots)$ and τ_n is defined in (22) and $\mu \in R$, and the system (34) can be written as FDE in $C = C([-1, 0], R^3)$ as

$$\dot{x}(t) = L_\mu(x_t) + F(\mu, x_t), \quad (36)$$

where $x_t(\theta) = x(t+\theta) \in C$, $L_\mu : C \to R$, $F : R \times C \to R$ are given, respectively, by

$$\begin{aligned}
L_\mu(\phi) &= (\tau_n + \mu)\begin{pmatrix} -l_1 & -l_2 & -l_3 \\ l_1 & m_2 & m_3 \\ 0 & 0 & -n_3 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\
&\quad + (\tau_n + \mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -n_2 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}, \\
F(\mu, x_t) &= (\tau_n + \mu)\begin{pmatrix} -\lambda_1\phi_1(0)\phi_2(0) - \lambda_2\phi_1(0)\phi_3(0) \\ \lambda_1\phi_1(0)\phi_2(0) + \lambda_2\phi_1(0)\phi_3(0) \\ -\lambda_3\phi_2(-1)\phi_3(0) \end{pmatrix},
\end{aligned} \quad (37)$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$. By the Riezs representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta), \quad \phi \in C. \quad (38)$$

In fact, we can choose

$$\begin{aligned}
\eta(\theta, \mu) &= (\tau_n + \mu)\begin{pmatrix} -l_1 & -l_2 & -l_3 \\ l_1 & m_2 & m_3 \\ 0 & 0 & -n_3 \end{pmatrix}\delta(\theta) \\
&\quad - (\tau_n + \mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -n_2 & 0 \end{pmatrix}\delta(\theta + 1),
\end{aligned} \quad (39)$$

where δ is Dirac delta function.

For $\phi \in C^1([-1, 0], R^3)$, define

$$\begin{aligned}
A(\mu)\phi &= \begin{cases} \dfrac{d\phi(\theta)}{d\theta} & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(s, \mu)\phi(s) & \theta = 0, \end{cases} \\
R(\mu)\phi &= \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases}
\end{aligned} \quad (40)$$

Then system (36) is equivalent to

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \quad (41)$$

where

$$x_t(\theta) = x(t+\theta) \quad \theta \in [-1, 0). \quad (42)$$

For $\varphi \in C^1([0, 1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in [0, 1), \\ \displaystyle\int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases} \quad (43)$$

and bilinear inner product

$$\begin{aligned}
\langle \psi(s), \phi(s) \rangle &= \overline{\psi^T}(0)\phi(0) \\
&\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi^T}(\xi - \theta)\, d\eta(\theta)\phi(\xi)\, d\xi,
\end{aligned} \quad (44)$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. From Section 3, we know that $\pm iw_0$ are eigenvalues of $A(0)$. Thu,s they are eigenvalues of A^*. We need to compute the eigenvector of $A(0)$ and A^* corresponding to iw_0 and $-iw_0$, respectively.

Suppose $q(\theta) = (1, \alpha, \beta)^T e^{i\theta w_0 \tau_n}$ is the eigenvector of $A(0)$ corresponding to iw_0. Then $A(0)q(\theta) = iw_0\tau_n q(\theta)$. It follows from the definition of $A(0)$ and $\eta(\theta, \mu)$ that

$$\tau_n \begin{pmatrix} -l_1 - iw_0 & -l_2 & -l_3 \\ l_1 & m_2 - iw_0 & m_3 \\ 0 & -n_2 e^{-iw_0\tau_n} & -n_3 - iw_0 \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (45)$$

Then we can get

$$\begin{aligned} q(0) &= (1, \alpha, \beta) \\ &= \left(1, \frac{l_1 n_3}{l_3 n_2 e^{-iw_0\tau_n} - l_2 n_3}, \frac{-l_1 n_2}{l_3 n_2 e^{-iw_0\tau_n} - l_2 n_3} e^{-iw_0\tau_n}\right)^T. \end{aligned} \quad (46)$$

