Discrete Dynamics in Nature and Society

Discrete Dynamics of Fractional Systems: Theory and Numerical Techniques

Lead Guest Editor: Jorge E. Macías-Díaz
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Editorial Board

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Editorial

Discrete Dynamics of Fractional Systems: Theory and Numerical Techniques

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In recent decades, fractional calculus has found a large number of profound applications, which have triggered the development of both the theory and methods for more reliable discretization and approximations of the dynamics of continuous systems. Fractional order models and discretized problems are nonlocal. They provide better descriptions of and, ultimately, deeper insights into underlying complex phenomena in sciences and technologies. Novel new analytical approaches have become a key to the study of qualitative properties of the aforementioned fractional systems and the existence and uniqueness of their nonlocal solutions.

Continuous models based on systems of ordinary or partial differential equations have been investigated under proper criteria of discretization in this special issue. Novel numerical approximations of solutions of fractional systems are also investigated. In fact, the search for discrete techniques which are faster and stable, that possess higher orders of convergence at lower computational costs, and that preserve the main features of the solutions of interest has been a constant pursuit in numerical analysis. To this end, this special issue pays a special attention to the discretization of continuous fractional systems that preserve important characteristics including the positivity, boundedness, convexity, monotonicity, and energy of the underlying systems.

The 8 research papers in this special issue are highly selective. These high-quality papers represent the latest developments in the theory of discrete fractional systems and the discretization of fractional differential equations arising from sciences and technologies. The final contributed papers focus on issues like

(i) Lebesgue-p norm convergence analysis of PD^{α}-type iterative learning control for fractional order nonlinear systems,

(ii) solution existence for initial-value problems of hybrid fractional sum-difference equations,

(iii) homotopy series solutions to time-space fractional coupled systems,

(iv) numerical simulations of one-dimensional fractional nonsteady heat transfer models based on the second kind Chebyshev wavelet,

(v) numerical analysis of fractional order epidemic models of childhood diseases,

(vi) weak solutions for partial random Hadamard fractional integral equations with multiple delays,

(vii) modified function projective synchronization for a partially linear and fractional order financial chaotic system with uncertain parameters,

(viii) two new approximations for variable order fractional derivatives.

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Research Article

Lebesgue-p Norm Convergence Analysis of PD$^{\alpha}$-Type Iterative Learning Control for Fractional-Order Nonlinear Systems

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The first-order and second-order PD$^{\alpha}$-type iterative learning control (ILC) schemes are considered for a class of Caputo-type fractional-order nonlinear systems. Due to the imperfection of the λ-norm, the Lebesgue-p (L^p) norm is adopted to overcome the disadvantage. First, a generalization of the Gronwall integral inequality with singularity is established. Next, according to the reached generalized Gronwall integral inequality and the generalized Young inequality, the monotonic convergence of the first-order PD$^{\alpha}$-type ILC is investigated, while the convergence of the second-order PD$^{\alpha}$-type ILC is analyzed. The resultant condition shows that both the learning gains and the system dynamics affect the convergence. Finally, numerical simulations are exploited to verify the results.

1. Introduction

Iterative learning control (ILC) is an effective control developed for target trajectory tracking [1]. The key feature of the ILC is to improve the quality of control iteratively by using proportional, integral, and/or derivative tracking errors obtained from previous operation and finally to generate the control input that causes the desired output trajectory. Due to its satisfactory tracking performance by using less a prior knowledge, ILC has been widely applied to repetitive operations including robot manipulations and batch processes [2–4].

Fractional calculus is a mathematical topic with more than three-hundred-year-old history, but its application to physics and engineering has attracted a lot of attention in the latest decades [5, 6]. They have been verified to be a powerful technique to model the memory and hereditary properties of many materials and processes [7–9]. Further, it has been acknowledged that a fractional-order controller performs well compared to an integer-order controller for a fractional-order system. This pushes the development of the fractional-order controllers [10, 11].

Among different fractional-order controllers, fractional-order iterative learning control (FOILC) is becoming one of active research areas. In the 2010s, for an αth-order linear system, the authors investigated the αth-order derivative-type (D^{α}-type) ILC in time domain [12]. This investigation showed that the optimal ILC for an αth-order linear system is the ILC order being α. In the following years, many FOILC problems are presented for various fractional-order systems. Up to now, the FOILC area has attracted much attention, of which the convergence analysis is one of key issues. For more details, readers can refer to the works [13–19] and the references therein. Despite the nice results of existing investigations, there still remain some undesirable problems between the theoretical development and its practical application.

The first one is that many nice results are derived under a questionable assumption that the desired control input exists [20–22]. However, from the engineering application perspective, the desired output trajectory should be predetermined by the target to be tracked, rather than constructed under the assumption that the desired control input existed. That is, the convergence analysis process relies on the information that seems to be known but is actually unknown to the desired control input.

The second one is that the existing FOILC investigations analyzed the convergence by using λ-norm-based analytical methods, and the convergence is guaranteed with the sufficiently large λ. As commented in [23, 24], the larger parameter λ may greatly inhibit the actual tracking error and ignore the influence of the state matrix and proportional learning gains to the condition. This conveys that the results of both the ILCs and FOILCs are mathematically good, but practically, it may result in the normal tracking error exceeding the practical tolerance even though the λ-norm tracking error is satisfactory. Thus, both the existing ILCs and FOILCs need to be refined. For this aspect, reference [25] has adopted L^p-norm to evaluate the tracking error and has derived the monotonic convergence of the conventional first-order PD-type ILC for a class of integer-order linear systems. The derivation tells that the convergence is not only dominated by the system input and output matrices and the derivative learning gain, but also by the system state matrix and the proportional learning gain. The result reflects the relationship between the system dynamics and the learning mechanism to the convergence. To be specific, as L^2-norm measures the tracking error in the concept of energy, the convergence result in the sense of L^p-norm may boost the theoretical development near to practical execution. However, the result of the FOILC is a hanging issue. It is necessary to state that the integer derivative is a local operator, while the fractional derivative is a nonlocal operator which has many different properties; thus many theoretical approaches based on the integer-order control systems cannot have been directly applied to the fractional ones. Therefore, it is worthwhile to apply the L^p-norm for convergence analysis of existing FOILCs.

In addition, as discussed in FOILCs [12, 15–17], the higher-order learning algorithms, which employ preceding control information of more than one iteration, have utility to lead a better performance in terms of both convergence rate and robustness, which is taken advantage of. As a matter of fact, with different choice of learning gains, the higher-order classical ILC algorithm can be perform slower and faster than or equivalent to the lower-order ones in terms of convergence rate [25]. However, such affirmation has not been seen valid for fractional-order iterative learning control systems.

Motivated by the aforementioned hanging issue regarding the fractional-order systems and FOILC schemes, this paper develops the first-order and the second-order proportional-Caputo-fractional-order-derivative-type (PD$^\alpha$-type) ILCs for a class of Caputo-type fractional-order nonlinear dynamic systems and then applies the L^p-norm to investigate their convergence in an objective manner. The main contributions of this paper are that we establish a theoretical analysis framework on the monotonic convergence of the first-order PD$^\alpha$-type ILC for a Caputo-type fractional-order nonlinear system in the sense of L^p-norm. In the theoretical analysis, there is no need for the questionable assumption that the desired input exists and a novel Growall integral inequality with singularity is established for the strict convergence analysis. And then, the convergence is derived for the case when the second-order PD$^\alpha$-type is implemented on the systems and the convergent speed comparison of the second-order law with the first-order one is generalized to FOILCs.

The rest of the paper is organized as follows. In Section 2, the basic concepts, properties, and lemmas are described. In Section 3, the monotonic convergence of the first-order PD$^\alpha$-type ILC scheme and the convergence of the second-order PD$^\alpha$-type ILC are given. In Section 4, examples are presented to validate the theoretical results. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we briefly give some basic definitions and properties related to fractional calculus [5, 6].

Definition 1. For an arbitrary integrable function $f(t)$, the definition of the fractional integrals of order $\alpha > 0$ is defined as

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t \in [0,\infty), \quad (1)$$

where $\Gamma(\cdot)$ is the Gamma function and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

Definition 2. For a given number $\alpha > 0$, the α-order Caputo-type derivative of the function $f(t)$ is defined as

$$ {}_0^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad n-1 < \alpha < n,\ t \in [0,\infty), \quad (2)$$

where n is an integer and $f^{(n)}(t) = (d^n/dt^n) f(t)$.

Property 3. If $0 < \alpha < 1$, then ${}_0D_t^{-\alpha}({}_0^CD_t^{\alpha} f(t)) = f(t) - f(0)$.

Definition 4. The Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0,\ \beta > 0,\ z \in C^{n\times n}, \quad (3)$$

where $E_\alpha(z) = E_{\alpha,1}(z)$ and $E_{1,1}(z) = e^z$.

Definition 5 (see [26]). For a given scalar function $f : [0,T] \to R$, its L^p-norm is defined as

$$ \|f(\cdot)\|_p = \left[\int_0^T |f(t)|^p dt \right]^{1/p}, \quad 1 \le p \le \infty. \quad (4)$$

For a time-varying vector function $f : [0,T] \to R^m$, $f(t) = [f^1(t), \ldots, f^m(t)]^T$, its L^p-norm is defined as

$$ \|f(\cdot)\|_p = \left[\int_0^T \left(\max_{1\le i\le m} \left| f^i(t) \right| \right)^p dt \right]^{1/p}, \quad 1 \le p \le \infty. \quad (5)$$

Lemma 6 (see [27]). *If the function $f(t) \in C^n[0,T]$, then the initial value problem,*

$$ {}_0^CD_t^{\alpha} x(t) = f(x(t), t), \quad 0 < \alpha < 1, \qquad x(0) = x_0, \quad (6)$$

is equivalent to the following nonlinear Volterra integral equation:

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x(\tau), \tau)\, d\tau, \quad (7)$$

and its solutions are continuous.

For brevity, we set ${}_0D_t^{\alpha} = {}_0^CD_t^{\alpha}$ in the following section.

Lemma 7 (generalized Young inequality of convolution integral [26]). *For Lebesgue integrable scalar functions $g, h : [0,T] \in R$, the generalized Young inequality of their convolution integral is*

$$\|g * h(\cdot)\|_r \leq \|g(\cdot)\|_q \|h(\cdot)\|_p , \tag{8}$$

where $1 \leq p,\ q, r \leq \infty$ satisfy $1/r = 1/p + 1/q - 1$. Particularly, when $r = p$ and thus $q = 1$, then the inequality of convolution integral is

$$\|g * h(\cdot)\|_p \leq \|g(\cdot)\|_1 \|h(\cdot)\|_p . \tag{9}$$

Lemma 8 (see [2]). *Let $\{a_n\}$ be a positive real sequence defined as*

$$a_n \leq \rho_1 a_{n-1} + \rho_2 a_{n-2}. \tag{10}$$

If $\rho_1,\ \rho_2$ are nonnegative numbers satisfying

$$\rho = \rho_1 + \rho_2 < 1, \tag{11}$$

then the following holds:

$$\lim_{n\to\infty} a_n = 0. \tag{12}$$

Let us establish an extended Gronwall integral inequality with singularity, which is important to the convergence analysis in the next section. The proof is based on an iteration argument.

Lemma 9. *Suppose $a, b \geq 0$ (constant) and $\alpha > 0$, $c(t)$, $x(t)$ and $y(t)$ are nonnegative and locally integrable on $[0, T_0)$ $(T_0 \leq +\infty)$. If*

$$x(t) \leq c(t) + \int_0^t (t-s)^{\alpha-1} [a y(s) + b x(s)]\, ds, \quad t \in [0, T_0). \tag{13}$$

Then

$$x(t) \leq c(t) + \int_0^t [\Gamma(\alpha) \cdot \Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)) \cdot (bc(s) + a y(s))]\, ds, \tag{14}$$

where $\Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)) = (t-s)^{\alpha-1} E_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)^{\alpha})$.

Proof. See Appendix. □

3. PD$^{\alpha}$-Type ILCs and Convergence Analysis for Fractional-Order Nonlinear Systems

Consider the following nonlinear α-order ($0 < \alpha < 1$) systems:

$$\begin{aligned} {}_0^C D_t^{\alpha} x_k(t) &= f(x_k(t), t) + B u_k(t), \\ y_k(t) &= C x_k(t), \quad t \in [0, T] \\ x_k(0) &= 0, \end{aligned} \tag{15}$$

where k refers to the operation number and $[0, T]$ is an operation time interval while $\alpha \in (0,1)$. $x_k(t) \in R^n$, $u_k(t) \in R$ and $y_k(t) \in R$ denote n-dimensional state vector, scalar control input, and output, respectively. B and C are matrices with appropriate dimensions. The function $f(x_k(t), t)$ satisfies the global Lipschitz condition:

$$\begin{aligned} &|C(f(x_{k+1}(t), t) - f(x_k(t), t))| \\ &\quad \leq L_0 |C(x_{k+1}(t) - x_k(t))| = L_0 |y_{k+1}(t) - y_k(t)|, \end{aligned} \tag{16}$$

where L_0 is positive Lipschitz constant.

In this section, the sufficient conditions are derived for convergence of the first-order and second-order PD$^{\alpha}$-type ILC algorithms for fractional-order nonlinear systems. Now, we give our main results.

4. Monotonic Convergence Analysis for First-Order PD$^{\alpha}$-Type ILC

To control the systems stated in (15), the first-order PD$^{\alpha}$-type ILC is given as follows:

(I_1)

$$\begin{aligned} &u_1(t) \text{ is given arbitrarily,} \\ &u_{k+1}(t) = u_k(t) + L_{p_1} e_k(t) + L_{d_1}\, {}_0D_t^{\alpha} e_k(t), \\ &\qquad t \in [0, T], \ k = 2, 3, 4, \ldots. \end{aligned} \tag{17}$$

Here, L_{p_1} and L_{d_1} are the first-order proportional and fractional-order derivative learning gains, respectively. The expression $e_k(t) = y_d(t) - y_k(t)$ denotes the tracking error between the desired trajectory $y_d(t)$ and the system output $y_k(t)$ of the system (15) driven by $u_k(t)$ at the kth iteration.

Theorem 10. *For the first-order PD$^{\alpha}$-type iterative learning control rue (I_1) is applied to system (15), if the system matrices B, C, the order α and the Lipchitz constant L_0 together with learning gains L_{p_1}, L_{d_1} satisfy the following condition:*

$$\begin{aligned} \rho_1 = &\left|1 - CBL_{d_1}\right| + \left(\left|CBL_{p_1}\right| + L_0 + L_0 \left|1 - CBL_{d_1}\right|\right) \\ &\cdot \|\Phi_{\alpha,\alpha}(L_0 \cdot (\cdot))\|_1 < 1. \end{aligned} \tag{18}$$

Here, $|\cdot|$ stands for the absolute value and $\|\cdot\|_1$ stands for the L^1-norm of the function defined on the operation time interval $[0, T]$.

Then, the output error is strictly monotonic convergence in L^p-norm; that is,

(1) $\|e_{k+1}(\cdot)\|_p < \|e_k(\cdot)\|_p$;

(2) $\lim_{k\to\infty} \|e_{k+1}(\cdot)\|_p = 0$.

Proof. For the dynamic system (15) and the PD$^{\alpha}$-type ILC scheme (I_1), from Lemma 6, we have

$$\begin{aligned} x_{k+1}(t) = x_{k+1}(0) + \frac{1}{\Gamma(\alpha)} \\ \cdot \int_0^t (t-\tau)^{\alpha-1} (f(x_{k+1}(\tau), \tau) + B u_{k+1}(\tau))\, d\tau, \end{aligned} \tag{19}$$

and, then, we get

$$
\begin{aligned}
e_{k+1}(t) &= y_d(t) - y_{k+1}(t) = y_d(t) - y_k(t) - [y_{k+1}(t) \\
&\quad - y_k(t)] = e_k(t) - C[x_{k+1}(t) - x_k(t)] = e_k(t) \\
&\quad - C\left[x_{k+1}(0) + \frac{1}{\Gamma(\alpha)}\right. \\
&\quad \left.\cdot \int_0^t (t-\tau)^{\alpha-1} \left(f(x_{k+1}(\tau),\tau) + Bu_{k+1}(\tau)\right) d\tau\right] \\
&\quad + C\left[x_k(0) + \frac{1}{\Gamma(\alpha)}\right. \\
&\quad \left.\cdot \int_0^t (t-\tau)^{\alpha-1} \left(f(x_k(\tau),\tau) + Bu_k(\tau)\right) d\tau\right] \\
&= e_k(t) - C \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\
&\quad \cdot \left(f(x_{k+1}(\tau),\tau) - f(x_k(\tau),\tau)\right) d\tau - CB \\
&\quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(u_{k+1}(\tau) - u_k(\tau)\right) d\tau = e_k(t) \\
&\quad - C \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\
&\quad \cdot \left(f(x_{k+1}(\tau),\tau) - f(x_k(\tau),\tau)\right) d\tau - CBL_{p_1} \\
&\quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_k(\tau)\, d\tau - CBL_{d_1} \cdot \frac{1}{\Gamma(\alpha)} \\
&\quad \cdot \int_0^t (t-\tau)^{\alpha-1} {}_0D_\tau^\alpha e_k(\tau)\, d\tau.
\end{aligned} \tag{20}
$$

Applying Property 3 to the last term on the right side of (20), we have

$$
\begin{aligned}
e_{k+1}(t) &= \left(1 - CBL_{d_1}\right) e_k(t) - C \cdot \frac{1}{\Gamma(\alpha)} \\
&\quad \cdot \int_0^t (t-\tau)^{\alpha-1} \left(f(x_{k+1}(\tau),\tau) - f(x_k(\tau),\tau)\right) d\tau \\
&\quad - CBL_{p_1} \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_k(\tau)\, d\tau.
\end{aligned} \tag{21}
$$

Taking absolution on both sides of (21) yields

$$
\begin{aligned}
\left|e_{k+1}(t)\right| &\le \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\
&\quad \cdot \left|C\left(f(x_{k+1}(\tau),\tau) - f(x_k(\tau),\tau)\right)\right| d\tau \\
&\quad + \left|CBL_{p_1}\right| \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau.
\end{aligned} \tag{22}
$$

Applying Lipschitz condition to the second term on the right side of (22), we get

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\
&\quad \cdot \left|C\left(f(x_{k+1}(\tau),\tau) - f(x_k(\tau),\tau)\right)\right| d\tau \le L_0 \\
&\quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_{k+1}(\tau) - y_d(\tau)\right| d\tau + L_0 \\
&\quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_d(\tau) - y_k(\tau)\right| d\tau = L_0 \\
&\quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k+1}(\tau)\right| d\tau + L_0 \cdot \frac{1}{\Gamma(\alpha)} \\
&\quad \cdot \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau.
\end{aligned} \tag{23}
$$

Taking (23) into (22) obtains

$$
\begin{aligned}
\left|e_{k+1}(t)\right| \le \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| &+ L_0 \\
&\cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k+1}(\tau)\right| d\tau \\
&+ \left(\left|CBL_{p_1}\right| + L_0\right) \\
&\cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau.
\end{aligned} \tag{24}
$$

Using Lemma 9 to inequality (24), we have

$$
\begin{aligned}
\left|e_{k+1}(t)\right| &\le \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| \\
&\quad + \int_0^t \left[\left(\left|CBL_{p_1}\right| + L_0 + L_0 \left|1 - CBL_{d_1}\right|\right)\right. \\
&\quad \left.\cdot \Phi_{\alpha,\alpha}\left(L_0 (t-\tau)\right)\right] \left|e_k(\tau)\right| d\tau.
\end{aligned} \tag{25}
$$

Taking the L^p-norm on both sides of (25) and adopting the generalized Young inequality of convolution integral, we get

$$
\left\|e_{k+1}(\cdot)\right\|_p \le \rho_1 \left\|e_k(\cdot)\right\|_p. \tag{26}
$$

This completes the proof of Theorem 10. □

Remark 11. We can see, from the above derivation, that the convergence condition is quantified directly from the L^p-norm, not by using the sufficiently large λ, and analyzed in terms of the tracking error rather than the control input error. Besides, the monotonic property of convergence can ensure the first-order PD$^\alpha$-type ILC rule to be practically implementable. Further, from condition (18), we can observe that the convergence is affected not only by the derivative learning gain and the system dynamics, but also by the proportional learning gain. That is, the result reflects the features of system dynamics and the mechanism of the algorithm to the convergence. Actually, the impact of the state dynamics and proportional learning gain, which are neglected in the existing

FOILC investigations, exists but is significantly suppressed by the sufficiently large parameter λ. It should be noted that, as mentioned in [23], a large value of λ may have a huge tracking error, which is not allowable in practice.

Remark 12. Note that the convergence analysis for fractional-order linear time-invariant system has been investigated in my early work [28], in which the derivation is made by means of the state transition matrix in an equality form, and the convergence condition is

$$\rho = \left|1 - CBL_{d_1}\right| + \left\|C\widetilde{\Phi}_{\alpha,\alpha}(\cdot)\left(BL_{p_1} + ABL_{d_1}\right)\right\|_1, \tag{27}$$

where A is the system matrix and $\widetilde{\Phi}_{\alpha,\alpha}(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})$ is the state transition matrix. However, in this paper, the investigated fractional-order system is nonlinear with its nonlinearity unknown; the proof of convergence is derived by means of inequality Lemma 9. Thus, the convergence condition ρ_1 cannot degenerate to the condition ρ when the fractional-order nonlinear system is reduced to the corresponding linear case.

4.1. Convergence Analysis for Second-Order PD$^{\alpha}$-Type ILC. In this section, we go on considering the second-order PD$^{\alpha}$-type ILC algorithm, which is constructed by employing the control inputs and their output errors of the latest previously adjacent operations in a weighting average form as follows:

$$\begin{aligned}
&(I_2)\\
&u_1(t) \text{ is given arbitrarily,}\\
&u_2(t) = u_1(t) + L_{p_1}e_1(t) + L_{d_1}\,{}_0D_t^{\alpha}e_1(t),\\
&u_{k+1}(t) = r_1\left[u_k(t) + L_{p_1}e_k(t) + L_{d_1}\,{}_0D_t^{\alpha}e_k(t)\right]\\
&\qquad + r_2\left[u_{k-1}(t) + L_{p_2}e_{k-1}(t) + L_{d_2}\,{}_0D_t^{\alpha}e_{k-1}(t)\right],\\
&\qquad\qquad k = 2, 3, 4, \ldots.
\end{aligned} \tag{28}$$

Here, L_{p_2} and L_{d_2} denote the second-order proportional and fractional-order derivative learning gains, respectively. The weighting coefficients r_1 and r_2 satisfy $0 \le r_1 < 1$, $0 \le r_2 \le 1$, and $r_1 + r_2 = 1$.

Theorem 13. *For the second-order PD$^{\alpha}$-type iterative learning control rule (I_2) is applied to system (15), if the system matrices B, C, the order α , and the Lipchitz constant L_0 together with the learning gains L_{p_1}, L_{p_2}, L_{d_1}, L_{d_2} satisfy the following conditions:*

(1) $\rho_1 = |1 - CBL_{d_1}| + (|CBL_{p_1}| + L_0 + L_0|1 - CBL_{d_1}|)\|\Phi_{\alpha,\alpha}(L_0\cdot(\cdot))\|_1 < 1.$

(2) $\rho_2 = |1 - CBL_{d_2}| + (|CBL_{p_2}| + L_0 + L_0|1 - CBL_{d_2}|)\|\Phi_{\alpha,\alpha}(L_0\cdot(\cdot))\|_1 < 1.$

Then, the learning scheme (I_2) is convergent, that is, $\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p = 0$.

Proof. From the dynamic system (15) and the PD$^{\alpha}$-type ILC scheme (I_2), we have

$$\begin{aligned}
&e_{k+1}(t) = y_d(t) - y_{k+1}(t) = r_1\left[y_d(t) - y_k(t)\right]\\
&\quad + r_2\left[y_d(t) - y_{k-1}(t)\right] - \left[y_{k+1}(t) - r_1y_k(t)\right.\\
&\quad \left.- r_2y_{k-1}(t)\right] = r_1e_k(t) + r_2e_{k-1}(t) - C\left[x_{k+1}(t)\right.\\
&\quad \left.- r_1x_k(t) - r_2x_{k-1}(t)\right] = r_1e_k(t) + r_2e_{k-1}(t)\\
&\quad - C\left[x_{k+1}(0) + \frac{1}{\Gamma(\alpha)}\right.\\
&\quad \left.\cdot\int_0^t(t-\tau)^{\alpha-1}\left(f\left(x_{k+1}(\tau),\tau\right) + Bu_{k+1}(\tau)\right)d\tau\right]\\
&\quad + r_1C\left[x_k(0) + \frac{1}{\Gamma(\alpha)}\right.\\
&\quad \left.\cdot\int_0^t(t-\tau)^{\alpha-1}\left(f\left(x_k(\tau),\tau\right) + Bu_k(\tau)\right)d\tau\right]\\
&\quad + r_2C\left[x_{k-1}(0) + \frac{1}{\Gamma(\alpha)}\right.\\
&\quad \left.\cdot\int_0^t(t-\tau)^{\alpha-1}\left(f\left(x_{k-1}(\tau),\tau\right) + Bu_{k-1}(\tau)\right)d\tau\right]\\
&\quad = r_1e_k(t) + r_2e_{k-1}(t) - r_1C\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right)d\tau - r_2C\\
&\quad \cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_{k-1}(\tau),\tau\right)\right)d\tau - CB\\
&\quad \cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(u_{k+1}(\tau) - r_1u_k(\tau) - r_2u_{k-1}(\tau)\right)d\tau = r_1e_k(t)\\
&\quad + r_2e_{k-1}(t) - r_1C\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right)d\tau - r_2C\\
&\quad \cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_{k-1}(\tau),\tau\right)\right)d\tau - CB\\
&\quad \cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(r_1L_{p_1}e_k(\tau) + r_2L_{p_2}e_{k-1}(\tau)\right)d\tau - CB\cdot\frac{1}{\Gamma(\alpha)}\\
&\quad \cdot\int_0^t(t-\tau)^{\alpha-1}\\
&\quad \cdot\left(r_1L_{d_1}\,{}_0D_\tau^{\alpha}e_k(\tau) + r_2L_{d_2}\,{}_0D_\tau^{\alpha}e_{k-1}(\tau)\right)d\tau.
\end{aligned} \tag{29}$$

Applying Property 3 to the last term on the right side of (29) yields

$$\begin{aligned} & CB\cdot\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \\ & \cdot\left(r_1L_{d_1}\,{}_0D_\tau^\alpha e_k(\tau)+r_2L_{d_2}\,{}_0D_\tau^\alpha e_{k-1}(\tau)\right)d\tau \\ & = r_1CBL_{d_1}e_k(t)+r_2CBL_{d_2}e_{k-1}(t). \end{aligned} \tag{30}$$

Submitting (30) into (29) and taking absolute values on both sides of (29) obtain

$$\begin{aligned} & \left|e_{k+1}(t)\right| \le r_1\left|1-CBL_{d_1}\right|\left|e_k(t)\right|+r_2\left|1-CBL_{d_2}\right| \\ & \cdot\left|e_{k-1}(t)\right|+r_1\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ & \cdot\left|C\left(f\left(x_{k+1}(\tau),\tau\right)-f\left(x_k(\tau),\tau\right)\right)\right|d\tau+r_2 \\ & \cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ & \cdot\left|C\left(f\left(x_{k+1}(\tau),\tau\right)-f\left(x_{k-1}(\tau),\tau\right)\right)\right|d\tau \\ & +r_1\left|CBL_{p_1}\right|\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_k(\tau)\right|d\tau \\ & +r_2\left|CBL_{p_2}\right|\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_{k-1}(\tau)\right|d\tau \\ & \le r_1\left|1-CBL_{d_1}\right|\left|e_k(t)\right|+r_2\left|1-CBL_{d_2}\right|\left|e_{k-1}(t)\right| \\ & +r_1L_0\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|y_{k+1}(\tau)-y_d(\tau)\right|d\tau \\ & +r_1L_0\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|y_d(\tau)-y_k(\tau)\right|d\tau \\ & +r_2L_0\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|y_{k+1}(\tau)-y_d(\tau)\right|d\tau \\ & +r_2L_0\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|y_d(\tau)-y_{k-1}(\tau)\right|d\tau \\ & +r_1\left|CBL_{p_1}\right|\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_k(\tau)\right|d\tau \\ & +r_2\left|CBL_{p_2}\right|\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_{k-1}(\tau)\right|d\tau \\ & = r_1\left|1-CBL_{d_1}\right|\left|e_k(t)\right|+r_2\left|1-CBL_{d_2}\right|\left|e_{k-1}(t)\right| \\ & +L_0\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_{k+1}(\tau)\right|d\tau+r_1\left(L_0\right. \\ & \left.+\left|CBL_{p_1}\right|\right)\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_k(\tau)\right|d\tau+r_2\left(L_0\right. \\ & \left.+\left|CBL_{p_2}\right|\right)\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|e_{k-1}(\tau)\right|d\tau. \end{aligned} \tag{31}$$

Using Lemma 9 to equality (31) derives that

$$\begin{aligned} & \left|e_{k+1}(t)\right| \le r_1\left|1-CBL_{d_1}\right|\left|e_k(t)\right|+r_2\left|1-CBL_{d_2}\right| \\ & \cdot\left|e_{k-1}(t)\right| \\ & +r_1\int_0^t\left[\left(\left|CBL_{p_1}\right|+L_0+L_0\left|1-CBL_{d_1}\right|\right)\right. \\ & \left.\cdot\Phi_{\alpha,\alpha}\left(L_0(t-\tau)\right)\right]\left|e_k(\tau)\right|d\tau \\ & +r_2\int_0^t\left[\left(\left|CBL_{p_2}\right|+L_0+L_0\left|1-CBL_{d_2}\right|\right)\right. \\ & \left.\cdot\Phi_{\alpha,\alpha}\left(L_0(t-\tau)\right)\right]\left|e_{k-1}(\tau)\right|d\tau. \end{aligned} \tag{32}$$

Adopting the L^p-norm on both sides of (32) implies that

$$\left\|e_{k+1}(\cdot)\right\|_p \le r_1\rho_1\left\|e_k(\cdot)\right\|_p+r_2\rho_2\left\|e_{k-1}(\cdot)\right\|_p, \tag{33}$$

and, then, according to Lemma 8 and assumptions (1) and (2), it is therefore finally evident that $\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p = 0$. □

Remark 14. It can be observed that when $r_2 = 0$, the second-order PD$^\alpha$-type ILC updating law (I_2) degenerates to the first-order PD$^\alpha$-type ILC updating law (I_1). Then the convergence becomes

$$\begin{aligned} \rho_1 &= \left|1-CBL_{d_1}\right|+\left(\left|CBL_{p_1}\right|+L_0+L_0\left|1-CBL_{d_1}\right|\right) \\ & \cdot\left\|\Phi_{\alpha,\alpha}\left(L_0\cdot(\cdot)\right)\right\|_1 < 1. \end{aligned} \tag{34}$$

Corollary 15. *Assume that* $\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p/\|e_k(\cdot)\|_p$ *exists. Analogous to the discussion regarding the convergent speed in [25], we can make assertions as follows:*

(1) *If* $\rho_2 < \rho_1^2 < 1$, *then the second-order algorithm* (I_2) *is* Q_p*-faster than the first-order* (I_1).

(2) *If* $\rho_1^2 = \rho_2 < 1$, *then the scheme* (I_2) *is* Q_p*-equivalent to the scheme* (I_1).

(3) *If* $\rho_1^2 < \rho_2 < 1$, *then the second-order strategy* (I_2) *is* Q_p*-slower than the first-order* (I_1).

Remark 16. From the view point of speed, comparing (1) and (3) of Corollary 15, we can conclude that if we choose a suitable learning gains, the second-order updating law (I_2) may not be a preferred candidate for the systems. However, if we pursue more freedom in choosing the learning gains and better robustness to noise, the second-order updating law (I_2) is a useful alternative.

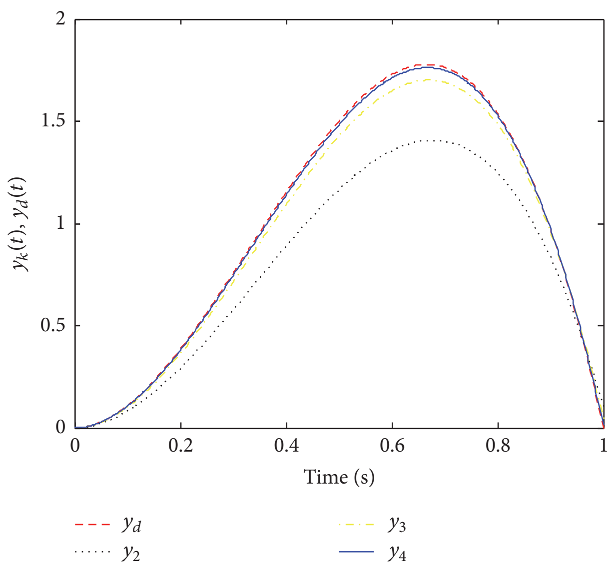

Figure 1: System outputs of (I_1).

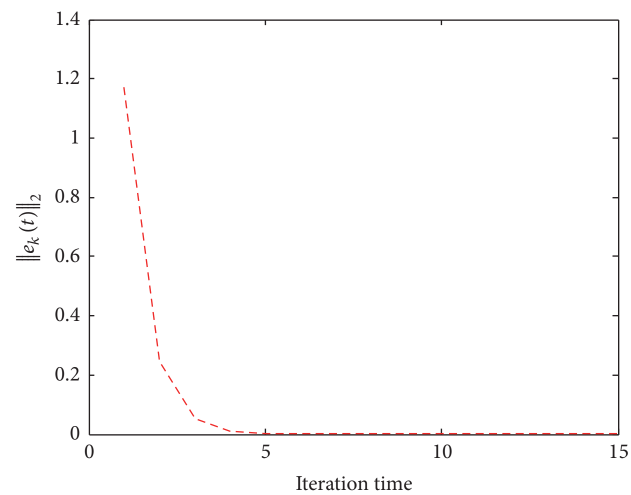

Figure 2: Tracking errors in the sense of L^2-norm.

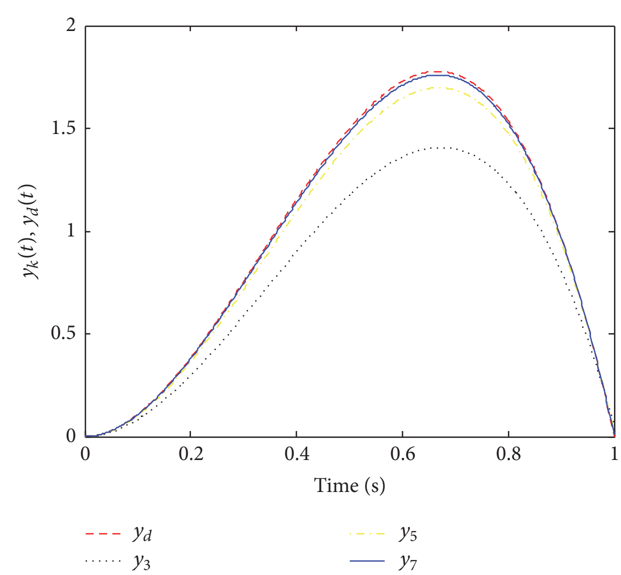

Figure 3: System outputs of (I_2).

5. Simulation Illustrations

In this simulation, we consider the following fractional-order nonlinear system:

$$\begin{gathered}
\begin{bmatrix} {}_0D_t^{0.5}\, x_1(t) \\ {}_0D_t^{0.5}\, x_2(t) \end{bmatrix} = \begin{bmatrix} 0.2x_1(t) \\ 0.1\sin x_1(t) + 0.2x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u(t), \\
y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\
\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{gathered} \tag{35}$$

The operation time period is $[0, 1]$, the desired trajectory is $y_d(t) = 12t^2(1-t)$, and the initial control is $u_1(t) = 0$.

For the first-order PD$^{0.5}$-type ILC scheme (I_1), the first-order proportional and fractional-order derivative learning gains are chosen as $L_{p_1} = 0.2$ and $L_{d_1} = 1.5$, respectively. It can be seen $L_0 = 0.2$ and $\|\Phi_{\alpha,\alpha}(L_0 \cdot (\cdot))\|_1 = 1.3630$; then it is easy to verify that $\rho_1 = 0.7271 < 1$, which means that the monotonic convergence condition (18) is satisfied. The outputs by the scheme (I_1) at the 2nd, 3rd, and 4th operations are shown in Figure 1, respectively. The monotonic tracking error in the sense of L^2-norm is shown in Figure 2.

For the second-order PD$^{0.5}$-type ILC (I_2), we consider two cases as follows.

Case 1. In schemes (I_1) and (I_2), the first-order learning gains are set as $L_{p_1} = 0.2$ and $L_{d_1} = 1.5$, respectively. In scheme (I_2), the weighting coefficients are assigned as $r_1 = 0.5$ and $r_2 = 0.5$, and the second-order learning gains are selected as $L_{p_2} = 0.1$ and $L_{d_2} = 1.2$, respectively. It is computed that $\rho_1 = 0.7271 < 1$ and $\rho_2 = 0.8498 < 1$, which is included in the case that $\rho_1^2 < \rho_2$. It shows that the output error of (I_1) convergence is faster than that of (I_2). The corresponding outputs by scheme (I_2) at the 3rd, 5th, and 7th operations are displayed in Figure 3. It shows that the output follows the desired trajectory well as the iteration increases. The comparison of the tracking error in the sense of L^2-norm made by the updating laws (I_1) and (I_2) is shown in Figure 4. It shows that the tracking errors of both the first-order and second-order laws are convergent and the first-order law (I_1) is convergence faster than second-order law (I_2).

Case 2. In schemes (I_1) and (I_2), the first-order learning gains are chosen as $L_{p_1} = 0.1$ and $L_{d_1} = 1$, respectively. In (I_2), the weighting coefficients are assigned as $r_1 = 0.3$ and $r_2 = 0.7$, and the second-order learning gains are set as $L_{p_2} = 0.1$ and $L_{d_2} = 1.7$, respectively. It is computed that $\rho_1 = 0.9771 < 1$

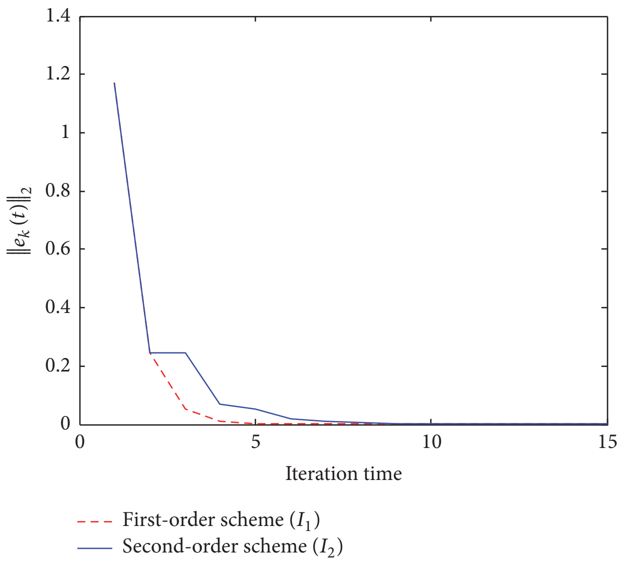

Figure 4: Comparison of tracking errors.

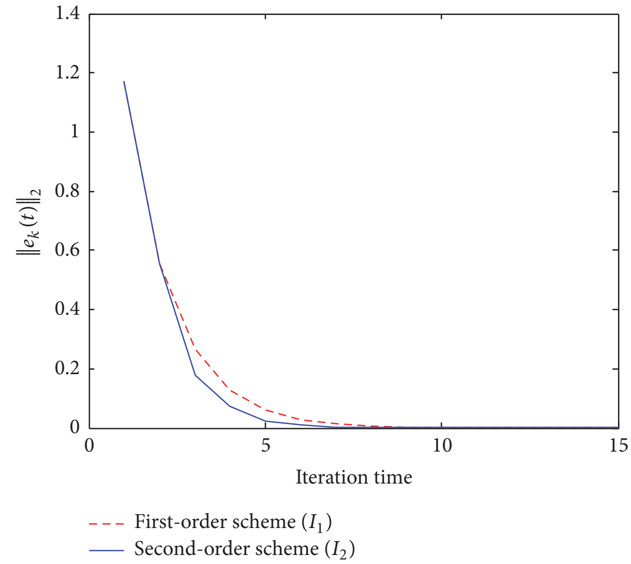

Figure 6: Comparison of tracking errors.

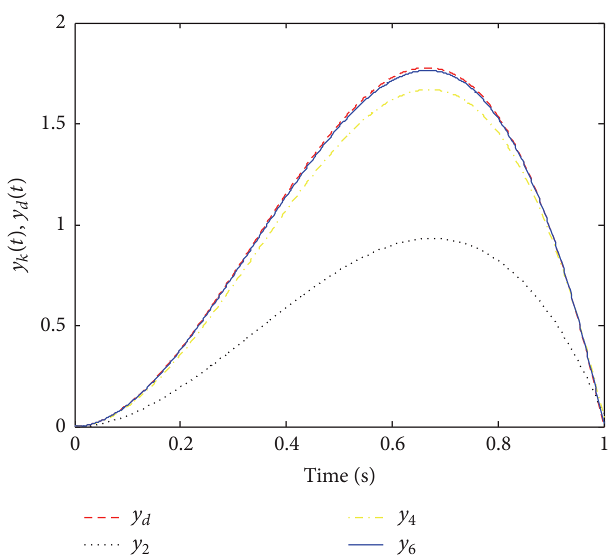

Figure 5: System outputs of (I_2).

and $\rho_2 = 0.5316 < 1$, which belongs to the case that $\rho_2 < \rho_1^2$. The outputs by scheme (I_2) at the 2nd, 4th, and 6th iterations are exhibited in Figure 5. Figure 6 shows that the tracking error of the updating law (I_1) convergence is slower than that of (I_2) in the sense of L^2-norm.

6. Conclusion

In this paper, for a class of fractional-order nonlinear systems, the first-order and second-order PD$^\alpha$-type ILC strategies are developed and the sufficiency for convergence is analyzed by means of evaluating the tracking error in the sense of L^p-norm. For analysis, it is found that the sufficient conditions of convergence not only depend on all of system dynamics, but also rely on all of learning gains. Moreover, the convergence speed comparison of the second-order law with the first-order one has been affirmed. We have clarified that, for a fractional-order nonlinear system, the results of the second-order law, that is, its convergence being faster than, equivalent to, or slower than first-order scheme, are validated. All theoretical results were conducted by simulations.

Appendix

Proof of Lemma 9

For the locally integrable function $r(t)$, denote $Br(t) = \int_0^t (t-s)^{\alpha-1} r(s)ds$. Then we have

$$x(t) \le c(t) + aBy(t) + bBx(t), \qquad \text{(A.1)}$$

and it can be written as

$$x(t) \le \sum_{k=0}^{n-1} b^k B^k c(t) + \sum_{k=0}^{n-1} ab^k B^{k+1} y(t) + b^n B^n x(t). \qquad \text{(A.2)}$$

Now, let us prove that

$$B^n x(t) \le \int_0^t \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} x(s)\,ds \qquad \text{(A.3)}$$

and $B^n x(t) \to 0$ as $n \to +\infty$ for each t in $[0, T_0)$.

Step 1. For $n = 1$, inequality (A.3) is true.

Step 2. Assume that when $n = k$, inequality (A.3) is true.