We can suppose that $q^*(s) = D(1, \alpha^*, \beta^*)e^{isw_0\tau_n}$ is the eigenvector of A^* corresponding to $-iw_0\tau_n$, and similarly we can obtain

$$\tau_n \begin{pmatrix} -l_1 + iw_0 & l_1 & 0 \\ -l_2 & m_2 + iw_0 & -n_2 e^{iw_0\tau_n} \\ -l_3 & m_3 & -n_3 + iw_0 \end{pmatrix} q^*(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\alpha^* = \frac{l_1 - iw_0}{l_1}, \qquad \beta^* = \frac{l_1 m_2 + w_0^2 - l_1 l_2 - iw_0(m_2 - l_1)}{l_1 n_2 e^{iw_0\tau_n}}. \quad (47)$$

By (44), we get

$$\begin{aligned} &\langle q^*(s), q(\theta)\rangle \\ &= \overline{D}\left(1, \overline{\alpha}^*, \overline{\beta}^*\right)(1, \alpha, \beta)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}\left(1, \overline{\alpha}^*, \overline{\beta}^*\right) e^{-iw_0(\xi-\theta)} d\eta(\theta)(1, \alpha, \beta)^T e^{iw_0\xi} d\xi \\ &= \overline{D}\left\{1 + \overline{\alpha}^*\alpha + \overline{\beta}^*\beta - \int_{-1}^{0}\left(1, \overline{\alpha}^*, \overline{\beta}^*\right)\theta e^{iw_0\theta} d\eta(\theta)(1, \alpha, \beta)^T\right\} \\ &= \overline{D}\left\{1 + \overline{\alpha}^*\alpha + \overline{\beta}^*\beta - \tau_n n_2 \alpha \overline{\beta}^* e^{-iw_0\tau_n}\right\}. \end{aligned} \quad (48)$$

Then we choose

$$\overline{D} = \frac{1}{1 + \overline{\alpha}^*\alpha + \overline{\beta}^*\beta - \tau_n n_2 \alpha \overline{\beta}^* e^{-iw_0\tau_n}}, \quad (49)$$

such that $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

In the following, we use the ideas by Ross [1] to compute the coordinates describing center manifold C_0 at $\mu = 0$. Define

$$z(t) = \langle q^*, x_t\rangle, \qquad W(t, \theta) = x_t - 2\operatorname{Re} z(t) q(\theta). \quad (50)$$

On the center manifold C_0, we have

$$\begin{aligned} W(t, \theta) &= W(z(t), \overline{z}(t), \theta) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots, \end{aligned} \quad (51)$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q^*}$. Note that W is real if x_t is real. We can only consider real solutions. For the solution $x_t \in C_0$, since $\mu = 0$ and (41), we have

$$\begin{aligned} \dot{z} &= iw_0\tau_n z + \langle q^*(\theta), F(0, W(\overline{z}, z, \theta) + 2\operatorname{Re} zq(\theta))\rangle \\ &= iw_0\tau_n z + \overline{q}^*(0) F(0, W(\overline{z}, z, 0) + 2\operatorname{Re} zq(0)) \\ &\overset{\text{def}}{=} iw_0\tau_n z + \overline{q}^*(0) F_0(z, \overline{z}) \\ &= iw_0\tau_n z + g(z, \overline{z}), \end{aligned} \quad (52)$$

where

$$\begin{aligned} g(z, \overline{z}) &= \overline{q}^*(0) F_0(z, \overline{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \quad (53)$$