Step 3. If $n = k+1$, then, from induction hypothesis, we derive that

$$\begin{aligned} B^{k+1} x(t) &= B\left(B^k x(t)\right) \le \int_0^t (t-s)^{\alpha-1} \\ &\cdot \left[\int_0^s \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} (s-\tau)^{k\alpha-1} x(\tau)\,d\tau\right] ds. \end{aligned} \qquad \text{(A.4)}$$

By interchanging the order of integration, we get

$$\begin{aligned} B^{k+1}x(t) &\le \int_0^t \left[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}\,ds\right]x(\tau)\,d\tau \\ &= \frac{(\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)}(t-s)^{(k+1)\alpha-1}x(s)\,ds, \end{aligned} \tag{A.5}$$

where the integral

$$\begin{aligned} &\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}\,ds \\ &\quad = (t-\tau)^{(k+1)\alpha-1}\int_0^1 (1-z)^{\alpha-1}z^{k\alpha-1}dz \\ &\quad = (t-\tau)^{(k+1)\alpha-1}B(k\alpha,\alpha) \\ &\quad = \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)}(t-\tau)^{(k+1)\alpha-1} \end{aligned} \tag{A.6}$$

with $s = \tau + z(t-\tau)$.

Inequality (A.3) is proved.

Since $B^n x(t) \le \int_0^t ((\Gamma(\alpha))^n/\Gamma(n\alpha))(t-s)^{n\alpha-1}x(s)ds \to 0$ as $n \to +\infty$ for each t in $[0,T_0)$, we have

$$\begin{aligned} x(t) &\le c(t) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}c(s)\right]ds \\ &\quad + \int_0^t \left[\sum_{n=0}^{\infty} a\Gamma(\alpha) \right.\\ &\quad \left.\cdot \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha+\alpha)}(t-s)^{(n+1)\alpha-1}y(s)\right]ds. \end{aligned} \tag{A.7}$$

Note that, the second term on the right side of (A.7) implies

$$\begin{aligned} &\int_0^t \left[\sum_{n=1}^{\infty}\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}c(s)\right]ds \\ &\quad = \int_0^t \left[\sum_{n=1}^{\infty}\frac{(b\Gamma(\alpha))^{n-1}\cdot b\Gamma(\alpha)}{\Gamma((n-1)\alpha+\alpha)}(t-s)^{(n-1)\alpha+\alpha-1}\right.\\ &\quad \left.\cdot c(s)\right]ds = \int_0^t \left[\sum_{n=0}^{\infty}\frac{(b\Gamma(\alpha))^n\cdot b\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}(t-s)^{n\alpha}\right.\\ &\quad \left.\cdot (t-s)^{\alpha-1}c(s)\right]ds = \int_0^t \left[b\Gamma(\alpha)\cdot E_{\alpha,\alpha}\left(b\Gamma(\alpha)\right.\right.\\ &\quad \left.\left.\cdot (t-s)^\alpha\right)\cdot (t-s)^{\alpha-1}c(s)\right]ds. \end{aligned} \tag{A.8}$$

Since the last term on the right side of (A.7) is

$$\begin{aligned} &\int_0^t \left[\sum_{n=0}^{\infty} a\Gamma(\alpha)\cdot \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha+\alpha)}(t-s)^{(n+1)\alpha-1}y(s)\right]ds \\ &\quad = \int_0^t \left[a\Gamma(\alpha)\cdot E_{\alpha,\alpha}\left(b\Gamma(\alpha)(t-s)^\alpha\right)\cdot (t-s)^{\alpha-1}\right.\\ &\quad \left.\cdot y(s)\right]ds, \end{aligned} \tag{A.9}$$

submitting (A.8) and (A.9) into (A.7) obtains

$$\begin{aligned} x(t) &\le c(t) + \int_0^t \left[\Gamma(\alpha)\cdot \Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s))\right.\\ &\quad \left.\cdot (bc(s)+ay(s))\right]ds, \end{aligned} \tag{A.10}$$

where $\Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)) = (t-s)^{\alpha-1}E_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)^\alpha)$.

The proof is complete.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Existence Results of Initial Value Problems for Hybrid Fractional Sum-Difference Equations

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We consider a hybrid fractional sum-difference initial value problem and a hybrid fractional sequential sum-difference initial value problem. The existence results of these two problems are proved by using the hybrid fixed point theorem for three operators in a Banach algebra and the generalized Krasnoselskii's fixed point theorem, respectively.

1. Introduction

As recognized that fractional difference calculus is a powerful tool used to describe many real world phenomena problems such as physics, chemistry, mechanics, control systems, flow in porous media, and electrical networks [1, 2], this is an impact on a researcher's motivation to develop the research works in this area. Basic definitions and properties of fractional difference calculus were proposed by Goodrich and Peterson [3]. The developments of the theory related to discrete fractional boundary value problems were studied by many authors (see [4–48]). In particular Sitthiwirattham [41, 42] studied three-point Caputo fractional difference-fractional sum boundary value problem for sequential Caputo fractional difference equation of the form

$$\begin{aligned} &\Delta_C^{\alpha}\left[\phi_p\left(\Delta_C^{\beta}x\right)\right](t) \\ &\quad = f\left(t+\alpha+\beta-1, x\left(t+\alpha+\beta-1\right)\right), \\ &\Delta_C^{\beta}x\left(\alpha-1\right) = 0, \\ &x\left(\alpha+\beta+T\right) = \rho\Delta^{-\gamma}x\left(\eta+\gamma\right), \end{aligned} \tag{1}$$

and three-point fractional sum boundary value problem for sequential Riemann-Liouville fractional difference equation of the form

$$\begin{aligned} &\Delta_{\alpha}^{\alpha}\left(\Delta_{\alpha+\beta-1}^{\beta}+\lambda E_{\beta}\right)x(t) \\ &\quad = f\left(t+\alpha+\beta-1, x\left(t+\alpha+\beta-1\right)\right), \\ &x\left(\alpha+\beta-2\right) = 0, \\ &x\left(\alpha+\beta+T\right) = \rho\Delta_{\alpha+\beta-1}^{-\gamma}x\left(\eta+\gamma\right), \end{aligned} \tag{2}$$

where $t \in \mathbb{N}_{0,T}$, $0 < \alpha, \beta \le 1$, $1 < \alpha+\beta \le 2$, $0 < \gamma \le 1$, $\eta \in \mathbb{N}_{\alpha+\beta-1,\alpha+\beta+T-1}$, ρ is a constant, $f : \mathbb{N}_{\alpha+\beta-2,\alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $E_{\beta}x(t) = x(t+\beta-1)$, and ϕ_p is the p-Laplacian operator.

Calculus which deals with derivatives and integrals of arbitrary orders is known as hybrid differential equations (i.e., quadratic perturbation of a nonlinear differential equation). Hybrid fractional differential equations are initialized to be used to model successfully several physical phenomena (see [49–51]). Apparently, this issue has found numerous miscellaneous applications connected with real world problems as they appear in many fields of engineering and science, including biology, chemistry, diffusion, control theory, electromagnetic theory, fluid flow, signal and image processing, fractals theory, fitting of experimental data, potential theory, and viscoelasticity. For some recent developments on the topic, see [52–57]. Recently, there are several research works related to boundary value problems for hybrid differential equations (see [58–65]). For example, Sun et al. [60] studied the existence of solutions for the boundary value problem of fractional hybrid differential equations

$$D^{\nu}\left[\frac{x(t)}{f(t,x(t))}\right]+g(t,x(t))=0,\quad 0<t<1, \qquad x(0)=x(1)=0, \tag{3}$$

where D^{ν} denotes the Riemann-Liouville fractional derivative of order ν, $1<\nu\le 2$.

Sitho et al. [64] studied existence results for the following initial value problem for hybrid fractional integrodifferential equations,

$$D^{\alpha}\left[\frac{x(t)-\sum_{i=1}^{m}I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right]=F(t,x(t)), \quad t\in J:[0,T], \qquad x(0)=0, \tag{4}$$

and the initial value problem for hybrid fractional sequential integrodifferential equations,

$$D^{\alpha}\left[\frac{D^{\omega}x(t)-\sum_{i=1}^{m}I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right] = G\left(t,x(t),I^{\gamma}x(t)\right),\quad t\in J, \qquad x(0)=D^{\omega}x(0)=0, \tag{5}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\ldots,\beta_m\}$, $0<\alpha,\omega\le 1$, $1<\alpha+\omega\le 2$, and functions are $F\in C(J\times\mathbb{R},\mathbb{R})$, $G\in C(J\times\mathbb{R}^2)$, $\mathbb{R}$, $f\in C(J\times\mathbb{R},\mathbb{R}-\{0\})$, and $h_i\in C(J\times\mathbb{R},\mathbb{R})$ with $h_i(0,0)=0$, $i=1,2,\ldots,m$.

While the boundary value problem for hybrid fractional difference equations has not been studied, to fill this gap, we study a hybrid fractional difference initial value problem of the form,

$$\Delta^{\alpha}\left[\frac{u(t)-\Delta^{-\gamma}p\left(t+\gamma,u(t+\gamma)\right)}{f(t,u(t))}\right] = F\left[t+\alpha,u(t+\alpha)\right], \qquad u(\alpha-1)=0, \tag{6}$$

and a hybrid fractional sequential sum-difference initial value problem of the form

$$\Delta^{\alpha}\left[\frac{\Delta^{\beta}u(t)-\Delta^{-\gamma}g\left(t+\beta+\gamma-1,u\left(t+\beta+\gamma-1\right)\right)}{h(t,u(t))}\right] = H\left[t+\alpha+\beta-1,u\left(t+\alpha+\beta-1\right),\Delta^{-\omega}u\left(t+\alpha+\beta+\omega-1\right)\right], \qquad u(\alpha+\beta-2)=\Delta^{\beta}u(\alpha-1)=0, \tag{7}$$

where $t\in\mathbb{N}_{0,T}:=\{0,1,\ldots,T\}$, $\alpha,\beta,\gamma,\omega\in(0,1]$, $1<\alpha+\beta\le 2$ are given constants, $f\in C(\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R},\mathbb{R}-\{0\})$, $h\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}\times\mathbb{R},\mathbb{R}-\{0\})$, $p\in C(\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R},\mathbb{R})$ with $p(\alpha-1,0)=0$, $g\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}\times\mathbb{R},\mathbb{R})$ with $g(\alpha+\beta-2,0)=0$, $F\in C(\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R},\mathbb{R})$, and $H\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

The article is organized as follows. In Section 2, we recall some definitions and basic lemmas used in this work. Then, we present the solutions of (6) and (7) by converting the problem to an equivalent summation equation. In Sections 3 and 4, we prove existence results of problems (6) and (7) by employing the hybrid fixed point theorem for three operators in a Banach algebra and the generalized Krasnoselskii's fixed point theorem, respectively. We end with some examples to illustrate our results in the last section.

2. Preliminaries

In what follows are the notations, definitions, and lemmas which are used in the main results.

Definition 1 (see [6]). Define the generalized falling function by $t^{\underline{\alpha}}:=\Gamma(t+1)/\Gamma(t+1-\alpha)$, for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}}=0$.

Lemma 2 (see [4]). *Assume that the factorial functions are well defined. If $t\le r$, then $t^{\underline{\alpha}}\le r^{\underline{\alpha}}$ for any $\alpha>0$.*

Definition 3 (see [6]). For $\alpha>0$ and f defined on $\mathbb{N}_a:=\{a,a+1,\ldots\}$, the α-order fractional sum of f is defined by

$$\Delta^{-\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s), \tag{8}$$

where $t\in\mathbb{N}_{a+\alpha}$ and $\sigma(s)=s+1$.

Definition 4 (see [6]). For $\alpha>0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by

$$\Delta^{\alpha}f(t):=\Delta^{N}\Delta^{-(N-\alpha)}f(t) = \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{t+\alpha}(t-\sigma(s))^{\underline{-\alpha-1}}f(s), \tag{9}$$

where $t\in\mathbb{N}_{a+N-\alpha}$ and $N\in\mathbb{N}$ is satisfied with $0\le N-1<\alpha<N$.

Lemma 5 (see [5]). *For any real number ν and any positive integer p, the following equality holds:*

$$\Delta^{-\nu}\Delta^{p}f(t)=\Delta^{p}\Delta^{-\nu}f(t)-\sum_{k=0}^{p-1}\frac{(t-a)^{\underline{\nu-p+k}}}{\Gamma(\nu-p+k+1)}\Delta^{k}f(a), \quad (10)$$

or f is defined on $\mathbb{N}_a$.

Lemma 6 (see [4]). *Let $0\le N-1<\alpha\le N$. Then,*

$$\Delta^{-\alpha}\Delta^{\alpha}y(t)=y(t)+C_1t^{\underline{\alpha-1}}+C_2t^{\underline{\alpha-2}}+\cdots+C_Nt^{\underline{\alpha-N}}, \quad (11)$$

for some $C_i\in\mathbb{R}$, with $1\le i\le N$.

We provide the following lemma dealing with linear variant of the boundary value problems (6) and (7) and give a representation of the solution.

Lemma 7. *Let $\alpha,\gamma\in(0,1]$, $p\in C(\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R},\mathbb{R})$ with $p(\alpha-1,0)=0$, $y\in C(\mathbb{N}_{\alpha-1,T+\alpha},\mathbb{R}-\{0\})$, and $h\in C(\mathbb{N}_{\alpha-1,T+\alpha},\mathbb{R})$. Then, for $t\in\mathbb{N}_{0,T}$, the problem*

$$\begin{gathered}\Delta^{\alpha}\left[\frac{u(t)-\Delta^{-\gamma}p(t+\gamma,u(t+\gamma))}{y(t)}\right]=h(t+\alpha),\\ u(\alpha-1)=0,\end{gathered} \quad (12)$$

has the unique solution

$$\begin{aligned}u(t)&=\frac{y(t)}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha)\\&\quad+\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-\gamma-1}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}p(s+\gamma,u(s+\gamma)),\end{aligned} \quad (13)$$

for $t\in\mathbb{N}_{\alpha-1,T+\alpha}$.

Proof. Using the fractional sum of order $\alpha:\Delta^{-\alpha}$ for (12) and Lemmas 5 and 6, we obtain

$$\begin{aligned}\Delta^{-\alpha}h(t+\alpha)&=\left[\frac{u(t)-\Delta^{-\gamma}p(t+\gamma,u(t+\gamma))}{y(t)}\right]-\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\\&\quad\cdot\Delta^{-(1-\alpha)}\left[\frac{u(\alpha-1)-\Delta^{-\gamma}p(\alpha-1+\gamma,u(\alpha-1+\gamma))}{y(\alpha-1)}\right]\\&=\left[\frac{u(t)-\Delta^{-\gamma}p(t+\gamma,u(t+\gamma))}{y(t)}\right]-\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\\&\quad\cdot\Delta^{-(1-\alpha)}\left\{\frac{1}{y(\alpha-1)}\times\left[u(\alpha-1)\right.\right.\\&\quad\left.\left.-\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-1}^{(\alpha+\gamma-1)-\gamma}(\alpha+\gamma-1-\sigma(s))^{\underline{\gamma-1}}p(s,u(s))\right]\right\}\\&=\left[\frac{u(t)-\Delta^{-\gamma}p(t+\gamma,u(t+\gamma))}{y(t)}\right]-\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\\&\quad\cdot\Delta^{-(1-\alpha)}\left\{\frac{1}{y(\alpha-1)}\left[u(\alpha-1)\right.\right.\\&\quad\left.\left.-\gamma p(\alpha-1,u(\alpha-1))\right]\right\},\end{aligned} \quad (14)$$

for $t\in\mathbb{N}_{\alpha-1,T+\alpha}$.

Since $u(\alpha-1)=0$, $p(\alpha-1,0)=0$, and $y(\alpha-1)\neq0$, it follows that

$$\left[\frac{u(t)-\Delta^{-\gamma}p(t+\gamma,u(t+\gamma))}{y(t)}\right]=\Delta^{-\alpha}h(t+\alpha). \quad (15)$$

Thus, (13) holds. The proof is completed. □

Lemma 8. *Let $\alpha,\beta,\gamma\in(0,1]$, $1<\alpha+\beta\le2$, $g\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}\times\mathbb{R},\mathbb{R})$ with $g(\alpha+\beta-2,0)=0$, $x\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta},\mathbb{R}-\{0\})$, and $k\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta},\mathbb{R})$ be given. Then, for $t\in\mathbb{N}_{0,T}$, the problem*

$$\begin{aligned}&\Delta^{\alpha}\left[\frac{\Delta^{\beta}u(t)-\Delta^{-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1))}{x(t)}\right]\\&\quad=k(t+\alpha+\beta-1),\\&u(\alpha+\beta-2)=\Delta^{\beta}u(\alpha-1)=0,\end{aligned} \quad (16)$$

has the unique solution

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=\alpha-1}^{t-\beta}\left[(t-\sigma(s))^{\underline{\beta-1}}x(s+\beta)\right.\\&\quad\left.\cdot\sum_{\xi=0}^{s-\alpha}(s-\sigma(\xi))^{\underline{\alpha-1}}k(\xi+\alpha+\beta-1)\right]+\frac{1}{\Gamma(\gamma)}\\&\quad\cdot\sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}g(s+\beta+\gamma,u(s+\beta+\gamma)),\end{aligned} \quad (17)$$

for $t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

Proof. Using the fractional sum of order $\alpha : \Delta^{-\alpha}$ for (16) and Lemmas 5 and 6, we obtain

$$\begin{aligned}
&\Delta^{-\alpha}k(t+\alpha+\beta-1)\\
&=\left[\frac{\Delta^{\beta}u(t)-\Delta^{-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1))}{x(t)}\right]\\
&\quad-\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\Delta^{-(1-\alpha)}\left[\frac{\Delta^{\beta}u(\alpha-1)-\Delta^{-\gamma}g(\alpha-1+\beta+\gamma-1,u(\alpha-1+\beta+\gamma-1))}{x(\alpha-1)}\right]\\
&=\left[\frac{\Delta^{\beta}u(t)-\Delta^{-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1))}{x(t)}\right]\\
&\quad-\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\Delta^{-(1-\alpha)}\left\{\frac{1}{x(\alpha-1)}\times\left[\Delta^{\beta}u(\alpha-1)-\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-2}^{(\alpha+\beta+\gamma-2)-\gamma}(\alpha+\beta+\gamma-2-\sigma(s))^{\underline{\gamma-1}}g(s,u(s))\right]\right\}\\
&=\left[\frac{\Delta^{\beta}u(t)-\Delta^{-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1))}{x(t)}\right]\\
&\quad-\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\Delta^{-(1-\alpha)}\left\{\frac{1}{x(\alpha-1)}\left[\Delta^{\beta}u(\alpha-1)-\gamma g(\alpha+\beta-2,u(\alpha+\beta-2))\right]\right\},
\end{aligned}\tag{18}$$

for $t\in\mathbb{N}_{\alpha-1,T+\alpha}$.

Since $u(\alpha+\beta-2)=\Delta^{\beta}u(\alpha-1)=0$, $g(\alpha+\beta-2,0)=0$, and $x(\alpha-1)\neq 0$, we have

$$\begin{aligned}
&\left[\frac{\Delta^{\beta}u(t)-\Delta^{-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1))}{x(t)}\right]\\
&\quad=\Delta^{-\alpha}k(t+\alpha+\beta-1).
\end{aligned}\tag{19}$$

That can be arranged in the form

$$\begin{aligned}
\Delta^{\beta}u(t)&=x(t)\Delta^{-\alpha}k(t+\alpha+\beta-1)\\
&\quad+\Delta^{-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1)).
\end{aligned}\tag{20}$$

Using the fractional sum of order β for (20), we obtain

$$\begin{aligned}
u(t)&=\Delta^{-\beta}\left[x(t)\Delta^{-\alpha}k(t+\alpha+\beta-1)\right]\\
&\quad+\Delta^{-\beta-\gamma}g(t+\beta+\gamma-1,u(t+\beta+\gamma-1)),
\end{aligned}\tag{21}$$

for $t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

Thus (17) holds. Our proof is completed. □

Lemma 9 (Arzelá-Ascoli theorem [66]). *A set of functions in $C[a,b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 10 (see [66]). *If a set is closed and relatively compact, then it is compact.*

3. Hybrid Fractional Sum-Difference Initial Value Problem (6)

In this section, we aim to show the existence results for problem (6). To accomplish this, we let $E=C(\mathbb{N}_{\alpha-1,T+\alpha},\mathbb{R})$ be a space of all functions u and defined a norm and a multiplication in E by

$$\begin{aligned}
\|u\|&=\max_{t\in\mathbb{N}_{\alpha-1,T+\alpha}}|u(t)|,\\
(uv)(t)&=u(t)v(t).
\end{aligned}\tag{22}$$

In addition, we define operator $\mathcal{F}:E\to E$ by

$$\begin{aligned}
&(\mathcal{F}u)(t)\\
&\quad=\frac{f(t,u(t))}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}F((s+\alpha),u(s+\alpha))\\
&\qquad+\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-\gamma-1}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}p(s+\gamma,u(s+\gamma)).
\end{aligned}\tag{23}$$

Clearly, problem (6) has solutions if and only if operator $\mathcal{F}$ has fixed points. The first shows the existence and uniqueness of a solution to problem (6) by using the Banach contraction principle.

Theorem 11. *Assume that $f\in C(\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R},\mathbb{R}-\{0\})$, $p\in C(\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R},\mathbb{R})$ with $p(\alpha-1,0)=0$, and $H\in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. In addition, suppose that*

(A_1) *there exist constants $L_1, L_2 > 0$ such that for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$ and $u, v \in \mathbb{R}$*

$$\begin{aligned} |F(t,u) - F(t,v)| &\le L_1 |u - v|, \\ |p(t,u) - p(t,v)| &\le L_2 |u - v|, \end{aligned} \tag{24}$$

(A_2) *there exists a positive function θ with bound $\|\theta\|$ such that for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$ and $u, v \in \mathbb{R}$*

$$|f(t,u) - f(t,v)| \le \theta(t). \tag{25}$$

If $L_1\|\theta\|(T^{\underline{\alpha}}/\Gamma(\alpha+1)) + L_2((T+\alpha)^{\underline{\gamma}}/\Gamma(\gamma+1)) < 1$, then problem (6) has a unique solution.

Proof. We shall show that $\mathcal{F}$ is a contraction. For any $u, v \in \mathcal{C}$ and for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$, we have

$$\begin{aligned} &|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \le \frac{\|f\|}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\underline{\alpha-1}} \\ &\quad \cdot |F((s+\alpha), u(s+\alpha)) - F((s+\alpha), v(s+\alpha))| \\ &\quad + \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t - \sigma(s))^{\underline{\gamma-1}} \\ &\quad \cdot |p(s+\gamma, u(s+\gamma)) - p(s+\gamma, v(s+\gamma))| \\ &\quad \le \frac{\|\theta\|}{\Gamma(\alpha)} L_1 \|u - v\| \sum_{s=0}^{T} (T + \alpha - \sigma(s))^{\underline{\alpha-1}} + \frac{1}{\Gamma(\gamma)} \\ &\quad \cdot L_2 \|u - v\| \sum_{s=\alpha-\gamma-1}^{T+\alpha-\gamma} (T + \alpha - \sigma(s))^{\underline{\gamma-1}} \le \|u \\ &\quad - v\| \left\{ L_1 \|\theta\| \frac{T^{\underline{\alpha}}}{\Gamma(\alpha+1)} + L_2 \frac{(T+\alpha)^{\underline{\gamma}}}{\Gamma(\gamma+1)} \right\}. \end{aligned} \tag{26}$$

Thus, we have $\|(\mathcal{F}u) - (\mathcal{F}v)\| < \|u - v\|$.

Consequently, $\mathcal{F}$ is a contraction. Therefore, by the Banach fixed point theorem, we get that $\mathcal{F}$ has a fixed point which is a unique solution of problem (6). □

In the second result, we deduce the existence of at least one solution of the initial value problem (6) by using the hybrid fixed point theorem for three operators in a Banach algebra. Clearly, E is a Banach algebra with respect to the above norm and multiplication in it.

Theorem 12 (hybrid fixed point theorem for three operators in a Banach algebra [67]). *Let S be a nonempty, closed convex, and bounded subset of the Banach algebra E, and let $A, C : E \to E$ and $B : S \to E$ be three operators such that*

(i) *A and C are Lipschitzian with Lipschitz constants δ and ρ, respectively,*

(ii) *B is completely continuous,*

(iii) *$x = AxBy + Cx \Rightarrow x \in S$ for all $y \in S$,*

(iv) *$\delta M + \rho < 1$, where $M = \|B(S)\|$.*

Then, the operator equation $AxBx + Cx = x$ has a solution.

Theorem 13. *Assume that $f \in C(\mathbb{N}_{\alpha-1,T+\alpha} \times \mathbb{R}, \mathbb{R} - \{0\})$, $p \in C(\mathbb{N}_{\alpha-1,T+\alpha} \times \mathbb{R}, \mathbb{R})$ with $p(\alpha-1, 0) = 0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that*

(H_1) *there exist two positive functions ϕ and ψ with bound $\|\phi\|$ and $\|\psi\|$, respectively, such that for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$ and $u, v \in \mathbb{R}$*

$$\begin{aligned} |f(t,u) - f(t,v)| &\le \phi(t) |u - v|, \\ |p(t,u) - p(t,v)| &\le \psi(t) |u - v|, \end{aligned} \tag{27}$$

(H_2) *there exists a function $\omega \in C(\mathbb{N}_{\alpha-1,T+\alpha}, \mathbb{R}^+)$ and a continuous nondecreasing function $\Psi : [0, \infty) \to (0, \infty)$ such that for each $(t, u) \in \mathbb{N}_{\alpha-1,T+\alpha} \times \mathbb{R}$*

$$|F(t,u)| \le \omega(t) \Psi(|u|), \tag{28}$$

(H_3) *there exists a number*

$$\begin{aligned} &R \\ &\ge \frac{\hat{f}_0 (\|\omega\| \Psi(R) T^{\alpha}/\Gamma(\alpha+1)) + \hat{p}_0 ((T+\alpha)^{\gamma}/\Gamma(\gamma+1))}{1 - \|\phi\| (\|\omega\| \Psi(R) T^{\alpha}/\Gamma(\alpha+1)) - \|\psi\| ((T+\alpha)^{\gamma}/\Gamma(\gamma+1))}, \end{aligned} \tag{29}$$

where $\hat{f}_0 = \max_{t \in \mathbb{N}_{\alpha-1,T+\alpha}} |f(t,0)|$, $\hat{p}_0 = \max_{t \in \mathbb{N}_{\alpha-1,T+\alpha}} |p(t,0)|$, and

$$\|\phi\| \frac{\|\omega\| \Psi(R) T^{\alpha}}{\Gamma(\alpha+1)} + \|\psi\| \frac{(T+\alpha)^{\gamma}}{\Gamma(\gamma+1)} < 1. \tag{30}$$

Then problem (6) has a unique solution on $\mathbb{N}_{\alpha-1,T+\alpha}$.

Proof. Define subset S of E as

$$S = \{u \in E : \|u\| \le R\}. \tag{31}$$

We see that S is closed, convex, and bounded subset of the Banach space $\mathcal{C}$. By Lemma 7, we define three operators $\mathcal{A} : E \to E$, $\mathcal{B} : S \to E$, and $\mathcal{C} : E \to E$ by

$$\begin{aligned} &\mathcal{A}u(t) := f(t, u(t)), \quad t \in \mathbb{N}_{\alpha-1,T+\alpha}, \\ &\mathcal{B}u(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t - \sigma(s))^{\underline{\alpha-1}} F(s+\alpha, u(s+\alpha)), \\ &\qquad t \in \mathbb{N}_{\alpha-1,T+\alpha}, \\ &\mathcal{C}u(t) \\ &\quad := \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t - \sigma(s))^{\underline{\gamma-1}} p(s+\gamma, u(s+\gamma)), \\ &\qquad t \in \mathbb{N}_{\alpha-1,T+\alpha}. \end{aligned} \tag{32}$$

Note that problem (6) has solutions if and only if the operator $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u$ has fixed points.

To show that all operators satisfy all the conditions of Theorem 12, we proceed with the following steps.

Step 1. Prove that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on E.

For any $u, v \in \mathbb{R}$ and for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$, then by (H_1), we have

$$\begin{aligned} |\mathscr{A}u(t) - \mathscr{A}v(t)| &= |f(t,u(t)) - f(t,v(t))| \le \phi(t)|u - v| \le \|\phi\|\|u-v\|, \\ |\mathscr{C}u(t) - \mathscr{C}v(t)| &= \frac{1}{\Gamma(\gamma)} \sum_{s=\alpha-\gamma-1}^{t-\gamma} (t-\sigma(s))^{\underline{\gamma-1}} \cdot |p(s+\gamma, u(s+\gamma)) - p(s+\gamma, v(s+\gamma))| \le \frac{1}{\Gamma(\gamma)} \\ &\cdot \sum_{s=\alpha-\gamma-1}^{T+\alpha-\gamma} (T+\alpha-\sigma(s))^{\underline{\gamma-1}} \psi(s)|u-v| \\ &\le \frac{\|\psi(s)\|(T+\alpha)^{\gamma}}{\Gamma(\gamma+1)}\|u-v\|. \end{aligned} \tag{33}$$

This implies that, for all $u, v \in \mathbb{R}$,

$$\begin{aligned} \|\mathscr{A}u(t) - \mathscr{A}v(t)\| &\le \|\phi\|\|u-v\|, \\ \|\mathscr{C}u(t) - \mathscr{C}v(t)\| &\le \frac{\|\psi(s)\|(T+\alpha)^{\gamma}}{\Gamma(\gamma+1)}\|u-v\|. \end{aligned} \tag{34}$$

Therefore, $\mathscr{A}$and $\mathscr{C}$ are Lipschitzian on E with Lipschitz constants $\|\phi\|$ and $\|\psi(s)\|(T+\alpha)^{\gamma}/\Gamma(\gamma+1)$.

Step 2. We prove that $\mathscr{B}$ is completely continuous on S.

Since F is continuous, the operator $\mathscr{B}$ is continuous on S. Next, we will prove that the set $\mathscr{B}(S)$ is uniformly bounded in S. For any $u \in S$, we find that

$$\begin{aligned} |\mathscr{B}u(t)| &\le \left| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} \omega(s)\Psi(R) \right| \\ &\le \frac{\|\omega\|\Psi(R)T^{\alpha}}{\Gamma(\alpha+1)} =: K, \end{aligned} \tag{35}$$

for all $t \in \mathbb{N}_{\alpha-1,T+\alpha}$. Therefore, $\|\mathscr{B}\| \le K$, which shows that $\mathscr{B}$ is uniformly bounded on S.

Next we show that $\mathscr{B}(S)$ is an equicontinuous set in E. For any $\epsilon > 0$, there exists a positive constant δ such that, for $t_1, t_2 \in \mathbb{N}_{\alpha-1,T+\alpha}$,

$$\left|t_2^{\alpha} - t_1^{\alpha}\right| < \frac{\Gamma(\alpha+1)}{\|\omega\|\Psi(R)}, \quad \text{whenever } \left|t_2 - t_1\right| < \delta. \tag{36}$$

Then we obtain

$$\begin{aligned} &\left|(\mathscr{B}x)(t_2) - (\mathscr{B}x)(t_1)\right| \\ &\le \frac{\|\omega\|\Psi(R)}{\Gamma(\alpha)} \left| \sum_{s=0}^{t_2-\alpha} (t_2-\sigma(s))^{\underline{\alpha-1}} - \sum_{s=0}^{t_1-\alpha} (t_1-\sigma(s))^{\underline{\alpha-1}} \right| \le \frac{\|\omega\|\Psi(R)}{\Gamma(\alpha+1)} \left|t_2^{\alpha} - t_1^{\alpha}\right| < \epsilon. \end{aligned} \tag{37}$$

This implies that the set $\mathscr{B}(S)$ is an equicontinuous set. From the Arzelá-Ascoli theorem, we find that $\mathscr{B}$ is completely continuous.

Step 3. $u = \mathscr{A}u\mathscr{B}v + \mathscr{C}u \Rightarrow u \in S$ for all $v \in S$.

Let $u \in E$ and $v \in S$ be arbitrary elements such that $u = \mathscr{A}u\mathscr{B}v + \mathscr{C}u$. Then,

$$\begin{aligned} |u(t)| &\le |\mathscr{A}u(t)||\mathscr{B}v(t)| + |\mathscr{C}u(t)| \le \frac{|f(t,u(t))|}{\Gamma(\alpha)} \\ &\cdot \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} |F(s+\alpha, u(s+\alpha))| + \frac{1}{\Gamma(\gamma)} \\ &\cdot \sum_{s=\alpha-1}^{t} (t+\gamma-\sigma(s))^{\underline{\gamma-1}} |p(s,u(s))| \\ &\le \frac{|f(t,u(t)) - f(t,0)| + |f(t,0)|}{\Gamma(\alpha)} \\ &\cdot \sum_{s=0}^{T} (T+\alpha-\sigma(s))^{\underline{\alpha-1}} \|\omega\|\Psi(R) + \frac{1}{\Gamma(\gamma)} \\ &\cdot \sum_{s=\alpha-1}^{T+\alpha} (T+\alpha+\gamma-\sigma(s))^{\underline{\gamma-1}} \\ &\cdot (|p(s,u(s)) - p(t,0)| + |p(t,0)|) \le (R\|\phi\| \\ &+ \widehat{f}_0) \frac{\|\omega\|\Psi(R)T^{\alpha}}{\Gamma(\alpha+1)} + (R\|\psi\| + \widehat{p}_0) \frac{(T+\alpha)^{\gamma}}{\Gamma(\gamma+1)}. \end{aligned} \tag{38}$$

We find that

$$\begin{aligned} \|u\| &\le (R\|\phi\| + \widehat{f}_0) \frac{\|\omega\|\Psi(R)T^{\alpha}}{\Gamma(\alpha+1)} \\ &+ (R\|\psi\| + \widehat{p}_0) \frac{(T+\alpha)^{\gamma}}{\Gamma(\gamma+1)} \le R. \end{aligned} \tag{39}$$

Therefore, $u \in S$.

Step 4. We prove that $\delta M + \rho < 1$. Since

$$\begin{aligned} M = \|\mathscr{B}(S)\| &= \sup_{u \in S} \left\{ \max_{t \in \mathbb{N}_{\alpha-1,T+\alpha}} |\mathscr{B}u(t)| \right\} \\ &\le \frac{\|\omega\|\Psi(R)T^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{40}$$

and by (H_3), we have

$$\|\phi\|M + \|\psi\| \frac{(T+\alpha)^{\gamma}}{\Gamma(\gamma+1)} < 1, \tag{41}$$

with $\delta = \|\phi\|$ and $\rho = \|\psi\|((T+\alpha)^{\gamma}/\Gamma(\gamma+1))$.

We see that all the conditions of Theorem 12 are satisfied. Hence, the operator equation $u = \mathscr{A}u\mathscr{B}v + \mathscr{C}u$ has a solution in S. In consequence, problem (6) has a solution on $\mathbb{N}_{\alpha-1,T+\alpha}$. This completes the proof. □

4. Hybrid Fractional Sequential Sum-Difference Initial Value Problem (7)

In this section, we prove existence results of problem (7). To accomplish this, we denote that $X = C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}, \mathbb{R})$ and define the space of all functions u with the norm by

$$\|u\| = \max_{t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}} |u(t)|. \tag{42}$$

In addition, we define the operator $\mathscr{H} : X \to X$ by

$$(\mathscr{H}u)(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} \Bigg[(t-\sigma(s))^{\underline{\beta-1}} \cdot h((s+\beta), u(s+\beta)) \times \sum_{\xi=0}^{s-\alpha} (s-\sigma(\xi))^{\underline{\alpha-1}} \cdot H((\xi+\alpha+\beta-1), u(\xi+\alpha+\beta-1)) \Bigg] + \frac{1}{\Gamma(\gamma)} \cdot \sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma} (t-\sigma(s))^{\underline{\gamma-1}} g(s+\beta+\gamma, u(s+\beta+\gamma)). \tag{43}$$

Clearly, problem (7) has solutions if and only if the operator $\mathscr{H}$ has fixed points. The first shows the existence and uniqueness of a solution to problem (7) by using the Banach contraction principle.

Theorem 14. *Assume that $h \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}, \mathbb{R} - \{0\})$, $g \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}, \mathbb{R})$ with $g(\alpha+\beta-2, 0) = 0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that*

(B_1) *there exist constants $M_1, M_2, N > 0$ such that for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ and $u, v, u^*, v^* \in \mathbb{R}$*

$$\begin{aligned} &\left|H(t,u,u^*) - H(t,v,v^*)\right| \\ &\quad \le M_1 |u-v| + M_2 \left|u^* - v^*\right|, \\ &\left|g(t,u) - g(t,v)\right| \le N|u-v|, \end{aligned} \tag{44}$$

(B_2) *there exists a positive function ϑ with bound $\|\vartheta\|$ such that for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ and $u, v \in \mathbb{R}$*

$$|h(t,u) - h(t,v)| \le \vartheta(t). \tag{45}$$

If $\Omega < 1$ then problem (7) has a unique solution, where

$$\Omega := \frac{\|\vartheta\|}{\Gamma(\alpha)\Gamma(\beta)} (T+\beta)^{\underline{\beta-1}} (T+\alpha+\beta)^{\underline{\alpha-1}} \cdot \left[M_1 + M_2 \frac{(T+\alpha+\beta)^{\underline{\omega}}}{\Gamma(\omega+1)} \right] + \frac{N(T+\alpha+\beta)^{\underline{\gamma}}}{\Gamma(\gamma+1)}. \tag{46}$$

Proof. We shall show that $\mathscr{H}$ is a contraction. For any $u, v \in \mathscr{C}$ and for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$, we have

$$\begin{aligned} &|(\mathscr{H}u)(t) - (\mathscr{H}v)(t)| \le \frac{\|h\|}{\Gamma(\alpha)\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} \Bigg[(t-\sigma(s))^{\underline{\beta-1}} \\ &\cdot \sum_{\xi=0}^{s-\alpha} (s-\sigma(\xi))^{\underline{\alpha-1}} \left|H((\xi+\alpha+\beta-1), u(\xi+\alpha+\beta-1)) - H((\xi+\alpha+\beta-1), v(\xi+\alpha+\beta-1))\right| \Bigg] + \frac{1}{\Gamma(\gamma)} \\ &\cdot \sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma} (t-\sigma(s))^{\underline{\gamma-1}} \left|g(s+\beta+\gamma, u(s+\beta+\gamma)) - g(s+\beta+\gamma, v(s+\beta+\gamma))\right| \le \frac{\|\vartheta\|}{\Gamma(\alpha)\Gamma(\beta)} \\ &\cdot \sum_{s=\alpha-1}^{T+\alpha} \sum_{\xi=0}^{s-\alpha} (T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} \left[M_1|u-v| + M_2\left|\Delta^{-\omega}u - \Delta^{-\omega}v\right|\right] + \frac{N}{\Gamma(\gamma)} \|u-v\| \sum_{s=\alpha+\beta-\gamma-2}^{T+\alpha+\beta-\gamma} (T+\alpha+\beta \\ &- \sigma(s))^{\underline{\gamma-1}} \le \frac{\|\vartheta\|}{\Gamma(\alpha)\Gamma(\beta)} \left[M_1 + M_2 \frac{(T+\alpha+\beta)^{\underline{\omega}}}{\Gamma(\omega+1)} \right] \|u-v\| (T+\beta)^{\underline{\beta-1}} (T+\alpha+\beta)^{\underline{\alpha-1}} + \frac{N}{\Gamma(\gamma)} \|u-v\| (T+\alpha+\beta)^{\underline{\gamma}} \\ &\le \|u-v\| \left\{ \frac{\|\vartheta\|}{\Gamma(\alpha)\Gamma(\beta)} (T+\beta)^{\underline{\beta-1}} (T+\alpha+\beta)^{\underline{\alpha-1}} \left[M_1 + M_2 \frac{(T+\alpha+\beta)^{\underline{\omega}}}{\Gamma(\omega+1)} \right] + \frac{N(T+\alpha+\beta)^{\underline{\gamma}}}{\Gamma(\gamma+1)} \right\} = \|u-v\| \Omega. \end{aligned} \tag{47}$$

Thus, we have $\|(\mathscr{H}u) - (\mathscr{H}v)\| < \|u-v\|$.

Consequently, $\mathscr{H}$ is a contraction. Therefore, by the Banach fixed point theorem, we get that $\mathscr{H}$ has a fixed point which is a unique solution of problem (7). □

In the second result, we deduce the existence of at least one solution of the initial value problem (7) by using the generalized Krasnoselskii's fixed point theorem.

Theorem 15 (generalized Krasnoselskii's fixed point theorem [68]). *Let M be a nonempty, closed convex, and bounded subset of the Banach space X. Let $A : X \to X$ and $B : M \to X$ be two operators such that*

(i) *A is a contraction,*

(ii) *B is completely continuous,*

(iii) *$x = Ax + By$ for all $y \in M \Rightarrow x \in M$.*

Then, the operator equation $Ax + Bx = x$ has a solution.

Theorem 16. *Assume that $h \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}, \mathbb{R}-\{0\})$, $g \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}, \mathbb{R})$ with $g(\alpha+\beta-2,0)=0$, and $H \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, suppose that*

(K_1) *there exist two positive functions ϑ and θ with bound $\|\vartheta\|$ and $\|\theta\|$, respectively, such that, for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$ and $u, v, \Delta^{-\omega}u, \Delta^{-\omega}v \in \mathbb{R}$,*

$$\begin{aligned}&|h(t,u)-h(t,v)| \le \vartheta(t)|u-v|,\\&\left|H\left[t,u,\Delta^{-\omega}u\right]-H\left[t,u,\Delta^{-\omega}v\right]\right|\\&\quad\le \theta(t)\left(|u-v|-\left|\Delta^{-\omega}u-\Delta^{-\omega}v\right|\right),\end{aligned} \tag{48}$$

(K_2) *there exist functions $\mu, \nu, \kappa \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}, \mathbb{R}^+)$ such that*

$$\begin{aligned}&|h(t,u)| \le \mu(t) \quad \forall (t,u) \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R},\\&\left|H\left(t,u,\Delta^{-\omega}u\right)\right| \le \nu(t)\\&\qquad \forall\left(t,u,\Delta^{-\omega}u\right) \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R} \times \mathbb{R},\\&|g(t,u)| \le \kappa(t) \quad \forall (t,u) \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta} \times \mathbb{R}.\end{aligned} \tag{49}$$

If

$$\begin{aligned}&\frac{(T+\alpha+\beta)^{\beta}}{\Gamma(\beta+1)}\left\{\frac{\|\vartheta\|\|\nu\|(T+\alpha+\beta)^{\alpha}}{\Gamma(\alpha+1)}\right.\\&\quad\left.+\|\mu\|\|\theta\|\left(1+\frac{(T+\alpha+\beta)^{\omega}}{\Gamma(\omega+1)}\right)\right\}<1,\end{aligned} \tag{50}$$

then problem (7) has a unique solution on $\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$.

Proof. Let $\max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}}|\mu(t)| = \|\mu\|$, $\max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}}|\nu(t)| = \|\nu\|$, and $\max_{t\in\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}}|\kappa(t)| = \|\kappa\|$, and choose a constant

$$R \ge \frac{\|\kappa\|(T+\alpha+\beta)^{\gamma}}{\Gamma(\gamma+1)} + \frac{\|\mu\|\|\nu\|(T+\alpha+\beta)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}. \tag{51}$$

We consider $B_R = \{u \in X : \|u\| \le R\}$. Define four operators $\mathscr{P} : X \to X$, $\mathscr{Q} : B_R \to X$, and $\mathscr{S} : X \to X$ by

$$\begin{aligned}&\mathscr{P}u(t) := h(t,u(t)),\\&\mathscr{Q}u(t) := \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\\&\qquad\cdot H(s+\alpha+\beta-1,u(s+\alpha+\beta-1)),\\&\mathscr{S}u(t) := \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}\\&\qquad\cdot g(s+\beta+\gamma,u(s+\beta+\gamma)),\end{aligned} \tag{52}$$

for $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$, and

$$\mathscr{T}u(t) := \frac{1}{\Gamma(\beta)}\sum_{s=0}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\mathscr{P}u(s)\,\mathscr{Q}u(s). \tag{53}$$

Problem (7) has solutions if and only if the operator $u = \mathscr{S}u + \mathscr{T}u$ has fixed points.