From (50), we have $x_t = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta)) = W(t, \theta) + zq(\theta) + \overline{zq(\theta)}$ and $q(\theta) = (1, \alpha, \beta)^T e^{iw_0\tau_n}$, and then

$$\begin{aligned} x_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\overline{z} \\ &\quad + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O\left(|(z, \overline{z})|^3\right), \\ x_{2t}(0) &= z\alpha + \overline{z\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\overline{z} \\ &\quad + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + O\left(|(z, \overline{z})|^3\right), \\ x_{3t}(0) &= z\beta + \overline{z\beta} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0) z\overline{z} \\ &\quad + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + O\left(|(z, \overline{z})|^3\right), \\ x_{2t}(-1) &= z\alpha e^{-iw_0\tau_n} + \overline{z\alpha} e^{iw_0\tau_n} + W_{20}^{(2)}(-1)\frac{z^2}{2} \\ &\quad + W_{11}^{(2)}(-1) z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + O\left(|(z, \overline{z})|^3\right). \end{aligned} \quad (54)$$

From the definition of $F(\mu, x_t)$, we have

$$\begin{aligned}
g(z,\overline{z}) &= \overline{q}^*(0)F_0(z,\overline{z}) \\
&= \overline{D}\tau_n\left(1,\overline{\alpha}^*,\overline{\beta}^*\right)\begin{pmatrix} -\lambda_1 x_{1t}(0)x_{2t}(0)-\lambda_2 x_{1t}(0)x_{3t}(0) \\ \lambda_1 x_{1t}(0)x_{2t}(0)+\lambda_2 x_{1t}(0)x_{3t}(0) \\ -\lambda_3 x_{2t}(-1)x_{3t}(0)\end{pmatrix} \\
&= \overline{D}\tau_n\Bigg\{z^2\left[\alpha(\overline{\alpha}^*-1)(\lambda_1+\beta\lambda_2)-\lambda_3\alpha\beta\overline{\beta}^* e^{-iw_0\tau_n}\right] \\
&\quad + 2z\overline{z}\Big[\lambda_1 \operatorname{Re}\{\alpha\}(\overline{\alpha}^*-1)+\lambda_2\operatorname{Re}\left\{\alpha\overline{\beta}\right\}(\overline{\alpha}^*-1) \\
&\quad -\lambda_3\overline{\beta}^*\operatorname{Re}\left\{\overline{\beta}\alpha e^{-iw_0\tau_n}\right\}\Big] \\
&\quad + \overline{z}^2\left[\lambda_1\overline{\alpha}(\overline{\alpha}^*-1)+\lambda_2\overline{\alpha\beta}(\overline{\alpha}^*-1)-\lambda_3\overline{\alpha\beta\beta^*}e^{iw_0\tau_n}\right] \\
&\quad + z^2\overline{z}\Bigg[\lambda_1(\overline{\alpha}^*-1)\left(W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)+\frac{\overline{\alpha}}{2}W_{20}^{(1)}(0)+\alpha W_{11}^{(1)}(0)\right) \\
&\quad + \lambda_2(\overline{\alpha}^*-1)\left(\alpha W_{11}^{(3)}(0)+\frac{\overline{\alpha}}{2}W_{20}^{(3)}(0)+\frac{\overline{\beta}}{2}W_{20}^{(2)}(0)+\beta W_{11}^{(2)}(0)\right) \\
&\quad - \lambda_3\overline{\beta}^*\Bigg(\beta W_{11}^{(2)}(-1)+\frac{\overline{\beta}}{2}W_{20}^{(2)}(-1)+\frac{\overline{\alpha}}{2}W_{20}^{(3)}(0)e^{iw_0\tau_n} \\
&\quad +\alpha W_{11}^{(3)}(0)e^{-iw_0\tau_n}\Bigg)\Bigg]+\cdots\Bigg\}.
\end{aligned} \tag{55}$$