The proof is divided into three steps as follows.

Step 1. Verify $u = \mathscr{S}u + \mathscr{T}u$ map bounded sets into bounded sets in B_R.

For each $v \in B_R$, we obtain

$$\begin{aligned}&|u(t)| = |\mathscr{S}u(t)+\mathscr{T}u(t)| \le \frac{1}{\Gamma(\gamma)}\\&\quad\cdot\sum_{s=\alpha+\beta-\gamma-2}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}\left|g(s+\beta+\gamma,u(s+\beta+\gamma))\right|\\&\quad+\frac{1}{\Gamma(\beta)}\sum_{s=0}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}|\mathscr{P}u(s)||\mathscr{Q}u(s)| \le \frac{\|\kappa\|}{\Gamma(\gamma)}\\&\quad\cdot\sum_{s=\alpha+\beta-\gamma-2}^{T+\alpha+\beta-\gamma}(T+\alpha+\beta-\sigma(s))^{\underline{\gamma-1}}+\frac{\|\mu\|\|\nu\|}{\Gamma(\alpha)\Gamma(\beta)}\\&\quad\cdot\sum_{s=0}^{T+\alpha}\left[(T+\alpha+\beta-\sigma(s))^{\underline{\beta-1}}\sum_{\xi=0}^{s-\alpha}(\xi-\sigma(\xi))^{\underline{\alpha-1}}\right]\\&\quad\le\frac{\|\kappa\|(T+\alpha+\beta)^{\gamma}}{\Gamma(\gamma+1)}+\frac{\|\mu\|\|\nu\|(T+\alpha)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \le R.\end{aligned} \tag{54}$$

Thus, $\|u\| \le R$. This implies that $\mathscr{S}u + \mathscr{T}u$ is uniformly bounded.

Hence, condition (iii) of Theorem 15 holds.

Step 2. Check that $\mathscr{T}$ is contraction mapping.

For any $u, v \in \mathbb{R}$ and for each $t \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta}$, by (K_1), we have

$$\begin{aligned}&|\mathscr{T}u(t)-\mathscr{T}v(t)| \le \frac{1}{\Gamma(\beta)}\left|\sum_{s=0}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\left[|\mathscr{P}u(s)|\right.\right.\\&\quad\left.\left.\cdot|\mathscr{Q}u(s)|-|\mathscr{P}v(s)||\mathscr{Q}v(s)|\right]\right| = \frac{1}{\Gamma(\beta)}\left|\sum_{s=0}^{t-\beta}(t\right.\\&\quad-\sigma(s))^{\underline{\beta-1}}\left[|\mathscr{Q}u(s)|(|\mathscr{P}u(s)|-|\mathscr{P}v(s)|)+|\mathscr{P}v(s)|\right.\\&\quad\left.\left.\cdot(|\mathscr{Q}u(s)|-|\mathscr{Q}v(s)|)\right]\right| = \frac{1}{\Gamma(\beta)}\left|\sum_{s=0}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\right.\\&\quad\cdot\left\{\frac{\|\vartheta\|\|\nu\|(T+\alpha+\beta)^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|+\|\mu\|\|\theta\|\right.\\&\quad\left.\left.\cdot\left(\|u-v\|+\|u-v\|\frac{1}{\Gamma(\omega)}\sum_{s=\alpha+\beta-2}^{t}(t-\sigma(s))^{\omega-1}\right)\right\}\right|\\&\quad\le\|u-v\|\frac{(T+\alpha+\beta)^{\beta}}{\Gamma(\beta+1)}\left\{\frac{\|\vartheta\|\|\nu\|(T+\alpha+\beta)^{\alpha}}{\Gamma(\alpha+1)}\right.\\&\quad\left.+\|\mu\|\|\theta\|\left(1+\frac{(T+\alpha+\beta)^{\omega}}{\Gamma(\omega+1)}\right)\right\}.\end{aligned} \tag{55}$$

By (50), $\mathcal{T}$ is a contraction mapping. Hence, condition (i) of Theorem 15 holds.

Step 3. Check that $\mathcal{S}$ is completely continuous on B_R.

The operator $\mathcal{S}$ is obviously continuous on B_R. Furthermore, $\mathcal{S}$ is uniformly bounded on B_R since

$$\|\mathcal{S}\| \leq \frac{\|\kappa\| (T+\alpha+\beta)^{\gamma}}{\Gamma(\gamma+1)}. \tag{56}$$

For any $\epsilon > 0$, there exists a positive constant δ such that for $t_1, t_2 \in \mathbb{N}_{\alpha+\alpha+\beta, T+\alpha+\beta}$

$$\left|t_2^{\gamma} - t_1^{\gamma}\right| < \frac{\Gamma(\gamma+1)}{\|\kappa\|}, \quad \text{whenever } \left|t_2 - t_1\right| < \delta. \tag{57}$$

Then,

$$\begin{aligned} &\left|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)\right| \\ &\leq \frac{1}{\Gamma(\gamma)} \left| \sum_{s=\alpha+\beta-\gamma-2}^{t_2-\gamma} (t_2 - \sigma(s))^{\underline{\gamma-1}} \right. \\ &\cdot \left|g(s+\beta+\gamma, u(s+\beta+\gamma))\right| \\ &- \sum_{s=\alpha+\beta-\gamma-2}^{t_1-\gamma} (t_1 - \sigma(s))^{\underline{\gamma-1}} \\ &\left. \cdot \left|g(s+\beta+\gamma, u(s+\beta+\gamma))\right| \right| \leq \frac{\|\kappa\|}{\Gamma(\gamma+1)} \left|t_2^{\gamma} \right. \\ &\left. - t_1^{\gamma}\right| < \epsilon. \end{aligned} \tag{58}$$

This implies that the set $\mathcal{S}(B_R)$ is an equicontinuous set. Therefore, by the Arzelá-Ascoli theorem, we find that $\mathcal{S}$ is completely continuous. Hence, condition (ii) of Theorem 15 holds.

We see that all the assumptions of Theorem 15 are satisfied. Therefore, we can conclude that problem (7) has at least one solution. The proof is completed. □

5. Examples

In this section, we provide some examples to illustrate our results.

Example 1. Consider the following fractional difference initial value problem:

$$\begin{aligned} &\Delta^{1/2}\left[\frac{u(t) - \Delta^{-1/3} p(t+1/3, u(t+1/3))}{f(t, u(t))}\right] \\ &= F\left[t + \frac{1}{2}, u\left(t+\frac{1}{2}\right)\right], \quad t \in \mathbb{N}_{0,10}, \\ &u\left(-\frac{1}{2}\right) = 0, \end{aligned} \tag{59}$$

where

$$\begin{aligned} p(t, u(t)) &= \frac{u^2(t) + 4|u(t)|}{3 + |u(t)|} \cdot \frac{t \sin^2 2\pi t}{5000(2 + e^{-t})}, \\ f(t, u(t)) &= \frac{10\left(2\cos^2 2\pi t + 3t\right)}{3(100\pi - 2t)^2} \cdot \frac{u^2(t) + 3|u(t)|}{|u(t)| + 10}, \\ F[t, u(t)] &= \frac{\left(t^2+5\right)(8|u(t)| + 10\pi)}{80(100\pi - t)}. \end{aligned} \tag{60}$$

We set $\alpha = 1/2$, $\gamma = 1/3$, and $T = 10$.

Noticing that (H_1)-(H_2) hold, for each $t \in \mathbb{N}_{-1/2,21/2}$, we have

$$\begin{aligned} \left|p(t,u) - p(t,v)\right| &= \frac{4t}{15000(2+e^{-t})} |u - v|, \\ \left|f(t,u) - f(t,v)\right| &= \frac{2+3t}{(100\pi - 2t)^2} |u - v|, \\ |F[t,u] - F[t,v]| &= \left(\frac{\left(t^2+5\right)}{100\pi - t}\right)\left(\frac{|u|}{10} + \frac{\pi}{8}\right). \end{aligned} \tag{61}$$

Thus,

$$\begin{aligned} \|\phi\| &= 0.000389, \\ \|\psi\| &= 0.00139, \\ \|\omega\| &= 0.441. \end{aligned} \tag{62}$$

Finally, we find that

$$\begin{aligned} \widehat{f}_0 &= 0.00389, \\ \widehat{p}_0 &= 0.00105. \end{aligned} \tag{63}$$

So, (H_3) holds with a number $R \in [730.5887, 892.9406]$.

Hence, by Theorem 13, problem (59) has a unique solution on $\mathbb{N}_{-1/2,21/2}$.

Example 2. Consider the following fractional difference boundary value problem:

$$\begin{aligned} &\Delta^{1/2}\left[\frac{\Delta^{2/5}u(t) - \Delta^{-1/3} g(t - 4/15, u(t - 4/15))}{h(t, u(t))}\right] \\ &= H\left[t - \frac{1}{10}, u\left(t - \frac{1}{10}\right), \Delta^{-3/4} u\left(t + \frac{13}{20}\right)\right], \\ &u\left(-\frac{11}{10}\right) = \Delta^{2/5} u\left(-\frac{1}{2}\right) = 0, \end{aligned} \tag{64}$$

for $t \in \mathbb{N}_{0,10}$, where

$$\begin{aligned} g(t, u(t)) &= \log\left(1 + \frac{|u(t)| e^t \sin 5\pi t}{1 + |u(t)|}\right), \\ h(t, u(t)) &= \left(\frac{|u(t)| + 1}{|u(t)| + 2}\right) \frac{e^{-\sin^2 \pi t}}{100e - t}, \end{aligned}$$

$$\begin{aligned}
&H\left[t,u(t),\Delta^{-3/4}u\left(t+\frac{3}{4}\right)\right] \\
&\quad=\frac{1}{3}\arctan\left(\frac{3|u(t)|\sin^2 5\pi t}{10+|u(t)|}\right) \\
&\quad-\frac{1}{3}\arctan\left(\frac{3\left|\Delta^{-3/4}u(t+3/4)\right|\sin^2 5\pi t}{10+\left|\Delta^{-3/4}u(t+3/4)\right|}\right).
\end{aligned} \tag{65}$$

We let $\alpha = 1/2$, $\beta = 2/5$, $\gamma = 1/3$, $\omega = 3/4$, and $T = 10$.

Noticing that (K_1)-(K_2) hold, for each $t \in \mathbb{N}_{-11/10,109/10}$, we obtain

$$\begin{aligned}
&|h(t,u)-h(t,v)| = \left(\frac{e^{-\sin^2\pi t}}{400e-4t}\right)|u-v|, \\
&\left|H\left[t,u,\Delta^{-3/4}u\right]-H\left[t,v,\Delta^{-3/4}v\right]\right| \\
&\quad=\frac{1}{10}\sin^2 5\pi t\left(|u-v|+\left|\Delta^{-3/4}u-\Delta^{-3/4}v\right|\right), \\
&|h(t,u)| = \frac{e^{-\sin^2\pi t}}{100e-t}+e^{-\pi t}, \\
&\left|H\left[t,u,\Delta^{-3/4}u\right]\right| = 3\sin^2 5\pi t.
\end{aligned} \tag{66}$$

Thus, we obtain

$$\begin{aligned}
\|\vartheta\| &= 0.000923, \\
\|\theta\| &= 0.1, \\
\|\mu\| &= 0.00369, \\
\|\nu\| &= 3.
\end{aligned} \tag{67}$$

Finally, we find that

$$\begin{aligned}
&\frac{(T+\alpha+\beta)^{\beta}}{\Gamma(\beta+1)}\left\{\frac{\|\vartheta\|\,\|\nu\|\,(T+\alpha+\beta)^{\alpha}}{\Gamma(\alpha+1)}\right. \\
&\quad\left.+\|\mu\|\,\|\theta\|\left(1+\frac{(T+\alpha+\beta)^{\omega}}{\Gamma(\omega+1)}\right)\right\} \approx 0.2277 < 1.
\end{aligned} \tag{68}$$

Hence, by Theorem 16, problem (64) has a unique solution on $\mathbb{N}_{-11/10,109/10}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Homotopy Series Solutions to Time-Space Fractional Coupled Systems

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We apply the homotopy perturbation Sumudu transform method (HPSTM) to the time-space fractional coupled systems in the sense of Riemann-Liouville fractional integral and Caputo derivative. The HPSTM is a combination of Sumudu transform and homotopy perturbation method, which can be easily handled with nonlinear coupled system. We apply the method to the coupled Burgers system, the coupled KdV system, the generalized Hirota-Satsuma coupled KdV system, the coupled WBK system, and the coupled shallow water system. The simplicity and validity of the method can be shown by the applications and the numerical results.

1. Introduction

Fractional calculus, compared to integer calculus, was mentioned in a letter from L'Hospital to Leibniz in 1695. In the letter, L'Hospital raised a question, "what is the result of $\mathrm{d}^n y/\mathrm{d}x^n$ if $n = 1/2$?" The answer of Leibniz was "$\mathrm{d}^{1/2}x$ will be equal to $x\sqrt{\mathrm{d}x : x}$. This is an apparent paradox, from which, one day useful consequences will be drawn" [1]. Furthermore, the generalization of this framework indicates that it is more appropriate to talk about integration and differentiation of arbitrary order, such as fractional order, real number order, and even complex number order just as the development of number system. Thus, there is a basic question: "what are the definitions of fractional integral and derivative?" Or "how to define the fractional integral and derivative?" More and more mathematicians focused on this problem, like Lagrange, Laplace, Fourier, and so on. Some different fractional integrals and derivatives have been given according to different needs, like Riemann-Liouville fractional integral, Caputo fractional derivative, Weyl fractional derivative, and so on [2]. But there are no uniform definitions of fractional integral and derivative, and the frequently used definitions are Riemann-Liouville integral and Caputo derivative.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so on, which involve derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much more attention. For example, in electromagnetism, Sebaa et al. [3] studied ultrasonic wave propagation in human cancellous bone by using fractional calculus to describe the viscous interactions between fluid and solid structure. In signal processing, Assaleh and Ahmad [4] proposed a new approach for speech signal modeling through using fractional calculus. Magin and Ovadia [5] molded the cardiac tissue electrode interface using fractional calculus. In control theory, Suarez et al. [6] applied fractional controllers to the path-tracking problem in an autonomous electric vehicle. In fluid mechanics, Kulish and Lage [7] applied fractional calculus to the solution of time-dependent, viscous-diffusion fluid mechanics problems.

In this paper, we intend to construct the approximate solutions to the nonlinear time-space fractional coupled systems. There are many effective methods to solve this problem, like Adomian decomposition method [8–10], variation iteration method [11], differential transform method [12], residual power series method [13, 14], iteration method [15], homotopy perturbation method [16], homotopy analysis method [17], and so on. Furthermore, for the nonlinear problem, the multiple exp-function method [18, 19], the transformed rational function method [20–22], and invariant subspace method [23, 24] are three systematical approaches to handle the nonlinear terms. The first one is to propose the exact solution of nonlinear partial differential equations by using rational function transformations. Its key point is to search for rational solutions to variable-coefficient ordinary differential equations transformed from given partial differential equations. The second one is to consider the form of solution as rational exponential functions with unknown coefficients whose advantage is direct applicability to underlying equation. The invariant subspace method is refined to present more unity and more diversity of exact solutions to evolution equations. The key idea is to take subspaces of solutions to linear ordinary differential equations as invariant subspaces that evolution equations admit. Motivated by these fruitful results, Singh et al. [25] proposed the homotopy perturbation Sumudu transform method based on the homotopy perturbation method and Sumudu transform method and applied it to nonlinear partial differential equations. The HPSTM was extended to the time-fractional PDEs in [26, 27]. It is worth mentioning that the HPSTM is applied without any using of Adomian polynomials, over restrictive assumption or linearization, and is capable of reducing the volume of computational work as compared to the classical numerical methods while still maintaining the high accuracy of the result. Meanwhile, it is appropriate not only for strongly nonlinear system but also for weakly nonlinear system.

The rest of the paper is organized as follows. In Section 2, we introduce some concepts on fractional calculus and the Sumudu transform. In Section 3, we illustrate the basic idea of HPSTM which is applied to the time-space fractional coupled systems. In Section 4, we apply HPSTM to obtain fractional power series solutions of nonlinear time-space fractional coupled systems with initial values, and some numerical results are presented as well.

2. Preliminaries

Definition 1 (see [2]). A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $\rho > \mu$, such that $f(x) = x^\rho f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said that $f(x) \in C_\mu^n$, if $f^{(n)}(x) \in C_\mu$, $n \in \mathbb{N}$.

Definition 2 (see [2]). The fractional integral of $f(t)$ in the Riemann-Liouville (left-sided) sense is defined as

$$I_t^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, & \alpha > 0,\ t > \tau \geqslant 0; \\ f(t), & \alpha = 0, \end{cases} \tag{1}$$

where $\alpha \geqslant 0$, $f \in C_\mu$, $\mu \geqslant -1$, and Γ is the Gamma function.

Definition 3 (see [2]). The fractional integral of $f(x)$ in the Riemann-Liouville sense is defined as

$$I_x^\alpha f(x) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{-\infty}^x (x-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, & \alpha > 0,\ -\infty < x < \infty; \\ f(x), & \alpha = 0, \end{cases} \tag{2}$$

where $\alpha \geqslant 0$, $f \in C_\mu$, $\mu \geqslant -1$, and Γ is the Gamma function.

Definition 4 (see [2]). The Caputo (left-sided) fractional derivative operator of order $\alpha \geq 0$, of a function $f \in C_\mu^n$ $(\mu \geqslant -1,\ n \in \mathbb{N})$, is defined as

$$D_t^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau, & n-1 < \alpha < n,\ t > 0; \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n. \end{cases} \tag{3}$$

Definition 5 (see [2]). The Caputo fractional derivative operator of order $\alpha \geq 0$, of a function $f \in C_\mu^n$ $(\mu \geqslant -1,\ n \in \mathbb{N})$, is defined as

$$D_x^\alpha f(x) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{-\infty}^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau, & n-1 < \alpha < n,\ -\infty < x < \infty; \\ \dfrac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}, & \alpha = n. \end{cases} \tag{4}$$

Lemma 6 (see [28]). *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, one has*

$$D_t^\alpha I_t^\alpha f(t) = f(t),$$

$$I_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}. \tag{5}$$

In 1998, a new integral transform, named Sumudu transform, was introduced by Watugala [29] to study solutions of ordinary differential equations in control engineering problems. The Sumudu transform is defined over the set of functions $A = \{f(t) : \exists M, \tau_1, \tau_2 > 0, \text{ s.t. } |f(t)| < Me^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, +\infty)\}$ by the following formula:

$$F(u) = S[f(t)] = \int_0^{\infty} f(ut) e^{-t} \mathrm{d}t, \quad u \in (-\tau_1, \tau_2). \tag{6}$$

Property 7 (see [30]). (i) The Sumudu transform satisfies linear property; that is,

$$S[af(t) + bh(t)] = aS[f(t)] + bS[h(t)], \tag{7}$$

where a, b are constants.

(ii)

$$S[t^n] = u^n \Gamma(n+1), \quad n \in \mathbb{N}. \tag{8}$$

Lemma 8 (see [31]). *The Sumudu transform of the Caputo fractional derivative is*

$$S[D_t^{\alpha} f](t) = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m} u^{-\alpha+k} f^{(k)}(0), \quad m < \alpha \le m+1, \ m \in \mathbb{N}. \tag{9}$$

3. Homotopy Perturbation Sumudu Transform Method

In this section, to illustrate the basic idea of this method, we consider a general nonhomogeneous fractional partial differential coupled system

$$\begin{aligned} &D_t^{\alpha} U^1(x,t) + \mathscr{R}^1(U^1, U^2, U^3) + \mathscr{N}^1(U^1, U^2, U^3) \\ &\quad = g^1(x,t), \\ &D_t^{\beta} U^2(x,t) + \mathscr{R}^2(U^1, U^2, U^3) + \mathscr{N}^2(U^1, U^2, U^3) \\ &\quad = g^2(x,t), \\ &D_t^{\gamma} U^3(x,t) + \mathscr{R}^3(U^1, U^2, U^3) + \mathscr{N}^3(U^1, U^2, U^3) \\ &\quad = g^3(x,t), \end{aligned} \tag{10}$$

with the initial conditions

$$\begin{aligned} U^1(x,0) &= f^1(x), \\ U^2(x,0) &= f^2(x), \\ U^3(x,0) &= f^3(x), \end{aligned} \tag{11}$$

where $\alpha, \beta, \gamma \in (0,1]$, $\mathscr{R}^i$, $\mathscr{N}^i$, $i = 1,2,3$, denote linear differential operators and nonlinear differential operators, respectively, and $g^i(x,t)$ are the source terms. Applying the Sumudu transform on both sides of (10) yields

$$\begin{aligned} &S[D_t^{\alpha} U^1(x,t)] + S[\mathscr{R}^1(U^1, U^2, U^3)] \\ &\quad + S[\mathscr{N}^1(U^1, U^2, U^3)] = S[g^1(x,t)], \\ &S[D_t^{\beta} U^2(x,t)] + S[\mathscr{R}^2(U^1, U^2, U^3)] \\ &\quad + S[\mathscr{N}^2(U^1, U^2, U^3)] = S[g^2(x,t)], \\ &S[D_t^{\gamma} U^3(x,t)] + S[\mathscr{R}^3(U^1, U^2, U^3)] \\ &\quad + S[\mathscr{N}^3(U^1, U^2, U^3)] = S[g^3(x,t)]. \end{aligned} \tag{12}$$

It follows from the property of the Sumudu transform in (9) that

$$\begin{aligned} &S[U^1(x,t)] = f^1(x) - u^{\alpha}\left(S[\mathscr{R}^1(U^1, U^2, U^3)]\right. \\ &\quad \left.+ S[\mathscr{N}^1(U^1, U^2, U^3)]\right) + u^{\alpha} S[g^1(x,t)], \\ &S[U^2(x,t)] = f^2(x) - u^{\beta}\left(S[\mathscr{R}^2(U^1, U^2, U^3)]\right. \\ &\quad \left.+ S[\mathscr{N}^2(U^1, U^2, U^3)]\right) + u^{\beta} S[g^2(x,t)], \\ &S[U^3(x,t)] = f^3(x) - u^{\gamma}\left(S[\mathscr{R}^3(U^1, U^2, U^3)]\right. \\ &\quad \left.+ S[\mathscr{N}^3(U^1, U^2, U^3)]\right) + u^{\gamma} S[g^3(x,t)]. \end{aligned} \tag{13}$$

Furthermore, applying the inverse Sumudu transform S^{-1} on both sides of (13) yields

$$\begin{aligned} &U^1(x,t) = M^1(x,t) - S^{-1}\left[u^{\alpha} S\left[\mathscr{R}^1(U^1, U^2, U^3)\right.\right. \\ &\quad \left.\left.+ \mathscr{N}^1(U^1, U^2, U^3)\right]\right], \\ &U^2(x,t) = M^2(x,t) - S^{-1}\left[u^{\beta} S\left[\mathscr{R}^2(U^1, U^2, U^3)\right.\right. \\ &\quad \left.\left.+ \mathscr{N}^2(U^1, U^2, U^3)\right]\right], \\ &U^3(x,t) = M^3(x,t) - S^{-1}\left[u^{\gamma} S\left[\mathscr{R}^3(U^1, U^2, U^3)\right.\right. \\ &\quad \left.\left.+ \mathscr{N}^3(U^1, U^2, U^3)\right]\right], \end{aligned} \tag{14}$$

where $M^i(x,t)$, $i = 1,2,3$, represent the terms arising from the source terms and prescribed initial conditions; that is,

$$\begin{aligned} M^1(x,t) &= S^{-1}\left[f^1(x) + u^{\alpha} S[g^1(x,t)]\right], \\ M^2(x,t) &= S^{-1}\left[f^2(x) + u^{\beta} S[g^2(x,t)]\right], \\ M^3(x,t) &= S^{-1}\left[f^3(x) + u^{\gamma} S[g^3(x,t)]\right]. \end{aligned} \tag{15}$$

Let us construct the homotopy perturbation equations

$$\begin{aligned} &U^1(x,t) = M^1(x,t) - p \times S^{-1}\left[u^{\alpha} S\left[\mathscr{R}^1(U^1, U^2, U^3)\right.\right. \\ &\quad \left.\left.+ \mathscr{N}^1(U^1, U^2, U^3)\right]\right], \end{aligned}$$

$$U^2(x,t) = M^2(x,t) - p \times S^{-1}\left[u^{\beta} S\left[\mathscr{R}^2\left(U^1,U^2,U^3\right) + \mathscr{N}^2\left(U^1,U^2,U^3\right)\right]\right],$$
$$U^3(x,t) = M^3(x,t) - p \times S^{-1}\left[u^{\gamma} S\left[\mathscr{R}^3\left(U^1,U^2,U^3\right) + \mathscr{N}^3\left(U^1,U^2,U^3\right)\right]\right], \tag{16}$$

where homotopy parameter $p \in [0,1]$. Suppose that $U^i(x,t)$ and the nonlinear terms $\mathscr{N}^j U^i(x,t)$ can be written as

$$U^i(x,t) = \sum_{n=0}^{\infty} p^n U_n^i(x,t), \quad i = 1,2,3,$$
$$\mathscr{N}^j\left(U^1,U^2,U^3\right) = \sum_{n=0}^{\infty} p^n H_n^j\left(U^1,U^2,U^3\right), \quad j = 1,2,3, \tag{17}$$

where the coefficient polynomials U_n^i can be determined below and $H_n^j(U^1,U^2,U^3)$ are given by the following formulae:

$$H_n^j\left(U^1,U^2,U^3\right) = \frac{1}{n!} \cdot \frac{\partial^n}{\partial p^n}\left(\mathscr{N}^j\left(\sum_{k=0}^{\infty} p^k U_k^1, \sum_{k=0}^{\infty} p^k U_k^2, \sum_{k=0}^{\infty} p^k U_k^3\right)\right)\Bigg|_{p=0}$$
$$= \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\mathscr{N}^j\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right) + \mathscr{N}^j\left(\sum_{k=n+1}^{\infty} p^k U_k^1, \sum_{k=n+1}^{\infty} p^k U_k^2, \sum_{k=n+1}^{\infty} p^k U_k^3\right)\right)\Bigg|_{p=0}$$
$$= \frac{1}{n!} \cdot \frac{\partial^n}{\partial p^n}\left(\mathscr{N}^j\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right)\Bigg|_{p=0}, \quad j = 1,2,3. \tag{18}$$

Substituting (17) into (16) gives

$$\sum_{n=0}^{\infty} p^n U_n^1(x,t) = M^1(x,t) - p \times S^{-1}\left[u^{\alpha} S\left[\mathscr{R}^1\left(\sum_{k=0}^{\infty} p^k U_k^1, \sum_{k=0}^{\infty} p^k U_k^2, \sum_{k=0}^{\infty} p^k U_k^3\right) + \sum_{n=0}^{\infty} p^n H_n^1\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right]\right],$$
$$\sum_{n=0}^{\infty} p^n U_n^2(x,t) = M^2(x,t) - p \times S^{-1}\left[u^{\beta} S\left[\mathscr{R}^2\left(\sum_{k=0}^{\infty} p^k U_k^1, \sum_{k=0}^{\infty} p^k U_k^2, \sum_{k=0}^{\infty} p^k U_k^3\right) + \sum_{n=0}^{\infty} p^n H_n^2\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right]\right], \tag{19}$$
$$\sum_{n=0}^{\infty} p^n U_n^3(x,t) = M^3(x,t) - p \times S^{-1}\left[u^{\gamma} S\left[\mathscr{R}^3\left(\sum_{k=0}^{\infty} p^k U_k^1, \sum_{k=0}^{\infty} p^k U_k^2, \sum_{k=0}^{\infty} p^k U_k^3\right) + \sum_{n=0}^{\infty} p^n H_n^3\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right]\right].$$

Comparing the coefficients of p, we obtain the following recurrence equations:

$$p^0: \quad U_0^1(x,t) = M^1(x,t), \quad U_0^2(x,t) = M^2(x,t), \quad U_0^3(x,t) = M^3(x,t),$$
$$\vdots$$
$$p^n: \quad U_n^1(x,t) = -S^{-1}\left[u^{\alpha} S\left[\mathscr{R}^1\left(\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r} U_r^1 U_s^2 U_{n-1-r-s}^3\right) + H_{n-1}^1\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right]\right],$$
$$U_n^2(x,t) = -S^{-1}\left[u^{\beta} S\left[\mathscr{R}^2\left(\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r} U_r^1 U_s^2 U_{n-1-r-s}^3\right) + H_{n-1}^2\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right]\right],$$
$$U_n^3(x,t) = -S^{-1}\left[u^{\gamma} S\left[\mathscr{R}^3\left(\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r} U_r^1 U_s^2 U_{n-1-r-s}^3\right) + H_{n-1}^3\left(\sum_{k=0}^{n} p^k U_k^1, \sum_{k=0}^{n} p^k U_k^2, \sum_{k=0}^{n} p^k U_k^3\right)\right]\right], \tag{20}$$

where $n = 1, 2, \ldots$.

According to the homotopy perturbation method, we assume that the solution of (10)-(11) can be written as

$$V^i(x,t) = \lim_{n\to\infty}\left(U_0^i + U_1^i p + U_2^i p^2 + \cdots + U_n^i p^n\right), \quad i = 1,2,3. \tag{21}$$

Setting $p \to 1$, the approximate solution to (10)-(11) is

$$\begin{aligned} U^i(x,t) &= \lim_{p\to 1} V^i(x_1,\ldots,x_m,t) \\ &= \lim_{n\to\infty}\left(U_0^i + U_1^i + U_2^i + \cdots + U_n^i\right), \quad i = 1,2,3. \end{aligned} \tag{22}$$

The convergence of series (21) depends on the nonlinear differential operator $\mathcal{N}$. Generally, the derivative with respect to U of the nonlinear part in the splitting must be sufficiently small, since the parameter p may be relatively large; in fact we take $p \to 1$. The series is convergent for most cases [32].

Remark 9. HPSTM is applied to construct homotophy series solutions for fractional coupled systems, which has not too many overstrict assumptions compared to some classical methods.

4. Application of HPSTM to Time-Space Fractional Coupled Systems

In this section, we apply HPSTM to nonlinear time-space fractional coupled systems with initial conditions.

4.1. The Time-Space Fractional Coupled Burgers System. The Burgers equation is one of the most important partial differential equations from fluid mechanics, which not only describes many phenomena, for example, modeling the motion of turbulence [33], but also has many applications in science and engineering [34]. Here we apply HPSTM to solve the following nonlinear time-space fractional coupled Burgers system:

$$\begin{aligned} D_t^\alpha U - D_x^2 U - 2UD_x^\beta U + D_x(UV) &= 0, \\ D_t^\gamma V - D_x^2 V - 2VD_x^\delta V + D_x(UV) &= 0, \end{aligned} \tag{23}$$

with the initial conditions

$$\begin{aligned} U(x,0) &= \sin x, \\ V(x,0) &= \sin x, \end{aligned} \tag{24}$$

where $0 < \alpha,\beta,\delta,\gamma \le 1$, $(x,t) \in \mathbb{R}\times[0,\infty)$.

Applying the Sumudu transform on both sides of (23) with the initial conditions, we can obtain

$$\begin{aligned} S[U(x,t)] &= \sin x + u^\alpha \left(S\left[D_x^2 U(x,t) \right.\right. \\ &\quad \left.\left. + 2U(x,t)D_x^\beta U(x,t) - D_x(U(x,t)V(x,t))\right]\right), \\ S[V(x,t)] &= \sin x + v^\gamma \left(S\left[D_x^2 V(x,t) \right.\right. \\ &\quad \left.\left. + 2V(x,t)D_x^\delta V(x,t) - D_x(U(x,t)V(x,t))\right]\right). \end{aligned} \tag{25}$$

The inverse Sumudu transform of (25) implies that

$$\begin{aligned} U(x,t) &= \sin x + S^{-1}\left[u^\alpha \left(S\left[D_x^2 U(x,t) \right.\right.\right. \\ &\quad \left.\left.\left. + 2U(x,t)D_x^\beta U(x,t) - D_x(U(x,t)V(x,t))\right]\right)\right], \\ V(x,t) &= \sin x + S^{-1}\left[v^\gamma \left(S\left[D_x^2 V(x,t) \right.\right.\right. \\ &\quad \left.\left.\left. + 2V(x,t)D_x^\delta V(x,t) - D_x(U(x,t)V(x,t))\right]\right)\right]. \end{aligned} \tag{26}$$

Now applying the homotopy perturbation method gives

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,t) &= \sin x + p \\ &\quad \times S^{-1}\left[u^\alpha S\left[D_x^2\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right) \right.\right. \\ &\quad \left.\left. + \sum_{n=0}^{\infty} p^n H_n^U(x,t)\right]\right], \\ \sum_{n=0}^{\infty} p^n V_n(x,t) &= \sin x + p \\ &\quad \times S^{-1}\left[v^\gamma S\left[D_x^2\left(\sum_{n=0}^{\infty} p^n V_n(x,t)\right) \right.\right. \\ &\quad \left.\left. + \sum_{n=0}^{\infty} p^n H_n^V(x,t)\right]\right], \end{aligned} \tag{27}$$

where $H_n^U(x,t)$ and $H_n^V(x,t)$ are polynomials which denote the homotopy coefficients of the nonlinear term and are given by

$$\begin{aligned} \sum_{n=0}^{\infty} p^n H_n^U(x,t) &= 2U(x,t)D_x^\beta U(x,t) \\ &\quad - D_x(U(x,t)V(x,t)), \\ \sum_{n=0}^{\infty} p^n H_n^V(x,t) &= 2V(x,t)D_x^\delta V(x,t) \\ &\quad - D_x(U(x,t)V(x,t)). \end{aligned} \tag{28}$$

Set

$$\begin{aligned} U_n(x,t) &:= \widetilde{U}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ V_n(x,t) &:= \widetilde{V}_n(x)\frac{t^{n\gamma}}{\Gamma(n\gamma+1)}, \end{aligned} \tag{29}$$

and then

$$\begin{aligned} H_n^U(x,t) &= 2\sum_{i=0}^{n}\widetilde{U}_i(x)D_x^\beta \widetilde{U}_{n-i}(x) \\ &\quad \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)} \\ &\quad - \sum_{i=0}^{n} D_x\left(\widetilde{U}_i(x)\widetilde{V}_{n-i}(x)\right) \end{aligned}$$

$$\cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)},$$

$$H_n^V(x,t) = 2\sum_{i=0}^{n} \widetilde{V}_i(x) D_x^{\delta} \widetilde{V}_{n-i}(x) \cdot \frac{t^{n\gamma}}{\Gamma(i\gamma+1)\Gamma((n-i)\gamma+1)} - \sum_{i=0}^{n} D_x\left(\widetilde{U}_i(x)\widetilde{V}_{n-i}(x)\right) \cdot \frac{t^{n\gamma}}{\Gamma(i\gamma+1)\Gamma((n-i)\gamma+1)}. \tag{30}$$

Comparing the coefficients of p, this gives

$$\begin{aligned}
p^0:\quad & U_0(x,t) = \widetilde{U}_0(x) = \sin x,\\
& V_0(x,t) = \widetilde{V}_0(x) = \sin x,\\
p^1:\quad & U_1(x,t) = S^{-1}\left[u^{\alpha} S\left[D_x^2 U_0(x,t) + H_0^u(x,t)\right]\right] = S^{-1}\left[u^{\alpha} S\left[D_x^2\widetilde{U}_0(x) + 2\widetilde{U}_0(x) D_x^{\beta}\widetilde{U}_0(x) - D_x\left(\widetilde{U}_0(x)\widetilde{V}_0(x)\right)\right]\right]\\
&= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left\{D_x^2\widetilde{U}_0(x) + 2\widetilde{U}_0(x) D_x^{\beta}\widetilde{U}_0(x) - D_x\left(\widetilde{U}_0(x)\widetilde{V}_0(x)\right)\right\} = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\widetilde{U}_1(x),\\
& V_1(x,t) = S^{-1}\left[u^{\gamma} S\left[D_x^2 V_0(x,t) + H_0^v(x,t)\right]\right] = S^{-1}\left[u^{\gamma} S\left[D_x^2\widetilde{V}_0(x) + 2\widetilde{V}_0(x) D_x^{\delta}\widetilde{V}_0(x) - D_x\left(\widetilde{U}_0(x)\widetilde{V}_0(x)\right)\right]\right]\\
&= \frac{t^{\gamma}}{\Gamma(\gamma+1)}\left\{D_x^2\widetilde{V}_0(x) + 2\widetilde{V}_0(x) D_x^{\delta}\widetilde{U}_0(x) - D_x\left(\widetilde{U}_0(x)\widetilde{V}_0(x)\right)\right\} = \frac{t^{\gamma}}{\Gamma(\gamma+1)}\widetilde{V}_1(x).
\end{aligned} \tag{31}$$

Generally, we have

$$\begin{aligned}
p^k:\quad & U_k(x,t)\\
&= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{D_x^2\widetilde{U}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(2\widetilde{U}_i(x) D_x^{\beta}\widetilde{U}_{k-1-i}(x) - D_x\left(\widetilde{U}_i(x)\widetilde{V}_{k-1-i}(x)\right)\right)\right\}\\
&= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{U}_k(x),\\
& V_k(x,t)\\
&= \frac{t^{k\gamma}}{\Gamma(k\gamma+1)}\left\{D_x^2\widetilde{V}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\gamma+1)}{\Gamma(i\gamma+1)\Gamma((k-i-1)\gamma+1)}\left(2\widetilde{V}_i(x) D_x^{\delta}\widetilde{V}_{k-1-i}(x) - D_x\left(\widetilde{U}_i(x)\widetilde{V}_{k-1-i}(x)\right)\right)\right\}\\
&= \frac{t^{k\gamma}}{\Gamma(k\gamma+1)}\widetilde{V}_k(x),
\end{aligned} \tag{32}$$

where

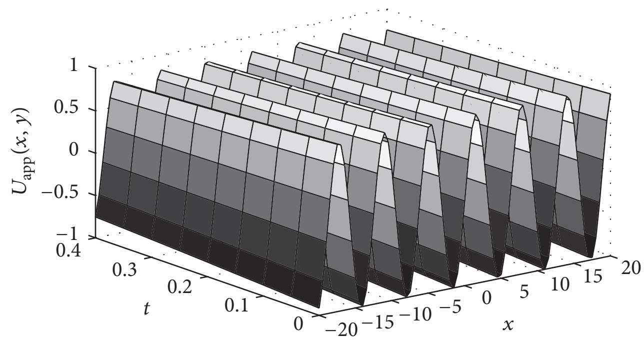
$$\begin{aligned}
\widetilde{U}_k(x) &= D_x^2\widetilde{U}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(2\widetilde{U}_i(x) \cdot D_x^{\beta}\widetilde{U}_{k-1-i}(x) - D_x\left(\widetilde{U}_i(x)\widetilde{V}_{k-1-i}(x)\right)\right),\\
\widetilde{V}_k(x) &= D_x^2\widetilde{V}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\gamma+1)}{\Gamma(i\gamma+1)\Gamma((k-i-1)\gamma+1)}\left(2\widetilde{V}_i(x) \cdot D_x^{\delta}\widetilde{V}_{k-1-i}(x) - D_x\left(\widetilde{U}_i(x)\widetilde{V}_{k-1-i}(x)\right)\right).
\end{aligned} \tag{33}$$

Hence, the series solution of (23) is

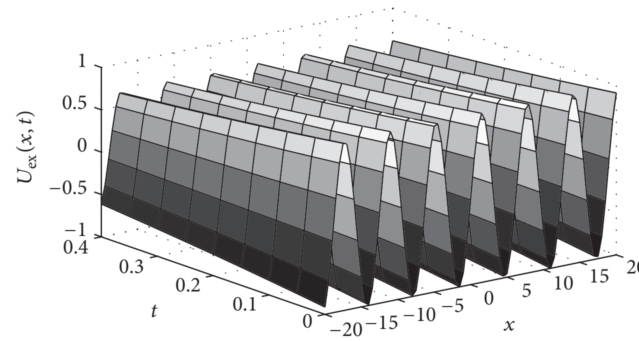
$$\begin{aligned}
U(x,t) &= \sum_{k=0}^{\infty} U_k(x,t) = \sum_{k=0}^{\infty}\widetilde{U}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},\\
V(x,t) &= \sum_{k=0}^{\infty} V_k(x,t) = \sum_{k=0}^{\infty}\widetilde{V}_k(x)\frac{t^{k\gamma}}{\Gamma(k\gamma+1)}.
\end{aligned} \tag{34}$$

Particularly, when $\alpha = \gamma = \beta = \delta = 1$, the exact solution of (23) is

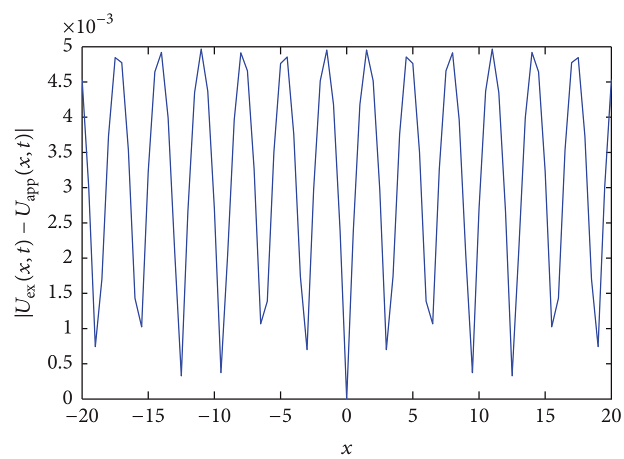
$$\begin{aligned}
U(x,t) &= e^{-t}\sin x,\\
V(x,t) &= e^{-t}\sin x.
\end{aligned} \tag{35}$$



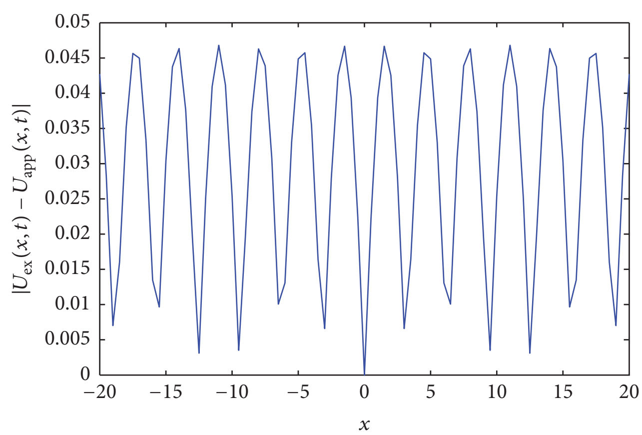
(a) The third-order approximate solution of $U(x,t)$



(b) The exact solution of $U(x,t)$



(c) The error $|U_{app} - U_{ex}|$ of $U(x,t)$ at $t = 0.01$



(d) The error $|U_{app} - U_{ex}|$ of $U(x,t)$ at $t = 0.1$

FIGURE 1

Using HPSTM, when $\alpha = \gamma = \beta = \delta = 1$, the third approximate solution of (23) is

$$\begin{aligned}
U_{app}(x,t) &= \sin x + \frac{t}{2!}(-\sin x) + \frac{t^2}{3!}(\sin x) \\
&\quad + \frac{t^3}{4!}(-\sin x) \\
&= \sin x\left\{1 + \frac{-t}{2!} + \frac{(-t)^2}{3!} + \frac{-t^3}{4!}\right\}, \\
V_{app}(x,t) &= \sin x + \frac{t}{2!}(-\sin x) + \frac{t^2}{3!}(\sin x) \\
&\quad + \frac{t^3}{4!}(-\sin x) \\
&= \sin x\left\{1 + \frac{-t}{2!} + \frac{(-t)^2}{3!} + \frac{-t^3}{4!}\right\}.
\end{aligned} \tag{36}$$

In general, the limit of the approximate solution is

$$\begin{aligned}
U(x,t) &= \sin x + \frac{t}{2!}(-\sin x) + \frac{t^2}{3!}(\sin x) \\
&\quad + \frac{t^3}{4!}(-\sin x) + \cdots \\
&= \sin x\left\{1 + \frac{-t}{2!} + \frac{(-t)^2}{3!} + \frac{-t^3}{4!} + \cdots\right\} \\
&= e^{-t}\sin x, \\
V(x,t) &= \sin x + \frac{t}{2!}(-\sin x) + \frac{t^2}{3!}(\sin x) \\
&\quad + \frac{t^3}{4!}(-\sin x) + \cdots \\
&= \sin x\left\{1 + \frac{-t}{2!} + \frac{(-t)^2}{3!} + \frac{-t^3}{4!} + \cdots\right\} \\
&= e^{-t}\sin x,
\end{aligned} \tag{37}$$

which is as same as the exact solution. However, if the initial values are too complex to find the limit of the approximated solution, then we replace the exact solution by the approximated solution within a certain scale, which is useful in the application of engineering.