Comparing the coefficients with (53), we obtain

$$\begin{aligned}
g_{20} &= 2\overline{D}\tau_n\left[\alpha(\overline{\alpha}^*-1)(\lambda_1+\beta\lambda_2)-\lambda_3\alpha\beta\overline{\beta}^* e^{-iw_0\tau_n}\right], \\
g_{11} &= 2\overline{D}\tau_n\Big[\lambda_1\operatorname{Re}\{\alpha\}(\overline{\alpha}^*-1)+\lambda_2\operatorname{Re}\left\{\alpha\overline{\beta}\right\}(\overline{\alpha}^*-1) \\
&\quad -\lambda_3\overline{\beta}^*\operatorname{Re}\left\{\overline{\beta}\alpha e^{-iw_0\tau_n}\right\}\Big], \\
g_{02} &= 2\overline{D}\tau_n\left[\lambda_1\overline{\alpha}(\overline{\alpha}^*-1)+\lambda_2\overline{\alpha\beta}(\overline{\alpha}^*-1)-\lambda_3\overline{\alpha\beta\beta^*}e^{iw_0\tau_n}\right], \\
g_{21} &= 2\overline{D}\tau_n\Bigg[\lambda_1(\overline{\alpha}^*-1)\left(W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)+\frac{\overline{\alpha}}{2}W_{20}^{(1)}(0)+\alpha W_{11}^{(1)}(0)\right) \\
&\quad + \lambda_2(\overline{\alpha}^*-1)\left(\alpha W_{11}^{(3)}(0)+\frac{\overline{\alpha}}{2}W_{20}^{(3)}(0)+\frac{\overline{\beta}}{2}W_{20}^{(2)}(0)+\beta W_{11}^{(2)}(0)\right) \\
&\quad - \lambda_3\overline{\beta}^*\Bigg(\beta W_{11}^{(2)}(-1)+\frac{\overline{\beta}}{2}W_{20}^{(2)}(-1)+\frac{\overline{\alpha}}{2}W_{20}^{(3)}(0)e^{iw_0\tau_n} \\
&\quad +\alpha W_{11}^{(3)}(0)e^{-iw_0\tau_n}\Bigg)\Bigg].
\end{aligned} \tag{56}$$

In order to determine g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (41) and (50), we have

$$\begin{aligned}
\dot{W} &= \dot{x}_t - \dot{z}q - \dot{\overline{z}}\,\overline{q} \\
&= \begin{cases} AW - 2\operatorname{Re}\left\{\overline{q^*}(0)F_0 q(\theta)\right\}, & \theta\in[-1,0), \\ AW - 2\operatorname{Re}\left\{\overline{q^*}(0)F_0 q(\theta)\right\}+F_0, & \theta = 0, \end{cases} \\
&\overset{\text{def}}{=} AW + H(z,\overline{z},\theta),
\end{aligned} \tag{57}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}\frac{\overline{z}^2}{2}+\cdots. \tag{58}$$

Note that, on the center manifold C_0 near to the origin,

$$\dot{W} = W_z\dot{z}+W_{\overline{z}}\dot{\overline{z}}. \tag{59}$$

Thus we obtain

$$(A-2iw_0\tau_n)W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta). \tag{60}$$

By (57), we know that, for $\theta\in[-1,0)$,

$$\begin{aligned}
H(z,\overline{z},\theta) &= -\overline{q}^*(0)F_0 q(\theta) - q^*(0)\overline{F}_0\overline{q}(\theta) \\
&= -gq(\theta) - \overline{gq}(\theta).
\end{aligned} \tag{61}$$

Comparing the coefficients with (58), we can get

$$\begin{aligned}
H_{20}(\theta) &= -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \\
H_{11}(\theta) &= -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta).
\end{aligned} \tag{62}$$

From (60), (62), and the definition of A, we have

$$\dot{W}_{20}(\theta) = 2iw_0\tau_n W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{63}$$

Noticing $q(\theta) = q(0)e^{i\theta w_0\tau_n}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\tau_n w_0}q(0)e^{i\tau_n w_0\theta}+\frac{i\overline{g}_{02}}{3\tau_n w_0}\overline{q}(0)e^{-i\tau_n w_0\theta}+E_1 e^{2i\tau_n w_0\theta}, \tag{64}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in R^3$ is a constant vector. Similarly, we can have

$$W_{11}(\theta) = -\frac{ig_{11}}{\tau_n w_0} q(0) e^{i\tau_n w_0 \theta} + \frac{i\overline{g}_{11}}{\tau_n w_0} \overline{q}(0) e^{-i\tau_n w_0 \theta} + E_2, \tag{65}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in R^3$ is a constant vector.