Thus we plot the images of the approximate solution (see Figure 1(a)), the exact solution (see Figure 1(b)), and the error function (see Figures 1(c) and 1(d)). It is clear that the error function $|U_{app} - U_{ex}|$ depends on time t. When time t is small (e.g., $t = 0.01$), the error function is in the scale of 10^{-3} (see

Figure 1(c)), which indicates that this is a good approximation in the neighbour of time 0 for system (23) with some explicit parameters. However, when time becomes large (e.g., $t = 0.1$), the error function tends to be large as well (see Figure 1(d)); that is to say, this method is only suitable for constructing the approximated solution around the initial data.

4.2. The Time-Space Fractional Coupled KdV System of Generalized Hirota-Satsuma Type. In this subsection, consider the time-space fractional generalization of the Hirota-Satsuma coupled KdV system

$$\begin{aligned} D_t^{\alpha}U - \frac{1}{2}D_x^{3\beta}U + 3UD_x^{\gamma}U - 3D_x^{\delta}(VW) &= 0, \\ D_t^{\alpha}V + D_x^{3\lambda}V - 3UD_x^{\tau}V &= 0, \\ D_t^{\alpha}W + D_x^{3\theta}W - 3UD_x^{\sigma}W &= 0, \end{aligned} \tag{38}$$

with respect to the initial conditions

$$\begin{aligned} U(x,0) &= a_0(x), \\ V(x,0) &= b_0(x), \\ W(x,0) &= c_0(x), \end{aligned} \tag{39}$$

where $0 < \alpha, \gamma, \delta, \lambda, \tau \leq 1$, $2/3 < \beta, \sigma, \theta \leq 1$, $U = U(x,t)$, $V = V(x,t)$, $W = W(x,t)$, $(x,t) \in \mathbb{R} \times [0,\infty)$. The Hirota-Satsuma coupled KdV equation describes the unidirectional propagation of shallow water waves, which was initiated by Wu et al. [35]. Further (38) becomes a generalized fractional KdV equation for $U = 0$ and a fractional MKdV equation for $V = 0$.

Applying the Sumudu transform on both sides of (38) with the initial conditions, we obtain

$$\begin{aligned} S[U(x,t)] &= a_0(x) + u^{\alpha}\left(S\left[\frac{1}{2}D_x^{3\beta}U(x,t)\right.\right. \\ &\quad \left.\left. - 3U(x,t)D_x^{\gamma}U(x,t) + 3D_x^{\delta}(V(x,t)W(x,t))\right]\right), \\ S[V(x,t)] &= b_0(x) - v^{\alpha}\left(S\left[D_x^{3\lambda}V(x,t)\right.\right. \\ &\quad \left.\left. - 3U(x,t)D_x^{\tau}V(x,t)\right]\right), \\ S[W(x,t)] &= c_0(x) - w^{\alpha}\left(S\left[D_x^{3\theta}W(x,t)\right.\right. \\ &\quad \left.\left. - 3U(x,t)D_x^{\sigma}W(x,t)\right]\right). \end{aligned} \tag{40}$$

The inverse Sumudu transform of (40) implies that

$$\begin{aligned} U(x,t) &= a_0(x) + S^{-1}\left[u^{\alpha}\left(S\left[\frac{1}{2}D_x^{3\beta}U(x,t)\right.\right.\right. \\ &\quad - 3U(x,t)D_x^{\gamma}U(x,t) \\ &\quad \left.\left.\left. + 3D_x^{\delta}(V(x,t)W(x,t))\right]\right)\right], \\ V(x,t) &= b_0(x) - S^{-1}\left[v^{\alpha}\left(S\left[D_x^{3\lambda}V(x,t)\right.\right.\right. \\ &\quad \left.\left.\left. - 3U(x,t)D_x^{\tau}V(x,t)\right]\right)\right], \\ W(x,t) &= c_0(x) - S^{-1}\left[w^{\alpha}\left(S\left[D_x^{3\theta}W(x,t)\right.\right.\right. \\ &\quad \left.\left.\left. - 3U(x,t)D_x^{\sigma}W(x,t)\right]\right)\right]. \end{aligned} \tag{41}$$

Via the homotopy perturbation method, it gives

$$\begin{aligned} \sum_{n=0}^{\infty}p^nU_n(x,t) &= a_0(x) + p \\ &\quad \times S^{-1}\left[u^{\alpha}S\left[\frac{1}{2}D_x^{3\beta}\left(\sum_{n=0}^{\infty}p^nU_n(x,t)\right)\right.\right. \\ &\quad \left.\left. + \sum_{n=0}^{\infty}p^nH_n^U(x,t)\right]\right], \\ \sum_{n=0}^{\infty}p^nV_n(x,t) &= b_0(x) + p \\ &\quad \times S^{-1}\left[v^{\alpha}S\left[-D_x^{3\lambda}\left(\sum_{n=0}^{\infty}p^nV_n(x,t)\right)\right.\right. \\ &\quad \left.\left. + \sum_{n=0}^{\infty}p^nH_n^V(x,t)\right]\right], \\ \sum_{n=0}^{\infty}p^nW_n(x,t) &= c_0(x) + p \\ &\quad \times S^{-1}\left[w^{\alpha}S\left[-D_x^{3\theta}\left(\sum_{n=0}^{\infty}W_n(x,t)\right)\right.\right. \\ &\quad \left.\left. + \sum_{n=0}^{\infty}p^nH_n^W(x,t)\right]\right], \end{aligned} \tag{42}$$

where $H_n^U(x,t)$, $H_n^V(x,t)$, and $H_n^W(x,t)$ are polynomials which denote the nonlinear term, and they are given by

$$\begin{aligned} \sum_{n=0}^{\infty}p^nH_n^U(x,t) &= 3D_x^{\delta}(V(x,t)W(x,t)) \\ &\quad - 3U(x,t)D_x^{\gamma}U(x,t), \\ \sum_{n=0}^{\infty}p^nH_n^V(x,t) &= 3U(x,t)D_x^{\tau}V(x,t), \\ \sum_{n=0}^{\infty}p^nH_n^W(x,t) &= 3U(x,t)D_x^{\sigma}W(x,t). \end{aligned} \tag{43}$$

Set

$$\begin{aligned} U_n(x,t) &:= \widetilde{U}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ V_n(x,t) &:= \widetilde{V}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ W_n(x,t) &:= \widetilde{W}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{44}$$

and then

$$H_n^U = 3D_x^\delta \left(\sum_{i=0}^{n} \widetilde{V}_i(x) \widetilde{W}_{n-i}(x) \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)} \right) - 3\sum_{i=0}^{n} \widetilde{U}_i(x) \cdot D_x^\gamma \widetilde{U}_{n-i}(x) \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)},$$
$$H_n^V = 3\sum_{i=0}^{n} \widetilde{U}_i(x) D_x^\tau \widetilde{V}_{n-i}(x) \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)},$$
$$H_n^W = 3\sum_{i=0}^{n} \widetilde{U}_i(x) D_x^\sigma \widetilde{W}_{n-i}(x) \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}. \tag{45}$$

Comparing the coefficients of p shows

$$\begin{aligned}
p^0: \quad & U_0(x,t) = \widetilde{U}_0(x) = a_0(x), \\
& V_0(x,t) = \widetilde{V}_0(x) = b_0(x), \\
& W_0(x,t) = \widetilde{W}_0(x) = c_0(x), \\
p^1: \quad & U_1(x,t) = S^{-1}\left[u^\alpha S\left[\frac{1}{2}D_x^{3\beta}U_0(x,t) + H_0^U(x,t)\right]\right] \\
& = S^{-1}\left[u^\alpha S\left[\frac{1}{2}D_x^{3\beta}\widetilde{U}_0(x) + 3D_x^\delta\left(\widetilde{V}_0(x)\widetilde{W}_0(x)\right) - 3\widetilde{U}_0(x)D_x^\gamma\widetilde{U}_0(x)\right]\right] \\
& = \frac{t^\alpha}{\Gamma(\alpha+1)}\left(\frac{1}{2}D_x^{3\beta}\widetilde{U}_0(x) + 3D_x^\delta\left(\widetilde{V}_0(x)\widetilde{W}_0(x)\right) - 3\widetilde{U}_0(x)D_x^\gamma\widetilde{U}_0(x)\right) = \frac{t^\alpha}{\Gamma(\alpha+1)}\widetilde{U}_1(x), \\
& V_1(x,t) = S^{-1}\left[v^\alpha S\left[-D_x^{3\lambda}V_0(x,t) + H_0^V(x,t)\right]\right] = S^{-1}\left[v^\alpha S\left[-D_x^{3\lambda}\widetilde{V}_0(x) + 3\widetilde{U}_0(x)D_x^\tau\widetilde{V}_0(x)\right]\right] \\
& = \frac{t^\alpha}{\Gamma(\alpha+1)}\left(-D_x^{3\lambda}\widetilde{V}_0(x) + 3\widetilde{U}_0(x)D_x^\gamma\widetilde{U}_0(x)\right) = \frac{t^\alpha}{\Gamma(\alpha+1)}\widetilde{V}_1(x), \\
& W_1(x,t) = S^{-1}\left[w^\alpha S\left[-D_x^{3\theta}W_0(x,t) + H_0^W(x,t)\right]\right] = S^{-1}\left[w^\alpha S\left[-D_x^{3\theta}\widetilde{W}_0(x) + 3\widetilde{U}_0(x)D_x^\sigma\widetilde{W}_0(x)\right]\right] \\
& = \frac{t^\alpha}{\Gamma(\alpha+1)}\left(-D_x^{3\theta}\widetilde{W}_0(x) + 3\widetilde{U}_0(x)D_x^\sigma\widetilde{W}_0(x)\right) = \frac{t^\alpha}{\Gamma(\alpha+1)}\widetilde{W}_1(x).
\end{aligned} \tag{46}$$

Generally, one has

$$\begin{aligned}
p^k: \quad & U_k(x,t) \\
& = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{\frac{1}{2}D_x^{3\beta}\widetilde{U}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(3D_x^\delta\left(\widetilde{V}_{k-1-i}(x)\widetilde{W}_i(x)\right) - 3\widetilde{U}_i(x)D_x^\gamma\widetilde{U}_{k-1-i}(x)\right)\right\} \\
& = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{U}_k(x), \\
& V_k(x,t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{-D_x^{3\lambda}\widetilde{V}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}3\widetilde{U}_i(x)D_x^\tau\widetilde{V}_{k-1-i}(x)\right\} = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{V}_k(x), \\
& W_k(x,t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{-D_x^{3\theta}\widetilde{W}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}3\widetilde{U}_i(x)D_x^\sigma\widetilde{W}_{k-1-i}(x)\right\} \\
& = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{W}_k(x),
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
\widetilde{U}_k(x) &= \frac{1}{2}D_x^{3\beta}\widetilde{U}_{k-1}(x) \\
&+ \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)} \\
&\cdot \left(3D_x^{\delta}\left(\widetilde{V}_{k-1-i}(x)\widetilde{W}_i(x)\right)\right. \\
&\left.- 3\widetilde{U}_i(x)D_x^{\gamma}\widetilde{U}_{k-1-i}(x)\right), \\
\widetilde{V}_k(x) &= -D_x^{3\lambda}\widetilde{V}_{k-1}(x) \\
&+ \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}3\widetilde{U}_i(x) \\
&\cdot D_x^{\tau}\widetilde{V}_{k-1-i}(x), \\
\widetilde{W}_k(x) &= -D_x^{3\theta}\widetilde{W}_{k-1}(x) \\
&+ \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}3\widetilde{U}_i(x) \\
&\cdot D_x^{\sigma}\widetilde{W}_{k-1-i}(x).
\end{aligned} \tag{48}$$

Therefore, the approximate series solution of (38) is

$$\begin{aligned}
U(x,t) &= \sum_{k=0}^{\infty}U_k(x,t) = \sum_{k=0}^{\infty}\widetilde{U}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \\
V(x,t) &= \sum_{k=0}^{\infty}V_k(x,t) = \sum_{k=0}^{\infty}\widetilde{V}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \\
W(x,t) &= \sum_{k=0}^{\infty}W_k(x,t) = \sum_{k=0}^{\infty}\widetilde{W}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.
\end{aligned} \tag{49}$$

Particularly, when $\alpha = \gamma = \delta = \lambda = 1$, $\tau = \beta = \sigma = \theta = 2/3$, and the special initial value of (38) is

$$\begin{aligned}
U(x,0) &= -2 + 4\tanh^2 x, \\
V(x,0) &= -4 + 4\tanh^2 x, \\
W(x,0) &= 1 + \tanh^2 x,
\end{aligned} \tag{50}$$

then the exact solutions are

$$\begin{aligned}
U(x,t) &= -2 + 4\tanh^2(x+2t), \\
V(x,t) &= -4 + 4\tanh^2(x+2t), \\
W(x,t) &= 1 + \tanh^2(x+2t).
\end{aligned} \tag{51}$$

Under these special conditions, via HPSTM, the first approximate solution of (38) is

$$\begin{aligned}
U_{\text{app}}(x,t) &= -2 + 4\tanh^2 x + 16\left(\tanh x - \tanh^3 x\right)t, \\
V_{\text{app}}(x,t) &= -4 + 4\tanh^2 x + 16\left(\tanh x - \tanh^3 x\right)t, \\
W_{\text{app}}(x,t) &= 1 + \tanh^2 x + 4\left(\tanh x - \tanh^3 x\right)t.
\end{aligned} \tag{52}$$

Similarly, we obtain the following numerical results: see Figures 2(a), 2(b), 2(c), 2(d), 3(a), 3(b), 3(c), 3(d), 4(a), 4(b), 4(c), and 4(d).

4.3. The Time-Space Fractional Coupled Shallow Water System. Shallow water systems are widely used in many areas of fluid dynamics, such as multiphase flows [36], turbulence [37], and viscoelasticity [38]. It is well known that the shallow water systems can accurately predict both the hydraulic parameters under conditions of slow erosion and low sediment concentration. Let us consider the time-space fractional coupled shallow water system

$$\begin{aligned}
D_t^{\alpha}U + UD_x^{\beta}U + D_x^{\gamma}V + aD_x^{2\delta}U &= 0, \\
D_t^{\alpha}V + VD_x^{\lambda}U + UD_x^{\tau}V - aD_x^{2\theta}V + bD_x^{3\sigma}U &= 0
\end{aligned} \tag{53}$$

with initial values

$$\begin{aligned}
U(x,0) &= a_0(x), \\
V(x,0) &= b_0(x),
\end{aligned} \tag{54}$$

where $0 < \alpha, \beta, \gamma, \lambda, \tau \le 1$, $1/2 < \delta, \theta \le 1$, $2/3 < \sigma \le 1$, $U = U(x,t)$, $V = V(x,t)$, $(x,t) \in \mathbb{R} \times [0, 1/3]$.

Applying the Sumudu transform on both sides of (53) with the initial conditions, we obtain

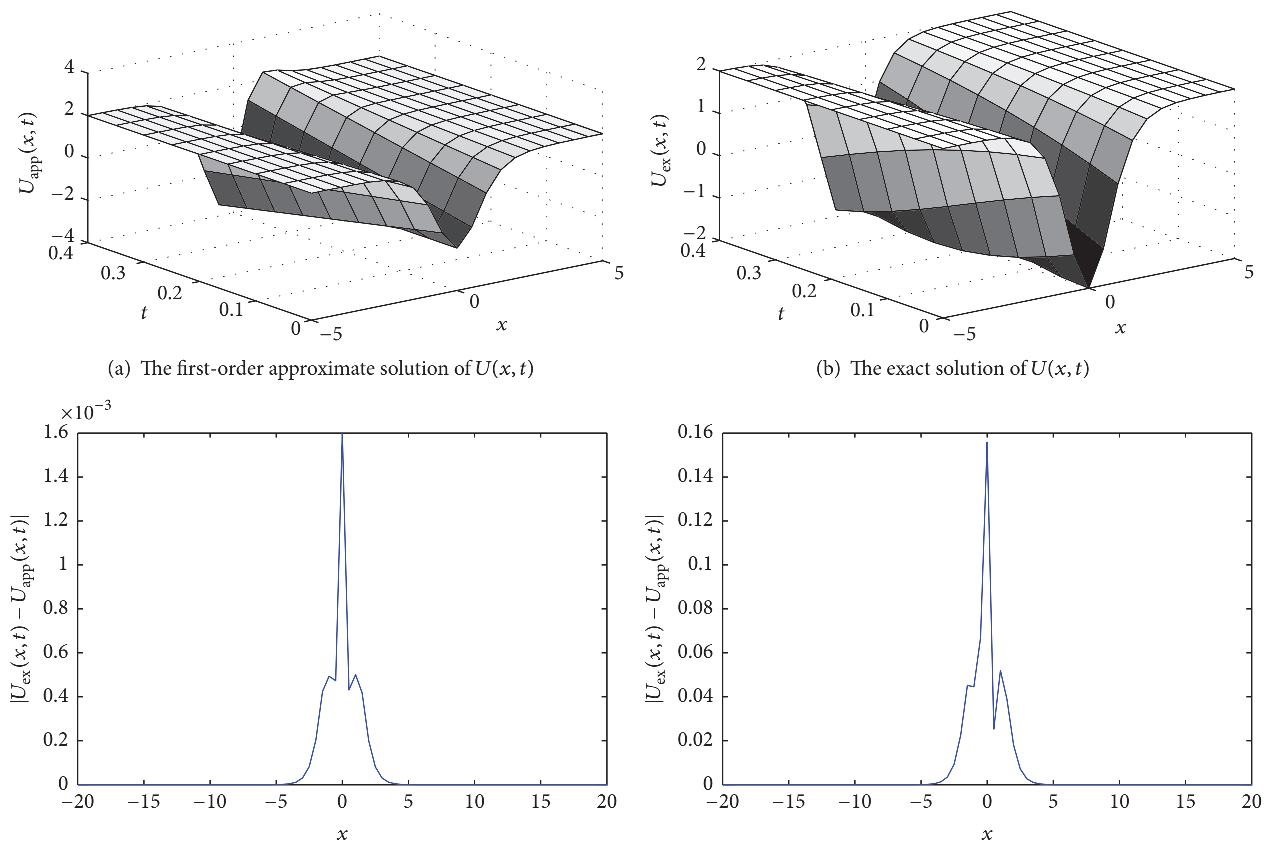
$$\begin{aligned}
S[U(x,t)] &= a_0(x) + u^{\alpha}\left(S\left[-U(x,t)D_x^{\beta}U(x,t)\right.\right. \\
&\left.\left.- D_x^{\gamma}V(x,t) - aD_x^{2\delta}U(x,t)\right]\right), \\
S[V(x,t)] &= b_0(x) + v^{\alpha}\left(S\left[-V(x,t)D_x^{\lambda}U(x,t)\right.\right. \\
&- U(x,t)D_x^{\tau}V(x,t) + aD_x^{2\theta}V(x,t) \\
&\left.\left.- bD_x^{3\sigma}U(x,t)\right]\right).
\end{aligned} \tag{55}$$

The inverse Sumudu transform of (55) implies that

$$\begin{aligned}
U(x,t) &= a_0(x) + S^{-1}\left[u^{\alpha}\left(S\left[-U(x,t)D_x^{\beta}U(x,t)\right.\right.\right. \\
&\left.\left.\left.- D_x^{\gamma}V(x,t) - aD_x^{2\delta}U(x,t)\right]\right)\right], \\
V(x,t) &= b_0(x) + S^{-1}\left[v^{\alpha}\left(S\left[-V(x,t)D_x^{\lambda}U(x,t)\right.\right.\right. \\
&- U(x,t)D_x^{\tau}V(x,t) + aD_x^{2\theta}V(x,t) \\
&\left.\left.\left.- bD_x^{3\sigma}U(x,t)\right]\right)\right].
\end{aligned} \tag{56}$$

According to homotopy perturbation method, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}p^nU_n(x,t) = a_0(x) + p \\
&\quad\times S^{-1}\left[u^{\alpha}S\left[-D_x^{\gamma}\left(\sum_{n=0}^{\infty}p^nV_n(x,t)\right)\right.\right.
\end{aligned}$$



(a) The first-order approximate solution of $U(x,t)$

(b) The exact solution of $U(x,t)$

(c) The error $|U_{app} - U_{ex}|$ of $U(x,t)$ at $t = 0.01$

(d) The error $|U_{app} - U_{ex}|$ of $U(x,t)$ at $t = 0.1$

FIGURE 2

$$
\begin{aligned}
&- aD_x^{2\delta}\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right) - \sum_{n=0}^{\infty} p^n H_n^U(x,t)\Bigg]\Bigg], \\
&\sum_{n=0}^{\infty} p^n V_n(x,t) = b_0(x) + p \\
&\quad \times S^{-1}\left[v^{\alpha} S\left[aD_x^{2\theta}\left(\sum_{n=0}^{\infty} p^n V_n(x,t)\right)\right.\right. \\
&\quad \left.\left. - bD_x^{3\sigma}\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right) - \sum_{n=0}^{\infty} p^n H_n^V(x,t)\right]\right],
\end{aligned}
\tag{57}
$$

where $H_n^U(x,t)$ and $H_n^V(x,t)$ are polynomials of the nonlinear term and are given by

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n H_n^U(x,t) &= U(x,t) D_x^{\beta} U(x,t), \\
\sum_{n=0}^{\infty} p^n H_n^V(x,t) &= V(x,t) D_x^{\lambda} U(x,t) \\
&\quad + U(x,t) D_x^{\tau} V(x,t).
\end{aligned}
\tag{58}
$$

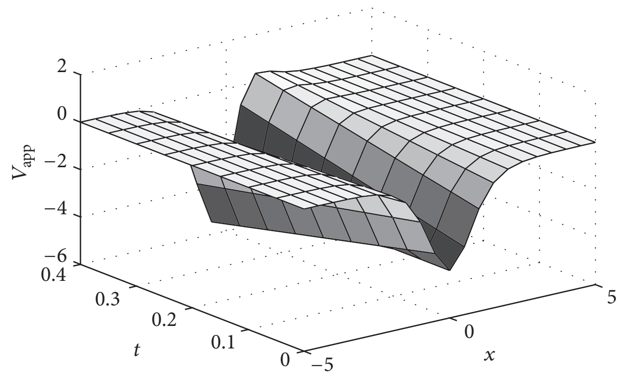
Setting

$$
\begin{aligned}
U_n(x,t) &:= \widetilde{U}_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \\
V_n(x,t) &:= \widetilde{V}_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)},
\end{aligned}
\tag{59}
$$

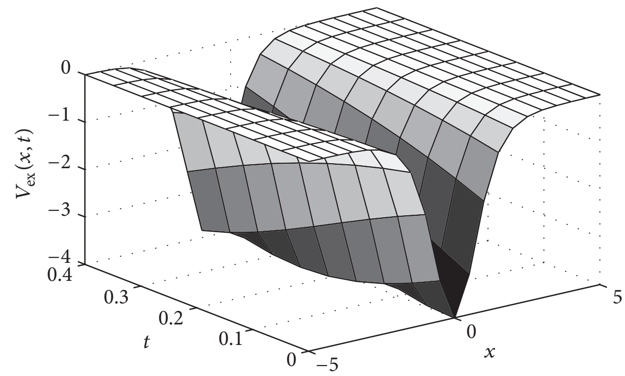
then we arrive at

$$
\begin{aligned}
&H_n^U(x,t) \\
&\quad = \sum_{i=0}^{n} \widetilde{U}_i(x) D_x^{\beta} \widetilde{U}_{n-i}(x) \frac{t^{n\alpha}}{\Gamma(i\alpha + 1)\Gamma((n-i)\alpha + 1)}, \\
&H_n^V(x,t) \\
&\quad = \left(\sum_{i=0}^{n} \widetilde{V}_i(x) D_x^{\lambda} \widetilde{U}_{n-i}(x) + \sum_{i=0}^{n} \widetilde{U}_i(x) D_x^{\tau} \widetilde{V}_{n-i}(x)\right) \\
&\quad \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha + 1)\Gamma((n-i)\alpha + 1)}.
\end{aligned}
\tag{60}
$$

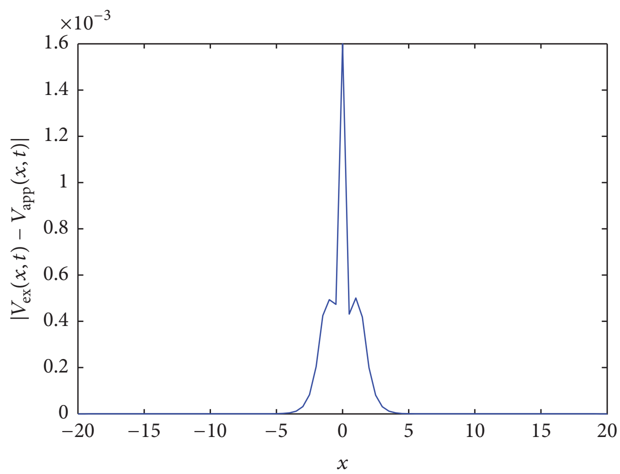
Comparing the coefficients of p yields



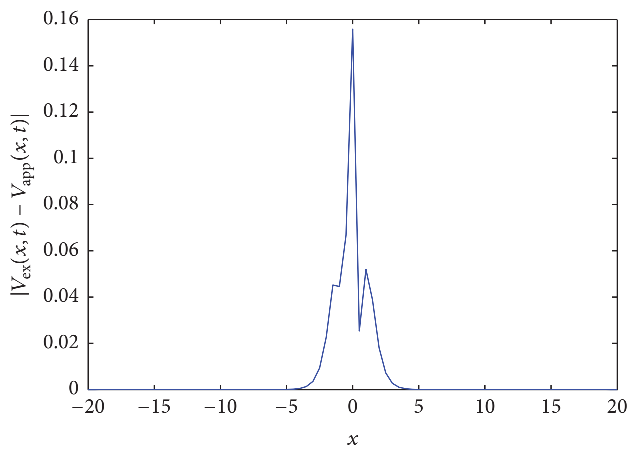
(a) The first-order approximate solution of $V(x,t)$



(b) The exact solution of $V(x,t)$



(c) The error $|V_{app} - V_{ex}|$ of $V(x,t)$ at $t = 0.01$



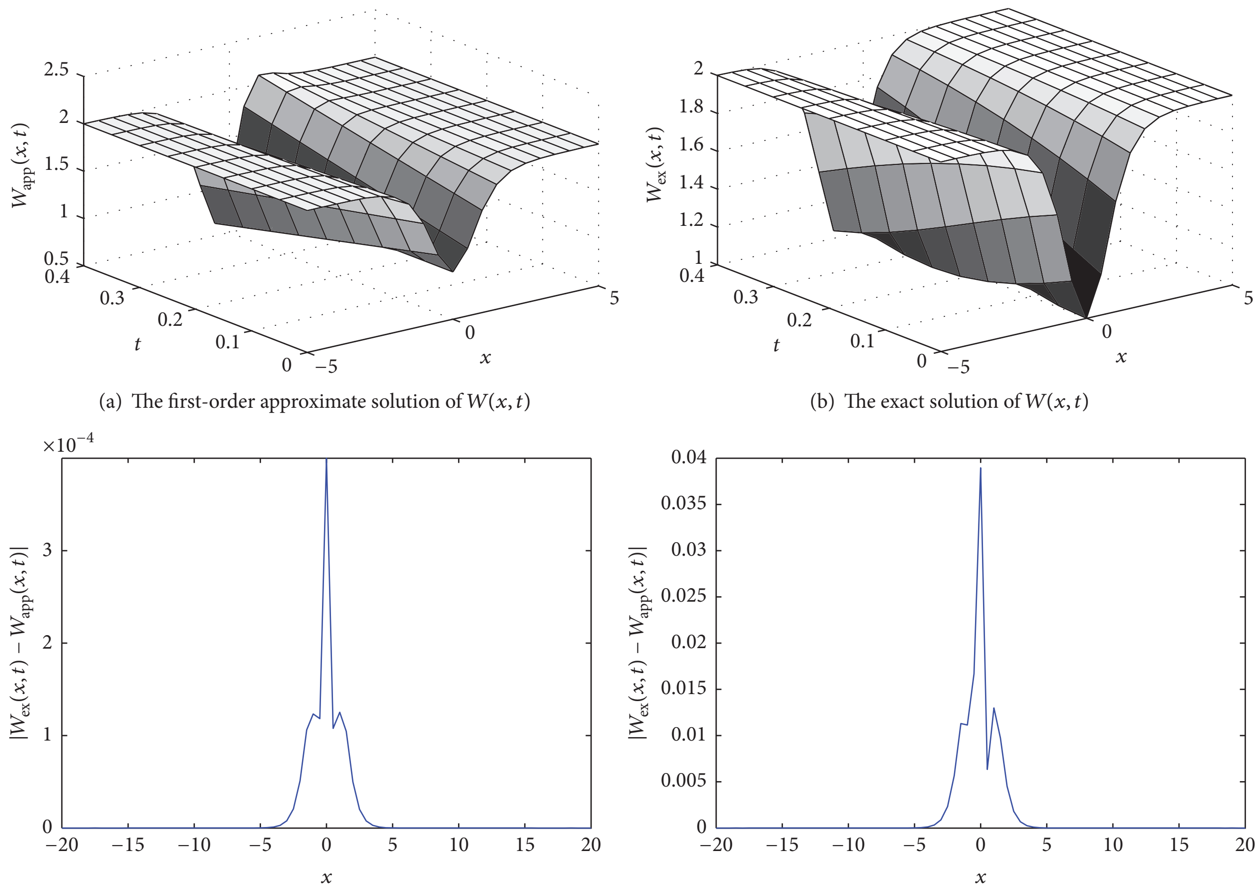
(d) The error $|V_{app} - V_{ex}|$ of $V(x,t)$ at $t = 0.1$

FIGURE 3

$$\begin{aligned}
p^0:\quad & U_0(x,t) = \widetilde{U}_0(x) = a_0(x), \\
& V_0(x,t) = \widetilde{V}_0(x) = b_0(x), \\
p^1:\quad & U_1(x,t) = S^{-1}\left[u^\alpha S\left[-D_x^\gamma V_0(x,t) - aD_x^{2\delta}U_0(x,t) - H_0^U(x)\right]\right] \\
& = S^{-1}\left[u^\alpha S\left[-D_x^\gamma \widetilde{U}_0(x) - aD_x^{2\delta}\widetilde{U}_0(x) - \widetilde{U}_0(x)D_x^\beta \widetilde{U}_0(x)\right]\right] \\
& = \frac{t^\alpha}{\Gamma(\alpha+1)}\left(-D_x^\gamma \widetilde{U}_0(x) - aD_x^{2\delta}\widetilde{U}_0(x) - \widetilde{U}_0(x)D_x^\beta \widetilde{U}_0(x)\right) = \frac{t^\alpha}{\Gamma(\alpha+1)}\widetilde{U}_1(x), \\
& V_1(x,t) = S^{-1}\left[v^\alpha S\left[-bD_x^{3\sigma}V_0(x,t) + aD_x^{2\theta}V_0(x,t) - H_0^V(x)\right]\right] \\
& = S^{-1}\left[v^\alpha S\left[-bD_x^{3\sigma}\widetilde{V}_0(x) + aD_x^{2\theta}\widetilde{V}_0(x) - \widetilde{V}_0(x)D_x^\lambda \widetilde{U}_0(x) - \widetilde{U}_0(x)D_x^\tau \widetilde{V}_0(x)\right]\right] \\
& = \frac{t^\alpha}{\Gamma(\alpha+1)}\left(-bD_x^{3\sigma}\widetilde{V}_0(x) + aD_x^{2\theta}\widetilde{V}_0(x) - \widetilde{V}_0(x)D_x^\lambda \widetilde{U}_0(x) - \widetilde{U}_0(x)D_x^\tau \widetilde{V}_0(x)\right) = \frac{t^\alpha}{\Gamma(\alpha+1)}\widetilde{V}_1(x).
\end{aligned} \tag{61}$$

Generally, we have

$$\begin{aligned}
p^k:\quad U_k(x,t) &= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{-D_x^\gamma \widetilde{V}_{k-1}(x) - aD_x^{2\delta}\widetilde{U}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\widetilde{U}_i(x)D_x^\beta \widetilde{U}_{k-1-i}(x)\right\} \\
&= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{U}_k(x),
\end{aligned}$$



(a) The first-order approximate solution of $W(x,t)$

(b) The exact solution of $W(x,t)$

(c) The error $|W_{\mathrm{app}} - W_{\mathrm{ex}}|$ of $W(x,t)$ at $t = 0.01$

(d) The error $|W_{\mathrm{app}} - W_{\mathrm{ex}}|$ of $W(x,t)$ at $t = 0.1$

FIGURE 4

$$\begin{aligned} &V_k(x,t) \\ &= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{-bD_x^{3\sigma}\widetilde{V}_{k-1}(x) + aD_x^{2\theta}\widetilde{V}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(\widetilde{U}_i(x)D_x^{\lambda}\widetilde{V}_{k-1-i}(x) + \widetilde{V}_i(x)D_x^{\tau}\widetilde{U}_{k-1-i}(x)\right)\right\} \\ &= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{V}_k(x), \end{aligned} \tag{62}$$

where

$$\begin{aligned} \widetilde{U}_k(x) &= -D_x^{\gamma}\widetilde{V}_{k-1}(x) - aD_x^{2\delta}\widetilde{U}_{k-1}(x) \\ &\quad - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\widetilde{U}_i(x) \\ &\quad \cdot D_x^{\beta}\widetilde{U}_{k-1-i}(x), \\ \widetilde{V}_k(x) &= -bD_x^{3\sigma}\widetilde{V}_{k-1}(x) + aD_x^{2\theta}\widetilde{V}_{k-1}(x) \\ &\quad - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(\widetilde{U}_i(x)\right. \\ &\quad \left.\cdot D_x^{\lambda}\widetilde{V}_{k-1-i}(x) + \widetilde{V}_i(x)D_x^{\tau}\widetilde{U}_{k-1-i}(x)\right). \end{aligned} \tag{63}$$

Hence, the series solution is

$$\begin{aligned} U(x,t) &= \sum_{k=0}^{\infty}U_k(x,t) = \sum_{k=0}^{\infty}\widetilde{U}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ V(x,t) &= \sum_{k=0}^{\infty}V_k(x,t) = \sum_{k=0}^{\infty}\widetilde{V}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned} \tag{64}$$

4.4. The Time-Space Fractional Coupled KdV System. KdV equation plays an important role in nonlinear equations for wide applications in physics and engineering. Hirota and Satsuma [39] firstly found coupled KdV system to describe the iterations of water waves; meanwhile, they claimed that the system exists with a soliton solution. In [40], Fan and

Zhang settled several kinds of solutions by an improved homogeneous method. The time-space fractional coupled KdV equation is a generalization of the classical coupled KdV equation. In this subsection, we consider the following time-space fractional coupled KdV system:

$$\begin{aligned} D_t^{\alpha}U - aD_x^{3\beta}U - 6aUD_x^{\gamma}U - 2bVD_x^{\delta}V &= 0, \\ D_t^{\alpha}V + D_x^{3\lambda}V + 3UD_x^{\tau}V &= 0, \end{aligned} \tag{65}$$

with respect to initial values

$$\begin{aligned} U(x,0) &= a_0(x), \\ V(x,0) &= b_0(x), \end{aligned} \tag{66}$$

where $0 < \alpha, \gamma, \delta, \tau \le 1$, $2/3 < \beta, \lambda \le 1$, $(x,t) \in \mathbb{R} \times [0,\infty)$, and the coefficients a, b are constants.

Applying the Sumudu transform on both sides of (65) with the initial conditions, we obtain

$$\begin{aligned} S[U(x,t)] &= a_0(x) + u^{\alpha}\left(S\left[aD_x^{3\beta}U(x,t)\right.\right. \\ &\quad \left.\left.+ 6aU(x,t)D_x^{\gamma}U(x,t) + 2bV(x,t)D_x^{\delta}V(x,t)\right]\right), \\ S[V(x,t)] &= b_0(x) - v^{\alpha}\left(S\left[D_x^{3\lambda}V(x,t)\right.\right. \\ &\quad \left.\left.+ 3U(x,t)D_x^{\tau}V(x,t)\right]\right). \end{aligned} \tag{67}$$

The inverse Sumudu transform of (67) implies that

$$\begin{aligned} U(x,t) &= a_0(x) + S^{-1}\left[u^{\alpha}\left(S\left[aD_x^{3\beta}U(x,t)\right.\right.\right. \\ &\quad \left.\left.\left.+ 6aU(x,t)D_x^{\gamma}U(x,t) + 2bV(x,t)D_x^{\delta}V(x,t)\right]\right)\right], \\ V(x,t) &= b_0(x) - S^{-1}\left[v^{\alpha}\left(S\left[D_x^{3\lambda}V(x,t)\right.\right.\right. \\ &\quad \left.\left.\left.+ 3U(x,t)D_x^{\tau}V(x,t)\right]\right)\right]. \end{aligned} \tag{68}$$

Analogously, using homotopy perturbation method gives

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,t) &= a_0(x) + p \\ &\quad \times S^{-1}\left[u^{\alpha}S\left[aD_x^{3\beta}\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)\right.\right. \\ &\quad \left.\left.+ \sum_{n=0}^{\infty} p^n H_n^U(x,t)\right]\right], \\ \sum_{n=0}^{\infty} p^n V_n(x,t) &= b_0(x) + p \\ &\quad \times S^{-1}\left[v^{\alpha}S\left[-D_x^{3\lambda}\left(\sum_{n=0}^{\infty} p^n V_n(x,t)\right)\right.\right. \\ &\quad \left.\left.+ \sum_{n=0}^{\infty} p^n H_n^V(x,t)\right]\right], \end{aligned} \tag{69}$$

where $H_n^U(x,t)$ and $H_n^V(x,t)$ are homotopy polynomials coefficients of the nonlinear term, which are given by

$$\begin{aligned} \sum_{n=0}^{\infty} p^n H_n^U(x,t) &= 6aU(x,t)D_x^{\gamma}U(x,t) \\ &\quad + 2bV(x,t)D_x^{\delta}V(x,t), \\ \sum_{n=0}^{\infty} p^n H_n^V(x,t) &= -3U(x,t)D_x^{\tau}V(x,t). \end{aligned} \tag{70}$$

Setting

$$\begin{aligned} U_n(x,t) &:= \widetilde{U}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ V_n(x,t) &:= \widetilde{V}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{71}$$

then

$$\begin{aligned} H_n^U(x,t) &= 6a\sum_{i=0}^{n}\widetilde{U}_i(x)D_x^{\gamma}\widetilde{U}_{n-i}(x) \\ &\quad \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)} \\ &\quad + 2b\sum_{i=0}^{n}\frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}\widetilde{V}_i(x) \\ &\quad \cdot D_x^{\delta}\widetilde{V}_{n-i}(x), \\ H_n^V(x,t) &= -3\sum_{i=0}^{n}\widetilde{U}_i(x)D_x^{\tau}\widetilde{V}_{n-i}(x) \\ &\quad \cdot \frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}. \end{aligned} \tag{72}$$

Comparing the coefficients of p,

$$\begin{aligned} p^0:\quad & U_0(x,t) = \widetilde{U}_0(x) = a_0(x), \\ & V_0(x,t) = \widetilde{V}_0(x) = b_0(x), \\ p^1:\quad & U_1(x,t) = S^{-1}\left[u^{\alpha}S\left[aD_x^{3\beta}U_0(x,t) + H_0^U(x,t)\right]\right] \\ & \quad = S^{-1}\left[u^{\alpha}S\left[aD_x^{3\beta}\widetilde{U}_0(x) + 6a\widetilde{U}_0(x)D_x^{\gamma}\widetilde{U}_0(x) + 2b\widetilde{V}_0(x)D_x^{\delta}\widetilde{V}_0(x)\right]\right] \end{aligned}$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(aD_x^{3\beta}\widetilde{U}_0(x) + 6a\widetilde{U}_0(x)D_x^{\gamma}\widetilde{U}_0(x) + 2b\widetilde{V}_0(x)D_x^{\delta}\widetilde{V}_0(x)\right) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\widetilde{U}_1(x),$$
$$V_1(x,t) = S^{-1}\left[v^{\alpha}S\left[-D_x^{3\lambda}V_0(x,t) + H_0^V(x,t)\right]\right] = S^{-1}\left[v^{\alpha}S\left[-D_x^{3\lambda}\widetilde{V}_0(x) - 3\widetilde{U}_0D_x^{\tau}\widetilde{V}_0(x)\right]\right]$$
$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(-D_x^{3\lambda}\widetilde{V}_0(x) - 3\widetilde{U}_0D_x^{\tau}\widetilde{V}_0(x)\right) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\widetilde{V}_1(x). \tag{73}$$

Generally, we get

$$p^k:\quad U_k(x,t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{aD_x^{3\beta}\widetilde{U}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(6a\widetilde{U}_i(x)D_x^{\gamma}\widetilde{U}_{k-1-i}(x) + 2b\widetilde{V}_i(x)D_x^{\delta}\widetilde{V}_{k-1-i}(x)\right)\right\}$$
$$= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{U}_k(x),$$
$$V_k(x,t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left(-D_x^{3\lambda}\widetilde{V}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}3\widetilde{U}_i(x)D_x^{\tau}\widetilde{V}_{k-1-i}(x)\right) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{V}_k(x), \tag{74}$$

where

$$\widetilde{U}_k(x) = aD_x^{3\beta}\widetilde{U}_{k-1}(x) + \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}6a\widetilde{U}_i(x) \cdot D_x^{\gamma}\widetilde{U}_{k-1-i}(x) + 2b\widetilde{V}_i(x)D_x^{\delta}\widetilde{V}_{k-1-i}(x),$$
$$\widetilde{V}_k(x) = -D_x^{3\lambda}\widetilde{V}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}3\widetilde{U}_i(x) \cdot D_x^{\tau}\widetilde{V}_{k-1-i}(x). \tag{75}$$

Thus, the series solution of (65) is

$$U(x,t) = \sum_{k=0}^{\infty}U_k(x,t) = \sum_{k=0}^{\infty}\widetilde{U}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},$$
$$V(x,t) = \sum_{k=0}^{\infty}V_k(x,t) = \sum_{k=0}^{\infty}\widetilde{V}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{76}$$

4.5. The Time-Space Fractional Coupled Whitham-Broer-Kaup (WBK) System. Under the Boussinesq approximation, Whitham [41], Broer [42], and Kaup [43] obtained the following nonlinear WBK system. In this subsection, we construct the approximate solution by the HPST method to the time-space fractional coupled WBK system.

Consider the time-space fractional coupled WBK system

$$D_t^{\alpha}U + UD_x^{\beta}U + D_x^{\gamma}V + aD_x^{2\delta}U = 0,$$
$$D_t^{\alpha}V + D_x^{\lambda}(UV) - aD_x^{2\tau}V + bD_x^{3\theta}U = 0 \tag{77}$$

with respect to the initial conditions

$$U(x,0) = a_0(x),$$
$$V(x,0) = b_0(x), \tag{78}$$

where $0 < \alpha,\beta,\gamma,\lambda \le 1$, $1/2 < \delta,\tau \le 1$, $2/3 < \theta \le 1$, $(x,t) \in \mathbb{R}\times[0,\infty)$, $a,b \in \mathbb{R}$ denote different dispersive power, $U = U(x,t)$ is the field of horizontal velocity, and $V = V(x,t)$ is the height deviating equilibrium position of liquid.

Applying the Sumudu transform on both sides of (77) with the initial conditions, we obtain

$$S[U(x,t)] = a_0(x) + u^{\alpha}\left(S\left[-U(x,t)D_x^{\beta}U(x,t) - D_x^{\gamma}V(x,t) - aD_x^{2\delta}U(x,t)\right]\right),$$
$$S[V(x,t)] = b_0(x) + v^{\alpha}\left(S\left[-D_x^{\lambda}(U(x,t)V(x,t)) + aD_x^{2\tau}V(x,t) - bD_x^{3\theta}U(x,t)\right]\right). \tag{79}$$

The inverse Sumudu transform of (79) implies that

$$U(x,t) = a_0(x) + S^{-1}\left[u^{\alpha}\left(S\left[-U(x,t)D_x^{\beta}U(x,t) - D_x^{\gamma}V(x,t) - aD_x^{2\delta}U(x,t)\right]\right)\right],$$

$$V(x,t) = b_0(x) + S^{-1}\left[v^{\alpha}\left(S\left[-D_x^{\lambda}(U(x,t)V(x,t)) + aD_x^{2\tau}V(x,t) - bD_x^{3\theta}U(x,t)\right]\right)\right]. \tag{80}$$

Using homotopy perturbation method, it leads to

$$\begin{aligned}\sum_{n=0}^{\infty}p^nU_n(x,t) &= a_0(x) + p \\ &\times S^{-1}\left[u^{\alpha}S\left[-D_x^{\gamma}\left(\sum_{n=0}^{\infty}p^nV_n(x,t)\right)\right.\right. \\ &\left.\left. - aD_x^{2\delta}\left(\sum_{n=0}^{\infty}p^nU_n(x,t)\right) - \sum_{n=0}^{\infty}p^nH_n^U(x,t)\right]\right], \\ \sum_{n=0}^{\infty}p^nV_n(x,t) &= b_0(x) + p \\ &\times S^{-1}\left[v^{\alpha}S\left[aD_x^{2\tau}\left(\sum_{n=0}^{\infty}p^nV_n(x,t)\right)\right.\right. \\ &\left.\left. - bD_x^{3\theta}\left(\sum_{n=0}^{\infty}p^nU_n(x,t)\right) - \sum_{n=0}^{\infty}p^nH_n^V(x,t)\right]\right],\end{aligned} \tag{81}$$

where $H_n^U(x,t)$ and $H_n^V(x,t)$ are polynomials of the nonlinear term and are given by

$$\begin{aligned}\sum_{n=0}^{\infty}p^nH_n^U(x,t) &= U(x,t)D_x^{\beta}U(x,t), \\ \sum_{n=0}^{\infty}p^nH_n^V(x,t) &= D_x^{\lambda}(V(x,t)U(x,t)).\end{aligned} \tag{82}$$

Setting

$$\begin{aligned}U_n(x,t) &:= \widetilde{U}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ V_n(x,t) &:= \widetilde{V}_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\end{aligned} \tag{83}$$

then

$$\begin{aligned}&H_n^U(x,t) \\ &\quad = \sum_{i=0}^{n}\widetilde{U}_i(x)D_x^{\beta}\widetilde{U}_{n-i}(x)\frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}, \\ &H_n^V(x,t) \\ &\quad = D_x^{\lambda}\sum_{i=0}^{n}\widetilde{U}_i(x)\widetilde{V}_{n-i}(x)\frac{t^{n\alpha}}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}.\end{aligned} \tag{84}$$

Comparing the coefficients of p, this gives

$$\begin{aligned}p^0:\quad & U_0(x,t) = \widetilde{U}_0(x) = a_0(x), \\ & V_0(x,t) = \widetilde{V}_0(x) = b_0(x), \\ p^1:\quad & U_1(x,t) = S^{-1}\left[u^{\alpha}S\left[-D_x^{\gamma}V_0(x,t) - aD_x^{2\delta}U_0(x,t) - H_0^U(x,t)\right]\right] \\ &= S^{-1}\left[u^{\alpha}S\left[-D_x^{\gamma}\widetilde{V}_0(x) - aD_x^{2\delta}\widetilde{U}_0(x) - \widetilde{U}_0(x)D_x^{\beta}\widetilde{U}_0(x)\right]\right] \\ &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(-D_x^{\gamma}\widetilde{V}_0(x) - aD_x^{2\delta}\widetilde{U}_0(x) - \widetilde{U}_0(x)D_x^{\beta}\widetilde{U}_0(x)\right) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\widetilde{U}_1(x), \\ & V_1(x,t) = S^{-1}\left[v^{\alpha}S\left[aD_x^{2\tau}V_0(x,t) - bD_x^{3\theta}U_0(x,t) - H_0^V(x,t)\right]\right] \\ &= S^{-1}\left[v^{\alpha}S\left[aD_x^{2\tau}\widetilde{V}_0(x) - bD_x^{3\theta}\widetilde{U}_0(x) - D_x^{\lambda}\left(\widetilde{U}_0(x)\widetilde{V}_0(x)\right)\right]\right] \\ &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(aD_x^{2\tau}\widetilde{V}_0(x) - bD_x^{3\theta}\widetilde{U}_0(x) - D_x^{\lambda}\left(\widetilde{U}_0(x)\widetilde{V}_0(x)\right)\right) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\widetilde{V}_1(x).\end{aligned} \tag{85}$$

Generally, we have

$$\begin{aligned}p^k:\quad U_k(x,t) &= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{-D_x^{\gamma}\widetilde{V}_{k-1}(x) - aD_x^{2\delta}\widetilde{U}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\widetilde{U}_i(x)D_x^{\beta}\widetilde{U}_{k-1-i}(x)\right\} \\ &= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{U}_k(x),\end{aligned}$$

$$V_k(x,t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left\{aD_x^{2\tau}\widetilde{V}_{k-1}(x) - bD_x^{3\theta}\widetilde{U}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}D_x^{\lambda}\left(\widetilde{U}_i(x)\widetilde{V}_{k-1-i}(x)\right)\right\}$$
$$= \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\widetilde{V}_k(x), \tag{86}$$

where

$$\widetilde{U}_k(x) = -D_x^{\gamma}\widetilde{V}_{k-1}(x) - aD_x^{2\delta}\widetilde{U}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\widetilde{U}_i(x) \cdot D_x^{\beta}\widetilde{U}_{k-1-i}(x),$$
$$\widetilde{V}_k(x) = aD_x^{2\tau}\widetilde{V}_{k-1}(x) - bD_x^{3\theta}\widetilde{U}_{k-1}(x) - \sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}D_x^{\lambda}\left(\widetilde{U}_i(x) \cdot \widetilde{V}_{k-1-i}(x)\right). \tag{87}$$

Hence, the series solution of (77) is

$$U(x,t) = \sum_{k=0}^{\infty}U_k(x,t) = \sum_{k=0}^{\infty}\widetilde{U}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},$$
$$V(x,t) = \sum_{k=0}^{\infty}V_k(x,t) = \sum_{k=0}^{\infty}\widetilde{V}_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{88}$$

Remark 10. When alpha $= \tau = \delta = \lambda = \theta = 1$, $\beta \neq 0$, and $\gamma = 0$, (77) reduces to the classical long-wave system that describes the shallow water wave with diffusion.

Remark 11. When $\alpha = \tau = \delta = \lambda = \theta = 1$, $\beta = 0$, and $\gamma = 1$, (77) reduces to the variant Boussinesq system.

5. Concluding Remarks

In this paper, we apply the HPSTM to the nonlinear time-space fractional coupled equations. Applying the HPSTM, we can obtain analytic and approximate solutions to different coupled systems, for example, the coupled Burgers system, the coupled KdV system, the generalized Hirota-Satsuma coupled KdV system, the coupled WBK system, and the coupled shallow water system. The advantage of the HPSTM is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear system. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation, or restrictive assumptions. The numerical results indicate that this method is effective and simple in constructing analytic or approximate solutions to fractional coupled systems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Numerical Simulation of One-Dimensional Fractional Nonsteady Heat Transfer Model Based on the Second Kind Chebyshev Wavelet

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In the current study, a numerical technique for solving one-dimensional fractional nonsteady heat transfer model is presented. We construct the second kind Chebyshev wavelet and then derive the operational matrix of fractional-order integration. The operational matrix of fractional-order integration is utilized to reduce the original problem to a system of linear algebraic equations, and then the numerical solutions obtained by our method are compared with those obtained by CAS wavelet method. Lastly, illustrated examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

Fractional calculus is a branch of mathematics that deals with generalization of the well-known operations of differentiations to arbitrary orders. Many papers on fractional calculus have been published for the real-world applications in science and engineering such as viscoelasticity [1], bioengineering [2], biology [3], and more can be found in [4, 5]. Moreover fractional partial differential equations also are widely used in the areas of signal processing [6], mechanics [7], econometrics [8], fluid dynamics [9], and electromagnetics [10]. As the analytical solutions of fractional partial differential equations are not easy to derive, the scholars are committed to obtain their numerical solutions of these equations.

In recent years, various numerical methods have been proposed for solving fractional diffusion equations, these methods include wavelets methods [11–17], Jacobi, Legendre, and Chebyshev polynomials methods [18–21], spectral methods [22, 23], finite element method [24], wavelet Galerkin method [25], and finite difference methods [26, 27]. In [28], a new matrix method is proposed to solve two-dimensional time-dependent diffusion equations with Dirichlet boundary conditions. In [29], the authors utilize the second kind Chebyshev wavelets to obtain the numerical solutions of the convection diffusion equations. Xie et al. use the Chebyshev operational matrix method to numerically solve one-dimensional fractional convection diffusion equations in [30]. In this paper, we apply the second kind Chebyshev wavelet method to obtain the numerical solutions of one-dimensional fractional nonsteady heat transfer model. The obtained numerical solutions by our method have been compared with those obtained by CAS wavelet method.

The current paper is organized as follows: Section 2 introduces the basic definitions of fractional calculus. In Section 3, the mathematical model of nonsteady heat transfer problem is proposed. Section 4 illustrates the second kind Chebyshev wavelets and their properties. In Section 5, we apply the second kind Chebyshev wavelet for solving fractional nonsteady heat transfer model. Numerical examples are presented to test the proposed method in Section 6. Finally, a conclusion is drawn in Section 7.

2. One-Dimensional Nonsteady Heat Transfer Model

For one infinite plate sample, as shown in Figure 1, the height is δ, the upper surface and the edge are adiabatic,

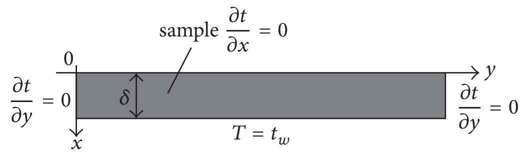

FIGURE 1: Nonsteady heat transfer model with constant temperature boundary condition.

and the lower surface is contacted with the fluid, which its temperature is t_w. The heat conductivity coefficient of the sample is λ, the density is ρ, and the specific heat capacity is c_p. The initial temperature is t_0, taking the origin of coordinates on the sample adiabatic surfaces, and the nonsteady heat transfer model with the initial-boundary condition can be defined as follows [31]:

$$\begin{aligned} \frac{\partial t}{\partial \tau} &= \frac{\lambda \partial^2 t}{\rho c_p \partial x^2}, \\ \tau &= 0, \\ t &= t_0, \\ x &= 0, \\ \frac{\partial t}{\partial x} &= 0, \\ x &= \sigma, \\ t &= t_w. \end{aligned} \tag{1}$$

Obviously, when the sample density ρ, heat conductivity coefficient λ, specific heat capacity c_p, and thickness δ are known, we can obtain the temperature distribution at any position x and any time τ, which is the nonsteady heat conduction model with constant temperature boundary condition. Based on the above-mentioned model, we give the fractional-order nonsteady heat transfer model of the following form:

$$\frac{\partial T}{\partial t} = \frac{\lambda \partial^\alpha T}{\rho c_p \partial x^\alpha} + g(x,t), \quad 0 \le x \le 1,\ t \ge 0,\ 1 < \alpha \le 2, \tag{2}$$

with the initial condition:

$$T(x,0) = f(x), \quad 0 \le x \le 1, \tag{3}$$

and the boundary conditions:

$$\begin{aligned} T(0,t) &= g_0(t), \\ T(1,t) &= g_1(t), \\ &0 \le t \le 1, \end{aligned} \tag{4}$$

where $g(x,t)$ denotes source term, $f(x)$ is a given function, and $g_0(t)$, $g_1(t)$ are continuous functions with first-order derivative.

3. Preliminaries of the Fractional Calculus

In this section, we give some necessary definitions and mathematical preliminaries on fractional calculus which will be used further in this paper.

Definition 1. The Riemann-Liouville fractional integral operator I^α $(\alpha > 0)$ of a function $f(t)$ is defined as follows [4]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0,\ \alpha \in \mathfrak{R}^+. \tag{5}$$

Some properties of the operator I^α are as follows:

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \quad (\alpha > 0,\ \beta > 0), \tag{6}$$

$$I^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}, \quad (\gamma > -1). \tag{7}$$

Definition 2. The Caputo fractional derivative ${}_0D_t^\alpha$ of a function $f(t)$ is defined as follows [4]:

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{n-\alpha+1}}\, d\tau, \quad (n-1 < \alpha \le n,\ n \in N). \tag{8}$$

Some properties of the Caputo fractional derivative are as follows:

$$\begin{aligned} {}_0D_t^\alpha t^\beta &= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0 < \alpha < \beta+1,\ \beta > -1, \\ I^\alpha D^\alpha f(t) &= f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad n-1 < \alpha \le n,\ n \in N. \end{aligned} \tag{9}$$

4. The Second Kind Chebyshev Wavelet and Its Operational Matrix of Fractional Integration

4.1. The Second Kind Chebyshev Wavelet and Its Properties.

The second kind Chebyshev wavelet $\psi_{nm}(t) = \psi(k,n,m,t)$ has four arguments, $n = 1,2,\ldots,2^{k-1}$, $k \in N^*$. They are defined on the interval $[0,1)$ as follows [19]:

$$\psi_{nm}(t) = \begin{cases} 2^{k/2}\widetilde{U}_m\left(2^k t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & \text{o.w.} \end{cases} \tag{10}$$

with

$$\widetilde{U}_m(t) = \sqrt{\frac{2}{\pi}} U_m(t), \quad m = 0,1,2,\ldots,M-1. \tag{11}$$

Here $U_m(t)$ are the second kind Chebyshev polynomials which are orthogonal with respect to the weight function $w(t) = \sqrt{1-t^2}$ and satisfy the following recursive formula:

$$\begin{aligned} U_0(t) &= 1, \\ U_1(t) &= 2t, \\ U_{m+1}(t) &= 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \ldots. \end{aligned} \tag{12}$$

A function $f(t)$ defined over $[0,1)$ may be expanded in terms of the second kind Chebyshev wavelet as follows:

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T \Psi(t), \tag{13}$$

where

$$c_{nm} = \left(f(t), \psi_{nm}(t)\right)_{\omega_n} = \int_0^1 \omega_n(t)\psi_{nm}(t)\,\mathrm{d}t, \tag{14}$$

and the weight function $w_n(t) = w(2^k t - 2n + 1)$. Moreover, C and $\Psi(t)$ are $\widehat{m} = (2^{k-1}M)$ column vectors given by

$$\begin{aligned} C &= \left[c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, c_{21}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}\right]^T, \\ \Psi(t) &= \left[\psi_{10}, \psi_{11}, \ldots, \psi_{1(M-1)}, \psi_{20}, \psi_{21}, \ldots, \psi_{2(M-1)}, \ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1}(M-1)}\right]^T. \end{aligned} \tag{15}$$

Take the collocation points as follows:

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M, \ \widehat{m} = 2^{k-1}M. \tag{16}$$

We define the second kind Chebyshev wavelet matrix $\Phi_{\widehat{m}\times\widehat{m}}$ as

$$\Phi_{\widehat{m}\times\widehat{m}} = \left[\Psi\left(\frac{1}{2\widehat{m}}\right), \Psi\left(\frac{3}{2\widehat{m}}\right), \ldots, \Psi\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right)\right]. \tag{17}$$

An arbitrary function of two variables $T(x,t)$ defined over $[0,1)\times[0,1)$ may be expanded into Chebyshev wavelets basis as follows:

$$T(x,t) \simeq \sum_{i=1}^{\widehat{m}} \sum_{j=1}^{\widehat{m}} d_{ij}\psi_i(x)\psi_j(t) = \Psi^T(x) D \Psi(t), \tag{18}$$

where $D = [d_{ij}]_{\widehat{m}\times\widehat{m}}$ and $d_{ij} = (\psi_i(x), (T(x,t), \psi_j(t)))$.

The following theorem discusses the convergence and accuracy estimation of the proposed method.

Theorem 3. *Let $f(t)$ be a second-order derivative square-integrable function defined over $[0,1)$ with bounded second-order derivative, satisfying $|f''(t)| \le B$ for some constants B; then*

(1) *$f(t)$ can be expanded as an infinite sum of the second kind Chebyshev wavelets and the series converge to $f(t)$ uniformly, that is,*

$$f(t) = \sum_{n=0}^{\infty} \sum_{m\in Z}^{\infty} c_{nm}\psi_{nm}(t), \tag{19}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t)\rangle_{L^2_\omega[0,1)}$.

(2)

$$\sigma_{f,k,M} < \frac{\sqrt{\pi}B}{2^3}\left(\sum_{n=2^{k-1}+1}^{\infty} \frac{1}{n^5} \sum_{m=M}^{\infty} \frac{1}{(m-1)^4}\right)^{1/2}, \tag{20}$$

where $\sigma_{f,k,M} = (\int_0^1 |f(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t)|^2 \omega_n(t) dt)^{1/2}$.

4.2. Operational Matrix of Fractional Integration. On the interval $[0,1)$, we defined a $\widehat{m}$ – set of block-pulse functions (BPFs) as

$$b_i(t) = \begin{cases} 1, & \dfrac{i}{\widehat{m}} \le t < \dfrac{i+1}{\widehat{m}}, \\ 0, & \text{o.w.} \end{cases} \quad i = 0, 1, 2, \ldots, \widehat{m}-1. \tag{21}$$

The functions $\{b_i(t)\}$ are disjoint and orthogonal:

$$\begin{aligned} b_i(t)\, b_j(t) &= \begin{cases} 0, & i \ne j, \\ b_i(t), & i = j, \end{cases} \\ \int_0^1 b_i(s)\, b_j(s)\,\mathrm{d}s &= \begin{cases} 0, & i \ne j, \\ \dfrac{1}{m}, & i = j. \end{cases} \end{aligned} \tag{22}$$

Similarly, the second kind Chebyshev wavelet may be expanded into an $\widehat{m}$-term block-pulse functions as

$$\Psi(t) = \Phi_{\widehat{m}\times\widehat{m}} B_{\widehat{m}}(t). \tag{23}$$

Kilicman has given the block-pulse functions operational matrix of fractional integration F^α of following form:

$$\left(I^\alpha B_{\widehat{m}}\right)(t) \approx F^\alpha B_{\widehat{m}}(t), \tag{24}$$

where

$$\begin{aligned} B_{\widehat{m}}(t) &= \left[b_0(t), b_1(t), \ldots, b_{\widehat{m}-1}(t)\right]^T, \\ F^\alpha &= \frac{1}{\widehat{m}^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{\widehat{m}-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{\widehat{m}-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{\widehat{m}-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \end{aligned} \tag{25}$$

Next, we derive the second kind Chebyshev wavelet operational matrix of fractional integration. Let

$$\left(I^\alpha \Psi\right)(t) = P^\alpha_{\widehat{m}\times\widehat{m}} \Psi(t), \tag{26}$$

where $P^\alpha_{\widehat{m}\times\widehat{m}}$ is called the second kind Chebyshev wavelet operational matrix of fractional integration and it can be given by

$$P^\alpha_{\widehat{m}\times\widehat{m}} = \Phi_{\widehat{m}\times\widehat{m}} F^\alpha \Phi^{-1}_{\widehat{m}\times\widehat{m}}. \tag{27}$$

For More details, see [29].

5. Numerical Implementation

In this section, we use the second kind Chebyshev wavelets method for numerically solving the nonsteady fractional-order heat transfer model with initial-boundary conditions. In order to solve this problem, we assume

$$\frac{\partial^3 T}{\partial t \partial x^2} = \Psi^T(x) D \Psi(t), \tag{28}$$

where $D = (d_{ij})_{\widehat{m}\times\widehat{m}}$ is an unknown matrix which should be determined, and $\Psi(\cdot)$ is the vector defined in (15). By integrating (28) from 0 to t, we obtain

$$\frac{\partial^2 T}{\partial x^2} = \left.\frac{\partial^2 T}{\partial x^2}\right|_{t=0} + \Psi^T(x) D P_{\widehat{m}\times\widehat{m}} \Psi(t). \tag{29}$$

Making use of the initial condition (3) enables one to put (29) in the following form:

$$\frac{\partial^2 T}{\partial x^2} = f''(x) + \Psi^T(x) D P_{\widehat{m}\times\widehat{m}} \Psi(t). \tag{30}$$

Then we have

$$\begin{aligned}\frac{\partial^\alpha T}{\partial x^\alpha} &= I_x^{2-\alpha}\left(\frac{\partial^2 T}{\partial x^2}\right) \\ &= I_x^{2-\alpha}\left(\left.\frac{\partial^2 T}{\partial x^2}\right|_{t=0} + \Psi^T(x) D P_{\widehat{m}\times\widehat{m}} \Psi(t)\right) \\ &= I_x^{2-\alpha} f''(x) + \Psi^T(x)\left(P^{2-\alpha}_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t).\end{aligned} \tag{31}$$

By integrating (30) two times from 0 to x, we obtain

$$\begin{aligned}T(x,t) &= T(0,t) + x\left.\frac{\partial T}{\partial x}\right|_{x=0} + f(x) - f(0) \\ &\quad - x f'(0) + \Psi^T(x)\left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t),\end{aligned} \tag{32}$$

and, by putting $x = 1$ in (32), we get

$$\begin{aligned}T(x,t) &= T(0,t) + xH(t) + f(x) - f(0) - x f'(0) \\ &\quad + \Psi^T(x)\left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t),\end{aligned} \tag{33}$$

where

$$\begin{aligned}H(t) &= T(1,t) - T(0,t) + f(0) + f'(0) - f(1) \\ &\quad - \Psi^T(1)\left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t).\end{aligned} \tag{34}$$

By one time differentiation of (33) with respect to t, we obtain

$$\begin{aligned}\frac{\partial T}{\partial t} &= T'(0,t) + xH'(t) \\ &\quad + \Psi^T(x)\left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t),\end{aligned} \tag{35}$$

where

$$\begin{aligned}H'(t) &= T'(1,t) - T'(0,t) \\ &\quad - \Psi^T(1)\left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t).\end{aligned} \tag{36}$$

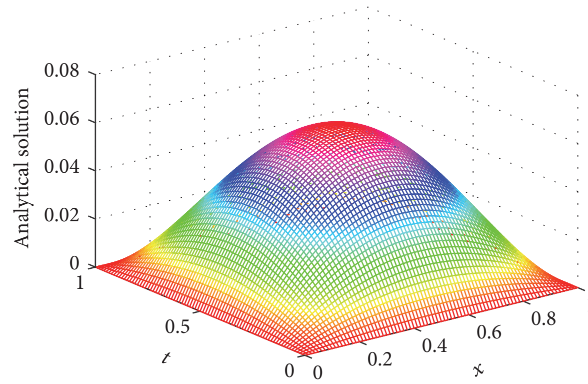

Figure 2: Analytical solution.

Now by substituting (31) and (35) into (2) and combining (4) and taking collocation points $x_i = (2i-1)/\widehat{m}, t_j = (2j-1)/\widehat{m}$, $i, j = 1, 2, 3, \ldots, \widehat{m}$, we obtain the following linear system of algebraic equations:

$$\begin{aligned}&T'(0,t_j) + x_i\Big(T'(1,t_j) - T'(0,t_j) \\ &\quad - \Psi^T(1)\left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t_j)\Big) + \Psi^T(x_i) \\ &\quad \cdot \left(P^2_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t_j) = a I_x^{2-\nu} f''(x_i) \\ &\quad + a\Psi^T(x_i)\left(P^{2-\nu}_{\widehat{m}\times\widehat{m}}\right)^T D P_{\widehat{m}\times\widehat{m}} \Psi(t_j) + g(x_i, t_j), \\ &\qquad i, j, = 1, 2, 3, \ldots, \widehat{m}.\end{aligned} \tag{37}$$

By solving this system to determine D, we can get the numerical solution of this problem by substituting D into (33).

6. Numerical Simulations

In this section, we use the proposed method to solve the initial-boundary problem of nonsteady heat transfer equations. The following numerical examples are given to show the effectiveness and practicability of the proposed method and the results have been compared with the analytical solution.

Example 4. Consider the following fractional-order nonsteady heat transfer model:

$$\frac{\partial T}{\partial t} = \frac{\lambda \partial^{1.5} T}{\rho \bar{c}_p \partial x^{1.5}} + g(x,t), \quad 0 \le x \le 1, \ t \ge 0, \tag{38}$$

where the parameters $\rho = 7500$, $\bar{c}_p = 0.795$, $\lambda = 800$, and $g(x,t) = x(x-1)(2t-1) - 0.302793571044498 x^{0.5} t(t-1)$ with initial-boundary condition $T(x,0) = T(0,t) = T(1,t) = 0$. The analytical solution of this problem is $T(x,t) = xt(x-1)(t-1)$. The graph of the analytical solution is shown in Figure 2. The graphs of the numerical solutions when $k = M = 3$, $k = M = 4$, $k = M = 5$ are shown in Figures 3–5. From Examples 4, 6, and 7, it can be concluded that the numerical solutions approximate to the analytical solution for

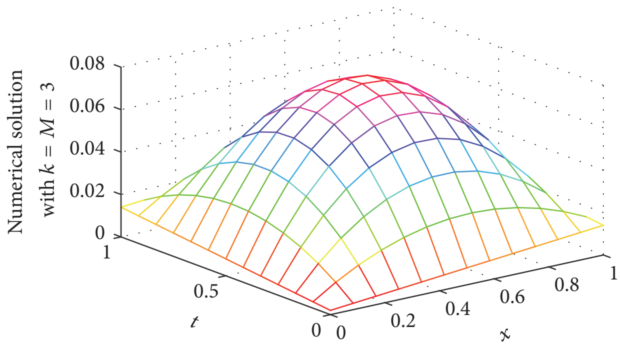

Figure 3: Numerical solution with $k = M = 3$.

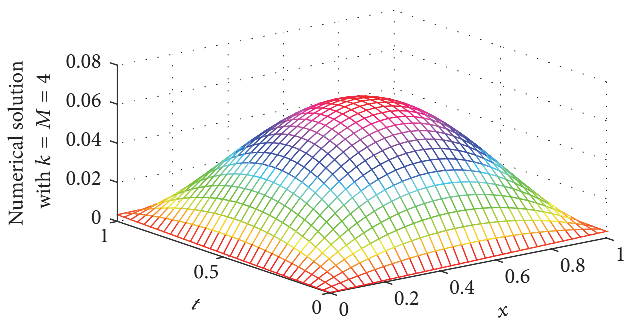

Figure 4: Numerical solution with $k = M = 4$.

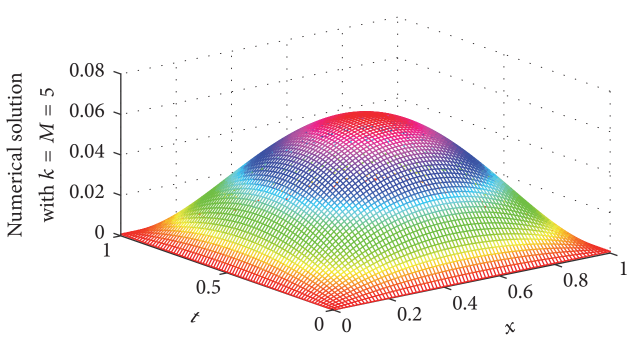

Figure 5: Numerical solution with $k = M = 5$.

a given value k, as M increases, or, for a given value M, as k increases.

Example 5. Consider the following fractional-order non-steady heat transfer equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^{1.8} T}{\partial x^{1.8}} + 2x^2 t - \frac{2x^{0.2}t^2}{\Gamma(1.2)}, \quad 0 \le x \le 1, \ t \ge 0, \tag{39}$$

with initial-boundary condition $T(x,0) = T(0,t) = 0$, $T(1,t) = t^2$. The analytical solution of this problem is $T(x,t) = x^2 t^2$. When $k = M = 3$, $k = M = 4$, $k = M = 5$, the numerical solutions obtained by our method and those obtained by CAS wavelet method at some values of x, t are listed in Table 1.

Example 6. We consider the following second-order non-steady heat transfer model:

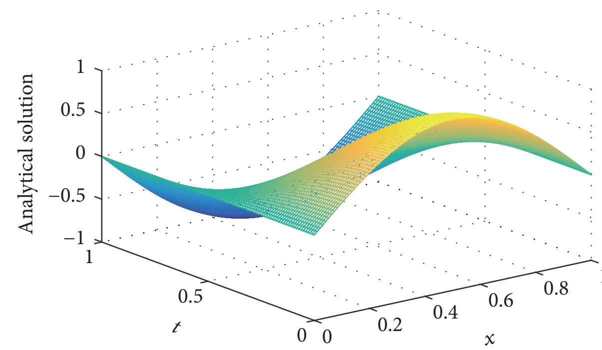

Figure 6: Analytical solution.

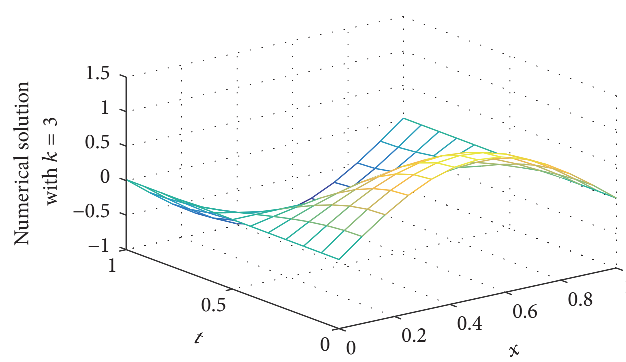

Figure 7: Numerical solution with $k = 3$.

$$\frac{\partial T}{\partial t} = 2\frac{\partial^2 T}{\partial x^2} + 3\sin(x) - \sin(t) - 2\cos(t), \tag{40}$$
$$0 \le x \le 1, \ t > 0,$$

in such a way that $T(x,0) = \sin(x) + 1$, $T(0,t) = \cos(t)$, $T(1,t) = \sin(1) + \cos(t)$. The analytical solution of the system is $T(x,t) = \sin(x) + \cos(t)$. The absolute errors between the numerical and analytical solutions obtained by our method and CAS wavelet method at some values of x, t when $k = 3$, $(M = 3, M = 4, M = 5)$ are shown in Table 2. Table 2 shows that our method has a better approximation than CAS wavelet method.

Example 7. Consider the following second-order nonsteady heat transfer model:

$$\frac{\partial T}{\partial t} = \frac{\lambda \partial^2 T}{\rho c_p \partial x^2} + g(x,t), \quad 0 \le x \le 1, \ t \ge 0, \tag{41}$$

where the parameters $\rho = 7500$, $\bar{c}_p = 0.795$, $\lambda = 1000$, and $g(x,t) = -\pi\sin(\pi x)\sin(\pi t) + 0.167714884696017\pi^2\sin(\pi x)\cos(\pi t)$, in such a way that $T(x,0) = \sin(\pi x)$, $T(0,t) = T(1,t) = 0$. The analytical solution of this problem is $T(x,t) = \sin(\pi x)\cos(\pi t)$. The graphs of the analytical and numerical solutions, when $M = 3$, $(k = 3, 4, 5)$, are shown in Figures 6–9.

Example 8. Consider (41), with $\alpha = 2, 1.9, 1.8, 1.7$; the numerical solutions when $k = M = 4$ at $t = 0.3, 0.6, 0.95$ are shown in Figure 10. This example is introduced to verify

Table 1: The numerical solutions obtained by our method and those obtained by CAS wavelet method when $k = M = 3$, $k = M = 4$, $k = M = 5$.

t	x	Anal. Sol.	$k = M = 3$		$k = M = 4$		$k = M = 5$	
			Our method	CAS wavelet	Our method	CAS wavelet	Our method	CAS wavelet
	0.3	0.0036000	0.00362673	0.01527126	0.00360257	0.00471281	0.00360019	0.00382719
0.2	0.6	0.0144000	0.01445390	0.03638127	0.01440370	0.01673180	0.01440048	0.01492319
	0.9	0.0324000	0.03248217	0.07371928	0.03240631	0.03631963	0.03240060	0.03273187
	0.3	0.0225000	0.02253176	0.04872121	0.02250487	0.02826189	0.02250046	0.02293819
0.5	0.6	0.0900000	0.09061074	0.12739812	0.09006721	0.09537428	0.09000059	0.09072347
	0.9	0.2025000	0.20257431	0.25873179	0.20250850	0.20736183	0.20250074	0.20301829
	0.3	0.0576000	0.05765362	0.09381981	0.05760489	0.60121872	0.05760062	0.05830218
0.8	0.6	0.2304000	0.23048904	0.28237189	0.23040790	0.23833829	0.23040074	0.23138192
	0.9	0.5184000	0.51851904	0.60381038	0.51841027	0.52478172	0.51840112	0.51953785

Table 2: The absolute errors obtained by our method and CAS wavelet method when $M = 3$, $M = 4$, $M = 5$.

(x, t)	Anal. Sol.	$k = 3, M = 3$		$k = 3, M = 4$		$k = 3, M = 5$	
		Our method	CAS wavelet	Our method	CAS wavelet	Our method	CAS wavelet
$(0, 0)$	1.00000000	$1.627162e-4$	$2.381923e-2$	$8.719295e-6$	$8.737819e-4$	$2.319280e-6$	$2.648278e-4$
$(0.1, 0.1)$	1.09483758	$1.738173e-4$	$2.731899e-2$	$5.371912e-6$	$6.271928e-4$	$2.842802e-6$	$3.748217e-4$
$(0.2, 0.2)$	1.17873590	$2.371827e-4$	$3.759289e-2$	$2.361827e-5$	$3.271929e-3$	$4.830209e-6$	$4.684278e-4$
$(0.3, 0.3)$	1.25085669	$2.731872e-4$	$4.542767e-2$	$4.731872e-5$	$4.281912e-3$	$5.371982e-6$	$6.472938e-4$
$(0.4, 0.4)$	1.31047933	$3.261772e-4$	$5.251757e-2$	$5.219289e-5$	$5.381018e-3$	$7.381928e-7$	$7.863982e-4$
$(0.5, 0.5)$	1.35700810	$8.271985e-5$	$4.378391e-2$	$6.319288e-5$	$6.379843e-3$	$6.238299e-6$	$7.635176e-4$
$(0.6, 0.6)$	1.38997808	$4.268278e-4$	$8.373456e-3$	$5.738273e-5$	$5.792808e-3$	$8.302930e-6$	$8.368386e-4$
$(0.7, 0.7)$	1.40905987	$4.791982e-4$	$7.371928e-2$	$7.382093e-5$	$7.728732e-3$	$1.983100e-5$	$9.673817e-4$
$(0.8, 0.8)$	1.41406280	$5.281928e-4$	$6.367643e-2$	$8.382938e-5$	$8.732763e-3$	$9.381098e-6$	$2.371927e-3$
$(0.9, 0.9)$	1.40493687	$6.782916e-4$	$7.371892e-2$	$9.381982e-5$	$9.738273e-3$	$2.381983e-5$	$4.281988e-3$
$(1.0, 1.0)$	1.38177329	$9.381928e-4$	$8.263828e-2$	$9.983787e-5$	$9.425146e-3$	$3.313910e-5$	$8.871999e-4$

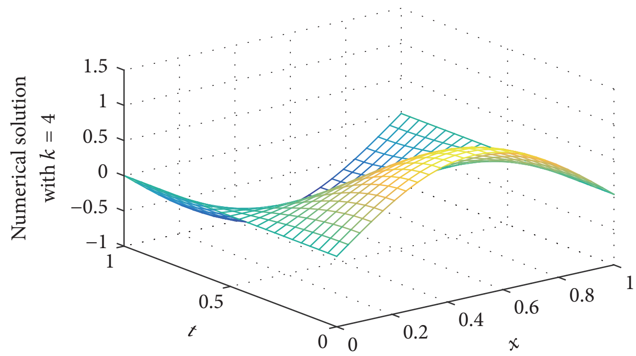

Figure 8: Numerical solution with $k = 4$.

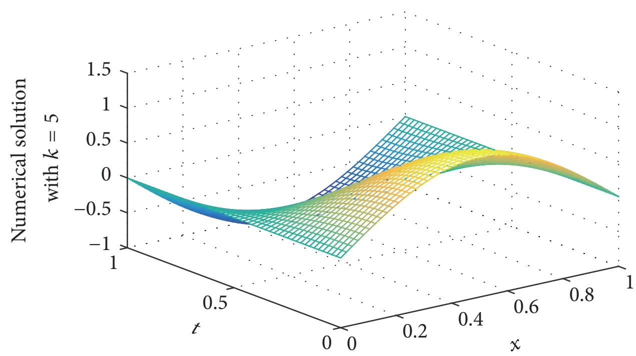

Figure 9: Numerical solution with $k = 5$.

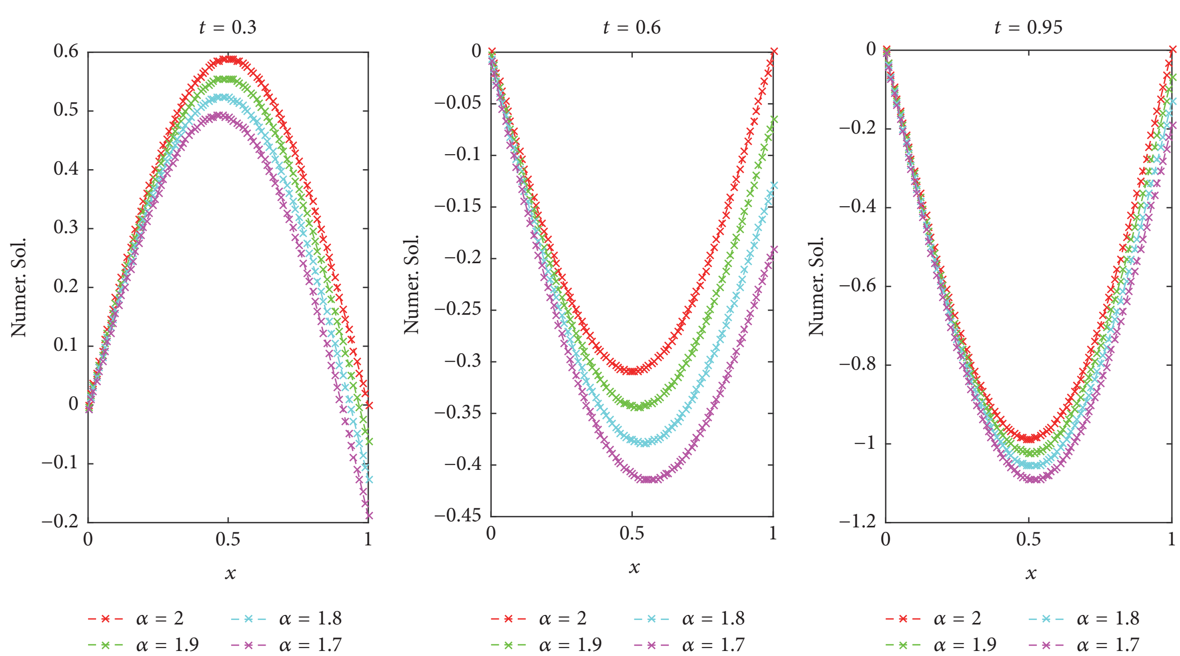

Figure 10: The numerical solutions with $\alpha = 2, 1.9, 1.8, 1.7$ when $k = M = 4$.

the robustness of the proposed method; when the fractional order gradually approaches to 2, the numerical solutions are in agreement with the analytical solution.

7. Conclusions

This paper presents a numerical technique for approximating solutions of one-dimensional fractional nonsteady heat transfer model by combining the second kind Chebyshev wavelet with its operational matrix of fractional-order integration. In the proposed method, a small number of grid points guarantee the necessary accuracy. The main advantage of wavelet method for solving the kinds of equations is that, after dispersing the coefficients, matrix of algebraic equations is sparse. The solution is convenient, even though the size of increment may be large. Several examples are given to demonstrate the powerfulness of the proposed method.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Numerical Analysis of Fractional Order Epidemic Model of Childhood Diseases

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The fractional order Susceptible-Infected-Recovered (SIR) epidemic model of childhood disease is considered. Laplace–Adomian Decomposition Method is used to compute an approximate solution of the system of nonlinear fractional differential equations. We obtain the solutions of fractional differential equations in the form of infinite series. The series solution of the proposed model converges rapidly to its exact value. The obtained results are compared with the classical case.

1. Introduction

Childhood diseases are most serious infectious diseases. Measles, poliomyelitis, and rubella are famous among them [1, 2]. Measles is a highly infectious disease, caused by respiratory infection by a *Morbillivirus*. These diseases normally affect the children, because child population is more prone to the disease as compared to the adults [1]. Therefore, the population can be divided into two major classes: premature and mature populations. Premature population takes a constant time to become mature, which is known as maturation delay. In disease dynamics, a disease cannot spread instantaneously but rather it will take some time in the body which is called latent period for the particular disease. For the control of childhood disease vaccination is a significant strategy being used all over the world. A universal effort to extend vaccination range to all children began in 1974, when the World Health Organization (WHO) founded the Expanded Program on Immunization [3]. Mathematical model plays an important role to comprehend the process of transmission of a disease and provides different techniques to control its propagation. Many mathematicians investigated childhood disease; for instance, Singh et al. [2] studied about vaccination of the childhood diseases. In addition, the authors presented numerical solution of the childhood disease model. Makinde [4] presented a Susceptible-Infected-Recovered model

$$\begin{aligned} \frac{ds}{dt} &= (1-p)\pi - \beta\frac{si}{n} - \pi s, \\ \frac{di}{dt} &= \beta\frac{si}{n} - (\gamma+\pi) i, \\ \frac{dr}{dt} &= p\pi + \gamma i - \pi r. \end{aligned} \tag{1}$$

The authors [1] rearranged model (1) using the relations $s/n = S$, $i/n = I$, and $r/n = R$ and obtained a new (SIR) model in the following form:

$$\begin{aligned} \frac{dS}{dt} &= (1-p)\pi - \beta SI - \pi S, \\ \frac{dI}{dt} &= \beta SI - (\gamma+\pi) I, \\ \frac{dR}{dt} &= p\pi + \gamma I - \pi R. \end{aligned} \tag{2}$$

The description of the preceding model is given below.