In the following, we wiiill find out E_1 and E_2. From the definition of A and (60), we can obtain

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2iw_0\tau_n W_{20}(0) - H_{20}(0), \tag{66}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{67}$$

where $\eta(\theta) = \eta(0, \theta)$. From (57) and (58), we have

$$H_{20}(0) = -g_{20} q(0) - \overline{g}_{02} \overline{q}(0) + 2\tau_n \begin{pmatrix} -\lambda_1\alpha - \lambda_2\alpha\beta \\ \lambda_1\alpha + \lambda_2\alpha\beta \\ -\lambda_3\alpha\beta e^{-iw_0\tau_n} \end{pmatrix}, \tag{68}$$

$$\begin{aligned} H_{11}(0) = {} & -g_{11} q(0) - \overline{g}_{11} \overline{q}(0) \\ & + 2\tau_n \begin{pmatrix} -2\lambda_1 \operatorname{Re}(\alpha) - 2\lambda_2 \operatorname{Re}(\alpha\overline{\beta}) \\ 2\lambda_1 \operatorname{Re}(\alpha) + 2\lambda_2 \operatorname{Re}(\alpha\overline{\beta}) \\ -2\lambda_3 \operatorname{Re}(\alpha\overline{\beta} e^{-iw_0\tau_n}) \end{pmatrix}. \end{aligned} \tag{69}$$

Substituting (68) into (66) and noticing that

$$\begin{aligned} \left(iw_0\tau_n I - \int_{-1}^{0} e^{i\theta w_0\tau_n} d\eta(\theta)\right) q(0) = 0, \\ \left(-iw_0\tau_n I - \int_{-1}^{0} e^{-i\theta w_0\tau_n} d\eta(\theta)\right) q(0) = 0, \end{aligned} \tag{70}$$

we then obtain

$$\left(2i\tau w_0 I - \int_{-1}^{0} e^{2i\theta\tau w_0} d\eta(\theta)\right) E_1 = 2\tau_n \begin{pmatrix} -\lambda_1\alpha - \lambda_2\alpha\beta \\ \lambda_1\alpha + \lambda_2\alpha\beta \\ -\lambda_3\alpha\beta e^{-iw_0\tau_n} \end{pmatrix}, \tag{71}$$

which is

$$\begin{pmatrix} 2iw_0 + l_1 & l_2 & l_3 \\ -l_1 & 2iw_0 - m_2 & -m_3 \\ 0 & n_2 e^{-2iw_0\tau_n} & 2iw_0 + n_3 \end{pmatrix} E_1 = 2\begin{pmatrix} -\lambda_1\alpha - \lambda_2\alpha\beta \\ \lambda_1\alpha + \lambda_2\alpha\beta \\ -\lambda_3\alpha\beta e^{-iw_0\tau_n} \end{pmatrix}. \tag{72}$$