The model shows that vaccination is 100 percent efficient and the natural death rate μ is unequal. Therefore the total population size N is not constant. The birth rate is represented by π while the rate of mortality of the childhood disease is very low. The parameter, p, represents a fraction of vaccinated population at birth, where $0 < p < 1$, considering that the rest of population is susceptible. A susceptible individual suffers from the disease through a contact with infected individuals at rate β. Infected individuals recover at a rate γ. There are many applications of fractional calculus [5, 6]. Mathematicians and researchers used fractional calculus to model real world problems. Fractional calculus has greater degree of freedom; therefore it helps to solve nonlinear problems [7, 8]. Also with the help of fractional derivative interdisciplinary applications can be studied. The nonlinear oscillation of earthquake can be modeled with the help of fractional derivative.

In 1980, Laplace–Adomian Decomposition Method (LADM) was introduced by Adomian, which is an effective method for finding the numerical and explicit solution of a system of differential equations representing physical problems. This method works efficiently for solving several kinds of differential equations. It includes nonlinear boundary value/initial value problems. Moreover, we can use it for the partial differential equations; in addition, it can be used to solve a system of stochastic differential equations also. In this method, no perturbation or linearization is required. The advantage of LADM is that it needs no extra memory and does not require any additional parameters, which wastes time and memory.

Studying the literature review of childhood disease model it has been found that Arafa et al. [1] presented the fractional order model for the childhood disease and provided numerical solution by using homotopy analysis method. The reason behind the use of fractional order differential equations (FDEs) is that FDEs are naturally related to systems that involve memory, which can be found in many biological systems. Also, they show the realistic behavior of infection of disease but at a slower rate. Motivated by the applications of fractional calculus and LADM we explore numerical solution of childhood disease model. The Caputo derivative, which is a modification of the Riemann-Liouville definition [9, 10], is considered as a differential operator in our model.

We now gather some well-known definitions and results from the literature, which we will use throughout this paper. For more details, we refer the reader to [1, 7–12].

Definition 1. The Caputo fractional order derivative of a function y on the interval $[0, T]$ is defined by

$$ {}^{c}D^{\alpha}_{0+} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds, \qquad (3)$$

where $n = [\alpha] + 1$ and $[\alpha]$ represents the integer part of α.

Definition 2. We recall the definition of Laplace transform of Caputo derivative as

$$\mathscr{L}\left\{{}^{c}D^{\alpha} y(t)\right\} = s^{\alpha} y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0), \qquad n-1 < \alpha < n,\ n \in N. \qquad (4)$$

2. The Laplace–Adomian Decomposition Method

Using the Caputo fractional derivative system (2) gets the following form:

$$\begin{aligned} {}^{c}D^{\alpha}S(t) &= (1-p)\pi - \beta SI - \pi S, \\ {}^{c}D^{\alpha}I(t) &= \beta SI - (\gamma+\pi) I, \\ {}^{c}D^{\alpha}R(t) &= p\pi + \gamma I - \pi R, \end{aligned} \qquad (5)$$

where $\alpha \in (0, 1]$ while π, γ, p, and β are positive parameters and the given initial conditions are $S(0) = N_1$, $I(0) = N_2$, and $R(0) = N_3$. To solve system (5), we use Laplace–Adomian Decomposition Method (LADM) [5, 13]. Moreover, the obtained solution will be compared with the integer order derivative case. Furthermore, we use Laplace transformation to convert the system of differential equations into a system of algebraic equations. Then, the algebraic equations are used to obtain the required solution in form of series. We will discuss the procedure for solving model (5) with given initial conditions. Applying Laplace transform on both sides of model (5), we obtain the following system:

$$\begin{aligned} \mathscr{L}\left\{{}^{c}D^{\alpha}S(t)\right\} &= \mathscr{L}\left\{(1-p)\pi - \beta SI - \pi S\right\}, \\ \mathscr{L}\left\{{}^{c}D^{\alpha}I(t)\right\} &= \mathscr{L}\left\{\beta SI - (\gamma+\pi) I\right\}, \\ \mathscr{L}\left\{{}^{c}D^{\alpha}R(t)\right\} &= \mathscr{L}\left\{p\pi + \gamma I - \pi R\right\}, \end{aligned} \qquad (6)$$

or

$$\begin{aligned} s^{\alpha}\mathscr{L}\{S(t)\} - s^{\alpha-1}S(0) &= \mathscr{L}\left\{(1-p)\pi - \beta SI - \pi S\right\}, \\ s^{\alpha}\mathscr{L}\{I(t)\} - s^{\alpha-1}I(0) &= \mathscr{L}\left\{\beta SI - (\gamma+\pi) I\right\}, \\ s^{\alpha}\mathscr{L}\{R(t)\} - s^{\alpha-1}R(0) &= \mathscr{L}\left\{p\pi + \gamma I - \pi R\right\}. \end{aligned} \qquad (7)$$

Using the initial conditions (7), we obtain the form

$$\begin{aligned} \mathscr{L}\{S(t)\} &= \frac{S_0}{s} + \left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{(1-p)\pi - \beta SI - \pi S\right\}\right], \\ \mathscr{L}\{I(t)\} &= \frac{I_0}{s} + \left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{\beta SI - (\gamma+\pi) I\right\}\right], \\ \mathscr{L}\{R(t)\} &= \frac{R_0}{s} + \left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{p\pi + \gamma I - \pi R\right\}\right]. \end{aligned} \qquad (8)$$

Assume that the solutions $S(t)$, $I(t)$, and $R(t)$ in the form of infinite series are given by

$$S(t) = \sum_{k=0}^{\infty} S_k(t),$$
$$I(t) = \sum_{k=0}^{\infty} I_k(t), \quad (9)$$
$$R(t) = \sum_{k=0}^{\infty} R_k(t),$$

while the nonlinear term $S(t)I(t)$ is decomposed as follows:

$$S(t)I(t) = \sum_{k=0}^{\infty} A_k(t), \quad (10)$$

where each A_k is the Adomian polynomials defined as

$$A_k = \frac{1}{\Gamma(k+1)} \frac{d^k}{d\lambda^k} \left[\sum_{j=0}^{k} \lambda^j S_j(t) \sum_{j=0}^{k} \lambda^j I_j(t) \right] \Bigg|_{\lambda=0}. \quad (11)$$

The first three polynomials are given by

$$A_0 = S_0(t) I_0(t),$$
$$A_1 = S_0(t) I_1(t) + S_1(t) I_0(t), \quad (12)$$
$$A_2 = 2S_0(t) I_2(t) + 2S_1(t) I_1(t) + 2S_2(t) I_0(t).$$

Substituting (9) and (10) into (8) results in

$$\mathscr{L}\left\{\sum_{k=0}^{\infty} S_k(t)\right\} = \frac{S_0}{s} + \left[\frac{1}{s^\alpha}\mathscr{L}\left\{(1-p)\pi - \beta\sum_{k=0}^{\infty} A_k(t) - \pi\sum_{k=0}^{\infty} S_k(t)\right\}\right],$$
$$\mathscr{L}\left\{\sum_{k=0}^{\infty} I_k(t)\right\} = \frac{I_0}{s} + \left[\frac{1}{s^\alpha}\mathscr{L}\left\{\beta\sum_{k=0}^{\infty} A_k(t) - (\gamma+\pi)\sum_{k=0}^{\infty} I_k(t)\right\}\right], \quad (13)$$
$$\mathscr{L}\left\{\sum_{k=0}^{\infty} R_k(t)\right\} = \frac{R_0}{s} + \left[\frac{1}{s^\alpha}\mathscr{L}\left\{p\pi + \gamma\sum_{k=0}^{\infty} I_k(t) - \pi\sum_{k=0}^{\infty} R_k(t)\right\}\right].$$

Matching the two sides of (13) yields the following iterative algorithm:

$$\mathscr{L}(S_0) = \frac{N_1}{s},$$
$$\mathscr{L}(S_1) = \frac{(1-p)\pi}{s^\alpha} - \frac{\beta}{s^\alpha}\mathscr{L}\{A_0\} - \frac{\pi}{s^\alpha}\mathscr{L}\{S_0\},$$
$$\mathscr{L}(S_2) = \frac{(1-p)\pi}{s^\alpha} - \frac{\beta}{s^\alpha}\mathscr{L}\{A_1\} - \frac{\pi}{s^\alpha}\mathscr{L}\{S_1\},$$
$$\vdots$$
$$\mathscr{L}(S_{k+1}) = \frac{(1-p)\pi}{s^\alpha} - \frac{\beta}{s^\alpha}\mathscr{L}\{A_k\} - \frac{\pi}{s^\alpha}\mathscr{L}\{S_k\}, \quad k \geq 1,$$
$$\mathscr{L}(I_0) = \frac{N_2}{s},$$
$$\mathscr{L}(I_1) = \frac{\beta}{s^\alpha}\mathscr{L}\{A_0\} - \frac{\gamma+\pi}{s^\alpha}\mathscr{L}\{I_0\}, \quad (14)$$
$$\mathscr{L}(I_2) = \frac{\beta}{s^\alpha}\mathscr{L}\{A_1\} - \frac{\gamma+\pi}{s^\alpha}\mathscr{L}\{I_1\},$$
$$\vdots$$
$$\mathscr{L}(I_{k+1}) = \frac{\beta}{s^\alpha}\mathscr{L}\{A_k\} - \frac{\gamma+\pi}{s^\alpha}\mathscr{L}\{I_k\}, \quad k \geq 1,$$
$$\mathscr{L}(R_0) = \frac{N_3}{s},$$
$$\mathscr{L}(R_1) = \frac{p\pi}{s^\alpha} + \frac{\gamma}{s^\alpha}\mathscr{L}\{I_0\} - \frac{\pi}{s^\alpha}\mathscr{L}\{R_0\},$$
$$\vdots$$
$$\mathscr{L}(R_{k+1}) = \frac{p\pi}{s^\alpha} + \frac{\gamma}{s^\alpha}\mathscr{L}\{I_k\} - \frac{\pi}{s^\alpha}\mathscr{L}\{R_k\}, \quad k \geq 1.$$

Taking Laplace inverse of (14) and considering first three terms, we get

$$S_0 = N_1,$$
$$S_1 = (1-p)\pi\frac{t^\alpha}{\Gamma(\alpha+1)} - \beta(N_1N_2 + N_1\pi)\frac{t^\alpha}{\Gamma(t^\alpha+1)} - (1-p)\pi^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$S_2 = (1-p)\pi\frac{t^\alpha}{\Gamma(\alpha+1)} - \beta^2\left(N_1^2N_2 + \beta(\gamma+\pi)N_1N_2\right)\frac{t^2\alpha}{\Gamma(t^2\alpha+1)} - \beta(N_1N_2 + N_1\pi)$$

$$\cdot (\beta N_2 \pi) \frac{t^{2\alpha}}{\Gamma(\alpha+1)} - (1-p)\pi^2(\beta N_2 + \pi) \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$I_0 = N_2,$$

$$I_1 = \beta N_1 N_2 \frac{t^{\alpha}}{\Gamma(t^{\alpha}+1)} - (\gamma+\pi) N_2 \frac{t^{\alpha}}{\Gamma(t^{\alpha}+1)},$$

$$I_2 = \beta N_1 N_2 (\beta N_1 N_2 - (\gamma+\pi) - (\gamma+\pi)\beta N_1 N_2 - (\gamma+\pi) N_2) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \beta N_2 (\beta N_1 N_2 + \pi N_1) \cdot \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left((1-p)\pi^2 - (\beta N_2 + \pi)\right) \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$R_0 = N_3,$$

$$R_1 = (p\pi - \gamma N_2 - \gamma N_3) \frac{t^{\alpha}}{\Gamma(\alpha+1)} - p\pi \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$R_2 = p\pi \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \gamma\left(\beta N_1 N_2 - (\gamma+\pi) + \left(\pi\gamma N_2 - \pi^2 N_3\right)\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - p\pi^2 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \quad (15)$$

3. Numerical Results and Discussion

Using $N_1 = 1$, $N_2 = 0.5$, $N_3 = 0$, $\mu = 0.4$, $\beta = 0.8$, $\gamma = 0.03$, $p = 0.9$, and $\pi = 0.4$, the LADM provides us with an approximate solution in the form of infinite series. Thus we calculate the first four terms of (5) to obtain

$$\begin{aligned} S(t) &= 1 + 0.4400 \frac{t^{\alpha}}{\Gamma(\alpha+1)} - 0.0868 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 0.0490 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ I(t) &= 0.2 + 0.2340 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 0.0923 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 0.0080 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ R(t) &= 1.0800 \frac{t^{\alpha}}{\Gamma(\alpha+1)} - 0.3240 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 0.0336 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \quad (16)$$

For $\alpha = 1$, (16) attains the form

$$\begin{aligned} S(t) &= 1 + 0.4400t - 0.1278000t^2 + 0.0081666668t^3, \\ I(t) &= 0.2 + 0.2340t + 0.0465000t^2 + 0.0003333668t^3, \\ R(t) &= 1.0800t - 0.324000t^2 + 0.0366000t^3. \end{aligned} \quad (17)$$

Similarly, we get the following system for $\alpha = 0.95$:

$$\begin{aligned} S(t) &= 1 + 0.44903t^{0.95} - 0.1398742t^{1.90} + 0.0098280395t^{2.85}, \\ I(t) &= 0.2 + 0.23880453t^{0.95} + 0.050899322t^{1.90} + 0.000449848t^{2.85}, \\ R(t) &= 1.1021503t^{0.95} - 0.35461087t^{1.90} + 0.0400435362t^{2.85}. \end{aligned} \quad (18)$$

Now, for $\alpha = 0.85$, one can obtain the following system:

$$\begin{aligned} S(t) &= 1 + 0.4653075t^{0.85} - 0.16547042t^{1.70} + 0.01394913t^{2.55}, \\ I(t) &= 0.2 + 0.2474402t^{0.85} + 0.06020641t^{1.70} + 0.00078925t^{2.55}, \\ R(t) &= 1.14211848t^{0.85} - 0.41950277t^{1.70} + 0.05738273t^{2.55}. \end{aligned} \quad (19)$$

Similarly, the solution after three terms for $\alpha = 0.75$ is calculated as follows:

$$\begin{aligned} S(t) &= 1 + 0.44787487t^{0.75} - 0.1922758t^{1.50} + 0.0122128708t^{2.55}, \\ I(t) &= 0.2 + 0.254607t^{0.75} + 0.0699959508t^{1.50} + 0.00133334t^{2.55}, \\ R(t) &= 1.1751104t^{0.75} - 0.4874598003t^{1.50} + 0.790818664t^{2.55}. \end{aligned} \quad (20)$$

From the graphical results it is clear that the result obtained by using LADM is very efficient. It also shows that the presented method can predict the behavior of the variables accurately for the region under consideration. It is also clear that the efficiency of this method can be dramatically increased by increasing the terms. Fractional order derivative provides a greater degree of freedom as compared to integer order derivative. The dynamics of various compartments have been shown in Figure 1, Figure 2, and Figure 3, respectively.

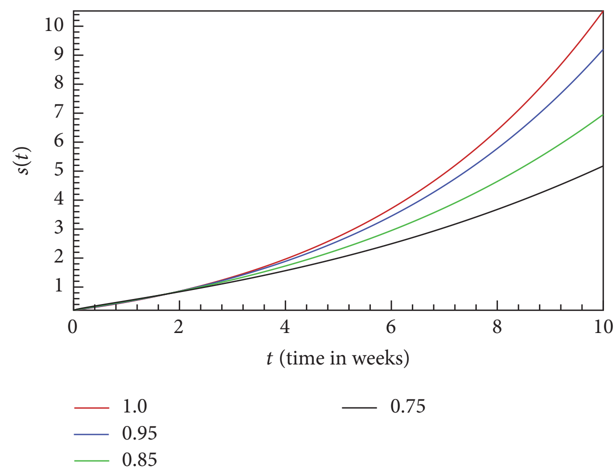

FIGURE 1: Plot of approximate solutions of susceptible class (s) corresponding to different fractional values of α_k for $k = 1, 2, 3$.

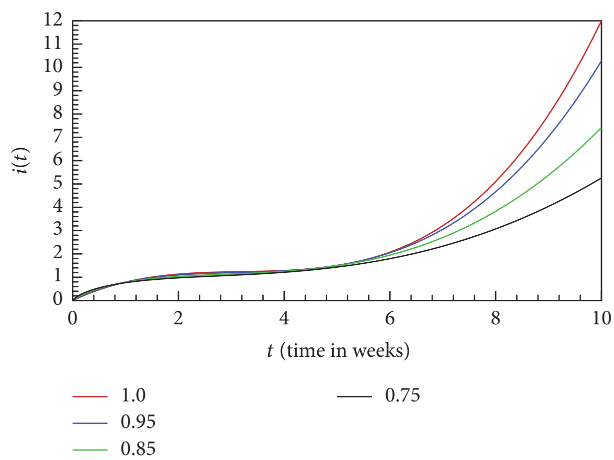

FIGURE 2: Plot of approximate solutions of infected class (i) corresponding to different fractional values of α_k for $k = 1, 2, 3$.

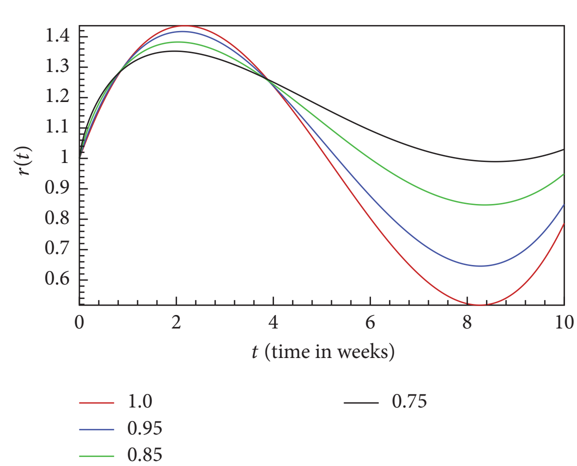

FIGURE 3: Plot of approximate solutions of removed class (r) corresponding to different fractional values of α_k for $k = 1, 2, 3$.

Table 1: Numerical solution of the proposed model using LADM at classical order $\alpha = 1$.

Time (week)	$S(t)$	$I(t)$	$R(t)$
$t = 0$	1.00000	0.20000	0.00
$t = 0.1$	0.57945	0.23226	0.29235
$t = 0.2$	0.34554	0.22790	0.49526
$t = 0.3$	0.22200	0.17660	0.50051
$t = 0.4$	0.15708	0.13220	0.63007
$t = 0.5$	0.12364	0.10420	0.65919
$t = 0.6$	0.10530	0.06465	0.70740
$t = 0.7$	0.19586	0.04311	0.70311
$t = 0.8$	0.10088	0.02750	0.75970
$t = 0.9$	0.08861	0.01640	0.74071

Table 2: Numerical solution of the proposed model using $RK4$ at classical order $\alpha = 1$.

Time (week)	$S(t)$	$I(t)$	$R(t)$
$t = 0$	1.00000	0.20000	0.00
$t = 0.1$	0.57735	0.24227	0.30235
$t = 0.2$	0.35667	0.22792	0.50527
$t = 0.3$	0.23300	0.18672	0.64052
$t = 0.4$	0.16808	0.14221	0.73009
$t = 0.5$	0.13374	0.10423	0.78910
$t = 0.6$	0.11546	0.07485	0.82784
$t = 0.7$	0.10581	0.05317	0.85318
$t = 0.8$	0.10088	0.03756	0.86971
$t = 0.9$	0.09853	0.02646	0.88047

In addition, we give a comparison of $RK4$ and LADM in Tables 1 and 2 for $\alpha = 1$, which shows that both the methods agree for short interval of time. The proposed method is better than $RK4$ method as it needs no extra predefined parameter which controls the method.

4. Convergence Analysis

Solution (16) is in the form of series, which converges uniformly to the exact solution. To check the convergence of series (16), we use techniques (see [14]). For sufficient conditions of convergence of this method, we give the following theorem by using [14].

Theorem 3 (see [13]). *Let $\mathcal{X}$ be the Banach space and $\phi : \mathcal{X} \to \mathcal{X}$ be a contractive nonlinear operator such that, for all $x, x' \in \mathcal{X}$, $\|\phi(x) - \phi(x')\| \le k\|x - x'\|$, $0 < k < 1$.*

By using Banach contraction principles ϕ has a unique point x such that $\phi x = x$, where $x = (S, I, R)$. The series given in (16) can be written by applying Adomian Decomposition Method as follows:

$$x_n = \phi x_{n-1}, \quad x_{n-1} = \sum_{i=1}^{n-1} x_i, \ n = 1, 2, 3, \ldots, \tag{21}$$

and assume that $x_0 = x_0 \in B_r(x)$, where $B_r(x) = x' \in \mathcal{X} : \|x' - x\| < r$; then, we have

(1) $x_n \in B_r(x)$;

(2) $\lim_{n\to\infty} x_n = x$.

Theorem 4 (see [15]). *Let f be an operator from a Hilbert space H into H and y be the exact solution of (5), where $y = (S(t), I(t), R(t))$. $\sum_{i=0}^{\infty} y_i$, which is obtained by (9), converges to y when $\exists$, $0 \le \alpha < 1$ and $\|y_{k+1}\| \le \alpha\|y_k\|$, $\forall k \in NU\{0\}$.*

5. Conclusion

In this paper, we have considered a fractional order childhood disease model. The concern model is investigated for numerical solution by using LADM. The LADM is effective tool to solve nonlinear models and is widely used in engineering and applied mathematics. Also we have provided convergence results for the aforesaid method. It is clear that this method provides good results. One can observe that the method provides excellent numerical solutions for nonlinear fractional order models as compared to other methods like homotopy analysis, homotopy perturbation method, and $RK4$. Since these methods involve an extra parameter h on which the solutions depend, therefore our proposed method needs no parameter and is easy to understand as well as to implement. For initial value problems, our proposed method is better than the PDQM and cubic-spline DQM methods. As PDQM, cubic-spline DQM methods are based on discretization technique which needs extra memory and time as compared to LADM. The method includes a pseudospectral method referred to as the quadrature discretization method (PQDM) and cubic-spline DQM, which are based on nonstandard polynomial basis sets. The aforesaid method is slowly converging as compared to LADM [16].

Conflicts of Interest

The authors declared that no conflicts of interest exist regarding this paper.

Authors' Contributions

All authors have equal contributions and they read and approved the final version of the paper. In the review process Dr. Shakoor Muhammad has provided valuable comments and suggestions and finally he checked and approved the final version. The review process without him was not possible.

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Research Article

Weak Solutions for Partial Random Hadamard Fractional Integral Equations with Multiple Delays

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We present some results concerning the existence of weak solutions for some functional integral equations of Hadamard fractional order with random effects and multiple delays by applying Mönch's and Engl's fixed point theorems associated with the technique of measure of weak noncompactness.

1. Introduction

Random differential equations arise in many applications and have been studied in the literature on bounded as well as unbounded internals of the real line for different aspects of the solution. See, for example, [1]. We refer the reader to the monograph [2] and the papers [1, 3] and the references therein. There are real-world phenomena with anomalous dynamics such as signals transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, and the effect of speculations on the profitability of stocks in financial markets, where the classical models are not sufficiently good to describe these features. In this case, the theory of fractional differential equations is a good tool for modeling such phenomena. Therefore, the study of the fractional differential equations with random parameters seems to be a natural one. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs of Abbas et al. [4, 5], Baleanu et al. [6], and Kilbas et al. [7] and a series of recent research articles [8–12] and the references therein.

The measure of weak noncompactness is introduced by De Blasi [13]. The strong measure of noncompactness was developed first by Banaś and Goebel [14] and subsequently developed and used in many papers; see, for example, Akhmerov et al. [15], Alvàrez [16], and Guo et al. [17] and the references therein. In [18], the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [5]. Existence of random solutions for functional differential and integral equations has extensively been studied in various papers; see [19, 20] and the references therein.

In this paper, we discuss the existence of random solutions for the partial Hadamard fractional integral equation of the following form:

$$\begin{aligned} u(x,y,w) &= \sum_{i=1}^{m} b_i(x,y,w)\,u(x-\xi_i, y-\mu_i, w) \\ &\quad + f\left(x,y,{}^{H}I_{\sigma}^{r}u(x,y,w),u(x,y,w),w\right); \\ &\qquad (x,y)\in J,\ w\in\Omega, \\ u(x,y,w) &= \Phi(x,y,w); \quad \text{if } (x,y)\in\widetilde{J},\ w\in\Omega, \end{aligned} \tag{1}$$

where $J := [1,a]\times[1,b]$, $\widetilde{J} := [-\xi,a]\times[-\mu,b]\setminus(1,a]\times(1,b]$, $a,b > 1$, $\xi_i,\mu_i \geq 1$ $(i = 1,\dots,m)$, $\xi = \max_{i=1,\dots,m}\{\xi_i\}$,

$\mu = \max_{i=1,\dots,m}\{\mu_i\}$, $\sigma = (1,1)$, $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $b_i : J \times \Omega \to \mathbb{R}$ $(i = 1,\dots,m)$, and $f : J \times E \times E \times \Omega \to E$ are given continuous functions, $(\Omega, \mathscr{A}, \nu)$ is a measurable space, and E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, ${}^{H}I_{\sigma}^{r}$ is the left-sided mixed Hadamard integral of order r, and $\Phi : \tilde{J} \times \Omega \to E$ is a given continuous and measurable function such that

$$\Phi(x,1,w) = \sum_{i=1}^{m} b_i(x,1,w)\,\Phi(x-\xi_i, 1-\mu_i, w); \quad x \in [1,a],\ w \in \Omega,$$
$$\Phi(1,y,w) = \sum_{i=1}^{m} b_i(1,y,w)\,\Phi(1-\xi_i, y-\mu_i, w); \quad y \in [1,b],\ w \in \Omega. \tag{2}$$

2. Preliminaries

Let $\mathscr{C}$ be the Banach space of all continuous functions from $[-\xi, a] \times [-\mu, b]$ into E with the supremum (uniform) norm $\|\cdot\|_\infty$. By $L^\infty(\Omega,\nu)$, we denote the Banach space of measurable functions $u : \Omega \to C$ which are essentially bounded equipped with the norm

$$\|u\|_{L^\infty} := \sup_{w\in\Omega} \operatorname{ess} \|u(w)\|_C = \inf\{c > 0 : \|u(w)\|_C \le c\ \nu\text{—a.e. in } \Omega\}. \tag{3}$$

Denote by $(E, w) = (E, \sigma(E, E^*))$ the Banach space E with its weak topology.

Definition 1. A Banach space X is called weakly compactly generated (WCG, in short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w), one has $h(u_n) \to h(u)$ in (E, w)).

Definition 3 (see [21]). The function $u : J \to E$ is said to be Pettis integrable on J if and only if there is an element $u_j \in E$ corresponding to each $j \subset J$ such that $\phi(u_j) = \int\int_j \phi(u(s,t))dt\,ds$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue (by definition, $u_j = \int\int_j u(s,t)dt\,ds$).

Let $P(J,E)$ be the space of all E-valued Pettis integrable functions on J, and let $L^1(J,\mathbb{R})$ be the Banach space of Lebesgue measurable functions $u : J \to \mathbb{R}$. Define the class $P_1(J,E)$ by

$$P_1(J,E) = \{u \in P(J,E) : \varphi(u) \in L^1(J,\mathbb{R});\ \text{for every } \varphi \in E^*\}. \tag{4}$$

The space $P_1(J,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi\in E^*, \|\varphi\|\le 1} \int_1^a \int_1^b |\varphi(u(x,y))|\, d\lambda(x,y), \tag{5}$$

where λ stands for a Lebesgue measure on J.

The following result is due to Pettis (see [21], Theorem 3.4 and Corollary 3.41).

Proposition 4 (see [21]). *If $u \in P_1(J,E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in P_1(J,E)$.*

For all what follows, the sign "$\int$" denotes the Pettis integral.

Let us recall the definitions of Pettis integral and Hadamard integral of fractional order.

Definition 5 (see [7]). The left-sided mixed Pettis Hadamard integral of order $q > 0$, for a function $g \in P_1([1,a],E)$, is defined as

$$\left({}^{H}I_1^{r} g\right)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln\frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds. \tag{6}$$

Remark 6. Let $g \in P_1([1,a],E)$. For every $\varphi \in E^*$, one has

$$\varphi\left({}^{H}I_1^{r} g\right)(x) = \left({}^{H}I_1^{r} \varphi g\right)(x); \quad \text{for a.e. } x \in [1,a]. \tag{7}$$

Definition 7. Let $r_1, r_2 \ge 0$, $\sigma = (1,1)$, and $r = (r_1, r_2)$. For $w \in P_1(J,E)$, define the left-sided mixed Pettis Hadamard partial fractional integral of order r by the expression

$$\left({}^{H}I_\sigma^{r} w\right)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{r_1-1} \left(\ln\frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dt\,ds. \tag{8}$$

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if, for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \in \mathscr{A}. \tag{9}$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 8. A function $T : \Omega \times E \to E$ is called jointly measurable if, for any $B \in \beta_E$, one has

$$T^{-1}(B) := \{(w,v) \in \Omega \times E : T(w,v) \in B\} \in \mathscr{A} \times \beta_E, \tag{10}$$

where $\mathscr{A} \times \beta_E$ is the direct product of the σ-algebras $\mathscr{A}$ and β_E, those defined in Ω and E, respectively.

Lemma 9 (see [22]). *A function $T : \Omega \times E \to E$ is jointly measurable if $T(\cdot,u)$ is measurable for all $u \in E$ and $T(w,\cdot)$ is continuous for all $w \in \Omega$.*

Definition 10. A function $f : J \times E \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(x, y, w) \to f(x, y, u, v, w)$ is jointly measurable for all $u, v \in E$.

(ii) The map $(u, v) \to f(x, y, u, v, w)$ is continuous for all $(x, y) \in J$ and $w \in \Omega$.

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it is expressed as $T(w)u = T(w, u)$. In this case, we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp., compact, totally bounded, and completely continuous) if $T(w, u)$ is continuous (resp., compact, totally bounded, and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [23].

Definition 11 (see [24]). Let $\mathscr{P}(Y)$ be the family of all nonempty subsets of Y and let C be a mapping from Ω into $\mathscr{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega,\ C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w),\ T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for $P-$ almost all $w \in \Omega$, $y(w) \in C(w)$, and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Definition 12 (see [13]). Let E be a Banach space, let Ω_E be the bounded subsets of E, and let B_1 be the unit ball of E. De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : X \subset \epsilon B_1 + \Omega\}. \tag{11}$$

De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$.

(b) $\beta(A) = 0 \Leftrightarrow A$ is weakly relatively compact.

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$.

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$ ($\overline{A}^{\omega}$ denotes the weak closure of A).

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$.

(f) $\beta(\lambda A) = |\lambda|\beta(A)$.

(g) $\beta(\mathrm{conv}(A)) = \beta(A)$.

(h) $\beta(\bigcup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 13. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : J \to E$, let us denote that

$$V(x, y) = \{v(x, y) : v \in V\}; \quad (x, y) \in J,$$
$$V(J) = \{v(x, y) : v \in V,\ (x, y) \in J\}. \tag{12}$$

Lemma 14 (see [17]). *Let $H \subset C$ be bounded and equicontinuous. Then the function $(x, y) \to \beta(H(x, y))$ is continuous on J, and*

$$\beta_C(H) = \max_{(x,y)\in J} \beta(H(x, y)),$$
$$\beta\left(\int\int_J u(s, t)\,dt\,ds\right) \leq \int\int_J \beta(H(s, t))\,dt\,ds, \tag{13}$$

where $H(x, y) = \{u(x, y) : u \in H\}$; $(x, y) \in J$, and β_C is De Blasi measure of weak noncompactness defined on the bounded sets of C.

We will need the following fixed point theorems.

Theorem 15 (see [25]). *Let Q be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose that $T : Q \to Q$ is weakly, sequentially continuous. If the implication*

$$\overline{V} = \overline{\mathrm{conv}}(\{0\} \cup T(V)) \Longrightarrow V \text{ is relatively weakly compact} \tag{14}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

Theorem 16 (see [24]). *Let $C : \Omega \to 2^Y$ be measurable with closed, convex, and solid $C(w)$ (i.e., $\mathrm{int}\, C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists measurable $y_0 : \Omega \to Y$ with $y_0 \in \mathrm{int}\, C(w)$ for all $w \in \Omega$. Let T be a continuous random operator with stochastic domain C such that, for every $w \in \Omega$, $\{y \in C(\omega) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

3. Existence Results

Let us start by defining what we mean by a random solution of problem (1).

Definition 17. By a random solution of problem (1), we mean a measurable function $u : \Omega \to C([-\xi, a]\times[-\mu, b])$ that satisfies the integral equation $u(x, y, w) = \sum_{i=1}^{m} b_i(x, y, w)u(x - \xi_i, y - \mu_i, w) + f(x, y, {}^{H}I_{\sigma}^{r} u(x, y, w), u(x, y, w), w)$ on $J \times \Omega$, as well as $u(x, y, w) = \phi(x, y, w)$ on $\widetilde{J} \times \Omega$.

The following hypotheses will be used in the sequel:

(H_1) The functions $w \mapsto b_i(x, y, w)$, $i = 1, \ldots, m$, are bounded for a.e. $(x, y) \in J$, and $b_i(\cdot, \cdot, w) \in L^{\infty}(J, \mathbb{R})$.

(H_2) The function f is random Carathéodory on $J \times E \times E \times \Omega$ for each $w \in \Omega$.

(H_3) For a.e. $(x, y) \in J$, as well as all $w \in \Omega$, the function $u \to f(x, y, {}^{H}I_{\sigma} u, u, w)$ is weakly, sequentially continuous.

(H_4) There exist functions $p_1, p_2, p_3 : J\times\Omega \to [0,\infty)$ with $p_i(.,w) \in L^\infty(J,[0,\infty))$, $i = 1,2,3$, for each $w \in \Omega$, such that, for all $\varphi \in E*$, one has

$$|\varphi(f(x,y,u,v,w))| \leq \frac{p_1(x,y,w)\|\varphi\| + p_2(x,y,w)\|u\|_E + p_3(x,y,w)\|v\|_E}{1+\|\varphi\|}, \quad (15)$$

for all $u, v \in E$ and a.e. $(x,y) \in J$.

(H_5) For all $u \in E$, there exists a continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$, such that, for each $\varphi \in E^*$, $(x_1,y_1), (x_2,y_2) \in J$ and any $w \in \Omega$, one has

$$\begin{aligned}\sum_{i=1}^{m} &|b_i(x_1,y_1,w)u(x_1-\xi_i,y_1-\mu_i,w) - b_i(x_2,y_2,w) \\ &\cdot u(x_2-\xi_i,y_2-\mu_i,w)| + |\varphi(f(x \\ &-1,y_1,u(x_1,y_1),v(x_1,y_1,w) \\ &- f(x_2,y_2,u(x_2,y_2),v(x_2,y_2),w)))| \\ &\leq \frac{\psi(|x_1-x_2|+|y_1-y_2|)\|\varphi\|}{1+\|\varphi\|+\|u\|_E+\|v\|_E}.\end{aligned} \quad (16)$$

(H_6) There exists a function $q : J\times\Omega \to [0,\infty)$ with $q(\cdot,\cdot,w) \in L^\infty(J,[0,\infty))$ for each $w \in \Omega$ such that, for any bounded $B \subset E$,

$$\alpha\left(f\left(x,y,{}^H I_\sigma B, B, w\right)\right) \leq q(x,y,w)\,\alpha(B), \quad \text{for a.e. } (x,y) \in J, \quad (17)$$

where ${}^H I_\sigma B := \{{}^H I_\sigma u(x,y) : u(x,y) \in B;\ (x,y) \in J\}$.

(H_7) There exists a random function $R : \Omega \to (0,\infty)$ such that

$$p_1^*(w) + (mb^* + p_3^*(w))R(w) + \frac{p_2^*(w)R(w)(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq R(w), \quad (18)$$

where

$$b^* = \max_{i=1,\dots,m}\left\{\operatorname*{ess\,sup}_{(x,y,w)\in J\times\Omega}|b_i(x,y)|\right\}, \qquad p_i^*(w) = \operatorname*{ess\,sup}_{(x,y)\in J} p_i(x,y,w);\quad i=1,2,3. \quad (19)$$

Set

$$q^* = \operatorname*{ess\,sup}_{(x,y,w)\in J\times\Omega} q(x,y,w). \quad (20)$$

Theorem 18. *Assume that hypotheses* (H_1)–(H_7) *hold. If*

$$\ell := mb^* + q^* < 1, \quad (21)$$

then problem (1) has a random solution defined on $[-\xi,a]\times[-\mu,b]$.

Proof. Define the operator $N : \Omega\times\mathscr{C} \to \mathscr{C}$ by

$$\begin{aligned}(N(w)u)(x,y) &= \sum_{i=1}^{m} b_i(x,y,w)u(x-\xi_i,y-\mu_i,w) \\ &\quad + f\left(x,y,{}^H I_\sigma^r u(x,y,w), u(x,y,w), w\right); \quad (x,y) \in J, \\ (N(w)u)(x,y) &= \Phi(x,y,w); \quad (x,y) \in \tilde{J}.\end{aligned} \quad (22)$$

The functions $\Phi, b_i, i = 1,\dots,m$, are continuous for all $w \in \Omega$. Again, as the function f is continuous on J, $N(w)$ defines a mapping $N : \Omega\times\mathscr{C} \to \mathscr{C}$. Thus, u is a solution for problem (1) if and only if $u = (N(w))u$. We shall show that the operator N satisfies all conditions of Theorem 16. The proof will be given in several steps.

Step 1 ($N(w)$ is a random operator with stochastic domain on $\mathscr{C}$). Since $f(x,y,u,v,w)$ is random Carathéodory, the map $w \to f(x,y,u,v,w)$ is measurable in view of Definition 10. Therefore, the map

$$w \longmapsto \sum_{i=1}^{m} b_i(x,y,w)u(x-\xi_i,y-\mu_i,w) + f\left(x,y,{}^H I_\sigma^r u(x,y,w), u(x,y,w), w\right) \quad (23)$$

is measurable. As a result, N is a random operator on $\Omega\times\mathscr{C}\times\mathscr{C}$ into $\mathscr{C}$.

Let $W : \Omega \to \mathscr{P}(C)$ be defined by

$$W(w) = \{u \in C : \|u\|_C \leq R(w),\ \|u(x_1,y_1,w) - u(x_2,y_2,w)\|_E \leq \psi(|x_1-x_2|+|y_1-y_2|)\}. \quad (24)$$

Clearly, the subset $W(w)$ is closed, convex, end equicontinuous for all $w \in \Omega$. Then W is measurable by Lemma 17 in [24]. Therefore, N is a random operator with stochastic domain W.

Step 2 ($N(w)$ is continuous). Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathscr{C}$. Then, there exists $\phi \in E^*$ such that $\|(N(w)u_n)(x,y)\|_E = \phi((N(w)u_n)(x,y))$, and $\|(N(w)u)(x,y)\|_E = \phi((N(w)u)(x,y))$.

Thus, for each $(x,y) \in J$ and $w \in \Omega$, one has

$$\begin{aligned}&\|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \\ &= \phi((N(w)u_n)(x,y) - (N(w)u)(x,y)) \\ &\leq \sum_{i=1}^{m}|b_i(x,y,w)|\,\|u_n(x-\xi_i,y-\mu_i,w) - u(x \\ &- \xi_i, y-\mu_i,w)\|_E \\ &+ \left|\phi\left(f\left(x,y,{}^H I_\sigma^r u_n(x,y,w), u_n(x,y,w), w\right)\right.\right. \\ &\left.\left.- f\left(x,y,{}^H I_\sigma^r u(x,y,w), u(x,y,w), w\right)\right)\right|.\end{aligned} \quad (25)$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\left\|N(w)u_n - N(w)u\right\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{26}$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain W, and $N(w)(W(w))$ is bounded.

Step 3 (for every $w \in \Omega, \{u \in W(w) : N(w)u = u\} \neq \emptyset$). For this, we apply Theorem 15. The proof will be given in several claims.

Claim 1 ($N(w)$ maps $W(w)$ into itself). Let $w \in \Omega$ be fixed, and let $u \in W(w), (x, y) \in J$. Assume that $(N(w)u)(x, y) \neq 0$. Then there exists $\phi \in E^*$ such that $\|(N(w)u)(x, y)\|_E = \phi((N(w)u)(x, y))$. Thus, we get

$$\begin{aligned}
\|(N(w)u)(x,y)\|_E &\leq \left\|\sum_{i=1}^m b_i(x,y,w) \cdot u(x-\xi_i, y-\mu_i, w)\right\|_E \\
&+ \left|\phi\left(f\left(x,y,{}^H I_\sigma^r u(x,y,w), u(x,y,w), w\right)\right)\right| \\
&\leq \sum_{i=1}^m \left|b_i(x,y,w)\right| \left\|u(x-\xi_i, y-\mu_i, w)\right\|_E \\
&+ p_1(x,y,w) + p_2(x,y,w)\left\|{}^H I_\sigma^r u(x,y,w)\right\|_E \\
&+ p_3(x,y,w)\|u(x,y,w)\|_E \leq mb^* \|u\|_\infty + p_1^*(w) \\
&+ \frac{p_2^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \\
&\cdot \|u(s,t,w)\|_E \, dt\, ds + p_3^*(w) R(w) \leq p_1^*(w) \\
&+ \left(mb^* + p_3^*(w)\right) R(w) \\
&+ \frac{p_2^*(w) R(w) (\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq R(w).
\end{aligned} \tag{27}$$

Next, for any fixed $w \in \Omega$, let $(x_1, y_1), (x_2, y_2) \in J$ such that $x_1 < x_2$ and $y_1 < y_2$, and let $u \in W(w)$, where $(N(w)u)(x_1, y_1) - (N(w)u)(x_2, y_2) \neq 0$. Then there exists $\phi \in E^*$ such that $\|(N(w)u)(x_1, y_1) - (N(w)u)(x_2, y_2)\|_E = \phi((N(w)u)(x_1, y_1) - (N(w)u)(x_2, y_2))$ and $\|\varphi\| = 1$. Thus, one has

$$\begin{aligned}
&\left\|(N(w)u)(x_2,y_2) - (N(w)u)(x_1,y_1)\right\|_E \\
&\quad = \phi\left((N(w)u)(x_2,y_2) - (N(w)u)(x_1,y_1)\right) \\
&\quad \leq \psi\left(|x_1 - x_2| + |y_1 - y_2|\right).
\end{aligned} \tag{28}$$

Hence $N(W(w)) \subset W(w)$. Therefore, $N(w) : W(w) \to N(w)$ maps $W(w)$ into itself.

Claim 2 ($N(w)$ is weakly, sequentially continuous). Let (u_n) be a sequence in $W(w)$ and let $(u_n(x, y, w)) \to u(x, y, w)$ in (E, ω) for any $w \in \Omega$, and each $(x, y) \in J$. Fix $(x, y) \in J$, and since f satisfies assumption (H_6), one has that $f(x, y, {}^H I_\sigma u_n(x, y, w), u_n(x, y, w), w)$ converges weakly uniformly to

$$f\left(x, y, {}^H I_\sigma u(x,y,w), u(x,y,w), w\right). \tag{29}$$

Hence, the Lebesgue Dominated Convergence Theorem for Pettis integral implies that $(Nu_n)(x, y, w)$ converges weakly uniformly to $(N(w)u)(x, y)$ in (E, ω). We do it for any $w \in \Omega$, and each $(x, y) \in J$, so $N(w)(u_n) \to N(w)(u)$. Then $N : W(w) \to W(w)$ is weakly, sequentially continuous.

Claim 3 (implication (14) holds). Let V be a subset of $W(w)$ such that $\overline{V} = \overline{\text{conv}}(N(w)(V) \cup \{0\})$. Obviously, $V(x, y, w) \subset \overline{\text{conv}}((N(w)V)(x, y) \cup \{0\})$. Further, as V is bounded and equicontinuous, by Lemma 3 in [26], the function $(x, y, w) \to u(x, y, w) = \beta(V(x, y, w))$ is continuous on $J \times \Omega$. Since the function μ is continuous on $J \times \Omega$, the set $\{\mu(x, y, w);\ (x, y) \in J,\ w \in \Omega\} \subset E$ is compact. From Lemma 14 and the properties of the measure β, for any $w \in \Omega$, as well as each $(x, y) \in J$, one has

$$\begin{aligned}
v(x,y,w) &\leq \beta\left((N(w)V)(x,y) \cup \{0\}\right) \\
&\leq \beta\left((N(w)V)(x,y)\right) \\
&= \beta\left(\sum_{i=1}^m b_i(x,y,w)\, u(x-\xi_i, y-\mu_i, w) + f\left(x,y,{}^H I_\sigma^r u(x,y,w), u(x,y,w), w\right)\right) \\
&\leq \sum_{i=1}^m \left|b_i(x,y,w)\right| \beta\left(V(x,y,w)\right) + q(x,y,w) \\
&\cdot \beta\left(V(x,y,w)\right) \leq mb^* v(x,y,w) + q^* v(x,y,w) \\
&\leq \left(mb^* + q^*\right) \|v\|_C.
\end{aligned} \tag{30}$$

Thus,

$$\|v\|_C \leq \ell \|v\|_C. \tag{31}$$

From (21), we get $\|v\|_C = 0$, that is; $v(x, y, w) = \beta(V(x, y, w)) = 0$, for any $w \in \Omega$, and each $(x, y) \in J$. Hence, Theorem 2 in [27] shows that V is weakly relatively compact in C.

As consequence of Claims 1–3 and from Theorem 15, it follows that, for every $w \in \Omega$, $\{u \in W(w) : N(w)u = u\} \neq \emptyset$. Apply now Theorem 16; Steps 1–3 show that, for each $w \in \Omega$, N has at least one fixed point in W. Since $\bigcap_{w\in\Omega} \operatorname{int} W(w) \neq \emptyset$, and a measurable selector of $\operatorname{int} W$ exists, the operator N has a stochastic fixed point; that is, problem (1) has at least one random solution defined on $[-\xi, a] \times [-\mu, b]$. □

4. An Example

Let

$$E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\} \quad (32)$$

be the Banach space with norm $\|w\|_E = \sum_{n=1}^{\infty} |w_n|$ and let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \to C([-7/2, e] \times [-5, e])$, consider the functional random integral problem of the following form:

$$\begin{aligned} u(x, y, w) &= \frac{x^3 e^{-y}}{17 + w^2} u\left(x - 3, y - \frac{4}{3}, w\right) \\ &\quad + \frac{xy^2}{10 + w^2} u(x - 2, y - 6, w) \\ &\quad + \frac{1}{11 + w^2 + x^2 + y^2} u\left(x - \frac{9}{2}, y - \frac{5}{4}, w\right) \\ &\quad + \frac{w^2 e^{-x-y-3}}{1 + w^2 + |u(x, y, w)| + |{}^H I_\sigma^r u(x, y, w)|}, \\ &\qquad (x, y) \in J = [1, e] \times [1, e], \quad w \in \Omega, \\ u(x, y, w) &= \Phi(x, y, w), \quad (x, y) \in \widetilde{J}, \quad w \in \Omega, \end{aligned} \quad (33)$$

where $\widetilde{J} := [-7/2, e] \times [-5, e] \setminus (1, e] \times (1, e]$, $m = 3$, and $r = (r_1, r_2)$; $r_1, r_2 \in (0, \infty)$, $u = (u_1, u_2, \ldots, u_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, and

$$\begin{aligned} &f_n\left(x, y, {}^H I_\sigma^r u, u, w\right) \\ &= \frac{w^2 e^{-x-y-3}}{1 + w^2 + |u_n(x, y, w)| + |{}^H I_\sigma^r u_n(x, y, w)|}; \end{aligned} \quad (34)$$

$(x, y) \in J = [0, 1] \times [0, 1]$, $w \in \Omega$, $n \in \mathbb{N}$, and $\Phi : \widetilde{J} \to E$ is a continuous and measurable function such that $\phi = (\phi_1, \phi_2, \ldots, \phi_n, \ldots)$, where

$$\begin{aligned} \phi_n(x, 1, w) &= \frac{x^3}{17 + w^2} \phi_n\left(x - 3, -\frac{1}{3}, w\right) \\ &\quad + \frac{1}{11 + w^2 + x^2} \phi_n\left(x - \frac{9}{2}, -\frac{1}{4}, w\right); \\ &\qquad x \in [0, 1], \quad n \in \mathbb{N}, \\ \phi_n(1, y, w) &= \frac{1}{11 + w^2 + y^2} \phi_n\left(-\frac{7}{2}, y - \frac{5}{4}, w\right); \\ &\qquad y \in [0, 1], \quad n \in \mathbb{N}. \end{aligned} \quad (35)$$

Set

$$\begin{aligned} b_1(x, y, w) &= \frac{x^3 e^{-y}}{17}, \\ b_2(x, y) &= \frac{xy^2}{10}, \\ b_3(x, y, w) &= \frac{1}{11 + x^2 + y^2}. \end{aligned} \quad (36)$$

Then, $b^* = 1/10$. For each $u, v \in E$, $(x, y) \in [0, 1] \times [0, 1]$, and $w \in \Omega$, one has

$$\|f(x, y, u, v, w)\|_E \le 1 + \frac{1}{e^3} (\|u\|_E + \|v\|_E). \quad (37)$$

Hence condition (H_4) is satisfied with $p_1^* = 1$ and $p_2^* = p_3^* = 1/e^3$.

A simple computation shows that all other conditions of Theorem 18 are satisfied. Consequently, Theorem 18 implies that problem (33) has a random solution defined on $[-7/2, e] \times [-5, e]$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Modified Function Projective Synchronization for a Partially Linear and Fractional-Order Financial Chaotic System with Uncertain Parameters

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This paper investigates the modified function projective synchronization between fractional-order chaotic systems, which are partially linear financial systems with uncertain parameters. Based on the stability theory of fractional-order systems and the Lyapunov matrix equation, a controller is obtained for the synchronization between fractional-order financial chaotic systems. Using the controller, the error systems converged to zero as time tends to infinity, and the uncertain parameters were also estimated so that the phenomenon of parameter distortion was effectively avoided. Numerical simulations demonstrate the validity and feasibility of the proposed method.

1. Introduction

In recent years, study on chaos is one of the most interesting research topics in real and physical systems [1–4]. In 1985, chaotic behavior was discovered in financial systems [5]. In fact, chaotic phenomena often appear in financial systems [6], such as the supernormal shock of the international oil prices, stock market crash, and financial crisis. Owing to its randomness and unpredictability, these chaotic phenomena usually lead to financial crisis and instability in the financial system. Since financial risks come from uncertainties, it is important to introduce unknown parameters in financial systems [7, 8]. Therefore, chaotic behaviors and uncertainties in financial systems must be taken into account [9]. It is necessary to investigate chaos control strategies for financial systems to address financial crisis and other related problems. Hence, it is important to achieve a synchronized and healthy development of financial markets.

Since Pecora and Carroll [10] proposed synchronization between two chaotic systems in 1990, chaotic synchronization has been extensively and intensively studied in various fields [11, 12]. Different types of chaotic synchronization have been reported, including generalized synchronization [13], lag synchronization [14], function projective synchronization [15], modified projective synchronization [16], and modified function projective synchronization (MFPS) [17, 18]. The modified function projective synchronization (MFPS) means that the drive and response systems could be synchronized up to a scaling function matrix, but not a constant matrix. Recent studies on complicated financial systems have shown satisfactory results using the nonlinear method [19]. Chaos synchronization in fractional-order financial systems has also been studied in recent years [20–22].

The synchronization of financial systems indicates that the financial systems in two different areas are to maintain synchronized development by the appropriate control conditions. That is, the drive system and the response system can be interpreted as a virtual economic target and a controlled objective, respectively. Hence, the goal of synchronization is to control a specific objective to a virtual economic target using the proposed controller. The dynamical behaviors of a financial system are more complex. Therefore, financial systems are not always completely synchronized, and some complex synchronization methods [20–22], such as function projective synchronization and MFPS, should be considered. Meanwhile, because of the complexity of the financial environment, this study highlights its theoretical value and ensures the comprehensiveness of the theory.

Also, in practice, the parameter distortion phenomenon frequently appears in fractional-order chaotic systems. In other words, parameters may be uncertain or drift with time in the chaotic synchronization between the response and drive systems because of various kinds of interferences. Many projective synchronization methods of chaotic systems with unknown parameters have been proposed in recent years [23]. Hence, achieving synchronization and identifying parameters are very important in financial chaotic systems.

In this study, we investigated fractional-order financial systems proposed by Chen [24]. A new controller is proposed to achieve synchronization. The controller makes the drive and response systems remain asymptotically stable. Based on the stability theory of fractional-order systems and the Lyapunov matrix equation, the MFPS was realized for partially linear and fractional-order (PLFO) financial chaotic systems, and the unknown parameters were also estimated so that the phenomenon of parameter distortion was avoided. Numerical simulation results showed that the proposed method effectively eliminated chaos and stabilized two financial systems.

The rest of the paper is organized as follows. In Section 2, the definition of MFPS, the PLFO chaotic system, and a useful lemma are presented. In Section 3, the scheme of MFPS is presented for PLFO chaotic systems with uncertain parameters. Numerical simulations are presented in Section 4. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

The definitions of PLFO chaotic system and coupled PLFO chaotic system were given in [25]. We will give the definition of PLFO chaotic system with uncertain parameters and coupled PLFO chaotic system with uncertain parameters as follows.

Definition 1. The fractional-order chaotic systems with uncertain parameters are defined as follows:

$$\begin{aligned}\frac{\mathrm{d}^q u}{\mathrm{d}t^q} &= M(z,\theta)\cdot u,\\ \frac{\mathrm{d}^q u}{\mathrm{d}t^q} &= f(u,z).\end{aligned} \tag{1}$$

It is called an uncertain PLFO chaotic system, where the state vector is $u = (u_1, u_2, \ldots, u_n)^T \in R^n$, $z \in R$ is a variable, and $f : R \times R^n \rightarrow R$ is a differentiable function. Its order is subject to $0 < q \leq 1$. $\theta \in R^n$ represents the vector of uncertain parameters. The coefficient matrix $M(z,\theta) \in R^{n\times n}$ is dependent on the variable z and the uncertain parameter θ.

System (1) can be written as

$$\begin{aligned}\frac{\mathrm{d}^q u_1}{\mathrm{d}t^q} &= m_{11}(z,\theta)u_1 + m_{12}(z,\theta)u_2 + \cdots\\ &\quad + m_{1n}(z,\theta)u_n,\\ \frac{\mathrm{d}^q u_2}{\mathrm{d}t^q} &= m_{21}(z,\theta)u_1 + m_{22}(z,\theta)u_2 + \cdots\\ &\quad + m_{2n}(z,\theta)u_n,\\ &\vdots\\ \frac{\mathrm{d}^q u_n}{\mathrm{d}t^q} &= m_{n1}(z,\theta)u_1 + m_{n2}(z,\theta)u_2 + \cdots\\ &\quad + m_{nn}(z,\theta)u_n,\\ \frac{\mathrm{d}^q z}{\mathrm{d}t^q} &= f(u,z),\end{aligned} \tag{2}$$

where $m_{ij}(z,\theta)$ is a coefficient of u_j in the ith differential equation, $i, j = 1, 2, \ldots, n$. Hence, $M(z,\theta)$ can be described by

$$M(z,\theta) = \begin{pmatrix} m_{11}(z,\theta) & m_{12}(z,\theta) & \cdots & m_{1n}(z,\theta)\\ m_{21}(z,\theta) & m_{22}(z,\theta) & \cdots & m_{2n}(z,\theta)\\ \vdots & \vdots & \vdots & \vdots\\ m_{n1}(z,\theta) & m_{n2}(z,\theta) & \cdots & m_{nn}(z,\theta)\end{pmatrix}. \tag{3}$$

Obviously, system (1) is an uncertain partially linear and integer-order chaotic system with $q = 1$.

Definition 2. Consider two uncertain PLFO chaotic systems (1), which can be described as

$$\begin{aligned}\frac{\mathrm{d}^q u_r}{\mathrm{d}t^q} &= M(z,\theta)\cdot u_r,\\ \frac{\mathrm{d}^q z}{\mathrm{d}t^q} &= f(u_r,z),\\ \frac{\mathrm{d}^q u_s}{\mathrm{d}t^q} &= \widetilde{M}\left(z,\widetilde{\theta}\right)\cdot u_s + \psi.\end{aligned} \tag{4}$$

The system is coupled by the variable z. The subscripts r and s in system (4) represent the drive and response systems, respectively. $\psi \in R^n$ is a controller. $\widetilde{\theta}$ is the estimated value of unknown parameters. The coefficient matrix $\widetilde{M}(z,\widetilde{\theta}) \in R^{n\times n}$ is dependent on z and $\widetilde{\theta}$. System (4) is the coupled PLFO chaotic system with uncertain parameters [26].

The synchronization error vector of the MFPS is defined as

$$e(t) = u_s - h(t) \cdot u_r, \tag{5}$$

where $h(t) = \mathrm{diag}\{h_1(t), h_2(t), \ldots, h_n(t)\}$, $h_i(t)$ are continuously differentiable functions, and $h(t)$ is a scaling function matrix. $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T \in R^n$ is an error state vector.

Definition 3. System (4) is said to be MFPS with respect to the scaling function matrix $h(t)$ if there exists a vector controller ψ such that

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|u_s - h(t) \cdot u_r\| = 0, \tag{6}$$

where $\|\cdot\|$ is the Euclidean norm. It implies that the MFPS between the drive and response system (4) can be achieved or the error dynamic systems are globally asymptotically stable.

Remark 4. If $h_1(t) = h_2(t) = \cdots = h_n(t)$, the MFPS converts to function projective synchronization. If $h_1(t), h_2(t), \ldots, h_n(t)$ represent identical or different constants, the MFPS is simplified to modified projective synchronization (such as antisynchronization and complete synchronization). If $h_1(t) = h_2(t) = \cdots = h_n(t) = 0$, the synchronization problem is transformed into a control problem.

Lemma 5 (see [27]). *Presume that the fractional-order autonomous system is*

$$\frac{\mathrm{d}^q e}{\mathrm{d}t^q} = A(e) \cdot e, \quad e(0) = e_0, \tag{7}$$

where $q_i \in (0,1]$ $(i = 1, 2, \ldots, n)$ are the orders of the fractional derivative and $q = (q_1, q_2, \ldots, q_n)^T$, $A(e) \in R^{n\times n}$ is a coefficient matrix which is a polynomial matrix and depends on the state vector e, and $e = (e_1, e_2, \ldots, e_n)^T \in R^n$ is an n-dimension state vector. System (7) is asymptotically stable if and only if

$$|\arg(\lambda)| \geq \frac{\pi\alpha}{2}, \quad \alpha = \max(q_1, q_2, \ldots, q_n), \tag{8}$$

where λ is an arbitrary eigenvalue of $A(e)$. In this case, each state vector decays toward 0, such as $t^{-\alpha}$. Furthermore, the system is stable if and only if $|\arg(\lambda)| \geq \pi\alpha/2$ and the critical eigenvalues that satisfy $|\arg(\lambda)| = \pi\alpha/2$ have geometric multiplicity one.

Then, the MFPS between the drive and response system (4) is transformed into the analysis of the asymptotical stability of zero solution of the error system (7).

Theorem 6 (see [28]). *Given the autonomous system (7), there exist a real symmetric positive definite matrix P and a positive definite matrix Q such that the MFPS between the drive and response system (4) can be achieved if $PA(e) + (A(e))^{\mathrm{H}}P = -Q$, where H is the conjugate transpose of a matrix and $PA(e) + (A(e))^{\mathrm{H}}P = -Q$ is called a continuous Lyapunov matrix equation.*

Proof. Assume that λ is one of the eigenvalues of the polynomial matrix $A(e)$ and the corresponding nonzero eigenvector is β; that is,

$$A(e)\beta = \lambda\beta. \tag{9}$$

For (9), the Hermitian transpose is

$$\overline{(A(e)\beta)^T} = \bar{\lambda}\beta^{\mathrm{H}}. \tag{10}$$

Multiplying the left side of (9) by $\beta^{\mathrm{H}}P$, we can obtain

$$\beta^{\mathrm{H}} PA(e)\beta = \lambda\beta^{\mathrm{H}} P\beta. \tag{11}$$

Then, multiplying the right side of (10) by $P\beta$, we can also obtain

$$\beta^{\mathrm{H}} (A(e))^{\mathrm{H}} P\beta = \bar{\lambda}\beta^{\mathrm{H}} P\beta. \tag{12}$$

From (11) and (12), we obtain

$$\beta^{\mathrm{H}} \left(PA(e) + (A(e))^{\mathrm{H}} P\right)\beta = \left(\lambda + \bar{\lambda}\right)\beta^{\mathrm{H}} P\beta. \tag{13}$$

Since $(PA(e) + (A(e))^{\mathrm{H}}P)^{\mathrm{H}} = PA(e) + (A(e))^{\mathrm{H}}P$, then $PA(e) + (A(e))^{\mathrm{H}}P$ is a Hermitian matrix. At the same time, since $PA(e) + (A(e))^{\mathrm{H}}P = -Q$, P is a real symmetric positive definite matrix, and Q is a positive definite matrix, then $\beta^{\mathrm{H}}P\beta > 0$ and $\beta^{\mathrm{H}}(-Q)\beta < 0$, so that

$$\lambda + \bar{\lambda} = \frac{\beta^{\mathrm{H}}(-Q)\beta}{\beta^{\mathrm{H}}P\beta} < 0. \tag{14}$$

From (14), we can obtain $|\arg(\lambda)| > \pi/2 \geq \pi\alpha/2$ $(\alpha \leq 1)$. According to Lemma 5, the equilibrium point of system (7) is asymptotically stable. That is,

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|u_s - h(t) \cdot u_r\| = 0, \tag{15}$$

which indicates that the MFPS between the drive and response system (4) can be achieved. The proof is completed. □

Remark 7. Since $A(e)$ is a polynomial matrix, it is difficult to solve for the eigenvalues. However, based on Theorem 6, the equilibrium point in system (7) is asymptotically stable if and only if $PA(e) + (A(e))^{\mathrm{H}}P = -Q$ is a continuous Lyapunov matrix equation.

3. Designing the Lyapunov Matrix Equation

Next, we introduce a scheme of MFPS for coupled PLFO chaotic systems with uncertain parameters (7).

Theorem 8. *For a given scaling function matrix $h(t)$, we can select a controller (16) and a synchronization error system (17) as follows:*

$$\psi = -h(t)\widetilde{M}\left(z,\widetilde{\theta}\right)u_r - M(z,\theta)e + Ke + \frac{\mathrm{d}^q\left(h(t)u_r\right)}{\mathrm{d}t^q}, \tag{16}$$

$$\frac{\mathrm{d}^q e}{\mathrm{d}t^q} = \left[\widetilde{M}\left(z,\widetilde{\theta}\right) - M(z,\theta) + K\right]e, \tag{17}$$

where $K = \operatorname{diag}\{k_1, k_2, \ldots, k_n\}$, $k_i < 0$, $0 < q \le 1$. The error system (17) can be equivalently written as

$$\begin{aligned}
\frac{\mathrm{d}^q e_1}{\mathrm{d}t^q} &= A_{11}e_{\theta_{11}} + A_{12}e_{\theta_{12}} + \cdots + A_{1l_1}e_{\theta_{1l_1}} + k_1 e_1,\\
\frac{\mathrm{d}^q e_2}{\mathrm{d}t^q} &= A_{2(l_1+1)}e_{\theta_{2(l_1+1)}} + A_{2(l_1+2)}e_{\theta_{2(l_1+2)}} + \cdots\\
&\quad + A_{2l_2}e_{\theta_{2l_2}} + k_2 e_2,\\
&\ \ \vdots\\
\frac{\mathrm{d}^q e_n}{\mathrm{d}t^q} &= A_{n(l_{n-1}+1)}e_{\theta_{n(l_{n-1}+1)}} + A_{n(l_{n-1}+2)}e_{\theta_{n(l_{n-1}+2)}} + \cdots\\
&\quad + A_{nl_n}e_{\theta_{nl_n}} + k_n e_n,
\end{aligned} \tag{18}$$

where $e_{\theta_{ij}} = \widetilde{\theta}_{ij} - \theta_{ij}$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, l_1, l_1 + 1, \ldots, l_{n-1}, l_{n-1} + 1, \ldots, l_n$, where l_1 stands for the number of unknown parameters in the first error equation, and $l_n - l_{n-1}$ stands for the number of unknown parameters in the nth error equation) and A_{ij} is a coefficient of $e_{\theta_{ij}}$ in the ith error equation. Therefore, we can obtain the systems with uncertain parameters as

$$\begin{aligned}
\frac{\mathrm{d}^q e_{\theta_{11}}}{\mathrm{d}t^q} &= -A_{11}e_1 + k'_1 e_{\theta_{11}},\\
\frac{\mathrm{d}^q e_{\theta_{12}}}{\mathrm{d}t^q} &= -A_{12}e_1 + k'_2 e_{\theta_{12}},\\
&\ \ \vdots\\
\frac{\mathrm{d}^q e_{\theta_{1l_1}}}{\mathrm{d}t^q} &= -A_{1l_1}e_1 + k'_{l_1} e_{\theta_{1l_1}},\\
\frac{\mathrm{d}^q e_{\theta_{2(l_1+1)}}}{\mathrm{d}t^q} &= -A_{2(l_1+1)}e_2 + k'_{(l_1+1)} e_{\theta_{2(l_1+1)}},\\
&\ \ \vdots\\
\frac{\mathrm{d}^q e_{\theta_{2l_2}}}{\mathrm{d}t^q} &= -A_{2l_2}e_2 + k'_{l_2} e_{\theta_{2l_2}},\\
&\ \ \vdots\\
\frac{\mathrm{d}^q e_{\theta_{nl_n}}}{\mathrm{d}t^q} &= -A_{nl_n}e_n + k'_{l_n} e_{\theta_{nl_n}},
\end{aligned} \tag{19}$$

where $k'_j < 0$ ($j = 1, 2, \ldots, l_n$). We can achieve the MFPS of system (7) through the error system of (18) and the estimate of the system with unknown parameters (19). That is, the error system and the estimate of unknown parameters are asymptotically stable.

Proof. Assume that $e(t) = u_s - h(t)\cdot u_r$; we have

$$\begin{aligned}
\frac{\mathrm{d}^q e(t)}{\mathrm{d}t^q} &= \frac{\mathrm{d}^q u_s}{\mathrm{d}t^q} - \frac{\mathrm{d}^q\left(h(t)u_r\right)}{\mathrm{d}t^q}\\
&= \widetilde{M}\left(z,\widetilde{\theta}\right)u_s + \psi - \frac{\mathrm{d}^q\left(h(t)u_r\right)}{\mathrm{d}t^q}.
\end{aligned} \tag{20}$$

We select $\psi = -h(t)\widetilde{M}(z,\widetilde{\theta})u_r - M(z,\theta)e + Ke + \mathrm{d}^q(h(t)u_r)/\mathrm{d}t^q$; then it has

$$\begin{aligned}
\frac{\mathrm{d}^q e(t)}{\mathrm{d}t^q} &= \widetilde{M}\left(z,\widetilde{\theta}\right)u_s + \psi - \frac{\mathrm{d}^q\left(h(t)u_r\right)}{\mathrm{d}t^q}\\
&= \widetilde{M}\left(z,\widetilde{\theta}\right)u_s - h(t)\widetilde{M}\left(z,\widetilde{\theta}\right)u_r - M(z,\theta)e\\
&\quad + Ke + \frac{\mathrm{d}^q\left(h(t)u_r\right)}{\mathrm{d}t^q} - \frac{\mathrm{d}^q\left(h(t)u_r\right)}{\mathrm{d}t^q}\\
&= \left[\widetilde{M}\left(z,\widetilde{\theta}\right) - M(z,\theta) + K\right]e.
\end{aligned} \tag{21}$$

Hence, we obtain a controller (16) and a synchronization error system (17).

From (18) and (19), the error system and the estimate of unknown parameters can be simply written as

$$\begin{pmatrix} \dfrac{\mathrm{d}^q e}{\mathrm{d}t^q}\\ \dfrac{\mathrm{d}^q e_\theta}{\mathrm{d}t^q} \end{pmatrix} = A\begin{pmatrix} e\\ e_\theta \end{pmatrix}, \tag{22}$$

where $e = (e_1, e_2, \ldots, e_n)^T$, $e_\theta = (e_{\theta_{11}}, e_{\theta_{12}}, \ldots, e_{\theta_{1l_1}}, e_{\theta_{2(l_1+1)}}, \ldots, e_{\theta_{2l_2}}, \ldots, e_{\theta_{nl_n}})^T$, and

$$A=\begin{pmatrix}
k_1 & 0 & \cdots & 0 & A_{11} & \cdots & A_{1l_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
0 & k_2 & \cdots & 0 & 0 & \cdots & 0 & A_{2(l_1+1)} & \cdots & A_{2l_2} & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & k_n & 0 & \cdots & 0 & 0 & \cdots & 0 & \vdots & A_{n(l_{n-1}+1)} & \cdots & A_{nl_n}\\
-A_{11} & 0 & \cdots & 0 & k'_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \vdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots\\
-A_{1l_1} & 0 & \cdots & 0 & 0 & \cdots & k'_{l_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
0 & -A_{2(l_1+1)} & \cdots & 0 & 0 & \cdots & 0 & k'_{l_1+1} & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & -A_{2l_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & k'_{l_2} & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots\\
0 & 0 & \vdots & -A_{n(l_{n-1}+1)} & 0 & \vdots & 0 & 0 & \cdots & 0 & \cdots & k'_{l_{n-1}+1} & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & -A_{nl_n} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & k'_{l_n}
\end{pmatrix}. \tag{23}$$

Hence, we can obtain $PA + A^{\mathrm{H}}P = -Q$, where P is a real symmetric positive definite matrix and Q is a positive definite matrix. We usually choose $P = E$, where E is an identity matrix. Then,

$$Q = \operatorname{diag}\left\{-2k_1, -2k_2, \ldots, -2k_n, -2k'_1, \ldots, -2k'_{l_n}\right\}. \tag{24}$$

According to Theorem 6, $PA + A^{\mathrm{H}}P = -Q$ is a continuous Lyapunov matrix equation. Meanwhile, we can realize the MFPS of system (7) through controller (16). Therefore, $\lim_{t\to\infty}\|e(t)\| = 0$, which indicates that the MFPS within system (17) can be achieved. The proof is completed. □

Remark 9. In the proof of Theorem 8, it is easy to see that the scaling function matrix $h(t)$ has no direct effect on $\mathrm{d}^q e/\mathrm{d}t^q$ and $\mathrm{d}^q e_\theta/\mathrm{d}t^q$. The results verify the validity and feasibility of the proposed control strategy.

Remark 10. Based on Theorem 8, we can choose the scaling function matrix to realize a variety of synchronizations by the method. For example, if $h_1(t) = h_2(t) = \cdots = h_n(t)$ are continuously differentiable functions, the MFPS converts to function projective synchronization, and if $h(t) = \operatorname{diag}\{h_1(t), h_2(t), \ldots, h_n(t)\}$, $h_i(t)$ $(i = 1, 2, \ldots, n)$ being real constants, the MFPS converts to modified projective synchronization (such as complete synchronization and antisynchronization).

Remark 11. This method has universal applicability if and only if the chaotic system is a PLFO chaotic system (1).

4. Numerical Simulations

Huang and Li [29] proposed a dynamic model of finance comprising three first-order differential equations. Then, following the dynamic model, Chen [24] proposed a fractional-order financial system to describe the running of a financial system as follows:

$$\begin{aligned}
\frac{\mathrm{d}^{q_1} x_1}{\mathrm{d}t^{q_1}} &= nx_3 + mx_1x_2 - ax_1,\\
\frac{\mathrm{d}^{q_2} x_2}{\mathrm{d}t^{q_2}} &= 1 - bx_2 - x_1^2,\\
\frac{\mathrm{d}^{q_3} x_3}{\mathrm{d}t^{q_3}} &= px_1 - cx_3.
\end{aligned} \tag{25}$$

It has three nonlinear equations of fractional-order chaotic systems. The state variables x_1, x_2, x_3 represent the interest rate, the investment demand, and the price index, respectively. Changes in x_1 are mostly influenced by two factors: contradictions from the investment market, which indicate a surplus between investment and savings, and structural adjustment from prices of goods. The changing rate of x_2 is proportional to the rate of investment but inversely proportional to the cost of investment and interest rates. Changes in x_3 are controlled by a contradiction between supply and demand in commercial markets and are affected by inflation rates. Particularly, the negative interest rate x_1, negative investment demand x_2, and negative price index x_3 mean that the banks' deposit rates are below the level

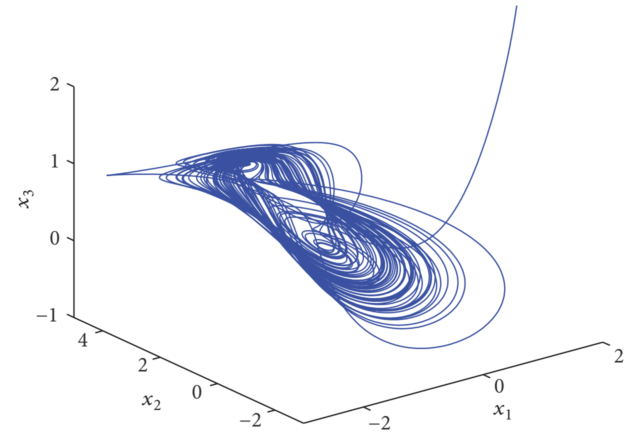

FIGURE 1: Three-dimensional phase diagram of fractional-order financial system (25).

of inflation, the investment market presents overinvestment, and the rate of deflation is increasing, respectively.

The constants a, b, c are representatives of the saving amount, the per-investment cost, and the elasticity of demand of the commercial markets. The fractional-order financial system (25) will exhibit chaotic behavior when $q_i = q \geq 0.85$, $i = 1, 2, 3$, and $n = 1$, $m = 1$, $p = -1$, $a = 3$, $b = 0.1$, and $c = 1$. When $q = 0.9$, the three-dimensional phase diagram of system (25) is illustrated in Figure 1 and the largest Lyapunov exponent is $L = 0.14$. The results highlight the chaotic attractor of fractional-order financial system (25).

It is obvious that system (25) is a PLFO chaotic system. Therefore, the drive and response systems can be constructed as follows:

$$\begin{aligned}
\frac{\mathrm{d}^{q_1} x_1}{\mathrm{d}t^{q_1}} &= nx_3 + mx_1x_2 - ax_1, \\
\frac{\mathrm{d}^{q_3} x_3}{\mathrm{d}t^{q_3}} &= px_1 - cx_3, \\
\frac{\mathrm{d}^{q_2} x_2}{\mathrm{d}t^{q_2}} &= 1 - bx_2 - x_1^2, \\
\frac{\mathrm{d}^{q_1} x_4}{\mathrm{d}t^{q_1}} &= \widetilde{n}x_5 + \widetilde{m}x_4x_2 - \widetilde{a}x_4 + \psi_1, \\
\frac{\mathrm{d}^{q_3} x_5}{\mathrm{d}t^{q_3}} &= \widetilde{p}x_4 - \widetilde{c}x_5 + \psi_2,
\end{aligned} \tag{26}$$

where ψ_i $(i = 1, 2)$ is the controller. In the coupled system (26), the drive system evolves independently, while the response system is governed by the drive system through the variable x_2. That is, the drive system is

$$\begin{aligned}
\frac{\mathrm{d}^{q_1} x_1}{\mathrm{d}t^{q_1}} &= nx_3 + mx_1x_2 - ax_1, \\
\frac{\mathrm{d}^{q_3} x_3}{\mathrm{d}t^{q_3}} &= px_1 - cx_3, \\
\frac{\mathrm{d}^{q_2} x_2}{\mathrm{d}t^{q_2}} &= 1 - bx_2 - x_1^2,
\end{aligned} \tag{27}$$

and the response system is

$$\begin{aligned}
\frac{\mathrm{d}^{q_1} x_4}{\mathrm{d}t^{q_1}} &= \widetilde{n}x_5 + \widetilde{m}x_4x_2 - \widetilde{a}x_4 + \psi_1, \\
\frac{\mathrm{d}^{q_3} x_5}{\mathrm{d}t^{q_3}} &= \widetilde{p}x_4 - \widetilde{c}x_5 + \psi_2.
\end{aligned} \tag{28}$$

System (26) becomes a coupled PLFO chaotic system with uncertain parameters. Based on Theorem 8 and q_i $(i = 1, 2, 3) = q$, we have $u_r = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$, $M(x_2, \theta) = \begin{pmatrix} mx_2 - a & n \\ p & -c \end{pmatrix}$, $\widetilde{M}(x_2, \widetilde{\theta}) = \begin{pmatrix} \widetilde{m}x_2 - \widetilde{a} & \widetilde{n} \\ \widetilde{p} & -\widetilde{c} \end{pmatrix}$, $h(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{pmatrix}$, $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} x_1 - h_1(t)x_4 \\ x_3 - h_2(t)x_5 \end{pmatrix}$, $e_m = \widetilde{m} - m$, $e_a = \widetilde{a} - a$, $e_n = \widetilde{n} - n$, $e_p = \widetilde{p} - p$, and $e_c = \widetilde{c} - c$. According to (16), we can describe the controller as

$$\begin{aligned}
\psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&= -h(t)\widetilde{M}(x_2, \widetilde{\theta}) u_r - M(x_2, \theta) e + Ke \\
&\quad + \frac{\mathrm{d}^q (h(t) u_r)}{\mathrm{d}t^q}.
\end{aligned} \tag{29}$$

From (17) to (19), we can obtain the error systems as

$$\begin{aligned}
\frac{\mathrm{d}^q e_1}{\mathrm{d}t^q} &= x_2e_1e_m + (-e_1)e_a + e_2e_n + k_1e_1, \\
\frac{\mathrm{d}^q e_2}{\mathrm{d}t^q} &= e_1e_p + (-e_2)e_c + k_2e_2.
\end{aligned} \tag{30}$$

It shows that $A_{11} = x_2e_1$, $A_{12} = -e_1$, $A_{13} = e_2$, $A_{24} = e_1$, and $A_{25} = -e_2$. Therefore, the unknown parameters systems are given by

$$\begin{aligned}
\frac{\mathrm{d}^q e_m}{\mathrm{d}t^q} &= -x_2e_1^2 + k_1'e_m, \\
\frac{\mathrm{d}^q e_a}{\mathrm{d}t^q} &= e_1^2 + k_2'e_a, \\
\frac{\mathrm{d}^q e_n}{\mathrm{d}t^q} &= -e_1e_2 + k_3'e_n, \\
\frac{\mathrm{d}^q e_p}{\mathrm{d}t^q} &= -e_1e_2 + k_4'e_p, \\
\frac{\mathrm{d}^q e_c}{\mathrm{d}t^q} &= e_2^2 + k_5'e_c,
\end{aligned} \tag{31}$$

where $K' = \mathrm{diag}(k_1', k_2', k_3', k_4', k_5')$ $(k_i' < 0)$.

The initial conditions are $x_1(0) = -0.5$, $x_2(0) = -0.2$, $x_3(0) = 8$, $x_4(0) = 3$, $x_5(0) = -1$, and $q = 0.9$ and the estimated parameters have initial conditions $\widetilde{n}(0) = -1$, $\widetilde{m}(0) = 3$, $\widetilde{a}(0) = -0.5$, $\widetilde{p}(0) = 3$, $\widetilde{c}(0) = 3.5$, and $k_i = -10$ $(i = 1, 2)$ and $k_i' = -20$ $(i = 1, 2, 3, 4, 5)$. Let the scaling function factors $h_1(t) = 0.1\cos(2t)$ and $h_2(t) = \sqrt[3]{t}$; then the simulation results are shown in Figures 2 and 3. Obviously, in Figure 2, the errors converge to zero as time goes to infinity, which implies that the MFPS of system (26) is achieved with

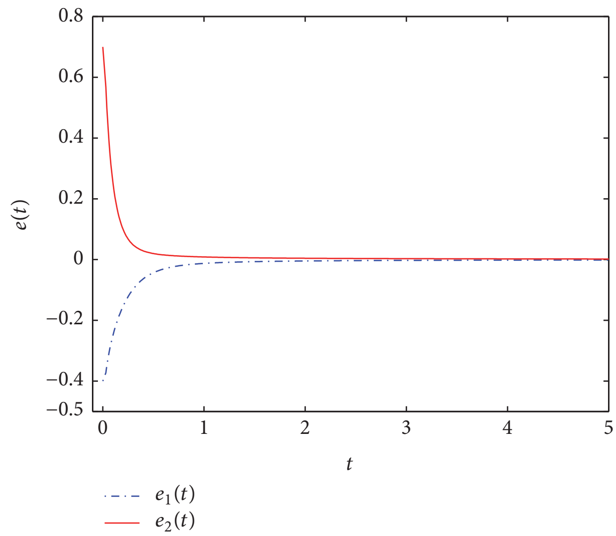

Figure 2: MFPS errors of system (30) with $h_1(t) = 0.1\cos(2t)$ and $h_2(t) = \sqrt[3]{t}$.

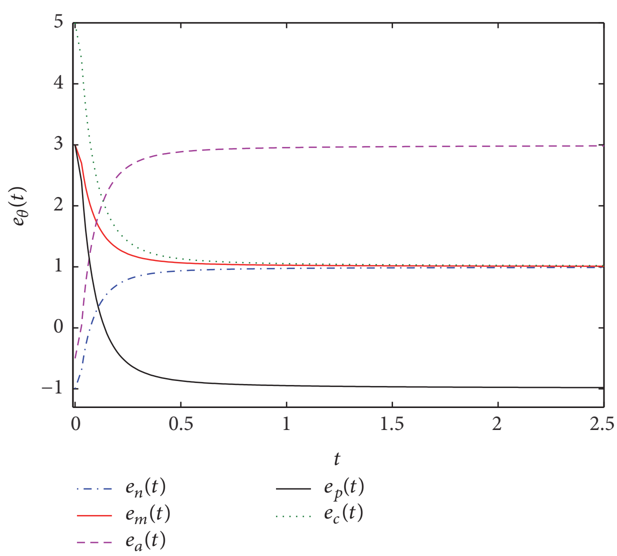

Figure 3: Unknown parameters of system (31) with $h_1(t) = 0.1\cos(2t)$ and $h_2(t) = \sqrt[3]{t}$.

the scaling function factors. Figure 3 shows that the estimated values of unknown parameters $\tilde{n}$, $\tilde{m}$, $\tilde{a}$, $\tilde{p}$, and $\tilde{c}$ converge to $n = 1$, $m = 1$, $a = 3$, $p = -1$, and $c = 1$, respectively, as $t \to \infty$. Therefore, the uncertain parameters n, m, a, p, and c are identified. Additionally, with the scaling function factors $h_1(t) = 0.1\cos(2t)$ and $h_2(t) = \sqrt[3]{t}$, phase portrait of the drive and response systems is illustrated in Figure 4.

5. Conclusion

In this paper, the MFPS was investigated for the coupled uncertain fractional-order financial system. Based on the stability criterion of the fractional-order system and the continuous Lyapunov matrix equation, a synchronization

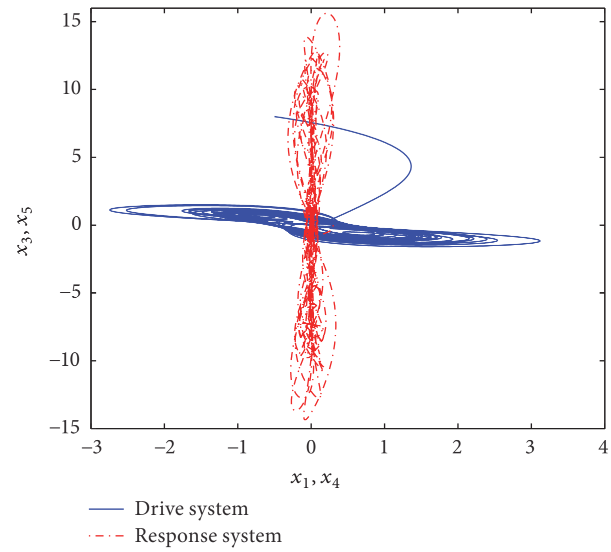

Figure 4: Phase portrait of the drive and response systems in $x_1 - x_3$ $(x_4 - x_5)$ plane with $h_1(t) = 0.1\cos(2t)$ and $h_2(t) = \sqrt[3]{t}$.

controller was proposed. Then, using an uncertain financial system, we verified the validity of the proposed controller. The unknown parameters were also estimated so that the phenomenon of parameter distortion was effectively avoided. The results showed that the controller was feasible and effective. In theory, the proposed approach can realize a synchronized and healthy development of financial markets. In the case of possessing homogeneous investment demand, we can keep profits with price index synchronous development so that interference factors can be avoided.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

Two New Approximations for Variable-Order Fractional Derivatives

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We introduced a parameter $\sigma(t)$ which was related to $\alpha(t)$; then two numerical schemes for variable-order Caputo fractional derivatives were derived; the second-order numerical approximation to variable-order fractional derivatives $\alpha(t) \in (0,1)$ and $3-\alpha(t)$-order approximation for $\alpha(t) \in (1,2)$ are established. For the given parameter $\sigma(t)$, the error estimations of formulas were proven, which were higher than some recently derived schemes. Finally, some numerical examples with exact solutions were studied to demonstrate the theoretical analysis and verify the efficiency of the proposed methods.

1. Introduction

Fractional differential equations include constant-order and variable-order equations; a great quantity of natural phenomena can be modeled by variable-order fractional differential equations; the study of such problems has attracted much attention. In recent years, profound background of physical applications for the variable-order fractional calculus has been already established; the definition of variable-order operator has been investigated in [1–4]. Numerous problems in mathematical physics and engineering have been modeled by variable-order fractional differential equations, such as successful applications in mechanics [5], in the simulation of linear and nonlinear viscoelasticity oscillators [6], and in other cases where the order of the derivative varies with time [7]. In addition, a physical experimental investigation of variable-order operators has been considered in [8]. Variable-order fractional derivatives can be used to model anomalous diffusion, as they can describe the time dependent diffusion process more specifically than fractional derivatives of constant order, just as shown in [9]. The difference schemes of fractional derivatives with constant and variable order are investigated in [10].

Due to the existence of variable fractional derivative, it is usually difficult to obtain the analytical solution of such equations; therefore, it is particularly significant to give numerical solutions to these problems. Cao and Qiu [11] derived a high order numerical method for variable-order fractional ordinary differential equation by establishing a second-order numerical approximation to variable-order Riemann-Liouville fractional derivative. Fu et al. [12] adopted the method of approximate particular solutions for both constant-order and variable-order time fractional diffusion models. Several finite difference methods for variable-order fractional partial diffusion equations were proposed in [13–18].