Solving the equation, we get

$$\begin{aligned} E_1^{(1)} &= \frac{2}{R} \begin{vmatrix} -\lambda_1\alpha - \lambda_2\alpha\beta & l_2 & l_3 \\ \lambda_1\alpha + \lambda_2\alpha\beta & 2iw_0 - m_2 & -m_3 \\ -\lambda_3\alpha\beta e^{-iw_0\tau_n} & n_2 e^{-2iw_0\tau_n} & 2iw_0 + n_3 \end{vmatrix}, \\ E_1^{(2)} &= \frac{2}{R} \begin{vmatrix} 2iw_0 + l_1 & -\lambda_1\alpha - \lambda_2\alpha\beta & l_3 \\ -l_1 & \lambda_1\alpha + \lambda_2\alpha\beta & -m_3 \\ 0 & -\lambda_3\alpha\beta e^{-iw_0\tau_n} & 2iw_0 + n_3 \end{vmatrix}, \\ E_1^{(3)} &= \frac{2}{R} \begin{vmatrix} 2iw_0 + l_1 & l_2 & -\lambda_1\alpha - \lambda_2\alpha\beta \\ -l_1 & 2iw_0 - m_2 & \lambda_1\alpha + \lambda_2\alpha\beta \\ 0 & n_2 e^{-2iw_0\tau_n} & -\lambda_3\alpha\beta e^{-iw_0\tau_n} \end{vmatrix}, \end{aligned} \tag{73}$$

where

$$R = \begin{vmatrix} -\lambda_1\alpha - \lambda_2\alpha\beta & l_2 & l_3 \\ \lambda_1\alpha + \lambda_2\alpha\beta & 2iw_0 - m_2 & -m_3 \\ -\lambda_3\alpha\beta e^{-iw_0\tau_n} & n_2 e^{-2iw_0\tau_n} & 2iw_0 + n_3 \end{vmatrix}. \tag{74}$$

Similarly, we can get E_2:

$$\begin{aligned} \begin{pmatrix} -l_1 & -l_2 & -l_3 \\ l_1 & m_2 & m_3 \\ 0 & -n_2 & -n_3 \end{pmatrix} E_2 &= 4\begin{pmatrix} \lambda_1 \operatorname{Re}(\alpha) + \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) \\ -\lambda_1 \operatorname{Re}(\alpha) - \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) \\ \lambda_3 \operatorname{Re}(\overline{\alpha}\beta e^{iw_0\tau_n}) \end{pmatrix}, \\ E_2^{(1)} &= \frac{4}{S} \begin{vmatrix} \lambda_1 \operatorname{Re}(\alpha) + \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) & -l_2 & -l_3 \\ -\lambda_1 \operatorname{Re}(\alpha) - \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) & m_2 & m_3 \\ \lambda_3 \operatorname{Re}(\overline{\alpha}\beta e^{iw_0\tau_n}) & -n_2 & -n_3 \end{vmatrix}, \\ E_2^{(2)} &= \frac{4}{S} \begin{vmatrix} -l_1 & \lambda_1 \operatorname{Re}(\alpha) + \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) & -l_3 \\ l_1 & -\lambda_1 \operatorname{Re}(\alpha) - \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) & m_3 \\ 0 & \lambda_3 \operatorname{Re}(\overline{\alpha}\beta e^{iw_0\tau_n}) & -n_3 \end{vmatrix}, \\ E_2^{(3)} &= \frac{4}{S} \begin{vmatrix} -l_1 & -l_2 & \lambda_1 \operatorname{Re}(\alpha) + \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) \\ l_1 & m_2 & -\lambda_1 \operatorname{Re}(\alpha) - \lambda_2 \operatorname{Re}(\alpha\overline{\beta}) \\ 0 & -n_2 & \lambda_3 \operatorname{Re}(\overline{\alpha}\beta e^{iw_0\tau_n}) \end{vmatrix}, \end{aligned} \tag{75}$$

where

$$S = \begin{vmatrix} -l_1 & -l_2 & -l_3 \\ l_1 & m_2 & m_3 \\ 0 & -n_2 & -n_3 \end{vmatrix}. \tag{76}$$

Therefor, all g_{ij} have been expressed in terms of parameters. And we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\tau_n w_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_n)\}}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\lambda'(\tau_n)}{\tau_n w_0}, \end{aligned} \tag{77}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_n; μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical); β_2 determines the stability of the bifurcating periodic solutions: the periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); T_2 determines the period of the bifurcating solutions: the periodic increases (decreases) if $T_2 > 0$ (< 0).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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