As what we can see above, there are many different definitions of the variable-order fractional derivatives. However, different from other definitions in mathematics, only initial condition is needed for the variable-order Caputo definition which can be easily used in physical field; this definition means that the memory rate of system is determined by the current time instant and changes with time. For simplicity, only variable-order Caputo definition was discussed in the whole paper.

A good approximation of the variable-order Caputo derivatives was observed in [19], it was defined at the points $t_{n+1/2}$ and t_n; then what will happen if it is redefined in a neighborhood at the point t_n? Approximation formula for the α-order $(0 \le \alpha \le 1)$ Caputo derivatives at the point $t = t_{n-1+\sigma}$ was investigated in [20], where $\sigma = 1 - \alpha/2$ is a superconvergence point which ensures that the convergence order improves from the local second order to the overall $3-\alpha$ order. In the current work, we applied the ideas in [20]; a parameter $\sigma(t)$ was given, which was changed with $\alpha(t)$, and the selection of $\sigma(t)$ played a vital role in the numerical results in the paper. We proposed two new approximation formulas of second-order and $3-\alpha(t)$ accuracy for variable-order time fractional operator with orders $0<\alpha(t)<1$ and $1<\alpha(t)<2$, respectively. Specifically, we adopted the following definition of variable-order Caputo fractional derivatives:

$$ {}_0^C D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha(t)-n+1}} ds, \quad n-1<\alpha(t)<n. \tag{1} $$

The paper is organized as follows. In the next section, we present two new formulas, select $\sigma(t)=1-\alpha(t)/2$ for $0<\alpha(t)<1$ and $\sigma(t)=3/2-\alpha(t)/2$ for $1<\alpha(t)<2$, respectively, and provide the corresponding error analysis. In Section 3, the numerical verifications are presented; as can be seen from the numerical results, minor changes to the selected parameters $\sigma(t)$ will have a significant impact on the error estimates. Finally, in Section 4, the conclusion is drawn.

2. Approximation Formulas

In this section, we present the following lemma, which plays a vital role in the later analysis.

Lemma 1 (see [19]). *Assuming that the derivatives of the function $f(x)$ exist to the order of n on $[a,b]$ and to the order of $n+1$ on (a,b), $a\le x_0<x_1<\cdots<x_n\le b$, $p_n(x)$ is the nth degree interpolation polynomial of $f(x)$ based on the points $x_0,x_1,\ldots,x_n$, then, for $x\in[a,b]$, the following holds:*

$$ f^{(k)}(x)-p_n^{(k)}(x) = \frac{f^{(n+1)}(\xi)}{(n-k+1)!}\left(x-x_0^{(k)}\right)\left(x-x_1^{(k)}\right)\cdots\left(x-x_{n-k}^{(k)}\right), \quad (k=0,1,2,\ldots), \tag{2} $$

where $\xi\in(a,b)$ depends on k and x, and $x_i<x_i^{(k)}<x_{i+k}$ $(i=0,1,\ldots,n-k)$.

For a given function $f(x)$ and an integer $N>0$, denote $t_n=n\tau$, $f^n=f(t_n)$, $n=0,1,\ldots,N$, where $\tau=T/N$ is the step. Introduce the following notation:

$$ \begin{aligned} t_{n+\sigma} &= (n+\sigma)\tau, \\ \delta_t f^{n+1/2} &= \frac{1}{\tau}\left(f^{n+1}-f^n\right), \\ \delta_t^2 f^n &= \frac{\delta_t f^{n+1/2}-\delta_t f^{n-1/2}}{\tau}, \end{aligned} \tag{3} $$

and, for convenience, denote $\sigma=\sigma(t_n)$.

2.1. Second-Order Formula for $\alpha(t)\in(0,1)$. Denote $\alpha_{n-1+\sigma}=\alpha(t_{n-1+\sigma})$, $\sigma=1-\alpha(t_{n-1+\sigma})/2$, $n=0,1,\ldots,N-1$.

Estimating the derivative ${}_0^C D_t^{\alpha(t)} f(t)$ with order $\alpha(t)\in(0,1)$ at the grid point $t_{n-1+\sigma}$, $n=0,1,\ldots,N-1$, from (1), we directly obtain

$$ {}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) = \frac{1}{\Gamma(1-\alpha_{n-1+\sigma})}\int_0^{t_{n-1+\sigma}} \frac{f'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}} ds. \tag{4} $$

Evaluate the integration on each subinterval, leading to

$$ {}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) = \frac{1}{\Gamma(1-\alpha_{n-1+\sigma})}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} \frac{f'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}} ds + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{f'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}} ds\right]. \tag{5} $$

For each interval $[t_{k-1},t_k]$, $k=1,\ldots,n-1$, denote the second-degree interpolation polynomial in the Lagrange form as follows:

$$ \begin{aligned} L_k^2 f(s) &= \frac{f^{k-1}(t-t_k)(t-t_{k+1})}{2\tau^2} \\ &\quad - \frac{f^{k}(t-t_{k-1})(t-t_{k+1})}{\tau^2} \\ &\quad + \frac{f^{k+1}(t-t_{k-1})(t-t_{k})}{2\tau^2}, \end{aligned} \tag{6} $$

and for the last interval $[t_{n-1},t_{n-1+\sigma}]$, denote the first-degree interpolation polynomial in the Lagrange form as

$$ L_n^1 f(s) = -\frac{f^{n-1}(t-t_n)}{\tau}+\frac{f^n(t-t_{n-1})}{\tau}. \tag{7} $$

As an approximation formula, the following result is obtained:

$$ f(s) = L_k^2 f(s) + r_k(s), \quad s\in[t_{k-1},t_k],\ k=1,\ldots,n-1, \tag{8} $$

where

$$ r_k(s) = \frac{f'''(\xi_k)}{6}(s-t_{k-1})(s-t_k)(s-t_{k+1}), \quad \xi_k\in(t_{k-1},t_{k+1}), \tag{9} $$

$$ f(s) = L_n^1 f(s) + r_n(s), \quad s\in[t_{n-1},t_{n-1+\sigma}], \tag{10} $$

where

$$ r_n(s) = \frac{f''(\xi_n)}{2}(s-t_{n-1})(s-t_n), \quad \xi_n\in(t_{n-1},t_n). \tag{11} $$

Substituting (8) and (10) into (5) yields

$$\begin{aligned}&{}_0^CD_t^{\alpha_{n-1+\sigma}} f\left(t_{n-1+\sigma}\right)\\&=\frac{1}{\Gamma\left(1-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{\left(L_k^2 f(s)\right)'}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right.\\&\left.+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\left(L_n^1 f(s)\right)'}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right]+R^{n-1+\sigma},\end{aligned}\tag{12}$$

and the truncation error is

$$\begin{aligned}R^{n-1+\sigma}&=\frac{1}{\Gamma\left(1-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{r_k'(s)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right.\\&\left.+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{r_n'(s)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right].\end{aligned}\tag{13}$$

Here we denote the discrete approximation formula for the variable-order derivative with order $\alpha_{n-1+\sigma}$ as ${}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma}$; thus from (12), it is easy to get the following result:

$$\begin{aligned}{}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma}&=\frac{1}{\Gamma\left(1-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{\left(L_k^2 f(s)\right)'}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\left(L_n^1 f(s)\right)'}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right]\\
&=\frac{1}{\Gamma\left(1-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{(1/2)\left(\delta_t f^{k+1/2}+\delta_t f^{k-1/2}\right)+\delta_t^2 f^k\left(s-t_k\right)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\delta_t f^{n-1/2}}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right]\\
&=\frac{1}{\Gamma\left(1-\alpha_{n-1+\sigma}\right)}\left\{\sum_{k=1}^{n-1}\frac{\left(\delta_t f^{k+1/2}+\delta_t f^{k-1/2}\right)\tau^{1-\alpha_{n-1+\sigma}}}{2\left(1-\alpha_{n-1+\sigma}\right)}\left[\left(n+\sigma-k\right)^{1-\alpha_{n-1+\sigma}}-\left(n+\sigma-k-1\right)^{1-\alpha_{n-1+\sigma}}\right]\right.\\
&\left.+\frac{\delta_t f^{n-1/2}\tau^{1-\alpha_{n-1+\sigma}}}{1-\alpha_{n-1+\sigma}}\sigma^{1-\alpha_{n-1+\sigma}}+\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{\delta_t^2 f^k\left(s-t_k\right)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}}}ds\right\}\\
&=\frac{\tau^{1-\alpha_{n-1+\sigma}}}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\left\{\sum_{k=1}^{n-1}\frac{\left(\delta_t f^{k+1/2}+\delta_t f^{k-1/2}\right)}{2}\left[\left(n+\sigma-k\right)^{1-\alpha_{n-1+\sigma}}-\left(n+\sigma-k-1\right)^{1-\alpha_{n-1+\sigma}}\right]\right.\\
&\left.+\sum_{k=1}^{n-1}\delta_t^2 f^k\tau\left[\frac{1}{2-\alpha_{n-1+\sigma}}\left(\left(n+\sigma-k\right)^{2-\alpha_{n-1+\sigma}}-\left(n+\sigma-k-1\right)^{2-\alpha_{n-1+\sigma}}\right)-\left(n+\sigma-k\right)^{1-\alpha_{n-1+\sigma}}\right]+\delta_t f^{n-1/2}\sigma^{1-\alpha_{n-1+\sigma}}\right\}\\
&=\frac{\tau^{1-\alpha_{n-1+\sigma}}}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=1}^{n-1}a_{n-k}^{(\alpha_{n-1+\sigma})}\left(\delta_t f^{k+1/2}+\delta_t f^{k-1/2}\right)+\sum_{k=1}^{n-1}b_{n-k}^{(\alpha_{n-1+\sigma})}\tau\delta_t^2 f^k+\delta_t f^{n-1/2}\sigma^{1-\alpha_{n-1+\sigma}}\right]\\
&=\frac{\tau^{1-\alpha_{n-1+\sigma}}}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=1}^{n-1}a_{n-k}^{(\alpha_{n-1+\sigma})}\left(\delta_t f^{k+1/2}+\delta_t f^{k-1/2}\right)+\sum_{k=1}^{n-1}b_{n-k}^{(\alpha_{n-1+\sigma})}\left(\delta_t f^{k+1/2}-\delta_t f^{k-1/2}\right)+\delta_t f^{n-1/2}\sigma^{1-\alpha_{n-1+\sigma}}\right]\\
&=\frac{\tau^{1-\alpha_{n-1+\sigma}}}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\left[\sum_{k=2}^{n-1}\left(a_{n-k+1}^{(\alpha_{n-1+\sigma})}+a_{n-k}^{(\alpha_{n-1+\sigma})}+b_{n-k+1}^{(\alpha_{n-1+\sigma})}-b_{n-k}^{(\alpha_{n-1+\sigma})}\right)\delta_t f^{k-1/2}+\left(a_1^{(\alpha_{n-1+\sigma})}+a_0^{(\alpha_{n-1+\sigma})}+b_1^{(\alpha_{n-1+\sigma})}\right)\delta_t f^{n-1/2}\right.\\
&\left.+\left(a_{n-1}^{(\alpha_{n-1+\sigma})}-b_{n-1}^{(\alpha_{n-1+\sigma})}\right)\delta_t f^{1/2}\right]=\frac{\tau^{1-\alpha_{n-1+\sigma}}}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\sum_{k=1}^{n}d_{n-k}^{(\alpha_{n-1+\sigma})}\delta_t f^{k-1/2}=\frac{\tau^{-\alpha_{n-1+\sigma}}}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\sum_{k=1}^{n}d_{n-k}^{(\alpha_{n-1+\sigma})}\left(f^k-f^{k-1}\right),\end{aligned}\tag{14}$$

where

$$\begin{aligned}&a_0^{(\alpha_{n-1+\sigma})}=\sigma^{1-\alpha_{n-1+\sigma}},\\
&a_k^{(\alpha_{n-1+\sigma})}=\frac{1}{2}\left[\left(k+\sigma\right)^{1-\alpha_{n-1+\sigma}}-\left(k+\sigma-1\right)^{1-\alpha_{n-1+\sigma}}\right],\\
&\qquad 1\le k\le n-1,\\
&b_k^{(\alpha_{n-1+\sigma})}\\
&=\frac{1}{2-\alpha_{n-1+\sigma}}\left[\left(k+\sigma\right)^{2-\alpha_{n-1+\sigma}}-\left(k+\sigma-1\right)^{2-\alpha_{n-1+\sigma}}\right]\\
&\quad-\left(k+\sigma\right)^{1-\alpha_{n-1+\sigma}},\quad 1\le k\le n-1.\\
&d_k^{(\alpha_{n-1+\sigma})}\\
&=\begin{cases}a_1^{(\alpha_{n-1+\sigma})}+a_0^{(\alpha_{n-1+\sigma})}+b_1^{(\alpha_{n-1+\sigma})}, & k=0,\\ a_{k+1}^{(\alpha_{n-1+\sigma})}+a_k^{(\alpha_{n-1+\sigma})}+b_{k+1}^{(\alpha_{n-1+\sigma})}-b_k^{(\alpha_{n-1+\sigma})}, & 1\le k\le n-2,\\ a_k^{(\alpha_{n-1+\sigma})}-b_k^{(\alpha_{n-1+\sigma})}, & k=n-1.\end{cases}\end{aligned}\tag{15}$$

Next, the analysis for the approximation error of formula (12) was given in detail.

Theorem 2. *Let $\alpha_{n-1+\sigma} \in (0,1)$, $f \in C^3([0,t_{n+1}])$; the following holds:*

$$\left|{}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) - {}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma}\right| \le \frac{1}{\Gamma(1-\alpha_{n-1+\sigma})}\left[\Theta_n \max_{t_0\le t\le t_{n-1}}\left|f''(t)\right| + \frac{\sigma^{1-\alpha_{n-1+\sigma}}\tau^{-\alpha_{n-1+\sigma}}}{1-\alpha_{n-1+\sigma}} \max_{t_{n-1}\le t\le t_n}\left|f''(t)\right|\right]\tau^2, \tag{16}$$

where $\Theta_1 = \tau^{-\alpha_\sigma}\sigma^{1-\alpha_\sigma}/(1-\alpha_\sigma)$, $\Theta_n = c_1 t_{n-1+\sigma}^{1-\alpha_{n-1+\sigma}}/(1-\alpha_{n-1+\sigma})$, $n = 2,\ldots,N-1$, c_1 is a positive constant.

Proof. From (12), obviously

$$R^{n-1+\sigma} = \frac{1}{\Gamma(1-\alpha_{n-1+\sigma})}\left[\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{r_k'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds + \int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{r_n'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds\right] = \frac{1}{\Gamma(1-\alpha_{n-1+\sigma})}(R_1+R_2). \tag{17}$$

Using Lemma 1, it produces

$$r_k'(s) = \frac{1}{2}f'''(\eta_k)\left(s-t_0^{(k)}\right)\left(s-t_1^{(k)}\right),$$
$$\eta_k \in (t_{k-1},t_{k+1}),\ t_0^{(k)} \in (t_{k-1},t_k),\ t_1^{(k)} \in (t_k,t_{k+1}),\ 1\le k\le n-1, \tag{18}$$

$$r_n'(s) = f''(\eta_n)\left(s-t_0^{(n)}\right),\quad \eta_n \in (t_{n-1},t_n),\ t_0^{(n)} \in (t_{n-1},t_n). \tag{19}$$

Next, the error on each interval is analyzed.

When $n=1$, from (19), it follows that

$$\left|\int_0^{t_\sigma}\frac{r_1'(s)}{(t_\sigma-s)^{\alpha_\sigma}}ds\right| \le \tau\max_{t_0\le t\le t_1}\left|f''(t)\right|\int_0^{t_\sigma}\frac{1}{(t_\sigma-s)^{\alpha_\sigma}}ds = \frac{\tau^{-\alpha_\sigma}\sigma^{1-\alpha_\sigma}}{1-\alpha_\sigma}\max_{t_0\le t\le t_1}\left|f''(t)\right|\tau^2. \tag{20}$$

When $n = 2,\ldots,N-1$, from (18), this leads to the following estimation:

$$\begin{aligned}\left|R_1\right| &= \left|\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{r_k'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds\right| = \left|\sum_{k=1}^{n-1}\frac{1}{2}\int_{t_{k-1}}^{t_k}\frac{f'''(\eta_k)\left(s-t_0^{(k)}\right)\left(s-t_1^{(k)}\right)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds\right| \\ &\le c_1\max_{t_0\le t\le t_{n-1}}\left|f'''(t)\right|\left|\int_{t_0}^{t_{n-1}}\frac{1}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds\right|\tau^2 = \frac{c_1 t_{n-1+\sigma}^{1-\alpha_{n-1+\sigma}}}{1-\alpha_{n-1+\sigma}}\max_{t_0\le t\le t_{n-1}}\left|f'''(t)\right|\tau^2,\end{aligned} \tag{21}$$

where c_1 is a positive constant.

From (19), it can be obtained in the same method.

$$\left|R_2\right| = \left|\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{r_n'(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds\right| \le \tau\max_{t_{n-1}\le t\le t_n}\left|f''(t)\right|\left|\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{1}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}}}ds\right| = \frac{\sigma^{1-\alpha_{n-1+\sigma}}\tau^{-\alpha_{n-1+\sigma}}}{1-\alpha_{n-1+\sigma}}\max_{t_{n-1}\le t\le t_n}\left|f''(t)\right|\tau^2. \tag{22}$$

Substituting (20)–(22) into (17) leads to the theorem. □

2.2. Second-Order Formula for $\alpha(t) \in (1,2)$. Denote $f_0' = f'(t_0)$, $\sigma = 3/2 - \alpha_{n-1+\sigma}/2$. Computing the variable-order derivative with order $\alpha(t)$ $(1<\alpha(t)<2)$ at $t_{n-1+\sigma}$,

$$\begin{aligned}{}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) &= \frac{1}{\Gamma(2-\alpha_{n-1+\sigma})}\cdot\int_{t_0}^{t_{n-1+\sigma}}\frac{f''(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds \\ &= \frac{1}{\Gamma(2-\alpha_{n-1+\sigma})}\left[\int_{t_0}^{t_1}\frac{f''(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds + \sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}}\frac{f''(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds + \int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{f''(s)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds\right].\end{aligned} \tag{23}$$

For the interval $[t_0,t_1]$, the cubic interpolation polynomial in the Hermite form is as

$$L_0^3 f(s) = f^0 + f_0'(s-t_0) + \frac{(1/\tau)\left(f^1-f^0\right)-f_0'}{\tau}(s-t_0)^2 + \frac{1}{2\tau}\left[\frac{(1/\tau)\left(f^1-f^0\right)-f_0'}{\tau} - \frac{(1/\tau)\left(f^2-f^1\right)-(1/\tau)\left(f^1-f^0\right)}{2\tau}\right](s-t_0)^2(s-t_1), \tag{24}$$

and it follows that

$$f(s) = L_0^3 f(s) + r_0(s); \tag{25}$$

the truncation error is

$$r_0(s) = \frac{f^{(4)}(\xi_0)}{24}(s-t_0)^2(s-t_1)(s-t_2),\quad \xi_0 \in (t_0,t_2). \tag{26}$$

For the interval $[t_j, t_{j+1}]$, $j = 1, \ldots, n-2$, the cubic interpolation polynomial at points t_{j-1}, t_j, t_{j+1}, and t_{j+2} in the Lagrange form is as

$$L_j^3 f(s) = \sum_{k=j-1}^{j+2} f^k \prod_{m=j-1, m\neq k}^{j+2} \frac{s-t_m}{t_k - t_m}, \tag{27}$$

leading to

$$f(s) = L_j^3 f(s) + r_j(s); \tag{28}$$

the truncation error is

$$r_j(s) = \frac{f^{(4)}(\xi_j)}{24} \prod_{k=-1}^{2} (s - t_{k+j}), \quad \xi_j \in (t_{j-1}, t_{j+2}); \tag{29}$$

using the cubic interpolation polynomial at points t_{n-2}, t_{n-1}, and t_n in the Lagrange form to approximate $f(s)$ on the last interval $[t_{n-1}, t_{n-1+\sigma}]$, we get

$$\begin{aligned} L_n^2 f(s) = {} & \frac{f^{n-2}(t-t_{n-1})(t-t_n)}{2\tau^2} \\ & - \frac{f^{n-1}(t-t_{n-2})(t-t_n)}{\tau^2} \\ & + \frac{f^n (t-t_{n-2})(t-t_{n-1})}{2\tau^2}, \end{aligned} \tag{30}$$

the following holds:

$$f(s) = L_n^2 f(s) + r_n(s); \tag{31}$$

the truncation error is

$$r_n(s) = \frac{f'''(\xi_n)}{6} (s-t_{n-2})(s-t_{n-1})(s-t_n), \quad \xi_n \in (t_{n-2}, t_n). \tag{32}$$

Substituting (25), (28), and (31) into (23), there is

$$\begin{aligned} {}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) = {} & \frac{1}{\Gamma(2-\alpha_{n-1+\sigma})} \left[\int_{t_0}^{t_1} \frac{\left(L_0^3 f(s) + r_0(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \right. \\ & + \sum_{j=1}^{n-2} \int_{t_j}^{t_{j+1}} \frac{\left(L_j^3 f(s) + r_j(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \\ & \left. + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{\left(L_n^2 f(s) + r_n(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \right]. \end{aligned} \tag{33}$$

Denote the approximation formula of the variable-order derivative with order $\alpha_{n-1+\sigma}$ as ${}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma}$ and obtain

$$\begin{aligned} {}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma} & = \frac{1}{\Gamma(2-\alpha_{n-1+\sigma})} \left[\sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \frac{\left(L_j^3 f(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{\left(L_n^2 f(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \right] \\ & = \frac{1}{\Gamma(2-\alpha_{n-1+\sigma})} \left[\int_{t_0}^{t_1} \frac{\left(L_0^3 f(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds + \sum_{j=1}^{n-2} \int_{t_j}^{t_{j+1}} \frac{\left(L_j^3 f(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \right. \\ & \quad \left. + \int_{t_{n-1}}^{t_{n-1+\sigma}} \frac{\left(L_n^2 f(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \right], \end{aligned} \tag{34}$$

where

$$\begin{aligned} \left(L_0^3 f(s)\right)'' & = \frac{s-t_0}{\tau} \frac{f^2 - 2f^1 + f^0}{\tau^2} + \frac{t_1 - s}{\tau} \frac{8f^1 - f^2 - 7f^0 - 6\tau f_0'}{2\tau^2}, \\ \left(L_j^3 f(s)\right)'' & = \frac{t_{j+1}-s}{\tau} \frac{f^{j-1} - 2f^j + f^{j+1}}{\tau^2} + \frac{s-t_j}{\tau} \frac{f^j - 2f^{j+1} + f^{j+2}}{\tau^2}, \quad 1 \le j \le N-2, \\ \left(L_n^2 f(s)\right)'' & = \delta_t^2 f^{n-1}. \end{aligned} \tag{35}$$

Substituting the above formula into (34), thus

$$\begin{aligned} {}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma} = \frac{1}{\Gamma(2-\alpha_{n-1+\sigma})} \left[\int_{t_0}^{t_1} \frac{((s-t_0)/\tau)\left(\left(f^2 - 2f^1 + f^0\right)/\tau^2\right) + ((t_1-s)/\tau)\left(\left(8f^1 - f^2 - 7f^0 - 6\tau f_0'\right)/2\tau^2\right)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}} ds \right. \end{aligned}$$

$$+\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}}\frac{\left(\left(t_{j+1}-s\right)/\tau\right)\left(\left(f^{j-1}-2f^j+f^{j+1}\right)/\tau^2\right)+\left(\left(s-t_j\right)/\tau\right)\left(\left(f^j-2f^{j+1}+f^{j+2}\right)/\tau^2\right)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds$$
$$\left.+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\delta_t^2 f^{n-1}}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds\right]=\frac{1}{\Gamma\left(2-\alpha_{n-1+\sigma}\right)}\left(I_1+I_2+I_3\right). \quad (36)$$

For the first term in the right hand side of (36), this leads to

$$\begin{aligned}
I_1 &= \int_{t_0}^{t_1}\frac{\left((s-t_0)/\tau\right)\left(\left(f^2-2f^1+f^0\right)/\tau^2\right)+\left((t_1-s)/\tau\right)\left(\left(8f^1-f^2-7f^0-6\tau f_0'\right)/2\tau^2\right)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds \\
&= -\frac{\delta_t^2 f^1}{\tau}\int_{t_0}^{t_1}\left(t_{n-1+\sigma}-s\right)^{2-\alpha_{n-1+\sigma}}ds+(n-1+\sigma)\,\delta_t^2 f^1\int_{t_0}^{t_1}\left(t_{n-1+\sigma}-s\right)^{1-\alpha_{n-1+\sigma}}ds \\
&\quad+\frac{8f^1-f^2-7f^0-6\tau f_0'}{2\tau^3}\int_{t_0}^{t_1}\left(t_{n-1+\sigma}-s\right)^{2-\alpha_{n-1+\sigma}}ds-(n+\sigma-2)\,\frac{8f^1-f^2-7f^0-6\tau f_0'}{2\tau^2}\int_{t_0}^{t_1}\left(t_{n-1+\sigma}-s\right)^{1-\alpha_{n-1+\sigma}}ds \\
&= \frac{\left(3f^2-12f^1+9f^0+6\tau f_0'\right)\tau^{-\alpha_{n-1+\sigma}}}{2\left(2-\alpha_{n-1+\sigma}\right)\left(3-\alpha_{n-1+\sigma}\right)}\left[(n+\sigma-1)^{3-\alpha_{n-1+\sigma}}-(n+\sigma-2)^{3-\alpha_{n-1+\sigma}}\right] \\
&\quad+\frac{\left(8f^1-f^2-7f^0-6\tau f_0'\right)\tau^{-\alpha_{n-1+\sigma}}}{2\left(2-\alpha_{n-1+\sigma}\right)}(n+\sigma-1)^{2-\alpha_{n-1+\sigma}}-\frac{\left(2f^2-4f^1+2f^0\right)\tau^{-\alpha_{n-1+\sigma}}}{2\left(2-\alpha_{n-1+\sigma}\right)}(n+\sigma-2)^{2-\alpha_{n-1+\sigma}}.
\end{aligned} \quad (37)$$

For the second term in the right hand side of (36), the following holds:

$$\begin{aligned}
I_2 &= \int_{t_j}^{t_{j+1}}\frac{\left(\left(t_{j+1}-s\right)/\tau\right)\left(\left(f^{j-1}-2f^j+f^{j+1}\right)/\tau^2\right)+\left(\left(s-t_j\right)/\tau\right)\left(\left(f^j-2f^{j+1}+f^{j+2}\right)/\tau^2\right)}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds \\
&= \frac{\delta_t^2 f^j}{\tau}\int_{t_j}^{t_{j+1}}\frac{t_{j+1}-s}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds+\frac{\delta_t^2 f^{j+1}}{\tau}\int_{t_j}^{t_{j+1}}\frac{s-t_j}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds \\
&= \frac{\left(f^{j+2}-3f^{j+1}+3f^j-f^{j-1}\right)\tau^{-\alpha_{n-1+\sigma}}}{\left(2-\alpha_{n-1+\sigma}\right)\left(3-\alpha_{n-1+\sigma}\right)}\left[(n+\sigma-j-1)^{3-\alpha_{n-1+\sigma}}-(n+\sigma-j-2)^{3-\alpha_{n-1+\sigma}}\right] \\
&\quad+\frac{\left(f^{j+1}-2f^j+f^{j-1}\right)\tau^{-\alpha_{n-1+\sigma}}}{2-\alpha_{n-1+\sigma}}(n+\sigma-j-1)^{2-\alpha_{n-1+\sigma}}-\frac{\left(f^{j+2}-2f^{j+1}+f^j\right)\tau^{-\alpha_{n-1+\sigma}}}{2-\alpha_{n-1+\sigma}}(n+\sigma-j-2)^{2-\alpha_{n-1+\sigma}}.
\end{aligned} \quad (38)$$

For the third term in the right hand side of (36), this yields

$$\begin{aligned}
I_3 &= \int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\delta_t^2 f^{n-1}}{\left(t_{n-1+\sigma}-s\right)^{\alpha_{n-1+\sigma}-1}}ds \\
&= \frac{\left(f^n-2f^{n-1}+f^{n-2}\right)\sigma^{2-\alpha_{n-1+\sigma}}}{2-\alpha_{n-1+\sigma}}\tau^{-\alpha_{n-1+\sigma}}.
\end{aligned} \quad (39)$$

Substituting (37)–(39) into (36), we obtain the approximate difference scheme of the ${}_{0}^{C}D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma})$:

$${}_{0}\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma}=\frac{\tau^{-\alpha_{n-1+\sigma}}}{\Gamma\left(3-\alpha_{n-1+\sigma}\right)}\left[\sum_{j=0}^{n-2}a_{n-j}^{(\alpha_{n-1+\sigma})}\left(f^{j+2}\right.\right.$$

$$-2f^{j+1}+f^{j}\Big)+\sum_{j=1}^{n-2}b_{n-j}^{(\alpha_{n-1+\sigma})}\left(f^{j+1}-2f^{j}+f^{j-1}\right)$$
$$+\frac{1}{2}b_{n}^{(\alpha_{n-1+\sigma})}\left(8f^{1}-f^{2}-7f^{0}-6\tau f_{0}'\right)$$
$$+\sigma^{2-\alpha_{n-1+\sigma}}\left(f^{n}-2f^{n-1}+f^{n-2}\right)\Bigg],\qquad(40)$$

where

$$a_{j}^{(\alpha_{n-1+\sigma})}=\frac{1}{3-\alpha_{n-1+\sigma}}\left[(j+\sigma-1)^{3-\alpha_{n-1+\sigma}}-(j+\sigma-2)^{3-\alpha_{n-1+\sigma}}\right]-(j+\sigma-2)^{2-\alpha_{n-1+\sigma}},\quad 1\le j\le n,\qquad(41)$$

$$b_{j}^{(\alpha_{n-1+\sigma})}=(j+\sigma-1)^{2-\alpha_{n-1+\sigma}}-\frac{1}{3-\alpha_{n-1+\sigma}}\left[(j+\sigma-1)^{3-\alpha_{n-1+\sigma}}-(j+\sigma-2)^{3-\alpha_{n-1+\sigma}}\right],\quad 2\le j\le n-1.\qquad(42)$$

Similar to the proof of the Theorem 2, the truncation error of formula (40) is given by following theorem.

Theorem 3. *For $\alpha_{n-1+\sigma}\in(1,2)$, $f\in C^{4}([0,t_{n+1}])$, the following holds:*

$$\left|{}_{0}^{C}D_{t}^{\alpha_{n-1+\sigma}}f(t_{n-1+\sigma})-{}_{0}\Delta_{t}^{\alpha_{n-1+\sigma}}f^{n-1+\sigma}\right|\le\frac{1}{\Gamma(3-\alpha_{n-1+\sigma})}\left(t_{n-1+\sigma}^{2-\alpha_{n-1+\sigma}}\max_{t_0\le t\le t_{n-1}}\left|f^{(4)}(t)\right|\tau^{2}+c_2\sigma^{2-\alpha_{n-1+\sigma}}\max_{t_{n-1}\le t\le t_n}\left|f'''(t)\right|\tau^{3-\alpha_{n-1+\sigma}}\right),\qquad(43)$$

where c_2 is a positive constant.

Proof. From (33), we obtain the truncation error

$$R^{n-1+\sigma}=\frac{1}{\Gamma(2-\alpha_{n-1+\sigma})}\left[\sum_{j=0}^{n-2}\int_{t_j}^{t_{j+1}}\frac{\left(r_j^{3}(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds+\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{\left(r_n^{2}(s)\right)''}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds\right].\qquad(44)$$

From Lemma 1, the result below is natural.

$$\left[r_0^{3}(s)\right]''=\frac{f^{(4)}(\eta_0)}{2}\left(s-t_0^{(0)}\right)\left(s-t_1^{(0)}\right),\quad \eta_0\in(t_0,t_2),\ t_0^{(0)}\in(t_0,t_1),\ t_1^{(0)}\in(t_0,t_2),$$
$$\left[r_j^{3}(s)\right]''=\frac{f^{(4)}(\eta_j)}{2}\left(s-t_0^{(j)}\right)\left(s-t_1^{(j)}\right),\quad \eta_j\in(t_{j-1},t_{j+2}),\ t_0^{(j)}\in(t_{j-1},t_{j+1}),\ t_1^{(j)}\in(t_j,t_{j+2}),$$
$$\left[r_n^{2}(s)\right]''=f'''(\eta_n)\left(s-t_0^{(n)}\right),\quad \eta_n\in(t_{n-2},t_n),\ t_0^{(n)}\in(t_{n-2},t_n).\qquad(45)$$

Substituting (45) into (44), it yields the estimation

$$\left|\int_{t_0}^{t_1}\frac{\left(f^{(4)}(\eta_0)/2\right)\left(s-t_0^{(0)}\right)\left(s-t_1^{(0)}\right)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds\right|\le\max_{t_0\le t\le t_2}\left|f^{(4)}(t)\right|\left|\int_{t_0}^{t_1}(t_{n-1+\sigma}-s)^{1-\alpha_{n-1+\sigma}}ds\right|\tau^{2}\le\frac{t_{n-1+\sigma}^{2-\alpha_{n-1+\sigma}}}{2-\alpha_{n-1+\sigma}}\max_{t_0\le t\le t_2}\left|f^{4}(t)\right|\tau^{2},$$
$$\left|\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}}\frac{\left(f^{(4)}(\eta_j)/2\right)\left(s-t_0^{(j)}\right)\left(s-t_1^{(j)}\right)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds\right|\le2\max_{t_0\le t\le t_{n-1}}\left|f^{(4)}(t)\right|\left|\int_{t_1}^{t_{n-1}}(t_{n-1+\sigma}-s)^{1-\alpha_{n-1+\sigma}}ds\right|\tau^{2}\le\frac{t_{n-1+\sigma}^{2-\alpha_{n-1+\sigma}}}{2-\alpha_{n-1+\sigma}}\max_{t_1\le t\le t_{n-1}}\left|f^{4}(t)\right|\tau^{2},\qquad(46)$$
$$\left|\int_{t_{n-1}}^{t_{n-1+\sigma}}\frac{f'''(\eta_n)\left(s-t_0^{(n)}\right)}{(t_{n-1+\sigma}-s)^{\alpha_{n-1+\sigma}-1}}ds\right|\le c_2\max_{t_{n-1}\le t\le t_n}\left|f'''(t)\right|\left|\int_{t_{n-1}}^{t_{n-1+\sigma}}(t_{n-1+\sigma}-s)^{1-\alpha_{n-1+\sigma}}ds\right|\tau\le\frac{c_2\sigma^{2-\alpha_{n-1+\sigma}}}{2-\alpha_{n-1+\sigma}}\max_{t_{n-1}\le t\le t_n}\left|f'''(t)\right|\tau^{3-\alpha_{n-1+\sigma}},$$

where c_2 is a positive constant.

Substituting (46) into (44) leads to the theorem. □

Remark 4. From the second term of the right hand of the truncation error estimate in Theorem 2, we can easily obtain the fact that formula (14) does not have exact second-order accuracy; it is related to the selection of $\alpha(t)$ and σ. This can be seen from Table 2 in the numerical example below; its accuracy is much higher than the second order.

3. Numerical Verification

In this section, the validity and numerical accuracy of the new presented formula (14) and (40) are demonstrated, respectively. Meanwhile, the corresponding computational results with the formula of [19] are given for contrast.

Table 1: Maximum errors and convergence rates for Example 1 at $T = 1$ and $\sigma = 1 - \alpha_1/2$.

$\alpha(t)$	τ	$E^N(\tau)$ (14)	Rate (14)	$E^N(\tau)$ [19]	Rate [19]
	1/40	$1.400e-3$	2.0753	$1.726e-3$	1.9354
	1/80	$3.560e-4$	1.9755	$4.512e-4$	1.9421
$\dfrac{t^2+1}{2}$	1/120	$1.576e-5$	2.0121	$2.053e-4$	1.9438
	1/160	$8.844e-5$	2.0091	$1.173e-4$	1.9468
	1/200	$5.653e-5$	—	$7.600e-5$	—

Table 2: Maximum errors and convergence rates for Example 1 at $T = 2$ and $\sigma = 1 - \alpha_2/2$.

$\alpha(t)$	τ	$E^N(\tau)$ (14)	Rate (14)	$E^N(\tau)$	Rate [19]
	1/40	$5.830e-4$	2.5859	$4.834e-2$	1.9724
	1/80	$9.138e-5$	2.6734	$1.232e-2$	1.9843
e^{-t}	1/120	$3.021e-5$	2.7145	$5.509e-3$	1.9889
	1/160	$1.362e-5$	2.7462	$3.109e-3$	1.9914
	1/200	$7.285e-6$	—	$1.993e-3$	—

Take a positive integer N, let $T = t_{n-1+\sigma}$, $\tau = T/N$, and denote

$$F^{n-1+\sigma} = {}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}),$$
$$f^{n-1+\sigma} = {}_0\Delta_t^{\alpha_{n-1+\sigma}} f^{n-1+\sigma}, \qquad 0 \le n \le N,$$
$$E^N(\tau) = \left|F^{n-1+\sigma} - f^{n-1+\sigma}\right|,$$
$$\text{Rate} = \log_2 \frac{E^N(\tau)}{E^N(\tau/2)}. \tag{47}$$

Example 1 (accuracy of formula (14) and [19]). Take $f(t) = t^5 - t^3$, $0 < t \le T$. Compute the Caputo fractional derivative of $f(t)$ at $T = 1, 2$ numerically.

The exact solution is given by

$$ {}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) = \frac{\Gamma(6)}{\Gamma(6-\alpha_{n-1+\sigma})} t^{5-\alpha_{n-1+\sigma}} - \frac{\Gamma(4)}{\Gamma(4-\alpha_{n-1+\sigma})} t^{3-\alpha_{n-1+\sigma}}. \tag{48}$$

From the results presented in Table 1, taking $\sigma = 1 - \alpha_1/2 = 1/2$, we found that although both formula (14) and [19] have second-order approximation accuracy, the numerical accuracy by the formula (14) is significantly higher than that by the formula of [19]; furthermore, the computational errors are also obviously smaller than the formula of [19]. Thus, the new formula (14) is valid for solving complex problems that require high accuracy, and better computational results can be obtained from the formula.

In Table 2, taking $\sigma = 1 - \alpha_2/2 = 0.9323$, it shows that the convergence order is higher as compared with the theoretical result, and the error is smaller than the theoretical result, but why this happens is not clear yet; it requires further investigation.

Moreover, on the basis of Table 1, if a slight disturbance to σ is added to Tables 3 and 4, the experimental results will be greatly different. It can be easily found that there is a very close relationship between the convergence order and the selection of σ. The experimental results show that the convergence order is the ideal result only when $\sigma = 1 - \alpha_1/2$. Tables 2, 5, and 6 also illustrate this fact.

Example 2 (accuracy of formula (40)). Take $f(t) = t^5$, $f'(0) = 0$, $0 < t \le T$. Compute the Caputo fractional derivative of $f(t)$ at $T = 1$ numerically.

The exact solution is given by

$$ {}_0^C D_t^{\alpha_{n-1+\sigma}} f(t_{n-1+\sigma}) = \frac{\Gamma(6)}{\Gamma(6-\alpha_{n-1+\sigma})} t^{5-\alpha_{n-1+\sigma}}. \tag{49}$$

Table 7 lists the maximum errors and convergence rates for the variable-order $\sin(t) + 1$. In particular, when $T = 1$, $\alpha(t_{n-1+\sigma}) = \sin(1) + 1 = 1.8415$, and $\sigma = 0.5793$, the numerical results demonstrate that formula (40) leads to $3 - \alpha(t)$ accuracy.

Similar to Example 1, we also give Tables 8 and 9 as a comparison; the results show that $\sigma = 3/2 - \alpha_1/2$ is the best choice.

Remark 3. Due to the fact that the selection of σ is very important, it is worth noting that different selected σ leads to the different numerical accuracy. From considerable trials, we observed that only if we take $\sigma = 1-\alpha(t)/2$ for $\alpha(t) \in (0, 1)$ and $\sigma = 3/2-\alpha(t)/2$ for $\alpha(t) \in (1, 2)$, will numerical results be closer to theoretical analysis. In addition, carefully observing Table 9, although the convergence order is not higher than in Table 7, the error is lower than in Table 7; further investigation is needed on this phenomenon.

4. Conclusion

In this paper, two approximations to variable-order Caputo fractional derivatives were developed, and the analysis for the truncation errors of new formulas was made. In addition, numerical examples support theoretical results. In further work, we also want to give a strict proof of the selection of

Table 3: Maximum errors and convergence rates for Example 1 at $T = 1$ and $\sigma_1 = \sigma - 0.1$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
$\frac{t^2+1}{2}$	1/40	$3.720e-2$	1.0847
	1/80	$1.800e-2$	1.0473
	1/120	$1.190e-2$	1.0300
	1/160	$8.900e-3$	1.0161
	1/200	$7.100e-3$	1.0101

Table 4: Maximum errors and convergence rates for Example 1 at $T = 1$ and $\sigma_2 = \sigma + 0.1$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
$\frac{t^2+1}{2}$	1/40	$3.380e-2$	0.9412
	1/80	$1.720e-2$	0.9746
	1/120	$1.150e-2$	0.9874
	1/160	$8.700e-3$	0.9833
	1/200	$7.000e-3$	0.9792

Table 5: Maximum errors and convergence rates for Example 1 at $T = 2$ and $\sigma_3 = \sigma + 0.1$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
e^{-t}	1/40	$3.070e-2$	1.8040
	1/80	$8.600e-3$	1.8358
	1/120	$4.100e-3$	1.8323
	1/160	$2.400e-3$	1.8413
	1/200	$1.600e-3$	1.8329

Table 6: Maximum errors and convergence rates for Example 1 at $T = 2$ and $\sigma_4 = \sigma - 0.23$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
e^{-t}	1/40	$5.390e-2$	1.8808
	1/80	$1.470e-2$	1.8745
	1/120	$6.900e-3$	1.8688
	1/160	$4.000e-3$	1.8777
	1/200	$2.600e-3$	1.8995

Table 7: Maximum errors and convergence rates for Example 2 at $T = 1$ and $\sigma = 3/2 - \alpha_1/2$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
$\sin(t) + 1$	1/40	$4.023e-1$	1.1310
	1/80	$1.821e-1$	1.1435
	1/160	$8.210e-2$	1.1493
	1/320	$3.690e-2$	1.1538
	1/640	$1.660e-2$	1.1524
	1/1280	$7.400e-3$	1.1656

Table 8: Maximum errors and convergence rates for Example 2 at $T = 1$ and $\sigma_5 = \sigma + 0.2$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
$\sin(t) + 1$	1/40	$5.644e-1$	1.1219
	1/80	$2.562e-1$	1.1394
	1/160	$1.156e-1$	1.1481
	1/320	$5.200e-2$	1.1526
	1/640	$2.33e-2$	1.1582
	1/1280	$1.050e-2$	1.1499

TABLE 9: Maximum errors and convergence rates for Example 2 at $T = 1$ and $\sigma_6 = \sigma - 0.3$.

$\alpha(t)$	τ	$E^N(\tau)$	Rate
$\sin(t) + 1$	1/40	$1.705e-1$	1.1216
	1/80	$7.760e-2$	1.1356
	1/160	$3.510e-2$	1.1446
	1/320	$1.580e-2$	1.1515
	1/640	$7.100e-3$	1.1540
	1/1280	$3.200e-3$	1.1497

σ and apply these new methods to solve more complex cases within a reasonable accuracy.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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